Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems

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Abstract. We propose a matrix model for two- and many-valued logic using families of observables in Hilbert space, the eigenvalues give the truth values of logical propositions where the atomic input proposition cases are represented by the respective eigenvectors. For binary logic using the truth values $\{0, 1\}$ logical observables are pairwise commuting projectors. For the truth values $\{+1, -1\}$ the operator system is formally equivalent to that of a composite spin $1/2$ system, the logical observables being isometries belonging to the Pauli group. Also in this approach fuzzy logic arises naturally when considering non-eigenvectors. The fuzzy membership function is obtained by the quantum mean value of the logical projector observable and turns out to be a probability measure in agreement with recent quantum cognition models. The analogy of many-valued logic with quantum angular momentum is then established. Logical observables for three-value logic are formulated as functions of the L_z observable of the orbital angular momentum $\ell = 1$. The representative 3-valued 2-argument logical observables for the Min and Max connectives are explicitly obtained.

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1 Introduction

Quantum logic developed by Birkhoff and von Neumann in their seminal article in 1936 [1] considers logical propositions as subspaces of a quantum state Hilbert space. As will be shown hereafter and also underlined in [2], these subspaces can be viewed as eigenspaces of projectors, the projectors corresponding to logical propositions. A true proposition is then associated to the eigenvalue +1. The representation of logical propositions in a vector space could be of interest in

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modern semantic theories such as distributional semantics, for example using the "Hyperspace Analogue to Language" algorithm as was done in [3], or in connectionist models of cognition [4].

In this work we show that a proposition in a logical system can be represented by an observable in Hilbert space. When interpreted in the context of quantum mechanics this model uses finite dimensional projectors and angular momentum observables. Conversely, a quantum system when considered in its eigenspace is formally equivalent to a logical propositional system. The view here, which comes under the name of "Eigenlogic" (for the original motivation and more detailed discussion see [5]), considers that the eigenvalues of the logical observables are the truth values of a proposition and the associated eigenvectors correspond to the different input atomic propositional cases. When considering vectors outside of the eigensystem this view leads to a "fuzzy" measure of the degree of truth of a logical proposition.

In our model for binary valued logic, using numbers $\{0, 1\}$, the logical observables are pairwise commuting projectors. The model is extended to the other binary system using numbers $\{+1, -1\}$, differences reside in the symmetry of the corresponding logical observables. In the latter case the observables are equivalent to quantum spin $1/2$ observables, no more idempotent projectors but isometric self-inverse reflection observables squaring to 1. These are equivalent to the recently proposed "quantum Boolean functions" [6] developed in the context of the research topic "Fourier analysis of Boolean functions" having many applications in theoretical computer science, information theory and also in social decision and voting theory. We then propose an algebraic generalization, based on the finite-elements method, that can be applied to whatever m-value n-argument logical system.

The paper is organized as follows: we start with Boolean two-valued $\{0, 1\}$ logic and we demonstrate important expressions for the projector observables in the 2-argument case indicating also the general method for n-arguments. The case for binary values $\{+1, -1\}$ is then presented. Then we consider the case for fuzzy logical propositions and give the method for calculating fuzzy membership functions by using the Born rule and show that these functions can be identified with probabilities. The last section is devoted to the many-valued systems ($m > 2$) the case of 3-valued 2-argument logic is discussed with some examples of applications.

2 Two-Valued Eigenlogic

2.1 Projector Two-Valued Logic

We will consider a two-dimensional rank-1 projector $\boldsymbol{\Pi}$ acting on a single set. What are the expected outcomes when applying this projector? If, for example, vector $|a>$ corresponds to an element of the set, the following matrix equation will be verified: $\boldsymbol{\Pi} \cdot |a> = 1 \cdot |a>$. The value 1 being the eigenvalue of the projector associated with the eigenvector $|a>$. Interpretable results [5] considered in a two-value $\{0, 1\}$ logical system will correspond to the possible eigenvalues

0 and 1, where 0 is the result for elements not belonging to the set. So in this way a question concerning the proposition of belonging or not to a particular set, will have as an answer one of the two eigenvalues. The "true" value 1 will correspond to the eigenvector $|a>$, now named $|1>$, and the "false" value 0 will correspond to the complementary eigenvector $|\overline{a}>$, named $|0>$. When these properties are expressed in matrix form: vectors $|1>$ and $|0>$ become 2 dimensional orthonormal column vectors and the projection operators 2×2 square matrices. This gives:

$$|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The choice of the position of the value 1 in the column vectors is arbitrary, here it follows the quantum information convention for a "qubit-1" [7]. As usual in Quantum Mechanics we can find the set of projectors that completely represent the quantum system, in particular by lifting the eventual degeneracy of the eigenvalues. Here eigenvalues are always equal to 0 or 1 and the question about the multiplicity of eigenvalues is natural. In this contribution we focus on different projective structures that completely define the logical system. In the very simple case where 0 and 1 are both not degenerate eigenvalues, the projectors relative to the eigenvector basis take the form:

$$\boldsymbol{\Pi}_1 = \boldsymbol{\Pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{\Pi}_0 = \boldsymbol{I} - \boldsymbol{\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

We systematically consider all the possible structures of such projectors. When representing logic with n atomic propositions using projectors various possibilities are intrinsically present in a unique structure with 2^{2^n} different projectors. Once the eigenbasis is chosen the remaining structure is intrinsic.

For example the two projectors shown in Eq. (1) are complementary and idempotent. One can give a general expression of a one-argument "logical observable" as an expansion over the commuting projectors $\boldsymbol{\Pi}_0$ and $\boldsymbol{\Pi}_1$ spanning the vector space:

$$\boldsymbol{F} = f(0)\,\boldsymbol{\Pi}_0 + f(1)\,\boldsymbol{\Pi}_1 = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} \tag{2}$$

the coefficients $f(0)$ and $f(1)$ in the expansion are the truth values of the corresponding $\{0, 1\}$ Boolean logical connective. Eq. (2) represents the spectral decomposition of the operator and because the eigenvalues are real the logical operator is Hermitian and can thus be considered as a quantum observable. In this way, in Eigenlogic, the truth values of the logical proposition are the eigenvalues of the logical observable. One can then construct the 4 logical observables corresponding to the 4 one-argument Boolean connectives: $\boldsymbol{A} = \boldsymbol{\Pi}_1$ is the "logical projector" and $\overline{\boldsymbol{A}} = \boldsymbol{I} - \boldsymbol{\Pi}_1 = \boldsymbol{\Pi}_0$ its complement. The "True" operator corresponds here to the identity operator $\boldsymbol{I}$. The "False" observable corresponds to the null operator. These four observables form a complete family of commuting projectors. The extension to more arguments is obtained by using the

Kronecker product $\otimes$ in the same way as for the composition of quantum systems (for technical details on this operation see for example [7]).

In the case of $n = 2$ arguments we will have an expansion over 4 commuting orthogonal rank-1 projectors. Some properties of the Kronecker product on projectors have to be specified: (i) The Kronecker product of two projectors is also a projector; (ii) If projectors are rank-1 projectors (a single eigenvalue is equal to 1, all the others are 0) then their Kronecker product is also a rank-1 projector. Using these properties, the 4 commuting orthogonal rank -1 projectors $\boldsymbol{\Pi}_{00}$, $\boldsymbol{\Pi}_{01}$, $\boldsymbol{\Pi}_{10}$, and $\boldsymbol{\Pi}_{11}$, spanning the 4 dimensional vector space are calculated in a straightforward way:

$$\begin{cases} \boldsymbol{\Pi}_{00} = (\boldsymbol{I} - \boldsymbol{\Pi}) \otimes (\boldsymbol{I} - \boldsymbol{\Pi}) \,, & \boldsymbol{\Pi}_{01} = (\boldsymbol{I} - \boldsymbol{\Pi}) \otimes \boldsymbol{\Pi} \,, \\ \boldsymbol{\Pi}_{10} = \boldsymbol{\Pi} \otimes (\boldsymbol{I} - \boldsymbol{\Pi}) \,, & \boldsymbol{\Pi}_{11} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} \,. \end{cases}$$

So one can write the logical observable for $n = 2$ arguments:

$$\boldsymbol{F} = f(0,0)\,\boldsymbol{\Pi}_{00} + f(0,1)\,\boldsymbol{\Pi}_{01} + f(1,0)\,\boldsymbol{\Pi}_{10} + f(1,1)\,\boldsymbol{\Pi}_{11} \,. \tag{3}$$

In an explicit way:

$$\boldsymbol{F} = \begin{pmatrix} f(0,\,0) & 0 & 0 & 0 \\ 0 & f(0,\,1) & 0 & 0 \\ 0 & 0 & f(1,\,0) & 0 \\ 0 & 0 & 0 & f(1,\,1) \end{pmatrix} .$$

Equation (3) represents a spectral decomposition with the eigenvalues being the truth values, in this case we will have a family of 16 possible different observables. All these observables are pairwise commuting projectors and in general their product (matrix product) is not equal to zero. This last point is essential in the model, because not only mutually exclusive projectors are representative for a logical system, the complete family of projectors must be used. For example the observables for conjunction, AND, and disjunction, OR, which have in common the truth value, $(1, 1)$, for the input combination (True $\equiv$ 1, True $\equiv$ 1), have their matrix product different from zero.

This method can be extended to whatever number of arguments n using the "seed" projector $\boldsymbol{\Pi}$, its complement $(\boldsymbol{I} - \boldsymbol{\Pi})$ and by applying the Kronecker product. So given the number of input arguments n and knowing the truth table of the logical connective one directly obtains the corresponding binary Eigenlogic observable.

Now let's develop the case for $n = 2$ arguments: one can express the connectives corresponding to a "logical projector" according to the composition rule, thus obtaining two commuting projector observables:

$$\boldsymbol{A} = \boldsymbol{\Pi} \otimes \boldsymbol{I} \,, \qquad \boldsymbol{B} = \boldsymbol{I} \otimes \boldsymbol{\Pi} \,, \qquad \boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} \tag{4}$$

the conjunction, AND, observable becomes simply the product of these two logical projectors $\boldsymbol{A} \cdot \boldsymbol{B}$. The disjunction, OR, and exclusive disjunction, XOR, observables are shown on Table 1, where the algebraic expansions for Boolean

Table 1. The sixteen two-argument two-valued logical connectives and the respective Eigenlogic observables for eigenvalues $\{0,1\}$ and $\{+1,-1\}$.

Connective for Boolean A, B	Truth table $\{\mathrm{F},\mathrm{T}\}$: $\{0,1\}$; $\{+1,-1\}$	$\{0,1\}$ projective logical observable	$\{+1,-1\}$ isometric logical observable
False F	F F F F	$\boldsymbol{0}$	$+\boldsymbol{I}$
NOR ; $\overline{A \vee B}$	F F F T	$\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(+\boldsymbol{I} - \boldsymbol{U} - \boldsymbol{V} - \boldsymbol{U}\cdot\boldsymbol{V})$
$A \nLeftarrow B$	F F T F	$\boldsymbol{B} - \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(+\boldsymbol{I} - \boldsymbol{U} + \boldsymbol{V} + \boldsymbol{U}\cdot\boldsymbol{V})$
$\overline{A}$	F F T T	$\boldsymbol{I} - \boldsymbol{A}$	$-\boldsymbol{U}$
$A \nRightarrow B$	F T F F	$\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(+\boldsymbol{I} + \boldsymbol{U} - \boldsymbol{V} + \boldsymbol{U}\cdot\boldsymbol{V})$
$\overline{B}$	F T F T	$\boldsymbol{I} - \boldsymbol{B}$	$-\boldsymbol{V}$
XOR; $A \oplus B$	F T T F	$\boldsymbol{A} + \boldsymbol{B} - 2\boldsymbol{A}\cdot\boldsymbol{B}$	$\boldsymbol{U}\cdot\boldsymbol{V} = \boldsymbol{Z} \otimes \boldsymbol{Z}$
NAND; $\overline{A \wedge B}$	F T T T	$\boldsymbol{I} - \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(-\boldsymbol{I} - \boldsymbol{U} - \boldsymbol{V} + \boldsymbol{U}\cdot\boldsymbol{V})$
AND; $A \wedge B$	T F F F	$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi}$	$\frac{1}{2}(+\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}\cdot\boldsymbol{V})$
$A \equiv B$	T F F T	$\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B} + 2\boldsymbol{A}\cdot\boldsymbol{B}$	$-\boldsymbol{U}\cdot\boldsymbol{V}$
B	T F T F	$\boldsymbol{B} = \boldsymbol{I} \otimes \boldsymbol{\Pi}$	$\boldsymbol{V} = \boldsymbol{I} \otimes \boldsymbol{Z}$
$A \Rightarrow B$	T F T T	$\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(-\boldsymbol{I} - \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}\cdot\boldsymbol{V})$
A	T T F F	$\boldsymbol{A} = \boldsymbol{\Pi} \otimes \boldsymbol{I}$	$\boldsymbol{U} = \boldsymbol{Z} \otimes \boldsymbol{I}$
$A \Leftarrow B$	T T F T	$\boldsymbol{I} - \boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(-\boldsymbol{I} + \boldsymbol{U} - \boldsymbol{V} - \boldsymbol{U}\cdot\boldsymbol{V})$
OR; $A \vee B$	T T T F	$\boldsymbol{A} + \boldsymbol{B} - \boldsymbol{A}\cdot\boldsymbol{B}$	$\frac{1}{2}(-\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} + \boldsymbol{U}\cdot\boldsymbol{V})$
True T	T T T T	$\boldsymbol{I}$	$-\boldsymbol{I}$

connectives explicitly derived in [5] are used. Negation (complementation) is obtained by subtracting from the identity operator for projective logical observables and by multiplying by -1 for isometric logical observables (see hereafter). Useful transformations are obtained by De Morgan's theorem (for general theorems in logic see for example Knuth [8]), for the negative conjunction, NAND one has the identity $\overline{A \wedge B} = \overline{A} \vee \overline{B}$ in the same way one can obtain NOR with the identity $\overline{A \vee B} = \overline{A} \wedge \overline{B}$. Implication observables are also shown on Table 1.

2.2 Isometric Reversible Two-Valued Logical Observables

There is a linear bijection (isomorphism) from the projector logical observables $\boldsymbol{F}$ towards reversible observables $\boldsymbol{G}$:

$$\boldsymbol{G} = \boldsymbol{I} - 2\boldsymbol{F}\,.$$

The two families of observables commute and have the same system of eigenvectors. Practically to obtain $\boldsymbol{G}$ from $\boldsymbol{F}$ one just has to substitute the eigenvalue 0 with $+1$ and 1 with -1. The observables $\boldsymbol{G}$ are "isometries": unitary reflection operators. From projector $\boldsymbol{\Pi}$ in Eq. (4) one obtains the observable $\boldsymbol{Z}$:

$$\boldsymbol{Z} = \boldsymbol{I} - 2\boldsymbol{\Pi} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

which is actually one of the Pauli matrices σ_z and corresponds in quantum mechanics, to the z component of a spin $1/2$ observable $\mathbf{S}_z = (\hbar/2)\, \sigma_z$ where $\hbar$ is the reduced Planck's constant. In the field of quantum information this operator is also named the "Pauli-Z" gate or "phase-π" gate [7]. Here, $\boldsymbol{U} = \boldsymbol{Z}$ designates the "logical projector" connective and $\overline{\boldsymbol{U}} = -\boldsymbol{Z}$ its complement (negation), *nota bene* in this case the connective "logical projector" is not a projection operator, in order to avoid ambiguity it is often named [6] "dictator".

For $n = 2$ arguments one can then write directly the expression for a logical isometric observable by using its spectral decomposition. The logical "dictators" $\boldsymbol{U}$ and $\boldsymbol{V}$ become:

$$\boldsymbol{U} \;=\; \boldsymbol{Z} \otimes \boldsymbol{I}\,, \qquad \boldsymbol{V} \;=\; \boldsymbol{I} \otimes \boldsymbol{Z}\,, \qquad \boldsymbol{U} \cdot \boldsymbol{V} \;=\; \boldsymbol{Z} \otimes \boldsymbol{Z}\,.$$

The exclusive disjunction XOR observable is here simply given by the product of the dictators: $\boldsymbol{U} \cdot \boldsymbol{V}$. Negation is obtained by multiplying by the number -1. From Table 1 one sees that there are more complicated relations, for example the conjunction, AND, observable is:

$$\frac{1}{2}(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U} \cdot \boldsymbol{V}) = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;=\; \boldsymbol{C}^{Z}\,.$$

Those familiar with the domain of quantum information can easily recognize the reversible logical gate "control-Z" or simply named C^Z [7].

3 From Deterministic Logic to Fuzzy Logic

Fuzzy logic deals with truth values that may be any number between 0 and 1, here the truth of a proposition may range between completely true and completely false. It is generally considered that probability theory and fuzzy logic are related to different forms of uncertainty, the first is concerned with how probable it is that a variable belongs to a given set and the second one uses the concept of fuzzy set membership, intended as the degree of membership. This was the first motivation of fuzzy logic [9]. But this distinction when considering the quantum probabilistic Born rule is not so strict from a formal point of view. We will start the discussion by giving the interpretation of a vector state in Eigenlogic.

In the preceding sections we considered operations on the eigenspace of a logical observable family. For example for $n = 2$ arguments a complete family of 16 commuting logical observables represents all possible logical connectives and becomes "interpretable" [5] when applied to one of the four possible canonical eigenvectors of the family. These vectors, corresponding to all the possible atomic input propositional cases, are represented by the vectors $|00>$, $|01>$, $|10>$ and $|11>$ forming a complete orthonormal basis. When applying a logical observable on one of these vectors the resulting eigenvalue will correspond to the truth value for the considered input.

Now what happens when the state-vector is not one of the eigenvectors of the logical system? In quantum mechanics, where vectors operate in Hilbert space, one can always express a state-vector as a decomposition on a complete orthonormal basis. In particular we can express it over the canonical eigenbasis of the logical observable family. For two-arguments this vector can be written as:

$$|\Psi> = \; C_{00}\,|00> \,+ C_{01}\,|01> \,+ C_{10}\,|10> \,+ C_{11}\,|11> \;.$$

We can interpret this in the following way: when only one of the coefficients is non-zero (in this case its absolute value must take the value 1) then we are back in the preceding situation of a determinate input atomic propositional case. But when more than one coefficient is non-zero we are in a "mixed" or "fuzzy" propositional case. Such a state could also possibly be interpreted as a quantum superposition of atomic propositional cases.

We can then calculate the "mean value" of a logical observable. In particular the logical projector observables $\boldsymbol{F}$ will give a "fuzzy measure" of the logical proposition in the form of the "fuzzy membership function" μ. Let's show this on some examples: in the case of one argument one can express an arbitrary 2-dimensional quantum state as: $|\varphi>= \sin\alpha\,|0> + e^{i\beta}\cos\alpha\,|1>$ where the "angles" α and β are real numbers. The quantum mean value of the "logical projector" observable $\boldsymbol{A} = \boldsymbol{\Pi}$ can then be calculated using the Born rule:

$$\mu(a) \;= <\varphi|\boldsymbol{\Pi}|\varphi> = \;\cos\alpha\; \mathrm{e}^{-\mathrm{i}\beta} <1|1><1|\cos\alpha \mathrm{e}^{\mathrm{i}\beta}|1> = \;\cos^2\alpha\,;$$

in the same way one can calculate the complement

$$\mu(\overline{a}) \;= <\varphi|\boldsymbol{I} - \boldsymbol{\Pi}|\varphi> = \;\sin^2\alpha \;=\; 1 - \mu(a)\,.$$

This verifies one of the requirements of fuzzy logic for the complement (negation) of a fuzzy set.

According to standard notations for spin $1/2$ quantum states, or qubits, on the Bloch sphere [7] we use the transformation $\alpha = (\pi-\theta)/2$ and $\beta = \varphi$. A quantum compound state can be built by taking the tensor product of two elementary states: $|\psi>= |\varphi_p> \otimes|\varphi_q>$, where $|\varphi_p>= \cos\frac{\theta_p}{2}|0> + e^{i\varphi_p}\sin\frac{\theta_p}{2}|1>$ (for $|\varphi_q>$ we have a similar expression). Now $\sin^2\frac{\theta_p}{2} = p$ and $\sin^2\frac{\theta_q}{2} = q$ represent the probabilities of being in the "True" state $|1>$ for spins $1/2$ oriented along two different axes θ_p and θ_q.

One can calculate the fuzzy membership function of the corresponding "logical projector" for the two-argument case using Eq. (4).

$$\mu(a) =<\psi|\boldsymbol{\Pi}\otimes\boldsymbol{I}|\psi>= p(1-q) + p\cdot q = p\,, \quad \mu(b) =<\psi|\boldsymbol{I}\otimes\boldsymbol{\Pi}|\psi>= q\,.$$

This shows that the mean values correspond to the respective probabilities. Now let's "measure" for example the conjunction and the disjunction, using the observables in Table 1, this gives:

$$\begin{cases} \mu(a{\wedge}b) \;= <\psi|\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}|\psi> = \; p\cdot q \;=\; \mu(a)\cdot\mu(b)\,, \\ \mu(a{\vee}b) \;=\; p + q - p\cdot q \;=\; \mu(a)+\mu(b)-\mu(a)\cdot\mu(b)\,. \end{cases}$$

Similar results for conjunction and disjunction have been outlined recently, also using projector operators, when considering concept combinations [10] for quantum-like experiments in the domain of quantum cognition.

What happens when the state-vector cannot be put in the form of a tensor product, that is when it corresponds to an entangled state? The problem is outside the scope of this paper but an interesting result can be shown: the mean value of whatever logical observable of the type $\boldsymbol{F}$ on an arbitrary quantum state $|\Psi>$ will always verify the inequality:

$$< \Psi|\boldsymbol{F}|\Psi> = \operatorname{Tr}(\rho_\Psi \cdot \boldsymbol{F}) \leq 1 , \qquad \text{with} \quad \rho_\Psi \equiv |\Psi><\Psi| ,$$

and can thus be interpreted as a probability measure.

4 From Two-Valued to Multi-Valued Logic

Multi-valued logic requires a different algebraic structure than an ordinary binary-valued one. Many properties of binary logic do not support set of values that do not have cardinality 2^n. Multi-valued logic is often used for the development of logical systems that are more expressive than Boolean systems for reasoning [11]. Particularly three and four valued systems, have been of interest with applications to digital circuits and computer science.

The total number of possible logical connectives for an m-valued n-argument system is the combinatorial number m^{m^n}, so in particular for a binary 2-valued 2-argument system, as shown above, the number of connectives will be $2^{2^2} = 16$, the complete list indicated on Table 1. For a binary three-argument system, the number increases to $2^{2^3} = 256$. For a 3-valued 1-argument system the number of connectives will be $3^{3^1} = 27$ and for a 3-valued 2-argument system: $3^{3^2} = 19683$. So it is clear that by increasing the values from two to three the possibilities of new connectives becomes intractable for a complete description of a logical system, but some special connectives play important roles and will be illustrated hereafter. We will proceed by showing the general algebraic method.

4.1 Interpolation with Finite Elements

The finite element method (see for example [12]) allows one to interpolate a function, *id est* to make explicit the values $f(x)$ from the given values of specific numbers, the (so-called) degrees of freedom.

Let's consider the following simple example: given the values $f(+1)$, $f(0)$ and $f(-1)$ of a function f at the particular points $x = +1, 0, -1$, and using the appropriate Dirac linear forms, we can write: $< \delta_{+1}, f >= f(+1)$, $< \delta_0, f >= f(0)$ and $< \delta_{-1}, f >= f(-1)$, where $\Sigma \equiv \{\delta_{+1}, \delta_0, \delta_{-1}\}$ is called the set of degrees of freedom. This linear structure shows that it is natural to consider a three-dimensional space. The "basis function" φ_i associated to the set of degrees of freedom Σ and to the polynomial space solves this problem. The three basis functions using degrees of freedom and second-degree polynomials are:

$$\varphi_{+1}(x) = \frac{1}{2}\, x\,(x+1) , \quad \varphi_0(x) = 1 - x^2 , \quad \varphi_{-1}(x) = \frac{1}{2}\, x\,(x-1) . \qquad (5)$$

So in general, an arbitrary function f can be written:

$$f(x) = \sum_{i=+1,0,-1} f(i)\ \varphi_i(x)\,, \qquad \sum_{i=+1,0,-1} \varphi_i(x) \equiv 1 \tag{6}$$

where the completeness of the basis functions is verified by their sum being 1.

4.2 Formalization of Three-Valued Eigenlogic

We use an operator system which is equivalent to the one of orbital angular momentum $\ell = 1$. In general angular momentum is characterized by two quantum numbers: j the angular momentum number and m_j the magnetic momentum number. Both these numbers must be integer or half integer. The rules are: $j \geq 0$, and attached to this value we have the condition: $-j \leq m_j \leq j$. The value $j = 0$ is possible and gives a single value $m_j = 0$ the next is $j = s = 1/2$ giving two values $m_s = \pm 1/2$ corresponding to the two-valued spin system. The value $j = 1$ gives three possible values $m_j = \{+1, 0, -1\}$ and so on. We consider for $j = \ell = 1$ the z-component orbital angular momentum observable [13]

$$\boldsymbol{L}_z = \hbar \boldsymbol{\Lambda} = \hbar \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

In the above matrix the three eigenvalues $\{+1, 0, -1\}$ will be considered as the logical values. A convention for these values, extending binary logic, is the following: $\quad \text{False}: \text{F} \equiv +1\,,\ \text{Neutral}: \text{N} \equiv 0\,,\ \text{True}: \text{T} \equiv -1\,.$
We can now express the three-value logical observables as spectral decompositions over the rank-1 projectors spanning the vector space: $\boldsymbol{\Pi}_{+1}$, $\boldsymbol{\Pi}_0$ and $\boldsymbol{\Pi}_{-1}$. These operators correspond to the pure state density matrices of the three eigenstates $|+1>$, $|0>$ and $|-1>$ of $\boldsymbol{L}_z$. The three projectors can be expressed as a function of the dimensionless observable $\boldsymbol{\Lambda}$, using directly the expressions given above in (5) where the basis functions φ_i become the projectors and the symbol x the observable $\boldsymbol{\Lambda}$ given in (7):

$$\boldsymbol{\Pi}_{+1} = \frac{1}{2}\boldsymbol{\Lambda}\,(\boldsymbol{\Lambda} + \boldsymbol{I}) \qquad \boldsymbol{\Pi}_0 = \boldsymbol{I} - \boldsymbol{\Lambda}^2 \qquad \boldsymbol{\Pi}_{-1} = \frac{1}{2}\boldsymbol{\Lambda}\,(\boldsymbol{\Lambda} - \boldsymbol{I}) \tag{8}$$

Then every one-argument "local projector" $\boldsymbol{F}(\boldsymbol{\Lambda})$ can be obtained using the relation (6).

4.3 Three-Valued, Two-Argument Examples: Min, Max

When considering a 2-argument 3-valued system we find the expansion by using the Kronecker product in the same way as for the binary system in Eq. (3):

$$\boldsymbol{F} \ = \sum_{i,\,j\ =+1,\,0,\,-1} f_{ij}\ \boldsymbol{\Pi}_i \otimes \boldsymbol{\Pi}_j\,, \qquad f_{ij} \in \{+1,\,0,\,-1\}\,. \tag{9}$$

these observables are now 9×9 matrices. We can define the two argument "dictators", $\boldsymbol{U}$ and $\boldsymbol{V}$, simply by the rule of composition, this leads to:

$$\boldsymbol{U} = \boldsymbol{\Lambda} \otimes \boldsymbol{I} \qquad \boldsymbol{V} = \boldsymbol{I} \otimes \boldsymbol{\Lambda} \qquad \boldsymbol{U} \cdot \boldsymbol{V} = \boldsymbol{\Lambda} \otimes \boldsymbol{\Lambda} . \tag{10}$$

In trivalent logic (see *e.g.* [11]) popular connectives are Min and Max, defined in the maps on Table 2.

Table 2. The Min and Max maps for a three-valued two-argument logic.

Min $U \setminus V$	F	N	T
F $\equiv +1$	$+1$	$+1$	$+1$
N $\equiv 0$	$+1$	0	0
T $\equiv -1$	$+1$	0	-1

Max $U \setminus V$	F	N	T
F $\equiv +1$	$+1$	0	-1
N $\equiv 0$	0	0	-1
T $\equiv -1$	-1	-1	-1

Here the connectives Min and Max are symmetric, they are equivalent for a complete inversion of signs on inputs and outputs. Using the relations (8), (9) and (10) in conjunction with reduction rules we obtain the following observables:

$$\begin{cases} \mathrm{Min}(\boldsymbol{U},\boldsymbol{V}) \;=\; \dfrac{1}{2}\left(\boldsymbol{U}+\boldsymbol{V}+\boldsymbol{U}^2+\boldsymbol{V}^2-\boldsymbol{U}\cdot\boldsymbol{V}-\boldsymbol{U}^2\cdot\boldsymbol{V}^2\right) \\ \mathrm{Max}(\boldsymbol{U},\boldsymbol{V}) \;=\; \dfrac{1}{2}\left(\boldsymbol{U}+\boldsymbol{V}-\boldsymbol{U}^2-\boldsymbol{V}^2+\boldsymbol{U}\cdot\boldsymbol{V}+\boldsymbol{U}^2\cdot\boldsymbol{V}^2\right) \end{cases} \tag{11}$$

The proof of the relations (11) is a direct consequence of relations (5) and (9). We have on one hand:

$$\begin{aligned} \mathrm{Min}\,(U,V) \;=\; & \varphi_1(U) \otimes \varphi_1(V) + \varphi_1(U) \otimes \varphi_0(V) + \varphi_1(U) \otimes \varphi_{-1}(V) \\ & + \varphi_0(U) \otimes \varphi_1(V) + \varphi_{-1}(U) \otimes \varphi_1(V) - \varphi_{-1}(U) \otimes \varphi_{-1}(V) \\ = \varphi_1(U) + \varphi_1(V) & - \varphi_1(U) \otimes \varphi_1(V) - \varphi_{-1}(U) \otimes \varphi_{-1}(V) \qquad \text{due to (6)} \\ = \tfrac{1}{2}\, U\,(U+I) & + \tfrac{1}{2}\, V\,(V+I) - \tfrac{1}{4}\, U\,(U+I)\, V\,(V+I) - \tfrac{1}{4}\, U\,(U-I)\, V\,(V-I) \\ = \tfrac{1}{2} & \left(U^2 + U + V^2 + V - U^2 V^2 - UV\right) \end{aligned}$$

and the first relation of (11) is proven. On the other hand, we have

$$\begin{aligned} \mathrm{Max}\,(U,V) \;=\; & \varphi_1(U) \otimes \varphi_1(V) - \varphi_1(U) \otimes \varphi_{-1}(V) - \varphi_0(U) \otimes \varphi_{-1}(V) \\ & - \varphi_{-1}(U) \otimes \varphi_{-1}(V) - \varphi_{-1}(U) \otimes \varphi_1(V) - \varphi_{-1}(U) \otimes \varphi_0(V) \\ = \varphi_1(U) \otimes \varphi_1(V) & - \varphi_{-1}(U) - \varphi_{-1}(V) + \varphi_{-1}(U) \otimes \varphi_{-1}(V) \qquad \text{due to (6)} \\ = \tfrac{1}{4}\, U\,(U+I)\, V\,(V+I) & - \tfrac{1}{2}\, U\,(U-I) - \tfrac{1}{2}\, V\,(V-I) + \tfrac{1}{4}\, U\,(U-I)\, V\,(V-I) \\ = \tfrac{1}{2} & \left(U^2 V^2 + UV - U^2 - V^2 + U + V\right) \end{aligned}$$

and the second relation of (11) is proven. □

The proof presented above exploits the properties of the Kronecker product and reduction rules due to the completeness of the finite projection space. Reduction of logical expressions is an important topic in logic. In binary logic it is formalized by using Karnaugh maps which represent canonical SOP (Sum Of Products) disjunctive normal forms [8].

Binary logic is “included” in ternary logic, we want to verify this by eliminating the “neutral” state, N $\equiv 0$, and considering only the two logical values $\{+1, -1\}$. In this case we have: $\boldsymbol{U}^2 = \boldsymbol{V}^2 = \boldsymbol{I}$ and so (11) reduces to:

$$\begin{cases} \text{Min}\,(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}\big(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}\cdot\boldsymbol{V}\big)\,, \\ \text{Max}\,(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}\big(-\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} + \boldsymbol{U}\cdot\boldsymbol{V}\big) \end{cases}$$

considering that for binary logic the Min connective becomes the conjunction, AND, and the Max connective the disjunction, OR, we find the previous results given on Table 1 for binary $\{+1, -1\}$ observables.

5 Discussion and Conclusion

We have presented an operational formalism named “Eigenlogic” using observables in Hilbert space. The original feature being that the eigenvalues of a logical observable represent the truth values of the corresponding logical connective, the associated eigenvectors corresponding to one of the fixed combination of the inputs (atomic propositions). This approach differs from other geometric formalizations of logic (for references and discussion see [5]). Here the outcome of a “measurement” or “observation” on a logical observable will give the truth value of the associated logical proposition, and becomes “interpretable” when applied to the eigenspace leading to a natural analogy with the measurement postulate in quantum mechanics. One of the referees proposed the following diagram to summarize the point of view presented in this contribution:

$$\begin{aligned} \text{eigenvectors in Hilbert space} &\longrightarrow \text{atomic propositional cases} \\ \text{projectors} &\longrightarrow \text{logical connectives} \\ \text{eigenvalues} &\longrightarrow \text{truth values.} \end{aligned}$$

At first sight this method could be viewed as “classical” because exactly the same results are obtained in Eigenlogic as in ordinary propositional logic. This is in itself an important result demonstrating a new method in logic based on linear algebra, the method being also developed in multivalued logic. But when considering vector states, *id est* input propositions, that are not eigenvectors, the measurement outcomes are governed by the quantum Born rule, and interpretable results are then given by the mean values. This fact led us to apply the method to Fuzzy logic.

Another important point is the general algebraic method, based on classical interpolation framework suggested by the finite-element method. Our method can be employed for whatever m-valued n-argument logical system and in each case the corresponding logical observables can be defined. Some observables can be formally compared with angular momentum observables in quantum mechanics. Because of the exponential increase of complexity, an analytical formulation is only tractable for a low number of logical values and arguments. We treated the two-argument binary case completely and the three-valued case using the logical observables Min and Max. An algorithmic approach for logical connectives with a large number of arguments could be interesting to develop using Eigenlogic observables in high-dimensional vector spaces. But because the space grows in dimension very quickly, it may not be particularly useful for practical implementation without logical reduction. It would be interesting to develop specific algebraic reduction methods for logical observables inspired from actual research in the field. For a good synthesis of the state of the art, see *e.g.* [14].

Eigenlogic could create a new perspective in the field of quantum computation because several of the observables turn out to be well-known quantum gates. Here we represent them as diagonal matrices, *id est* in their eigenbasis, other "normal" forms being easily recovered by unitary transformations. It would be interesting to operate quantum gates in our framework. Many-valued logic is being investigated in quantum computation for example with ternary-logic quantum gates using "qutrits". Our formulation of multivalued logical observables could be used for the design of new quantum gates.

Dynamical evolution of the logical system could be included in the model by identifying the appropriate Hamiltonian operators. Standard procedures for expressing interaction Hamiltonians as a function of angular momentum observables could be used [13].

More generally we think that this view of logic could add some insight on more fundamental issues. Boolean functions are nowadays considered as a "toolbox" for resolving many problems in theoretical computer science, information theory and even fundamental mathematics. In the same way Eigenlogic can be considered as a new "toolbox" and could be of interest for the "Quantum Interaction" community where quantum-like approaches in human and social sciences need to be founded on a logical basis.

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