Chapter 3
Structures on Valuations

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Abstract In recent years on the space of translation invariant continuous valuations there have been discovered several canonical structures. Some of them turned out to be important for applications in integral geometry. In this chapter we review the relevant background and the main properties of the following new structures: product, convolution, Fourier type transform, and pull-back and push-forward of valuations under linear maps.

3.1 Preliminaries

Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathrm{Val}(V)$ denote the space of translation invariant continuous valuations on $\mathscr{K}(V)$. We have the following important result called McMullen's decomposition [17] with respect to degrees of homogeneity:

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{n} \mathrm{Val}_i(V).$$

It turns out that one can classify valuations in degrees of homogeneity 0, n, and $n-1$:

Theorem 3.1

(i) *(Obvious)* $\mathrm{Val}_0(V) = \mathbb{C} \cdot \chi$.
(ii) *(Hadwiger [14])* $\mathrm{Val}_n(V) = \mathbb{C} \cdot \mathrm{vol}_n$.
(iii) *(McMullen [18]) Let us describe* $\mathrm{Val}_{n-1}(V)$. *Fix a Euclidean metric on* V *for convenience. For any* $\phi \in \mathrm{Val}_{n-1}(V)$ *there exists a continuous function*

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$f: S^{n-1} \to \mathbb{C}$ such that for any $K \in \mathcal{K}(V)$

$$\phi(K) = \int_{S^{n-1}} f(\omega) dS(K,\omega). \tag{3.1}$$

Moreover the function f is defined uniquely by ϕ up to addition of a linear functional. Conversely any above expression belongs to $\mathrm{Val}_{n-1}(V)$. *(Here $S(K,\cdot)$ denotes the surface area measure of a convex body K, see [21].)*

Let us also state a very important characterization of *simple* translation invariant continuous valuations due to Klain [15] and Schneider [20] from 1995 which is used a lot in the theory. A valuation is called simple if it vanishes on convex sets of dimension less than $\dim V$.

Theorem 3.2 (Klain–Schneider) *Let ϕ be a simple continuous translation invariant valuation on $\mathcal{K}(V)$. Fix a Euclidean metric on V for convenience; $n := \dim V$. Then ϕ can be presented*

$$\phi(K) = a \cdot \mathrm{vol}(K) + \int_{S^{n-1}} f(\omega) dS(K,\omega) \text{ for any } K \in \mathcal{K}(V),$$

where $a \in \mathbb{C}$, f is a continuous odd function on S^{n-1}; the constant a is determined uniquely, and f is unique up to a linear functional. Furthermore any such expression (with f being odd) is a simple translation invariant continuous valuation.

We have further decomposition with respect to parity:

$$\mathrm{Val}_i(V) = \mathrm{Val}_i^+(V) \oplus \mathrm{Val}_i^-(V).$$

The group $\mathrm{GL}(V)$ acts linearly and continuously on $\mathrm{Val}(V)$ preserving the above decompositions:

$$g(\phi)(K) = \phi(g^{-1}K),$$

for any $g \in \mathrm{GL}(V), \phi \in \mathrm{Val}(V), K \in \mathcal{K}(V)$.

Theorem 3.3 (Irreducibility Theorem, Alesker [1]) *For any i the spaces* $\mathrm{Val}_i^{\pm}(V)$ *are topologically irreducible representations of* $\mathrm{GL}(V)$*, i.e. they have no proper* $\mathrm{GL}(V)$*-invariant closed subspaces.*

Remark 3.4 This theorem easily implies the so called McMullen's conjecture which says that linear combinations of valuations of the form $\mathrm{vol}(\cdot + A)$ where $A \in \mathcal{K}(V)$ are dense in $\mathrm{Val}(V)$.

Definition 3.5 A valuation $\phi \in \mathrm{Val}(V)$ is called *smooth* if the map $\mathrm{GL}(V) \to \mathrm{Val}(V)$ given by $g \mapsto g(\phi)$ is C^∞-differentiable.

It is well known in representation theory (and not hard to see) that the subset $\mathrm{Val}^\infty(V)$ of smooth valuations is a linear $\mathrm{GL}(V)$-invariant subspace dense in $\mathrm{Val}(V)$. Moreover it has a canonical Fréchet topology which is stronger than that induced from $\mathrm{Val}(V)$. The action of $\mathrm{GL}(V)$ on $\mathrm{Val}^\infty(V)$ is still continuous. Versions of the McMullen's decomposition and irreducibility theorem still hold for $\mathrm{Val}^\infty(V)$.

Example 3.6

(1) Let $A \in \mathscr{K}(V)$ has infinitely smooth boundary and strictly positive Gauss curvature. Then the valuation $\mathrm{vol}(\cdot + A)$ is smooth.
(2) (Alesker [3]) Let $G \subset \mathrm{O}(n)$ be a compact subgroup acting transitively on the unit sphere S^{n-1}. Then $\mathrm{Val}(V)^G \subset \mathrm{Val}^\infty(V)$; actually $\mathrm{Val}(V)^G$ is also finite dimensional in this case.
(3) Let us give an example of non-smooth valuation. Fix a proper linear subspace $E \subset V$. Let $p: V \to E$ be a linear projection. Fix a Lebesgue measure vol_E on E. Then $K \mapsto \mathrm{vol}_E(p(K))$ is a continuous, but not smooth valuation.

Remark 3.7 There is an equivalent description of smooth translation invariant valuations in terms of differential forms [4]: a valuation $\phi \in \mathrm{Val}(V)$ is smooth if and only if it can be presented in the form

$$\phi(K) = \int_{\mathbf{nc}(K)} \omega + a \cdot vol(K),$$

where ω in an infinitely smooth differential $(n-1)$-form on spherical bundle $V \times \mathbb{P}_+(V)$ (here $\mathbb{P}_+(V) := (V \setminus \{0\})/\mathbb{R}_{>0}$), $\mathbf{nc}(K) \subset V \times \mathbb{P}_+(V)$ is the normal cycle of K defined in Sect. 2.6 in this book, and a is a constant. This description turned out to be very useful for subsequent developments.

We will also need the notion of the *Klain imbedding* for even valuations. For convenience we will fix again a Euclidean metric on V. Let us construct a linear continuous map

$$Kl: \mathrm{Val}_k^+(V) \to C(\mathrm{Gr}_k(V))$$

as follows. Let $\phi \in \mathrm{Val}_k^+(V)$. For any $E^k \in \mathrm{Gr}_k(V)$ the restriction $\phi|_{E^k} \in \mathrm{Val}_k(E^k)$. By the mentioned above Hadwiger theorem $\phi|_{E^k} = c(E)\,\mathrm{vol}_E$. The map $\phi \mapsto c$ is the required Klain map. The main theorem proved by Klain [16] (based on [15]) is that this map is injective. Sometimes c is called the Klain function of ϕ and is denoted by Kl_ϕ.

The Klain map on smooth valuations $Kl: \mathrm{Val}_k^{\infty+}(V) \to C^\infty(\mathrm{Gr}_k(V))$ has a closed image which can be characterized in terms of decomposition under the $\mathrm{SO}(n)$-action [7]. (Note that it is harder to describe exactly the image of Kl of continuous valuations in continuous functions: very recently it was shown not to be a closed subspace (see Parapatits and Wannerer [19] and Alesker and Faifman [8]).)

3.2 Product on Valuations

The goal of this section is to introduce the canonical product on $\mathrm{Val}^\infty(V)$ and describe some if its properties. We will start with a slightly more refined notion: exterior product.

Theorem 3.8 *There exists a bilinear map, called exterior product,*

$$\boxtimes\colon \mathrm{Val}^\infty(V)\times \mathrm{Val}^\infty(W)\to \mathrm{Val}(V\times W)$$

which is uniquely characterized by the following properties:

- *it is continuous with the usual topology on* Val *and the Garding topology on* Val^∞*;*
- *if* $\phi(\cdot)=\mathrm{vol}_V(\cdot+A)$, $\psi(\cdot)=\mathrm{vol}_W(\cdot+B)$ *then*

$$(\phi\boxtimes\psi)(\cdot)=(\mathrm{vol}_V\boxtimes\mathrm{vol}_W)(\cdot+(A\times B)).$$

Note that the uniqueness follows from the McMullen's conjecture. But existence is a non-trivial statement which is based not only on the irreducibility theorem. The general idea of the proof is that any smooth valuation can be presented as a rapidly convergent (in $\mathrm{Val}^\infty(V)$) series of the form $\sum_p \alpha_p \mathrm{vol}_V(\cdot+A_p)$. For two such expressions there is only one way to define their exterior product satisfying the properties of the theorem. However a presentation of a valuation as such a series is non-unique, and one has to check that the product is independent of a presentation. This last step we will demonstrate now assuming for simplicity that all series are in fact finite sums. Assume that ϕ has two presentations

$$\phi=\sum_p \alpha_p \mathrm{vol}_V(\cdot+A_p)=\sum_p \alpha'_p \mathrm{vol}_V(\cdot+A'_p).$$

For ψ we fix a similar presentation and show that $\phi\boxtimes\psi$ is independent of the presentation of ϕ. Thus we may assume that ψ has a single summand

$$\psi=\mathrm{vol}_W(\cdot+B).$$

For any $K \in \mathscr{K}(V \times W)$ we have

$$\begin{aligned}(\phi \boxtimes \psi)(K) &= \sum_p \alpha_p (\mathrm{vol}_V \times \mathrm{vol}_W)(K + (A_p \times B)) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \sum_p \alpha_p \,\mathrm{vol}_V \big([K + (A_p \times B)] \cap [V \times \{y\}]\big) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \sum_p \alpha_p \,\mathrm{vol}_V \big(\{(K + (\{0\} \times B)) \cap (V \times \{y\})\} + A_p\big) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \phi\left([K + (\{0\} \times B)] \cap (V \times \{y\})\right),\end{aligned}$$

where the second equation is based on Fubini's theorem. From the last expression we see that the exterior product does not depend on presentation of ϕ.

Let us define the product on $\mathrm{Val}^\infty(V)$.

Definition 3.9 For $\phi, \psi \in \mathrm{Val}^\infty(V)$ let us define the product

$$(\phi \cdot \psi)(K) := (\phi \boxtimes \psi)(\Delta(K)),$$

where $K \in \mathscr{K}(V)$, $\Delta \colon V \hookrightarrow V \times V$ is the diagonal imbedding.

Theorem 3.10 (Alesker [3])

(1) *The product is a bilinear continuous map*

$$\mathrm{Val}^\infty(V) \times \mathrm{Val}^\infty(V) \to \mathrm{Val}^\infty(V).$$

(2) *Equipped with this product,* $\mathrm{Val}^\infty(V)$ *becomes a commutative associative graded algebra with a unit (unit is the Euler characteristic; the grading is given by the McMullen's decomposition).*

(3) *For any* $0 \le i \le n$ *the bilinear map given by the product*

$$\mathrm{Val}_i^\infty(V) \times \mathrm{Val}_{n-i}^\infty(V) \to \mathrm{Val}_n(V) = \mathbb{C} \cdot \mathrm{vol}_V$$

is a perfect pairing, i.e. the induced map $\mathrm{Val}_i^\infty(V) \to (\mathrm{Val}_{n-i}^\infty(V))^* \otimes \mathrm{Val}_n(V)$ *is injective with image dense in the weak* topology.*

Remark 3.11 When the two valuations are given by differential forms on the spherical bundle as in Remark 3.7 then their product also can be described by an differential form expressed explicitly via the given forms by a rather complicated formula [6].

The following result is a version of the hard Lefschetz type theorem. In this form it was proved by Alesker [5], but the proof is heavily based on a different version of the hard Lefschetz theorem which in full generality was proved by Bernig and Bröcker [9] and earlier in the even case by Alesker [2].

Theorem 3.12 *Fix a Euclidean metric on V. Let* $0 \leq i < n/2$. *The map*

$$\mathrm{Val}_i^\infty(V) \to \mathrm{Val}_{n-i}^\infty(V)$$

given by $\phi \mapsto V_1^{n-2i} \cdot \phi$, *is an isomorphism. (Here* V_1 *is the first intrinsic volume as usual.)*

Remark 3.13 This theorem immediately implies that the operator $\mathrm{Val}_i^\infty(V) \to \mathrm{Val}_{i+j}^\infty(V)$ given by $\phi \mapsto V_1^j \cdot \phi$ is injective for $j \leq n-2i$ and surjective for $i \geq n-2i$.

The product structure has been computed in some cases.

Example 3.14

(1) (Alesker [3]) Let

$$\phi(K) = V(K[i], A_1, \ldots, A_{n-i}),\ \psi(K) = V(K[n-i], B_1, \ldots, B_i).$$

Then

$$\phi \cdot \psi = V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) \cdot \mathrm{vol}\,.$$

(2) $\mathrm{Val}^{O(n)}(\mathbb{R}^n)$ is isomorphic as a graded algebra to $\mathbb{C}[t]/(t^{n+1})$ where $t = V_1$.
(3) A geometric description of the space $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ of unitarily invariant valuations was obtained by Alesker [2] in 2003. Fu [12] has obtained in 2006 the following beautiful description of the algebra structure of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ in terms of generators and relations:

$$\mathrm{Val}^{U(n)}(\mathbb{C}^n) = \mathbb{C}[s,t]/(f_{n+1}, f_{n+2}),$$

where $\deg s = 2, \deg t = 1$ and the polynomial f_i is the degree i term of the power series $\log(1 + s + t)$.
(4) Some non-trivial examples of the product of tensor valued valuations were recently computed by Bernig and Hug [11]; see also Chap. 3 in this book.

3.3 Convolution of Valuations

We denote by $D(V^*)$ the space of complex valued Lebesgue measures on V^*.

Theorem 3.15 (Bernig-Fu [10]) *There exists a bilinear map called convolution*

$$*\colon \big(\mathrm{Val}^\infty(V) \otimes D(V^*)\big) \times \big(\mathrm{Val}^\infty(V) \otimes D(V^*)\big) \to \mathrm{Val}^\infty(V) \otimes D(V^*)$$

which is uniquely characterized by the following properties:

- *continuity in the Garding topology;*
- *if* $\phi(\cdot) = \mathrm{vol}(\cdot + A) \otimes \mathrm{vol}^{-1}$, $\psi(\cdot) = \mathrm{vol}(\cdot + B) \otimes \mathrm{vol}^{-1}$, *then*

$$(\phi * \psi)(\cdot) = \mathrm{vol}(\cdot + A + B) \otimes \mathrm{vol}^{-1},$$

where vol^{-1} *is the Lebesgue measure on* V^* *such that for any basis* $e_1, \dots, e_n$ *of* V *spanning the parallelepiped of unit volume with respect to* vol, *the parallelepiped in* V^* *spanned by the dual basis* $e_1^*, \dots, e_n^*$ *has the unit volume with respect to* vol^{-1}.

Equipped with this product, $\mathrm{Val}^\infty(V) \otimes D(V^*)$ *becomes a commutative associative graded algebra with the unit, when the unit is* $\mathrm{vol} \otimes \mathrm{vol}^{-1}$, *and the grading is* ($n$ – *deg of homogeneity*).

The uniqueness again follows immediately from McMullen's conjecture. The existence is non-trivial. Later we will deduce it from existence of exterior product on valuations.

Remark 3.16 If two valuations are given by differential forms as in Remark 3.7 then their convolution can be given by a differential form expressed by an explicit formula via the two given forms [10].

3.4 Fourier Type Transform on Valuations

It turns out that the algebras $(\mathrm{Val}^\infty(V), \cdot)$ and $(\mathrm{Val}^\infty(V^*) \otimes D(V), *)$ are isomorphic. We are going to discuss a specific isomorphism between them, called a Fourier type transform, which has some additional interesting properties.

Theorem 3.17 (Alesker [5]) *There exists an isomorphism of algebras*

$$\mathbb{F} \colon \mathrm{Val}^\infty(V) \xrightarrow{\sim} \mathrm{Val}^\infty(V^*) \otimes D(V)$$

which has the following extra properties:

- $\mathbb{F}$ *is an isomorphism of linear topological spaces.*
- $\mathbb{F}$ *commutes with the natural action of* $\mathrm{GL}(V)$ *on both spaces.*
- *(Plancherel type inversion formula) Consider the composition* $\mathscr{E}_V$*:*

$$\mathrm{Val}^\infty(V) \xrightarrow{\mathbb{F}_V} \mathrm{Val}^\infty(V^*) \otimes D(V) \xrightarrow{\mathbb{F}_{V^*} \otimes \mathrm{Id}_{D(V)}} \mathrm{Val}^\infty(V) \otimes \underbrace{D(V^*) \otimes D(V)}_{\simeq \mathbb{C}} = \mathrm{Val}^\infty(V).$$

Then $(\mathscr{E}_V \phi)(K) = \phi(-K)$.

The construction of the Fourier transform is rather difficult and uses some more of representation theory. Nevertheless in few examples the Fourier transform can be computed. In the case of even valuations there is another description using the Klain function. In the rest of this section we will discuss this material.

Below for simplicity we will fix a Euclidean metric on V. Hence we get identifications $V^* \simeq V$ and $D(V) \simeq \mathbb{C}$. Thus $\mathbb{F}\colon Val^\infty(V) \xrightarrow{\sim} Val^\infty(V)$ commutes with $O(n)$, but not with $GL(V)$.

Example 3.18

(1) $\mathbb{F}_V(\chi) = vol_V$.
(2) $\mathbb{F}_V(vol_V) = \chi$.
(3) $\mathbb{F}(V_i) = c_{i,n} V_{n-i}$ and the constant $c_{i,n}$ can be written down explicitly. Indeed this (without the exact value of the constant) follows from the Hadwiger theorem.
(4) Assume $\dim V = 2$. Given the first two examples and McMullen's decomposition, it remains to describe $\mathbb{F}$ on 1-homogeneous smooth valuations. Fix a Euclidean metric and an orientation on V. Let $J\colon V \to V$ be the operator of rotation by $\pi/2$ counterclockwise.

By the Hadwiger's theorem [13] (which now follows from McMullen's description of $(n-1)$-homogeneous valuations from Sect. 3.1) any such valuation ϕ has the form

$$K \mapsto \int_{S^1} f(\omega) dS(K,\omega),$$

where $f \in C^\infty(S^1)$ is defined uniquely up to a linear functional. Decompose f into the even and odd parts:

$$f = f_+ + f_-.$$

Furthermore let us decompose the odd part $f_- = f_-^{hol} + f_-^{anti}$ into holomorphic and anti-holomorphic parts as follows. First decompose f_- into the usual Fourier series on S^1:

$$f_-(\omega) = \sum_{k\in\mathbb{Z}} \hat{f}_-(k) e^{ik\omega}.$$

Then define

$$f_-^{hol}(\omega) := \sum_{k>0} \hat{f}_-(k) e^{ik\omega}, \qquad f_-^{anti}(\omega) := \sum_{k<0} \hat{f}_-(k) e^{ik\omega}.$$

Then the Fourier transform of ϕ is

$$(\mathbb{F}\phi)(K)$$
$$= \int_{S^1} f_+(J\omega)dS(K,\omega) + \int_{S^1} f_-^{\text{hol}}(J\omega)dS(K,\omega) - \int_{S^1} f_-^{\text{anti}}(J\omega)dS(K,\omega).$$

(5) For even smooth valuations there is a simple description of the Fourier transform in terms of the Klain functions; historically this was the first construction of the Fourier transform (Alesker [2]). Fix a Euclidean metric on V for the simplicity of notation. Let $\phi \in Val_k^{\infty+}(V)$. Then for any $F^{n-k} \in Gr_{n-k}(V)$ one has

$$Kl_{\mathbb{F}\phi}(F) = Kl_\phi(F^\perp).$$

Thus the Fourier transform can be easily described on the language of functions on Grassmannians. The non-trivial point is that given a smooth Klain function of a valuation then the transformed function indeed corresponds to some valuation (the uniqueness follows from the Klain's theorem). This follows from the description of the image of the Klain map obtained by Alesker and Bernstein [7].

(6) Recently Bernig and Hug [11] have made some explicit non-trivial computations of the Fourier transform on odd valuations in dimensions higher than 2 in order to obtain kinematic formulas for tensor valuations; see also Chap. 3 of this book.

3.5 Pull-Back and Push-Forward on Valuations

In this section we discuss, following [5], operations of pull-back and push-forward on valuations under linear mappings and their relations to product, convolution and the Fourier transform. In particular we claim that the convolution on valuations can be presented as composition of the exterior product and push-forward under the addition map $a: V \times V \to V$; that will provide another explanation why convolution is well defined (given the exterior product).

Let us start with the notion of pull-back under a linear map $f: V \to W$. Define the pull-back map

$$f^*: Val(W) \to Val(V) \tag{3.2}$$

by $(f^*\phi)(K) = \phi(f(K))$. Obviously f^* is linear and continuous, it preserves degree of homogeneity. Clearly

$$(f \circ g)^* = g^* \circ f^*.$$

A formal simple remark is that if $\Delta\colon V \hookrightarrow V \times V$ is the diagonal imbedding then

$$\phi \cdot \psi = \Delta^*(\phi \boxtimes \psi).$$

The push-forward map

$$f_*\colon \mathrm{Val}(V) \otimes D(V^*) \to \mathrm{Val}(W) \otimes D(W^*)$$

is going to be a linear continuous map. In order to motivate somehow its introduction, let us have some non-rigorous remarks. f_* is going to be dual to f^* in the following not very precise sense.

Consider the bilinear map $\mathrm{Val}^\infty(V) \times \mathrm{Val}^\infty(V) \to \mathrm{Val}_n(V) = D(V)$ given by the product and taking the n-th homogeneous component. By the Poincaré duality the induced map

$$\mathrm{Val}^\infty(V) \otimes D(V^*) \to (\mathrm{Val}^\infty(V))^*$$

is injective and has a dense image in the weak* topology. Informally speaking, up to a completion in appropriate topology, the dual of $\mathrm{Val}^?(V)$ is equal to $\mathrm{Val}^?(V) \otimes D(V^*)$, where $\mathrm{Val}^?(V)$ is a class of valuations of unspecified class of smoothness. Hence, with these identifications, the dual of f^* from (3.2) should lead to a linear map which we call push-forward and denote f_*:

$$f_*\colon \mathrm{Val}^?(V) \otimes D(V^*) \to \mathrm{Val}^?(W) \otimes D(W^*).$$

A closer investigation of this map shows that in fact f_* is a continuous linear map between spaces of continuous (!) valuations (twisted by densities):

$$f_*\colon \mathrm{Val}(V) \otimes D(V^*) \to \mathrm{Val}(W) \otimes D(W^*).$$

It does satisfy the property

$$(f \circ g)_* = f_* \circ g_* \tag{3.3}$$

as it should be by dualizing the corresponding property of the pull-back.

Now we have to describe f_* more explicitly. By the property (3.3) and since every linear map can be presented as composition of injective and surjective linear maps, it suffices to do that only in these two cases.

Assume first $f\colon V \to W$ is onto. To simplify the notation, we may assume that W is a subspace of V, and may choose a Euclidean metric on V such that f is the orthogonal projection. This choice of metric also induces isomorphisms

$$D(V) \simeq D(W) \simeq D(W^\perp) \simeq \mathbb{C}$$

and the same for dual of $V, W, W^{\perp}$. Let $\phi \in \mathrm{Val}(V) \otimes D(V^*) \simeq \mathrm{Val}(V)$. Fix any $K \in \mathcal{K}(W)$. Let us choose $\tilde{K} \in \mathcal{K}(V)$ such that $f(\tilde{K}) = K$. For any $\lambda \geq 0$ consider the valuation on $\mathcal{K}(W^{\perp})$

$$R \mapsto \phi(\lambda R + \tilde{K}).$$

By McMullen's decomposition, this is a polynomial in λ of degree at most $k := \dim W^{\perp}$. The highest degree term is the k-homogeneous valuation on $W^{\perp}$, hence by Hadwiger's theorem it is proportional to $\mathrm{vol}_k(R)$. The coefficient depends on ϕ and $\tilde{K}$ (but not on R of course). Moreover one can show that it depends only on K rather than on $\tilde{K}$ (the proof I know uses the McMullen's conjecture). More precisely we have

$$\phi(\lambda R + K) = \frac{1}{k!}\lambda^k \,\mathrm{vol}_k(R) \cdot (f_*\phi)(K) + \mathrm{O}(\lambda^{k-1}).$$

Thus we got a description of f_* for surjective maps.

Before we describe f_* for injective maps, let us say that the convolution on valuations can be describe as

$$\phi * \psi = a_*(\phi \boxtimes \psi),$$

where $a\colon V \times V \to V$ is the addition map (which is of course surjective).

Let now $f\colon V \to W$ be an injective map. It is convenient to assume without loss of generality that V is a subspace of W, and f is the identity imbedding. We fix a Euclidean metric on W and use various identifications it induces. Let $\phi \in \mathrm{Val}(V)$ and $K \in \mathcal{K}(W)$. Then

$$(f_*\phi)(K) = \int_{y \in V^{\perp}} \phi(K \cap (y + V))d\,\mathrm{vol}_{V^{\perp}}(y).$$

Finally let us discuss the relation of pull-back and push-forward to the Fourier transform. We will do it here in a non-rigorous way for the sake of simplicity. Let $f\colon V \to W$ be a linear map, and $f^{\vee}\colon W^* \to V^*$ be the dual map. Then we should have the following non-rigorously stated identity

$$\mathbb{F}_V \circ f^* = (f^{\vee})_* \circ \mathbb{F}_W.$$

This identity is non-rigorous because the Fourier transform is defined on the class of smooth valuations which is not preserved under the pull-back and push-forward maps.

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