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Welschinger Invariants Revisited

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To the memory of Mikael Passare, remarkable mathematician and beautiful personality

Abstract. We establish the enumerativity of (original and modified) Welschinger invariants for every real divisor on any real algebraic del Pezzo surface and give an algebro-geometric proof of the invariance of that count both up to variation of the point constraints on a given surface and variation of the complex structure of the surface itself.

- Мы говорим с тобой на разных языках,
как всегда, - отозвался Воланд,
- но вещи, о которых мы говорим,
от этого не меняются.

*М. Булгаков. Мастер и Маргарита.**

Introduction

The discovery of Welschinger invariants [27, 28] has revolutionized real enumerative geometry. Since then much effort was devoted to the numerical study of Welschinger invariants, especially in the case of real del Pezzo surfaces, which allowed one to prove long time stated conjectures on existence of real solutions in corresponding enumerative problems and to observe a new, unexpected phenomena of abundance (see [2, 12, 14, 16, 17, 21]); it also led to introducing certain modified Welschinger invariants (see [16]). This development raised several natural questions: first, for which real del Pezzo surfaces the Welschinger invariants are strongly enumerative (*i.e.*, provided by a count, with weights $\pm$ 1, of real rational curves in a given divisor class, passing through a suitable number of real and complex conjugated points) and, second, to what extent such a count is invariant under deformations of the surface. The enumerative nature of the invariants in the

* "We speak different languages, as usual," responded Woland, "but this does not change the things we speak about."- *M. Bulgakov. The Master and Margarita.*

symplectic setting is the key point of [28], but it does not imply their enumerative nature in the algebro-geometric setting because of stronger genericity assumptions. The deformation invariance in the symplectic setting implies the deformation invariance in the algebro-geometric setting, but in [28] the symplectic deformation invariance is declared without proof. Therefore, our principal motivation has been to answer the question on algebro-geometric enumerativity of Welschinger invariants on real del Pezzo surfaces, and to prove the deformation invariance in the algebro-geometric setting. Our second motivation is an expectation that a good understanding of enumeration of real rational curves on real del Pezzo surfaces can help to extend the results to other types of surfaces and to curves of higher genus (such an expectation is confirmed now by [18, 24]). The algebro-geometric framework can be also helpful in the study of algorithmic and complexity aspects.

In most of the papers on the subject, the algebro-geometric enumerativity of Welschinger invariants on del Pezzo surfaces is considered as known. Indeed, it follows from enumerativity of Gromov–Witten invariants for such surfaces, and in the literature on Gromov–Witten invariants the latter enumerativity is considered as known. However, a careful analysis, see Lemma 9, has shown that there is one, and luckily only one, exception (apparently not mentioned in the literature): that is the case of del Pezzo surfaces of canonical degree 1 and $D = -K_\Sigma$; for any other pair of a real del Pezzo surface and a real divisor on it, the Welschinger invariants, original and modified, are strongly enumerative (in the above exceptional case, the number of solutions is still finite, but certain solutions may acquire some nontrivial multiplicity).

To prove the deformation invariance, we split the task into two parts. First, we fix the complex structure and vary the position of the points. Here, our strategy is close to that of the original proof of Welschinger in [28], but uses algebro-geometric tools instead of symplectic ones. In fact, already some time ago in [15] we have undertaken an attempt to give a purely algebro-geometric proof of such an invariance. However, that proof appears to be incomplete, since one type of local bifurcations in the set of counted curves was missing; it shows up for del Pezzo surfaces of canonical degree 1 and $D = -2K_\Sigma$ (see Lemma 11 (i) below, which states, in particular, that the closure of the one-dimensional family of rational curves in $|-2K_\Sigma|$ contains non-reduced curves). To the best of our knowledge, up to now this bifurcation has not been addressed in the literature, but it is unavoidable even in the symplectic setting (contrary to [28, Remark 2.12]). This step is summed up in Proposition 4, which states the invariance of the Welschinger count under the variation of points for any real divisor on each real del Pezzo surface.

The crucial point of the next step is the invariance under crossing the walls that correspond to, so-called, uninodal del Pezzo surfaces. Here, our proof is based on a real version of the Abramovich–Bertran–Vakil formula (note that adapting the formula to the symplectic setting one can prove the symplectic deformation invariance following the same lines). In addition, as in the study of the enumerativity, there appears a case not to miss and to investigate separately, here this is the case of del Pezzo surfaces of canonical degree 1 and $D = -K_\Sigma$.

The paper is organized as follows. In Section 1 we recall a few basic facts concerning del Pezzo surfaces and their deformations, introduce Welschinger invariants in their modified version and formulate the main results. Section 2 develops technical tools needed for the proof of the main results. There, we study moduli spaces of stable maps of pointed genus zero curves to del Pezzo surfaces and uninodal del Pezzo surfaces, describe generic elements of these moduli spaces and generic elements of the codimension one strata. We show also that Welschinger numbers extend by continuity from the case of immersions to the case of birational stable maps with arbitrary singularities. Section 3 is devoted to the proof of the main results.

1. Definitions and main statements

1.1. Surfaces under consideration

Over $\mathbb{C}$, a del Pezzo surface is either $(\mathbb{P}^1)^2$ or $\mathbb{P}^2$ blown up at $0 \le k \le 8$ points. Conversely, blowing up $0 \le k \le 8$ points of $\mathbb{P}^2$ yields a del Pezzo surface if and only if no 3 points lie on a straight line, no 6 lie on a conic, and no 8 points lie on a rational cubic having a singularity at one of these 8 points.

Del Pezzo surfaces of degree $d = K^2 = 9 - k \ge 5$ have no moduli. If $d = 9$ or $7 \ge d \ge 5$, then there is only one, up to isomorphism, del Pezzo surface of degree d and it can be seen as a blown up $\mathbb{P}^2$. If $d = 8$, then there are 2 isomorphism classes: $(\mathbb{P}^1)^2$ and $\mathbb{P}^2$ blown up at a point. The latter two surfaces are not deformation equivalent. For $4 \ge d \ge 1$ the moduli space of del Pezzo surfaces of degree $d = 9-k$ is an irreducible $(2k - 8)$-dimensional variety.

All del Pezzo surfaces of given degree $d \ne 8$ are deformation equivalent to each other, and, for our purpose, it will be more convenient to use, instead of the moduli spaces, the deformation spaces, that is, to fix in each deformation class one of the del Pezzo surfaces (say, a blow up of $\mathbb{P}^2$ at a certain generic collection of points) and consider the Kodaira–Spencer–Kuranishi space, *i.e.*, the space of all complex structures on the underlying smooth 4-manifold factorized by the action of diffeomorphisms isotopic to identity. Naturally, we awake this space only when $d \le 4$. We denote it by $\mathcal{D}_d$. Del Pezzo surfaces of degree d form in $\mathcal{D}_d$ an open dense subset, which we denote by $\mathcal{D}_d^{DP}$.

The problem of deformations of complex structures on rational surfaces is not obstructed, since $H^2(X, \mathcal{T}_X) = 0$ for any smooth rational surface X (here and further on, we denote by $\mathcal{T}_X$ the tangent sheaf). In addition, for degree $d \le 4$ del Pezzo surfaces as well as for any generic smooth rational surface X with $K_X^2 \le 4$, we have $H^0(X, \mathcal{T}_X) = 0$, so that at such points the Kodaira–Spencer–Kuranishi space is smooth (but not necessarily Hausdorff).

In fact, the only properties of this space which we use further on are the following. We call a surface $\Sigma \in \mathcal{D}_d$ *uninodal del Pezzo* if it contains a smooth rational (-2)-curve E_Σ, and $-K_\Sigma C > 0$ for each irreducible curve $C \ne E_\Sigma$ (in particular, $C^2 \geq -1$). For $d \leq 4$, denote by $\mathcal{D}_d(A_1) \subset \mathcal{D}_d$ the subspace formed by uninodal del Pezzo surfaces.

Proposition 1. *All but finite number of surfaces in a generic one-parameter Kodaira–Spencer family of rational surfaces with $1 \leq K_\Sigma^2 \leq 4$ are unnodal (i.e., del Pezzo), while the exceptional members of the family are uninodal del Pezzo.*

Proof. Let us denote by $\mathcal{T}_{X\|D}$ the subsheaf of the sheaf $\mathcal{T}_X$ generated by vectors fields tangent to D, and by $\mathcal{N}'_{D/X}$ their quotient, so that we obtain the following short exact sequence of sheafs:

$$0 \to \mathcal{T}_{X\|D} \to \mathcal{T}_X \to \mathcal{N}'_{D/X} \to 0.$$

According to the well-known theory of deformations of pairs (see [22, Section 3.4.4]), and due to the long exact cohomology sequence associated to the above short sequence, it is sufficient to show that $h^1(\mathcal{N}'_{D/X}) \geq 2$ if D is either a rational irreducible curve with $D^2 \leq -3$ or $D = D_1 \cup D_2$ where $D_i^2 \leq -2$. In the first case, it follows from Serre–Riemann–Roch duality. In the second case, from the exactness of the fragment $H^0(\mathcal{N}_{D_2/X}) \to H^1(\mathcal{N}_{D_1/X}) \to H^1(\mathcal{N}'_{D/X}) \to H^1(\mathcal{N}_{D_2/X})$ of the long cohomology sequence associated with the exact sequence of sheaves $0 \to \mathcal{N}_{D_1/X} \to \mathcal{N}'_{D/X} \to \mathcal{N}_{D_2/X} \to 0$.[†] $\square$

By a *real algebraic surface* we understand a pair (Y, c), where Y is a complex algebraic surface and $c : Y \to Y$ is an antiholomorphic involution. The classification of minimal real rational surfaces and the classification of real del Pezzo surfaces are well known: they are summarized in the two propositions below, respectively (see, e.g., [7, Theorems 6.11.11 and 17.3]).

Proposition 2. *Each minimal real rational surface Y is one of the following:*

(1) *$\mathbb{P}^2$ with its standard real structure ($d = 9$), the real part $\mathbb{R}Y$ of Y is homeomorphic to $\mathbb{RP}^2$.*
(2) *$\mathbb{P}^1 \times \mathbb{P}^1$ with one of its four nonequivalent real structures ($d = 8$): $\mathbb{R}Y = (S^1)^2$, $\mathbb{R}Y = S^2$, and two structures with $\mathbb{R}Y = \emptyset$;*
(3) *rational geometrically ruled surfaces $\mathcal{F}_a$, $a \geq 2$, with $\mathbb{R}Y = \#_2\mathbb{RP}^2$ and the standard real structure, if a is odd, and with $\mathbb{R}Y = (S^1)^2$ or $\emptyset$ and one of the two respective nonequivalent structures, if a is even ($d = 8$);*
(4) *real conic bundles over $\mathbb{P}^1$ with $2m \geq 4$ reducible fibers, which are all real and consist of pairs of complex conjugate exceptional curves ($d = 8 - 2m$), $\mathbb{R}Y = mS^2$;*
(5) *del Pezzo surfaces of degree $d = 1$ or 2: $\mathbb{R}Y = \mathbb{RP}^2 \sqcup 4S^2$, if $d = 1$, and $\mathbb{R}Y = 3S^2$ or $4S^2$, if $d = 2$.*

[†]In both cases, we use the equality $H^2(\mathcal{T}_{X\|D}) = 0$, which can be deduced, for example, from Serre duality, $H^2(\mathcal{T}_{X\|D}) = (H^0(\Omega^1_X(\log D) \otimes K))^*$, and Bogomolov–Sommese vanishing $H^0(\Omega^1_X(\log D) \otimes K) = 0$; the latter holds in our case since X is a rational surface with $K^2 \geq 1$, and thus its anticanonical Iitaka–Kodaira dimension is equal to 2.

Proposition 3. *With one exception, a real del Pezzo surface (Y, c) of degree $d \geq 1$ is determined up to deformation by the topology of $\mathbb{R}Y$. In the exceptional case $d = 8$ and $\mathbb{R}Y = \varnothing$, there are two deformation classes, distinguished by whether Y/c is Spin or not.*

The topological types of $\mathbb{R}Y$ are the following extremal types and their derivatives, which are obtained from the extremal ones by sequences of topological Morse simplifications of $\mathbb{R}Y$:

$$
\begin{array}{ll}
d = 9 & \mathbb{R}Y = \mathbb{R}\mathbb{P}^2; \\
d = 8 & \mathbb{R}Y = \#_2\mathbb{R}\mathbb{P}^2 \ \textit{or}\ (S^1)^2; \\
d = 7 & \mathbb{R}Y = \#_3\mathbb{R}\mathbb{P}^2; \\
d = 6 & \mathbb{R}Y = \#_4\mathbb{R}\mathbb{P}^2 \ \textit{or}\ (S^1)^2; \\
d = 5 & \mathbb{R}Y = \#_5\mathbb{R}\mathbb{P}^2; \\
d = 4 & \mathbb{R}Y = \#_6\mathbb{R}\mathbb{P}^2,\ (S^1)^2,\ \textit{or}\ 2S^2; \\
d = 3 & \mathbb{R}Y = \#_7\mathbb{R}\mathbb{P}^2 \ \textit{or}\ \mathbb{R}\mathbb{P}^2 \sqcup S^2; \\
d = 2 & \mathbb{R}Y = \#_8\mathbb{R}\mathbb{P}^2,\ 2\mathbb{R}\mathbb{P}^2,\ \#_2\mathbb{R}\mathbb{P}^2 \sqcup S^2,\ (S^1)^2,\ \textit{or}\ 4S^2; \\
d = 1 & \mathbb{R}Y = \#_9\mathbb{R}\mathbb{P}^2,\ \#_2\mathbb{R}\mathbb{P}^2 \sqcup \mathbb{R}\mathbb{P}^2,\ \#_3\mathbb{R}\mathbb{P}^2 \sqcup S^2,\ \textit{or}\ \mathbb{R}\mathbb{P}^2 \sqcup 4S^2.
\end{array}
$$

1.2. Main results

Let us consider a real del Pezzo surface (Σ, c), and assume that its real point set $\mathbb{R}\Sigma = \mathrm{Fix}(c)$ is nonempty. Pick a real divisor class $D \in \mathrm{Pic}(\Sigma)$, satisfying $-DK_\Sigma > 0$ and $D^2 \geq -1$, and put $r = -DK_\Sigma - 1$. Fix an integer m such that $0 \leq 2m \leq r$ and introduce a real structure $c_{r,m}$ on Σ^r that maps $(w_1, \ldots, w_r) \in \Sigma^r$ to $(w'_1, \ldots, w'_r) \in \Sigma^r$ with $w'_i = c(w_i)$ if $i > 2m$, and $(w'_{2j-1}, w'_{2j}) = (c(w_{2j}), c(w_{2j-1}))$ if $j \leq m$. With respect to this real structure a point $\boldsymbol{w} = (w_1, \ldots, w_r)$ is real, *i.e.*, $c_{r,m}$-invariant, if and only if w_i belongs to the real part $\mathbb{R}\Sigma$ of Σ for $i > 2m$ and w_{2j-1}, w_{2j} are conjugate to each other for $j \leq m$. In what follows we work with an open dense subset $\mathcal{P}_{r,m}(\Sigma)$ of $\mathbb{R}\Sigma^r = \mathrm{Fix}\, c_{r,m}$ consisting of $c_{r,m}$-invariant r-tuples $\boldsymbol{w} = (w_1, \ldots, w_r)$ with pairwise distinct $w_i \in \Sigma$.

Observe that, if a real irreducible rational curve $C \in |D|$ can be traced through all the points w_i of $\boldsymbol{w}$ and $2m < r = -CK_\Sigma - 1$, the real points of $\boldsymbol{w}$ must lie on the unique one-dimensional connected component of the real part of C, hence must belong to the same connected component of $\mathbb{R}\Sigma$. In the case $2m = r$, each real rational curve $C \in |D|$ passing through a collection of m pairs of complex conjugate points of Σ has an odd intersection with the real divisor K_Σ, hence C has a homologically non-trivial real part in $\mathbb{R}\Sigma$.

Thus, we fix a connected component F of $\mathbb{R}\Sigma$ and put

$$\mathcal{P}_{r,m}(\Sigma, F) = \{\boldsymbol{w} = (w_1, \ldots, w_r) \in \mathcal{P}_{r,m}(\Sigma)\ :\ w_i \in F \text{ for } i > 2m\}\ .$$

Denote by $\mathcal{M}_{0,r}(\Sigma, D)$ the set of isomorphism classes of pairs $(\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p})$, where $\nu : \mathbb{P}^1 \to \Sigma$ is a holomorphic map such that $\nu_*(\mathbb{P}^1) \in |D|$, and $\boldsymbol{p}$ is a sequence of r pairwise distinct points in $\mathbb{P}^1$. Put

$$\begin{aligned}\mathcal{R}(\Sigma, D, F, \boldsymbol{w}) \quad = \quad & \{[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathcal{M}_{0,r}(\Sigma, D)\ : \\ & \quad \nu \circ \mathrm{Conj} = c \circ \nu,\ \nu(\mathbb{RP}^1) \subset F,\ \nu(\boldsymbol{p}) = \boldsymbol{w}\}\,,\end{aligned}$$

where $\mathrm{Conj} : \mathbb{P}^1 \to \mathbb{P}^1$ is the complex conjugation. If either the degree of Σ is greater than 1, or $D \ne -K_\Sigma$, then for any generic r-tuple $\boldsymbol{w} \in \mathcal{P}_{r,m}(\Sigma, F)$, the set $\mathcal{R}(\Sigma, D, F, \boldsymbol{w})$ is finite and presented by immersions (see Lemma 9). In such a case, pick a conjugation-invariant class $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ and put

$$W_m(\Sigma, D, F, \varphi, \boldsymbol{w}) = \sum_{[\nu,\boldsymbol{p}] \in \mathcal{R}(\Sigma,D,F,\boldsymbol{w})} (-1)^{C_+ \circ C_- + C_+ \circ \varphi}\ , \tag{1}$$

where $C_\pm = \nu(\mathbb{P}^1_\pm)$ with $\mathbb{P}^1_+, \mathbb{P}^1_-$ being the two connected components of $\mathbb{P}^1 \setminus \mathbb{RP}^1$.

If the degree of Σ is equal to 1 and $D = -K_\Sigma$, then for any generic r-tuple $\boldsymbol{w} \in \mathcal{P}_{r,m}(\Sigma, F)$ and any conjugation-invariant class $\varphi \in H_2(\Sigma \setminus F; \mathbb{Z}/2)$ we define the number $W_m(\Sigma, D, F, \varphi, \boldsymbol{w})$ by the formula (1) retaining in it only the classes $[\nu, \boldsymbol{p}]$ presented by immersions.

If $\varphi = 0$, we get the original definition of Welschinger [27, 28].

Proposition 4. *The number $W_m(\Sigma, D, F, \varphi, \boldsymbol{w})$ does not depend on the choice of a generic element $\boldsymbol{w} \in \mathcal{P}_{r,m}(\Sigma, F)$.*

Proposition 4 is in fact a special case of more general deformation invariance statements. Consider a smooth real surface X_0 with $\mathbb{R}X_0 \ne \varnothing$, a real divisor class $D_0 \in \mathrm{Pic}(X_0)$, a connected component F_0 of $\mathbb{R}X_0$, a conjugation-invariant class $\varphi_0 \in H_2(X_0 \setminus F_0; \mathbb{Z}/2)$, and a conjugation invariant collection $\boldsymbol{w}_0$ of points in X_0. By an *elementary deformation* of the tuple $(X_0, D_0, F_0, \varphi_0)$ (respectively, $(X_0, D_0, F_0, \varphi_0, \boldsymbol{w}_0)$) we mean a one-parameter smooth family of smooth surfaces X_t, $t \in [-1, 1]$, extended to a continuous family of tuples $(X_t, D_t, F_t, \varphi_t)$ (respectively, $(X_t, D_t, F_t, \varphi_t, \boldsymbol{w}_t)$). Two tuples $T = (X, D, F, \varphi)$ and $T' = (X', D', F', \varphi')$ are called *deformation equivalent* if they can be connected by a chain $T = T^{(0)}$, $\ldots$, $T^{(k)} = T'$ so that any two neighboring tuples in the chain are isomorphic to fibers of an elementary deformation.

Proposition 5. *Let $(\Sigma_t, D_t, F_t, \varphi_t, \boldsymbol{w}_t)$, $t \in [-1, 1]$, be an elementary deformation of tuples such that all surfaces Σ_t, $t \ne 0$, belong to $\mathcal{D}_d^{\mathrm{DP}}$ for some $1 \le d \le 9$, and the collections $\boldsymbol{w}_{\pm 1}$ belong to $\mathcal{P}_{r,m}(\Sigma_{\pm 1}, F_{\pm 1})$ and are generic. Then,*

$$W_m(\Sigma_{-1}, D_{-1}, F_{-1}, \varphi_{-1}, \boldsymbol{w}_{-1}) = W_m(\Sigma_1, D_1, F_1, \varphi_1, \boldsymbol{w}_1)\ . \tag{2}$$

We skip $\boldsymbol{w}$ in the notation of the numbers $W_m(\Sigma, D, F, \varphi, \boldsymbol{w})$ and call them *Welschinger invariants.*

Proposition 5 plays a central role in the proof of the following statement.

Theorem 6. *If tuples (Σ, D, F, φ) and $(\Sigma', D', F', \varphi')$ are deformation equivalent, then $W_m(\Sigma, D, F, \varphi) = W_m(\Sigma', D', F', \varphi')$.*

Proofs of Propositions 4 and 5, as well as the proof of Theorem 6, are found in Section 3.

2. Families of rational curves on rational surfaces

2.1. General setting

Let Σ be a smooth rational surface, and $D \in \mathrm{Pic}(\Sigma)$ a divisor class. Denote by $\overline{\mathcal{M}}_{0,n}(\Sigma, D)$ the space of the isomorphism classes of pairs $(\nu : \hat{C} \to \Sigma, \boldsymbol{p})$, where $\hat{C}$ is either $\mathbb{P}^1$ or a connected reducible nodal curve of arithmetic genus zero, $\nu_*\hat{C} \in |D|$, $\boldsymbol{p} = (p_1, \dots, p_n)$ is a sequence of distinct smooth points of $\hat{C}$, and each component of $\hat{C}$ contracted by ν contains at least three special points. This moduli space is a projective scheme (see [9]), and there are natural morphisms

$$\Phi_{\Sigma,D} : \overline{\mathcal{M}}_{0,n}(\Sigma, D) \to |D|, \quad [\nu : \hat{C} \to \Sigma, \boldsymbol{p}] \mapsto \nu_*\hat{C} \ ,$$
$$\mathrm{Ev} : \overline{\mathcal{M}}_{0,n}(\Sigma, D) \to \Sigma^n, \quad [\nu : \hat{C} \to \Sigma, \boldsymbol{p}] \mapsto \nu(\boldsymbol{p}) \ .$$

For any subscheme $\mathcal{V} \subset \overline{\mathcal{M}}_{0,n}(\Sigma, D)$, define the *intersection dimension* $\mathrm{idim}\mathcal{V}$ of $\mathcal{V}$ as follows:

$$\mathrm{idim}\mathcal{V} = \dim(\Phi_{\Sigma,D} \times \mathrm{Ev})(\mathcal{V}) \ ,$$

where the latter value is the maximum over the dimensions of all irreducible components.

Put

$$\mathcal{M}^{br}_{0,n}(\Sigma, D) = \{[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathcal{M}_{0,n}(\Sigma, D) \ : \ \nu \text{ is birational onto } \nu(\mathbb{P}^1)\},$$
$$\mathcal{M}^{im}_{0,n}(\Sigma, D) = \{[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathcal{M}_{0,n}(\Sigma, D) \ : \ \nu \text{ is an immersion}\} \ .$$

Denote by $\overline{\mathcal{M}^{br}_{0,n}}(\Sigma, D)$ the closure of $\mathcal{M}^{br}_{0,n}(\Sigma, D)$ in $\overline{\mathcal{M}}_{0,n}(\Sigma, D)$, and introduce also the space

$$\mathcal{M}'_{0,n}(\Sigma, D) = \{[\nu : \hat{C} \to \Sigma, \boldsymbol{p}] \in \overline{\mathcal{M}^{br}_{0,n}}(\Sigma, D) \ : \ \hat{C} \simeq \mathbb{P}^1\} \ .$$

The following statement will be used below.

Lemma 7. *For any element*

$$[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \mathcal{M}^{br}_{0,n}(\Sigma, D) \ \ \textit{such that} \ \ \nu(\boldsymbol{p}) \cap \mathrm{Sing}\,(\nu(\mathbb{P}^1)) = \emptyset,$$

the map $\Phi_{\Sigma,D} \times \mathrm{Ev}$ *is injective in a neighborhood of that element, and, for the germ at* $[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}]$ *of any irreducible subscheme* $\mathcal{V} \subset \mathcal{M}^{br}_{0,n}(\Sigma, D)$, *we have*

$$\dim \mathcal{V} = \mathrm{idim}\mathcal{V} \ .$$

Proof. The inequality $\mathrm{idim}\mathcal{V} \le \dim \mathcal{V}$ is immediate from the definition. The opposite inequality and the injecitvity of $\Phi_{\Sigma,D} \times \mathrm{Ev}$ follow from the observation that, for an irreducible rational curve $C \in |D|$ and a tuple $\boldsymbol{z} \subset C \setminus \mathrm{Sing}\,(C)$ of n distinct points, the normalization map $\nu : \mathbb{P}^1 \to C$ and the lift $\boldsymbol{p} = \nu^{-1}(\boldsymbol{z})$ represent the unique preimage of $(C, \boldsymbol{z}) \in |D| \times \Sigma^n$ in $\mathcal{M}^{br}_{0,n}(\Sigma, D)$. $\square$

2.2. Curves on del Pezzo and uninodal del Pezzo surfaces

We establish here certain properties of the spaces $\mathcal{M}^{im}_{0,0}(\Sigma, D)$, $\mathcal{M}^{br}_{0,0}(\Sigma, D)$, and $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$, notably, compute dimension and describe generic members of these spaces as well as of some divisors therein. These properties basically follow from [10, Theorem 4.1 and Lemma 4.10]. However, the cited paper considers the plane blown up at generic points, whereas we work with arbitrary del Pezzo or uninodal del Pezzo surfaces. For this reason, we supply all claims with complete proofs.

Through all this section we use the notation

$$r = -DK_\Sigma - 1.$$

Lemma 8. *If Σ is a smooth rational surface and $-DK_\Sigma > 0$, then the space $\mathcal{M}^{im}_{0,0}(\Sigma, D)$ is either empty, or a smooth variety of dimension r.*

Proof. Let $[\nu : \mathbb{P}^1 \to \Sigma] \in \mathcal{M}^{im}_{0,0}(\Sigma, D)$. The Zariski tangent space to $\mathcal{M}^{im}_{0,0}(\Sigma, D)$ at $[\nu : \mathbb{P}^1 \to \Sigma]$ can be identified with $H^0(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1})$, where $\mathcal{N}^\nu_{\mathbb{P}^1} = \nu^*\mathcal{T}\Sigma/\mathcal{T}\mathbb{P}^1$ is the normal bundle. Since

$$\deg \mathcal{N}^\nu_{\mathbb{P}^1} = -DK_\Sigma - 2 \ge -1 > (2g-2)\big|_{g=0} = -2 , \tag{3}$$

we have

$$h^1(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1}) = 0 , \tag{4}$$

and hence $\mathcal{M}^{im}_{0,0}(\Sigma, D)$ is smooth at $[\nu : \mathbb{P}^1 \to \Sigma]$ and is of dimension

$$h^0(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1}) = \deg \mathcal{N}^\nu_{\mathbb{P}^1} - g + 1 = -DK_\Sigma - 1 = r . \tag{5}$$

$\square$

Lemma 9.

(1) *Let $\Sigma \in \mathcal{D}^{\mathrm{DP}}_d$ and $-DK_\Sigma > 0$. Then, the following holds:*

- (i) *The space $\mathcal{M}^{br}_{0,0}(\Sigma, D)$ is either empty or a variety of dimension r, and* $\mathrm{idim}(\mathcal{M}_{0,0}(\Sigma, D) \setminus \mathcal{M}^{br}_{0,0}(\Sigma, D)) < r$.
- (ii) *If either $d > 1$ or $D \ne -K_\Sigma$, then $\mathcal{M}^{im}_{0,0}(\Sigma, D) \subset \mathcal{M}^{br}_{0,0}(\Sigma, D)$ is an open dense subset.*
- (iii) *There exists an open dense subset $U_1 \subset \mathcal{D}^{\mathrm{DP}}_1$ such that, if $\Sigma \in U_1$, then $\mathcal{M}_{0,0}(\Sigma, -K_\Sigma)$ consists of 12 elements, each corresponding to a rational nodal curve.*

(2) *Let $d \le 4$. There exists an open dense subset $U_d(A_1) \subset \mathcal{D}_d(A_1)$ such that if $\Sigma \in U_d(A_1)$ and $-DK_\Sigma > 0$, then*

- (i) $\mathrm{idim}\mathcal{M}_{0,0}(\Sigma, D) \le r$;
- (ii) *a generic element $[\nu : \mathbb{P}^1 \to \Sigma]$ of any irreducible component $\mathcal{V}$ of $\mathcal{M}_{0,0}(\Sigma, D)$ such that $\mathrm{idim}\mathcal{V} = r$, is an immersion, and the divisor $\nu^*(E_\Sigma)$ consists of DE_Σ distinct points.*

Proof. Let $\Sigma \in \mathcal{D}_d^{\rm DP} \cup \mathcal{D}_d(A_1)$. All the statements in the case of an effective $-K_\Sigma - D$ immediately follow from elementary properties of plane lines, conics, and cubics. Thus, we suppose that $-K_\Sigma - D$ is not effective.

Let $\mathcal{V}_1$ be an irreducible component of $\mathcal{M}_{0,0}^{br}(\Sigma, D)$ and $[\nu : \mathbb{P}^1 \to \Sigma]$ its generic element. Then by [19, Theorem II.1.2]

$$\dim \operatorname{Hom}(\mathbb{P}^1, \Sigma)_\nu \geq -DK_\Sigma + 2\chi(\mathcal{O}_{\mathbb{P}^1}) = -DK_\Sigma + 2 \ . \tag{6}$$

Reducing by the automorphisms of $\mathbb{P}^1$, we get

$$\dim \mathcal{V}_1 \geq -DK_\Sigma + 2 - 3 = r \ . \tag{7}$$

Hence, in view of Lemma 8, to prove that $\dim \mathcal{M}_{0,0}^{br}(\Sigma, D) = r$ and $\mathcal{M}_{0,0}^{im}(\Sigma, D)$ is dense in $\mathcal{M}_{0,0}^{br}(\Sigma, D)$, it is enough to show that $\dim(\mathcal{M}_{0,0}^{br}(\Sigma, D) \setminus \mathcal{M}_{0,0}^{im}(\Sigma, D)) < r$.

Notice, first, that, in the case $r = 0$, the curves $C \in \Phi_{\Sigma,D}(\mathcal{M}_{0,0}^{br}(\Sigma, D))$ are nonsingular due to the bound

$$-DK_\Sigma \geq (C \cdot C')(z) \geq s \ , \tag{8}$$

coming from the intersection of C with a curve $C' \in |-K_\Sigma|$ passing through a point $z \in C$, where C has multiplicity s. Thus, we suppose that $r > 0$. Let $\mathcal{V}_2$ be an irreducible component of $\mathcal{M}_{0,0}^{br}(\Sigma, D) \setminus \mathcal{M}_{0,0}^{im}(\Sigma, D)$, $[\nu : \mathbb{P}^1 \to \Sigma] \in \mathcal{V}_2$ a generic element, and let ν have $s \geq 1$ critical points of multiplicities $m_1 \geq \cdots \geq m_s \geq 2$. In particular, bound (8) gives

$$-DK_\Sigma \geq m_1 \ . \tag{9}$$

Then (*cf.* [5, First formula in the proof of Corollary 2.4]),

$$\dim \mathcal{V}_2 \leq h^0(\mathbb{P}^1, \mathcal{N}_{\mathcal{P}^1}^\nu / \operatorname{Tors}(\mathcal{N}_{\mathbb{P}^1}^\nu)) \ ,$$

where the normal sheaf $\mathcal{N}_{\mathbb{P}^1}^\nu$ on $\mathbb{P}^1$ is defined as the cokernel of the map $d\nu : \mathcal{T}\mathbb{P}^1 \to \nu^*\mathcal{T}\Sigma$, and $\operatorname{Tors}(*)$ is the torsion sheaf. It follows from [5, Lemma 2.6] (*cf.* also the computation in [5, Page 363]) that $\deg \operatorname{Tors}(\mathcal{N}_{\mathbb{P}^1}^\nu) = \sum_i (m_i - 1)$, and hence

$$\deg \mathcal{N}_{\mathbb{P}^1}^\nu / \operatorname{Tors}(\mathcal{N}_{\mathbb{P}^1}^\nu) = -DK_\Sigma - 2 - \sum_{i=1}^s (m_i - 1) \tag{10}$$

which yields

$$\begin{aligned} \dim \mathcal{V}_2 &\leq h^0(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1}^\nu / \operatorname{Tors}(\mathcal{N}_{\mathbb{P}^1}^\nu)) \\ &= \max\{\deg \mathcal{N}_{\mathbb{P}^1}^\nu / \operatorname{Tors}(\mathcal{N}_{\mathbb{P}^1}^\nu) + 1, \ 0\} \overset{(9)}{\leq} r - (m_1 - 1) < r, \end{aligned} \tag{11}$$

Let us show that $\operatorname{idim} \mathcal{V} < r$ for any irreducible component $\mathcal{V}$ of $\mathcal{M}_{0,0}(\Sigma, D) \setminus \mathcal{M}_{0,0}^{br}(\Sigma, D)$. Indeed, if a generic element $[\nu : \mathbb{P}^1 \to \Sigma] \in \mathcal{V}$ satisfies $\nu_*(\mathbb{P}^1) = sC$ for some $s \geq 2$, then

$$\operatorname{idim} \mathcal{V} \leq -\frac{1}{s} DK_\Sigma - 1 < -DK_\Sigma - 1 = r \ .$$

To complete the proof of (2ii), let us assume that $\dim \mathcal{V} = r$ and the divisor $\nu^*(E_\Sigma)$ contains a multiple point sz, $s \geq 2$. In view of $DE_\Sigma \geq s$ and $(-K_\Sigma - E_\Sigma)D \geq 0$ (remind that D is irreducible and $-K - D$ is not effective), we have $-DK_\Sigma \ge s$. Furthermore, $T_{[\nu]}\mathcal{V}$ can be identified with a subspace of $H^0(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1}(-(s-1)z))$ (*cf.* [5, Remark in page 364]). Since

$$\deg \mathcal{N}^\nu_{\mathbb{P}^1}(-(s-1)z)) = -DK_\Sigma - 1 - s \ge -1 > -2 \ ,$$

we have

$$H^1(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1}(-(s-1)z)) = 0 \ ,$$

and hence

$$\dim \mathcal{V} \le h^0(\mathbb{P}^1, \mathcal{N}^\nu_{\mathbb{P}^1}(-(s-1)z)) = r - (s-1) < r$$

contrary to the assumption $\dim \mathcal{V} = r$. □

Lemma 10. *There exists an open dense subset $U_2 \subset \mathcal{D}_1^{\mathrm{DP}}$ such that, for each $\Sigma \in U_2$, the set of effective divisor classes $D \in \mathrm{Pic}(\Sigma)$ satisfying $-DK_\Sigma = 1$ is finite, the set of rational curves in the corresponding linear systems $|D|$ is finite, and any two such rational curves C_1, C_2 either coincide, or are disjoint, or intersect in C_1C_2 distinct points.*

Proof. For any $\Sigma \in \mathcal{D}_1^{\mathrm{DP}}$, we have $\dim |-2K_\Sigma| = 3$. Hence, the condition $-DK_\Sigma = 1$ yields that $-2K_\Sigma - D$ is effective, which in turn implies the finiteness of the set of effective divisors such that $-DK_\Sigma = 1$. The finiteness of the set of rational curves in these linear systems $|D|$ follows from Lemma 9(i). At last, for a generic $\Sigma \in \mathcal{D}_1^{\mathrm{DP}}$, these curves are either singular elements in the elliptic pencil $|-K_\Sigma|$ or the (-1)-curves, and as it follows easily, for example, from considering Σ as a projective plane blown up at 8 generic points, any two of these curves intersect transversally and in distinct smooth points. □

Lemma 11. *Let U_1, U_2 be the subsets of $\mathcal{D}_1^{\mathrm{DP}}$ introduced in Lemmas 9 and 10, respectively. For each $\Sigma \in U_1 \cap U_2$, each $D \in \mathrm{Pic}(\Sigma)$ with $-DK_\Sigma > 0$ and $D^2 \ge -1$, and for each irreducible component $\mathcal{V}$ of $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D) \setminus \mathcal{M}^{br}_{0,0}(\Sigma, D)$ with* $\mathrm{idim}\mathcal{V} = r - 1$*, one has:*

(i) *A generic element $[\nu : \hat{C} \to \Sigma] \in \mathcal{V}$ is as follows*
- *$\hat{C} = \hat{C}_1 \cup \hat{C}_2$ with $\hat{C}_i \simeq \mathbb{P}^1$, $[\nu|_{\hat{C}_i} : \hat{C}_i \to \Sigma] \in \mathcal{M}^{im}_{0,0}(\Sigma, D_i)$, where $D_1D_2 > 0$ and $-D_iK_\Sigma > 0$, $D_i^2 \ge -1$ for each $i = 1, 2$;*
- *$\nu(\hat{C}_1) \ne \nu(\hat{C}_2)$, except for the only case when $D_1 = D_2 = -K_\Sigma$ and $\nu(\hat{C}_1) = \nu(\hat{C}_2)$ is one of the 12 uninodal curves in $|-K_\Sigma|$;*
- *ν is an immersion (i.e., a local isomorphism onto the image).*

Moreover, each element $[\nu : \hat{C} \to \Sigma] \in \overline{\mathcal{M}}_{0,0}(\Sigma, D)$ as above does belong to $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$.

(ii) *The germ of $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$ at a generic element of $\mathcal{V}$ is smooth.*

Proof. Show, first, that $\mathrm{idim}(\mathcal{M}'_{0,0}(\Sigma, D) \setminus \mathcal{M}^{br}_{0,0}(\Sigma, D)) \le r - 2$. Assume on the contrary that there exists a component $\mathcal{V}$ of $\mathcal{M}'_{0,0}(\Sigma, D) \setminus \mathcal{M}^{br}_{0,0}(\Sigma, D)$ with $\mathrm{idim}\mathcal{V} =$

$r-1$ (idim$\mathcal{V}$ cannot be bigger by Lemma 9(i)). Then its generic element $[\nu : \mathbb{P}^1 \to \Sigma]$ is such that $\nu_*(\mathbb{P}^1) = sC$ with C an irreducible rational curve, $s \geq 2$. Thus,

$$r - 1 = -sCK_\Sigma - 2 \leq -CK_\Sigma - 1 = \dim \mathcal{M}^{br}_{0,0}(\Sigma, C) \ ,$$

which yields $s = 2$ and $-CK_\Sigma = 1$. By adjunction formula, either $C^2 = -1$, or $C^2 \geq 1$. The former case is excluded by the assumption $D^2 \geq -1$. In the case $C^2 \geq 1$, since $K^2_\Sigma = 1$ and $-CK_\Sigma = 1$, the only possibility is $C \in |-K_\Sigma|$. However, in such a case the map ν cannot be deformed into an element of $\mathcal{M}^{br}_{0,0}(\Sigma, -2K_\Sigma)$, since C has a node, and hence the deformed map would birationally send $\mathbb{P}^1$ onto a curve with δ-invariant ≥ 4, which is bigger than its arithmetic genus, $((-2K_\Sigma)^2 + (-2K_\Sigma)K_\Sigma)/2 + 1 = 2$.

Let $[\nu : \hat{C} \to \Sigma]$ be a generic element of an irreducible component $\mathcal{V}$ of $\overline{\mathcal{M}^{br}_{0,0}(\Sigma, D)} \setminus \mathcal{M}'_{0,0}(\Sigma, D)$ with idim$\mathcal{V} = r - 1$. Then $\hat{C}$ has at least 2 components. On the other side, if $\hat{C}$ had ≥ 3 components, Lemma 9(1) would yield idim$\mathcal{V} \leq -DK_\Sigma - 3 < r - 1$. Hence $\hat{C} = \hat{C}_1 \cup \hat{C}_2$, $\hat{C}_1 \simeq \hat{C}_2 \simeq \mathbb{P}^1$, and, according to Lemma 8 and Lemma 9(1), for each $i = 1, 2$ we have: $\nu_i = \nu\big|_{\hat{C}_i}$ is an immersion, $\dim \mathcal{M}_{0,0}(\Sigma, D_i)_{[\nu_i]} = -D_iK_\Sigma - 1$, and $-D_iK_\Sigma > 0$, $D_i^2 \geq -1$.

If $-DK_\Sigma = 2$ and $\nu(\hat{C}_1) \neq \nu(\hat{C}_2)$, then the intersection points of these curves are nodes, which follows from the definition of the set U_2 (see Lemma 10), and hence ν is an immersion at the node $\hat{z}$ of $\hat{C}$.

If $-DK_\Sigma = 2$ and $\nu(\hat{C}_1) = \nu(\hat{C}_2)$, then $D_1 = D_2$ and $D_1^2 = D_2^2 \geq 1$ in view of the adjunction formula and the condition $D^2 \geq -1$. It is easy to see that this is only possible, when $D_1 = D_2 = -K_\Sigma$. In particular, by the definition of the set U_1 (see Lemma 9(iii)), the curve $C = \nu(\hat{C}_1) = \nu(\hat{C}_2) \in |-K_\Sigma|$ has one node z. We then see that, ν takes the germ $(\hat{C}, \hat{z})$ isomorphically onto the germ (C, z), since, otherwise we would get a deformed map ν with the image whose δ-invariant ≥ 4, which is bigger than its arithmetic genus, $((-2K_\Sigma)^2 + (-2K_\Sigma)K_\Sigma)/2 + 1 = 2$.

Suppose, now, that $-DK_\Sigma > 2$, thus, $-D_1K_\Sigma > 1$. Then

$$\dim \mathcal{M}_{0,0}(\Sigma, D_1)_{[\nu_1]} > 0,$$

and hence $C_1 \neq C_2$. To prove that ν is an immersion at the node $\hat{z} \in \hat{C}$, we will show that any two local branches of ν_1 and ν_2 either are disjoint, or intersect transversally. Indeed, assume on the contrary that there exist $z_i \in \hat{C}_i$, $i = 1, 2$, such that $\nu_1(z_1) = \nu_2(z_2) = z \in \Sigma$, and $\nu_1(\hat{C}_1, z_1)$ intersects $\nu_2(\hat{C}_2, z_2)$ at z with multiplicity ≥ 2. Then

$$\dim \mathcal{M}_{0,0}(\Sigma, D_1)_{[\nu_1]} \leq h^0(\hat{C}_1, \mathcal{N}^{\nu_1}_{\hat{C}_1}(-z_1)) \ . \tag{12}$$

Since

$$\deg \mathcal{N}^{\nu_1}_{\hat{C}_1}(-z_1) = -D_1K_\Sigma - 2 - 1 = -D_1K_\Sigma - 3 > -2 \ ,$$

we get $h^1(\hat{C}_1, \mathcal{N}^{\nu_1}_{\hat{C}_1}(-z_1)) = 0$. Therefore,

$$\begin{aligned}\deg \mathcal{N}^{\nu_1}_{\hat{C}_1}(-z_1) &\le \deg \mathcal{N}^{\nu_1}_{\hat{C}_1}(-z_1) + 1 = -D_1K_\Sigma - 2 \\ &< -D_1K_\Sigma - 1 = \dim \mathcal{M}_{0,0}(\Sigma, D_1)_{[\nu_1]} \ ,\end{aligned}$$

which contradicts (12).

The smoothness of $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$ at $[\nu : \hat{C} \to \Sigma]$, where $\nu_*\hat{C}$ is a reduced nodal curve, follows from [25, Lemma 2.9], where the requirements are $D_iK_\Sigma < 0$, $i = 1, 2$. We will show that the same requirements suffice under assumption that ν is an immersion. Let us show that

$$T_{[\nu]}\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D) \simeq H^0(\hat{C}, \mathcal{N}^\nu_{\hat{C}}) \ , \tag{13}$$

where the normal sheaf $\mathcal{N}^\nu_{\hat{C}}$ comes from the exact sequence

$$0 \to \mathcal{T}_{\hat{C}} \to \nu^*\mathcal{T}_\Sigma \to \mathcal{N}^\nu_{\hat{C}} \to 0 \ , \tag{14}$$

$\mathcal{T}_\Sigma$ being the tangent bundle of Σ, and $\mathcal{T}_{\hat{C}}$ being the tangent sheaf of $\hat{C}$ viewed as the push-forward by the normalization $\pi : \hat{C}_1 \sqcup \hat{C}_2 \to \hat{C}$ of the subsheaf $\mathcal{T}'_{\hat{C}_1\sqcup\hat{C}_2} \subset \mathcal{T}_{\hat{C}_1\sqcup\hat{C}_2}$ generated by the sections vanishing at the preimages of the node $z \in \hat{C}$.

Indeed, the Zariski tangent space to $\mathrm{Hom}(\hat{C}, \Sigma)$ at ν is naturally isomorphic to $H^0(\hat{C}, \nu^*\mathcal{T}_\Sigma)$ (see [19, Theorem 1.7, Section II.1]). Next, we take the quotient by action of the germ of $\mathrm{Aut}(\hat{C})$ at the identity. This germ is smooth and acts freely on the germ of $\mathrm{Hom}(\hat{C}, \Sigma)$ at ν. The tangent space to $\mathrm{Aut}(\hat{C})$ at the identity is isomorphic to $H^0(\hat{C}, \mathcal{T}_{\hat{C}})$ (*cf.* [19, 2.16.4, Section I.2]). Since

$$H^1(\hat{C}, \mathcal{T}_{\hat{C}}) = H^1(\hat{C}_1 \sqcup \hat{C}_2, \mathcal{T}'_{\hat{C}_1\sqcup\hat{C}_2}) = H^1(\hat{C}_1, \mathcal{O}_{\hat{C}_1}(1)) \oplus H^1(\hat{C}_2, \mathcal{O}_{\hat{C}_2}(1)) = 0 \ , \tag{15}$$

the associated to (14) cohomology exact sequence yields

$$\begin{aligned}T_{[\nu]}\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D) &\simeq T_\nu \mathrm{Hom}(\hat{C}, \Sigma)/T_{\mathrm{Id}}\mathrm{Aut}(\hat{C}) \\ &\simeq H^0(\hat{C}, \nu^*\mathcal{T}_\Sigma)/H^0(\hat{C}, \mathcal{T}_{\hat{C}}) \simeq H^0(\hat{C}, \mathcal{N}^\nu_{\hat{C}}) \ .\end{aligned}$$

We will verify that

$$h^0(\hat{C}, \mathcal{N}^\nu_{\hat{C}}) = r \ , \tag{16}$$

which in view of $\dim_{[\nu]} \overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D) = r$ (see Lemma 9(i)) will imply the smoothness of $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$ at $[\nu]$. There exists a natural morphism of sheaves on $\hat{C}$:

$$\alpha : \pi_*\mathcal{N}^{\nu\circ\pi}_{\hat{C}_1\sqcup\hat{C}_2} \longrightarrow \mathcal{N}^\nu_{\hat{C}} \ ,$$

where α is an isomorphism outside z and acts at z as follows: since ν embeds the germ of $\hat{C}$ at z into Σ, one can identify the stalk $\left(\pi_*\mathcal{N}^{\nu\circ\pi}_{\hat{C}_1\sqcup\hat{C}_2}\right)_z$ with $\mathbb{C}\{x\} \oplus \mathbb{C}\{y\}$, the stalk $(\mathcal{N}^\nu_{\hat{C}})_z$ with $\mathbb{C}\{x, y\}/\langle xy\rangle$, and write

$$\alpha_z(f(x), g(y)) = xf(x) + yg(y) \in \left(\mathcal{N}^\nu_{\hat{C}}\right)_z \cong \mathbb{C}\{x, y\}/\langle xy\rangle \ .$$

Hence we obtain an exact sequence of sheaves

$$0 \to \pi_*\mathcal{N}^{\nu\circ\pi}_{\hat{C}_1\sqcup\hat{C}_2} \xrightarrow{\alpha} \mathcal{N}^\nu_{\hat{C}} \to \mathcal{O}_z \to 0 \ , \tag{17}$$

whose cohomology sequence vanishes at

$$h^1(z,\mathcal{O}_z)=0,\quad h^1(\hat{C}_1\sqcup\hat{C}_2,\mathcal{N}^{\nu\circ\pi}_{\hat{C}_1\sqcup\hat{C}_2})=0\ ,$$

(the latter one is equivalent to (4)); hence $h^1(\hat{C},\mathcal{N}^\nu_{\hat{C}})=0$ and, furthermore,

$$\begin{aligned} h^0(\hat{C},\mathcal{N}^\nu_{\hat{C}}) &= h^0(\hat{C}_1\sqcup\hat{C}_2,\mathcal{N}^{\nu\circ\pi}_{\hat{C}_1\sqcup\hat{C}_2}))+h^0(z,\mathcal{O}_z)\\ &= h^0(\hat{C}_1,\mathcal{N}^{\nu_1}_{\hat{C}_1})+h^0(\hat{C}_2,\mathcal{N}^{\nu_2}_{\hat{C}_2})+h^0(z,\mathcal{O}_z)\\ &\overset{cf.\ (5)}{=} (-D_1K_\Sigma-1)+(-D_2K_\Sigma-1)+1=r \end{aligned}$$

as predicted in (14).

Finally, let us show that any element $[\nu:\hat{C}\to\Sigma]\in\overline{\mathcal{M}_{0,0}}(\Sigma,D)$, satisfying conditions of 1(i)–1(iii), belongs to $\overline{\mathcal{M}^{br}_{0,0}}(\Sigma,D)$, or, equivalently, admits a deformation into a map $\mathbb{P}^1\to\Sigma$ birational onto its image. Indeed, it follows from [1, Theorem 15] under the condition $h^1(\hat{C},\nu^*\mathcal{T}_\Sigma)=0$, which one obtains from the cohomology exact sequence associated with (14) and from vanishing relations (15) and (16). $\square$

Lemma 12. *Consider the subsets U_1, U_2 of $\mathcal{D}^{\rm DP}_1$ introduced in Lemmas* 9 *and* 10, *respectively, a surface $\Sigma\in U_1\cap U_2\subset\mathcal{D}^{\rm DP}_1$, and an effective divisor class $D\in\mathrm{Pic}(\Sigma)$ such that $-DK_\Sigma\ge 2$. Let $\boldsymbol{w}=(w_1,\dots,w_r)$ be a sequence of r distinct points in Σ, let σ_i be smooth curve germs in Σ centered at w_i, $r'<i\le r$, for some $r'<r$, $\boldsymbol{w}'=(w_i)_{1\le i\le r'}$, and let*

$$\begin{aligned} &\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D;\boldsymbol{w}',\{\sigma_i\}_{r'<i\le r})\\ &\quad=\{[\nu:\hat{C}\to\Sigma,\boldsymbol{p}]\in\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D)\ :\\ &\qquad\quad \nu(p_i)=w_i\,\mathit{for}\ 1\le i\le r',\ \nu(p_i)\in\sigma_i,\ \mathit{for}\ r'<i\le r\}\ . \end{aligned}$$

(1) *Suppose that $[\nu:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]\in\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D;\boldsymbol{w})\cap\mathcal{M}^{im}_{0,r}(\Sigma,D)$. Then* Ev *sends the germ of $\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D;\boldsymbol{w}',\{\sigma_i\}_{r'<i\le r})$ at $[\nu:\mathbb{P}^1\to\Sigma,\boldsymbol{p}]$ diffeomorphically onto $\prod_{r'<i\le r}\sigma_i$.*

(2) *Suppose that $[\nu:\hat{C}\to\Sigma,\boldsymbol{p}]\in\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D;\boldsymbol{w})$ is such that*
   - $[\nu:\hat{C}\to\Sigma]\in\overline{\mathcal{M}^{br}_{0,0}}(\Sigma,D)$ *is as in Lemma* 11(i),
   - *$r'\ge -D_1K_\Sigma-1$, $\#(\boldsymbol{p}\cap\hat{C}_1)=-D_1K_\Sigma-1$, $\#(\boldsymbol{p}\cap\hat{C}_2)=-D_2K_\Sigma$, the point sequences $(w_i)_{1\le i<-D_1K_\Sigma}$, $(w_i)_{-D_1K_\Sigma\le i\le r}$ are generic on $C_1=\nu_*\hat{C}_1$, $C_2=\nu_*\hat{C}_2$, respectively, and the germs σ_i, $r'<i\le r$, cross C_2 transversally.*

   Then Ev *sends the germ of $\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D;\boldsymbol{w}',\{\sigma_i\}_{r'<i\le r})$ at $[\nu:\hat{C}\to\Sigma,\boldsymbol{p}]$ diffeomorphically onto $\prod_{r'<i\le r}\sigma_i$.*

Proof. Both statements follow from the fact that Ev diffeomorphically sends the germ of $\overline{\mathcal{M}^{br}_{0,r}}(\Sigma,D)$ at $[\nu:\hat{C}\to\Sigma,\boldsymbol{p}]$ onto the germ of Σ^r at $\boldsymbol{w}=\nu(\boldsymbol{p})$.

In view of $\dim \overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D) = 2r$ Lemma 9(i)), it is sufficient to show that the Zariski tangent space to $\mathrm{Ev}^{-1}(\boldsymbol{w})$ is zero-dimensional. In view of relation (13) this is equivalent to

$$h^0(\hat{C}, \mathcal{N}^{\nu}_{\hat{C}}(-\boldsymbol{p})) = 0 \ . \tag{18}$$

In the case of $[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}] \in \overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D; \boldsymbol{w}', \{\sigma_i\}_{r'<i\le r}) \cap \mathcal{M}^{im}_{0,r}(\Sigma, D)$, we have

$$\deg \mathcal{N}^{\nu}_{\hat{C}}(-\boldsymbol{p}) = (-DK_\Sigma - 2) - (-DK_\Sigma - 1) = -1 > -2 \ ,$$

and hence (18) follows by Riemann–Roch.

In the second case, put $\widetilde{\boldsymbol{p}} = \boldsymbol{p} \setminus \{p_r\}$ and twist the exact sequence (17) to get

$$0 \to \pi_* \mathcal{N}^{\nu\circ\pi}_{\hat{C}_1 \sqcup \hat{C}_2}(-\widetilde{\boldsymbol{p}}) \to \mathcal{N}^{\nu}_{\hat{C}}(-\widetilde{\boldsymbol{p}}) \to \mathcal{O}_z \to 0.$$

Since

$$\deg \mathcal{N}^{\nu_1}_{\hat{C}_i}(-\widetilde{\boldsymbol{p}} \cap \hat{C}_i) = (-D_i K_\Sigma - 2) - (-D_i K_\Sigma - 1) = -1 > -2, \quad i = 1, 2 \ ,$$

we have $h^1(\pi_* \mathcal{N}^{\nu\circ\pi}_{\hat{C}_1 \sqcup \hat{C}_2}(-\widetilde{\boldsymbol{p}})) = 0$, and $h^0(\hat{C}, \pi_* \mathcal{N}^{\nu\circ\pi}_{\hat{C}_1 \sqcup \hat{C}_2}(-\widetilde{\boldsymbol{p}})) = 0$, which yields that $H^0(\hat{C}, \mathcal{N}^{\nu}_{\hat{C}}(-\widetilde{\boldsymbol{p}}))$ is isomorphically mapped onto $H^0(z, \mathcal{O}_z) \simeq \mathbb{C}$. It implies that a non-zero global section of the sheaf $\mathcal{N}^{\nu}_{\hat{C}}(-\widetilde{\boldsymbol{p}})$ does not vanish at z, and hence, it does not vanish at p_r chosen on $\hat{C}_2$ in a generic way. Thus, (18) follows. $\square$

2.3. Deformation of isolated curve singularities

Let us recall a few facts on deformations of curve singularities (see, for example, [6]). Let Σ be a smooth algebraic surface, z an isolated singular point of a curve $C \subset \Sigma$, and $B_{C,z}$ the base of a semiuniversal deformation of the germ (C, z). This base can be viewed as a germ $(\mathbb{C}^N, 0)$ and can be identified with $\mathcal{O}_{C,z}/J_{C,z}$, where $J_{C,z} \subset \mathcal{O}_{C,z}$ is the Jacobian ideal.

The equigeneric locus $B^{eg}_{C,z} \subset B_{C,z}$ parametrizes local deformations of (C, z) with constant δ-invariant equal to $\delta(C, z)$. This locus is irreducible and has codimension $\delta(C, z)$ in $B_{C,z}$. The subset $B^{eg,im}_{C,z} \subset B^{eg}_{C,z}$ that parametrizes the immersed deformations is open and dense in $B^{eg}_{C,z}$, and consists only of smooth points of $B^{eg}_{C,z}$. The tangent cone $T_0 B^{eg}_{C,z}$ (defined as the limit of the tangent spaces at points of $B^{eg,im}_{C,z}$) can be identified with $J^{\mathrm{cond}}_{C,z}/J_{C,z}$, where $J^{\mathrm{cond}}_{C,z} \subset \mathcal{O}_{C,z}$ is the conductor ideal. The subset $B^{eg,nod}_{C,z} \subset B^{eg}_{C,z}$ that parameterizes the nodal deformations is also open and dense. Furthermore, $B^{eg}_{C,z} \setminus B^{eg,nod}_{C,z}$ is the closure of three codimension-one strata: $B^{eg}_{C,z}(A_2)$ that parameterizes deformations with one cusp A_2 and $\delta(C, z) - 1$ nodes, $B^{eg}_{C,z}(A_3)$ that parameterizes deformations with one tacnode A_3 and $\delta(C, z) - 2$ nodes, and $B^{eg}_{C,z}(D_4)$ that parameterizes deformations with one ordinary triple point D_4 and $\delta(C, z) - 3$ nodes.

If $C \subset \Sigma$ is a curve with isolated singularities, we consider the joint semiuniversal deformation for all singular points of C. The base of this deformation, the equigeneric locus, and the tangent cone to this locus at the point corresponding to C are as follows:

$$B_C = \prod_{z\in \mathrm{Sing}\,(C)} B_{C,z}, \quad B_C^{eg} = \prod_{z\in \mathrm{Sing}\,(C)} B_{C,z}^{eg}, \quad T_0 B_C^{eg} = \prod_{z\in \mathrm{Sing}\,(C)} T_0 B_{C,z}^{eg}\ .$$

Lemma 13. *Let $\nu : \mathbb{P}^1 \to \Sigma$ be birational onto its image $C = \nu(\mathbb{P}^1)$. Assume that $C \in |D|$, where D is a divisor class such that $r = -DK_\Sigma - 1 > 0$. Let $\boldsymbol{p}$ be an r-tuple of distinct points of $\mathbb{P}^1$ such that $\boldsymbol{w} = \nu(\boldsymbol{p})$ is an r-tuple of distinct nonsingular points of C. Let $|D|_{\boldsymbol{w}} \subset |D|$ be the linear subsystem of curves passing through $\boldsymbol{w}$, and $\Lambda(\boldsymbol{w}) \subset B_C$ be the natural image of $|D|_{\boldsymbol{w}}$.*

1. *One has $\mathrm{codim}_{B_C}\Lambda(\boldsymbol{w}) = \dim B_C^{eg}$, and $\Lambda(\boldsymbol{w})$ intersects $T_0 B_C^{eg}$ transversally.*
2. *For any r-tuple $\widetilde{\boldsymbol{w}} \in \Sigma^r$ sufficiently close to $\boldsymbol{w}$ and such that $\Lambda(\widetilde{\boldsymbol{w}})$ intersects B_C^{eg} transversally and only at smooth points, the natural map from the germ $\mathcal{M}_{0,r}(\Sigma, D)_{[\nu,\boldsymbol{p}]}$ of $\mathcal{M}_{0,r}(\Sigma, D)$ at $[\nu : \mathbb{P}^1 \to \Sigma, \boldsymbol{p}]$ to B_C^{eg} gives rise to a bijection between the set of elements $[\widetilde{\nu} : \mathbb{P}^1 \to \Sigma, \widetilde{\boldsymbol{p}}] \in \mathcal{M}_{0,r}(\Sigma, D)_{[\nu,\boldsymbol{p}]}$ such that $\widetilde{\nu}(\widetilde{\boldsymbol{p}}) = \widetilde{\boldsymbol{w}}$ on one side and the set $\Lambda(\widetilde{\boldsymbol{w}}) \cap B_C^{eg}$ on the other side.*

Proof. (1) The dimension and the transversality statements reduce to the fact that the pull-back of $T_0 B_C^{eg}$ to $|D|$ intersects $|D|_{\boldsymbol{w}}$ transversally and only at one point. In view of the identification of $T_0 B_C^{eg}$ with $\prod_{z\in \mathrm{Sing}\,(C)} J_{C,z}^{\mathrm{cond}}/J_{C,z}$ [6, Theorem 4.15], both required claims read as

$$H^0(C, \mathcal{J}_C^{\mathrm{cond}}(-\boldsymbol{w}) \otimes \mathcal{O}_\Sigma(D)) = 0\ , \tag{19}$$

where $\mathcal{J}_C^{\mathrm{cond}} = \mathrm{Ann}(\nu_*\mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_C)$ is the conductor ideal sheaf, since $\mathcal{J}_C^{\mathrm{cond}}$ can be equivalently regarded as the ideal sheaf of the zero-dimensional subscheme of C defined at all singular points $z \in \mathrm{Sing}\,(C)$ by the conductor ideals $J_{C,z}^{\mathrm{cond}} = \mathrm{Ann}(\nu_* \bigoplus_{q\in\nu^{-1}(z)} \mathcal{O}_{\mathbb{P}^1,q})/\mathcal{O}_{C,z}$.

It is known that $\mathcal{J}_C^{\mathrm{cond}} = \nu_*\mathcal{O}_{\mathbb{P}^1}(-\Delta)$, where $\Delta \subset \mathbb{P}^1$ is the so-called double-point divisor, whose degree is $\deg\Delta = 2\sum_{z\in \mathrm{Sing}\,(C)} \delta(C, z)$ (see, for example, [5, Section 2.4] or [8, Section 4.2.4]). Hence, the relations (19) can be rewritten as

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\boldsymbol{d} - \Delta - \boldsymbol{p})) = 0\ , \tag{20}$$

where $\deg \boldsymbol{d} = D^2$. Since

$$\begin{aligned}\deg \mathcal{O}_{\mathbb{P}^1}(\boldsymbol{d} - \Delta - \boldsymbol{p})) &= D^2 - 2\sum_{z\in \mathrm{Sing}\,(C)} \delta(C, z) - r \\ &= D^2 - 2\left(\frac{D^2 + DK_\Sigma}{2} + 1\right) - (-DK_\Sigma - 1) = -1 > -2\ ,\end{aligned}$$

we obtain $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\boldsymbol{d} - \Delta - \boldsymbol{p})) = 0$, and hence by Riemann–Roch

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\boldsymbol{d} - \Delta - \boldsymbol{p})) = \deg \mathcal{O}_{\mathbb{P}^1}(\boldsymbol{d} - \Delta - \boldsymbol{p})) + 1 = 0\ .$$

(2) The second statement of Lemma 13 immediately follows from the first one due to the fact that the tangent spaces to the stratum B_C^{eg} at its smooth

points close to the origin converge to the same linear space of dimension $\dim B^{eg}_C$ [6, Theorem 4.15]. $\square$

Suppose now that Σ possesses a real structure, C is a real curve, and z is its real singular point. Let $b \in B^{eg,im}_{C,z}$ be a real point, and let C_b be the corresponding fiber of the semiuniversal deformation of the germ (C, z). Define the Welschinger sign W_b as follows. Let $\pi : \hat{C}_b \to C_b \hookrightarrow \Sigma$ be the normalization of C_b. Here $\hat{C}_b$ is the union of discs, some of them being real (*i.e.*, invariant with respect to the complex conjugation), the others forming complex conjugate pairs. Put $W_b = (-1)^{C_{b,+} \circ C_{b,-}}$, where $C_{b,\pm} = \pi(\hat{C}_{b,\pm})$ and $\hat{C}_b \setminus \mathbb{R}\hat{C}_b = \hat{C}_{b,+} \sqcup \hat{C}_{b,-}$ is a splitting into disjoint complex conjugate halves.

Lemma 14. *The Welschinger sign W_b is equal to $(-1)^s$, where s is the number of solitary nodes in a small real nodal perturbation of C_b.*

Proof. Straightforward from the definition. $\square$

Lemma 15. *Let L_t, $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$, be a smooth one-parameter family of conjugation-invariant affine subspaces of $B_{C,z}$ of dimension $\delta(C, z)$ such that*

- *L_0 passes through the origin and is transversal to $T_0 B^{eg}_{C,z}$,*
- *$L_t \cap B^{eg}_{C,z} \subset B^{eg,im}_{C,z}$ for each $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$.*

Then,

(i) *the intersection $L_t \cap B^{eg}_{C,z}$ is finite for each $t \in (-\varepsilon', \varepsilon') \setminus \{0\}$, where $\varepsilon' > 0$ is sufficiently small.*

(ii) *the function $W(t) = \sum_{b \in L_t \cap \mathbb{R}B^{eg}_{C,z}} W_b$ is constant in $(-\epsilon', \epsilon') \setminus \{0\}$, where $\varepsilon' > 0$ is sufficiently small.*

Proof. The finiteness of the intersection follows from the transversality of L_0 and $T_0 B^{eg}_{C,z}$ in $B_{C,z}$. To prove the second statement, assume, first, that the germ (C, z) represents an ordinary cusp A_2. Then $\mathbb{R}B_{C,z} = (\mathbb{R}^2, 0)$ and $\mathbb{R}B^{eg}_{C,z}$ is a semicubical parabola with vertex at the origin. For the points b belonging to one of the two connected components of $\mathbb{R}B^{eg}_{C,z} \setminus \{0\}$, the curve C_b has a non-solitary real node; for the points b from the other component, C_b has a solitary node. Since, in addition, the line L_0 crosses the tangent to the parabola at the origin transversally we have $W(t) = 0$ for each $t \in (-\varepsilon', \varepsilon') \setminus \{0\}$ for sufficiently small $\varepsilon' > 0$.

In the general case, if $\varepsilon' > 0$ is sufficiently small, then for any two points $t_1 < t_2$ in $(-\epsilon', \epsilon') \setminus \{0\}$ we can connect L_{t_1} with L_{t_2} by a family of $\delta(C, z)$-dimensional conjugation-invariant affine subspaces $L'_t \subset B_{C,z}$, $t \in [t_1, t_2]$, such that

- the subspaces L'_t, $t \in [t_1, t_2]$, are transversal to $B^{eg}_{C,z}$,
- the intersection number of L'_t and $B^{eg}_{C,z}$ is constant in $[t_1, t_2]$,
- for all but finitely many values of t the intersection $L'_t \cap B^{eg}_{C,z}$ is contained in $B^{eg,nod}_{C,z}$, and for the remaining values of t, the subspace L'_t intersects $\mathbb{R}B^{eg}_{C,z}$ within $B^{eg}_{C,z}(A_2) \cup B^{eg}_{C,z}(A_3) \cup B^{eg}_{C,z}(D_4)$.

The bifurcations through the immersed singularities A_3 and D_4 do not affect $W(t)$, as well as the cuspidal bifurcation, which we have treated above. $\square$

Remark 16. In fact, Lemma 15 allows one to extend the definition of Welschinger signs and attribute a *Welschinger weight* to any map $\nu : \mathbb{P}^1 \to \Sigma$ birational onto its image.

3. Proof of Theorem 6

3.1. Preliminary observations

We start with two remarks.

(1) If Y is an irreducible complex variety, equipped with a real structure, and $\mathbb{R}Y$ contains nonsingular points of Y, then $\mathbb{R}Y \cap U \neq \emptyset$ for any Zariski open subset $U \subset Y$. In particular, a generic element of $\mathcal{P}_{r,m}(\Sigma, F)$ is generic in Σ^r.

(2) By blowing up extra real points we can reduce the consideration to the case of del Pezzo surfaces of degree 1.

The following statement will be used in the sequel.

Lemma 17. *Let $t \in (\mathbb{R}, 0) \mapsto \Sigma_t$ be a germ of an elementary deformation $(\Sigma_t, D_t, F_t, \varphi_t, \boldsymbol{w}_t)$ of a tuple $(\Sigma_0, D_0, F_0, \varphi_0, \boldsymbol{w}_0)$, where Σ_0 is a del Pezzo surface of degree* 1*, $D_0 \in \mathrm{Pic}(\Sigma_0)$ is a real effective divisor such that $r = -D_0 K_{\Sigma_0} - 1 > 0$, and $\boldsymbol{w}_0$ belongs to $\mathcal{P}_{r,m}(\Sigma_0, F_0)$ and is generic. Then*

$$W_m(\Sigma_t, D_t, F_t, \varphi_t, \boldsymbol{w}_t) = W_m(\Sigma_0, D_0, F_0, \varphi_0, \boldsymbol{w}_0) \ .$$

Proof. Since $D_0 K_{\Sigma_0} > 1$ and $\boldsymbol{w}_0$ is generic, Lemma 9 implies that all the curves under count are immersed. Thus, each of these curves contributes 1 to the Gromov–Witten invariant, and the required equality follows from Lemma 14. $\square$

3.2. Proof of Proposition 4

The only situation to consider is the one where $\Sigma \in \mathcal{D}_1^{\mathrm{DP}}$ and $r = -DK_\Sigma - 1 > 0$. Due to Lemma 17, we can fix any dense subset in $\mathcal{D}_1^{\mathrm{DP}}$ and check the statement for the surfaces belonging to this subset. Throughout this section, we assume that $\Sigma \in U_1 \cap U_2$.

We prove the invariance of Welschinger numbers by studying wall-crossing events when moving either one real point of the given collection, or a pair of complex conjugate points.

3.2.1. Moving a real point of configuration. Suppose that $2m < r$. Let tuples $\boldsymbol{w}' \cup \{w^{(0)}\}, \boldsymbol{w}' \cup \{w^{(1)}\} \in \mathcal{P}_{r,m}(\Sigma, F)$, where $\boldsymbol{w}' \in \mathcal{P}_{r-1,m}(\Sigma, F)$, be such that the sets $\mathcal{R}(\Sigma, D, F, \boldsymbol{w}' \cup \{w^{(0)}\})$ and $\mathcal{R}(\Sigma, D, F, \boldsymbol{w}' \cup \{w^{(1)}\})$ are finite and presented by immersions (see Lemma 9). We prove that

$$W_m(\Sigma, D, F, \varphi, \boldsymbol{w}' \cup \{w^{(0)}\}) = W_m(\Sigma, D, F, \varphi, \boldsymbol{w}' \cup \{w^{(1)}\}) \ . \tag{21}$$

Due to Lemma 11, by a small deformation of $\boldsymbol{w}'$ we can reach the following: whenever an element $[\nu : \widehat{C} \to \Sigma] \in \overline{\mathcal{M}_{0,0}^{br}}(\Sigma, D) \setminus \mathcal{M}_{0,0}^{br}(\Sigma, D)$ is such that $\nu(\widehat{C}) \supset$

$\boldsymbol{w}'$, the element $[\nu : \widehat{C} \to \Sigma]$ satisfies the conditions of Lemma 11(i), $-D_1K_\Sigma - 1$ points of $\boldsymbol{w}'$ lie on $C_1 \setminus (\mathrm{Sing}\,(C_1) \cup C_2)$, and the remaining $-D_2K_\Sigma - 1$ points of $\boldsymbol{w}'$ lie on $C_2 \setminus (\mathrm{Sing}\,(C_2) \cup C_1)$.

There exists a smooth real-analytic path $\sigma : [0,1] \to F$ lying in the real part of some smooth real algebraic curve $\sigma(\mathbb{C}) \subset \Sigma$, such that σ is disjoint from all the points of $\boldsymbol{w}'$, $\sigma(0) = w^{(0)}$, $\sigma(1) = w^{(1)}$, and in the family

$$\overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D; \boldsymbol{w}', \sigma) = \{[\nu : \hat{C} \to \Sigma, \boldsymbol{p}] \in \overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D)\ :\ \nu(\boldsymbol{p}') = \boldsymbol{w}',\ \nu(p_r) \in \sigma\}\ ,$$

where $\boldsymbol{p}' = \boldsymbol{p} \setminus \{p_r\}$, all but finitely many elements belong to $\mathcal{M}^{im}_{0,r}(\Sigma, D)$, and the remaining elements $[\nu : \hat{C} \to \Sigma, \boldsymbol{p}]$ (corresponding to some values $t \in I_0 \subset [0,1]$, $|I_0| < \infty$) are such that:

- (D1$_{re}$) either $[\nu : \hat{C} \to \Sigma] \in \overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$ is as in Lemma 11(i), the point $w^{(t)} \in \sigma \cap C_2$ belongs to $C_2 \setminus (\mathrm{Sing}\,(C_2) \cup C_1 \cup \boldsymbol{w}')$, and the germ of $\sigma(\mathbb{C})$ at $w^{(t)} \in C$ intersects C_2 transversally;
- (D2$_{re}$) or $[\nu : \hat{C} \to \Sigma] \in \mathcal{M}^{br}_{0,0}(\Sigma, D) \setminus \mathcal{M}^{im}_{0,0}(\Sigma, D)$, the point $w^{(t)} \in \sigma \cap C$, where $C = \nu(\hat{C})$, belongs to $C \setminus (\mathrm{Sing}\,(C) \cup \boldsymbol{w}')$, and the germ of $\sigma(\mathbb{C})$ at $w^{(t)}$ intersect C transversally.

Denote by $M_{[\nu,\boldsymbol{p}]}$ the germ of $\overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D; \boldsymbol{w}', \sigma)$ at an element $[\nu : \hat{C} \to \Sigma, \boldsymbol{p}]$.

If $[\nu : \hat{C} \to \Sigma] \in \mathcal{M}^{im}_{0,0}(\Sigma, D)$, or $[\nu : \hat{C} \to \Sigma]$ satisfies condition (D1$_{re}$), then, by Lemma 12, the germ $M_{[\nu,\boldsymbol{p}]}$ is diffeomorphically mapped by Ev onto the germ $(\sigma, w^{(t)})$. Moreover, the Welschinger sign $\mu(\nu, \varphi)$ does not change along $M_{[\nu,\sigma]}$. This is evident for $[\nu : \hat{C} \to \Sigma] \in \mathcal{M}^{im}_{0,0}(\Sigma, D)$, and, under condition (D1$_{re}$), immediately follows from the fact that ν maps the germ of $\hat{C}$ at the node to a pair of real smooth branches that intersect transversally and undergo a standard smoothing in the considered bifurcation.

Under the hypotheses of condition (D2$_{re}$), the required constancy of the Welschinger number $W_m(\Sigma, D, F, \varphi, \boldsymbol{w}' \cup \{w^{(t)}\})$ immediately follows from Lemmas 13, 14 and 15.

3.2.2. Moving a pair of imaginary conjugate points. Assume that $m \geq 1$. Let tuples $\boldsymbol{w}' \cup \{w^{(0)}, \mathrm{Conj}\, w^{(0)}\}, \boldsymbol{w}' \cup \{w^{(1)}, \mathrm{Conj}\, w^{(1)}\} \in \mathcal{P}_{r,m}(\Sigma, F)$, where $\boldsymbol{w}' \in \mathcal{P}_{r-2,m-1}(\Sigma, F)$, be such that the sets

$$\mathcal{R}(\Sigma, D, F, \boldsymbol{w}' \cup \{w^{(0)}, \mathrm{Conj}\, w^{(0)}\}) \ \text{ and } \ \mathcal{R}(\Sigma, D, F, \boldsymbol{w}' \cup \{w^{(1)}, \mathrm{Conj}\, w^{(1)}\})$$

are finite and presented by immersions (see Lemma 9). We prove that

$$W_m(\Sigma, D, F, \varphi, \boldsymbol{w}' \cup \{w^{(0)}, \mathrm{Conj}\, w^{(0)}\}) = W_m(\Sigma, D, F, \varphi, \boldsymbol{w}' \cup \{w^{(1)}, \mathrm{Conj}\, w^{(1)}\})\,. \tag{22}$$

Due to Lemma 11, by a small deformation of $\boldsymbol{w}'$ we can reach the following: for any point w of a certain Zariski open subset $\Sigma_{\boldsymbol{w}'} \subset \Sigma \setminus \boldsymbol{w}'$, whenever for an element $[\nu : \widehat{C} \to \Sigma] \in \overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D) \setminus \mathcal{M}^{br}_{0,0}(\Sigma, D)$ we have $\nu(\widehat{C}) \supset \boldsymbol{w}' \cup \{w\}$, this element $[\nu : \widehat{C} \to \Sigma]$ satisfies the conditions of Lemma 11(i), $-D_1K_\Sigma - 1$ points of $\boldsymbol{w}'$ lie on $C_1 \setminus (\mathrm{Sing}\,(C_1) \cup C_2)$, and the remaining $-D_2K_\Sigma - 2$ points of $\boldsymbol{w}'$ and the point w lie on $C_2 \setminus (\mathrm{Sing}\,(C_2) \cup C_1)$. Further on, assuming this property of $\boldsymbol{w}'$, we can find a smooth real-analytic path $\sigma : [0,1] \to \mathrm{Sing} \setminus \mathbb{R}\Sigma$ lying in some smooth real algebraic curve $\sigma(\mathbb{C}) \subset \Sigma \setminus \mathbb{R}\Sigma$, such that σ starts at $w^{(0)}$ and ends up at $w^{(1)}$, avoids all the points of $\boldsymbol{w}'$, and satisfies the following condition (*cf.* section 3.2.1): for all but finitely many elements of the family

$$\overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D; \boldsymbol{w}', \sigma, \mathrm{Conj}\,\sigma) = \{[\nu : \hat{C} \to \Sigma, \boldsymbol{p}] \in \overline{\mathcal{M}^{br}_{0,r}}(\Sigma, D)\ : \\ \nu(\boldsymbol{p}') = \boldsymbol{w}',\ \nu(p_{r-1}) \in \sigma,\ \nu(p_r) \in \mathrm{Conj}\,\sigma\}\ ,$$

where $\boldsymbol{p}' = \boldsymbol{p} \setminus \{p_{r-1}, p_r\}$, we have $[\nu : \hat{C} \to \Sigma] \in \mathcal{M}^{im}_{0,0}(\Sigma, D)$, while the remaining elements (which correspond to some values $t \in I_0 \subset [0,1]$, $|I_0| < \infty$) are such that:

($\mathrm{D1}_{im}$) either $[\nu : \hat{C} \to \Sigma] \in \overline{\mathcal{M}^{br}_{0,0}}(\Sigma, D)$ is as in Lemma 11(i), where $\nu_i : \hat{C}_i \to \Sigma$ commutes with the real structure, $-D_1K_\Sigma - 1$ points of $\boldsymbol{w}'$ lie on $C_1 \setminus (\mathrm{Sing}\,(C_1) \cup C_2)$, the remaining $-D_2K_\Sigma - 2$ points of $\boldsymbol{w}'$ lie in $C_2 \setminus (\mathrm{Sing}\,(C_2) \cup C_1)$, the point $w^{(t)} \in \sigma$ belongs to $C_2 \setminus (\mathrm{Sing}\,(C_2) \cup C_1 \cup \boldsymbol{w}')$, and the germ of $\sigma(\mathbb{C})$ at $w^{(t)} \in C_2$ intersects C_2 transversally;

($\mathrm{D2}_{im}$) or $[\nu : \hat{C} \to \Sigma] \in \mathcal{M}^{br}_{0,0}(\Sigma, D) \setminus \mathcal{M}^{im}_{0,0}(\Sigma, D)$, the point $w^{(t)} \in \sigma \cap C$, where $C = \nu(\hat{C})$, belongs to $C \setminus (\mathrm{Sing}\,(C) \cup \boldsymbol{w}')$, and the germ of $\sigma(\mathbb{C})$ at $w^{(t)}$ intersects C transversally.

Notice that, in ($\mathrm{D1}_{im}$), the case of $C_1 = C_2$ is not relevant due to $-DK_\Sigma > 2$, and the case of complex conjugate C_1 and C_2 does not occur either, since any real rational curve in $|D|$ must have a non-trivial one-dimensional real branch (see Section 1.2).

Then the proof of (22) literally follows the argument of the preceding section.

3.3. Proof of Proposition 5 and Theorem 6

In view of Proposition 4 and Lemmas 9(ii) and 17, Theorem 6 follows from Proposition 5, and, in its turn, to prove Proposition 5 it is sufficient to check the constancy of the Welschinger number in the following families:

- a germ of elementary deformation $\{\Sigma_t\}_{t \in (\mathbb{R},0)}$, where $\Sigma_0 \in U_1(A_1)$, $\Sigma_t \in U_1 \cap U_2$ for each $t \neq 0$, and $D_t \neq -K_{\Sigma_t}$;
- a germ of elementary deformation $\{\Sigma_t\}_{t \in (\mathbb{R},0)}$, where $\Sigma_0 \in \mathcal{D}^{\mathrm{DP}}_1 \setminus U_1$, $\Sigma_t \in U_1 \cap U_2$ for each $t \neq 0$, and $D_t = -K_{\Sigma_t}$.

Let $\Sigma_0 \in U_1(A_1)$, $\Sigma_t \in U_1 \cap U_2$ for $t \neq 0$, and $D_t \neq -K_{\Sigma_t}$. Extend the family $\{\Sigma_t\}_{t \in (\mathbb{R},0)}$ to a conjugation invariant family $\{\Sigma_t\}_{(\mathbb{C},0)}$. By Lemma 9(2), there exists $\boldsymbol{w}_0 \in \mathcal{P}_{r,m}(\Sigma_0, F_0)$ such that, for any $k \geq 0$, all elements $[\nu : \mathbb{P}^1 \to \Sigma_0, \boldsymbol{p}_0] \in \mathcal{M}_{0,r}(\Sigma_0, D - kE, \boldsymbol{w}_0)$ satisfy the properties indicated in Lemma 9(2ii). These elements appear only for a finite number of values of k and form a finite set. Let us associate with each of them a comb-like curve $[\nu : \hat{C} \to \Sigma_0, \boldsymbol{p}] \in \overline{\mathcal{M}_{0,r}(\Sigma_0, D, \boldsymbol{w}_0)}$ such that:

- either $\hat{C} \simeq \mathbb{P}^1$, or $\hat{C} = \hat{C}' \cup \hat{E}_1 \cup \cdots \cup \hat{E}_k$ for some $k > 0$, where $\hat{C}' \simeq \hat{E}_1 \simeq \cdots \simeq \hat{E}_k \simeq \mathbb{P}^1$, $\hat{E}_i \cap \hat{E}_j = \emptyset$ for all $i \neq j$, and $\#(\hat{C}' \cap \hat{E}_i) = 1$ for all $i = 1, \ldots, k$;
- $\boldsymbol{p} \subset \hat{C}'$ and $[\nu : \hat{C}' \to \Sigma_0, \boldsymbol{p}] \in \mathcal{M}^{im}_{0,r}(\Sigma_0, D - kE, \boldsymbol{w}_0)$, and each of $\hat{E}_1, \ldots, \hat{E}_k$ is isomorphically mapped onto E.

Then, complement $\boldsymbol{w}_0$ to a conjugation invariant family of r-tuples $\boldsymbol{w}_t \in (\Sigma_t)^r$, $t \in (\mathbb{C}, 0)$, so that $\boldsymbol{w}_t \in \mathcal{P}_{r,m}(\Sigma_t, F_t)$ for each real t. It follows from [26, Theorem 4.2] that each of the introduced elements $[\nu : \hat{C} \to \Sigma_0, \boldsymbol{p}] \in \overline{\mathcal{M}_{0,r}(\Sigma_0, D, \boldsymbol{w}_0)}$ extends to a smooth family $[\nu_t : \hat{C}_t \to \Sigma_t, \boldsymbol{p}_t] \in \overline{\mathcal{M}^{br}_{0,r}}(\Sigma_t, D, \boldsymbol{w}_t)$, $t \in (\mathbb{C}, 0)$, where $\hat{C}_t \simeq \mathbb{P}^1$ and ν_t is an immersion for all $t \neq 0$, and, furthermore, each element of $\mathcal{M}_{0,r}(\Sigma_t, D, \boldsymbol{w}_t)$, $t \in (\mathbb{C}, 0) \setminus \{0\}$ is included into some of the above families. Thus, the Welschinger number $W(\Sigma_t, D, F_t, \varphi_t, \boldsymbol{w}_t)$ remains constant in $t \in (\mathbb{R}, 0) \setminus \{0\}$, since the only change of the topology in the real loci of the curves under the count consists in smoothing of non-solitary nodes, while the difference between the homology classes of the halves $[C_\pm(t)]$ in $H_2(\Sigma_t, F_t; \mathbb{Z}/2) = H_2(\Sigma_0, F_0; \mathbb{Z}/2)$ with $t < 0$ and those with $t > 0$ belongs to $(1 + \mathrm{Conj}_*) H_2(\Sigma_0, F_0; \mathbb{Z}/2)$ and, hence $[C_\pm(t)] \circ \phi_t$ does not depend on t.

Assume that $\Sigma_0 \in \mathcal{D}^{\mathrm{DP}}_1 \setminus U_1$ $\Sigma_t \in U_1 \cap U_2$ for $t \neq 0$, and $D_t = -K_{\Sigma_t}$. In this case we deal with a family of real elliptic pencils $|-K_{\Sigma_t}|$, $t \in (\mathbb{R}, 0)$, such that the central one $|-K_{\Sigma_0}|$ has a real cuspidal curve $C_0 \in |-K_{\Sigma_0}|$ and, otherwise, the family is generic. As it can be seen from the local Weierstrass normal form, due to the above genericity the image of $|-K_{\Sigma_0}|$ in the base $(\mathbb{C}^2, 0)$ of the versal deformation of the cuspidal point intersects the tangent space to the discriminant locus, that is the cusp curve $27p^2 + 4q^3 = 0$ in terms of Weierstrass coordinates p, q, transversally at one point. Therefore, for $t \in (\mathbb{R}, 0)$ on one side of $t = 0$ the singular curves in $|-K_{\Sigma_t}|$ close to C_0 form a pair of complex conjugate curves, while for $t \in (\mathbb{R}, 0)$ on the opposite side of $t = 0$ they are real, one with a solitary node, and the other one with a cross point. Thus, the total Welschinger number is the same on the both sides.

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