Analysis Meets Geometry: A Tribute to Mikael Passare
Trends in Mathematics, 191–212

Coamoebas of Polynomials Supported on Circuits

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Abstract. We study coamoebas of polynomials supported on circuits. Our results include an explicit description of the space of coamoebas, a relation between connected components of the coamoeba complement and critical points of the polynomial, an upper bound on the area of a planar coamoeba, and a recovered bound on the number of positive solutions of a fewnomial system.

1. Introduction

A possibly degenerate circuit is a point configuration $A \subset \mathbb{Z}^n$ of cardinality $n+2$ which span a sublattice $\mathbb{Z}A$ of rank n. That is, such that the Newton polytope $\mathcal{N}_A = \operatorname{Conv}(A)$ is of full dimension. A polynomial system $f(z) = 0$ is said to be supported on a circuit A if each polynomial occurring in $f(z)$ is supported on A. Polynomial systems supported on circuits have recently been been studied in the context of, e.g., real algebraic geometry [4, 6], complexity theory [5], and amoeba theory [18]. The name "circuit" originate from matroid theory; see [17] and [20] for further background.

The aim of this article is to describe geometrical and topological properties of coamoebas of polynomials supported on circuits. Such an investigation is motivated not only by the vast number of applications of circuits in different areas of geometry, but also since circuits provide an ideal testing ground for open problems in coamoeba theory.

This paper is organized as follows. In Section 2 we will give a brief overview of coamoeba theory. In Section 3 we will discuss the relation between real polynomials and the coamoeba of the A-discriminant. The main results of this paper are contained in Sections 4–7, each of which can be read as a standalone text.

In Section 4 we will give a complete description of the space of coamoebas. That is, we will describe how the topology of the coamoeba $\mathcal{C}_f$ depends on the coefficients of f. Describing the space of amoebas is the topic of the articles [18] and [19], and to fully appreciate our result one should consider these spaces simultaneously, see, e.g., Figure 2. The geometry of the space of coamoebas is closely related to the A-discriminantal variety, see Theorems 4.1 and 4.2.

In Section 5 we will prove that the area of a planar circuit coamoeba is bounded from above by $2\pi^2$. That is, a planar circuit coamoeba covers at most half of the torus $\mathbf{T}^2$. Furthermore, we will prove that a circuit admits a coamoeba of maximal area if and only if it admits an equimodular triangulation. Note that we calculate area without multiplicities, in contrast to [11]. However, the relation between (co)amoebas of maximal area and Harnack curves is made visible also in this setting.

In Section 6 we will prove that, under certain assumptions on A, the critical points of $f(z)$ are projected by the componentwise argument mapping into distinct connected component of the complement of the coamoeba $\mathcal{C}_f$. Furthermore, this projection gives a bijective relation between the set of critical points and the set of connected components of the complement of the closed coamoeba. This settles a conjecture used in [10] when computing monodromy in the context of dimer models and mirror symmetry.

In Section 7 we will consider bivariate systems supported on a circuit. If such a system is real, then it admits at most three roots in $\mathbb{R}^2_+$. The main contribution of this section is that we offer a new approach to fewnomial theory. Using our method, we will prove that if $\mathcal{N}_A$ is a simplex, then, for each $\theta \in \mathbf{T}^2$, a complex bivariate system supported on A has at most two roots in the *sector* $\operatorname{Arg}^{-1}(\theta)$.

2. Coamoebas and lopsidedness

Let A denote a point configuration $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_{N-1}\} \subset \mathbb{Z}^n$, where $N = \#A$. By abuse of notation, we identify A with the $(1+n) \times N$-matrix

$$A = \begin{pmatrix} 1 & \ldots & 1 \\ \mathbf{a}_0 & \ldots & \mathbf{a}_{N-1} \end{pmatrix}. \tag{1}$$

The *codimension* of A is the integer $m = N-1-n$. A *circuit* is a point configuration of codimension one. A circuit is said to be nondegenerate if it is not a pyramid over a circuit of smaller dimension. That is, if all maximal minors of the matrix A are nonvanishing. We will partition the set of circuits into two classes; *simplex circuits*, for which $\mathcal{N}_A$ is a simplex, and *vertex circuits*, for which $A = \operatorname{Vert}(\mathcal{N}_A)$.

We associate to A the family $\mathbb{C}^A_*$ consisting of all polynomials

$$f(z) = \sum_{k=0}^{N-1} f_k\, z^{\mathbf{a}_k},$$

where $f(z)$ is identified with the point $f = (f_0, \ldots, f_{N-1}) \in \mathbb{C}^A_*$. By slight abuse of notation, we will denote by $f_k(z)$ the monomial function $z \mapsto f_k\, z^{\mathbf{a}_k}$. We denote the algebraic set defined by f by $Z(f) \subset \mathbb{C}^n_*$. The *coamoeba* $\mathcal{C}_f$ is the image of $Z(f)$ under the componentwise argument mapping $\operatorname{Arg}\colon \mathbb{C}^n_* \to \mathbf{T}^n$ defined by

$$\operatorname{Arg}(z) = (\arg(z_1), \ldots, \arg(z_n)),$$

where $\mathbf{T}^n$ denotes the real n-torus. It is sometimes beneficial to consider the multi-valued argument mapping, which gives the coamoeba as a multiply periodic subset of $\mathbb{R}^n$. Coameobas were introduced by Passare and Tsikh as a dual object, in an imprecise sense, of the amoeba $\mathcal{A}_f$.

We will say that a point $z \in \mathbb{C}_*^n$ is a *critical point* of f if it solves the system

$$\partial_1 f(z) = \cdots = \partial_n f(z) = 0. \tag{2}$$

If in addition $z \in Z(f)$ then z will be called a *singular point* of f. The A-discriminant $\Delta(f) = \Delta_A(f)$ is an irreducible polynomial with domain $\mathbb{C}_*^A$ which vanishes if and only if f has a singular point in $\mathbb{C}_*^n$ [9].

A *Gale dual* of A is an integer matrix B whose columns span the right $\mathbb{Z}$-kernel of A. That is, B is an integer $N \times m$-matrix, of full rank, such that its maximal minors are relatively prime. A Gale dual is unique up to the action of $\mathrm{SL}_m(\mathbb{Z})$. The rows $\mathbf{b}$ of B are indexed by the points $\mathbf{a}_k \in A$. To each Gale dual we associate a zonotope

$$\mathcal{Z}_B = \left\{ \frac{\pi}{2} \sum_{k=0}^{N-1} \lambda_k \mathbf{b}_k \,\middle|\, |\lambda_k| \le 1 \right\} \subset \mathbb{R}^m.$$

We will say that a triangulation T of $\mathcal{N}_A$ is a triangulation of A if $\mathrm{Vert}(T) \subset A$. Such a triangulation is said to be *equimodular* if all maximal simplices has equal volume.

Let h be a *height function* $h\colon A \to \mathbb{R}$. The function h induces a triangulation T_h of A in the following manner. Let $\mathcal{N}_h$ denote the polytope in $\mathbb{R}^{n+1}$ with vertices $(\mathbf{a}, h(\mathbf{a}))$. The lower facets of $\mathcal{N}_h$ are the facets whose outward normal vector has negative last coordinate. Then, T_h is the triangulation of A whose maximal simplices are the images of the lower facets of $\mathcal{N}_h$ under the projection onto the first n coordinates. A triangulation T of A is said to be *coherent* if there exists a height function h such that $T = T_h$.

If A is a circuit then B is a column vector, unique up to sign. Hence, the zonotope $\mathcal{Z}_B$ is an interval. Let $A_k = A \setminus \{\mathbf{a}_k\}$, with associated matrix A_k, and let $V_k = \mathrm{Vol}(A_k)$. If A is a nondegenerate circuit, so that $V_k > 0$ for all k, then $\mathcal{N}_A$ admits exactly two coherent triangulations with vertices in A [9]. Denote these two triangulations by T_δ for $\delta \in \{\pm 1\}$. Each simplex $\mathcal{N}_{A_k}$ occurs in exactly one of the triangulations T_δ. Hence, there is a well-defined assignment of signs $k \mapsto \delta_k$, where $\delta_k \in \{\pm 1\}$, such that

$$T_\delta = \left\{ \mathcal{N}_{A_k} \right\}_{\delta_k = \delta}, \quad \delta = \pm 1.$$

Here, we have identified a triangulation with its set of maximal simplices. As shown in [9, Ch. 7 and Ch. 9] and [7, Sec. 5], a Gale dual of A is given by

$$\mathbf{b}_k = (-1)^k |A_k| = \delta_k V_k. \tag{3}$$

Thus, the zonotope $\mathcal{Z}_B$ is an interval of length $2\pi\,\mathrm{Vol}(A)$.

The A-discriminant Δ has $n+1$ homogeneities, one for each row of the matrix A. Each Gale dual correspond to a dehomogenization of Δ. To be specific, introducing the variables

$$\xi_j = \prod_{k=0}^{N-1} f_k^{\mathbf{b}_{kj}}, \quad j = 1, \ldots, m, \tag{4}$$

there is a Laurent monomial $M(c)$ and a polynomial $\Delta_B(\xi)$ such that

$$\Delta_B(\xi) = M(f)\Delta_A(f).$$

We will say that Δ_B is the *reduced form* of Δ. Such a reduction yields a projection $\mathrm{pr}_B \colon \mathbb{C}_*^A \to \mathbb{C}_*^m$, and we will say that $\mathbb{C}_*^m$ is the reduced family associated to A, and that $\mathrm{pr}_B(f)$ is the reduced form of f.

Example 2.1. Let $A = \{0, 1, 2\}$, so that $\mathbb{C}_*^A$ is the family of quadratic univariate polynomials

$$f(z) = f_0 + f_1 z + f_2 z^2.$$

Consider the Gale dual $B = (1, -2, 1)^t$, and introduce the variable $\xi = f_0 f_1^{-2} f_2$. In this case the A-discriminant Δ_A is well known, and we find that

$$f_1^{-2}\Delta_A(f) = f_1^{-2}\left(f_1^2 - 4f_0 f_2\right) = 1 - 4\xi = \Delta_B(\xi).$$

The projection pr_B correspond to performing the change of variables $z \mapsto f_0 f_1^{-1} z$, and multiplying $f(z)$ by f_0^{-1}, after with we obtain the reduced family consisting of all polynomials of the form

$$f(z) = 1 + z + \xi z^2.$$

Let S denote a subset of A. The *truncated polynomial* f_S is the image of f under the projection $\mathrm{pr}_S \colon \mathbb{C}_*^A \to \mathbb{C}_*^S$. Of particular interest is the case when $S = \Gamma \cap A$ for some face Γ of the Newton polytope $\mathcal{N}_A$ (denoted by $\Gamma \prec \mathcal{N}_A$). We will write $f_\Gamma = f_{\Gamma \cap A}$. It was shown in [14] that

$$\overline{\mathcal{C}}_f = \bigcup_{\Gamma \prec \mathcal{N}_A} \mathcal{C}_{f_\Gamma},$$

Let $\mathcal{E}$ denote the set of edges of $\mathcal{N}_A$, then the *shell* of the coamoeba is defined by

$$\mathcal{H}_f = \bigcup_{\Gamma \in \mathcal{E}} \mathcal{C}_{f_\Gamma}.$$

As an edge Γ is one-dimensional, the shell $\mathcal{H}_f$ is a hyperplane arrangement. Its importance can be seen in that each full-dimensional cell of $\mathcal{H}_f$ contain at most one connected component of the complement of $\overline{\mathcal{C}}_f$, see [7].

Example 2.2. The coamoeba of $f(z) = 1 + z_1 + z_2$, as described in [7] and [14], can be seen in Figure 1, where it is drawn in the fundamental domains $[-\pi, \pi]^2$ and $[0, 2\pi]^2$. The shell $\mathcal{H}_f$ consist of the hyperplane arrangement drawn in black. In this case, it is equal to the boundary of $\mathcal{C}_f$. The Newton polytope $\mathcal{N}_A$ and its outward normal vectors are drawn in the rightmost picture. If $\mathcal{H}_f$ is given orientations in accordance with the outward normal vectors of $\mathcal{N}_A$, then the interior of the coamoeba consist of the oriented cells.

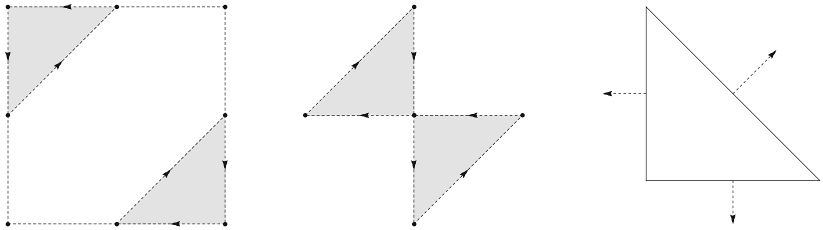

FIGURE 1. The coamoeba of $f(z) = 1 + z_1 + z_2$ in two fundamental regions, and the Newton polytope $\mathcal{N}_A$.

Acting on A by an *integer affine transformation* is equivalent to performing a monomial change of variables and multiplying f by a Laurent monomial. Such an action induces a linear transformation of the coamoeba $\mathcal{C}_f$, when viewed in $\mathbb{R}^n$ [7]. We will repeatedly use this fact to impose assumptions on A, e.g., that it contains the origin.

The polynomial f is said to be *colopsided* at a point $\theta \in \mathbb{R}^n$ if there exist a phase φ such that

$$\Re\left(e^{i\varphi} f_k(e^{i\theta})\right) \geq 0, \quad k = 0, \dots, N-1, \tag{5}$$

with at least one of the inequalities (5) being strict. The motivation for this definition is as follows. If f is colopsided at θ, then

$$\Re\left(e^{i\varphi} f(re^{i\theta})\right) = \sum_{k=0}^{N-1} r^{\mathbf{a}_k} \, \Re\left(e^{i\varphi} f_k(e^{i\theta})\right) > 0, \quad \forall\, r \in \mathbb{R}_+^n,$$

since at least one term of the sum is strictly positive. Hence, colopsidedness at θ implies that $\theta \in \mathbf{T}^n \setminus \mathcal{C}_f$. The colopsided coamoeba, denoted $\mathcal{L}_f$, is defined as the set of all θ such that (5) does not hold for any phase φ [7]. Hence, $\mathcal{C}_f \subset \mathcal{L}_f$.

Each monomial $f_k(z)$ defines an affinity (i.e., a group homomorphism composed with a translation) $f_k \colon \mathbb{C}_*^n \to \mathbb{C}_*$ by $z \mapsto f_k\, z^{\mathbf{a}_k}$. We thus obtain unique affinities $|f_k|$ and $\hat{f}_k$ such that the following diagram of short exact sequences commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}_+^n & \longrightarrow & \mathbb{C}_*^n & \longrightarrow & (S^1)^n & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle |f_k|} & & \big\downarrow {\scriptstyle f_k} & & \big\downarrow {\scriptstyle \hat{f}_k} & & \\
0 & \longrightarrow & \mathbb{R}_+ & \longrightarrow & \mathbb{C}_* & \longrightarrow & S^1 & \longrightarrow & 0.
\end{array}$$

Notice that $\mathbf{T} \simeq S^1 \subset \mathbb{C}$. We denote by $\hat{f}(\theta) \subset (S^1)^A \subset \mathbb{C}_*^A$ the vector with components $\hat{f}_k(\theta)$. Assume that f contains the constant monomial 1, and consider the map $\operatorname{ord}_B(f) \colon \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\operatorname{ord}_B(f)(\theta) = \operatorname{Arg}_\pi\left(\hat{f}(\theta)\right) \cdot B, \tag{6}$$

where Arg_π denotes the componentwise principal argument map. It was shown in [7] that the map $\operatorname{ord}_B(f)$ induces a map

$$\operatorname{ord}_B(f)\colon \mathbf{T}^n \setminus \overline{\mathcal{L}}_f \to \{\operatorname{Arg}_\pi(f)B + 2\pi\mathbb{Z}^m\} \cap \operatorname{int}\mathcal{Z}_B. \tag{7}$$

which in turn induces a bijection between the set of connected components of the complement of $\overline{\mathcal{L}}_f$ and the finite set in the right-hand side of (7). The map $\operatorname{ord}_B(f)$ is known as the *order map* of the lopsided coamoeba.

Remark 2.3. The requirement that f contains the monomial 1 is related to the choice of branch cut of the function Arg; in order to obtain a well-defined map, we need the right-hand side of (6) to be discontinuous only for θ such that two components of $\hat{f}(\theta)$ are antipodal, see [7]. If f does not contain the constant monomial 1, then one should fix a point $\mathbf{a}_k \in A$ and multiply the vector $\hat{f}(\theta)$ by the scalar $\hat{f}_k(\theta)^{-1}$ before taking principal arguments. It is shown in [7, Thm. 4.3] that the obtained map is independent of the choice of $\mathbf{a}_k$.

If $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{L}}_f$, then we can choose φ such that

$$\Re\left(e^{i\varphi} f_k(e^{i\theta})\right) > 0, \quad k = 0, \dots, N-1.$$

That is, the boundary of $\mathcal{L}_f$ is contained in the hyperplane arrangement consisting of all θ such that two components of $\hat{f}(\theta)$ are antipodal.

It has been conjectured that the number of connected components of the complement of $\overline{\mathcal{C}}_f$ is at most $\operatorname{Vol}(A)$.[1] A proof in arbitrary dimension has been proposed by Nisse in [13], and an independent proof in the case $n = 2$ was given in [8]. That the number of connected components of the complement of $\overline{\mathcal{L}}_f$ is at most $\operatorname{Vol}(A)$ follows from the theory of Mellin–Barnes integral representations of A-hypergeometric functions, see [2] and [3].

A finite set $\mathcal{I} \subset \mathbf{T}^n$ which is in a bijective correspondence with the set of connected components of the complement of $\overline{\mathcal{C}}_f$ by inclusion, will be said to be an *index set* of the coamoeba complement. This notation will be slightly abused; a set $\mathcal{I}$ of cardinality $\operatorname{Vol}(A)$ will be said to be an index set of the coamoeba if each connected component of its complement contains exactly one element of $\mathcal{I}$.

The term "lopsided" was first used by Purbhoo in [15], denoting the corresponding condition to (5) for amoebas: the polynomial f is said to be *lopsided* at a point $x \in \mathbb{R}^n$ if there is a $\mathbf{a}_k \in A$ such that the moduli $|f_k|(x)$ is greater than the sum of the remaining modulis. As a comparison, note that the polynomial f is colopsided at $\theta \in \mathbf{T}$ if and only if the greatest intermediate angle of the components of $\hat{f}(\theta)$ is greater than the sum of the remaining intermediate angles.

[1] This conjecture has commonly been attributed to Mikael Passare, however, it seems to originate from a talk given by Mounir Nisse at Stockholm University in 2007.

3. Real points and the coameoba of the A-discriminant

We will say that f is *real at* θ, if there is a real subvector space $\ell \subset \mathbb{C}$ such that $\hat{f}_k(\theta) \in \ell$ for all $k = 0, \dots, N-1$. If such a θ exist then f is real, that is, after a change of variables and multiplication with a Laurent monomial $f \in \mathbb{R}_*^A$. In this section, we will study the function $\hat{f}$ from the viewpoint of real polynomials. Our main result is the following characterization of the coamoeba of the A-discriminant of a circuit.

Proposition 3.1. *Let A be a nondegenerate circuit, and let δ_k be as in* (3)*. Then,* $\operatorname{Arg}(f) \in \mathcal{C}_\Delta$ *if and only if after possibly multiplying f with a constant, there is a $\theta \in \mathbb{R}^n$ such that $\hat{f}_k(\theta) = \delta_k$ for all k.*

If A is a circuit and B is a Gale dual of A then the Horn–Kapranov parametrization of the reduced discriminant Δ_B can be lifted to a parametrization of the discriminant surface Δ as

$$z \mapsto \left(\mathbf{b}_0 z^{\mathbf{a}_0}, \dots, \mathbf{b}_{N-1} z^{\mathbf{a}_{N-1}}\right).$$

Taking componentwise arguments, we obtain a simple proof Proposition 3.1. In particular, the proposition can be interpreted as a coamoeba version of the Horn–Kapranov parametrization valid for circuits. Our proof of Proposition 3.1 will be more involved, however, for our purposes the lemmas contained in this section are of equal importance.

Lemma 3.2. *Assume that the polynomial f is real at $\theta_0 \in \mathbb{R}^n$. Then, f is real at $\theta \in \mathbb{R}^n$ if and only if $\theta \in \theta_0 + \pi L$, where L is the dual lattice of $\mathbb{Z}A$.*

Proof. After translating θ and multiplying f with a Laurent monomial, we can assume that $\theta_0 = 0$, that $\ell_0 = \mathbb{R}$, and that f contains the monomial 1. That is, all coefficients of f are real, in particular proving *if*-part of the statement. To show the *only if*-part, notice first that $\hat{f}(\theta) \subset \ell$ implies that ℓ contains both the origin and 1. That is, $\ell = \mathbb{R}$. Furthermore, $\hat{f}(\theta) \subset \mathbb{R}$ only if for each $\mathbf{a} \in A$ there is a $k \in \mathbb{Z}$ such that $\langle \mathbf{a}, \theta \rangle = \pi k$, which concludes the proof. $\square$

The A-discriminant Δ related to a circuit has been described in [9, Chp. 9, Pro. 1.8] where the formula

$$\Delta(f) = \prod_{\delta_k = 1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k = -1} f_k^{-\mathbf{b}_k} - \prod_{\delta_k = -1} \mathbf{b}_k^{-\mathbf{b}_k} \prod_{\delta_k = 1} f_k^{\mathbf{b}_k} \tag{8}$$

was obtained. In particular, Δ is a binomial. As the zonotope $\mathcal{Z}_B$ is a symmetric interval of length $2\pi \operatorname{Vol}(A)$, the image of the map $\operatorname{ord}_B(f)$ is of cardinality $\operatorname{Vol}(A)$ unless

$$\operatorname{Arg}_\pi(f) B \equiv 2\pi \operatorname{Vol}(A) \mod 2\pi. \tag{9}$$

In particular, the complement of $\overline{\mathcal{C}}_f$ has the maximal number of connected components (i.e., $\operatorname{Vol}(A)$-many) unless the equivalence (9) holds.

Lemma 3.3. *For each* $\kappa = 0, 1, \ldots, n+1$*, there are exactly* $\mathrm{Vol}(A_\kappa)$*-many points* $\theta \in \mathbf{T}$ *such that*

$$\hat{f}_k(\theta) = \delta_k, \quad \forall\, k \neq \kappa. \tag{10}$$

Proof. By applying an integer affine transformation, the statement follows from the case when A_κ consist of the vertices of the standard simplex. □

Lemma 3.4. *Fix* $\kappa \in \{0, \ldots, n+1\}$*. For each* θ *fulfilling* (10)*, let* $\varphi_\theta \in \mathbf{T}$ *be defined by the condition that if* $\arg_\pi(f_\kappa) = \varphi_\theta$ *then*

$$\hat{f}_\kappa(\theta) = \delta_\kappa. \tag{11}$$

Assume that $\mathbb{Z}A = \mathbb{Z}^n$*. Then, the numbers* φ_θ *are distinct.*

Proof. We can assume that $\mathbf{a}_0 = \mathbf{0}$ and that $f_0 = 1$. Assume that $\varphi_{\theta_1} = \varphi_{\theta_2}$. Then,

$$\langle \mathbf{a}, \theta_2 \rangle = \langle \mathbf{a}, \theta_1 \rangle + 2\pi r, \quad \forall \mathbf{a} \in A.$$

By translating, we can assume that $\theta_1 = 0$, and hence, since 1 is a monomial of f, that all coefficients are real. Consider the lattice L consisting of all points $\theta \in \mathbb{R}^n$ such that f is real at θ. Since $\mathbb{Z}A = \mathbb{Z}^n$, Lemma 3.2 shows that $L = \pi\mathbb{Z}^n$. However, we find that

$$\left\langle \mathbf{a}, \frac{\theta_2}{2} \right\rangle = \pi r,$$

and hence $\frac{\theta_2}{2} \in L$. This implies that $\theta_2 \in 2\pi\mathbb{Z}^n$, and hence $\theta_2 = 0$ in $\mathbf{T}^n$. □

Proof of Proposition 3.1. Assume first that there is a θ as in the statement of the proposition, where we can assume that $\theta = 0$. Then, $\arg(f_k) = \arg(\delta_k)$. It follows that the monomials

$$\prod_{\delta_k = 1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k = -1} f_k^{-\mathbf{b}_k} \quad \text{and} \quad \prod_{\delta_k = -1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k = 1} f_k^{-\mathbf{b}_k}$$

have equal signs. Therefor, Δ vanishes for $f_k = \delta_k |\mathbf{b}_k|$, implying that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$.

For the converse, fix κ, and reduce f by requiring that $f_k = \delta_k|\mathbf{b}_k| = \mathbf{b}_k$ for $k \neq \kappa$. Let $\mathcal{I}$ denote the set of points $\theta \in \mathbf{T}^n$ such that $\hat{f}_k(\theta) = \delta_k$ for $k \neq \kappa$, which by Lemma 3.3 has cardinality V_κ. By Lemma 3.4, the set $\mathcal{I}$ is in a bijective correspondence with values of $\arg(f_\kappa)$ such that $\hat{f}_\kappa(\theta) = \delta_\kappa$. Therefor, we find that Δ vanishes at $f_\kappa = V_\kappa e^{i\varphi}$ for each $\varphi \in \mathcal{I}$. However, the discriminant Δ specializes, up to a constant, to the binomial

$$\Delta_\kappa(f_\kappa) = f_\kappa^{|\mathbf{b}_\kappa|} - \mathbf{b}_\kappa^{|\mathbf{b}_\kappa|} = f_\kappa^{V_\kappa} - \mathbf{b}_\kappa^{V_\kappa},$$

which has exactly V_κ-many solutions in $\mathbb{C}_*$ of distinct arguments. Hence, since $\Delta(f) = 0$ by assumption, and comparing the number of solutions, it holds that $\hat{f}_\kappa(\theta) = \delta_\kappa$ for one of the points $\theta \in \mathcal{I}$. □

4. The space of coamoebas

Let $U_k \subset \mathbb{C}_*^A$ denote the set of all f such that the number of connected components of the complement of $\overline{\mathcal{C}}_f$ is $\mathrm{Vol}(A) - k$. Describing the sets U_k is known as the problem of describing the *space of coamoebas* of $\mathbb{C}_*^A$. In this section, we will give explicit descriptions of the sets U_k in the case when A is a circuit. As a first observation we note that the image of the map $\mathrm{ord}_B(f)$ is at least of cardinality $\mathrm{Vol}(A) - 1$, implying that

$$\mathbb{C}_*^A = U_0 \cup U_1,$$

and in particular $U_k = \emptyset$ for $k \geq 2$. Hence, it suffices for us to give an explicit description of the set U_1. Our main result is the following two theorems, highlighting also the difference between vertex circuits and simplex circuits. Note that Δ is a real polynomial [9].

Theorem 4.1. *Assume that A is a nondegenerate simplex circuit, with $\mathbf{a}_{n+1}$ as an interior point. Choose B such that $\delta_{n+1} = -1$, and let Δ be as in* (8)*. Then, $f \in U_1$ if and only if* $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$ *and*

$$(-1)^{\mathrm{Vol}(A)}\, \Delta\big(\delta_0|f_0|, \ldots, \delta_{n+1}|f_{n+1}|\big) \leq 0. \tag{12}$$

Theorem 4.2. *Assume that A is a vertex circuit. Then, $f \in U_1$ if and only if* $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$.

The article [18] describes the *space of amoebas* in the case when A is a simplex circuit in dimension at least two. In this case, the number of connected components of the amoeba complement is either equal to the number of vertices of $\mathcal{N}_A$ or one greater. One implication of [18, thm. 4.4 and thm. 5.4] is that, if the amoeba complement has the minimal number of connected components, then

$$(-1)^{\mathrm{Vol}(A)}\, \Delta\big(\delta_0|f_0|, \ldots, \delta_{n+1}|f_{n+1}|\big) \geq 0.$$

Furthermore, this set intersect U_1 only in the discriminant locus $\Delta(f) = 0$. The space of amoebas in the case when A is a simplex circuit in dimension $n = 1$ has been studied in [19], and is a more delicate problem. On the other hand, if A is a vertex circuit, then each $f \in \mathbb{C}_*^A$ is maximally sparse and hence has a solid amoeba. That is, the components of the complement of the amoeba is in a bijective correspondance with the vertices of the Newton polytope $\mathcal{N}_A$. In particular, the number of connected components of the amoeba complement does not depend on f. From Theorems 4.1 and 4.2 we see that a similar discrepancy between simplex circuits and vertex circuits occurs for coamoebas.

Example 4.3. The reduced family

$$f(z) = 1 + z_1^3 + z_2^3 + \xi\, z_1 z_2.$$

was considered in [16, ex. 6, p. 59], where the study of the space of amoebas was initiated. We have drawn the space of amoebas and coamoebas jointly in the left picture in Figure 2. The blue region, whose boundary is a hypocycloid, marks values of ξ for which the amoeba complement has no bounded component. The set U_1 is seen in orange. The red dots is the discriminant locus $\Delta(\xi) = 0$, which is contained in the circle $|\xi| = 3$ corresponding to an equality in (12).

Example 4.4. The reduced family

$$f(z) = 1 + z_1 + z_2^3 + \xi\, z_1^3 z_2$$

is a vertex circuit. In this case, the topology of the amoeba does not depend on the coefficient ξ. The space of coamoebas is drawn in the right picture in Figure 2. The set U_1 comprises the three orange lines emerging from the origin. The red dots is the discriminant locus $\Delta(\xi) = 0$. It might seem like the set U_0 is disconnected, however this a consequence of that we consider f in reduced form. In $\mathbb{C}_*^A$ the set U_0 is connected, though not simply connected.

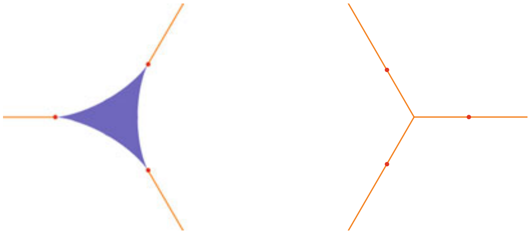

FIGURE 2. The amoeba and coamoeba spaces of Examples 4.3 and 4.4.

4.1. Proof of Theorem 4.1

Impose the assumptions of Theorem 4.1. Then,

$$\mathbf{b}_0 + \cdots + \mathbf{b}_n = -\mathbf{b}_{n+1} = \mathrm{Vol}(A),$$

and in particular $V_{n+1} = \mathrm{Vol}(A)$. By Lemma 3.3 there is a set $\mathcal{I}$ of cardinality $\mathrm{Vol}(A)$ consisting of all points θ such that $\hat{f}_0(\theta) = \cdots = \hat{f}_n(\theta) = \delta_k = 1$ In particular, f is colopsided at $\theta \in \mathcal{I}$ unless $\hat{f}_{n+1}(\theta) = -1$. It was shown in [7, sec. 5] that, if $f \notin \mathcal{C}_\Delta$, then $\mathcal{I}$ is an index set for the complement of $\overline{\mathcal{C}}_f$. In fact, $\mathcal{I}$ is an index set of the complement of $\overline{\mathcal{C}}_f$ for arbitrary f.

Proposition 4.5. *Let A be a simplex circuit. Assume that* $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$*, i.e., that there exists a $\theta \in \mathcal{I}$ with $\hat{f}_{n+1}(\theta) = \delta_{n+1}$. Then, the complement of $\overline{\mathcal{C}}_f$ has* $\mathrm{Vol}(A)$*-many connected components if and only if it contains θ.*

Proof. We can assume that $\theta = 0$. To prove the *if*-part, assume that $0 \in \Theta$ for some connected component Θ of the complement of $\overline{\mathcal{C}}_f$. We wish to show that f is never colopsided in Θ, for this implies that the complement of $\overline{\mathcal{C}}_f$ has $\mathrm{Vol}(A)$-many connected components. Assume to the contrary that there exist a point $\hat{\theta} \in \Theta$ such that f is colopsided at $\hat{\theta}$. Then, $\mathrm{ord}_B(f)(\hat{\theta}) = m\pi$ for some integer m, with $|m| < \mathrm{Vol}(A)$, see (7). Let $f^\varepsilon = (f_0, \ldots, f_n, f_{n+1}e^{i\varepsilon})$. Then f^ε is colopsided at 0 for $\varepsilon \notin 2\pi\mathbb{Z}$. By continuity of roots, for $\varepsilon > 0$ sufficiently small, the points 0 and $\hat\theta$ are contained in the same connected component of the complement of $\overline{\mathcal{C}}(f^\varepsilon)$. Hence, by [7, pro. 3.9], they are contained in the same connected component of the complement of $\overline{\mathcal{L}}(f^\varepsilon)$. However,

$$\mathrm{ord}_B(f^\varepsilon)(0) = \mathrm{Vol}(A)(\pi - \varepsilon) \neq m(\pi - \varepsilon) = \mathrm{ord}_B(f^\varepsilon)(\hat\theta),$$

contradicting that $\mathrm{ord}_B(f^\varepsilon)$ is constant on connected components of the complement of $\mathcal{L}(f^\varepsilon)$.

To prove the *only if*-part, assume that there exists a connected component Θ of the complement of $\overline{\mathcal{C}}_f$ in which f is never colopsided. We wish to prove that $0 \in \Theta$. As f^ε is colopsided at 0 for $\varepsilon > 0$ sufficiently small, we find that $0 \in \overline{\Theta}$. Indeed, if this was not the case, then the complement of $\overline{\mathcal{C}}(f^\varepsilon)$ has $(\mathrm{Vol}(A) + 1)$-many connected components, a contradiction. As $0 \notin \mathcal{H}_f$, and by [7, lem. 2.3], there exists a disc D_0 around 0 such that

$$D_0 \cap (\mathbf{T}^n \setminus \overline{\mathcal{C}}_f) = D_0 \cap \Theta.$$

Furthermore, $D_0 \cap \Theta \neq \emptyset$, since $0 \in \overline{\Theta}$. Let $\theta \in D_0 \cap \Theta$. As f is a real polynomial, conjugation yields that $-\theta \in D_0 \cap \Theta$. However, $\Theta \subset \mathbb{R}^n$ is convex, implying that $0 \in \Theta$. □

Proof of Theorem 4.1. If $\mathrm{Arg}(f) \notin \mathcal{C}_\Delta$ then the image of $\mathrm{ord}_B(f)$ is of cardinality $\mathrm{Vol}(A)$, and hence $f \in U_0$. Thus, we only need to consider $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$, where we can assume that $\hat f(0) = \delta_k$ for all k. In particular, f is a real polynomial. By Proposition 4.5, it holds that the complement of $\overline{\mathcal{C}}_f$ has $\mathrm{Vol}(A)$-many connected components if and only if it contains 0. Keeping $f_0, \dots, f_n$ and $\arg(f_{n+1})$ fixed, let us consider f as a function of $|f_{n+1}|$. As f is a real polynomial, it restricts to a map $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$, whose image depends nontrivially on $|f_{n+1}|$. Notice that $0 \in \overline{\mathcal{C}}_f$ if and only if $f(\mathbb{R}^n_{\geq 0})$ contains the origin. Since $\hat f_k(0) = \delta_k = 1$ for $k \neq n+1$, and since $\mathbf{a}_{n+1}$ is an interior point of A, the map f takes the boundary of $\mathbb{R}^n_{\geq 0}$ to $[1, \infty)$. In particular, if $0 \in f(\mathbb{R}^n_{\geq 0})$, then $0 \in f(\mathbb{R}^n_+)$. The boundary of the set of all $|f_{n+1}|$ for which $0 \in f(\mathbb{R}^n_{\geq 0})$ is the set of all values of $|f_{n+1}|$ for which $f(\mathbb{R}^n_+) = [0, \infty)$. Furthermore, $f(\mathbb{R}^n_+) = [0, \infty)$ holds if and only if there exists an $r \in \mathbb{R}^n_+$ such that $f(r) = 0$, while $f(r) \geq 0$ in a neighborhood of r, implying that r is a critical point of f. That is,

$$\Delta\big(\delta_1|f_1|, \dots, \delta_{n+1}|f_{n+1}|\big) = 0.$$

Since Δ is a binomial, there is exactly one such value of $|f_{n+1}|$. Finally, we note that $0 \in \overline{\mathcal{C}}_f$ if $|f_{n+1}| \to \infty$, which finishes the proof. □

4.2. Proof of Theorem 4.2

If $\mathrm{Arg}(f) \notin \mathcal{C}_\Delta$, then the image of $\mathrm{ord}_B(f)$ is of cardinality $\mathrm{Vol}(A)$ and hence $f \in U_0$. Assume that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$, and that $\hat f_k(0) = \delta_k$ for all k. It holds that $0 \in \mathcal{H}_f$ since there exists two adjacent vertices $\mathbf{a}_0$ and $\mathbf{a}_1$ of A such that $\delta_0 = 1$ and $\delta_1 = -1$. Let,

$$H = \{\theta \mid \langle \mathbf{a}_0 - \mathbf{a}_1, \theta\rangle = 0\}$$

be the hyperplane of $\mathcal{H}_f$ containing 0. Assume that exists connected component Θ of the complement of $\overline{\mathcal{C}}_f$ in which f is nowhere colopsided. As in the proof of Proposition 4.5, we conclude that $0 \in \overline{\Theta}$, for otherwise we could construct a coamoeba with $(\mathrm{Vol}(A) + 1)$-many connected components of its complement. As $H \subset \overline{\mathcal{C}}_f$, we find that Θ is contained in one of the half-spaces

$$H_\pm = \{\theta \,|\, \pm \langle \mathbf{a}_0 - \mathbf{a}_1, \theta\rangle > 0\},$$

say that $\Theta \subset H_+$. Let $f^\varepsilon = (f_0 e^{i\varepsilon}, f_1, \ldots, f_{n+1})$, and let H^ε denote the corresponding hyperplane

$$H^\varepsilon = \{\theta \,|\, \langle \mathbf{a}_0 - \mathbf{a}_1, \theta\rangle = -\varepsilon\}.$$

For $|\varepsilon|$ sufficiently small, continuity of roots implies that there is a connected component $\Theta^\varepsilon \subset H_+^\varepsilon$ in which f^ε is never colopsided. However, by choosing the sign of ε, we can force $0 \in H_-^\varepsilon$. This implies that the coamoeba $\overline{\mathcal{C}}_{f^\varepsilon}$ has $(\mathrm{Vol}(A)+1)$-many connected components of its complement, a contradiction.

5. The maximal area of planar circuit coamoebas

In this section, we will prove the following bound.

Theorem 5.1. *Let A be a planar circuit, and let $f \in \mathbb{C}_*^A$. Then* $\mathrm{Area}(\mathcal{C}_f) \leq 2\pi^2$.

Furthermore, we provide the following classification of for which circuits the bound of Theorem 5.1 is sharp.

Theorem 5.2. *Let A be a planar circuit. Then there exist a polynomial $f \in \mathbb{C}_*^A$ such that* $\mathrm{Area}(\mathcal{C}_f) = 2\pi^2$ *if and only if A admits an equimodular triangulation.*

Example 5.3. Let $f(z) = 1 + z_1 + z_2 - rz_1z_2$ for $r \in \mathbb{R}_+$. Notice that A admits a unimodular triangulation. The shell $\mathcal{H}_f$ consist of the families $\theta_1 = k_1\pi$ and $\theta_2 = k_2\pi$ for $k_1, k_2 \in \mathbb{Z}$. Hence, the shell $\mathcal{H}_f$ divides $\mathbf{T}^2$ into four regions of equal area. Exactly two of these regions are contained in the coamoeba, which implies that $\mathrm{Area}(\mathcal{C}_f) = 2\pi^2$. See the left picture of Figure 3.

Example 5.4. Let $f(z) = 1 + zw^2 + z^2w - rzw$ for $r \in \mathbb{R}_+$. Also in this case A admits a unimodular triangulation. Notice that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$. The coamoeba of the trinomial $g(z) = 1 + zw^2 + z^2w$ has three components of its complement, of which f is colopsided in two. We have that $\mathcal{H}_f = \mathcal{H}_g$. Thus, if the complement of $\overline{\mathcal{C}}_f$ has two connected components, i.e., if $r \geq 3$, then one of the three components of the complement of $\overline{\mathcal{C}}_g$ is contained in $\overline{\mathcal{C}}_f$, which in turn implies that $\mathrm{Area}(\mathcal{C}_f) = 2\pi^2$. See the right picture of Figure 3.

Let us compare our results to the corresponding results of planar circuit amoebas. It was shown in [16, Thm. 12, p. 30] that the sharp upper bound on the number of connected components of the amoeba complement is $\#A$. In [12], a bound on the area of the amoeba was given as $\pi^2\,\mathrm{Vol}(A)$, and it was shown that maximal area was obtained for Harnack curves. For coamoebas, to roles of the

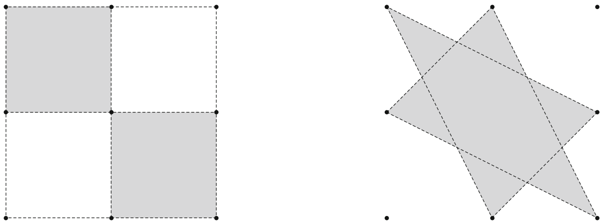

FIGURE 3. The coamoebas of Examples 5.3 and 5.4.

integers $\mathrm{Vol}(A)$ and $\#A$ are reversed. The upper bound on the number of connected components of the coamoeba complement is given by $\mathrm{Vol}(A)$. While, at least for codimension $m \leq 1$, the maximal area of the coameoba is $\pi^2(m+1) = \pi^2(\#A - n)$. Note also that the coamoebas of Examples 5.3 and 5.4 are both coamoebas of Harnack curves.

Consider a bivariate trinomial f, with one marked monomial. Let $\Sigma = \Sigma(f)$ denote the quadruple of polynomials obtained by flipping signs of the unmarked monomials. Furthermore, let

$$\mathcal{H}_\Sigma = \bigcup_{g \in \Sigma} \mathcal{H}_g,$$

which is a hyperplane arrangement in $\mathbb{R}^2$ (or $\mathbf{T}^2$). Let $\mathcal{P}_\Sigma$ denote the set of all intersection points of distinct hyperplanes in $\mathcal{H}_\Sigma$.

Proposition 5.5. *Let $f(z)$ be a bivariate trinomial. Then, the union*

$$\overline{\mathcal{C}}_\Sigma = \bigcup_{g \in \Sigma} \overline{\mathcal{C}}_g,$$

covers $\mathbb{R}^2$. To be specific, $\mathcal{P}_\Sigma$ is covered thrice, $\mathcal{H}_\Sigma \setminus \mathcal{P}_\Sigma$ is covered twice, and $\mathbb{R}^2 \setminus \mathcal{H}_\Sigma$ is covered once.

Proof. After applying an integer affine transformations, we reduce to the case when A consist of the vertices of the standard simplex. This case that follows from the description in [7] and [14], see also Figure 1. □

Corollary 5.6. *If $f(z)$ is a bivariate trinomial, then* $\mathrm{Area}(\overline{\mathcal{C}}_f) = \pi^2$.

Proof. The coamoebas appearing in the union $\overline{\mathcal{C}}_\Sigma$, when considered in $\mathbb{R}^2$, are merely translations of each other. Hence, they have equal area. As they cover the torus once a.e., and $\mathrm{Area}(\mathbf{T}^2) = 4\pi^2$, the result follows. □

Notice that a bivariate trinomial is not supported on a circuit, but on the vertex set of a simplex. Let $f_{\hat{k}}$ denote the image of f under the projection $\mathrm{pr}_k \colon \mathbb{C}_*^A \to \mathbb{C}_*^{A_k}$. As shown in [7] the family of trinomials $f_{\hat{k}}$, $k = 1, \ldots, 4$, contains all necessary information about the lopsided coamoeba $\mathcal{L}_f$.

Lemma 5.7. *Let A be a planar circuit, and let $f \in \mathbb{C}_*^A$. Assume that $\theta \in \mathbf{T}$ is generic in the sense that no two components of $\hat{f}(\theta)$ are antipodal, and assume* *furthermore that f is* not *colopsided at θ. Then, exactly two of the trinomials $f_{\hat{1}}, \dots, f_{\hat{4}}$ are colopsided at θ.*

Proof. Fix an arbitrary point $\mathbf{a}_1 \in A$, and let $\ell \subset \mathbb{C}$ denote the real subvector space containing $\hat{f}_1(\theta)$. As f is not colopsided at θ, both half-spaces relative ℓ contains at least one component of $\hat{f}(\theta)$. There is no restriction to assume that the upper half-space contains the two components $\hat{f}_2(\theta)$ and $\hat{f}_3(\theta)$, and that the latter is of greatest angular distance from $\hat{f}_1(\theta)$. Then, $f_{\hat{4}}$ is colopsided at θ. Furthermore, we find that $f_{\hat{2}}$ is not colopsided at θ, for if it where then so would f. As $\mathbf{a}_1 \in A_4$ and $\mathbf{a}_1 \in A_2$, there is at least one trinomial obtained from f containing $\mathbf{a}_1$ which is not colopsided at θ, and at least one which is colopsided at θ. As $\mathbf{a}_1$ was arbitrary, it follows that exactly two of the trinomials $f_{\hat{1}}, \dots, f_{\hat{4}}$ are colopsided at θ, and exactly two are not. □

Proof of Theorem 5.1. By containment, it holds that $\mathrm{Area}(\mathcal{C}_f) \leq \mathrm{Area}(\mathcal{L}_f)$, and thus it suffices to calculate the area of $\mathcal{L}_f$. By [7, Prop. 3.4], we have that

$$\mathcal{L}_f = \bigcup_{k=1}^{4} \mathcal{C}_{f_{\hat{k}}}. \tag{13}$$

For a generic point $\theta \in \mathcal{L}_f$, Lemma 5.7 gives that θ (and, in fact, a small neighborhood of θ) is contained in the interior of exactly two out of the four coamoebas in the right-hand side of (13). Hence,

$$\mathrm{Area}(\mathcal{L}_f) = \frac{1}{2}\big(\mathrm{Area}(\mathcal{C}_{f_{\hat{1}}}) + \dots + \mathrm{Area}(\mathcal{C}_{f_{\hat{4}}})\big) = 2\pi^2. \qquad \square$$

Proof of Theorem 5.2. To prove the *if* part, we will prove that A admits an equimodular triangulation only if, after applying an integer affine transformation, it is equal to the point configuration of either Example 5.3 or Example 5.4. Assume that $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are vertices of $\mathcal{N}_A$. After applying an integer affine transformation, we can assume that $\mathbf{a}_1 = k_1\mathbf{e}_1$, that $\mathbf{a}_2 = k_2\mathbf{e}_2$ with $k_1 \geq k_2$, and that $\mathbf{a}_3 = \mathbf{0}$. Notice that such a transformation rescales A, though it does not affect the area of the coamoeba $\mathcal{C}_f$ [7]. Let $\mathbf{a}_4 = m_1\mathbf{e}_1 + m_2\mathbf{e}_2$.

If A is a vertex circuit, then each triangulation of A consist of two simplices, which are of equal area by assumption. Comparing the areas of the subsimplices of A, we obtain the relations

$$|k_1k_2 - k_1m_2 - k_2m_1| = k_1k_2 \quad \text{and} \quad k_1m_2 = k_2m_1.$$

In m, this system has (k_1, k_2) as the only nontrivial solution, and we conclude that A is the unit square, up to integer affine transformations.

If A is a simplex circuit, then A has one triangulation with three simplices of equal area. Comparing areas, we obtain the relations

$$3k_1m_2 = 3k_2m_1 = k_1k_2.$$

Thus, $3m_1 = k_1$ and $3m_2 = k_2$, and we conclude that A is the simplex from Example 5.4, up to integer affine transformations.

To prove the *only if*-statement, consider $f \in \mathbb{C}^A_*$. Let $S = \{\mathbf{a}_1, \mathbf{a}_2\} \subset A$ be such that the line segment $[\mathbf{a}_1, \mathbf{a}_2]$ is interior to $\mathcal{N}_A$. Applying an integer affine transformation, we can assume that $[\mathbf{a}_1, \mathbf{a}_2] \subset \mathbb{R}\mathbf{e}_1$, and that $\mathbf{a}_3$ and $\mathbf{a}_4$ lies in the upper and lower half-space respectively. Then, the hyperplane arrangement $\mathcal{C}_{f_S} \subset \mathbf{T}$ consist of Length$[\mathbf{a}_1, \mathbf{a}_2]$-many lines, each parallel to the θ_2-axis. If $\mathbf{a}_3 = m_{31}\mathbf{e}_1 + m_{32}\mathbf{e}_2$ and $\mathbf{a}_4 = m_{41}\mathbf{e}_1 + m_{42}\mathbf{e}_2$, then $\hat{f}_3(\theta)$ and $\hat{f}_4(\theta)$ takes m_{32} respectively m_{42} turns around the origin when θ traverses once a line of $\mathcal{C}_{f_S}$. Notice that $\mathcal{C}_{f_S} \subset \overline{\mathcal{L}}_f$, as $\hat{f}_1(\theta)$ and $\hat{f}_2(\theta)$ are antipodal for $\theta \in \mathcal{C}_{f_S}$. That is, for such θ, $\hat{f}_S(\theta)$ is contained in a real subvector space $\ell_\theta \subset \mathbb{C}$.

Assume that f is colopsided for some $\theta \in \mathcal{C}_{f_S}$, so that in particular $\theta \notin \mathcal{C}_f$. If $\theta \in \mathcal{H}_f$, then at exactly one of the points $\hat{f}_3(\theta)$ and $\hat{f}_4(\theta)$ is contained in ℓ_θ, for otherwise f would not be colopsided at θ. By wiggling θ in $\mathcal{C}_{f_S}$ we can assume that $\theta \notin \mathcal{H}_f$. Under this assumption, we find that $\theta \notin \overline{\mathcal{C}}(f)$. Thus, there is a neighborhood N_θ which is separated from $\overline{\mathcal{C}}_f$. As $\theta \in \overline{\mathcal{L}}_f$, the intersection $N_\theta \cap \overline{\mathcal{L}}_f$ has positive area, implying that $\text{Area}(\overline{\mathcal{C}}_f) < \text{Area}(\overline{\mathcal{L}}_f)$.

Thus, if f is such that $\text{Area}(\overline{\mathcal{C}}_f) = 2\pi^2$, then f can never be colopsided in $\mathcal{C}_{f_S}$. In particular, for $\theta \in \mathcal{C}_{f_S}$ such that $\hat{f}_3(\theta) \in \ell$, it must hold $\hat{f}_4(\theta) \in \ell$, and vice versa. As there are $2m_{32}$ points of the first kind, and $2m_{42}$ points of the second kind, it holds that $m_{32} = m_{42}$. Hence, the simplices with vertices $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ have equal area.

If A is a vertex circuit, this suffices in order to conclude that A admits an equimodular triangulation. If A is a simplex circuit, then we can assume that $\mathbf{a}_1$ is an interior point of $\mathcal{N}_A$. Repeating the argument for either $S = \{\mathbf{a}_1, \mathbf{a}_3\}$ or $S = \{\mathbf{a}_1, \mathbf{a}_4\}$ yields that A has a triangulation with three triangles of equal area. That is, it admits an equimodular triangulation. □

6. Critical points

Let $C(f)$ denote the critical points of f, that is, the variety defined by (2). Let $\mathcal{I} = \text{Arg}(C(f))$ denote the coamoeba of $C(f)$. We will say that $\mathcal{I}$ is the set of *critical arguments* of f. In this section we will prove that, under certain assumptions on A, the set $\mathcal{I}$ is an index set of the coamoeba complement. That it is necessary to impose assumptions on A is related to the fact that an integer affine transformation acts nontrivially on the set of critical points $C(f)$.

Let A be a circuit, with the elements $\mathbf{a} \in A$ ordered so that it has a Gale dual $B = (B_1, B_2)^t$ such that $B_1 \in \mathbb{R}_+^{m_1+1}$ and that $B_2 \in \mathbb{R}_-^{m_2+1}$. That is, B_1 has only positive entries, while B_2 has only negative entries. We have that $m_1 + m_2 = n$. Let $A = (A_1, A_2)$ denote the corresponding decomposition of the matrix A. We will say that A is in *orthogonal form* if

$$A = \begin{pmatrix} 1 & 1 \\ \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}, \tag{14}$$

where $\tilde{A}_1$ is an $m_1 \times (m_1+1)$-matrix and $\tilde{A}_2$ is an $m_2 \times (m_2+1)$-matrix. In particular, the Newton polytopes $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has $\mathbf{0}$ as a relatively interior point, and as their only intersection point.

With A in the form (14), we can act by integer affine transformations affecting $\tilde{A}_1$ and $\tilde{A}_2$ separately. Therefor, if A is in orthogonal form, then we can assume that

$$\tilde{A}_k = (-p_1\mathbf{e}_1, \dots, -p_{m_k}\mathbf{e}_{m_k}, \mathbf{a}_{m_k+1}), \tag{15}$$

where $p_1, \dots, p_{m_k}$ are positive integers, and hence $\mathbf{a}_{m_k+1}$ has only positive coordinates. We will say that A is in *special orthogonal form* if (15) holds. The main result of this section is the following lemma and theorem.

Lemma 6.1. *Each circuit A can be put in* (*special*) *orthogonal form by applying an integer affine transformation.*

Theorem 6.2. *Let A be a circuit in special orthogonal form. Then, for each $f \in \mathbb{C}_*^A$, the set of critical arguments is an index set of the complement of $\overline{\mathcal{C}}_f$.*

The conditions of Theorem 6.2 can be relaxed in small dimensions. When $n=1$, it is enough to require that $\mathbf{0}$ is an interior point of $\mathcal{N}_A$. When $n=2$, for generic f, it is enough to require that each quadrant Q fulfills that $\overline{Q}\setminus\{\mathbf{0}\}$ has nonempty intersection with A.

Proof of Lemma 6.1. Let $\mathbf{u}_1, \dots, \mathbf{u}_{m_2}$ be a basis for the left kernel $\ker(A_1)$, and let $\mathbf{v}_1, \dots, \mathbf{v}_{m_1}$ be a basis for the left kernel $\ker(A_2)$. Multiplying A from the left by

$$T = \big(\mathbf{e}_1, \mathbf{v}_1, \dots, \mathbf{v}_{m_1}, \mathbf{u}_1, \dots, \mathbf{u}_{m_2}\big)^t,$$

it takes the desired form. We need only to show that $\det(T) \neq 0$.

Notice that $\ker(A_1)\cap\ker(A_2) = 0$, since A is assumed to be of full dimension. Assume that there is a linear combination

$$\lambda_0\mathbf{e}_1 + \sum_{i=1}^{m_1}\lambda_i\mathbf{v}_i + \sum_{j=1}^{m_2}\lambda_j\mathbf{u}_j = 0.$$

Then, since B is a Gale dual of A,

$$0 = \left(\sum_{j=1}^{m_2}\lambda_j\mathbf{u}_j\right)AB = (0,\dots,0,-\lambda_0,\dots,-\lambda_0)B = -\lambda_0\sum_{\mathbf{a}\in A_2}\mathbf{b}_{\mathbf{a}} = \lambda_0\,\mathrm{Vol}(A),$$

and hence $\lambda_0 = 0$. This implies that $\sum_{i=1}^{m_1}\lambda_i\mathbf{v}_i \in \ker(A_2)$, and hence $\sum_{i=1}^{m_1}\lambda_i\mathbf{v}_i = \mathbf{0}$. Thus, $\lambda_i = 0$ for all i by linear independence of the vectors $\mathbf{v}$. Then, linear independence of the vectors $\mathbf{u}_j$ imply that $\lambda_j = 0$. □

Proof of Theorem 6.2. We find that

$$z_i\partial_i f(z) = -p_i f_i z^{\mathbf{a}_i} + \langle\mathbf{a}_{m_k}, \mathbf{e}_i\rangle f_{m_1} z^{\mathbf{a}_{m_1}}, \quad i = 0, \dots, m_1$$
$$z_j\partial_j f(z) = -p_j f_i z^{\mathbf{a}_j} + \langle\mathbf{a}_{n+1}, \mathbf{e}_j\rangle f_{n+1} z^{\mathbf{a}_{n+1}}, \quad j = m_1+1, \dots, n.$$

Hence, for each $\theta \in \mathcal{I}$, it holds that

$$\hat{f}_0(\theta) = \cdots = \hat{f}_{m_1}(\theta) \quad \text{and} \quad \hat{f}_{m_1+1}(\theta) = \cdots = \hat{f}_{n+1}(\theta). \tag{16}$$

In particular, f is colopsided at θ unless, after a rotation, $\hat{f}_k(\theta) = \delta_k$ for all k. In the latter case, we refer to Theorems 4.1 and 4.2.

To see that the points $\theta \in \mathcal{I}$ for which f is colopsided at θ are contained in distinct connected components of the complement of $\overline{\mathcal{L}}_f$, consider a line segment ℓ in $\mathbb{R}^n$ with endpoints in $\mathcal{I}$. Then, not all identities of (16) can hold identically along ℓ. Since the argument of each monomial is linear in θ, this implies that for a pair such that the identity in (16) does not hold identically along ℓ, there is an intermediate point $\theta \in \ell$ for which the corresponding monomials are antipodal, and hence $\theta \in \overline{\mathcal{L}}_f$. □

7. On systems supported on a circuit

In this section we will consider a system

$$F_1(z) = F_2(z) = 0 \tag{17}$$

of two bivariate polynomials. We will write $f(z) = 0$ for the system (17). The system is said to be generic if it has finitely many roots in $\mathbb{C}_*^2$, and it is said to be supported on a circuit A if the supports of F_1 and F_2 are contained in, but not necessarily equal to, A. That is, we allow coefficients in $\mathbb{C}$ rather than $\mathbb{C}_*$. By the Bernstein–Kushnirenko theorem, a generic system $f(z) = 0$ has at most $\mathrm{Vol}(A)$-many roots in $\mathbb{C}_*^2$. However, if f is real, then fewnomial theory states that a generic system $f(z)$ has at most three roots in $\mathbb{R}_+^2 = \mathrm{Arg}^{-1}(0)$. We will solve the complexified fewnomial problem, i.e., for $f(z)$ with complex coefficients we will bound the number of roots in each sector $\mathrm{Arg}^{-1}(\theta)$. Our intention is to offer a new approach to fewnomial theory. We will restrict to the case of simplex circuits, for the following two reasons. Firstly, it allows for a simpler exposition. Secondly, for vertex circuits our method recovers the known (sharp) bound, while for simplex circuits we obtain a sharpening of the fewnomial bound.

Theorem 7.1. *Let $f(z) = 0$ be a generic system of two bivariate polynomials supported on a planar simplex circuit $A \subset \mathbb{Z}^2$. Then, each sector $\mathrm{Arg}^{-1}(\theta)$ contains at most two solutions of $f(z) = 0$.*

7.1. Reducing $f(z)$ to a system of trinomials

A generic system $f(z)$ is, by taking appropriate linear combinations, equivalent to a system of two trinomials whose support intersect in a dupleton. That is, we can assume that $f(z)$ is in the form

$$\begin{cases} F_1(z) = & f_1 z^{\mathbf{a}_0} & + & z^{\mathbf{a}_2} & + & f_2 z^{\mathbf{a}_3} & = 0 \\ F_2(z) = & f_3 z^{\mathbf{a}_1} & + & z^{\mathbf{a}_2} & + & f_4 z^{\mathbf{a}_3} & = 0, \end{cases} \tag{18}$$

with coefficients in $\mathbb{C}_*$. We will use the notation

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} \quad \text{and} \quad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ A_1 & A_2 \end{pmatrix},$$

where A_k denotes the support of F_k (notice that this differs from the notation used in previous sections). Notice that we can identify a system $f(z)$ in the form (18) with its corresponding vector in $\mathbb{C}_*^{\hat{A}}$.

When reducing $f(z)$ to the form (18) by taking linear combinations, there is a choice of which monomials to eliminate in F_1 and F_2 respectively. In order for the arguments of the roots of $f(z) = 0$ to depend continuously on the coefficients, we need to be careful with which choice to make.

Lemma 7.2. *Let ℓ denote the line through $\mathbf{a}_2$ and $\mathbf{a}_3$, and let γ be a compact path in $\mathbb{C}_*^{\hat{A}}$. If ℓ intersect the interior of $\mathcal{N}_A$, then the arguments of the solutions to $f(z) = 0$ vary continuously along γ.*

Proof. It is enough to show that along a compact path γ, the set

$$\bigcup_{f \in \gamma} \mathcal{A}_f = \bigcup_{f \in \gamma} \operatorname{Log}(Z(f)) \tag{19}$$

is bounded, for it implies that for $f \in \gamma$, the roots of f are uniformly separated from the boundary of $\mathbb{C}_*^{\hat{A}}$.

We first claim that our assumptions imply that the normal fans of $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has no coinciding one-dimensional cones. Indeed, these fans has a coinciding one-dimensional cone if and only if the Newton polytopes $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has facets Γ_1 and Γ_2 with a common outward normal vector $\mathbf{n}$. As A is a circuit, it holds that $\Gamma_1 = \Gamma_2 = [\mathbf{a}_2, \mathbf{a}_3] \subset \ell$. Since the normal vector $\mathbf{n}$ is common for $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$, we find that Γ_1 (and Γ_2) is a facet of $\mathcal{N}_A$. But then ℓ contains a facet of $\mathcal{N}_A$, and hence it cannot intersect the interior of $\mathcal{N}_A$, a contradiction.

Consider a point $f \in \mathbb{C}_*^A$. Since the normal fans $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has no coinciding one-dimensional cones, the intersection of the amoebas $\mathcal{A}_{F_1}$ and $\mathcal{A}_{F_2}$ is bounded (this follows, e.g., from the fact the amoeba has finite Hausdorff distance from the Archimedean tropical variety, see [1]). Thus, the amoeba $\mathcal{A}_f$ is bounded, say that $\mathcal{A}_f \subset D(R_f)$ where $D(R_f)$ denotes the disk of radii R_f centered around the origin. By continuity of roots, $\mathcal{A}_{\tilde{f}} \subset D(R_f)$ for all $\tilde{f}$ in some neighborhood N_f of f. The compactness of γ implies our result. □

In order for the assumptions of Lemma 7.2 to be fulfilled, for a simplex circuit A, we need that $\mathbf{a}_0$ and $\mathbf{a}_1$ are vertices of $\mathcal{N}(A)$, see Figure 4.

Proposition 7.3. *If f is nonreal at θ, then there is at most one zero of $f(z) = 0$ contained in the sector $\operatorname{Arg}^{-1}(\theta)$.*

Proof. If F_k is nonreal, then the fiber in $Z(F_k)$ over a point $\theta \in \mathcal{C}_{F_k}$ is a singleton. Hence, if the number of roots of $f(z) = 0$ in $\operatorname{Arg}^{-1}(\theta)$ is greater than one, then both F_1 and F_2 are real at θ. □

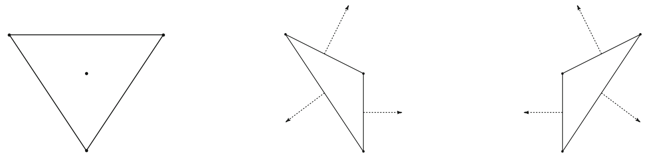

FIGURE 4. The Newton polytopes $\mathcal{N}_A$, $\mathcal{N}_{A_1}$, and $\mathcal{N}_{A_2}$.

The implication of Proposition 7.3 is that the complexified fewnomial problem reduces to the real fewnomial problem. However, our approach is dependent on allowing coefficients to be nonreal. In fact, we will consider a partially complexified problem, allowing $f_1, f_3 \in \mathbb{C}_*$ but requiring $f_2, f_4 \in \mathbb{R}_*$.

7.2. Colopsidedness

We define the colopsided coamoeba of the system $f(z)$ by

$$\mathcal{L}_f = \mathcal{L}_{F_1} \cap \mathcal{L}_{F_2} = \mathcal{C}_{F_1} \cap \mathcal{C}_{F_2},$$

where the last equality follows from [7, cor. 3.3]. That is, f is said to be colopsided at θ if either F_1 or F_2 is colopsided at θ. We will say that f is real at θ if both F_1 and F_2 are real at θ.

The lopsided coamoeba $\mathcal{L}_f$ consist of a number of polygons on $\mathbf{T}^2$, possibly degenerated to singletons. The following two lemmas will allow us to count the number of such polygons.

Lemma 7.4. *Assume that f is nonreal. Let g be a binomial constructed by choosing two monomials from* (18), *possibly alternating signs. If f_2 and f_4 are of opposite signs, then $\mathcal{C}_g \subset \mathbf{T}^2 \setminus \mathcal{L}_f$. If f_2 and f_4 are of equal signs, then $\mathcal{C}_g \subset \mathbf{T}^2 \setminus \mathcal{L}_f$* except *for $g(z) = \pm(f_1 z^{\mathbf{a}_0} - f_3 z^{\mathbf{a}_1})$.*

Proof. If, for $\theta \in \mathbf{T}^2$, two components of $\hat{F}_1(\theta)$ is contained in a real subvector space $\ell \subset \mathbb{C}$, then either F_1 is colopsided at θ or $\hat{F}_1(\theta) \subset \ell$. However, the latter implies that two components of $\hat{F}_2(\theta)$ are contained in ℓ. Repeating the argument yields that either f is real, or it is colopsided at θ.

Thus, the only binomials we need to consider is $g_\pm(z) = f_1 z^{\mathbf{a}_0} \pm f_3 z^{\mathbf{a}_1}$. For each $\theta \in \mathcal{C}_{g_+}$ the vectors $\hat{F}_1(\theta)$ and $\hat{F}_2(\theta)$ differ in sign in their first component, and hence at least one is colopsided at θ, unless f is real. For each $\theta \in \mathcal{C}_{g_-}$, the vectors $\hat{F}_1(\theta)$ and $\hat{F}_2(\theta)$ differ in signs in the the last component only if f_2 and f_4 differ in signs. If this is the case, then at least one is colopsided at θ unless f is real. $\square$

Lemma 7.5. *Let $\theta \in \mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ for truncated binomials g_1 and g_2 of F_1 and F_2 respectively. If the Newton polytopes* (*i.e., line segments*) *of g_1 and g_2 are nonparallel, then $\theta \in \overline{\mathcal{L}}_f$.*

Proof. If F_1 and F_2 are both real at θ, then $\theta \in \mathcal{L}_f$. If F_1 is nonreal at θ, then for a sufficiently small neighborhood $N_\theta \subset \mathbb{R}^2$, it holds that

$$\mathcal{C}_{F_1} \cap N_\theta = \{\varphi \,|\, \langle \varphi, \mathbf{n} \rangle > \langle \theta, \mathbf{n} \rangle\} \cap N_\theta,$$

where $\mathbf{n}$ is a normal vector of $\mathcal{N}(g_1)$. Since connected components of the complement of $\mathcal{C}_{F_2}$ are convex, either $\mathcal{C}_{F_2}$ intersect $\mathcal{C}_{F_1}$ in N_θ, or the boundary of $\mathcal{C}_{F_2}$ is contained in the line $\ell = \{\varphi \,|\, \langle \varphi, \mathbf{n} \rangle = \langle \theta, \mathbf{n} \rangle\}$. As the boundary of $\mathcal{C}_{F_2}$ contains $\mathcal{C}_{g_2}$, it holds in the latter case that $\mathcal{C}_{g_2} \subset \ell$, which in turn implies that $\mathbf{n}$ is a normal vector of $\mathcal{N}(g_2)$, contradicting our assumptions. We conclude that $\mathcal{C}_{F_2} \cap \mathcal{C}_{F_1} \cap N_\theta \neq \emptyset$. Since this holds for any sufficiently small neighborhood N_θ, the result follows. □

Example 7.6. Consider the system

$$f(z) = \begin{cases} f_1 z_1 z_2^2 + 1 + f_2 z_1 z_2 \\ f_3 z_1^2 z_2 + 1 + f_4 z_1 z_2. \end{cases}$$

We have that $\mathrm{Vol}(A) = 3$. Hence H divides $\mathbf{T}^2$ into three cells. The lopsided coamoeba $\mathcal{L}_f$, and the hyperplane arrangement H, can be seen in Figure 5. In the first two picture, the generic respectively real situation when f_2 and f_4 differs in signs. In last two pictures, the generic respectively real situation when f_2 and f_4 have equal signs. In the generic case, the lopsided coamoeba $\mathcal{L}_f$ consist of three polygons. When deforming from the generic to the real case, we observe the following behavior. Some polygons of $\mathcal{L}_f$ deform into single points – by necessity points contained in the lattice P. Some pairs of polytopes of $\mathcal{L}_f$ deforms to nonconvex polygons, typically with a single intersection point. Our proof of Theorem 7.1 is based on the observation that, when deforming from a generic to a real system, at most two polytopes of $\mathcal{L}_F$ deforms a nonconvex polygon intersecting H.

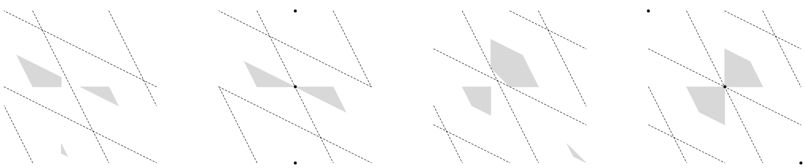

FIGURE 5. The lopsided coamoebas from Example 7.6.

7.3. Proof of Theorem 7.1

Let us consider the auxiliary binomials

$$g_1(z) = f_1 z^{\mathbf{a}_0} - z^{\mathbf{a}_2}, \qquad g_2(z) = f_3 z^{\mathbf{a}_1} - z^{\mathbf{a}_2},$$
$$h_1(z) = f_1 z^{\mathbf{a}_0} + z^{\mathbf{a}_2}, \quad \text{and} \quad h_2(z) = f_3 z^{\mathbf{a}_1} + z^{\mathbf{a}_2}.$$

The vectors $\mathbf{a}_2 - \mathbf{a}_0$ and $\mathbf{a}_2 - \mathbf{a}_1$ span the simplex $\mathcal{N}_A$, hence the hyperplane arrangement $H = \mathcal{C}_{g_1} \cup \mathcal{C}_{g_2}$ divides $\mathbf{T}^2$ into $\mathrm{Vol}(A)$-many parallelograms with the points $P = \mathcal{C}_{h_1} \cap \mathcal{C}_{h_2}$ as their centers of mass.

If f is nonreal, then Lemma 7.4 shows that $H \subset \mathbf{T}^2 \setminus \mathcal{L}_f$, and Lemma 7.5 shows that $P \subset \overline{\mathcal{L}}_f$. By Lemma 7.2 we find that $\mathcal{L}_f$ has at most $\mathrm{Vol}(A)$-many connected components. Hence, $\mathcal{L}_f$ has at exactly one connected component in each of the cells of H, and the number of roots of $f(z) = 0$ projected by the argument map into each such component is exactly one.

Consider now the real case when f_2 and f_4 differs in signs. Then, at least one of F_1 and F_2 are colopsided at the intersection points $\mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$. Thus, if $\operatorname{Arg}^{-1}(\theta)$ contains a root of $f(z) = 0$, then a sufficiently small neighborhood N_θ intersect at most two of the cells of the hyperplane arrangement H. Hence, using Lemma 7.2 and wiggling the arguments of coefficients of f by ε, N_θ intersect at most two of the polygons of $\mathcal{L}_{f^\varepsilon}$. Hence, there can be at most two roots contained in $\operatorname{Arg}^{-1}(\theta)$.

Consider now the case when f real with f_2 and f_4 of equal signs. In this case, a point $\theta \in \mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ can be contained in $\mathcal{L}_f$. See the left picture of Figure 6, where the hyperplane arrangement H is given in black, and the shells $\mathcal{H}_{F_1}$ and $\mathcal{H}_{F_2}$ are given in red and blue respectively, with indicated orientation. Wiggling the arguments of coefficient f_1 and/or f_3 by ε, we claim the we obtain a situation as in the right picture of Figure 6. That is, at most two polygons of $\mathcal{L}_{f^\varepsilon}$ will intersect a small neighborhood N_θ of θ. Let us prove this last claim.

Let f be generic, with f_2 and f_4 real and of equal signs. The hyperplanes $\mathcal{C}_{g_1}$ and $\mathcal{C}_{g_2}$ (locally) divides the plane into four regions. We can assume that $\mathbf{a}_2 = \mathbf{0}$. Then, $\mathcal{C}_{g_1}$ consist of all θ such that $\hat{f}_1(\theta) = 1$, and $\mathcal{C}_{g_1}$ consist of all θ such that $\hat{f}_3(\theta) = 1$. Thus, locally, the cells of H can be indexed by the signs of the imaginary parts of $\hat{f}_1(\theta)$ and $\hat{f}_3(\theta)$. Assume that $\tilde{\theta} \in \mathcal{L}_f \cap N_\theta$. Then neither F_1 nor F_2 is colopsided at $\tilde{\theta}$. Observe that $\hat{f}_2(\tilde{\theta}) = \hat{f}_4(\tilde{\theta})$, since f_2 and f_4 has equal sign. We find that

$$\operatorname{sgn}(\Im(\hat{f}_1(\tilde{\theta}))) = -\operatorname{sgn}(\Im(\hat{f}_2(\tilde{\theta}))) = -\operatorname{sgn}(\Im(\hat{f}_4(\tilde{\theta}))) = \operatorname{sgn}(\Im(\hat{f}_4(\tilde{\theta}))),$$

where the first and the last equality holds since neither F_1 nor F_2 is colopsided at $\tilde{\theta}$. This implies that polygons of $\mathcal{L}_f$ intersecting a small neighbourhood of θ are necessarily contained in the cells of H which corresponds to that the imaginary parts of $\hat{f}_1(\tilde{\theta})$ and $\hat{f}_4(\tilde{\theta})$ have equal signs. As there are two such cells, we find that there are at most two polygons of $\mathcal{L}_f$ intersecting a small neighbourhood of θ.

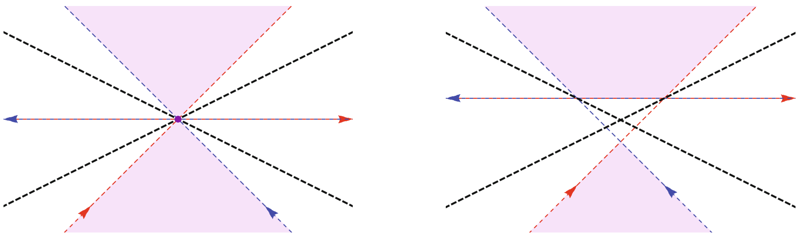

FIGURE 6. To the left: the coamoeba $\mathcal{L}_f$ close to a point of $\mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ when f_2 and f_4 have equal in signs and f_1 and f_3 are real. To the right: the same picture after wiggling the argument of f_1 or f_3.

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