Chapter 8
Valuations on Lattice Polytopes

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Abstract This survey is on classification results for valuations defined on lattice polytopes that intertwine the special linear group over the integers. The basic real valued valuations, the coefficients of the Ehrhart polynomial, are introduced and their characterization by Betke and Kneser is discussed. More recent results include classification theorems for vector and convex body valued valuations.

8.1 From the Pick Theorem to the Ehrhart Polynomial

A (full-dimensional) lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup spanned by n independent vectors. Given a basis of Λ, the automorphisms of Λ are transformations of the form $x \mapsto Ax + b$ with $b \in \Lambda$ and $A \in \mathrm{GL}_n(\mathbb{Z})$, that is, A is an $n \times n$ integer matrix with determinant ± 1. Such transformations are called *unimodular*. A *lattice polytope* is the convex hull of a finite subset of Λ and we write $\mathscr{P}(\Lambda)$ for the family of lattice polytopes. Since every lattice is a linear image of $\mathbb{Z}^n$, in general we just consider the lattice $\mathbb{Z}^n$.

This section concentrates on the lattice point enumerator $L(P)$ for a bounded set $P \subset \mathbb{R}^n$, where

$$L(P) := \sum_{x \in P \cap \mathbb{Z}^n} 1. \tag{8.1}$$

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E.B.V. Jensen, M. Kiderlen (eds.), *Tensor Valuations and Their Applications in Stochastic Geometry and Imaging*, Lecture Notes in Mathematics 2177,
DOI 10.1007/978-3-319-51951-7_8

Hence, $L(P)$ is the number of lattice points in P and $P \mapsto L(P)$ is a valuation on $\mathcal{P}(\mathbb{Z}^n)$. For basic properties of lattices related to this chapter from various aspects, see Barvinok [3], Beck and Robins [4], Gruber [20] or Gruber and Lekkerkerker [21].

The starting point is a formula [51] due to Georg Alexander Pick (1859–1942). For $P \in \mathcal{P}(\mathbb{Z}^2)$, write $B(P)$ for the number of lattice points on the boundary of P if P is two-dimensional, and $B(P) := 2|P \cap \mathbb{Z}^2| - 2$ if P is a segment or a point, where $|\cdot|$ denotes the cardinality of a finite set. Note that $P \mapsto B(P)$ is a valuation.

Theorem 8.1 (Pick) *For $P \in \mathcal{P}(\mathbb{Z}^2)$ non-empty,*

$$L(P) = V_2(P) + \frac{1}{2}B(P) + 1.$$

Here $V_2(P)$ is the two-dimensional volume of the polytope P. The core fact behind Pick's theorem is that if $P \in \mathcal{P}(\mathbb{Z}^2)$ is a triangle with $L(P) = 3$, then $V_2(P) = 1/2$. Thus the essential two-dimensional case can be proved for example by induction on $L(P)$, dissecting P into triangles sharing a common vertex if $L(P) \geq 4$. The Pick theorem has various proofs (see e.g. [9, 22]).

In higher dimensions, there is no simple formula as in Pick's theorem, as was noted by Reeve [54, 55]. The reason is that the volume of an n-dimensional simplex $S \in \mathcal{P}(\mathbb{Z}^n)$ with $L(S) = n + 1$ can be any non-negative integer multiple of $1/n!$ However, Eugène Ehrhart (1906–2000), a French high school teacher, found the following fundamental formula in [17] which works in all dimensions. We write $\mathbb{N}_0$ for the set of non-negative integers and call a valuation *unimodular* if it is invariant with respect to unimodular transformations.

Theorem 8.2 (Ehrhart) *There exist rational numbers $L_i(P)$ for $i = 0, \ldots, n$ such that*

$$L(kP) = \sum_{i=0}^{n} L_i(P)k^i$$

for every $k \in \mathbb{N}_0$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. For each i, the functional $L_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}$ is a unimodular valuation which is homogeneous of degree i.

Note that $L_n(P)$ is the n-dimensional volume $V_n(P)$ and that $L_0(P)$ is the Euler characteristic of P, that is, $L_0(P) := 1$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ non-empty and $L_0(\emptyset) := 0$. Also note that $L_i(P) = 0$ for $i > \dim P$, where $\dim P$ is the dimension (of the affine hull) of P.

Let $\det_{n-1} \Lambda$ denote the determinant of an $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$. In addition, for an n-dimensional polytope $P \in \mathcal{P}(\mathbb{Z}^n)$, let $\mathcal{F}_{n-1}(P)$ be the family of $(n-1)$-dimensional faces and write aff for affine hull. For $n \geq 2$, we have

$$L_{n-1}(P) = \begin{cases} \frac{1}{2} \sum_{F \in \mathscr{F}_{n-1}(P)} \frac{V_{n-1}(F)}{\det_{n-1}(\mathbb{Z}^n \cap \operatorname{aff} F)} & \text{if } \dim(P) = n, \\ \frac{V_{n-1}(P)}{\det_{n-1}(\mathbb{Z}^n \cap \operatorname{aff} P)} & \text{if } \dim(P) = n-1, \\ 0 & \text{if } \dim(P) \leq n-2. \end{cases}$$

Thus $L_{n-1}(P)$ is a *lattice surface area* of P. Note, in particular, that $L_1(P) = \frac{1}{2}B(P)$ in accordance with Pick's Theorem for $n = 2$.

The coefficient $L_i(P)$ may not be an integer for $i = 1, \dots, n$, but $n!L_i(P) \in \mathbb{Z}$ for $P \in \mathscr{P}(\mathbb{Z}^n)$. There seems to be no known "geometric interpretation" for $L_i(P)$ if $n \geq 3$ and $1 \leq i \leq n-2$, and actually $L_i(P)$ might be negative in this case (see [30] for a strong result in this direction). If $P \in \mathscr{P}(\mathbb{Z}^n)$ is n-dimensional and $i = 1, \dots, n-1$, then good bounds of the form

$$a(n,i)V_n(P) + b(n,i) \leq L_i(P) \leq c(n,i)V_n(P) + d(n,i)$$

involving the so-called Stirling numbers are known. Here the optimal upper bound on $L_i(P)$ for $i = 1, \dots, n-1$ is due to Betke and McMullen [8]. A lower bound is due to Henk and Tagami [29] and Tsuchiya [64], and it is known to be optimal if $i = 1, 2, 3, n-3, n-2$, and if $n-i$ is even.

There is a representation of the Ehrhart polynomial via projective toric varieties associated to a lattice polytope (see, e.g., [13, 15, 18]). Using this representation, or combinatorial analogues of the algebraic geometric approach, formulas for $L_i(P)$ were established by Pommersheim [52] in terms of Dedekind sums if $P \in \mathscr{P}(\mathbb{Z}^3)$ is a tetrahedron, by Kantor and Khovanskii [32] if $n = 3, 4$, by Brion and Vergne [12] if P is simple, by Diaz and Robins [16] using Fourier analysis for any P and by Chen [14] if P is a simplex.

We note that inspired by the algebraic geometric representation of the Ehrhart polynomial, Barvinok [2] provided a polynomial time algorithm to calculate $L_i(P)$ for $P \in \mathscr{P}(\mathbb{Z}^n)$ and $i = 1, \dots, n$, if the dimension n is fixed.

Ehrhart's Theorem 8.2 was extended to non-negative integer linear combinations of lattice polytopes by Bernstein [5] and McMullen [46].

Theorem 8.3 *Let $P_1, \dots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ be given. If $k_1, \dots, k_m \in \mathbb{N}_0$, then $L(k_1P_1 + \dots + k_mP_m)$ is a polynomial in $k_1, \dots, k_m$ of total degree at most n. Moreover, the coefficient of $k_1^{r_1} \cdots k_m^{r_m}$ in this polynomial is a translation invariant valuation in P_i which is homogeneous of degree r_i.*

To prove this result, McMullen [46] uses induction on the number of summands, while Bernstein [5] considers intersections of algebraic hypersurfaces in $(\mathbb{C}\backslash\{0\})^n$ determined by Laurent polynomials with given Newton polytope. Here the Newton polytope associated to a Laurent polynomial is the convex hull of the lattice points corresponding to the exponents of its non-zero coefficients. Note that Theorems 8.2 and 8.3 imply that L_1 is additive.

Corollary 8.4 *If* $P, Q \in \mathscr{P}(\mathbb{Z}^n)$, *then* $L_1(P+Q) = L_1(P) + L_1(Q)$.

For the lattice point enumerator, the following important reciprocity relation was established by Ehrhart [17] and Macdonald [45]. For $P \in \mathscr{P}(\mathbb{Z}^n)$, write relint P for the relative interior of P (with respect to the affine hull of P).

Theorem 8.5 *If* $P \in \mathscr{P}(\mathbb{Z}^n)$, *then* $L(\operatorname{relint} P) = (-1)^{\dim P} \sum_{i=0}^n L_i(P)(-1)^i$.

This is also called the Ehrhart-Macdonald reciprocity law. The right side of the formula in Theorem 8.5 is, up to multiplication with the factor $(-1)^{\dim P}$, the Ehrhart polynomial $k \mapsto L(kP)$ evaluated at $k = -1$. For a multivariate version, that is, a version using the polynomial from Theorem 8.3, see [31].

One may choose other bases for the vector space of polynomials of degree at most n instead of the monomials and obtains other representations for the Ehrhart polynomial. In particular, for $k \in \mathbb{N}_0$,

$$L(kP) = \sum_{i=0}^n H_i^*(P) \binom{k+n-i}{n}.$$

For $i = 0, \dots, n$, the functional H_i^* is a unimodular valuation on $\mathscr{P}(\mathbb{Z}^n)$ (which is not homogeneous). More commonly used are the functionals h_i^*, defined by

$$L(kP) = \sum_{i=0}^m h_i^*(P) \binom{k+m-i}{m} \tag{8.2}$$

for $k \in \mathbb{N}_0$, where $m = \dim P$. The vector $(h_0^*(P), \dots, h_n^*(P))$, where we set $h_i^*(P) := 0$ for $i > \dim P$, is called the Ehrhart h^*-vector of P. Stanley [61] showed that the Ehrhart h^*-vector of P coincides with the combinatorial h-vector of a unimodular triangulation of P, if such a triangulation exists. Betke [6] and Stanley [61] showed that for $i = 0, \dots, n$, the functional h_i^* is integer-valued and non-negative on $\mathscr{P}(\mathbb{Z}^n)$. Stanley [62] showed that each h_i^* is monotone with respect to set inclusion. Clearly, we have $H_i^*(P) = h_i^*(P)$ for n-dimensional polytopes P. However, the functionals h_i^* are not valuations on $\mathscr{P}(\mathbb{Z}^n)$ while the valuations H_i^* are not monotone or non-negative.

Another representation of the Ehrhart polynomial, introduced by Breuer [11], is

$$L(kP) = \sum_{i=0}^n f_i^*(P) \binom{k-1}{i} \tag{8.3}$$

for $k \in \mathbb{N}_0$. For $i = 0, \dots, n$, the functional f_i^* is a unimodular valuation on $\mathscr{P}(\mathbb{Z}^n)$ (which again is not homogeneous). Note that $f_i^*(P) = 0$ for $i > \dim P$. The vector $(f_0^*(P), \dots, f_n^*(P))$ is called the Ehrhart f^*-vector of P and coincides with the combinatorial f-vector of a unimodular triangulation of P, if such a triangulation exists. Breuer [11] showed that for $i = 0, \dots, n$, the valuation f_i^* is integer-valued and non-negative on $\mathscr{P}(\mathbb{Z}^n)$ and that these properties extend to polyhedral complexes.

8.2 The Inclusion-Exclusion Principle

The inclusion-exclusion principle is a fundamental property of valuations on lattice polytopes, which was first established in the case of translation invariant and real valued valuations by Stein [63] and for general real valued valuations by Betke (Das Einschließungs-Ausschließungsprinzip für Gitterpolytope. Unpublished manuscript). The first published proof is by McMullen [48], who also established the more general extension property. Since the family of lattice polytopes is not intersectional, that is, the intersection of two lattice polytopes is in general not a lattice polytope, results for valuations on polytopes (see Theorem 1.3) could not easily be generalized.

For $m \geq 1$, we write $P_J := \cap_{i \in J} P_i$ for $\emptyset \neq J \subset \{1, \dots, m\}$ and given polytopes $P_1, \dots, P_m \in \mathscr{P}(\mathbb{Z}^n)$. Let $\mathbb{G}$ be an abelian group. The *inclusion-exclusion formula* for lattice polytopes is the following result.

Theorem 8.6 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation, then for lattice polytopes* $P_1, \dots, P_m$,

$$\mathrm{Z}(P_1 \cup \dots \cup P_m) = \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{|J|-1}\, \mathrm{Z}(P_J).$$

whenever $P_1 \cup \dots \cup P_m \in \mathscr{P}(\mathbb{Z}^n)$ *and* $P_J \in \mathscr{P}(\mathbb{Z}^n)$ *for all* $\emptyset \neq J \subset \{1, \dots, m\}$.

It is often helpful to extend valuations defined on lattice polytopes to finite unions of lattice polytopes whose intersections are again lattice polytopes. McMullen [48] showed that this is always possible. This is the *extension property*.

Theorem 8.7 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation, then there exists a function* $\bar{\mathrm{Z}}$ *defined on finite unions of lattice polytopes such that for lattice polytopes* $P_1, \dots, P_m$,

$$\bar{\mathrm{Z}}(P_1 \cup \dots \cup P_m) = \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{|J|-1}\, \mathrm{Z}(P_J),$$

whenever $P_J \in \mathscr{P}(\mathbb{Z}^n)$ *for all* $\emptyset \neq J \subset \{1, \dots, m\}$.

For a given valuation Z, we denote its extension by $\bar{\mathrm{Z}}$ and will use this notation throughout the chapter.

The inclusion-exclusion formula and the extension property are frequently needed for cell decompositions. We call a dissection of the polytope Q into polytopes $P_1, \dots, P_m$ a cell decomposition if $P_i \cap P_j$ is either empty or a common face of P_i and P_j for every $1 \le i < j \le m$. The faces of the cell decomposition are the faces of all P_i for $i = 1, \dots, m$.

Theorem 8.8 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation and* $Q \in \mathscr{P}(\mathbb{Z}^n)$*, then*

$$\bar{\mathrm{Z}}(Q) = (-1)^{\dim Q} \sum_{\substack{F \in \mathscr{F} \\ F \cap \operatorname{int} Q \neq \emptyset}} (-1)^{\dim F}\, \mathrm{Z}(F),$$

where $\mathscr{F}$ *is the set of all faces of a cell decomposition of* Q.

In particular, Theorem 8.7 implies the following. Write $\mathscr{F}(P)$ for the family of all non-empty faces of $P \in \mathscr{P}(\mathbb{Z}^n)$ (including the face P) and set $\bar{\mathrm{Z}}(\operatorname{relint} P) = \mathrm{Z}(P) - \bar{\mathrm{Z}}(\operatorname{relbd} P)$, where relbd stands for relative boundary. Expressing relbd P as the union of its faces, we obtain

$$\bar{\mathrm{Z}}(\operatorname{relint} P) = (-1)^{\dim P} \sum_{F \in \mathscr{F}(P)} (-1)^{\dim F}\, \mathrm{Z}(F) \tag{8.4}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

For a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$, Sallee [56] introduced the associated function $\mathrm{Z}^\circ : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ defined by

$$\mathrm{Z}^\circ(P) := \sum_{F \in \mathscr{F}(P)} (-1)^{\dim F}\, \mathrm{Z}(F) \tag{8.5}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$, which by (8.4) is closely related to $\bar{\mathrm{Z}}(\operatorname{relint} P)$. He showed that Z° is a valuation on $\mathscr{P}(\mathbb{Z}^n)$ (while $P \mapsto \bar{\mathrm{Z}}(\operatorname{relint} P)$ is not a valuation) and that $(\mathrm{Z}^\circ)^\circ = \mathrm{Z}$. McMullen [46] gave simple proofs for these facts. We will use the notation (8.5) and the valuation property of Z° throughout the chapter. Using this, we can write the Ehrhart-Macdonald reciprocity law (Theorem 8.5) also as

$$L^\circ(P) = \sum_{i=0}^{n} L_i(P)(-1)^i \tag{8.6}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

We note that many of the results related to the inclusion-exclusion principle have a variant if $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{A}$ is a valuation with $\mathbb{A}$ a cancellative abelian semigroup. For example, the analogue of Theorem 8.6 is that if $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{A}$ is a valuation, and $P_1, \dots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ satisfy that $P_1 \cup \dots \cup P_m \in \mathscr{P}(\mathbb{Z}^n)$ and $P_J \in \mathscr{P}(\mathbb{Z}^n)$ for all $\emptyset \neq J \subset \{1, \dots, m\}$, then

$$\mathrm{Z}(P_1 \cup \dots \cup P_m) + \sum_{\substack{\emptyset \neq J \subset \{1,\dots,m\} \\ |J| \text{ even}}} \mathrm{Z}(P_J) = \sum_{\substack{\emptyset \neq J \subset \{1,\dots,m\} \\ |J| \text{ odd}}} \mathrm{Z}(P_J).$$

A typical case when $\mathbb{A}$ is only a semigroup is the case of Minkowski valuations, which will be discussed in Sect. 8.5.

8.3 Translation Invariant Valuations

Let $\mathbb{V}$ be a vector space over $\mathbb{Q}$. A valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ is *translation invariant* if $\mathrm{Z}(P + x) = \mathrm{Z}(P)$ for every $P \in \mathcal{P}(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$. Translation invariant valuations on $\mathcal{P}(\mathbb{Z}^n)$ behave similarly to the lattice point enumerator in many ways, as was proved by McMullen [46]. The paper [46] assumes that the valuation Z on $\mathcal{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle, which always holds by Theorem 8.6.

Theorem 8.9 *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ *be a translation invariant valuation. There exist* $\mathrm{Z}_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ *for* $i = 0, \dots, n$ *such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$. *Moreover,* $\mathrm{Z}_i(P) = 0$ *for* $i > \dim P$.

The corresponding result for valuations on polytopes is described in Theorem 1.13.

Combining results in McMullen [46] and [48] leads to an analogue of the Ehrhart-Macdonald reciprocity law (8.6).

Theorem 8.10 *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ *is a translation invariant valuation, then*

$$\mathrm{Z}^\circ(-P) = \sum_{i=0}^{n} \mathrm{Z}_i(P)(-1)^i$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$.

The Ehrhart-Macdonald reciprocity law (8.6) is easily deduced from Theorem 8.10 because in addition to translation invariance, the lattice point enumerator also satisfies $L(\operatorname{relint}(-P)) = L(\operatorname{relint} P)$.

Taking Theorem 8.9 as starting point, Jochemko and Sanyal [31] consider analogues of the coefficients $h_i^*(P)$ in (8.2) for translation invariant valuations. For a translation invariant valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ and $P \in \mathcal{P}(\mathbb{Z}^n)$, they define $h_0^{\mathrm{Z}}(P), \dots, h_n^{\mathrm{Z}}(P)$ by

$$\mathrm{Z}(kP) = \sum_{i=0}^{m} h_i^{\mathrm{Z}}(P) \binom{k+m-i}{m},$$

where $m = \dim P$. A translation invariant valuation Z is called *h^*-nonnegative*, if $h_i^{\mathrm{Z}} \geq 0$ on $\mathscr{P}(\mathbb{Z}^n)$ for $i = 0, \ldots, n$. It is called *h^*-monotone* if h_i^{Z} is monotone (with respect to set inclusion) on $\mathscr{P}(\mathbb{Z}^n)$ for $i = 0, \ldots, n$. Using the extended valuation $\bar{\mathrm{Z}}$, Jochemko and Sanyal [31] establish a version of Stanley's theorem on the non-negativity and monotonicity of h_i^* for any translation invariant valuation.

Theorem 8.11 *For a translation invariant valuation* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$, *the following three statements are equivalent.*

1. *Z is h^*-nonnegative.*
2. *Z is h^*-monotone.*
3. $\bar{\mathrm{Z}}(\operatorname{relint} P) \geq 0$ *for every* $P \in \mathscr{P}(\mathbb{Z}^n)$.

Since for the lattice point enumerator we have $L(\operatorname{relint} P) \geq 0$ for every $P \in \mathscr{P}(\mathbb{Z}^n)$, the non-negativity and monotonicity of h_i^* on $\mathscr{P}(\mathbb{Z}^n)$ is a simple consequence of Theorem 8.11. Jochemko and Sanyal [31] also obtain the following result.

Theorem 8.12 *A functional* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a unimodular and h^*-nonnegative valuation if and only if there exist constants* $c_0, \ldots, c_n \geq 0$ *such that*

$$\mathrm{Z}(P) = c_0 f_0^*(P) + \cdots + c_n f_n^*(P)$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

In the proof, essential use is made of the Betke-Kneser theorem, which is described in the following section.

8.4 The Betke-Kneser Theorem

The classical classification result for valuations on lattice polytopes concerns real valued and unimodular valuations and is due to Betke [6]. It was first published in Betke and Kneser [7]. It shows that the coefficients of the Ehrhart polynomial form a basis of the vector space of unimodular valuations.

Theorem 8.13 (Betke) *A functional* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a unimodular valuation if and only if there exist constants* $c_0, \ldots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_0\, L_0(P) + \cdots + c_n\, L_n(P)$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

We remark that by Corollary 8.16 below, it is sufficient to assume that Z is an $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant valuation to obtain the same result, where $\mathrm{SL}_n(\mathbb{Z})$ denotes the group of $n \times n$ integer matrices with determinant 1.

The Euclidean counterpart of Theorem 8.13 is the celebrated classification of rigid motion invariant and continuous valuations on convex bodies by Hadwiger [27] (see Theorem 1.23). A classification of $\mathrm{SL}_n(\mathbb{R})$ invariant, Borel measurable valuations on convex polytopes containing the origin in their interiors was recently established by Haberl and Parapatits [24] extending results from [25, 33]. For a complete classification of $\mathrm{SL}_n(\mathbb{R})$ invariant valuations on convex polytopes, see [38].

We say that a j-dimensional $S \in \mathscr{P}(\mathbb{Z}^n)$ is a unimodular simplex if $j = 0$ or $S = [x_0, \dots, x_j]$ for $j \geq 1$ and $\{x_1 - x_0, \dots, x_j - x_0\}$ is part of a basis of $\mathbb{Z}^n$. Here $[\dots]$ stands for convex hull. We define a particular set of unimodular simplices by setting $T_0 := \{0\}$ and $T_j := [0, e_1, \dots, e_j]$ for $j = 1, \dots, n$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{Z}^n$. Betke and Kneser [7] also established the following result for an abelian group $\mathbb{G}$.

Theorem 8.14 (Betke-Kneser) *Every unimodular valuation* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is uniquely determined by its values on* $T_0, \dots, T_n$ *and these values can be chosen arbitrarily in* $\mathbb{G}$.

Again, by Corollary 8.16 below, it is sufficient to assume that Z is an $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant valuation.

The following statement is the core of the argument in Betke and Kneser [7]. It is proved using dissection into simplices and suitable complementation by simplices.

Proposition 8.15 *For* $P \in \mathscr{P}(\mathbb{Z}^n)$*, there exist unimodular simplices* $S_1, \dots, S_m$ *and integers* $l_1, \dots, l_m$ *such that for any abelian group* $\mathbb{G}$*,*

$$\mathrm{Z}(kP) = \sum_{j=1}^{m} l_j \, \mathrm{Z}(kS_j)$$

for every valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *and* $k \in \mathbb{N}_0$.

This proposition implies Ehrhart's theorem. Just note that for $k \geq 1$,

$$L(kT_i) = \binom{k+i}{i} \qquad \text{for } i = 0, \dots, n,$$

that each unimodular simplex S_j is an image under a unimodular transformation of some T_i, and that for each i, the above binomial coefficient is a polynomial in k of degree i.

The following statement is another direct consequence of Proposition 8.15.

Corollary 8.16 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *and* $\mathrm{Z}' : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *are* $\mathrm{SL}_n(\mathbb{Z})$ *and translation invariant valuations such that*

$$\mathrm{Z}(T_i) = \mathrm{Z}'(T_i) \qquad \textit{for } i = 0, \dots, n,$$

then $\mathrm{Z}(P) = \mathrm{Z}'(P)$ *for every* $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.5 Minkowski Valuations

Let $\mathscr{F}$ be a family of subsets of $\mathbb{R}^n$ and write $\mathscr{K}^n$ for the set of convex bodies, that is, compact convex sets, in $\mathbb{R}^n$. The subset of convex polytopes is denoted by $\mathscr{P}^n$. An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is a *Minkowski valuation* if Z satisfies

$$\mathrm{Z}K + \mathrm{Z}L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathscr{F}$ with $K \cup L, K \cap L \in \mathscr{F}$ and addition on $\mathscr{K}^n$ is Minkowski addition; that is,

$$K + L := \{x + y : x \in K,\ y \in L\}.$$

Let $\mathrm{SL}_n(\mathbb{R})$ be the special linear group on $\mathbb{R}^n$, that is, the group of real matrices of determinant 1. An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is called $\mathrm{SL}_n(\mathbb{R})$ equivariant if

$$\mathrm{Z}(\phi P) = \phi\, \mathrm{Z} P \qquad \text{for } \phi \in \mathrm{SL}_n(\mathbb{R}) \text{ and } P \in \mathscr{F}.$$

Define $\mathrm{SL}_n(\mathbb{Z})$ equivariance of operators on $\mathscr{P}(\mathbb{Z}^n)$ analogously. For recent results on $\mathrm{SL}_n(\mathbb{R})$ equivariant operators on convex bodies and their associated inequalities, see, for example, [26, 40–43].

For $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant Minkowski valuations defined on convex polytopes, the following complete classification was established in [35]. It provides a characterization of the difference body operator

$$P \mapsto P - P := \{x - y : x, y \in P\},$$

which assigns to P its *difference body*. For more information on difference bodies and their associated inequalities, see [19, 59]. Let $n \geq 2$.

Theorem 8.17 *An operator* $\mathrm{Z} : \mathscr{P}^n \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *equivariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z} P = c(P - P)$$

for every $P \in \mathscr{P}^n$.

Further results on the classification of $\mathrm{SL}_n(\mathbb{R})$ equivariant Minkowski valuations can be found, for example, in [23, 36, 50, 65].

The following result, taken from [10], is an analogue for lattice polytopes of Theorem 8.17.

Theorem 8.18 *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist* $a, b \geq 0$ *such that*

$$\mathrm{Z}P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here for a lattice polytope P, the point $\ell_1(P)$ is its discrete Steiner point that was introduced in [10]. See Sect. 8.6 for the definition and characterization theorems. The proof of Theorem 8.18 uses constructions from Betke and Kneser [7] as well as results on Minkowski summands and it also exploits the large symmetry group of the standard simplex T_n.

For operators mapping $\mathscr{P}(\mathbb{Z}^n)$ to $\mathscr{P}(\mathbb{Z}^n)$, the following result was established in [10]. Write LCM for least common multiple.

Theorem 8.19 *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{P}(\mathbb{Z}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist integers* $a, b \geq 0$ *with* $b - a \in \mathrm{LCM}(2, \dots, n+1)\mathbb{Z}$ *such that*

$$\mathrm{Z}P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here it is used that the discrete Steiner point of a lattice polytope is a vector with rational coordinates.

An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is $\mathrm{SL}_n(\mathbb{R})$ *contravariant* if

$$\mathrm{Z}(\phi P) = \phi^{-t}\,\mathrm{Z}P \qquad \text{for } \phi \in \mathrm{SL}_n(\mathbb{R}) \text{ and } P \in \mathscr{F},$$

where ϕ^{-t} is the inverse of the transpose of ϕ. We define $\mathrm{SL}_n(\mathbb{Z})$ contravariance of operators on $\mathscr{P}(\mathbb{Z}^n)$ analogously. For recent results on $\mathrm{SL}_n(\mathbb{R})$ contravariant operators on convex bodies, see, for example, [26, 41, 44].

An important $\mathrm{SL}_n(\mathbb{R})$ contravariant operator on $\mathscr{K}^n$ is the operator $K \mapsto \Pi K$, that associates with a convex body its projection body. To define this operator, we describe a convex body L by its support function $h(L, \cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ where $h(L, u) := \max\{u \cdot x : x \in L\}$.

For a convex body K, the *projection body* ΠK is given by

$$h(\Pi K, u) = V_{n-1}(K|u^{\perp}),$$

for $u \in \mathbb{S}^{n-1}$, where $K|u^{\perp}$ is the orthogonal projection of K onto the hyperplane orthogonal to u. We refer to [19, 59] for more information on projection bodies and their associated inequalities. For a polytope P with facets (that is, $(n-1)$-dimensional faces) $F_1, \dots, F_m$, the projection body ΠP is given as the following

Minkowski sum,

$$\Pi P = \tfrac{1}{2}\big([-v_1, v_1] + \cdots + [-v_m, v_m]\big),$$

where v_i is the scaled normal corresponding to the facet F_i, that is, v_i is a normal vector to the facet F_i with length equal to $V_{n-1}(F_i)$. Here $[-v_i, v_i]$ is the segment with endpoints $-v_i$ and v_i.

For $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on $\mathscr{P}^n$, the following complete classification was established in [35]. Let $n \geq 2$.

Theorem 8.20 *An operator* $\mathrm{Z} : \mathscr{P}^n \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z} P = c\,\Pi P$$

for every $P \in \mathscr{P}^n$.

Further classification theorems for $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on convex bodies can be found in [23, 34, 36, 37, 49, 60].

The following analogue of Theorem 8.20 for lattice polytopes is from [10].

Theorem 8.21

(i) *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^2) \to \mathscr{K}^2$ *is an* $\mathrm{SL}_2(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exist constants* $a, b \geq 0$ *such that*

$$\mathrm{Z} P = a\,\varrho_{\pi/2}(P - \ell_1(P)) + b\,\varrho_{\pi/2}(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^2)$.

(ii) *For* $n \geq 3$, *an operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z} P = c\,\Pi P$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here $\varrho_{\pi/2}$ denotes the rotation by an angle $\pi/2$ in $\mathbb{R}^2$. Note that for $n = 2$, the projection body is obtained from the difference body by applying this rotation.

The projection body of a lattice polytope is a rational polytope. For operators mapping $\mathscr{P}(\mathbb{Z}^n)$ to $\mathscr{P}(\mathbb{Z}^n)$, the following result was established in [10].

Theorem 8.22

(i) *An operator* $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^2) \to \mathscr{P}(\mathbb{Z}^2)$ *is an* $\mathrm{SL}_2(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exist integers* $a, b \geq 0$ *with* $b - a \in 6\,\mathbb{Z}$ *such that*

$$\mathrm{Z}\,P = a\,\varrho_{\pi/2}(P - \ell_1(P)) + b\,\varrho_{\pi/2}(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^2)$.

(ii) *For* $n \geq 3$, *an operator* $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathscr{P}(\mathbb{Z}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \in (n-1)!\,\mathbb{N}_0$ *such that*

$$\mathrm{Z}\,P = c\,\Pi P$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.6 Vector Valuations

In analogy to (8.1), for $P \in \mathscr{P}(\mathbb{Z}^n)$, the *discrete moment vector* was introduced in [10] as

$$\ell(P) := \sum_{x \in P \cap \mathbb{Z}^n} x. \tag{8.7}$$

The discrete moment vector $\ell : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Z}^n$ is a valuation that is equivariant with respect to unimodular linear transformations. In addition, if $y \in \mathbb{Z}^n$, then

$$\ell(P + y) = \ell(P) + L(P)\,y. \tag{8.8}$$

In general, a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is called *translation covariant* if for all $P \in \mathscr{P}(\mathbb{Z}^n)$ and $y \in \mathbb{Z}^n$,

$$\mathrm{Z}(P + y) = \mathrm{Z}(P) + \mathrm{Z}^0(P)y$$

with some $\mathrm{Z}^0 : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$. Note that it easily follows from this definition that the associated functional Z^0 is also a valuation.

McMullen [46] established the following analogue of Theorem 8.9.

Theorem 8.23 *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *be a translation covariant valuation. There exist* $\mathrm{Z}_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *for* $i = 0, \dots, n+1$ *such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n+1} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$*. For each* i*, the function* Z_i *is a translation covariant valuation which is homogeneous of degree* i*.*

Note that if the valuation Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then so are $\mathrm{Z}_0, \dots, \mathrm{Z}_{n+1}$. Using this homogeneous decomposition, McMullen [46] established the following more general result.

Theorem 8.24 *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *be a translation covariant valuation and let* $P_1, \dots, P_m \in \mathcal{P}(\mathbb{Z}^n)$ *be given. If* $k_1, \dots, k_m \in \mathbb{N}_0$*, then* $\mathrm{Z}(k_1 P_1 + \dots + k_m P_m)$ *is a polynomial of total degree at most* $(n+1)$ *in* $k_1, \dots, k_m$*. Moreover, the coefficient of* $k_1^{r_1} \cdots k_m^{r_m}$ *in this polynomial is a translation covariant valuation in* P_i *which is homogeneous of degree* r_i*.*

The discrete moment vector is a translation covariant valuation. Hence, we obtain as a special case of Theorem 8.23 the following result.

Corollary 8.25 *There exist* $\ell_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *for* $i = 1, \dots, n+1$ *such that*

$$\ell(kP) = \sum_{i=1}^{n+1} \ell_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$*. For each* i*, the function* ℓ_i *is a translation covariant valuation which is equivariant with respect to unimodular linear transformations and homogeneous of degree* i*.*

Note that $\ell_{n+1}(P)$ is the moment vector of P, that is, $\ell_{n+1}(P) = \int_P x \, \mathrm{d}x$. We call the vector $\ell_1(P)$ the *discrete Steiner point* of P. From Theorem 8.24, we deduce as in Corollary 8.4 the following result.

Corollary 8.26 *The function* $\ell_1 : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *is additive.*

It is shown in [10] that the discrete Steiner point of a unimodular simplex is its centroid. Hence, by using suitable dissections and complementations, it is possible to obtain $\ell_1(P)$ for a given lattice polytope P.

The following results, Theorems 8.27 and 8.29, both from [10], are the reason for calling ℓ_1 the discrete Steiner point map. A function $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is called *translation equivariant* if $\mathrm{Z}(P + x) = \mathrm{Z}(P) + x$ for $x \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$.

Theorem 8.27 *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant and additive if and only if* Z *is the discrete Steiner point map.*

Theorem 8.27 corresponds to the following characterization of the classical Steiner point by Schneider [57]. The classical Steiner point, $s(K)$, is defined by

$$s(K) := \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} u\, h(K, u)\, \mathrm{d}u,$$

where κ_n is the n-dimensional volume of the n-dimensional unit ball and $\mathrm{d}u$ denotes integration with respect to $(n-1)$-dimensional Hausdorff measure on the unit sphere.

Theorem 8.28 *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is continuous, rigid motion equivariant and additive if and only if* Z *is the Steiner point map.*

Note that Wannerer [66] recently obtained a corresponding characterization of vector valuations in the Hermitian setting (see Corollary 6.15).

The discrete Steiner point is also characterized in the following result.

Theorem 8.29 *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant valuation if and only if* Z *is the discrete Steiner point map.*

This theorem corresponds to the following characterization of the classical Steiner point by Schneider [58].

Theorem 8.30 *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a continuous and rigid motion equivariant valuation if and only if* Z *is the Steiner point map.*

By (8.8), the discrete moment vector is translation covariant. Note that

$$\ell_i(P + x) = \ell_i(P) + L_{i-1}(P)\, x$$

for $i = 1, \ldots, n+1$, where the case $i = 1$ is just the translation equivariance of ℓ_1. Hence ℓ_i is translation covariant for each i. The following result is from [39].

Theorem 8.31 *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there exist constants* $c_1, \ldots, c_{n+1} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_1 \ell_1(P) + \cdots + c_{n+1}\, \ell_{n+1}(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

The Euclidean counterpart of this result is the classification of rotation equivariant and translation covariant, continuous valuations $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ by Hadwiger and Schneider [28] (see Theorem 2.4). A classification of $\mathrm{SL}_n(\mathbb{R})$ equivariant, Borel measurable vector valuations on convex polytopes containing the origin in their interiors was recently established by Haberl and Parapatits [25].

8.7 Polynomial Valuations

To discuss polynomial valuations, let us review what we mean by polynomial in our context. Let $\mathbb{G}$ be an abelian group and Λ a lattice in $\mathbb{R}^n$. We say that $p : \Lambda \to \mathbb{G}$ is polynomial of degree 0, if p is constant on Λ. We say that p is polynomial of degree $d \geq 1$ if for any $y \in \Lambda$, the map $x \mapsto p(x + y) - p(x)$ is polynomial of degree at most $d - 1$. If $w_1, \ldots, w_n$ form a basis of Λ, then this implies that there are $b_i \in \mathbb{G}$ and integer polynomials $p_i : \mathbb{Z}^n \to \mathbb{Z}$ of degree at most d for $i = 1, \ldots, r$ such that for $k_i \in \mathbb{N}_0$

$$p(k_1 w_1 + \cdots + k_n w_n) = \sum_{i=1}^{r} p_i(k_1, \ldots, k_n)\, b_i.$$

Now a valuation $\mathrm{Z} : \mathscr{P}(\Lambda) \to \mathbb{G}$ is polynomial of degree d if for every $P \in \mathscr{P}(\Lambda)$, the function, defined on Λ by $x \mapsto \mathrm{Z}(P + x)$ is a polynomial of degree d.

Clearly, a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ is translation invariant if and only if it is polynomial of degree 0. If $q : \mathbb{Z}^n \to \mathbb{G}$ is a polynomial of degree at most d, then $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ defined by

$$\mathrm{Z}(P) := \sum_{x \in P \cap \mathbb{Z}^n} q(x) \tag{8.9}$$

is a polynomial valuation of degree at most d.

McMullen [46] considered polynomial valuations of degree at most one and Pukhlikov and Khovanskii [53] proved Theorem 8.32 in the general case. Another proof, following the approach of [46], is due to Alesker [1]. These papers assume that the valuation Z on $\mathscr{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle, which holds by Theorem 8.6.

Theorem 8.32 *Let* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *be a polynomial valuation of degree at most d and let* $P_1, \ldots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ *be given. If* $k_1, \ldots, k_m \in \mathbb{N}_0$*, then* $\mathrm{Z}(k_1 P_1 + \cdots + k_m P_m)$ *is a polynomial of total degree at most* $(d+n)$ *in* $k_1, \ldots, k_m$*. Moreover, the coefficient of* $k_1^{r_1} \cdots k_m^{r_m}$ *in this polynomial is a polynomial valuation in* P_i *of degree at most d which is homogeneous of degree* r_i*.*

This result implies that a homogeneous decomposition for polynomial valuations exists. Let $\mathbb{V}$ be a vector space over $\mathbb{Q}$.

Corollary 8.33 *Let* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *be a polynomial valuation of degree at most d. There exist valuations* $\mathrm{Z}_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *for* $i = 0, \ldots, n + d$ *which are polynomial of degree at most* $d + n$ *and homogeneous of degree i such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{d+n} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$*.*

If a polynomial valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ respects the action of linear unimodular transformations, then so do $\mathrm{Z}_0, \dots, \mathrm{Z}_{n+d}$. Important cases include $\mathrm{SL}_n(\mathbb{Z})$ invariant valuations and $\mathrm{SL}_n(\mathbb{Z})$ equivariant as well as $\mathrm{SL}_n(\mathbb{Z})$ contravariant valuations.

A version of the Ehrhart-Macdonald reciprocity law for polynomial valuations of type (8.9) was established by Brion and Vergne [12]. The following more general result is from [39] and was proved along the lines of reciprocities laws from [46].

Theorem 8.34 *If $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ is a polynomial valuation which is homogeneous of degree j, then*

$$\mathrm{Z}^\circ(-P) = (-1)^j \,\mathrm{Z}(P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.8 Tensor Valuations

In analogy to (8.1) and (8.7), for $P \in \mathscr{P}(\mathbb{Z}^n)$, we define for $r \in \mathbb{N}_0$, the *discrete moment tensor of rank r* by

$$L^r(P) := \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r,$$

where x^r denotes the r-fold symmetric tensor product of x. Let $\mathbb{T}^r$ denote the vector space of symmetric tensors of rank r on $\mathbb{R}^n$. Note that $\mathbb{T}^0 = \mathbb{R}$ and $L^0 = L$ and that $\mathbb{T}^1 = \mathbb{R}^n$ and $L^1 = \ell$.

We view each element of $\mathbb{T}^r$ as a symmetric r linear functional on $(\mathbb{R}^n)^r$. So, in particular,

$$L^r(P)(v_1, \dots, v_r) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} (x \cdot v_1) \cdots (x \cdot v_r)$$

for $v_1, \dots, v_r \in \mathbb{R}^n$, where $x \cdot v$ is the inner product of x and v.

The discrete moment tensor $L^r : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ has the following behavior with respect to unimodular linear transformations. For $v_1, \dots, v_r \in \mathbb{R}^n$,

$$L^r(\phi P)(v_1, \dots, v_r) = L^r(P)(\phi^t v_1, \dots, \phi^t v_r)$$

for all $\phi \in \mathrm{GL}_n(\mathbb{Z})$ and $P \in \mathscr{P}(\mathbb{Z}^n)$. In general, a tensor valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is called $\mathrm{SL}_n(\mathbb{Z})$ equivariant if for $v_1, \dots, v_r \in \mathbb{R}^n$,

$$\mathrm{Z}(\phi P)(v_1, \dots, v_r) = \mathrm{Z}(P)(\phi^t v_1, \dots, \phi^t v_r)$$

for all $\phi \in \mathrm{GL}_n(\mathbb{Z})$ and $P \in \mathscr{P}(\mathbb{Z}^n)$.

In addition, if $y \in \mathbb{Z}^n$, then

$$L^r(P+y) = \sum_{m=0}^{r} L^{r-m}(P)\frac{y^m}{m!},$$

where we use the convention that $y^0 = 1 \in \mathbb{R}$. Following McMullen [47], a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is called *translation covariant* if there exist associated functions $\mathrm{Z}^m : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^m$ for $m = 0, \dots, r$ such that

$$\mathrm{Z}(P+y) = \sum_{m=0}^{r} \mathrm{Z}^m(P)\frac{y^{r-m}}{(r-m)!}$$

for all $y \in \mathbb{Z}^n$ and $P \in \mathscr{P}(\mathbb{Z}^n)$. It follows from this definition that Z^m is a valuation for $m = 0, \dots, r$ and that $\mathrm{Z}^r = \mathrm{Z}$. Note that the associated valuation Z^m is translation covariant for $m = 0, \dots, r$, since we have

$$\mathrm{Z}^m(P+y) = \sum_{j=0}^{m} \mathrm{Z}^j(P)\frac{y^{m-j}}{(m-j)!}.$$

For given $v_1, \dots, v_r \in \mathbb{R}^n$, associate with the translation covariant tensor valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$, the real valued valuation $P \mapsto \mathrm{Z}(P)(v_1, \dots, v_r)$, which is easily seen to be polynomial of degree at most r. Hence we obtain the following result from Theorem 8.32.

Theorem 8.35 *Let $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a translation covariant valuation and let $P_1, \dots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ be given. If $k_1, \dots, k_m \in \mathbb{N}_0$, then $\mathrm{Z}(k_1P_1 + \cdots + k_mP_m)$ is a polynomial of total degree at most $(n+r)$ in $k_1, \dots, k_m$. Moreover, the coefficient of $k_1^{r_1} \cdots k_m^{r_m}$ in this polynomial is a translation covariant valuation in P_i which is homogeneous of degree r_i.*

As a special case, we obtain the following homogeneous decomposition.

Theorem 8.36 *Let $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a translation covariant valuation. There exist $\mathrm{Z}_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ for $i = 0, \dots, n+r$ such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n+r} \mathrm{Z}_i(P)k^i$$

for every $k \in \mathbb{N}_0$ and $P \in \mathscr{P}(\mathbb{Z}^n)$. For each i, the function Z_i is a translation covariant valuation which is homogeneous of degree i.

Note that if Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then so are the homogeneous components $\mathrm{Z}_0, \dots, \mathrm{Z}_{n+r}$.

We apply these results to the discrete moment tensor and obtain the following result.

Corollary 8.37 *There exist* $L_i^r : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *for* $i = 1, \ldots, n + r$ *such that*

$$L^r(kP) = \sum_{i=1}^{n+r} L_i^r(P)k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$*. For each i, the function* L_i^r *is a translation covariant valuation which is equivariant with respect to unimodular linear transformations and homogeneous of degree i.*

Note that $L_{n+r}^r(P)$ is the rth moment tensor of the lattice polytope P, that is, $L_{n+r}^r(P) = \frac{1}{r!}\int_P x^r \mathrm{d}x$ [cf. (2.4)]. See [39], for results on the classification of tensor valuations.

Using the approach from [46], we can extend the reciprocity laws to tensor valuations and obtain the following result, which is proved in [39].

Theorem 8.38 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation which is homogeneous of degree j, then*

$$\mathrm{Z}^\circ(P) = (-1)^j \mathrm{Z}(-P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

Since Z° is again a translation covariant valuation, Theorem 8.36 implies that there are homogeneous decompositions for Z and Z°. Hence the following result is a simple consequence of Theorem 8.38.

Corollary 8.39 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation, then*

$$\mathrm{Z}^\circ(P) = \sum_{i=0}^{n+r} (-1)^i \mathrm{Z}_i(-P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

So, in particular, using that $L^r(-P) = (-1)^r L^r(P)$, we obtain

Corollary 8.40 *For* $P \in \mathscr{P}(\mathbb{Z}^n)$,

$$L^r(\operatorname{relint} P) = (-1)^{m+r} \sum_{i=1}^{m+r} (-1)^i L_i^r(P),$$

where $m = \dim P$.

Acknowledgements The authors thank Raman Sanyal for pointing out Corollary 8.4 and its proof to them. They also thank Martin Henk for helpful remarks. The work of Károly J. Böröczky was supported, in part, by the Hungarian Scientific Research Fund No 109789 and No 116451, and the work of Monika Ludwig was supported, in part, by Austrian Science Fund (FWF) Project P25515-N25.

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