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On the Optimal Regularity of Weak Geodesics in the Space of Metrics on a Polarized Manifold

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To the memory of Mikael Passare

Abstract. Let (X, L) be a polarized compact manifold, i.e., L is an ample line bundle over X and denote by $\mathcal{H}$ the infinite-dimensional space of all positively curved Hermitian metrics on L equipped with the Mabuchi metric. In this short note we show, using Bedford–Taylor type envelope techniques developed in the authors previous work [3], that Chen's weak geodesic connecting any two elements in $\mathcal{H}$ are $C^{1,1}$-smooth, i.e., the real Hessian is bounded, for any fixed time t, thus improving the original bound on the Laplacians due to Chen. This also gives a partial generalization of Blocki's refinement of Chen's regularity result. More generally, a regularity result for complex Monge–Ampère equations over $X \times D$, for D a pseudoconvex domain in $\mathbb{C}^n$ is given.

1. Introduction

Let X be an n-dimensional compact complex manifold equipped with a Kähler form ω and denote by $[\omega]$ the corresponding cohomology class in $H^2(X, \mathbb{R})$. The space of all Kähler metrics in $[\omega]$ may be identified with the space $\mathcal{H}(X, \omega)$ of all Kähler potentials, modulo constants, i.e., the space of all functions u on X such that

$$\omega_u := \omega + dd^c u, \quad \left(dd^c := \frac{i}{2\pi} \partial\bar{\partial} \right)$$

is positive, i.e., defines a Kähler form on X. Mabuchi introduced a natural Riemannian metric on $\mathcal{H}(X, \omega)$ [21], where the squared norm of a tangent vector $v \in C^\infty(X)$ at u is defined by

$$g_{|u}(v, v) := \int_X v^2 \omega_u^n \tag{1.1}$$

The main case of geometric interest is when the cohomology class $[\omega]$ is integral, which equivalently means that it can be realized as the first Chern class

$c_1(L)$ of an ample line bundle L over the projective algebraic manifold X. Then the space $\mathcal{H}(X,\omega)$ may be identified with the space $\mathcal{H}(L)$ of all positively curved metrics ϕ on the line bundle L and as pointed by Donaldson [15] the space $\mathcal{H}(L)$ may then be interpreted as the symmetric space dual of the group $\mathrm{Ham}(X,\omega)$ of Hamiltonian diffeomorphisms of (X,ω). Under this (formal) correspondence the geodesics in $\mathcal{H}(X,\omega)$ correspond to one-parameter subgroups in the (formal) complexification of $\mathrm{Ham}(X,\omega)$ and this motivated Donaldson's conjecture concerning the existence of geodesics in $\mathcal{H}(X,\omega)$, connecting any two given elements.

However, Donaldson's existence problem has turned out to be quite subtle. In fact, according to the recent counter-examples in [20, 11] the existence of bona fide geodesic segments fails in general. On the other hand, there always exists a (unique) *weak* geodesic u_t connecting given points u_0 and u_1 in $\mathcal{H}(X,\omega)$ defined as follows. First recall that, by an important observation of Semmes [23] and Donaldson [15], after a complexification of the variable t, the geodesic equation for u_t on $X\times[0,1]$ may be written as the following complex Monge–Ampère equation on a domain $M:=X\times D$ in $X\times\mathbb{C}$ for the function $U(x,t):=u_t(x)$:

$$(\pi^*\omega+dd^cU)^{n+1}=0. \tag{1.2}$$

As shown by Chen [9], with complements by Blocki [8], for any smoothly bounded domain D in $\mathbb{C}$ the corresponding boundary value problem on M admits a unique solution U such that $\pi^*\omega+dd^cU$ is a positive current with coefficients in L^∞, satisfying the equation 1.2 almost everywhere. In particular, when D is an annulus in $\mathbb{C}$ this construction gives rise to the notion of a weak geodesic curve u_t in the space of all functions u such that ω_u is a positive current with coefficients in L^∞ (the latter regularity equivalently means that the Laplacian of u is in L^∞). In particular, by standard linear elliptic estimates, U is "almost $C^{1,1}$" in the sense that U is in the Hölder class $C^{1,\alpha}$ for any $\alpha<1$. As shown by Blocki [8], in the case when X admits a Kähler metric with non-negative holomorphic bisectional curvature Chen's regularity result can be improved to give that U is $C^{1,1}$-smooth. However, the assumption on X appearing in Blocki's result is very strong and essentially implies that X is a homogeneous manifold. In this short note we point out that, in the case when the given Kähler class $[\omega]$ is an integral the function u_t on X is in general, for any fixed t, in $C^{1,1}(X)$, i.e., its first derivatives are Lipschitz continuous. More precisely, the real Hessian of u_t has bounded coefficients with a bound which is independent of t :

Theorem 1.1. *For any integral Kähler class $[\omega]$ the weak geodesic u_t connecting any two points u_0 and u_1 in the space $\mathcal{H}(X,\omega)$ of ω-Kähler potentials has the property that, for any fixed t, the function u_t is in $C^{1,1}(X)$. More precisely, the upper bound on the sup norm on X of the real Hessian of u_t only depending on an upper bound of sup norms of the real Hessians of u_0 and u_1.*

This regularity result should be compared with recent results of Darvas–Lempert [11] showing that the solution $U(x,t):=u_t(x)$ is not, in general, C^2-smooth up to the boundary of M in (more precisely dd^cU is not represented by a continuous form). However, the argument in [11], which is inspired by a similar argument in the case of $M = D$ for a pseudoconvex domain D in $\mathbb{C}^2$ due to Bedford–Fornaess [1], does not seem to exclude the possibility that U be C^2-smooth in the *interior* of M. Anyway, the latter scenario appears to be highly unlikely in view of the explicit counter-example of Gamelin–Sibony [17] to interior C^2-regularity for the case when D is the unit-ball in $\mathbb{C}^2$. Note also that, since the bounds on the real Hessian of u_t are controlled by the Hessians of u_0 and u_1 the previous theorem shows that $PSH(X, \omega) \cap C^{1,1}(X)$ is closed with respect to weak geodesics. By the very recent work of Darvas [10] and Guedj [18] this the latter property equivalently means that $PSH(X, \omega) \cap C^{1,1}(X)$ defines a geodesic subspace of the metric completion of the space $\mathcal{H}$ equipped with the Mabuchi metric.

The starting point of the proof of Theorem 1.1 is the well-known Perron type envelope representation of the solution to the Dirichlet problem for the complex Monge–Ampère operator. The proof, which is inspired by Bedford–Taylor's approach in their seminal paper [2], proceeds by a straightforward generalization of the technique used in [3] to establish the corresponding regularity result for certain envelopes of positively curved metrics in a line bundle $L \to X$ (which can be viewed as solutions to a free boundary value problem for the complex Monge–Ampère equation on X). In fact, the situation here is considerably simpler than the one in [3] which covers the case when the line bundle L is merely big (the $C^{1,1}$-regularity then holds on the ample locus of L in X) and one of the motivations for the present note is to highlight the simplicity of the approach in [3] in the present situation (see also [22] for other generalizations of [3]). But it should be stressed that, just as in [3], the results can be generalized to more general line bundles. For example, by passing to a smooth resolution, Theorem 1.1 be generalized to show that the weak geodesic connecting any two smooth metrics with non-negative curvature current on an ample line bundle L over a singular compact normal complex variety X is $C^{1,1}$-smooth on the regular locus of X (for a fixed "time").

As it turns out one can formulate a general result (Theorem 2.1 below) which contains both Theorem 1.1 and the corresponding regularity result in [3]. In particular, the latter result covers the case when the domain D is the unit disc (or more generally, the unit ball in $\mathbb{C}^n$, where the following more precise regularity result holds:

Theorem 1.2. *For any integral Kähler class* $[\omega]$ *on a compact complex manifold* X *the solution* U *to the Dirichlet problem for the complex Monge–Ampère equation* 1.2 *with* C^2*-boundary data,* ω*-psh along the slices* $\{t\} \times X$, *is* $C^{1,1}$*-smooth in the interior of* $X \times D$, *if* D *is the unit disc in* $\mathbb{C}$.

As pointed out by Donaldson [15] the boundary value problem appearing in the previous theorem can be viewed as an infinite-dimensional analog of a standard boundary value problem for holomorphic discs in the complexification of a compact Lie group G or more precisely the classical factorization theorem for loops in G (recall that the role of G in the present infinite-dimensional setting is played by the group Ham(X,ω) of Hamiltonian diffeomorphisms). As shown by Donaldson [16] the solution U is in general not smooth and Donaldson raised the problem of studying the singularities of Chen's weak solution; the paper can thus be seen as one step in this direction.

One potentially useful consequence of the regularity results in Theorems 1.1, 1.2 is that, for a fixed "time" t the differential of u_t (which geometrically represents the connection one form of the corresponding metric on the line bundle L) is Lipschitz continuous and in particular differentiable on $X - E$, where the exceptional set E is a null set for the Lebesgue measure. For example, it then follows from the results in [3] that the corresponding scaled Bergman kernel $B_k(x,x)/k^n$, attached to high tensor powers $L^{\otimes k}$, converges when $k \to \infty$ point-wise on $X - E$ to the density of $\omega_{u_t}^n$. By a circle of ideas going back to Yau such Bergman kernels can be used to approximate differential geometric objects in Kähler geometry. Accordingly, the precise $C^{1,1}$-regularity established in the present paper will hopefully find applications in Kähler geometry in the future. In fact, one of the initial motivations for writing the present note came from a very recent joint work with Bo Berndtsson [5] where Bergman kernel asymptotics are used to establish the convexity of Mabuchi's K-energy along weak geodesics and where the precise $C^{1,1}$-regularity was needed at an early stage of the work. Eventually it turned that Chen's regularity, or more precisely the fact that u_t has a bounded Laplacian, is sufficient to get the point-wise convergence of B_k/k^n for some *subsequence* away from some (non-explicit) null set E (see Theorem 2.1 in [5]) which is enough to run the approximation argument. But with a bit of imagination one could envisage future situations where the more precise $C^{1,1}$-regularity would be needed.

Let us finally point out that in a very recent article Darvas and Rubinstein [12] consider psh-envelopes of functions of the form $f = \min\{f_1, f_2, \ldots, f_m\}$. Such envelopes appear in the Legendre transform type formula for weak geodesics introduced in [12] which has remarkable applications to the study of the completion of the Mabuchi metric space [10]. The same technique from [3] we describe here implies $C^{1,1}$-regularity of such envelopes in the case the Kähler class is integral (see the first point in Section 2.3). In [12] the authors give a different proof of this result (still using [3]) and also prove a Laplacian bound in the case of a general Kähler class.

2. $C^{1,1}$-regularity of solutions to complex Monge–Ampère equations over products

2.1. Notation: quasi-psh functions vs metrics on line bundles

Here we will briefly recall the notion for (quasi-) psh functions and metrics on line bundles that we will use. Let (X,ω_0) be a compact complex manifold of dimension n equipped with a fixed Kähler form ω_0, i.e., a smooth real positive closed $(1,1)$-form on X. Denote by $PSH(X,\omega_0)$ be the space of all ω_0-psh functions u on X, i.e., $u \in L^1(X)$ and u is (strongly) upper-semicontinuous (usc) and

$$\omega_u := \omega_0 + \frac{i}{2\pi}\partial\bar{\partial}u := \omega_0 + dd^c u \geq 0,$$

holds in the sense of currents.

We will write $\mathcal{H}(X, \omega_0)$ for the interior of $PSH(X, \omega_0) \cap \mathcal{C}^\infty(X)$, i.e., the space of all Kähler potentials (w.r.t ω_0). In the *integral case*, i.e., when $[\omega] = c_1(L)$ for a holomorphic line bundle $L \to X$, the space $PSH(X, \omega_0)$ may be identified with the space $\mathcal{H}_L$ of (singular) Hermitian metrics on L with positive curvature current. We will use additive notion for metrics on L, i.e., we identify a Hermitian metric $\|\cdot\|$ on L with its "weight" ϕ. Given a covering (U_i, s_i) of X with local trivializing sections s_i of $L_{|U_i}$ the object ϕ is defined by the collection of open functions $\phi_{|U_i}$ defined by

$$\|s_i\|^2 = e^{-\phi_{|U_i}}.$$

The (normalized) curvature ω of the metric $\|\cdot\|$ is the globally well-defined $(1,1)$-current defined by the following local expression:

$$\omega = dd^c\phi_{|U_i}.$$

The identification between $\mathcal{H}_L$ and $PSH(X, \omega_0)$ referred to above is obtained by fixing ϕ_0 and identifying ϕ with the function $u := \phi - \phi_0$, so that $dd^c\phi = \omega_u$.

2.2. The $C^{1,1}$-regularity of weak geodesics

Let (X, ω) be a compact Kähler manifold and D a domain in $\mathbb{C}^n$. Set $M := X \times D$ and denote by π the natural projection from M to X. Given a continuous function f on $\partial M (= X \times \partial D)$ we define the following point-wise Perron type upper envelope on the interior of M :

$$U := P(f) := \sup\{V : \ V \in \mathcal{F}\}, \tag{2.1}$$

where $\mathcal{F}$ denotes the set of all $V \in PSH(M, \pi^*\omega)$ such that $V_{|\partial M} \leq f$ on the boundary ∂M (in a point-wise limiting sense). In the case when D is a smoothly bounded pseudoconvex domain and f is ω-psh in the "X-directions", i.e., $f(\cdot, t) \in PSH(X, \omega)$ it was shown in [4] that $P(f)$ is continuous up to the boundary of M and U then coincides with the unique solution of the Dirichlet problem for the corresponding complex Monge–Ampère operator with boundary data f, in the weak sense of pluripotential theory [2]. Here we will establish the following higher-order regularity result for the envelope $P(f)$:

Theorem 2.1. *Let (X, ω_0) be an n-dimensional integral compact Kähler manifold manifold and D a bounded domain in $\mathbb{C}^m$ and set $M := X \times D$. Then, given f a function on ∂M such that $f(\cdot, \tau)$ is in $C^{1,1}(X)$, with a uniform bound on the corresponding real Hessians, the function $u_\tau := P(f)_{|X \times \{\tau\}}$ is in $C^{1,1}(X)$ and satisfies*

$$\sup_X |\nabla^2 u_\tau|_{\omega_0} \leq C,$$

where $|\nabla^2 u_\tau|_{\omega_0}$ denotes the point-wise norm of the real Hessian matrix of the function u_τ on X defined with respect to the Kähler metric ω_0. Moreover, the constant C only depends on an upper bound on the sup norm of the real Hessians of f_τ for $\tau \in \partial D$. In the case when D is the unit ball the function $U(x,\tau)$ is in $C^{1,1}_{\mathrm{loc}}$ in the interior of M.

2.2.1. Proof of Theorem 2.1. In the course of the proof of the theorem we will identify an $\pi^*\omega$-psh function U on M with a positively curved metric Φ on the line bundle $\pi^*L \to M$. The case when D is a point is the content of Theorem 1.1 in [3] and as will be next explained the general case can be proved in completely analogous manner. First recall that the argument in [3] is modelled on Bedford–Taylor's proof of the case when X is a point and D is the unit-ball [2] (see also Demailly's simplifications [13]). The latter proof uses that B is a homogeneous domain. In order to explain the idea of the proof of Theorem 2.1 first consider the case when (X, L) is *homogeneous*, i.e., the group Aut (X, L) of all biholomorphic automorphisms of X lifting to L acts transitively on X. In particular, there exists a family F_λ in Aut (X, L) parametrized by $\lambda \in \mathbb{C}^n$ such that, for any fixed point $x \in X$, the map $\lambda \mapsto F_\lambda(x)$ is a biholomorphism (onto its image) from a sufficiently small ball centered at the origin in $\mathbb{C}^n$. Given a metric ϕ on L we set $\phi^\lambda := F_\lambda^*\phi$. Similarly, given a metric $\Phi(= \Phi(x,\tau))$ on π^*L we set

$$\Phi^\lambda := (F_\lambda \times I)^*\Phi.$$

Since F_λ is holomorphic the metric Φ^λ has positive curvature iff Φ has positive curvature. Now to first prove a Lipschitz bound on $P\Phi_f$, where Φ_f is the metric on $L \to \partial M$ corresponding to the given boundary data f, we take any candidate Ψ for the sup defining $P\Phi_f$ and note that, on ∂M, i.e., for $\tau \in \partial D$:

$$\Psi^\lambda \leq \Phi_f^\lambda \leq \Phi_f + C_1|\lambda|, \tag{2.2}$$

where C_1 only depends on the Lipschitz bounds in the "X-direction" of the given function f on $X \times \partial D$. But this means that $\Psi^\lambda - C_1|\lambda|$ is also a candidate for sup defining $P\Phi_f$ and hence $\Psi^\lambda - C_1|\lambda| \leq P\Phi_f$ on all of $X \times D$. Finally, taking the sup over all candidates Ψ gives, on $X \times D$, that

$$(P\Phi_f)^\lambda \leq (P\Phi_f) + C_1|\lambda|.$$

Since this holds for any λ and in particular for $-\lambda$ this concludes the proof of the desired Lipschitz bound on $P\Phi_f$. Next, to prove the bound on the real Hessian one first replaces Ψ^λ in the previous argument with $\frac{1}{2}(\Psi^\lambda + \Psi^{-\lambda})$ and deduces, precisely as before, that

$$\frac{1}{2}\left((P\Phi_f)^\lambda + (P\Phi_f)^{-\lambda}\right) \leq (P\Phi_f) + C_2|\lambda|^2,$$

where now C_2 depends on the upper bound in the "X-direction" of the real Hessian of the function f on $X \times \partial D$. The previous inequality implies an upper bound on the real Hessians of the local regularizations Ψ_ϵ of $P\Phi_f$ defined by local convolutions. Moreover, since $dd^c\Psi_\epsilon \geq 0$ it follows from basic linear algebra that a lower bound on the real Hessians also holds. Hence, letting $\epsilon \to 0$ shows that $P\Phi_f$ is in $C^{1,1}_{\text{loc}}$ in the "X-direction" with a uniform upper bound on the real Hessians (compare [2, 13]).

Of course, a general polarized manifold (X, L) may not admit even a single (non-trivial) holomorphic vector field. But as shown in [3] this problem can be circumvented by passing to the total space Y of the dual line bundle $L^* \to X$, which does admit an abundance of holomorphic vector fields. The starting point is the standard correspondence between positively curved metrics ϕ on L and psh "log-homogeneous" functions χ on Y induced by the following formula:

$$\chi(z, w) = \phi(z) + \log |w|^2,$$

where z denotes a vector of local coordinates on X and (z, w) denote the corresponding local coordinates on Y induced by a local trivialization of L. Accordingly, the envelope $P\Phi_f$ on X corresponds to an envelope construction on Y, defined w.r.t the class of psh log-homogenous functions on Y. Fixing a metric ϕ_0 on L we denote by K the compact set in Y defined by the corresponding unit circle bundle. By homogeneity any function χ as above is uniquely determined by its restriction to K. Now, for any fixed point y_0 in K there exists an $(n + 1)$-tuple of global holomorphic vector fields V_i on Y defining a frame in a neighborhood of y_0:

Lemma 2.2. *Given any point y_0 in the space Y^* defined as the complement of the zero-section in the total space of L^* there exist holomorphic vector fields $V_1, \ldots,$ V_{n+1} on Y^* which are linearly independent close to y_0.*

Proof. This follows from Lemma 3.7 in [3]. For completeness and since we do not need the explicit estimates furnished by Lemma 3.7 in [3] we give a short direct proof here. Set $Z := \mathbb{P}(L^* \oplus \mathbb{C})$, viewed as the fiber-wise $\mathbb{P}^1$-compactification of Y. Denote by π the natural projection from Z to X and by $\mathcal{O}(1)$ the relative (fiber-wise) hyper plane line bundle on Z. As is well known, for any sufficiently positive integer the line bundle $L_m := (\pi^* L) \otimes \mathcal{O}(1)^{\otimes m}$ on Z is ample and holomorphically trivial on Y^*. As a consequence, the rank $n + 1$-vector bundle $E := TZ \otimes L_m^{\otimes k}$ is globally generated for k sufficiently large, i.e., any point z_0 in Z there exists global holomorphic sections $S_1, \ldots, S_{n+1}$ spanning $E_{|z_0}$. Since, L_m is holomorphically trivial on $Y^* \subset Z$ this concludes the proof. □

Now, integrating the (short-time) flow of the holomorphic vector field $V(\lambda) := \sum \lambda_i V_i$ gives a family of holomorphic maps $F_\lambda(y)$ defined for $y \in K$ and λ in a sufficiently small ball B centered at the origin in $\mathbb{C}^{n+1}$ such that $\lambda \mapsto F_\lambda(y_0)$ is a biholomorphism. However, the problem is that the corresponding function $\chi^\lambda := F_\lambda^* \chi$ is only defined in a neighborhood of K in Y (and not log-homogeneous). But this issue can be bypassed by replacing χ^λ with a new function that we will denote by $T(\chi^\lambda)$, where $T(f)$, for f a function on K, is obtained by first taking the sup of f over the orbits of the standard S^1-action on Y to get an S^1-invariant function $g := \hat{f}$ and then replacing g with its log-homogeneous extension $\tilde{g}$, i.e.,

$$T(f) := \widetilde{\left(\widehat{\chi^\lambda}\right)}.$$

The following lemma follows from basic properties of plurisubharmonic functions (see [3] for a proof):

Lemma 2.3. *If f is the restriction to the unit circle bundle $K \subset Y$ of a psh function, then $T(f)$ is a psh log-homogeneous function on Y*

Now performing the previous constructions for any fixed $\tau \in D$ and identifying a candidate Ψ with a function χ on $Y \times D$, as above, gives

$$\chi^\lambda(y_0) \leq \widehat{\chi^\lambda}(y_0) = \widetilde{\left(\widehat{\chi^\lambda}\right)}(y_0) := T(\chi^\lambda)(y_0). \tag{2.3}$$

But, by construction, for $\tau \in \partial D$ we have $T(\chi^\lambda) \leq T(\chi^\lambda_{\Phi_f})$ and since f_τ is assumed Lipschitz for $\tau \in \partial D$ we also have that

$$T(\chi^\lambda_{\Phi_f}) \leq T(\chi_{\Phi_f}) + C_1|\lambda| = \chi_{\Phi_f} + C_1|\lambda|.$$

But this means that $T(\chi^\lambda) - C_1|\lambda|$ is a candidate for the sup in question and hence bounded from above by $\chi_{P\Phi_f}$, which combined with the inequality 2.3 gives

$$\chi^\lambda(y_0) - C_1|\lambda| \leq \chi_{\Phi_f}(y_0).$$

Taking the sup over all candidates χ and replacing λ with $-\lambda$ hence gives the desired Lipschitz bound on $P\Phi_f$ at the given point y_0 and hence, by compactness, for any point in K. The estimate on the Hessian then proceeds precisely as above.

Finally, in the case when B is the unit ball one can exploit that B is homogeneous (under the action of the Möbius group), replacing the holomorphic maps $(x, \tau) \mapsto (F_\lambda(x), \tau)$ used above with $(x, \tau) \mapsto (F_\lambda(x), G_a(\tau))$, where G_a is a suitable family of Möbius transformations (the case when X is point is precisely the original situation in [2]). Then the proof proceeds precisely as before.

2.3. Further remarks

- The proof of the previous theorem also applies in the more general situation where f may be written as $f = \inf_{\alpha \in A} f_\alpha$ for a given family of functions f_α, as long as the Hessians of $f_\alpha(\tau, \cdot)$ are uniformly bounded on X (by a constant C independent of τ and α)) and similarly for the Lipschitz bound. Indeed, then equation 2.2 holds with f replaced by f_α for any $\alpha \in A$ with the same constant C. For D equal to a point this result has been obtained in [12] using a different proof.
- As shown in [4] (using a different pluripotential method), in the case of a general, possibly non-integral, Kähler class $[\omega]$ a bounded Laplacian in the X-directions of the boundary data f results in a bounded Laplacian of the corresponding envelope. In the case of geodesics this result has also recently been obtained in [19] by refining Chen's proof.
- By the proof of the previous theorem, the Lipschitz norm $\|u_t\|_{C^{0,1}(X)}$ of a weak geodesic u_t only depends on an upper bound on the Lipschitz norms of u_0 and u_1. Since the Lipschitz norm in the t-variable is controlled by the C^0-norm of $u_0 - u_1$ [6] it follows that the Lipschitz norm $\|U\|_{C^{0,1}(X \times A)}$ of the

corresponding solution U on $X \times A$ is controlled by the Lipschitz norms of u_0 and u_1 and the C^0-norm of $u_0 - u_1$. For a general Kähler class this result also follows from Blocki's gradient estimate [7, 8].

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