Analysis Meets Geometry: A Tribute to Mikael Passare
Trends in Mathematics, 63–80

Amoebas and Coamoebas of Linear Spaces

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Abstract. We give a complete description of amoebas and coamoebas of k-dimensional very affine linear spaces in $(\mathbb{C}^*)^n$. This include an upper bound of their dimension, and we show that if a k-dimensional very affine linear space in $(\mathbb{C}^*)^n$ is generic, then the dimension of its (co)amoeba is equal to $\min\{2k, n\}$. Moreover, we prove that the volume of its coamoeba is equal to π^{2k}. In addition, if the space is generic and real, then the volume of its amoeba is equal to $\pi^{2k} / 2^k$.

Mathematics Subject Classification (2010). 14T05, 32A60.

Keywords. Very affine linear spaces, amoebas, coamoebas, logarithmic Gauss map.

1. Introduction

Amoebas and coamoebas are very fascinating notions in mathematics, the first has been introduced by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in 1994 [3], and the second by the second author in a talk in 2004. They are natural projections of complex varieties, and which turn out to have relations to several other fields: tropical geometry, real algebraic geometry, generalized hypergeometric functions, mirror symmetry, and others (e.g., [6], [7], [13], [12], [17], [19]). More precisely, the amoebas (respectively coamoebas) of complex algebraic and generally analytic varieties in the complex algebraic torus $(\mathbb{C}^*)^n$ are defined as their image under the logarithmic mapping Log : $(z_1, \dots, z_n) \mapsto (\log|z_1|, \dots, \log|z_n|)$ (respectively the argument mapping Arg : $(z_1, \dots, z_n) \mapsto (\frac{z_1}{|z_1|}, \dots, \frac{z_1}{|z_1|})$). Amoebas (respectively coamoebas) are the link between classical complex algebraic geometry and tropical (respectively complex tropical) geometry. More precisely, amoebas degenerate to piecewise-linear objects called tropical varieties (see [13], and [19]), and comoebas degenerate to a non-Archimedean coamoebas which are the coamoebas of some lifting in the complex algebraic torus of tropical varieties. See [18] for

Research of the first author is partially supported by NSF MCS grant DMS-0915245.

more details about non-Archimedean coamoebas, and [16] about this degeneration in case of hypersurfaces. Whereas the theory of (co)amoebas of complex hypersurfaces is by now reasonably well understood (see, e.g., [2], [11], [16], and [19]), much less is known about the structure of (co)amoebas coming from varieties of higher codimension. A natural first step in this direction is to explore the case of linear spaces.

Being of a logarithmic nature, it is natural that coamoebas are closely related to the exponents of the defining functions of V, and to the associated Newton polytopes. This connection is extensively explored in the thesis of the first author [3], [15], and [19]. Another important connection is to the currently very active field of tropical geometry, a piecewise linear incarnation of classical algebraic geometry where the varieties can be seen as non-Archimedean versions of amoebas (see [7], [12], [13] and others).
A fundamental theorem was shown by K. Purbhoo [20] for the general study of amoebas that do not come from hypersurfaces. The theorem states that the amoeba of an algebraic variety V is equal to the intersection of all hypersurface amoebas corresponding to functions in the defining ideal $\mathcal{I}(V)$ of the variety V. We give a simple proof of this theorem with an extension to coamoebas.

Theorem 1.1. *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety with defining ideal $\mathcal{I}(V)$. Then the amoeba (respectively coamoeba) of V is given as follows:*

$$\mathscr{A}(V) = \bigcap_{f\in\mathcal{I}(V)} \mathscr{A}(V_f) \qquad \text{and} \qquad co\mathscr{A}(V) = \bigcap_{f\in\mathcal{I}(V)} co\mathscr{A}(V_f).$$

In [19], Rullgård and the second author showed that the area of complex plane curve amoebas is finite and the bound is given in terms of the Newton polygon of the defining polynomial. They, also compute the area of the amoeba of a plane line. It was shown by Mikhalkin and Rullgård that this bound is always sharp [14]. In [8], Madani and the first author generalized this result and showed that the volume of the amoeba of a k-dimensional algebraic variety of $(\mathbb{C}^*)^n$ with $n \geq 2k$ is finite. Moreover, they proved in [9] that the finiteness of the volume of the amoeba of a generic analytic variety is equivalent to the variety being algebraic. Theorem 1.1 and Proposition 3.1 was shown separately and in the same time by Petter Johansson in [4].

Let V be a variety in the projective space $\mathbb{CP}^n$. We choose homogeneous coordinates $[Z_0 : \cdots : Z_n]$ so that V is transverse to coordinate hyperplanes $Z_j = 0$ and all their intersections. The complement of the arrangement of coordinate hyperplanes in $\mathbb{CP}^n$ is $(\mathbb{C}^*)^n$. Then the variety $\mathscr{V} = V \cap (\mathbb{C}^*)^n$ is called a *very affine variety*, and in the case where $P(k)$ is a k-dimensional linear subspace of $\mathbb{CP}^n$ we say that $\mathscr{P}(k) = P(k) \cap (\mathbb{C}^*)^n$ is a *very affine linear space*, and by abuse of language we will call it just affine linear space. Moreover, $\mathscr{P}(k)$ can be presented as a complete intersection of hyperplanes given by first degree equations $f_1(z) = \cdots = f_{n-k}(z) = 0$, where $z = (z_1, \ldots, z_n) = (Z_1/Z_0, \ldots, Z_n/Z_0)$ stands for the affine coordinates in $(\mathbb{C}^*)^n$.

Theorem 1.2. *Let* $\mathscr{P}(k)$ *be a generic affine linear subspace of* $(\mathbb{C}^*)^{2k}$*. Then we have the following:*

(i) *The volume of the coamoeba* $co\mathscr{A}(\mathscr{P}(k))$ *is equal to* π^{2k}*;*
(ii) *Moreover, if* $\mathscr{P}(k)$ *is real, then the volume of its amoeba* $\mathscr{A}(\mathscr{P}(k))$ *is equal to* $\frac{\pi^{2k}}{2^k}$*.*

The present paper is organized as follows. We give definitions, background, and some known results in connection with this paper in Section 2. We prove Theorem 1.1 in Section 3, and detailed description of amoebas and coamoebas of lines in n-dimensional complex algebraic torus in Section 4.1 for any $n \geq 2$. We prove Theorem 1.2 in Section 5.

Remark. My first meeting and mathematical discussion with Michael was during the summer school in Paris in 2006 where he gave a series of lectures on amoebas. We talked a lot on the geometric and topological properties of these objects in particular the solidness of some of them. Moreover, at Stockholm University, when I visited him in the same year, we discussed their similarity to other objects called coamoebas. At that time we did not know exactly what kind of similarities because the ambient spaces of these two objects are different: one is compact and the other is not compact. Amoebas are closed subsets in the Euclidean space but coamoebas are not closed and not open subsets of the real torus. However, both of them have a similar (dual in some sense) combinatorial properties, and strongly related to the combinatorial type of the Newton polytopes of the defining polynomial in the hypersurface case. At that time we did not know a lot of things in higher codimension. This work was started on June 2011, but after the tragic death of Mikael Passare on 15 September 2011, the completion and writing of this paper was done by the first author.

2. Preliminaries

In this section, we review some known results related to this paper, and give some notations and definitions. Let V be an algebraic variety in $(\mathbb{C}^*)^n$. The *amoeba* $\mathscr{A}$ of V is by definition the image of V under the logarithmic map defined as follows (see M. Gelfand, M. M. Kapranov and A. V. Zelevinsky [3]):

$$\begin{array}{rccl} \mathrm{Log} : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_1, \ldots, z_n) & \longmapsto & (\log|z_1|, \ldots, \log|z_n|). \end{array}$$

The argument map is the map defined as follows:

$$\begin{array}{rccl} \mathrm{Arg} : & (\mathbb{C}^*)^n & \longrightarrow & (S^1)^n \\ & (z_1, \ldots, z_n) & \longmapsto & (\frac{z_1}{|z_1|}, \ldots, \frac{z_1}{|z_1|}). \end{array}$$

The *coamoeba* of V, denoted by $co\mathscr{A}$, is its image under the argument map (defined by the second author in 2004).

Purbhoo shows that the amoeba of an algebraic variety V is equal to the intersection of all hypersurface amoebas corresponding to functions in the defining ideal $\mathcal{I}(V)$ of the variety V (see [20], Corollary 5.2). Passare and Rullgård prove the following (see [19]):

Theorem 2.1 (Passare–Rullgård, (2000)). *Let f be a Laurent polynomial in two variables. Then the area of the amoeba of an algebraic plane curve with defining polynomial f is not greater than π^2 times the area of the Newton polytope of f.*

In [14], Mikhalkin and Rullgård showed that up to multiplication by a constant in $(\mathbb{C}^*)^2$, the algebraic plane curves with Newton polygon Δ with maximal amoeba area are defined over $\mathbb{R}$. Furthermore, their real loci are isotopic to the so-called Harnack curves (possibly singular with ordinary real isolated double points). Moreover, Rullgård and the second author compute the area of the amoeba of a line in the plane.

Madani and the first author showed that if the dimension n of the ambient space is at least the double of the dimension of V (i.e., $n \geq 2\dim_{\mathbb{C}}(V) = 2k$), then the map $\mathrm{Log} \circ \mathrm{Arg}^{-1}$ conserves the $2k$-volume, i.e., the absolute value of the determinant of its Jacobian, when it exists, is equal to one (see [9], Proposition 3.1). Moreover, the same proposition shows that the set of critical points of the logarithmic and the argument maps restricted to V coincide. Hence, if the argument map restricted to the set of regular points is injective, and the cardinality d of the inverse image under the logarithmic map of a regular value in the amoeba is constant, then the volume of the amoeba will be the volume of the coamoeba divided by d. So, first we show that if V is a generic k-dimensional linear space in $(\mathbb{C}^*)^{2k}$, then the argument map restricted to the set of regular points is injective, and we compute the volume of its coamoeba. Moreover, if the linear space is real, we show that the cardinality of the inverse image under the logarithmic map of a regular value in the amoeba is constant and equal to 2^k. Finally, we compute the amoeba volume using the conservation of the volume by the map $\mathrm{Log} \circ \mathrm{Arg}^{-1}$.

In the following paragraph, we will recall the definition of the logarithmic Gauss map for hypersurface, and its generalization. We will present some known relations between this map and (co)amoebas. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic hypersurface with defining polynomial f, and denote by V_{reg} the subset of its smooth points. The *logarithmic Gauss map* of the hypersurface V is the holomorphic map defined by (see Kapranov [5]):

$$\begin{array}{rccl} \gamma: & V_{\mathrm{reg}} & \longrightarrow & \mathbb{CP}^{n-1} \\ & z & \longmapsto & \gamma(z) = [v(z)], \end{array}$$

where $[v(z)] = [z_1 \frac{\partial f}{\partial z_1}(z) : \cdots : z_n \frac{\partial f}{\partial z_n}(z)]$ denotes the class of the vector $v(z)$ in $\mathbb{CP}^{n-1}$.

Madani and the first author generalize this map to any codimension, and extract some relations between the set of its critical points and (co)amoebas, and they generalized an earlier result of Mikhalkin [11] on critical points of the logarithmic map (see [10]). More precisely, let $V \subset (\mathbb{C}^*)^n$ be an algebraic variety of dimension k with defining ideal $\mathcal{I}(V)$ generated by $\{f_1, \dots, f_l\}$. A holomorphic map γ_G from the set of smooth points of V to the complex Grassmannian $\mathbb{G}_{n-k,\,n}$ was defined as follows: If we denote by V_{reg} the subset of smooth points of V as before, and $M(l \times n)$ denotes the set of $l \times n$ matrices. Let g_G be the following map:

$$\begin{array}{cccc} g_G & : & V_{\text{reg}} & \longrightarrow & M(l \times n) \\ & & z = (z_1, \dots, z_n) & \longmapsto & \begin{pmatrix} z_1 \frac{\partial f_1}{\partial z_1}(z) & \dots & z_n \frac{\partial f_1}{\partial z_n}(z) \\ \vdots & \vdots & \vdots \\ z_1 \frac{\partial f_l}{\partial z_1}(z) & \dots & z_n \frac{\partial f_l}{\partial z_n}(z) \end{pmatrix}. \end{array}$$

Since z is a smooth point of V, then the complex vector space L_z generated by the rows of the matrix $g_G(z)$ is of dimension $n-k$, and orthogonal to the tangent space to V at z. Indeed, the problem is local and V_{reg} is locally a complete intersection. Moreover, the tangent space to V at a regular point is contained in the tangent space of all the hypersurfaces defined by the polynomials f_i, and each row vector of index i is orthogonal to the hypersurface defined by the polynomial f_i which contains V. This means that the image of V_{reg} by g_G is contained in the subvariety of $M(l \times n)$ consisting of $l \times n$ matrices of rank $n-k$, which we map to the complex Grassmannian $\mathbb{G}_{n-k,\,n}$. Composing this identification with g_G we obtain the desired map:

$$\gamma_G : V_{\text{reg}} \to \mathbb{G}_{n-k,\,n}$$

called the *generalized logarithmic Gauss map.*

If $V \subset (\mathbb{C}^*)^n$ is a hypersurface, Mikhalkin showed that the set of critical points of $\text{Log}_{|V}$ coincides with $\gamma_G^{-1}(\mathbb{RP}^{n-1})$ (see Lemma 3 in [11], and Lemma 4.3 in [12]). This result was generalized by Madani and the first author for higher codimension in [10].

Throughout all this paper, the genericity of an algebraic variety $V \subset (\mathbb{C}^*)^n$ is defined as follows:

Definition 2.1. An irreducible algebraic variety $V \subset (\mathbb{C}^*)^n$ of dimension k is generic if it satisfies the following:

(1) The variety V contains an open dense subset U such that the Jacobian of the restriction to U of the logarithmic map $\text{Jac}(\text{Log}_{|U})$ has maximal rank, i.e., $\min\{2k, n\}$;
(2) The variety V lies in no affine subgroup, otherwise we may replace $(\mathbb{C}^*)^n$ by the smallest affine subgroup containing V.

We denote by $\mathscr{L}og_{|V}$ the complex logarithmic map, and Re the real part of a complex vector. In this case, we have $\text{Log}_{|V} = \text{Re} \circ \mathscr{L}og_{|V}$. This means that the amoeba of V is the real part of $\mathscr{L}og_{|V}(V)$ (by taking the imaginary part we obtain the same conclusion for the coamoeba),

$$\begin{array}{ccc} V \subset (\mathbb{C}^*)^n & \xrightarrow{\mathscr{L}og_{|V}} & \mathbb{C}^n \supset \mathscr{L}og(V) \\ & {\scriptstyle \mathrm{Log}_{|V}} \searrow \quad \swarrow {\scriptstyle \mathrm{Re}} & \\ & \mathscr{A}(V) \subset \mathbb{R}^n. & \end{array}$$

We can check that for any $r \in \mathbb{R}^n$, the set $T_r := \mathrm{Log}^{-1}(r)$ is an n-dimensional real torus, and $r \in \mathscr{A}(V)$ if and only if $T_r \cap V \neq \emptyset$.

3. (Co)amoebas of complex algebraic varieties

In this section, we describe the amoeba (respectively coamoeba) of a complex variety V with defining ideal $\mathcal{I}(V)$ as the intersection of the amoebas (respectively coamoebas) of the complex hypersurfaces with defining polynomials in $\mathcal{I}(V)$.

The first part of Theorem 1.1 concerning amoebas was shown by Purbhoo in 2008 (see Corollary 5.2 in [20]). We present a very simple proof of this fact, and extend it to coamoebas.

Our first observation, is the following proposition about the dimension of (co)amoebas:

Proposition 3.1. *Let $V \subset (\mathbb{C}^*)^n$ be an irreducible algebraic variety of dimension k. Then, the dimension of the (co)amoeba $\mathscr{A}(V)$ of V satisfies the following:*

$$\dim((co)\mathscr{A}(V)) \leq \min\{2k, n\}.$$

In particular, if V is generic, then the dimension of its amoeba is $\min\{2k, n\}$.

Proof. The rank of the Jacobian of the logarithmic (respectively argument) map restricted to V at a regular point is equal to $\min\{2k, n\}$. So, the dimension of the (co)amoeba cannot exceed $\min\{2k, n\}$. Moreover, if the dimension of the amoeba (respectively coamoeba) of a k-dimensional irreducible variety V in $(\mathbb{C}^*)^n$ is strictly less than $\min\{2k, n\}$, then the map Re is not an immersion (respectively submersion) if $n \geq 2k$ (respectively $n < 2k$). Hence, the set of critical points of the logarithmic (respectively argument) map is equal to all the variety (see [10] for more details about critical values of the logarithmic Gauss map in higher codimension case). □

Let $V_f \subset (\mathbb{C}^*)^n$ be a hypersurface with defining polynomial f. Then, by definition, the amoeba of V_f is the image by the logarithmic map of the subset $\mathscr{S}_f$ of $(\mathbb{R}_+^*)^n$ defined as follows:

$$\mathscr{S}_f := \{(x_1, \ldots, x_n) \in (\mathbb{R}_+^*)^n \,|\, \exists\, z \in (\mathbb{C}^*)^n \text{ such that } x_i = |z_i|, \text{ and } f(z) = 0\}.$$

Since $\mathscr{L}og : (\mathbb{R}_+^*)^n \to \mathbb{R}^n$ is a diffeomorphism, we have the following:

$$\bigcap_{f\in\mathcal{I}(V)} \mathrm{Log}\,(\mathscr{S}_f) = \mathrm{Log}\left(\bigcap_{f\in\mathcal{I}(V)} \mathscr{S}_f\right),$$

where $\mathrm{Log}\,(\mathscr{S}_f)$ is used with abuse of notation.

Lemma 3.1. *We have the following equality:*

$$\bigcap_{f\in\mathcal{I}(V)} \mathscr{S}_f = \{(x_1,\dots,x_n)\in(\mathbb{R}_+^*)^n |\ x_i = |z_i|,\ \text{and}\ (z_1,\dots,z_n)\in V\}.$$

Proof. Let r be in

$$(\mathbb{R}_+^*)^n \setminus \{(x_1,\dots,x_n)\in(\mathbb{R}_+^*)^n |\, x_i = |z_i| \text{ and } (z_1,\dots,z_n)\in V\},$$

and T_r be the real torus $\mathrm{Log}^{-1}(r)$. So, $T_r\cap V$ is empty. Let $f\in\mathcal{I}(V)$ with $f(z) = \sum c_\alpha z^\alpha$ and g be the Laurent polynomial defined by $g(z) = \sum \overline{c}_\alpha w^\alpha$ with $w = (\frac{r_1^2}{z_1},\dots,\frac{r_n^2}{z_n})$ where the r_j's are the coordinates of r, and $\overline{c}_\alpha$ denotes the conjugate of the coefficient c_α. The value of the Laurent polynomial $h(z) = f(z)g(z)$ is equal to the value of $|f(z)|^2$ for every $z\in T_r$. By construction, the hypersurface V_h with defining polynomial h contains V (because $h\in\mathcal{I}(V)$). Let $\langle f_1,\dots,f_s\rangle$ be a set of generators of the ideal $\mathcal{I}(V)$, and for any j let g_j be the Laurent polynomial defined as before. We can check the hypersurface defined by the polynomial $G = \sum f_j g_j$ contains V and does not intersect the torus T_r. This proves that $r\in(\mathbb{R}_+^*)^n\setminus\bigcap_{f\in\mathcal{I}(V)}\mathscr{S}_f$. Hence, we have the inclusion:

$$\bigcap_{f\in\mathcal{I}(V)} \mathscr{S}_f \subset \{(x_1,\dots,x_n)\in(\mathbb{R}_+^*)^n \,|\, x_i = |z_i|,\ \text{and}\ (z_1,\dots,z_n)\in V\}.$$

Now let $(x_1,\dots,x_n)\in(\mathbb{R}_+^*)^n$ such that $x_i = |z_i|$ and $(z_1,\dots,z_n)\in V$, then for all $f\in\mathcal{I}(V)$ we have $f(z_1,\dots,z_n) = 0$. This means that $(x_1,\dots,x_n)\in\bigcap_{f\in\mathcal{I}(V)}\mathscr{S}_f$. □

Proof of Theorem 1.1. The first equality of Theorem 1.1 is a consequence of Lemma 3.1. In fact, by applying the logarithmic map to both sides of the equality of Lemma 3.1 we obtain: $\mathrm{Log}\left(\bigcap_{f\in\mathcal{I}(V)}\mathscr{S}_f\right) = \mathscr{A}(V)$, and then

$$\mathscr{A}(V) = \bigcap_{f\in\mathcal{I}(V)} \mathscr{A}(V_f).$$

Let us prove the second equality of Theorem 1.1. Let $w\in\bigcap_{f\in\mathcal{I}(V)} co\mathscr{A}(V_f)$, then there exists a fundamental domain $\mathscr{D} = ([a; a+2\pi[)^n$ in the universal covering of the real torus $(S^1)^n$ and a unique $\widetilde{w}\in\mathscr{D}$ such that $w = \exp(i\widetilde{w})$. In this domain, the exponential map is a diffeomorphism between $\mathscr{D}$ and $(S^1)^n\setminus(S^1)^{n-1}\wedge\cdots\wedge(S^1)^{n-1}$ where $(S^1)^{n-1}\wedge\cdots\wedge(S^1)^{n-1}$ denotes the bouquet of n tori of dimension $n-1$. Let us define the subset $co\mathscr{S}_f$ of $\mathscr{D}$ as follow:

$$co\mathscr{S}_f := \{\theta\in\mathscr{D}|\ \text{there exists } z\in V_f \text{ and } \exp(i\theta) = \mathrm{Arg}\,(z)\}.$$

So, we have:

$$\bigcap_{f\in\mathcal{I}(V)} \exp\left(ico\mathscr{S}_f\right) = \exp\left(i\bigcap_{f\in\mathcal{I}(V)} co\mathscr{S}_f\right)$$

because the exponential map is a diffeomorphism from $\mathscr{D}$ into its image. Moreover, $\widetilde{w}$ is contained in the intersection $\bigcap_{f\in\mathcal{I}(V)} co\mathscr{S}_f$. But the last intersection, using the same argument as in Lemma 3.1, can be described as follows:

$$\begin{aligned}\bigcap_{f\in\mathcal{I}(V)} co\mathscr{S}_f &= \bigcap_{f\in\mathcal{I}(V)} \{\theta\in\mathscr{D} \mid \text{there exists } z\in V_f \text{ and } \exp(i\theta)=\operatorname{Arg}(z)\}\\ &= \{\theta\in\mathscr{D} \mid \text{there exists } z\in V \text{ and } \exp(i\theta)=\operatorname{Arg}(z)\}.\end{aligned}$$

Indeed, to prove the last equality, let $e^{i\theta}\notin co\mathscr{A}(V)$, and for each generator $f_j(z)=\sum c_\alpha z^\alpha$ of $\mathcal{I}(V)$ we define the polynomial g_j as follows:

$$g_j(z)=\sum \overline{c}_\alpha (e^{-2i\theta})^\alpha z^\alpha.$$

If $z\in\operatorname{Arg}^{-1}(e^{i\theta})$, then we have $f_jg_j(z)=|f_j(z)|^2$. The polynomial $G=\sum_j f_jg_j$ is in $\mathcal{I}(V)$, but $e^{i\theta}\notin co\mathscr{A}(V_G)$ because $|f_j(z)|^2>0$ and hence $G(z)=\sum_j f_jg_j(z)>0$ for every j and every $z\in\operatorname{Arg}^{-1}(e^{i\theta})$. Namely, we have the following inclusion:

$$\bigcap_{f\in\mathcal{I}(V)} co\mathscr{S}_f \subset \{\theta\in\mathscr{D} \mid \text{there exists } z\in V \text{ and } \exp(i\theta)=\operatorname{Arg}(z)\}.$$

In other words, $\bigcap_{f\in\mathcal{I}(V)} co\mathscr{A}_f \subset co\mathscr{A}(V)$. □

4. (Co)Amoebas of linear spaces

Throughout this section, $\mathscr{P} := P(k)\cap(\mathbb{C}^*)^{k+m}$ where $P(k)$ is the k-dimensional affine linear subspace of $\mathbb{C}^{k+m}$ given by the parametrization ρ as follows:

$$\begin{array}{cccc}\rho\ : & \mathbb{C}^k & \longrightarrow & \mathbb{C}^{k+m}\\ & (t_1,\dots,t_k) & \longmapsto & (t_1,\dots,t_k,f_1(t_1,\dots,t_k),\dots,f_m(t_1,\dots,t_k)),\end{array} \qquad (1)$$

where $f_j(t_1,\dots,t_k)=b_j+\sum_{i=1}^k a_{ji}t_i$, and $a_{ji},\ b_j$ are complex numbers for $i=1,\dots,k$, and $j=1,\dots,m$. By abuse of language, we call $\mathscr{P}$ an affine linear space instead of very affine linear space. First of all, if $\mathscr{P}$ is generic then all the coefficients b_j are different than zero. Otherwise $\mathscr{P}$ will be contained in an affine subgroup of $(\mathbb{C}^*)^{k+m}$. Indeed, if there exits j such that $b_j=0$, then there is an action of $\mathbb{C}^*$ on $\mathscr{P}$, and then $\mathscr{P}$ can be viewed as a product of $\mathbb{C}^*$ with an affine linear space of dimension $k-1$. Namely, $\mathscr{P}$ lies in no affine subgroup, i.e., $\rho(\mathbb{C}^k)$ meets each of the n coordinate hyperplanes of $\mathbb{C}^n$ in distinct hyperplanes, otherwise we may replace $(\mathbb{C}^*)^n$ by the smallest affine subgroup containing $\mathscr{P}$.

Lemma 4.1. *If $\mathscr{P}$ is generic, then we can assume that $f_1(t_1,\dots,t_k)=1+\sum_{i=1}^k t_i$.*

Proof. In fact, if we make a translation by $\frac{1}{b_1}$ in the algebraic multiplicative torus $(\mathbb{C}^*)^{k+m}$, we get

$$\left(\frac{t_1}{b_1},\dots,\frac{t_1}{b_1},\frac{f_1(t_1,\dots,t_k)}{b_1},\dots,\frac{f_m(t_1,\dots,t_k)}{b_1}\right).$$

We translate again by $a=(a_{11},a_{21},\dots,a_{1k},1,\dots,1)$ to obtain:

$$\left(\frac{a_{11}t_1}{b_1},\dots\frac{a_{1k}t_k}{b_1},1+\sum_{i=1}^{k}\frac{a_{1i}t_i}{b_1},\frac{f_2(t_1,\dots,t_k)}{b_1},\dots,\frac{f_m(t_1,\dots,t_k)}{b_1}\right).$$

For any point z in $(\mathbb{C}^*)^{k+m}$, we denote by τ_z the translation by z in the multiplicative group $(\mathbb{C}^*)^{k+m}$, and denote by ρ' the required parametrization, i.e.,

$$\rho'(t_1,\dots,t_k)=\left(t_1,\dots,t_k,1+\sum_{i=1}^{k}t_i,f_2(t_1,\dots,t_k),\dots,f_m(t_1,\dots,t_k)\right).$$

Hence, we obtain $\tau_a\circ\tau_{\frac{1}{b_1}}\circ\rho=\rho'\circ\tau_c$, where $c=(\frac{a_{11}}{b_1},\dots,\frac{a_{1k}}{b_1})$, and then, for any $(t_1,\dots,t_k)$ in $(\mathbb{C}^*)^k$ we have:

$$\mathrm{Arg}\left(\rho(t_1,\dots,t_k)\right)-\mathrm{Arg}\,(b_1)+\mathrm{Arg}\,(a)=\mathrm{Arg}\left(\rho'(\tau_c(t_1,\dots,t_k))\right).$$

We obtain the same relation if we replace the argument map by the logarithmic map. This means that the amoeba (respectively coamoeba) of a generic complex affine linear space $\mathscr{P}$ given by the parametrization (1) is the translation in the real space $\mathbb{R}^{k+m}$ (respectively the real torus $(S^1)^{k+m}$) by a vector v in $\mathbb{R}^{k+m}$ (respectively a point in the real torus) of an affine linear space given by a parametrization such that $f_1(t_1,\dots,t_k)=1+\sum_{i=1}^{k}t_i$. Hence, $co\mathscr{A}(\mathscr{P})=\tau_v\circ co\mathscr{A}(\mathscr{P}_{\rho'})$ where $\mathscr{P}_{\rho'}$ is the affine linear space given by the required parametrization, and we have a similar equality for their amoebas. In the last formula, v is the argument of the vector $b_1^{-1}a$. □

To be more precise, $\mathscr{P}$ can be seen as the image by ρ of the complement in $\mathbb{C}^k$ of an arrangement of n hyperplanes $\mathscr{H}:=\cup_{i=1}^{k}\{t_i=0\}\cup_{j=1}^{m}\{f_j=0\}$.

4.1. (Co)Amoebas of lines in $(\mathbb{C}^*)^{1+m}$

In this subsection we give a complete description of (co)amoebas of generic lines in $(\mathbb{C}^*)^{1+m}$ (we mean a complex subvariety of complex dimension one defined by an ideal generated by polynomials of degree one). Moreover, we describe the (co)amoebas of real lines, i.e., lines those are invariant under the involution given by the conjugation of complex numbers. In other word, lines given by a parametrization with real coefficients. But first, let L be a generic line in $(\mathbb{C}^*)^{1+m}$ parametrized as follows:

$$\begin{array}{rcccl}\rho & : & \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^{1+m}\\ & & t & \longmapsto & (t,t+1,a_2t+b_2,\dots,a_mt+b_m),\end{array}\tag{2}$$

where a_j and b_j are non-vanishing complex numbers.

Lemma 4.2. *There are two types of amoebas of lines in* $(\mathbb{C}^*)^{1+m}$ *for* $m \geq 3$. *There are amoebas with boundary and other without boundary (we mean topological boundary). The amoebas of generic lines given by the parametrization* (2) *have boundary if and only if* $\frac{a_i}{b_i} \in \mathbb{R}^*$ *for all* $j = 2, \ldots, m$.

Proof. Since the boundary of an amoeba is a subset of the set of critical values of the logarithmic map, then an amoeba has a boundary means that the set of critical points of the logarithmic map restricted to the variety is nonempty (see [10], and [11] for more details about the critical points). The Jacobian of the logarithmic map restricted to the line L is given by:

$$\mathrm{Jac}(\mathrm{Log}_{|L})(t) = \frac{\partial \mathrm{Log}}{\partial(t, \bar{t})} = \frac{1}{2}\begin{pmatrix} 1/t & 1/\bar{t} \\ 1/(t+1) & 1/(\overline{t+1}) \\ a_2/(a_2t+b_2) & \bar{a}_2/(\overline{a_2t+b_2}) \\ \vdots & \vdots \\ a_m/(a_mt+b_m) & \bar{a}_m/(\overline{a_mt+b_m}) \end{pmatrix}.$$

Hence, a point $\rho(t)$ is critical for $\mathrm{Log}_{|L}$ if and only if all the 2×2-minors of the Jacobian matrix have determinant equal to zero. Let us write down these relations. The determinant of the 2×2-minor given by the two first rows:

$$\frac{1}{2}\begin{pmatrix} 1/t & 1/\bar{t} \\ 1/(t+1) & 1/(\bar{t}+1) \end{pmatrix}$$

is equal to zero, means the following equality holds:

$$\frac{1}{t}\frac{1}{\bar{t}+1} = \frac{1}{\bar{t}}\frac{1}{t+1}.$$

This implies that t should be real. For all $i = 2, \ldots, m$, the 2×2-minor:

$$\frac{1}{2}\begin{pmatrix} 1/t & 1/\bar{t} \\ a_i/(a_it+b_i) & \bar{a}_i/(\overline{a_it+b_i}) \end{pmatrix}$$

gives the following relation:

$$\frac{1}{t}\frac{\bar{a}_i}{(\overline{a_it+b_i})} = \frac{1}{\bar{t}}\frac{a_i}{a_it+b_i}.$$

But t is real, so $\bar{a}_i(a_it+b_i) = a_i(\overline{a_it+b_i})$, and hence $\frac{a_i}{b_i} = \overline{(\frac{a_i}{b_i})}$, i.e., $\frac{a_i}{b_i} \in \mathbb{R}^*$ (because L is generic, all the coefficients are different than zero). So, if $\frac{a_i}{b_i} \in \mathbb{R}^*$ for $i = 2, \ldots, m$, then the set of critical points of $\mathrm{Log}_{|L}$ is the image under ρ of the real part of $\mathbb{C}^*$, where this image intersects $(m+2)$ quadrants of $\mathbb{R}^{1+m}$ because L is generic. Moreover, this shows that the set of critical values of $\mathrm{Log}_{|L}$ is the image under $\mathrm{Log} \circ \rho$ of the real part of $\mathbb{C}^*$, and the number of its connected components is $(m+2)$. So, a generic complex line given by the parametrization (2) with $\frac{a_i}{b_i} \in \mathbb{R}^*$ for $i = 2, \ldots, m$ is real up to a translation by a complex number, and its amoeba is a surface with boundary, and the boundary has $(m+2)$ connected components. Also, we can check in this case that the cardinality of the inverse image of a regular (respectively critical) value is two (respectively one). □

This motivates the following definition (see [14] for real plane curves):

Definition 4.1. A generic affine line given by the following parametrization:

$$\begin{array}{rcccl}\rho & : & \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^{1+m} \\ & & t & \longmapsto & (t, a_1t+b_1, a_2t+b_2, \dots, a_mt+b_m),\end{array} \tag{3}$$

where a_j and b_j are in $\mathbb{C}^*$ is called real up to a translation by a vector in $(\mathbb{C}^*)^{1+m}$ if and only if $[\frac{a_1}{b_1} : \cdots : \frac{a_m}{b_m}] \in \mathbb{RP}^{m-1}$.

If a line L in $(\mathbb{C}^*)^{1+m}$ with $m \geq 2$ is not real, then its amoeba is a surface without boundary homeomorphic to the Riemann sphere without $(m+2)$ points (see proof of Lemma 4.2), and the map $\mathrm{Log}_{|L}$ is a one-to-one map.

The following lemma gives a description of the coamoeba of a generic line in $(\mathbb{C}^*)^{1+m}$ with $m \geq 1$

Lemma 4.3. *Let $L \subset (\mathbb{C}^*)^{1+m}$ be a generic line given by the parametrization* (3)*. The restriction of the argument map to the set of its regular points in L is injective, and the inverse image under the argument map of a critical value has real dimension one.*

Proof. To see injectivity, let $(e^{i\theta}, e^{i\psi_1}, \dots, e^{i\psi_m})$ be a fixed regular value in $co\mathscr{A}(L)$. In other word, we have $t = |t|e^{i\theta}$, and $f_j(t) = (a_jt + b_j) = |a_jt + b_j|e^{i\psi_1}$ for $j = 1, \dots, m$, and consider a_jt, b_j, and $f_j(t)$ as a vectors in the complex plane. Hence, for each $j = 1, \dots, m$ we obtain a parallelogram with vertices the origin, and the extremities of the three vectors a_jt, b_j, and $f_j(t)$. If one of these vectors is fixed, and the arguments of the two others are fixed (which is our case, because b_j is given and the arguments of a_jt and $f_j(t)$ are fixed by assumption), then there exists at most one parallelogram with those vertices. This implies the injectivity.

The second part of the lemma comes from the fact that the set of critical points of the logarithmic map and the argument map coincide (see Proposition 3.1 in [9]). Indeed, the set of critical points is equal to $(m+2)$ connected components of dimension one (each one corresponds to the intersection of the real part of L with some quadrant of $(\mathbb{R}^*)^{m+1}$). □

The set of critical points of the argument map restricted to L given by the parametrization (3) is the image by ρ of the real part of $\mathbb{C}^*$ translated by $(1, b_1, \dots, b_m)$ in $(\mathbb{C}^*)^{1+m}$ as a multiplicative group. So, the set of critical values consists of the translation by $(1, \frac{b_1}{|b_1|}, \dots, \frac{b_m}{|b_m|})$ of $(m+2)$ points in the real torus $(S^1)^{1+m}$ from the 2^{m+1} real points corresponding to the arguments of the 2^{m+1} quadrants of $\mathrm{Re}((\mathbb{C}^*)^{1+m}) = (\mathbb{R}^*)^{1+m}$. The closure of the coamoeba of L contains an arrangement of $(m+1)$ geodesic circles. Each circle corresponds to an end of the line (i.e., where L meets a coordinate axis). The union of these circles is the set of accumulation points of arguments of sequences in L with unbounded logarithm, and is called the phase limit set of L (see [17] for more details). It is the counterpart of the logarithmic limit set introduced by Bergman in 1971 (see [1] and [7] for more details), which consists of $(m+2)$ points is our case. In Figure 1

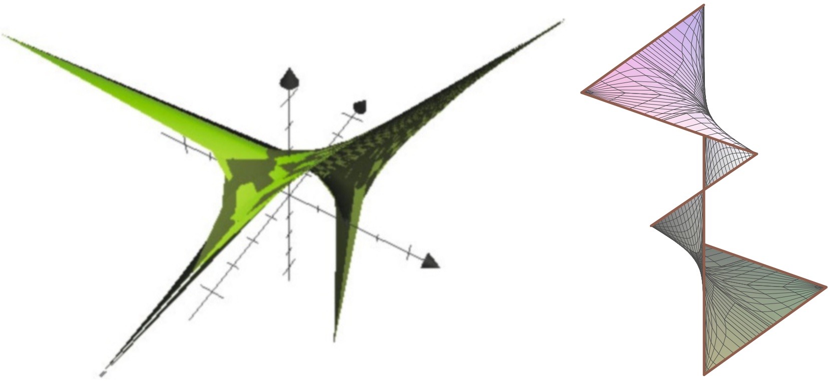

FIGURE 1. The amoeba and the coamoeba of the real line in $(\mathbb{C}^*)^3$ given by the parametrization $\rho(z) = (z, z + \frac{1}{2}, z - \frac{3}{2})$. The amoeba is topologically the closed disk without four points of its boundary.

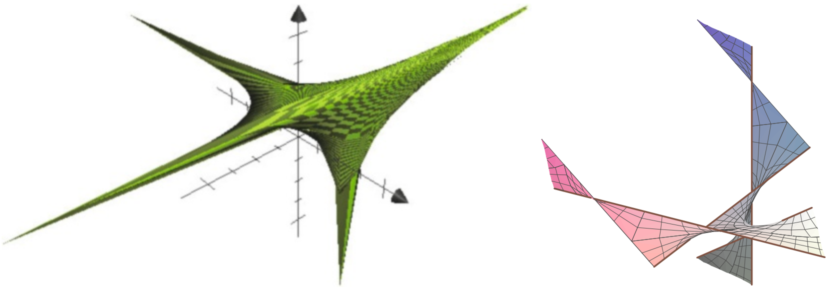

FIGURE 2. The amoeba and the coamoeba of the complex line (i.e., not real) in $(\mathbb{C}^*)^3$ given by the parametrization $\rho(z) = (z, z+1, z-2i)$. The amoeba is topologically the Riemann sphere without four points.

(respectively Figure 2), we draw the amoeba and the coamoeba of a real (respectively non real) line in $(\mathbb{C}^*)^3$. The coamoebas in Figure 1, and Figure 2 are made with collaboration with F. Sottile.

5. Volume of (co)amoebas of k-dimensional very affine linear spaces in $(\mathbb{C}^*)^{2k}$

It was shown by Rullgård and the second author in [19] that the area of the amoeba of a complex algebraic plane curve is always finite, and the bound is given in terms of the area of the Newton polygon of the defining polynomial. Mikhalkin and Rullgård proved that this bound is always sharp for (possibly singular) Harnack curves (see [14]). It was shown by Madani and the first author in [8] that the volume of the amoeba of a k-dimensional algebraic variety in $(\mathbb{C}^*)^n$ with $n \geq 2k$ is finite. This generalizes the result of Rullgård and the second author about the finiteness of the volume of the amoeba of plane curves. In this section, we compute the volume of the amoeba of a generic real k-dimensional very affine linear space in $(\mathbb{C}^*)^{2k}$. We will proceed as follows: (i) We show that the argument map restricted to the subset of regular points in the very affine linear space is injective; (ii) We compute the volume of the coamoeba of any k-dimensional very affine linear space in $(\mathbb{C}^*)^{2k}$; (iii) We compute the cardinality of the inverse image under the logarithmic map of any regular value in the amoeba of a real affine space, and prove that this cardinality is a constant and equal to 2^k; (iv) We use that the map $\mathrm{Log} \circ \mathrm{Arg}^{-1}$ conserves the volume, i.e., the determinant of its Jacobian has absolute value equal one (see Proposition 3.1 in [9]), and finally we compute the volume of the amoeba, which is equal to the coamoeba volume divided by 2^k if the plane is real. We will use the following lemma proved in [10], which is a generalization of Mikhalkin's Lemma 4.3 in [12] for hypersurface:

Lemma 5.1 (Madani–Nisse). *Let $V \subset (\mathbb{C}^*)^n$ be a k-dimensional algebraic variety, and z be a smooth point of V. Then z is a critical point for the map $\mathrm{Log}_{|V}$ if and only if the image of the tangent space T_zV to V at z by the derivative of the complex logarithm $d\mathscr{L}og$ contains at least s purely imaginary linearly independent vectors with $s = \max\{1, 2k - n + 1\}$.*

Also, we will use the following proposition proved in [10]:

Proposition 5.1 (Madani–Nisse). *Let $\mathscr{P} \subset (\mathbb{C}^*)^n$ be a generic k-dimensional very affine linear space with $n \geq 2k$. Suppose that the complex dimension of $\mathscr{P} \cap \overline{\mathscr{P}}$ is equal to l, with $0 \leq l \leq k$. Then, for any regular value x in the amoeba $\mathscr{A}(\mathscr{P})$ of $\mathscr{P}$, the cardinality of $\mathrm{Log}^{-1}(x)$ is at least 2^l.*

Let $\mathscr{P} \subset (\mathbb{C}^*)^{2k}$ be a generic k-dimensional very affine linear space. Suppose $\mathscr{P}$ is given by the parametrization ρ:

$$\begin{array}{cccc} \rho : & (\mathbb{C}^*)^k & \longrightarrow & (\mathbb{C}^*)^{2k} \\ & (t_1, \ldots, t_k) & \longmapsto & (t_1, \ldots, t_k, f_1(t_1, \ldots, t_k), \ldots, f_k(t_1, \ldots, t_k)), \end{array} \tag{4}$$

with $f_j(t_1, \ldots, t_k) = b_j + \sum_{i=1}^k a_{ji}t_i$, where a_{ji}, and b_j are complex numbers for $i = 1, \ldots, k$ and $j = 1, \ldots, k$. Since the space $\mathscr{P}$ is generic, then there is no $b_j = 0$.

Definition 5.1. A generic k-dimensional very affine linear space $\mathscr{P}(k) \subset (\mathbb{C}^*)^{2k}$ given by the parametrization (4) is said to be real up to a translation by a complex vector in the multiplicative group $(\mathbb{C}^*)^{k+m}$ if and only if the $(m \times k)$-matrix given by

$$\begin{pmatrix} \frac{a_{11}}{b_1} & \cdots & \frac{a_{1k}}{b_1} \\ \vdots & \vdots & \vdots \\ \frac{a_{k1}}{b_k} & \cdots & \frac{a_{kk}}{b_k} \end{pmatrix}$$

has rank k and all of its entries are real.

Let $\mathbb{Z}_2 := \{\pm 1\}$ be the real subgroup of the multiplicative group $\mathbb{C}^*$, and $\mathbb{Z}_2^{2k}$ be the finite real subgroup of $(\mathbb{C}^*)^{2k}$. For each $s \in \mathbb{Z}_2^{2k}$, let ρ_s be the parametrization given by $\rho_s(t_1, \dots, t_k) = s.\rho(t_1, \dots, t_k)$ where

$$s.(z_1, \dots, z_{2k}) = (s_1 z_1, \dots, s_{2k} z_{2k})$$

for any $(z_1, \dots, z_{2k}) \in (\mathbb{C}^*)^{2k}$, and $s = (s_1, \dots, s_{2k}) \in \mathbb{Z}_2^{2k}$. Let $\mathscr{P}_s$ be the k-dimensional very affine linear space in $(\mathbb{C}^*)^{2k}$ parametrized by ρ_s. Let us denote by $\mathrm{Reg}(co\mathscr{A}(\mathscr{P}_s))$ the set of regular values of $co\mathscr{A}(\mathscr{P}_s)$. Remark that if 1 denotes the identity element of the group $\mathbb{Z}_2^{2k}$, then $\mathscr{P} = \mathscr{P}_1$.

Let $u \in \mathbb{Z}_2^{2k}$ and denote by $\mathrm{Reg}(co\mathscr{A}(\mathscr{P}_u))$ the set of regular values of the coamoeba $co\mathscr{A}(\mathscr{P}_u)$.

Proposition 5.2. *With the above notations, the following statements hold:*

(i) *For all s, the argument map from the subset of regular points of $\mathscr{P}_s$ to the set of regular values of its coamoeba $co\mathscr{A}(\mathscr{P}_s)$ is injective;*

(ii) *Let s and r in $\mathbb{Z}_2^{2k}$ with $s \neq r$, then the set*

$$\mathrm{Reg}(co\mathscr{A}(\mathscr{P}_s)) \cap \mathrm{Reg}(co\mathscr{A}(\mathscr{P}_r))$$

is empty;

(iii) *The union $\bigcup_{s \in \mathbb{Z}_2^{2k}} \mathrm{Reg}(co\mathscr{A}(\mathscr{P}_s))$ is an open dense subset of the real torus $(S^1)^{2k}$.*

First of all, we denote by $z := (z_1, \dots, z_{2k})$ the coordinates of $\mathbb{C}^{2k}$. So, if z is a point in $\mathscr{P}$, then $z_i = t_i$ and $z_{k+i} = f_i(z_1, \dots, z_k)$ for $1 \leq i \leq k$. Let $\Theta = (e^{i\theta_1}, \dots, e^{i\theta_k}, e^{i\psi_1}, \dots, e^{i\psi_k})$ be a point in the set of regular values of $co\mathscr{A}(\mathscr{P})$. This means that the linear system (E) of $2k$ equations and $2k$ variables $(x_1, \dots, x_k, y_1, \dots, y_k)$ in $(\mathbb{R}_+^*)^{2k}$:

$$\left\{ \begin{array}{lcl} \mathrm{Re}(b_j + \sum_{l=1}^{k} a_{jl} x_l e^{i\theta_l}) & = & \mathrm{Re}\,(y_j e^{i\psi_j}) \\ \mathrm{Im}(b_j + \sum_{l=1}^{k} a_{jl} x_l e^{i\theta_l}) & = & \mathrm{Im}\,(y_j e^{i\psi_j}) \end{array} \right. \qquad (E)$$

with $j = 1, \dots, k$, has a solution in $(\mathbb{R}_+^*)^{2k}$. Moreover, if $\mathbb{Z}_2^{2k}$ is viewed as a subgroup of the real torus $(S^1)^{2k}$, then $s.\Theta \in \bigcup_{u \in \mathbb{Z}_2^{2k}} \mathrm{Reg}(co\mathscr{A}(\mathscr{P}_u(k)))$ means that the system (E) has a solution in $(\mathbb{R}^*)^{2k}$.

Since the matrix $A(z)$ defined by:

$$A(z) = \begin{pmatrix} a_{11}z_1 & a_{12}z_2 & \dots & a_{1k}z_k & -z_{k+1} & 0 & 0 & \dots & 0 \\ a_{21}z_1 & a_{22}z_2 & \dots & a_{2k}z_k & 0 & -z_{k+2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1}z_1 & a_{k2}z_2 & \dots & a_{kk}z_k & 0 & 0 & 0 & \dots & -z_{2k} \end{pmatrix}$$

is the image under the logarithmic Gauss map of the point z in $\mathscr{P}$, and the matrix $A(z)$ has rank k when z is a regular point.

Claim I. If $\overline{A}$ denotes the matrix conjugate to A, then for any regular point z of $\mathscr{P}$ the matrix $\widehat{A}(z) = \begin{pmatrix} A(z) \\ \overline{A(z)} \end{pmatrix}$ is of rank $2k$.

Proof. In fact, the rows of the matrix $A(z)$ form a basis of the orthogonal space to $\mathscr{L}og(\mathscr{P})$ at the point $\mathscr{L}og(z)$. So, if the rank of $\widehat{A}(z)$ is less than $2k$, then the orthogonal space to $\mathscr{L}og(\mathscr{P})$ at $\mathscr{L}og(z)$ contains at least one real vector v different than zero. This is equivalent to saying that the tangent space to $\mathscr{L}og(\mathscr{P})$ at $\mathscr{L}og(z)$ contains at least one purely imaginary vector. Indeed, since v is a vector different than zero orthogonal to both $T_{\mathscr{L}og(z)}(\mathscr{L}og(\mathscr{P}))$ and $\mathrm{Im}(\mathbb{C}^{2k})$, then $T_{\mathscr{L}og(z)}(\mathscr{L}og(\mathscr{P})) \cap \mathrm{Im}(\mathbb{C}^{2k})$ must be of dimension at least one. By Lemma 5.1, this implies that z is a critical point for the logarithmic map, which is in contradiction with our assumption on z. □

The matrix defining the system (E) is $\widetilde{B}(\Theta) = \begin{pmatrix} \mathrm{Re}B(\Theta) \\ \mathrm{Im}B(\Theta) \end{pmatrix}$ where $B(\Theta)$ is

$$\begin{pmatrix} a_{11}e^{i\theta_1} & a_{12}e^{i\theta_2} & \dots & a_{1k}e^{i\theta_k} & -e^{i\psi_1} & 0 & 0 & \dots & 0 \\ a_{21}e^{i\theta_1} & a_{22}e^{i\theta_2} & \dots & a_{2k}e^{i\theta_k} & 0 & -e^{i\psi_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1}e^{i\theta_1} & a_{k2}e^{i\theta_2} & \dots & a_{kk}e^{i\theta_k} & 0 & 0 & 0 & \dots & -e^{i\psi_k} \end{pmatrix}.$$

We can check that the rank of $\widetilde{B}(\Theta)$ is the same as the rank of the matrix $\widetilde{A}(z) = \begin{pmatrix} \mathrm{Re}A(z) \\ \mathrm{Im}A(z) \end{pmatrix}$ with $z = (x_1e^{i\theta_1}, \dots, x_ke^{i\theta_k}, y_1e^{\psi_1}, \dots, y_ke^{i\psi_k})$, because the variables x_i and y_j are non zero for all $i, j = 1, \dots, k$.

Claim II. The rank of the matrix $\widetilde{A}(z)$ is equal to $2k$.

Proof. Suppose we have a non trivial linear combination of the rows of the matrix $\widetilde{A}(z)$ that is equal to zero. Hence, there exist a real numbers λ_l, and μ_l not all equal to zero, with $l = 1, \dots, k$ such that:

$$\sum_{l,j=1}^{k} \frac{\lambda_l}{2}\Big((z_ja_{lj} + \bar{z}_j\bar{a}_{lj}) - (z_{k+l} + \bar{z}_{k+l})\Big) + \frac{\mu_l}{2i}\Big((z_ja_{lj} - \bar{z}_j\bar{a}_{lj}) - (z_{k+l} - \bar{z}_{k+l})\Big) = 0.$$

We get:

$$\sum_{l=1}^{k}\left(\frac{\lambda_l - i\mu_l}{2}\right)\left(\sum_{j=1}^{k} z_ja_{lj} - z_{k+l}\right) + \sum_{l=1}^{k}\left(\frac{\lambda_l + i\mu_l}{2}\right)\left(\sum_{j=1}^{k} \bar{z}_j\bar{a}_{lj} - \bar{z}_{k+l}\right) = 0.$$

Since the matrix $\widehat{A}(z)$ is of rank $2k$ by Claim I, this implies that $\lambda_l - i\mu_l = 0$, and $\lambda_l + i\mu_l = 0$ for all $l = 1, \dots, k$. This means that all the λ_l's and the μ_l's vanish. This contradicts the fact that some of the real numbers λ_l's and μ_l's are different than zero by hypothesis. Hence, the real rank of the matrix $\widetilde{A}(z)$ is equal to $2k$. □

Proof of Proposition 5.2. Since the k-dimensional linear space $\mathscr{P}$ is generic, the coefficients b_j are different than zero, and the system (E) is consistent. Claim II shows that the system (E) has a unique solution for any Θ in the set of regular values of $co\mathscr{A}(\mathscr{P})$, which proves the first and the second statements of the proposition. The third statement comes from the fact that the set of $\Theta = (\theta_1, \dots, \theta_k, \psi_1, \dots, \psi_k)$ for which the determinant of $\widetilde{B}(\Theta)$ vanishes is a hypersurface in the real torus and then its $2k$-volume is zero. In other words, the union $\bigcup_{s\in\mathbb{Z}_2^{2k}} \mathrm{Reg}(co\mathscr{A}(\mathscr{P}_s))$ is an open dense subset of the real torus $(S^1)^{2k}$. □

Corollary 5.1. *The volume of the coamoeba of any generic k-dimensional linear space in $(\mathbb{C}^*)^{2k}$ is equal to π^{2k}.*

Proof. By Proposition 5.2 (iii), the volume of the disjoint union

$$\bigcup_{s\in\mathbb{Z}_2^{2k}} \mathrm{Reg}(co\mathscr{A}(\mathscr{P}_s))$$

is equal to the volume of all the real torus $(S^1)^{2k}$. Moreover, they have the same volume, because they are obtained from each other by translation (i.e., isometry of the real torus equipped with the flat metric). So, the volume of one of them must be equal to $(2\pi)^{2k}/2^{2k} = \pi^{2k}$. □

We compute the cardinality of the inverse image under the logarithmic map of any regular value in the amoeba of a generic k-dimensional real very affine linear space $\mathscr{P}(k) \subset (\mathbb{C}^*)^{2k}$.

Proposition 5.3. *Let $\mathscr{P}$ be a generic real affine k-dimensional linear subspace of $(\mathbb{C}^*)^{2k}$, and x be a regular value of its amoeba. Then, the cardinality of* $\mathrm{Log}^{-1}(x)$ *is equal* 2^k.

Proof. We assume that $\mathscr{P}$ is given by a parametrization ρ as in (4), where all the coefficients are real numbers. The matrix A defined by:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

is invertible, otherwise the image of ρ is a linear space of dimension strictly less than k. The following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{C}^*)^k & \xrightarrow{\rho} & (\mathbb{C}^*)^{2k} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A\times Id} \\ (\mathbb{C}^*)^k & \xrightarrow{\rho'} & (\mathbb{C}^*)^{2k}, \end{array}$$

where ρ' is the parametrization given by:

$$\begin{array}{cccc} \rho' : & (\mathbb{C}^*)^k & \longrightarrow & (\mathbb{C}^*)^{2k} \\ & (T_1, \dots, T_k) & \longmapsto & (T_1, \dots, T_k, b_1 + T_1, \dots, b_k + T_k). \end{array}$$

Each regular value of the amoeba of the k-dimensional linear space $\mathscr{L} := \rho'((\mathbb{C}^*)^k)$ is covered 2^k times under the logarithmic mapping. Indeed, $\mathscr{L}$ is a product of lines $L_1, \ldots, L_k$ in $\mathbb{C}^2$. The matrix A is real, so the image of the set of critical points of the logarithmic mapping restricted to $\mathscr{P}$ is the set of critical points of the logarithmic mapping restricted to $\mathscr{L}$. By Lemma 5.1, if z is a critical in $\mathscr{P}$, then the tangent space to $\mathscr{L}og(\mathscr{P})$ at $\mathscr{L}og(z)$ contains at least one purely imaginary vector v. Since A is real, then the image of v in the tangent space to $\mathscr{L}og(\mathscr{L})$ at $\mathscr{L}og((A \times Id)(z)$ is also purely imaginary tangent vector, and then, the point $(A \times Id)(z)$ is critical. Let $\text{Critp}(\text{Log}_{|\mathscr{P}})$ and $\text{Critp}(\text{Log}_{|\mathscr{L}})$ be the set of critical points of the restriction of the logarithmic map to $\mathscr{P}$ and $\mathscr{L}$ respectively. Since the volume of their amoebas is finite (see [8]), this means that the set of critical values in their amoebas contains a subset of dimension $2k-1$ (at least the topological boundary of the amoeba). Hence, the number of connected components of $\mathscr{P} \setminus \text{Critp}(\text{Log}_{|\mathscr{P}})$ is equal to the number of connected components of $\mathscr{L} \setminus \text{Critp}(\text{Log}_{|\mathscr{L}})$. The fact that the set of critical points of the argument and the logarithmic maps coincide (see, e.g., [9]), and by Proposition 5.2, the restriction of the argument map to the set of regular points is injective, then, the cardinality of $\text{Log}^{-1}(x)$ is at most 2^k. Since $\mathscr{P}$ is real, then by Proposition 5.1, for any regular value $x \in \mathscr{A}(\mathscr{P}(k))$, the cardinality of $\text{Log}^{-1}(x)$ is at least 2^k. Hence, the cardinality of the inverse image of a regular value is equal to 2^k. □

Proof of Theorem 1.2. The first statement of Theorem 1.2 is Corollary 5.1. The second statement of Theorem 1.2 is because the cardinality of the inverse image of a regular value in the amoeba is constant and equal to 2^k, and the map $\text{Log} \circ \text{Arg}^{-1}$ conserve the volume (see [9], Proposition 3,1). Hence, the volume of the amoeba in this case is equal to the volume of the coamoeba divided by 2^k.

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