Lecture Notes in Mathematics 2177

Eva B. Vedel Jensen
Markus Kiderlen *Editors*

Tensor Valuations and Their Applications in Stochastic Geometry and Imaging

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Preface

The purpose of this volume is to give an up-to-date introduction to tensor valuations and their applications on a graduate level. A valuation is a finitely additive mapping. Since Dehn's use of real-valued valuations to give a negative answer to Hilbert's third problem in 1900, the theory of valuations has been extended considerably and found widespread use in applied sciences like biology, materials sciences, medicine and physics. At the end of the twentieth century, McMullen and several other authors started to consider tensor-valued valuations, but the research only gained momentum when Alesker could show a characterization theorem for tensor valuations which satisfy certain natural geometric and topological requirements, thus mirroring the corresponding result for real-valued valuations by Hadwiger from the 1950s. In the last decades, the intensive use of algebraic methods and representation theory yielded a wealth of new results, including important integral geometric formulae for tensor valuations. At the same time, the application of tensor valuations was starting in stochastic geometry and a number of applied research areas, primarily with the purpose of quantifying the morphology and anisotropy of complex spatial structures.

In 2011 a very close collaboration between the *Centre for Stochastic Geometry and Advanced Bioimaging* (CSGB) and the DFG funded research group *Geometry and Physics of Spatial Random Systems* (GPSRS) started, where one of the main goals was to advance theoretical research and applications of tensor valuations. The GPSRS, with research groups from Karlsruhe Institute of Technology and the University of Erlangen-Nürnberg, combined fundamental mathematical research with studies in the physical sciences, enabling fruitful interdisciplinary collaborations. The CSGB group from the University of Aarhus, supported by the Danish Villum Foundation, made a number of contributions to the basic theory of tensor valuations and developed applications in biology and imaging. In view of the many new developments, these two research groups jointly organized the *Workshop on Tensor Valuations in Stochastic Geometry and Imaging* during September 21–26, 2014, at the Sandbjerg Estate in Southern Denmark. Most of the eight invited speakers of this workshop gave lectures of twice 45 min introducing their field of expertise on an accessible graduate level. Extended transcripts of these lectures form the backbone of this book and have been complemented by invited contributions to broaden the scope.

The book develops around the central notion of Minkowski tensors, which are tensor-valued valuations, typically defined on the family of convex bodies in n-dimensional Euclidean space. They are isometry covariant and continuous with respect to the Hausdorff metric. The most important special cases—also those that have attracted most attention in the past—are the Minkowski tensors of rank zero, the intrinsic volumes. Many of the results presented in this volume were historically formulated for these scalar-valued valuations and later extended to general Minkowski tensors.

Since Blaschke introduced integral geometry as a subject of its own, integral geometric formulae have been playing a prominent role in the theory of valuations. Federer even stated that for a theory of curvature measures 'to be worthwhile', it must contain versions of the principal kinematic formula (and the Gauss-Bonnet theorem). Integral geometric formulae are also crucial for applications because they can be used as tools in image reconstruction and stereology, as exemplified in Chap. 14. Integral geometric relations play also a prominent role in this volume. In Chaps. 4 and 5, versions of the classical Crofton formula are stated for Minkowski tensors. They are kinematic in nature, as they involve the intersection of a convex body with an invariantly translated and rotated flat. Important stereological applications in confocal microscopy are often based on flat sections through a reference point and thus require rotational Crofton formulae, where the integration over translations is omitted. Rotational Crofton formulae are presented in Chap. 7. Hadwiger's general integral geometric theorem allows to derive principal kinematic formulae from (kinematic) Crofton formulae, and this is outlined in Chap. 4. To treat non-isotropic Boolean models, a translative version of the principal kinematic formula is needed and thus provided in Chap. 11, even in an iterated form. Finally, rotation sum formulae for certain tensor valuations are given in Chap. 4.

The book is organized as follows. The first two chapters lay the foundations by introducing valuations and giving characterization theorems. Chapter 1 gives an overview of the status of the field prior to the recent advances first by Alesker and later by others who exploited algebraic methods in integral geometry. It gives a smooth introduction into the classical facts. It also introduces support, curvature and area measures as important examples of measure-valued valuations. In Chap. 2, Minkowski tensors are introduced and their properties are explained. In particular, Alesker's characterization theorem is given, stating that the vector space of all continuous isometry covariant tensor valuations on convex bodies is spanned by combinations of Minkowski tensors and the metric tensor. A similar characterization theorem is then established for local versions of the Minkowski tensors, where it surprisingly turns out that the latter class is richer than in the global case.

The next four chapters give an introduction to aspects of 'algebraic integral geometry'. Chapters 3 and 4 introduce algebraic structures on certain subspaces of tensor valuations. Significant in their own right, these concepts are particularly valuable to derive integral geometric results. Chapter 3 introduces product, convolution, the Alesker-Fourier transform and pull-back and push-forward of valuations under linear maps. This chapter only treats the case of scalar-valued valuations. Tensor-valued versions are given in Chap. 4 and their close connection to integral geometric relations is revealed. This is then exploited to obtain rotation sum formulae and a considerably simplified Crofton formula for translation invariant tensor valuations. More general Crofton formulae, also for surface area measures, are outlined as well. Chapter 5 varies and extends the Crofton formulae of the foregoing chapter in several respects. Firstly, it states intrinsic versions, where the intersection of the convex body with the integration plane E is considered as a subset of E. Secondly, it shows that Crofton formulae can also be stated for tensorial curvature measures, which are versions of the Minkowski tensors localized in $\mathbb{R}^n$. The algebraic methods outlined in Chaps. 3 and 4 are also principal tools of Chap. 6. A decomposition of the space of continuous and translation invariant valuations into a sum of $\mathrm{SO}(n)$-irreducible subspaces is discussed. This result leads to a Hadwiger-type theorem for translation invariant and $\mathrm{SO}(n)$-equivariant valuations with values in an arbitrary finite dimensional $\mathrm{SO}(n)$-module. The class of these valuations includes those with values in general tensor spaces. In Chap. 7 rotational Crofton formulae and versions of a principal rotational formula are presented. This chapter also describes a Hadwiger-type characterization theorem for continuous, rotation invariant polynomial valuations.

Chapters 8–10 are devoted to valuations on domains other than convex bodies, although valuations on convex polytopes already played a role in the first two chapters of the volume. In Chap. 8, a theory of valuations on lattice polytopes is outlined, including a Hadwiger-type characterization, the Betke-Kneser theorem for certain real-valued valuations and more recent results on valuations with values in the families of tensors or convex bodies. In Chap. 9, instead of treating valuations on convex bodies of the Euclidean space $\mathbb{R}^n$, (smooth) valuations on subsets of n-dimensional manifolds are considered. The role of convex bodies is now played by simple differential polyhedra. A theory, including local and global kinematic formulae on space forms and a transfer principle, is explained with particular emphasis on the Hermitian case. Chapter 10 generalizes the domains of valuations in a different direction: while still working with the flat case of the Euclidean n-space, it investigates what regularity of its subsets actually is required in order to develop a theory of valuations that allows for integral geometric kinematic formulae. The very general class of WDC sets is introduced and the construction of the normal cycle for these sets is discussed.

The last five chapters are devoted to applications of tensor valuations in stochastic geometry, biology and imaging. Chapters 11 and 12 describe properties of tensor valuations of Boolean models with convex or polyconvex grains. While isotropy (together with stationarity) is often a standard assumption in the literature, none of these chapters requires isotropy. In Chap. 11 mean value formulae for scalar- and tensor-valued valuations applied to Boolean models are given and explained in the stationary case with an outlook to the newer developments and the non-stationary case. Second-order formulae for valuations of the Boolean model in an observation window, that is, covariances, can be derived in the asymptotic regime when the window is expanding. This is done in Chap. 12 together with the formulation of central limit theorems. Chapter 13 describes the analysis of random tessellations with the help of tensor valuations applied to individual cells. The goal is to assess properties of the underlying stochastic process that generated the tessellation. The chapter is a theory-based simulation study and compares Voronoi tessellations of standard point process models, STIT- and hyperplane mosaics, and tessellations derived from hard sphere and hard ellipsoid models from particle physics. In applied image analysis of structured synthetic and biological materials, the described methods have been used to infer information about the formation process from spatial measurements of an observed random structure. Chapter 14 is motivated by the applied problem of estimating volume tensors from observations in planar sections in conventional microscopy using local stereological methods. It presents a new estimator of mean particle volume tensors in three-dimensional space from vertical sections. Also the last chapter is devoted to the determination of tensor valuations in practical applications. It gives an overview over algorithms that approximate tensor valuations of an object from binary and grey-valued images and discusses in particular the asymptotic properties of these algorithms when the resolution of the images tends to infinity.

The editors have aimed for a collection of self-contained contributions allowing the reader to select chapters without the necessity of reading the whole volume from its beginning. Therefore, chapters often start with a short summary of concepts and notions possibly already introduced earlier in the book. This, and the comprehensive cross-referencing to other chapters, may give the reader with a sound background a deeper understanding of a specific subarea. At the same time, the newcomer can read these lecture notes as a comprehensive introduction to tensor valuations and important applications.

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Chapter 1
Valuations on Convex Bodies: The Classical Basic Facts

Rolf Schneider

Abstract The purpose of this chapter is to give an elementary introduction to valuations on convex bodies. The goal is to serve the newcomer to the field, by presenting basic notions and collecting fundamental facts, which have proved of importance for the later development, either as technical tools or as models and incentives for widening and deepening the theory. We also provide hints to the literature where proofs can be found. It is not our intention to duplicate the existing longer surveys on valuations, nor to update them. We restrict ourselves to classical basic facts and geometric approaches, which also means that we do not try to describe the exciting developments of valuation theory in the last 15 years, which involve deeper methods and will be the subject of later chapters. The sections of the present chapter treat, in varying detail, general valuations, valuations on polytopes, examples of valuations from convex geometry, continuous valuations on convex bodies, measure-valued valuations, valuations on lattice polytopes.

1.1 General Valuations

The natural domain for a valuation, as it is understood here, would be a lattice (in the sense of Birkhoff [4]; see p. 230, in particular). However, many important functions arising naturally in convex geometry have a slightly weaker property, and they become valuations on a lattice only after an extension procedure. For that reason, valuations on intersectional families are the appropriate object to study here. A family $\mathscr{S}$ of sets is called *intersectional* if $A, B \in \mathscr{S}$ implies $A \cap B \in \mathscr{S}$.

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Definition 1.1 A function φ from an intersectional family $\mathscr{S}$ into an Abelian group (with composition $+$ and zero element 0) is *additive* or a *valuation* if

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \tag{1.1}$$

for all $A, B \in \mathscr{S}$ with $A \cup B \in \mathscr{S}$, and if $\varphi(\emptyset) = 0$ in case $\emptyset \in \mathscr{S}$.

The Abelian group in the definition may be replaced by an Abelian semigroup with cancellation law, because the latter can be embedded in an Abelian group. A trivial example of a valuation on $\mathscr{S}$ is given by $\varphi(A) := \mathbf{1}_A$, where $\mathbf{1}_A$ is the characteristic function of A, defined on $S := \bigcup_{A \in \mathscr{S}} A$ by

$$\mathbf{1}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in S \setminus A. \end{cases}$$

For the Abelian group appearing in Definition 1.1 one can take in this case, for example, the additive group of all real functions on S.

It would generally be too restrictive to assume that the intersectional family $\mathscr{S}$ is also closed under finite unions. However, we can always consider the family $\mathrm{U}(\mathscr{S})$ consisting of all finite unions of elements from $\mathscr{S}$. Then $(\mathrm{U}(\mathscr{S}), \cup, \cap)$ is a lattice. If φ is a valuation on $\mathrm{U}(\mathscr{S})$ (not only on $\mathscr{S}$), then (1.1) is easily extended by induction to the formula

$$\varphi(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1} \varphi(A_J) \tag{1.2}$$

for $m \in \mathbb{N}$ and $A_1, \ldots, A_m \in \mathrm{U}(\mathscr{S})$; here $A_J := \bigcap_{j \in J} A_j$ and $|J| := \operatorname{card} J$. Relation (1.2) is known as the *inclusion-exclusion formula*. This gives rise to another definition.

Definition 1.2 A function φ from the intersectional family $\mathscr{S}$ into an Abelian group is called *fully additive* if (1.2) holds for $m \in \mathbb{N}$ and all $A_1, \ldots, A_m \in \mathscr{S}$ with $A_1 \cup \cdots \cup A_m \in \mathscr{S}$.

Thus, a valuation on $\mathscr{S}$ that has an additive extension to the lattice $\mathrm{U}(\mathscr{S})$, is fully additive. It is a nontrivial fact that the converse is also true. We formulate a more general extension theorem. For this, we denote by $U^\bullet(\mathscr{S})$ the $\mathbb{Z}$-module spanned by the characteristic functions of the elements of $\mathscr{S}$.

Theorem 1.3 (Groemer's First Extension Theorem) *Let φ be a function from an intersectional family of sets (including $\emptyset$) into an Abelian group, such that $\varphi(\emptyset) = 0$. Then the following conditions* (a)–(d) *are equivalent.*

(a) *φ is fully additive;*
(b) *If*

$$n_1 \mathbf{1}_{A_1} + \cdots + n_m \mathbf{1}_{A_m} = 0$$

with $A_i \in \mathscr{S}$ *and* $n_i \in \mathbb{Z}$ $(i = 1, \dots, m)$, *then*

$$n_1\varphi(A_1) + \cdots + n_m\varphi(A_m) = 0;$$

(c) *The functional* $\varphi^\bullet$ *defined by* $\varphi^\bullet(\mathbf{1}_A) := \varphi(A)$ *for* $A \in \mathscr{S}$ *has a* $\mathbb{Z}$*-linear extension to* $U^\bullet(\mathscr{S})$*;*
(d) φ *has an additive extension to the lattice* $\mathrm{U}(\mathscr{S})$.

This theorem is due to Groemer [11]. His proof is reproduced in [42, Theorem 6.2.1]. Actually, Groemer formulated a slightly different version. In his version, φ maps into a real vector space. The preceding theorem then remains true with $\mathbb{Z}$ replaced by $\mathbb{R}$, $U^\bullet(\mathscr{S})$ replaced by the real vector space $V(\mathscr{S})$ that is spanned by the characteristic functions of the elements of $\mathscr{S}$, and '$\mathbb{Z}$-linear' replaced by '$\mathbb{R}$-linear'.

In this case, if φ is fully additive, then Groemer defined the *φ-integral* of a function $f \in V(\mathscr{S})$ in the following way. If

$$f = a_1\mathbf{1}_{A_1} + \cdots + a_m\mathbf{1}_{A_m}, \qquad a_1, \dots, a_m \in \mathbb{R},$$

then

$$\int f \,\mathrm{d}\varphi := a_1\varphi(A_1) + \cdots + a_m\varphi(A_m).$$

This definition makes sense, since by Theorem 1.3 the right-hand side does not depend on the chosen representation of the function f.

Such integrals with respect to a valuation were later rediscovered by Viro [46], and they were applied by him and other authors in various ways, mainly in the case where φ is the Euler characteristic on suitable sets.

Results on general valuations, as mentioned in this section, were preceded by concrete geometric applications of valuations. We give two historic examples in subsequent sections.

1.2 Valuations on Polytopes

From now on, we work in n-dimensional Euclidean space $\mathbb{R}^n$, with scalar product denoted by $\cdot$ and induced norm $\|\cdot\|$. The domain of the considered valuations will be either the set $\mathscr{K}^n$ of convex bodies (nonempty, compact, convex sets) or the set $\mathscr{P}^n$ of convex polytopes in $\mathbb{R}^n$. We consider the latter case first.

Real valuations on polytopes (by which we always mean convex polytopes) are closely related to dissections of polytopes.

Definition 1.4 A *dissection* of the polytope $P \in \mathscr{P}^n$ is a set $\{P_1, \dots, P_m\}$ of polytopes such that $P = \bigcup_{i=1}^m P_i$ and $\dim(P_i \cap P_j) < n$ for $i \neq j$.

Let G be a subgroup of the affine group of $\mathbb{R}^n$. The polytopes $P, Q \in \mathscr{P}^n$ are called *G-equidissectable* if there are a dissection $\{P_1, \dots, P_m\}$ of P, a dissection $\{Q_1, \dots, Q_m\}$ of Q, and elements $g_1, \dots, g_m \in G$ such that $Q_i = g_i P_i$ for $i = 1, \dots, m$.

The most frequently considered cases are those where G is the group T_n of translations of $\mathbb{R}^n$ or the group G_n of rigid motions of $\mathbb{R}^n$. Here a *rigid motion* is an isometry of $\mathbb{R}^n$ that preserves the orientation, thus, a mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ of the form $gx = \vartheta x + t$, $x \in \mathbb{R}^n$, with $\vartheta \in \mathrm{SO}(n)$ and $t \in \mathbb{R}^n$.

The following is a classical result of elementary geometry.

Theorem 1.5 (Bolyai-Gerwien 1833/35) *In $\mathbb{R}^2$, any two polygons of the same area are G_2-equidissectable.*

The theorem remains true if the motion group G_2 is replaced by the group consisting of translations and reflections in points (Hadwiger and Glur [22]).

Hilbert's third problem from 1900 asked essentially whether a result analogous to the Bolyai-Gerwien theorem holds in three dimensions. The negative answer given by Dehn [8] is apparently the first use of valuations in convexity. We describe the essence of his answer, though in different terms and using later modifications. This gives us an opportunity to introduce some further notions and facts about valuations.

On polytopes, the valuation property follows from a seemingly weaker assumption.

Definition 1.6 A function φ on $\mathscr{P}^n$ with values in an Abelian group is called *weakly additive* (or a *weak valuation*) if (setting $\varphi(\emptyset) := 0$) for each $P \in \mathscr{P}^n$ and each hyperplane H, bounding the two closed halfspaces H^+, H^-, the relation

$$\varphi(P) = \varphi(P \cap H^+) + \varphi(P \cap H^-) - \varphi(P \cap H) \tag{1.3}$$

holds.

Every valuation on $\mathscr{P}^n$ is weakly additive, but also the converse is true, even more.

Theorem 1.7 *Every weakly additive function on $\mathscr{P}^n$ with values in an Abelian group is fully additive on $\mathscr{P}^n$.*

A proof can be found in [42, Theorem 6.2.3], and Note 1 there gives hints to the origins of this result.

Together with Groemer's first extension theorem (Theorem 1.3), the preceding theorem shows that every weakly additive function on $\mathscr{P}^n$ has an additive extension to the lattice $\mathrm{U}(\mathscr{P}^n)$. The elements of $\mathrm{U}(\mathscr{P}^n)$ are the finite unions of convex polytopes; we call them *polyhedra*.

We need two other important notions.

Definition 1.8 A valuation φ on a subset of $\mathcal{K}^n$ is called *simple* if $\varphi(A) = 0$ whenever $\dim A < n$.

Definition 1.9 Let G be a subgroup of the affine group of $\mathbb{R}^n$. A valuation φ on a subset of $\mathcal{K}^n$ (which together with A contains gA for $g \in G$) is called *G-invariant* if $\varphi(gA) = \varphi(A)$ for all A in the domain of φ and all $g \in G$.

The following is easy, but important.

Lemma 1.10 *Let G be a subgroup of the affine group of $\mathbb{R}^n$. If φ is a G-invariant simple valuation on $\mathcal{P}^n$ and if the polytopes $P, Q \in \mathcal{P}^n$ are G-equidissectable, then $\varphi(P) = \varphi(Q)$.*

In fact, by Theorems 1.7 and 1.3, the valuation φ has an additive extension to $\mathrm{U}(\mathcal{P}^n)$, hence the inclusion-exclusion formula (1.2) can be applied to dissections $\{P_1, \dots, P_m\}$ of P and $\{Q_1, \dots, Q_m\}$ of Q, satisfying $g_i P_i = Q_i$ for $g_i \in G$. Since φ is simple, the terms in (1.2) with $|J| > 1$ vanish, and what remains is

$$\begin{aligned}\varphi(P) &= \varphi(P_1 \cup \cdots \cup P_m) = \varphi(P_1) + \cdots + \varphi(P_m)\\ &= \varphi(g_1 P_1) + \cdots + \varphi(g_m P_m) = \varphi(g_1 P_1 \cup \cdots \cup g_m P_m)\\ &= \varphi(Q_1 \cup \cdots \cup Q_m) = \varphi(Q).\end{aligned}$$

Dehn's negative answer to Hilbert's third problem can now be obtained as follows. We have to show that there are three-dimensional polytopes of equal volume that are not G_3-equidissectable. For this, we construct a simple, G_3-invariant valuation φ on $\mathcal{P}^3$ such that $\varphi(C) = 0$ for all cubes C and $\varphi(T) \neq 0$ for all regular tetrahedra T. Denote by $\mathcal{F}_1(P)$ the set of edges of $P \in \mathcal{P}^3$, by $V_1(F)$ the length of the edge $F \in \mathcal{F}_1(P)$, and by $\gamma(P, F)$ the outer angle of P at F. Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of Cauchy's functional equation

$$f(x + y) = f(x) + f(y) \qquad \text{for } x, y \in \mathbb{R} \tag{1.4}$$

which satisfies

$$f(\pi/2) = 0 \tag{1.5}$$

and

$$f(\alpha) \neq 0, \tag{1.6}$$

where α denotes the external angle of a regular tetrahedron T at one of its edges. That such a solution f exists, can be shown by using a Hamel basis of $\mathbb{R}$ and the fact that $\pi/2$ and α are rationally independent. Then we define

$$\varphi(P) := \sum_{F \in \mathcal{F}_1(P)} V_1(F) f(\gamma(P, F)) \qquad \text{for } P \in \mathcal{P}^3.$$

Because of (1.4), it can be shown that φ is weakly additive and hence a valuation, and as a consequence of (1.5) (which implies $f(\pi) = 0$) it is simple. Clearly, it is G_3-invariant. A cube C has outer angle $\pi/2$ at its edges, hence $\varphi(C) = 0$, whereas $\varphi(T) \neq 0$, due to (1.6). Now it follows from Lemma 1.10 that C and T cannot be G_3-equidissectable (even if they have the same volume). For this approach, see Hadwiger [13], and for an elementary exposition, Boltyanskii [6].

The interrelations between the dissection theory of polytopes and valuations have been developed in great depth. For a general account, we refer to the book of Sah [37] and to the survey articles [34, Sect. II] and [32, Sect. 4]. For a recent contribution, see Kusejko and Parapatits [27].

While Dehn's result shows that, in dimension $n \geq 3$, two polytopes of equal volume need not be G_n-equidissectable, the following result of Hadwiger [15] is rather surprising. The proof (following Hadwiger) can also be found in [42, Lemma 6.4.2]. The result plays a role in the further study of valuations.

Theorem 1.11 *Any two parallelotopes of equal volume in $\mathbb{R}^n$ are T_n-equidissectable.*

The first main goals of a further study of valuations on polytopes will be general properties of such valuations and representation or classification results, possibly under additional assumptions, such as invariance properties or continuity.

A further extension theorem can be helpful. As we have seen, the inclusion-exclusion formula is easy to use for simple valuations, but it is a bit clumsy in the general case. We can circumvent this by decomposing a polytope into a finite disjoint union of relatively open polytopes. A *relatively open polytope*, briefly *ro-polytope*, is the relative interior of a convex polytope. We denote the set of ro-polytopes in $\mathbb{R}^n$ by $\mathscr{P}^n_{\mathrm{ro}}$ and the set of finite unions of ro-polytopes by $\mathrm{U}(\mathscr{P}^n_{\mathrm{ro}})$. The elements of the latter are called *ro-polyhedra*. Every convex polytope $P \in \mathscr{P}^n$ is the disjoint union of the relative interiors of its faces (including P) and hence belongs to $\mathrm{U}(\mathscr{P}^n_{\mathrm{ro}})$.

Theorem 1.12 *Any weakly additive function on $\mathscr{P}^n$ with values in an Abelian group has an additive extension to* $\mathrm{U}(\mathscr{P}^n_{ro})$.

This can be deduced from Theorems 1.7 and 1.3; see [42, Corollary 6.2.4]. The result facilitates the proof of the following theorem, which is fundamental for many of the further investigations. Here φ is called *homogeneous* of degree r if

$$\varphi_r(\lambda P) = \lambda^r \varphi(P) \qquad \text{for all } P \in \mathscr{P}^n \text{ and all real } \lambda \geq 0,$$

and *rational homogeneous* of degree r if this holds for rational $\lambda \geq 0$.

Theorem 1.13 *Let φ be a translation invariant valuation on $\mathscr{P}^n$ with values in a rational vector space X. Then*

$$\varphi(\lambda P) = \sum_{r=0}^{n} \lambda^r \varphi_r(P) \qquad \textit{for } P \in \mathscr{P}^n \textit{ and rational } \lambda \geq 0. \tag{1.7}$$

Here $\varphi_r : \mathscr{P}^n \to X$ is a translation invariant valuation which is rational homogeneous of degree r ($r = 0, \ldots, n$).

Setting $\lambda = 1$ in (1.7) gives

$$\varphi = \varphi_0 + \cdots + \varphi_n, \tag{1.8}$$

which is known as the *McMullen decomposition*. It has the important consequence that for the investigation of translation invariant valuations on $\mathscr{P}^n$ with values in a rational vector space X one need only consider such valuations which are rational homogeneous of some degree $r \in \{0, \ldots, n\}$.

Another consequence of Theorem 1.13 is a polynomial expansion with respect to Minkowski addition. Recall that the Minkowski sum (or vector sum) of $K, L \in \mathscr{K}^n$ is defined by

$$K + L = \{x + y : x \in K,\ y \in L\},$$

and that $K + L \in \mathscr{K}^n$. A function φ from $\mathscr{K}^n$ to some Abelian group is *Minkowski additive* if

$$\varphi(K + L) = \varphi(K) + \varphi(L) \qquad \text{for all } K, L \in \mathscr{K}^n.$$

By repeatedly applying (1.7), it is not difficult to deduce the following.

Theorem 1.14 *Let $\varphi : \mathscr{P}^n \to X$ (with X a rational vector space) be a translation invariant valuation which is rational homogeneous of degree $m \in \{1, \ldots, n\}$. Then there is a polynomial expansion*

$$\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k) = \sum_{r_1, \ldots, r_k = 0}^{m} \binom{m}{r_1 \ldots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k}\, \overline{\varphi}(\underbrace{P_1, \ldots, P_1}_{r_1}, \ldots, \underbrace{P_k, \ldots, P_k}_{r_k}),$$

valid for all $P_1, \ldots, P_k \in \mathscr{P}^n$ and all rational $\lambda_1, \ldots, \lambda_k \geq 0$. Here $\overline{\varphi} : (\mathscr{P}^n)^m \to X$ is a symmetric mapping, which is translation invariant and Minkowski additive in each variable.

Historical Note The result of Theorem 1.13, even in a more general version, was stated by Hadwiger [12] (his first publication on valuations), as early as 1945, but without proof. His later work gives a proof of the decomposition (1.8) for simple valuations only, see [21, p. 54]. The question for a result as stated in Theorem 1.14 was posed by Peter McMullen, at an Oberwolfach conference in 1974. He gave a proof the same year, see [28, 29]. Different proofs were provided by Meier [35] and Spiegel [45]. A variation of Spiegel's proof, using Theorem 1.12 instead of the inclusion-exclusion formula, is found in [42, Sect. 6.3]. Proofs of more general versions of the polynomiality theorem were given by Pukhlikov and Khovanskii [36] and Alesker [1].

A consequence of Theorem 1.14 is the fact that a valuation $\varphi : \mathscr{P}^n \to \mathbb{R}$ that is translation invariant and rational homogeneous of degree 1 is Minkowski additive. A variant of this result was first proved by Spiegel [44].

We turn to representation results for translation invariant, real valuations on $\mathscr{P}^n$. Without additional assumptions, little is known about these. Setting $\lambda = 0$ in (1.7), we see that any such valuation which is homogeneous of degree zero, is constant. Then we mention two classical characterizations of the volume on polytopes, which are due to Hadwiger. The volume is denoted by V_n.

Theorem 1.15 *Let $\varphi : \mathscr{P}^n \to \mathbb{R}$ be a translation invariant valuation which is simple and nonnegative. Then $\varphi = cV_n$ with a constant c.*

The proof can be found in Hadwiger's book [21, Sect. 2.1.3]. The following result is also due to Hadwiger (see [21, p. 79]; also [42, Theorem 6.4.3]). The proof makes use of Theorem 1.11.

Theorem 1.16 *Let $\varphi : \mathscr{P}^n \to \mathbb{R}$ be a translation invariant valuation which is homogeneous of degree n. Then $\varphi = cV_n$ with a constant c.*

For translation invariant and *simple* valuations on polytopes, more general representations are possible. Under a weak continuity assumption, these go back to Hadwiger [18], and without that assumption to recent work of Kusejko and Parapatits [27]. We consider Hadwiger's result first, but use the terminology of [27].

For $k \in \{0, \dots, n\}$, let $\mathscr{U}^k$ denote the set of all ordered orthonormal k-tuples of vectors from the unit sphere $\mathbb{S}^{n-1}$. $\mathscr{U}^0$ contains only the empty tuple (). For $P \in \mathscr{P}^n$ and $u \in \mathbb{S}^{n-1}$, let $F(P, u)$ be the face of P with outer normal vector u. For $U = (u_1, \dots, u_k) \in \mathscr{U}^k$ and $P \in \mathscr{P}^n$ we define recursively the face P_U of P by

$$P_{()} := P, \quad P_{(u_1,\dots,u_r)} := F(P_{(u_1,\dots,u_{r-1})}, u_k), \qquad r = 1, \dots, k.$$

The orthonormal frame $U = (u_1, \dots, u_k) \in \mathscr{U}^k$ is *P-tight* if $\dim P_{(u_1,\dots,u_r)} = n - r$ for $r = 0, \dots, k$. Let $\mathscr{U}^k_P$ denote the (evidently finite) set of all P-tight frames in $\mathscr{U}^k$. Then $V_{n-k}(P_U) > 0$ for $U \in \mathscr{U}^k_P$, where V_{n-k} denotes the $(n-k)$-dimensional volume.

A function $f : \mathscr{U}^k \to \mathbb{R}$ is called *odd* if

$$f(\varepsilon_1 u_1, \dots, \varepsilon_k u_k) = \varepsilon_1 \cdots \varepsilon_k f(u_1, \dots, u_k)$$

for $\varepsilon_i = \pm 1$.

A valuation $\varphi : \mathscr{P}^n \to \mathbb{R}$ is *weakly continuous* if it is continuous under parallel displacements of the facets of a polytope. To make this more precise, we consider the set of polytopes whose system of outer normal vectors of facets belongs to a given finite set $U = \{u_1, \dots, u_m\}$; these vectors positively span $\mathbb{R}^n$. Now a function

φ on $\mathscr{P}^n$ is called weakly continuous if for any such U the function

$$(\eta_1, \ldots, \eta_m) \mapsto \varphi(\{x \in \mathbb{R}^n : x \cdot u_i \leq \eta_i,\ i = 1, \ldots, m\})$$

is continuous on the set of all $(\eta_1, \ldots, \eta_m)$ for which the argument of φ is not empty.

The following is Hadwiger's [18] result. For a version of his proof, we refer to [42, Theorem 6.4.6]. The proof given in [27] appears to be simpler. We write $\mathscr{U} := \bigcup_{k=0}^{n-1} \mathscr{U}^k$.

Theorem 1.17 *A function $\varphi : \mathscr{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant, simple valuation if and only if for each $U \in \mathscr{U}$ there is a constant $c_U \in \mathbb{R}$ such that $U \mapsto c_U$ is odd and*

$$\varphi(P) = \sum_{k=0}^{n-1} \sum_{U \in \mathscr{U}_P^k(P)} c_U V_{n-k}(P_U) \tag{1.9}$$

for $P \in \mathscr{P}^n$.

For non-simple valuations, the following result holds. As usual, $\mathscr{F}_r(P)$ denotes the set of r-dimensional faces of a polytope P, and $N(P, F)$ is the cone of normal vectors of P at its face F.

Theorem 1.18 *A function $\varphi : \mathscr{P}^n \to \mathbb{R}$ is a weakly continuous, translation invariant valuation if and only if there are a constant c and for each $r \in \{1, \ldots, n-1\}$ a simple real valuation θ_r on the system of convex polyhedral cones in $\mathbb{R}^n$ of dimension at most $n - r$ such that*

$$\varphi(P) = \varphi(\{0\}) + \sum_{r=1}^{n-1} \sum_{F \in \mathscr{F}_r(P)} \theta_r(N(P, F)) V_r(F) + cV_n(P) \tag{1.10}$$

for $P \in \mathscr{P}^n$.

McMullen [31] has deduced this from Hadwiger's result on simple valuations. For a different approach, see in [32] the remark after Theorem 5.19.

Satisfactory as these results are in the realm of polytopes, they seem, at present, not to lead much further in the investigation of continuous valuations on general convex bodies. Conditions on the functions θ_r, which do or do not allow a continuous extension of a valuation φ represented by (1.10) to general convex bodies, were investigated in [23].

Without the assumption of weak continuity, Kusejko and Parapatits [27] have obtained the following result.

Theorem 1.19 *A function $\varphi : \mathscr{P}^n \to \mathbb{R}$ is a translation invariant, simple valuation if and only if for each $U \in \mathscr{U}$ there exists an additive function $f_U : \mathbb{R} \to \mathbb{R}$ such that $U \mapsto f_U$ is odd and*

$$\varphi(P) = \sum_{k=0}^{n-1} \sum_{U \in \mathscr{U}_P^k(P)} f_U(V_{n-k}(P_U)) \tag{1.11}$$

for $P \in \mathscr{P}^n$.

The implications of this result for translative equidecomposability are explained in [27].

1.3 Examples of Valuations from Convex Geometry

The theory of convex bodies is replete with natural examples of valuations. We explain the most important of these, before turning to classification and characterization results.

A first example is given by the identity mapping $\mathscr{K}^n \to \mathscr{K}^n$. This makes sense, since $\mathscr{K}^n$, as usual equipped with Minkowski addition, is an Abelian semigroup with cancellation law. The identity mapping is a valuation, since the relation

$$(K \cup L) + (K \cap L) = K + L \tag{1.12}$$

holds for convex bodies $K, L \in \mathscr{K}^n$ with $K \cup L \in \mathscr{K}^n$ (as first pointed out by Sallee [38]; the easy proof can be found in [42, Lemma 3.1.1]). Consequently, also the support function defines a valuation. The *support function* $h(K, \cdot) = h_K$ of the convex body $K \in \mathscr{K}^n$ is defined by

$$h(K, u) := \max\{\langle u, x\rangle : x \in K\} \qquad \text{for } u \in \mathbb{R}^n.$$

The function h is Minkowski additive in the first argument. The Minkowski additivity of the support function together with (1.12) yields

$$h(K \cup L, \cdot) + h(K \cap L, \cdot) = h(K, \cdot) + h(L, \cdot) \qquad \text{if } K \cup L \text{ is convex,}$$

hence the map $K \mapsto h(K, \cdot)$, from $\mathscr{K}^n$ into (say) the vector space of real continuous functions on $\mathbb{R}^n$, is a valuation. Using the support function, the following can be shown (see, e.g., [42, Theorem 6.1.2] and, for the history, Note 2 on p. 332).

Theorem 1.20 *Every Minkowski additive function on $\mathscr{K}^n$ with values in an Abelian group is fully additive.*

Minkowski addition plays a role in valuation theory of convex bodies in more than one way. As one example, we mention a way to construct new valuations from a given one. Let φ be a valuation on $\mathcal{K}^n$. If $C \in \mathcal{K}^n$ is a fixed convex body, then

$$\varphi_C(K) := \varphi(K + C) \qquad \text{for } K \in \mathcal{K}^n$$

defines a new valuation φ_C on $\mathcal{K}^n$. If φ is translation invariant, then the same holds for φ_C.

A basic example of a valuation on $\mathcal{K}^n$ is, of course, the volume V_n. Being the restriction of a measure, the function $V_n : \mathcal{K}^n \to \mathbb{R}$ is a valuation, and since lower-dimensional convex bodies have volume zero, it is simple. Moreover, the valuation V_n is invariant under rigid motions and continuous (continuity of functions on $\mathcal{K}^n$ always refers to the Hausdorff metric). Via the construction (1.13) below, it gives rise to many other (non-simple) valuations. The following fact, which goes back to Minkowski at the beginning of the twentieth century, was, actually, the template for Theorem 1.14. There is a nonnegative, symmetric function $V : (\mathcal{K}^n)^n \to \mathbb{R}$, called the *mixed volume*, such that

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n})$$

for all $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$. (For proofs and more information, we refer to [42, Sect. 5.1].) We write

$$V(\underbrace{K_1, \ldots, K_1}_{r_1}, \ldots, \underbrace{K_m, \ldots, K_m}_{r_m}) =: V(K_1[r_1], \ldots, K_m[r_m]).$$

For arbitrary $p \in \{1, \ldots, n\}$ and fixed convex bodies $M_{p+1}, \ldots, M_n \in \mathcal{K}^n$, the function φ defined by

$$\varphi(K) := V(K[p], M_{p+1}, \ldots, M_n), \qquad K \in \mathcal{K}^n, \tag{1.13}$$

is a valuation on $\mathcal{K}^n$. It is translation invariant, continuous, and homogeneous of degree p. Often in the literature, these functionals φ are also called ‘mixed volumes’, but we find that slightly misleading (since the mixed volume is a function of n variables) and prefer to call them *mixed volume valuations*.

Of particular importance are the special cases of the mixed volume valuations where the fixed bodies are equal to the unit ball B^n. First we recall two frequently used constants: κ_n is the volume of the unit ball in $\mathbb{R}^n$ and ω_n is its surface area; explicitly,

$$\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}, \qquad \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{1.14}$$

We define

$$V_j(K) := \frac{\binom{n}{j}}{\kappa_{n-j}} V(K[j], B^n[n-j]) \tag{1.15}$$

for $K \in \mathcal{K}^n$. The functional V_j is called the j-th *intrinsic volume*. In addition to the properties that all mixed volume valuations share, it is invariant under rotations and thus under rigid motions. The normalizing factor has the effect that the intrinsic volume is independent of the dimension of the ambient space in which it is computed. In particular, if the convex body K has dimension $\dim K \leq m$, then $V_m(K)$ is the m-dimensional volume of K.

As a special case of the above approach to mixed volumes, we see that the intrinsic volumes are uniquely defined by the coefficients in the expansion

$$V_n(K + \rho B^n) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K), \qquad \rho \geq 0. \tag{1.16}$$

Here, $K + \rho B^n$ is the *outer parallel body* of K at distance $\rho \geq 0$, that is, the set of all points of $\mathbb{R}^n$ that have distance at most ρ from K. Equation (1.16) is known as the *Steiner formula*.

The concept of the parallel body can be localized. There is a local Steiner formula, which leads to measure-valued valuations. For this, we need a few more definitions. For $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$, there is a unique point $p(K, x) \in K$ with

$$\|x - p(K, x)\| \leq \|x - y\| \qquad \text{for all } y \in K.$$

The map $p(K, \cdot) : \mathbb{R}^n \to K$ is known as the *metric projection* of K. The map $K \mapsto p(K, x)$, for fixed x, is another example of a valuation, from $\mathcal{K}^n$ to $\mathbb{R}^n$. The distance of a point x from K is defined by $d(K, x) := \|x - p(K, x)\|$ and, for $x \in \mathbb{R}^n \setminus K$,

$$u(K, x) := \frac{x - p(K, x)}{d(K, x)}$$

denotes the unit vector pointing from $p(K, x)$ to x. The pair $(p(K, x), u(K, x))$ is a support element of K. Generally, a *support element* of K is a pair (x, u), where $x \in \operatorname{bd} K$ and u is an outer unit normal vector of K at x. The set $\mathbf{nc}(K)$ of all support elements of K is called the (generalized) *normal bundle* or the *normal cycle* of K. It is a subset of the product space

$$\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1} \tag{1.17}$$

(which is equipped with the product topology). Now for $\eta \in \mathcal{B}(\Sigma^n)$, the σ-algebra of Borel sets of Σ^n, for $K \in \mathcal{K}^n$ and $\rho > 0$, we define the *local parallel set*

$$M_\rho(K, \eta) := \{x \in \mathbb{R}^n : 0 < d(K, x) \leq \rho \text{ and } (p(K, x), u(K, x)) \in \eta\}.$$

This is a Borel set. By $\mathcal{H}^n$ we denote n-dimensional Hausdorff measure. Again, one has a polynomial expansion, namely

$$\mathcal{H}^n(M_\rho(K,\eta)) = \sum_{j=0}^{n-1} \rho^{n-j}\kappa_{n-j}\Lambda_j(K,\eta) \qquad \text{for } \rho \geq 0.$$

This defines finite Borel measures $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ on Σ^n. One calls $\Lambda_j(K,\cdot)$ the j-th *support measure* of K. From the valuation property of the nearest point map, one can deduce that

$$\Lambda_j(K \cup L,\cdot) + \Lambda_j(K \cap L,\cdot) = \Lambda_j(K,\cdot) + \Lambda_j(L,\cdot)$$

for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. Thus, the mapping $K \mapsto \Lambda_j(K,\cdot)$ is a valuation on $\mathcal{K}^n$, with values in the vector space of finite signed Borel measures on Σ^n.

From the support measures we get two series of marginal measures. They appear in the literature with two different normalizations. For Borel sets $\beta \subset \mathbb{R}^n$, we define

$$\frac{\binom{n}{j}}{n\kappa_{n-j}} C_j(K,\beta) = \Phi_j(K,\beta) := \Lambda_j(K, \beta \times \mathbb{S}^{n-1}).$$

The measures $C_0(K,\cdot),\dots,C_{n-1}(K,\cdot)$ are the *curvature measures* of K. They are measures on $\mathbb{R}^n$, concentrated on the boundary of K. The definition is supplemented by

$$\tfrac{1}{n} C_n(K,\beta) = \Phi_n(K,\beta) := \mathcal{H}^n(K \cap \beta).$$

For Borel sets $\omega \subset \mathbb{S}^{n-1}$, we define

$$\frac{\binom{n}{j}}{n\kappa_{n-j}} S_j(K,\omega) = \Psi_j(K,\omega) := \Lambda_j(K, \mathbb{R}^n \times \omega).$$

The measures $S_0(K,\cdot),\dots,S_{n-1}(K,\cdot)$ are the *area measures* of K. They are measures on the unit sphere $\mathbb{S}^{n-1}$.

1.4 Continuous Valuations on Convex Bodies

The continuous valuations on the space $\mathcal{K}^n$ of general convex bodies in $\mathbb{R}^n$ are of particular interest. These comprise those which have their values in a (here always real) topological vector space (such as $\mathbb{R}$, $\mathbb{R}^n$, tensor spaces, spaces of functions or measures) and are continuous with respect to the topology on $\mathcal{K}^n$ that is induced by the Hausdorff metric.

Before describing consequences of continuity, we wish to point out that general valuations on $\mathscr{K}^n$ can show rather irregular behaviour. For example, if we choose a non-continuous solution f of Cauchy's functional equation, $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$, then $\varphi := f \circ V_j$ with $j \in \{1, \dots, n\}$ is a valuation on $\mathscr{K}^n$ which is not continuous, in fact not even locally bounded, since f is unbounded on every nondegenerate interval. For $j = 1$, the function φ is Minkowski additive and hence, by Theorem 1.20, even fully additive.

As a first consequence of continuity, we mention another extension theorem of Groemer [11]. It needs only a weaker version of continuity. A function φ from $\mathscr{K}^n$ into some topological (Hausdorff) vector space is called *σ-continuous* if for every decreasing sequence $(K_i)_{i\in\mathbb{N}}$ in $\mathscr{K}^n$ one has

$$\lim_{i\to\infty} \varphi(K_i) = \varphi\Big(\bigcap_{i\in\mathbb{N}} K_i\Big).$$

If φ is continuous with respect to the Hausdorff metric, then it is σ-continuous. This follows from Lemma 1.8.2 in [42].

Theorem 1.21 (Groemer's Second Extension Theorem) *Let φ be a function on $\mathscr{K}^n$ with values in a topological vector space. If φ is weakly additive on $\mathscr{P}^n$ and is σ-continuous on $\mathscr{K}^n$, then φ has an additive extension to the lattice* $\mathrm{U}(\mathscr{K}^n)$.

Groemer's proof is reproduced in [43, Theorem 14.4.2]. The formulation of the theorem here is slightly more general, and we give a slightly shorter proof, based on the following lemma.

Lemma 1.22 *Let $K_1, \dots, K_m \in \mathscr{K}^n$ be convex bodies such that $K_1 \cup \dots \cup K_m$ is convex. Let $\varepsilon > 0$. Then there are polytopes $P_1, \dots, P_m \in \mathscr{P}^n$ with $K_i \subset P_i \subset K_i + \varepsilon B^n$ for $i = 1, \dots, m$ such that $P_1 \cup \dots \cup P_m$ is convex.*

For the proof and the subsequent argument, we refer to Weil [47, Lemma 8.1]. With this lemma, Theorem 1.21 can be proved as follows (following a suggestion of Daniel Hug). Let φ satisfy the assumptions of Theorem 1.21. Let $K_1, \dots, K_m \in \mathscr{K}^n$ be convex bodies such that $K_1 \cup \dots \cup K_m$ is convex. We apply Lemma 1.22 with K_i replaced by $K_i + 2^{-k}B^n$, $k \in \mathbb{N}$, and $\varepsilon = 2^{-k}$ (note that $\bigcup_{i=1}^m (K_i + 2^{-k}B^n) = \left(\bigcup_{i=1}^m K_i\right) + 2^{-k}B^n$ is convex). This yields polytopes $P_1^{(k)}, \dots, P_m^{(k)}$ with convex union and such that $K_i + 2^{-k}B^n \subset P_i^{(k)} \subset K_i + 2^{1-k}B^n$. Each sequence $(P_i^{(k)})_{k\in\mathbb{N}}$ is decreasing. By Theorem 1.7, the function φ is fully additive on $\mathscr{P}^n$, hence

$$\varphi(P_1^{(k)} \cup \dots \cup P_m^{(k)}) = \sum_{\emptyset \neq J \subset \{1,\dots,m\}} (-1)^{|J|-1} \varphi(P_J^{(k)}).$$

Since

$$\bigcap_{k\in\mathbb{N}} \big(P_1^{(k)} \cup \dots \cup P_m^{(k)}\big) = K_1 \cup \dots \cup K_m$$

and

$$\bigcap_{k\in\mathbb{N}} P_J^{(k)} = K_J \qquad \text{if } K_J \neq \emptyset,$$

the σ-continuity of φ yields

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{\emptyset\neq J\subset\{1,\dots,m\}} (-1)^{|J|-1}\varphi(K_J).$$

Thus, φ is fully additive on $\mathcal{K}^n$. By Theorem 1.3, it has an additive extension to $\mathrm{U}(\mathcal{K}^n)$. This proves Theorem 1.21.

The elements of the lattice $\mathrm{U}(\mathcal{K}^n)$, which has been termed the *convex ring*, are finite unions of convex bodies and are known as *polyconvex sets*.

It seems to be unknown whether every valuation on $\mathcal{K}^n$ (without a continuity assumption) has an additive extension to $\mathrm{U}(\mathcal{K}^n)$.

One consequence of Theorem 1.21 is the fact that the trivial valuation on $\mathcal{K}^n$, which is constantly equal to 1, has an additive extension to polyconvex sets. This extension is called the *Euler characteristic* and is denoted by χ, since it coincides, on this class of sets, with the equally named topological invariant. It should be mentioned that for the existence of the Euler characteristic on polyconvex sets, there is a very short and elegant proof due to Hadwiger [19]; it is reproduced in [42, Theorem 4.3.1].

Next, we point out that the polynomiality results from Sect. 1.2 can immediately be extended by continuity. Let φ be a translation invariant, continuous valuation on $\mathcal{K}^n$ with values in a topological vector space X. Then it follows from Theorem 1.13 that there are continuous, translation invariant valuations $\varphi_0,\dots,\varphi_n$ on $\mathcal{K}^n$, with values in X, such that φ_i is homogeneous of degree i ($i = 0,\dots,n$) and

$$\varphi(\lambda K) = \sum_{i=0}^{n} \lambda^i \varphi_i(K) \qquad \text{for } K \in \mathcal{K}^n \text{ and } \lambda \geq 0.$$

In particular, the McMullen decomposition $\varphi = \varphi_0 + \cdots + \varphi_n$ holds, where each φ_i has the same properties as φ and is, moreover, homogeneous (not only rationally homogeneous) of degree i.

If φ is, in addition, homogeneous of degree m, then it follows from Theorem 1.14 that there is a continuous symmetric mapping $\overline{\varphi} : (\mathcal{K}^n)^m \to X$ which is translation invariant and Minkowski additive in each variable, such that

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_k K_k)$$
$$= \sum_{r_1,\dots,r_k=0}^{m} \binom{m}{r_1 \dots r_k} \lambda_1^{r_1}\cdots\lambda_k^{r_k}\, \overline{\varphi}(\underbrace{K_1,\dots,K_1}_{r_1},\dots,\underbrace{K_k,\dots,K_k}_{r_k})$$

holds for all $K_1, \dots, K_k \in \mathscr{K}^n$ and all real $\lambda_1, \dots, \lambda_k \geq 0$. Further, one can deduce that for $r \in \{1, \dots, m\}$ the mapping

$$K \mapsto \overline{\varphi}(\underbrace{K, \dots, K}_{r}, M_{r+1}, \dots, M_m), \tag{1.18}$$

with fixed convex bodies $M_{r+1}, \dots, M_m$, is a continuous, translation invariant valuation, which is homogeneous of degree r.

Now that we have the classical examples of valuations on convex bodies at our disposal, we can have a look at the second historical incentive for the development of the theory of valuations. This came from the early history of integral geometry. In his booklet on integral geometry, Blaschke [5, Sect. 43], asked a question, which we explain here in a modified form. For convex bodies $K, M \in \mathscr{K}^n$, consider the 'kinematic integral'

$$\psi(K, M) := \int_{G_n} \chi(K \cap gM)\, \mu(\mathrm{d}g).$$

Here μ denotes the (suitably normalized) Haar measure on the motion group G_n, and χ is the Euler characteristic, that is, $\chi(K) = 1$ for $K \in \mathscr{K}^n$ and $\chi(\emptyset) = 0$. In other words, $\psi(K, M)$ is the rigid motion invariant measure of the set of all rigid motions g for which gM intersects K. There are different approaches to the computation of $\psi(K, M)$, and the result is that

$$\psi(K, M) = \sum_{i,j=0}^{n} c_{ij} V_i(K) V_j(M) \tag{1.19}$$

with explicit constants c_{ij}. This throws new light on the importance of the intrinsic volumes. Blaschke investigated this formula in a slightly different context (three-dimensional polytopal complexes). He made the important observation that some formal properties of the above functionals were essential for his proof of such formulas. Specifically, these were the valuation property, rigid motion invariance and, in his case, the local boundedness. He claimed that these properties characterize, 'to a certain extent', the linear combinations of intrinsic volumes. He proved a result in this direction, where, however, he had to introduce an additional assumption in the course of the proof, namely the invariance under volume preserving affine transformations for the 'volume part' of his considered functional. Whether a characterization theorem for valuations on polyhedra satisfying Blaschke's original conditions is possible, seems to be unknown. Later, Hadwiger considered valuations on general convex bodies and introduced the assumption of continuity. The following is his celebrated characterization theorem.

Theorem 1.23 (Hadwiger's Characterization Theorem) *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation, then there are constants $c_0, \dots, c_n$ such that*

$$\varphi(K) = \sum_{j=0}^{n} c_j V_j(K)$$

for all $K \in \mathcal{K}^n$.

For the three-dimensional case, Hadwiger gave a proof in [16], and for general dimensions in [17]; his proof is also found in his book [21, Sect. 6.1.10]. Hadwiger expressed repeatedly ([14, p. 346], and [16], footnote 3 on p. 69) that a characterization theorem for the intrinsic volumes with the assumption of local boundedness instead of continuity would be desirable. However, the following counterexample, given in [34, p. 239], shows that this is not possible. For $K \in \mathcal{K}^n$, let

$$\varphi(K) := \sum_{u \in \mathbb{S}^{n-1}} \mathcal{H}^{n-1}(F(K, u)),$$

where $F(K, u)$ is the support set of K with outer normal vector u. This has non-zero $\mathcal{H}^{n-1}$ measure for at most countably many vectors u, hence the sum is well-defined, and its value is bounded by the surface areaof K. Thus, φ is a rigid motion invariant valuation which is locally bounded, but it is not continuous and hence not a linear combination of intrinsic volumes.

Hadwiger showed in [14, 20] how his theorem immediately leads to integral-geometric results. For instance, to prove (1.19), one notes that for fixed K the function $\psi(K, \cdot)$ satisfies the assumptions of Theorem 1.23 and hence is a linear combination of the intrinsic volumes of the variable convex body, with real constants that are independent of this body, thus $\psi(K, M) = \sum_{j=0}^{n} c_j(K) V_j(M)$. Then one repeats the argument with variable K and obtains that ψ must be of the form $\psi(K, M) = \sum_{i,j=0}^{n} c_{ij} V_i(K) V_j(M)$. The constants c_{ij} can then be determined by applying the obtained formula to balls of different radii. There are also different approaches to integral geometric formulas. For one result, however, called 'Hadwiger's general integral-geometric theorem' (it is reproduced in [43, Theorem 5.1.2]), the proof via the characterization theorem is the only one known.

Hadwiger's proof of his characterization theorem used a fair amount of dissection theory of polytopes. A slightly simplified version of his proof was published by Chen [7]. A considerably shorter, elegant proof of Hadwiger's theorem is due to Klain [24]. This proof is reproduced in the book by Klain and Rota [26], which presents a neat introduction to integral geometry, with some emphasis on discrete aspects. Klain's proof is also reproduced in [42, Theorem 6.4.14].

An essential aspect of Hadwiger's characterization theorem is the fact that the real vector space spanned by the continuous, motion invariant real valuations on $\mathcal{K}^n$ has finite dimension. This is no longer true if the valuations under consideration are only translation invariant. We turn to these valuations, whose investigation is a central part of the theory. By Val we denote the real vector space of translation invariant, continuous real valuations on $\mathcal{K}^n$, and by Val_m the subspace of valuations that are homogeneous of degree m. The McMullen decomposition tells us that

$$\mathrm{Val} = \bigoplus_{m=0}^{n} \mathrm{Val}_m .$$

Further, a valuation φ (on $\mathcal{K}^n$ or $\mathcal{P}^n$) is called *even* (*odd*) if $\varphi(-K) = \varphi(K)$ (respectively, $\varphi(-K) = -\varphi(K)$) holds for all K in the domain of φ. We denote by Val^+ and Val^- the subspace of even, respectively odd, valuations in Val, and Val_m^+ and Val_m^- are the corresponding subspaces of m-homogeneous valuations. Since we can always write

$$\varphi(K) = \tfrac{1}{2}(\varphi(K) + \varphi(-K)) + \tfrac{1}{2}(\varphi(K) - \varphi(-K)),$$

we have

$$\mathrm{Val}_m = \mathrm{Val}_m^+ \oplus \mathrm{Val}_m^- .$$

It would be nice to have a simple explicit description of the valuations in each space Val_m. Only special cases are known. So it follows from the results on polytopes (Theorem 1.16, in particular), together with continuity, that the spaces Val_m are one-dimensional for $m = 0$ and $m = n$.

Corollary 1.24 *The space* Val_0 *is spanned by the Euler characteristic, and the space* Val_n *by the volume functional.*

An explicit description is also known for the elements of Val_{n-1}. The following result is due to McMullen [30].

Theorem 1.25 *Each* $\varphi \in \mathrm{Val}_{n-1}$ *has a representation*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \textit{for } K \in \mathcal{K}^n,$$

with a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$. *This function is uniquely determined up to adding the restriction of a linear function.*

More complete results are known for simple valuations. The following result of Klain [24] was an essential step in his proof of Hadwiger's characterization theorem.

Theorem 1.26 (Klain's Volume Characterization) *If* $\varphi \in \mathrm{Val}^+$ *is simple, then* $\varphi(K) = cV_n(K)$ *for* $K \in \mathcal{K}^n$*, with some constant* c.

A counterpart for odd simple valuations was proved in [41] (the proof can also be found in [42, Theorem 6.4.13]):

Theorem 1.27 *If $\varphi \in \mathrm{Val}^-$ is simple, then*

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} g(u)\, S_{n-1}(K, \mathrm{d}u) \qquad \text{for } K \in \mathscr{K}^n,$$

with an odd continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$.

A different approach to Theorems 1.25 and 1.27 was provided by Kusejko and Parapatits [27].

Klain's volume characterization (Theorem 1.26) has a consequence for even valuations, which has turned out to be quite useful. By $G(n, m)$ we denote the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^n$. Now let $m \in \{1, \dots, n-1\}$, and let $\varphi \in \mathrm{Val}_m$. Let $L \in G(n, m)$. It follows from Corollary 1.24 that the restriction of φ to the convex bodies in L is a constant multiple of the m-dimensional volume. Thus, $\varphi(K) = c_\varphi(L) V_m(K)$ for the convex bodies $K \subset L$, where $c_\varphi(L)$ is a real constant. Since φ is continuous, this defines a continuous function c_φ on $G(n, m)$. It is called the *Klain function* of the valuation φ. This function determines even valuations uniquely, as Klain [25] has deduced from his volume characterization.

Theorem 1.28 *A valuation in Val_m^+ ($m \in \{1, \dots, n-1\}$) is uniquely determined by its Klain function.*

The proofs given by Klain for Theorems 1.26 and 1.28 are reproduced in [42, Theorems 6.4.10 and 6.4.11].

1.5 Measure-Valued Valuations

We leave the translation invariant, real valuations and turn to some natural extensions of the intrinsic volumes. We have already seen the measure-valued localizations of the intrinsic volumes, the support, curvature, and area measures. Another natural extension (in the next chapter) will be that from real-valued to vector- and tensor-valued functions. In both cases, invariance (or rather, equivariance) properties with respect to the group of rigid motions play an important role.

First we recall that with each convex body $K \in \mathscr{K}^n$ we have associated its support measures

$$\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$$

and, by marginalization and renormalization, the curvature measures $C_j(K, \cdot)$ and the area measures $S_j(K, \cdot)$, $j = 0, \dots, n-1$. Each mapping $K \mapsto \Lambda_j(K, \cdot)$ is a valuation, with values in the vector space of finite signed Borel measures on $\Sigma^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$, and it is weakly continuous. The latter means that $K_i \to K$ in the Hausdorff metric implies $\Lambda_j(K_i, \cdot) \xrightarrow{w} \Lambda_j(K, \cdot)$, where the weak convergence $\xrightarrow{w}$ is equivalent to

$$\lim_{i\to\infty} \int_{\Sigma^n} f \, \mathrm{d}\Lambda_j(K_i, \cdot) = \int_{\Sigma^n} f \, \mathrm{d}\Lambda_j(K, \cdot)$$

for every continuous function $f : \Sigma^n \to \mathbb{R}$. The measure $\Lambda_j(K, \cdot)$ is concentrated on the normal bundle $\mathbf{nc}(K)$ of K. The valuation property and the weak continuity carry over to the mappings C_j and S_j. The measure $C_j(K, \cdot)$ is a Borel measure on $\mathbb{R}^n$, concentrated on $\mathrm{bd}\, K$ for $j \le n-1$. The area measure $S_j(K, \cdot)$ is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$.

The behaviour of these measures under the motion group is as follows. First, if $g \in G_n$, we denote the rotation part of g by g_0 (that is, $gx = g_0x + t$ for all $x \in \mathbb{R}^n$, with a fixed translation vector t). Then we define

$$g\eta := \{(gx, g_0u) : (x, u) \in \eta\} \qquad \text{for } \eta \subset \Sigma^n,$$
$$g\beta := \{gx : x \in \beta\} \qquad \text{for } \beta \subset \mathbb{R}^n,$$
$$g\omega := \{g_0u : u \in \omega\} \qquad \text{for } \omega \subset \mathbb{S}^{n-1}.$$

For $K \in \mathcal{K}^n$, $g \in G_n$ and Borel sets $\eta \subset \Sigma^n$, $\beta \subset \mathbb{R}^n$ and $\omega \subset \mathbb{S}^{n-1}$ we then have

$$\Lambda_j(gK, g\eta) = \Lambda_j(K, \eta), \quad C_j(gK, g\beta) = C_j(K, \beta), \quad S_j(gK, g\omega) = S_j(K, \omega).$$

In each case, we talk of this behaviour as *rigid motion equivariant*.

One may ask whether, for these measure-valued extensions of the intrinsic volumes, there are classification results similar to Hadwiger's characterization theorem. It turns out that in addition to the valuation, equivariance, and continuity properties we need, because we are dealing with measures, some assumption of local determination, saying roughly that the value of the relevant measure of K at a Borel set α depends only on a local part of K determined by α. With an appropriate assumption of this kind, the following characterization theorems have been obtained. If $\varphi(K)$ is a measure, we write here $\varphi(K)(\alpha) =: \varphi(K, \alpha)$.

Theorem 1.29 *Let φ be a map from $\mathcal{K}^n$ into the set of finite Borel measures on $\mathbb{R}^n$, satisfying the following conditions.*

(a) *φ is a valuation;*
(b) *φ is rigid motion equivariant;*
(c) *φ is weakly continuous;*
(d) *φ is locally determined, in the following sense: if $\beta \subset \mathbb{R}^n$ is open and $K \cap \beta = L \cap \beta$, then $\varphi(K, \beta') = \varphi(L, \beta')$ for every Borel set $\beta' \subset \beta$.*

Then there are real constants $c_0, \dots, c_n \geq 0$ *such that*

$$\varphi(K, \beta) = \sum_{i=0}^{n} c_i C_i(K, \beta)$$

for $K \in \mathcal{K}^n$ *and* $\beta \in \mathcal{B}(\mathbb{R}^n)$.

In the following theorem, $\tau(K, \omega)$ denotes the inverse spherical image of K at ω, that is, the set of all boundary points of the convex body K at which there is an outer normal vector belonging to the given set $\omega \subset \mathbb{S}^{n-1}$.

Theorem 1.30 *Let* φ *be a map from* $\mathcal{K}^n$ *into the set of finite signed Borel measures on* $\mathbb{S}^{n-1}$, *satisfying the following conditions.*

(a) φ *is a valuation;*
(b) φ *is rigid motion equivariant;*
(c) φ *is weakly continuous;*
(d) φ *is locally determined, in the following sense: if* $\omega \subset \mathbb{S}^{n-1}$ *is a Borel set and if* $\tau(K, \omega) = \tau(L, \omega)$, *then* $\varphi(K, \omega) = \varphi(L, \omega)$.

Then there are real constants $c_0, \dots, c_{n-1}$ *such that*

$$\varphi(K, \omega) = \sum_{i=0}^{n-1} c_i S_i(K, \omega)$$

for $K \in \mathcal{K}^n$ *and* $\omega \in \mathcal{B}(\mathbb{S}^{n-1})$.

Theorem 1.29 was proved in [40] and Theorem 1.30 in [39]. The following result is due to Glasauer [10].

Theorem 1.31 *Let* φ *be a map from* $\mathcal{P}^n$ *into the set of finite signed Borel measures on* Σ^n, *satisfying the following conditions.*

(a) φ *is rigid motion equivariant;*
(b) φ *is locally determined, in the following sense: if* $\eta \in \mathcal{B}(\Sigma^n)$ *and* $K, L \in \mathcal{K}^n$ *satisfy* $\eta \cap \mathbf{nc}(K) = \eta \cap \mathbf{nc}(L)$, *then* $\varphi(K, \eta) = \varphi(L, \eta)$.

Then there are real constants $c_0, \dots, c_{n-1}$ *such that*

$$\varphi(K, \eta) = \sum_{j=0}^{n-1} c_j \Lambda_j(K, \eta)$$

for $K \in \mathcal{K}^n$ *and* $\eta \in \mathcal{B}(\Sigma^n)$.

Here the valuation property has not been forgotten! Indeed, the last theorem has a character different from the two previous ones: the assumption that $\varphi(K, \cdot)$ is a locally determined measure on Σ^n, is strong enough to allow a simpler proof, without assuming the valuation property. The latter point will be important in the treatment of local tensor valuations (in Chap. 2).

1.6 Valuations on Lattice Polytopes

We denote by $\mathscr{P}(\mathbb{Z}^n)$ the set of all polytopes with vertices in $\mathbb{Z}^n$. In contrast to $\mathscr{P}^n$ and $\mathscr{K}^n$ considered so far, $\mathscr{P}(\mathbb{Z}^n)$ is not an intersectional family. For that reason, we modify the definition of a valuation in this case and say that a mapping φ from $\mathscr{P}(\mathbb{Z}^n)$ into some Abelian group is a valuation if

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q) \tag{1.20}$$

holds whenever $P, Q, P \cup Q, P \cap Q \in \mathscr{P}(\mathbb{Z}^n)$; moreover, we define that $\emptyset \in \mathscr{P}(\mathbb{Z}^n)$ and assume that $\varphi(\emptyset) = 0$. In a similar vein, we say that φ satisfies the *inclusion-exclusion principle* if

$$\varphi(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1} \varphi(A_J)$$

holds whenever $m \in \mathbb{N}$, $A_1 \cup \cdots \cup A_m \in \mathscr{P}(\mathbb{Z}^n)$ and $A_J \in \mathscr{P}(\mathbb{Z}^n)$ for all nonempty $J \subset \{1, \ldots, m\}$. Further, a valuation φ on $\mathscr{P}(\mathbb{Z}^n)$ is said to have the *extension property* if there is a function $\widetilde{\varphi}$ on the family of finite unions of polytopes in $\mathscr{P}(\mathbb{Z}^n)$ such that

$$\widetilde{\varphi}(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1} \varphi(A_J)$$

whenever $A_J \in \mathscr{P}(\mathbb{Z}^n)$ for all nonempty $J \subset \{1, \ldots, m\}$. The following theorem was proved by McMullen [33].

Theorem 1.32 *A valuation on $\mathscr{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle and has the extension property.*

For polytopes in $\mathscr{P}(\mathbb{Z}^n)$, the natural counterpart to the volume functional is the *lattice point enumerator L*. It is defined by

$$L(P) := \operatorname{card}(P \cap \mathbb{Z}^n) \qquad \text{for } P \in \mathscr{P}(\mathbb{Z}^n).$$

It was first proved by Ehrhart [9] that there is a polynomial expansion

$$L(kP) = \sum_{i=0}^{n} k^i L_i(P), \qquad P \in \mathscr{P}(\mathbb{Z}^n),\ k \in \mathbb{N}. \tag{1.21}$$

We refer to the surveys [32, 34] for information about how this fact embeds into the general polynomiality theorems proved later.

The expansion (1.21) defines valuations $L_0, \dots, L_n (= V_n)$ on $\mathscr{P}(\mathbb{Z}^n)$, which are invariant under unimodular transformations, that is, volume preserving affine maps of $\mathbb{R}^n$ into itself that leave $\mathbb{Z}^n$ invariant. A result of Betke [2] (see also Betke and Kneser [3]), together with Theorem 1.32, gives the following characterization theorem.

Theorem 1.33 *If φ is a real valuation on $\mathscr{P}(\mathbb{Z}^n)$ which is invariant under unimodular transformations, then*

$$\varphi(P) = \sum_{i=0}^{n} c_i L_i(P) \qquad \text{for } P \in \mathscr{P}(\mathbb{Z}^n),$$

with real constants $c_0, \dots, c_n$.

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Chapter 2
Tensor Valuations and Their Local Versions

Daniel Hug and Rolf Schneider

Abstract The intrinsic volumes, recalled in the previous chapter, provide an array of size measurements for a convex body, one for each integer degree of homogeneity from 0 to n. For measurements and descriptions of other aspects, such as position, moments of the volume and of other size functionals, or anisotropy, tensor-valued functionals on convex bodies are useful. The classical approach leading to the intrinsic volumes, namely the Steiner formula for parallel bodies, can be extended by replacing the volume by higher moments of the volume. This leads, in a natural way, to a series of tensor-valued valuations. These so-called Minkowski tensors are introduced in the present chapter, and their properties are studied. A version of Hadwiger's theorem for tensor valuations is stated. The next natural step is a localization of the Minkowski tensors, in the form of tensor-valued measures. The essential valuation, equivariance and continuity properties of these local Minkowski tensors are collected. The main goal is then a description of the vector space of all tensor valuations on convex bodies sharing these properties. Continuity properties of local Minkowski tensors and of support measures follow from continuity properties of normal cycles of convex bodies. We establish Hölder continuity of the normal cycles of convex bodies, which provides a quantitative improvement of the aforementioned continuity property.

2.1 The Minkowski Tensors

We use the notation introduced in Chap. 1. We recall that the intrinsic volumes, certainly the most important valuations in the theory of convex bodies in Euclidean space, all arise from one basic valuation, the volume functional. In fact, they are

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generated by the Steiner formula (1.16),

$$V_n(K + \rho B^n) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K), \qquad \rho \geq 0. \tag{2.1}$$

Here and in the following, $K \in \mathscr{K}^n$ denotes a convex body. The point to be kept in mind is that the evaluation of the volume of parallel bodies leads to a polynomial expansion and that the coefficients yield new valuations, which inherit some essential properties of the volume functional, but are no longer simple.

The volume functional, which we may write as

$$V_n(K) = \int_K \mathrm{d}x,$$

where dx indicates integration with respect to Lebesgue measure, has a natural vector-valued analogue, the moment vector

$$\int_K x \,\mathrm{d}x,$$

which is needed to define the centre of gravity,

$$c(K) := \frac{1}{V_n(K)} \int_K x \,\mathrm{d}x,$$

of convex bodies K with positive volume. If one wants to study moments of inertia, for example, one has to consider matrices with entries of type

$$\int_K \xi_i \xi_j \,\mathrm{d}x,$$

where $\xi_1, \dots, \xi_n$ are the coordinates of $x \in \mathbb{R}^n$ with respect to an orthonormal basis. This can be continued and leads to a series of simple valuations with values in spaces of symmetric tensors. Application to parallel bodies and polynomial expansion then reveals more general tensor-valued valuations. In the present section, we introduce these tensor valuations.

First we fix some conventions how to deal with tensors. We use the scalar product of $\mathbb{R}^n$ to identify $\mathbb{R}^n$ with its dual space. Thus, each vector $a \in \mathbb{R}^n$ is identified with the linear functional $x \mapsto a \cdot x$ from $\mathbb{R}^n$ to $\mathbb{R}$. For $r \in \mathbb{N}_0$, an *r-tensor*, or tensor of rank r, on $\mathbb{R}^n$ is defined as an r-linear mapping from $(\mathbb{R}^n)^r$ to $\mathbb{R}$. It is *symmetric* if it is invariant under permutations of its arguments. By $\mathbb{T}^r$ we denote the real vector space (with its standard topology) of symmetric r-tensors on $\mathbb{R}^n$. By definition, $\mathbb{T}^0 = \mathbb{R}$, and by the identification made above, $\mathbb{T}^1 = \mathbb{R}^n$. The *symmetric tensor product* of the symmetric tensors $a_i \in \mathbb{T}^{r_i}$, $i = 1, \dots, k$, is defined as follows. We write $s_0 = 0$,

$s_i = r_1 + \cdots + r_i$ for $i = 1, \ldots, k$, then

$$(a_1 \odot \cdots \odot a_k)(x_1, \ldots, x_{s_k}) := \frac{1}{s_k!} \sum_{\sigma \in \mathscr{S}(s_k)} \prod_{i=1}^{k} a_i(x_{\sigma(s_{i-1}+1)}, \ldots, x_{\sigma(s_i)})$$

for $x_1, \ldots, x_{s_k} \in \mathbb{R}^n$, where $\mathscr{S}(m)$ denotes the group of permutations of the set $\{1, \ldots, m\}$. Then $a_1 \odot \cdots \odot a_k \in \mathbb{T}^{r_1+\cdots+r_k}$. Thus the space of symmetric tensors (of arbitrary rank) becomes an associative, commutative graded algebra with unit. We shall always use the abbreviations $a \odot b =: ab$,

$$a_1 \odot \cdots \odot a_k =: a_1 \cdots a_k, \qquad \underbrace{a \odot \cdots \odot a}_{r} =: a^r, \qquad a^0 := 1.$$

For instance, for a vector $a \in \mathbb{R}^n$, the r-tensor a^r with $r \geq 1$ is given by

$$a^r(x_1, \ldots, x_r) = (a \cdot x_1) \cdots (a \cdot x_r), \qquad x_1, \ldots, x_r \in \mathbb{R}^n.$$

The scalar product,

$$Q(x, y) = x \cdot y, \qquad x, y \in \mathbb{R}^n,$$

is a symmetric tensor of rank two; we call Q the *metric tensor*.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$. Then the tensors $e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1 \leq \cdots \leq i_r \leq n$ form a basis of $\mathbb{T}^r$. The corresponding coordinate representation of $T \in \mathbb{T}^r$ is given by

$$T = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} t_{i_1 \ldots i_r} e_{i_1} \cdots e_{i_r} \tag{2.2}$$

with

$$t_{i_1 \ldots i_r} = \binom{r}{m_1 \ldots m_n} T(e_{i_1}, \ldots, e_{i_r}), \tag{2.3}$$

where m_k counts how often the number k appears among the indices $i_1, \ldots, i_r$ ($k = 1, \ldots, n$). (We remark that (2.3) should replace the formula given in [16, p. 463, line −8].)

Now we define the moment tensors, which generalize the volume. Integrals of tensor-valued functions can, of course, be defined coordinate-wise. For $r \in \mathbb{N}_0$, let

$$\Psi_r(K) := \frac{1}{r!} \int_K x^r \,\mathrm{d}x, \qquad K \in \mathscr{K}^n. \tag{2.4}$$

Thus, $\Psi_r(K) \in \mathbb{T}^r$, and explicitly

$$\Psi_r(K)(y_1,\dots,y_r) = \frac{1}{r!}\int_K (x\cdot y_1)\cdots(x\cdot y_r)\,\mathrm{d}x$$

for $y_1,\dots,y_r \in \mathbb{R}^n$. The factor $1/r!$ in (2.4) is only for convenience. It is clear that $\Psi_r : \mathscr{K}^n \to \mathbb{T}^r$ is a simple valuation.

Immediately from (2.4) we see how Ψ_r behaves under translations. Since the binomial theorem holds for the symmetric tensor product, for $t \in \mathbb{R}^n$ we get

$$\Psi_r(K+t) = \sum_{j=0}^{r} \frac{1}{j!}\Psi_{r-j}(K)t^j. \tag{2.5}$$

Formally, this looks like an ordinary polynomial, but we have to keep in mind that here, according to our notational conventions,

$$\Psi_{r-j}(K)t^j = \Psi_{r-j}(K)\odot \underbrace{t\odot\cdots\odot t}_{j}.$$

Nevertheless, in view of (2.5) one says that Ψ_r has *polynomial behaviour* under translations.

Also the behaviour under rotations is easy to see. Let $\mathrm{O}(n)$ be the orthogonal group of $\mathbb{R}^n$. Its elements are called *rotations* of $\mathbb{R}^n$; thus, rotations in our terminology can be proper (orientation preserving) or improper. For $\vartheta \in \mathrm{O}(n)$ and for $y_1,\dots,y_r \in \mathbb{R}^n$ we have

$$\begin{aligned}
\Psi_r(\vartheta K)(y_1,\dots,y_r) &= \frac{1}{r!}\int_{\vartheta K}(x\cdot y_1)\cdots(x\cdot y_r)\,\mathrm{d}x\\
&= \frac{1}{r!}\int_K(\vartheta x\cdot y_1)\cdots(\vartheta x\cdot y_r)\,\mathrm{d}x\\
&= \frac{1}{r!}\int_K(x\cdot \vartheta^{-1}y_1)\cdots(x\cdot \vartheta^{-1}y_r)\,\mathrm{d}x\\
&= \Psi_r(K)(\vartheta^{-1}y_1,\dots,\vartheta^{-1}y_r) = (\vartheta\Psi_r(K))(y_1,\dots,y_r).
\end{aligned}$$

Thus,

$$\Psi_r(\vartheta K) = \vartheta\Psi_r(K),$$

where the usual operation of $\mathrm{O}(n)$ on $\mathbb{T}^r$ is defined by

$$(\vartheta a)(y_1,\dots,y_r) = a(\vartheta^{-1}y_1,\dots,\vartheta^{-1}y_r)$$

for $a \in \mathbb{T}^r$.

The tensor functional Ψ_r also satisfies a Steiner formula. To express it in a convenient way, we have to introduce further tensor functionals. In the following, we use the support measures Λ_k (see Sect. 1.3), which are Borel measures on $\Sigma^n = \mathbb{R}^n \times \mathbb{S}^{n-1}$. The constants κ_j, ω_j were introduced in (1.14).

Definition 2.1 The *Minkowski tensors* are defined by

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u)) \tag{2.6}$$

for $k \in \{0, \dots, n-1\}$ and $r, s \in \mathbb{N}_0$. Further, we define

$$\Phi_n^{r,0}(K) := \Psi_r(K) \tag{2.7}$$

and

$$\Phi_k^{r,s} := 0 \qquad \text{if } k \notin \{0, \dots, n\} \text{ or } r \notin \mathbb{N}_0 \text{ or } s \notin \mathbb{N}_0 \text{ or } k = n, s \neq 0.$$

The latter definition will allow us later to extend some summations formally over all nonnegative integers.

Now we can formulate a Steiner-type formula.

Theorem 2.2 *For $r \in \mathbb{N}_0$, $K \in \mathscr{K}^n$ and $\rho \geq 0$, the formula*

$$\Psi_r(K + \rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} V_k^{(r)}(K) \tag{2.8}$$

holds, where

$$V_k^{(r)} = \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s}. \tag{2.9}$$

For $r = 0$, formula (2.8) reduces to the ordinary Steiner formula (2.1) for the volume.

We indicate the proof of formula (2.8). For this, we need to compute an integral $\int_{\mathbb{R}^n} f(x) \, \mathrm{d}x$ by a procedure that generalizes the transformation to polar coordinates, with the role of the unit sphere played by the boundary of a general convex body. Since such a general convex body need neither be smooth nor strictly convex, this generalized transformation formula makes use of the support measures. These satisfy themselves a Steiner formula [17, Theorem 4.2.7], of which here the following special case is relevant. We write $K_\rho := K + \rho B^n$, for $\rho \geq 0$, and define the mapping

$$\tau_\rho : \Sigma^n \to \Sigma^n, \qquad \tau_\rho(x,u) := (x + \rho u, u).$$

Then

$$2\Lambda_{n-1}(K_\rho, \cdot) = \sum_{k=0}^{n-1} \rho^{n-k-1} \omega_{n-k} \tau_\rho \Lambda_k(K, \cdot),$$

where $\tau_\rho \Lambda_k(K, \cdot)$ is the image measure (pushforward) of $\Lambda_k(K, \cdot)$ under τ_ρ. Using this, the following formula can be proved [17, Theorem 4.2.8].

Lemma 2.3 *Let $K \in \mathscr{K}^n$, and let $f : \mathbb{R}^n \setminus K \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int_{\mathbb{R}^n \setminus K} f(x)\, \mathrm{d}x = \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\infty t^{n-j-1} \int_{\Sigma^n} f(x + tu)\, \Lambda_j(K, \mathrm{d}(x, u))\, \mathrm{d}t. \tag{2.10}$$

To prove now formula (2.8), we first write

$$\Psi_r(K_\rho) = \Psi_r(K) + \frac{1}{r!} \int_{K_\rho \setminus K} x^r\, \mathrm{d}x.$$

To the last term we apply the transformation (2.10) coordinate-wise and obtain

$$\begin{aligned}
\int_{K_\rho \setminus K} x^r\, \mathrm{d}x &= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_{\Sigma^n} (x + tu)^r\, \Lambda_j(K, \mathrm{d}(x, u))\, \mathrm{d}t \\
&= \sum_{j=0}^{n-1} \omega_{n-j} \int_0^\rho t^{n-j-1} \int_{\Sigma^n} \sum_{s=0}^{r} \binom{r}{s} x^{r-s} u^s t^s\, \Lambda_j(K, \mathrm{d}(x, u))\, \mathrm{d}t \\
&= \sum_{j=0}^{n-1} \sum_{s=0}^{r} \omega_{n-j} \binom{r}{s} \frac{\rho^{n-j+s}}{n-j+s} \int_{\Sigma^n} x^{r-s} u^s\, \Lambda_j(K, \mathrm{d}(x, u)).
\end{aligned}$$

Introducing the index $k = j + r - s$ and using the definition (2.6), we obtain the assertion (2.8).

2.2 A Classification of Tensor Valuations

To describe our next goals, we recall Hadwiger's characterization theorem, that is, Theorem 1.23. It determines the real vector space of all mappings $\varphi : \mathscr{K}^n \to \mathbb{R}$ which are

- valuations,
- rigid motion invariant,
- continuous.

Hadwiger proved that this vector space has dimension $n + 1$ and is spanned by the intrinsic volumes $V_0, \dots, V_n$. These intrinsic volume functionals are linearly independent, because they have different degrees of homogeneity and are not identically zero.

As the intrinsic volumes have been generalized to Minkowski tensors, it is natural to ask whether, respectively in which form, Hadwiger's characterization theorem can be extended. For tensor valuations of rank one, there is a closely analogous result.

Theorem 2.4 *The real vector space of all mappings $\psi : \mathscr{K}^n \to \mathbb{R}^n$ which are*

- *valuations,*
- *rotation equivariant, and such that $\psi(K + t) - \psi(K)$ is parallel to t,*
- *continuous,*

is spanned by the mappings

$$K \mapsto \int_K x\, C_j(K, \mathrm{d}x), \qquad j = 0, \dots, n.$$

Recall from Sect. 1.3 the relation between the support measures $\Lambda_j(K, \cdot)$ and the curvature measures $C_j(K, \cdot)$. The integral $\int_K x\, C_j(K, \mathrm{d}x)$ is the moment vector of the curvature measure $C_j(K, \cdot)$. Again, the vector space in question has dimension $n+1$, because the moment vectors have different degrees of homogeneity and are not identically zero. The result was proved by Hadwiger and Schneider [7]. Although it looks similar to Hadwiger's characterization theorem, its proof uses a different approach. One might wonder why the dimension of the vector space is still $n + 1$. The Steiner formula for the moment vector $\int_K x\, \mathrm{d}x$ has, in fact, $n+2$ terms. However, one of these, namely $\int_{B^n} x\, \mathrm{d}x$, is identically zero.

For tensor valuations of ranks larger than one, the situation is more complicated. It remains true that each Minkowski tensor $\Phi_k^{r,s}$ defines a mapping $\Gamma : \mathscr{K}^n \to \mathbb{T}^p$, for $p = r + s$, which is a valuation and is continuous. The behaviour under isometries (combinations of rotations and translations) can be described as follows. First, we point out that in Hadwiger's theorem, 'rigid motions' are orientation preserving, whereas in the following, a 'rotation' is an element of $\mathrm{O}(n)$ and thus can be improper. The mapping Γ is *rotation covariant*, that is, if $\vartheta \in \mathrm{O}(n)$ is a rotation, then $\Gamma(\vartheta K) = \vartheta\Gamma(K)$ for all $K \in \mathscr{K}^n$. We recall that the operation of the orthogonal group appearing here is defined by

$$(\vartheta T)(y_1, \dots, y_p) = T(\vartheta^{-1} y_1, \dots, \vartheta^{-1} y_p) \qquad \text{for } y_1, \dots, y_p \in \mathbb{R}^n,\ T \in \mathbb{T}^p.$$

Further, Γ has *polynomial translation behaviour*, by which we mean that

$$\Gamma(K + t) = \sum_{j=0}^{p} \frac{1}{j!} \Gamma_{p-j}(K) t^j \qquad \text{for } K \in \mathscr{K}^n,\ t \in \mathbb{R}^n,$$

with tensors $\Gamma_{p-j}(K) \in \mathbb{T}^{p-j}$, which are independent of t. (By convention, $0^0 = 1$ here.) We say that Γ is *isometry covariant* if it has both properties, rotation covariance and polynomial behaviour under translations.

One new aspect appearing for higher ranks is the following. For rank two, there is a constant mapping $\Gamma : \mathcal{K}^n \to \mathbb{T}^2$ that has all the properties listed above, namely $\Gamma(K) = Q$, the metric tensor. Since Q does not depend on K, this mapping Γ is trivially a valuation, continuous, and has polynomial behaviour under translations. For $\vartheta \in \mathrm{O}(n)$,

$$Q(y_1, y_2) = y_1 \cdot y_2 = \vartheta^{-1} y_1 \cdot \vartheta^{-1} y_2 = (\vartheta Q)(y_1, y_2),$$

hence Γ is also rotation covariant. As the considered properties are preserved under symmetric products, it follows that also the mappings $K \mapsto Q^m \Phi_k^{r,s}(K)$, for any $m \in \mathbb{N}_0$, share these properties with the Minkowski tensors. But this is as far as we can go, as the following characterization theorem due to Alesker [1] shows.

Theorem 2.5 (Alesker) *Let $p \in \mathbb{N}_0$. The real vector space of all mappings $\Gamma : \mathcal{K}^n \to \mathbb{T}^p$ which are*

- *valuations,*
- *isometry covariant,*
- *continuous,*

is spanned by the tensor valuations

$$Q^m \Phi_k^{r,s}, \tag{2.11}$$

where $m, r, s \in \mathbb{N}_0$ satisfy $2m + r + s = p$, where $k \in \{0, \ldots, n\}$, and where $s = 0$ if $k = n$.

The characterizations given in Theorem 1.23 (i.e., Hadwiger's characterization theorem) and Theorem 2.4 are special cases of this result. However, there is an essential difference: for $p \geq 2$, the spanning tensor valuations (2.11) are no longer linearly independent. They satisfy a series of linear relations, known as the *McMullen relations*. We prove these now.

The crucial relation is the identity

$$Q\Phi_n^{r-1,0} = 2\pi \Phi_{n-1}^{r,1}. \tag{2.12}$$

Explicitly, this reads

$$Q\Psi_{r-1}(K) = \frac{2}{r!} \int_{\Sigma^n} x^r u \, \Lambda_{n-1}(K, \mathrm{d}(x, u)). \tag{2.13}$$

It suffices to prove this identity for smooth convex bodies, because the general case can then be obtained by approximation. If K is smooth, we denote by $u(K, x)$

the unique outer unit normal vector of K at its boundary point x. For a smooth convex body K, the measure $2\Lambda_{n-1}(K,\cdot)$ is the image measure of the Hausdorff measure $\mathscr{H}^{n-1}$ on ∂K, the boundary of K, under the measurable mapping $x \mapsto (x, u(K,x))$ from ∂K to Σ^n. Therefore, Eq. (2.13) is equivalent to

$$Q\Psi_{r-1}(K) = \frac{1}{r!} \int_{\partial K} x^r u(K,x) \, \mathscr{H}^{n-1}(\mathrm{d}x). \tag{2.14}$$

To prove this, we use coordinates. We introduce an orthonormal basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ and write $x \in \mathbb{R}^n$ in the form $x = x_1 e_1 + \dots + x_n e_n$ (so $x_1, \dots, x_n$ are now Cartesian coordinates). For given $i_1, \dots, i_r, j \in \{1, \dots, n\}$, we define the vector field v by

$$v(x) := x_{i_1} \cdots x_{i_r} e_j, \qquad x \in \mathbb{R}^n.$$

To this and the convex body K we apply the divergence theorem. It says that

$$\int_K \operatorname{div} v(x) \, \mathrm{d}x = \int_{\partial K} v(x) \cdot u(K,x) \, \mathscr{H}^{n-1}(\mathrm{d}x).$$

To write this explicitly in a concise form, we use the Kronecker symbol δ and indicate by $\check{x}_m$ that x_m has to be deleted. Then we get

$$\int_K \sum_{k=1}^r \delta_{i_k j} \, x_{i_1} \cdots \check{x}_{i_k} \cdots x_{i_r} \, \mathrm{d}x = \int_{\partial K} x_{i_1} \cdots x_{i_r} (e_j \cdot u(K,x)) \, \mathscr{H}^{n-1}(\mathrm{d}x).$$

Using tensor notation, this can equivalently be written as

$$\begin{aligned} &\sum_{k=1}^r Q(e_{i_k}, e_j) \Psi_{r-1}(K)(e_{i_1}, \dots, \check{e}_{i_k}, \dots, e_{i_r}) \\ &\qquad = \frac{1}{(r-1)!} \int_{\partial K} x^r (e_{i_1}, \dots, e_{i_r}) u(K,x)(e_j) \, \mathscr{H}^{n-1}(\mathrm{d}x). \end{aligned} \tag{2.15}$$

This identity holds for arbitrary $(r+1)$-tuples $(i_1, \dots, i_r, j)$ from $\{1, \dots, n\}$.

To prove the identity (2.14), we have to check (only) that the $(r+1)$-tensors on either side attain the same value at any $(r+1)$-tuple $(e_{i_1}, \dots, e_{i_{r+1}})$ of basis vectors. Now, by the definition of the symmetric tensor product, for the left side of (2.14) we have

$$\begin{aligned} &(r+1)!(Q\Psi_{r-1}(K))(e_{i_1}, \dots, e_{i_{r+1}}) \\ &\qquad = \sum_{\sigma \in \mathscr{S}(r+1)} Q(e_{i_{\sigma(1)}}, e_{i_{\sigma(2)}}) \Psi_{r-1}(K)(e_{i_{\sigma(3)}}, \dots, e_{i_{\sigma(r+1)}}). \end{aligned} \tag{2.16}$$

For the right side of (2.14) we obtain from (2.15) that

$$
\begin{aligned}
&(r+1)!\,\frac{1}{r!}\int_{\partial K}(x^r u(K,x))(e_{i_1},\dots,e_{i_{r+1}})\,\mathscr{H}^{n-1}(\mathrm{d}x)\\
&\quad=\frac{1}{r!}\sum_{\sigma\in\mathscr{S}(r+1)}\int_{\partial K}x^r(e_{i_{\sigma(1)}},\dots,e_{i_{\sigma(r)}})u(K,x)(e_{i_{\sigma(r+1)}})\,\mathscr{H}^{n-1}(\mathrm{d}x)\\
&\quad=\frac{1}{r}\sum_{k=1}^{r}\sum_{\sigma\in\mathscr{S}(r+1)}Q(e_{i_{\sigma(k)}},e_{i_{\sigma(r+1)}})\Psi_{r-1}(K)(e_{i_{\sigma(1)}},\dots,\check{e}_{i_{\sigma(k)}},\dots,e_{i_{\sigma(r)}})\\
&\quad=\sum_{\sigma\in\mathscr{S}(r+1)}Q(e_{i_{\sigma(1)}},e_{i_{\sigma(2)}})\Psi_{r-1}(K)(e_{i_{\sigma(3)}},\dots,e_{i_{\sigma(r+1)}}).
\end{aligned}
$$

The latter agrees with (2.16). This completes the proof of (2.12).

From (2.12), further identities can be derived by applying (2.12) to the parallel bodies of a given convex body. For this, we write (2.12) in another explicit form, which is a counterpart to (2.14) for strictly convex bodies. If the convex body K is strictly convex, then to each unit vector $u \in \mathbb{S}^{n-1}$ there is a unique boundary point of K at which u is attained as outer normal vector. We denote this boundary point by $x(K,u)$. For a strictly convex body K, the measure $2\Lambda_{n-1}(K,\cdot)$ is the image measure of the area measure $S_{n-1}(K,\cdot)$ under the measurable mapping $u \mapsto (x(K,u),u)$ from $\mathbb{S}^{n-1}$ to Σ^n (for the area measure, see Sect. 1.3 or [17, Sect. 4.2]). Therefore, Eq. (2.13) is transformed into

$$Q\Psi_{r-1}(K)=\frac{1}{r!}\int_{\mathbb{S}^{n-1}}x(K,u)^r u\,S_{n-1}(K,\mathrm{d}u). \tag{2.17}$$

We apply this to a parallel body $K+\rho B^n$, for $\rho \geq 0$, which is also strictly convex if K is strictly convex. For the left side we get, using the Steiner formula (2.8),

$$Q\Psi_{r-1}(K+\rho B^n)=\sum_{k=0}^{n+r-1}\rho^{n+r-1-k}\kappa_{n+r-1-k}QV_k^{(r-1)}(K). \tag{2.18}$$

To compute the right side of (2.17) for $K+\rho B^n$, we note that

$$x(K+\rho B^n,u)=x(K,u)+\rho u,$$

and hence

$$x(K+\rho B^n,u)^r=\sum_{j=0}^{r}\binom{r}{j}\rho^{r-j}x(K,u)^j u^{r-j}.$$

Further, we have to use the Steiner-type formula

$$S_{n-1}(K+\rho B^n,\,\cdot\,)=\sum_{i=0}^{n-1}\rho^{n-1-i}\binom{n-1}{i}S_i(K,\,\cdot\,)$$

(see [17, (4.36)]). Therefore, we also have

$$\begin{aligned}
&Q\Psi_{r-1}(K+\rho B^n)\\
&\quad=\frac{1}{r!}\sum_{i=0}^{n-1}\binom{n-1}{i}\int_{\mathbb{S}^{n-1}}\sum_{j=0}^{r}\binom{r}{j}x(K,u)^j u^{r-j+1}\,S_i(K,\mathrm{d}u)\,\rho^{n+r-1-i-j}\\
&\quad=\frac{1}{r!}\sum_{k=0}^{n+r-1}\rho^{n+r-1-k}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s} \qquad (2.19)\\
&\qquad\times\int_{\Sigma^n}x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

Here we have introduced new indices by $s=r+1-j$ and $k=i+j$, and instead of the measure $\Lambda_m(K,\,\cdot\,)$ we have used its re-normalization

$$\Theta_m(K,\,\cdot\,)=\frac{n\kappa_{n-m}}{\binom{n}{m}}\Lambda_m(K,\,\cdot\,).$$

Comparing the coefficients in (2.18) and (2.19), we now get

$$\begin{aligned}
&\kappa_{n+r-1-k}QV_k^{(r-1)}(K)\\
&\quad=\frac{1}{r!}\sum_{s=1}^{r+1}\binom{r}{s-1}\binom{n-1}{k-r-1+s}\int_{\Sigma^n}x^{r+1-s}u^s\,\Theta_{k-r-1+s}(K,\mathrm{d}(x,u)).
\end{aligned}$$

With the help of the identity $2\pi\kappa_m=\omega_{m+2}$, this can be simplified. Replacing $r+1$ by r, we obtain

$$QV_k^{(r-2)}(K)=2\pi\sum_{s\in\mathbb{N}_0}s\Phi_{k-r+s}^{r-s,s}(K). \qquad (2.20)$$

So far, this identity has been proved for strictly convex bodies K. By approximation, this result can be extended to general convex bodies.

Now, multiplying (2.9) (with r replaced by $r-2$) by Q and comparing with (2.20), we immediately get the McMullen relations. McMullen [12] proved these relations in a different way, namely first for polytopes.

Theorem 2.6 (McMullen) *For $r \in \mathbb{N}$ with $r \geq 2$ and $k \in \{0, \ldots, n+r-2\}$,*

$$Q \sum_{s \in \mathbb{N}_0} \Phi_{k-r+s}^{r-s,s-2} = 2\pi \sum_{s \in \mathbb{N}_0} s \Phi_{k-r+s}^{r-s,s}. \tag{2.21}$$

For $r = 1$, relation (2.21) also holds, but only expresses the well-known fact that

$$\int_{\mathbb{S}^{n-1}} u\, S_j(K, \mathrm{d}u) = 0$$

for $j = 0, \ldots, n-1$. For rank two, the McMullen relations are given by

$$Q\Phi_k^{0,0} = 2\pi \Phi_{k-1}^{1,1} + 4\pi \Phi_k^{0,2}, \qquad k = 0, \ldots, n.$$

We recall that

$$\begin{aligned}
\Phi_k^{0,0}(K) &= V_k, \\
\Phi_{k-1}^{1,1}(K) &= a_k \int_{\Sigma^n} xu\, \Lambda_{k-1}(K, \mathrm{d}(x,u)) \qquad \text{for } k \geq 1,\ \Phi_{-1}^{1,1}(K) = 0, \\
\Phi_k^{0,2}(K) &= b_k \int_{\Sigma^n} u^2\, \Lambda_k(K, \mathrm{d}(x,u)) \qquad \text{for } k \leq n-1,\ \Phi_n^{0,2}(K) = 0,
\end{aligned}$$

with positive constants a_k, b_k.

Now the question arises whether the McMullen relations are essentially the only linear dependences between the basic tensor valuations $Q^m \Phi_k^{r,s}$. This is, in fact, true. The following was proved by Hug et al. [9].

Theorem 2.7 *Any nontrivial linear relation between basic tensor valuations $Q^m \Phi_k^{r,s}$ can be obtained by multiplying suitable McMullen relations by powers of Q and by taking linear combinations of relations obtained in this way.*

This result opened the way to determine bases and dimensions of the vector spaces in question. Let $T_{p,k}$ denote the real vector space of all mappings $\mathcal{K}^n \to \mathbb{T}^p$ that are continuous, isometry covariant valuations and homogeneous of degree k. Theorem 3.1 of [9] gives an explicit formula for the dimension of $T_{p,k}$. As an example for explicit bases, we present here the case of rank two:

- $T_{2,0}$: a basis is $\{Q\Phi_0^{0,0}\}$.
- $T_{2,1}$: a basis is $\{\Phi_1^{0,2}, Q\Phi_1^{0,0}\}$.
- $T_{2,k}$ for $k = 2, \ldots, n-1$: a basis is $\{\Phi_k^{0,2}, \Phi_{k-2}^{2,0}, Q\Phi_k^{0,0}\}$.
- $T_{2,n}$: a basis is $\{\Phi_{n-2}^{2,0}, Q\Phi_n^{0,0}\}$.
- $T_{2,k}$ for $k = n+1, n+2$: a basis is $\{\Phi_{k-2}^{2,0}\}$.

Thus, the vector space of continuous, isometry covariant tensor valuations of rank two has dimension $3n + 1$.

2.3 Local Tensor Valuations

In the same way as the intrinsic volumes have local versions, the support measures, so the Minkowski tensors have natural measure-valued extensions. We abbreviate now the normalizing factor appearing in (2.6) by

$$c_{n,k}^{r,s} := \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}$$

and define the *local Minkowski tensors* by

$$\phi_k^{r,s}(K,\eta) := c_{n,k}^{r,s}\int_\eta x^r u^s\, \Lambda_k(K, \mathrm{d}(x,u)) \tag{2.22}$$

for $\eta \in \mathscr{B}(\Sigma^n)$, the σ-algebra of Borel sets in Σ^n, and for $r, s \in \mathbb{N}_0$, $k \in \{0,\dots,n-1\}$. These local tensor valuations can also be introduced in a way that generalizes the introduction of the support measures by means of a local Steiner formula (see [17, Theorem 4.2.1]). For this, we define, for $K \in \mathscr{K}^n$ and $\eta \in \mathscr{B}(\Sigma^n)$, a tensor in $\mathbb{T}^{r+s}$ by

$$\mathscr{V}_\rho^{r,s}(K,\eta) := \int_{K_\rho\setminus K} \mathbf{1}_\eta\big(p_K(x), u_K(x)\big)p_K(x)^r\big(x - p_K(x)\big)^s\,\mathrm{d}x \tag{2.23}$$

for $\rho \geq 0$ and $r, s \in \mathbb{N}_0$. Here $\mathbf{1}_\eta$ is the characteristic function of the set η and $p_K(x)$ denotes the point in K nearest to x; the vector $u_K(x) := (x - p_K(x))/\|x - p_K(x)\|$ points from $p_K(x)$ to x, for $x \notin K$. (Variants of the tensor (2.23) have been introduced in [13] and [11], aiming at applications.) Noting that for (x,u) in the support of the measure $\Lambda_j(K,\cdot)$ and $t > 0$ the relations $p_K(x + tu) = x$ and $u_K(x + tu) = u$ hold, we obtain from Lemma 2.3 that

$$\mathscr{V}_\rho^{r,s}(K,\eta) = r!s!\sum_{j=0}^{n-1}\rho^{n-j+s}\kappa_{n-j+s}\phi_j^{r,s}(K,\eta). \tag{2.24}$$

Equation (2.22) defines a mapping $\phi_k^{r,s}$ from $\mathscr{K}^n \times \mathscr{B}(\Sigma^n)$ into $\mathbb{T}^{r+s}$. We want to list the properties of this mapping and collect, therefore, the most important properties which a general mapping $\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ may have. For $\eta \in \mathscr{B}(\Sigma^n)$, $t \in \mathbb{R}^n$ and $\vartheta \in \mathrm{O}(n)$, we write $\eta + t := \{(x + t, u) : (x,u) \in \eta\}$ and $\vartheta\eta := \{(\vartheta x, \vartheta u) : (x,u) \in \eta\}$. Moreover, recall from Sect. 1.3 that $\mathbf{nc}(K) = \{(p_K(x), u_K(x)) : x \in \mathbb{R}^n \setminus K\}$ denotes the normal bundle of K. The following properties will play an important role.

- Γ has *polynomial translation behaviour of degree q*, where $0 \leq q \leq p$, if

$$\Gamma(K + t, \eta + t) = \sum_{j=0}^{q}\frac{1}{j!}\Gamma_{p-j}(K,\eta)t^j \tag{2.25}$$

with tensors $\Gamma_{p-j}(K,\eta) \in \mathbb{T}^{p-j}$, for all $K \in \mathscr{K}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$ (the factor $1/j!$ is convenient); here $\Gamma_p = \Gamma$. In particular, Γ is called *translation invariant* if it is translation covariant of degree zero.
- Γ is *rotation covariant* if $\Gamma(\vartheta K, \vartheta\eta) = \vartheta\Gamma(K,\eta)$ for all $K \in \mathscr{K}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $\vartheta \in \mathrm{O}(n)$.
- Γ is *isometry covariant* (of degree q) if it has polynomial translation behaviour of some degree $q \le p$ (and hence of degree p) and is rotation covariant.
- Γ is *locally defined* if for $\eta \in \mathscr{B}(\Sigma^n)$ and $K, K' \in \mathscr{K}^n$ with $\eta \cap \mathbf{nc}(K) = \eta \cap \mathbf{nc}(K')$ the equality $\Gamma(K,\eta) = \Gamma(K',\eta)$ holds.
- If $\Gamma(K,\cdot)$ is a $\mathbb{T}^p$-valued measure for each $K \in \mathscr{K}^n$, then Γ is *weakly continuous* if for each sequence $(K_i)_{i\in\mathbb{N}}$ of convex bodies in $\mathscr{K}^n$ converging to a convex body K the relation

$$\lim_{i\to\infty}\int_{\Sigma^n} f\,\mathrm{d}\Gamma(K_i,\cdot) = \int_{\Sigma^n} f\,\mathrm{d}\Gamma(K,\cdot)$$

holds for all continuous functions $f : \Sigma^n \to \mathbb{R}$.

We point out that 'locally defined', also called 'locally determined', appears with different interpretations in different situations; compare Theorems 1.29–1.31.

In the previous definitions, the set $\mathscr{K}^n$ may be replaced by $\mathscr{P}^n$.

Returning to the local Minkowski tensors, we note that from the properties of the support measures, the following can be deduced for each $\Gamma = \phi_k^{r,s}$.

- For each $K \in \mathscr{K}^n$, $\Gamma(K,\cdot)$ is a $\mathbb{T}^{r+s}$-valued measure.
- Γ is weakly continuous.
- For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is measurable.
- For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is a valuation.
- The mapping Γ is isometry covariant.
- The mapping Γ is locally defined.

It will be the main goal of the rest of this chapter to determine all mappings with these properties. In fact, it will turn out that some properties are consequences of the others.

2.4 A Characterization Result for Local Tensor Valuations on Polytopes

In a first step to achieve the goal just formulated, we consider local tensor valuations on the space $\mathscr{P}^n$ of polytopes.

Let $P \in \mathscr{P}^n$ be a polytope. By $\mathscr{F}_k(P)$ we denote the set of k-dimensional faces of P, for $k \in \{0,\dots,n\}$. For $F \in \mathscr{F}_k(P)$, the set $\nu(P,F) = N(P,F) \cap \mathbb{S}^{n-1}$ is the set of outer unit normal vectors of P at its face F (see [17, Sect. 2.4] for the normal cone $N(P,F)$). From a representation of the support measures for polytopes (see [17, (4.3)]), one can deduce that the local Minkowski tensors of a polytope P have the explicit representation

$$\phi_k^{r,s}(P,\eta) = C_{n,k}^{r,s} \sum_{F\in\mathscr{F}_k(P)} \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathscr{H}^{n-k-1}(\mathrm{d}u)\, \mathscr{H}^k(\mathrm{d}x), \tag{2.26}$$

for $k \in \{0,\dots,n-1\}$ and $r,s \in \mathbb{N}_0$, where

$$C_{n,k}^{r,s} := (r!s!\omega_{n-k+s})^{-1}. \tag{2.27}$$

We point out that the integrations in (2.26) are only with respect to Hausdorff measures. The structure of (2.26) should be well understood, since it plays an important role in the following.

If one studies valuations on polytopes, it is always advisable to see how far one gets without the assumption of continuity. Theorem 1.31, for example, does not need any continuity assumption. However, without this assumption, there are mappings on $\mathscr{P}^n$ which share the preceding properties with the local Minkowski tensors, but are far more general. Hence, a possible classification theorem has to take these into account.

To define these generalizations, we associate with each face F of a polytope the linear subspace that is a translate of the affine hull of F. We denote this subspace by $L(F)$ and call it the *direction space* of F. For a linear subspace L of $\mathbb{R}^n$, we denote by $\pi_L : \mathbb{R}^n \to L$ the orthogonal projection. Then we define $Q_L \in \mathbb{T}^2$ by

$$Q_L(a,b) := \pi_L a \cdot \pi_L b \qquad \text{for } a,b \in \mathbb{R}^n.$$

We note that $Q_{\vartheta L} = \vartheta Q_L$ for $\vartheta \in \mathrm{O}(n)$.

Now we define the *generalized local Minkowski tensors* by extending (2.26) in the following way:

$$\phi_k^{r,s,j}(P,\eta) := C_{n,k}^{r,s} \sum_{F\in\mathscr{F}_k(P)} Q_{L(F)}^j \int_F \int_{\nu(P,F)} \mathbf{1}_\eta(x,u) x^r u^s \, \mathscr{H}^{n-k-1}(\mathrm{d}u)\, \mathscr{H}^k(\mathrm{d}x), \tag{2.28}$$

for $r,s,j,k \in \mathbb{N}_0$ with $1 \le k \le n-1$. This definition is supplemented by

$$\phi_0^{r,s,0} := \phi_0^{r,s},$$

but $\phi_0^{r,s,j}$ remains undefined for $j \ge 1$. Each mapping $\Gamma = \phi_k^{r,s,j}$ has the following properties. It is isometry covariant and locally defined. For each $P \in \mathscr{P}^n$, $\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure, with $p = 2j + r + s$. For each $\eta \in \mathscr{B}(\Sigma^n)$, $\Gamma(\cdot,\eta)$ is a valuation. The first of these properties are easy to see; the proof of the last one uses Theorem 1.7; we refer to [8, Theorem 3.3] for the details.

Now we can state a characterization theorem. It is motivated by Theorems 2.5 and 1.31.

Theorem 2.8 *For $p \in \mathbb{N}_0$, let $T_p(\mathscr{P}^n)$ denote the real vector space of all mappings*

$$\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$$

with the following properties.

(a) *$\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathscr{P}^n$;*
(b) *Γ is isometry covariant;*
(c) *Γ is locally defined.*

Then a basis of $T_p(\mathscr{P}^n)$ is given by the mappings

$$Q^m \phi_k^{r,s,j},$$

where $m, r, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + r + s = p$, where $k \in \{0, \dots, n-1\}$, and where $j = 0$ if $k \in \{0, n-1\}$.

That only $j = 0$ appears if $k = n - 1$, is due to the easily proved identity

$$\phi_{n-1}^{r,s,j} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(s+2i)!\omega_{1+s+2i}}{s!\omega_{1+s}} Q^{j-i} \phi_{n-1}^{r,s+2i}. \tag{2.29}$$

Theorem 2.8 is a stronger version of a theorem proved in [16]. Some modifications, including the linear independence result, were proved in [8]. We state this linear independence as a separate theorem.

Theorem 2.9 *Let $p \in \mathbb{N}_0$. On $\mathscr{P}^n$, the generalized local Minkowski tensors $Q^m \phi_k^{r,s,j}$ with*

$$m, r, s, j \in \mathbb{N}_0, \; 2m + 2j + r + s = p, \; k \in \{0, \dots, n-1\},$$

$$\text{and } j = 0 \text{ if } k \in \{0, n-1\},$$

are linearly independent.

The proof starts with a general linear relation

$$\sum_{m,r,s,j,k} a_{kmrsj} Q^m \phi_k^{r,s,j} = 0,$$

where $a_{kmrsj} \in \mathbb{R}$ with $a_{0mrsj} = a_{(n-1)mrsj} = 0$ for $j \neq 0$ and the summation extends over all m, r, s, j, k such that $2m + 2j + r + s = p$. This relation is evaluated for a

k-dimensional polytope F and arbitrary Borel sets $\beta \subset \operatorname{relint} F$, $\omega \subset L(F)^{\perp} \cap \mathbb{S}^{n-1}$. Using the generality of these Borel sets, homogeneity considerations, and the fact that the symmetric tensor algebra has no zero divisors, it follows for any fixed r and k that

$$\sum_{m,s,j} a_{kmrsj} Q^m Q^j_{L(F)} C^{r,s}_{n,k} u^s = 0$$

for all $u \in \mathbb{S}^{n-1} \cap L(F)^{\perp}$. This simplified relation is then applied to a tuple $(x, \ldots, x)$ of vectors $x = x_1 e_1 + \cdots + x_n e_n$, and from the fact that the resulting polynomial in $x_1, \ldots, x_n$ is zero, one can deduce that all coefficients must be zero.

We shall now describe the main steps and ideas of the proof of Theorem 2.8 (the details are found in [8] and [16]). For this, we suppose that

$$\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$$

is a mapping which has the following properties.

- For each $P \in \mathscr{P}^n$, $\Gamma(P, \cdot)$ is a $\mathbb{T}^p$-valued measure.
- Γ is isometry covariant.
- Γ is locally defined.

That Γ is isometry covariant, includes that it has polynomial translation behaviour of some degree q. Thus, there are mappings $\Gamma_{p-j} : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^{p-j}$, $j = 0, \ldots, q$, (possibly zero for some j and with $\Gamma_p = \Gamma$) such that

$$\Gamma(P + t, \eta + t) = \sum_{j=0}^{q} \frac{1}{j!} \Gamma_{p-j}(P, \eta) t^j$$

for all $P \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$. This implies similar behaviour of the coefficient tensors, namely

$$\Gamma_{p-j}(P + t, \eta + t) = \sum_{r=0}^{q-j} \frac{1}{r!} \Gamma_{p-j-r}(P, \eta) t^r \tag{2.30}$$

for $j = 0, \ldots, q$ and all $P \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$, in particular (case $j = q$),

$$\Gamma_{p-q}(P + t, \eta + t) = \Gamma_{p-q}(P, \eta).$$

Properties of Γ_{p-j} can be derived from those of Γ, by means of the following relation. There are constants a_{jm} ($j = 0, \ldots, q$, $m = 1, \ldots, q + 1$), depending only on q, j, m, such that

$$\Gamma_{p-j}(P,\eta)t^j = \sum_{m=1}^{q+1} a_{jm}\Gamma(P+mt,\eta+mt) \tag{2.31}$$

for all $P \in \mathscr{P}^n$, $\eta \in \mathscr{B}(\Sigma^n)$ and $t \in \mathbb{R}^n$. In particular, we can deduce that $\Gamma_{p-j}(P,\cdot)$ is a $\mathbb{T}^{p-j}$-valued measure and that

$$\Gamma_{p-j}(\vartheta P, \vartheta\eta) = \vartheta\Gamma_{p-j}(P,\eta) \tag{2.32}$$

for $\vartheta \in \mathrm{O}(n)$. Together with (2.30) this shows that also Γ_{p-j} is isometry covariant.

Lemma 2.10 *For each $P \in \mathscr{P}^n$, the measure $\Gamma(P,\cdot)$ is concentrated on* $\mathbf{nc}(P)$.

The proof is based on the fact that Γ is locally defined and has polynomial translation behaviour. Further it uses that the only translation invariant finite signed measure on the bounded Borel sets of $\mathbb{R}^n$ is Lebesgue measure, up to a constant factor.

The essential step to prove Theorem 2.8 is the translation invariant case, that is, the following result.

Theorem 2.11 *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathscr{P}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the following properties.*

(a) *$\Gamma(P,\cdot)$ is a $\mathbb{T}^p$-valued measure, for each $P \in \mathscr{P}^n$;*
(b) *Γ is translation invariant and rotation covariant;*
(c) *Γ is locally defined.*

Then Γ is a linear combination, with constant coefficients, of the mappings

$$Q^m\phi_k^{0,s,j},$$

where $m, s, j \in \mathbb{N}_0$ satisfy $2m + 2j + s = p$, where $k \in \{0,\dots,n-1\}$, and where $j = 0$ if $k \in \{0, n-1\}$.

If this has been proved, then one can use the properties of the coefficient tensors Γ_{p-j} mentioned above, to give for Theorem 2.8 an inductive proof, which step by step reduces the degree of the polynomial translation behaviour of Γ.

Now we indicate some ideas in the proof of Theorem 2.11. To show, as we have to do, an equality for measures on $\mathscr{B}(\Sigma^n)$, it is sufficient to prove equality on product sets $\beta \times \omega$ with $\beta \in \mathscr{B}(\mathbb{R}^n)$ and $\omega \in \mathscr{B}(\mathbb{S}^{n-1})$. Let $P \in \mathscr{P}^n$. By Lemma 2.10, $\Gamma(P,\cdot)$ is concentrated on $\mathbf{nc}(P) \subset \partial P \times \mathbb{S}^{n-1}$. The boundary ∂P of the polytope P is the disjoint union of the relative interiors of the proper faces of P. Therefore,

$$\Gamma(P,\beta\times\omega) = \sum_{k=0}^{n-1}\sum_{F\in\mathscr{F}_k(P)} \Gamma\big(P,(\beta\cap\operatorname{relint}F)\times(\omega\cap\nu(P,F))\big). \tag{2.33}$$

Consequently, it is sufficient to determine $\Gamma(P, \beta \times \omega)$ for the case where $\beta \subset \operatorname{relint} F$ and $\omega \subset \nu(P, F)$, for some face $F \in \mathscr{F}_k(P)$.

Therefore, and because Γ is locally defined, we may restrict ourselves to the following situation. We are given a number $k \in \{0, \dots, n-1\}$, a k-dimensional linear subspace $L \subset \mathbb{R}^n$, a bounded Borel set $\beta \subset L$, a Borel set $\omega \subset \mathbb{S}^{n-1} \cap L^\perp$, and a k-dimensional polytope $P \subset L$ with $\beta \subset \operatorname{relint} P$. It suffices to determine $\Gamma(P, \beta \times \omega)$ in this case.

First, we fix ω and use the standard characterization of Lebesgue measure in L to show that

$$\Gamma(P, \beta \times \omega) = a(L, \omega)\mathscr{H}^k(\beta),$$

where the constant $a(L, \omega)$ is a tensor in $\mathbb{T}^p$ that depends on the subspace L and the Borel set ω. The main task is to determine this tensor function. It has an important covariance property, namely

$$a(\vartheta L, \vartheta \omega) = \vartheta a(L, \omega) \qquad \text{for } \vartheta \in \mathrm{O}(n)$$

and

$$\vartheta a(L, \omega) = a(L, \omega) \qquad \text{if } \vartheta \text{ fixes } L^\perp \text{ pointwise.}$$

From this, it is deduced in [16] that

$$a(L, \omega) = \sum_{j=0}^{\lfloor p/2 \rfloor} Q_L^j \sum_{i=0}^{\lfloor p/2 \rfloor} c_{pkij} Q_{L^\perp}^i \int_\omega u^{p-2j-2i} \, \mathscr{H}^{n-k-1}(\mathrm{d}u) \tag{2.34}$$

with real constants c_{pkij}. Once this has been proved, things can be put together to finish the proof of Theorem 2.11.

The only hints we can give here to the proof of (2.34) is the formulation of two lemmas. The first exhibits the crucial point where the tensors Q_L enter the scene.

Lemma 2.12 *Let $L \subset \mathbb{R}^n$ be a linear subspace. Let $r \in \mathbb{N}_0$, let $T \in \mathbb{T}^r$ be a tensor satisfying $\vartheta T = T$ for each $\vartheta \in \mathrm{O}(n)$ that fixes $L^\perp$ pointwise. Then*

$$T = \sum_{j=0}^{\lfloor r/2 \rfloor} Q_L^j \pi_{L^\perp}^* T^{(r-2j)}$$

with tensors $T^{(r-2j)} \in \mathbb{T}^{r-2j}(L^\perp)$, $j = 0, \dots, \lfloor r/2 \rfloor$.

Here $\mathbb{T}^p(L^\perp)$ denotes the space of p-tensors on $L^\perp$, and for $T \in \mathbb{T}^p(L^\perp)$ we have used the notation

$$(\pi_{L^\perp}^* T)(x_1, \dots, x_p) := T(\pi_{L^\perp} x_1, \dots, \pi_{L^\perp} x_p) \qquad \text{for } x_1, \dots, x_p \in \mathbb{R}^n.$$

The proof of Lemma 2.12 is based on the fact that the algebra of symmetric tensors on $\mathbb{R}^n$ is isomorphic to the polynomial algebra on $\mathbb{R}^n$, and it uses some manipulations with polynomials.

The second crucial lemma deals with rotation covariant tensor measures on the sphere.

Lemma 2.13 *Let $r \in \mathbb{N}_0$, and let $\mu : \mathscr{B}(\mathbb{S}^{n-1}) \to \mathbb{T}^r$ be a $\mathbb{T}^r$-valued measure satisfying*

$$\mu(\vartheta\omega) = (\vartheta\mu)(\omega) \qquad \textit{for all } \omega \in \mathscr{B}(\mathbb{S}^{n-1}) \textit{ and all } \vartheta \in \mathrm{O}(n).$$

Then

$$\mu(\omega) = \sum_{j=0}^{\lfloor r/2 \rfloor} a_j Q^j \int_\omega u^{r-2j}\, \mathscr{H}^{n-1}(\mathrm{d}u), \qquad \omega \in \mathscr{B}(\mathbb{S}^{n-1}),$$

with real constants a_j, $j = 0, \dots, \lfloor r/2 \rfloor$.

A first step of the proof uses that the total variation measure of μ is rotation invariant and hence a constant multiple of spherical Lebesgue measure. Then the Radon-Nikodym theorem, applied coordinate-wise, yields a representation

$$\mu(\omega) = \int_\omega f \,\mathrm{d}\mathscr{H}^{n-1}, \qquad \omega \in \mathscr{B}(\mathbb{S}^{n-1}),$$

with an almost everywhere defined measurable mapping $f : \mathbb{S}^{n-1} \to \mathbb{T}^r$. A special case of Lemma 2.12 together with the covariance property and Lebesgue's differentiation theorem can then be used to determine the function f.

2.5 The Characterization Result on General Convex Bodies

If we want to extend Theorem 2.8 from polytopes to general convex bodies, we certainly need some continuity assumption. This raises the question whether $\phi_k^{r,s,j}$ has a weakly continuous extension from polytopes to general convex bodies. To make this question more precise, let

$$\Gamma : \mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^p \tag{2.35}$$

be a mapping and consider the following properties, which it may or may not have:

(A) $\Gamma(K, \cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$;
(B) Γ is isometry covariant;
(C) Γ is locally defined;
(D) Γ is weakly continuous.

Question For given $k \in \{0, \ldots, n-1\}$ and $r, s, j \in \mathbb{N}_0$, is there a mapping Γ as in (2.35) having properties (A)–(D) and satisfying $\Gamma(P, \cdot) = \phi_k^{r,s,j}(P, \cdot)$ for $P \in \mathcal{P}^n$?

This is trivially true for $k = 0$, since $\phi_0^{r,s,0} = \phi_0^{r,s}$ by definition, and $\phi_0^{r,s,j}$ is not defined for $j \geq 1$. It is also true for $k = n-1$, since we may define

$$\phi_{n-1}^{r,s,j}(K, \cdot) := \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(s+2i)!\omega_{1+s+2i}}{s!\omega_{1+s}} Q^{j-i} \phi_{n-1}^{r,s+2i}(K, \cdot)$$

for $K \in \mathcal{K}^n$; by (2.29), this is consistent with the case of polytopes. The weak continuity of $\phi_{n-1}^{r,s+2i}$ follows from (2.22) and the weak continuity of the support measures. Further, the answer is affirmative if $j = 0$, since $\phi_k^{r,s,0}(P, \cdot) = \phi_k^{r,s}(P, \cdot)$ for $P \in \mathcal{P}^n$, and we can define $\phi_k^{r,s,0}(K, \cdot) = \phi_k^{r,s}(K, \cdot)$ for $K \in \mathcal{K}^n$. It remains to consider the cases of $\phi_k^{r,s,j}$ where $1 \leq k \leq n-2$ and $j \geq 1$.

Proposition 2.14 *For $k \in \{1, \ldots, n-2\}$ and $r, s \in \mathbb{N}_0$, the answer to the question above is affirmative for $j = 1$.*

Postponing the proof of this proposition to Sect. 2.6, we can now state the following characterization theorem. It includes the fact that the statement of Proposition 2.14 does not extend to $j > 1$.

Theorem 2.15 *For $p \in \mathbb{N}_0$, let $T_p(\mathcal{K}^n)$ denote the real vector space of all mappings $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ with properties* (A)–(D).

A basis of $T_p(\mathcal{K}^n)$ is given by the mappings

$$Q^m \phi_k^{r,s,j}, \qquad k \in \{0, \ldots, n-1\},\ m, r, s \in \mathbb{N}_0,\ j \in \{0, 1\},$$

where $2m + 2j + r + s = p$ and $j = 0$ if $k \in \{0, n-1\}$.

As in the case of polytopes, where Theorem 2.8 follows from Theorem 2.11, it suffices to consider the translation invariant case. By an inductive argument, which was already used by Alesker [1] in his proof of Theorem 2.5, our Theorem 2.15 can be deduced from the following result. We also observe that linear independence is implied by Theorem 2.9.

Theorem 2.16 *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma^n) \to \mathbb{T}^p$ be a mapping with the properties* (A), (C), (D) *and*

(B') *Γ is translation invariant and rotation covariant.*

Then Γ is a linear combination, with constant coefficients, of the mappings

$$Q^m \phi_k^{0,s,j}, \qquad k \in \{0, \dots, n-1\},\ m, s \in \mathbb{N}_0,\ j \in \{0, 1\},$$

where $2m + 2j + s = p$ and $j = 0$ if $k \in \{0, n-1\}$.

For the proof, some further simplifications are possible. If Γ satisfies the assumptions of Theorem 2.16, then it is not difficult to see (cf. [8, Lemma 3.5]) that $\Gamma = \sum_{k=0}^{n-1} \Gamma_k$, where Γ_k is a mapping with the same properties which is, moreover, homogeneous of degree k. Therefore, to prove Theorem 2.16, we can and will assume in addition that Γ is homogeneous of some degree $k \in \{0, \dots, n-1\}$. If $k \in \{0, n-1\}$, then Theorem 2.11 shows that the restriction of Γ to $\mathcal{P}^n$ is a linear combination of mappings $Q^m \phi_k^{0,s}$, and by weak continuity this holds also for Γ on $\mathcal{K}^n$. Hence, we can assume now that Γ is homogeneous of some degree $k \in \{1, \dots, n-2\}$ (and, therefore, $n \geq 3$). Under these assumptions, Theorem 2.11 implies that there are constants c_{mjs} (only finitely many of them different from zero) such that

$$\Gamma(P, \cdot) = \sum_{\substack{m,j,s \geq 0 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot) \qquad \text{for } P \in \mathcal{P}^n.$$

Since Γ and $\phi_k^{0,s,0}$, and by the postponed Proposition 2.14 also $\phi_k^{0,s,1}$, are weakly continuous, the mapping Γ' defined by

$$\Gamma' := \Gamma - \sum_{\substack{m,j,s \geq 0, j \leq 1 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j} \tag{2.36}$$

has the properties (A), (B'), (C), (D), and for $P \in \mathcal{P}^n$ it satisfies

$$\Gamma'(P, \cdot) = \sum_{\substack{m,s \geq 0, j \geq 2 \\ 2m+2j+s=p}} c_{mjs} Q^m \phi_k^{0,s,j}(P, \cdot).$$

Theorem 2.16 and thus Theorem 2.15 is proved if we show that Γ' is identically zero. We sketch the main ideas leading to this result and refer to [8] for the details.

The strategy of the proof is indicated by the following lemma. We write

$$\Gamma'(K, f) := \int_{\Sigma^n} f(u)\, \Gamma'(K, \mathrm{d}(x, u))$$

for $K \in \mathcal{K}^n$ and continuous real functions f on the unit sphere $\mathbb{S}^{n-1}$.

Lemma 2.17 *If the function Γ' defined by (2.36) is not identically zero, then there exist a convex body $K \in \mathcal{K}^n$, a continuous function f on $\mathbb{S}^{n-1}$, a p-tuple E of vectors from $\mathbb{R}^n$, and a rotation $\vartheta \in \mathrm{O}(n)$ such that K and f are invariant under ϑ, but $\Gamma'(K,f)(\vartheta E) \neq \Gamma'(K,f)(E)$.*

If this is proved, then it follows from the invariance of K and f under ϑ and from the rotation covariance of Γ' that

$$\Gamma'(K,f)(\vartheta E) = \Gamma'(\vartheta K, \vartheta f)(\vartheta E) = \Gamma'(K,f)(E),$$

which is a contradiction. The conclusion is that $\Gamma' \equiv 0$, which proves the theorem.

The (lengthy) proof of Lemma 2.17 constructs a sequence $(P_i)_{i\in\mathbb{N}}$ of polytopes converging to a convex body K, such that K has a symmetry ϑ (a rotation mapping K into itself) with the following property. For each i, the rotation ϑ is not a symmetry of P_i, and this fact can be strengthened as follows. If Γ' is not identically zero, then there are a continuous function f on $\mathbb{S}^{n-1}$, invariant under ϑ, and a p-tuple E of vectors from $\mathbb{R}^n$, such that

$$|\Gamma'(P_i,f)(\vartheta E) - \Gamma'(P_i,f)(E)| \geq c > 0. \tag{2.37}$$

The function f, the p-tuple E and the constant c are independent of i. By the weak continuity of Γ', it then follows that $|\Gamma'(K,f)(\vartheta E) - \Gamma'(K,f)(E)| \geq c > 0$.

The polytopes P_i are constructed as follows (we describe the construction for $n \geq 4$; a modification is necessary for $n = 3$). Let $(e_1, \dots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$, and identify $\mathrm{lin}\{e_1, \dots, e_{n-1}\}$ with $\mathbb{R}^{n-1}$. In $\mathbb{R}^{n-1}$, we consider the lattice

$$\mathbb{Z}^{n-1} := \{m_1 e_1 + \dots + m_{n-1} e_{n-1} : m_1, \dots, m_{n-1} \in \mathbb{Z}\}.$$

Its points are the vertices of a tessellation of $\mathbb{R}^{n-1}$ into $(n-1)$-cubes. We lift the homothets of this lattice to a paraboloid of revolution. For this, we define the lifting map $\ell : \mathbb{R}^{n-1} \to \mathbb{R}^n$ by $\ell(x) := x + \|x\|^2 e_n$ for $x \in \mathbb{R}^{n-1}$. For $t > 0$ we define the polyhedral set

$$R_t := \mathrm{conv}\, \ell(2t\mathbb{Z}^{n-1}).$$

It is well known and easy to see that under orthogonal projection to $\mathbb{R}^{n-1}$, the facets of R_t project into the cubes of the tessellation induced by $2t\mathbb{Z}^{n-1}$. With $H_h^- := \{y \in \mathbb{R}^n : y \cdot e_n \leq h\}$ for suitable $h > 0$, we define

$$P_i := R_{1/i} \cap H_h^- \quad \text{and} \quad K := \mathrm{epi}\, \ell \cap H_h^-,$$

where epi denotes the epigraph. Then P_i is a convex polytope, and $P_i \to K$ for $i \to \infty$.

The details of the estimates leading to (2.37) (if $h > 0$ is sufficiently small) are found in [8].

We point out, however, that the last argument of the proof given there (which concerns the case $n = 3$) needs a correction, and we replace the reasoning on page 1561 of [8] by the following.

Let

$$F(\lambda) := \sum_{j=2}^{d} c_j \sum_{r=0}^{d-1} \big(\lambda \cos r\beta_d + \sqrt{1-\lambda^2} \sin r\beta_d\big)^{2j}, \qquad \lambda \in [0,1],$$

where $d \in \mathbb{N}$, $d \geq 2$, $c_j \in \mathbb{R}$, $c_d \neq 0$, $\beta_d = \pi/d$. We have to show that F is not constant. First we note that $F(\lambda) = P(\lambda) + \sqrt{1-\lambda^2}\, Q(\lambda)$ with polynomials P and Q, in particular,

$$P(\lambda) = \sum_{j=2}^{d} c_j \sum_{r=0}^{d-1} \sum_{\ell=0}^{j} \binom{2j}{2\ell} \lambda^{2\ell} (\cos r\beta_d)^{2\ell} (1-\lambda^2)^{j-\ell} (\sin r\beta_d)^{2j-2\ell}.$$

Suppose, to the contrary, that $F(\lambda) = c$ for $\lambda \in [0,1]$, with a constant c. Then

$$(P(\lambda) - c)^2 = (1-\lambda)(1+\lambda) Q(\lambda)^2 \tag{2.38}$$

for $\lambda \in [0,1]$ and hence for all $\lambda \in \mathbb{R}$. If $P - c$ and Q are not identically zero, then the multiplicity of 1 as a root of either $(P-c)^2$ or Q^2 is even, but according to (2.38), for $(P-c)^2$ it is odd. This contradiction shows, in particular, that P is constant. However, we have

$$\begin{aligned}
\lim_{\lambda \to \infty} \lambda^{-2d} P(\lambda) &= c_d \sum_{r=0}^{d-1} \sum_{\ell=0}^{d} \binom{2d}{2\ell} (-1)^{d-\ell} (\cos r\beta_d)^{2\ell} (\sin r\beta_d)^{2d-2\ell} \\
&= c_d \sum_{r=0}^{d-1} \mathrm{Re}\, (\cos r\beta_d + \mathrm{i} \sin r\beta_d)^{2d} \\
&= c_d\, \mathrm{Re} \sum_{r=0}^{d-1} \exp\left(r \frac{\pi}{d} \mathrm{i} \cdot 2d\right) = d c_d \neq 0,
\end{aligned}$$

a contradiction.

2.6 A Weakly Continuous Extension

The main purpose of this section is to sketch the proof of Proposition 2.14, which was formulated in the previous section. Moreover, for an arbitrary convex body K we shall give an explicit description of $\phi_k^{r,s,1}(K,\cdot)$ as an integral over the normal bundle of K involving generalized curvatures and principal directions of curvature, which is then specialized for smooth convex bodies.

It is well known that the map $K \mapsto \Lambda_j(K,\cdot)$ is weakly continuous on $\mathcal{K}^n$. This follows most easily from the weak continuity of the local parallel volume map $K \mapsto \mathcal{H}^n(M_\rho(K,\cdot))$, for all $\rho > 0$. As an immediate consequence we obtain that $K \mapsto \phi_k^{r,s,0}(K,\cdot)$ is weakly continuous. In order to show that $P \mapsto \phi_k^{r,s,1}(P,\cdot)$ has a weakly continuous extension from polytopes to general convex bodies, we shall proceed in a different way. The starting point is a description of the support measure $\Lambda_k(K,\cdot)$ of K by means of a current, the normal cycle T_K of K, evaluated at suitably chosen differential forms φ_k (the Lipschitz-Killing curvature forms), as first explained in [18]. From the continuity of the map $K \mapsto T_K$ (in a suitable topology), it follows again that the support measures are weakly continuous. The main task then is to find suitable tensor-valued differential forms $\varphi_k^{r,s}$ such that T_P evaluated at $\varphi_k^{r,s}$ yields $\phi_k^{r,s,1}(P,\cdot)$, for an arbitrary polytope P.

We start with some basic terminology and facts of multilinear algebra and geometric measure theory (see [5]), which will also be useful in the final section. Let V be a finite-dimensional real vector space. Then $\bigwedge_m V$, for $m \in \mathbb{N}_0$, denotes the vector space of *m-vectors* of V, and $\bigwedge^m V$ is the vector space of all m-linear alternating maps from V^m to $\mathbb{R}$, whose elements are called *m-covectors*. The map $\bigwedge^m V \to \mathrm{Hom}(\bigwedge_m V,\mathbb{R})$, which assigns to $f \in \bigwedge^m V$ the homomorphism $v_1 \wedge \cdots \wedge v_m \mapsto f(v_1,\dots,v_m)$, allows us to identify $\bigwedge^m V$ and $\mathrm{Hom}(\bigwedge_m V,\mathbb{R})$. By this identification, the dual pairing of elements $a \in \bigwedge_m V$ and $\varphi \in \bigwedge^m V$ can be defined by $\langle a,\varphi\rangle := \varphi(a)$. If V' is another finite-dimensional vector space and $f : V \to V'$ is a linear map, then a linear map $\bigwedge_m f : \bigwedge_m V \to \bigwedge_m V'$ is determined by $(\bigwedge_m f)(v_1 \wedge \cdots \wedge v_m) = f(v_1) \wedge \cdots \wedge f(v_m)$, for all $v_1,\dots,v_m \in V$.

To introduce the normal cycle T_K of $K \in \mathcal{K}^n$, we remark that the normal bundle $\mathbf{nc}(K) \subset \mathbb{R}^{2n}$ of K is an $(n-1)$-rectifiable set. To see this, observe that the map $F : \partial K_1 \to \mathbb{R}^n \times \mathbb{S}^{n-1}$ given by $F(x) := (p_K(x), u_K(x))$ is bi-Lipschitz, and hence the image $\mathbf{nc}(K)$ is an $(n-1)$-rectifiable subset of $\mathbb{R}^{2n}$. Therefore, for $\mathcal{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$, the set of $(\mathcal{H}^{n-1} \llcorner \mathbf{nc}(K), n-1)$-approximate tangent vectors at $(x,u) \in \mathbf{nc}(K)$ is an $(n-1)$-dimensional linear subspace of $\mathbb{R}^{2n}$, which is denoted by $\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \mathbf{nc}(K),(x,u))$. The symbol $\llcorner$ describes the restriction of a measure to a subset (see [5, p. 54]), and approximate tangent vectors are defined and characterized in [5, p. 252, 3.2.16]. We also refer to [5, p. 253, 3.2.16] for the notions of the *approximate differentiability* and the *approximate differential* of a map, which are used in the following. In the present context, the *approximate tangent space*, which has just been introduced, is spanned by an orthonormal basis

$(a_1(x,u),\dots,a_{n-1}(x,u))$, where

$$a_i(x,u) := \left(\frac{1}{\sqrt{1+k_i(x,u)^2}}\, b_i(x,u), \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}}\, b_i(x,u) \right)$$

and where $(b_1(x,u),\dots,b_{n-1}(x,u))$ is a suitable orthonormal basis of $u^\perp$, which is chosen so that $(b_1(x,u),\dots,b_{n-1}(x,u),u)$ has the same orientation as the standard basis $(e_1,\dots,e_n)$ of $\mathbb{R}^n$. Here, $k_i(x,u) \in [0,\infty]$ for $i = 1,\dots,n-1$ with the usual convention

$$\frac{1}{\sqrt{1+k_i(x,u)^2}} = 0 \quad \text{and} \quad \frac{k_i(x,u)}{\sqrt{1+k_i(x,u)^2}} = 1 \qquad \text{if } k_i(x,u) = \infty.$$

The dependence of a_i, b_i, k_i on K is not made explicit by our notation. We remark that b_i, k_i, $i = 1,\dots,n-1$, are essentially uniquely determined (see [15, Proposition 3, Lemma 2]). The numbers $k_i(x,u)$ can be interpreted as generalized curvatures with corresponding generalized principal directions of curvature $b_i(x,u)$. Moreover, we can assume that $b_i(x+\varepsilon u, u) = b_i(x,u)$, independent of $\varepsilon > 0$, where $(x,u) \in \mathbf{nc}(K)$ and $(x+\varepsilon u, u) \in \mathbf{nc}(K_\varepsilon)$. For $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$,

$$a_K(x,u) := a_1(x,u) \wedge \dots \wedge a_{n-1}(x,u)$$

is an $(n-1)$-vector, which fixes an orientation of the approximate tangent space $\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K), (x,u))$. Then

$$T_K := \left(\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K)\right) \wedge a_K$$

defines an $(n-1)$-dimensional current in $\mathbb{R}^{2n}$ (see [5, Chap. 4.1]), the *normal cycle* of K. Explicitly, we have

$$T_K(\varphi) = \int_{\mathbf{nc}(K)} \langle a_K(x,u), \varphi(x,u)\rangle \, \mathscr{H}^{n-1}(d(x,u)),$$

for all $\mathscr{H}^{n-1} \llcorner \mathbf{nc}(K)$-integrable functions $\varphi : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$. Note that T_K is a rectifiable current (see [5, Theorem 4.1.28] for a characterization of such currents), which has compact support, and thus T_K can be defined for a larger class of functions than just for the class of smooth differential forms.

In order to define the *Lipschitz-Killing forms* φ_k, $k \in \{0,\dots,n-1\}$, we write $\Pi_1 : \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x,u) \mapsto x$, and $\Pi_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x,u) \mapsto u$, for the projections to the components of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Let Ω_n be the volume form on $\mathbb{R}^n$ with the orientation chosen so that $\Omega_n(e_1,\dots,e_n) = \langle e_1 \wedge \dots \wedge e_n, \Omega_n\rangle = 1$. Then differential forms $\varphi_k : \mathbb{R}^{2n} \to \bigwedge^{n-1} \mathbb{R}^{2n}$, $k \in \{0,\dots,n-1\}$, of degree $n-1$ on $\mathbb{R}^{2n}$

are defined by

$$\varphi_k(x,u)(\xi_1,\dots,\xi_{n-1}) := \frac{1}{k!(n-1-k)!\omega_{n-k}}$$
$$\times \sum_{\sigma\in\mathscr{S}(n-1)} \operatorname{sgn}(\sigma)\Big\langle \bigwedge_{i=1}^{k} \Pi_1\xi_{\sigma(i)} \wedge \bigwedge_{i=k+1}^{n-1} \Pi_2\xi_{\sigma(i)} \wedge u, \Omega_n\Big\rangle,$$

where $(x,u) \in \mathbb{R}^{2n}$ and $\xi_1,\dots,\xi_{n-1} \in \mathbb{R}^{2n}$. Note that the right-hand side is independent of x, and recall that $\mathscr{S}(n-1)$ denotes the set of all permutations of $\{1,\dots,n-1\}$. Then, writing

$$\mathbb{K}(x,u) := \prod_{i=1}^{n-1} \sqrt{1+k_i(x,u)^2},$$

we have

$$\langle a_K(x,u), \varphi_k(x,u)\rangle = \frac{1}{\omega_{n-k}} \sum_{|I|=n-1-k} \frac{\prod_{i\in I} k_i(x,u)}{\mathbb{K}(x,u)}$$

for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$. The summation extends over all subsets I of $\{1,\dots,n-1\}$ of cardinality $n-1-k$, where a product over an empty set is defined as 1. Then, for $\eta \in \mathscr{B}(\Sigma^n)$,

$$T_K(\mathbf{1}_\eta\varphi_k) = \Lambda_k(K,\eta),$$

which provides a representation of the k-th support measure of K in terms of the normal cycle of K, evaluated at the k-th Lipschitz-Killing form φ_k.

The construction of suitable tensor-valued differential forms $\varphi_k^{r,s}$, for $n \geq 3$ and $k \in \{1,\dots,n-2\}$, is slightly more involved. By a tensor-valued differential form we mean the following. If W is the vector space of symmetric tensors of a given rank, we identify $\bigwedge^{n-1}(\mathbb{R}^{2n}, W)$ and $\operatorname{Hom}(\bigwedge_{n-1}\mathbb{R}^{2n}, W)$. Then the map

$$\varphi_k^{r,s} : \mathbb{R}^{2n} \to \bigwedge^{n-1}(\mathbb{R}^{2n}, \mathbb{T}^{r+s+2}), \qquad (x,u) \mapsto \varphi_k^{r,s}(x,u),$$

is a differential form of degree $n-1$ on $\mathbb{R}^{2n}$ with coefficients in $\mathbb{T}^{r+s+2}$ (see [5, p. 351] for this terminology). In particular, this means that $\langle a, \varphi_k^{r,s}(x,u)\rangle \in \mathbb{T}^{r+s+2}$, for all $(x,u) \in \mathbb{R}^{2n}$ and $a \in \bigwedge_{n-1}\mathbb{R}^{2n}$. For the explicit definition of $\varphi_k^{r,s}$, we refer to [8, Sect. 4]. A straightforward calculation shows that

$$\langle \vartheta a, \varphi_k^{r,s}(\vartheta x, \vartheta u)\rangle = \vartheta\langle a, \varphi_k^{r,s}(x,u)\rangle,$$

for all $\vartheta \in \mathrm{O}(n)$, where in each case the natural operation of the rotation group is used (in particular, $\vartheta\xi := (\vartheta p, \vartheta q)$ for $\xi = (p,q) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$). As a result of the construction and by some calculations, which use that for a polytope P and a face F of P, for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$ with $x \in \operatorname{relint} F$ we have $k_i(x,u) = 0$ if and only if $b_i(x,u) \in L(F)$ and $k_i(x,u) = \infty$ if and only if $b_i(x,u) \in L(F)^\perp \cap u^\perp$, we obtain

$$T_P(\mathbf{1}_\eta \varphi_k^{r,s}) = \phi_k^{r,s,1}(P,\eta)$$

for all $P \in \mathscr{P}^n$ and $\eta \in \mathscr{B}(\Sigma^n)$.

It is known that

- T_K is a cycle for $K \in \mathscr{K}^n$ (see [14, Proposition 2.6]);
- the map $K \mapsto T_K$ is a valuation on $\mathscr{K}^n$ (see [14, Theorem 2.2]);
- $T_{K_i} \to T_K$ in the dual flat seminorm for currents, if $K_i, K \in \mathscr{K}^n$, $i \in \mathbb{N}$, and $K_i \to K$ in the Hausdorff metric, as $i \to \infty$ (see [14, Theorem 3.1], and for the dual flat seminorm, [5, Sect. 4.1.12, p. 367]).

In the next section, we prove a strengthened form of the continuity assertion stated in the third point, namely local Hölder continuity of the normal cycles of convex bodies with respect to the Hausdorff metric and the dual flat seminorm.

The third point above implies that if $f : \mathbb{R}^{2n} \to \mathbb{R}$ is of class C^∞, then the map

$$\mathscr{K}^n \to \mathbb{R}, \qquad K \mapsto T_K(f\varphi_k^{r,s}),$$

is continuous. But then the same is true if f is merely continuous. Hence, $(K,\eta) \mapsto T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$ is the weakly continuous extension of the map $(P,\eta) \mapsto \phi_k^{r,s,1}(P,\eta)$ from polytopes P to general convex bodies. Moreover, we have the following result (with (A)–(D) as formulated at the beginning of Sect. 2.5).

Theorem 2.18 *The map $\mathscr{K}^n \times \mathscr{B}(\Sigma^n) \to \mathbb{T}^{r+s+2}$, $(K,\eta) \mapsto T_K\left(\mathbf{1}_\eta \varphi_k^{r,s}\right)$, satisfies the properties* (A)–(D).

The next corollary then is an immediate consequence.

Corollary 2.19 *Let $r, s \in \mathbb{N}_0$ and $k \in \{1, \dots, n-2\}$. Then, for each $\eta \in \mathscr{B}(\Sigma^n)$, the map $K \mapsto \phi_k^{r,s,1}(K,\eta)$ is a valuation and Borel measurable on $\mathscr{K}^n$.*

Since the global functionals $\phi_k^{r,s,1}(P,\Sigma^n)$ are continuous, Alesker's characterization theorem must yield a representation for them. Such a representation was explicitly known before. In fact, for $r = 0$ it follows from another relation by McMullen (see [12, p. 269] and [10, Lemma 3.3]) that

$$\phi_k^{0,s,1}(P,\Sigma^n) = Q\Phi_k^{0,s}(P) - 2\pi(s+2)\,\Phi_k^{0,s+2}(P).$$

The general case is covered by Hug et al. [10, p. 505].

It is instructive to express the new local tensor valuations $\phi_k^{r,s,1}(K,\cdot)$ for a general convex body K in terms of the generalized curvatures $k_i(x,u)$ and the corresponding principal directions of curvature $b_i(x,u)$, $i=1,\dots,n-1$. Let $n \geq 3$ and $k \in \{1,\dots,n-2\}$. Then a short calculation shows that

$$\phi_k^{r,s,1}(K,\eta) \tag{2.39}$$
$$= C_{n,k}^{r,s} \int_{\eta\cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \sum_{\substack{|I|=n-1-k\\ i\notin I}} \frac{\prod_{j\in I} k_j(x,u)}{\mathbb{K}(x,u)} \mathscr{H}^{n-1}(\mathrm{d}(x,u)).$$

If $k=1$, then

$$\phi_1^{r,s,1}(K,\eta)$$
$$= C_{n,1}^{r,s} \int_{\eta\cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \frac{\prod_{j:j\neq i} k_j(x,u)}{\mathbb{K}(x,u)} \mathscr{H}^{n-1}(\mathrm{d}(x,u)),$$

and for $k=n-2$, we have

$$\phi_{n-2}^{r,s,1}(K,\eta)$$
$$= C_{n,n-2}^{r,s} \int_{\eta\cap \mathbf{nc}(K)} x^r u^s \sum_{i=1}^{n-1} b_i(x,u)^2 \sum_{j:j\neq i} \frac{k_j(x,u)}{\mathbb{K}(x,u)} \mathscr{H}^{n-1}(\mathrm{d}(x,u)).$$

For $n=3$, these two special cases coincide and we get

$$\phi_1^{r,s,1}(K,\eta)$$
$$= C_{3,1}^{r,s} \int_{\eta\cap \mathbf{nc}(K)} x^r u^s \frac{k_1(x,u)b_2(x,u)^2 + k_2(x,u)b_1(x,u)^2}{\mathbb{K}(x,u)} \mathscr{H}^{2}(\mathrm{d}(x,u)).$$

For a given convex body K of class C^2, we write $u(x)$ for the unique exterior unit normal of K at the boundary point $x \in \partial K$ of K. (We omit the reference to K in our notation.) An application of the coarea formula then yields

$$\phi_k^{r,s,1}(K,\eta)$$
$$= C_{n,k}^{r,s} \int_{\partial K} \mathbf{1}_\eta(x,u(x)) x^r u(x)^s \sum_{i=1}^{n-1} b_i(x)^2 \sum_{\substack{|I|=n-1-k\\ i\notin I}} \prod_{j\in I} k_j(x) \mathscr{H}^{n-1}(\mathrm{d}x),$$

where the $k_j(x)$ are the principal curvatures and the unit vectors $b_j(x)$ give the principal directions of curvature of K at $x \in \partial K$ (again the dependence on K is not indicated by our notation). In particular, for a convex body K in $\mathbb{R}^3$ with a C^2 boundary we get

$$\phi_1^{r,s,1}(K,\eta)$$
$$= C_{3,1}^{r,s}\int_{\partial K} \mathbf{1}_\eta(x,u(x))x^r u(x)^s\big(k_1(x)b_2(x)^2 + k_2(x)b_1(x)^2\big)\,\mathscr{H}^2(\mathrm{d}x).$$

2.7 Hölder Continuity of Normal Cycles of Convex Bodies

The normal cycle T_K of a convex body K in $\mathbb{R}^n$ has a useful continuity property, which we have used in the previous section. If K_i, $i \in \mathbb{N}$, and K are convex bodies in $\mathbb{R}^n$ and $K_i \to K$ in the Hausdorff metric, as $i \to \infty$, then $T_{K_i} \to T_K$ in the dual flat seminorm for currents (cf. [5, Sect. 1.12, p. 367]). This was stated without proof in [19, p. 251] and was proved in [14, Theorem 3.1]; see also [6, Theorem 3.1]. The continuity property has been used in the theory of valuations on manifolds (see, for instance, [2]). It is also a crucial ingredient in [8], in the course of the proof of a classification theorem for local tensor valuations on the space of convex bodies, as we have seen in the previous section.

The purpose of this section is to obtain a quantitative improvement of the preceding continuity result, in the form of a Hölder estimate. As usual we equip $\mathscr{K}^n$ with the Hausdorff metric d_H. We denote by $\mathscr{E}^{n-1}(\mathbb{R}^{2n}) = \mathscr{E}(\mathbb{R}^{2n}, \bigwedge^{n-1}\mathbb{R}^{2n})$ the vector space of all differential forms of degree $n-1$ on $\mathbb{R}^{2n}$ with real coefficients of class C^∞.

Theorem 2.20 *Let $K, L \in \mathscr{K}^n$, and let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_1 \times \mathbb{S}^{n-1}$ and $L_1 \times \mathbb{S}^{n-1}$. Then, for each $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$,*

$$|T_K(\varphi) - T_L(\varphi)| \le C(M,\varphi)\, d_H(K,L)^{\frac{1}{2n+1}},$$

where $C(M,\varphi)$ is a constant which depends (for given dimension) on M and on the Lipschitz constant and the sup-norm of φ on M.

According to the definition of the dual flat seminorm, this result can be interpreted as local Hölder continuity of the normal cycles of convex bodies with respect to the Hausdorff metric and the dual flat seminorm. A similar, but essentially different quantitative result is obtained in [3, Theorem 2]. It refers to more general sets and is, therefore, less explicit. On the other hand, its restriction to convex bodies does not yield the present result, since at least one of the sets in [3] has to be bounded by a submanifold of class C^2. We have not been able to decide whether the stability exponent $1/(2n+1)$ in Theorem 2.20 can be improved.

It remains to prove Theorem 2.20. We continue to use the same notation as in Federer's [5] book, in order to facilitate the comparison. For the scalar product of vectors $x, y \in \mathbb{R}^n$, however, we continue to write $x \cdot y$; the induced norm is denoted by $|\cdot|$. The same notation is used also for other Euclidean spaces which will come up in the following. We identify $\mathbb{R}^n$ and its dual space via the given scalar product.

Given an inner product space $(V, \cdot)$ with norm $|\cdot|$ we obtain an inner product on $\bigwedge_m V$. For $\xi, \eta \in \bigwedge_m V$ with $\xi = v_1 \wedge \cdots \wedge v_m$ and $\eta = w_1 \wedge \cdots \wedge w_m$, where $v_i, w_j \in V$, we define $\xi \cdot \eta = \det(\langle v_i, w_j\rangle_{i,j=1}^m)$. This is independent of the particular representation of ξ, η. For general $\xi, \eta \in \bigwedge_m V$ the inner product is defined by linear extension, and then we put $|\xi| := \sqrt{\xi \cdot \xi}$ for $\xi \in \bigwedge_m V$. If $(b_1, \ldots, b_n)$ is an orthonormal basis of V, then the m-vectors $b_{i_1} \wedge \cdots \wedge b_{i_m}$ with $1 \leq i_1 < \cdots < i_m \leq n$ form an orthonormal basis of $\bigwedge_m V$. Moreover, if $\xi \in \bigwedge_p V$ or $\eta \in \bigwedge_q V$ is simple, then

$$|\xi \wedge \eta| \leq |\xi|\,|\eta|. \tag{2.40}$$

Let $(b_1, \ldots, b_n)$ be an orthonormal basis of V, and let $(b_1^*, \ldots, b_n^*)$ be the dual basis in $V^* = \bigwedge^1 V$. We endow $\bigwedge^m V$ (which is identified with $\bigwedge_m V^*$) with the inner product for which the vectors $b_{i_1}^* \wedge \cdots \wedge b_{i_m}^*$, for $1 \leq i_1 < \cdots < i_m \leq n$, are an orthonormal basis. Then

$$|\langle \xi, \Phi\rangle| \leq |\xi|\,|\Phi| \tag{2.41}$$

for $\xi \in \bigwedge_m V$ and $\Phi \in \bigwedge^m V$. The preceding facts are essentially taken from [5, Sect. 1.7].

Finally, if V is an n-dimensional inner product space, then *comass* and *mass* are defined as in [5, Sect. 1.8]. In particular, for $\Phi \in \bigwedge^m V$ the comass $\|\Phi\|$ of Φ satisfies $\|\Phi\| = |\Phi|$ if Φ is simple. Moreover, for $\xi \in \bigwedge_m V$ the mass $\|\xi\|$ of ξ satisfies $\|\xi\| = |\xi|$ if ξ is simple.

The proof of Theorem 2.20 will be preceded by a sequence of lemmas. In order to obtain an upper bound for $|T_K - T_L|$, we first establish an upper bound for $|T_{A_\varepsilon} - T_A|$, for $A \in \{K, L\}$ and $\varepsilon \in [0, 1]$, which is done in Lemma 2.21. Then we derive an upper bound for $|T_{K_\varepsilon} - T_{L_\varepsilon}|$ under the assumption that the Hausdorff distance of K and L is sufficiently small. This bound is provided in Lemma 2.26, which in turn is based on four preparatory lemmas.

Lemma 2.21 *Let $K \in \mathcal{K}^n$ and $\varepsilon \in [0, 1]$. Let $\varphi \in \mathcal{E}^{n-1}(\mathbb{R}^{2n})$. Then*

$$|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \leq C(K, \varphi)\, \varepsilon,$$

where $C(K, \varphi)$ is a real constant, which depends on the maximum and the Lipschitz constant of φ on $K_1 \times \mathbb{S}^{n-1}$ and on $\mathcal{H}^{n-1}(\partial K_1)$.

Proof We consider the bi-Lipschitz map

$$F_\varepsilon : \mathbf{nc}(K) \to \mathbf{nc}(K_\varepsilon), \qquad (x,u) \mapsto (x+\varepsilon u, u).$$

The extension of F_ε to all $(x,u) \in \mathbb{R}^{2n}$ by $F_\varepsilon(x,u) := (x+\varepsilon u, u)$ is differentiable for all $(x,u) \in \mathbb{R}^{2n}$. By Federer [5, Theorem 3.2.22 (1)], for $\mathscr{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$ the approximate $(n-1)$-dimensional Jacobian of F_ε (see [5, p. 256, Corollary 3.2.20]) satisfies

$$\operatorname{ap} J_{n-1} F_\varepsilon(x,u) = \left\| \bigwedge\nolimits_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u) \right\| > 0, \tag{2.42}$$

and the simple orienting $(n-1)$-vectors $a_K(x,u)$ and $a_{K_\varepsilon}(x+\varepsilon u, u)$ are related by

$$a_{K_\varepsilon}(x+\varepsilon u, u) = \frac{\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u)}{\left\|\bigwedge_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u)\right\|}. \tag{2.43}$$

The orientations coincide, since

$$\left\langle \bigwedge\nolimits_{n-1} (\varPi_1 + \varrho\, \varPi_2) a_K(x,u) \wedge u, \varOmega_n \right\rangle > 0$$

for all $\varrho > 0$. Here, as before $\varPi_1, \varPi_2$ are the projections to the components of $\mathbb{R}^n \times \mathbb{R}^n$. Thus, first using the coarea theorem [5, Theorem 3.2.22] and then (2.42) and (2.43), we get

$$\begin{aligned} T_{K_\varepsilon}(\varphi) &= \int_{\mathbf{nc}(K_\varepsilon)} \langle a_{K_\varepsilon}, \varphi\rangle \, \mathrm{d}\mathscr{H}^{n-1} \\ &= \int_{\mathbf{nc}(K)} \langle a_{K_\varepsilon} \circ F_\varepsilon(x,u), \varphi \circ F_\varepsilon(x,u)\rangle \ \operatorname{ap} J_{n-1} F_\varepsilon(x,u)\, \mathscr{H}^{n-1}(\mathrm{d}(x,u)) \\ &= \int_{\mathbf{nc}(K)} \left\langle \bigwedge\nolimits_{n-1} \operatorname{ap} DF_\varepsilon(x,u) a_K(x,u), \varphi \circ F_\varepsilon(x,u) \right\rangle \mathscr{H}^{n-1}(\mathrm{d}(x,u)). \end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned} &|T_{K_\varepsilon}(\varphi) - T_K(\varphi)| \\ &\quad \le \int_{\mathbf{nc}(K)} \Big\{ \Big| \Big\langle \Big(\bigwedge\nolimits_{n-1} \operatorname{ap} DF_\varepsilon(x,u) - \bigwedge\nolimits_{n-1} \mathrm{id} \Big) a_K(x,u), \varphi \circ F_\varepsilon(x,u) \Big\rangle \Big| \\ &\qquad + |\langle a_K(x,u), \varphi(x+\varepsilon u, u) - \varphi(x,u)\rangle| \Big\}\, \mathscr{H}^{n-1}(\mathrm{d}(x,u)). \end{aligned}$$

We have

$$\begin{aligned}&\left|\left\langle\left(\bigwedge\nolimits_{n-1}\operatorname{ap}DF_\varepsilon(x,u)-\bigwedge\nolimits_{n-1}\operatorname{id}\right)a_K(x,u),\varphi\circ F_\varepsilon(x,u)\right\rangle\right|\\&\qquad\le\left|\varphi(x+\varepsilon u,u)\|\left(\bigwedge\nolimits_{n-1}\operatorname{ap}DF_\varepsilon(x,u)-\bigwedge\nolimits_{n-1}\operatorname{id}\right)a_K(x,u)\right|,\end{aligned}$$

where we used (2.41). Now $a_K(x,u)$ is of the form $\bigwedge_{i=1}^{n-1}(v_i,w_i)$ with suitable $(v_i,w_i)\in\mathbb{R}^{2n}$ and $|v_i|^2+|w_i|^2=1$. Moreover, we have $DF_\varepsilon(x,u)(v,w)=(v+\varepsilon w,w)$, for all $(v,w)\in\mathbb{R}^{2n}$. Writing $z_i^0:=v_i$, $z_i^1:=w_i$, we have

$$\begin{aligned}&\left|\left(\bigwedge\nolimits_{n-1}\operatorname{ap}DF_\varepsilon(x,u)-\bigwedge\nolimits_{n-1}\operatorname{id}\right)a_K(x,u)\right|\\&\quad=\left|\bigwedge_{i=1}^{n-1}(v_i+\varepsilon w_i,w_i)-\bigwedge_{i=1}^{n-1}(v_i,w_i)\right|\\&\quad=\left|\sum_{\alpha_1,\dots,\alpha_{n-1}\in\{0,1\}}\varepsilon^{\sum_{j=1}^{n-1}\alpha_j}\bigwedge_{i=1}^{n-1}(z_i^{\alpha_i},w_i)-\bigwedge_{i=1}^{n-1}(z_i^0,w_i)\right|\\&\quad\le\varepsilon\sum_{\substack{\alpha_1,\dots,\alpha_{n-1}\in\{0,1\}\\ \sum_{j=1}^{n-1}\alpha_j\ge1}}\left|\bigwedge_{i=1}^{n-1}(z_i^{\alpha_i},w_i)\right|\\&\quad\le c(n)\varepsilon,\end{aligned}$$

where we used (2.40) and the fact that $|(v_i,w_i)|=1$ and $|(w_i,w_i)|\le 2$. We deduce that

$$|\varphi(x+\varepsilon u,u)|\left|(\bigwedge\nolimits_{n-1}\operatorname{ap}DF_\varepsilon(x,u)-\bigwedge\nolimits_{n-1}\operatorname{id})a_K(x,u)\right|\le C_1(K,\varphi)\varepsilon.$$

Furthermore, again by (2.41) we get

$$|\langle a_K(x,u),\varphi(x+\varepsilon u,u)-\varphi(x,u)\rangle|\le|\varphi(x+\varepsilon u,u)-\varphi(x,u)|\le C_2(K,\varphi)\,\varepsilon.$$

Thus we conclude that

$$|T_{K_\varepsilon}(\varphi)-T_K(\varphi)|\le C_3(K,\varphi)\,\varepsilon\,\mathscr{H}^{n-1}(\mathbf{nc}(K)).$$

Since $F:\partial K_1\to\mathbf{nc}(K)$, $z\mapsto(p(K,z),z-p(K,z))$, is Lipschitz with Lipschitz constant bounded from above by 3, the assertion follows. □

A convex body $K\in\mathscr{K}^n$ is said to be ε-*smooth* (for some $\varepsilon>0$), if $K=K'+\varepsilon B^n$ for some $K'\in\mathscr{K}^n$. For a nonempty set $A\subset\mathbb{R}^n$, we define the distance from A to $x\in\mathbb{R}^n$ by $d(A,x):=\inf\{|a-x|:a\in A\}$. The signed distance is defined by $d^*(A,x):=d(A,x)-d(\mathbb{R}^n\setminus A,x)$, $x\in\mathbb{R}^n$, if $A,\mathbb{R}^n\setminus A\neq\emptyset$. If K is ε-smooth, then

∂K has positive reach. More precisely, if $x \in \mathbb{R}^n$ satisfies $d(\partial K, x) < \varepsilon$, then there is a unique point $p(\partial K, x) \in \partial K$ such that $d(\partial K, x) = |p(\partial K, x) - x|$.

Lemma 2.22 *Let* $\varepsilon \in (0, 1)$ *and* $\delta \in (0, \varepsilon/2)$. *Let* $K, L \in \mathcal{K}^n$ *be* ε*-smooth and assume that* $d_H(K, L) \leq \delta$. *Then*

$$p : \partial K \to \partial L, \qquad x \mapsto p(\partial L, x),$$

is well-defined, bijective, bi-Lipschitz with $\mathrm{Lip}(p) \leq \varepsilon/(\varepsilon - \delta)$, *and* $|p(x) - x| \leq \delta$ *for all* $x \in \partial K$.

Proof Since $d_H(K, L) \leq \delta$, we have $K \subset L + \delta B^n$, $L \subset K + \delta B^n$, and a separation argument yields that

$$\{x \in L : d(\partial L, x) \geq \delta\} \subset K. \tag{2.44}$$

This shows that $\partial K \subset \{z \in \mathbb{R}^n : d(\partial L, z) \leq \delta\}$ and therefore the map p is well-defined on ∂K and $|p(x) - x| \leq \delta$ for all $x \in \partial K$. By Federer [4, Theorem 4.8 (8)] it follows that $\mathrm{Lip}(p) \leq \varepsilon/(\varepsilon - \delta)$. Since L is ε-smooth, for $y \in \partial L$ there is a unique exterior unit normal of L at y, which we denote by $u =: u_L(y)$ (here we slightly deviate from our previous notation where this vector was denoted by $u(L, y)$). Put $y_0 := y - \varepsilon u$ and note that $y_0 + (\varepsilon - \delta)B^n \subset K \cap L$ by (2.44). Then $x \in \partial K$ is uniquely determined by the condition $\{x\} = (y_0 + [0, \infty)u) \cap \partial K$ and satisfies $p(x) = y$. This shows that p is surjective.

Now let $x_1, x_2 \in \partial K$ satisfy $p(x_1) = p(x_2) =: p_0 \in \partial L$. Since there is a ball B of radius ε with $p_0 \in B \subset L$, the points $x_1, x_2 \in \partial K$ are on the line through p_0 and the center of B. By (2.44), they cannot be on different sides of p_0, hence $x_1 = x_2$. This shows that the map p is also injective. If $d^*(\partial K, \cdot) : \mathbb{R}^n \to \partial K$ denotes the signed distance function of ∂K, then $q : \partial L \to \partial K$, $z \mapsto z - d^*(\partial K, z)u_L(z)$, is the inverse of p. Since the signed distance function is Lipschitz, Lemma 2.23 below shows that q is Lipschitz as well. □

The following lemma provides a simple argument for the fact that the spherical image map of an ε-smooth convex body is Lipschitz with Lipschitz constant at most $1/\varepsilon$.

Lemma 2.23 *Let* $K \in \mathcal{K}^n$ *be* ε*-smooth,* $\varepsilon > 0$. *Then the spherical image map* u_K *is Lipschitz with Lipschitz constant* $1/\varepsilon$.

Proof Let $x, y \in \partial K$, and define $u := u_K(x)$, $v := u_K(y)$. Then $x - \varepsilon u + \varepsilon v \in x - \varepsilon u + \varepsilon B^n \subset K$, and hence $(x - \varepsilon u + \varepsilon v - y) \cdot v \leq 0$. This yields

$$\varepsilon(v - u) \cdot v \leq (y - x) \cdot v. \tag{2.45}$$

By symmetry, we also have $\varepsilon(u - v) \cdot u \leq (x - y) \cdot u$, and therefore

$$\varepsilon(v - u) \cdot (-u) \leq (y - x) \cdot (-u). \tag{2.46}$$

Addition of (2.45) and (2.46) yields

$$\varepsilon|v-u|^2 \le (y-x)\cdot(v-u) \le |y-x|\,|v-u|,$$

which implies the assertion. □

Lemma 2.24 *Let* $\varepsilon \in (0,1)$ *and* $\delta \in (0,\varepsilon/2)$*. Let* $K, L \in \mathscr{K}^n$ *be* ε*-smooth and assume that* $d_H(K,L) \le \delta$*. Put* $p(x) := p(\partial L, x)$ *for* $x \in \partial K$*. Then*

$$G : \mathbf{nc}(K) \to \mathbf{nc}(L), \qquad (x,u) \mapsto (p(x), u_L(p(x))),$$

is bijective, bi-Lipschitz with $\mathrm{Lip}(G) \le 2/(\varepsilon-\delta) \le 4/\varepsilon$*, and*

$$|G(x,u)-(x,u)| \le \delta + 2\sqrt{\delta/\varepsilon}$$

for all $(x,u) \in \mathbf{nc}(K)$*.*

Proof It follows from Lemma 2.22 that G is bijective. Then, for $(x,u),(y,v) \in \mathbf{nc}(K)$ we get

$$\begin{aligned}
|G(x,u)-G(y,v)| &\le |p(x)-p(y)| + |u_L(p(x))-u_L(p(y))| \\
&\le \frac{\varepsilon}{\varepsilon-\delta}|x-y| + \frac{1}{\varepsilon}\frac{\varepsilon}{\varepsilon-\delta}|x-y| \\
&\le \frac{\varepsilon+1}{\varepsilon-\delta}|x-y| \\
&\le \frac{2}{\varepsilon-\delta}|(x,u)-(y,v)|,
\end{aligned}$$

where we have used again Lemmas 2.22 and 2.23. Let $x \in \partial K$ and $z := p(x) \in \partial L$. We want to bound $u_L(z)\cdot u_K(x)$ from below. If $x \notin L$, then

$$\mathrm{conv}\big(\{x\} \cup (z-\varepsilon u_L(z) + (\varepsilon-\delta)B^n)\big) \subset K,$$

and therefore

$$u_L(z)\cdot u_K(x) \ge \frac{\varepsilon-\delta}{\varepsilon+\delta} \ge 1 - \frac{2\delta}{\varepsilon}.$$

If $x \in L$, then in a similar way we obtain

$$u_L(z)\cdot u_K(x) \ge \frac{\varepsilon-\delta}{\varepsilon} \ge 1 - \frac{\delta}{\varepsilon},$$

hence

$$u_L(z) \cdot u_K(x) \geq 1 - \frac{2\delta}{\varepsilon} \tag{2.47}$$

holds for all $x \in \partial K$. Thus

$$|u_L(z) - u_K(x)| \leq 2\sqrt{\delta/\varepsilon},$$

which finally implies that, for all $(x, u) \in \mathbf{nc}(K)$,

$$\begin{aligned} |G(x,u) - (x,u)| &\leq |p(x) - x| + |u_L(p(x)) - u_K(x)| \\ &\leq \delta + 2\sqrt{\delta/\varepsilon}. \end{aligned}$$

Since $G^{-1} : \mathbf{nc}(L) \to \mathbf{nc}(K)$ is given by $G^{-1}(z,u) = (q(z), u_K(q(z)))$ (with q as defined in the proof of Lemma 2.22), it follows that also G^{-1} is Lipschitz. □

Next we show that under suitable assumptions $\bigwedge_{n-1} DG(x,u)$ is an orientation preserving map from the approximate tangent space of $\mathbf{nc}(K)$ to the approximate tangent space of $\mathbf{nc}(L)$. It seems that a corresponding fact is not provided in the proofs of related assertions in the literature.

Lemma 2.25 *Let $\varepsilon \in (0,1)$ and $\delta \in (0, \varepsilon/(4n))$. Let $K, L \in \mathcal{K}^n$ be ε-smooth and assume that $d_H(K,L) \leq \delta$. Then, for $\mathcal{H}^{n-1}$-almost all $(x,u) \in \mathbf{nc}(K)$, the $(n-1)$-vector $\bigwedge_{n-1} DG(x,u) a_K(x,u) \in \mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \mathbf{nc}(L), G(x,u))$ has the same orientation as $a_L(G(x,u))$.*

Proof Let $x \in \partial K$, $u := u_K(x)$, and $\bar{x} := p(x)$, hence $d(\partial L, x) = |x - \bar{x}|$. The orientation of $\mathrm{Tan}^{n-1}(\partial K, x)$ is determined by an arbitrary orthonormal basis $(b_1(x), \dots, b_{n-1}(x))$ of $u^\perp$ with $\Omega_n(b_1(x), \dots, b_{n-1}(x), u) = 1$. Similarly, any orthonormal basis $(\bar{b}_1(\bar{x}), \dots, \bar{b}_{n-1}(\bar{x}), \bar{u})$ with $\bar{u} := u_L(p(x))$ determines the orientation of the space $\mathrm{Tan}^{n-1}(\partial L, p(x))$. Since G is bi-Lipschitz, we can assume that $(x,u) \in \mathbf{nc}(K)$ is such that all differentials exist that are encountered in the proof. Moreover, we can also assume that $\bigwedge_{n-1} DG(x,u) a_K(x,u)$ spans $\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \mathbf{nc}(L), G(x,u))$, where we write again G for a Lipschitz extension of the given map G to $\mathbb{R}^{2n}$. In the following, we put $b_i := b_i(x)$ and $\bar{b}_i := \bar{b}_i(\bar{x})$ for $i = 1, \dots, n-1$.

By our previous discussion, the differentials of the maps $\mathbf{nc}(K) \to \partial K$, $(x,u) \mapsto x$, and $\partial L \to \mathbf{nc}(L)$, $z \mapsto (z, u_L(z))$, are orientation preserving. Hence, it remains to be shown that the differential of $p : \partial K \to \partial L$, $x \mapsto p(x)$, is orientation preserving, that is,

$$\Delta := \Omega_n(Dp(x)(b_1), \dots, Dp(x)(b_{n-1}), \bar{u}) > 0.$$

First, we assume that $x \neq \bar{x}$, that is, $x \notin \partial L$. Since $Dp(x)(\bar{u}) = 0$, we get

$$Dp(x)(b_i) = \sum_{j=1}^{n-1} b_i \cdot \bar{b}_j \, Dp(x)(\bar{b}_j),$$

and thus

$$\Delta = \det(B)\, \Omega_n(Dp(x)(\bar{b}_1), \ldots, Dp(x)(\bar{b}_{n-1}), \bar{u}),$$

where $B = (B_{ij})$ with $B_{ij} := b_i \cdot \bar{b}_j$ for $i, j \in \{1, \ldots, n-1\}$. We choose $\bar{b}_1, \ldots, \bar{b}_{n-1}$ as principal directions of curvature of ∂L at $\bar{x} = p(x)$. Then $Dp(x)(\bar{b}_i) = \tau_i \, \bar{b}_i$ with

$$\tau_i := 1 - d(\partial L, x) k_i \left(\partial L, \bar{x}, \frac{x - \bar{x}}{|x - \bar{x}|} \right) > 0,$$

for $i = 1, \ldots, n-1$, where $k_1(\partial L, \cdot), \ldots, k_{n-1}(\partial L, \cdot)$ are the generalized curvatures of ∂L as functions on the normal bundle of ∂L. Here we use that L is ε-smooth, hence ∂L has positive reach, $d(\partial L, x) < \varepsilon$ and

$$\left| k_i \left(\partial L, \bar{x}, \frac{x - \bar{x}}{|x - \bar{x}|} \right) \right| \leq 1/\varepsilon.$$

Hence it follows that $\Delta > 0$ if we can show that $\det(B) > 0$. Let $\tilde{B} = (\tilde{B}_{ij})$ be defined by $\tilde{B}_{ij} := B_{ij}$, $\tilde{B}_{in} := b_i \cdot \bar{u}$, $\tilde{B}_{nj} := u \cdot \bar{b}_j$, and $\tilde{B}_{nn} := u \cdot \bar{u}$, for $i, j \in \{1, \ldots, n-1\}$. Then

$$\begin{aligned} 1 &= \Omega_n(b_1, \ldots, b_{n-1}, u)\, \Omega_n(\bar{b}_1, \ldots, \bar{b}_{n-1}, \bar{u}) = \det(\tilde{B}) \\ &\leq u \cdot \bar{u} \det(B) + \sum_{i=1}^{n-1} |b_i \cdot \bar{u}| \cdot 1 \\ &\leq u \cdot \bar{u} \det(B) + \sqrt{n-1}\sqrt{1 - (u \cdot \bar{u})^2}. \end{aligned}$$

From (2.47) and our assumptions, we get $u \cdot \bar{u} \geq 1 - (2\delta)/\varepsilon \geq 1 - 1/(2n)$, and therefore

$$\sqrt{1 - (u \cdot \bar{u})^2} \leq \sqrt{1/n}.$$

Thus

$$1 < u \cdot \bar{u} \det(B) + 1,$$

which implies that $\det(B) > 0$.

Finally, we have to consider the case where $x \in \partial L$. For $\mathscr{H}^{n-1}$-almost all $x \in \partial K \cap \partial L$, we have $\mathrm{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner (\partial K \cap \partial L), x) = u^\perp$ and $Dp(x) = \mathrm{id}_{u^\perp}$, since $p(z) = z$ for all $z \in \partial K \cap \partial L$. Hence, $\Delta = \Omega_n(b_1, \ldots, b_{n-1}, \bar{u}) = u \cdot \bar{u} > 0$. □

Lemma 2.26 *Let $\varepsilon \in (0,1)$ and $\delta \in (0, \varepsilon/(4n))$. Let $K, L \in \mathscr{K}^n$ be ε-smooth and assume that $d_H(K,L) \leq \delta$. Let $M \subset \mathbb{R}^{2n}$ be a compact convex set containing $K_{1-\varepsilon} \times \mathbb{S}^{n-1}$ and $L_{1-\varepsilon} \times \mathbb{S}^{n-1}$ in its interior. Then*

$$|T_K(\varphi) - T_L(\varphi)| \leq C(M,\varphi)(4/\varepsilon)^{n-1}(\delta + 2\sqrt{\delta/\varepsilon})$$

for $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$, where $C(M,\varphi)$ is a constant which depends on the sup-norm and the Lipschitz constant of φ on M, and on $\mathscr{H}^{n-1}(\partial K_1)$.

Proof Let G be as before (or a Lipschitz extension to the whole space with the same Lipschitz constant [5, Theorem 2.10.43]). Then [5, Theorem 4.1.30] implies that the pushforward $G_\sharp T_K$ of T_K under the Lipschitz map G (see [5, Sect. 4.1.7, p. 359] for the pushforward of a current under a smooth map and [5, Sect. 4.1.14] for the extension to Lipschitz maps) satisfies

$$T_L = G_\sharp T_K,$$

since $\bigwedge_{n-1} DG$ preserves the orientation of the orienting $(n-1)$-vectors, by Lemma 2.25. (In [14] a corresponding fact is stated without further comment.) Recall the definitions of the dual flat seminorm $\mathbf{F}_M$ from [5, 4.1.12] and of the mass $\mathbf{M}$ (of a current) from [5, p. 358]. Using [5, 4.1.14] (since G is not necessarily smooth of class ∞), $\partial T_K = 0$ (that is, T_K is a cycle), the fact that T_K has compact support contained in the interior of M and Lemma 2.24, we get

$$\begin{aligned} \mathbf{F}_M(T_L - T_K) &= \mathbf{F}_M(G_\sharp T_K - T_K) \\ &\leq \mathbf{M}(T_K) \cdot \|G - \mathrm{id}\|_{\mathbf{nc}(K),\infty} \cdot (4/\varepsilon)^{n-1} \\ &\leq \mathscr{H}^{n-1}(\partial K_1)(4/\varepsilon)^{n-1}(\delta + 2\sqrt{\delta/\varepsilon}), \end{aligned}$$

where $\|G - \mathrm{id}\|_{\mathbf{nc}K),\infty} := \sup\{|G(x,u) - (x,u)| : (x,u) \in \mathbf{nc}(K)\}$. The assertion now follows from the definition of $\mathbf{F}_M$, since $\|d\varphi\|$ can be bounded in terms of the sup-norm and the Lipschitz constant of φ on M. □

Now we are in a position to complete the proof of Theorem 2.20.

Proof of Theorem 2.20 Let $\varphi \in \mathscr{E}^{n-1}(\mathbb{R}^{2n})$. Let $\delta := d_H(K,L) > 0$ and $\varepsilon := \delta^{\frac{1}{2n+1}}$. Assume that $\delta < (4n)^{-\frac{2n+1}{2n}}$. Then $\delta < \varepsilon/(4n)$. Lemma 2.21 implies that

$$|T_K(\varphi) - T_{K_\varepsilon}(\varphi)| \leq C_1(M,\varphi)\,\varepsilon,$$

$$|T_L(\varphi) - T_{L_\varepsilon}(\varphi)| \leq C_2(M,\varphi)\,\varepsilon.$$

Since K_ε and L_ε are ε-smooth, $d_H(K_\varepsilon, L_\varepsilon) = \delta$, $(K_\varepsilon)_{1-\varepsilon} = K_1$ and $(L_\varepsilon)_{1-\varepsilon} = L_1$, Lemma 2.26 shows that

$$|T_{K_\varepsilon}(\varphi) - T_{L_\varepsilon}(\varphi)| \leq C_3(M,\varphi)(4/\varepsilon)^{n-1}(\delta + 2\sqrt{\delta/\varepsilon}).$$

The triangle inequality then yields

$$\begin{aligned}|T_K(\varphi) - T_L(\varphi)| &\leq C_4(M,\varphi)(\varepsilon + \varepsilon^{1-n}\delta + \varepsilon^{1-n}\sqrt{\delta/\varepsilon}) \\ &\leq C_5(M,\varphi)\,\delta^{\frac{1}{2n+1}}.\end{aligned}$$

If $\delta \geq (4n)^{-\frac{2n+1}{2n}}$, we simply adjust the constant. □

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Chapter 3
Structures on Valuations

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Abstract In recent years on the space of translation invariant continuous valuations there have been discovered several canonical structures. Some of them turned out to be important for applications in integral geometry. In this chapter we review the relevant background and the main properties of the following new structures: product, convolution, Fourier type transform, and pull-back and push-forward of valuations under linear maps.

3.1 Preliminaries

Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathrm{Val}(V)$ denote the space of translation invariant continuous valuations on $\mathscr{K}(V)$. We have the following important result called McMullen's decomposition [17] with respect to degrees of homogeneity:

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{n} \mathrm{Val}_i(V).$$

It turns out that one can classify valuations in degrees of homogeneity 0, n, and $n-1$:

Theorem 3.1

(i) *(Obvious)* $\mathrm{Val}_0(V) = \mathbb{C} \cdot \chi$.
(ii) *(Hadwiger [14])* $\mathrm{Val}_n(V) = \mathbb{C} \cdot \mathrm{vol}_n$.
(iii) *(McMullen [18]) Let us describe* $\mathrm{Val}_{n-1}(V)$. *Fix a Euclidean metric on V for convenience. For any $\phi \in \mathrm{Val}_{n-1}(V)$ there exists a continuous function*

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$f\colon S^{n-1}\to \mathbb{C}$ such that for any $K\in \mathscr{K}(V)$

$$\phi(K)=\int_{S^{n-1}} f(\omega)dS(K,\omega). \tag{3.1}$$

Moreover the function f is defined uniquely by ϕ up to addition of a linear functional. Conversely any above expression belongs to $\mathrm{Val}_{n-1}(V)$. *(Here $S(K,\cdot)$ denotes the surface area measure of a convex body K, see [21].)*

Let us also state a very important characterization of *simple* translation invariant continuous valuations due to Klain [15] and Schneider [20] from 1995 which is used a lot in the theory. A valuation is called simple if it vanishes on convex sets of dimension less than $\dim V$.

Theorem 3.2 (Klain–Schneider) *Let ϕ be a simple continuous translation invariant valuation on $\mathscr{K}(V)$. Fix a Euclidean metric on V for convenience; $n:=\dim V$. Then ϕ can be presented*

$$\phi(K)=a\cdot \mathrm{vol}(K)+\int_{S^{n-1}} f(\omega)dS(K,\omega) \textit{ for any } K\in\mathscr{K}(V),$$

where $a\in\mathbb{C}$, f is a continuous odd function on S^{n-1}; the constant a is determined uniquely, and f is unique up to a linear functional. Furthermore any such expression (with f being odd) is a simple translation invariant continuous valuation.

We have further decomposition with respect to parity:

$$\mathrm{Val}_i(V)=\mathrm{Val}_i^+(V)\oplus \mathrm{Val}_i^-(V).$$

The group $\mathrm{GL}(V)$ acts linearly and continuously on $\mathrm{Val}(V)$ preserving the above decompositions:

$$g(\phi)(K)=\phi(g^{-1}K),$$

for any $g\in \mathrm{GL}(V), \phi\in \mathrm{Val}(V), K\in\mathscr{K}(V)$.

Theorem 3.3 (Irreducibility Theorem, Alesker [1]) *For any i the spaces* $\mathrm{Val}_i^{\pm}(V)$ *are topologically irreducible representations of* $\mathrm{GL}(V)$, *i.e. they have no proper* $\mathrm{GL}(V)$*-invariant closed subspaces.*

Remark 3.4 This theorem easily implies the so called McMullen's conjecture which says that linear combinations of valuations of the form $\mathrm{vol}(\cdot+A)$ where $A\in\mathscr{K}(V)$ are dense in $\mathrm{Val}(V)$.

Definition 3.5 A valuation $\phi \in \mathrm{Val}(V)$ is called *smooth* if the map $\mathrm{GL}(V) \to \mathrm{Val}(V)$ given by $g \mapsto g(\phi)$ is C^∞-differentiable.

It is well known in representation theory (and not hard to see) that the subset $\mathrm{Val}^\infty(V)$ of smooth valuations is a linear $\mathrm{GL}(V)$-invariant subspace dense in $\mathrm{Val}(V)$. Moreover it has a canonical Fréchet topology which is stronger than that induced from $\mathrm{Val}(V)$. The action of $\mathrm{GL}(V)$ on $\mathrm{Val}^\infty(V)$ is still continuous. Versions of the McMullen's decomposition and irreducibility theorem still hold for $\mathrm{Val}^\infty(V)$.

Example 3.6

(1) Let $A \in \mathscr{K}(V)$ has infinitely smooth boundary and strictly positive Gauss curvature. Then the valuation $\mathrm{vol}(\cdot + A)$ is smooth.
(2) (Alesker [3]) Let $G \subset \mathrm{O}(n)$ be a compact subgroup acting transitively on the unit sphere S^{n-1}. Then $\mathrm{Val}(V)^G \subset \mathrm{Val}^\infty(V)$; actually $\mathrm{Val}(V)^G$ is also finite dimensional in this case.
(3) Let us give an example of non-smooth valuation. Fix a proper linear subspace $E \subset V$. Let $p \colon V \to E$ be a linear projection. Fix a Lebesgue measure vol_E on E. Then $K \mapsto \mathrm{vol}_E(p(K))$ is a continuous, but not smooth valuation.

Remark 3.7 There is an equivalent description of smooth translation invariant valuations in terms of differential forms [4]: a valuation $\phi \in \mathrm{Val}(V)$ is smooth if and only if it can be presented in the form

$$\phi(K) = \int_{\mathbf{nc}(K)} \omega + a \cdot vol(K),$$

where ω in an infinitely smooth differential $(n-1)$-form on spherical bundle $V \times \mathbb{P}_+(V)$ (here $\mathbb{P}_+(V) := (V \backslash \{0\})/\mathbb{R}_{>0}$), $\mathbf{nc}(K) \subset V \times \mathbb{P}_+(V)$ is the normal cycle of K defined in Sect. 2.6 in this book, and a is a constant. This description turned out to be very useful for subsequent developments.

We will also need the notion of the *Klain imbedding* for even valuations. For convenience we will fix again a Euclidean metric on V. Let us construct a linear continuous map

$$Kl \colon \mathrm{Val}_k^+(V) \to C(\mathrm{Gr}_k(V))$$

as follows. Let $\phi \in \mathrm{Val}_k^+(V)$. For any $E^k \in \mathrm{Gr}_k(V)$ the restriction $\phi|_{E^k} \in \mathrm{Val}_k(E^k)$. By the mentioned above Hadwiger theorem $\phi|_{E^k} = c(E)\,\mathrm{vol}_E$. The map $\phi \mapsto c$ is the required Klain map. The main theorem proved by Klain [16] (based on [15]) is that this map is injective. Sometimes c is called the Klain function of ϕ and is denoted by Kl_ϕ.

The Klain map on smooth valuations $Kl \colon \mathrm{Val}_k^{\infty +}(V) \to C^\infty(\mathrm{Gr}_k(V))$ has a closed image which can be characterized in terms of decomposition under the $\mathrm{SO}(n)$-action [7]. (Note that it is harder to describe exactly the image of Kl of continuous valuations in continuous functions: very recently it was shown not to be a closed subspace (see Parapatits and Wannerer [19] and Alesker and Faifman [8]).)

3.2 Product on Valuations

The goal of this section is to introduce the canonical product on $\mathrm{Val}^\infty(V)$ and describe some if its properties. We will start with a slightly more refined notion: exterior product.

Theorem 3.8 *There exists a bilinear map, called exterior product,*

$$\boxtimes\colon \mathrm{Val}^\infty(V) \times \mathrm{Val}^\infty(W) \to \mathrm{Val}(V \times W)$$

which is uniquely characterized by the following properties:

- *it is continuous with the usual topology on* Val *and the Garding topology on* Val^∞*;*
- *if* $\phi(\cdot) = \mathrm{vol}_V(\cdot + A)$, $\psi(\cdot) = \mathrm{vol}_W(\cdot + B)$ *then*

$$(\phi \boxtimes \psi)(\cdot) = (\mathrm{vol}_V \boxtimes \mathrm{vol}_W)(\cdot + (A \times B)).$$

Note that the uniqueness follows from the McMullen's conjecture. But existence is a non-trivial statement which is based not only on the irreducibility theorem. The general idea of the proof is that any smooth valuation can be presented as a rapidly convergent (in $\mathrm{Val}^\infty(V)$) series of the form $\sum_p \alpha_p \,\mathrm{vol}_V(\cdot + A_p)$. For two such expressions there is only one way to define their exterior product satisfying the properties of the theorem. However a presentation of a valuation as such a series is non-unique, and one has to check that the product is independent of a presentation. This last step we will demonstrate now assuming for simplicity that all series are in fact finite sums. Assume that ϕ has two presentations

$$\phi = \sum_p \alpha_p \,\mathrm{vol}_V(\cdot + A_p) = \sum_p \alpha'_p \,\mathrm{vol}_V(\cdot + A'_p).$$

For ψ we fix a similar presentation and show that $\phi \boxtimes \psi$ is independent of the presentation of ϕ. Thus we may assume that ψ has a single summand

$$\psi = \mathrm{vol}_W(\cdot + B).$$

For any $K \in \mathcal{K}(V \times W)$ we have

$$\begin{aligned}(\phi \boxtimes \psi)(K) &= \sum_p \alpha_p (\mathrm{vol}_V \times \mathrm{vol}_W)(K + (A_p \times B)) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \sum_p \alpha_p \,\mathrm{vol}_V \left([K + (A_p \times B)] \cap [V \times \{y\}]\right) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \sum_p \alpha_p \,\mathrm{vol}_V \left(\{(K + (\{0\} \times B)) \cap (V \times \{y\})\} + A_p\right) \\ &= \int_{y \in W} d\,\mathrm{vol}_W(y) \phi\left([K + (\{0\} \times B)] \cap (V \times \{y\})\right),\end{aligned}$$

where the second equation is based on Fubini's theorem. From the last expression we see that the exterior product does not depend on presentation of ϕ.

Let us define the product on $\mathrm{Val}^\infty(V)$.

Definition 3.9 For $\phi, \psi \in \mathrm{Val}^\infty(V)$ let us define the product

$$(\phi \cdot \psi)(K) := (\phi \boxtimes \psi)(\Delta(K)),$$

where $K \in \mathcal{K}(V)$, $\Delta \colon V \hookrightarrow V \times V$ is the diagonal imbedding.

Theorem 3.10 (Alesker [3])

(1) *The product is a bilinear continuous map*

$$\mathrm{Val}^\infty(V) \times \mathrm{Val}^\infty(V) \to \mathrm{Val}^\infty(V).$$

(2) *Equipped with this product,* $\mathrm{Val}^\infty(V)$ *becomes a commutative associative graded algebra with a unit (unit is the Euler characteristic; the grading is given by the McMullen's decomposition).*

(3) *For any* $0 \le i \le n$ *the bilinear map given by the product*

$$\mathrm{Val}_i^\infty(V) \times \mathrm{Val}_{n-i}^\infty(V) \to \mathrm{Val}_n(V) = \mathbb{C} \cdot \mathrm{vol}_V$$

is a perfect pairing, i.e. the induced map $\mathrm{Val}_i^\infty(V) \to (\mathrm{Val}_{n-i}^\infty(V))^* \otimes \mathrm{Val}_n(V)$ *is injective with image dense in the weak* topology.*

Remark 3.11 When the two valuations are given by differential forms on the spherical bundle as in Remark 3.7 then their product also can be described by an differential form expressed explicitly via the given forms by a rather complicated formula [6].

The following result is a version of the hard Lefschetz type theorem. In this form it was proved by Alesker [5], but the proof is heavily based on a different version of the hard Lefschetz theorem which in full generality was proved by Bernig and Bröcker [9] and earlier in the even case by Alesker [2].

Theorem 3.12 *Fix a Euclidean metric on V. Let $0 \leq i < n/2$. The map*

$$\mathrm{Val}_i^\infty(V) \to \mathrm{Val}_{n-i}^\infty(V)$$

given by $\phi \mapsto V_1^{n-2i} \cdot \phi$, is an isomorphism. (Here V_1 is the first intrinsic volume as usual.)

Remark 3.13 This theorem immediately implies that the operator $\mathrm{Val}_i^\infty(V) \to \mathrm{Val}_{i+j}^\infty(V)$ given by $\phi \mapsto V_1^j \cdot \phi$ is injective for $j \leq n-2i$ and surjective for $i \geq n-2i$.

The product structure has been computed in some cases.

Example 3.14

(1) (Alesker [3]) Let

$$\phi(K) = V(K[i], A_1, \ldots, A_{n-i}),\ \psi(K) = V(K[n-i], B_1, \ldots, B_i).$$

Then

$$\phi \cdot \psi = V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) \cdot \mathrm{vol}\,.$$

(2) $\mathrm{Val}^{O(n)}(\mathbb{R}^n)$ is isomorphic as a graded algebra to $\mathbb{C}[t]/(t^{n+1})$ where $t = V_1$.
(3) A geometric description of the space $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ of unitarily invariant valuations was obtained by Alesker [2] in 2003. Fu [12] has obtained in 2006 the following beautiful description of the algebra structure of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ in terms of generators and relations:

$$\mathrm{Val}^{U(n)}(\mathbb{C}^n) = \mathbb{C}[s,t]/(f_{n+1}, f_{n+2}),$$

where $\deg s = 2, \deg t = 1$ and the polynomial f_i is the degree i term of the power series $\log(1+s+t)$.
(4) Some non-trivial examples of the product of tensor valued valuations were recently computed by Bernig and Hug [11]; see also Chap. 3 in this book.

3.3 Convolution of Valuations

We denote by $D(V^*)$ the space of complex valued Lebesgue measures on V^*.

Theorem 3.15 (Bernig-Fu [10]) *There exists a bilinear map called convolution*

$$*\colon \big(\mathrm{Val}^\infty(V) \otimes D(V^*)\big) \times \big(\mathrm{Val}^\infty(V) \otimes D(V^*)\big) \to \mathrm{Val}^\infty(V) \otimes D(V^*)$$

which is uniquely characterized by the following properties:

- *continuity in the Garding topology;*
- *if* $\phi(\cdot) = \mathrm{vol}(\cdot + A) \otimes \mathrm{vol}^{-1}$, $\psi(\cdot) = \mathrm{vol}(\cdot + B) \otimes \mathrm{vol}^{-1}$, *then*

$$(\phi * \psi)(\cdot) = \mathrm{vol}(\cdot + A + B) \otimes \mathrm{vol}^{-1},$$

  where vol^{-1} *is the Lebesgue measure on* V^* *such that for any basis* $e_1, \dots, e_n$ *of* V *spanning the parallelepiped of unit volume with respect to* vol, *the parallelepiped in* V^* *spanned by the dual basis* $e_1^*, \dots, e_n^*$ *has the unit volume with respect to* vol^{-1}.

Equipped with this product, $\mathrm{Val}^\infty(V) \otimes D(V^*)$ *becomes a commutative associative graded algebra with the unit, when the unit is* $\mathrm{vol} \otimes \mathrm{vol}^{-1}$, *and the grading is* ($n$ – *deg of homogeneity*).

The uniqueness again follows immediately from McMullen's conjecture. The existence is non-trivial. Later we will deduce it from existence of exterior product on valuations.

Remark 3.16 If two valuations are given by differential forms as in Remark 3.7 then their convolution can be given by a differential form expressed by an explicit formula via the two given forms [10].

3.4 Fourier Type Transform on Valuations

It turns out that the algebras $(\mathrm{Val}^\infty(V), \cdot)$ and $(\mathrm{Val}^\infty(V^*) \otimes D(V), *)$ are isomorphic. We are going to discuss a specific isomorphism between them, called a Fourier type transform, which has some additional interesting properties.

Theorem 3.17 (Alesker [5]) *There exists an isomorphism of algebras*

$$\mathbb{F}\colon \mathrm{Val}^\infty(V) \xrightarrow{\sim} \mathrm{Val}^\infty(V^*) \otimes D(V)$$

which has the following extra properties:

- $\mathbb{F}$ *is an isomorphism of linear topological spaces.*
- $\mathbb{F}$ *commutes with the natural action of* $GL(V)$ *on both spaces.*
- *(Plancherel type inversion formula) Consider the composition* $\mathcal{E}_V$*:*

$$\mathrm{Val}^\infty(V) \xrightarrow{\mathbb{F}_V} \mathrm{Val}^\infty(V^*) \otimes D(V) \xrightarrow{\mathbb{F}_{V^*} \otimes \mathrm{Id}_{D(V)}} \mathrm{Val}^\infty(V) \otimes \underbrace{D(V^*) \otimes D(V)}_{\simeq \mathbb{C}} = \mathrm{Val}^\infty(V).$$

Then $(\mathcal{E}_V \phi)(K) = \phi(-K)$.

The construction of the Fourier transform is rather difficult and uses some more of representation theory. Nevertheless in few examples the Fourier transform can be computed. In the case of even valuations there is another description using the Klain function. In the rest of this section we will discuss this material.

Below for simplicity we will fix a Euclidean metric on V. Hence we get identifications $V^* \simeq V$ and $D(V) \simeq \mathbb{C}$. Thus $\mathbb{F}\colon Val^\infty(V) \xrightarrow{\sim} Val^\infty(V)$ commutes with $O(n)$, but not with $GL(V)$.

Example 3.18

(1) $\mathbb{F}_V(\chi) = vol_V$.
(2) $\mathbb{F}_V(vol_V) = \chi$.
(3) $\mathbb{F}(V_i) = c_{i,n} V_{n-i}$ and the constant $c_{i,n}$ can be written down explicitly. Indeed this (without the exact value of the constant) follows from the Hadwiger theorem.
(4) Assume $\dim V = 2$. Given the first two examples and McMullen's decomposition, it remains to describe $\mathbb{F}$ on 1-homogeneous smooth valuations. Fix a Euclidean metric and an orientation on V. Let $J\colon V \to V$ be the operator of rotation by $\pi/2$ counterclockwise.

By the Hadwiger's theorem [13] (which now follows from McMullen's description of $(n-1)$-homogeneous valuations from Sect. 3.1) any such valuation ϕ has the form

$$K \mapsto \int_{S^1} f(\omega) dS(K, \omega),$$

where $f \in C^\infty(S^1)$ is defined uniquely up to a linear functional. Decompose f into the even and odd parts:

$$f = f_+ + f_-.$$

Furthermore let us decompose the odd part $f_- = f_-^{hol} + f_-^{anti}$ into holomorphic and anti-holomorphic parts as follows. First decompose f_- into the usual Fourier series on S^1:

$$f_-(\omega) = \sum_{k \in \mathbb{Z}} \hat{f}_-(k) e^{ik\omega}.$$

Then define

$$f_-^{hol}(\omega) := \sum_{k>0} \hat{f}_-(k) e^{ik\omega}, \qquad f_-^{anti}(\omega) := \sum_{k<0} \hat{f}_-(k) e^{ik\omega}.$$

Then the Fourier transform of ϕ is

$$(\mathbb{F}\phi)(K) = \int_{S^1} f_+(J\omega)dS(K,\omega) + \int_{S^1} f_-^{\text{hol}}(J\omega)dS(K,\omega) - \int_{S^1} f_-^{\text{anti}}(J\omega)dS(K,\omega).$$

(5) For even smooth valuations there is a simple description of the Fourier transform in terms of the Klain functions; historically this was the first construction of the Fourier transform (Alesker [2]). Fix a Euclidean metric on V for the simplicity of notation. Let $\phi \in Val_k^{\infty+}(V)$. Then for any $F^{n-k} \in Gr_{n-k}(V)$ one has

$$Kl_{\mathbb{F}\phi}(F) = Kl_\phi(F^\perp).$$

Thus the Fourier transform can be easily described on the language of functions on Grassmannians. The non-trivial point is that given a smooth Klain function of a valuation then the transformed function indeed corresponds to some valuation (the uniqueness follows from the Klain's theorem). This follows from the description of the image of the Klain map obtained by Alesker and Bernstein [7].

(6) Recently Bernig and Hug [11] have made some explicit non-trivial computations of the Fourier transform on odd valuations in dimensions higher than 2 in order to obtain kinematic formulas for tensor valuations; see also Chap. 3 of this book.

3.5 Pull-Back and Push-Forward on Valuations

In this section we discuss, following [5], operations of pull-back and push-forward on valuations under linear mappings and their relations to product, convolution and the Fourier transform. In particular we claim that the convolution on valuations can be presented as composition of the exterior product and push-forward under the addition map $a: V \times V \to V$; that will provide another explanation why convolution is well defined (given the exterior product).

Let us start with the notion of pull-back under a linear map $f: V \to W$. Define the pull-back map

$$f^*: Val(W) \to Val(V) \tag{3.2}$$

by $(f^*\phi)(K) = \phi(f(K))$. Obviously f^* is linear and continuous, it preserves degree of homogeneity. Clearly

$$(f \circ g)^* = g^* \circ f^*.$$

A formal simple remark is that if $\Delta\colon V \hookrightarrow V \times V$ is the diagonal imbedding then

$$\phi \cdot \psi = \Delta^*(\phi \boxtimes \psi).$$

The push-forward map

$$f_*\colon \mathrm{Val}(V) \otimes D(V^*) \to \mathrm{Val}(W) \otimes D(W^*)$$

is going to be a linear continuous map. In order to motivate somehow its introduction, let us have some non-rigorous remarks. f_* is going to be dual to f^* in the following not very precise sense.

Consider the bilinear map $\mathrm{Val}^\infty(V) \times \mathrm{Val}^\infty(V) \to \mathrm{Val}_n(V) = D(V)$ given by the product and taking the n-th homogeneous component. By the Poincaré duality the induced map

$$\mathrm{Val}^\infty(V) \otimes D(V^*) \to (\mathrm{Val}^\infty(V))^*$$

is injective and has a dense image in the weak* topology. Informally speaking, up to a completion in appropriate topology, the dual of $\mathrm{Val}^?(V)$ is equal to $\mathrm{Val}^?(V) \otimes D(V^*)$, where $\mathrm{Val}^?(V)$ is a class of valuations of unspecified class of smoothness. Hence, with these identifications, the dual of f^* from (3.2) should lead to a linear map which we call push-forward and denote f_*:

$$f_*\colon \mathrm{Val}^?(V) \otimes D(V^*) \to \mathrm{Val}^?(W) \otimes D(W^*).$$

A closer investigation of this map shows that in fact f_* is a continuous linear map between spaces of continuous (!) valuations (twisted by densities):

$$f_*\colon \mathrm{Val}(V) \otimes D(V^*) \to \mathrm{Val}(W) \otimes D(W^*).$$

It does satisfy the property

$$(f \circ g)_* = f_* \circ g_* \tag{3.3}$$

as it should be by dualizing the corresponding property of the pull-back.

Now we have to describe f_* more explicitly. By the property (3.3) and since every linear map can be presented as composition of injective and surjective linear maps, it suffices to do that only in these two cases.

Assume first $f\colon V \to W$ is onto. To simplify the notation, we may assume that W is a subspace of V, and may choose a Euclidean metric on V such that f is the orthogonal projection. This choice of metric also induces isomorphisms

$$D(V) \simeq D(W) \simeq D(W^\perp) \simeq \mathbb{C}$$

and the same for dual of $V, W, W^\perp$. Let $\phi \in \mathrm{Val}(V) \otimes D(V^*) \simeq \mathrm{Val}(V)$. Fix any $K \in \mathscr{K}(W)$. Let us choose $\tilde{K} \in \mathscr{K}(V)$ such that $f(\tilde{K}) = K$. For any $\lambda \geq 0$ consider the valuation on $\mathscr{K}(W^\perp)$

$$R \mapsto \phi(\lambda R + \tilde{K}).$$

By McMullen's decomposition, this is a polynomial in λ of degree at most $k := \dim W^\perp$. The highest degree term is the k-homogeneous valuation on $W^\perp$, hence by Hadwiger's theorem it is proportional to $\mathrm{vol}_k(R)$. The coefficient depends on ϕ and $\tilde{K}$ (but not on R of course). Moreover one can show that it depends only on K rather than on $\tilde{K}$ (the proof I know uses the McMullen's conjecture). More precisely we have

$$\phi(\lambda R + K) = \frac{1}{k!}\lambda^k \,\mathrm{vol}_k(R) \cdot (f_*\phi)(K) + \mathrm{O}(\lambda^{k-1}).$$

Thus we got a description of f_* for surjective maps.

Before we describe f_* for injective maps, let us say that the convolution on valuations can be describe as

$$\phi * \psi = a_*(\phi \boxtimes \psi),$$

where $a \colon V \times V \to V$ is the addition map (which is of course surjective).

Let now $f \colon V \to W$ be an injective map. It is convenient to assume without loss of generality that V is a subspace of W, and f is the identity imbedding. We fix a Euclidean metric on W and use various identifications it induces. Let $\phi \in \mathrm{Val}(V)$ and $K \in \mathscr{K}(W)$. Then

$$(f_*\phi)(K) = \int_{y \in V^\perp} \phi(K \cap (y + V))d\,\mathrm{vol}_{V^\perp}(y).$$

Finally let us discuss the relation of pull-back and push-forward to the Fourier transform. We will do it here in a non-rigorous way for the sake of simplicity. Let $f \colon V \to W$ be a linear map, and $f^\vee \colon W^* \to V^*$ be the dual map. Then we should have the following non-rigorously stated identity

$$\mathbb{F}_V \circ f^* = (f^\vee)_* \circ \mathbb{F}_W.$$

This identity is non-rigorous because the Fourier transform is defined on the class of smooth valuations which is not preserved under the pull-back and push-forward maps.

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Chapter 4
Integral Geometry and Algebraic Structures for Tensor Valuations

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Abstract In this survey, we consider various integral geometric formulas for tensor-valued valuations that have been obtained by different methods. Furthermore we explain in an informal way recently introduced algebraic structures on the space of translation invariant, smooth tensor valuations, including convolution, product, Poincaré duality and Alesker-Fourier transform, and their relation to kinematic formulas for tensor valuations. In particular, we describe how the algebraic viewpoint leads to new intersectional kinematic formulas and substantially simplified Crofton formulas for translation invariant tensor valuations. We also highlight the connection to general integral geometric formulas for area measures.

4.1 Introduction

An important part of integral geometry is devoted to the investigation of integrals (mean values) of the form

$$\int_{\mathrm{G}} \varphi(K \cap gL)\, \mu(\mathrm{d}g),$$

where $K, L \subset \mathbb{R}^n$ are sets from a suitable intersection stable class of sets, G is a group acting on $\mathbb{R}^n$ and thus on its subsets, μ is a Haar measure on G, and φ is a functional with values in some vector space W. Common choices for W are the reals or the space of signed Radon measures. Instead of the intersection, Minkowski addition is another natural choice for a set operation which has

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been studied. The principle aim then is to express such integrals by means of basic geometric functionals of K and L. Depending on the specific framework, such as the class of sets or the type of functional under consideration, different methods have been developed to establish integral geometric formulas, ranging from classical convexity, differential geometry, geometric measure theory to the theory of valuations. The interplay between the theory of valuations and integral geometry, although a classical topic in convexity, has been expanded and deepened considerably in recent years. In the present survey, we explore the integral geometry of tensor-valued functionals. This study suggests and requires generalizations in the theory of valuations which are of independent interest.

Therefore, we describe how some algebraic operations known for smooth translation invariant scalar-valued valuations (product, convolution, Alesker-Fourier transform) can be extended to smooth translation invariant tensor-valued valuations. Although these extensions are straightforward to define, they encode various integral geometric formulas for tensor valuations, like Crofton-type formulas, rotation sum formulas (also called additive kinematic formulas) and intersectional kinematic formulas. Even in the easiest case of translation invariant and $\mathrm{O}(n)$-covariant tensor valuations, explicit formulas are hard to obtain by classical methods. With the present algebraic approach, we are able to simplify the constants in Crofton-type formulas for tensor valuations, and to formulate a new type of intersectional kinematic formulas for tensor valuations. For the latter we show how such formulas can be explicitly calculated in the $\mathrm{O}(n)$-covariant case. As an important byproduct, we compute the Alesker-Fourier transform on a certain class of smooth valuations, called spherical valuations. This result is of independent interest and is the technical heart of the computation of the product of tensor valuations.

4.2 Tensor Valuations

The present chapter is based on the general introduction to valuations in Chap. 1 and on the description and structural analysis of tensor valuations contained in Chap. 2. The algebraic framework for the investigation of scalar valuations, which has already proved to be very useful in integral geometry, is outlined in Chap. 3. In these chapters relevant background information is provided, including references to previous work, motivation and hints to applications. The latter are also discussed in other parts of this volume, especially in Chaps. 11–15.

Let us fix our notation and recall some basic structural facts. We will write V for a finite-dimensional real vector space. Sometimes we fix a Euclidean structure on V, which allows us to identify V with Euclidean space $\mathbb{R}^n$. The space of compact convex sets (including the empty set) is denoted by $\mathscr{K}(V)$ (or $\mathscr{K}^n$ if $V = \mathbb{R}^n$). The vector space of translation invariant, continuous scalar valuations is denoted by $\mathrm{Val}(V)$ (or simply by Val if the vector space V is clear from the context). The smooth valuations in $\mathrm{Val}(V)$ constitute an important subspace for which we write $\mathrm{Val}^\infty(V)$; see Definition 3.5 and Remark 3.7, Definition 9.5 and Proposition 9.8, and

Sect. 6.3. There is a natural decomposition of $\mathrm{Val}(V)$ (and then also of $\mathrm{Val}^\infty(V)$) into subspaces of different parity and different degrees of homogeneity, hence

$$\mathrm{Val}(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{Val}_m^{\varepsilon}(V),$$

if V has dimension n, and similarly

$$\mathrm{Val}^\infty(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{Val}_m^{\varepsilon,\infty}(V);$$

see Sects. 1.4, 3.1 and Theorem 9.1.

Our main focus will be on valuations with values in the space of symmetric tensors of a given rank $p \in \mathbb{N}_0$, for which we write $\mathrm{Sym}^p\,\mathbb{R}^n$ or simply Sym^p if the underlying vector space is clear from the context (resp., $\mathrm{Sym}^p\,V$ in case of a general vector space V). Here we deviate from the notation $\mathbb{T}^p$ used in Chaps. 1 and 2. The spaces of symmetric tensors of different ranks can be combined to form a graded algebra in the usual way. By a tensor valuation we mean a valuation on $\mathscr{K}(V)$ with values in the vector space of tensors of a fixed rank, say $\mathrm{Sym}^p(V)$. For the space of translation invariant, continuous tensor valuations with values in $\mathrm{Sym}^p(V)$ we write $\mathrm{TVal}^p(V)$; cf. the notation in Chaps. 3, 6 and Definition 9.38. This vector space can be identified with $\mathrm{Val}(V) \otimes \mathrm{Sym}^p(V)$ (or $\mathrm{Val} \otimes \mathrm{Sym}^p$, for short). If we restrict to smooth tensor valuations, we add the superscript ∞, that is $\mathrm{TVal}^{p,\infty}(V)$. It is clear that McMullen's decomposition extends to tensor valuations, hence

$$\mathrm{TVal}^p(V) = \bigoplus_{\substack{m=0\\ \varepsilon=\pm}}^{n} \mathrm{TVal}_m^{p,\varepsilon}(V),$$

if $\dim(V) = n$. The corresponding decomposition is also available for smooth tensor-valued valuations or valuations covariant (or invariant) with respect to a compact subgroup G of the orthogonal group which acts transitively on the unit sphere. The vector spaces of tensor valuations satisfying an additional covariance condition with respect to such a group G is finite-dimensional and consists of smooth valuations only (cf. Example 3.6 and Theorem 9.15). In the following, we will only consider rotation covariant valuations (see Chap. 2).

4.2.1 *Examples of Tensor Valuations*

In the following, we mainly consider translation invariant tensor valuations. However, we start with recalling general Minkowski tensors, which are translation covariant but not necessarily translation invariant. For Minkowski tensors, and hence for all isometry covariant continuous tensor valuations, we first state a general Crofton formula. The major part of this contribution is then devoted to translation invariant, rotation covariant, continuous tensor valuations. In this framework, we explain how algebraic structures can be introduced and how they are related to Crofton formulas as well as to additive and intersectional kinematic formulas. Crofton formulas for tensor-valued curvature measures are the subject of Chap. 5.

For $k \in \{0,\dots,n-1\}$ and $K \in \mathcal{K}^n$, let $\Lambda_0(K,\cdot),\dots,\Lambda_{n-1}(K,\cdot)$ denote the support measures associated with K (see Sect. 1.3). They are Borel measures on $\Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ which are concentrated on the normal bundle $\mathbf{nc}\,K$ of K. Let κ_n denote the volume of the unit ball and $\omega_n = n\kappa_n$ the volume of its boundary, the unit sphere. Using the support measures, we recall from Sects. 1.3 or 2.1 that the Minkowski tensors are defined by

$$\Phi_k^{r,s}(K) := \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} x^r u^s \,\Lambda_k(K,\mathrm{d}(x,u)),$$

for $k \in \{0,\dots,n-1\}$ and $r,s \in \mathbb{N}_0$, and

$$\Phi_n^{r,0}(K) := \frac{1}{r!}\int_K x^r \,\mathrm{d}x.$$

In addition, we define $\Phi_k^{r,s} := 0$ for all other choices of indices. Clearly, the tensor valuations $\Phi_k^{0,s}$ and $\Phi_n^{0,0}$, which are obtained by choosing $r = 0$, are translation invariant. However, these are not the only translation invariant examples, since e.g. $\Phi_{k-1}^{1,1}$, for $k \in \{1,\dots,n\}$, also satisfies $\Phi_{k-1}^{1,1}(K+t) = \Phi_{k-1}^{1,1}(K)$ for all $K \in \mathcal{K}^n$ and $t \in \mathbb{R}^n$.

Further examples of continuous, isometry covariant tensor valuations are obtained by multiplying the Minkowski tensors with powers of the metric tensor Q and by taking linear combinations. As shown by Alesker [1, 2], no other examples exist (see also Theorem 2.5). In the following, we write

$$\begin{aligned}\Phi_k^{s}(K) := \Phi_k^{0,s}(K) &= \frac{1}{s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} u^s \,\Lambda_k(K,\mathrm{d}(x,u))\\ &= \binom{n-1}{k}\frac{1}{\omega_{n-k+s}s!}\int_{\mathbb{S}^{n-1}} u^s \,S_k(K,\mathrm{d}u),\end{aligned}$$

for $k \in \{0,\dots,n-1\}$, where we used the k-th area measure $S_k(K,\cdot)$ of K, a Borel measure on $\mathbb{S}^{n-1}$ defined by

$$S_k(K,\cdot) := \frac{n\kappa_{n-k}}{\binom{n}{k}}\Lambda_k(K,\mathbb{R}^n \times \cdot).$$

In addition, we define $\Phi_n^0 := V_n$ and $\Phi_n^s := 0$ for $s > 0$. The normalization is such that $\Phi_k^0 = V_k$, for $k \in \{0, \ldots, n\}$, where V_k is the k-th intrinsic volume. Clearly, the tensor valuations $Q^i \Phi_k^s$, for $k \in \{0, \ldots, n\}$ and $i, s \in \mathbb{N}_0$, are continuous, translation invariant, O(n)-covariant, homogeneous of degree k and have tensor rank $2i + s$. We have $\Phi_n^s \equiv 0$ for $s \neq 0$, and $\Phi_0^s(K)$ is independent of K. Hence, we usually exclude these trivial cases. Apart from these, Alesker showed that for each fixed $k \in \{1, \ldots, n-1\}$ the valuations

$$Q^i \Phi_k^s, \qquad i, s \in \mathbb{N}_0, 2i + s = p, s \neq 1,$$

form a basis of the vector space of all continuous, translation invariant, O(n)-covariant tensor valuations of rank p which are homogeneous of degree k. The fact that these valuations span the corresponding vector space is implied by [1, Proposition 4.9] (and [2]), the proof is based in particular on basic representation theory. A result of Weil [17, Theorem 3.5] states that differences of area measure of order k, for any fixed $k \in \{1, \ldots, d-1\}$, are dense in the vector space of differences of finite, centered Borel measures on the unit sphere. From this the asserted linear independence of the tensor valuations can be inferred. We also refer to Sect. 6.5 where the present case is discussed as an example of a very general representation theoretic theorem.

The situation for general tensor valuations (which are not necessarily translation invariant) is more complicated. As explained in Chap. 2, the valuations $Q^i \Phi_k^{r,s}$ span the corresponding vector space, but there exist linear dependences between these functionals. Although all linear relations are known and the dimension of the corresponding vector space (for fixed rank and degree of homogeneity) has been determined, the situation here is not perfectly understood.

In the following, it will often (but not always) be sufficient to neglect the metric tensor powers Q^i and just consider the tensor valuations Φ_k^s, since the metric tensor commutes with the algebraic operations to be considered.

4.2.2 Integral Geometric Formulas

Let $A(n, k)$, for $k \in \{0, \ldots, n\}$, denote the affine Grassmannian of k-flats in $\mathbb{R}^n$, and let μ_k denote the motion invariant measure on $A(n, k)$ normalized as in [13, 14]. The Crofton formulas to be discussed below relate the integral mean

$$\int_{A(n,k)} \Phi_j^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

of the tensor valuation $\Phi_j^{r,s}(K \cap E)$ of the intersection of K with flats $E \in A(n, k)$ to tensor valuations of K. Guessing from the scalar case, one would expect that only tensor valuations of the form $Q^i\Phi_{n-k+j}^{r',s'}(K)$ are required. It turns out, however, that for general r the situation is more involved.

The following Crofton formulas for Minkowski tensors have been established in [7]. Since $\Phi_j^{r,s}(K\cap E)=0$ if $k<j$, we only have to consider the cases where $k\geq j$.

We start with the basic case $k=j$, in which the Crofton formula has a particularly simple form.

Theorem 4.1 *For $K\in\mathcal{K}^n$, $r,s\in\mathbb{N}_0$ and $0\leq k\leq n-1$,*

$$\int_{\mathrm{A}(n,k)}\Phi_k^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E)=\begin{cases}\tilde{\alpha}_{n,k,s}\,Q^{\frac{s}{2}}\Phi_n^{r,0}(K), & \text{if } s \text{ is even,}\\ 0, & \text{if } s \text{ is odd,}\end{cases}$$

where

$$\tilde{\alpha}_{n,k,s}:=\frac{1}{(4\pi)^{\frac{s}{2}}\left(\frac{s}{2}\right)!}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}.$$

This result essentially follows from Fubini's theorem, combined with a relation due to McMullen, which connects the Minkowski tensors of $K\cap E$ and the Minkowski tensors of $K\cap E$, defined with respect to the flat E as the ambient space (see (4.4) for a precise statement).

The main case $j<k$ is considered in the next theorem.

Theorem 4.2 *Let $K\in\mathcal{K}^n$ and $k,j,r,s\in\mathbb{N}_0$ with $0\leq j<k\leq n-1$. Then*

$$\begin{aligned}&\int_{\mathrm{A}(n,k)}\Phi_j^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E)\\&=\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor}\chi^{(1)}_{n,j,k,s,z}Q^z\Phi_{n+j-k}^{r,s-2z}(K)+\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor-1}\chi^{(2)}_{n,j,k,s,z}Q^z\\&\quad\times\sum_{l=0}^{s-2z-1}\left(2\pi l\Phi_{n+j-k-s+2z+l}^{r+s-2z-l,l}(K)-Q\Phi_{n+j-k-s+2z+l}^{r+s-2z-l,l-2}(K)\right),\end{aligned}\tag{4.1}$$

where $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ are explicitly known constants.

The constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ only depend on the indicated lower indices. It is remarkable that they are independent of r. Moreover, the right-hand side of this Crofton formula also involves other tensor valuations than $\Phi_{n-k+j}^{r',s'}(K)$. For instance, in the special case where $n=3$, $k=2$, $j=0$, $r=1$ and $s=2$, Theorem 4.2 yields that

$$\int_{A(3,2)}\Phi_0^{1,2}(K\cap E)\,\mu_2(\mathrm{d}E)=\tfrac{1}{3}\Phi_1^{1,2}(K)+\tfrac{1}{24\pi}Q\Phi_1^{1,0}(K)+\tfrac{1}{6}\Phi_0^{2,1}(K).$$

It can be shown that it is not possible to write $\Phi_0^{2,1}$ as a linear combination of $\Phi_1^{1,2}$ and $Q\Phi_1^{1,0}$, which are the only other Minkowski tensors of rank 3 and homogeneity degree 2.

The explicit expressions obtained for the constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ in [7] require a multiple (fivefold) summation over products and ratios of binomial coefficients and Gamma functions. Some progress which can be made in simplifying this representation is described in Chap. 5.

Since the tensor valuations on the right-hand side of the Crofton formula (4.1) are not linearly independent, the specific representation is not unique. Using the linear relation due to McMullen, the result can also be expressed in the form

$$\begin{aligned}\int_{\mathrm{A}(n,k)} & \Phi_j^{r,s}(K\cap E)\,\mu_k(\mathrm{d}E)\\ &= \sum_{z=0}^{\lfloor \frac{s}{2}\rfloor} \chi^{(1)}_{n,j,k,s,z} Q^z \Phi^{r,s-2z}_{n+j-k}(K) + \sum_{z=0}^{\lfloor \frac{s}{2}\rfloor -1} \chi^{(2)}_{n,j,k,s,z} Q^z \\ &\quad \times \sum_{l\geq s-2z} \Big(Q\Phi^{r+s-2z-l,l-2}_{n+j-k-s+2z+l}(K) - 2\pi l \Phi^{r+s-2z-l,l}_{n+j-k-s+2z+l}(K)\Big) \end{aligned} \tag{4.2}$$

with the same constants as before. From (4.2) we now deduce the Crofton formula for the translation invariant tensor valuations Φ_j^s. For $r = 0$, the sum $\sum_{l\geq s-2z}$ on the right-hand side of (4.2) is non-zero only if $l = s - 2z$. Therefore, after some index shift (and discussion of the 'boundary cases' $z = 0$ and $z = \lfloor \frac{s}{2}\rfloor$), we obtain

$$\int_{\mathrm{A}(n,k)} \Phi_j^s(K\cap E)\,\mu_k^n(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2}\rfloor} \chi^{(*)}_{n,j,k,s,z} Q^z \Phi^{s-2z}_{n+j-k}(K) \tag{4.3}$$

for $j < k$, where

$$\chi^{(*)}_{n,j,k,s,z} = \chi^{(1)}_{n,j,k,s,z} + \chi^{(2)}_{n,j,k,s,z-1} - 2\pi(s-2z)\chi^{(2)}_{n,j,k,s,z}.$$

Since the right-hand side of (4.3) is uniquely determined by the left-hand side and the tensor valuations on the right-hand side are linearly independent, the constant $\chi^{(*)}_{n,j,k,s,z}$ is uniquely determined. Using the expression which is obtained for $\chi^{(*)}_{n,j,k,s,z}$ from the constants $\chi^{(1)}_{n,j,k,s,z}$ and $\chi^{(2)}_{n,j,k,s,z}$ provided in [7], it seems to be a formidable task to get a reasonably simple expression for this constant. If $j = k$, then Theorem 4.1 shows that (4.3) remains true if we define $\chi_{n,k,k,s,\lfloor \frac{s}{2}\rfloor} := \tilde{\alpha}_{n,k,s}$ if s is even, and as zero in all other cases. As we will see, the approach of algebraic integral geometry to (4.3) will reveal that $\chi^{(*)}_{n,j,k,s,z}$ has indeed a surprisingly simple expression.

To compare the algebraic approach with the one used in [7], and extended to tensorial curvature measures in Chap. 5, we point out that the result of Theorem 4.2 is complemented by and in fact is based on an intrinsic Crofton formula, where the tensor valuation $\Phi_j^{r,s}(K \cap E)$ is replaced by $\Phi_{j,E}^{r,s}(K \cap E)$. The latter is the tensor valuation of the intersection $K \cap E$, determined with respect to E as the ambient space but considered as a tensor in $\mathbb{R}^n$ (see Sect. 5.2 or [7] for an explicit definition). The two tensors are connected by the relation

$$\Phi_j^{r,s}(K \cap E) = \sum_{m \geq 0} \frac{Q(E^\perp)^m}{(4\pi)^m m!} \Phi_{j,E}^{r,s-2m}(K \cap E), \tag{4.4}$$

due to McMullen [11, Theorem 5.1] (see also [7]), where $Q(E^\perp)$ is the metric tensor of the linear subspace orthogonal to the direction space of E but again considered as a tensor in $\mathbb{R}^n$, that is, $Q(E^\perp) = e_{k+1}^2 + \cdots + e_n^2$, where $e_{k+1}, \ldots, e_n$ is an orthonormal basis of $E^\perp$. Note that for $s = 0$ we get $\Phi_j^{r,0}(K \cap E) = \Phi_{j,E}^{r,0}(K \cap E)$, since the intrinsic volumes and the suitably normalized curvature measures are independent of the ambient space.

The intrinsic Crofton formula for

$$\int_{\mathrm{A}(n,k)} \Phi_{j,E}^{r,s}(K \cap E)\, \mu_k(\mathrm{d}E)$$

has the same structure as the extrinsic Crofton formula stated in Theorem 4.2, but the constants are different. Apart from reducing the number of summations required for determining the constants, progress in understanding the structure of these (intrinsic and extrinsic) integral geometric formulas can be made by localizing the Minkowski tensors. This is the topic of Chap. 5.

Crofton and intersectional kinematic formulas for Minkowski tensors $\Phi_j^{r,s}$ with $s = 0$ are special cases of corresponding (more general) integral geometric formulas for curvature measures. For example, we have

$$\int_{\mathrm{A}(n,k)} \Phi_j^{r,0}(K \cap E)\, \mu_k(\mathrm{d}E) = a_{njk}\, \Phi_{n+j-k}^{r,0}(K) \tag{4.5}$$

and

$$\int_{\mathrm{G}_n} \Phi_j^{r,0}(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} a_{njk}\, \Phi_{n+j-k}^{r,0}(K) V_k(M), \tag{4.6}$$

where G_n is the Euclidean motion group, μ is the suitably normalized Haar measure and the (simple) constants a_{njk} are known explicitly. Therefore, we focus on the case $s \neq 0$ (and $r = 0$) in the following.

A close connection between Crofton formulas and intersectional kinematic formulas follows from Hadwiger's general integral geometric theorem (see [14, Theorem 5.1.2]). It states that for any continuous valuation φ on the space of convex bodies and for all $K, M \in \mathcal{K}^n$, we have

$$\int_{\mathrm{G}_n} \varphi(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=0}^{n} \int_{\mathrm{A}(n,k)} \varphi(K \cap E)\, \mu_k(\mathrm{d}E)\, V_k(M). \tag{4.7}$$

Hence, if a Crofton formula for the functional φ is available, then an intersectional kinematic formula is an immediate consequence. This statement includes also tensor-valued functionals, since (4.7) can be applied coordinate-wise. In particular, this shows that (4.6) can be obtained from (4.5). In the same way, Theorem 4.2 and the special case shown in (4.3) imply kinematic formulas for intersections of convex bodies, one fixed the other moving. Thus, for instance, we obtain

$$\int_{\mathrm{G}_n} \Phi_j^s(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{k=j}^{n} \chi^{(*)}_{n,j,k,s,z}\, Q^z \Phi^{s-2z}_{n+j-k}(K) V_k(M). \tag{4.8}$$

These results are related to and in fact inspired general integral geometric formulas for area measures (see [10]). The starting point is a local version of Hadwiger's general integral geometric theorem for measure-valued valuations. To state it, let $\mathcal{M}^+(\mathbb{S}^{n-1})$ be the cone of non-negative measures in the vector space $\mathcal{M}(\mathbb{S}^{n-1})$ of finite Borel measures on the unit sphere.

Theorem 4.3 *Let $\varphi : \mathcal{K}^n \to \mathcal{M}^+(\mathbb{S}^{n-1})$ be a continuous and additive mapping with $\varphi(\emptyset, \cdot) = 0$ (the zero measure). Then, for $K, M \in \mathcal{K}^n$ and Borel sets $A \subset \mathbb{S}^{n-1}$,*

$$\int_{\mathrm{G}_n} \varphi(K \cap gM, A)\, \mu(\mathrm{d}g) = \sum_{k=0}^{n} [T_{n,k}\varphi(K, \cdot)](A) V_k(M), \tag{4.9}$$

with (the Crofton operator) $T_{n,k}$ on weakly continuous measure-valued valuations given by

$$T_{n,k}\varphi(K, \cdot) := \int_{\mathrm{A}(n,k)} \varphi(K \cap E, \cdot)\, \mu_k(\mathrm{d}E), \qquad k = 0, \dots, n.$$

We want to apply this result to area measures of convex bodies, hence we need a Crofton formula for area measures. The statement of such a Crofton formula is based on Fourier operators I_p, for $p \in \{-1, 0, 1, \dots, n\}$, which act on C^∞ functions on $\mathbb{S}^{n-1}$. For $f \in C^\infty(\mathbb{S}^{n-1})$, let f_p be the extension of f to $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree $-n + p$, and let $\hat{f}_p$ be the distributional Fourier transform of f_p. For $0 < p < n$, the restriction $I_p(f)$ of $\hat{f}_p$ to the unit sphere is again a smooth function. Let $\mathcal{H}^n_s$ denote the space of spherical harmonics of degree s. Recall that a spherical harmonic of degree s is the restriction to the unit sphere of a homogeneous polynomial p of degree s on $\mathbb{R}^n$ which satisfies $\Delta p = 0$ (and hence is called harmonic), where Δ is the Laplace operator. We refer to [13] for more information on spherical harmonics. Since I_p intertwines the group action of SO(n), we have $I_p(f_s) = \lambda_s(n,p) f_s$ for $f_s \in \mathscr{H}_s^n$ and some $\lambda_s(n,p) \in \mathbb{C}$. It is known that

$$\lambda_s(n,p) = \pi^{\frac{n}{2}} 2^p \, \mathbf{i}^s \frac{\Gamma\left(\frac{s+p}{2}\right)}{\Gamma\left(\frac{s+n-p}{2}\right)}.$$

Note that $\lambda_s(n,p)$ is purely imaginary if s is odd, and real if s is even. See [10] for a summary of the main properties of this Fourier operator and [8, 9] for a detailed exposition.

Using the connection to mean section bodies (see [8]) and the Fourier operators I_p, the following Crofton formula for area measures has been established in [10, Theorem 3.1].

Theorem 4.4 *Let* $1 \leq j < k \leq n$ *and* $K \in \mathscr{K}^n$. *Then*

$$\int_{\mathrm{A}(n,k)} S_j(K \cap E, \cdot)\, \mu_k(\mathrm{d}E) = a(n,j,k) I_j I_{k-j} S_{n+j-k}(-K, \cdot) \tag{4.10}$$

with

$$a(n,j,k) := \frac{j}{2^n \pi^{(n+k)/2}(n+j-k)} \frac{\Gamma(\frac{k+1}{2})\Gamma(n-j)}{\Gamma(\frac{n+1}{2})\Gamma(k-j)}.$$

Let I^* be the reflection operator $(I^* f)(u) = f(-u)$, $u \in \mathbb{S}^{n-1}$, for a function f on the unit sphere. The operator $T_{n,j,k} := a(n,j,k) I_j I_{k-j} I^*$, for $1 \leq j < k \leq n$, and the identity operator $T_{n,j,n}$ act as linear operators on $\mathscr{M}(\mathbb{S}^{n-1})$ and can be used to express (4.10) in the form

$$\int_{\mathrm{A}(n,k)} S_j(K \cap E, \cdot)\, \mu_k(\mathrm{d}E) = T_{n,j,k} S_{n+j-k}(K, \cdot). \tag{4.11}$$

This is also true for $k = j < n$ if we define

$$T_{n,j,j} S_n(K, \cdot) := \binom{n-1}{j}^{-1} \frac{\omega_{n-j}}{\omega_n} V_n(K) \sigma,$$

where σ is spherical Lebesgue measure. Combining Eqs. (4.9) and (4.11), we obtain a kinematic formula for area measures. Using again the operator $T_{n,j,k}$, it can be stated in the form

$$\int_{\mathrm{G}_n} S_j(K \cap gM, A)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} [T_{n,j,k} S_{n+j-k}(K, \cdot)](A) V_k(M),$$

for $j = 1, \dots, n-1$. Since the Fourier operators act as multiplier operators on spherical harmonics, it follows that Theorem 4.4 can be rewritten in the form

$$\int_{\mathrm{A}(n,k)} \int_{\mathbb{S}^{n-1}} f_s(u)\, S_j(K \cap E, \mathrm{d}u)\, \mu_k(\mathrm{d}E)$$
$$= a_s(n,j,k) \int_{\mathbb{S}^{n-1}} f_s(u)\, S_{n+j-k}(K, \mathrm{d}u), \tag{4.12}$$

where $f_s \in \mathscr{H}_s^n$ and $a_s(s,j,k) := a(n,j,k)b_s(n,j,k)$ with

$$b_s(n,j,k) := 2^k \pi^n \frac{\Gamma\left(\frac{s+j}{2}\right) \Gamma\left(\frac{s+k-j}{2}\right)}{\Gamma\left(\frac{s+n-j}{2}\right) \Gamma\left(\frac{s+n-k+j}{2}\right)}.$$

In addition to Crofton and intersectional kinematic formulas, there is another classical type of integral geometric formula. Since they involve rotations and Minkowski sums of convex bodies, it is justified to call them rotation sum formulas. Let $\mathrm{SO}(n)$ denote the group of rotations and let ν denote the Haar probability measure on this group. A general form of such a formula can again be stated for area measures. Let $K, M \in \mathscr{K}^n$ be convex bodies and let $\alpha, \beta \subset \mathbb{S}^{n-1}$ be Borel sets. Then [13, Theorem 4.4.6] can be written in the form

$$\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} \mathbf{1}_\alpha(u) \mathbf{1}_\beta(\rho^{-1}u)\, S_j(K + \rho M, \mathrm{d}u)\, \nu(\mathrm{d}\rho)$$
$$= \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} S_k(K, \alpha) S_{j-k}(M, \beta). \tag{4.13}$$

More generally, by the inversion invariance of the Haar measure ν, by basic measure theoretic extension arguments, and by an application of (4.13) to the coordinate functions of an arbitrary continuous function $f : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathrm{Sym}^{s_1} \otimes \mathrm{Sym}^{s_2}$, for given $s_1, s_2 \in \mathbb{N}_0$, we obtain

$$\int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} f(u, \rho u)\, S_j(K + \rho^{-1} M, \mathrm{d}u)\, \nu(\mathrm{d}\rho)$$
$$= \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} f(u, v) \left(S_k(K, \cdot\,) \times S_{j-k}(M, \cdot\,)\right)(\mathrm{d}(u, v)).$$

To simplify constants, we define

$$\phi_k^s(K) := \int_{\mathbb{S}^{n-1}} u^s\, S_k(K, \mathrm{d}u). \tag{4.14}$$

Choosing $f(u, v) = u^{s_1} \otimes v^{s_2}$, we thus get

$$\begin{aligned}
&\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2})\phi_j^{s_1+s_2}(K + \rho^{-1}M)\, \nu(\mathrm{d}\rho) \\
&\qquad = \int_{\mathrm{SO}(n)} \int_{\mathbb{S}^{n-1}} u^{s_1} \otimes (\rho u)^{s_2}\, S_j(K + \rho^{-1}M, \mathrm{d}u)\, \nu(\mathrm{d}\rho) \\
&\qquad = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \int_{(\mathbb{S}^{n-1})^2} u^{s_1} \otimes v^{s_2}\, \big(S_k(K, \cdot) \times S_{j-k}(M, \cdot)\big)\, (\mathrm{d}(u, v)),
\end{aligned} \tag{4.15}$$

and hence

$$\begin{aligned}
&\int_{\mathrm{SO}(n)} (\mathrm{id}^{\otimes s_1} \otimes \rho^{\otimes s_2})\phi_j^{s_1+s_2}(K + \rho^{-1}M)\, \nu(\mathrm{d}\rho) \\
&\qquad = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^{s_1}(K) \otimes \phi_{j-k}^{s_2}(M).
\end{aligned}$$

Up to the different normalization, this is the additive kinematic formula for tensor valuations stated in [6, Theorem 5]. In particular,

$$\int_{\mathrm{SO}(n)} \phi_j^s(K + \rho M)\, \nu(\mathrm{d}\rho) = \frac{1}{\omega_n} \sum_{k=0}^{j} \binom{j}{k} \phi_k^s(K) S_{j-k}(M),$$

where $S_i(M) := S_i(M, \mathbb{S}^{n-1}) = n\kappa_{n-i}\binom{n}{i}^{-1} V_i(M)$.

In the following section, we develop basic algebraic structures for tensor valuations and provide applications to integral geometry. From this approach, we will obtain a Crofton formula for the tensor valuations Φ_k^s, but also for another set of tensor valuations, denoted by Ψ_k^s, for which the Crofton formula has 'diagonal form'. Moreover, we will study more general intersectional kinematic formulas than the one considered in (4.8) and describe the connection between intersectional and additive kinematic formulas. In the course of our analysis, we determine Alesker's Fourier operator for spherical valuations, that is, valuations obtained by integration of a spherical harmonic (or, more generally, any smooth spherical function) against an area measure.

4.3 Algebraic Structures on Tensor Valuations

Recall that Val $=$ Val$(\mathbb{R}^n)$ denotes the Banach space of translation invariant continuous valuations on $V = \mathbb{R}^n$, and $\mathrm{Val}^\infty = \mathrm{Val}^\infty(\mathbb{R}^n)$ is the dense subspace of smooth valuations; see Chaps. 3 and 9 for more information. In this section, we first discuss the extension of basic operations and transformations from scalar valuations to tensor-valued valuations. The scalar case is described in Chap. 3.

In the following, we usually work in Euclidean space $\mathbb{R}^n$ with the Lebesgue measure and the volume functional V_n on convex bodies. Since some of the results are also stated in invariant terms, we write vol for a volume measure on V, that is, a choice of a translation invariant locally finite Haar measure on an n-dimensional vector space V. Of course, in case $V = \mathbb{R}^n$ we always use V_n as a specific choice of the restriction of a volume measure vol to $\mathcal{K}^n$ (the corresponding choice is made for $V = \mathbb{R}^n \times \mathbb{R}^n$).

4.3.1 Product

Existence and uniqueness of the product of smooth valuations is provided by the following result; see also Sect. 3.2 for the more general construction of an exterior product between smooth scalar-valued valuations on possibly different vector spaces.

Proposition 4.5 *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i), \qquad K \in \mathcal{K}^n,$$

where $A_1, A_2 \in \mathcal{K}^n$ are smooth convex bodies with positive Gauss curvature at every boundary point. Let $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding. Then

$$\phi_1 \cdot \phi_2(K) := \mathrm{vol}(\Delta K + A_1 \times A_2), \qquad K \in \mathcal{K}^n,$$

extends by continuity and bilinearity to a product on Val^∞.

The product is compatible with the degree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 \cdot \phi_2$ has degree $k_1 + k_2$ if $k_1 + k_2 \le n$), and more generally with the action of the group $\mathrm{GL}(n)$.

We can extend the product component-wise from smooth scalar-valued valuations to smooth tensor-valued valuations. To see this, let V be a finite-dimensional vector space, $V = \mathbb{R}^n$ say, and $s_1, s_2 \in \mathbb{N}_0$. Let $\Phi_i \in \mathrm{TVal}^{s_i,\infty}(V)$ for $i = 1, 2$. Let $w_1, \ldots, w_m$ be a basis of $\mathrm{Sym}^{s_1} V$, and let $u_1, \ldots, u_l$ be a basis of $\mathrm{Sym}^{s_2} V$. Then there are $\phi_i, \psi_j \in \mathrm{Val}^\infty(V)$, $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$, such that

$$\Phi_1(K) = \sum_{i=1}^{m} \phi_i(K) w_i \quad \text{and} \quad \Phi_2(K) = \sum_{j=1}^{l} \psi_j(K) u_j$$

for $K \in \mathcal{K}(V)$. Now we would like to define (omitting the obvious ranges of the indices)

$$(\Phi_1 \cdot \Phi_2)(K) := \sum_{i,j} (\phi_i \cdot \psi_j)(K) w_i u_j.$$

The dot on the right-hand side is the product of the smooth valuations ϕ_i, ψ_j, and $w_i u_j \in \mathrm{Sym}^{s_1+s_2} V$ denotes the symmetric tensor product of the symmetric tensors $w_i \in \mathrm{Sym}^{s_1} V$ and $u_j \in \mathrm{Sym}^{s_2} V$.

Let us verify that this definition is independent of the chosen bases. For this, let $w_i' = \sum_j c_{ij} w_j$ with some invertible matrix (c_{ij}), and let $u_i' = \sum_j e_{ij} u_j$ with an invertible matrix (e_{ij}).

If

$$\Phi_1(K) = \sum_i \phi_i'(K) w_i' = \sum_i \phi_i(K) w_i,$$

then a comparison of coefficients yields that $\phi_i' = \sum_j c^{ji} \phi_j$, where (c^{ji}) denotes the matrix inverse. Similarly, from

$$\Phi_2(K) = \sum_i \psi_j'(K) u_i' = \sum_i \psi_i(K) u_i,$$

we conclude that $\psi_i' = \sum_j e^{ji} \psi_j$, where (e^{ji}) denotes the matrix inverse. Therefore, we have

$$\begin{aligned}
\sum_{i,j} (\phi_i' \cdot \psi_j') w_i' u_j' &= \sum_{i,j,b_1,b_2} \Big(\sum_{a_1,b_1} c^{a_1 i} \phi_{a_1} \cdot e^{b_1 j} \psi_{b_1} \Big) \sum_{a_2,b_2} c_{ia_2} w_{a_2} e_{jb_2} u_{b_2} \\
&= \sum_{a_1,a_2,b_1,b_2} \underbrace{\Big(\sum_{i,j} c^{a_1 i} c_{ia_2} e^{b_1 j} e_{jb_2} \Big)}_{=\delta^{a_1}_{a_2} \delta^{b_1}_{b_2}} (\phi_{a_1} \cdot \psi_{b_1}) w_{a_2} \cdot u_{b_2} \\
&= \sum_{a,b} (\phi_a \cdot \psi_a) w_a \cdot u_b,
\end{aligned}$$

which proves the asserted independence of the representation.

Thus, recalling that $\mathrm{TVal}^s_m(V)$ denotes the vector space of translation invariant continuous valuations on $\mathcal{K}(V)$ which are homogeneous of degree m and take values in the vector space $\mathrm{Sym}^s V$ of symmetric tensors of rank s over V, and that $\mathrm{TVal}^{s,\infty}_m(V)$ is the subspace consisting of the smooth elements of this vector space, we have

$$\Phi_1 \cdot \Phi_2 \in \mathrm{TVal}^{s_1+s_2,\infty}_{k+l}(V), \qquad k + l \leq n,$$

for $\Phi_1 \in \mathrm{TVal}^{s_1,\infty}_k(V)$, $\Phi_2 \in \mathrm{TVal}^{s_2,\infty}_l(V)$ and $k, l \in \{0, \dots, n\}$.

A similar description and similar arguments can be given for the operations considered in the following sections.

4.3.2 Convolution

Similarly as for the product of valuations, an explicit definition of the convolution of two valuations (as defined in [5]) is given only for a suitable subclass of valuations (cf. Sect. 3.3).

Proposition 4.6 *Let $\phi_1, \phi_2 \in \mathrm{Val}^\infty$ be smooth valuations on $\mathbb{R}^n$ given by*

$$\phi_i(K) = \mathrm{vol}(K + A_i), \qquad K \in \mathscr{K}^n,$$

where A_1, A_2 are smooth convex bodies with positive Gauss curvature at every boundary point. Then

$$\phi_1 * \phi_2(K) := \mathrm{vol}(K + A_1 + A_2), \qquad K \in \mathscr{K}^n,$$

extends by continuity and bilinearity to a product (which is called convolution) on Val^∞.

Written in invariant terms, the convolution is a bilinear map

$$(\mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*)) \times (\mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*)) \to \mathrm{Val}^\infty(V) \otimes \mathrm{Dens}(V^*),$$

where $\mathrm{Dens}(V^*)$ is the one-dimensional space of translation invariant, locally finite complex-valued Haar measures (Lebesgue measures, see Sect. 3.3) on the dual space V^*. It is compatible with the action of the group $\mathrm{GL}(n)$ and with the codegree of a valuation (i.e., if ϕ_i has degree k_i, then $\phi_1 * \phi_2$ has degree $k_1 + k_2 - n$ if $k_1 + k_2 \geq n$).

The convolution can be extended component-wise to a convolution on the space of translation invariant smooth tensor valuations. Hence we have

$$\Phi_1 * \Phi_2 \in \mathrm{TVal}_{k+l-n}^{s_1+s_2,\infty}(V), \qquad k + l \geq n,$$

for $\Phi_1 \in \mathrm{TVal}_k^{s_1,\infty}(V)$, $\Phi_2 \in \mathrm{TVal}_l^{s_2,\infty}(V)$ and $k, l \in \{0, \dots, n\}$. This is analogous to the definition and computation in the previous section.

4.3.3 Alesker-Fourier Transform

Alesker introduced an operation on smooth valuations, now called Alesker-Fourier transform (cf. Sect. 3.4). It is a map $\mathbb{F} : \mathrm{Val}^\infty(\mathbb{R}^n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$ which reverses the degree of homogeneity, that is,

$$\mathbb{F} : \mathrm{Val}_k^\infty(\mathbb{R}^n) \to \mathrm{Val}_{n-k}^\infty(\mathbb{R}^n),$$

and which transforms product into convolution of smooth valuations, more precisely, we have

$$\mathbb{F}(\phi_1 \cdot \phi_2) = \mathbb{F}(\phi_1) * \mathbb{F}(\phi_2). \tag{4.16}$$

On valuations which are smooth and even, the Alesker-Fourier transform can easily be described in terms of Klain functions as follows. Let $\phi \in \mathrm{Val}_k^{\infty,+}(\mathbb{R}^n)$ (the space of smooth and even valuations which are homogeneous of degree k). Then the restriction of ϕ to a k-dimensional subspace E is a multiple $\mathrm{Kl}_\phi(E)$ of the volume, and the resulting function (Klain function) Kl_ϕ determines ϕ. Then

$$\mathrm{Kl}_{\mathbb{F}\phi}(E) = \mathrm{Kl}_\phi(E^\perp)$$

for every $(n-k)$-dimensional subspace E.

As an example (and consequence of the relation to Klain functions), the intrinsic volumes satisfy

$$\mathbb{F}(V_k) = V_{n-k}, \tag{4.17}$$

where $V_0, \dots, V_n$ denote the intrinsic volumes on $\mathcal{K}^n$.

The description in the odd case is more involved and it is preferable to describe it in invariant terms (i.e., without referring to a Euclidean structure).

Let V be an n-dimensional real vector space. Then

$$\mathbb{F} : \mathrm{Val}_k^\infty(V) \to \mathrm{Val}_{n-k}^\infty(V) \otimes \mathrm{Dens}(V^*),$$

where Dens denotes the one-dimensional space of densities (Lebesgue measures). This map commutes with the action of $\mathrm{GL}(V)$ on both sides. Applying it twice (and using the identification $\mathrm{Dens}(V^*) \otimes \mathrm{Dens}(V) \cong \mathbb{C}$), it satisfies the Plancherel type formula

$$(\mathbb{F}^2\phi)(K) = \phi(-K), \qquad K \in \mathcal{K}(V).$$

Working again on Euclidean space $V = \mathbb{R}^n$, we can extend the Alesker-Fourier transform component-wise to a map $\mathbb{F} : \mathrm{TVal}^{s,\infty} \to \mathrm{TVal}^{s,\infty}$ such that

$$\mathbb{F} : \mathrm{TVal}_k^{s,\infty} \to \mathrm{TVal}_{n-k}^{s,\infty}.$$

It is not an easy task to determine the Fourier transform of valuations other than the intrinsic volumes.

4.3.4 Example: Intrinsic Volumes

As an example, let us compute the Alesker product of intrinsic volumes $V_0, \dots, V_n$ in $\mathbb{R}^n$. We complement the definition of the intrinsic volumes by $V_l := 0$ for $l < 0$. Let $\mathrm{vol} = V_n$ denote the volume measure on $\mathcal{K}^n$.

Recall Steiner's formula (1.16) which states that

$$\mathrm{vol}(K + rB) = \sum_{i=0}^{n} V_{n-i}(K)\kappa_i r^i, \qquad r \geq 0.$$

Now we fix $r \geq 0$ and $s \geq 0$ and define the smooth valuations $\phi_1(K) := \mathrm{vol}(K+rB)$ and $\phi_2(K) := \mathrm{vol}(K + sB)$. Then

$$\begin{aligned}\phi_1 * \phi_2(K) &= \mathrm{vol}(K + rB + sB) = \mathrm{vol}(K + (r + s)B)\\ &= \sum_{k=0}^{n} V_{n-k}(K)\kappa_k (r + s)^k,\end{aligned}$$

hence

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} V_{n-i-j}\kappa_{i+j}\binom{i+j}{i} r^i s^j.$$

On the other hand, since $\phi_1 = \sum_{i=0}^{n} V_{n-i}\kappa_i r^i$ and $\phi_2 = \sum_{i=0}^{n} V_{n-i}\kappa_i s^i$, we obtain

$$\phi_1 * \phi_2 = \sum_{i,j=0}^{n} V_{n-i} * V_{n-j}\kappa_i\kappa_j r^i s^j.$$

Now we compare the coefficient of $r^i s^j$ in these equations and get

$$V_{n-i-j}\kappa_{i+j}\binom{i+j}{i} = V_{n-i} * V_{n-j}\kappa_i\kappa_j.$$

Writing i instead of $n - i$ and j instead of $n - j$, we obtain

$$V_i * V_j = \begin{bmatrix} 2n - i - j \\ n - i \end{bmatrix} V_{i+j-n}, \tag{4.18}$$

where we used the flag coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k}\frac{\kappa_n}{\kappa_k\kappa_{n-k}}, \qquad k \in \{0, \ldots, n\}.$$

Taking Alesker-Fourier transform on both sides yields

$$V_i \cdot V_j = \begin{bmatrix} i + j \\ i \end{bmatrix} V_{i+j}. \tag{4.19}$$

The computation of convolution and product of tensor valuations follows the same scheme: first one computes the convolution of tensor valuations, which can be considered easier. Then one applies the Alesker-Fourier transform to obtain the product. However, in the tensor-valued case it is much harder to write down the Alesker-Fourier transform in an explicit way. This step is the technical heart of our approach.

4.3.5 *Poincaré Duality*

The product of smooth translation invariant valuations as well as the convolution both satisfy a version of Poincaré duality, which moreover are identical up to a sign.

To state this more precisely, recall that the vector spaces $\mathrm{Val}_k = \mathrm{Val}_k(\mathbb{R}^n)$, $k \in \{0, n\}$, are one-dimensional and spanned by the Euler-characteristic $\chi = V_0$ and the volume functional $V_n = \mathrm{vol}$, that is, $\mathrm{Val}_0 \cong \mathbb{R} \cdot \chi$ and $\mathrm{Val}_n \cong \mathbb{R} \cdot \mathrm{vol}$. We denote by $\phi_0, \phi_n \in \mathbb{R}$ the component of $\phi \in \mathrm{Val}$ of degree 0 and n, respectively.

Proposition 4.7 *The pairings*

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \qquad (\phi_1, \phi_2) \mapsto (\phi_1 \cdot \phi_2)_n,$$

and

$$\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathbb{R}, \qquad (\phi_1, \phi_2) \mapsto (\phi_1 * \phi_2)_0,$$

are perfect, that is, the induced maps

$$\mathrm{pd}_m, \mathrm{pd}_c : \mathrm{Val}_k^\infty \to \mathrm{Val}_{n-k}^{\infty,*}$$

are injective with dense image. Moreover,

$$\mathrm{pd}_c = \begin{cases} \mathrm{pd}_m & \text{on } \mathrm{Val}_k^+, \\ -\mathrm{pd}_m & \text{on } \mathrm{Val}_k^- . \end{cases}$$

To illustrate this proposition and to highlight the difference between the two pairings, let us compute them on an easy example. Let $\phi_i(K) := \mathrm{vol}(K + A_i)$, where A_i, $i \in \{1, 2\}$, are smooth convex bodies with positive Gauss curvature. Then $\phi_1 * \phi_2(K) = \mathrm{vol}(K + A_1 + A_2)$, and hence $(\phi_1 * \phi_2)_0 = \mathrm{vol}(A_1 + A_2)$.

On the other hand, $\phi_1 \cdot \phi_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2)$. Using Fubini's theorem, one rewrites this as

$$\phi_1 \cdot \phi_2(K) = \int_{\mathbb{R}^n} \phi_2((x - A_1) \cap K)\, \mathrm{d}x.$$

Taking for K a large ball reveals that $(\phi_1 \cdot \phi_2)_n = \phi_2(-A_1) = \mathrm{vol}(A_2 - A_1)$. If $A_1 = -A_1$, then ϕ_1 is even and both pairings agree indeed.

The extension of Poincaré duality to tensor-valued valuations is postponed to Sect. 4.4.1 where it is required for the description of the relation between additive and intersectional kinematic formulas for tensor valuations.

4.3.6 Explicit Computations in the O(*n*)-Equivariant Case

In this section, we outline the explicit computation of product, convolution and Alesker-Fourier transform in the $\mathrm{O}(n)$-equivariant case. Depending on the situation, we will either use the basis consisting of the tensor valuations $Q^i\Phi_k^{s-2i}$ or the basis consisting of the tensor valuations $Q^i\Psi_k^{s-2i}$. The latter are defined in the following proposition.

Proposition 4.8 *The following statements hold.*

(i) *For* $0 \leq k < n$ *and* $s \neq 1$*, define*

$$\Psi_k^s := \Phi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j \Gamma(\frac{n-k+s}{2}) \Gamma(\frac{n}{2} + s - 1 - j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - 1)} Q^j \Phi_k^{s-2j}$$

and let $\Psi_n^0 := \Phi_n^0$*. Then* Ψ_k^s *is the trace free part of* Φ_k^s*. In particular,* $\Psi_k^s \equiv \Phi_k^s$ mod Q.

(ii) *For* $0 \leq k < n$ *and* $s \neq 1$*,* Φ_k^s *can be written in terms of* $\Psi_k^{s'}$ *as*

$$\Phi_k^s = \Psi_k^s + \sum_{j=1}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{n-k+s}{2}\right) \Gamma(\frac{n}{2} + s - 2j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - j)} Q^j \Psi_k^{s-2j}.$$

The inversion which is needed to derive (ii) from (i) can be accomplished with the help of Zeilberger's algorithm.

The first and easier step in the explicit calculations of algebraic structures for tensor valuations is to compute the convolution of two tensor valuations. Since Φ_k^s is smooth (i.e., each component is a smooth valuation), we may write

$$\Phi_k^s(K) = \int_{\mathrm{nc}(K)} \omega_{k,s},$$

where $\omega_{k,s}$ is a smooth $(n-1)$-form on the sphere bundle $\mathbb{R}^n \times S^{n-1}$ with values in $\mathrm{Sym}^s \mathbb{R}^n$. Next, for valuations represented by differential forms, there is an easy formula for the convolution, which involves only some linear and bilinear operations (a kind of Hodge star and a wedge product). The resulting formula states that, for $k, l \leq n$ with $k + l \geq n$ and $s_1, s_2 \neq 1$, we have

$$\begin{aligned}\Phi_k^{s_1} * \Phi_l^{s_2} = & \frac{\omega_{s_1+s_2+2n-k-l}}{\omega_{s_1+n-k}\omega_{s_2+n-l}} \frac{(n-k)(n-l)}{2n-k-l} \\ & \cdot \binom{2n-k-l}{n-k}\binom{s_1+s_2}{s_1}\frac{(s_1-1)(s_2-1)}{1-s_1-s_2}\Phi_{k+l-n}^{s_1+s_2},\end{aligned}$$

or, using the normalization (4.14) which is more convenient for this purpose,

$$\phi_k^{s_1} * \phi_l^{s_2} = n\frac{\binom{k+l}{n}}{\binom{k+l}{k}}\frac{(s_1-1)(s_2-1)}{1-s_1-s_2}\phi_{k+l-n}^{s_1+s_2}.$$

The computation of the Alesker-Fourier transform of tensor valuations is the main step and will be explained in the next section. For $0 \leq k \leq n$ and $s \neq 1$, the result is

$$\begin{aligned}\mathbb{F}(\Psi_k^s) &= \mathbf{i}^s \, \Psi_{n-k}^s, \\ \mathbb{F}(\Phi_k^s) &= \mathbf{i}^s \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{(-1)^j}{(4\pi)^j j!} Q^j \Phi_{n-k}^{s-2j}.\end{aligned}$$

Finally, the product of two tensor valuations can be computed once the convolution and the Alesker-Fourier transform are known, see (4.16). The result is a bit more involved than the formulas for convolution and Alesker-Fourier transform. The reason is that the formula for the convolution is best described in terms of the tensor valuations Φ_k^s, while the description of the Alesker-Fourier transform has a simpler expression for the Ψ_k^s.

After some algebraic manipulations (which make use of Zeilberger's algorithm), we arrive at

$$\begin{aligned}\Phi_k^{s_1} \cdot \Phi_l^{s_2} = & \frac{kl}{k+l}\binom{k+l}{k} \sum_{\substack{a=0 \\ 2a \neq s_1+s_2-1}}^{\lfloor \frac{s_1+s_2}{2} \rfloor} \frac{1}{(4\pi)^a a!} \\ & \cdot \left(\sum_{m=0}^{a} \sum_{i=\max\{0, m-\lfloor \frac{s_2}{2} \rfloor\}}^{\min\{m, \lfloor \frac{s_1}{2} \rfloor\}} (-1)^{a-m} \binom{a}{m}\binom{m}{i} \frac{\omega_{s_1+s_2-2m+k+l}}{\omega_{s_1-2i+k}\omega_{s_2-2m+2i+l}} \right. \\ & \left. \cdot \binom{s_1+s_2-2m}{s_1-2i} \frac{(s_1-2i-1)(s_2-2m+2i-1)}{1-s_1-s_2+2m} \right) Q^a \Phi_{k+l}^{s_1+s_2-2a}.\end{aligned} \tag{4.20}$$

Here $0 \leq k, l$ with $k + l \leq n$ and $s_1, s_2 \neq 1$. It seems that there is no simple closed expression for the inner sum.

4.3.7 Tensor Valuations Versus Scalar-Valued Valuations

The interplay between tensor valuations and scalar-valued valuations will be essential in the computation of the Alesker-Fourier transform. We therefore explain this now is some more detail.

We first need some facts from representation theory. It is well-known that equivalence classes of complex irreducible (finite-dimensional) representations of $\mathrm{SO}(n)$ are indexed by their highest weights. The possible highest weights are tuples $(\lambda_1, \lambda_2, \dots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ of integers such that

1. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\lfloor \frac{n}{2} \rfloor} \geq 0$ if n is odd,
2. $\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_{\frac{n}{2}}| \geq 0$ if n is even.

Given $\lambda = (\lambda_1, \dots, \lambda_{\lfloor \frac{n}{2} \rfloor})$ satisfying this condition, we will denote the corresponding equivalence class of representations by Γ_λ.

The decomposition of the $\mathrm{SO}(n)$-module Val_k has been obtained in [3].

Theorem 4.9 ([3]) *There is an isomorphism of* $\mathrm{SO}(n)$*-modules*

$$\mathrm{Val}_k \cong \bigoplus_\lambda \Gamma_\lambda,$$

where λ ranges over all highest weights such that $|\lambda_2| \leq 2$, $|\lambda_i| \neq 1$ for all i and $\lambda_i = 0$ for $i > \min\{k, n-k\}$. In particular, these decompositions are multiplicity-free.

Let Γ be an irreducible representation of $\mathrm{SO}(n)$ and Γ^* its dual. The space of k-homogeneous $\mathrm{SO}(n)$-equivariant Γ-valued valuations (i.e., maps $\Phi : \mathscr{K} \to \Gamma$ such that $\Phi(gK) = g\Phi(K)$ for all $g \in \mathrm{SO}(n)$) is $(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma^*, \mathrm{Val}_k)$. By Theorem 4.9, Γ^* appears in the decomposition of Val_k precisely if Γ appears, and in this case the multiplicity is 1. By Schur's lemma it follows that $\dim(\mathrm{Val}_k \otimes \Gamma)^{\mathrm{SO}(n)} = 1$ in this case.

Let us construct the (unique up to scale) equivariant Γ-valued valuation explicitly. Denote by $\mathrm{Val}_k(\Gamma)$ the Γ-isotypical summand, which is isomorphic to Γ since Val_k is multiplicity free.

Let $\phi_1, \dots, \phi_m$ be a basis of $\mathrm{Val}_k(\Gamma)$. These elements play two different roles: first we can look at them as valuations, i.e., elements of Val_k. Second, we may think of $\phi_1, \dots, \phi_m$ as basis of the irreducible representation Γ. The action of $\mathrm{SO}(n)$ on this basis is given by

$$g\phi_i = \sum_j c_i^j(g)\phi_j,$$

where $(c_i^j(g))_{i,j}$ is a matrix depending on g. The map $g \mapsto (c_i^j(g))_{i,j}$ is a homomorphism of Lie groups $\mathrm{SO}(n) \to \mathrm{GL}(m)$.

Let $\phi_1^*, \ldots, \phi_m^*$ be the dual basis of Γ^*. Then

$$g\phi_i^* = \sum_j (c_i^j(g))^{-t}\phi_j = \sum_j c_j^i(g^{-1})\phi_j,$$

Using the double role played by the ϕ_i mentioned above, we set

$$\Phi(K) := \sum_i \phi_i(K)\phi_i^* \in \Gamma^*. \tag{4.21}$$

We claim that Φ is an $\mathrm{O}(n)$-equivariant valuation with values in Γ^*. Indeed, we compute

$$\begin{aligned}\Phi(gK) &= \sum_i \phi_i(gK)\phi_i^* = \sum_i (g^{-1}\phi_i)(K)\phi_i^* \\ &= \sum_{i,j} c_i^j(g^{-1})\phi_j(K)\phi_i^* = \sum_j \phi_j(K)\sum_i c_i^j(g^{-1})\phi_i^* \\ &= \sum_j \phi_j(K)g\phi_j^* = g(\Phi(K)).\end{aligned}$$

Conversely, we now start with an equivariant Γ^*-valued continuous translation invariant valuation Φ of degree k. Let $w_1, \ldots, w_m$ be a basis of Γ^*. Then we may look at the components of Φ, i.e., we decompose

$$\Phi(K) = \sum_i \phi_i(K)w_i$$

with $\phi_i \in \mathrm{Val}_k$. Let the action of $\mathrm{SO}(n)$ on Γ^* be given by

$$gw_i = \sum_j a_i^j(g)w_j.$$

We have

$$\begin{aligned}\Phi(gK) &= \sum_i \phi_i(gK)w_i = \sum_i (g^{-1}\phi_i)(K)w_i, \\ g(\Phi(K)) &= \sum_j \phi_j(K)gw_j = \sum_{i,j} \phi_j(K)a_j^i(g)w_i.\end{aligned}$$

Comparing coefficients yields $g^{-1}\phi_i = \sum_j a_j^i(g)\phi_j$, or

$$g\phi_i = \sum_j a_j^i(g^{-1})\phi_j.$$

This shows that the subspace of Val_k spanned by $\phi_1, \ldots, \phi_m$ is isomorphic to Γ.

In summary, we have shown the following fact.

Each $\mathrm{SO}(n)$*-irreducible representation* Γ *appearing in the decomposition of* Val_k *corresponds to the (unique up to scale)* Γ^**-valued continuous translation invariant valuation* Φ *from* (4.21)*. Conversely, the coefficients of a* Γ^**-valued continuous translation invariant valuation span a subspace of* Val_k *isomorphic to* Γ*.*

Let us now discuss the special case of symmetric tensor valuations. The $\mathrm{SO}(n)$-representation space Sym^s is (in general) not irreducible. Indeed, the trace map $\mathrm{tr} : \mathrm{Sym}^s \to \mathrm{Sym}^{s-2}$ commutes with $\mathrm{SO}(n)$, hence its kernel is an invariant subspace. This subspace turns out to be the irreducible representation $\Gamma_{(s,0,\ldots,0)}$ and can be identified with the space $\mathscr{H}_s^n$ of spherical harmonics of degree s.

Since the trace map is onto, we get the following decomposition.

$$\mathrm{Sym}^s \cong \bigoplus_j \mathscr{H}_{s-2j}^n.$$

Instead of studying Sym^s-valued valuations, we can therefore study $\mathscr{H}_s^n$-valued valuations. For $s \neq 1$ and $1 \leq k \leq n-1$, the representation $\mathscr{H}_s^n$ appears in Val_k with multiplicity 1. Since $\mathscr{H}_s^n$ is self-dual, the construction sketched above yields in the special case $\Gamma := \mathscr{H}_s^n$ a unique (up to scale) $\mathscr{H}_s^n$-valued equivariant continuous translation invariant valuation homogeneous of degree k, which we denoted by Ψ_k^s.

4.3.8 The Alesker-Fourier Transform

As we have seen in the previous section, the study of (symmetric) tensor valuations and the study of the $\mathscr{H}^s$-isotypical summand of Val_k are equivalent. For the computation of the Alesker-Fourier transform, it is easier to work with scalar-valued valuations. Let us first define a particular class of valuations, called spherical valuations.

Let f be a smooth function on S^{n-1}. For $k \in \{0, \ldots, n-1\}$, we define a valuation $\mu_{k,f} \in \mathrm{Val}_k(\mathbb{R}^n)$ by

$$\mu_{k,f}(K) := \binom{n-1}{k} \frac{1}{\omega_{n-k}} \int_{S^{n-1}} f(y)\, S_k(K, \mathrm{d}y).$$

Such valuations are called spherical (see also [15]). Here the normalization is chosen such that for $f \equiv 1$ we have $\mu_{k,f} = V_k$, $k \in \{0, \ldots, n-1\}$. By Sect. 4.3.7, the components of an $\mathrm{SO}(n)$-equivariant tensor valuation are spherical. Since the Alesker-Fourier transform of such a tensor valuation is defined component-wise, it suffices to compute the Alesker-Fourier transform of spherical valuations.

In this section, we sketch this (rather involved) computation. The first and easy observation is that, by Schur's lemma, there exist constants $c_{n,k,s} \in \mathbb{C}$ which only depend on n, k, s such that

$$\mathbb{F}(\mu_{k,f}) = c_{n,k,s}\mu_{n-k,f}, \qquad f \in \mathscr{H}_s^n. \tag{4.22}$$

The multipliers $c_{n,k,s}$ of the Alesker-Fourier transform can be computed in the even case (i.e., if s is even) by looking at Klain functions. In the odd case, there seems to be no easy way to compute them. We adapt ideas from [12], where the multipliers of the α-cosine transform were computed, to our situation. The main point is that the Alesker-Fourier transform is not only an SO(n)-equivariant operator, but (if written in intrinsic terms) is equivariant under the larger group GL(n). Using elements from the Lie algebra $\mathfrak{gl}(n)$ allows us to pass from one irreducible SO(n)-representation to another and to obtain a recursive formula for the constants $c_{n,k,s}$, which states that

$$\frac{c_{n,k,s+2}}{c_{n,k,s}} = -\frac{k+s}{n-k+s}. \tag{4.23}$$

This step requires extensive computations using differential forms, and we refer to [6] for the details.

Next, one can use induction over s, k, n to prove that

$$c_{n,k,s} = \mathbf{i}^s \frac{\Gamma(\frac{n-k}{2})\Gamma(\frac{s+k}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{s+n-k}{2})}.$$

More precisely, in the even case, we may use as induction start the case $s = 0$, which corresponds to intrinsic volumes, whose Alesker-Fourier transform is known by (4.17).

In the odd case, we use as induction start $s = 3$. In order to compute $c_{n,k,3}$, we use a special case of a Crofton formula from [7] (see also Chap. 4) to compute the quotients $\frac{c_{n,k+1,3}}{c_{n,k,3}}$. This fixes all constants up to a scaling which may depend on n. More precisely,

$$c_{n,k,s} = \varepsilon_n \mathbf{i}^s \frac{\Gamma(\frac{n-k}{2})\Gamma(\frac{s+k}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{s+n-k}{2})}, \tag{4.24}$$

where ε_n depends only on n. Using functorial properties of the Alesker-Fourier transform, we find that ε_n is independent of n. In the two-dimensional case, however, there is a very explicit description of the Alesker-Fourier transform (see also Example 3.18 (4)) which finally allows us to deduce that $\varepsilon_n = 1$ for all $n \geq 2$.

A variant of this approach to determining the constants $c_{n,k,s}$ might be to prove independently a Crofton formula for the tensor valuations Ψ_k^s. But still this will leave the task of determining $c_{n,1,s}$ or $c_{n,n-1,s}$. This point of view suggests to relate the Fourier operator for spherical valuations to the Fourier operators for spherical functions via the relation

$$\mathbb{F}(\bar{\mu}_{k,f}) = (2\pi)^{-\frac{d}{2}}\, \bar{\mu}_{d-k,I_k f},$$

for $f \in C^\infty(S^{d-1})$, where

$$\bar{\mu}_{k,f}(K) = \binom{d-1}{k}(2\pi)^{\frac{k}{2}} \int_{S^{d-1}} f(u)\, S_k(K, \mathrm{d}u),$$

is just a renormalization of $\mu_{k,f}(K)$.

4.4 Kinematic Formulas

In this section, we first describe the interplay between algebraic structures and kinematic formulas in general (i.e., for tensor valuations which are equivariant under a group G acting transitively on the unit sphere). Then we will specialize to the $\mathrm{O}(n)$-covariant case.

4.4.1 Relation Between Kinematic Formulas and Algebraic Structures

Let G be a subgroup of $\mathrm{O}(n)$ which acts transitively on the unit sphere. Then the space $\mathrm{TVal}^{s,G}(V)$ of G-covariant, translation invariant continuous $\mathrm{Sym}^s(V)$-valued valuations is finite-dimensional. Next we define two integral geometric operators. We start with the one for rotation sum formulas.

Let $\Phi \in \mathrm{TVal}^{s_1+s_2,G}(V)$. We define a bivaluation with values in the tensor product $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$a^G_{s_1,s_2}(\Phi)(K,L) := \int_G (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2}) \Phi(K + g^{-1}L)\, \nu(\mathrm{d}g)$$

for $K, L \in \mathscr{K}(V)$, where G is endowed with the Haar probability measure ν (see [16]). (This notation is consistent with the case $V = \mathbb{R}^n$ and $G = \mathrm{O}(n)$.)

Let $\Phi_1, \ldots, \Phi_{m_1}$ be a basis of $\mathrm{TVal}^{s_1,G}(V)$, and let $\Psi_1, \ldots, \Psi_{m_2}$ be a basis of $\mathrm{TVal}^{s_2,G}(V)$. Arguing as in the classical Hadwiger argument (cf. [16, Theorem 4.3]), it can be seen that there are constants c_{ij}^Φ such that

$$a^G_{s_1,s_2}(\Phi)(K,L) = \sum_{i,j} c_{ij}^\Phi\, \Phi_i(K) \otimes \Psi_j(L)$$

for $K, L \in \mathscr{K}(V)$. The *additive kinematic operator* is the map

$$a^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) \to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V)$$
$$\Phi \mapsto \sum_{i,j} c^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j,$$

which is independent of the choice of the bases.

In view of intersectional kinematic formulas, we define a bivaluation with values in $\mathrm{Sym}^{s_1} V \otimes \mathrm{Sym}^{s_2} V$ by

$$k^G_{s_1,s_2}(\Phi)(K, L) := \int_{\bar{G}} (\mathrm{id}^{\otimes s_1} \otimes g^{\otimes s_2})\Phi(K \cap \bar{g}^{-1}L)\, \mu(\mathrm{d}\bar{g})$$

for $K, L \in \mathscr{K}(V)$, where $\bar{G}$ is the group generated by G and the translation group of V, endowed with the product measure μ of ν and a translation invariant Haar measure on V, and where g is the rotational part of $\bar{g}$. Again this notation is consistent with the special case where $\bar{G} = \mathrm{G}_n$ is the motion group, $G = \mathrm{O}(n)$ and μ is the motion invariant Haar measure with its usual normalization as a 'product measure'. Choosing bases and arguing as above, we find

$$k^G_{s_1,s_2}(\Phi)(K, L) = \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i(K) \otimes \Psi_j(L) \tag{4.25}$$

for $K, L \in \mathscr{K}(V)$. Of course, the constants d^{Φ}_{ij} depend on the chosen bases and on Φ, but the operator, called *intersectional kinematic operator*,

$$k^G_{s_1,s_2} : \mathrm{TVal}^{s_1+s_2,G}(V) \to \mathrm{TVal}^{s_1,G}(V) \otimes \mathrm{TVal}^{s_2,G}(V)$$
$$\Phi \mapsto \sum_{i,j} d^{\Phi}_{ij}\, \Phi_i \otimes \Psi_j,$$

is independent of these choices.

In the following, we explain the connection between these operators and then provide explicit examples.

For this we first lift the Poincaré duality maps to tensor-valued valuations. Let V be a Euclidean vector space with scalar product $\langle \cdot\,,\cdot \rangle$. For $s \leq r$ we define the contraction map by

$$\mathrm{contr} : V^{\otimes s} \times V^{\otimes r} \to V^{\otimes (r-s)},$$
$$(v_1 \otimes \cdots \otimes v_s, w_1 \otimes \cdots \otimes w_r) \mapsto \langle v_1, w_1\rangle\langle v_2, w_2\rangle \cdots \langle v_s, w_s\rangle w_{s+1} \otimes \cdots \otimes w_r,$$

and linearity. This map restricts to a map $\mathrm{contr} : \mathrm{Sym}^s V \times \mathrm{Sym}^r V \to \mathrm{Sym}^{r-s} V$. In particular, if $r = s$, the map $\mathrm{Sym}^s V \times \mathrm{Sym}^s V \to \mathbb{R}$ is the usual scalar product on $\mathrm{Sym}^s V$, which will also be denoted by $\langle \cdot\,,\cdot \rangle$.

The trace map $\mathrm{tr} : \mathrm{Sym}^s V \to \mathrm{Sym}^{s-2} V$ is defined by restriction of the map $V^{\otimes s} \to V^{\otimes(s-2)}$, $v_1 \otimes \cdots \otimes v_s \mapsto \langle v_1, v_2\rangle v_3 \otimes \cdots \otimes v_s$, for $s \geq 2$.

The scalar product on $\mathrm{Sym}^s V$ induces an isomorphism $q^s : \mathrm{Sym}^s V \to (\mathrm{Sym}^s V)^*$ and we set

$$\mathrm{pd}_c^s : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V \xrightarrow{\mathrm{pd}_c \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*,$$

$$\mathrm{pd}_m^s : \mathrm{TVal}^{s,\infty} = \mathrm{Val}^\infty \otimes \mathrm{Sym}^s V \xrightarrow{\mathrm{pd}_m \otimes q^s} (\mathrm{Val}^\infty)^* \otimes (\mathrm{Sym}^s V)^* = (\mathrm{TVal}^{s,\infty})^*.$$

From Proposition 4.7 it follows easily that

$$\mathrm{pd}_m^s = (-1)^s \,\mathrm{pd}_c^s\,. \tag{4.26}$$

Finally, we write

$$m, c : \mathrm{TVal}^{s_1,\infty}(V) \otimes \mathrm{TVal}^{s_2,\infty}(V) \to \mathrm{TVal}^{s_1+s_2,\infty}(V)$$

for the maps corresponding to the product and the convolution. Moreover, we write m_G, c_G for the restrictions of these maps to the corresponding spaces of G-covariant tensor valuations.

Theorem 4.10 *Let G be a compact subgroup of $\mathrm{O}(n)$ acting transitively on the unit sphere. Then the diagram*

$$\begin{array}{ccc}
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{a^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G} \\
\Big\downarrow{\scriptstyle \mathrm{pd}_c^{s_1+s_2}} & & \Big\downarrow{\scriptstyle \mathrm{pd}_c^{s_1} \otimes \mathrm{pd}_c^{s_2}} \\
(\mathrm{TVal}^{s_1+s_2,G})^* & \xrightarrow{c_G^*} & (\mathrm{TVal}^{s_1,G})^* \otimes (\mathrm{TVal}^{s_2,G})^* \\
\Big\downarrow{\scriptstyle \mathbb{F}^*} & & \Big\downarrow{\scriptstyle \mathbb{F}^* \otimes \mathbb{F}^*} \\
(\mathrm{TVal}^{s_1+s_2,G})^* & \xrightarrow{m_G^*} & (\mathrm{TVal}^{s_1,G})^* \otimes (\mathrm{TVal}^{s_2,G})^* \\
\Big\uparrow{\scriptstyle \mathrm{pd}_m^{s_1+s_2}} & & \Big\uparrow{\scriptstyle \mathrm{pd}_m^{s_1} \otimes \mathrm{pd}_m^{s_2}} \\
\mathrm{TVal}^{s_1+s_2,G} & \xrightarrow{k^G_{s_1,s_2}} & \mathrm{TVal}^{s_1,G} \otimes \mathrm{TVal}^{s_2,G}
\end{array}$$

commutes and encodes the relations between product, convolution, Alesker-Fourier transform, intersectional and additive kinematic formulas.

This diagram allows us to express the additive kinematic operator in terms of the intersectional kinematic operator, and vice versa, with the Fourier transform as the link between these operators.

Corollary 4.11 *Intersectional and additive kinematic formulas are related by the Alesker-Fourier transform in the following way:*

$$a^G = (\mathbb{F}^{-1} \otimes \mathbb{F}^{-1}) \circ k^G \circ \mathbb{F},$$

or equivalently

$$k^G = (\mathbb{F} \otimes \mathbb{F}) \circ a^G \circ \mathbb{F}^{-1}.$$

This follows by looking at the outer square in Theorem 4.10, by carefully taking into account the signs coming from (4.26).

4.4.2 Some Explicit Examples of Kinematic Formulas

We start with a description of a Crofton formula for tensor valuations. Combining the connection between Crofton formulas and the product of valuations (see [4, (2) and (16)]) and the explicit formulas for the product of tensor valuations given in (4.20), we obtain

$$\begin{aligned}\int_{\mathrm{A}(n,n-l)} \Phi_k^s(K \cap E)\,\mu_{n-l}(\mathrm{d}E) &= \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \left(\Phi_k^s \cdot \Phi_l^0\right)(K) \\ &= \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{k+l} \sum_{a=0, 2a \neq s-1}^{\lfloor \frac{s}{2} \rfloor} \frac{1}{(4\pi)^a a!} \\ &\quad \times \sum_{m=0}^{a} (-1)^{a-m} \binom{a}{m} \frac{\omega_{s-2m+k+l}}{\omega_{s-2m+k}\omega_l} Q^a \Phi_{k+l}^{s-2a}.\end{aligned}$$

After simplification of the inner sum by means of Zeilberger's algorithm, we obtain the Crofton formula in the Φ-basis which was obtained in [6].

Theorem 4.12 *If $k, l \geq 0$ with $k + l \leq n$ and $s \in \mathbb{N}_0$, then*

$$\begin{aligned}\int_{\mathrm{A}(n,n-l)} \Phi_k^s(K \cap E)\,\mu_{n-l}(\mathrm{d}E) &= \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \binom{k+l}{k} \frac{kl}{2(k+l)} \frac{1}{\Gamma\left(\frac{k+l+s}{2}\right)} \\ &\quad \times \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma\left(\frac{l}{2}+j\right) \Gamma\left(\frac{k+s}{2}-j\right)}{(4\pi)^j j!} Q^j \Phi_{k+l}^{s-2j}(K).\end{aligned}$$

The result is also true in the cases $k, l \in \{0, n\}$, if the right-hand side is interpreted properly; see the comments after [6, Theorem 3]. The same is true for the next result.

Comparing the trace-free part of this formula (or by inversion), we deduce the Crofton formula for the Ψ-basis, in which the result has a particularly convenient form.

Corollary 4.13 *If* $k, l \geq 0$ *and* $k + l \leq n$*, then*

$$\int_{\mathrm{A}(n,n-l)} \Psi_k^s(K \cap E)\, \mu_{n-l}(\mathrm{d}E) = \frac{\omega_{s+k+l}}{\omega_{s+k}\omega_l} \binom{k+l}{k} \frac{kl}{k+l} \begin{bmatrix} n \\ l \end{bmatrix}^{-1} \Psi_{k+l}^s(K).$$

Alternatively, as observed in [10], Corollary 4.13 can be deduced from (4.12), and then Theorem 4.12 can be obtained as a consequence.

Thus, having now a convenient Crofton formula for tensor valuations, we deduce from Hadwiger's general integral geometric theorem an intersectional kinematic formula in the Ψ-basis.

Theorem 4.14 *Let* $K, M \in \mathscr{K}^n$ *and* $j \in \{0, \dots, n\}$*. Then*

$$\int_{\mathrm{G}_n} \Psi_j^s(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} \frac{\omega_{s+k}}{\omega_{s+j}\kappa_{k-j}} \binom{k-1}{j-1} \begin{bmatrix} n \\ k-j \end{bmatrix}^{-1} \Psi_k^s(K)\, V_{n-k+j}(M).$$

Let us now prove some more refined intersectional kinematic formulas. In principle, we could also use Corollary 4.11 to find the intersectional kinematic formulas once we know the additive formulas. The problem is that (4.15) only gives us the value of a_{s_1,s_2} on the basis element $\phi_j^{s_1+s_2}$, but not on multiples of such basis elements with powers of the metric tensors. However, such terms appear naturally in the Fourier transform.

We therefore use Theorem 4.10 with $V = \mathbb{R}^n$ and $G = \mathrm{O}(n)$, more precisely the lower square in the diagram.

In (4.20) we have computed the product of two tensor valuations. For fixed (small) ranks s_1, s_2, the formula simplifies and can be evaluated in a closed form. For instance, if $1 \leq k, l$ with $k + l \leq n$ and $s_1 = s_2 = 3$, we get

$$\begin{aligned} \Phi_k^3 \cdot \Phi_l^3 = & \frac{(k+1)(l+1)\Gamma\left(\frac{k+l+1}{2}\right)}{\pi^{\frac{5}{2}}(k+l+4)(k+l+2)(k+l)\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)} \\ & \cdot \Big(-32\Phi_{k+l}^6\pi^3 + 8Q\Phi_{k+l}^4\pi^2 - Q^2\Phi_{k+l}^2\pi + \tfrac{1}{12}Q^3\Phi_{k+l}^0 \Big). \end{aligned} \tag{4.27}$$

Let us next work out the vertical arrows in the diagram of Theorem 4.10, that is, the Poincaré duality pd_m. Again, this is a computation involving differential forms. The result (see [6, Corollary 5.3]) is

$$\langle \mathrm{pd}_m^s(\Phi_k^s), \Phi_{n-k}^s \rangle = (-1)^s \frac{1-s}{\pi^s s!^2} \binom{n}{k} \frac{k(n-k)}{4} \frac{\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{4.28}$$

We now explain how to compute the intersectional kinematic formula $k_{3,3}^{\mathrm{O}(n)}$ with this knowledge.

Since $\Phi_m^1 \equiv 0$, it is clear that there is a formula of the form

$$k_{3,3}^{\mathrm{O}(n)}(\Phi_i^6) = \sum_{k+l=n+i} a_{n,i,k} \Phi_k^3 \otimes \Phi_l^3$$

with some constants $a_{n,i,k}$ which remain to be determined. Fix k, l with $k+l = n+i$. Using (4.28), we find

$$\langle \mathrm{pd}_m^3\, \Phi_k^3, \Phi_{n-k}^3 \rangle = \frac{1}{72\pi^3} \binom{n}{k} k(n-k) \frac{\Gamma(\frac{k+3}{2})\Gamma(\frac{n-k+3}{2})}{\Gamma(\frac{n}{2}+1)},$$

$$\langle \mathrm{pd}_m^3\, \Phi_l^3, \Phi_{n-l}^3 \rangle = \frac{1}{72\pi^3} \binom{n}{l} l(n-l) \frac{\Gamma(\frac{l+3}{2})\Gamma(\frac{n-l+3}{2})}{\Gamma(\frac{n}{2}+1)},$$

and therefore

$$\begin{aligned} &\langle (\mathrm{pd}_m^3 \otimes \mathrm{pd}_m^3) \circ k_{3,3}^{\mathrm{O}(n)}(\Phi_i^6), \Phi_{n-k}^3 \otimes \Phi_{n-l}^3 \rangle \\ &\quad = a_{n,i,k} \frac{1}{72\pi^3} \binom{n}{k} k(n-k) \frac{\Gamma(\frac{k+3}{2})\Gamma(\frac{n-k+3}{2})}{\Gamma(\frac{n}{2}+1)} \\ &\qquad \cdot \frac{1}{72\pi^3} \binom{n}{l} l(n-l) \frac{\Gamma(\frac{l+3}{2})\Gamma(\frac{n-l+3}{2})}{\Gamma(\frac{n}{2}+1)}. \end{aligned}$$

On the other hand, by (4.27) and (4.28),

$$\begin{aligned} \langle m_{\mathrm{O}(n)}^* \circ \mathrm{pd}_m^6(\Phi_i^6), \Phi_{n-k}^3 \otimes \Phi_{n-l}^3 \rangle &= \langle \mathrm{pd}_m^6(\Phi_i^6), \Phi_{n-k}^3 \cdot \Phi_{n-l}^3 \rangle \\ = &\frac{(n-k+1)(n-l+1)\Gamma(\frac{n-i+1}{2})}{\pi^{\frac{5}{2}}(n-i+4)(n-i+2)(n-i)\Gamma(\frac{n-l}{2})\Gamma(\frac{n-k}{2})} \\ &\cdot \left\langle \mathrm{pd}_m^6(\Phi_i^6), -32\Phi_{n-i}^6\pi^3 + 8Q\Phi_{n-i}^4\pi^2 - Q^2\Phi_{n-i}^2\pi + \tfrac{1}{12}Q^3\Phi_{n-i}^0 \right\rangle \\ = &\frac{1}{207360} \frac{(k-n-1)(i-k-1)\Gamma(\frac{n+1}{2})(i+1)(i-1)(i-3)}{\pi^5\Gamma(\frac{i+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{k-i}{2})}. \end{aligned}$$

Comparing these expressions, we find that

$$a_{n,i,k} = \frac{(i+1)(i-1)(i-3)}{40\Gamma(\frac{n+1}{2})\Gamma(\frac{i+1}{2})} \frac{\Gamma(\frac{k}{2})\Gamma(\frac{l}{2})}{(k+1)(l+1)}.$$

We summarize the result in the following theorem.

Theorem 4.15 *Let $K, M \in \mathcal{K}^n$ and $i \in \{0, \dots, n-1\}$. Then*

$$\int_{G_n} (\mathrm{id}^{\otimes 3} \otimes g^{\otimes 3}) \Phi_i^6(K \cap g^{-1}M)\, \mu(\mathrm{d}g)$$
$$= \frac{(i+1)(i-1)(i-3)}{40\Gamma(\frac{n+1}{2})\Gamma(\frac{i+1}{2})} \sum_{k+l=n+i} \frac{\Gamma(\frac{k}{2})\Gamma(\frac{l}{2})}{(k+1)(l+1)} \Phi_k^3(K) \otimes \Phi_l^3(M).$$

The same technique can be applied to all bidegrees, but it seems hard to find a closed formula which is valid simultaneously in all cases.

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Chapter 5
Crofton Formulae for Tensor-Valued Curvature Measures

Daniel Hug and Jan A. Weis

Abstract The tensorial curvature measures are tensor-valued generalizations of the curvature measures of convex bodies. We prove a set of Crofton formulae for such tensorial curvature measures. These formulae express the integral mean of the tensorial curvature measures of the intersection of a given convex body with a uniform affine k-flat in terms of linear combinations of tensorial curvature measures of the given convex body. Here we first focus on the case where the tensorial curvature measures of the intersection of the given body with an affine flat is defined with respect to the affine flat as its ambient space. From these formulae we then deduce some new and also recover known special cases. In particular, we substantially simplify some of the constants that were obtained in previous work on Minkowski tensors. In a second step, we explain how the results can be extended to the case where the tensorial curvature measure of the intersection of the given body with an affine flat is determined with respect to the ambient Euclidean space.

5.1 Introduction

The *classical Crofton formula* is a major result in integral geometry. Its name originates from works of the Irish mathematician Crofton [4] on integral geometry in $\mathbb{R}^2$ in the late nineteenth century. For a convex body K (a non-empty, convex and compact set) in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$, the classical Crofton formula (see [17, (4.59)]) states that

$$\int_{A(n,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) = \alpha_{njk} V_{n-k+j}(K), \tag{5.1}$$

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for $k \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, k\}$, where $A(n,k)$ is the affine Grassmannian of k-flats in $\mathbb{R}^n$, μ_k denotes the motion invariant Haar measure on $A(n,k)$, normalized as in [18, p. 588], and $\alpha_{njk} > 0$ is an explicitly known constant.

Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$. The functionals $V_i : \mathcal{K}^n \to \mathbb{R}$, for $i \in \{0, \ldots, n\}$, appearing in (5.1), are the *intrinsic volumes*, which occur as the coefficients of the monomials in the *Steiner formula*

$$V_n(K + \epsilon B^n) = \sum_{j=0}^{n} \kappa_{n-j} V_j(K) \epsilon^{n-j}, \tag{5.2}$$

for a convex body $K \in \mathcal{K}^n$ and $\epsilon \geq 0$ (cf. (1.16)); here, as usual, $+$ denotes the Minkowski addition in $\mathbb{R}^n$ and κ_n is the volume of the Euclidean unit ball B^n in $\mathbb{R}^n$. Properties of the V_i such as continuity, isometry invariance and additivity are derived from corresponding properties of the volume. A key result for the intrinsic volumes is *Hadwiger's characterization theorem* (see [7, 2. Satz] and Theorem 1.23), which states that $V_0, \ldots, V_n$ form a basis of the vector space of continuous and isometry invariant real-valued valuations on $\mathcal{K}^n$.

A natural way to extend the Crofton formula is to apply the integration over the affine Grassmannian $A(n,k)$ to functionals which generalize the intrinsic volumes. One of these generalizations concerns the class of continuous and isometry covariant $\mathbb{T}^p$-valued valuations on $\mathcal{K}^n$, where $\mathbb{T}^p$ denotes the vector space of symmetric tensors of rank $p \in \mathbb{N}_0$ over $\mathbb{R}^n$.

The $\mathbb{T}^0$-valued valuations are simply the well-known and extensively studied intrinsic volumes. For the $\mathbb{T}^1$-valued (i.e. vector-valued) valuations, Hadwiger and Schneider [8, Hauptsatz] proved in 1971 a characterization theorem similar to the aforementioned real-valued case due to Hadwiger. In addition, they also established integral geometric formulae, including a Crofton formula [8, (5.4)]. In 1997, McMullen [14] initiated a systematic investigation of this class of $\mathbb{T}^p$-valued valuations for general $p \in \mathbb{N}_0$. Only 2 years later Alesker generalized Hadwiger's characterization theorem (see [2, Theorem 2.2] and Theorem 2.5) by showing that the vector space of continuous and isometry covariant $\mathbb{T}^p$-valued valuations on $\mathcal{K}^n$ is spanned by the tensor-valued versions of the intrinsic volumes, the *Minkowski tensors* $\Phi_j^{r,s}$, where $j, r, s \in \mathbb{N}_0$ and $j < n$, multiplied with suitable powers of the metric tensor in $\mathbb{R}^n$. In 2008, Hug, Schneider and Schuster proved a set of Crofton formulae for these Minkowski tensors (see [11, Theorems 2.1–2.6]).

Localizations of the intrinsic volumes yield other types of generalizations. The *support measures* are weakly continuous, locally defined and motion equivariant valuations on convex bodies with values in the space of finite measures on Borel subsets of $\mathbb{R}^n \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the Euclidean unit sphere in $\mathbb{R}^n$. These are determined by a local version of (5.2). Therefore, they are a crucial example of localizations of the intrinsic volumes. Furthermore, their marginal measures on Borel subsets of $\mathbb{R}^n$ are called *curvature measures* and the ones on Borel subsets of $\mathbb{S}^{n-1}$ are called *surface area measures*. In 1959, Federer [5] proved Crofton formulae for curvature measures, even in the more general setting of sets with positive reach.

For further details and references, see also Sects. 1.3 and 1.5. Certain Crofton formulae for support measures were proved by Glasauer in 1997 [6, Theorem 3.2].

The combination of Minkowski tensors and localizations leads to another generalization of the intrinsic volumes. This topic has been explored by Schneider [16] and Hug and Schneider [9, 10] in recent years. They introduced particular tensorial support measures, the *generalized local Minkowski tensors*, and proved that they essentially span the vector space of isometry covariant and locally defined valuations on the space of convex polytopes $\mathscr{P}^n$ with values in the $\mathbb{T}^p$-valued measures on $\mathscr{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ (see Sect. 2.4). Under the additional assumption of weak continuity they extended this result to valuations on $\mathscr{K}^n$; a summary of the required arguments is given in Sect. 2.5.

The aim of the present chapter is to prove a set of Crofton formulae for similar functionals, which are localized in $\mathbb{R}^n$, the *tensorial curvature measures* or *tensor-valued curvature measures*. Here we first focus on the case where the tensorial curvature measures of the intersection of the given body with an affine flat are defined with respect to the affine flat as the ambient space (intrinsic viewpoint). For precise definitions and references we refer to Sect. 5.2. In a second step, we demonstrate how the arguments can be extended to the case where the curvature measures are considered in $\mathbb{R}^n$ (extrinsic viewpoint). The current approach combines main ideas of the previous works [11] and [9] and also links it to [3]. A major advantage of the localization is that it naturally leads to a suitable choice of local tensor-valued measures for which the constants in the Crofton formulae are reasonably simple. From the general local results, we finally deduce various special consequences for the total measures (obtained by globalization), which are the Minkowski tensors that have been studied in [11]. For the latter, we restrict ourselves to the translation invariant case, which simplifies the involved constants, but the general case can be treated similarly. In the case of the results for the extrinsic tensorial Crofton formulae, the connection to the approach in [3] via the methods of algebraic integral geometry is used and deepened. This interplay will be explored further in future work.

It is a remarkable feature of the present work that results of algebraic integral geometry, which originally have been developed for translation invariant tensor valuations (see [3] and Chap. 4), are applied in the context of tensorial curvature measures. So far it is an open question (and perhaps doubtful) whether basic algebraic structures for translation invariant valuations (such as the product structure) can be introduced for smooth curvature measures (see the comment in Sect. 9.2.2). However, the space of smooth curvature measures forms a module over the algebra of smooth valuations and this structure is compatible with globalization (see Theorem 9.13 and the references given there). The results of the present work also contribute to the explicit determination of this module structure. As part of work in progress, we will develop yet another approach which applies to nonsmooth tensorial curvature measures as well and will lead to a full set of local integral geometric formulae. Such results are crucial for applications in stereology (see for instance [13] and the references given therein) and stochastic geometry, in particular for obtaining localized versions of mean value formulae for Boolean models, as briefly discussed in Sect. 11.7, or for the derivation of Crofton formulae for random sets and point processes, as described in [18, Sect. 9.4], and thus for the derivation of (unbiased) estimators for densities of functionals of random closed sets.

The structure of this contribution is as follows. In Sect. 5.2, we fix our notation and collect various auxiliary results which will be needed. Section 5.3 contains the main results. We first state our findings for intrinsic tensorial curvature measures, then we discuss some special cases and finally explain the extension to extrinsic tensorial curvature measures where the involved constants turn out to be surprisingly simple. Motivated by the results in [3] we also introduce another system of tensorial curvature measures for which the Crofton formulae even have "diagonal form". The proofs of the results for the intrinsic case are given in Sect. 5.4. Section 5.5 contains the arguments in the extrinsic setting. Some auxiliary results on sums of Gamma functions are provided in the final section.

5.2 Some Basic Tools

In the following, we work in the n-dimensional Euclidean space $\mathbb{R}^n$, equipped with its usual topology generated by the standard scalar product $\cdot$ and the corresponding Euclidean norm $\|\cdot\|$. Recall that the unit ball centered at the origin is denoted by B^n, its boundary (the unit sphere) is denoted by $\mathbb{S}^{n-1}$. For a topological space X, we denote the Borel σ-algebra on X by $\mathscr{B}(X)$.

By $G(n,k)$, for $k \in \{0,\dots,n\}$, we denote the Grassmannian of k-dimensional linear subspaces in $\mathbb{R}^n$, and we write ν_k for the (rotation invariant) Haar probability measure on $G(n,k)$. The directional space of an affine k-flat $E \in A(n,k)$ is denoted by $L(E) \in G(n,k)$, its orthogonal complement by $E^\perp \in G(n,n-k)$, and the translate of E by a vector $t \in \mathbb{R}^n$ is denoted by $E_t := E + t$. For $k \in \{0,\dots,n\}$, $l \in \{0,\dots,k\}$ and $F \in G(n,k)$, we define $G(F,l) := \{L \in G(n,l) : L \subset F\}$. On $G(F,l)$ there exists a unique Haar probability measure ν_l^F invariant under rotations of $\mathbb{R}^n$ mapping F into itself and leaving $F^\perp$ pointwise fixed. The orthogonal projection of a vector $x \in \mathbb{R}^n$ to a linear subspace L of $\mathbb{R}^n$ is denoted by $p_L(x)$ and its direction by $\pi_L(x) \in \mathbb{S}^{n-1}$, if $x \notin L^\perp$. For two linear subspaces L, L' of $\mathbb{R}^n$, the generalized sine function $[L,L']$ is defined as follows. One extends an orthonormal basis of $L \cap L'$ to an orthonormal basis of L and to one of L'. Then $[L,L']$ is the volume of the parallelepiped spanned by all these vectors.

The vector space of symmetric tensors of rank $p \in \mathbb{N}_0$ over $\mathbb{R}^n$ is denoted by $\mathbb{T}^p$. The symmetric tensor product of two vectors $x, y \in \mathbb{R}^n$ is denoted by xy and the p-fold tensor product of a vector $x \in \mathbb{R}^n$ by x^p. Identifying $\mathbb{R}^n$ with its dual space via its scalar product, we interpret a symmetric tensor $a \in \mathbb{T}^p$ as a symmetric p-linear map from $(\mathbb{R}^n)^p$ to $\mathbb{R}$. One special tensor is the *metric tensor* $Q \in \mathbb{T}^2$, defined by $Q(x,y) := x \cdot y$ for $x, y \in \mathbb{R}^n$. For an affine k-flat $E \in A(n,k)$, $k \in \{0,\dots,n\}$, the metric tensor $Q(E)$ in E is defined by $Q(E)(x,y) := p_{L(E)}(x) \cdot p_{L(E)}(y)$ for $x, y \in \mathbb{R}^n$.

Defining the tensorial curvature measures requires some preparation (see also Sect. 1.3). For a convex body $K \in \mathscr{K}^n$, we call the pair $(x,u) \in \mathbb{R}^{2n}$ a *support* *element* whenever x is a boundary point of K and u is an outer unit normal vector of K at x. The set of all these support elements of K is denoted by $\mathbf{nc}\, K \subset \Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ and called the *normal bundle* of K. For $x \in \mathbb{R}^n$, we denote the metric projection of x onto K by $p(K, x)$, and define $u(K, x) := (x - p(K, x))/\|x - p(K, x)\|$ for $x \in \mathbb{R}^n \setminus K$, the unit vector pointing from $p(K, x)$ to x. For $\epsilon > 0$ and a Borel set $\eta \subset \Sigma^n$,

$$M_\epsilon(K, \eta) := \{x \in (K + \epsilon B^n) \setminus K \colon (p(K, x), u(K, x)) \in \eta\}$$

is a local parallel set of K which satisfies a *local Steiner formula*

$$V_n(M_\epsilon(K, \eta)) = \sum_{j=0}^{n-1} \kappa_{n-j} \Lambda_j(K, \eta)\epsilon^{n-j}, \qquad \epsilon \geq 0. \tag{5.3}$$

This relation determines the *support measures* $\Lambda_0(K, \cdot), \ldots, \Lambda_{n-1}(K, \cdot)$ of K, which are finite Borel measures on $\mathscr{B}(\Sigma^n)$. Obviously, a comparison of (5.2) and (5.3) yields $V_j(K) = \Lambda_j(K, \Sigma^n)$.

Now, for a convex body $K \in \mathscr{K}^n$, a Borel set $\beta \in \mathscr{B}(\mathbb{R}^n)$ and $j, r, s \in \mathbb{N}_0$, the *tensorial curvature measures* are given by

$$\phi_j^{r,s,0}(K, \beta) := \omega_{n-j} \int_{\beta \times \mathbb{S}^{n-1}} x^r u^s \, \Lambda_j(K, \mathrm{d}(x, u)),$$

for $j \in \{0, \ldots, n-1\}$, where ω_n denotes the $(n-1)$-dimensional volume of $\mathbb{S}^{n-1}$, and by

$$\phi_n^{r,0,0}(K, \beta) := \int_{K \cap \beta} x^r \, \mathscr{H}^n(\mathrm{d}x).$$

If $K \subset E \in A(n, k)$ with $j < k \leq n$, we denote the j-th support measure of K defined with respect to E as the ambient space by $\Lambda_j^{(E)}(K, \cdot)$, which is a Borel measure on $\mathscr{B}(\mathbb{R}^n \times (L(E) \cap \mathbb{S}^{n-1}))$, concentrated on $\Sigma^{(E)} := E \times (L(E) \cap \mathbb{S}^{n-1})$ with $L(E) \in G(n, k)$ being the linear subspace parallel to E. Then, we define the *intrinsic tensorial curvature measures*

$$\phi_{j,E}^{r,s,0}(K, \beta) := \omega_{k-j} \int_{\beta \times (L(E) \cap \mathbb{S}^{n-1})} x^r u^s \, \Lambda_j^{(E)}(K, \mathrm{d}(x, u))$$

and

$$\phi_{k,E}^{r,0,0}(K, \beta) := \int_{K \cap \beta} x^r \, \mathscr{H}^k(\mathrm{d}x).$$

For the sake of convenience, we extend the definition by $\phi_j^{r,s,0} := 0$ (resp. $\phi_{j,E}^{r,s,0} := 0$) for $j \notin \{0, \dots, n\}$ (resp. $j \notin \{0, \dots, k\}$) or $r \notin \mathbb{N}_0$ or $s \notin \mathbb{N}_0$ or $j = n$ (resp. $j = k$) and $s \neq 0$. We adopt the same convention for the Minkowski tensors and the generalized tensorial curvature measures introduced below.

The tensorial curvature measures are natural local versions of the *Minkowski tensors*. For a convex body $K \in \mathscr{K}^n$ and $j, r, s \in \mathbb{N}_0$, the latter are just the total measures $\Phi_j^{r,s}(K) := \phi_j^{r,s,0}(K, \mathbb{R}^n)$ and, if $K \subset E \in A(n, k)$, an intrinsic version is given by $\Phi_{j,E}^{r,s}(K) := \phi_{j,E}^{r,s,0}(K, \mathbb{R}^n)$. These definitions of the Minkowski tensors differ slightly from the ones commonly used in the literature, as we change the usual normalization (compare with the normalization used in Definition 2.1). The purpose of this change is to simplify the presentation of the main results of this chapter (and of future work).

For a polytope $P \in \mathscr{P}^n$ and $j \in \{0, \dots, n\}$, we denote the set of j-dimensional faces of P by $\mathscr{F}_j(P)$ and the normal cone of P at a face $F \in \mathscr{F}_j(P)$ by $N(P, F)$. For a polytope $P \in \mathscr{P}^n$ and a Borel set $\eta \subset \Sigma^n$, the j-th support measure is explicitly given by

$$\Lambda_j(P, \eta) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathscr{F}_j(P)} \int_F \int_{N(P,F) \cap \mathbb{S}^{n-1}} \mathbf{1}_\eta(x, u) \, \mathscr{H}^{n-j-1}(\mathrm{d}u) \, \mathscr{H}^j(\mathrm{d}x)$$

for $j \in \{0, \dots, n-1\}$. For $\beta \in \mathscr{B}(\mathbb{R}^n)$, this yields

$$\phi_j^{r,s,0}(P, \beta) = \sum_{F \in \mathscr{F}_j(P)} \int_{F \cap \beta} x^r \, \mathscr{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathscr{H}^{n-j-1}(\mathrm{d}u)$$

and, if $P \subset E \in A(n, k)$ and $j < k \leq n$,

$$\phi_{j,E}^{r,s,0}(P, \beta) = \sum_{F \in \mathscr{F}_j(P)} \int_{F \cap \beta} x^r \, \mathscr{H}^j(\mathrm{d}x) \int_{N_E(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathscr{H}^{k-j-1}(\mathrm{d}u),$$

where $N_E(P, F) = N(P, F) \cap L(E)$ is the normal cone of P at the face F, taken with respect to the subspace $L(E)$. Of course, analogous representations are obtained for the (global) intrinsic and extrinsic Minkowski tensors.

The Crofton formulae, which are stated in the next section, will naturally also involve the *generalized tensorial curvature measures* (see formula (2.28))

$$\phi_j^{r,s,1}(P, \beta) := \sum_{F \in \mathscr{F}_j(P)} Q(F) \int_{F \cap \beta} x^r \, \mathscr{H}^j(\mathrm{d}x) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathscr{H}^{n-j-1}(\mathrm{d}u),$$

for $j \in \{1, \dots, n-1\}$, and, if $P \subset E \in A(n, k)$ and $0 < j < k \leq n$,

$$\phi_{j,E}^{r,s,1}(P, \beta) := \sum_{F \in \mathscr{F}_j(P)} Q(F) \int_{F \cap \beta} x^r \, \mathscr{H}^j(\mathrm{d}x) \int_{N_E(P,F) \cap \mathbb{S}^{n-1}} u^s \, \mathscr{H}^{k-j-1}(\mathrm{d}u).$$

Due to Hug and Schneider [9] there exists a weakly continuous extension of the generalized tensorial curvature measures to $\mathscr{K}^n$. In fact, they proved such an extension for the generalized local Minkowski tensors, which are measures on $\mathscr{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$. Globalizing this in the $\mathbb{S}^{n-1}$-coordinate yields the result for the tensorial curvature measures.

Apart from the easily verified relation

$$\phi_{n-1}^{r,s,1} = Q\phi_{n-1}^{r,s,0} - \phi_{n-1}^{r,s+2,0}, \tag{5.4}$$

the tensorial curvature measures and the generalized tensorial curvature measures are linearly independent. In contrast, McMullen [14] discovered basic linear relations for the (global) Minkowski tensors (see also Theorem 2.6), and it was shown in [12] that these are essentially all linear dependences between the Minkowski tensors (see also Theorem 2.7). Furthermore, McMullen [14, p. 269] found relations for the global counterparts of the generalized tensorial curvature measures. In fact, the globalized form of (5.4) is a very special example of one of these relations. For the translation invariant Minkowski tensors $\Phi_j^{0,s}$, these relations take a very simple form, nevertheless for our purpose they are essential in the proof of Theorem 5.5. To have a short notation for these translation invariant Minkowski tensors, we omit the first superscript and put

$$\Phi_j^s := \Phi_j^{0,s}, \qquad \Phi_{j,E}^s := \Phi_{j,E}^{0,s}.$$

Then we can state the following very special case of McMullen's relations.

Lemma 5.1 (McMullen) *Let $P \in \mathscr{P}^n$ and $j, s \in \mathbb{N}_0$ with $j \leq n-1$. Then*

$$\tfrac{n-j+s}{s+1}\Phi_j^{s+2}(P) = \sum_{F \in \mathscr{F}_j(P)} Q(F^\perp)\mathscr{H}^j(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^s \, \mathscr{H}^{n-j-1}(\mathrm{d}u).$$

Note that this lemma is essentially a global result which is derived by applying a version of the divergence theorem.

5.3 Crofton Formulae

In this chapter, for $0 \leq j \leq k < n$ and $i, s \in \mathbb{N}_0$, we are first concerned with the Crofton integrals

$$\int_{A(n,k)} Q(E)^i \, \phi_{j,E}^{r,s,0}(K \cap E, \beta \cap E) \, \mu_k(\mathrm{d}E), \tag{5.5}$$

which involve the intrinsic tensorial curvature measures, and the Crofton integrals

$$\int_{A(n,k)} Q(E)^i \, \Phi_{j,E}^s(K \cap E) \, \mu_k(\mathrm{d}E) \tag{5.6}$$

for the global versions of the translation invariant intrinsic tensorial curvature measures, the translation invariant intrinsic Minkowski tensors obtained by setting $r = 0$. In the global case, we restrict our investigations mainly to these translation invariant intrinsic Minkowski tensors, general Crofton formulae have already been established in [11].

Using the substantial simplifications of the formulae obtained in the present work, the extrinsic formulae in [11], that is, Crofton formulae for the integrals

$$\int_{A(n,k)} \phi_j^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \tag{5.7}$$

can be simplified accordingly. We explain this in detail in the case where $j = k - 1$. The connection to [3] turns out to be crucial for simplifying the constants if s is odd. However, for even s the current approach works completely independently. More general results will be derived in subsequent work via a different approach.

5.3.1 Crofton Formulae for Intrinsic Tensorial Curvature Measures

In this section we state the formulae for the integrals given in (5.5) and (5.6). We start with the local versions, where we distinguish the cases $j = k$ and $j < k$.

Theorem 5.2 *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i, k, r, s \in \mathbb{N}_0$ with $k < n$. Then*

$$\int_{A(n,k)} Q(E)^i\, \phi_{k,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2} + i)}{\Gamma(\frac{n}{2} + i)\Gamma(\frac{k}{2})}\, Q^i \phi_n^{r,0,0}(K, \beta)$$

if $s = 0$; for $s \neq 0$ the integral is zero.

If $s = 0$ in Theorem 5.2, then we interpret the coefficient of the tensor on the right-hand side as 0, if $k = 0$ and $i \neq 0$, and as 1, if $k = i = 0$. A global version of Theorem 5.2 is obtained by simply setting $\beta = \mathbb{R}^n$.

Next we turn to the case $j < k$.

Theorem 5.3 *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $i, j, k, r, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$. Then*

$$\int_{A(n,k)} Q(E)^i\, \phi_{j,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$

$$= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} Q^z \big(\lambda_{n,k,j,s,i,z}^{(0)}\, \phi_{n-k+j}^{r,s+2i-2z,0}(K, \beta) + \lambda_{n,k,j,s,i,z}^{(1)}\, \phi_{n-k+j}^{r,s+2i-2z-2,1}(K, \beta)\big),$$

where for $\varepsilon \in \{0,1\}$ *we set*

$$\gamma_{n,k,j} := \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi},$$

$$\lambda^{(\varepsilon)}_{n,k,j,s,i,z} := \sum_{p=0}^{i} \sum_{q=(z-p+\varepsilon)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q-\varepsilon}{z} \Gamma(q+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)} \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+p+q)} \vartheta^{(\varepsilon)}_{n,k,j,p,q},$$

$$\vartheta^{(0)}_{n,k,j,p,q} := (n-k+j)\big(\tfrac{k-1}{2}+p\big), \qquad \vartheta^{(1)}_{n,k,j,p,q} := p(n-k) - q(k-1).$$

If $j = k-1$, then the tensorial curvature measures and the generalized tensorial curvature measures are linearly dependent. In this case, the right-hand side can be expressed as a linear combination of the tensor-valued curvature measures $Q^z \phi^{r,s+2i-2z,0}_{n-1}(K,\cdot)$, whereas the measures $Q^z \phi^{r,s+2i-2z,1}_{n-1}(K,\cdot)$ are not needed. An explicit description of this case is given in Corollary 5.10 for $i=0$ and in (5.15) for $i \in \mathbb{N}_0$.

If the additional metric tensor is omitted as a weight function, that is in the case $i = 0 = p$, then the coefficients $\lambda^{(\varepsilon)}_{n,k,j,s,0,z}$ in Theorem 5.3 simplify to a single sum.

Apparently, the coefficients in Theorem 5.3 are not well defined in the (excluded) case $k=1$ and $j=0$, as $\Gamma(0)$ is involved in the numerator of $\lambda^{(\varepsilon)}_{n,1,0,s,i,z}$. Although this issue can be resolved by a proper interpretation of the (otherwise ambiguous) expression $\Gamma(p)\cdot p = \Gamma(p+1)$ as 1 for $p=0$, we prefer to state and derive this case separately. In fact, our analysis leads to substantial simplifications of the constants, as our next result shows.

Theorem 5.4 *Let* $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ *and* $i,r,s \in \mathbb{N}_0$. *Then*

$$\int_{A(n,1)} Q(E)^i \, \phi^{r,s,0}_{0,E}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$
$$= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z}\, Q^{\frac{s}{2}+i-z} \phi^{r,2z,0}_{n-1}(K,\beta)$$

for even s. If s is odd, then

$$\int_{A(n,1)} Q(E)^i \, \phi^{r,s,0}_{0,E}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$
$$= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\,\Gamma(\frac{n+s+1}{2}+i)}\, Q^{\frac{s-1}{2}+i} \phi^{r,1,0}_{n-1}(K,\beta).$$

Note that in Theorem 5.4 the Crofton integral is expressed only by tensorial curvature measures $\phi^{r,z,0}_{n-1}$ (multiplied with suitable powers of the metric tensor), whereas generalized tensorial curvature measures are not needed. A global version of Theorem 5.4 is obtained by simply setting $\beta = \mathbb{R}^n$.

A translation invariant, global version of Theorem 5.3 allows us to combine several of the summands on the right-hand side of the formula.

Theorem 5.5 *Let $K \in \mathcal{K}^n$ and $i, j, k, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$. Then*

$$\int_{A(n,k)} Q(E)^i \, \Phi_{j,E}^{s}(K \cap E) \, \mu_k(\mathrm{d}E) = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} \, Q^z \Phi_{n-k+j}^{s+2i-2z}(K),$$

where $\gamma_{n,k,j}$ and $\lambda_{n,k,j,s,i,z}^{(0)}$ are defined as in Theorem 5.3, but

$$\vartheta_{n,k,j,s,i,z,p,q}^{(0)} := (n-k+j)\left(\tfrac{k-1}{2} + p\right) - \left(p(n-k) - q(k-1)\right)\left(1 + \tfrac{k-j-1}{s+2i-2z-1}\left(1 - \tfrac{z}{p+q}\right)\right)$$

replaces $\vartheta_{n,k,j,p,q}^{(0)}$, except if s is odd and $z = \lfloor \frac{s}{2} \rfloor + i$, where $\lambda_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i}^{(0)} := 0$.

In Theorem 5.5, if $p = q = 0$, then the definition of $\lambda_{n,k,j,s,i,z}^{(0)}$ implies that also $z = 0$ and thus, $\vartheta_{n,k,j,s,i,0,0,0}^{(0)}$ is well-defined with $\frac{z}{p+q} = 1$.

5.3.2 Some Special Cases

In the following, we restrict to the case $i = 0$ of Crofton formulae for unweighted intrinsic Minkowski tensors or tensorial curvature measures.

Corollary 5.6 *Let $K \in \mathcal{K}^n$ and $k, j, s \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{A(n,k)} \Phi_{j,E}^{s}(K \cap E) \, \mu_k(\mathrm{d}E) = \delta_{n,k,j,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \eta_{n,k,j,s,z} \, Q^z \Phi_{n-k+j}^{s-2z}(K),$$

where

$$\delta_{n,k,j,s} := \binom{n-k+j-1}{j} \frac{\Gamma(\frac{n-k+1}{2})\Gamma(\frac{k+1}{2})}{\pi \Gamma(\frac{n-k+j+s}{2} + 1)},$$

$$\eta_{n,k,j,s,z} := \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{j+s}{2} - q + 1)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \times \left(\tfrac{n-k+j}{2} + q + \tfrac{(k-j-1)(q-z)}{s-2z-1}\right),$$

but $\eta_{n,k,j,s,\lfloor \frac{s}{2} \rfloor} := 0$ if s is odd.

5.3.2.1 Specific Choices of *s*

Next we collect some special cases of Corollary 5.6, which are obtained for specific choices of $s \in \mathbb{N}_0$ by applications of Legendre's duplication formula and elementary calculations.

Corollary 5.7 *Let $K \in \mathcal{K}^n$ and $k, j \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{A(n,k)} \Phi^2_{j,E}(K \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j+1}{2})}{\Gamma(\frac{n+3}{2})\Gamma(\frac{j+1}{2})}\Big(\tfrac{n-k}{4(n-k+j)}\, Q\Phi^0_{n-k+j}(K) + \tfrac{n-k+nj+j}{2(n-k+j)}\, \Phi^2_{n-k+j}(K)\Big).$$

Corollary 5.8 *Let $K \in \mathcal{K}^n$ and $k, j \in \mathbb{N}_0$ with $0 \le j < k < n$. Then*

$$\int_{A(n,k)} \Phi^3_{j,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \frac{j+1}{n-k+j+1}\, \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+j}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j}{2})}\, \Phi^3_{n-k+j}(K).$$

As $\Gamma(\frac{j}{2})^{-1} = 0$, for $j = 0$, the integral in Corollary 5.8 equals 0 in this case. However, as the integrand on the left-hand side is already 0, this is not surprising. The same is true for any odd number $s \in \mathbb{N}$ and $j = 0$.

Corollary 5.8 immediately leads to a result which was obtained and applied by Bernig and Hug in [3, Lemma 4.13].

Corollary 5.9 *Let $K \in \mathcal{K}^n$. Then*

$$\int_{A(n,2)} \Phi^3_{1,E}(K \cap E)\, \mu_k(\mathrm{d}E) = \binom{n}{2}^{-1} \Phi^3_{n-1}(K).$$

5.3.2.2 The Choice $j = k - 1$

Furthermore, we obtain simple Crofton formulae for the specific choice $j = k - 1$ in the local and in the global case.

Corollary 5.10 *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then*

$$\int_{A(n,k)} \phi^{r,s,0}_{k-1,E}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \delta_{n,k,k-1,s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \xi_{n,k,s,z}\, Q^z \phi^{r,s-2z,0}_{n-1}(K, \beta),$$

where

$$\xi_{n,k,s,z} := \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma\left(q + \tfrac{1}{2}\right) \frac{\Gamma(\frac{k+s+1}{2} - q)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + q)}.$$

Corollary 5.10 will be derived from Theorem 5.3 in the same way as Theorem 5.5 is proved. More specifically, relation (5.4) is applied, which can be considered as a local version of Lemma 5.1 in the particular case $j = n - 1$. Although $k = 1$ is excluded in Corollary 5.10, the result is formally consistent with Theorem 5.4 (for $i = 0$), which can be checked by simplifying the coefficients $\xi_{n,1,s,z}$ with the help of Zeilberger's algorithm.

A global version of Corollary 5.10 is obtained by setting $\beta = \mathbb{R}^n$.

Finally, Theorem 5.4 can be globalized to give a result, which was obtained in [13] by a completely different approach.

Corollary 5.11 *Let $K \in \mathscr{K}^n$ and $s \in \mathbb{N}_0$. Then*

$$\int_{A(n,1)} \Phi_{0,E}^{s}(K \cap E)\, \mu_k(\mathrm{d}E) = \frac{2\omega_{n+s+1}}{\pi\omega_{s+1}\omega_n} \sum_{z=0}^{\frac{s}{2}} \frac{(-1)^z}{1-2z} \binom{\frac{s}{2}}{z} Q^{\frac{s}{2}-z} \Phi_{n-1}^{2z}(K)$$

for even s. For odd s the integral on the left-hand side equals 0.

Note that if $s \in \mathbb{N}$ is odd, then the Crofton integral in Theorem 5.4 is a non-zero measure, as the tensorial curvature measures $\phi_{n-1}^{r,1,0}(K, \cdot)$ are non-zero (if the underlying set K is at least $(n-1)$-dimensional), whereas $\Phi_{n-1}^{1} \equiv 0$ in the global case considered in Corollary 5.11.

5.3.3 Crofton Formulae for Extrinsic Tensorial Curvature Measures

In the following, we state Crofton formulae for tensorial curvature measures for $j = k - 1$. The method also applies to the cases where $j \leq k - 2$, but it remains to be explored to which extent the constants can be simplified then by the current approach.

As for the intrinsic versions, we have to distinguish between the cases $k > 1$ and $k = 1$. We start with the former.

Theorem 5.12 *Let $K \in \mathscr{K}^n$, $\beta \in \mathscr{B}(\mathbb{R}^n)$ and $k, r, s \in \mathbb{N}_0$ with $1 < k < n$. Then*

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z}\, Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

where

$$\kappa_{n,k,s,z} := \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2}+z)\Gamma(\frac{k+s-1}{2}-z)}{\Gamma(\frac{s}{2}-z+1) z!}$$

if $z \neq \frac{s-1}{2}$, *and*

$$\kappa_{n,k,s,\frac{s-1}{2}} := \pi^{\frac{n-k-1}{2}} \frac{2k(n+s-2)}{(n-1)(n-k+s-1)} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n+s+1}{2})}. \tag{5.8}$$

In Theorem 5.12, if s is odd the coefficient $\kappa_{n,k,s,(s-1)/2}$ has to be defined separately, as the proof shows. In fact, one easily checks that the difference amounts to a factor $k(n+s-2)[(k-1)(n+s-1)]^{-1}$. For even s, the constants involved in the proof of Theorem 5.12 can be simplified by a direct calculation to arrive at the asserted result. However, if s is odd, we need the connection to the work [3] to simplify the constants. Since this connection breaks down for $z = (s-1)/2$, s odd, a separate direct calculation is required for this case, and this finally yields the correct constant in (5.8). The result is also consistent with the special case $k = 1$ which is considered next. A more structural viewpoint, which will be developed in future work, will provide another explanation for the case distinction required for the coefficients in the preceding Crofton formula.

For $k = 1$ the Crofton integrals can be represented with a single functional, as the following theorem shows.

Theorem 5.13 *Let* $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$ *and* $r, s \in \mathbb{N}_0$. *Then*

$$\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$
$$= \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\lfloor\frac{s+1}{2}\rfloor + \frac{1}{2})}{\Gamma(\frac{n}{2} + \lfloor\frac{s+1}{2}\rfloor)} Q^{\lfloor\frac{s}{2}\rfloor} \phi_{n-1}^{r,s-2\lfloor\frac{s}{2}\rfloor,0}(K,\beta).$$

It can be easily checked that the result for $k = 1$ can be obtained from the one for $k > 1$ by a formal specialization and proper interpretation of expressions which a priori are not well defined. For this to work, it is indeed crucial that for odd values of s and $z = (s-1)/2$ the definition in (5.8) applies.

In [3, Proposition 4.10], an alternative basis of the vector space of continuous, translation invariant and rotation covariant $\mathbb{T}^p$-valued valuations on $\mathcal{K}^n$ was introduced, based on the trace free part of the Minkowski tensors, which was called the Ψ-basis. In the same spirit (but locally and with the current normalization), we now define

$$\psi_k^{r,s,0} := \phi_k^{r,s,0} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\lfloor\frac{s}{2}\rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j \phi_k^{r,s-2j,0}$$

for $r, s \in \mathbb{N}_0$ and $k \in \{0, \dots, n-1\}$. Interpreting this definition in the right way if $n = 2$ and $s = 0$ (where $\psi_k^{r,0,0} = \phi_k^{r,0,0}$), we can also write

$$\psi_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor\frac{s}{2}\rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j \phi_k^{r,s-2j,0}. \tag{5.9}$$

In particular, $\psi_k^{r,s,0} = \phi_k^{r,s,0}$ for $s \in \{0,1\}$. Conversely, we have

$$\phi_k^{r,s,0} = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-2j)}{\Gamma(\frac{n}{2}+s-j)} Q^j \psi_k^{r,s-2j,0}. \tag{5.10}$$

Although this will not be needed explicitly, it shows how we can switch between a ϕ-representation and a ψ-representation of tensorial curvature measures.

The main advantage of the new local tensor valuations given in (5.9) is that the Crofton formula takes a particularly simple form.

Corollary 5.14 *Let $K \in \mathcal{K}^n$, $\beta \in \mathcal{B}(\mathbb{R}^n)$, and let $k, r, s \in \mathbb{N}_0$ with $0 < k < n$.*
If $s \notin \{0,1\}$, then

$$\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \pi^{\frac{n-k}{2}} \frac{k-1}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n-k+s+1}{2})}\, \psi_{n-1}^{r,s,0}(K,\beta).$$

If $s = 0$, then

$$\int_{A(n,k)} \psi_{k-1}^{r,0,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \pi^{\frac{n-k+1}{2}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+1}{2})\Gamma(\frac{n+1}{2})}\, \psi_{n-1}^{r,0,0}(K,\beta).$$

If $s = 1$, then

$$\int_{A(n,k)} \psi_{k-1}^{r,1,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \pi^{\frac{n-k}{2}} \frac{k}{n} \frac{1}{\Gamma(\frac{n-k+2}{2})}\, \psi_{n-1}^{r,1,0}(K,\beta).$$

For $r = 0$ and $\beta = \mathbb{R}^n$, Corollary 5.14 coincides with [3, Corollary 6.1] (in the case corresponding to $j = k-1$). If $s \in \{0,1\}$, then $\psi_k^{r,s,0} = \phi_k^{r,s,0}$ and Corollary 5.14 coincides with Theorem 5.12 (resp. Theorem 5.13, for $k = 1$). If $k = 1$, then the integral in Corollary 5.14 vanishes, except for $s \in \{0,1\}$.

5.4 Proofs of the Main Results

In this section, we first recall some results from [11]. Then we prove an integral formula which is required in the following. Finally, all ingredients are combined for the proofs of our main theorems.

A basic tool is the following transformation formula (see [11, Corollary 4.2]). It can be used to carry out an integration over linear Grassmann spaces recursively.

The result is also true for $k = 1$, but in this case the outer integration on the right-hand side is trivial.

Lemma 5.15 *Let $u \in \mathbb{S}^{n-1}$ and let $h : G(n,k) \to \mathbb{T}^p$ be an integrable function for $k, p \in \mathbb{N}_0$, $0 < k < n$. Then*

$$\int_{G(n,k)} h(L)\,\nu_k(\mathrm{d}L) = \frac{\omega_k}{2\omega_n}\int_{G(u^\perp,k-1)}\int_{-1}^{1}\int_{U^\perp\cap u^\perp\cap\mathbb{S}^{n-1}} |t|^{k-1}(1-t^2)^{\frac{n-k-2}{2}}$$
$$\times\, h\big(\mathrm{span}\big\{U, tu + \sqrt{1-t^2}w\big\}\big)\,\mathscr{H}^{n-k-1}(\mathrm{d}w)\,\mathrm{d}t\,\nu_{k-1}^{u^\perp}(\mathrm{d}U).$$

The next results are derived from the previous one (see [11, Lemma 4.3 and Corollary 4.6]).

Lemma 5.16 *Let $i, k \in \mathbb{N}_0$ with $k \le n$. Then*

$$\int_{G(n,k)} Q(L)^i\,\nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i)\Gamma(\frac{k}{2})}Q^i.$$

In Lemma 5.16, we interpret the coefficient of the tensor on the right-hand side as 0, if $k = 0$ and $i \neq 0$, and as 1, if $k = i = 0$, as $\Gamma(0)^{-1} := 0$ and $\frac{\Gamma(a)}{\Gamma(a)} = 1$ for all $a \in \mathbb{R}$.

Lemma 5.17 *Let $i \in \mathbb{N}_0$, $k, r \in \{0, \dots, n\}$ with $k + r \ge n$, and let $F \in G(n,r)$. Then*

$$\int_{G(n,k)} [F,L]^2 Q(L)^i\,\nu_k(\mathrm{d}L) = \frac{r!k!}{n!(k+r-n)!}\,\frac{\Gamma(\frac{n}{2}+1)\Gamma(\frac{k}{2}+i)}{\Gamma(\frac{n}{2}+i+1)\Gamma(\frac{k}{2}+1)}$$
$$\times\,\Big((\tfrac{k}{2}+i)Q^i + i\tfrac{k-n}{r}Q^{i-1}Q(F)\Big).$$

We interpret the second summand on the right-hand side of Lemma 5.17 as 0, if $i = 0$, which is consistent with [11, Lemma 4.4]. If $r = 0$, we also interpret the second summand as 0 and the integral on the left equals Q^i.

Finally, we state the following integral formula (see [11, p. 503]), which is a special case of [15, Theorem 3.1].

Lemma 5.18 *Let $P \in \mathscr{P}^n$ be a polytope, $L \in G(n,k)$ for $0 \le j < k < n$ and let $g : \mathbb{R}^n \times (\mathbb{S}^{n-1} \cap L) \to \mathbb{T}$ be a measurable bounded function. Then*

$$\int_{L^\perp}\int_{L_t\times(L\cap\mathbb{S}^{n-1})} g(x,u)\,\Lambda_j^{(L_t)}(P\cap L_t, \mathrm{d}(x,u))\,\mathscr{H}^{n-k}(\mathrm{d}t)$$
$$= \frac{1}{\omega_{k-j}}\sum_{F\in\mathscr{F}_{n-k+j}(P)}\int_{F\times(N(P,F)\cap\mathbb{S}^{n-1})} g(x,\pi_L(u))$$
$$\times\,\|p_L(u)\|^{j-k}[F,L]^2\,\mathscr{H}^{n-1}(\mathrm{d}(x,u)).$$

5.4.1 Auxiliary Integral Formulae

With the preliminary results from [11] we are able to establish the following integral formula, which is a slightly modified version of [11, Proposition 4.7].

Proposition 5.19 *Let $i, j, k, s \in \mathbb{N}_0$ with $j < k < n$ and $k > 1$, $F \in G(n, n-k+j)$ and $u \in F^\perp \cap \mathbb{S}^{n-1}$. Then*

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \big(\lambda^{(0)}_{n,k,j,s,i,z} u^2 + \lambda^{(1)}_{n,k,j,s,i,z} Q(F)\big) Q^z u^{s+2i-2z-2},$$

where the coefficients are defined as in Theorem 5.3.

Proof Lemma 5.15 yields

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\omega_k}{2\omega_n} \int_{G(u^\perp, k-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{k-1} (1-t^2)^{\frac{n-k-2}{2}} \pi_{\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}}(u)^s$$
$$\times Q\big(\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}\big)^i \|p_{\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}}(u)\|^{j-k}$$
$$\times \big[F, \mathrm{span}\{U, tu + \sqrt{1-t^2} w\}\big]^2 \, \mathscr{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu^{u^\perp}_{k-1}(\mathrm{d}U).$$

As

$$Q\big(\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}\big) = Q(U) + \big(|t|u + \sqrt{1-t^2} \,\mathrm{sign}(t) w\big)^2,$$
$$\pi_{\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}}(u) = |t|u + \sqrt{1-t^2} \,\mathrm{sign}(t) w,$$
$$\|p_{\mathrm{span}\{U, tu + \sqrt{1-t^2} w\}}(u)\| = |t|,$$
$$\big[F, \mathrm{span}\{U, tu + \sqrt{1-t^2} w\}\big] = [F, U]^{(u^\perp)} |t|$$

hold for all $t \in [-1, 1] \setminus \{0\}$, we obtain

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F, L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\omega_k}{2\omega_n} \int_{G(u^\perp, k-1)} \int_{-1}^{1} \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} |t|^{j+1} (1-t^2)^{\frac{n-k-2}{2}} \big([F, U]^{(u^\perp)}\big)^2$$
$$\times \big(|t|u + \sqrt{1-t^2} w\big)^s \big(Q(U) + (|t|u + \sqrt{1-t^2} w)^2\big)^i$$
$$\times \mathscr{H}^{n-k-1}(\mathrm{d}w) \, \mathrm{d}t \, \nu^{u^\perp}_{k-1}(\mathrm{d}U),$$

where we used the fact that the integration with respect to w is invariant under reflections in the origin. Then we apply the binomial theorem to the terms $(Q(U) + (|t|u + \sqrt{1-t^2}w)^2)^i$ and $(|t|u + \sqrt{1-t^2}w)^{s+2p}$ and get

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\omega_k}{2\omega_n} \sum_{p=0}^{i} \sum_{q=0}^{s+2p} \binom{i}{p} \binom{s+2p}{q} \int_{G(u^\perp,k-1)} \int_{-1}^{1} |t|^{j+s+2p-q+1} (1-t^2)^{\frac{n-k+q-2}{2}} \, \mathrm{d}t$$
$$\times \int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \, \mathscr{H}^{n-k-1}(\mathrm{d}w) \big([F,U]^{(u^\perp)}\big)^2 u^{s+2p-q} Q(U)^{i-p} \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).$$

Since

$$\int_{U^\perp \cap u^\perp \cap \mathbb{S}^{n-1}} w^q \, \mathscr{H}^{n-k-1}(\mathrm{d}w) = \mathbf{1}\{q \text{ even}\} 2 \frac{\omega_{n-k+q}}{\omega_{q+1}} Q(U^\perp \cap u^\perp)^{\frac{q}{2}},$$

we deduce from the definition of the Beta function and its relation to the Gamma function that

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\omega_k}{\omega_n} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \frac{\Gamma(\frac{j+s}{2} + p - q + 1)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + p + 1)} \frac{\omega_{n-k+2q}}{\omega_{2q+1}}$$
$$\times u^{s+2p-2q} \int_{G(u^\perp,k-1)} Q(U^\perp \cap u^\perp)^q \big([F,U]^{(u^\perp)}\big)^2 Q(U)^{i-p} \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).$$

Applying the binomial theorem to $Q(U^\perp \cap u^\perp)^q = (Q(u^\perp) - Q(U))^q$ yields

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{k}{2})} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^{q} (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q + \tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{j+s}{2} + p - q + 1)}{\Gamma(\frac{n-k+j+s}{2} + p + 1)} u^{s+2p-2q} Q\big(u^\perp\big)^{q-y}$$
$$\times \int_{G(u^\perp,k-1)} \big([F,U]^{(u^\perp)}\big)^2 Q(U)^{i-p+y} \, \nu_{k-1}^{u^\perp}(\mathrm{d}U). \tag{5.11}$$

We conclude from Lemma 5.17, which is applied in $u^\perp$ to the remaining integral on the right-hand side of (5.11),

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$

$$= \frac{(n-k+j)!(k-1)!}{\sqrt{\pi}(n-1)!j!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \binom{i}{p} \binom{s+2p}{2q} \Gamma(q+\tfrac{1}{2})$$

$$\times \frac{\Gamma(\frac{j+s}{2}+p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+p+1)} u^{s+2p-2q} \sum_{y=0}^{q} (-1)^y \binom{q}{y} \frac{\Gamma(\frac{k-1}{2}+i-p+y)}{\Gamma(\frac{n+1}{2}+i-p+y)}$$

$$\times \Big(\big(\tfrac{k-1}{2}+i-p+y\big) Q\big(u^\perp\big)^{i-p+q}$$

$$+ \tfrac{k-n}{n-k+j}(i-p+y) Q\big(u^\perp\big)^{i-p+q-1} Q(F) \Big).$$

Lemma 5.22 from Sect. 5.6 applied twice to the summations with respect to y and Legendre's duplication formula applied three times to the Gamma functions involving n, k and $n-k$ yield together with the definitions of $\gamma_{n,k,j}$ and $\vartheta^{(\varepsilon)}_{n,k,j,p,q}$, $\varepsilon \in \{0,1\}$,

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$

$$= \gamma_{n,k,j} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q+\tfrac{1}{2})$$

$$\times \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)} \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+p+q)}$$

$$\times u^{s+2i-2p-2q} \Big(\vartheta^{(0)}_{n,k,j,p,q} Q\big(u^\perp\big)^{p+q} - \vartheta^{(1)}_{n,k,j,p,q} Q\big(u^\perp\big)^{p+q-1} Q(F) \Big),$$

where we changed the order of summation with respect to p. From the binomial theorem applied to $Q(u^{\perp})^{p+q} = (Q - u^2)^{p+q}$ we obtain

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \gamma_{n,k,j} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - p} \binom{i}{p} \binom{s+2i-2p}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{j+s}{2}+i-p-q+1)}{\Gamma(\frac{n-k+j+s}{2}+i-p+1)}$$
$$\times \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+1}{2}+p+q)} \Bigg(\sum_{z=0}^{p+q} (-1)^{p+q-z} \binom{p+q}{z} \vartheta_{n,k,j,p,q}^{(0)} Q^z u^{s+2i-2z}$$
$$+ \sum_{z=0}^{p+q-1} (-1)^{p+q-z} \binom{p+q-1}{z} \vartheta_{n,k,j,p,q}^{(1)} Q^z u^{s+2i-2z-2} Q(F) \Bigg).$$

A change of the order of summation, such that we sum with respect to z first, gives

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \big(\lambda_{n,k,j,s,i,z}^{(0)} u^2 + \lambda_{n,k,j,s,i,z}^{(1)} Q(F) \big) Q^z u^{s+2i-2z-2},$$

which concludes the proof. □

Next we state the special case of Proposition 5.19 where $k = 1$.

Proposition 5.20 *Let $i, s \in \mathbb{N}_0$, $F \in G(n, n-1)$ and $u \in F^{\perp} \cap \mathbb{S}^{n-1}$. Then*

$$\int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \, \nu_1(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi \Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z} u^{2z} Q^{\frac{s}{2}+i-z}$$

for even s. If s is odd, then

$$\int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \, \nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\, \Gamma(\frac{n+s+1}{2}+i)} u Q^{\frac{s-1}{2}+i}.$$

Proof The proof basically works as the proof of Proposition 5.19. But we do not need to apply Lemma 5.17 as (5.11) simplifies to

$$\int_{G(n,1)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{-1} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{p=0}^{i} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + p} \sum_{y=0}^{q} (-1)^y \binom{i}{p} \binom{s+2p}{2q} \binom{q}{y} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+p-q+1)}{\Gamma(\frac{n+s+1}{2}+p)}$$
$$\times\, u^{s+2p-2q} Q\big(u^\perp\big)^{q-y} \int_{G(u^\perp,0)} \big([F,U]^{(u^\perp)}\big)^2 Q(U)^{i-p+y} \, \nu_{k-1}^{u^\perp}(\mathrm{d}U).$$

Since the remaining integral on the right-hand side equals 1, if $p = i$ and $y = 0$, and in all the other cases it equals 0, we obtain

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \binom{s+2i}{2q} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+i-q+1)}{\Gamma(\frac{n+s+1}{2}+i)} u^{s+2i-2q} Q\big(u^\perp\big)^q.$$

Applying the binomial theorem to $Q(u^\perp)^q = (Q-u^2)^q$ yields

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})}{\pi} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i} \sum_{z=0}^{q} (-1)^{q-z} \binom{s+2i}{2q} \binom{q}{z} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{s}{2}+i-q+1)}{\Gamma(\frac{n+s+1}{2}+i)} u^{s+2i-2z} Q^z.$$

A change of the order of summation and Legendre's duplication formula applied to the Gamma functions involving q give

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{(s+2i)!\,\Gamma(\frac{n}{2})}{2^{s+2i}\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \frac{1}{z!} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2}+i-q)(q-z)!} u^{s+2i-2z} Q^z.$$

If s is even, we conclude from Lemma 5.23 applied to the summation with respect to q and from another application of Legendre's duplication formula that

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L)$$
$$= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^{\frac{s}{2}+i-z+1} \binom{\frac{s}{2}+i}{z} \frac{1}{s+2i-2z-1} u^{s+2i-2z} Q^z.$$

A change of the order of summation with respect to z then yields the assertion.

On the other hand, if s is odd, the binomial theorem gives, for $\lfloor \frac{s}{2} \rfloor + i \neq z$,

$$\sum_{q=z}^{\lfloor \frac{s}{2} \rfloor + i} \frac{(-1)^{q-z}}{\Gamma(\frac{s+1}{2}+i-q)(q-z)!} = \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} \sum_{q=0}^{\lfloor \frac{s}{2} \rfloor + i - z} (-1)^q \binom{\lfloor \frac{s}{2} \rfloor + i - z}{q}$$
$$= \frac{1}{(\lfloor \frac{s}{2} \rfloor + i - z)!} (1-1)^{\lfloor \frac{s}{2} \rfloor + i - z}$$
$$= 0. \tag{5.12}$$

For $\lfloor \frac{s}{2} \rfloor + i = z$, the sum on the left-hand side of (5.12) equals 1. Hence, we finally obtain

$$\int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2 \, \nu_k(\mathrm{d}L) = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\Gamma(\frac{n+s+1}{2}+i)} u Q^{\lfloor \frac{s}{2} \rfloor + i},$$

if s is odd. □

5.4.2 *The Proofs for the Intrinsic Case*

Now all tools are available which are needed to prove the main theorems.

We start with the proof of Theorem 5.2.

Proof (Theorem 5.2) Let $L \in G(n,k)$ and $t \in L^\perp$. Then we have

$$\phi_{k,L_t}^{r,s,0}(K \cap L_t, \beta \cap L_t) = \mathbf{1}\{s=0\} \int_{K \cap \beta \cap L_t} x^r \, \mathscr{H}^k(\mathrm{d}x)$$

and thus, for $s \neq 0$,

$$\int_{A(n,k)} Q(E)^i \phi_{k,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \int_{G(n,k)} \int_{L^\perp} Q(L_t)^i \phi_{k,L_t}^{r,s,0}(K \cap L_t, \beta \cap L_t)\, \mathscr{H}^{n-k}(\mathrm{d}t)\, \nu_k(\mathrm{d}L) = 0.$$

Furthermore, for $s = 0$ Fubini's theorem yields

$$\int_{A(n,k)} Q(E)^i \phi_{k,E}^{r,0,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \int_{G(n,k)} Q(L)^i \int_{L^\perp} \int_{K\cap\beta\cap L_t} x^r\, \mathscr{H}^k(\mathrm{d}x)\, \mathscr{H}^{n-k}(\mathrm{d}t)\, \nu_k(\mathrm{d}L)$$
$$= \int_{G(n,k)} Q(L)^i\, \nu_k(\mathrm{d}L) \int_{K\cap\beta} x^r\, \mathscr{H}^n(\mathrm{d}x).$$

Then we conclude the proof with Lemma 5.16 and the definition of $\phi_n^{r,0,0}$. □

We turn to the proof of Theorem 5.3.

Proof (Theorem 5.3) First, we prove the formula for a polytope $P \in \mathscr{P}^n$. The general result then follows by an approximation argument.

As a matter of convenience, we name the integral of interest I. Then Lemma 5.18 yields

$$I = \omega_{k-j} \int_{G(n,k)} Q(L)^i \int_{L^\perp} \int_{L_t\times(L\cap\mathbb{S}^{n-1})} \mathbf{1}_\beta(x) x^r u^s$$
$$\times \Lambda_j^{(L_t)}(P \cap L_t, \mathrm{d}(x,u))\, \mathscr{H}^{n-k}(\mathrm{d}t)\, \nu_k(\mathrm{d}L)$$
$$= \sum_{F\in\mathscr{F}_{n-k+j}(P)} \int_{F\cap\beta} x^r\, \mathscr{H}^{n-k+j}(\mathrm{d}x) \int_{G(n,k)} Q(L)^i$$
$$\times \int_{N(P,F)\cap\mathbb{S}^{n-1}} \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2\, \mathscr{H}^{k-j-1}(\mathrm{d}u)\, \nu_k(\mathrm{d}L).$$

With Fubini's theorem we conclude

$$I = \sum_{F\in\mathscr{F}_{n-k+j}(P)} \int_{F\cap\beta} x^r\, \mathscr{H}^{n-k+j}(\mathrm{d}x) \int_{N(P,F)\cap\mathbb{S}^{n-1}}$$
$$\times \int_{G(n,k)} Q(L)^i \pi_L(u)^s \|p_L(u)\|^{j-k} [F,L]^2\, \nu_k(\mathrm{d}L)\, \mathscr{H}^{k-j-1}(\mathrm{d}u). \tag{5.13}$$

Then we obtain from Proposition 5.19

$$I = \gamma_{n,k,j} \sum_{F\in\mathscr{F}_{n-k+j}(P)} \int_{F\cap\beta} x^r \mathscr{H}^{n-k+j}(\mathrm{d}x)$$
$$\times \Bigg(\sum_{z=0}^{\lfloor\frac{s}{2}\rfloor+i} \lambda^{(0)}_{n,k,j,s,i,z} Q^z \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z} \mathscr{H}^{k-j-1}(\mathrm{d}u)$$
$$+ \sum_{z=0}^{\lfloor\frac{s}{2}\rfloor+i-1} \lambda^{(1)}_{n,k,j,s,i,z} Q^z Q(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z-2} \mathscr{H}^{k-j-1}(\mathrm{d}u)\Bigg).$$

With the definition of the tensorial curvature measures we get

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor\frac{s}{2}\rfloor+i} \lambda^{(0)}_{n,k,j,s,i,z} Q^z \phi^{r,s+2i-2z,0}_{n-k+j}(P,\beta)$$
$$+ \gamma_{n,k,j} \sum_{z=0}^{\lfloor\frac{s}{2}\rfloor+i-1} \lambda^{(1)}_{n,k,j,s,i,z} Q^z \phi^{r,s+2i-2z-2,1}_{n-k+j}(P,\beta).$$

Combining the two sums yields the assertion in the polytopal case.

As pointed out before, there exists a weakly continuous extension of the generalized tensorial curvature measures $\phi^{r,s+2i-2z-2,1}_{n-k+j}$ from the set of all polytopes to $\mathscr{K}^n$. The same is true for the tensorial curvature measures $\phi^{r,s+2i-2z,0}_{n-k+j}$. Hence, approximating a convex body $K \in \mathscr{K}^n$ by polytopes yields the assertion in the general case. □

Now we prove Theorem 5.4, which deals with the case $k = 1$ excluded in the statement of Theorem 5.3.

Proof (Theorem 5.4) The proof basically works as the one of Theorem 5.3. Again, we prove the formula for a polytope $P \in \mathscr{P}^n$. We call the integral of interest I and proceed as in the previous proof in order to obtain (5.13). Now we apply Proposition 5.20 and obtain

$$I = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2}+i)}{\pi\Gamma(\frac{n+s+1}{2}+i)} \sum_{z=0}^{\frac{s}{2}+i} (-1)^z \binom{\frac{s}{2}+i}{z} \frac{1}{1-2z} Q^{\frac{s}{2}+i-z}$$
$$\times \sum_{F\in\mathscr{F}_{n-1}(P)} \int_{F\cap\beta} x^r \mathscr{H}^{n-k+j}(\mathrm{d}x) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{2z} \mathscr{H}^0(\mathrm{d}u),$$

if s is even. Hence, we conclude the assertion with the definition of $\phi^{r,2z,0}_{n-1}$.

If s is odd, Proposition 5.20 yields

$$I = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s}{2}+i+1)}{\sqrt{\pi}\,\Gamma(\frac{n+s+1}{2}+i)} Q^{\frac{s-1}{2}+i} \phi_{n-1}^{r,1,0}(P,\beta).$$

As sketched in the proof of Theorem 5.3, the general result follows by an approximation argument. □

For the proof of Theorem 5.5, we first globalize Theorem 5.3 and then apply Lemma 5.1 to treat the appearing tensors $\phi_{n-k+j}^{0,s+2i-2z-2,1}$.

Proof (Theorem 5.5) We only prove the formula for a polytope $P \in \mathscr{P}^n$. As before, the general result follows by an approximation argument.

We briefly write I for the Crofton integral under investigation. Starting from the special case of Theorem 5.3 where $r = 0$ and $\beta = \mathbb{R}^n$, we obtain

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \Phi_{n-k+j}^{s+2i-2z}(P) + \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i-1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z$$
$$\times \sum_{F \in \mathscr{F}_{n-k+j}(P)} Q(F) \mathscr{H}^{n-k+j}(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z-2} \, \mathscr{H}^{k-j-1}(\mathrm{d}u).$$

With $Q(F) = Q - Q(N(P,F))$ and Lemma 5.1 we get

$$\sum_{F \in \mathscr{F}_{n-k+j}(P)} Q(F) \mathscr{H}^{n-k+j}(F) \int_{N(P,F)\cap\mathbb{S}^{n-1}} u^{s+2i-2z-2} \, \mathscr{H}^{k-j-1}(\mathrm{d}u)$$
$$= Q\Phi_{n-k+j}^{s+2i-2z-2}(P) - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \Phi_{n-k+j}^{s+2i-2z}(P)$$

and thus

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,j,s,i,z}^{(0)} Q^z \Phi_{n-k+j}^{s+2i-2z}(P)$$
$$+ \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i-1} \lambda_{n,k,j,s,i,z}^{(1)} Q^z \Big(Q\Phi_{n-k+j}^{s+2i-2z-2}(P) - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \Phi_{n-k+j}^{s+2i-2z}(P) \Big).$$

Combining these sums yields

$$I = \gamma_{n,k,j} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \big(\lambda_{n,k,j,s,i,z}^{(0)} + \lambda_{n,k,j,s,i,z-1}^{(1)} - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \lambda_{n,k,j,s,i,z}^{(1)} \big) Q^z \Phi_{n-k+j}^{s+2i-2z}(P).$$

In fact, we have $\lambda^{(1)}_{n,k,j,s,i,-1} = 0$ and, furthermore for even s, as the sum with respect to q is empty, $\lambda^{(1)}_{n,k,j,s,i,\lfloor \frac{s}{2} \rfloor + i}$ also vanishes. On the other hand, for odd s, as $\Phi^1_{n-k+j} \equiv 0$, the last summand of the sum with respect to z actually vanishes and thus its coefficient does not have to be determined and is defined as zero.

Hence, we obtained a representation of the integral with the desired Minkowski tensors. It remains to determine the coefficients explicitly. First, we consider the case where ($k > 1$ and) $z \in \{1, \ldots, \lfloor \frac{s}{2} \rfloor + i - 1\}$. We get

$$\begin{aligned}
&\lambda^{(0)}_{n,k,j,s,i,z} + \lambda^{(1)}_{n,k,j,s,i,z-1} \\
&\quad = \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\
&\qquad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)\Gamma(\frac{k-1}{2} + p)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)\Gamma(\frac{n+1}{2} + p + q)} \\
&\qquad \times \Big((n-k+j)(\tfrac{k-1}{2} + p) - \tfrac{z}{p+q}\big(p(n-k) - q(k-1)\big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\lambda^{(1)}_{n,k,j,s,i,z} = &\sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\
&\times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)\Gamma(\frac{k-1}{2} + p)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)\Gamma(\frac{n+1}{2} + p + q)} \\
&\times \tfrac{p+q-z}{p+q}\big(p(n-k) - q(k-1)\big).
\end{aligned} \tag{5.14}$$

Hence we conclude

$$\begin{aligned}
&\lambda^{(0)}_{n,k,j,s,i,z} + \lambda^{(1)}_{n,k,j,s,i,z-1} - \tfrac{k-j+s+2i-2z-2}{s+2i-2z-1} \lambda^{(1)}_{n,k,j,s,i,z} \\
&\quad = \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \Gamma(q + \tfrac{1}{2}) \\
&\qquad \times \frac{\Gamma(\frac{j+s}{2} + i - p - q + 1)\Gamma(\frac{k-1}{2} + p)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-k+j+s}{2} + i - p + 1)\Gamma(\frac{n+1}{2} + p + q)} \\
&\qquad \times \bigg((n-k+j)(\tfrac{k-1}{2} + p) - \tfrac{p(n-k)-q(k-1)}{p+q} \Big(p + q + \tfrac{(k-j-1)(p+q-z)}{s+2i-2z-1}\Big)\bigg).
\end{aligned}$$

The case $z = \lfloor \frac{s}{2} \rfloor + i$, for even s, follows similarly. For $z = 0$, we have $\lambda^{(1)}_{n,k,j,s,i,-1} = 0$ and (5.14) still holds, if one cancels the remaining $\frac{p+q-z}{p+q} = 1$. □

Finally, we provide the argument for Corollary 5.10, which is the special case of Theorem 5.3 obtained for $i = 0$ and $j + 1 = k \geq 2$.

Proof (Corollary 5.10) With the specific choices of the indices, we obtain

$$\lambda^{(\varepsilon)}_{n,k,k-1,s,0,z} = \sum_{q=z+\varepsilon}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q-\varepsilon}{z} \Gamma(q + \tfrac{1}{2}) \\ \times \frac{\Gamma(\frac{k+s+1}{2} - q)}{\Gamma(\frac{n+s+1}{2})} \frac{\Gamma(\frac{k-1}{2}) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \vartheta^{(\varepsilon)}_{n,k,k-1,0,q},$$

with

$$\vartheta^{(0)}_{n,k,k-1,0,q} = \tfrac{1}{2}(n-1)(k-1), \qquad \vartheta^{(1)}_{n,k,k-1,0,q} := -q(k-1),$$

and

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}.$$

Let us denote the Crofton integral by I. Then Theorem 5.3 implies that

$$\begin{aligned} I &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z \big(\lambda^{(0)}_{n,k,k-1,s,0,z} - \lambda^{(1)}_{n,k,k-1,s,0,z}\big) \phi^{r,s-2z,0}_{n-1}(K, \beta) \\ &\quad + \gamma_{n,k,k-1} \sum_{z=1}^{\lfloor \frac{s}{2} \rfloor + 1} Q^z \lambda^{(1)}_{n,k,k-1,s,0,z-1} \phi^{r,s-2z,0}_{n-1}(K, \beta) \\ &= \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} Q^z \underbrace{\big(\lambda^{(0)}_{n,k,k-1,s,0,z} + \lambda^{(1)}_{n,k,k-1,s,0,z-1} - \lambda^{(1)}_{n,k,k-1,s,0,z}\big)}_{=:\lambda} \phi^{r,s-2z,0}_{n-1}(K, \beta), \end{aligned}$$

where

$$\begin{aligned} \lambda = \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} &\sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)} \\ &\times \bigg[\binom{q}{z} \frac{1}{2}(n-1)(k-1) - \binom{q-1}{z-1} (-1) q (k-1) \\ &\qquad - \binom{q-1}{z} (-1) q (k-1) \bigg] \end{aligned}$$

$$= \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n+1}{2} + q)}$$
$$\times \binom{q}{z} (k-1) \left(\tfrac{n-1}{2} + q\right)$$
$$= 2\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+s+1}{2})} \sum_{q=z}^{\lfloor \frac{s}{2} \rfloor} (-1)^{q-z} \binom{s}{2q} \binom{q}{z} \Gamma(q + \tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2} - q)\Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + q)},$$

from which the assertion follows. □

5.5 The Proofs for the Extrinsic Case

Our starting point is a relation, due to McMullen, which relates the intrinsic and the extrinsic Minkowski tensors (see [14, 5.1 Theorem]). Its proof can easily be localized (see [19, Korollar 2.2.2]). Combining this localization with the relation $Q = Q(E) + Q(E^{\perp})$, where $E \subset \mathbb{R}^n$ is any k-flat, we obtain the following lemma.

Lemma 5.21 *Let $j, k, r, s \in \mathbb{N}_0$ with $j < k < n$, let $K \in \mathscr{K}^n$ with $K \subset E \in A(n,k)$ and $\beta \in \mathscr{B}(\mathbb{R}^n)$. Then*

$$\phi_j^{r,s,0}(K,\beta) = \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-j+s}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{k-j+s}{2} - m)}{4^m\, m!(s-2m)!}$$
$$\times Q^l Q(E)^{m-l} \phi_{j,E}^{r,s-2m,0}(K,\beta).$$

We start with the proof of Theorem 5.12, for which we use Theorem 5.3 after an application of Lemma 5.21.

Proof (Theorem 5.12) Lemma 5.21 for $j = k - 1$ gives

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l} Q^l$$
$$\times \int_{A(n,k)} Q(E)^{m-l} \phi_{k-1,E}^{r,s-2m,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E).$$

For $j = k-1$ we can argue as in the proof of Corollary 5.10 to see that Theorem 5.3 implies that

$$\int_{A(n,k)} Q(E)^i \phi_{k-1,E}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \gamma_{n,k,k-1} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor + i} \lambda_{n,k,k-1,s,i,z}\, Q^z \phi_{n-1}^{r,s+2i-2z,0}(K \cap E, \beta \cap E), \tag{5.15}$$

where

$$\lambda_{n,k,k-1,s,i,z} = (k-1) \sum_{p=0}^{i} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor + i - p} (-1)^{p+q-z} \binom{i}{p} \binom{s+2i-2p}{2q} \binom{p+q}{z} \times \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}+i-p-q)}{\Gamma(\frac{n+s+1}{2}+i-p)} \frac{\Gamma(\frac{k-1}{2}+p)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q)}.$$

(Of course, for $i = 0$ we recover Corollary 5.10.) Hence, we obtain

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - l} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} \times \lambda_{n,k,k-1,s-2m,m-l,z}\, Q^{l+z} \phi_{n-1}^{r,s-2l-2z,0}(K, \beta).$$

An index shift of the summation with respect to z yields

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} \sum_{z=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} \binom{m}{l} \times \lambda_{n,k,k-1,s-2m,m-l,z-l}\, Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).$$

Changing the order of summation gives

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k}{2}} s!}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{z} \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l}$$
$$\times \lambda_{n,k,k-1,s-2m,m-l,z-l}\, Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta). \tag{5.16}$$

The coefficients of the tensorial curvature measures on the right-hand side of (5.16) do not depend on the choice of $r \in \mathbb{N}_0$ or $\beta \in \mathscr{B}(\mathbb{R}^n)$. Thus, we can set

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k,s,z}\, Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

where the coefficient $\kappa_{n,k,s,z}$ is uniquely defined in the obvious way. By choosing $r = 0$ and $\beta = \mathbb{R}^n$, we can compare this to the Crofton formula for translation invariant Minkowski tensors in [3]. In fact, since the functionals $Q^z \phi_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n)$, $z \in \{0, \ldots, \lfloor s/2 \rfloor\} \setminus \{(s-1)/2\}$, are linearly independent, we can conclude from the Crofton formula for the translation invariant Minkowski tensors in [3, Theorem 3] that

$$\kappa_{n,k,s,z} = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2}) \Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2}) \Gamma(\frac{s}{2} + 1)}{\Gamma(\frac{n-k+s+1}{2}) \Gamma(\frac{n+s-1}{2})} \frac{\Gamma(\frac{n-k}{2} + z) \Gamma(\frac{k+s-1}{2} - z)}{\Gamma(\frac{s}{2} - z + 1) z!}$$

for $z \neq (s-1)/2$. If $z = (s-1)/2$, then $\phi_{n-1}^{0,s-2z,0}(K, \mathbb{R}^n) = \Phi_{n-1}^1(K) = 0$, and hence we do not get any information about the corresponding coefficient from the global theorem. Consequently, we have to calculate $\kappa_{n,k,s,(s-1)/2}$ directly, which is what we do later in the proof.

But first we demonstrate that the coefficients of the tensorial curvature measures in (5.16) can be determined also by a direct calculation if s is even. In fact, we obtain

$$S := \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} (-1)^{m-l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \binom{m}{l} \lambda_{n,k,k-1,s-2m,m-l,z-l}$$
$$= (k-1) \sum_{m=l}^{\lfloor \frac{s}{2} \rfloor} \sum_{p=l}^{m} \sum_{q=(z-p)^+}^{\lfloor \frac{s}{2} \rfloor - p} (-1)^{m+l+p+q-z} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!}$$
$$\times \binom{m}{l} \binom{m-l}{p-l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q + \tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{k+s+1}{2} - p - q)}{\Gamma(\frac{n+s+1}{2} - p)} \frac{\Gamma(\frac{k-1}{2} + p - l) \Gamma(\frac{n-k}{2} + q)}{\Gamma(\frac{n-1}{2} + p + q - l)}.$$

Changing the order of summation gives

$$S = (k-1)\sum_{p=l}^{\lfloor \frac{s}{2}\rfloor}\sum_{q=(z-p)^+}^{\lfloor \frac{s}{2}\rfloor - p}(-1)^{l+q-z}\binom{s-2p}{2q}\binom{p+q-l}{z-l}\Gamma(q+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)}\frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}$$
$$\times \sum_{m=p}^{\lfloor \frac{s}{2}\rfloor}(-1)^{m+p}\binom{m}{l}\binom{m-l}{p-l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!}.$$

We denote the sum with respect to m by T and conclude

$$T = \sum_{m=p}^{\lfloor \frac{s}{2}\rfloor}(-1)^{m+p}\binom{m}{l}\binom{m-l}{p-l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!}$$
$$= \frac{1}{l!(p-l)!}\sum_{m=p}^{\lfloor \frac{s}{2}\rfloor}(-1)^{m+p}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,(m-p)!(s-2m)!}.$$

An index shift yields

$$T = \frac{1}{2^s l!(p-l)!}\sum_{m=0}^{\lfloor \frac{s}{2}\rfloor - p}(-1)^m\frac{2^{s-2p-2m}\Gamma(\frac{s+1}{2}-p-m)}{m!(s-2p-2m)!}.$$

Legendre's duplication formula gives

$$T = \frac{\sqrt{\pi}}{2^s l!(p-l)!}\sum_{m=0}^{\lfloor \frac{s}{2}\rfloor - p}(-1)^m\frac{1}{m!\Gamma(\frac{s}{2}-p-m+1)}.$$

If s is even, the binomial theorem yields

$$T = \frac{\sqrt{\pi}}{2^s l!(p-l)!(\frac{s}{2}-p)!}\sum_{m=0}^{\frac{s}{2}-p}(-1)^m\binom{\frac{s}{2}-p}{m}$$
$$= \frac{\sqrt{\pi}}{2^s l!(p-l)!(\frac{s}{2}-p)!}(1-1)^{\frac{s}{2}-p}$$
$$= \mathbf{1}\{p=\tfrac{s}{2}\}\frac{\sqrt{\pi}}{2^s l!(\frac{s}{2}-l)!}.$$

Hence, we obtain

$$S = \frac{(k-1)\sqrt{\pi}}{2^s l! (\frac{s}{2}-l)!} \sum_{q=(z-\frac{s}{2})^+}^{0} (-1)^{l+q-z} \binom{\frac{s}{2}+q-l}{z-l} \Gamma(q+\tfrac{1}{2})$$
$$\times \frac{\Gamma(\frac{k+1}{2}-q)}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2}-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n+s-1}{2}+q-l)}$$
$$= (-1)^{l-z} \frac{(k-1)\sqrt{\pi}\,\Gamma(\frac{1}{2})}{2^s l! (\frac{s}{2}-l)!} \binom{\frac{s}{2}-l}{z-l} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k+s-1}{2}-l)\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+s-1}{2}-l)}$$
$$= (-1)^{l-z} \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+1}{2})} \frac{(k-1)\pi}{2^s l! (\frac{s}{2}-l)!} \binom{\frac{s}{2}-l}{z-l} \frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}.$$

Furthermore, Legendre's duplication formula yields

$$s! S = (-1)^{l-z} \frac{(k-1)\sqrt{\pi}\,\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})} \underbrace{\binom{\frac{s}{2}}{l}\binom{\frac{s}{2}-l}{z-l}}_{=\binom{\frac{s}{2}}{z}\binom{z}{l}} \frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}.$$

Thus, we obtain

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})} \frac{(k-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})} \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z}$$
$$\times \sum_{l=0}^{z} (-1)^{l-z} \binom{z}{l} \frac{\Gamma(\frac{k+s-1}{2}-l)}{\Gamma(\frac{n+s-1}{2}-l)}\, Q^z \phi_{n-1}^{r,s-2z,0}(K,\beta).$$

From Lemma 5.22 we conclude

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \gamma_{n,k,k-1} \frac{\pi^{\frac{n-k+1}{2}}}{\Gamma(\frac{n-k+s+1}{2})} \frac{(k-1)\Gamma(\frac{k+1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+s-1}{2})}$$
$$\times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\tfrac{k+s-1}{2}-z)\Gamma(\tfrac{n-k}{2}+z) Q^z \phi_{n-1}^{r,s-2z,0}(K,\beta).$$

With

$$\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}$$

we get

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \frac{(n-2)!}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!} \frac{\Gamma(\frac{k+1}{2})}{(k-2)!} \frac{\pi^{\frac{n-k-1}{2}} \Gamma(\frac{s+1}{2})}{2\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})}$$
$$\times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\tfrac{k+s-1}{2} - z)\Gamma(\tfrac{n-k}{2} + z) Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta).$$

Legendre's formula applied three times gives

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$
$$= \frac{k-1}{n-1} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{n-k+s+1}{2})}$$
$$\times \sum_{z=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{z} \Gamma(\tfrac{k+s-1}{2} - z)\Gamma(\tfrac{n-k}{2} + z) Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta),$$

which confirms the coefficients for even s.

On the other hand, if s is odd, then Lemma 5.23 yields

$$T = \frac{\sqrt{\pi}}{2^s l!(p-l)!} \sum_{m=0}^{\frac{s-1}{2}-p} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2} - p - m + 1)}$$
$$= \frac{\sqrt{\pi}}{2^s l!(p-l)!} \Bigg(\sum_{m=0}^{\frac{s+1}{2}-p} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2} - p - m + 1)} - (-1)^{\frac{s+1}{2}-p} \frac{1}{(\frac{s+1}{2} - p)!\Gamma(\frac{1}{2})} \Bigg)$$
$$= \frac{\sqrt{\pi}}{2^s l!(p-l)!} \Bigg((-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi}(-s+2p)(\frac{s+1}{2} - p)!} - (-1)^{\frac{s+1}{2}-p} \frac{1}{\sqrt{\pi}(\frac{s+1}{2} - p)!} \Bigg)$$

$$
\begin{aligned}
&= (-1)^{\frac{s-1}{2}-p} \frac{\sqrt{\pi}}{2^s l!(p-l)!} \frac{1}{\sqrt{\pi}(\frac{s+1}{2}-p)!} (\tfrac{1}{s-2p}+1) \\
&= (-1)^{\frac{s-1}{2}-p} \quad \frac{1}{2^{s-1}(s-2p)(\frac{s-1}{2}-p)!l!(p-l)!} \\
&= (-1)^{\frac{s-1}{2}-p} \frac{2\Gamma(\frac{s}{2}+1)}{\sqrt{\pi}(s-2p)s!} \binom{\frac{s-1}{2}}{p} \binom{p}{l}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
s! \sum_{l=0}^{z} S = \frac{2(k-1)\Gamma(\frac{s}{2}+1)}{\sqrt{\pi}} \sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \\
\times \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \\
\times \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&= 2(k-1)\gamma_{n,k,k-1} \frac{\pi^{\frac{n-k-1}{2}} \Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})} \sum_{z=0}^{\frac{s-1}{2}} Q^z \phi_{n-1}^{r,s-2z,0}(K, \beta) \\
&\quad \times \sum_{l=0}^{z} \sum_{p=l}^{\frac{s-1}{2}} \sum_{q=(z-p)^+}^{\frac{s-1}{2}-p} (-1)^{\frac{s-1}{2}+l+p+q-z} \frac{1}{(s-2p)} \binom{\frac{s-1}{2}}{p} \binom{p}{l} \binom{s-2p}{2q} \\
&\quad \times \binom{p+q-l}{z-l} \Gamma(q+\tfrac{1}{2}) \frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)} \frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.
\end{aligned}
$$

With

$$
\gamma_{n,k,k-1} = \binom{n-2}{k-1} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi} = \frac{(n-2)!}{(n-k-1)!(k-1)!} \frac{\Gamma(\frac{n-k+1}{2})}{2\pi}
$$

we get

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K\cap E,\beta\cap E)\,\mu_k(\mathrm{d}E)$$

$$= \frac{(n-2)!}{(n-k-1)!(k-2)!}\frac{\pi^{\frac{n-k-3}{2}}\Gamma(\frac{n-k+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})}\sum_{z=0}^{\frac{s-1}{2}} Q^z\phi_{n-1}^{r,s-2z,0}(K,\beta)$$

$$\times\sum_{l=0}^{z}\sum_{p=l}^{\frac{s-1}{2}}\sum_{q=(z-p)^+}^{\frac{s-1}{2}-p}(-1)^{\frac{s-1}{2}+l+p+q-z}\frac{1}{(s-2p)}$$

$$\times\binom{\frac{s-1}{2}}{p}\binom{p}{l}\binom{s-2p}{2q}\binom{p+q-l}{z-l}$$

$$\times\Gamma(q+\tfrac{1}{2})\frac{\Gamma(\frac{k+s+1}{2}-p-q)}{\Gamma(\frac{n+s+1}{2}-p)}\frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k}{2}+q)}{\Gamma(\frac{n-1}{2}+p+q-l)}.$$

We denote the threefold sum with respect to l, p and q by R. Hence, R multiplied with the factor in front of the sum with respect to z equals $\kappa_{n,k,s,z}$. A direct calculation for R still remains an open task. However, for the proof this is not required.

Finally, if s is odd we calculate the only so far unknown coefficient $\kappa_{n,k,s,(s-1)/2}$. For $z=(s-1)/2$ we see that the sum over q only contains one summand, namely $q=(s-1)/2-p$. Hence, we obtain

$$R = \Gamma(\tfrac{k}{2}+1)\sum_{l=0}^{z}\sum_{p=l}^{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}+l}\binom{\frac{s-1}{2}}{p}\binom{p}{l}\Gamma(\tfrac{s}{2}-p)$$

$$\times\frac{\Gamma(\frac{k-1}{2}+p-l)\Gamma(\frac{n-k+s-1}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)\Gamma(\frac{n+s}{2}-l-1)}$$

$$=\Gamma(\tfrac{k}{2}+1)\sum_{p=0}^{\frac{s-1}{2}}(-1)^{\frac{s-1}{2}}\binom{\frac{s-1}{2}}{p}\Gamma(\tfrac{s}{2}-p)\frac{\Gamma(\frac{n-k+s-1}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)}$$

$$\times\sum_{l=0}^{p}(-1)^l\binom{p}{l}\frac{\Gamma(\frac{k-1}{2}+p-l)}{\Gamma(\frac{n+s}{2}-l-1)}.$$

Then Lemma 5.22 yields

$$R = \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}+p} \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{s}{2}-p)}{\Gamma(\frac{n+s+1}{2}-p)}$$

$$= \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)} \sum_{p=0}^{\frac{s-1}{2}} (-1)^{p} \binom{\frac{s-1}{2}}{p} \frac{\Gamma(\frac{1}{2}+p)}{\Gamma(\frac{n}{2}+1+p)}.$$

Again, we apply Lemma 5.22 and obtain

$$R = \sqrt{\pi}\frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+s}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s}{2}-1)\Gamma(\frac{n+s+1}{2})}.$$

Thus, we conclude

$$\begin{aligned}\kappa_{n,k,s,\frac{s-1}{2}} &= \frac{(n-2)!}{(n-k-1)!(k-2)!} \frac{\pi^{\frac{n-k-3}{2}}\Gamma(\frac{n-k+1}{2})\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n-k+s+1}{2})} R \\ &= \pi^{\frac{n-k-2}{2}} \frac{(n-2)!}{\Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{k-1}{2})}{(k-2)!} \frac{\Gamma(\frac{n-k+1}{2})}{(n-k-1)!} \\ &\quad \times \frac{(n+s-2)\Gamma(\frac{s}{2}+1)}{(n-k+s-1)\Gamma(\frac{n+s+1}{2})}.\end{aligned}$$

Applying three times Legendre's formula gives

$$\kappa_{n,k,s,\frac{s-1}{2}} = \pi^{\frac{n-k-1}{2}} \frac{2k(n+s-2)}{(n-1)(n-k+s-1)} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{n+s+1}{2})},$$

which completes the argument. □

Next we prove Theorem 5.13. As in the previous proof, one can compare the Crofton integral to the global one obtained in [3, Theorem 3]. However, we deduce it directly from Theorem 5.4.

Proof (Theorem 5.13) Lemma 5.21 yields

$$\begin{aligned}&\int_{A(n,1)} \phi_0^{r,s,0}(K\cap E, \beta\cap E)\,\mu_1(\mathrm{d}E) \\ &= \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2}-m)}{4^m\, m!(s-2m)!} Q^l \\ &\quad \times \int_{A(n,1)} Q(E)^{m-l} \phi_{0,E}^{r,s-2m,0}(K\cap E, \beta\cap E)\,\mu_1(\mathrm{d}E).\end{aligned}$$

If $s \in \mathbb{N}_0$ is even, we conclude from Theorem 5.4

$$\begin{aligned}
&\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\
&\quad = \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{m=0}^{\frac{s}{2}} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \\
&\qquad \times \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2} - l)}{\pi \Gamma(\frac{n+s+1}{2} - l)} \sum_{z=0}^{\frac{s}{2}-l} (-1)^z \binom{\frac{s}{2} - l}{z} \frac{1}{1-2z}\, Q^{\frac{s}{2}-z} \phi_{n-1}^{r,2z,0}(K, \beta).
\end{aligned}$$

A change of the order of summation yields

$$\begin{aligned}
&\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E) \\
&\quad = \frac{\pi^{\frac{n-1}{2}} s!}{\Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s}{2}} \sum_{m=l}^{\frac{s}{2}} (-1)^{m-l} \binom{m}{l} \frac{\Gamma(\frac{s+1}{2} - m)}{4^m\, m!(s-2m)!} \\
&\qquad \times \frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2} - l)}{\pi \Gamma(\frac{n+s+1}{2} - l)} \sum_{z=0}^{\frac{s}{2}-l} (-1)^z \binom{\frac{s}{2} - l}{z} \frac{1}{1-2z}\, Q^{\frac{s}{2}-z} \phi_{n-1}^{r,2z,0}(K, \beta).
\end{aligned}$$

Legendre's duplication formula gives for the sum with respect to m, which we denote by S,

$$\begin{aligned}
S &= \frac{\sqrt{\pi}}{2^s} \sum_{m=l}^{\frac{s}{2}} (-1)^{m-l} \binom{m}{l} \frac{1}{m!\Gamma(\frac{s}{2} - m + 1)} \\
&= \frac{\sqrt{\pi}}{2^s l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \frac{1}{m!\Gamma(\frac{s}{2} - l - m + 1)}.
\end{aligned}$$

As seen before, we conclude from the binomial theorem

$$\begin{aligned}
S &= \frac{\sqrt{\pi}}{2^s(\frac{s}{2} - l)!l!} \sum_{m=0}^{\frac{s}{2}-l} (-1)^m \binom{\frac{s}{2} - l}{m} \\
&= \mathbf{1}\{l = \tfrac{s}{2}\} \frac{\Gamma(\frac{s+1}{2})}{s!}.
\end{aligned}$$

Hence, we obtain

$$\int_{A(n,1)} \phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E) = \frac{\pi^{\frac{n-3}{2}}\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s}{2})}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}Q^{\frac{s}{2}}\phi_{n-1}^{r,0,0}(K,\beta)$$
$$= \pi^{\frac{n-2}{2}}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{n+s}{2})\Gamma(\frac{n+1}{2})}Q^{\frac{s}{2}}\phi_{n-1}^{r,0,0}(K,\beta).$$

On the other hand, if $s\in\mathbb{N}$ is odd, we conclude from Theorem 5.4

$$\int_{A(n,1)} \phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E)$$
$$= \frac{\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2})s!}{\Gamma(\frac{n+s}{2})}\sum_{m=0}^{\frac{s-1}{2}}\sum_{l=0}^{m}(-1)^{m-l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,m!(s-2m)!}\binom{m}{l}$$
$$\times\frac{\Gamma(\frac{s}{2}-l+1)}{\Gamma(\frac{n+s+1}{2}-l)}\,Q^{\frac{s-1}{2}}\phi_{n-1}^{r,1,0}(K,\beta).$$

A change of the order of summation yields

$$\int_{A(n,1)} \phi_0^{r,s,0}(K\cap E,\beta\cap E)\,\mu_1(\mathrm{d}E)$$
$$= \frac{\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2})s!}{\Gamma(\frac{n+s}{2})}\sum_{l=0}^{\frac{s-1}{2}}\sum_{m=l}^{\frac{s-1}{2}}(-1)^{m-l}\frac{\Gamma(\frac{s+1}{2}-m)}{4^m\,m!(s-2m)!}\binom{m}{l}$$
$$\times\frac{\Gamma(\frac{s}{2}-l+1)}{\Gamma(\frac{n+s+1}{2}-l)}\,Q^{\frac{s-1}{2}}\phi_{n-1}^{r,1,0}(K,\beta).$$

Legendre's duplication formula gives for the sum with respect to m, which we denote by S,

$$S=\frac{\sqrt{\pi}}{2^s l!}\sum_{m=0}^{\frac{s-1}{2}-l}(-1)^m\frac{1}{m!\Gamma(\frac{s}{2}-l-m+1)}.$$

Then Lemma 5.23 yields

$$S=\frac{\sqrt{\pi}}{2^s l!}\left(\sum_{m=0}^{\frac{s+1}{2}-l}(-1)^m\frac{1}{m!\Gamma(\frac{s}{2}-l-m+1)}-(-1)^{\frac{s+1}{2}-l}\frac{1}{(\frac{s+1}{2}-l)!\Gamma(\frac{1}{2})}\right)$$
$$=\frac{\sqrt{\pi}}{2^s l!}\left((-1)^{\frac{s-1}{2}-l}\frac{1}{\sqrt{\pi}(s-2l)(\frac{s+1}{2}-l)!}-(-1)^{\frac{s+1}{2}-l}\frac{1}{\sqrt{\pi}(\frac{s+1}{2}-l)!}\right)$$
$$=(-1)^{\frac{s-1}{2}-l}\frac{1}{2^{s-1}l!(s-2l)(\frac{s-1}{2}-l)!}.$$

Hence, we obtain

$$\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$

$$= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{2^s \Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^{\frac{s-1}{2}-l} \frac{1}{l!(\frac{s-1}{2}-l)!} \frac{\Gamma(\frac{s}{2}-l)}{\Gamma(\frac{n+s+1}{2}-l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K,\beta)$$

$$= \frac{\pi^{\frac{n-2}{2}} \Gamma(\frac{n}{2}) s!}{2^s \Gamma(\frac{s+1}{2}) \Gamma(\frac{n+s}{2})} \sum_{l=0}^{\frac{s-1}{2}} (-1)^l \binom{\frac{s-1}{2}}{l} \frac{\Gamma(l+\frac{1}{2})}{\Gamma(\frac{n+2}{2}+l)} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K,\beta).$$

Then Lemma 5.22 gives

$$\int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_1(\mathrm{d}E)$$

$$= \frac{s!}{2^s \Gamma(\frac{s+1}{2})} \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{n+s+1}{2}) \Gamma(\frac{n+1}{2})} Q^{\frac{s-1}{2}} \phi_{n-1}^{r,1,0}(K,\beta).$$

Now, the assertion follows from Legendre's duplication formula. □

Finally, we show that the Crofton formula has a very simple form in the ψ-representation of tensorial curvature measures.

Proof (Corollary 5.14) The cases $s \in \{0, 1\}$ are checked directly, hence we can assume $s \geq 2$ in the following. Using (5.9) we get

$$\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$

$$= \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2}) \Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} Q^j$$

$$\times \int_{A(n,k)} \phi_{k-1}^{r,s-2j,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E). \tag{5.17}$$

Then, for $k \neq 1$, Theorem 5.12 yields

$$\begin{aligned}
&\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} \\
&\qquad \times \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - j} \kappa_{n,k,s-2j,z}\, Q^{z+j} \phi_{n-1}^{r,s-2j-2z,0}(K,\beta) \\
&\quad = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{z=j}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \frac{\Gamma(j+\frac{1}{2})\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+s-1)} \\
&\qquad \times \kappa_{n,k,s-2j,z-j}\, Q^{z} \phi_{n-1}^{r,s-2z,0}(K,\beta),
\end{aligned}$$

where

$$\begin{aligned}
\kappa_{n,k,s-2j,z-j} &= \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{\Gamma(\frac{s+1}{2}-j)\Gamma(\frac{s}{2}-j+1)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)} \\
&\quad \times \frac{\Gamma(\frac{n-k}{2}+z-j)\Gamma(\frac{k+s-1}{2}-z)}{\Gamma(\frac{s}{2}-z+1)(z-j)!},
\end{aligned}$$

if $z \neq (s-1)/2$. On the other hand, if $z = (s-1)/2$, then the coefficient needs to be multiplied by the factor $\frac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)}$ (see the comment after Theorem 5.12).

Applying Legendre's duplication formula twice, we thus obtain

$$\begin{aligned}
&\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\
&\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}} \Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \frac{s!}{2^s} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma(\frac{k+s-1}{2}-z)}{z!\,\Gamma(\frac{n}{2}+s-1)\Gamma(\frac{s}{2}-z+1)} Q^z \phi_{n-1}^{r,s-2z,0}(K,\beta) \\
&\qquad \times \sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\frac{n-k}{2}+z-j)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)} \\
&\qquad \times \left(1 - \mathbf{1}\{z = \tfrac{s-1}{2}\} \left(1 - \tfrac{k(n+s-2j-2)}{(k-1)(n+s-2j-1)}\right)\right).
\end{aligned}$$

Denoting the sum with respect to j by S_z, an application of Lemma 5.24 shows that

$$S_z = \sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\frac{n-k}{2}+z-j)}{\Gamma(\frac{n-k+s+1}{2}-j)\Gamma(\frac{n+s-1}{2}-j)}$$

$$= (-1)^z \frac{\Gamma(\frac{n-k}{2})\Gamma(\frac{s+1}{2})\Gamma(\frac{k+s-1}{2})\Gamma(\frac{n}{2}+s-z-1)}{\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})\Gamma(\frac{s+1}{2}-z)\Gamma(\frac{k+s-1}{2}-z)}, \tag{5.18}$$

for $z \neq (s-1)/2$ and $k > 1$. On the other hand, for $z = (s-1)/2 =: t$, we obtain from Lemmas 5.24 and 5.25 (since $s > 1$ and thus $t > 0$) that

$$\begin{aligned} S_t &= \tfrac{k}{k-1} \sum_{j=0}^{t} (-1)^j \binom{t}{j} \left(1 - \tfrac{1}{n+2t-2j}\right) \frac{\Gamma(\frac{n}{2}+2t-j)\Gamma(\frac{n-k}{2}+t-j)}{\Gamma(\frac{n-k}{2}+t-j+1)\Gamma(\frac{n}{2}+t-j)} \\ &= \tfrac{k}{k-1} \Bigg(\sum_{j=0}^{t} (-1)^j \binom{t}{j} \frac{\Gamma(\frac{n}{2}+2t-j)\Gamma(\frac{n-k}{2}+t-j)}{\Gamma(\frac{n-k}{2}+t-j+1)\Gamma(\frac{n}{2}+t-j)} \\ &\qquad - \sum_{j=0}^{t} (-1)^j \binom{t}{j} \tfrac{1}{\frac{n-k}{2}+t-j} \frac{\Gamma(\frac{n}{2}+2t-j)}{\Gamma(\frac{n}{2}+t-j+1)} \Bigg) \\ &= (-1)^t \frac{\Gamma(\frac{n-k}{2})\Gamma(t+1)\Gamma(\frac{k}{2}+t)}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2}+t+1)}, \end{aligned}$$

which coincides with (5.18) for $z = (s-1)/2$.

Thus, we have

$$\begin{aligned} &\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\ &\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k+1}{2}} \Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \frac{s!\,\Gamma(\frac{s+1}{2})}{2^s} \\ &\qquad \times \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \frac{\Gamma(\frac{n}{2}+s-z-1)}{z!\,\Gamma(\frac{n}{2}+s-1)\Gamma(\frac{s}{2}-z+1)\Gamma(\frac{s+1}{2}-z)} Q^z \phi_{n-1}^{r,s-2z,0}(K,\beta). \end{aligned}$$

Applying Legendre's duplication formula twice, we get

$$\begin{aligned} &\int_{A(n,k)} \psi_{k-1}^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E) \\ &\quad = \frac{k-1}{n-1} \frac{\pi^{\frac{n-k}{2}} \Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+s+1}{2})\Gamma(\frac{n+s-1}{2})} \\ &\qquad \times \frac{1}{\sqrt{\pi}} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^z \binom{s}{2z} \frac{\Gamma(z+\frac{1}{2})\Gamma(\frac{n}{2}+s-z-1)}{\Gamma(\frac{n}{2}+s-1)} Q^z \phi_{n-1}^{r,s-2z,0}(K,\beta). \end{aligned}$$

With (5.9) we obtain the assertion for $k \neq 1$.

On the other hand, if $k = 1$, then Theorem 5.13 yields for (5.17) that

$$\int_{A(n,1)} \psi_0^{r,s,0}(K \cap E, \beta \cap E)\, \mu_k(\mathrm{d}E)$$

$$= \frac{\pi^{\frac{n-3}{2}}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+s-1)} \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{s}{2j} \Gamma(j+\tfrac{1}{2})$$

$$\times \frac{\Gamma(\frac{n}{2}+s-j-1)\Gamma(\lfloor \frac{s+1}{2} \rfloor - j + \frac{1}{2})}{\Gamma(\frac{n}{2}+\lfloor \frac{s+1}{2} \rfloor - j)} Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K,\beta).$$

Denoting the sum with respect to j by S and applying Legendre's duplication formula three times, we conclude that

$$S = \sqrt{\pi}\,\Gamma(\lfloor \tfrac{s+1}{2} \rfloor + \tfrac{1}{2}) \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} (-1)^j \binom{\lfloor \frac{s}{2} \rfloor}{j} \frac{\Gamma(\frac{n}{2}+s-j-1)}{\Gamma(\frac{n}{2}+\lfloor \frac{s+1}{2} \rfloor - j)}.$$

Since $s \geq 2$, Lemma 5.22 yields $S = 0$ due to (5.19), and hence the assertion. □

5.6 Sums of Gamma Functions

In this section, we state four basic identities involving sums of Gamma functions.

Lemma 5.22 *Let $q \in \mathbb{N}_0$ and $a, b > 0$. Then*

$$\sum_{y=0}^{q} (-1)^y \binom{q}{y} \frac{\Gamma(a+y)}{\Gamma(b+y)} = \frac{\Gamma(a)\Gamma(b-a+q)}{\Gamma(b+q)\Gamma(b-a)}.$$

Under the additional assumption $a < b$, this lemma can be found as Lemma 15.6.4 in [1], which is also proved there. Since this case is not sufficient for our purposes, we deduce the current more general version via Zeilberger's algorithm.

The factor $\Gamma(b-a+q)$ in Lemma 5.22 does not cause any problems in case $a-b-q \in \mathbb{N}_0$, as the also appearing $\Gamma(b-a)$ cancels out the singularity, see (5.19).

Proof We set

$$F(q,y) := (-1)^y \binom{q}{y} \frac{\Gamma(a+y)}{\Gamma(b+y)},$$

for which we see that $F(q,y) = 0$ if $y \notin \{0,\ldots,q\}$, and

$$f(q) := \sum_{y=0}^{q} F(q,y).$$

Furthermore, we define the function

$$G(q,y) := \begin{cases} \frac{y(b+y-1)}{q-y+1} F(q,y), & \text{for } y \in \{0,\dots,q\}, \\ G(q,q) - (b+q)F(q+1,q) & \\ \quad + (b-a+q)F(q,q), & \text{for } y = q+1, \\ 0, & \text{else.} \end{cases}$$

A direct calculation yields

$$\begin{aligned} &-(b+q-1)F(q,y) + (b-a+q-1)F(q-1,y) \\ &\quad = G(q-1,y+1) - G(q-1,y) \end{aligned}$$

for $y \in \mathbb{N}_0$. Summing this relation over $y \in \{0,\dots,q\}$ gives

$$-(b+q-1)f(q) + (b-a+q-1)f(q-1) = 0$$

and thus

$$\begin{aligned} f(q) &= \frac{(b-a+q-2)(b-a+q-1)}{(b+q-2)(b+q-1)} f(q-2) \\ &\vdots \\ &= \frac{\Gamma(b-a+q)\Gamma(b)}{\Gamma(b+q)\Gamma(b-a)} f(0), \end{aligned}$$

where

$$\frac{\Gamma(b-a+q)}{\Gamma(b-a)} = (b-a)\cdots(b-a+q-1) \tag{5.19}$$

is well-defined, even for $a-b \in \mathbb{N}$. With

$$f(0) = \frac{\Gamma(a)}{\Gamma(b)}$$

we obtain the assertion. □

Lemma 5.23 *Let* $a \in \mathbb{N}_0$. *Then*

$$\sum_{q=0}^{a} \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} = \frac{(-1)^a}{\sqrt{\pi}(1-2a)a!}.$$

Proof For the sum S on the left-hand side of the asserted equation, we obtain

$$S = \sum_{q=0}^{a} \left(\frac{2q}{2a-1} \frac{(-1)^q}{\Gamma(a-q+\frac{1}{2})q!} + \frac{2q+2}{2a-1} \frac{(-1)^q}{\Gamma(a-q-\frac{1}{2})(q+1)!} \right),$$

where we use that $(-\frac{1}{2})\Gamma(-\frac{1}{2}) = \sqrt{\pi}$. Due to cancellation in this telescoping sum, the assertion follows immediately. □

Finally, we establish the following lemmas.

Lemma 5.24 *Let $a, b, c \in \mathbb{R}$ and $z \in \mathbb{N}_0$ with $a > z \geq 0$ and $b > 0$. Then*

$$\sum_{j=0}^{z} (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)}$$
$$= (-1)^z \frac{\Gamma(a-z)\Gamma(b)}{\Gamma(a+b-c+1)\Gamma(c)} \frac{\Gamma(a-c+1)}{\Gamma(a-c+1-z)} \frac{\Gamma(c-b)}{\Gamma(c-b-z)}.$$

The factor $\Gamma(a-c+1)$ (resp. $\Gamma(c-b)$) in Lemma 5.24 does not cause any problems for $c-a \in \mathbb{N}$ (resp. $b-c \in \mathbb{N}_0$), as the also appearing $\Gamma(a-c+1-z)$ (resp. $\Gamma(c-b-z)$) cancels out the singularity. On the other hand, in our applications of the lemma, we only need the cases where $a-c+1 > z$ and $c-b > z$.

Proof We set

$$F(z,j) := (-1)^j \binom{z}{j} \frac{\Gamma(a-j)\Gamma(b+z-j)}{\Gamma(c-j)\Gamma(a+b-c-j+1)},$$

for $j \in \{0, \ldots, z\}$, and $F(z,j) = 0$ in all other cases, and

$$f(z) := \sum_{j=0}^{z} F(z,j).$$

Furthermore, we define the function

$$G(z,j) := \begin{cases} -\frac{j(a-j)(b+z-j)}{z-j+1} F(z,j), & \text{for } j \in \{0, \ldots, z\}, \\ G(z,z) + (a-z-1)F(z+1,z) & \\ \quad + (c-b-z-1)(a-c-z)F(z,z), & \text{for } j = z+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$(a-z)F(z,j) + (c-b-z)(a-c-z+1)F(z-1,j)$$
$$= G(z-1,j+1) - G(z-1,j)$$

for $j \in \mathbb{N}_0$. Summing this relation over $j \in \{0, \dots, z\}$ gives

$$(a-z)f(z) + (c-b-z)(a-c-z+1)f(z-1) = 0$$

and thus

$$\begin{aligned} f(z) &= \frac{(c-b-z)(c-b-z+1)(a-c-z+1)(a-c-z+2)}{(a-z)(a-z+1)} f(z-2) \\ &\vdots \\ &= (-1)^z \frac{\Gamma(c-b)\Gamma(a-c+1)\Gamma(a-z)}{\Gamma(c-b-z)\Gamma(a-c+1-z)\Gamma(a)} f(0), \end{aligned}$$

where

$$\frac{\Gamma(c-b)}{\Gamma(c-b-z)} = (c-b-z)\cdots(c-b-1)$$

is well-defined, even for $b-c \in \mathbb{N}_0$, and a similar statement holds for $\frac{\Gamma(a-c+1)}{\Gamma(a-c+1-z)}$. With

$$f(0) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c+1)}$$

we obtain the assertion. □

Lemma 5.25 *Let $a, b \in \mathbb{R}$ with $a, b > 0$ and $t \in \mathbb{N}$. Then*

$$\sum_{j=0}^{t} (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)} = \frac{\Gamma(a-b+t)\Gamma(b)\Gamma(t+1)}{\Gamma(a-b+1)\Gamma(b+t+1)}.$$

The factor $\Gamma(a-b+t)$ in Lemma 5.25 does not cause any problems for $b-a-t \in \mathbb{N}_0$, as the also appearing $\Gamma(a-b+1)$ cancels out the singularity. In our application of the lemma, we will additionally know that $a > b$.

Proof We set

$$F(t,j) := (-1)^j \frac{1}{b+j} \binom{t}{j} \frac{\Gamma(a+t+j)}{\Gamma(a+1+j)},$$

for which we see that $F(t,j) = 0$ if $j \notin \{0, \dots, t\}$, and

$$f(t) := \sum_{j=0}^{t} F(t,j).$$

Furthermore, we define the function

$$G(t,j) := \begin{cases} \frac{j(a+j)(a+2t+1)(t^2+t(a+1)-j+1)(b+j)}{t(t-j+1)(a+t)(a+t+1)} F(t,j), & \text{for } j \in \{0,\dots,t\}, \\ G(t,t) - (b+t+1)F(t+1,t) \\ \quad + (t+1)(a-b+t)F(t,t), & \text{for } j = t+1, \\ 0, & \text{otherwise.} \end{cases}$$

A direct calculation yields

$$\begin{aligned} &-(b+t)F(t,j) + t(a-b+t-1)F(t-1,j) \\ &\quad = G(t-1,j+1) - G(t-1,j) \end{aligned}$$

for $j \in \mathbb{N}_0$. Summing this relation over $j \in \{0,\dots,t\}$ gives

$$-(b+t)f(t) + t(a-b+t-1)f(t-1) = 0$$

and thus

$$\begin{aligned} f(t) &= \frac{(t-1)t(a-b+t-2)(a-b+t-1)}{(b+t-1)(b+t)} f(t-2) \\ &\vdots \\ &= \frac{\Gamma(t+1)\Gamma(a-b+t)\Gamma(b+2)}{\Gamma(a-b+1)\Gamma(b+t+1)} f(1). \end{aligned}$$

With

$$f(1) = \frac{1}{b} - \frac{1}{b+1} = \frac{1}{b(b+1)}$$

we obtain the assertion. □

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Chapter 6
A Hadwiger-Type Theorem for General Tensor Valuations

Franz E. Schuster

Abstract Hadwiger's characterization of continuous rigid motion invariant real valued valuations has been the starting point for many important developments in valuation theory. In this chapter, the decomposition of the space of continuous and translation invariant valuations into a sum of SO(n) irreducible subspaces, derived by S. Alesker, A. Bernig and the author, is discussed. It is also explained how this result can be reformulated in terms of a Hadwiger-type theorem for translation invariant and SO(n) equivariant valuations with values in an arbitrary finite dimensional SO(n) module. In particular, this includes valuations with values in general tensor spaces. The proofs of these results will be outlined modulo a couple of basic facts from representation theory. In the final part, we survey a number of special cases and applications of the main results in different contexts of convex and integral geometry.

6.1 Statement of the Principal Results

Let $\mathcal{K}^n$ denote the space of convex bodies in Euclidean n-space $\mathbb{R}^n$, where $n \geq 3$, endowed with the Hausdorff metric. In this chapter we consider valuations ϕ defined on $\mathcal{K}^n$ and taking values in an Abelian semigroup $\mathcal{A}$, that is,

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K \cup L$ is convex and $+$ denotes the operation of $\mathcal{A}$.

The most famous and important classical result on scalar-valued valuations (where $\mathcal{A} = \mathbb{R}$ or $\mathbb{C}$) is the characterization of continuous rigid motion invariant valuations by Hadwiger [38] (which was slightly improved later by Klain [44]).

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Theorem ([38, 44]) *A basis for the vector space of all continuous, translation- and* $\mathrm{SO}(n)$ *invariant scalar valuations on* $\mathcal{K}^n$ *is given by the intrinsic volumes.*

The characterization theorem of Hadwiger had a transformative effect on integral geometry. It not only allows for an effortless proof of the principal kinematic formula (see, e.g., [46]) but almost all classical integral-geometric results can be derived from this landmark theorem. It also motivated subsequent characterizations of rigid motion equivariant vector-valued valuations (where $\mathscr{A} = \mathbb{R}^n$) (see [40]), valuations taking values in the set of finite Borel measures on $\mathbb{R}^n$ or $\mathbb{S}^{n-1}$ which intertwine rigid motions (see [67, 68]) and, more recently, Minkowski valuations (where $\mathscr{A} = \mathcal{K}^n$ endowed with Minkowski addition) which are translation invariant and $\mathrm{SO}(n)$ equivariant (see [43, 66, 71, 72, 74, 75]). Important parts of modern integral geometry also deal with variants of Hadwiger's characterization theorem, where either the group $\mathrm{SO}(n)$ is replaced by a subgroup acting transitively on the unit sphere (see [6, 15, 16, 18, 20, 23]) or the valuations are invariant under the larger group $\mathrm{SL}(n)$ but neither assumed to be continuous nor translation invariant (see, e.g., [35, 50, 54]).

Here we focus on continuous and translation invariant valuations which take values in a general (finite dimensional) *tensor space* Γ and are equivariant with respect to $\mathrm{SO}(n)$. The case of *symmetric* tensors, where $\Gamma = \mathrm{Sym}^k(\mathbb{R}^n)$, was first investigated by McMullen [58], who considered instead of translation invariant more general *isometry covariant* tensor valuations. Alesker [3, 4] showed that the space of all such continuous isometry covariant $\mathrm{Sym}^k(\mathbb{R}^n)$-valued valuations (of a fixed rank and given degree of homogeneity) is spanned by the Minkowski tensors. More recently, Hug, Schneider and Schuster [41, 42] explicitly determined the dimension of this space and obtained a full set of kinematic formulas for Minkowski tensors. Following a more algebraic approach, these kinematic formulas could be further simplified in the translation invariant case by Bernig and Hug [19]. For applications of the integral geometry of tensor valuations in different areas, see Chaps. 11–15 and the references therein. We also mention that $\mathrm{Sym}^k(\mathbb{R}^n)$-valued valuations were also investigated in the context of affine and centro-affine geometry by Ludwig [51] and Haberl and Parapatits [36, 37].

Bernig [14] constructed an interesting translation invariant valuation with values in $\Lambda^k(\mathbb{R}^n) \otimes \Lambda^k(\mathbb{R}^n)$ which can be interpreted as a natural curvature tensor. Apart from this, not much was known for general, non-symmetric tensor valuations until recently Alesker, Bernig and the author [11] established a Hadwiger-type theorem for continuous, translation invariant and $\mathrm{SO}(n)$ equivariant valuations with values in an *arbitrary finite dimensional complex* representation space Γ of $\mathrm{SO}(n)$. In order to state this result first recall that given a Lie group G and a topological vector space Γ (finite or infinite dimensional), a (continuous) representation of G on Γ is a continuous left action $G \times \Gamma \to \Gamma$ such that for each $g \in G$ the map $v \mapsto g \cdot v$ is linear. Note that we assume throughout that all representations are continuous.

For a finite dimensional complex vector space Γ, we denote by $\Gamma\mathrm{Val}$ the vector space of all continuous and translation invariant valuations with values in Γ and write $\Gamma\ \mathrm{Val}_i$ for its subspace of all valuations of degree i. If $\Gamma = \mathbb{C}$, then we simply write Val and Val_i, respectively. McMullen's decomposition theorem [56] implies that

$$\Gamma\ \mathrm{Val} = \bigoplus_{0\le i\le n} \Gamma\ \mathrm{Val}_i. \tag{6.1}$$

We also recall the parametrization of the isomorphism classes of irreducible representations of $\mathrm{SO}(n)$ in terms of their highest weights. These can be identified with $\lfloor n/2\rfloor$-tuples of integers $(\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$ such that

$$\begin{cases} \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{\lfloor n/2\rfloor} \ge 0 & \text{for odd } n, \\ \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{n/2-1} \ge |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{6.2}$$

We write Γ_λ for any isomorphic copy of an irreducible representation of $\mathrm{SO}(n)$ with highest weight $\lambda = (\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$. Note that since $\mathrm{SO}(n)$ is a compact Lie group, every Γ_λ is finite dimensional. Moreover, any finite dimensional representation of $\mathrm{SO}(n)$ can be decomposed into a direct sum of irreducible representations. In particular, we have a decomposition of our representation space Γ of the form

$$\Gamma = \bigoplus_{\lambda} m(\Gamma,\lambda)\,\Gamma_\lambda, \tag{6.3}$$

where the sum ranges over a finite number of highest weights $\lambda = (\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$ satisfying (6.2). Here and in the following $m(\Theta,\lambda)$ denotes the *multiplicity* of Γ_λ in an arbitrary $\mathrm{SO}(n)$ module Θ which, by Schur's lemma, is given by

$$m(\Theta,\lambda) = \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Theta,\Gamma_\lambda),$$

where $\mathrm{Hom}_{\mathrm{SO}(n)}$ denotes as usual the space of continuous linear $\mathrm{SO}(n)$ equivariant maps. If $m(\Theta,\lambda)$ is 0 or 1 for all highest weights λ satisfying (6.2), we say that the $\mathrm{SO}(n)$ module Θ is *multiplicity free*. For explicit examples of decompositions of the form (6.3) and more background material as well as references on representation theory of compact Lie groups, see Sect. 6.2.

We are now ready to state the main result of [11] which is the topic of this chapter.

Theorem 6.1 *Let* Γ *be a finite dimensional complex* $\mathrm{SO}(n)$ *module and let* $0 \leq i \leq n$. *The dimension of the subspace of* $\mathrm{SO}(n)$ *equivariant valuations in* Γ *Val*$_i$ *is given by*

$$\sum_{\lambda} m(\Gamma, \lambda),$$

where the sum ranges over all highest weights $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ *satisfying* (6.2) *and the following additional conditions:*

(i) $\lambda_j = 0$ *for* $j > \min\{i, n-i\}$;
(ii) $|\lambda_j| \neq 1$ *for* $1 \leq j \leq \lfloor n/2 \rfloor$;
(iii) $|\lambda_2| \leq 2$.

Theorem 6.1 follows from an equivalent result about the decomposition of the space Val_i into $\mathrm{SO}(n)$ irreducible subspaces.

Theorem 6.2 *Let* $0 \leq i \leq n$. *Under the action of* $\mathrm{SO}(n)$ *the space* Val_i *is multiplicity free. Moreover,* $m(\mathrm{Val}_i, \lambda) = 1$ *if and only if the highest weight* $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ *satisfies* (6.2) *and the conditions (i)–(iii) from Theorem 6.1.*

In order to outline how Theorem 6.1 can be deduced from Theorem 6.2 (for the precise argument, see [11, p. 765]), first note that we may assume that Γ is irreducible, that is, $\Gamma = \Gamma_\mu$ for some highest weight $\mu = (\mu_1, \dots, \mu_{\lfloor n/2 \rfloor})$ satisfying (6.2). Next, observe that the linear map $\iota : \mathrm{Val}_i \otimes \Gamma \to \Gamma\ \mathrm{Val}_i$, induced by

$$\iota(\phi \otimes v)(K) = \phi(K)v,$$

is an isomorphism and that the subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\ \mathrm{Val}_i$ corresponds under this isomorphism to the subspace of $\mathrm{SO}(n)$ *invariant* elements in $\mathrm{Val}_i \otimes \Gamma$. (As usual we will use the superscript $\mathrm{SO}(n)$ to denote subspaces of $\mathrm{SO}(n)$ invariant elements.) Now, if S denotes the set of highest weights of $\mathrm{SO}(n)$ satisfying conditions (i)–(iii), then, by Theorem 6.2,

$$\dim(\mathrm{Val}_i \otimes \Gamma)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim(\Gamma_\lambda \otimes \Gamma_\mu)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda^*, \Gamma_\mu).$$

It follows from Lemma 6.3 below, that the $\mathrm{SO}(n)$ irreducible subspaces Γ_λ for $\lambda \in S$ are *not* necessarily isomorphic as $\mathrm{SO}(n)$ modules to their dual representations Γ_λ^* (see Sect. 6.2 for details). However, Lemma 6.3 also implies that if $\lambda \in S$, then also $\lambda' \in S$, where λ' is the highest weight of Γ_λ^*. Thus, from an application of Schur's lemma, we obtain

$$\dim(\mathrm{Val}_i \otimes \Gamma)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda, \Gamma_\mu) = \begin{cases} 1 & \text{if } \mu \in S, \\ 0 & \text{otherwise} \end{cases}$$

which is precisely the statement of Theorem 6.1 in the case $\Gamma = \Gamma_\mu$. We also remark that the argument outlined here can be easily reversed to deduce Theorem 6.2 from Theorem 6.1.

A proof of Theorem 6.2 for *even* valuations was first given by Alesker and Bernstein [10] (based on the Irreducibility Theorem of Alesker [5]). They used the Klain embedding of continuous, translation invariant and even valuations and its relation to the cosine transform on Grassmannians to deduce Theorem 6.2 in this special case. We will discuss this approach in more detail in the last section of this chapter, where we also survey a number of other special cases and applications of Theorems 6.1 and 6.2.

In Sect. 6.2 we collect more background material about representations of $\mathrm{SO}(n)$ which is needed for the analysis of the action of $\mathrm{SO}(n)$ on the space of translation invariant differential forms on the sphere bundle. Combining this with a description of smooth translation invariant valuations via integral currents by Alesker [8] and, in a refined form, by Bernig and Bröcker [17] and Bernig [16], this will allow us to give an essentially complete proof of Theorem 6.2 in Sect. 6.4.

6.2 Irreducible Representations of SO(n)

For an introduction to the representation theory of compact Lie groups we refer to the books by Bröcker and tom Dieck [21], Fulton and Harris [24], Goodman and Wallach [32], and Knapp [48]. These books, in particular, contain all the material on irreducible representations of $\mathrm{SO}(n)$ which are needed in this chapter.

In this and the next section let V be an n-dimensional Euclidean vector space and write $V_{\mathbb{C}} = V \otimes \mathbb{C}$ for its complexification. For later reference we state here a number of examples of irreducible $\mathrm{SO}(n)$ modules as well as reducible ones together with their direct sum decomposition into $\mathrm{SO}(n)$ irreducible subspaces.

Examples

(a) Up to isomorphism, the trivial representation is the only one dimensional (complex) representation of $\mathrm{SO}(n)$. It corresponds to the $\mathrm{SO}(n)$ module $\Gamma_{(0,\dots,0)}$. The standard representation of $\mathrm{SO}(n)$ on $V_{\mathbb{C}}$ is isomorphic to $\Gamma_{(1,0,\dots,0)}$.

(b) The exterior power $\Lambda^i V_{\mathbb{C}}$ is $\mathrm{SO}(n)$ irreducible for every $0 \leq i \leq \lfloor n/2 \rfloor - 1$. If $n = 2i + 1$ is odd, then $\Lambda^i V_{\mathbb{C}}$ is also irreducible under the action of $\mathrm{SO}(n)$. In these cases the highest weight tuple of $\Lambda^i V_{\mathbb{C}}$ is given by $\lambda = (1,\dots,1,0,\dots,0)$, where 1 appears i times. If $n = 2i$ is even, then $\Lambda^i V_{\mathbb{C}}$ is *not* irreducible but is a direct sum of two irreducible subspaces, namely, $\Lambda^i V_{\mathbb{C}} = \Gamma_{(1,\dots,1)} \oplus \Gamma_{(1,\dots,1,-1)}$. Moreover, for every $i \in \{0,\dots,n\}$, there is a natural isomorphism of $\mathrm{SO}(n)$ modules

$$\Lambda^i V_{\mathbb{C}} \cong \Lambda^{n-i} V_{\mathbb{C}}. \tag{6.4}$$

The spaces $\Lambda^i V_{\mathbb{C}}$ are called *fundamental representations* since they can be used to construct arbitrary irreducible representations of $\mathrm{SO}(n)$ (cf. [24]).

(c) The symmetric power $\mathrm{Sym}^k V_{\mathbb{C}}$ is *not* irreducible as SO(n) module when $k \geq 2$. Its direct sum decomposition into irreducible subspaces takes the form

$$\mathrm{Sym}^k V_{\mathbb{C}} = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \Gamma_{(k-2j,0,\ldots,0)}. \tag{6.5}$$

(d) The decomposition of $L^2(\mathbb{S}^{n-1})$ into an *orthogonal* sum of SO(n) irreducible subspaces is given by

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k \in \mathbb{N}} \mathscr{H}_k^n, \tag{6.6}$$

where $\mathscr{H}_k^n$ is the space of spherical harmonics of dimension n and degree k. The highest weight tuple of the space $\mathscr{H}_k^n$ is given by $\lambda = (k, 0, \ldots, 0)$.

(e) For $1 \leq i \leq n-1$, the space $L^2(G(n,i))$ is an orthogonal sum of SO(n) irreducible subspaces whose highest weights $(\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfy (6.2) and the following two additional conditions (see, e.g., [48, Theorem 8.49]):

$$\begin{cases} \lambda_j = 0 & \text{for all } j > \min\{i, n-i\}, \\ \lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor} & \text{are all even.} \end{cases} \tag{6.7}$$

Let Θ be a complex finite dimensional SO(n) module which is not necessarily irreducible. The *dual representation* of SO(n) on the dual space Θ^* is defined by

$$(\vartheta\, u^*)(v) = u^*(\vartheta^{-1} v), \qquad \vartheta \in \mathrm{SO}(n),\ u^* \in \Theta^*,\ v \in \Theta.$$

We say that Θ is *self-dual* if Θ and Θ^* are isomorphic representations. The module Θ is called *real* if there exists a non-degenerate symmetric SO(n) invariant bilinear form on Θ. In particular, if Θ is real, then Θ is also self-dual.

Lemma 6.3 ([21]) *Let* $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ *be a tuple of integers satisfying* (6.2)*.*

(a) *If* $n \equiv 2 \mod 4$*, then the irreducible representation* Γ_λ *of* SO(n) *is real if and only if* $\lambda_{n/2} = 0$*.*
(b) *If* $n \equiv 2 \mod 4$ *and* $\lambda_{n/2} \neq 0$*, then the dual of* Γ_λ *is isomorphic to* $\Gamma_{\lambda'}$*, where* $\lambda' = (\lambda_1, \ldots, \lambda_{n/2-1}, -\lambda_{n/2})$*.*
(c) *If* $n \not\equiv 2 \mod 4$*, then all representations of* SO(n) *are real.*

Now, let Γ be again a finite dimensional complex SO(n) module and denote by $\varrho : \mathrm{SO}(n) \to \mathrm{GL}(\Gamma)$ the corresponding representation. The *character* of Γ is the function $\mathrm{char}\,\Gamma : \mathrm{SO}(n) \to \mathbb{C}$ defined by

$$(\mathrm{char}\,\Gamma)(\vartheta) = \mathrm{tr}\,\varrho(\vartheta),$$

where $\mathrm{tr}\,\varrho(\vartheta)$ is the trace of the linear map $\varrho(\vartheta) : \Gamma \to \Gamma$.

The most important property of the character of a complex representation is that it determines the module Γ up to isomorphism. Moreover, several well known properties of the trace map immediately carry over to useful arithmetic properties of characters. For example, if Γ and Θ are finite dimensional SO(n) modules, then

$$\operatorname{char}(\Gamma \oplus \Theta) = \operatorname{char}\Gamma + \operatorname{char}\Theta \tag{6.8}$$

and

$$\operatorname{char}(\Gamma \otimes \Theta) = \operatorname{char}\Gamma \cdot \operatorname{char}\Theta. \tag{6.9}$$

A description of the characters of irreducible representations of compact Lie groups in terms of their highest weights is provided by Weyl's character formula. For our purposes, that is, the case of the special orthogonal group SO(n), a consequence of this description, known as the *second determinantal formula*, is crucial. In order to state this result let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ be a tuple of *non-negative* integers satisfying (6.2). We define the SO(n) module $\overline{\Gamma}_\lambda$ by

$$\overline{\Gamma}_\lambda := \begin{cases} \Gamma_\lambda \oplus \Gamma_{\lambda'} & \text{if } n \text{ is even and } \lambda_{n/2} \neq 0, \\ \Gamma_\lambda & \text{otherwise.} \end{cases}$$

The *conjugate* of λ is the $s := \lambda_1$ tuple $\mu = (\mu_1, \dots, \mu_s)$ defined by letting μ_j be the number of terms in λ that are greater than or equal j.

The second determinantal formula expresses the character of $\overline{\Gamma}_\lambda$ as a polynomial in the characters $E_i := \operatorname{char} \Lambda^i V_{\mathbb{C}}$, $i \in \mathbb{Z}$. (Here and in the following, we use the convention $E_i = 0$ for $i < 0$ or $i > n$.)

Theorem 6.4 ([24]) *Let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ be a tuple of non-negative integers satisfying* (6.2) *and let $\mu = (\mu_1, \dots, \mu_s)$ be the conjugate of λ. The character of $\overline{\Gamma}_\lambda$ is equal to the determinant of the $s \times s$-matrix whose ith row is given by*

$$\left(E_{\mu_i - i+1} \quad E_{\mu_i - i+2} + E_{\mu_i - i} \quad \cdots \quad E_{\mu_i - i+s} + E_{\mu_i - i - s+2} \right). \tag{6.10}$$

In the definition of the conjugate of λ we will later also allow $s > \lambda_1$. Note that this introduces additional zeros at the end of the conjugate tuple but does not change the determinant of the matrix defined by (6.10).

In order to analyze the action of SO(n) on the *infinite dimensional* space Val, we need to briefly discuss the construction of a class of such infinite dimensional representations of a Lie group G induced from closed subgroups $H \subseteq G$ (although we only need the case $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$). To this end, we denote by $C^\infty(G; \Gamma)$ the space of all smooth functions from G to an arbitrary finite dimensional (complex) H module Γ. The *induced representation* of G by H on the space

$$\mathrm{Ind}_H^G\Gamma := \{f \in C^\infty(G;\Gamma) : f(gh) = h^{-1}f(g) \text{ for all } g \in G, h \in H\} \subseteq C^\infty(G;\Gamma)$$

is given by left translation, that is, $(gf)(u) = f(g^{-1}u)$, $g, u \in G$. Conversely, if Θ is any representation of G, we obtain a representation $\mathrm{Res}_H^G\Theta$ of H by restriction. The fundamental *Frobenius Reciprocity Theorem* establishes a connection between induced and restricted representations.

Theorem 6.5 ([32]) *If Θ is a G module and Γ is an H module, then there is a canonical vector space isomorphism*

$$\mathrm{Hom}_G(\Theta, \mathrm{Ind}_H^G\Gamma) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G\Theta, \Gamma).$$

A for our purposes crucial consequence of the Frobenius Reciprocity Theorem (and the definition of multiplicity) is the fact that if Θ and Γ are irreducible, then the multiplicity of Θ in $\mathrm{Ind}_H^G\Gamma$ equals the multiplicity of Γ in $\mathrm{Res}_H^G\Theta$.

In order to apply Theorem 6.5 in the case $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$, we require the following *branching formula* for decomposing $\mathrm{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}\Gamma$ into irreducible $\mathrm{SO}(n-1)$ modules.

Theorem 6.6 ([24]) *If Γ_λ is an irreducible representation of $\mathrm{SO}(n)$ with highest weight tuple $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (6.2), then*

$$\mathrm{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}\Gamma_\lambda = \bigoplus_\mu \Gamma_\mu, \tag{6.11}$$

where the sum ranges over all $\mu = (\mu_1, \dots, \mu_k)$ with $k := \lfloor (n-1)/2 \rfloor$ satisfying

$$\begin{cases} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor} \geq |\mu_k| & \textit{for odd } n, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_k \geq |\lambda_{n/2}| & \textit{for even } n. \end{cases}$$

6.3 Smooth Valuations and the Rumin-de Rham Complex

We discuss in this section a description of translation invariant *smooth* valuations via integral currents and how it relates to induced representations. This will allow us in the next section to apply the machinery from representation theory presented in Sect. 6.2 to prove Theorem 6.2.

First recall that by McMullen's decomposition theorem

$$\mathrm{Val} = \bigoplus_{0 \leq i \leq n} \mathrm{Val}_i^+ \oplus \mathrm{Val}_i^-, \tag{6.12}$$

where $\mathrm{Val}_i^{\pm}$ denote the subspaces of valuations of degree i and even or odd parity, respectively. In the cases $i = 0$ and $i = n$ a simple description of the valuations in Val_i is possible (cf. Chap. 1, Theorem 1.16 and Corollary 1.24).

Proposition 6.7 ([39])

(a) *The space* Val_0 *is one-dimensional and spanned by the Euler characteristic.*
(b) *The space* Val_n *is one-dimensional and spanned by the volume functional.*

Note that statement (a) of Proposition 6.7 is trivial while the non-trivial statement (b) was proved by Hadwiger [39, p. 79]. We also note that Proposition 6.7 directly implies Theorem 6.2 for the cases $i = 0$ and $i = n$.

There is also an explicit description of the valuations in Val_{n-1} going back to McMullen [57] (cf. Chap. 1, Theorem 1.25). However, in this chapter we will not make use of this result and we will therefore not repeat it here. Instead we turn to the notion of smooth valuations. To this end first recall that the space Val becomes a Banach space when endowed with the norm

$$\|\phi\| = \sup\{|\phi(K)| : K \subseteq B^n\}.$$

Here, B^n denotes as usual the Euclidean unit ball. On the Banach space Val there is a natural continuous action of the group $\mathrm{GL}(n)$ given by

$$(A\phi)(K) = \phi(A^{-1}K), \qquad A \in \mathrm{GL}(n),\ \phi \in \mathrm{Val}.$$

Clearly, the subspaces $\mathrm{Val}_i^{\pm} \subseteq \mathrm{Val}$ are $\mathrm{GL}(n)$ invariant. In fact, they are *irreducible* as was shown by Alesker [5] (but we will not use this deep result directly).

Smooth translation invariant valuations were first introduced by Alesker [6]. By definition, they are precisely the smooth vectors (see, e.g. [77]) of the natural representation of $\mathrm{GL}(n)$ on Val.

Definition A valuation $\phi \in \mathrm{Val}$ is called *smooth* if the map $\mathrm{GL}(n) \to \mathrm{Val}$, defined by $A \mapsto A\phi$, is infinitely differentiable.

As usual, we denote by Val^∞ the Fréchet space of smooth translation invariant valuations endowed with the Gårding topology (see, e.g., [80, Section 4.4]) and write Val_i^∞ for its subspaces of smooth valuations of degree i.

By general properties of smooth vectors (cf. [77]), the spaces Val_i^∞ are dense $\mathrm{GL}(n)$ invariant subspaces of Val_i and from (6.12) it is easy to deduce that

$$\mathrm{Val}^\infty = \bigoplus_{0 \le i \le n} \mathrm{Val}_i^\infty.$$

The advantage of considering smooth translation invariant valuations instead of merely continuous ones is that the Fréchet space Val^∞ admits additional algebraic structures. Since these are precisely the topic of Chap. 3, we will discuss here only one structural property of Val^∞ which is crucial for us. To this end, first recall that

McMullen's decomposition (6.12) implies that for any $\phi \in \mathrm{Val}$ and $K \in \mathcal{K}^n$, the function $t \mapsto \phi(K + tB^n)$ is a polynomial of degree at most n. This, in turn, gives rise to a derivation operator $\Lambda : \mathrm{Val} \to \mathrm{Val}$, defined by

$$(\Lambda\phi)(K) = \frac{d}{dt}\Big|_{t=0}\phi(K + tB^n). \tag{6.13}$$

From this definition it follows that if $\phi \in \mathrm{Val}_i$, then $\Lambda\phi \in \mathrm{Val}_{i-1}$, that Λ is continuous, $\mathrm{SO}(n)$ equivariant, and that Λ maps smooth valuations to smooth ones. Moreover, the following *hard Lefschetz theorem* for Λ was proved by Alesker [6] for even and by Bernig and Bröcker [17] for general valuations.

Theorem 6.8 ([6, 17]) *For $\frac{n}{2} < i \leq n$, the map $\Lambda^{2i-n} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$ is an $\mathrm{SO}(n)$ equivariant isomorphism of Fréchet spaces.*

The main tool used in [17] to establish Theorem 6.8 was a description of smooth valuations in terms of the normal cycle map by Alesker [8]. Since a refined version of this result by Bernig [16] (stated below as Theorem 6.9) is critical for the proof of Theorem 6.2, we discuss these results and the necessary background in the following.

Let $SV = V \times S^{n-1}$ denote the unit sphere bundle on the n-dimensional Euclidean vector space V. The natural (smooth) action of $\mathrm{SO}(n)$ on SV is given by

$$l_\vartheta(x, u) := (\vartheta x, \vartheta u), \qquad \vartheta \in \mathrm{SO}(n), (x, u) \in SV. \tag{6.14}$$

Similarly, each $y \in V$ determines a smooth map $t_y : SV \to SV$ by

$$t_y(x, u) = (x + y, u), \qquad (x, u) \in SV. \tag{6.15}$$

The canonical *contact form* α on SV is the one form given by

$$\alpha|_{(x,u)}(w) = \langle u, d_{(x,u)}\pi(w)\rangle, \qquad w \in T_{(x,u)}SV,$$

where $\pi : SV \to V$ denotes the canonical projection and $d_{(x,u)}\pi$ is its differential at $(x, u) \in SV$. Endowed with the contact form α the manifold SV becomes a $2n - 1$ dimensional *contact manifold*. The kernel of α defines the so-called *contact distribution* $Q := \ker\alpha$. Note that the restriction of $d\alpha$ to Q is a non-degenerate two form and, therefore, each space $Q_{(x,u)}$ becomes a symplectic vector space.

Since SV is a product manifold, the vector space $\Omega^*(SV)$ of complex valued smooth differential forms on SV admits a bigrading given by

$$\Omega^*(SV) = \bigoplus_{i,j} \Omega^{i,j}(SV),$$

where $\Omega^{i,j}(SV)$ are the subspaces of $\Omega^*(SV)$ of forms of bidegree (i,j). We denote by $\Omega^{i,j}$ the subspace of translation invariant forms in $\Omega^{i,j}(SV)$, that is,

$$\Omega^{i,j} = \{\omega \in \Omega^{i,j}(SV) : t_y^*\omega = \omega \text{ for all } y \in V\}.$$

The natural (continuous) action of $\mathrm{SO}(n)$ on the vector space $\Omega^{i,j}$ is given by

$$\vartheta\omega = l_{\vartheta^{-1}}^*\omega, \qquad \vartheta \in \mathrm{SO}(n), \omega \in \Omega^{i,j}.$$

Here, t_y^* and $l_{\vartheta^{-1}}^*$ are the pullbacks of the maps defined in (6.14) and (6.15). We also note that the restriction of the exterior derivative d to $\Omega^{i,j}$ has bidegree $(0, 1)$.

For $K \in \mathscr{K}^n$ and $x \in \partial K$, let $N(K, x)$ denote the normal cone of K at x. The *normal cycle* of K is the Lipschitz submanifold of SV defined by

$$\mathbf{nc}(K) = \{(x, u) \in SV : x \in \partial K, u \in N(K, x)\}.$$

For $0 \leq i \leq n-1$, Alesker [8, Theorem 5.2.1] proved that the $\mathrm{SO}(n)$ equivariant map $\nu : \Omega^{i,n-i-1} \to \mathrm{Val}_i^\infty$, defined by

$$\nu(\omega)(K) = \int_{\mathbf{nc}(K)} \omega, \tag{6.16}$$

is *surjective*. However, for our purposes we need a more precise version of this statement which includes, in particular, a description of the kernel of ν first obtained by Bernig and Bröcker [17]. In order to state this refinement, we first have to recall the notion of primitive forms.

Let $\mathfrak{I}^{i,j}$ denote the $\mathrm{SO}(n)$ submodule of $\Omega^{i,j}$ defined by

$$\mathfrak{I}^{i,j} := \{\omega \in \Omega^{i,j} : \omega = \alpha \wedge \xi + d\alpha \wedge \psi,\ \xi \in \Omega^{i-1,j}, \psi \in \Omega^{i-1,j-1}\}.$$

The $\mathrm{SO}(n)$ module $\Omega_p^{i,j}$ of *primitive* forms is defined as the quotient

$$\Omega_p^{i,j} := \Omega^{i,j}/\mathfrak{I}^{i,j}. \tag{6.17}$$

Primitive forms are very important for the study of translation invariant valuations since, by a theorem of Bernig [16], the space Val_i^∞, $0 \leq i \leq n$, fits into an exact sequence of the spaces $\Omega_p^{i,j}$. In order to state Bernig's result precisely, first note that $d\mathfrak{I}^{i,j} \subseteq \mathfrak{I}^{i,j+1}$. Thus, by (6.17), on one hand the exterior derivative induces a linear $\mathrm{SO}(n)$ equivariant operator $d_Q : \Omega_p^{i,j} \to \Omega_p^{i,j+1}$. On the other hand, integration over the normal cycle (6.16) induces a linear map $\nu : \Omega_p^{i,n-i-1} \to \mathrm{Val}_i^\infty$ which, clearly, is also $\mathrm{SO}(n)$ equivariant.

Theorem 6.9 ([16]) *For every $0 \leq i \leq n$, the* SO(n) *equivariant sequence of* SO(n) *modules*

$$0 \longrightarrow \Lambda^i V_{\mathbb{C}} \longrightarrow \Omega_p^{i,0} \xrightarrow{d_Q} \Omega_p^{i,1} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega_p^{i,n-i-1} \xrightarrow{\nu} \mathrm{Val}_i^{\infty} \longrightarrow 0$$

is exact.

In order to apply Theorem 6.9 in the proof of Theorem 6.2, we require an equivalent description of primitive forms involving horizontal forms. To this end, let R denote the *Reeb vector field* on SV defined by $R_{(x,u)} = (u, 0)$. Note that $\alpha(R) = 1$ and that $i_R d\alpha = 0$, where i_R denotes the interior product with the vector field R. The SO(n) submodule $\Omega_h^{i,j} \subseteq \Omega^{i,j}$ of *horizontal* forms is defined by

$$\Omega_h^{i,j} := \{\omega \in \Omega^{i,j} : i_R\omega = 0\}.$$

From this definition it is not difficult to see that the multiplication by the symplectic form $-d\alpha$ induces an SO(n) equivariant linear operator $L : \Omega_h^{i,j} \to \Omega_h^{i+1,j+1}$ which is injective if $i + j \leq n - 2$. Moreover, it follows from (6.17) that in this case

$$\Omega_p^{i,j} = \Omega_h^{i,j} / L\Omega_h^{i-1,j-1}. \tag{6.18}$$

Let us now fix an arbitrary point $u_0 \in S^{n-1}$ and let SO($n-1$) denote the stabilizer of SO(n) at u_0. For $u \in S^{n-1}$, we denote by $W_u := T_u S^{n-1} \otimes \mathbb{C}$ the complexification of the tangent space $T_u S^{n-1}$ and we write W_0 to denote W_{u_0}. The advantage of using description (6.18) instead of definition (6.17) of primitive forms becomes clear from the next lemma which relates horizontal and primitive forms to certain SO(n) representations induced from SO($n-1$).

Lemma 6.10 ([11]) *For $i, j \in \mathbb{N}$, there is an isomorphism of* SO(n) *modules*

$$\Omega_h^{i,j} \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} (\Lambda^i W_0^* \otimes \Lambda^j W_0^*). \tag{6.19}$$

Moreover, if $i + j \leq n - 1$, then there is an isomorphism of SO(n) *modules*

$$\Omega_p^{i,j} \oplus \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} (\Lambda^{i-1} W_0^* \otimes \Lambda^{j-1} W_0^*) \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} (\Lambda^i W_0^* \otimes \Lambda^j W_0^*). \tag{6.20}$$

Note that (6.20) is an immediate consequence of (6.19) and (6.18).

6.4 Proof of the Main Result

With the auxiliary results from the last two sections at hand, we are now in a position to complete the proof of Theorem 6.2. To this end, first recall that the cases $i = 0$ and $i = n$ are immediate consequences of Proposition 6.7. Hence, by Theorem 6.8, we may assume that $n/2 \leq i < n$.

Let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ be a highest weight tuple of $\mathrm{SO}(n)$. As a consequence of (6.20), the multiplicity $m(\Omega_p^{i,j}, \lambda)$ is finite for all $i, j \in \mathbb{N}$ such that $i + j \leq n - 1$. Since, by Theorem 6.9, the spaces Val_i^∞ are quotients of $\Omega_p^{i,n-i-1}$, it follows that also $m(\mathrm{Val}_i^\infty, \lambda)$ is finite. Moreover, since $m(\mathrm{Val}_i, \lambda) = m(\mathrm{Val}_i^\infty, \lambda)$, we deduce from Theorem 6.9 that

$$m(\mathrm{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}}, \lambda) + \sum_{j=0}^{n-i-1} (-1)^{n-1-i-j} m(\Omega_p^{i,j}, \lambda) \tag{6.21}$$

and another application of (6.20) yields

$$m(\Omega_p^{i,j}, \lambda) = m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W \otimes \Lambda^j W), \lambda\big) - m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^{i-1} W \otimes \Lambda^{j-1} W), \lambda\big),$$

where $W \cong W^*$ denotes the complex standard representation of $\mathrm{SO}(n-1)$. In order to further simplify the last expression, we require a consequence of the second determinantal formula, Theorem 6.4. In order to state this simple auxiliary result, let $\#(\lambda, j)$ denote the number of integers in λ which are equal to j.

Lemma 6.11 *If $i, j \in \mathbb{N}$ are such that $n/2 \leq i \leq n$ and $i + j \leq n$, then*

$$E_i E_j - E_{i-1} E_{j-1} = \sum_{\lambda} \mathrm{char}\, \overline{\Gamma}_\lambda, \tag{6.22}$$

where the sum ranges over all tuples of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (6.2) and

$$\lambda_1 \leq 2, \qquad \#(\lambda, 1) = n - i - j, \qquad \#(\lambda, 2) \leq j. \tag{6.23}$$

Proof The conjugate of an $\lfloor n/2 \rfloor$-tuple of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (6.2) and (6.23) is given by $\mu = (\mu_1, \mu_2)$, where $\mu_2 = \#(\lambda, 2) \leq j$ and $\mu_1 - \mu_2 = \#(\lambda, 1) = n - i - j$. Thus, by Theorem 6.4,

$$\mathrm{char}\, \overline{\Gamma}_\lambda = \det \begin{pmatrix} E_{\mu_2+k} & E_{\mu_2+k+1} + E_{\mu_2+k-1} \\ E_{\mu_2-1} & E_{\mu_2} + E_{\mu_2-2} \end{pmatrix},$$

where $k = n - i - j$. Since, by (6.4), $E_{n-i} = E_i$, we therefore obtain for the right hand side of (6.22),

$$\begin{aligned} \sum_{\lambda} \mathrm{char}\, \overline{\Gamma}_\lambda &= \sum_{\mu_2=0}^{j} \big(E_{\mu_2+k}(E_{\mu_2} + E_{\mu_2-2}) - E_{\mu_2-1}(E_{\mu_2+k+1} + E_{\mu_2+k-1})\big) \\ &= E_{n-i} E_j - E_{n-(i-1)} E_{j-1} = E_i E_j - E_{i-1} E_{j-1}. \end{aligned}$$

□

An application of Lemma 6.11 with n replaced by $n-1$ and $0 \leq j \leq n-1-i$ now yields

$$m(\Omega_p^{i,j}, \lambda) = \sum_{\sigma} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_{\sigma}, \lambda\big), \tag{6.24}$$

where the sum ranges over all $k := \lfloor (n-1)/2 \rfloor$-tuples of non-negative highest weights $\sigma = (\sigma_1, \ldots, \sigma_k)$ of $\mathrm{SO}(n-1)$ such that

$$\sigma_1 \leq 2, \qquad \#(\sigma, 1) = n-1-i-j, \qquad \#(\sigma, 2) \leq j.$$

Let $\mathscr{W}_i$ denote the union of these k-tuples of non-negative highest weights of $\mathrm{SO}(n-1)$. By (6.21) and (6.24), we now have

$$m(\mathrm{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}}, \lambda) + \sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_{\sigma}, \lambda\big). \tag{6.25}$$

The Frobenius Reciprocity Theorem (Theorem 6.5), the branching formula from Theorem 6.6, and the definition of $\overline{\Gamma}_{\sigma}$ yield

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_{\sigma}, \lambda\big) = \sum_{\mu} (-1)^{|\mu|},$$

where the sum on the right ranges over all tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_{n-i} = 0$ and

$$\begin{cases} \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq \ldots \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor}^* \geq |\mu_k| & \text{for odd } n, \\ \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq \ldots \geq \mu_k \geq \lambda_{n/2}^* & \text{for even } n. \end{cases}$$

Here, $\lambda_1^* := \min\{\lambda_1, 2\}$ and $\lambda_j^* := |\lambda_j|$ for every $1 < j \leq \lfloor n/2 \rfloor$. Thus, if $\lambda_{n-i+1}^* > 0$, then there is no such tuple μ. However, if $\lambda_{n-i+1}^* = 0$, then

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_{\sigma}, \lambda\big) = \prod_{j=1}^{n-i-1} \sum_{\mu_j = \lambda_{j+1}^*}^{\lambda_j^*} (-1)^{\mu_j}.$$

This product vanishes if the $\lambda_j^*, j = 1, \ldots, n-i$, do not all have the same parity. If the λ_j^* all do have the same parity, then the product above equals $(-1)^{(n-i-1)\lambda_1^*}$. Hence, we obtain for $i > n/2$,

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_{\sigma}, \lambda\big) = \begin{cases} (-1)^{n-i-1} & \text{if } \Gamma_\lambda \cong \Lambda^{n-i} V_{\mathbb{C}}, \\ 1 & \text{if } \lambda \text{ satisfies (i), (ii), (iii)}, \\ 0 & \text{otherwise.} \end{cases}$$

If $i = n/2$, in which case n must be even, then

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\big) = \begin{cases} (-1)^{i-1} & \text{if } \lambda = (1, \ldots, 1, \pm 1), \\ 1 & \text{if } \lambda \text{ satisfies (i), (ii) and (iii)}, \\ 0 & \text{otherwise}. \end{cases}$$

Plugging these expressions into (6.25) and using that $\Lambda^{n/2} V_{\mathbb{C}} = \Gamma_{(1,\ldots,1)} \oplus \Gamma_{(1,\ldots,1,-1)}$ if n is even and $\Lambda^{n-i} V_{\mathbb{C}} \cong \Lambda^i V_{\mathbb{C}}$ for every $i \in \{0, \ldots, n\}$, completes the proof of Theorem 6.2.

6.5 Special Cases and Applications

In this final section we discuss numerous special cases and recent applications of Theorems 6.1 and 6.2. In particular, these results should illustrate the variety of implications that the study of valuations has for different areas.

6.5.1 Special Cases

The following is a list of special cases and immediate consequences of Theorem 6.1.

- If $\Gamma = \Gamma_{(0,\ldots,0)} \cong \mathbb{C}$ is the trivial representation, then the subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}$ coincides with the vector space $\mathrm{Val}^{\mathrm{SO}(n)}$ of all continuous and rigid motion invariant scalar valuations on $\mathscr{K}^n$. By (6.1) and Theorem 6.1, we have

$$\dim \mathrm{Val}^{\mathrm{SO}(n)} = \sum_{i=0}^{n} \dim \mathrm{Val}_i^{\mathrm{SO}(n)} = n + 1.$$

  Together with the fact that intrinsic volumes of different degrees of homogeneity are linearly independent, this yields Hadwiger's characterization theorem.
- Let $\Gamma = \Gamma_{(1,0,\ldots,0)} \cong V_{\mathbb{C}}$ be the complex standard representation of $\mathrm{SO}(n)$. By Theorem 6.1, there is *no* non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant vector valued valuation. While this result seems of no particular interest at first, it directly implies a classical characterization of the Steiner point map by Schneider [65]. Recall that the Steiner point $s(K)$ of a convex body $K \in \mathscr{K}^n$ is defined by

$$s(K) = \frac{1}{n} \int_{S^{n-1}} u h(K, u)\, du,$$

where $h(K,\cdot)$ is the support function of K and du denotes integration with respect to the rotation invariant probability measure on the unit sphere.

Theorem **([65])** *A map* $\phi : \mathscr{K}^n \to \mathbb{R}^n$ *is a continuous, rigid motion equivariant valuation if and only if* ϕ *is the Steiner point map.*

Proof It is well known that $s : \mathscr{K}^n \to \mathbb{R}^n$ has the asserted properties (cf. [69]). Assume that ϕ is another such valuation. Then $\phi - s$ is a continuous, translation *invariant*, and $\mathrm{SO}(n)$ equivariant valuation and, hence, $\phi - s = 0$. ☐

- Next, let $\Gamma = \mathrm{Sym}^k V_{\mathbb{C}}$ be the space of symmetric tensors of rank $k \geq 2$ over $V_{\mathbb{C}}$. The subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}$ is then just the vector space $\mathrm{TVal}^{k,\mathrm{SO}(n)}$ of all continuous, translation invariant and $\mathrm{SO}(n)$ equivariant valuations on $\mathscr{K}^n$ with values in $\mathrm{Sym}^k V_{\mathbb{C}}$.

  By Theorem 6.1 and (6.5), we have, for $1 \leq i \leq n-1$,

$$\dim \mathrm{TVal}_i^{k,\mathrm{SO}(n)} = \begin{cases} k/2 + 1 & \text{if } k \text{ is even,} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

  In order to recall a basis of the space $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$, let $e_1, \ldots, e_n$ be an orthonormal basis of $V_{\mathbb{C}}$ and denote by $Q = \sum_{l=1}^n e_l^2 \in \mathrm{Sym}^2 V_{\mathbb{C}}$ the metric tensor. Moreover, for $s \in \mathbb{N}$, $1 \leq i \leq n-1$ and $K \in \mathscr{K}^n$, let

$$\Psi_{i,s}(K) = \int_{S^{n-1}} u^s \, dS_i(K,u),$$

  where, as usual, u^s denotes the s-fold symmetric tensor product of $u \in S^{n-1}$ and $S_i(K,\cdot)$ denotes the ith area measure of the body K. Then the valuations $Q^r\Psi_{i,s}$, where $r, s \geq 0$, $s \neq 1$, and $2r + s = k$, form a basis of the space $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$. The dimensions and bases of the spaces $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$ and more general spaces of *isometry covariant* tensor valuations were first determined in the articles [3, 41]. For more information, we refer to Chaps. 2, 4, and 9 of this volume.
- In [81], Yang posed the problem to classify valuations compatible with some subgroup of affine transformations with values in *skew-symmetric* tensors of rank two. Using Theorem 6.1, we can give a partial solution to Yang's problem. Taking $\Gamma = \Gamma_{(1,1,0\ldots,0)} = \Lambda^2 V_{\mathbb{C}}$, it follows that there is *no* non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuation with values in Γ.

  In contrast to this negative result, we note that Bernig [14] constructed for each $0 \leq k \leq i \leq n-1$ a family of continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuations of degree i with values in $\Lambda^k V_{\mathbb{C}} \otimes \Lambda^k V_{\mathbb{C}} = \Gamma_{(2,\ldots,2,0,\ldots,0)}$. By Theorem 6.1, Bernig's curvature tensor valuations are (up to scalar multiples) the uniquely determined $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}_i$.

6.5.2 Even Valuations and the Cosine Transform

As already mentioned in the introduction, Theorem 6.2 was first proved for *even* valuations by Alesker and Bernstein [10] using a fundamental relation between even translation invariant valuations and the cosine transform on Grassmannians. This relation will be the topic of this subsection.

First recall that the cosine of the angle between $E, F \in G(n,i)$, $1 \le i \le n-1$, is given by $|\cos(E,F)| = \mathrm{vol}_i(M|E)$, where M is an arbitrary subset of F with $\mathrm{vol}_i(M) = 1$. The *cosine transform* on smooth functions is the $\mathrm{SO}(n)$ equivariant linear operator $\mathrm{C}_i : C^\infty(G(n,i)) \to C^\infty(G(n,i))$ defined by

$$(\mathrm{C}_i f)(F) = \int_{G(n,i)} |\cos(E,F)| f(E)\, dE,$$

where integration is with respect to the Haar probability measure on $G(n,i)$.

Next, we also briefly recall the *Klain map* (for more information, see Chap. 1). For $1 \le i \le n-1$, Klain defined a map $\mathrm{Kl}_i : \mathrm{Val}_i^+ \to C(G(n,i))$, $\phi \mapsto \mathrm{Kl}_i\phi$, as follows: For $\phi \in \mathrm{Val}_i^+$ and every $E \in G(n,i)$, consider the restriction ϕ_E of ϕ to convex bodies in E. This is a continuous translation invariant valuation of degree i in E and, thus, a constant multiple of i-dimensional volume, that is, $\phi_E = (\mathrm{Kl}_i\phi)(E)\,\mathrm{vol}_i$. This gives rise to a function $\mathrm{Kl}_i\phi \in C(G(n,i))$, called the *Klain function* of the valuation ϕ. It is not difficult to see that Kl_i is $\mathrm{SO}(n)$ equivariant and maps smooth valuations to smooth ones. Moreover, by an important result of Klain [45], the Klain map Kl_i is *injective* for every $i \in \{1, \dots, n-1\}$ (see also [62]).

Now, for $1 \le i \le n-1$, consider the map $\mathrm{Cr}_i : C^\infty(G(n,i)) \to \mathrm{Val}_i^{+,\infty}$, defined by

$$(\mathrm{Cr}_i f)(K) = \int_{G(n,i)} \mathrm{vol}_i(K|E) f(E)\, dE.$$

Clearly, Cr_i is an $\mathrm{SO}(n)$ equivariant linear operator. Moreover, if $F \in G(n,i)$, then, for any $f \in C^\infty(G(n,i))$ and convex body $K \subseteq F$,

$$(\mathrm{Cr}_i f)(K) = \mathrm{vol}_i(K) \int_{G(n,i)} |\cos(E,F)| f(E)\, dE.$$

In other words, the Klain function of the valuation $\mathrm{Cr}_i f$ is the cosine transform $\mathrm{C}_i f$ of f. Hence, the image of the cosine transform is contained in the image of the Klain map. From the main result of [10] and an application of the Casselman-Wallach Theorem [22], Alesker [6] proved that, in fact, these images coincide.

Theorem 6.12 ([6, 10]) *Let* $1 \leq i \leq n-1$. *The image of the restriction of the Klain map to smooth valuations* $\mathrm{Kl}_i : \mathrm{Val}_i^{+,\infty} \to C^\infty(G(n,i))$ *coincides with the image of the cosine transform* $\mathrm{C}_i : C^\infty(G(n,i)) \to C^\infty(G(n,i))$.

Theorem 6.12 was essential in the discovery of algebraic structures on the space of continuous translation invariant even valuations (see [6, 7] and Chap. 3 for more detailed information). Using a variant of Theorem 6.12 combined with certain computations from the proof of Alesker's Irreducibility Theorem [5], Alesker and Bernstein [10] gave the following precise description of the range of the cosine transform in terms of the decomposition under the action of $\mathrm{SO}(n)$. This description is equivalent to Theorem 6.2 for even valuations.

Theorem 6.13 ([10]) *Let* $1 \leq i \leq n-1$. *The image of the cosine transform consists of irreducible representations of* $\mathrm{SO}(n)$ *with highest weights* $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ *satisfying* (6.2), (6.7), *and* $|\lambda_2| \leq 2$.

As a concluding remark for this subsection we note that the structural analysis of intertwining transforms on Grassmannians, such as Radon- and cosine transforms, has a long tradition in integral geometry and is still to this day a focus of research (see, e.g., [12, 25–27, 31, 33, 59, 60, 63, 82]).

6.5.3 Unitary Vector Valued Valuations

We have seen in Sect. 6.5.1 that there exists no non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuation from $\mathcal{K}^n$ to $\mathbb{R}^n$. As Wannerer [79] discovered, the situation changes when the translation invariant vector valued valuations are no longer required to be equivariant with respect to $\mathrm{SO}(n)$ but merely with respect to the smaller group $\mathrm{U}(n)$ (for classifications of vector valued valuations in the non-translation invariant case, see [36, 40, 49]).

Since the natural domain of the unitary group $\mathrm{U}(n)$ is $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we consider in this (and only in this) subsection valuations defined on the space $\mathcal{K}^{2n}$ of convex bodies in $\mathbb{R}^{2n}$. In particular, in the following also the spaces Val, Val_i, ... will refer to translation invariant continuous valuations on $\mathcal{K}^{2n}$.

We denote by Vec the (real) vector space of continuous and translation invariant valuations $\phi : \mathcal{K}^{2n} \to \mathbb{C}^n$ and we write $\mathrm{Vec}^{\mathrm{U}(n)}$ for its subspace of $\mathrm{U}(n)$ equivariant valuations. It follows from McMullen's decomposition (6.1) that

$$\mathrm{Vec} = \bigoplus_{0 \leq i \leq 2n} \mathrm{Vec}_i,$$

where as usual Vec_i denotes the subspace of valuations of degree i.

Theorem 6.14 ([79]) *Suppose that* $0 \leq i \leq 2n$. *Then*

$$\dim_{\mathbb{R}} \mathrm{Vec}_i^{\mathrm{U}(n)} = 2 \min \left\{ \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor \right\}. \tag{6.26}$$

Proof We put $V = \mathbb{R}^{2n}$ and write again $V_{\mathbb{C}}$ for the complexification of V. Since Vec_i is isomorphic as vector space to $\mathrm{Val}_i \otimes V$, we have, by Theorem 6.2,

$$\dim_{\mathbb{C}}(\mathrm{Vec}_i \otimes \mathbb{C})^{\mathrm{U}(n)} = \dim_{\mathbb{C}}(\mathrm{Val}_i \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = \sum_{\lambda} \dim_{\mathbb{C}}(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}, \tag{6.27}$$

where the sum ranges over all highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $\mathrm{SO}(2n)$ satisfying

(i) $\lambda_j = 0$ for $j > \min\{i, 2n-i\}$;
(ii) $|\lambda_j| \neq 1$ for $1 \leq j \leq n$;
(iii) $|\lambda_2| \leq 2$.

In order to determine the sum on the right hand side of (6.27), we first apply a formula of Klimyk [47] to $\Gamma_\lambda \otimes V_{\mathbb{C}}$ to obtain the decomposition of this tensor product into $\mathrm{SO}(2n)$ irreducible subspaces:

$$\Gamma_\lambda \otimes V_{\mathbb{C}} = \bigoplus_{\nu} \Gamma_\nu, \tag{6.28}$$

where the sum ranges over all $\nu = \lambda \pm e_k$ for some n-tuple $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$.

Next, a theorem of Helgason (see, e.g., [76, p. 151]) applied to the symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$ implies that the highest weights $\nu = (\nu_1, \ldots, \nu_n)$ we need to consider have to satisfy the following additional condition

$$\begin{cases} \nu_1 = \nu_2 \geq \nu_3 = \nu_4 \geq \cdots \geq \nu_{n-1} = \nu_n & \text{if } n \text{ is even,} \\ \nu_1 = \nu_2 \geq \nu_3 = \nu_4 \geq \cdots \geq \nu_{n-2} = \nu_{n-1} \geq \nu_n = 0 & \text{if } n \text{ is odd.} \end{cases} \tag{6.29}$$

We have

$$\dim_{\mathbb{C}} \Gamma_\nu^{\mathrm{U}(n)} = \begin{cases} 1 & \text{if } \nu \text{ satisfies (6.29),} \\ 0 & \text{otherwise.} \end{cases}$$

From this, conditions (i), (ii), (iii), and (6.28), it follows now that $(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}$ is non-trivial if and only if λ if of the form

$$\lambda_1 = 3, \qquad \lambda_2 = \cdots = \lambda_{2m} = 2, \qquad \lambda_j = 0 \text{ for } j > 2m$$

for some integer $1 \leq m \leq \min\{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{2n-i}{2} \rfloor\}$ and that in this case

$$\dim_{\mathbb{C}}(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = 2.$$

To see this, fix some λ satisfying (i), (ii), and (iii) and suppose that $\nu = \lambda + e_k$ for some k. If ν satisfies in addition (6.29), then necessarily $k = 2$, $\nu_1 = \nu_2 = 3$, and, thus, $\lambda_1 = 3$, $\lambda_2 = \cdots = \lambda_{2m} = 2$, and $\lambda_j = 0$ for $j > 2m$. If $\nu = \lambda - e_k$, then (6.29) forces $k = 1$, $\nu_1 = \nu_2 = 2$, and, again, $\lambda_1 = 3$, $\lambda_2 = \cdots = \lambda_{2m} = 2$, and $\lambda_j = 0$ for $j > 2m$.

Finally, since $\dim_{\mathbb{R}} \mathrm{Vec}_i^{\mathrm{U}(n)} = \dim_{\mathbb{C}}(\mathrm{Vec}_i \otimes \mathbb{C})^{\mathrm{U}(n)}$, we obtain now from (6.27) the desired dimension formula. □

As an application of Theorem 6.14, Wannerer [79] obtained the following new characterization of the Steiner point map in Hermitian vector spaces.

Corollary 6.15 ([79]) *A map $\phi : \mathcal{K}^{2n} \to \mathbb{C}^n$ is a continuous, translation and* U(n) *equivariant map satisfying $\phi(K + L) = \phi(K) + \phi(L)$ for all $K, L \in \mathcal{K}^{2n}$ if and only if ϕ is the Steiner point map.*

Corollary 6.15 is a generalization of a similar result by Schneider [64], where the unitary group U(n) is replaced by the lager group SO($2n$).

6.5.4 The Symmetry of Bivaluations

In this subsection we outline how Theorem 6.2 can be used to prove a remarkable symmetry property of rigid motion invariant continuous bivaluations which in turn has important consequences in geometric tomography and the study of geometric inequalities for Minkowski valuations.

Definition A map $\varphi : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{C}$ is called a *bivaluation* if φ is a valuation in both arguments. We call φ *translation biinvariant* if φ is invariant under independent translations of its arguments and say that φ has *bidegree* (i, j) if $\varphi(aK, bL) = a^i b^j \varphi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $a, b > 0$. If G is some group of linear transformations of $\mathbb{R}^n$, we say φ is G *invariant* provided $\varphi(gK, gL) = \varphi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $g \in G$.

The problem to classify invariant bivaluations was already posed in the book by Klain and Rota [46] on geometric probability. A first such classification was obtained by Ludwig [53] in connection with notions of surface area in normed spaces. However, here we want to discuss a structural property of rigid motion invariant bivaluations. To this end we denote by BVal the vector space of all continuous translation biinvariant complex valued bivaluations. An immediate consequence of McMullen's decomposition (6.1) of the space Val is an analogous result for the space BVal:

$$\mathrm{BVal} = \bigoplus_{i,j=0}^{n} \mathrm{BVal}_{i,j}, \tag{6.30}$$

where $\mathrm{BVal}_{i,j}$ denotes the subspace of all bivaluations of bidegree (i,j). In turn, (6.30) can be used to show that BVal becomes a Banach space when endowed with the norm

$$\|\varphi\| = \sup\{|\varphi(K,L)| : K,L \subseteq B^n\}.$$

The following symmetry property of rigid motion invariant bivaluations was established in [11].

Theorem 6.16 *Let* $0 \leq i \leq n$. *Then*

$$\varphi(K,L) = \varphi(L,K) \qquad \textit{for all } K,L \in \mathcal{K}^n \tag{6.31}$$

holds for all $\mathrm{SO}(n)$ *invariant* $\varphi \in \mathrm{BVal}_{i,i}$ *if and only if* $(i,n) \neq (2k+1, 4k+2)$, $k \in \mathbb{N}$. *Moreover,* (6.31) *holds for all* $\mathrm{O}(n)$ *invariant* $\varphi \in \mathrm{BVal}_{i,i}$.

In the following we outline the proof of the 'if' part of the first statement of Theorem 6.16 using Theorem 6.2. We refer to [11, p. 768], for the construction of an $\mathrm{SO}(n)$ invariant bivaluation $\zeta \in \mathrm{BVal}_{i,i}$, where $(i,n) = (2k+1, 4k+2)$ for some $k \in \mathbb{N}$, such that $\zeta(K,L) \neq \zeta(L,K)$ for some pair of convex bodies. Similarly, we will not treat $\mathrm{O}(n)$ invariant bivaluations here. For the proof of (6.31) in this case, a description of the irreducible representations of $\mathrm{O}(n)$ in terms of the irreducible representations of $\mathrm{SO}(n)$ is needed and we also refer to [11] for that.

Now, assume that $\varphi \in \mathrm{BVal}_{i,i}$ is $\mathrm{SO}(n)$ invariant and that $(i,n) \neq (2k+1, 4k+2)$. Moreover, since for $i = 0$ or $i = n$, (6.31) follows easily from Proposition 6.7, we may assume that $0 < i < n$. Now, by Theorem 6.2,

$$\mathrm{BVal}_{i,i}^{\mathrm{SO}(n)} \cong (\mathrm{Val}_i \otimes \mathrm{Val}_i)^{\mathrm{SO}(n)} \cong \bigoplus_{\gamma,\lambda} (\Gamma_\gamma \otimes \Gamma_\lambda)^{\mathrm{SO}(n)}, \tag{6.32}$$

where the sum ranges of all highest weights γ and λ of $\mathrm{SO}(n)$ satisfying conditions (i), (ii), and (iii) from Theorem 6.1. (In fact, in order to make the isomorphisms in (6.32) precise, we have to consider the dense subset of all bivaluations with finite $\mathrm{SO}(n) \times \mathrm{SO}(n)$ orbit, compare [11, p. 766]).

Since $(i,n) \neq (2k+1, 4k+2)$, it follows from condition (i) and Lemma 6.3, that all irreducible representations of $\mathrm{SO}(n)$ which appear in (6.32) are real and, thus, self-dual. Hence, we have

$$(\Gamma_\gamma \otimes \Gamma_\lambda)^{\mathrm{SO}(n)} \cong \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma, \Gamma_\lambda) \cong \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma \otimes \Gamma_\lambda, \mathbb{C}).$$

Since Γ_γ and Γ_λ are irreducible, Schur's lemma implies that

$$\dim \operatorname{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma, \Gamma_\lambda) = \begin{cases} 1 & \text{if } \gamma = \lambda, \\ 0 & \text{if } \gamma \neq \lambda. \end{cases}$$

Using again that the $\mathrm{SO}(n)$ irreducible representations which we consider are real, the space

$$\operatorname{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda \otimes \Gamma_\lambda, \mathbb{C}) = (\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)} \oplus (\Lambda^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$$

of $\mathrm{SO}(n)$ invariant bilinear forms on Γ_λ must coincide with $(\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$. Hence,

$$\mathrm{BVal}_{i,i}^{\mathrm{SO}(n)} \cong \bigoplus_\lambda (\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$$

which implies (6.31).

Using partial derivation operators on bivaluations [the definition of which is motivated by the operator $\Lambda : \mathrm{Val} \to \mathrm{Val}$ defined in (6.13)], one can easily obtain a corollary of Theorem 6.16 which is particularly useful for applications. To state this result, define for $m = 1, 2$ the operators $\Lambda_m : \mathrm{BVal} \to \mathrm{BVal}$ by

$$(\Lambda_1 \phi)(K, L) = \frac{d}{dt}\Big|_{t=0} \phi(K + tB^n, L)$$

and

$$(\Lambda_2 \phi)(K, L) = \frac{d}{dt}\Big|_{t=0} \phi(K, L + tB^n).$$

Clearly, if $\phi \in \mathrm{BVal}_{i,j}$, then $\Lambda_1 \phi \in \mathrm{BVal}_{i-1,j}$ and $\Lambda_2 \phi \in \mathrm{BVal}_{i,j-1}$.

Also define an operator $\mathrm{T} : \mathrm{BVal} \to \mathrm{BVal}$ by

$$(\mathrm{T}\phi)(K, L) = \phi(L, K).$$

Note that, by Theorem 6.16, the restriction of T to $\mathrm{BVal}_{i,i}^{\mathrm{O}(n)}$ acts as the identity.

Corollary 6.17 *Suppose that $0 \le j \le n$ and $0 \le i \le j$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{BVal}_{j,j}^{\mathrm{O}(n)} & \xrightarrow{\mathrm{T}=\mathrm{Id}} & \mathrm{BVal}_{j,j}^{\mathrm{O}(n)} \\ \downarrow{\scriptstyle \Lambda_2^{j-i}} & & \downarrow{\scriptstyle \Lambda_1^{j-i}} \\ \mathrm{BVal}_{j,i}^{\mathrm{O}(n)} & \xrightarrow{\mathrm{T}} & \mathrm{BVal}_{i,j}^{\mathrm{O}(n)}. \end{array}$$

Corollary 6.17 has found several interesting applications in connection with *Minkowski valuations*, that is, maps $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ such that

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K \cup L$ is convex and addition here is the usual Minkowski addition. Recently, Ludwig [52] started an important line of research concerned with the classification of Minkowski valuations intertwining *linear* transformations, see [1, 2, 34, 50, 53, 73, 78] and Chap. 8. However, first investigations of Minkowski valuations by Schneider [66] in 1974 were concentrating on rigid motion compatible valuations which are still a focus of intensive research, see [43, 71, 72, 74, 75].

In order to explain how Corollary 6.17 can be used in the theory of Minkowski valuations, let MVal denote the set of all continuous and translation invariant Minkowski valuations. Parapatits and the author [61] proved that for any $\Phi \in$ MVal, there exist $\Phi^{(j)} \in$ MVal, where $0 \le j \le n$, such that for every $K \in \mathscr{K}^n$ and $t \ge 0$,

$$\Phi(K + tB^n) = \sum_{j=0}^{n} t^{n-j} \Phi^{(j)}(K).$$

This Steiner type formula shows that an analogue of the operator Λ from (6.13) can be defined for Minkowski valuations $\Lambda :$ MVal $\to$ MVal by

$$h((\Lambda\Phi)(K), u) = \frac{d}{dt}\Big|_{t=0} h(\Phi(K + tB^n, u), \qquad u \in S^{n-1}.$$

For $K, L \in \mathscr{K}^n$, we use $W_i(K, L)$ to denote the mixed volume $V(K[n-i-1], B^n[i], L)$.

Corollary 6.18 ([61]) *Suppose that* $\Phi_j \in \mathbf{MVal}_j$, $2 \le j \le n-1$, *is* $\mathrm{O}(n)$ *equivariant. If* $1 \le i \le j+1$, *then*

$$W_{n-i}(K, \Phi_j(L)) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K))$$

for every $K, L \in \mathscr{K}^n$.

Proof For $K, L \in \mathscr{K}^n$, define $\phi \in \mathbf{BVal}_{j,j}^{\mathrm{O}(n)}$ by $\phi(K, L) = W_{n-1-j}(K, \Phi_j(L))$. Then it is not difficult to show that

$$W_{n-i}(K, \Phi_j(L)) = \frac{(i-1)!}{j!} (\Lambda_1^{j+1-i}\phi)(K, L).$$

Thus, an application of Corollary 6.17 completes the proof. □

Corollary 6.18 as well as variants and generalizations of this result have been a critical tool in the proof of log-concavity properties of rigid motion compatible Minkowski valuations (see [2, 11, 13, 55, 61, 70, 72]). Corollary 6.18 was also important in the solution of injectivity questions for certain Minkowski valuations arising naturally from tomographic data, more precisely, the so-called mean section operators, defined and investigated by Goodey and Weil [28–30].

6.5.5 Miscellaneous Applications

In this short final subsection we collect three more applications of Theorem 6.2 in various contexts. We will not outline proofs here but rather refer to the original source material. We begin with the following fact about Minkowski valuations.

Proposition 6.19 ([11]) *If* $\Phi \in \mathrm{MVal}$ *is* $\mathrm{SO}(n)$ *equivariant, then* Φ *is also* $\mathrm{O}(n)$ *equivariant.*

Note that, by Proposition 6.19, Corollary 6.18 in fact holds for $\mathrm{SO}(n)$ equivariant Minkowski valuations.

The proof of Proposition 6.19 is based on the simple fact that any continuous Minkowski valuation which is translation invariant and $\mathrm{SO}(n)$ equivariant is uniquely determined by a *spherical* valuation.

Definition For $0 \leq i \leq n$, the subspaces $\mathrm{Val}_i^{\mathrm{sph}}$ and $\mathrm{Val}_i^{\infty,\mathrm{sph}}$ of translation invariant continuous and smooth *spherical* valuations of degree i are defined as the closure (w.r.t. the respective topologies) of the direct sum of all $\mathrm{SO}(n)$ irreducible subspaces in Val_i and Val_i^{∞} which contain a non-zero $\mathrm{SO}(n-1)$ invariant valuation.

Since, by Theorem 6.2, the space Val_i is multiplicity free under the action of $\mathrm{SO}(n)$, it follows from basic facts about spherical representations (see [75]) that

$$\mathrm{Val}_i^{\infty,\mathrm{sph}} = \mathrm{cl}_{\infty} \bigoplus_{k \in \mathbb{N}} \Gamma_{(k,0,\ldots,0)}. \tag{6.33}$$

Spherical valuations also play an important role in the recent article [19] by Bernig and Hug, where they compute kinematic formulas for translation invariant and $\mathrm{SO}(n)$ equivariant tensor valuations. It follows from Theorem 6.2 and (6.5) that tensor valuations from $\mathrm{TVal}_i^{\mathrm{SO}(n)}$ are also determined by spherical valuations. In order to bring the algebraic machinery from modern integral geometry into play in the computation of the kinematic formulas in [19], the main step was to determine the Alesker–Fourier transform $\mathbb{F}$ (see [9]) of spherical valuations. Note that since $\mathbb{F}$ is a linear and $\mathrm{SO}(n)$ equivariant map, (6.33) and Schur's lemma imply that the restriction of $\mathbb{F}$ to $\mathrm{Val}_i^{\infty,\mathrm{sph}}$ is determined by a sequence of *multipliers* which was computed in [19].

As a final application of Theorem 6.2, we mention that it was used by Bernig and Solanes [20] to give a complete classification of valuations on the quaternionic plane which are invariant under the action of the group $\mathrm{Sp}(2)\mathrm{Sp}(1)$. Note that since $\mathrm{Sp}(2)\mathrm{Sp}(1)$ contains $-\mathrm{Id}$ all such valuations are even and, thus, determined by their Klain functions. For the proof of their classification theorem, Bernig and Solanes now identify certain Sp(2)Sp(1) invariant functions on the Grassmannian as eigenfunctions of the Laplace-Beltrami operator on $G(n, i)$ and determine the $\mathrm{SO}(n)$ irreducible subspaces that they are contained in. Then Theorem 6.2 is applied to show that these subspaces also appear in Val_i. Finally, the computation of $\dim \mathrm{Val}_i^{\mathrm{Sp}(2)\mathrm{Sp}(1)}$ from [16] is used to show that the so-constructed valuations form a basis.

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Chapter 7
Rotation Invariant Valuations

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Abstract In this chapter, we focus on rotation invariant valuations. We give an overview of the results available in the literature, concerning characterization of such valuations. In particular, we discuss the characterization theorem, derived in Alesker (Ann Math 149:977–1005, 1999), for continuous rotation invariant polynomial valuations on $\mathscr{K}^n$. Next, rotational Crofton formulae are presented. Using the new kinematic formulae for trace-free tensor valuations presented in Chap. 4, it is possible to extend the rotational Crofton formulae for tensor valuations, available in the literature. Principal rotational formulae for tensor valuations are also discussed. These formulae can be derived using locally defined tensor valuations, as introduced in Chap. 2. A number of open questions in rotational integral geometry are presented.

7.1 Preliminaries

The Grassmannian of q-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $G(n,q)$, $0 \leq q \leq n$. For $L \in G(n,q)$, the set $G(L,p)$ is the family of all p-dimensional linear subspaces M incident with L, that is, $M \subset L$ when $p \leq q$ and $L \subset M$, otherwise. The invariant probability measures on these spaces are denoted by ν_q and ν_p^L, respectively. Similarly the space $A(n,q)$ of q-dimensional flats is endowed with the motion invariant measure μ_q, normalized in such a way that

$$\mu_q\left(\{E \in A(n,q) : E \cap B^n \neq \emptyset\}\right) = \kappa_{n-q},$$

where κ_j is the volume of the Euclidean unit ball B^j in $\mathbb{R}^j$. For $E \in A(n,q)$ the family of all p-dimensional flats incident with E is denoted by $A(E,p)$ and endowed with the invariant measure μ_p^E. When $q \geq p$ the measure μ_p^E is obtained by identifying E with $\mathbb{R}^q$ and taking the image measure of μ_p in $\mathbb{R}^q$ using this identification. When

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$q < p$ and $E = L + x$ with $x \in L^\perp$, the measure μ_p^E is the image measure of $\mu_{n-q}^{L^\perp}$ under the mapping $N \mapsto N^\perp + x$.

The subspace determinant $[L, M]$ of two flats L and M is defined in [29, Sect. 14.1]. Let $\mathscr{K}^n$ be the family of convex bodies, that is, of all non-empty compact convex subsets of $\mathbb{R}^n$. For $E \in A(n,q)$ we let $\mathscr{K}_E^q$ be the family of all convex bodies in E. The unit normal bundle of a set of positive reach X is $\mathbf{nc}(X)$. For a definition of the unit normal bundle in the convex case, see Chap. 1, Sect. 1.3.

We will need a norm on the space $\mathbb{T}^p$ of symmetric tensors of rank $p \in \mathbb{N}_0$ and define

$$\|T\| := \sup\{|T(v_1, \dots, v_p)| : \|v_1\|, \dots, \|v_p\| \le 1\}$$

for $T \in \mathbb{T}^p$. Symmetric tensors are defined in Chap. 2, Sect. 2.1.

We will make use of Gauss' hypergeometric function

$$F_{\alpha,\beta;\gamma}(z) := \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, \tag{7.1}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$, $-\gamma \notin \mathbb{N}_0$, where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. The series in (7.1) converges absolutely for $z \in (-1, 1)$ and if $\alpha + \beta < \gamma$ even for $z \in [-1, 1]$. We will later use one of Euler's transformation rules

$$F_{\alpha,\beta;\gamma}(z) = (1-z)^{\gamma-(\alpha+\beta)} F_{\gamma-\alpha,\gamma-\beta;\gamma}(z) \tag{7.2}$$

and the fact that

$$F_{\alpha,\alpha+\frac{1}{2};2\alpha}(z) = (1-z)^{-1/2} \left(\frac{1+\sqrt{1-z}}{2} \right)^{1-2\alpha}, \qquad |z| < 1; \tag{7.3}$$

see, for instance, [7, (8.2.11) and p. 296].

7.2 Rotation Invariant Continuous Valuations on Star Sets

Before describing rotation invariant valuations on the family of convex bodies, we shortly describe a theory of rotation invariant tensor valuations for star sets. With the appropriate definition of star sets, this theory turns out to be rather complete and can serve as a reference for the convex case that still contains a number of open questions. In fact, the richness of the class of star sets considered greatly limits the form of rotation invariant valuations.

A set $S \subset \mathbb{R}^n$ is called *star shaped* if its intersection with an arbitrary line through the origin o is a (possibly degenerate) line-segment. Clearly, a star shaped set S is determined by its *radial function*

$$\rho(S, u) := \sup\{\alpha \in \mathbb{R} : \alpha u \in S\},$$

$u \in \mathbb{S}^{n-1}$. Usually one only works with geometrically defined subclasses of the family of all star shaped sets, and results depend crucially on the subclass chosen. In this survey we restrict considerations exclusively to star shaped sets containing the origin. Note that the results on star bodies in Gardner's monograph [12] do not require this assumption. We base our review on Klain's [23] definition of an $L^n - star$, which is a star shaped set $S \subset \mathbb{R}^n$ that contains the origin and has a finite volume, that is, its radial function is a non-negative element of $L^n(\mathscr{H}^{n-1})$. Here, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Recall that a measurable function f on $\mathbb{S}^{n-1}$ belongs to $L^n(\mathscr{H}^{n-1})$ if

$$\int_{\mathbb{S}^{n-1}} |f(u)|^n \mathrm{d}u < \infty.$$

Here and in the following we write 'du' when integrating with respect to the Hausdorff measure of appropriate dimension. The family of all L^n-stars will be denoted by $\mathscr{S}^n$, and endowed with the topology that is induced by the norm in $L^n(\mathscr{H}^{n-1})$ on $\{\rho(S,\cdot) : S \in \mathscr{S}^n\}$. As usual, one thus identifies L^n-stars when their radial functions coincide up to a set of $\mathscr{H}^{n-1}$-measure zero.

We now discuss examples of continuous $\mathrm{SO}(n)$-invariant valuations on $\mathscr{S}^n$ that take values in the space of tensors of rank $p \in \mathbb{N}_0$. We thus consider continuous tensor valued valuations φ on $\mathscr{S}^n$ satisfying $\varphi(\vartheta S) = \varphi(S)$ for all $\vartheta \in \mathrm{SO}(n)$ and $S \in \mathscr{S}^n$. The first examples that come to mind are the Euler-Poincaré characteristic

$$\chi(S) := 1, \tag{7.4}$$

and, of course, the volume

$$\lambda_n(S) = \int_S 1 \mathrm{d}x, \qquad S \in \mathscr{S}^n, \tag{7.5}$$

yielding tensor valued valuations of rank 0. To obtain higher rank tensors, the constant 1 in (7.4) and in the integrand of (7.5) can be replaced by tensors—in the second case possibly one that depends on x. To retain the rotation invariance, this tensor must depend on x only through $\|x\|$, so we may put

$$\varphi(S) = T + \int_S f(\|x\|)\mathrm{d}x, \tag{7.6}$$

with some fixed $T \in \mathbb{T}^p$ and a suitable function $f : [0,\infty) \to \mathbb{T}^p$. Rewriting (7.6) using polar coordinates gives

$$\varphi(S) = \int_{\mathbb{S}^{n-1}} \theta(\rho(S,u))\,\mathrm{d}u, \tag{7.7}$$

where $\theta : [0, \infty) \to \mathbb{T}^p$ must be continuous, as the restriction of φ on $\{\alpha B^n : \alpha \geq 0\}$ is continuous. Finally, to assure that (7.7) defines a tensor valued mapping on $\mathscr{S}^n$, $\theta(t)$ may not grow faster than t^n as $t \to \infty$; see [24, Lemma 2.2]. With these conditions on θ, we have found all rotation invariant continuous valuations on $\mathscr{S}^n$.

Theorem 7.1 (Klain [24, Theorem 2.8]) *For every* SO(n)*-invariant continuous valuation* $\varphi : \mathscr{S}^n \to \mathbb{T}^p$ *there is a continuous function* $\theta : [0, \infty) \to \mathbb{T}^p$ *with* $\|\theta(t)\| \leq at^n + b$, $t \geq 0$, *for some* $a, b \geq 0$ *such that* (7.7) *holds.*

Conversely, for any θ *as above,* (7.7) *defines an* SO(n)*-invariant continuous valuation on* $\mathscr{S}^n$ *with values in* $\mathbb{T}^p$.

Klain stated this result only for $p = 0$ but it can easily be extended to positive p by pointwise application to the tensors involved. The proof of Theorem 7.1 relies on the fact that the family $\mathscr{S}^n$ is very large. To illustrate the main idea, we restrict considerations to the case where $p = 0$. For any $r > 0$, the functional μ given by

$$\mu(A) = \varphi(\{ta : 0 \leq t \leq r, a \in A\}), \qquad A \in \mathscr{B}(\mathbb{S}^{n-1}), \tag{7.8}$$

is finitely additive. Here and in the following we write $\mathscr{B}(T)$ for the σ-algebra of Borel sets in a topological space T. Continuity and the valuation property of φ imply that μ is σ-additive, and hence μ is a (possibly signed) measure on $\mathbb{S}^{n-1}$. The measure μ inherits the rotation invariance from φ, so μ must be a multiple $\theta(r)$ of the uniform measure on $\mathbb{S}^{n-1}$. The proof is concluded by observing that any element of $\mathscr{S}^n$ can be approximated by finite unions of bounded cones with different r, as defined on the right hand side of (7.8). Concluding, the theory of invariant measures is the backbone of Theorem 7.1.

Consider an SO(n)-invariant continuous valuation $\varphi : \mathscr{S}^n \to \mathbb{T}^p$ with $p = 0$. If φ is homogeneous of degree $\alpha \in \mathbb{R}$, Theorem 7.1 implies that $0 \leq \alpha \leq n$. If $\alpha = i$ is an integer, the associated function θ in (7.7) must be proportional to t^i, and hence

$$\varphi = \frac{\varphi(B^n)}{\kappa_n} \tilde{W}_{n-i}$$

is proportional to the $(n-i)$th *dual quermassintegral*

$$\tilde{W}_{n-i}(S) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(S, u)^i \mathrm{d}u, \qquad S \in \mathscr{S}^n.$$

Of course, Theorem 7.1 also applies to valuations φ_L on the subfamily $\mathscr{S}_L^q$ of all L^q-stars in a fixed subspace $L \in G(n, q)$, when we identify L with $\mathbb{R}^q$, where $q \in \{1, \ldots, n-1\}$. Hence, if $\varphi_L : \mathscr{S}_L^q \to \mathbb{T}^p$ is a continuous valuation that is SO(q)-invariant (with respect to all rotations leaving L fixed), we have

$$\varphi_L(S) = \int_{\mathbb{S}^{n-1} \cap L} \theta_L(\rho(S, u)) \mathrm{d}u, \qquad S \in \mathscr{S}_L^q. \tag{7.9}$$

This opens the door to applications in rotational integral geometry. In many practically relevant cases, φ_L are given for all $L \in G(n, q)$, and are compatible with rotations in the following sense:

$$\varphi_L(S) = \varphi_{\vartheta L}(\vartheta S), \qquad \vartheta \in \mathrm{SO}(n), L \in G(n, q), S \in \mathscr{S}_L^q. \tag{7.10}$$

A family $\{\varphi_L : L \in G(n, q)\}$ of mappings $\varphi_L : \mathscr{S}_L^q \to \mathbb{T}^p$ that satisfies (7.10) is called SO(n)-*compatible*. Note that in this case all φ_L are O(n)-invariant on $\mathscr{S}_L^q$, and if all φ_L are continuous valuations on $\mathscr{S}_L^q$, their associated functions θ_L in (7.9) all coincide. In the following we use that when $S \in \mathscr{S}^n$, then $S \cap L \in \mathscr{S}_L^q$ for almost all $L \in G(n, q)$.

Corollary 7.2 *Let $q \in \{1, \ldots, n-1\}$ and let $\{\varphi_L : L \in G(n, q)\}$ be an* SO(n)-*compatible family of continuous valuations with values in $\mathbb{T}^p$. Let $\theta = \theta_L$ be the joint associated function in* (7.9)*. Then*

$$\int_{G(n,q)} \varphi_L(S \cap L)\nu_q(\mathrm{d}L) = \frac{\omega_q}{\omega_n} \int_{\mathbb{S}^{n-1}} \theta(\rho(S, u))\mathrm{d}u, \qquad S \in \mathscr{S}^n. \tag{7.11}$$

Example 7.3 For fixed $q \in \{1, \ldots, n-1\}$ and $i \in \{0, \ldots, q\}$ the family $\{\tilde{W}_{q-i,L} : L \in G(n, q)\}$ of $(q-i)$th dual quermassintegrals

$$\tilde{W}_{q-i,L} := \frac{1}{q} \int_{\mathbb{S}^{n-1} \cap L} \rho(\,\cdot\,, u)^i \mathrm{d}u,$$

is SO(n)-compatible. Equation (7.11) now reads

$$\int_{G(n,q)} \tilde{W}_{q-i,L}(S \cap L)\nu_q(\mathrm{d}L) = \frac{\kappa_q}{\kappa_n} \tilde{W}_{n-i}(S), \qquad S \in \mathscr{S}^n.$$

This is the dual Kubota integral recursion essentially due to Lutwak [26].

Corollary 7.4 *Let $q \in \{1, \ldots, n-1\}$ and assume that $\varphi : \mathscr{S}^n \to \mathbb{T}^p$ is an* SO(n)-*invariant continuous valuation on $\mathscr{S}^n$ such that the associated function in* (7.7) *satisfies $\|\theta(t)\| \le at^q + b$ for some $a, b \ge 0$. Then φ can be written as a rotational Crofton integral with q-planes:*

$$\int_{G(n,q)} \varphi_L(S \cap L)\nu_q(\mathrm{d}L) = \varphi(S), \qquad S \in \mathscr{S}^n, \tag{7.12}$$

where

$$\varphi_L(S') = \frac{\omega_n}{\omega_q} \int_{\mathbb{S}^{n-1} \cap L} \theta(\rho(S', u))\mathrm{d}u, \qquad S' \in \mathscr{S}_L^q. \tag{7.13}$$

Note that if $\{\varphi_L : L \in G(n,q)\}$ is an SO(n)-compatible family of continuous valuations satisfying (7.12), it must be given by (7.13), as such valuations are determined by their values on balls.

It should be mentioned that there are other, more geometrically motivated notions of star sets in the literature. One common class is defined as the family of all star shaped sets containing the origin and having a continuous radial function. Its elements are called star bodies in [33]. Let $\mathscr{S}$ be the family of all star bodies, endowed with the L^∞-topology, which is induced by the supremum norm of the radial functions. As $\mathscr{S} \subsetneq \mathscr{S}^n$ the above results do not readily apply to valuations on $\mathscr{S}$. However, Villanueva [33] showed that a real-valued SO(n)-invariant L^∞-continuous valuation φ that is in addition non-negative and satisfies $\varphi(\{o\}) = 0$, can be written in the form (7.7) with a non-negative continuous function θ satisfying $\theta(0) = 0$. The converse being obvious, this gives a strengthened version of Theorem 7.1 for star bodies and $p = 0$, but only for non-negative valuations with $\varphi(\{o\}) = 0$. If the latter two conditions are necessary is an open question.

Example 7.5 Let λ_n be the n-dimensional Lebesgue measure. The associated function of the real-valued continuous SO(n)-invariant valuation $\varphi(S) = \lambda_n(S)$ is $\theta(t) = t^n/n$. For $q < n$ there cannot be an SO(n)-compatible family $\{\varphi_L : L \in G(n,q)\}$ of continuous valuations satisfying (7.12), as the joint associated function θ would be $(\omega_n/(n\omega_q))t^n$, which increases faster than t^q as $t \to \infty$. We thus consider φ only on the class of star bodies.

With the same arguments that led to Corollary 7.4, we have for any $q \in \{1,\ldots,n-1\}$ that

$$\int_{G(n,q)} \varphi_L(S \cap L)\nu_q(\mathrm{d}L) = \lambda_n(S),$$

for any star body S, where

$$\varphi_L(S \cap L) = \frac{\omega_n}{n\omega_q} \int_{\mathbb{S}^{n-1}\cap L} \rho(S,u)^n \mathrm{d}u = \frac{\omega_n}{\omega_q} \int_{S\cap L} \|x\|^{n-q} \mathrm{d}x.$$

This is a special case of the rotational Crofton formula for intrinsic volumes in [4]. □

7.3 Rotation Invariant Continuous Valuations on Convex Bodies

We now turn to rotation invariant continuous valuations on the family of convex bodies, endowed with the Hausdorff metric. Throughout the rest of this chapter we assume $n \geq 2$ to avoid peculiarities of the one-dimensional setting.

Clearly, valuations of the form (7.6), restricted to $\mathscr{K}^n$, are examples of continuous $\mathrm{SO}(n)$-invariant valuations, but the family of continuous $\mathrm{SO}(n)$-invariant valuations on $\mathscr{K}^n$ is much richer. One simple example are the intrinsic volumes V_j, $0 < j < n$, they are even motion invariant, but not of the form (7.6).

In the seminal paper [1] by Alesker, characterization theorems for rotation invariant continuous polynomial valuations are derived. A valuation $\varphi : \mathscr{K}^n \to \mathbb{T}^p$ is called *polynomial of degree at most* k if for all $K \in \mathscr{K}^n$ the function $\varphi(K + x)$ is a polynomial in x of degree at most k with coefficients in $\mathbb{T}^p$. If φ is polynomial of degree at most k and $\varphi(K + x)$ is a polynomial in x of exact degree k for at least one $K \in \mathscr{K}^n$, φ is called *polynomial of degree* k.

In [1], a characterization theorem for continuous polynomial rotation invariant valuations is derived, involving the family of valuations given by

$$\varphi_{p,j}(K) := \int_{\mathbf{nc}(K)} p(\|x\|^2, \langle x, u\rangle)\, \Lambda_j(K, \mathrm{d}(x, u)), \tag{7.14}$$

where p is a polynomial in two variables with values in $\mathbb{T}^p$ and $j \in \{0, \dots, n-1\}$. The properties of the support measures $\Lambda_j(K, \cdot)$, listed for instance in [28, Sect. 4.2], imply that $\varphi_{p,j} : \mathscr{K}^n \to \mathbb{T}^p$ is an $\mathrm{O}(n)$-invariant continuous valuation. In addition, $\varphi_{p,j}$ is a polynomial valuation of degree at most $2 \deg p$.

Theorem 7.6 (Alesker [1, Theorem B (i)]) *For every continuous polynomial valuation $\varphi : \mathscr{K}^n \to \mathbb{T}^p$, which is* $\mathrm{SO}(n)$*-invariant if $n \geq 3$ and* $\mathrm{O}(n)$*-invariant if $n = 2$, there exist polynomials $p_0, \dots, p_{n-1}$ in two variables with values in $\mathbb{T}^p$ such that*

$$\varphi = \sum_{j=0}^{n-1} \varphi_{p_j,j}. \tag{7.15}$$

Conversely, any expression of the form (7.15) *defines a continuous polynomial* $\mathrm{O}(n)$*-invariant valuation on $\mathscr{K}^n$ with values in $\mathbb{T}^p$.*

Note that as (7.14) defines an $\mathrm{O}(n)$-invariant valuation, every continuous polynomial $\mathrm{SO}(n)$-invariant valuation is also $\mathrm{O}(n)$-invariant when $n \geq 3$. A characterization theorem for the particular case of continuous polynomial $\mathrm{SO}(2)$-invariant valuations on $\mathscr{K}^2$ can also be found in [1]. As we do not require that the valuations are translation invariant, McMullen's decomposition, one of the main tools of the theory for the class Val in Chap. 1, Sect. 1.4, is not readily available. However, polynomiality of degree at most k implies that φ can be decomposed into a sum of homogeneous valuations with homogeneity degrees in $\{0, 1, \dots, n + k\}$. This follows from a more general result in [21] and is used extensively in [1].

Example 7.7 A very simple example of a continuous polynomial O(n)-invariant valuation on $\mathscr{K}^n$ is

$$\varphi_n^k(K) := \int_K \|x\|^{2k} \mathrm{d}x,$$

where $k \in \mathbb{N}_0$. This valuation is of the form (7.15), since the divergence theorem implies

$$(n+2k)\int_K \|x\|^{2k}\mathrm{d}x = 2\int_{\mathbf{nc}(K)} \|x\|^{2k}\langle x,u\rangle\, \Lambda_{n-1}(K,\mathrm{d}(x,u)),$$

see e.g. [28, p. 316].

The space $V_{n,k}$ of all *real-valued* continuous O(n)-invariant (or, equivalently, SO(n)-invariant when $n \geq 3$) valuations in $\mathbb{R}^n$ that are polynomial of degree at most $k \in \mathbb{N}_0$ is finite dimensional. In fact, Alesker has shown the following decomposition

$$V_{n,k} = W_{n,0} \oplus W_{n,1} \oplus \cdots \oplus W_{n,k},$$

where each subspace $W_{n,q}$ is spanned by valuations of (exact) polynomial degree q. This yields the recursive formula

$$\dim V_{n,k} = \dim V_{n,k-1} + \dim W_{n,k}.$$

As $\dim W_{n,2q+1} = q(n-1)$ and $\dim W_{n,2q} = q(n-1)+(n+1)$, see [1, p. 997], this implies

$$\dim V_{n,2i} = i^2(n-1)+(i+1)(n+1),$$
$$\dim V_{n,2i+1} = i(i+1)(n-1)+(i+1)(n+1),$$

for all $i \in \mathbb{N}_0$, $n \geq 3$. The fact that $\dim V_{n,0} = \dim W_{n,0} = n+1$ is a direct consequence of Hadwiger's theorem, as valuations of polynomial degree zero are translation invariant, and thus $V_0,\ldots,V_n$ forms a basis of $W_{n,0}$. Furthermore, we see $\dim V_{n,1} = \dim V_{n,0} = n+1$, so $W_{n,1}$ is trivial—continuous SO(n)-invariant valuations of polynomial degree exactly one do not exist. Explicit bases for $W_{n,k}$ and hence for $V_{n,k}$ can be constructed from the family of valuations

$$\varphi_j^{r,s}(K) := \int_{\mathbf{nc}(K)} \|x\|^{2r}\langle x,u\rangle^s\, \Lambda_j(K,\mathrm{d}(x,u)), \tag{7.16}$$

for $r,s \in \mathbb{N}_0$, $j=0,\ldots,n-1$. For odd polynomial degree $k=2q+1$, $q \in \mathbb{N}$, the valuations $\varphi_j^{q-i,2i+1}$, $j=1,\ldots,n-1$, $i=1,\ldots,q$, form a basis of $W_{n,2q+1}$. For even polynomial degree $k=2q$, $q \in \mathbb{N}_0$, the valuations $\varphi_j^{q-i,2i}$, $j=1,\ldots,n-1$, $i=0,\ldots,q$, (note that $i=0$ is included now) together with φ_n^q and $\varphi_0^{q,0}$ form a basis of $W_{n,2q}$. (For the definition of φ_n^q, see Example 7.7.) This follows from the facts that the exact polynomial degree of any of these valuations is $2q+1$ and $2q$, respectively, and that these valuations can replace the less explicit ones in [1, Lemma 4.8]. More explicitly for the planar case, a basis of all O(2)-invariant continuous valuations of degree at most $k \in 2\mathbb{N}_0 + 1$ is given by the valuations $\varphi_1^{i,j}$, where the non-negative integers i and j satisfy $2i + j \leq k$, together with $\varphi_0^{2i,0}$, $0 \leq 2i < k$.

From the above it is straightforward to find a basis of the space $V_{n,k}^p$ of all $\mathbb{T}^p$-*valued* continuous O(n)-invariant (or, equivalently, SO(n)-invariant when $n \geq 3$) valuations in $\mathbb{R}^n$, as any $\varphi \in V_{n,k}^p$ can be written as a linear combination of basis vectors in $\mathbb{T}^p$, where the coefficients are in $V_{n,k}$. We only note here that this implies

$$\dim V_{n,k}^p = \dim V_{n,k} \cdot \dim \mathbb{T}^p,$$

where $\dim \mathbb{T}^p = \binom{n+p-1}{p}$.

The valuations in (7.14) are all quasi-smooth. A continuous valuation $\varphi : \mathscr{K}^n \to \mathbb{R}$ is called *quasi-smooth*, if the map on $\mathscr{K}^n$ given by

$$K \mapsto [(t, x) \mapsto \varphi(tK + x)], \qquad t \in [0, 1], x \in \mathbb{R}^n,$$

is a continuous map from $\mathscr{K}^n$ into the space $C^n([0, 1] \times \mathbb{R}^n)$ of n-times continuously differentiable functions on $[0, 1] \times \mathbb{R}^n$. This notion is extended to $\mathbb{T}^p$-valued valuations by assuming quasi-smoothness pointwise i.e. for all real-valued valuations $K \mapsto \varphi(K)(x_1, \dots, x_p)$, $x_1, \dots, x_p \in \mathbb{R}^n$. (Recall that $\varphi(K)$ is a p-linear map from $(\mathbb{R}^n)^p$ to $\mathbb{R}$.)

Alesker [2, 3] showed that any quasi-smooth valuation can be approximated uniformly on any compact subset of $\mathscr{K}^n$ by continuous polynomial valuations. For the understanding of SO(n)-invariant quasi-smooth valuations it is thus sufficient to investigate the valuations $\varphi_{p,j}$, defined in (7.14). There are SO(n)-invariant continuous valuations that are not quasi-smooth, but it is an open problem if all of them can be approximated by continuous polynomial valuations.

Example 7.8 On $\mathscr{K}^2$ the functional

$$\varphi(K) = \int_K \|x\|^{-1} \mathrm{d}x$$

is a real-valued O(2)-invariant continuous valuation (the finiteness of which can be seen by introducing polar coordinates). The valuation φ is a special case of the valuations appearing in Theorem 7.11 below. The valuation is not quasi-smooth. In fact, for $K = [0, 1]^2$ and $s > 0$ an application of the divergence theorem like in Example 7.7 shows that

$$\varphi(K + (s, s)) = -2s \int_s^{1+s} \|(s, y)\|^{-1} \mathrm{d}y + 2(1 + s) \int_s^{1+s} \|(1 + s, y)\|^{-1} \mathrm{d}y.$$

The second derivative of this function of s has a pole at 0, so φ is not quasi-smooth. However, it can be shown that φ can be approximated uniformly on any compact subset of $\mathscr{K}^2$ by continuous polynomial valuations.

Rotational integral geometry for the valuations appearing in the characterization theorems in [1] appears largely unexplored. Below we show, as a new result, how the valuation $\varphi_{n-1}^{r,s}$ defined in (7.16) with s even can be expressed as a rotational average. The assumption that s is even can be omitted when $o \in K$.

Theorem 7.9 *Let $q \in \{2, \dots, n-1\}$, r, s non-negative integers with s even. Then, the valuation $\varphi_{n-1}^{r,s}$ in (7.16) can be written as a rotational Crofton integral with q-planes:*

$$\int_{G(n,q)} \varphi_L^{r,s}(K \cap L)\, \nu_q(\mathrm{d}L) = \varphi_{n-1}^{r,s}(K). \tag{7.17}$$

for all $K \in \mathscr{K}^n$. Here

$$\varphi_L^{r,s}(K') = \frac{\omega_n}{\omega_q} \int_{\mathbf{nc}(K')} \|x\|^{2r+n-q} \langle x, u\rangle^s \times F_{\frac{s-1}{2}, -\frac{n-q}{2}; \frac{q-1}{2}}(\sin^2 \angle(x,u))\, \Lambda_{q-1}^L(K', \mathrm{d}(x,u))$$

is an integral with respect to the generalized curvature measure $\Lambda_{q-1}^L(K', \cdot)$ of $K' \in \mathscr{K}_L^q$ relative to L, ω_n is the surface area of $\mathbb{S}^{n-1}$ and $\angle(x,u)$ is the angle between x and u.

Proof As support measures are weakly continuous and the integrand in the definition of $\varphi_L^{r,s}$ is continuous in (x, u), one can apply an approximation argument. It is thus enough to show the claim for a polytope K for which the union of all support planes of K at the facets does not contain the origin. The variable s is even, so it does not matter if one works with the exterior or the interior normal vectors. It is thus enough to show the claim for one facet, or, equivalently, for all $(n-1)$-dimensional sets $K \in \mathscr{K}^n$. Let $u \in \mathbb{S}^{n-1}$ be one of the unit normals of K at a relative interior point. Then

$$\varphi_{n-1}^{r,s}(K) = \int_K \|x\|^{2r} \langle x, u\rangle^s \mathscr{H}^{n-1}(\mathrm{d}x),$$

and using [16, Proposition 5.4] we find

$$\varphi_{n-1}^{r,s}(K) = \frac{\omega_n}{2} \int_{G(n,1)} \int_{K \cap M} \|x\|^{2r+s+n-1} [u^\perp, M]^{s-1}\, \mathscr{H}^0(\mathrm{d}x)\, \nu_1(\mathrm{d}M).$$

The only analytic function h that satisfies

$$\int_{G(M,q)} h([u^{\perp} \cap L, M]^{s-1})\, \nu_q^M(\mathrm{d}L) = [u^{\perp}, M]^{s-1} \tag{7.18}$$

for all $M \in G(n, 1)$ is given by

$$h(z) = z\, F_{\frac{s-1}{2}, -\frac{n-q}{2}; \frac{q-1}{2}}(1 - z^{\frac{2}{s-1}}). \tag{7.19}$$

The proof of this claim follows closely [16, Sect. 5.6], where the case $s = 0$ is treated. Using (7.18) and interchanging the order of integration we find

$$\varphi_{n-1}^{r,s}(K) = \int_{G(n,q)} \varphi_L^{r,s}(K \cap L)\, \nu_q(\mathrm{d}L),$$

with

$$\begin{aligned}\varphi_L^{r,s}(K \cap L) &= \frac{\omega_n}{2} \int_{G(L,1)} \int_{(K\cap L)\cap M} \|x\|^{2r+s+n-1} h([u^{\perp} \cap L, M]^{s-1})\, \mathscr{H}^0(\mathrm{d}x)\, \nu_1^L(\mathrm{d}M)\\ &= \frac{\omega_n}{\omega_q} \int_{K\cap L} \|x\|^{2r+s+n-q} [u^{\perp} \cap L, M_x]\, h([u^{\perp} \cap L, M_x]^{s-1})\, \mathscr{H}^{q-1}(\mathrm{d}x),\end{aligned}$$

where, at the last equality sign, we have used again [16, Proposition 5.4], but now in L, and we wrote M_x for span$\{x\}$. As $[u^{\perp} \cap L, M_x]$ is the cosine of the angle between x and the unit normal vector of $K \cap L$ in L, this function $\varphi_L^{r,s}$ coincides with the one in the statement of the theorem. □

Rotational integral geometry of intrinsic volumes has been developed during the last decade in a series of papers [4, 5, 14, 19], motivated by the strong interest in such results from local stereology [16]. In the theorem below, we show in the spirit of Corollary 7.4 how the intrinsic volumes can be expressed as rotational averages. A central element in the proof of the theorem is the classical Crofton formula for affine subspaces

$$\int_{A(n,q)} V_j(K \cap E)\, \mu_q(\mathrm{d}E) = \alpha_{n,j,q} V_{n+j-q}(K), \tag{7.20}$$

where

$$\alpha_{n,j,q} := \frac{\binom{q}{j} \kappa_q \kappa_{n+j-q}}{\binom{n}{q-j} \kappa_j \kappa_n},$$

and $0 \le j \le q \le n$; see [28, Sect. 4.4].

Theorem 7.10 (Auneau and Jensen [4], Gual-Arnau et al. [14]) *For $q = 1,\dots,n-1$ and $j = 1,\dots,q$, let $\varphi = V_{n+j-q}$ be the intrinsic volume of homogeneity degree $n+j-q$. Then,*

$$\int_{G(n,q)} \varphi_L(K\cap L)\,\nu_q(\mathrm{d}L) = \varphi(K), \qquad K \in \mathscr{K}^n,$$

where

$$\varphi_L(K') = \frac{\omega_{n-q+1}}{\omega_1}\frac{1}{\alpha_{n,j-1,q-1}} \times \int_{A(L,q-1)} d(o,E)^{n-q}\,V_{j-1}(K'\cap E)\,\mu_{q-1}(\mathrm{d}E), \qquad K' \in \mathscr{K}_L^q, \tag{7.21}$$

and $d(o,E)$ is the distance from o to E. For $j = q$, (7.21) takes the following explicit form

$$\varphi_L(K') = \frac{\omega_n}{\omega_q}\int_{K'} \|x\|^{n-q}\,\mathrm{d}x, \tag{7.22}$$

while for $j = q-1$, (7.21) can equivalently be expressed as

$$\varphi_L(K') = \frac{\omega_n}{\omega_q}\int_{\mathbf{nc}(K')} \|x\|^{n-q} F_{-\frac{1}{2},-\frac{n-q}{2};\frac{q-1}{2}}(\sin^2\angle(x,u))\,\Lambda_{q-1}^L(K',\mathrm{d}(x,u)). \tag{7.23}$$

Note that (7.22) also appears in Example 7.5, while (7.23) is obtained by setting $r = s = 0$ in Theorem 7.9 and noting that $\varphi_{n-1}^{0,0}(K) = 2V_{n-1}(K)$.

Besides the classical Crofton formula, the proof of Theorem 7.10 uses the following Blaschke-Petkantschin formula for a non-negative measurable function f on $A(n,r)$, see [22, Theorem 2.7],

$$\int_{A(n,r)} f(E)\,\mu_r(\mathrm{d}E) = \frac{\omega_{n-r}}{\omega_{q-r}}\int_{G(n,q)}\int_{A(L,r)} d(o,E)^{n-q} f(E)\,\mu_r^L(\mathrm{d}E)\,\nu_q(\mathrm{d}L), \tag{7.24}$$

for $q = 1,\dots,n-1$, $r = 0,\dots,q-1$. This formula, also called the invariator principle in stereology [32], is used to translate the classical Crofton formula, dealing with affine subspaces, into a result for linear subspaces. The details of the proof may be found in [18, p. 239].

The formula for φ_L in (7.21) is not very explicit, but actually useful in local stereology, because a stereological estimator of $V_{n+j-q}(K)$ can be constructed from this formula, involving motion invariant random flats within isotropic random linear subspaces, as explained in Sect. 7.7 below. However, from a theoretical point of view, a more explicit expression for (7.21) would be desirable. To the best of our knowledge, this is an open problem in rotational integral geometry.

In the spirit of Corollary 7.2, we now consider the SO(n)-compatible family $\{\varphi_L : L \in G(n,q)\}$ where

$$\varphi_L(K') = V_j(K'), \qquad K' \in \mathscr{K}_L^q, \tag{7.25}$$

for $q = 1,\dots,n-1$, $j = 0,\dots,q$. In [5, 19], the rotational averages of these sectional valuations are derived. The result is presented in the theorem below.

Theorem 7.11 (Auneau et al. [5], Jensen and Rataj [19]) *Let $q = 1,\dots,n-1$, $j = 0,\dots,q$ and $\{\varphi_L : L \in G(n,q)\}$ be the* SO(n)*-compatible family given by* (7.25)*. Then,*

$$\int_{G(n,q)} \varphi_L(K \cap L)\, \nu_q(\mathrm{d}L) = \varphi(K), \tag{7.26}$$

where for $j = q$

$$\varphi(K) = \frac{\omega_q}{\omega_n} \int_K \|x\|^{-(n-q)}\, \mathrm{d}x.$$

If $o \notin \mathrm{bd}\, K$, then for $j < q$

$$\varphi(K) = \frac{2\omega_q}{\omega_n \omega_{q-j}} \int_{\mathbf{nc}(K)} \|x\|^{-(n-q)} \times \sum_{\substack{I \subset \{1,\dots,n-1\} \\ |I| = q-j-1}} Q_q(x,u,A_I) \frac{\prod_{i \in I} \kappa_i(x,u)}{\prod_{i=1}^{n-1} \sqrt{1+\kappa_i^2(x,u)}} \Lambda_{n-1}(K, \mathrm{d}(x,u)), \tag{7.27}$$

where $\kappa_i(x,u)$, $i = 1,\dots,n-1$, are the principal curvatures of $\mathbf{nc}(K)$ at (x,u). Furthermore, $A_I = A_I(x,u)$ is the $(n-1-|I|)$-dimensional subspace spanned by the principal directions $a_i(x,u)$, $i \notin I$, at $(x,u) \in \mathbf{nc}(K)$, and

$$Q_q(x,u,A_I) := \int_{G(\mathrm{span}\{x\},q)} \frac{[L, A_I]^2}{\|p_L u\|^{q-j}} \nu_q^{\mathrm{span}\{x\}}(\mathrm{d}L),$$

where $p_L u$ is the orthogonal projection of u onto L. If $q = 1$ and $x \perp u$, we set $Q_1(x,u,M) := 0$. For $j = q-1$, (7.27) *takes the following explicit form*

$$\varphi(K) = \frac{\omega_q}{\omega_n} \int_{\mathbf{nc}(K)} \|x\|^{-(n-q)} F_{-\frac{1}{2},\frac{n-q}{2};\frac{n-1}{2}}(\sin^2 \angle(x,u)) \Lambda_{n-1}(K, \mathrm{d}(x,u)). \tag{7.28}$$

The proof of the theorem involves extensive geometric measure theory.

In [5], the explicit form of Q_q has been derived. Generally, $Q_q(x,u,A_I)$ depends on the angle between x and u, *and* the angle between x and A_I. As an example, let

$j = 0$ and $q = n - 1$. Then, by [19, Proposition 3],

$$\varphi(K) = \frac{2}{(n-1)\omega_n} \int_{\mathbf{nc}(K)} \|x\|^{-1} \times \Bigg[\sum_{i=1}^{n-1} R(x,u,a_i(x,u)) \frac{\Pi_{j\neq i}\kappa_j(x,u)}{\prod_{l=1}^{n-1} \sqrt{1+\kappa_l^2(x,u)}} \Bigg] \Lambda_{n-1}(K, \mathrm{d}(x,u)),$$

where

$$R(x,u,a) := \sin^2 \angle(x,a)\Big[\sin^2\theta \, F_{\frac{n-1}{2},\frac{1}{2};\frac{n+1}{2}}(\sin^2\angle(x,u)) + \cos^2\theta \, F_{\frac{n-1}{2},\frac{3}{2};\frac{n+1}{2}}(\sin^2\angle(x,u))\Big],$$

with $\theta = \angle(p_{x^\perp}a, p_{x^\perp}u)$. An application of the Euler-transformation (7.2) yields

$$F_{1,\frac{1}{2};2}(\sin^2\angle(x,u)) = \cos\angle(x,u)\, F_{1,\frac{3}{2};2}(\sin^2\angle(x,u)),$$

and we find for $n = 3$

$$R(x,u,a) = F_{1,\frac{3}{2};2}(\sin^2\angle(x,u)) \sin^2\angle(x,a)[(\sin^2\theta)\cos\angle(x,u) + \cos^2\theta].$$

As

$$\cos\theta = \frac{\cos\angle(a,x)\cos\angle(x,u)}{\sin\angle(a,x)\sin\angle(x,u)},$$

trigonometric identities give

$$R(x,u,a) = F_{1,\frac{3}{2};2}(\sin^2\angle(x,u)) \times \left[(\sin^2\angle(x,a))\cos\angle(x,u) + 2\cos^2\angle(x,a)\frac{\cos^2\angle(x,u)}{\sin^2\angle(x,u)} \sin^2\frac{\angle(x,u)}{2} \right],$$

where $F_{1,\frac{3}{2};2}$ can be simplified using (7.3) with $\alpha = 1$ as

$$F_{1,\frac{3}{2};2}(z) = (1-z)^{-1/2} \left(\frac{1+\sqrt{1-z}}{2} \right)^{-1}.$$

Summarizing, we find for $n = 3$, $q = 2$ and $j = 0$ that (7.27) reduces to

$$\varphi(K) = \frac{1}{8\pi}\int_{\mathbf{nc}(K)} \|x\|^{-1}\Bigg[\sum_{i=1}^{2}\frac{\kappa_{3-i}(x,u)}{\Pi_{l=1}^{2}\sqrt{1+\kappa_l^2(x,u)}}\Bigg[\sin^2\angle(x,a_i(x,u))\cos^{-2}\frac{\angle(x,u)}{2} + 2\cos^2\angle(x,a_i(x,u))\frac{\cos\angle(x,u)}{\sin^2\angle(x,u)}\tan^2\frac{\angle(x,u)}{2}\Bigg]\Bigg]\mathscr{H}^2(\mathrm{d}(x,u)). \tag{7.29}$$

We conclude these considerations with a remark on SO(n)-invariant valuations in the context of the above rotational formulae. When $\varphi_L = V_j$ is an intrinsic volume, the left hand side of (7.26) defines a real-valued SO(n)-invariant valuation φ. In the case of the Euler characteristic, $j = 0$, the valuation φ is not continuous, as can be seen considering a non-constant sequence of singletons converging to $\{o\}$. Using the upper semi-continuity of the intersection operation one can show that φ is continuous for $j \geq 1$. One may ask if this valuation can be approximated by polynomial ones. Due to Weierstrass' approximation theorem the hypergeometric function in (7.28) can uniformly be approximated by polynomials on $[-1, 1]$. As a consequence, the valuation in (7.28) is a locally uniform limit of continuous SO(n)-invariant polynomial valuations by Theorem 7.6. In contrast to this, the valuation in (7.27) is for $j \leq q-2$ an integral over the unit normal bundle where the integrand depends on the principal directions of $\mathbf{nc}(K)$. It was therefore conjectured in [5] that such valuations are not locally uniform limits of continuous SO(n)-invariant polynomial valuations even if $j \geq 1$. The lowest dimensional example of this kind occurs for $n = 4$, $q = 3$ and $j = 1$. The mentioned problem is still open.

7.4 Rotational Crofton Formulae for Minkowski Tensors

Rotational Crofton formulae for Minkowski tensors have recently been derived in [6, 31].

To express Minkowski tensors as rotational averages, we need to generalize Theorem 7.10. An important element in the proof of Theorem 7.10 is the classical Crofton formula (7.20). In [15], (7.20) is generalized to the case of Minkowski tensors. (For the definition of Minkowski tensors, see Chap. 2, Sect. 2.1.) It turns out that the formula for Minkowski tensors derived in [15] is considerably more complicated than the classical Crofton formula, but for Minkowski tensors $\Phi_k^{r,0}$ it takes a sufficiently simple form so that the proof of Theorem 7.10 carries over. For a convex body K contained in a flat $E \subset \mathbb{R}^n$, there are variants of the Minkowski tensors denoted by $\Phi_j^{r,s(E)}(K)$. These Minkowski tensors are again tensor valuations of rank $r + s$ in $\mathbb{R}^n$, but they are calculated with respect to the support measures of K in E; see the beginning of [15, Sect. 3] for details. They coincide with $\Phi_{j,E}^{r,s}(K)$ in Sect. 5.2 apart from a different normalization.

Theorem 7.12 (Auneau-Cognacq et al. [6, Corollary 4.4]) *For $q = 1, \dots, n-1$, $j = 1, \dots, q$ and r a non-negative integer, let $\varphi = \Phi^{r,0}_{n-q+j}$ be the tensor of rank r with $s = 0$ and index $n - q + j$. Then,*

$$\int_{G(n,q)} \varphi_L(K \cap L)\, \nu_q(\mathrm{d}L) = \varphi(K), \qquad K \in \mathscr{K}^n,$$

where

$$\varphi_L(K') = \frac{\omega_{n-q+1}}{\omega_1} \frac{1}{\alpha_{n,j-1,q-1}} \times \int_{A(L,q-1)} d(o,E)^{n-q}\, \Phi^{r,0(L)}_{j-1}(K' \cap E)\, \mu_{q-1}(\mathrm{d}E), \qquad K' \in \mathscr{K}^q_L. \tag{7.30}$$

For $j = q$, (7.30) takes the following explicit form

$$\varphi_L(K') = \frac{\omega_n}{\omega_q} \frac{1}{r!} \int_{K'} x^r \|x\|^{n-q}\, \mathrm{d}x, \tag{7.31}$$

while for $j = q - 1$, (7.30) can equivalently be expressed as

$$\varphi_L(K') = \frac{\omega_n}{\omega_q} \frac{1}{r!} \int_{\mathbf{nc}(K')} x^r \|x\|^{n-q} F_{-\frac{1}{2}, -\frac{n-q}{2}; \frac{q-1}{2}} (\sin^2 \angle(x,u))\, \Lambda^L_{q-1}(K', \mathrm{d}(x,u)). \tag{7.32}$$

A result of the type (7.30) can also be established for $\Phi^{r,1}_{n-q+j}$, see [6, Corollary 4.4], but here explicit expressions for φ_L for $j = q$ and $j = q - 1$ are not available.

Surface tensors $\Phi^{0,s}_k$ are studied in [25]. In [25, Theorem 3.4], $\Phi^{0,s}_{n-1}(K)$ is expressed for even s as a Crofton-integral with respect to lines $E \in A(n,1)$, involving an explicitly known tensor $G_s(\pi(E))$ of rank s. Here, $\pi(E)$ is the line through the origin parallel to E. By combining this result with (7.24), $\Phi^{0,s}_{n-1}(K)$ can for even s be expressed as a rotational integral. We get for $q = 2, \dots, n-1$

$$\Phi^{0,s}_{n-1}(K) = \int_{G(n,q)} \varphi_L(K \cap L)\, \nu_q(\mathrm{d}L),$$

where

$$\varphi_L(K') = \frac{\omega_{n-1}}{\omega_{q-1}} \int_{A(L,1)} d(o,E)^{n-q} G_s(\pi(E)) V_0(K \cap E)\, \mu^L_1(\mathrm{d}E).$$

As is apparent from the discussion above, it is an open problem to express Minkowski tensors with general indices as rotational averages. One possible route to follow for the tensors $\Phi^{0,s}_k$ with arbitrary non-negative integer s is to use the recently established kinematic formula [8, Corollary 6.1] for trace-free tensors Ψ_k^s in combination with the Blaschke-Petkantschin formula (7.24). For $k, l \geq 0$, $k + l \leq n$ and $n < l + p$, we get

$$\frac{\omega_{s+k+l}}{\omega_{s+k}\omega_l}\binom{k+l}{k}\frac{kl}{k+l}\begin{bmatrix} n \\ l \end{bmatrix}^{-1}\Psi_{k+l}^s(K) = \int_{A(n,n-l)} \Psi_k^s(K \cap E)\mu_{n-l}(\mathrm{d}E)$$
$$= \int_{G(n,p)} \alpha_{p,k,l}^s(K, L)\nu_p(\mathrm{d}L),$$

where

$$\begin{bmatrix} n \\ l \end{bmatrix} := \binom{n}{l}\frac{\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{n-l}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

and

$$\alpha_{p,k,l}^s(K, L) = \frac{\omega_l}{\omega_{p-n+l}}\int_{A(L,n-l)} \Psi_k^s(K \cap E)d(o, E)^{n-p}\mu_L(\mathrm{d}E).$$

Combining this with the fact that $\Phi_k^{0,s}$ can be expressed in terms of $\Psi_k^0, \dots, \Psi_k^s$ [8, Proposition 4.16] it can be seen that any translation invariant Minkowski tensor $\Phi_k^{0,s}$, $2 \leq k \leq n-1$, $s \in \mathbb{N}_0$, can be written as a non-trivial rotational Crofton integral. To the best of our knowledge, explicit general formulae cannot be found in the literature. It is an open problem to express the more general Minkowski tensors $\Phi_k^{r,s}$ as rotational averages.

The situation is much more clear for rotational averages of Minkowski tensors, due to the recent work of Svane and Jensen [31]. Using the same techniques as in [19], Theorem 7.11 can be generalized as follows, where it should be noted that the integrand of the function Q_q in Theorem 7.13 below takes a more general form than in Theorem 7.11.

Theorem 7.13 (Svane and Jensen [31]) *Let $q = 1, \dots, n-1$, $j = 0, \dots, q$, r, s non-negative integers and let $\{\varphi_L : L \in G(n, q)\}$ be the* SO(n)*-compatible family given by*

$$\varphi_L(K') = \Phi_j^{r,s(L)}(K'), \qquad K' \in \mathcal{K}_L^q.$$

Then,

$$\int_{G(n,q)} \varphi_L(K \cap L)\,\nu_q(\mathrm{d}L) = \varphi(K), \qquad K \in \mathcal{K}^n,$$

where for $j = q$ *and* $s = 0$

$$\varphi(K) = \frac{1}{r!}\frac{\omega_q}{\omega_n}\int_K x^r \|x\|^{-(n-q)}\,\mathrm{d}x.$$

If $K \in \mathcal{K}^n$ *contains* o *in its interior, then for* $j < q$

$$\varphi(K) = \frac{1}{r!s!}\frac{2\omega_q}{\omega_n\omega_{q-j+s}}\int_{\mathbf{nc}(K)} x^r \|x\|^{-(n-q)} \times \sum_{\substack{I\subset\{1,\dots,n-1\}\\ |I|=q-j-1}} Q_q(x,u,A_I)\frac{\prod_{i\in I}\kappa_i(x,u)}{\prod_{i=1}^{n-1}\sqrt{1+\kappa_i^2(x,u)}}\Lambda_{n-1}\left(\mathrm{d}(x,u)\right), \tag{7.33}$$

where

$$Q_q(x,u,A_I) = \int_{G(\mathrm{span}\{x\},q)} (p_L u)^s \frac{[L,A_I]^2}{\|p_L u\|^{q-j+s}}\nu_q^{\mathrm{span}\{x\}}(\mathrm{d}L).$$

For $j = q-1$*, (7.33) takes the following explicit form*

$$\varphi(K) = \frac{2}{r!s!\omega_{s+1}}\frac{\omega_q\omega_{q-1}\omega_{n-q}}{\omega_n\omega_{n-1}\omega_{n-2}}\sum_{a+b+c+2l=s}\binom{s}{a,b,c,2l}\frac{\omega_{2l+n-2}}{\omega_{2l+1}} \times \sum_{e+f+t+v=l}\binom{l}{e,f,t,v}(-1)^{f+v+b}2^{t+1}Q^e \times \int_{\mathbf{nc}(K)} u^{c+2f+t}\frac{x^{r+a+b+2v+t}}{\|x\|^{n-q+a+b+2v+t}}g(\sin^2\angle(x,u))\Lambda_{n-1}(K;\mathrm{d}(x,u)),$$

where $Q \in \mathbb{T}^2$ *is the metric tensor and*

$$g(\alpha^2) = \frac{\omega_{n-1+2b+2c+4l}}{\omega_{q-1+2b+2c+2l}\omega_{n-q+2l}}\alpha^{2e}(1-\alpha^2)^{\frac{a+b+t}{2}}F_{\frac{s-1}{2},\frac{n-q}{2}+l;\frac{n-1}{2}+b+c+2l}(\alpha^2).$$

We finally mention that a recently derived kinematic Crofton formula for area measures (see Theorem 4.4 or [13]) can also be combined with the Blaschke-Petkantschin formula (7.24) in order to obtain a rotational Crofton-type representation of the surface area measure $S_k(K,\cdot)$ of K with index $2 \le k \le n-1$.

7.5 Uniqueness of the Measurement Function

Let $K \in \mathscr{K}^n$ and $q \in \{1, \dots, n-1\}$ be given. The rotational Crofton formulae in Sect. 7.4 all read

$$\int_{G(n,q)} \varphi_L(K \cap L)\, \nu_q(\mathrm{d}L) = \varphi(K), \tag{7.34}$$

where φ is some tensor valued valuation and the functionals φ_L are tensor valued valuations on $\mathscr{K}_L^q$ for all $L \in G(n,q)$. As φ_L is the quantity we have to measure in order to obtain a desired isotropic average, we refer to φ_L as the *measurement function*. In [14] it was asked if this measurement function is unique under appropriate additional assumptions when the right hand side of (7.34) is an intrinsic volume of K. This question was motivated by the observation that two apparently different measurement functions that satisfy (7.34) with $\varphi = V_n$ actually coincide. In fact, also the following result on surface area estimation appears to support uniqueness of the measurement function. Theorem 7.10 implies that (7.34) with $\varphi = V_{n-1}$ holds with φ_L given by (7.23). About a decade before Theorem 7.10 was established, a Blaschke-Petkantschin formula was used in [16, Sect. 5.6] to show that the apparently different measurement function

$$\varphi_L(K') = \frac{1}{2}\frac{\omega_n}{\omega_q}\int_{\mathbb{S}^{n-1}\cap L} \rho_K^{n-1}(u)\frac{1}{\cos\gamma_L(u)} F_{-\frac{1}{2},-\frac{n-q}{2},\frac{q-1}{2}}(\sin^2\gamma_L(u))\mathrm{d}u, \quad K' \in \mathscr{K}_L^q, \tag{7.35}$$

also satisfies (7.34) if $o \in \operatorname{int} K$. Here $\gamma_L(u)$ is the angle between u and the (almost everywhere unique) outer unit normal in L of K' at its boundary point $u\rho_{K'}(u)$. A closer examination reveals that the measurement functions (7.23) and (7.35) actually coincide when $q = 2$; see [9] for a proof in the case of strictly convex and smooth $K \subset \mathbb{R}^3$ and [32] for the general case.

Using the linearity of the integral, the original uniqueness question can equivalently be rephrased by asking under what conditions

$$\int_{G(n,q)} \varphi_L(K \cap L)\, \nu_q(\mathrm{d}L) = 0 \tag{7.36}$$

implies that all measurement functions φ_L are vanishing.

In contrast to the convex case, the corresponding question for measurement functions on L^n-stars is not difficult: We have already noted after Corollary 7.4 that an $\mathrm{SO}(n)$-compatible family $\{\varphi_L : L \in G(n,q)\}$ of continuous valuations is uniquely determined when φ_L is known on all balls in L, so (7.36) implies that all φ_L vanish. When $q = 1$ any member of an $\mathrm{SO}(n)$-compatible family $\{\varphi_L : L \in G(n,q)\}$ of functionals $\varphi_L : \mathscr{S}_L^n \to \mathbb{R}$ (without any further assumptions) must vanish when (7.36) holds. In fact, one only has to show that φ_L vanishes on all line-segments in L that contain the origin. However, this is a direct consequence of (7.36) applied to the sets

$$K = rB^n \cup \{x \in RB^n : \langle x, w\rangle \geq 0\}, \tag{7.37}$$

for $0 \leq r \leq R$, $w \in \mathbb{S}^{n-1}$, and the $\mathrm{SO}(n)$-compatibility.

The question under what conditions (7.36) determines φ_L *in the convex case* is widely open apart from the following result on one-dimensional sections.

Theorem 7.14 *Let* $\{\varphi_L : L \in G(n,1)\}$ *be an* $\mathrm{SO}(n)$*-compatible family of functionals* φ_L *on* $\mathscr{K}_L^n$*. Then* (7.36) *implies that* $\varphi_L = 0$ *for all* $L \in \mathscr{L}_1^n$*.*

The proof of this new result uses the convex hull K_1 of K in (7.37) and the intersection K_2 of all closed supporting half spaces of K_1 that contain a point of $\{x \in RB^n : \langle x, w\rangle \geq 0\}$ in their boundaries. An explicit calculation and comparison of (7.36) with K_1 and K_2 replacing K then yields the assertion.

7.6 Principal Rotational Formulae

A principal rotational formula for Minkowski tensors may involve integrals of the form

$$\int_{\mathrm{SO}(n)} \Phi_k^{r,s}(K \cap \vartheta M)\, \nu(\mathrm{d}\vartheta),$$

for $k = 0,\ldots,n$, $r,s \in \mathbb{N}_0$, where $K, M \in \mathscr{K}^n$ and ν is the unique rotation invariant probability measure on $\mathrm{SO}(n)$. In local stereology, principal rotational formulae are used in cases where an unknown spatial structure K is studied via the intersection with a randomly rotated set M. In such applications, M is a known 'sampling window' constructed by the observer.

In this section, we consider principal rotational formulae for general Minkowski tensors. Some formulae of this type already appeared in [17]. The more complete formulae presented below are new.

It turns out that local Minkowski tensors are an important tool in the derivation of such formulae. We use a slightly more general definition than the one given in Chap. 2, Sect. 2.3. Thus, for $K \in \mathscr{K}^n$, r,s non-negative integers and $k = 0,\ldots, n-1$, the local Minkowski tensors are defined by

$$\Phi_k^{r,s}(K,\psi) := \frac{\omega_{n-k}}{r!\,s!\,\omega_{n-k+s}} \int_{\mathbf{nc}(K)} \psi(x,u)\, x^r u^s \Lambda_k(K, \mathrm{d}(x,u)) \tag{7.38}$$

and

$$\Phi_n^{r,0}(K,\phi) := \frac{1}{r!} \int_K \phi(x) x^r \,\mathrm{d}x, \tag{7.39}$$

where ψ and ϕ are non-negative measurable functions on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ and $\mathbb{R}^n$, respectively. The classical Minkowski tensors are obtained in (7.38) and (7.39) by choosing the functions ψ and ϕ identically equal to 1. We remark for later use that the rotation group acts on the corresponding function spaces in the natural way: for $\vartheta \in \mathrm{SO}(n)$, let $(\vartheta\psi)(x,u) = \psi(\vartheta^{-1}x, \vartheta^{-1}u)$ and $(\vartheta\phi)(x) = \phi(\vartheta^{-1}x)$, $x \in \mathbb{R}^n, u \in \mathbb{S}^{n-1}$. We define the rotational average

$$\overline{\psi}(x,u) := \int_{\mathrm{SO}(n)} \psi(\vartheta x, \vartheta u)\, \nu(\mathrm{d}\vartheta)$$

and likewise for $\bar{\phi}$. The same definition can also be applied to functions ψ and ϕ with values in $\mathbb{T}^p$.

A simple application of Tonelli's theorem yields the following result for local Minkowski tensors.

Proposition 7.15 *Let ψ and ϕ be non-negative measurable functions on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ and $\mathbb{R}^n$, respectively. Then, for $K \in \mathscr{K}^n$, r, s non-negative integers and $k = 0, \ldots, n-1$,*

$$\int_{\mathrm{SO}(n)} \Phi_n^{r,0}(K, \vartheta\phi)\, \nu(\mathrm{d}\vartheta) = \Phi_n^{r,0}(K, \bar{\phi})$$

and

$$\int_{\mathrm{SO}(n)} \Phi_k^{r,s}(K, \vartheta\psi)\, \nu(\mathrm{d}\vartheta) = \Phi_k^{r,s}(K, \overline{\psi}).$$

As a consequence of Proposition 7.15, we have the following principal rotational formula for local Minkowski tensors. We slightly abuse notation using $\mathbf{1}_A(x,u) := \mathbf{1}_A(x)$ for the indicator function of a set $A \subset \mathbb{R}^n$ and $\phi_M(x,u) := \phi_M(x)$ for the function ϕ_M defined below, where $(x,u) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$.

Theorem 7.16 *Let $K, M \in \mathscr{K}^n$ and*

$$\phi_M(x) := \frac{\mathscr{H}^{n-1}(\mathrm{int}\, M \cap \|x\|\mathbb{S}^{n-1})}{\mathscr{H}^{n-1}(\|x\|\mathbb{S}^{n-1})},$$

if $x \in \mathbb{R}^n \setminus \{o\}$, and $\phi_M(o) := \mathbf{1}_{\mathrm{int}\, M}(o)$. Then, for any non-negative integer r we have

$$\int_{\mathrm{SO}(n)} \Phi_n^{r,0}(K \cap \vartheta M)\, \nu(\mathrm{d}\vartheta) = \Phi_n^{r,0}(K, \phi_M). \tag{7.40}$$

If, in addition, $k = 0, \dots, n-1$, *and* $s \in \mathbb{N}_0$ *we have*

$$\int_{\mathrm{SO}(n)} \Phi_k^{r,s}(K \cap \vartheta M, \vartheta \mathbf{1}_{\mathrm{int}\,M})\, \nu(\mathrm{d}\vartheta) = \Phi_k^{r,s}(K, \phi_M). \tag{7.41}$$

When $k = n-1$ *and* $\mathscr{H}^{n-1}(\mathrm{bd}\, K \cap \vartheta\, \mathrm{bd}\, M) = 0$ *for almost all* $\vartheta \in \mathrm{SO}(n)$*, this implies*

$$\int_{\mathrm{SO}(n)} \Phi_{n-1}^{r,s}(K \cap \vartheta M)\, \nu(\mathrm{d}\vartheta)$$
$$= \Phi_{n-1}^{r,s}(K, \phi_M) + \frac{1}{r!s!}\frac{2}{\omega_{s+1}} \int_{\mathbf{nc}(M)} \overline{\mathbf{1}_{\mathrm{int}\,K} x^r u^s}\, \Lambda_{n-1}(M, \mathrm{d}(x,u)). \tag{7.42}$$

Proof As $\overline{\mathbf{1}_{\mathrm{int}\,M}} = \phi_M$ and

$$\Phi_n^{r,0}(K \cap \vartheta M) = \Phi_n^{r,0}(K, \vartheta \mathbf{1}_{\mathrm{int}\,M}),$$

(7.40) follows directly from Proposition 7.15. Support measures are locally defined (see Sect. 2.3), so $\Lambda_k(K \cap \vartheta M, \eta) = \Lambda_k(K, \eta)$ for the open set $\eta = (\mathrm{int}\,\vartheta M) \times \mathbb{S}^{n-1}$. This implies

$$\Phi_k^{r,s}(K \cap \vartheta M, \vartheta \mathbf{1}_{\mathrm{int}\,M}) = \Phi_k^{r,s}(K, \vartheta \mathbf{1}_{\mathrm{int}\,M})$$

and Proposition 7.15 yields (7.41). To show (7.42) an application of the facts that support measures are locally defined together with the additional assumption yields

$$\Lambda_{n-1}(K \cap \vartheta M, \cdot\,) = \Lambda_{n-1}(K, \cdot \cap (\vartheta(\mathrm{int}\,M) \times \mathbb{S}^{n-1}))$$
$$+ \Lambda_{n-1}(\vartheta M, \cdot \cap ((\mathrm{int}\,K) \times \mathbb{S}^{n-1}))$$

for almost all ϑ. Integrating $x^r u^s$ with this measure, applying (7.41) and using again the fact that support measures are locally defined to simplify the second term, yields (7.42). □

Proposition 7.15 may also be used to derive a principal rotational formula where Minkowski tensors are expressed as rotational averages. The result is given in the theorem below.

Theorem 7.17 *Let* $K, M \in \mathscr{K}^n$*. Suppose that* M *is chosen such that* $o \in \mathrm{int}\,M$ *and that*

$$\mathscr{H}^{n-1}(\mathrm{int}\,M \cap \|x\|\mathbb{S}^{n-1}) > 0$$

for all $o \neq x \in K$. Let

$$\phi_M^\circ(x) := \frac{\mathscr{H}^{n-1}(\|x\|\mathbb{S}^{n-1})}{\mathscr{H}^{n-1}(\operatorname{int} M \cap \|x\|\mathbb{S}^{n-1})} \mathbf{1}_{\operatorname{int} M}(x),$$

if $\mathscr{H}^{n-1}(\operatorname{int} M \cap \|x\|\mathbb{S}^{n-1}) > 0$, and $\phi_M^\circ(x) := 0$, otherwise. Then, for r, s non-negative integers and $k = 0, \dots, n-1$,

$$\int_{SO(n)} \Phi_k^{r,s}(K \cap M, \vartheta\phi_M^\circ)\, \nu(d\vartheta) = \Phi_k^{r,s}(K)$$

and

$$\int_{SO(n)} \Phi_n^{r,0}(K \cap M, \vartheta\phi_M^\circ)\, \nu(d\vartheta) = \Phi_n^{r,0}(K).$$

The theorem follows from Proposition 7.15 as $\overline{\phi_M^\circ}(x) = 1$ for $x \in K$. □

In order to use Theorem 7.17 for estimating $\Phi_k^{r,s}(K)$ from an observation in $K \cap \vartheta M$, where ϑ is a random rotation, requires to determine the weight function ϕ_M° yielding a Horvitz-Thompson-type correction. This is possible when ϑM is known which is often the case in optical microscopy, see e.g. [30].

But from a basic science point of view, it is important to develop principal rotational formulae of the type (7.40) and (7.42) with integrands only depending on $K \cap \vartheta M$ without any further knowledge. To the best of our knowledge, this is an open problem in rotational integral geometry for the measurement function $\Phi_k^{r,s}$ with $k < n - 1$.

7.7 Local Stereology Applications

The aim of local stereology [16] is the estimation of quantitative parameters (volume, surface area, Minkowski tensors, ...) of spatial structures from sections through fixed points, called reference points.

Using a rotational Crofton formula, local stereological estimators of Minkowski tensors $\Phi_k^{r,s}(K)$ have recently been derived [20], based on measurements on random sections passing through a fixed point of K. More specifically, such local estimators are available for (i) $s = 0, 1$ and r, k arbitrary and for (ii) $r = 0$, s even and $k = n - 1$. In (i), the rotational Crofton formula presented in [6, Corollary 4.4] is used while (ii) follows by combining [25, Theorem 3.4] with the Blaschke-Petkantschin formula (7.24). The details were given in Sect. 7.4. The most common stereological application of rotational Crofton formulae is the estimation of intrinsic volumes ($r = s = 0$). For volume and surface area estimation, that is, when $k = n$ or

$k = n-1$, different forms of measurement functions have been suggested. In [32] several surface area estimators are discussed and a measurement function based on Morse theory is established. The works of Cruz-Orive and Gual-Arnau on this subject are summarized in the recent paper [10].

Alternatively, measurements for local estimation of $\Phi_k^{r,s}(K)$ may be performed on the intersection $K \cap \mathbf{M}$ of K with a randomly rotated convex body $\mathbf{M}$. Here, a principal rotational formula is used; see Sect. 7.6.

In this section, we will investigate to what extent these results can be transferred to particle processes. Let X be a particle process of full-dimensional convex particles in $\mathbb{R}^n$ that we represent as a stationary marked point process. The marked point process is given by

$$\{[x(K); K - x(K)] : K \in X\},$$

where $x(K) \in K$ is a reference point associated to each particle $K \in X$ while the mark $K - x(K)$ is the particle translated such that its reference point is the origin o. The particle mark distribution is denoted by $\mathbb{Q}$. We let $\mathbf{K}_0$ be a random convex set with distribution $\mathbb{Q}$. We may regard $\mathbf{K}_0$ as a randomly chosen particle or a typical particle with o put at its reference point.

Inference about the distribution of $\Phi_k^{r,s}(\mathbf{K}_0)$ may be based on a sample of particles, collected as those particles with reference point in a sampling window. More specifically, we consider a sample of the form

$$\{K \in X : x(K) \in W\}, \tag{7.43}$$

where $W \in \mathscr{B}(\mathbb{R}^n)$ is a full-dimensional sampling window with $0 < \lambda_n(W) < \infty$. The distribution of $\Phi_k^{r,s}(\mathbf{K}_0)$ may be studied via the empirical distribution of

$$\{\Phi_k^{r,s}(K - x(K)) : K \in X, x(K) \in W\}. \tag{7.44}$$

If complete access to the sampled particles is not possible, the distribution of $\Phi_k^{r,s}(\mathbf{K}_0)$ may still be studied via (7.44) if a precise estimate $\widehat{\Phi}_k^{r,s}(K - x(K))$ of $\Phi_k^{r,s}(K - x(K))$ is available, e.g. from replicated local sectioning of $K - x(K)$.

We will now discuss the situation where such precise estimates are not available. This situation is frequently encountered in optical microscopy where it is difficult to obtain a precise 3D image of K, due to overprojection at the peripheral parts of K. For this discussion, it turns out to be useful to consider the following $n+1$ probability measures $P_{X,k}$, $k = 0, \dots, n$, associated to the particle process X. The probability measure $P_{X,n}$ is concentrated on $\mathbb{R}^n$ and is absolutely continuous with respect to the Lebesgue measure λ_n with probability density

$$f_{\mathbf{K}_0}(x) := P(x \in \mathbf{K}_0)/\mathbb{E}\lambda_n(\mathbf{K}_0), \qquad x \in \mathbb{R}^n,$$

called the *cover density*. The density $f_{\mathbf{K}_0}$ may be envisaged as a kind of probability cloud. If $\mathbf{K}_0$ is deterministic, then $f_{\mathbf{K}_0}$ is proportional to $\mathbf{1}_{\mathbf{K}_0}$, so $\mathbf{K}_0$ can be reconstructed from $f_{\mathbf{K}_0}$. If $\mathbb{Q}$ is invariant under rotations, then $f_{\mathbf{K}_0}$ is also *rotation invariant*.

The remaining probability measures $P_{X,k}$, $k = 0, \ldots, n-1$, are concentrated on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ and are normalized versions of the mean support measures

$$P_{X,k}(A) := \frac{\mathbb{E}\Lambda_k(\mathbf{K}_0, A)}{\mathbb{E}\Lambda_k(\mathbf{K}_0, \mathbb{R}^n \times \mathbb{S}^{n-1})}, \qquad A \in \mathscr{B}(\mathbb{R}^n \times \mathbb{S}^{n-1}).$$

The probability measures $P_{X,k}$, $0 \leq k \leq n-1$, contain information about the probabilistic properties of the boundary of $\mathbf{K}_0$. As an example, $P_{X,n-1}(\mathbb{R}^n \times \cdot)$ is proportional to the surface area measure of the so-called *Blaschke body* $B(X)$ of the particle process, see [29, p. 149]. If $\mathbb{Q}$ is invariant under rotations, then $B(X)$ is a ball.

The theorem below shows that for particle processes, normalized mean Minkowski tensors determine the moments of arbitrary order in the distributions $P_{X,k}$, $k = 0, \ldots, n$.

Theorem 7.18 *Let X be a stationary particle process of full-dimensional convex particles in $\mathbb{R}^n$ with particle mark distribution $\mathbb{Q}$. Let $\mathbf{K}_0$ be a random convex set with distribution $\mathbb{Q}$. Then, for non-negative integers r, s and $k = 0, \ldots, n-1$*

$$\frac{\mathbb{E}\Phi_k^{r,s}(\mathbf{K}_0)}{\mathbb{E}\Phi_k^{0,0}(\mathbf{K}_0)} = \frac{\omega_{n-k}}{r!s!\omega_{n-k+s}} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} x^r u^s P_{X,k}(\mathrm{d}(x,u)).$$

For $k = n$, we get

$$\frac{\mathbb{E}\Phi_n^{r,0}(\mathbf{K}_0)}{\mathbb{E}\Phi_n^{0,0}(\mathbf{K}_0)} = \frac{1}{r!} \int_{\mathbb{R}^n} x^r f_{\mathbf{K}_0}(x) \lambda_n(\mathrm{d}x).$$

For $k = n-1$ and $r = 0$ we have the following connection to the surface area measure of the Blaschke body:

$$\frac{\mathbb{E}\Phi_{n-1}^{0,s}(\mathbf{K}_0)}{\mathbb{E}\Phi_{n-1}^{0,0}(\mathbf{K}_0)} = \frac{1}{s!\omega_{s+1}} \int_{\mathbb{S}^{n-1}} u^s \frac{S_{n-1}(B(X), \mathrm{d}u)}{V_{n-1}(B(X))}.$$

The second identity in Theorem 7.18 has earlier been presented in [34, Sect. 4.3], otherwise the results in Theorem 7.18 appear to be new. They are easily proved, using the definitions of $P_{X,k}$ and the Blaschke body $B(X)$, see [29, pp. 148–149]. Similar results may be derived for the characteristic functions of $P_{X,k}$ and $P_{X,n}$.

Let us now return to the problem of drawing inference about the distribution of $\Phi_k^{r,s}(\mathbf{K}_0)$ from a sample of particles. Using Campbell's theorem for marked point processes, we have

$$\frac{\mathbb{E}\sum_{K \in X, x(K) \in W} \Phi_k^{r,s}(K - x(K))}{\mathbb{E}\sum_{K \in X, x(K) \in W} \Phi_k^{0,0}(K - x(K))} = \frac{\mathbb{E}\Phi_k^{r,s}(\mathbf{K}_0)}{\mathbb{E}\Phi_k^{0,0}(\mathbf{K}_0)}. \tag{7.45}$$

Combining this result with Theorem 7.18, it follows under weak assumptions about the particle process that

$$\frac{r!s!\omega_{n-k+s}}{\omega_{n-k}} \frac{\sum_{K \in X, x(K) \in W} \Phi_k^{r,s}(K - x(K))}{\sum_{K \in X, x(K) \in W} \Phi_k^{0,0}(K - x(K))}$$

is a consistent (in a probabilistic sense) estimator of the moment of order (r,s) of $P_{X,k}$, also in the case where $\Phi_k^{r,s}(K - x(K))$ is substituted with an unbiased estimator $\widehat{\Phi}_k^{r,s}(K - x(K))$, subject to non-negligible variability. For instance, consistency follows in an expanding window regime if the particle process is ergodic, see [11, Corollary 12.2.V].

These ideas have been pursued in detail in [27, 34] for volume tensors and the resulting methods have been implemented in optical microscopy. For a sampled particle K, the volume tensor $\Phi_n^{r,0}(K - x(K))$ is here unbiasedly estimated using a local stereological design, involving measurements from the central part of K.

The design used in [27, 34] is a so-called vertical slice design. Let us consider a slice of the form $T = L + B(o,t)$ where $L \in G(M,q)$, $q > 1$, is a q-dimensional linear subspace containing a fixed line $M \in G(n,1)$ passing through o and $t > 0$ is the thickness of the slice. The line M is called the vertical axis. The set of such slices is denoted $T(n,q,M)$. We let ρ_q^M denote the unique probability measure on $T(n,q,M)$, invariant under rotations that keep M fixed.

The unbiased estimator of $\Phi_n^{r,0}(K - x(K))$ is obtained by replacing K by $K - x(K)$ in the lemma below.

Lemma 7.19 *Let* $\mathbf{T}$ *be a random vertical slice with distribution* ρ_q^M. *Let* $K \in \mathscr{K}^n$ *be a fixed convex set and* $G_{a,b}$ *the distribution function of the Beta distribution with parameters* $(a/2, b/2)$. *Then,*

$$\widehat{\Phi}_n^{r,0}(K;\mathbf{T}) := \frac{1}{r!} \int_{K \cap \mathbf{T}} x^r G_{n-q,q-1}(t^2/\|p_{M^\perp}(x)\|^2)^{-1} \lambda_n(\mathrm{d}x)$$

is an unbiased estimator of $\Phi_n^{r,0}(K)$.

The lemma is a direct consequence of [16, Proposition 6.3].

Combining Lemma 7.19 with Theorem 7.18 and (7.45), we obtain the following result.

Theorem 7.20 *Let* $W \in \mathscr{B}(\mathbb{R}^n)$ *with* $0 < \lambda_n(W) < \infty$. *Let* X *be a stationary particle process of convex particles in* $\mathbb{R}^n$ *with particle mark distribution* $\mathbb{Q}$. *Let* $\mathbf{K}_0$ *be a random convex set with distribution* $\mathbb{Q}$. *Finally, let* $\mathbf{T}$ *be a random vertical slice, independent of the particle process* X, *with distribution* ρ_q^M. *Then, for any non-negative integer* r,

$$\frac{\mathbb{E}\sum_{K \in X, x(K) \in W} \widehat{\Phi}_n^{r,0}(K - x(K);\mathbf{T})}{\mathbb{E}\sum_{K \in X, x(K) \in W} \widehat{\Phi}_n^{0,0}(K - x(K);\mathbf{T})} = \frac{1}{r!} \int_{\mathbb{R}^n} x^r f_{\mathbf{K}_0}(x) \lambda_n(\mathrm{d}x),$$

where $\widehat{\Phi}_n^{r,0}$ *is given in Lemma 7.19.*

If the particle mark distribution $\mathbb{Q}$ is invariant under rotations that keep the vertical axis fixed, then it is not needed to randomize the slice.

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Chapter 8
Valuations on Lattice Polytopes

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Abstract This survey is on classification results for valuations defined on lattice polytopes that intertwine the special linear group over the integers. The basic real valued valuations, the coefficients of the Ehrhart polynomial, are introduced and their characterization by Betke and Kneser is discussed. More recent results include classification theorems for vector and convex body valued valuations.

8.1 From the Pick Theorem to the Ehrhart Polynomial

A (full-dimensional) lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup spanned by n independent vectors. Given a basis of Λ, the automorphisms of Λ are transformations of the form $x \mapsto Ax + b$ with $b \in \Lambda$ and $A \in \mathrm{GL}_n(\mathbb{Z})$, that is, A is an $n \times n$ integer matrix with determinant ± 1. Such transformations are called *unimodular*. A *lattice polytope* is the convex hull of a finite subset of Λ and we write $\mathscr{P}(\Lambda)$ for the family of lattice polytopes. Since every lattice is a linear image of $\mathbb{Z}^n$, in general we just consider the lattice $\mathbb{Z}^n$.

This section concentrates on the lattice point enumerator $L(P)$ for a bounded set $P \subset \mathbb{R}^n$, where

$$L(P) := \sum_{x \in P \cap \mathbb{Z}^n} 1. \tag{8.1}$$

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Hence, $L(P)$ is the number of lattice points in P and $P \mapsto L(P)$ is a valuation on $\mathcal{P}(\mathbb{Z}^n)$. For basic properties of lattices related to this chapter from various aspects, see Barvinok [3], Beck and Robins [4], Gruber [20] or Gruber and Lekkerkerker [21].

The starting point is a formula [51] due to Georg Alexander Pick (1859–1942). For $P \in \mathcal{P}(\mathbb{Z}^2)$, write $B(P)$ for the number of lattice points on the boundary of P if P is two-dimensional, and $B(P) := 2|P \cap \mathbb{Z}^2| - 2$ if P is a segment or a point, where $|\cdot|$ denotes the cardinality of a finite set. Note that $P \mapsto B(P)$ is a valuation.

Theorem 8.1 (Pick) *For $P \in \mathcal{P}(\mathbb{Z}^2)$ non-empty,*

$$L(P) = V_2(P) + \frac{1}{2}B(P) + 1.$$

Here $V_2(P)$ is the two-dimensional volume of the polytope P. The core fact behind Pick's theorem is that if $P \in \mathcal{P}(\mathbb{Z}^2)$ is a triangle with $L(P) = 3$, then $V_2(P) = 1/2$. Thus the essential two-dimensional case can be proved for example by induction on $L(P)$, dissecting P into triangles sharing a common vertex if $L(P) \geq 4$. The Pick theorem has various proofs (see e.g. [9, 22]).

In higher dimensions, there is no simple formula as in Pick's theorem, as was noted by Reeve [54, 55]. The reason is that the volume of an n-dimensional simplex $S \in \mathcal{P}(\mathbb{Z}^n)$ with $L(S) = n + 1$ can be any non-negative integer multiple of $1/n!$ However, Eugène Ehrhart (1906–2000), a French high school teacher, found the following fundamental formula in [17] which works in all dimensions. We write $\mathbb{N}_0$ for the set of non-negative integers and call a valuation *unimodular* if it is invariant with respect to unimodular transformations.

Theorem 8.2 (Ehrhart) *There exist rational numbers $L_i(P)$ for $i = 0, \dots, n$ such that*

$$L(kP) = \sum_{i=0}^{n} L_i(P)k^i$$

for every $k \in \mathbb{N}_0$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. For each i, the functional $L_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{Q}$ is a unimodular valuation which is homogeneous of degree i.

Note that $L_n(P)$ is the n-dimensional volume $V_n(P)$ and that $L_0(P)$ is the Euler characteristic of P, that is, $L_0(P) := 1$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ non-empty and $L_0(\emptyset) := 0$. Also note that $L_i(P) = 0$ for $i > \dim P$, where $\dim P$ is the dimension (of the affine hull) of P.

Let $\det_{n-1} \Lambda$ denote the determinant of an $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$. In addition, for an n-dimensional polytope $P \in \mathcal{P}(\mathbb{Z}^n)$, let $\mathcal{F}_{n-1}(P)$ be the family of $(n-1)$-dimensional faces and write aff for affine hull. For $n \geq 2$, we have

$$L_{n-1}(P) = \begin{cases} \frac{1}{2} \sum_{F \in \mathscr{F}_{n-1}(P)} \dfrac{V_{n-1}(F)}{\det_{n-1}(\mathbb{Z}^n \cap \operatorname{aff} F)} & \text{if } \dim(P) = n, \\ \dfrac{V_{n-1}(P)}{\det_{n-1}(\mathbb{Z}^n \cap \operatorname{aff} P)} & \text{if } \dim(P) = n-1, \\ 0 & \text{if } \dim(P) \leq n-2. \end{cases}$$

Thus $L_{n-1}(P)$ is a *lattice surface area* of P. Note, in particular, that $L_1(P) = \frac{1}{2}B(P)$ in accordance with Pick's Theorem for $n = 2$.

The coefficient $L_i(P)$ may not be an integer for $i = 1, \ldots, n$, but $n!L_i(P) \in \mathbb{Z}$ for $P \in \mathscr{P}(\mathbb{Z}^n)$. There seems to be no known "geometric interpretation" for $L_i(P)$ if $n \geq 3$ and $1 \leq i \leq n-2$, and actually $L_i(P)$ might be negative in this case (see [30] for a strong result in this direction). If $P \in \mathscr{P}(\mathbb{Z}^n)$ is n-dimensional and $i = 1, \ldots, n-1$, then good bounds of the form

$$a(n,i)V_n(P) + b(n,i) \leq L_i(P) \leq c(n,i)V_n(P) + d(n,i)$$

involving the so-called Stirling numbers are known. Here the optimal upper bound on $L_i(P)$ for $i = 1, \ldots, n-1$ is due to Betke and McMullen [8]. A lower bound is due to Henk and Tagami [29] and Tsuchiya [64], and it is known to be optimal if $i = 1, 2, 3, n-3, n-2$, and if $n-i$ is even.

There is a representation of the Ehrhart polynomial via projective toric varieties associated to a lattice polytope (see, e.g., [13, 15, 18]). Using this representation, or combinatorial analogues of the algebraic geometric approach, formulas for $L_i(P)$ were established by Pommersheim [52] in terms of Dedekind sums if $P \in \mathscr{P}(\mathbb{Z}^3)$ is a tetrahedron, by Kantor and Khovanskii [32] if $n = 3, 4$, by Brion and Vergne [12] if P is simple, by Diaz and Robins [16] using Fourier analysis for any P and by Chen [14] if P is a simplex.

We note that inspired by the algebraic geometric representation of the Ehrhart polynomial, Barvinok [2] provided a polynomial time algorithm to calculate $L_i(P)$ for $P \in \mathscr{P}(\mathbb{Z}^n)$ and $i = 1, \ldots, n$, if the dimension n is fixed.

Ehrhart's Theorem 8.2 was extended to non-negative integer linear combinations of lattice polytopes by Bernstein [5] and McMullen [46].

Theorem 8.3 *Let $P_1, \ldots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ be given. If $k_1, \ldots, k_m \in \mathbb{N}_0$, then $L(k_1 P_1 + \cdots + k_m P_m)$ is a polynomial in $k_1, \ldots, k_m$ of total degree at most n. Moreover, the coefficient of $k_1^{r_1} \cdots k_m^{r_m}$ in this polynomial is a translation invariant valuation in P_i which is homogeneous of degree r_i.*

To prove this result, McMullen [46] uses induction on the number of summands, while Bernstein [5] considers intersections of algebraic hypersurfaces in $(\mathbb{C}\backslash\{0\})^n$ determined by Laurent polynomials with given Newton polytope. Here the Newton polytope associated to a Laurent polynomial is the convex hull of the lattice points corresponding to the exponents of its non-zero coefficients. Note that Theorems 8.2 and 8.3 imply that L_1 is additive.

Corollary 8.4 *If* $P, Q \in \mathscr{P}(\mathbb{Z}^n)$, *then* $L_1(P+Q) = L_1(P) + L_1(Q)$.

For the lattice point enumerator, the following important reciprocity relation was established by Ehrhart [17] and Macdonald [45]. For $P \in \mathscr{P}(\mathbb{Z}^n)$, write relint P for the relative interior of P (with respect to the affine hull of P).

Theorem 8.5 *If* $P \in \mathscr{P}(\mathbb{Z}^n)$, *then* $L(\operatorname{relint} P) = (-1)^{\dim P} \sum_{i=0}^{n} L_i(P)(-1)^i$.

This is also called the Ehrhart-Macdonald reciprocity law. The right side of the formula in Theorem 8.5 is, up to multiplication with the factor $(-1)^{\dim P}$, the Ehrhart polynomial $k \mapsto L(kP)$ evaluated at $k = -1$. For a multivariate version, that is, a version using the polynomial from Theorem 8.3, see [31].

One may choose other bases for the vector space of polynomials of degree at most n instead of the monomials and obtains other representations for the Ehrhart polynomial. In particular, for $k \in \mathbb{N}_0$,

$$L(kP) = \sum_{i=0}^{n} H_i^*(P) \binom{k+n-i}{n}.$$

For $i = 0, \dots, n$, the functional H_i^* is a unimodular valuation on $\mathscr{P}(\mathbb{Z}^n)$ (which is not homogeneous). More commonly used are the functionals h_i^*, defined by

$$L(kP) = \sum_{i=0}^{m} h_i^*(P) \binom{k+m-i}{m} \tag{8.2}$$

for $k \in \mathbb{N}_0$, where $m = \dim P$. The vector $(h_0^*(P), \dots, h_n^*(P))$, where we set $h_i^*(P) := 0$ for $i > \dim P$, is called the Ehrhart h^*-vector of P. Stanley [61] showed that the Ehrhart h^*-vector of P coincides with the combinatorial h-vector of a unimodular triangulation of P, if such a triangulation exists. Betke [6] and Stanley [61] showed that for $i = 0, \dots, n$, the functional h_i^* is integer-valued and non-negative on $\mathscr{P}(\mathbb{Z}^n)$. Stanley [62] showed that each h_i^* is monotone with respect to set inclusion. Clearly, we have $H_i^*(P) = h_i^*(P)$ for n-dimensional polytopes P. However, the functionals h_i^* are not valuations on $\mathscr{P}(\mathbb{Z}^n)$ while the valuations H_i^* are not monotone or non-negative.

Another representation of the Ehrhart polynomial, introduced by Breuer [11], is

$$L(kP) = \sum_{i=0}^{n} f_i^*(P) \binom{k-1}{i} \tag{8.3}$$

for $k \in \mathbb{N}_0$. For $i = 0, \dots, n$, the functional f_i^* is a unimodular valuation on $\mathscr{P}(\mathbb{Z}^n)$ (which again is not homogeneous). Note that $f_i^*(P) = 0$ for $i > \dim P$. The vector $(f_0^*(P), \dots, f_n^*(P))$ is called the Ehrhart f^*-vector of P and coincides with the combinatorial f-vector of a unimodular triangulation of P, if such a triangulation exists. Breuer [11] showed that for $i = 0, \ldots, n$, the valuation f_i^* is integer-valued and non-negative on $\mathscr{P}(\mathbb{Z}^n)$ and that these properties extend to polyhedral complexes.

8.2 The Inclusion-Exclusion Principle

The inclusion-exclusion principle is a fundamental property of valuations on lattice polytopes, which was first established in the case of translation invariant and real valued valuations by Stein [63] and for general real valued valuations by Betke (Das Einschließungs-Ausschließungsprinzip für Gitterpolytope. Unpublished manuscript). The first published proof is by McMullen [48], who also established the more general extension property. Since the family of lattice polytopes is not intersectional, that is, the intersection of two lattice polytopes is in general not a lattice polytope, results for valuations on polytopes (see Theorem 1.3) could not easily be generalized.

For $m \geq 1$, we write $P_J := \cap_{i \in J} P_i$ for $\emptyset \neq J \subset \{1, \ldots, m\}$ and given polytopes $P_1, \ldots, P_m \in \mathscr{P}(\mathbb{Z}^n)$. Let $\mathbb{G}$ be an abelian group. The *inclusion-exclusion formula* for lattice polytopes is the following result.

Theorem 8.6 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation, then for lattice polytopes* $P_1, \ldots, P_m$,

$$\mathrm{Z}(P_1 \cup \cdots \cup P_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1}\, \mathrm{Z}(P_J).$$

whenever $P_1 \cup \cdots \cup P_m \in \mathscr{P}(\mathbb{Z}^n)$ *and* $P_J \in \mathscr{P}(\mathbb{Z}^n)$ *for all* $\emptyset \neq J \subset \{1, \ldots, m\}$.

It is often helpful to extend valuations defined on lattice polytopes to finite unions of lattice polytopes whose intersections are again lattice polytopes. McMullen [48] showed that this is always possible. This is the *extension property*.

Theorem 8.7 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation, then there exists a function* $\bar{\mathrm{Z}}$ *defined on finite unions of lattice polytopes such that for lattice polytopes* $P_1, \ldots, P_m$,

$$\bar{\mathrm{Z}}(P_1 \cup \cdots \cup P_m) = \sum_{\emptyset \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1}\, \mathrm{Z}(P_J),$$

whenever $P_J \in \mathscr{P}(\mathbb{Z}^n)$ *for all* $\emptyset \neq J \subset \{1, \ldots, m\}$.

For a given valuation Z, we denote its extension by $\bar{\mathrm{Z}}$ and will use this notation throughout the chapter.

The inclusion-exclusion formula and the extension property are frequently needed for cell decompositions. We call a dissection of the polytope Q into polytopes $P_1, \dots, P_m$ a cell decomposition if $P_i \cap P_j$ is either empty or a common face of P_i and P_j for every $1 \le i < j \le m$. The faces of the cell decomposition are the faces of all P_i for $i = 1, \dots, m$.

Theorem 8.8 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation and* $Q \in \mathscr{P}(\mathbb{Z}^n)$*, then*

$$\bar{\mathrm{Z}}(Q) = (-1)^{\dim Q} \sum_{\substack{F \in \mathscr{F} \\ F \cap \operatorname{int} Q \neq \emptyset}} (-1)^{\dim F}\, \mathrm{Z}(F),$$

where $\mathscr{F}$ *is the set of all faces of a cell decomposition of* Q*.*

In particular, Theorem 8.7 implies the following. Write $\mathscr{F}(P)$ for the family of all non-empty faces of $P \in \mathscr{P}(\mathbb{Z}^n)$ (including the face P) and set $\bar{\mathrm{Z}}(\operatorname{relint} P) = \mathrm{Z}(P) - \bar{\mathrm{Z}}(\operatorname{relbd} P)$, where relbd stands for relative boundary. Expressing $\operatorname{relbd} P$ as the union of its faces, we obtain

$$\bar{\mathrm{Z}}(\operatorname{relint} P) = (-1)^{\dim P} \sum_{F \in \mathscr{F}(P)} (-1)^{\dim F}\, \mathrm{Z}(F) \tag{8.4}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

For a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$, Sallee [56] introduced the associated function $\mathrm{Z}^\circ : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ defined by

$$\mathrm{Z}^\circ(P) := \sum_{F \in \mathscr{F}(P)} (-1)^{\dim F}\, \mathrm{Z}(F) \tag{8.5}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$, which by (8.4) is closely related to $\bar{\mathrm{Z}}(\operatorname{relint} P)$. He showed that Z° is a valuation on $\mathscr{P}(\mathbb{Z}^n)$ (while $P \mapsto \bar{\mathrm{Z}}(\operatorname{relint} P)$ is not a valuation) and that $(\mathrm{Z}^\circ)^\circ = \mathrm{Z}$. McMullen [46] gave simple proofs for these facts. We will use the notation (8.5) and the valuation property of Z° throughout the chapter. Using this, we can write the Ehrhart-Macdonald reciprocity law (Theorem 8.5) also as

$$L^\circ(P) = \sum_{i=0}^{n} L_i(P)(-1)^i \tag{8.6}$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

We note that many of the results related to the inclusion-exclusion principle have a variant if $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{A}$ is a valuation with $\mathbb{A}$ a cancellative abelian semigroup. For example, the analogue of Theorem 8.6 is that if $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{A}$ is a valuation, and $P_1, \dots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ satisfy that $P_1 \cup \dots \cup P_m \in \mathscr{P}(\mathbb{Z}^n)$ and $P_J \in \mathscr{P}(\mathbb{Z}^n)$ for all $\emptyset \neq J \subset \{1, \dots, m\}$, then

$$\mathrm{Z}(P_1 \cup \dots \cup P_m) + \sum_{\substack{\emptyset \neq J \subset \{1,\dots,m\} \\ |J| \text{ even}}} \mathrm{Z}(P_J) = \sum_{\substack{\emptyset \neq J \subset \{1,\dots,m\} \\ |J| \text{ odd}}} \mathrm{Z}(P_J).$$

A typical case when $\mathbb{A}$ is only a semigroup is the case of Minkowski valuations, which will be discussed in Sect. 8.5.

8.3 Translation Invariant Valuations

Let $\mathbb{V}$ be a vector space over $\mathbb{Q}$. A valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ is *translation invariant* if $\mathrm{Z}(P + x) = \mathrm{Z}(P)$ for every $P \in \mathscr{P}(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$. Translation invariant valuations on $\mathscr{P}(\mathbb{Z}^n)$ behave similarly to the lattice point enumerator in many ways, as was proved by McMullen [46]. The paper [46] assumes that the valuation Z on $\mathscr{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle, which always holds by Theorem 8.6.

Theorem 8.9 *Let* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *be a translation invariant valuation. There exist* $Z_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *for* $i = 0, \dots, n$ *such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$. *Moreover,* $\mathrm{Z}_i(P) = 0$ *for* $i > \dim P$.

The corresponding result for valuations on polytopes is described in Theorem 1.13.

Combining results in McMullen [46] and [48] leads to an analogue of the Ehrhart-Macdonald reciprocity law (8.6).

Theorem 8.10 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *is a translation invariant valuation, then*

$$\mathrm{Z}^{\circ}(-P) = \sum_{i=0}^{n} \mathrm{Z}_i(P)(-1)^i$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

The Ehrhart-Macdonald reciprocity law (8.6) is easily deduced from Theorem 8.10 because in addition to translation invariance, the lattice point enumerator also satisfies $L(\mathrm{relint}(-P)) = L(\mathrm{relint}\, P)$.

Taking Theorem 8.9 as starting point, Jochemko and Sanyal [31] consider analogues of the coefficients $h_i^*(P)$ in (8.2) for translation invariant valuations. For a translation invariant valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$ and $P \in \mathscr{P}(\mathbb{Z}^n)$, they define $h_0^{\mathrm{Z}}(P), \dots, h_n^{\mathrm{Z}}(P)$ by

$$\mathrm{Z}(kP) = \sum_{i=0}^{m} h_i^{\mathrm{Z}}(P) \binom{k+m-i}{m},$$

where $m = \dim P$. A translation invariant valuation Z is called *h^*-nonnegative*, if $h_i^{\mathrm{Z}} \geq 0$ on $\mathscr{P}(\mathbb{Z}^n)$ for $i = 0, \dots, n$. It is called *h^*-monotone* if h_i^{Z} is monotone (with respect to set inclusion) on $\mathscr{P}(\mathbb{Z}^n)$ for $i = 0, \dots, n$. Using the extended valuation $\bar{\mathrm{Z}}$, Jochemko and Sanyal [31] establish a version of Stanley's theorem on the non-negativity and monotonicity of h_i^* for any translation invariant valuation.

Theorem 8.11 *For a translation invariant valuation* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$*, the following three statements are equivalent.*

1. *Z is h^*-nonnegative.*
2. *Z is h^*-monotone.*
3. $\bar{\mathrm{Z}}(\operatorname{relint} P) \geq 0$ *for every* $P \in \mathscr{P}(\mathbb{Z}^n)$.

Since for the lattice point enumerator we have $L(\operatorname{relint} P) \geq 0$ for every $P \in \mathscr{P}(\mathbb{Z}^n)$, the non-negativity and monotonicity of h_i^* on $\mathscr{P}(\mathbb{Z}^n)$ is a simple consequence of Theorem 8.11. Jochemko and Sanyal [31] also obtain the following result.

Theorem 8.12 *A functional* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a unimodular and h^*-nonnegative valuation if and only if there exist constants* $c_0, \dots, c_n \geq 0$ *such that*

$$\mathrm{Z}(P) = c_0 f_0^*(P) + \cdots + c_n f_n^*(P)$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

In the proof, essential use is made of the Betke-Kneser theorem, which is described in the following section.

8.4 The Betke-Kneser Theorem

The classical classification result for valuations on lattice polytopes concerns real valued and unimodular valuations and is due to Betke [6]. It was first published in Betke and Kneser [7]. It shows that the coefficients of the Ehrhart polynomial form a basis of the vector space of unimodular valuations.

Theorem 8.13 (Betke) *A functional* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a unimodular valuation if and only if there exist constants* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_0 L_0(P) + \cdots + c_n L_n(P)$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

We remark that by Corollary 8.16 below, it is sufficient to assume that Z is an $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant valuation to obtain the same result, where $\mathrm{SL}_n(\mathbb{Z})$ denotes the group of $n \times n$ integer matrices with determinant 1.

The Euclidean counterpart of Theorem 8.13 is the celebrated classification of rigid motion invariant and continuous valuations on convex bodies by Hadwiger [27] (see Theorem 1.23). A classification of $\mathrm{SL}_n(\mathbb{R})$ invariant, Borel measurable valuations on convex polytopes containing the origin in their interiors was recently established by Haberl and Parapatits [24] extending results from [25, 33]. For a complete classification of $\mathrm{SL}_n(\mathbb{R})$ invariant valuations on convex polytopes, see [38].

We say that a j-dimensional $S \in \mathscr{P}(\mathbb{Z}^n)$ is a unimodular simplex if $j = 0$ or $S = [x_0, \ldots, x_j]$ for $j \geq 1$ and $\{x_1 - x_0, \ldots, x_j - x_0\}$ is part of a basis of $\mathbb{Z}^n$. Here $[\ldots]$ stands for convex hull. We define a particular set of unimodular simplices by setting $T_0 := \{0\}$ and $T_j := [0, e_1, \ldots, e_j]$ for $j = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. Betke and Kneser [7] also established the following result for an abelian group $\mathbb{G}$.

Theorem 8.14 (Betke-Kneser) *Every unimodular valuation* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is uniquely determined by its values on* $T_0, \ldots, T_n$ *and these values can be chosen arbitrarily in* $\mathbb{G}$.

Again, by Corollary 8.16 below, it is sufficient to assume that Z is an $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant valuation.

The following statement is the core of the argument in Betke and Kneser [7]. It is proved using dissection into simplices and suitable complementation by simplices.

Proposition 8.15 *For* $P \in \mathscr{P}(\mathbb{Z}^n)$*, there exist unimodular simplices* $S_1, \ldots, S_m$ *and integers* $l_1, \ldots, l_m$ *such that for any abelian group* $\mathbb{G}$*,*

$$\mathrm{Z}(kP) = \sum_{j=1}^{m} l_j \, \mathrm{Z}(kS_j)$$

for every valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *and* $k \in \mathbb{N}_0$.

This proposition implies Ehrhart's theorem. Just note that for $k \geq 1$,

$$L(kT_i) = \binom{k+i}{i} \qquad \text{for } i = 0, \ldots, n,$$

that each unimodular simplex S_j is an image under a unimodular transformation of some T_i, and that for each i, the above binomial coefficient is a polynomial in k of degree i.

The following statement is another direct consequence of Proposition 8.15.

Corollary 8.16 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *and* $\mathrm{Z}' : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *are* $\mathrm{SL}_n(\mathbb{Z})$ *and translation invariant valuations such that*

$$\mathrm{Z}(T_i) = \mathrm{Z}'(T_i) \qquad \textit{for } i = 0, \ldots, n,$$

then $\mathrm{Z}(P) = \mathrm{Z}'(P)$ *for every* $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.5 Minkowski Valuations

Let $\mathscr{F}$ be a family of subsets of $\mathbb{R}^n$ and write $\mathscr{K}^n$ for the set of convex bodies, that is, compact convex sets, in $\mathbb{R}^n$. The subset of convex polytopes is denoted by $\mathscr{P}^n$. An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is a *Minkowski valuation* if Z satisfies

$$\mathrm{Z}K + \mathrm{Z}L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L)$$

for all $K, L \in \mathscr{F}$ with $K \cup L, K \cap L \in \mathscr{F}$ and addition on $\mathscr{K}^n$ is Minkowski addition; that is,

$$K + L := \{x + y : x \in K,\ y \in L\}.$$

Let $\mathrm{SL}_n(\mathbb{R})$ be the special linear group on $\mathbb{R}^n$, that is, the group of real matrices of determinant 1. An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is called $\mathrm{SL}_n(\mathbb{R})$ equivariant if

$$\mathrm{Z}(\phi P) = \phi\, \mathrm{Z}P \qquad \text{for } \phi \in \mathrm{SL}_n(\mathbb{R}) \text{ and } P \in \mathscr{F}.$$

Define $\mathrm{SL}_n(\mathbb{Z})$ equivariance of operators on $\mathscr{P}(\mathbb{Z}^n)$ analogously. For recent results on $\mathrm{SL}_n(\mathbb{R})$ equivariant operators on convex bodies and their associated inequalities, see, for example, [26, 40–43].

For $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant Minkowski valuations defined on convex polytopes, the following complete classification was established in [35]. It provides a characterization of the difference body operator

$$P \mapsto P - P := \{x - y : x, y \in P\},$$

which assigns to P its *difference body*. For more information on difference bodies and their associated inequalities, see [19, 59]. Let $n \geq 2$.

Theorem 8.17 *An operator* $\mathrm{Z} : \mathscr{P}^n \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *equivariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z}P = c(P - P)$$

for every $P \in \mathscr{P}^n$.

Further results on the classification of $\mathrm{SL}_n(\mathbb{R})$ equivariant Minkowski valuations can be found, for example, in [23, 36, 50, 65].

The following result, taken from [10], is an analogue for lattice polytopes of Theorem 8.17.

Theorem 8.18 *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist* $a, b \geq 0$ *such that*

$$\mathrm{Z}P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here for a lattice polytope P, the point $\ell_1(P)$ is its discrete Steiner point that was introduced in [10]. See Sect. 8.6 for the definition and characterization theorems. The proof of Theorem 8.18 uses constructions from Betke and Kneser [7] as well as results on Minkowski summands and it also exploits the large symmetry group of the standard simplex T_n.

For operators mapping $\mathscr{P}(\mathbb{Z}^n)$ to $\mathscr{P}(\mathbb{Z}^n)$, the following result was established in [10]. Write LCM for least common multiple.

Theorem 8.19 *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{P}(\mathbb{Z}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist integers* $a, b \geq 0$ *with* $b - a \in \mathrm{LCM}(2, \dots, n+1)\mathbb{Z}$ *such that*

$$\mathrm{Z}P = a(P - \ell_1(P)) + b(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here it is used that the discrete Steiner point of a lattice polytope is a vector with rational coordinates.

An operator $\mathrm{Z} : \mathscr{F} \to \mathscr{K}^n$ is $\mathrm{SL}_n(\mathbb{R})$ *contravariant* if

$$\mathrm{Z}(\phi P) = \phi^{-t}\,\mathrm{Z}P \qquad \text{for } \phi \in \mathrm{SL}_n(\mathbb{R}) \text{ and } P \in \mathscr{F},$$

where ϕ^{-t} is the inverse of the transpose of ϕ. We define $\mathrm{SL}_n(\mathbb{Z})$ contravariance of operators on $\mathscr{P}(\mathbb{Z}^n)$ analogously. For recent results on $\mathrm{SL}_n(\mathbb{R})$ contravariant operators on convex bodies, see, for example, [26, 41, 44].

An important $\mathrm{SL}_n(\mathbb{R})$ contravariant operator on $\mathscr{K}^n$ is the operator $K \mapsto \Pi K$, that associates with a convex body its projection body. To define this operator, we describe a convex body L by its support function $h(L,\cdot) : \mathbb{S}^{n-1} \to \mathbb{R}$ where $h(L, u) := \max\{u \cdot x : x \in L\}$.

For a convex body K, the *projection body* ΠK is given by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp),$$

for $u \in \mathbb{S}^{n-1}$, where $K|u^\perp$ is the orthogonal projection of K onto the hyperplane orthogonal to u. We refer to [19, 59] for more information on projection bodies and their associated inequalities. For a polytope P with facets (that is, $(n-1)$-dimensional faces) $F_1, \dots, F_m$, the projection body ΠP is given as the following

Minkowski sum,

$$\Pi P = \tfrac{1}{2}\big([-v_1, v_1] + \cdots + [-v_m, v_m]\big),$$

where v_i is the scaled normal corresponding to the facet F_i, that is, v_i is a normal vector to the facet F_i with length equal to $V_{n-1}(F_i)$. Here $[-v_i, v_i]$ is the segment with endpoints $-v_i$ and v_i.

For $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on $\mathscr{P}^n$, the following complete classification was established in [35]. Let $n \geq 2$.

Theorem 8.20 *An operator* $\mathrm{Z} : \mathscr{P}^n \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z} P = c\,\Pi P$$

for every $P \in \mathscr{P}^n$.

Further classification theorems for $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on convex bodies can be found in [23, 34, 36, 37, 49, 60].

The following analogue of Theorem 8.20 for lattice polytopes is from [10].

Theorem 8.21

(i) *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^2) \to \mathscr{K}^2$ *is an* $\mathrm{SL}_2(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exist constants* $a, b \geq 0$ *such that*

$$\mathrm{Z} P = a\,\varrho_{\pi/2}(P - \ell_1(P)) + b\,\varrho_{\pi/2}(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^2)$.

(ii) *For* $n \geq 3$, *an operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{K}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z} P = c\,\Pi P$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

Here $\varrho_{\pi/2}$ denotes the rotation by an angle $\pi/2$ in $\mathbb{R}^2$. Note that for $n = 2$, the projection body is obtained from the difference body by applying this rotation.

The projection body of a lattice polytope is a rational polytope. For operators mapping $\mathscr{P}(\mathbb{Z}^n)$ to $\mathscr{P}(\mathbb{Z}^n)$, the following result was established in [10].

Theorem 8.22

(i) *An operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^2) \to \mathscr{P}(\mathbb{Z}^2)$ *is an* $\mathrm{SL}_2(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exist integers* $a, b \geq 0$ *with* $b - a \in 6\,\mathbb{Z}$ *such that*

$$\mathrm{Z}\,P = a\,\varrho_{\pi/2}(P - \ell_1(P)) + b\,\varrho_{\pi/2}(-P + \ell_1(P))$$

for every $P \in \mathscr{P}(\mathbb{Z}^2)$.

(ii) *For* $n \geq 3$, *an operator* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathscr{P}(\mathbb{Z}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \in (n-1)!\,\mathbb{N}_0$ *such that*

$$\mathrm{Z}\,P = c\Pi P$$

for every $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.6 Vector Valuations

In analogy to (8.1), for $P \in \mathscr{P}(\mathbb{Z}^n)$, the *discrete moment vector* was introduced in [10] as

$$\ell(P) := \sum_{x \in P \cap \mathbb{Z}^n} x. \tag{8.7}$$

The discrete moment vector $\ell : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Z}^n$ is a valuation that is equivariant with respect to unimodular linear transformations. In addition, if $y \in \mathbb{Z}^n$, then

$$\ell(P + y) = \ell(P) + L(P)\,y. \tag{8.8}$$

In general, a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is called *translation covariant* if for all $P \in \mathscr{P}(\mathbb{Z}^n)$ and $y \in \mathbb{Z}^n$,

$$\mathrm{Z}(P + y) = \mathrm{Z}(P) + \mathrm{Z}^0(P)y$$

with some $\mathrm{Z}^0 : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}$. Note that it easily follows from this definition that the associated functional Z^0 is also a valuation.

McMullen [46] established the following analogue of Theorem 8.9.

Theorem 8.23 *Let* $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *be a translation covariant valuation. There exist* $\mathrm{Z}_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *for* $i = 0, \ldots, n+1$ *such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n+1} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$. *For each i, the function* Z_i *is a translation covariant valuation which is homogeneous of degree i.*

Note that if the valuation Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then so are $\mathrm{Z}_0, \ldots, \mathrm{Z}_{n+1}$. Using this homogeneous decomposition, McMullen [46] established the following more general result.

Theorem 8.24 *Let* $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *be a translation covariant valuation and let* $P_1, \ldots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ *be given. If* $k_1, \ldots, k_m \in \mathbb{N}_0$, *then* $\mathrm{Z}(k_1 P_1 + \cdots + k_m P_m)$ *is a polynomial of total degree at most* $(n+1)$ *in* $k_1, \ldots, k_m$. *Moreover, the coefficient of* $k_1^{r_1} \cdots k_m^{r_m}$ *in this polynomial is a translation covariant valuation in* P_i *which is homogeneous of degree* r_i.

The discrete moment vector is a translation covariant valuation. Hence, we obtain as a special case of Theorem 8.23 the following result.

Corollary 8.25 *There exist* $\ell_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *for* $i = 1, \ldots, n+1$ *such that*

$$\ell(kP) = \sum_{i=1}^{n+1} \ell_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$. *For each i, the function* ℓ_i *is a translation covariant valuation which is equivariant with respect to unimodular linear transformations and homogeneous of degree i.*

Note that $\ell_{n+1}(P)$ is the moment vector of P, that is, $\ell_{n+1}(P) = \int_P x \,\mathrm{d}x$. We call the vector $\ell_1(P)$ the *discrete Steiner point* of P. From Theorem 8.24, we deduce as in Corollary 8.4 the following result.

Corollary 8.26 *The function* $\ell_1 : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{Q}^n$ *is additive.*

It is shown in [10] that the discrete Steiner point of a unimodular simplex is its centroid. Hence, by using suitable dissections and complementations, it is possible to obtain $\ell_1(P)$ for a given lattice polytope P.

The following results, Theorems 8.27 and 8.29, both from [10], are the reason for calling ℓ_1 the discrete Steiner point map. A function $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is called *translation equivariant* if $\mathrm{Z}(P + x) = \mathrm{Z}(P) + x$ for $x \in \mathbb{Z}^n$ and $P \in \mathscr{P}(\mathbb{Z}^n)$.

Theorem 8.27 *A function* $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant and additive if and only if* Z *is the discrete Steiner point map.*

Theorem 8.27 corresponds to the following characterization of the classical Steiner point by Schneider [57]. The classical Steiner point, $s(K)$, is defined by

$$s(K) := \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} u\, h(K, u)\, \mathrm{d}u,$$

where κ_n is the n-dimensional volume of the n-dimensional unit ball and $\mathrm{d}u$ denotes integration with respect to $(n-1)$-dimensional Hausdorff measure on the unit sphere.

Theorem 8.28 *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is continuous, rigid motion equivariant and additive if and only if* Z *is the Steiner point map.*

Note that Wannerer [66] recently obtained a corresponding characterization of vector valuations in the Hermitian setting (see Corollary 6.15).

The discrete Steiner point is also characterized in the following result.

Theorem 8.29 *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant valuation if and only if* Z *is the discrete Steiner point map.*

This theorem corresponds to the following characterization of the classical Steiner point by Schneider [58].

Theorem 8.30 *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a continuous and rigid motion equivariant valuation if and only if* Z *is the Steiner point map.*

By (8.8), the discrete moment vector is translation covariant. Note that

$$\ell_i(P + x) = \ell_i(P) + L_{i-1}(P)\, x$$

for $i = 1, \ldots, n+1$, where the case $i = 1$ is just the translation equivariance of ℓ_1. Hence ℓ_i is translation covariant for each i. The following result is from [39].

Theorem 8.31 *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there exist constants* $c_1, \ldots, c_{n+1} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_1 \ell_1(P) + \cdots + c_{n+1}\, \ell_{n+1}(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

The Euclidean counterpart of this result is the classification of rotation equivariant and translation covariant, continuous valuations $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ by Hadwiger and Schneider [28] (see Theorem 2.4). A classification of $\mathrm{SL}_n(\mathbb{R})$ equivariant, Borel measurable vector valuations on convex polytopes containing the origin in their interiors was recently established by Haberl and Parapatits [25].

8.7 Polynomial Valuations

To discuss polynomial valuations, let us review what we mean by polynomial in our context. Let $\mathbb{G}$ be an abelian group and Λ a lattice in $\mathbb{R}^n$. We say that $p : \Lambda \to \mathbb{G}$ is polynomial of degree 0, if p is constant on Λ. We say that p is polynomial of degree $d \geq 1$ if for any $y \in \Lambda$, the map $x \mapsto p(x+y) - p(x)$ is polynomial of degree at most $d-1$. If $w_1, \ldots, w_n$ form a basis of Λ, then this implies that there are $b_i \in \mathbb{G}$ and integer polynomials $p_i : \mathbb{Z}^n \to \mathbb{Z}$ of degree at most d for $i = 1, \ldots, r$ such that for $k_i \in \mathbb{N}_0$

$$p(k_1 w_1 + \cdots + k_n w_n) = \sum_{i=1}^{r} p_i(k_1, \ldots, k_n)\, b_i.$$

Now a valuation $\mathrm{Z} : \mathscr{P}(\Lambda) \to \mathbb{G}$ is polynomial of degree d if for every $P \in \mathscr{P}(\Lambda)$, the function, defined on Λ by $x \mapsto \mathrm{Z}(P+x)$ is a polynomial of degree d.

Clearly, a valuation $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ is translation invariant if and only if it is polynomial of degree 0. If $q : \mathbb{Z}^n \to \mathbb{G}$ is a polynomial of degree at most d, then $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ defined by

$$\mathrm{Z}(P) := \sum_{x \in P \cap \mathbb{Z}^n} q(x) \tag{8.9}$$

is a polynomial valuation of degree at most d.

McMullen [46] considered polynomial valuations of degree at most one and Pukhlikov and Khovanskii [53] proved Theorem 8.32 in the general case. Another proof, following the approach of [46], is due to Alesker [1]. These papers assume that the valuation Z on $\mathscr{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle, which holds by Theorem 8.6.

Theorem 8.32 *Let* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{G}$ *be a polynomial valuation of degree at most* d *and let* $P_1, \ldots, P_m \in \mathscr{P}(\mathbb{Z}^n)$ *be given. If* $k_1, \ldots, k_m \in \mathbb{N}_0$*, then* $\mathrm{Z}(k_1 P_1 + \cdots + k_m P_m)$ *is a polynomial of total degree at most* $(d+n)$ *in* $k_1, \ldots, k_m$*. Moreover, the coefficient of* $k_1^{r_1} \cdots k_m^{r_m}$ *in this polynomial is a polynomial valuation in* P_i *of degree at most* d *which is homogeneous of degree* r_i*.*

This result implies that a homogeneous decomposition for polynomial valuations exists. Let $\mathbb{V}$ be a vector space over $\mathbb{Q}$.

Corollary 8.33 *Let* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *be a polynomial valuation of degree at most* d*. There exist valuations* $\mathrm{Z}_i : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ *for* $i = 0, \ldots, n+d$ *which are polynomial of degree at most* $d+n$ *and homogeneous of degree* i *such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{d+n} \mathrm{Z}_i(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$*.*

If a polynomial valuation $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ respects the action of linear unimodular transformations, then so do $\mathrm{Z}_0, \dots, \mathrm{Z}_{n+d}$. Important cases include $\mathrm{SL}_n(\mathbb{Z})$ invariant valuations and $\mathrm{SL}_n(\mathbb{Z})$ equivariant as well as $\mathrm{SL}_n(\mathbb{Z})$ contravariant valuations.

A version of the Ehrhart-Macdonald reciprocity law for polynomial valuations of type (8.9) was established by Brion and Vergne [12]. The following more general result is from [39] and was proved along the lines of reciprocities laws from [46].

Theorem 8.34 *If $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{V}$ is a polynomial valuation which is homogeneous of degree j, then*

$$\mathrm{Z}^\circ(-P) = (-1)^j\, \mathrm{Z}(P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$.

8.8 Tensor Valuations

In analogy to (8.1) and (8.7), for $P \in \mathscr{P}(\mathbb{Z}^n)$, we define for $r \in \mathbb{N}_0$, the *discrete moment tensor of rank r* by

$$L^r(P) := \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r,$$

where x^r denotes the r-fold symmetric tensor product of x. Let $\mathbb{T}^r$ denote the vector space of symmetric tensors of rank r on $\mathbb{R}^n$. Note that $\mathbb{T}^0 = \mathbb{R}$ and $L^0 = L$ and that $\mathbb{T}^1 = \mathbb{R}^n$ and $L^1 = \ell$.

We view each element of $\mathbb{T}^r$ as a symmetric r linear functional on $(\mathbb{R}^n)^r$. So, in particular,

$$L^r(P)(v_1, \dots, v_r) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} (x \cdot v_1) \cdots (x \cdot v_r)$$

for $v_1, \dots, v_r \in \mathbb{R}^n$, where $x \cdot v$ is the inner product of x and v.

The discrete moment tensor $L^r: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ has the following behavior with respect to unimodular linear transformations. For $v_1, \dots, v_r \in \mathbb{R}^n$,

$$L^r(\phi P)(v_1, \dots, v_r) = L^r(P)(\phi^t v_1, \dots, \phi^t v_r)$$

for all $\phi \in \mathrm{GL}_n(\mathbb{Z})$ and $P \in \mathscr{P}(\mathbb{Z}^n)$. In general, a tensor valuation $\mathrm{Z}: \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is called $\mathrm{SL}_n(\mathbb{Z})$ equivariant if for $v_1, \dots, v_r \in \mathbb{R}^n$,

$$\mathrm{Z}(\phi P)(v_1, \dots, v_r) = \mathrm{Z}(P)(\phi^t v_1, \dots, \phi^t v_r)$$

for all $\phi \in \mathrm{GL}_n(\mathbb{Z})$ and $P \in \mathscr{P}(\mathbb{Z}^n)$.

In addition, if $y \in \mathbb{Z}^n$, then

$$L^r(P+y) = \sum_{m=0}^{r} L^{r-m}(P)\frac{y^m}{m!},$$

where we use the convention that $y^0 = 1 \in \mathbb{R}$. Following McMullen [47], a valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is called *translation covariant* if there exist associated functions $\mathrm{Z}^m : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^m$ for $m = 0, \dots, r$ such that

$$\mathrm{Z}(P+y) = \sum_{m=0}^{r} \mathrm{Z}^m(P)\frac{y^{r-m}}{(r-m)!}$$

for all $y \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. It follows from this definition that Z^m is a valuation for $m = 0, \dots, r$ and that $\mathrm{Z}^r = \mathrm{Z}$. Note that the associated valuation Z^m is translation covariant for $m = 0, \dots, r$, since we have

$$\mathrm{Z}^m(P+y) = \sum_{j=0}^{m} \mathrm{Z}^j(P)\frac{y^{m-j}}{(m-j)!}.$$

For given $v_1, \dots, v_r \in \mathbb{R}^n$, associate with the translation covariant tensor valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$, the real valued valuation $P \mapsto \mathrm{Z}(P)(v_1, \dots, v_r)$, which is easily seen to be polynomial of degree at most r. Hence we obtain the following result from Theorem 8.32.

Theorem 8.35 *Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a translation covariant valuation and let $P_1, \dots, P_m \in \mathcal{P}(\mathbb{Z}^n)$ be given. If $k_1, \dots, k_m \in \mathbb{N}_0$, then $\mathrm{Z}(k_1P_1 + \dots + k_mP_m)$ is a polynomial of total degree at most $(n+r)$ in $k_1, \dots, k_m$. Moreover, the coefficient of $k_1^{r_1} \cdots k_m^{r_m}$ in this polynomial is a translation covariant valuation in P_i which is homogeneous of degree r_i.*

As a special case, we obtain the following homogeneous decomposition.

Theorem 8.36 *Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a translation covariant valuation. There exist $\mathrm{Z}_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ for $i = 0, \dots, n+r$ such that*

$$\mathrm{Z}(kP) = \sum_{i=0}^{n+r} \mathrm{Z}_i(P)k^i$$

for every $k \in \mathbb{N}_0$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. For each i, the function Z_i is a translation covariant valuation which is homogeneous of degree i.

Note that if Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then so are the homogeneous components $\mathrm{Z}_0, \dots, \mathrm{Z}_{n+r}$.

We apply these results to the discrete moment tensor and obtain the following result.

Corollary 8.37 *There exist* $L_i^r : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *for* $i = 1, \dots, n + r$ *such that*

$$L^r(kP) = \sum_{i=1}^{n+r} L_i^r(P) k^i$$

for every $k \in \mathbb{N}_0$ *and* $P \in \mathscr{P}(\mathbb{Z}^n)$*. For each i, the function* L_i^r *is a translation covariant valuation which is equivariant with respect to unimodular linear transformations and homogeneous of degree i.*

Note that $L_{n+r}^r(P)$ is the rth moment tensor of the lattice polytope P, that is, $L_{n+r}^r(P) = \frac{1}{r!}\int_P x^r \mathrm{d}x$ [cf. (2.4)]. See [39], for results on the classification of tensor valuations.

Using the approach from [46], we can extend the reciprocity laws to tensor valuations and obtain the following result, which is proved in [39].

Theorem 8.38 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation which is homogeneous of degree j, then*

$$\mathrm{Z}^\circ(P) = (-1)^j \, \mathrm{Z}(-P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$*.*

Since Z° is again a translation covariant valuation, Theorem 8.36 implies that there are homogeneous decompositions for Z and Z°. Hence the following result is a simple consequence of Theorem 8.38.

Corollary 8.39 *If* $\mathrm{Z} : \mathscr{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation, then*

$$\mathrm{Z}^\circ(P) = \sum_{i=0}^{n+r} (-1)^i \, \mathrm{Z}_i(-P)$$

for $P \in \mathscr{P}(\mathbb{Z}^n)$*.*

So, in particular, using that $L^r(-P) = (-1)^r L^r(P)$, we obtain

Corollary 8.40 *For* $P \in \mathscr{P}(\mathbb{Z}^n)$*,*

$$L^r(\operatorname{relint} P) = (-1)^{m+r} \sum_{i=1}^{m+r} (-1)^i L_i^r(P),$$

where $m = \dim P$*.*

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Chapter 9
Valuations and Curvature Measures on Complex Spaces

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Abstract We survey recent results in Hermitian integral geometry, i.e. integral geometry on complex vector spaces and complex space forms. We study valuations and curvature measures on complex space forms and describe how the global and local kinematic formulas on such spaces were recently obtained. While the local and global kinematic formulas in the Euclidean case are formally identical, the local formulas in the Hermitian case contain strictly more information than the global ones. Even if one is only interested in the flat Hermitian case, i.e. $\mathbb{C}^n$, it is necessary to study the family of all complex space forms, indexed by the holomorphic curvature 4λ, and the dependence of the formulas on the parameter λ. We will also describe Wannerer's recent proof of local additive kinematic formulas for unitarily invariant area measures.

9.1 Introduction

Hermitian integral geometry is a relatively old subject, with early contributions by Blaschke [23], Rohde [39], Santaló [41], Shifrin [44], Gray [30], Griffiths [31] and others, and more recent works by Tasaki [45, 46], Park [38], and Abardia, Gallego and Solanes [2].

A systematic study of this subject was completed only recently. This is partly due to the fact that Alesker's algebraic theory of translation invariant valuations as well as his theory of valuations on manifolds, both being indispensable tools in this line of research, were not available before. The main recent results of Hermitian integral geometry are the explicit description of global kinematic formulas in Hermitian spaces [19], local and global kinematic formulas on complex space forms [20] and kinematic formulas for unitarily invariant area measures [47].

These three papers cover almost 200 pages. The aim of the present chapter is to state most of the main theorems and to give some ideas about their proofs. More

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generally, we want to give a taste of this beautiful theory, which involves tools and results from convex geometry, differential geometry, geometric measure theory, representation theory and algebraic geometry.

In Sect. 9.2 we sketch some fundamental facts and definitions of Alesker's theory of translation invariant valuations, in particular we introduce product, convolution and Alesker-Fourier transform. Section 9.3 deals with valuations on manifolds; and introduces the important notion of a smooth curvature measure on a manifold. In Sect. 9.4 we introduce the general framework to study local and global kinematic formulas on space forms and state the important transfer principle.

The remaining sections deal with the Hermitian case. Section 9.5 is a survey on the results from [19], the most important theorem in this section being Theorem 9.24. The curved case, i.e. the case of complex space forms, is treated in Sect. 9.6. The main results are Theorems 9.30 and 9.34. Finally, the theory of unitarily invariant area measures is sketched in Sect. 9.7. This theory has some similar features as the theory of curvature measures. We do not explicitly state Wannerer's theorem (the local additive kinematic formulas for unitarily invariant area measures), but sketch the way they are obtained.

9.2 Translation Invariant Valuations on Vector Spaces

In this section, we will introduce the basic notation and state some fundamental theorems, some of which will be explained in more detail in [13].

9.2.1 McMullen's Decomposition and Hadwiger's Theorem

Let $\mathcal{K}^n$ be the space of compact convex bodies in $\mathbb{R}^n$. A (real-valued) valuation on $\mathbb{R}^n$ is a functional $\mu : \mathcal{K}^n \to \mathbb{R}$ which satisfies

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$.

Continuity of valuations is with respect to the Hausdorff topology. A valuation is said to be translation invariant, if $\mu(K + x) = \mu(K)$ for all $K \in \mathcal{K}^n, x \in \mathbb{R}^n$. The vector space of all continuous translation invariant valuations is denoted by Val.

If G is some group acting linearly on $\mathbb{R}^n$, a valuation μ is called *G-invariant* if $\mu(gK) = \mu(K)$ for all $K \in \mathcal{K}^n, g \in G$.

A valuation μ is called homogeneous of degree k if $\mu(tK) = t^k \mu(K), t > 0$. It is even/odd if $\mu(-K) = \mu(K)$ (resp. $\mu(-K) = -\mu(K)$) for all K. The corresponding space is denoted by $\mathrm{Val}_k^{\pm}$.

A fundamental theorem concerning translation invariant valuations is McMullen's decomposition.

Theorem 9.1 (McMullen [37])

$$\mathrm{Val} = \bigoplus_{\substack{k=0,\dots,n \\ \epsilon=\pm}} \mathrm{Val}_k^\epsilon .$$

The space Val_n is one-dimensional and spanned by the volume, while Val_0 is spanned by the Euler characteristic χ.

The space of $\mathrm{SO}(n)$-invariant valuations was described by Hadwiger. Let μ_k denote the k-th intrinsic volume, see [36, 42].

Theorem 9.2 (Hadwiger [32], See Also [34, 36]) *The intrinsic volumes $\mu_0, \dots, \mu_n$ form a basis of* $\mathrm{Val}^{\mathrm{SO}(n)}$.

Note that without any G-invariance, the spaces $\mathrm{Val}_k^\pm$ are infinite-dimensional (except for $k = 0, n$). Using McMullen's theorem, one can show that they admit a Banach space structure.

Even valuations are easier to understand than odd ones thanks to Klain's embedding result [35]. If $\phi \in \mathrm{Val}_k^+$, then the restriction of ϕ to a k-dimensional plane $E \in \mathrm{Gr}_k$ is a multiple $\mathrm{Kl}_\phi(E)$ of the Lebesgue measure. Klain proved that the map $\mathrm{Val}_k^+ \to C(\mathrm{Gr}_k), \phi \mapsto \mathrm{Kl}_\phi$ is injective.

9.2.2 Alesker's Irreducibility Theorem, Product and Convolution of Valuations

The group $\mathrm{GL}(n)$ acts on Val by

$$(g\mu)(K) := \mu(g^{-1}K).$$

Obviously, degree and parity are preserved under this action.

One of the most fundamental and influential theorems of modern integral geometry is the following.

Theorem 9.3 (Alesker [3]) *The spaces* $\mathrm{Val}_k^\epsilon, k = 0, \dots, n, \epsilon = \pm$ *are irreducible* $\mathrm{GL}(n)$*-representations.*

Since we are in an infinite-dimensional situation, this means that every non-trivial $\mathrm{GL}(n)$-invariant subspace of Val_k^ϵ is dense.

Corollary 9.4 *Valuations of the form* $K \mapsto \mathrm{vol}(K + A)$ *with* $A \in \mathscr{K}^n$ *span a dense subspace inside* Val.

Definition 9.5 A valuation $\mu \in \mathrm{Val}$ is called smooth, if the map

$$\mathrm{GL}(n) \to \mathrm{Val}$$

$$g \mapsto g\mu$$

is smooth as a map from the Lie group $\mathrm{GL}(n)$ to the Banach space Val.

Smooth valuations form a dense subspace $\mathrm{Val}^\infty \subset \mathrm{Val}$, which has a natural Fréchet space structure.

An example of a smooth valuation is a valuation of the form $K \mapsto \mathrm{vol}(K+A)$, where A is a convex body with smooth boundary and positive curvature.

Based on Alesker's Irreducibility theorem, a rich algebraic structure was introduced on Val^∞ in recent years.

Theorem 9.6 (Alesker [4, 5, 11], Bernig-Fu [18])

(Alesker) *There exists a unique continuous bilinear product on* Val^∞ *such that if*

$$\mu_i(K) = \mathrm{vol}_n(K + A_i), \qquad i = 1, 2,$$

with convex bodies A_i with smooth boundary and positive curvature, then

$$\mu_1 \cdot \mu_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2),$$

where $\Delta : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is the diagonal embedding.

(Bernig-Fu) *There exists a unique continuous bilinear product on* Val^∞ *such that if*

$$\mu_i(K) = \mathrm{vol}_n(K + A_i), \qquad i = 1, 2,$$

with convex bodies A_i with smooth boundary and positive curvature, then

$$\mu_1 * \mu_2(K) = \mathrm{vol}_n(K + A_1 + A_2).$$

(Alesker) *There exists a Fourier-type transform* $\mathbb{F} : \mathrm{Val}^\infty \to \mathrm{Val}^\infty$ *(called Alesker-Fourier transform) such that*

$$\mathbb{F}(\mu_1 \cdot \mu_2) = \mathbb{F}\mu_1 * \mathbb{F}\mu_2, \qquad \mu_1, \mu_2 \in \mathrm{Val}^\infty$$

and

$$\mathbb{F}^2 = \epsilon \qquad \textit{on } \mathrm{Val}_k^\epsilon.$$

We note that, up to some minor modifications, these algebraic structures do not depend on the choice of the Euclidean scalar product. We refer to [13] for more details.

In Sect. 9.4 it will become transparent that product and convolution are very closely related to kinematic formulas.

Theorem 9.7 ([5]) *The map*

$$\mathrm{Val}^\infty \otimes \mathrm{Val}^\infty \to \mathrm{Val}_n \cong \mathbb{R}, \qquad \phi_1 \otimes \phi_2 \mapsto (\phi_1 \cdot \phi_2)_n$$

is perfect and thus induces an injective map with dense image

$$\mathrm{PD} : \mathrm{Val}^\infty \to (\mathrm{Val}^\infty)^*$$

A smooth valuation can be described by a pair of differential forms as follows. Let K be a convex body and $N(K)$ its normal cycle [49]. As a set, it consists of all pairs (x, v), where $x \in \partial K$ and v is an outer normal vector of K at x. This set is an $(n-1)$-dimensional Lipschitz submanifold of the sphere bundle $S\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$, which can be endowed in a canonical way with an orientation. Given a smooth $(n-1)$-form ω on $S\mathbb{R}^n$, we can integrate it over $N(K)$.

Proposition 9.8 *A valuation $\mu \in \mathrm{Val}$ is smooth if and only if there are translation invariant differential forms $\omega \in \Omega^{n-1}(S\mathbb{R}^n), \phi \in \Omega^n(\mathbb{R}^n)$ such that*

$$\mu(K) = \int_{N(K)} \omega + \int_K \phi$$

for all K.

These forms are not unique, see Theorem 9.10 below.

9.3 Valuations and Curvature Measures on Manifolds

In this section, we define smooth valuations on manifolds and describe them in terms of differential forms. Then we extend the product from the previous section to such valuations. Smooth curvature measures on valuations are introduced as measure-valued smooth valuations. They form a *module* over the space of smooth valuations. We refer to [6–10, 12, 14, 16, 17] for valuations on manifolds and to [20, 38] for curvature measures on manifolds.

9.3.1 Smooth Valuations on Manifolds, the Rumin Operator and the Product Structure

If we want to define the notion of valuation on a smooth manifold of dimension n, an obvious obstacle is that the notion of convex set is not available. In the presence of a Riemannian metric, one can define convex sets, but their behaviour is too wild to be a good substitute of the notion of convexity on Euclidean space. Instead, we are led to define valuations on some other class of reasonable sets. One possibility is to use sets of positive reach or variants of it. Another possibility is to use compact submanifolds with corners, also called simple differentiable polyhedra. Each simple differential polyhedron has a conormal cycle, which is an $(n-1)$-dimensional Lipschitz manifold in the cosphere bundle $S^*M = \{(x,[\xi]) : x \in M, \xi \in T_x^*M \setminus \{0\}\}$, where the equivalence classes are taken with respect to the relation $\xi_1 \sim \xi_2$ if and only if $\xi_2 = \lambda \xi_1$ for some $\lambda > 0$.

Here we will not pay much attention to the precise class of sets, since the kinematic formulas in the different settings are formally identical. We refer to [28] for a more thorough study of this question.

The second difference with the flat case is that we do not really define valuations, but only smooth valuations. The definition is inspired by Proposition 9.8.

Definition 9.9 A smooth valuation on an n-dimensional manifold M is a functional μ on the space $\mathscr{P}(M)$ of simple differentiable polyhedra which has the form

$$\mu(P) = \int_{N(P)} \omega + \int_P \phi$$

with smooth forms $\omega \in \Omega^{n-1}(S^*M), \phi \in \Omega^n(M)$. Here $N(P) \subset S^*M$ denotes the conormal cycle of P. The space of smooth valuations on M is denoted by $\mathscr{V}(M)$.

Note that no invariance is assumed in this definition.

Taking $\omega = 0$ and ϕ a volume form (say with respect to a Riemannian metric), we see that the Riemannian volume is a smooth valuation. The Euler characteristic is another example. In fact, Chern [24, 25], when proving the famous Chern-Gauss-Bonnet theorem, constructed a pair (ω, ϕ) as above. By Proposition 9.8, each smooth translation invariant valuation on the vector space $\mathbb{R}^n$ can also be considered as a smooth valuation on the manifold $\mathbb{R}^n$.

An important point to note is that the pair of forms (ω, ϕ) is not unique. Since the conormal cycle is closed, it annihilates exact differential forms. Moreover, it is Legendrian, i.e. it annihilates forms which vanish on the contact distribution in S^*M. The kernel of the map which associates to a pair of forms the corresponding smooth valuation is given in the following theorem.

Theorem 9.10 ([17]) *A pair of forms (ω, ϕ) induces the trivial valuation if and only if*

1. $D\omega + \pi^*\phi = 0$,
2. $\pi_*\omega = 0$.

Here $D : \Omega^{n-1}(S^*M) \to \Omega^n(S^*M)$ is a certain second order differential operator, called Rumin operator [40], π^* denotes pull-back and π_* push-forward (or fiber integration) with respect to the projection map $\pi : S^*M \to M$.

Any operation on pairs of forms (ω, ϕ) which is compatible with the kernel described in Theorem 9.10 thus induces an operation on smooth valuations. An easy example is given by the Euler-Verdier involution, which on the level of forms is given by $(\omega, \phi) \mapsto ((-1)^n a^*\omega, (-1)^n \phi)$, where $a : S^*M \to S^*M, (x, [\xi]) \mapsto (x, [-\xi])$ is the natural involution (anti-podal map).

A much more involved example is the product of smooth valuations on manifolds. The complete formula is rather technical and uses certain blow-up spaces whose definition we prefer to omit.

Theorem 9.11 ([14, 16]) *The space of smooth valuations on a manifold admits a product structure which, on the level of forms, is given by a formula of the type*

$$(\omega_1, \phi_1) \cdot (\omega_2, \phi_2) \mapsto (Q_1(\omega_1, \omega_2), Q_2(\omega_1, \phi_1, \omega_2, \phi_2))$$

Here Q_1, Q_2 are certain explicitly known operators on differential forms involving some Gelfand transform on a blow-up space. This product is commutative, associative, has the Euler characteristic as unit, and is compatible with restrictions to submanifolds.

A nice interpretation of this formula was recently given in [29]. Of course, in the (very) special case of smooth translation invariant valuations on $\mathbb{R}^n$, the new product coincides with the product from Theorem 9.6.

Let us now describe a version of Poincaré duality on the level of valuations on manifolds, which was introduced in [9]. Given any compactly supported smooth valuation μ on a manifold M, we may evaluate it at the manifold M to obtain a real number denoted by $\int \mu$.

If we denote by $\mathcal{V}_c(M)$ the compactly supported smooth valuations on M, then we get a pairing

$$\mathcal{V}(M) \times \mathcal{V}_c(M) \to \mathbb{R}, (\mu_1, \mu_2) \mapsto \int \mu_1 \cdot \mu_2. \tag{9.1}$$

Alesker has shown that this pairing is perfect, i.e. it induces an injective map $\mathcal{V}(M) \to \mathcal{V}_c(M)^*$ with dense image. This fact is important in connection with *generalized valuations on manifolds* (see [9, 14, 15]), but also for kinematic formulas [14, 20].

9.3.2 Curvature Measures, Module Structure

Roughly speaking, a curvature measure is a valuation with values in the space of signed measures. An example are Federer's curvature measures $\Phi_0, \dots, \Phi_n$ in $\mathbb{R}^n$, which (up to scaling) are localizations of the intrinsic volumes.

Definition 9.12 ([20]) A smooth curvature measure on an n-dimensional manifold M is a functional Φ of the form

$$\Phi(P,B)=\int_{N(P)\cap\pi^{-1}B}\omega+\int_{P\cap B}\phi,$$

where P is a simple differentiable polyhedron, $B \subset M$ a Borel subset, and $\omega \in \Omega^{n-1}(S^*M), \phi \in \Omega^n(M)$. The globalization of Φ is the smooth valuation $\operatorname{glob}\Phi(P) := \Phi(P,M)$. The space of all smooth curvature measures on M is denoted by $\mathscr{C}(M)$.

Let us explain where the name "curvature measure" comes from. For this, suppose that P is a compact smooth submanifold of a Riemannian manifold M. The geometry of the second fundamental form of P is then encoded in the conormal cycle $N(P)$. If Φ is a smooth curvature measure, then the measure $\Phi(P,\cdot)$ is obtained by integration of some polynomial function on the second fundamental form, hence by some curvature expression.

As in Theorem 9.10, one may describe the kernel of the map which associates to a pair of differential forms a curvature measure. The pair (ω,ϕ) induces the trivial curvature measure if and only if $\phi=0$ and ω belongs to the ideal generated by α and $d\alpha$, where α is the contact form on S^*M.

It follows that the map $\operatorname{glob} : \mathscr{C}(M) \to \mathscr{V}(M)$ is surjective, but not injective. For example, if ω is an exact form which is not contained in the ideal generated by α and $d\alpha$, then $(\omega,0)$ defines a non-zero curvature measure whose globalization vanishes.

A more general globalization map is obtained as follows. If f is a smooth function on M, we may define a smooth valuation $\operatorname{glob}_f\Phi$ by

$$\operatorname{glob}_f\Phi(P):=\int_{N(P)}\pi^*f\omega+\int_P f\phi.$$

In the particular case $f\equiv 1$, this is just the map glob.

While we can multiply smooth valuations on a manifold, there seems to be no reasonable product structure on the space of smooth curvature measures. However, curvature measures form a module over valuations.

Theorem 9.13 ([20], Based on [14]) *The space $\mathscr{C}(M)$ is a module over the algebra $\mathscr{V}(M)$. More precisely, given $\mu\in\mathscr{V}(M)$, $\Phi\in\mathscr{C}(M)$, there exists a unique curvature measure $\mu\cdot\Phi\in\mathscr{C}(M)$ such that* $\operatorname{glob}_f(\mu\cdot\Phi)=\mu\cdot\operatorname{glob}_f\Phi$ *for all smooth functions f on M.*

The proof follows rather easily from Theorem 9.11.

9.4 Global and Local Kinematic Formulas, the Transfer Principle

In this section, we study the global and local kinematic operators on isotropic spaces, their link to the algebra and module structure from the previous section, and the important transfer principle relating local kinematic formulas on flat and curved spaces.

9.4.1 Fundamental Theorem of Algebraic Integral Geometry I: Flat Case

Definition 9.14 The space of smooth and translation invariant curvature measures on $\mathbb{R}^n$ is denoted by Curv. If G acts linearly on $\mathbb{R}^n$, then Curv^G is the subspace of G-invariant elements.

As an example, $\mathrm{Curv}^{\mathrm{SO}(n)}$ has Federer's curvature measures $\Phi_0, \ldots, \Phi_n$ as basis. In particular, the restricted globalization map glob : $\mathrm{Curv}^{\mathrm{SO}(n)} \to \mathrm{Val}^{\mathrm{SO}(n)}$ is a bijection.

Theorem 9.15 ([8]) *Let G be a subgroup of $\mathrm{O}(n)$. Then Val^G is finite-dimensional if and only if G acts transitively on the unit sphere. In this case, $\mathrm{Val}^G \subset \mathrm{Val}^\infty$ and Curv^G is finite-dimensional as well.*

Let G be such a group and denote by $\bar{G}$ the group generated by G and translations (endowed with a convenient invariant measure). Let $\phi_1, \ldots, \phi_N$ be a basis of Val^G. Then there are constants c_{kl}^i such that the kinematic formulas

$$\int_{\bar{G}} \phi_i(K \cap \bar{g}L) d\bar{g} = \sum_{k,l} c_{k,l}^i \phi_k(K)\phi_l(L), \qquad K, L \in \mathcal{K}^n$$

hold.

Similarly, if $\Phi_1, \ldots, \Phi_M$ is a basis of Curv^G, there are constants $d_{k,l}^i$ such that

$$\int_{\bar{G}} \Phi_i(K \cap \bar{g}L, B_1 \cap \bar{g}B_2) d\bar{g} = \sum_{k,l} d_{k,l}^i \Phi_k(K, B_1)\Phi_l(L, B_2)$$

holds for all convex bodies K, L and all Borel subsets $B_1, B_2 \subset \mathbb{R}^n$. The proof is contained in [26].

We call the corresponding operators

$$k_G : \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$$
$$\phi_i \mapsto \sum_{k,l} c_{k,l}^i \phi_k \otimes \phi_l$$

and

$$K_G : \mathrm{Curv}^G \to \mathrm{Curv}^G \otimes \mathrm{Curv}^G$$
$$\Phi_i \mapsto \sum_{k,l} d^i_{k,l} \Phi_k \otimes \Phi_l$$

the global and local kinematic operators. They are independent of the choice of the bases. Moreover, Fubini's theorem and the unimodularity of G imply that k_G, K_G are cocommutative coassociative coproducts.

Since the diagram

$$\begin{array}{ccc} \mathrm{Curv}^G & \xrightarrow{K_G} & \mathrm{Curv}^G \otimes \mathrm{Curv}^G \\ \downarrow{\scriptstyle \mathrm{glob}} & & \downarrow{\scriptstyle \mathrm{glob} \otimes \mathrm{glob}} \\ \mathrm{Val}^G & \xrightarrow{k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \end{array}$$

clearly commutes and glob is surjective, but usually non-injective, the local kinematic operator contains more information than the global one.

There is another kinematic operator which lies between these two, called semi-local kinematic operator, which is defined by

$$\bar{k}_G := (\mathrm{id} \otimes \mathrm{glob}) \circ K_G : \mathrm{Curv}^G \to \mathrm{Curv}^G \otimes \mathrm{Val}^G.$$

The following theorem, although rather easy to prove, is fundamental for our algebraic understanding of kinematic formulas.

Theorem 9.16 ([18, 27]) *Let* $m : \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathrm{Val}^G$ *denote the restriction of the product to G-invariant valuations and* $m^* : \mathrm{Val}^{G*} \to \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*}$ *its adjoint. Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{k_G} & \mathrm{Val}^G \otimes \mathrm{Val}^G \\ \downarrow{\scriptstyle \mathrm{PD}} & & \downarrow{\scriptstyle \mathrm{PD} \otimes \mathrm{PD}} \\ \mathrm{Val}^{G*} & \xrightarrow{m^*} & \mathrm{Val}^{G*} \otimes \mathrm{Val}^{G*} \end{array} \tag{9.2}$$

Thus knowing the product structure, we can (at least in principle) write down the global kinematic formulas.

9.4.2 The Transfer Principle

Definition 9.17 An isotropic space is a pair (M, G), where M is a Riemannian manifold and G is a Lie subgroup of the isometry group of M which acts transitively on the sphere bundle SM.

Theorem 9.18 ([8, 20]) *Let (M, G) be an isotropic space. Then $\mathscr{V}(M)^G$ and $\mathscr{C}(M)^G$ are finite-dimensional.*

If (M, G) is an isotropic space, there exist local kinematic formulas as follows. Let X, Y be sufficiently nice sets (compact submanifolds with corners will be enough for our purpose) and let $B_1, B_2 \subset M$ be Borel subsets. If $\Phi_1, \ldots, \Phi_N$ is a basis of $\mathscr{C}(M)^G$, then there are constants $d^i_{k,l}$ such that

$$\int_G \Phi_i(X \cap gY, B_1 \cap gB_2)dg = \sum_{k,l} d^i_{k,l}\Phi_k(X, B_1)\Phi_l(Y, B_2).$$

The existence of such formulas is harder to prove than in the flat case (compare [20, 26]). One reason for this is that it is unknown whether there are G-invariant continuous, but non-smooth valuations (even in the simplest case of the sphere).

As in the flat case, we obtain a map $K_G : \mathscr{C}(M)^G \to \mathscr{C}(M)^G \otimes \mathscr{C}(M)^G$. Semi-local and global kinematic formulas are defined similarly, the corresponding operators are denoted by $\bar{k}_G, k_G$. Then $(\mathscr{V}(M)^G, k_G)$ and $(\mathscr{C}(M)^G, K_G)$ are cocommutative coassociative coalgebras and the following diagram is commutative.

$$\begin{array}{ccc}
\mathcal{C}(M)^G & \xrightarrow{K_G} & \mathcal{C}(M)^G \otimes \mathcal{C}(M)^G \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} \otimes \mathrm{glob} \\
\mathcal{C}(M)^G & \xrightarrow{\bar{k}_G} & \mathcal{C}(M)^G \otimes \mathcal{V}(M)^G \\
\downarrow \mathrm{glob} & & \downarrow \mathrm{glob} \otimes \mathrm{id} \\
\mathcal{V}(M)^G & \xrightarrow{k_G} & \mathcal{V}(M)^G \otimes \mathcal{V}(M)^G
\end{array} \tag{9.3}$$

The transfer principle below will apply to the following families of isotropic spaces.

Table 9.1

Division algebra	Positively curved	Flat	Negatively curved
$\mathbb{R}$	$(S^n, \mathrm{SO(n+1)})$	$(\mathbb{R}^n, \overline{\mathrm{SO}(n)})$	$(\mathbb{H}^n, \mathrm{SO}(n,1))$
$\mathbb{C}$	$(\mathbb{C}P^n, \mathrm{PU}(n+1))$	$(\mathbb{C}^n, \overline{\mathrm{U}(n)})$	$(\mathbb{C}H^n, \mathrm{PU}(n,1))$
$\mathbb{H}$	$(\mathbb{H}P^n, \mathrm{Sp}(n+1))$	$(\mathbb{H}^n, \overline{\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)})$	$(\mathbb{H}H^n, \mathrm{Sp}(n,1))$
$\mathbb{O}$	$(\mathbb{O}P^2, \mathrm{F}_4)$	$(\mathbb{O}^2, \overline{\mathrm{Spin}(9)})$	$(\mathbb{O}H^2, \mathrm{F}_4^{-20})$

Here $\mathbb{H}$ stands for the skew field of quaternions and $\mathbb{O}$ for the division algebra of octonions. In the second column stand the simply connected positively curved space forms, in the third column the corresponding flat space and in the last column the corresponding hyperbolic space form. Scaling by the curvature yields in all four cases a family (M_λ, G_λ) indexed by the curvature.

Theorem 9.19 ([20, 33]) *Let $(M_\lambda, G_\lambda), \lambda \in \mathbb{R}$ be one of the families from the table. Then the coalgebras $(\mathscr{C}(M_\lambda)^{G_\lambda}, K_{G_\lambda}), \lambda \in \mathbb{R}$ are naturally isomorphic to each other.*

In particular, knowing the local kinematic formulas in the flat case, we can write down the local kinematic formulas in the curved cases, and from these formulas we can derive the global kinematic formulas in the curved cases by using (9.3).

9.4.3 Fundamental Theorem of Algebraic Integral Geometry II: Curved Case

We now formulate an analogue of Theorem 9.16 in the curved case. Let (M, G) be an isotropic space. Then $\mathscr{V}(M)^G$ is finite-dimensional and admits a product structure. Hence the horizontal maps in (9.2) generalize to the curved case. It is less obvious how to substitute the vertical map, i.e. the Poincaré duality.

In the compact case, we can use (9.1) to define a map $\mathrm{PD} : \mathscr{V}(M)^G \to \mathscr{V}(M)^{G*}$. To generalize this to the non-compact case, one has to give another interpretation. It turns out that there exists a unique element $\mathrm{vol}^* \in \mathscr{V}(M)^{G*}$ with the property that $\langle \mathrm{vol}^*, \mathrm{vol}\rangle = 1$ and such that vol^* annihilates all smooth valuations given by a pair of forms of the type $(\omega, 0)$. Up to some normalization factor, vol^* corresponds to $\int_M$ in the compact case and to PD in the flat case.

In all cases, we obtain a map (called normalized Poincaré duality)

$$\mathrm{pd} : \mathscr{V}^G(M) \to \mathscr{V}^G(M)^*, \qquad \langle \mathrm{pd}(\phi), \mu\rangle := \langle \mathrm{vol}^*, \mu \cdot \phi\rangle.$$

The analogue of Theorem 9.16 is the following.

Theorem 9.20 *Let (M, G) be an isotropic space. Let $m : \mathscr{V}^G(M) \otimes \mathscr{V}^G(M) \to \mathscr{V}^G(M)$ be the restricted multiplication map, $\mathrm{pd} : \mathscr{V}^G(M) \to \mathscr{V}^G(M)^*$ the normalized Poincaré duality and $k_G : \mathscr{V}^G(M) \to \mathscr{V}^G(M) \otimes \mathscr{V}^G(M)$ the kinematic coproduct. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{V}^G(M) & \xrightarrow{k_G} & \mathcal{V}^G(M) \otimes \mathcal{V}^G(M) \\ \downarrow \mathrm{pd} & & \downarrow \mathrm{pd}\otimes\mathrm{pd} \\ \mathcal{V}^G(M)^* & \xrightarrow{m^*} & \mathcal{V}^G(M)^* \otimes \mathcal{V}^G(M)^* \end{array}$$

As before, this can be used in two ways. Information on the algebra structure on $\mathscr{V}^G(M)$ can be translated into information on the (global) kinematic operator. In the Hermitian case, we will rather go the other way and translate information from the kinematic operator into information on the algebra structure.

The ultimate goal, however, is not to compute the global formulas only, but to compute the local kinematic formulas. One reason is that they are curvature independent (see Theorem 9.19), the other is that they contain more information than the global formulas. Unfortunately, an analogue of Theorem 9.20 for curvature measures and local kinematic formulas is unknown (and it seems unlikely to exist). Nevertheless, there is an improvement of Theorem 9.20 which relates local formulas and the module structure from Theorem 9.13.

Let $\bar{m} : \mathscr{V}^G(M) \otimes \mathscr{C}^G(M) \to \mathscr{C}^G(M), \mu \otimes \Phi \mapsto \mu \cdot \Phi$ denote the module product. Alternatively, we may think of this as a map $\bar{m} : \mathscr{C}^G(M) \to \mathscr{C}^G(M) \otimes \mathscr{V}^G(M)^*$.

Theorem 9.21 *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{C}^G(M) & \xrightarrow{\bar{k}_G} & \mathcal{C}^G(M) \otimes \mathcal{V}^G(M) \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \otimes \mathrm{pd} \\ \mathcal{C}^G(M) & \xrightarrow{\bar{m}} & \mathcal{C}^G(M) \otimes \mathcal{V}^G(M)^* \end{array} \tag{9.4}$$

Moreover, for $\phi \in \mathscr{V}^G(M)$ and $\Phi \in \mathscr{C}^G(M)$ we have

$$K_G(\phi \cdot \Phi) = (\phi \otimes \chi) \cdot K_G(\Phi) = (\chi \otimes \phi) \cdot K_G(\Phi).$$

The knowledge of the module structure will thus tell us a lot, although not everything about the local kinematic formulas.

9.5 Hermitian Integral Geometry: The Flat Case

In Sect. 9.4 we have laid out the theoretical ground for the study of the local kinematic formulas for the spaces from Table 9.1. Needless to say that to work out this program in practice requires some additional effort and ideas. In fact, only in the real and in the complex case the local kinematic formulas are known so far.

Before going to the complex case, let us say a few words about the real case. The main point here is that the globalization map $\mathrm{glob} : \mathrm{Curv}^{\mathrm{SO}(n)} \to \mathrm{Val}^{\mathrm{SO}(n)}$ is bijective. From (9.3) we can thus read off the local kinematic formulas from the global ones, and the latter are just the well-known classical Chern-Blaschke-Santaló formulas. Once the local formulas are determined, one can globalize again to write down global kinematic formulas for spheres and hyperbolic spaces.

The situation in the complex (and quaternionic and octonionic) case is different. The dimension of $\mathrm{Curv}^{\mathrm{U}(n)}$ is roughly twice that of $\mathrm{Val}^{\mathrm{U}(n)}$, hence the globalization map has a large kernel and the local formulas contain much more information than the global ones. But even to work out the global formulas is not an easy task. In this section we describe the ideas leading to a complete determination of the global kinematic formulas in the flat case $(\mathbb{C}^n, \mathrm{U}(n))$.

9.5.1 Vector Space Structure

Intrinsic volumes on a Euclidean vector space can be introduced or characterized in a number of ways, for instance averaging some functional applied to projections onto linear subspaces, averaging some functional applied to intersections with affine subspaces, or by prescribing their Klain functions.

In the complex case, one can do analogous constructions to define some interesting valuations. However, in contrast to the real case, this gives *different* bases for the space of invariant valuations.

Alesker showed in [3] that

$$\dim \mathrm{Val}_k^{\mathrm{U}(n)} = \min\left\{\left\lfloor \frac{k}{2}\right\rfloor, \left\lfloor \frac{2n-k}{2}\right\rfloor\right\} + 1. \tag{9.5}$$

Using intersections with affine complex planes, Alesker defined

$$U_{k,p}(K) := \int_{A_{\mathbb{C}}(n,n-p)} \mu_{k-2p}(K \cap \bar{E})\, d\bar{E},$$

where $A_{\mathbb{C}}(n, n-p)$ is the affine complex Grassmannian, endowed with a translation invariant and $\mathrm{U}(n)$-invariant measure.

The $U_{k,p}$, as p ranges over $0, 1, \ldots, \min\{\lfloor \frac{k}{2}\rfloor, \lfloor \frac{2n-k}{2}\rfloor\}$, constitute a basis of $\mathrm{Val}_k^{\mathrm{U}(n)}$.

Fu [27] renormalized these valuations by setting

$$t := \tfrac{2}{\pi}\mu_1 = \tfrac{2}{\pi} U_{1,0} \in \mathrm{Val}_1^{\mathrm{U}(n)} \tag{9.6}$$

$$s := n U_{2,1} \in \mathrm{Val}_2^{\mathrm{U}(n)} \tag{9.7}$$

which implies that

$$s^p t^{k-2p} = \frac{(k-2p)!n!\omega_{k-2p}}{(n-p)!\pi^{k-2p}} U_{k,p},$$

where the monomial on the left hand side refers to the Alesker product.

The second basis given by Alesker uses projections onto linear complex subspaces instead of intersections:

$$C_{k,q}(K) := \int_{G_{\mathbb{C}}(n,q)} \mu_k(\pi_E(K))\, dE,$$

where $G_{\mathbb{C}}(n,q)$ is the complex Grassmannian.

As q ranges over all values from $n - \min\{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{2n-k}{2} \rfloor\}$ to n, the $C_{k,q}$ constitute a basis of $\mathrm{Val}_k^{\mathrm{U}(n)}$. Up to a normalizing constant, the Alesker-Fourier transform of $U_{k,p}$ is $C_{2n-k,n-p}$.

A third basis, consisting of *Hermitian intrinsic volumes*, will be particularly useful. Recall that a real subspace E of $\mathbb{C}^n$ is called *isotropic* if the restriction of the symplectic form to E vanishes. Then the dimension of E does not exceed n, and an isotropic subspace of dimension n is called *Lagrangian*. We call E of type (k,q) if E can be written as the orthogonal sum of a complex subspace of (complex) dimension q and an isotropic subspace of dimension $k-2q$. Then $k - q \leq n$.

Theorem 9.22 *There is a unique valuation $\mu_{k,q} \in \mathrm{Val}_k^{\mathrm{U}(n)}$ whose Klain function evaluated at a subspace of type (k,q') equals $\delta_{qq'}$. Moreover,*

$$\mathbb{F}\mu_{k,q} = \mu_{2n-k,n-k+q}.$$

There are two more interesting bases related to Hermitian intrinsic volumes. The first one is quite useful from a geometric point of view. It consists of so called *Tasaki valuations*. As Hermitian intrinsic volumes, they are defined via their Klain function. The orbit space $G_{\mathbb{C}}(n,k)/U(n)$ can be described in terms of $\min\{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{2n-k}{2} \rfloor\}$ Kähler angles, and the Klain function of a Tasaki valuation is an elementary symmetric polynomial in the cosines of the Kähler angles. Comparison of the Klain functions then easily yields the explicit expression

$$\tau_{k,q} = \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q} \mu_{k,i}.$$

The last basis to be mentioned here is very useful for computational purposes. As is explained in [43], the $\mathrm{SO}(2n)$-module $\mathrm{Val}_k(\mathbb{C}^n)$ may be decomposed into a multiplicity free direct sum of irreducible representations. It is well known which of these irreducible representations contain $\mathrm{U}(n)$-invariant vectors, moreover these vectors are unique up to scale. Explicitly, this yields the valuations

$$\pi_{k,r} = (-1)^r (2n-4r+1)!! \sum_{i=0}^{r} (-1)^i \frac{(k-2i)!}{(2r-2i)!} \frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!} \tau_{k,i}, \qquad (9.8)$$

which form the so called *primitive basis*.

An interesting new line of research was opened by Abardia and Wannerer [1], who studied versions of the classical isoperimetric inequality, but with the usual intrinsic volumes replaced by unitarily invariant valuations. In small degrees ($k \leq 3$) they study which linear combinations of Hermitian intrinsic volumes satisfy an Alexandrov-Fenchel-type inequality, from which several other isoperimetric inequalities can be derived.

9.5.2 Algebra Structure

The next step is to write down the algebra structure, i.e. the Alesker product. Depending on which basis we take (Akesker's valuation $U_{k,p}, C_{k,p}$, Hermitian intrinsic volumes, Tasaki valuations, primitive basis), this task may be more or less difficult. The main point of entry is to determine the product of t and s with a Hermitian intrinsic volume. We will not go into the details of these computations. A more or less complete set of formulas was written down in [19].

The computation of the algebra $\mathrm{Val}^{\mathrm{U}(n)}$ was obtained earlier by Fu [27].

Theorem 9.23 (Fu [27]) *Let t, s be variables of degree 1 and 2 respectively. Let f_i be the component of total degree i in the expansion of $\log(1 + t + s)$. Then*

$$\mathrm{Val}^{\mathrm{U}(n)} \cong \mathbb{R}[t, s]/(f_{n+1}, f_{n+2}).$$

9.5.3 Global Kinematic Formulas

As stated in Theorem 9.16, the product structure may be translated into kinematic formulas. To write down explicit closed formulas valid in all dimensions is not straightforward, since we have to invert some matrices. However, in the primitive basis, the corresponding matrices are anti-diagonal and therefore easy to invert. One can then write down the principal kinematic formula in explicit form as follows.

Theorem 9.24 (Principal Kinematic Formula [19]) *The principal kinematic formula for $(\mathbb{C}^n, \mathrm{U}(n))$ is given by*

$$k_{\mathrm{U}(n)}(\chi) = \sum_{k=0}^{2n} \sum_{r=0}^{\min\{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{2n-k}{2} \rfloor\}} a_{n,k,r} \pi_{k,r} \otimes \pi_{2n-k,r}$$

with

$$a_{n,k,r} := \frac{\omega_k \omega_{2n-k}}{\pi^n} \frac{(n-r)!}{8^r (2n-4r)!} \frac{(2n-2r+1)!!}{(2n-4r+1)!!} \binom{n}{2r}^{-1}.$$

Together with the knowledge of the product structure, other kinematic formulas, as well as kinematic formulas in other bases, may be derived from the principal kinematic formula. It is not clear which of these formulas can be written in such a neat and closed form as the formula above.

9.6 Hermitian Integral Geometry: The Curved Case

In this section, we describe more recent work from [20]. Two of the previously open major problems in Hermitian integral geometry were to compute local kinematic formulas and to compute global kinematic formulas in the curved case (say on $\mathbb{C}P^n$ and $\mathbb{C}H^n$). These two problems are interrelated by Theorem 9.19: the local kinematic formula is independent of the curvature and globalizes simultaneously to the different global kinematic formulas in the space forms. Roughly speaking, the knowledge of the local kinematic formulas and the knowledge of the different global kinematic formulas is equivalent. Some partial results were known before [20]: Park established local kinematic formulas in small degrees ($n \leq 3$), while Abardia, Gallego and Solanes [2] proved Crofton-type formulas, which are special cases of the general kinematic formulas.

We write $\mathbb{C}P^n_\lambda$ for the complex space form with holomorphic curvature 4λ. If $\lambda > 0$, this is the complex projective space with an appropriate scaling of the Fubini-Study metric; if $\lambda < 0$ this is complex hyperbolic space with an appropriate scaling of the Bergman metric; and if $\lambda = 0$, this is just the Hermitian space $\mathbb{C}^n$. The holomorphic isometry group of $\mathbb{C}P^n_\lambda$ will be denoted by G_λ. Hence $G_\lambda \cong \mathrm{PU}(n+1)$ if $\lambda > 0$; $G_\lambda \cong \mathrm{PU}(n,1)$ if $\lambda < 0$ and $G_\lambda \cong \mathbb{C}^n \rtimes \mathrm{U}(n)$ if $\lambda = 0$.

Before describing the solution, let us write down some intermediate steps. The first step is to write down a basis for $\mathrm{Curv}^{\mathrm{U}(n)}$. It was basically achieved by Park [38]. He determined a basis

$$B_{k,q} \qquad (k > 2q, n \geq k-q) \tag{9.9}$$

$$\Gamma_{k,q} \qquad (n > k-q, k \geq 2q) \tag{9.10}$$

of curvature measures by writing down explicit invariant differential forms on the sphere bundle.

It is rather easy to describe the kernel of the globalization map $\mathrm{glob}_\lambda : \mathrm{Curv}^{\mathrm{U}(n)} \cong \mathscr{C}^{G_\lambda}(\mathbb{C}P^n_\lambda) \to \mathscr{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$. More precisely, this kernel is spanned by all curvature measures of the form

$$N_{k,q} + \lambda \frac{q+1}{\pi} B_{k+2,q+1}, \qquad k > 2q, q > k-n,$$

where $N_{k,q} := \frac{2(n-k+q)}{2n-k}(\Gamma_{k,q} - B_{k,q})$. Note that this kernel depends on λ. In curvature 0, both $B_{k,q}$ and $\Gamma_{k,q}$ globalize to the Hermitian intrinsic volume $\mu_{k,q}$.

The next problems are

1. Compute the module structure of $\mathrm{Curv}^{\mathrm{U}(n)}$ over $\mathrm{Val}^{\mathrm{U}(n)}$.
2. Compute the module structure of $\mathrm{Curv}^{\mathrm{U}(n)}$ over $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$.
3. Compute the local and semi-local formulas in curvature λ.
4. Compute the product structure of $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$.

All of these problems were solved in [20], but not in the order given here. In fact, none of these problems could be solved independently of the others, but all had to be solved simultaneously.

9.6.1 Module Structure

Even if it is not obvious how to determine the module structure of $\mathrm{Curv}^{\mathrm{U}(n)}$ over $\mathrm{Val}^{\mathrm{U}(n)}$, some a priori information can be obtained by geometric means.

Definition 9.25 Let V be a euclidean space of dimension m. A translation-invariant curvature measure $\Phi \in \mathrm{Curv}(V)$ is called *angular* if, for any compact convex polytope $P \subset V$,

$$\Phi(P, \cdot) = \sum_{k=0}^{m} \sum_{F \in \mathfrak{F}_k(P)} c_\Phi(\mathbf{F}) \angle(F, P) \operatorname{vol}_k|_F \tag{9.11}$$

where $c_\Phi(\mathbf{F})$ depends only on the k-plane $\mathbf{F} \in G(n, k)$ parallel to F and where $\angle(F, P)$ denotes the outer angle of F in P. The space of such translation-invariant curvature measures will be denoted by $\mathrm{Ang}(V)$.

Theorem 9.26 (Angularity Theorem) *The product of the first intrinsic volume with an angular measure is again angular.*

Let us return to the complex case. Define curvature measures $\Delta_{kq} \in \mathrm{Curv}^{U(n)}$, $\max\{0, k-n\} \le q \le \frac{k}{2} < n$ by

$$\Delta_{kq} := \frac{1}{2n-k}\big(2(n-k+q)\Gamma_{kq} + (k-2q)B_{kq}\big) \tag{9.12}$$

$$\Delta_{2n,n} := \operatorname{vol}_{2n}. \tag{9.13}$$

Thus $\Delta_{2q,q} = \Gamma_{2q,q}$ and $\Delta_{k,k-n} = B_{k,k-n}$. Then the subspace of angular elements in $\mathrm{Curv}^{U(n)}$ is precisely the span of the Δ_{kq}. This gives us some a priori information about the product of t with an invariant measure, but of course does not determine it completely.

The next piece of information concerns multiplication by s.

Proposition 9.27 *Let* $\mathrm{Beta}_k \subset \mathrm{Curv}_k^{U(n)}$ *be the subspace spanned by all* $B_{k,q}$*. Then* $s \cdot \mathrm{Beta}_k \subset \mathrm{Beta}_{k+2}$.

The proof of this proposition, as well as the proof of the angularity theorem, only uses elementary geometric considerations.

Apparently, the previous proposition gives us only information about the module structure of $\mathrm{Curv}^{U(n)}$ over $\mathrm{Val}^{\mathrm{U}(n)}$ in the flat case. In reality, the next proposition tells us that a similar statement holds true in the curved case as well. For this, we first have to define s as an element of $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$. Suppose first that $\lambda > 0$ and fix a complex hyperplane P_λ. Then we define

$$s := \frac{\lambda^{n-1} n!}{\pi^n} \int_{G_\lambda} \chi(g P_\lambda \cap \cdot) dg.$$

Regarding λ as a parameter, one can then show that this definition extends analytically to all $\lambda \in \mathbb{R}$ and, in the case $\lambda = 0$, coincides with the definition of s from (9.7).

Recall that we have a canonical identification of the space $\mathcal{C}^{G_\lambda}(\mathbb{C}P^n_\lambda)$ with $\mathrm{Curv}^{\mathrm{U}(n)}$.

Proposition 9.28 *The multiplication by s on* $\mathcal{C}^{G_\lambda}(\mathbb{C}P^n_\lambda) \cong \mathrm{Curv}^{\mathrm{U}(n)}$ *is independent of* λ.

The proof is basically an application of the transfer principle.

Propositions 9.27 and 9.28, together with some elementary properties, determine completely the multiplication by s.

The situation for t is a bit different. First we define t as an element of $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$. In fact, there is a unique assignment of a valuation t to a Riemannian manifold with the two properties that in the Euclidean case $t = \frac{2}{\pi}\mu_1$ and that the assignment is compatible with isometric immersions. This element is the generator of the so called Lipschitz-Killing algebra [8, 20].

For each k, the valuation t^k can be expressed by integration over a pair of forms (ω, ϕ), both of which are defined in terms of the Riemann curvature tensor. In the special case of $\mathbb{C}P^n_\lambda$, the Riemann curvature tensor is of course well-known, and modulo some combinatorial difficulties it is straightforward to express t^k as the globalization of some linear combination of basic curvature measures $B_{k,q}, \Gamma_{k,q}$.

9.6.2 The Isomorphism Theorem

The next aim is to determine the product structure in the curved case, i.e. the algebra structure of $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$.

We will need a basis of $\mathcal{V}^{G_\lambda}(\mathbb{C}P^n_\lambda)$ which will play the role of Hermitian intrinsic volumes.

Definition 9.29 For $\max\{0, k-n\} \leq q \leq \frac{k}{2} \leq n$ we set

$$\mu_{kq}^{\lambda} := \sum_{i \geq 0} \frac{\lambda^i (q+i)!}{\pi^i q!} \operatorname{glob}_\lambda(\Delta_{k+2i,q+i}) \in \mathcal{V}^{G_\lambda}(\mathbb{C}P_\lambda^n),$$

where $\Delta_{k+2i,q+i} \in \operatorname{Curv}^{\mathrm{U}(n)}$ was defined in (9.12), (9.13). These valuations form a basis of $\mathcal{V}^{G_\lambda}(\mathbb{C}P_\lambda^n)$.

If $k > 2q$, then μ_{kq}^{λ} is just the globalization of B_{kq}. In the case $\lambda = 0$, we obtain the usual Hermitian intrinsic volumes: $\mu_{kq}^0 = \mu_{kq}$.

Using that we can write both $s^i t^{k-2i}$ and μ_{kq}^{λ} as globalizations of some curvature measures, it is then possible to establish a complete dictionary between these two bases. The result is

$$\mu_{kq}^{\lambda} = (1-\lambda s) \sum_{i=q}^{\lfloor \frac{k}{2} \rfloor} (-1)^{i+q} \binom{i}{q} \frac{\pi^k}{\omega_k (k-2i)!(2i)!} v^{\frac{k}{2}-i} u^i,$$

where $v := t^2(1-\lambda s), u := 4s - v$. Here $v^{\frac{r}{2}}$ has to be understood in the sense of power series if r is odd, but only finitely many terms will be non-zero for a given dimension n.

From these computations follows the next, rather surprising, theorem.

Theorem 9.30 (Isomorphism Theorem) *The algebras $\mathcal{V}^{G_\lambda}(\mathbb{C}P_\lambda^n), \lambda \in \mathbb{R}$ are pairwise isomorphic. More precisely, the map $s \mapsto s, t \mapsto t\sqrt{1-\lambda s}$ induces an isomorphism $\operatorname{Val}^{\mathrm{U}(n)} \to \mathcal{V}^{G_\lambda}(\mathbb{C}P_\lambda^n)$.*

A related statement concerns the principal kinematic formula for $\mathbb{C}P_\lambda^n$.

Theorem 9.31 (Global Principal Kinematic Formula) *The principal kinematic formula $k_\lambda(\chi)$, expressed in the basis μ_{kq}^{λ}, is independent of λ.*

Since we know the principal kinematic formula in the case $\lambda = 0$ (see Theorem 9.24), we can therefore write it down for all λ without any additional effort.

9.6.3 Local Kinematic Formulas

By Proposition 9.28, multiplication by s on $\operatorname{Curv}^{\mathrm{U}(n)}$ is curvature independent. This is not true for t.

However, in the flat case $\lambda = 0$ one can completely determine the action of t on $\operatorname{Curv}^{\mathrm{U}(n)}$ by using Theorem 9.26 and some elementary properties (for instance that the actions of t and s commute). Hence the module structure of $\operatorname{Curv}^{\mathrm{U}(n)}$ over $\operatorname{Val}^{\mathrm{U}(n)}$ is known. It turns out a posteriori that a version of the angularity theorem is valid in the curved case as well, but an a priori proof of this theorem seems to be missing.

We now have enough information to find out the local kinematic formulas. More precisely, we first know that the operator K is curvature independent by Theorem 9.19. From the global kinematic formulas (which are known by Theorem 9.31), we can compute $(\mathrm{glob}_\lambda \otimes \mathrm{glob}_\lambda) \circ K$. The module structure in the flat case implies the knowledge of $(\mathrm{glob}_0 \otimes \mathrm{id}) \circ K$. These properties fix K.

To write down explicit formulas, we have to introduce some more notation.

Definition 9.32 Let $N_{1,0} := \Delta_{1,0} - B_{1,0}$. Define two natural maps from $\mathrm{Val}^{U(n)}$ to $\mathrm{Curv}^{U(n)}$ by

$$\mathfrak{l}(\phi) := \phi \Delta_{0,0}, \qquad \mathfrak{n}(\phi) := \phi N_{1,0}.$$

The importance of these maps comes from the following proposition.

Proposition 9.33 *The* $\mathrm{Val}^{\mathrm{U}(n)}$*-module* $\mathrm{Curv}^{\mathrm{U}(n)}$ *is generated by* $\Delta_{0,0}$ *and* $N_{1,0}$.

Since the module structure is completely known, the proof is easy.

Theorem 9.21 and Proposition 9.33 imply that the local kinematic operator K is completely determined by $K(\Delta_{0,0})$ and $K(N_{1,0})$. We have already sketched above how to compute K. The main result is the following.

Recall the valuations $\pi_{k,r}$ from (9.8) and the constants $a_{n,k,r}$ from Theorem 9.24. Define

$$\begin{aligned}\rho_{kr} := {} & \frac{2(-1)^r(2n-4r+1)!!\pi^{k-1}}{\omega_k} \\ & \times \left(\frac{(2r-1)!!(k+1)!}{(2n-2r+1)!!(2r)!} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^{i+1}}{(2i+3)!(k-2i-1)!} t^{k-2i-1} u^i \right. \\ & \left. + \sum_{i=0}^{r-1} \frac{(-1)^i (2r-2i-3)!!}{(2n-2r-2i-1)!!(2r-2i-2)!(2i+2)!} t^{k-2i-1} u^i \right) \in \mathrm{Val}^{\mathrm{U}(n)},\end{aligned} \tag{9.14}$$

where $u = 4s - t^2$.

Theorem 9.34 (Principal Local Kinematic Formulas [20])

$$K(\Delta_{00}) = \sum a_{nkr} \left[\mathfrak{l}(\pi_{kr}) \otimes \mathfrak{l}(\pi_{2n-k,r}) - \mathfrak{n}(\rho_{kr}) \otimes \mathfrak{n}(\rho_{2n-k,r}) \right]. \tag{9.15}$$

$$\begin{aligned} K(N_{10}) = \sum a_{nkr} \big[& \mathfrak{n}(\pi_{kr}) \otimes \mathfrak{l}(\pi_{2n-k,r}) + \mathfrak{l}(\pi_{kr}) \otimes \mathfrak{n}(\pi_{2n-k,r}) \\ & - \mathfrak{n}(\pi_{kr}) \otimes \mathfrak{n}(\rho_{2n-k,r}) - \mathfrak{n}(\rho_{kr}) \otimes \mathfrak{n}(\pi_{2n-k,r}) \big]. \end{aligned} \tag{9.16}$$

9.7 Local Additive Kinematic Formulas for Hermitian Area Measures and Tensor Valuations

In this section, we describe a recent deep theorem by Wannerer [47, 48], which gives another type of localization of the global kinematic formulas. The theory of unitarily invariant area measures has some similarities with that of unitarily invariant curvature measures.

9.7.1 *Area Measures and Local Additive Kinematic Formulas*

Roughly speaking, an area measure is a valuation on a Euclidean vector space with values in the space of signed measures on the unit sphere. An example is *the* area measure $S_{n-1}(K,\cdot)$ in an n-dimensional Euclidean space, or more generally the kth area measure $S_k(K,\cdot)$, compare [42].

Definition 9.35 A smooth area measure on an n-dimensional Euclidean vector space V is a functional of the form

$$\Phi(K,B)=\int_{N(K)\cap\pi_2^{-1}B}\omega,$$

where ω is a smooth translation invariant $(n-1)$-form on the sphere bundle $SV=V\times S^{n-1}$, K is a convex body, $B\subset S^{n-1}$ a Borel subset of the unit sphere and $\pi_2:SV\to S^{n-1}$ the projection on the sphere. The space of smooth area measures is denoted by $\operatorname{Area}(V)$ or just by Area.

Plugging $B:=S^{n-1}$ into a smooth area measure, we obtain a smooth valuation. The corresponding map is called globalization map and denoted by glob.

As was the case for Curv, Area is a module over Val. However, the algebra structure on Val is not the Alesker product, but the convolution product.

Theorem 9.36 ([48]) *There is a unique module structure of* Area *over* $(\mathrm{Val}^\infty,*)$ *such that if* $\phi(K)=\operatorname{vol}(K+A)$ *with A a smooth convex body with strictly positive curvature, then*

$$\phi*\Phi(K,B)=\Phi(K+A,B).$$

Let G be a subgroup of $\mathrm{SO}(n)$ acting transitively on the unit sphere. Then Area^G, the space of G-invariant area measures, is finite-dimensional. Wannerer proved the existence of local additive kinematic formulas as follows.

Theorem 9.37 ([47]) *There exists a linear map* $A : \mathrm{Area}^G \to \mathrm{Area}^G \otimes \mathrm{Area}^G$ *such that*

$$A(\Phi)(K, B_1; L, B_2) = \int_G \Phi(K + gL, B_1 \cap gB_2)dg.$$

It is called local additive kinematic operator.

The local additive kinematic formula, its semi-local version $\bar{a} := (\mathrm{id} \otimes \mathrm{glob}) \circ A$, and the global additive kinematic formula $a_G : \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$ fit into a diagram analogous to (9.3). Moreover, the relation between the semi-local formula and the module structure is as in (9.4).

9.7.2 The Moment Map and Additive Kinematic Formulas for Tensor Valuations

In Sect. 9.6 we have described how to obtain local kinematic formulas for curvature measures in the Hermitian case. One main ingredient was the passage to complex space forms and the use of the transfer principle. For area measures, this strategy does not work, since there are no area measures on space forms. Instead, Wannerer uses the moment map, which relates local additive kinematic formulas and additive kinematic formulas for tensor valuations.

Definition 9.38 Let $\mathrm{Val}^r := \mathrm{Val} \otimes \mathrm{Sym}^r$ denote the space of tensor valuations of rank r. The r-th moment map is the map

$$\begin{aligned} M^r : \mathrm{Area} &\to \mathrm{Val}^r \\ \Phi &\mapsto \int_{S^{n-1}} u^r d\Phi(\cdot, u) \end{aligned}$$

Theorem 9.39 (Wannerer [47]) *Let* A *be the local additive kinematic operator, and* a^{r_1,r_2} *the additive kinematic operator for tensor valuations (see [21, 22]). The following diagram commutes*

$$\begin{array}{ccc} \mathrm{Area}^G & \xrightarrow{A} & \mathrm{Area}^G \otimes \mathrm{Area}^G \\ \downarrow{\scriptstyle M^{r_1+r_2}} & & \downarrow{\scriptstyle M^{r_1} \otimes M^{r_2}} \\ \mathrm{Val}^{r_1+r_2,G} & \xrightarrow{a^{r_1,r_2}} & \mathrm{Val}^{r_1,G} \otimes \mathrm{Val}^{r_2,G} \end{array}$$

From the theorem we can derive a strategy to compute the operator A, i.e. the local additive kinematic formulas: we have to choose r_1, r_2 in such a way that M^{r_1} and M^{r_2} are injective, and then a^{r_1,r_2} will determine A.

9.7.3 Hermitian Case

Park's result [38], which was already used in Sect. 9.6 to determine the unitarily invariant curvature measures, gives the following characterization.

Proposition 9.40 *The area measures $B_{k,q}, \Gamma_{k,q}$ with the same restrictions on k, q as in (9.9) and with $k < 2n$, form a basis of the space $\mathrm{Area}^{\mathrm{U}(n)}$ of smooth $\mathrm{U}(n)$-invariant area measures.*

The module structure of $\mathrm{Area}^{\mathrm{U}(n)}$ over $(\mathrm{Val}^{\mathrm{U}(n)}, *)$ was determined earlier by Wannerer in [48].

Proposition 9.41 *The module structure of $\mathrm{Area}^{\mathrm{U}(n)}$ over $(\mathrm{Val}^{\mathrm{U}(n)}, *)$ has the following properties.*

1. *The subspace generated by all $\Gamma_{k,q}$ is a submodule.*
2. *If $\hat{s}$ denotes the Alesker-Fourier transform of s, then $\hat{s} * B_{k,q}$ is a linear combination of $B_{k',q'}$'s.*

Both properties follow from a careful inspection of the formula from [18]. Since the module product is compatible with the globalization map, the first property allows to determine $\hat{t} * \Gamma_{k,q}$ and $\hat{s} * \Gamma_{k,q}$, and the second property allows to determine $\hat{s} * B_{k,q}$. Finally, $\hat{t} * B_{k,q}$ can be easily written down using some Lie derivative computation. We refer to the original paper [48] for the statement of the theorem.

Finally, let us consider the local additive kinematic formulas for unitarily invariant area measures. In order to work out the strategy sketched in the previous section, Wannerer showed in [47] that the second moment map $M^2 : \mathrm{Area}^{\mathrm{U}(n)} \to \mathrm{Val}^{2,\mathrm{U}(n)}$ is injective. He then went on to compute the relevant parts of the additive kinematic formula $a^{2,2}$ for unitarily invariant tensor valuations. He first proved an additive version of Theorem 9.16 for tensor valuations, relating a and the convolution product of tensor valuations (compare also [22]). The convolution product of unitarily invariant tensor valuations of rank 2 can be computed using a formula from [18]. This formula is much easier to use than the corresponding formula for the product of valuations from Theorem 9.11 since it does not involve any fiber integrations.

To state his theorem, Wannerer gives a precise description of the adjoint of A, which is a commutative associative product on the dual space $\mathrm{Area}^{\mathrm{U}(n),*}$. It turns out that this algebra is a polynomial algebra with three generators t, s, v. We refer to [47] for the precise statement of the theorem.

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Chapter 10
Integral Geometric Regularity

Joseph H.G. Fu

Abstract Smooth scalar-valued valuations may be thought of as curvature integrals that are robust enough to apply to objects with convex singularities. It turns out that certain kinds of nonconvex singularities are also included. The distinguishing feature is the existence of a normal cycle, which is an integral current that stands in for the manifold of unit normals in case they do not exist in the usual sense. We describe the elements of the normal cycle construction, and sketch how it may be used to establish the fundamental relations of integral geometry, with emphasis on the class of WDC sets recently introduced by Pokorný and Rataj.

10.1 Introduction

> Those subsets of *n* space which are to be the objects of such a theory must be singled out by some simple geometric property. [...] Whatever differentiability may be required for an auxiliary analytic or algebraic argument must be implied by the geometric properties. Of course, in order to be worth while, such a theory must contain natural generalizations of the principal kinematic formula and of the Gauss-Bonnet Theorem [...] [Geometric measure theory has] greatly contributed to the understanding of first order tangential properties of point sets, and one can hope for similar success in dealing with second order differential geometric concepts such as curvature.
>
> – Herbert Federer, *Curvature measures*

It is natural to think of the intrinsic volumes, the most famous of all valuations, as integrals of curvature that are sufficiently robust to continue to make sense for convex singularities. Both the valuations and the underlying integrals of curvature apply also to certain nonconvex objects, for example the convex ring. Thus it is natural to ask how far their domain may extend, and in what sense they may be regarded as true curvatures. These questions belong to the study of *integral geometric regularity*. The first serious attempt to understand integral geometric regularity was Federer's theory [14] of the curvature measures of sets with positive

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reach in Euclidean space. From the viewpoint of valuation theory, a curvature measure is a smooth valuation that takes values among signed measures, giving rise to a scalar-valued valuation by taking the total variation.

Federerer understood too that the issue of integral geometric regularity is inseparable from the integral geometry originally introduced by Blaschke, revolving around the kinematic formulas. This understanding lies in turn at the root of an important mental picture of what a valuation is, viz. a set function that associates to a set B the integral of the Euler characteristics $\chi(B \cap A_p)$ of the intersection with B of a measured family of sets (A_p, dp) (cf. Sect. 10.5.3 below). McMullen's Conjecture, proved by Alesker in [2], is a prominent formal statement in this direction.

Motivated by the relative success, described in Sect. 10.5 below, of the concept of the *normal cycle* in understanding kinematic formulas, we adopt here the viewpoint that the fundamental issues surrounding integral geometric regularity are encapsulated by this notion. Whereas in the smooth case the normal cycle $\mathbf{nc}(A)$ is easy to construct directly, viewing it formally as an integral current we may tie it axiomatically to the underlying object A even if the singularities of A become severe. Note that once the normal cycle is defined as a current in this way, the sets A under consideration become amenable to any valuations (such as tensor valuations) that are given in terms of integration against a smooth differential form.

In the present chapter we sketch the theory of the normal cycle and present the state of the art regarding the regularity needed to resolve the primary issues raised by Federer. We discuss a raft of further technical problems surrounding the elusive nature, entailed by the axiomatic approach, of the relation between an object and its normal cycle. We also raise a family of fundamental issues that have been largely ignored to date, having to do with the relation between the intrinsic volumes of a singular space X and the structure of X as a length space.

10.1.1 Notation and Basic Concepts

10.1.1.1 Tangents and Cotangents

Our main constructions occur within the tangent and sphere bundles TM, SM, and the cotangent and cosphere bundles T^*M, S^*M, of a smooth manifold M. For convenience we will often conflate them, in particular sometimes ignoring the need for a Riemannian metric to make sense of SM. By the same token, our main object of interest is the normal cycle of a singular subspace of M, living in SM. Although the *co*normal cycle in S^*M is formally more natural, we do not make this distinction.

10.1.1.2 Currents

We will make essential use of established tools from geometric measure theory, taken mostly from Chap. 4 of [15]. A *current* of dimension k on a smooth manifold

M is a linear functional T on the space $\Omega^k(M)$ of compactly supported smooth differential forms of degree k on M, continuous with respect to C^∞ convergence with common compact support. The support of T is denoted $\operatorname{spt} T$.

The pairing of T with a form ϕ will be denoted either by $T(\phi)$ or $\int_T \phi$. The boundary ∂T of such T is the current of dimension $k-1$ given by $\partial T(\psi) := T(d\psi)$. Given a proper smooth map $F : M \to N$ to another smooth manifold N, we may form the pushforward current F_*T by $F_*T(\phi) = T(F^*\phi)$. Here $F^*\phi$ denotes the usual pullback of the differential k-form ϕ to M, i.e.

$$(F^*\phi)_x(v_1, \dots, v_k) = \phi_{F(x)}(D_xF(v_1), \dots, D_xF(v_k))$$

for $x \in M$ and $v_1, \dots, v_k \in T_xM$. A measurable subset E of a smooth oriented submanifold $V^k \subset M$ gives rise to a k-dimensional current $[[E]]$ given by $[[E]](\phi) = \int_E \phi$.

Such T is an *integral current* if there exists a sequence $g_1, g_2, \dots : \mathbb{R}^k \to M$ of C^1 maps, and measurable subsets $E_1, E_2, \dots \subset \mathbb{R}^k$, such that

$$T(\phi) := \sum_i g_{i*}[[E_i]](\phi) = \sum_i \int_{E_i} g_i^* \phi,$$

and such that ∂T may be similarly expressed. By Rademacher's theorem, an integral current may be pushed forward by maps F that are merely locally Lipschitz. Thus an integral current T is one that resembles integration over an oriented submanifold, and so for convenience we may at times conflate T with a subset of M.

If ψ is a differential form of deg d then

$$T \llcorner \psi(\phi) := T(\psi \wedge \phi).$$

Since the integral current T has the special form above, here ψ may be merely Borel measurable and this will still make sense. If $U \subset M$ we put $(T \llcorner U) := T \llcorner 1_U$, where 1_U is the characteristic function of U.

A rudimentary intersection theory for integral currents exists in the form of *slicing*: given a Lipschitz map $F : M \to N^d =$ an oriented manifold of dimension d, and $y \in N^d$, one constructs $\langle T, F, y\rangle$ (the slice of T by F at y) as an integral current of dimension $k - d$ in M, well-defined for a.e. $y \in N^d$. This slice may be thought of as the oriented intersection of T with the fiber $F^{-1}(y)$.

For smooth oriented submanifolds $V, W \subset M$ of complementary dimension that meet transversely, we denote by $V \bullet W$ their intersection product, i.e. the sum of the multiplicities at their various points of intersection. Note that if W arises as $W = F^{-1}(y)$ for some smooth map $F : M \to N^d$ and regular value $y \in N^d$, where an orientation of N^d induces the orientation of W, then the sum of the delta functions at the points of $V \cap W$, weighted with their intersection multiplicities, equals the slice $\langle [[V]], F, y\rangle$. Thus we adopt the $\bullet$ notation also in the scenario of the last paragraph: if $d = k$ then the slice of T by F at y is a 0-dimensional integral current, i.e. a sum of delta functions with integer multiplicities. In this case we put $T \bullet F^{-1}(y)$ for the sum of these multiplicities, and the following form of the change of variables formula for integrals applies:

$$F_*T = [[N^d]] \llcorner (T \bullet F^{-1}), \tag{10.1}$$

where $T \bullet F^{-1}$ denotes the integer-valued function $y \mapsto T \bullet F^{-1}(y)$. For subsets $U \subset M$ we put $(T \bullet F^{-1}(y))(U)$ or $T \bullet F^{-1}(y)\big|_U$ for the sum of the multiplicities attached to points lying in U. Observe that if T is closed (i.e. $\partial T = 0$) then so is F_*T, so that by the constancy theorem [15, 4.1.7] the latter must be a constant multiple of $[[N]]$, and therefore $T \bullet F^{-1}(y)$ is independent of y.

10.2 Classical Theory

In this section we give an account of Federer's curvature measures for sets with positive reach through the lens of the normal cycle.

10.2.1 Tube Formulas

The mother of all integral geometric theorems is the Steiner-Weyl tube formula. Let $\mathscr{K}^n$ denote the space of all compact convex subsets of $\mathbb{R}^n$, equipped with the Hausdorff metric. Let κ_k denote the volume of the unit ball in $\mathbb{R}^k$. For $r \geq 0$ and $A \subset \mathbb{R}^n$ put

$$A_r := \{x \in \mathbb{R}^n : \mathrm{dist}(A, x) \leq r\}.$$

The first statement of the following theorem, and a localized version of it, are discussed in detail in Chap. 1, which restricts attention entirely to the convex case.

Theorem 10.1

1. *(Steiner 1840) Let* $A \in \mathscr{K}^n$*, Then there are constants* $V_0(A), \dots, V_n(A)$ *such that*

$$\mathrm{V_n}(A_r) = \sum_{i=0}^{n} \kappa_{n-i} V_i(A) r^{n-i} \tag{10.2}$$

   for all $r \geq 0$.
2. *(Weyl 1939) If* $M \subset \mathbb{R}^n$ *is a compact smoothly embedded submanifold then there exists* $r_0 > 0$ *such that* (10.2) *holds for all* $r \in [0, r_0]$.

Proof Consider first the case 2. of a submanifold M and define the $(n-1)$-dimensional C^1 submanifold $\mathbf{nc}(M) \subset S\mathbb{R}^n := \mathbb{R}^n \times S^{n-1}$ given by

$$\mathbf{nc}(M) := \{(x, v) : v \perp T_x M\}. \tag{10.3}$$

For sufficiently small $t_0 > 0$, the restriction to $\mathbf{nc}(M)$ of the map $(x, v) \mapsto x + t_0 v$ yields a diffeomorphism with the boundary of the tubular neighborhood of M of radius t_0. The natural orientation of the latter thus induces a canonical orientation of $\mathbf{nc}(M)$. Recall that $S\mathbb{R}^n$ is a *contact manifold*, with contact 1-form α given by

$$\alpha_{x,v} := \langle v, dx\rangle = \sum_{i=1}^{n} v_i dx_i. \tag{10.4}$$

Clearly $\mathbf{nc}(M)$ is *Legendrian* in the usual sense, i.e. $\alpha|_{\mathbf{nc}(M)} = 0$ and has maximal dimension subject to this property.

Define the map

$$\begin{aligned} &\exp : S\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \\ &\exp(x, v, t) := x + tv. \end{aligned}$$

For $t_0 > 0$ sufficiently small, the restriction of exp to $\mathbf{nc}(M) \times (0, t_0)$ is a diffeomorphism onto the tubular neighborhood $M_{t_0} \setminus M$. Thus by the change of variables formula for integrals

$$\mathrm{V_n}(M_{t_0}) = \int_{\mathbf{nc}(M)\times(0,t_0)} \exp^*(dx_1 \wedge \cdots \wedge dx_n) \tag{10.5}$$

where

$$\begin{aligned} \exp^*(dx_1 \wedge \cdots \wedge dx_n) &= d(x_1 + tv_1) \wedge \cdots \wedge d(x_n + tv_n) \\ &= \Big(\sum_{i=0}^{n-1} \kappa_{n-i}(n-i)t^{n-i-1}\mu_i\Big) \wedge (dt + \alpha). \end{aligned}$$

The differential forms μ_i appearing here are determined by the condition that μ_i contains i factors dx and $n-i-1$ factors dv. This may be seen by evaluating at a representative point $(x, e_n) \in S\mathbb{R}^n$, where

$$v_1 = v_2 = \cdots = v_{n-1} = 0 = dv_n, \qquad v_n = 1, \qquad \alpha = dx_n.$$

The tube formula (10.2) follows, with

$$V_i(M) = \int_{\mathbf{nc}(M)} \mu_i, \qquad i = 0, \ldots, n-1. \tag{10.6}$$

The original Steiner formula (10.2) for convex sets $A \in \mathscr{K}^n$ may be proved in exactly the same way. One considers the normal cycle

$$\mathbf{nc}(A) := \{(x, v) \in S\mathbb{R}^n : x \in \operatorname{bd} A, \langle v, x - y \rangle \geq 0 \text{ for all } y \in A\}. \tag{10.7}$$

Since A is convex, the distance function $\delta_A(x) := \operatorname{dist}(x, A)$ is $C^{1,1}$ (continuously differentiable, with locally Lipschitz gradient) in the complement of A, with gradient

$$\nabla \delta_A(x) = \frac{x - p(A,x)}{|x - p(A,x)|} = \frac{x - p(A,x)}{\delta_A(x)}$$

where $p(A,x) \in \operatorname{bd} A$ is the point of A minimizing the distance to x. It follows that each level set $\delta_A^{-1}(t)$ is a $C^{1,1}$ hypersurface of $\mathbb{R}^n$, oriented as the boundary of the tubular neighborhood A_t. The gradient $\nabla \delta_A(x)$ is of course normal to this level set, and moreover it is easy to see that $(p(A,x), \nabla \delta(x)) \in \mathbf{nc}(A)$ for all $x \notin A$. The map $x \mapsto (p(A,x), \nabla \delta_A(x))$ is then a biLipschitz homeomorphism $\delta_A^{-1}(t) \to \mathbf{nc}(A)$, thus inducing an orientation on $\mathbf{nc}(A)$. □

In the convex setting, the V_i are precisely the intrinsic volumes; see Sect. 1.3. In the smooth setting, the integrals (10.6) may also be expressed as integrals of the traces of the second fundamental form of M. For example, if M is a compact domain with smooth boundary then $\mathbf{nc}(M)$ is the manifold of outward unit normals. The integral $\int_{\mathbf{nc}(M)} \mu_i = \text{const.} \int_{\operatorname{bd} M} \sigma_{n-i-1}(k_1, \ldots, k_{n-1})\, d\mathscr{H}^{n-1}$, where σ_j is the j-th elementary symmetric function and the k_i are the principal curvatures. To see this we consider the outward normal vector field ν, which gives a diffeomorphism $\operatorname{bd} M \to \mathbf{nc}(M)$. Given any point of the boundary we choose coordinates so that $e_1, \ldots, e_{n-1}$ are principal directions, so that $dv_i = k_i dx_i, i = 1, \ldots, n-1$.

Both cases of Theorem 10.1 are subsumed by a theorem of Federer. Following Federer we define the *reach* of a closed set $A \subset \mathbb{R}^n$ as the supremal $r \geq 0$ such that for all $x \in A_r$ there exists a unique point $p \in A$ such that $\operatorname{dist}(x, A) = |x - p|$. Denote this nearest point by $p(A,x) := p$. If this $r > 0$ (i.e. A has positive reach) then we say that A is PR.

Theorem 10.2 (Federer [14]) *There exist signed Radon measures*

$$\Phi_0(A, \cdot), \ldots, \Phi_n(A, \cdot)$$

such that for $r > 0$ less than the reach r_0 of A, and for every bounded Borel set $E \subset A$,

$$V_n(A_r \cap p(A, \cdot)^{-1}(E)) = \sum_{i=0}^{n} \kappa_{n-i} \Phi_i(A, E) r^{n-i}. \tag{10.8}$$

Again in this setting the measures $\Phi_i(A, \cdot)$ are artifacts of the normal cycle. The distance function δ_A is $C^{1,1}$, with no critical points, in the open set $A_{r_0} \setminus A$. Thus each level set $\delta_A^{-1}(s), s \in (0, r_0)$ is a $C^{1,1}$ hypersurface, oriented as the boundary of A_s. As in the convex case, if we fix such s the map

$$x \mapsto \left(p(A,x), \frac{x - p(A,x)}{\delta_A(x)} \right)$$

is a biLipschitz homeomorphism from $\delta_A^{-1}(s)$ to its image in the sphere bundle $S\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$, and moreover the latter is independent of s. Thus this image carries the structure of a closed oriented Lipschitz submanifold $\mathbf{nc}(A) \subset S\mathbb{R}^n$, and the Federer curvature measures may be expressed

$$\Phi_i(A,E) = \int_{\mathbf{nc}(A)\cap\pi^{-1}(E)} \mu_i, \qquad i = 0, \dots, n-1, \tag{10.9}$$

where $\pi : S\mathbb{R}^n \to \mathbb{R}^n$ is the projection to the first factor.

10.2.2 Euler-Morse Theory, the Gauss-Bonnet Theorem, and the Principal Kinematic Formula

The normal cycle lends itself via the intersection product of cycles to a reduced form of Morse theory that is sensitive only to the Euler characteristic. As we will see below, the Euler-Morse theory of a set A determines the normal cycle $\mathbf{nc}(A)$ completely provided it exists.

If $f : A \to \mathbb{R}$ is a function on a "nice" space A and $c \in \mathbb{R}$, we define the *Euler-Morse index*

$$\Delta\chi(f,c) := \chi\big(f^{-1}(-\infty, c+\varepsilon]\big) - \chi\big(f^{-1}(-\infty, c-\varepsilon]\big)$$

provided this is well-defined and independent of the choice of (small) $\varepsilon > 0$. Here χ denotes the Euler characteristic; we will assume that these sublevel sets are compact *absolute neighborhood retracts* (ANRs), and hence all standard notions of (co)homology coincide (cf. [12]).

If $A = M$ is an oriented compact smooth manifold and f is a Morse function in the usual sense, then

$$\Delta\chi(f,c) = \begin{cases} 0, & \text{if } c \text{ is a regular value of } f, \\ (-1)^\lambda, & \text{if } c \text{ is a critical value of } f, \end{cases}$$

where λ is the usual Morse index of f at the (unique, by hypothesis) critical point $x \in f^{-1}(c)$. The Morse condition on f is equivalent to the statement that the graph $\Gamma_{df} \subset T^*M$ of the differential df intersects the graph Γ_0 of the zero section $z : M \to T^*M$ transversely at the single point $z(x)$. The multiplicity of the intersection

$\Gamma_0 \bullet \Gamma_{df}$ at this point is precisely $\Delta\chi(f,c)$. Here the orientations of the two factors are induced by the orientation of M. More generally, if $N \subset M$ is a submanifold, and $g \in C^\infty(M)$, then the Morse condition on $f := g|_N$ is equivalent to the transversality of Γ_{dg} and the manifold $T^*_N M$ of conormals to N, together with the assumption that the critical values are distinct. The Euler-Morse indices of f are then given by the intersection multiplicities of $T^*_N M \bullet \Gamma_{dg}$, or equivalently (assuming that N contains no critical points of f) by those of $S^*_N M \bullet \Gamma_{[dg]}$, where $[dg]$ denotes the image of dg under the normalization map $T^*M \setminus$ (zero section) $\to S^*M$ and $S^*_N M$ the manifold of unit conormals to N.

If the ambient manifold M is Euclidean space $\mathbb{R}^n$ then we apply this fact to height functions $g(x) = h_v(x) := v \cdot x$, $v \in S^{n-1}$. In this case it is not difficult to see that $h_v|_N$ is Morse for a.e. v, and that the Euler-Morse index $\Delta\chi(h_v, h_v(x))$ agrees with the sign $\sigma(x,v)$ of the Jacobian determinant of the Gauss map $\mathbf{nc}(N) \to S^{n-1}$ at $(x,-v)$ whenever x is a critical point of h_v. This interpretation yields a proof of an elementary version of the Gauss-Bonnet theorem:

Theorem 10.3 *If $N \subset \mathbb{R}^n$ is a compact smooth submanifold without boundary, then*

$$V_0(N) = \chi(N).$$

Proof By the change of variables formula, taking dv to be the invariant probability measure on S^{n-1}, we have

$$\begin{aligned}
V_0(N) &= \int_{\mathbf{nc}(N)} \mu_0 \\
&= \int_{S^{n-1}} \sum_{(x,v)\in\mathbf{nc}(N)} \sigma(x,v)\, dv \\
&= \int_{S^{n-1}} \sum_{c\in\mathbb{R}} \Delta\chi(h_v|_N, c)\, dv \\
&= \int_{S^{n-1}} \chi(N)\, dv \\
&= \chi(N).
\end{aligned}$$

□

Suitably interpreted, the discussion above applies also to PR sets A. We refer to [17] for the formal definitions underlying the following statement.

Theorem 10.4 ([17]) *Let $f \in C^\infty(\mathbb{R}^n)$. Suppose that $A \subset \mathbb{R}^n$ is PR, that the graph of the normalized differential of f is transverse to $\mathbf{nc}(A)$, and that f has no critical points in A. If c is a regular value of $f|_A$ then*

$$\chi(A \cap f^{-1}(-\infty,c]) = \left[\mathbf{nc}(A) \bullet \Gamma_{[-df]}\right](\pi^{-1} f^{-1}(-\infty,c]).$$

Here the right hand side denotes the sum of all the multiplicities of the oriented intersection lying in the region of T^*M lying above $f^{-1}(-\infty, c]$.

In light of this statement it is worthwhile to consider what happens in the tube formula (10.8) when the radius exceeds the reach r_0 of A. For a.e. $x \notin A$, restriction to A of $\delta_x := |\cdot - x|$ is Morse, and if $\exp(p, v, t) = x$ for $(p, v) \in \mathbf{nc}(A)$ and $t > 0$ then the sign of the Jacobian determinant $\bar{\sigma}(p, v, t)$ of $\exp|_{\mathbf{nc}(A)\times\mathbb{R}}$ at (p, v, t) equals the Euler-Morse index $\Delta\chi(\delta_x|_A, t)$. Thus we may apply the change of variables formula to the right hand side of (10.5) to obtain for all $r \geq 0$

$$\begin{aligned}
\sum_{i=0}^{n-1} \kappa_{n-i} V_i(A) r^{n-i} &= \int_{\mathbf{nc}(M)\times(0,r)} \exp^*(dx_1 \wedge \cdots \wedge dx_n) \\
&= \int_{\mathbb{R}^n} \sum_{(p,v,t)\in\exp^{-1}(x)\cap[\mathbf{nc}(A)\times(0,r)]} \bar{\sigma}(p, v, t)\, dx \\
&= \int_{\mathbb{R}^n} \sum_{t\in(0,r]} \Delta\chi(\delta_x|_A, t)\, dx \\
&= \int_{\mathbb{R}^n} \left[\mathbf{nc}(A) \bullet \Gamma_{[-d\delta_x]}\right](\pi^{-1}\delta_x^{-1}(-\infty, r])\, dx \\
&= \int_{\mathbb{R}^n} \chi(\bar{B}(x, r) \cap A))\, dx.
\end{aligned}$$

If $r < r_0$ then this last integrand is either 0 (if $\delta_A(x) \leq r$) or 1 (if $\delta_A(x) > r$).

One computes that $V_i(\bar{B}(x, r)) = \frac{\kappa_n}{\kappa_{n-i}}\binom{n}{i} r^i$, so the tube formula in its extended form above is a special case of the Principal Kinematic Formula:

Theorem 10.5 (Federer [14]) *For suitable constants c_i, if $A, B \subset \mathbb{R}^n$ are compact and PR then*

$$\int_{\overline{\mathrm{SO}(n)}} \chi(A \cap \bar{g}B)\, d\bar{g} = \sum_{i=0}^{n} c_i V_i(A) V_{n-i}(B). \tag{10.10}$$

Here $\overline{\mathrm{SO}(n)}$ denotes the group of orientation-preserving Euclidean isometries, generated by $\mathrm{SO}(n)$ and translations, equipped with its usual Haar measure.

Formulas of this type are valid also if $V_i, i = 1, \ldots, n$, replaces $\chi = V_0$ in the integrand, and moreover local versions also hold:

Theorem 10.6 (Federer [14]) *Given $k = 0, \ldots, n$ there exist constants c_{ij}^k such that whenever $A, B \subset \mathbb{R}^n$ are compact and PR, and $E, F \subset \mathbb{R}^n$ are Borel, then*

$$\int_{\overline{\mathrm{SO}(n)}} \Phi_k(A \cap \bar{g}B, E \cap \bar{g}F)\, d\bar{g} = \sum_{i+j=n+k} c_{ij}^k \Phi_i(A, E) \Phi_j(B, F). \tag{10.11}$$

10.3 The Normal Cycle and the Differential Cycle

As we have seen above, many of the basic operations of elementary differential geometry may be framed in terms of the manifold of normals to a smooth object. Similarly, the differential calculus of a function f of several variables may be carried out in terms of the graph of its differential. Whereas one constructs these objects of the smooth category directly by differentiation, this is not possible once they become singular. Thus it is necessary to introduce them by indirect means. It turns out that one may use the language of the Federer-Fleming theory of integral currents (cf. [15, Chap. 4]) to frame a set of axioms that associates a unique current to a singular subspace or function. Due to the compactness theorem for integral currents, this correspondence applies to an impressively broad range of such objects.

10.3.1 Axioms for Differential and Normal Cycles

Differential and normal cycles of nonsmooth sets and functions are determined by means of a few simple and inevitable axioms, reproduced here in Theorems 10.7 and 10.9. We sketch their proofs in Sect. 10.3.3 below.

The following may be paraphrased as: the normal cycle of a compact subset of $\mathbb{R}^n$ is determined by the Euler-Morse theory of height functions. Denote by S^{n-1*} the unit cosphere, i.e. the space of all linear functionals of norm 1 on $\mathbb{R}^n$.

Theorem 10.7 ([17, 21, 38]) *Let $A \subset \mathbb{R}^n$ be compact, and suppose that $A \cap \lambda^{*-1}(-\infty, c]$ is an ANR for a.e. $(\lambda^*, c) \in S^{n-1*} \times \mathbb{R}$. Then there is at most one compactly supported, closed Legendrian integral current T in the cosphere bundle $S^*\mathbb{R}^n$ such that for a.e. $(\lambda^*, c) \in S^{n-1*} \times \mathbb{R}$*

$$\chi(A \cap \lambda^{*-1}(-\infty, c]) = \left[T \bullet \Gamma_{[-d\lambda^*]}\right]\Big|_{\pi^{-1}\lambda^{*-1}(-\infty,c]}. \tag{10.12}$$

If this T exists we call it the *normal cycle* of A, denoted $\mathbf{nc}(A)$. The definition is awkward in certain respects. One issue is diffeomorphism invariance (cf. e.g. Sect. 10.3.2.1). Another is the ANR condition, which seems likely to be a consequence of a more comprehensive, yet substantively equivalent, definition.

From a formal point of view it is more natural to deduce Theorem 10.7 from the following more natural and general statement.

Theorem 10.8 *If T is a compactly supported, closed, Legendrian integral current, and*

$$[T \bullet \Gamma_{[-d\lambda^*]}](\pi^{-1}\lambda^{*-1}(-\infty, c]) = 0 \qquad \textit{for a.e. } (\lambda^*, c) \in S^{n-1*} \times \mathbb{R} \tag{10.13}$$

then $T = 0$.

Note that in order for uniqueness to hold in Theorem 10.7 it is not necessary to specify the critical *points* of the $\lambda|_A$, but only their critical *values*. This corresponds to the classical idea of reconstruction of a hypersurface as the envelope of its tangent hyperplanes.

Theorem 10.7 is about an integral current that encodes the 2nd order properties of a nonsmooth (albeit integral geometrically regular) geometric object in Euclidean space. The following companion theorem is about *functions* instead. Recall that the cotangent bundle T^*M of a C^2 manifold M always admits a natural symplectic 2-form ω. An integral current T of dimension $n = \dim M$, living in T^*M, is *Lagrangian* if $T \llcorner \omega = 0$.

Theorem 10.9 ([17, 31]) *Let $U \subset \mathbb{R}^n$ be open and $f \in W^{1,1}_{\mathrm{loc}}(U)$, i.e. a function on U with differential locally in L^1. Then there is at most one closed Lagrangian integral current $\mathbb{D}(f)$ in T^*U such that* $\mathrm{mass}(\mathbb{D}(f) \llcorner \pi^{-1}K) < \infty$ *for all compact $K \subset U$ and*

$$\int_{\mathbb{D}(f)} g \cdot \pi^*(dx_1 \wedge \cdots \wedge dx_n) = \int_U g(x, df(x))\, dx \tag{10.14}$$

whenever g is continuous and compactly supported in T^*U.

If such a current exists then f is said to be a *Monge-Ampère (MA) function*, and $\mathbb{D}(f)$ its *differential cycle*. Again it is formally more natural to state the theorem as follows:

Theorem 10.10 *If T is a closed Lagrangian integral current satisfying the local mass condition above, and T annihilates all compactly supported functional multiples of $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$, then $T = 0$.*

The proof of Theorem 10.9 implies that the support of $\mathbb{D}(f)$ is included in the graph of the Clarke differential of f [11]: we recall that for Lipschitz f and a point x, the Clarke differential $\partial f(x)$ is defined to be the convex hull of the set of all limits $df(x_i)$ over all sequences $x_i \to x$ at which $df(x_i)$ is defined (these are plentiful by Rademacher's theorem). It is clear that ∂f is upper semicontinuous as a set function, i.e. if $x_i \to x_0, v_i \in \partial f(x_i), v_i \to v_0$ then $v_0 \in \partial f(x_0)$.

Under suitable topological regularity assumptions it is well known that the Euler characteristic is a valuation, i.e. $\chi(\emptyset) = 0$ and χ is finitely additive,

$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B).$$

Thus if these assumptions hold for generic intersections of A, B with halfspaces then the existence of normal cycles for any three of $A, B, A \cap B, A \cup B$ implies its existence for the fourth, with

$$\mathbf{nc}(A \cup B) + \mathbf{nc}(A \cap B) = \mathbf{nc}(A) + \mathbf{nc}(B). \tag{10.15}$$

Cf. Proposition 10.38 below for a precise statement of this type. Similarly, if we put $f \wedge g, f \vee g$ for the minimum and the maximum, respectively, of f, g, then $f \wedge g, f \vee g \in W^{1,1}_{\text{loc}}$ whenever $f, g \in W^{1,1}_{\text{loc}}$, with

$$d(f \wedge g) + d(f \vee g) = df + dg$$

a.e., from which it follows that if any three of $f, g, f \wedge g, f \vee g$ are MA then so is the fourth, with

$$\mathbb{D}(f \wedge g) + \mathbb{D}(f \vee g) = \mathbb{D}(f) + \mathbb{D}(g). \tag{10.16}$$

10.3.2 Some Basic Questions

10.3.2.1 Invariance Under Ambient Diffeomorphisms

While the reliance on the family of height functions (or, equivalently, linear functions) in Theorem 10.7 makes it relatively easy to test whether a given Legendrian cycle is the normal cycle $\mathbf{nc}(A)$ of some given set A, it also raises some basic unresolved questions.

Given a C^2 diffeomorphism ϕ of $\mathbb{R}^n$, put $\tilde{\phi} = (\phi^{-1})^* : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ for the induced symplectomorphism that covers it, and abuse notation to denote the induced contact transformation $S^*\mathbb{R}^n \to S^*\mathbb{R}^n$ by the same symbol. By the change of variables formula the following statement is easy to prove.

Proposition 10.11 *If f is MA then so is $f \circ \phi^{-1}$, with*

$$\mathbb{D}(f \circ \phi^{-1}) = \tilde{\phi}_* \mathbb{D}(f).$$

The corresponding statement for normal cycles is not known:

Conjecture 10.12 If $A \subset \mathbb{R}^n$ is a compact set admitting a normal cycle then so is $\phi(A)$, with

$$\mathbf{nc}(\phi(A)) = \tilde{\phi}_* \, \mathbf{nc}(A).$$

Thus we have no fully satisfactory definition at the present moment for the normal cycle of a subset of a general smooth manifold. On the other hand, since normal cycles in practice seem always to be given in more concrete terms, this has not yet presented any difficulties in applications (cf. e.g. Corollary 10.34 below).

It is not hard to see that the current on the right hand side is Legendrian, so it only remains to prove that it satisfies the Euler Morse condition (10.12) for linear functions for $\phi(A)$. Since linear functions are not stabilized by ϕ, the path to a proof is not clear.

10.3.2.2 Morse Theory of MA Functions

One more glaring gap lies in the relation between the differential cycle of a function f and the normal cycle of its graph (or epigraph). The question is basically whether the Euler-Morse theory of f can be read off from the differential cycle $\mathbb{D}(f)$ in the classical way.

Conjecture 10.13 Let $U \subset \mathbb{R}^n$ be open and $f \in MA(U)$. Suppose that $f, \partial f$ are proper maps (i.e. the sets $f^{-1}[a,b]$ and $\{x :$ there exists $v \in \partial f(x), |v| \leq c\}$ are relatively compact for all $a,b,c \in \mathbb{R}$) and f is bounded below. Then for a.e. $(\lambda, c) \in \mathbb{R}^{n*} \times \mathbb{R}$

$$\chi((f-\lambda)^{-1}(-\infty,c]) = \left[\mathbb{D}(f) \bullet q^{-1}(\lambda)\right]\Big|_{\left((f-\lambda)^{-1}(-\infty,c]\right)\times\mathbb{R}^{n*}}$$

where $q : T^*U \simeq U \times \mathbb{R}^{n*} \to \mathbb{R}^{n*}$ denotes the projection to the second factor.

Implicit in this statement is the conjecture that the sets on the left are ANRs. The properness hypothesis ensures that they are compact, and that the integers on the right hand side are well defined.

10.3.2.3 Continuity

In order to make sense of this last discussion it seems necessary that f be continuous, but this is not known of a general MA function f. However, a theorem of Ponce and Van Schaftingen [39] implies that this is true if $n = 2$.

Jerrard [32] gives examples showing that there exist MA functions that fail to be α-Hölder continuous for any $\alpha > \frac{2}{n+1}$.

10.3.3 Proofs of the Uniqueness Theorems

We prove Theorem 10.9 first. The idea is based on a characterization of certain simple Lagrangian submanifolds of a cotangent bundle.

Proposition 10.14 ([29]) *Let $M \subset T^*\mathbb{R}^n$ be a smooth Lagrangian submanifold such that the projection to the base restricts to a submersion $M \to \mathbb{R}^n$ of rank k. Then there exists a smoothly immersed k-dimensional submanifold $V \subset \mathbb{R}^n$ such that M is a perturbation of the conormal bundle to V by a closed 1-form β on V, i.e.*

$$M = \{(x, \eta + \beta(x)) : \eta \in T_xV^\perp\}.$$

The main point of the proof of Theorem 10.9 is to obtain a weak form of this characterization with M replaced by a carrier of T, an n-rectifiable Lagrangian subset of T^*M. The condition (10.14) implies that the projection to $\mathbb{R}^n$ is essentially a submersion on a large subset of the carrier, so the conclusion of Proposition 10.14 implies that the fibers of the projection are affine subspaces of dimension $k > 0$, hence of infinite k-volume. The coarea formula then implies that the mass of T must be locally infinite in the neighborhood of any such fiber, contradicting the local mass condition. The measure-theoretic fibration result [15, Theorem 3.2.22], together with the theory of slicing encapsulated in [15, Theorem 4.3.2], are enough to make this all work.

The proof of Theorem 10.7 proceeds in two steps. To state the first, denote by

$$q : S\mathbb{R}^n \simeq \mathbb{R}^n \times S^{n-1} \to S^{n-1}$$

the projection. We claim that any compactly supported, closed, Legendrian integral current T with

$$T \llcorner q^* d\,\mathrm{Vol}_{S^{n-1}} = 0 \tag{10.17}$$

must vanish, where $d\,\mathrm{Vol}_{S^{n-1}}$ is the volume form of the sphere. In fact this is a consequence of the argument of the last paragraph: we replace T by the n-dimensional conical current $\overrightarrow{T}$ in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, the image of $T \times (0, \infty)$ under the map $(x, v; t) \mapsto (x, tv)$. Interchanging the two factors we may regard the ambient space as the cotangent bundle of $U := \mathbb{R}^n \setminus \{0\})$, and $\overrightarrow{T}$ becomes a Lagrangian current satisfying the previous hypotheses—the locally finite mass condition is replaced here by the stronger condition that the intersections of the support of T with the fibers of the projection to U all lie in a common compact subset of $\mathbb{R}^n$.

To complete the proof of Theorem 10.7 it remains to show that the half-space vanishing condition (10.13) implies the stronger vanishing statement (10.17). We introduce the "envelope map"

$$p : S\mathbb{R}^n \to S^{n-1} \times \mathbb{R}, \qquad (x, v) \mapsto (v, x \cdot v).$$

If target space is identified with the space of oriented hyperplanes in $\mathbb{R}^n$, the image under p of the normal cycle of a smooth object $A \subset \mathbb{R}^n$ is then the space of all hyperplanes tangent to A. The relation (10.13) is equivalent to

$$p_*(T \llcorner q^* d\,\mathrm{Vol}_{S^{n-1}}) = 0.$$

Thus it is enough to prove the following.

Proposition 10.15 *Let $X \subset S\mathbb{R}^n$ be a Legendrian $(n-1)$-rectifiable subset. Then there is a set $C \subset S^{n-1}$ of measure zero such that $p|_{X \setminus p^{-1}(C \times \mathbb{R})}$ is injective.*

Proof The proof is simple and pleasant enough to give in its entirety. We may assume that X is included in a C^1 submanifold $\tilde{X} \subset S\mathbb{R}^n$ of dimension $n-1$. Then, by hypothesis, at a.e. $(x, v) \in X$ the tangent plane $T_{(x,v)}\tilde{X}$ is Legendrian. By Sard's theorem we may assume that $q|_{\tilde{X}}$ has nonvanishing Jacobian determinant everywhere, so we choose a countable neighborhood base $\{V_i\}$ for $\tilde{X}$ such that each $q|_{V_i}$ is a diffeomorphism with its image in the sphere. It is enough to show that if $V_i \cap V_j = \emptyset$ then the projection $R = R_{ij}$ of $p(V_i) \cap p(V_j)$ to the sphere has measure zero.

To this end, suppose that $(x_0, v_0) \in V_i, (y_0, v_0) \in V_j, t_0 := x_0 \cdot v_0 = y_0 \cdot v_0$. We show that the density of $R \subset S^{n-1}$ at v is zero. Suppose $v_k \in R, v_k \to v_0$, so that there exist x_k, y_k with $(x_k, v_k) \in V_i, (y_k, v_k) \in V_j$, and $p(x_k, v_k) = p(y_k, v_k)$, i.e. $(x_k - y_k) \cdot v_k = 0$. Since q is a diffeomorphism on each of V_i, V_j, it follows that

$$|x_k - x_0| = O(|v_k - v_0|) = |y_k - y_0|.$$

Thus the Legendrian condition implies that

$$\begin{aligned}\lim_{k\to\infty} v_k \cdot \frac{x_k - x_0}{|v_k - v_0|} &= \lim_{k\to\infty} v_0 \cdot \frac{x_k - x_0}{|v_k - v_0|} = \lim_{k\to\infty} v_0 \cdot \frac{y_k - y_0}{|v_k - v_0|} \\ &= \lim_{k\to\infty} v_k \cdot \frac{y_k - y_0}{|v_k - v_0|} = 0,\end{aligned}$$

so that

$$\begin{aligned}\lim_{k\to\infty} \frac{v_k - v_0}{|v_k - v_0|} \cdot (x_0 - y_0) &= \lim_{k\to\infty} v_k \cdot \frac{(x_0 - y_0)}{|v_k - v_0|} \\ &= \lim_{k\to\infty} v_k \cdot \frac{((x_0 - x_k) - (y_0 - y_k))}{|v_k - v_0|} \\ &= 0.\end{aligned}$$

In other words the v_k must approach v_0 tangentially along the great hypersphere perpendicular to $x_0 - y_0$, which implies that the density vanishes as claimed. □

10.3.4 Basic Properties and Examples

The axioms for differential cycles are more convenient than those for normal cycles, hence we begin the discussion there. In fact it is a simple matter to adapt this notion to functions on any C^2 (or even $C^{1,1}$) manifold M. For ease of notation, however, we state the results for the case $M = \mathbb{R}^n$.

10.3.4.1 Direct Constructions

In a few cases, differential cycles may be constructed directly.

Any $f \in C^{1,1}(\mathbb{R}^n)$ is MA, with $\mathbb{D}(f)$ given by integration over the Lipschitz submanifold of $T^*\mathbb{R}^n$ representing the graph of its differential. The same is true of a function that is convex or semiconvex (i.e. locally expressible as the sum of a smooth and a convex function): in this case the (sub)differential is multi-valued, but the graph Γ_{df} nonetheless fits together as a single Lipschitz submanifold: in the convex case it is not hard to see [1] that the map $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \mapsto (x + y, x - y)$ transforms Γ_{df} into the graph of a function $\mathbb{R}^n \to \mathbb{R}^n$ with Lipschitz constant 1.

Any piecewise linear function p is MA. If p is affine on each simplex of the triangulation $\mathscr{T}$ of $\mathbb{R}^n$, we may construct $\mathbb{D}(p)$ recursively by specifying it over simplices of successively smaller dimension. The leading term is the sum of constant graphs $\tau^n \times \{dp_{\tau^n}\}$ where τ^n is an n-dimensional simplex of $\mathscr{T}$ and dp_{τ^n} is the constant value of dp there. The resulting current satisfies all of the axioms for $\mathbb{D}(f)$ except for closure, since it has boundary living over the $(n-1)$-skeleton of $\mathscr{T}$; thus we wish to cancel the boundary without altering the other features. If τ_0^n, τ_1^n are the n-simplices adjacent to some $(n-1)$-simplex τ^{n-1}, then the difference $dp_{\tau_1^n} - dp_{\tau_0^n}$ is perpendicular to τ^{n-1}: another way of saying this is that these two differentials must restrict to the same value on τ^{n-1}, namely the differential of the restriction of p. It follows that the Cartesian product of τ^{n-1} with the segment $[dp_{\tau_1^n}, dp_{\tau_0^n}]$ is Lagrangian, and up to orientations its boundary is

$$(\partial\tau^{n-1} \times [dp_{\tau_1^n}, dp_{\tau_0^n}]) + \left(\tau^{n-1} \times ([[dp_{\tau_1^n}]] - [[dp_{\tau_0^n}]])\right).$$

If oriented correctly the second terms precisely cancel the boundary of the sum of the leading terms. However, we have introduced new boundary terms, living now over the $(n-2)$-skeleton. In fact it is not hard to see that the boundary is a sum of terms $\tau^{n-2} \times P$, where P is a closed polygon. The vertices of P are precisely the values of dp_{τ^n} corresponding to n-dimensional faces incident to τ^{n-2}; since these must restrict to the same value in τ^{n-2}, it follows that P lies in an affine 2-plane perpendicular to τ^{n-2}. Since P is closed, there exists a unique compact polyhedral chain C in this 2-plane with $\partial C = P$. Adding in all the terms $\tau^{n-2} \times C$ now annihilates the boundary produced at the last step, but gives rise to boundary terms living over the $(n-3)$-skeleton of the form $\tau^{n-3} \times Q$, where Q is a 2-dimensional polyhedral cycle lying in an affine 3-plane perpendicular to τ^{n-3}. The procedure may thus be repeated until finally all of the boundary is cancelled by the terms lying over the vertices of $\mathscr{T}$. Cf. [31] for a more careful account of this process.

It is easy to see that a function f of one variable is MA iff its derivative f' has bounded variation (BV), with $\mathbb{D}(f)$ given as the connected rectifiable curve representing the graph of f' (i.e. the literal graph of f' together with the countable union of line segments needed to connect the graph at the jump discontinuities). The corresponding statement in higher dimensions is false: there are functions f of two variables with BV differential that are not MA. For example, if $c > 0$ then the function $F_c(x) := \max(0, c - |x|)$ has BV differential, and the mass of the vector measure ∇F_c is proportional to c. Thus if $\sum_{i=1}^{\infty} c_i < \infty$ and $x_1, x_2, \dots \in \mathbb{R}^2$ then, putting $G(x) := \sum F_{c_i}(x - x_i)$, the vector distribution ∇G is a vector measure with mass proportional to this sum. In other words, $G \in \mathrm{BV}$. However, $\mathbb{D}(F_c)$ includes $\{0\} \times B(0, 1) \subset \mathbb{R}^2 \times \mathbb{R}^2$ as long as $c > 0$, and therefore such a countable sum cannot be MA as long as cancellations are avoided.

10.3.4.2 Normal Cycles

With regard to normal cycles, any compact subset $A \subset \mathbb{R}^n$ with positive reach admits a normal cycle which is again given by integration over a Lipschitz submanifold of $S\mathbb{R}^n$ [16]. In this case the distance function $\delta_A = \mathrm{dist}(\cdot, A)$ is $C^{1,1}$ and has no critical points in a small region $\delta_A^{-1}(0, r_0)$. Thus the level sets $\delta_A^{-1}(r)$ are $C^{1,1}$ hypersurfaces, $0 < r < r_0$, and the map $x \mapsto (p(A, x), \nabla \delta_A(x))$ gives a biLipschitz homeomorphism from each of these hypersurfaces to $\mathbf{nc}(A)$. If B is a second such PR set such $A \cap B$ is again PR (as is true for generic positions of B with respect to A; cf. Proposition 10.37 below) then the additivity property (10.15) may be used to define $\mathbf{nc}(A \cup B)$. These are the U_{PR} sets of Zähle [43].

The only known general procedure for constructing normal cycles applies only in case the underlying set A is a sublevel set of an MA function. The most successful instance to date is the case of the WDC sets introduced in [38] and described in detail below. The idea is simply to adapt the analogous procedure for smooth objects to the singular case: if $A \subset \mathbb{R}^n$ is a smooth domain then it may be described as a sublevel set $A = f^{-1}(-\infty, c]$ for some regular value c of a smooth function f, and the manifold of outward normals to A appears as the image of the gradient field of f along $f^{-1}(c)$. The tools of the theory of integral currents are enough to carry out the basic constructions in the MA setting; the challenge is then to show that the result makes sense, by applying the criteria pronounced by Federer in the epigram to this chapter.

10.3.4.3 The Subanalytic Case

Subanalytic objects are integral geometrically regular. Fundamentally the reason is the local topological triviality of subanalytic families of sets, together with the Lojasiewicz inequality (cf. [21]). Thus the assertions of this section apply also to objects definable with respect to an o-minimal structure with suitable properties (cf. [34]).

Theorem 10.16 ([21])

1. *Every locally Lipschitz subanalytic function defined on a real analytic manifold is MA.*
2. *Any compact subanalytic subset $X \subset \mathbb{R}^n$ admits a normal cycle. If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an analytic diffeomorphism then $\tilde{\phi}_* \mathbf{nc}(X) = \mathbf{nc}(\phi(X))$, where $\tilde{\phi} : S\mathbb{R}^n \to S\mathbb{R}^n$ is the induced diffeomorphism.*

Proof

1. We may assume that the domain of the Lipschitz subanalytic function f is $\mathbb{R}^n$. The Moreau-Yosida smoothing procedure

$$f_r(x) := \sup_y \left[f(y) - \frac{|y-x|^2}{2r} \right], \qquad r > 0$$

yields a subanalytic family of semiconvex functions converging to f. By subanalyticity, the masses of their differential cycles $\mathbb{D}(f_r)$ are locally bounded near $r = 0$, and therefore a subsequence converges. It is not hard to confirm that the axioms of Theorem 10.9 hold for any such limit.

2. We may display X as a sublevel set $f^{-1}(-\infty, 0]$ of subanalytic function f that is semiconcave off of X, for example the distance function from X. Subanalyticity implies that f has no small positive critical values. Thus the nearby sublevel sets $X_r := f^{-1}(-\infty, r]$, $r > 0$, admit normal cycles, which are again bounded in mass. Any subsequential limit satisfies the axioms of Theorem 10.7. □

10.3.4.4 Strong Approximations

The Federer-Fleming compactness theorem for integral currents, together with the uniqueness Theorem 10.9, provides a powerful method for producing new classes of MA functions. In fact all known examples of MA functions, including those described above, may be produced in this way.

Theorem 10.17 *Let* $f_1, f_2, \dots \in C^2(\mathbb{R}^n)$*, converging in* L^1_{loc} *to* f_0*, and suppose that for every* $K \subset\subset \mathbb{R}^n$ *there is a constant* $C(K)$ *such that*

$$\int_K \sum_{\substack{I,J \subset \{1,\dots,n\} \\ \#I = \#J}} \left| \det \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} \right)_{i \in I, j \in J} \right| \le C(K). \tag{10.18}$$

Then f_0 *is MA, with* $\mathbb{D}(f_0) = \lim_{k\to\infty} \mathbb{D}(f_k)$ *in the local flat metric topology.*

Proof The $\mathbb{D}(f_k)$ are simply the graphs of the differentials of the f_k, so by the area formula the mass of $\mathbb{D}(f_k) \llcorner \pi^{-1}(K)$ is

$$\int_K \left| \bigwedge\nolimits^n (Id_n | D^2 f_k) \right|$$

where $(Id_n|D^2 f_k)$ denotes the $n \times 2n$ matrix obtained by concatenating the $n \times n$ identity matrix with the Hessian matrix of f_k. From this one calculates easily that this mass is bounded above and below by constant multiples of the left hand side of (10.18).

If the masses corresponding to all $K \subset\subset \mathbb{R}^n$ are all bounded, then the Federer-Fleming compactness theorem for integral currents implies that there exists a subsequence $f_{k'}$ and an integral current T such that $\mathbb{D}(f_{k'}) \rightharpoonup T$. Using the BV compactness theorem, it is easy to confirm that the current T satisfies the hypotheses of Theorem 10.9 for the function f_0, so $T = \mathbb{D}(f_0)$. Since this result is independent of the choice of subsequence it follows that the original sequence must converge, as claimed. □

A sequence of functions as above is called a C^2 *strong approximation* of f_0; the C^2 condition on the approximating functions may be replaced by $C^{1,1}$, or by semiconvexity, with equivalent content. All functions known to be MA are in fact C^2 strongly approximable: for example, this is true not only of PL functions but more generally of all locally Lipschitz subanalytic functions (or, more generally still, all locally Lipschitz functions definable with respect to any o-minimal structure). There is another version of Theorem 10.17 with the C^2 functions f_k replaced by PL functions p_k, and the bounds (10.18) replaced by uniform local mass bounds on the $\mathbb{D}(p_k)$; this is called a *PL strong approximation*. The proof is again a direct consequence of Theorem 10.9. It is not difficult to show [19] that any C^2 function is PL strongly approximable.

It is natural to conjecture that the C^2 and the PL notions are equivalent. A slightly stronger and more natural conjecture is the following.

Conjecture 10.18 Given a bounded open set $U \subset \mathbb{R}^n$, there exists a constant $C = C(U) < \infty$ with the following properties: if $V \subset \mathbb{R}^n$ is open and $U \subset\subset V$ then

- if for any $f \in C^2(V)$ there is a sequence $p_1, p_2, \dots \in PL(V)$ such that $p_k \to f$ and $\text{mass}\, \mathbb{D}(p_k|_U) \le C \,\text{mass}\, \mathbb{D}(f|_U)$, and
- for any $p \in PL(U)$ there is a sequence $f_1, f_2, \dots \in C^2(U)$ such that $f_k \to p$ and $\text{mass}\, \mathbb{D}(f_k|_U) \le C \,\text{mass}\, \mathbb{D}(p|_U)$.

Of course the two statements may also be separated. Both parts of the conjecture are known to hold for $n \le 2$ [9, 27] but the cases $n \ge 3$ are open.

Any $f_0 \in W^{2,n}(\mathbb{R}^n)$ is C^2 strongly approximable, and hence MA: smoothing by convolution with an approximate identity yields a sequence of smooth functions that converges to f_0 and is locally uniformly bounded with respect to the $W^{2,n}$ norm. Now the Hölder inequality implies that the bounds (10.18) all hold.

10.3.5 DC Functions

We say that a function f defined on an open subset $U \subset \mathbb{R}^n$ is *DC* if it may be expressed locally as a difference of convex functions. This class enjoys some remarkable properties, and has appeared as a tool in other avenues of metric geometry (e.g. [5, 37]). The properties that will be most useful here are the following.

Theorem 10.19 (Hartman [28]) *Suppose $f : \mathbb{R}^k \supset U \to \mathbb{R}$ and $g_1, \ldots, g_k : \mathbb{R}^n \supset V \to \mathbb{R}$ are DC functions. Then $x \mapsto f(g_1(x), \ldots, g_k(x))$ is a DC function on V.*

Since any C^2 function g is clearly DC this implies the following two statements.

Corollary 10.20 *The class DC is stable under C^2 diffeomorphisms. In particular the concept of DC function makes sense on C^2 manifolds.*

Corollary 10.21 *If f, g are DC functions then so are* $\max(f, g), \min(f, g)$.

The next statement may be paraphrased as: the set of all tangent planes $\subset \mathbb{R}^n \times \mathbb{R}$ to the graph of a DC function $f : \mathbb{R}^n \supset U \to \mathbb{R}$ has Hausdorff dimension n. It is convenient to express these planes in "slope-intercept form".

Theorem 10.22 (Pavlica-Zajíček, [36]) *If $f : \mathbb{R}^n \supset U \to \mathbb{R}$ is DC then the set*

$$\{(v, t) \in \mathbb{R}^n \times \mathbb{R} : v \in \partial f(x),\ t = f(x) - v \cdot x \text{ for some } x \in U\} \tag{10.19}$$

has Hausdorff dimension at most n.

This theorem may be strengthened as follows. Recall that the d-dimensional *Minkowski content* of a subset $S \subset \mathbb{R}^k$ may be defined as $\limsup_{\varepsilon \downarrow 0} N(S, \varepsilon)\varepsilon^d$, where $N(S, \varepsilon)$ denotes the minimal number of balls of radius ε required to cover S. If this number is finite then clearly the Hausdorff dimension of S is no greater than d.

Theorem 10.23 ([26]) *For f as above, the set $\{(x, v) : v \in \partial f(x)\}$ (i.e. the graph of the Clarke differential of f) has locally finite n-dimensional Minkowski content.*

Remark 10.24 Theorem 10.22 is essentially equivalent to a theorem of Ewald et al. [13] that gives a parallel statement about the Hausdorff dimension of the subset of S^{n-1} consisting of all line segments lying in the boundary of a given convex body in $\mathbb{R}^n$. It is not known whether this set—or, equivalently, the set (10.19)—must be rectifiable. One may ask the same question about the rectifiability of the graph of the Clarke differential; a positive answer would of course yield the same statement about (10.19). As the reader will see from the discussion below, it would also imply that the support of the normal cycle of a WDC set is rectifiable, which is again an open question.

Finally, it turns out that the DC functions are also MA. Since the proof is remarkably simple we include it here.

Theorem 10.25 ([38]) *Any difference of convex (DC) functions is strongly approximable, and therefore MA.*

Proof The proof is based on the following fact from [38]: if A, B are $n \times n$ matrices then

$$\det(A - B) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \det((n-i)A + iB) \tag{10.20}$$

Suppose $f_0 = g_0 - h_0$, with g_0, h_0 convex. Smoothing by convolution we obtain a sequence $f_k = g_k - h_k$, where the g_k, h_k are smooth convex functions with locally uniformly bounded derivatives. It is well known and not hard to see that the volume of the graph of the differential of a smooth convex function u is bounded by a function of the diameter of its domain and the L^∞ norm $\| du \|_\infty$ of its differential. Since we can bound the latter quantities locally uniformly for the functions $u = u_{i,k} := ig_k + (n-i)h_k, i = 0, \dots, n$, the integrals

$$\int_K \left| \det \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i \in I, j \in J} \right| \le C(K),$$

independent of i, k. Thus (10.20) shows that the hypothesis of Theorem 10.17 is fulfilled for f_0. □

As noted above, the classes DC and MA coincide if $n = 1$. However this is definitely not the case for $n \ge 2$: the well known function

$$f_{HM}(x_1, \dots, x_n) = x_1 \sin \log \log(x_1^2 + \dots + x_n^2)^{-1}$$

of [30] is smooth away from the origin and belongs to $W^{2,n}_{\text{loc}} \subset$ MA for $n \ge 2$, but the restriction of $\frac{\partial f_{HM}}{\partial x_1}$ to the x_1 axis is not BV. It follows that $f_{HM} \notin DC$.

10.4 WDC Sets

We apply the ideas above to construct normal cycles for certain sublevel sets of DC functions, called WDC sets.

10.4.1 Regular and Weakly Regular Values of Lipschitz Functions

We wish to describe compact subsets of $\mathbb{R}^n$ as sublevel sets of Lipschitz functions f, which we therefore assume in the following discussion to be proper and bounded below.

We say that $c \in \mathbb{R}$ is a *regular value* of f if $0 \notin \partial f(x)$ whenever $f(x) = c$. We define similarly the notion of critical value, and of regular and critical points. The fundamental implicit function theorem of Clarke states that if c is such a regular value then the sublevel set $f^{-1}(-\infty, c]$ is a Lipschitz domain, i.e locally expressible as the set of points lying below the graph of a Lipschitz function of $n-1$ variables. A vector field V on $U \subset \mathbb{R}^n$ is said to be *gradient-like* for f if $V(x) \cdot v > 0$ for all $v \in \partial f(x), x \in U$.

Lemma 10.26 *If $U \subset \mathbb{R}^n$ consists entirely of regular points of f then there exists a smooth gradient-like vector field for $f|_U$.*

Proof For each point $x \in U$ there is a vector v_x such that v_x is gradient-like at x (for example, v_x may be taken to be the point of $\partial f(x) \subset \mathbb{R}^n$ that lies closest to the origin). Since ∂f is upper semicontinuous, the constant vector field v_x is also gradient-like in a neighborhood of x. Since the gradient-like condition is convex, a global smooth gradient-like field may then be constructed from these locally constant fields using a partition of unity. □

Recall that $q : S\mathbb{R}^n \simeq \mathbb{R}^n \times S^{n-1} \to S^{n-1}$ denotes the projection to the S^{n-1} factor.

Lemma 10.27 *Let c be a regular value of the Lipschitz function f. Suppose $T \in \mathbb{I}_{n-1}(S\mathbb{R}^n)$ is an integral current with*

- $\partial T = 0$,
- $\operatorname{spt} T \subset \pi^{-1} f^{-1}(c) \subset S\mathbb{R}^n$,
- *if $v \in \operatorname{spt} T \cap \pi^{-1}(x)$ then v is gradient-like for f at x,*
- $\pi_* T = \partial[[f^{-1}(-\infty, c]]]$ *where $[[f^{-1}(-\infty, c]]]$ denotes the n-dimensional integral current given by integration over this region.*

Then

$$T \bullet q^{-1}(v) = \chi(f^{-1}(-\infty, c]). \tag{10.21}$$

Proof The usual proof of the Poincaré-Hopf theorem implies that if V is smooth, unit, and gradient-like in the neighborhood of $f^{-1}(c)$ then the degree of $V|_{f^{-1}(c)}$ is equal to $\chi(f^{-1}(-\infty, c])$. Using (10.1) we find that if T' is the integral current of dimension $n-1$ given by integration over the graph of $V|_{f^{-1}(c)}$ then

$$q_* T' = \chi(f^{-1}(-\infty, c])[[S^{n-1}]].$$

The gradient-like condition on $\operatorname{spt} T$ implies that there exists a smooth homotopy $H : [0, 1] \times S\mathbb{R}^n \to S\mathbb{R}^n$ such that $H(0, \cdot)$ is the identity map and $H(1, (x, v)) = V(x)$ for $(x, v) \in \operatorname{spt} T$. Since the n-dimensional current $q_* H_*([0, 1] \times T)$ on S^{n-1} must vanish, we find that

$$q_* T' - q_* T = \partial(q_* H_*([0, 1] \times T)) = 0.$$

Now (10.1) implies (10.21). □

We say that $c \in \mathbb{R}$ is a *weakly regular value* of f if there exists $\varepsilon > 0$ such that $|v| \geq \varepsilon$ whenever $v \in \partial f(x)$ and $c < f(x) < c + \varepsilon$. In particular if $c' - c > 0$ is small enough then $f^{-1}(c')$ is a Clarke regular value of f; furthermore this is true in a controlled fashion. By smoothing the gradient field of f in the region $f^{-1}(c, c')$ and integrating, it is easy to see that any such weakly regular sublevel set is a deformation retract of a neighborhood.

Lemma 10.28 *If c is a weakly regular value of f then $f^{-1}(-\infty, c]$ is a deformation retract of $f^{-1}(-\infty, c + \varepsilon]$ for all for $\varepsilon > 0$ sufficiently small. In particular*

$$\chi\big(f^{-1}(-\infty, c]\big) = \chi\big(f^{-1}(-\infty, c + \varepsilon]\big).$$

Proof In view of the weak regularity condition, the construction above of a gradient-like vector field for f may be used to construct such a field V on $f^{-1}(c, c + \varepsilon)$ with $|V| \equiv 1$ and directional derivatives $D_V f \geq \varepsilon$. The backwards flow of V may then be reparametrized so that f decreases at rate 1 along each trajectory in $f^{-1}(c, c + \varepsilon)$, and moreover so that each trajectory has speed at least $(\mathrm{Lip} f)^{-1}$ and at most ε^{-1}. Stopping each trajectory when it reaches $f^{-1}(c)$ yields the required deformation retraction. □

The retraction map constructed above has the property that any curve $\gamma \subset f^{-1}(-\infty, c + \varepsilon]$ retracts to a curve of diameter $\leq C\varepsilon(\mathrm{length}\, \gamma)$, although it is not clear whether the retraction may be taken to be Lipschitz (the proof of Proposition 1.2 of [21] is not valid), even if the function f is DC.

10.4.2 WDC Sets

The concept of weakly regular value was introduced originally in the following result of Kleinjohann and Bangert, giving an alternate characterization of sets with positive reach.

Theorem 10.29 (Kleinjohann[33], Bangert [6]) *Let $A \subset \mathbb{R}^n$ be compact. Then A is PR iff there exists a proper semiconvex function f and a weakly regular value c of f such that $A = f^{-1}(-\infty, c]$.*

Since the notion of semiconvex function makes sense on a general C^2 manifold M, we may speak also of PR subsets of such M. Since the same is true of DC, we may also make the following definition in ambient spaces of the same generality.

Definition 10.30 We say that a compact set $A \subset M$ is a *WDC set* if the function f in the Kleinjohann-Bangert condition above is merely DC.

Thus every PR set is WDC, but the latter is a strictly larger class: for example the relative boundary of a general convex subset of $\mathbb{R}^n$ is WDC but not PR.

By Theorem 10.19, the function $\max(f - c, 0)$ is again DC, with 0 as a weakly regular value. Thus we may assume that $f \geq 0, c = 0, A = f^{-1}(0)$ above. In this case we say that f is an *aura* for A. Next we show that the normal cycle of a compact WDC subset A of Euclidean space exists in the sense of Theorem 10.7, and may be constructed from any aura for A, as follows.

Theorem 10.31 *Let $A \subset \mathbb{R}^n$ be a compact WDC set, and f an aura for A. Put $\nu : T^*\mathbb{R}^n \setminus (zero{-}section) \to S\mathbb{R}^n$ for the fiberwise radial projection to the cosphere bundle, and $r : T^*\mathbb{R}^n \to [0,\infty)$ the length function $r(x,y) := |y|$. Then for all sufficiently small neighborhoods $U \supset A$ and $\varepsilon > 0$*

$$\lim_{\delta \downarrow 0} \nu_* \langle \mathbb{D}(f), f \circ \pi, \delta \rangle = \nu_* \langle \mathbb{D}(f) \llcorner \pi^{-1}(U), r, \varepsilon \rangle \qquad (10.22)$$
$$= \nu_* \partial (\mathbb{D}(f) \llcorner \pi^{-1}(A)).$$

Furthermore, this current satisfies the conditions of Theorem 10.7, and we may therefore define it to be $\mathbf{nc}(A)$.

Proof The equivalence of the three expressions (10.22) follows from formal considerations, and it is easy to see that this current—let us call it $\mathbf{nc}(f)$—is Legendrian, closed, and has compact support. Thus it remains only to establish (10.12).

First we prove that the Gauss curvature measure associated to $\mathbf{nc}(f)$ yields a Gauss-Bonnet theorem for A, i.e. that $\int_{\mathbf{nc}(f)} \mu_0 = V_{n-1}(S^{n-1})\chi(A)$, where μ_0 is the pullback to $S\mathbb{R}^n$ of the volume form of S^{n-1} under the projection $q : S\mathbb{R}^n \to S^{n-1}$. By the remarks at the end of Sect. 10.1.1, this is equivalent to the statement in Lemma 10.32, below. We then replace f by $h_{\lambda,c}$ in Lemma 10.32 and apply Lemma 10.33, below, which yields (10.12). □

Lemma 10.32 *For a.e. $\lambda \in S^{n-1}$*

$$\mathbf{nc}(f) \bullet q^{-1}(\lambda) = \chi(A).$$

Proof By the first expression of (10.22) for $\mathbf{nc}(f)$, for a.e. $\lambda \in S^{n-1}$

$$\mathbf{nc}(f) \bullet q^{-1}(\lambda) = \lim_{r \downarrow 0} \nu_* \langle \mathbb{D}(f), f \circ \pi, r \rangle \bullet q^{-1}(\lambda)$$
$$= \lim_{r \downarrow r} \chi(f^{-1}[0,r])$$
$$= \chi(A),$$

where the second and third equality follows from Lemmas 10.27 and 10.28, respectively. □

Together with the following, replacing f by $h_{\lambda,c}$ in Lemma 10.32 now yields (10.12). It will be convenient here to conflate vectors $\lambda \in S^{n-1}$ with their associated covectors $v \mapsto \lambda \cdot v$. In order to form a mental image of the statement it may be helpful to consider the case in which A is a closed ball.

Lemma 10.33 *For a.e.* $(\lambda, c) \in S^{n-1} \times \mathbb{R}$*, the function*

$$h_{\lambda,c} := f + \max(\lambda - c, 0)$$

is an aura for $A \cap \lambda^{-1}(-\infty, c]$*, with*

$$\mathbf{nc}(h_{\lambda,c}) \bullet q^{-1}(-\lambda) = \big(\mathbf{nc}(f) \bullet q^{-1}(-\lambda)\big)\big|_{\lambda^{-1}(-\infty,c]}\,.$$

Proof Clearly the zero set of $h_{\lambda,c}$ is the intersection of A with the halfspace $H_{\lambda,c} := \{x : \lambda \cdot x \le c\}$. We claim that 0 is a weakly regular value of $h_{\lambda,c}$ for a.e. (λ, c), i.e. that $h_{\lambda,c}$ is an aura for this intersection. We claim that $h_{\lambda,c}$ is an aura for generic (λ, c), and that the slices by q at $-\lambda$ of $\mathbf{nc}(f)$, $\mathbf{nc}(h_{\lambda,c})$ agree above $H_{\lambda,c}$.

Obviously $f, h_{\lambda,c}$ agree in the open halfspace $\lambda^{-1}(-\infty, c)$, so the weak regularity condition holds there, and the differential cycles also agree there. Since $dh_{\lambda,c} \equiv \lambda$ in the complement of $\lambda^{-1}(-\infty, c]$), it remains only to show that the Clarke differential $\partial h_{\lambda,c}(x)$ is bounded away from the negative ray $(-\infty, 0] \cdot \lambda$ for $\lambda(x) = c$ and x near A.

To see this, recall that the calculus of the Clarke differential yields for such x

$$\partial h_{\lambda,c}(x) \subset \mathrm{conv}(\partial f(x) \cup \{\lambda\}). \tag{10.23}$$

Condition (10.23) is fulfilled if $\partial f(x)$ contains no nonpositive multiple of λ for x as above. Say that an $(n-1)$-plane $P \subset \mathbb{R}^n$ is *tangent* to A if there exists an n-plane $\bar{P} \subset \mathbb{R}^n \times \mathbb{R}$ that is tangent (in the sense of Theorem 10.22) to the graph of f at a point $(x, 0) \in \mathbb{R}^n \times \mathbb{R}$, $x \in A$, with $P = \bar{P} \cap (\mathbb{R}^n \times \{0\})$. Thus condition (10.23) holds if the hyperplane $\{x : \lambda \cdot x = c\}$ is not tangent to A.

Each such $\bar{P}$ corresponds to a nonzero element v of the Clarke differential $\partial f(x)$ that is perpendicular to P. Since 0 is the minimum value of f, it follows from the general theory of the Clarke differential that $0 \in \partial f(x)$; thus by convexity $rv \in \partial f(x)$ for $0 < r < 1$, so that in fact any such $(n-1)$-plane P tangent to A corresponds to an interval of corresponding n-planes $\bar{P}$ tangent to the graph of f. Since Theorem 10.22 states that the set of all such $\bar{P}$ is closed and of Hausdorff dimension n, it follows that the set of all planes P tangent to A is closed and of dimension $n-1$, i.e. has codimension 1 in the space of all affine hyperplanes in $\mathbb{R}^n$. Thus the desired conclusion in fact holds off of a closed set of Hausdorff codimension 1 in $S^{n-1} \times \mathbb{R}$. □

Thus by Theorem 10.7 the normal cycle constructed in (10.22) is independent of the choice of aura f for A. Since the axioms for differential cycles transform naturally, it follows that the same is true of the differential cycles themselves. Having constructed the normal cycle of a WDC set from the differential cycle of an aura we may now draw the following conclusion.

Corollary 10.34 *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 diffeomorphism and $\tilde{\phi} : S\mathbb{R}^n \to S\mathbb{R}^n$ the induced contact transformation. If f is an aura for the WDC set $A \subset \mathbb{R}^n$ then $f \circ \phi^{-1}$ is an aura for the WDC set $\phi(A)$, and*

$$\mathbf{nc}(\phi(A)) = \tilde{\phi}_* \, \mathbf{nc}(A).$$

Passing to local coordinates, it follows that the normal cycle of a WDC subset of a C^2 manifold M may be constructed from an aura in the same fashion as above.

10.4.3 Questions

Let $A \subset \mathbb{R}^n$ be a WDC set.

Is the support of $\mathbf{nc}(A)$ rectifiable? (Recalling Remark 10.24, this question is closely related to that of the rectifiability of the set of directions of line segments lying in the boundary of a given convex body.) Is the distance from A a DC function? Must the masses of the normal cycles $\mathbf{nc}(f^{-1}[0,\varepsilon))$ sublevel sets at small ε remain bounded? If not, must there exist a sequence of values $\varepsilon = \varepsilon_1, \varepsilon_2, \dots$ for which they are bounded?

These last three questions are to a certain degree intertwined: Pokorný has recently given an example of a Lipschitz graph $S \subset \mathbb{R}^3$ that admits a normal cycle, but for which the normal cycles of the distance tubes S_ε are unbounded in mass as $\varepsilon \downarrow 0$.

10.5 The Kinematic Formula for WDC Sets

Since the Gauss-Bonnet Theorem is already built into the axioms for the normal cycle, in order to satisfy the Federer criteria we must prove that the kinematic formulas hold for pairs of such sets. This we accomplish using a general scheme developed in [18, 21] which relies on representing the sets as sublevel sets of MA functions. Although there exists, under the same hypotheses, a form of the kinematic formula that applies to curvature measures, for simplicity we confine our discussion to scalar valuations.

10.5.1 The Formal Setup

Let (M, G) be a Riemannian isotropic space, in which we wish to establish the kinematic formulas.The formal part of the scheme follows the double fibration approach familiar to much of classical integral geometry: we construct an integral over a large space that represents the two respective sides of the equation when sliced in different ways. In this case the large space occurs as a subspace of the total space E of the pullback $\mathscr{E}$ a natural fiber bundle, and the two fibrations appear (1) as the projection Π of the pulled back bundle, and (2) as the tautological map Φ of the total spaces:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & SM \times G \\ \Pi \downarrow & & \downarrow \\ SM \times SM & \longrightarrow & M \times M \end{array} \tag{10.24}$$

Here the bottom map is the product of the two projections $\pi : SM \to M$ and the right hand map (bundle projection) is given by

$$(\xi, \gamma) \mapsto (\pi(\xi), \gamma^{-1}\pi(\xi)).$$

Thus the fiber above $(\xi, \eta) \in SM \times SM$ of the pullback bundle is

$$E_{\xi,\eta} := \{(\zeta, \gamma) : \pi(\zeta) = \pi(\xi), \gamma\pi(\eta) = \pi(\xi)\}.$$

Within $E_{\xi,\eta}$ lies the special subset $\mathbf{C}_{\xi,\eta}$, consisting of the pairs above for which ζ lies on some spherical geodesic joining $\gamma\eta$ to ξ. This geodesic is of course unique unless $\gamma\eta = -\xi$. Choosing reference points $o \in M, \bar{o} \in SM$ with $\pi(\bar{o}) = o$, the model fibers are then

$$E_0 := S_oM \times G_o \supset \mathbf{C}_0 := \{(\zeta, \gamma) \in E_0 : \zeta \in \overline{\bar{o}, \gamma\bar{o}}\}$$

where $G_o \subset G$ is the stabilizer of o. Thus $\mathbf{C}_0$ is a compact semialgebraic set of dimension $\dim G_o + 1$, and its top-dimensional stratum inherits a natural orientation from G_o.

There are obvious natural actions of $G \times G$ on all four spaces of (10.24), commuting with all of the maps. Thus this picture entails the following Proposition. Observe that the vector space of G-invariant differential forms on M is isomorphic to the space of $G_{\bar{o}}$-invariant elements of $\bigwedge^* T_{\bar{o}}SM$, and therefore finite dimensional. Put Z, Γ for the projections of $SM \times G$ to the respective factors, and similarly for $\Xi, H : SM \times SM \to SM$. Denote fiber integration with respect to the fibers $\mathbf{C}$ by $\Pi_{\mathbf{C}*} : \Omega^*(E) \to \Omega^*(SM)^2$.

Proposition 10.35 *Put $\Pi_{\mathbf{C}*} : \Omega^*(E) \to \Omega^*(SM \times SM)$ for the fiber integral with respect to the fibers $\mathbf{C}_{\xi,\eta}$. Let $d\,\mathrm{Vol}_G$ be an invariant volume form for G and β a G-invariant differential form on SM. The form $\Pi_{\mathbf{C}*}(\Phi^*(Z^*\beta \wedge \Gamma^* d\,\mathrm{Vol}_G))$ is then $G \times G$-invariant. Therefore it may be expressed as a finite sum $\sum_{ij} \Xi * \alpha_i \wedge H^*\alpha_j$, where the α_i constitute a basis for the space of G-invariant forms on SM.*

Given currents S, T in SM, we denote by $S \times T \times_{\mathscr{E}} \mathbf{C}$ the current in E given by taking the indicated fiber product with $\mathbf{C}$.

Corollary 10.36 *Suppose $\{\beta_1,\dots,\beta_N\}$ is a basis for the vector space of G-invariant forms of degree $n-1$ on SM. Then there are constants c_{ij}^k such that whenever $A,B \subset M$ are compact WDC subsets of M*

$$\int_G \left(\int_{Z_* \Phi_* \langle \mathbf{nc}(A) \times \mathbf{nc}(B) \times_{\mathscr{E}} \mathbf{C}, \Gamma, \gamma \rangle} \beta_k \right) d\,\mathrm{Vol}_G = \sum_{i,j} c_{ij}^k \int_{\mathbf{nc}(A)} \beta_i \int_{\mathbf{nc}(A)} \beta_j. \tag{10.25}$$

Proof This is a formal consequence of Proposition 10.35 and the slicing theory of [15, Sect. 4.2]. □

This formal picture is most easily interpreted in the simple case of two smooth domains $A,B \subset M$. We take $\xi = n_A(x), \eta = n_B(y)$ to be normal vectors to these domains at the boundary points $x \in A, y \in B$ respectively, and consider the set G_{xy} of all motions $\gamma \in G$ that map y to x, which is a coset of G_o. If ξ and $\gamma\eta$ are not antipodal then A,B meet transversely at x, and the intersection $A \cap \gamma B$ locally resembles a wedge. The normals to the wedge constitute an arc interpolating $\xi, \gamma\eta$, i.e. the geodesic arc that connects them in the fiber sphere S_xM. Thus, given any such boundary points x,y the set $\mathbf{C}_{n_A(x),n_B(y)}$ simultaneously depicts all of the fibers at x of the normal cycles $\mathbf{nc}(A \cap \gamma B)$ as γ varies over G_{xy}. Assembling all of these together as above and slicing at generic $\gamma \in G$ as on the left hand side of (10.25), we obtain the part of the normal cycle of $A \cap \gamma B$ that lies above the intersection of the boundaries of the two sets.

Thus the left hand integral of (10.25) represents one part of the kinematic formula as a whole, though of course not all of the normal vectors to $A \cap \gamma B$ arise in this way: if the boundary point x lies in the interior of γB then $A \cap \gamma B$ is locally identical to A, and $n_A(x)$ is the only normal vector also to $A \cap \gamma B$ there. A similar situation prevails if γy lies in the interior of A. Thus there are two terms of the kinematic formula missing above, corresponding to these geometrically trivial scenarios, but a much more elementary double fibration argument implies that these yield precisely $\int_{\mathbf{nc}(A)} \beta_k \cdot V_n(B)$ and $\int_{\mathbf{nc}(B)} \beta_k \cdot V_n(A)$.

10.5.2 Auras and Normal Cycles for Generic Intersections

In order to interpret this formalism as geometrically meaningful for pairs A,B of WDC sets, we argue below that (1) the set of all motions γ which map normals of B to vectors antipodal to normals of A has measure zero, and (2) the case of smooth domains is nonetheless representative even for singular spaces of this type. We address each point in turn.

Proposition 10.37 *Let $A,B \subset M$ be WDC sets with auras f,g respectively. Then $A \cap \gamma B$ is WDC for a.e. $\gamma \in G$. In fact, there is a closed set $C \subset G$ of measure zero such that if $\gamma \in G \setminus C$ then $h_\gamma := f + g \circ \gamma^{-1}$ is an aura for $A \cap \gamma B$.*

Proof It is clear that every h_γ is DC and nonnegative, with $A \cap \gamma B = h_\gamma^{-1}(0)$. It remains to show that 0 is a weakly regular value for γ as stated. By Theorem 10.23, the supports of $\mathbf{nc}(A), \mathbf{nc}(B)$ are $(n-1)$-dimensional in the Minkowski sense, which is well-behaved with respect to Cartesian products.

This follows from Theorem 10.23 in much the same way as Theorem 10.31 follows from the prototype Theorem 10.22. Put $N_f \subset SM$ for the subset of all elements (x, v) such that $f(x) = 0$ (i.e. $x \in A$) and $rv \in \partial f(x)$ for some $r > 0$, and define N_g similarly. Then N_f, N_g are compact and have finite $(n-1)$-dimensional Minkowski content, and hence the Cartesian product $N_f \times N_g$ has finite $(2n-2)$-dimensional Minkowski content. It is crucial that the dimensionality be measured in this way here: the corresponding statement for Hausdorff measure or dimension is false!

Consider now the fiber bundle $\mathscr{F}$

$$\{(\xi, \eta, \gamma) \in SM \times SM \times G : \gamma\eta = -\gamma\xi\}$$

over $(\xi, \eta) \in SM \times SM$. The fibers $\mathbf{F}$ are clearly diffeomorphic to the stabilizer $G_{\bar{o}} \subset G$ of a representative point $\bar{o} \in SM$, so that $N_f \times N_g \times_{\mathscr{F}} \mathbf{F}$ is compact and of finite $(2n-2+\dim G_{\bar{o}})$-dimensional Minkowski content. Since $2n-2+\dim G_{\bar{o}} = \dim SM + \dim G_{\bar{o}} - 1 = \dim G - 1$, the natural projection C to G of this set is compact and of Minkowski codimension 1. This set C has the desired property. □

Proposition 10.38 *Under the hypotheses of Proposition 10.37, for a.e. $\gamma \in G$ both of $A \cap \gamma B$, $A \cup \gamma B$ are WDC sets, with*

$$\mathbf{nc}(A \cup \gamma B) + \mathbf{nc}(A \cap \gamma B) = \mathbf{nc}(A) + \mathbf{nc}(\gamma B). \tag{10.26}$$

Proof It is not difficult to show that $\min(f, g \circ \gamma^{-1})$ is an aura for $A \cup \gamma B$ for $\gamma \in G \setminus C$. Therefore Lemmas 10.28 and 10.33 above imply that for a.e. halfspace H the respective intersections with H of $A, B, A \cup B, A \cap B$ are all neighborhood retracts. By VIII.6.12 and VIII.6.13 of [12] the Euler characteristics of these intersections are additive in the expected way. Therefore (10.26) follows from the Uniqueness Theorem 10.7. □

In view of Corollary 10.36, the proof of the kinematic formula for WDC sets will be completed by the following.

Proposition 10.39 *For such $\gamma \in G \setminus C$:*

$$\mathbf{nc}(A \cap \gamma B) = \mathbf{nc}(A) \llcorner \pi^{-1}(\gamma(B)) + \gamma_* \mathbf{nc}(B) \llcorner \pi^{-1} A + Z_* \langle \mathbf{nc}(A) \times \mathbf{nc}(B) \times_{\mathscr{E}} \mathbf{C}, \Gamma, \gamma \rangle.$$

Proof The first two terms on the right correspond to the points of the boundary of A lying in the interior of γB and vice versa. We claim that the last term arises from the expressions (10.22), with f replaced by h_γ.

First we observe that the differential cycles of the h_γ can be constructed from those of f, g in a straightforward way. Consider the bundle $\mathscr{F}$ over $T^*M \times T^*M$, with fibers $\simeq G_o$ and total space

$$F := \{(\xi, \eta, \gamma) : \gamma\pi(\eta) = \pi(\xi)\}.$$

Put $\Gamma : F \to G$ for the (restricted) projection, and define the mapping $\Sigma : F \to T^*M$ by $\Sigma(\xi, \eta, \gamma) := \xi + \gamma\eta$. Then for a.e. $\gamma \in G$

$$\mathbb{D}(h_\gamma) = \Sigma_* \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathscr{F}} G_o, \Gamma, \gamma \rangle. \tag{10.27}$$

The expressions (10.22) may be interpreted to say that $\mathbb{D}(f) \llcorner \mathrm{bd} A$ is a bundle of line segments $[0, r\xi]$ over $\xi \in \mathbf{nc}(A)$, for some function $r = r(\xi) > 0$. In view of (10.27) for a.e. $\gamma \in G$ any $\xi \in nc(A)$, $\eta \in \mathbf{nc}(B)$ with $\gamma\pi(\eta) = \pi(\xi)$ gives rise to the parallelogram (Minkowski sum)

$$[0, r\xi] + [0, r'\gamma\eta]$$

in the fiber of $\mathbb{D}(h_\gamma)$ over $\pi(\xi)$. Now we apply (10.22) to the aura h_γ: this parallelogram in $\mathbb{D}(h_\gamma)$ corresponds to the geodesic segment from ξ to $\gamma\eta$ in $\mathbf{nc}(A \cap \gamma B)$. □

10.5.3 Kinematic Valuations from WDC Sets and the Alesker-Bernig Formula

The kinematic formulas offer a prototype for the more general concept of Alesker product of smooth valuations: in [24] we propose as a model for a general smooth valuation in a smooth manifold M functionals of the following type, each associated to an appropriately rich smooth measured family of objects $A_p \subset M, p \in P$:

$$\nu(B) := \int_P \chi(B \cap A_p)\, dp. \tag{10.28}$$

This generalizes the construction in the kinematic formalism above, in which $P = G$ and A_p is replaced by gA. The Alesker product of such ν with a general smooth valuation μ is then given by

$$(\mu \cdot \nu)(B) := \int_P \mu(B \cap A_p)\, dp. \tag{10.29}$$

The proof in [24] of (10.29) applies to measured families of *smooth polyhedra* A_p, and proceeds by comparison with the celebrated product formula of Alesker and Bernig [3] for the product of two smooth valuations in terms of smooth differential forms that define them. The key point is to identify the term of (10.29) corresponding to the part of the normal cycle $\mathbf{nc}(B \cap A_p)$ lying above the intersection of the boundaries of A_p, B. In the proof of the kinematic formula, this corresponds to the part of the normal cycle analyzed in the proof of Proposition 10.39. It should be possible to execute the same analysis in the more general context, i.e. replacing the smooth polyhedra of Fu [24] with WDC sets.

10.5.4 Towards a Definitive Notion of Integral Geometric Regularity

There are numerous formally incompatible domains in which integral geometric regularity prevails (or should): WDC (which includes PR), piecewise $W^{2,n}$ domains (cf. Sect. 10.5.5 below), and one for each o-minimal structure. In order to make some sense out of this situation let us introduce the following notion.

Definition 10.40 A class $\mathscr{C}$ of compact subsets of $\mathbb{R}^n$ is an *integral geometric regularity (igregularity) class* if

1. $\mathscr{C}$ includes all compact smooth domains.
2. Every $X \in \mathscr{C}$ admits a normal cycle.
3. If $X \in \mathscr{C}$ and ϕ is a C^2 diffeomorphism of $\mathbb{R}^n$ then $\phi(X) \in \mathscr{C}$, with $\mathbf{nc}(\phi(X)) = \tilde{\phi}_* \, \mathbf{nc}(X)$.
4. If $X, Y \in \mathscr{C}$ then $X \cap \gamma Y \in \mathscr{C}$ for a.e. Euclidean motion γ, with

$$\begin{aligned}\mathbf{nc}(X \cap \gamma Y) = \; &\mathbf{nc}(X) \llcorner \pi^{-1}(\gamma Y) + \gamma_* \, \mathbf{nc}(Y) \llcorner \pi^{-1}(X) \\ &+ Z_* \langle \mathbf{nc}(X) \times \mathbf{nc}(Y \times_{\mathscr{E}} \mathbf{C}, \Gamma, \gamma \rangle.\end{aligned}$$

The last axiom implies that the Euclidean kinematic formulas apply to any pair of subsets from a given igregularity class. The second axiom implies that the class could also be extended to subsets of C^2 manifolds, and therefore the last axiom could also be stated to include kinematic formulas in all Riemannian isotropic spaces, but it is not intended that this definition be set in stone.

Thus PR and WDC are both igregularity classes. So is the class of all compact subanalytic subsets of $\mathbb{R}^n$, provided the second axiom above is relaxed to apply only to real analytic diffeomorphisms ϕ, and it seems more than plausible that the class of all C^2 images of compact subanalytic sets is an igregularity class. A similar statement holds for each o-minimal structure.

Since the definition is finitary, it follows from Zorn's lemma that every igregularity class is included in a maximal one.

Conjecture 10.41 There exists a unique maximal igregularity class.

What property might characterize such a class? One might postulate that it consists of all weakly regular sublevel sets of MA functions, but this rules out the interior cusps that may occur in the subanalytic case (e.g. the subset $\{(x, y) \in \mathbb{R}^2 : y \leq \sqrt{|x|}\}$). The only plausible suggestion known to us is the following.

Conjecture 10.42 Consider the class $\mathscr{N}$ consisting of all $X \subset \mathbb{R}^n$ such that there exists a constant C and a sequence $X_1 \supset X_2 \supset \cdots \supset X$ of compact smooth domains such that

$$\bigcap_{i=1}^{\infty} X_i = X, \qquad \text{mass}(\mathbf{nc}(X_i)) \leq C.$$

Then $\mathscr{N}$ is the unique maximal igregularity class.

One might be tempted to believe that the normal cycle of a set X can always be constructed by taking the sets X_i above to be tubular neighborhoods of X, at least if interior cusps are excluded (this exclusion could be formalized by assuming that X is a weakly regular sublevel sets of some Lipschitz function). However, a recent example of Pokorný shows that this is not the case. It is not known whether such examples may be taken to be WDC.

The class $\mathscr{N}$ satisfies axioms 1 and 3, and also the first part of 4. Since the Federer-Fleming compactness theorem implies that there is a convergent subsequence $\mathbf{nc}(X_{i'}) \rightharpoonup T$, it is tempting to take $\mathbf{nc}(X) = T$. However, this limit may not be unique, or might fail to have the right properties, for example if X is a point and the X_i are nested annuli: in this case $T = 0$. However, if integral geometric regularity is truly a kind of regularity then $\mathbf{nc}(X)$ must surely exist: the failure to converge must be attributable to extraneous pieces of the normal cycles $\mathbf{nc}(X_i)$ arising from unnecessary twists and turns in the approximating sets. Presumably the convergence could be made to work by eliminating these.

The sequence X_i may be imagined as a process of covering X with shrinkwrap. The process begins by enclosing X loosely by a bag X_1 of this wrapping material, then distorting it so that more and more of it is in contact with X, yielding the approximating sets $X_2, X_3, \ldots$. Distorting the shrinkwrap X_i—introducing curvature—is expensive, with cost measured by the mass of $\mathbf{nc}(X_i)$ as a kind of total curvature integral. In order for X to be wrapped in this way with finite cost, the total curvature of the successive stages X_i must remain bounded.

10.5.5 Questions

Is WDC $\subset \mathscr{N}$? This is related to the questions of Sect. 10.4.3 above. Does the class of $W^{2,n}$ domains (i.e. compact sets that are locally given as sublevel sets of $W^{2,n}$ functions) generate an integral geometric regularity class? Basically this would mean that a generic intersection of a finite collection of such domains admits a normal cycle. Such intersections might be regarded as "piecewise $W^{2,n}$ domains", although this description may implicitly understate their complexity.

10.6 How Smart is the Normal Cycle?

The (co)normal bundle of a smooth submanifold may be used to calculate a wide array of geometric quantities associated with a smooth manifold. Since the normal cycle is a measure-theoretic generalization, it is natural to ask whether $\mathbf{nc}(X)$ may be used to extend these quantities to singular, but integral geometrically regular, subspaces X. There is significant evidence for an affirmative answer.

10.6.1 Characteristic Classes

The conormal cycle may be used to construct the fundamental characteristic classes applicable to singular spaces, and to establish their basic properties. First we mention the generalized Stiefel-Whitney classes, constructed first by Sullivan, applicable to (sufficiently regular) *mod 2 Euler spaces*, i.e. spaces X with the property that the link of any $p \in X$ has even Euler characteristic.

Theorem 10.43 ([25]) *Let M be a smooth real analytic manifold, S^*M its cosphere bundle, and $\mathbb{RP}T^*M$ its real-projectivized cosphere bundle, so that S^*M is naturally an S^0 bundle $\mathscr{P}_\mathbb{R}$ over $\mathbb{RP}T^*M$. Let $s : S^*M \to S^*M$ denote the antipodal map.*

1. *A closed subanalytic set $X \subset M$ is a mod 2 Euler space iff its conormal cycle is antipodally symmetric mod 2, i.e. there exists an integral current T such that $\mathbf{nc}(X) + s_* \mathbf{nc}(X) = 2T$. Thus in this case there exists a mod 2 integral cycle $\mathbb{RP}\,\mathbf{nc}(X)$ of dimension $n-1$ in $\mathbb{RP}T^*M$ such that the mod 2 reduction of $\mathbf{nc}(X)$ may be expressed as $\mathbb{RP}\,\mathbf{nc}(X) \times_{\mathscr{P}_\mathbb{R}} S^0$.*
2. *There exist mod 2 cohomology classes $\beta_i \in H^i(\mathbb{RP}T^*M)$ such that the mod 2 cycles $[\mathbb{RP}\,\mathbf{nc}(X)] \cap \beta_i, i = 0, \ldots, n-1$, represent the Sullivan-Stiefel-Whitney classes of any mod 2 Euler subspace $X \subset M$.*

Next are the Chern-Schwartz-MacPherson homology classes of a singular complex analytic subvariety X of a smooth complex analytic manifold M; introduced originally in [35, 40], these generalize the Poincaré duals of the usual Chern cohomology classes of the tangent bundle of a smooth variety.

Theorem 10.44 ([20]) *Let M be a smooth complex analytic manifold of complex dimension n, S^*M its cosphere bundle and $\mathbb{C}\mathbb{P}T^*M$ its complex-projectivized cotangent bundle, so that S^*M is naturally an S^1 bundle $\mathscr{P}_{\mathbb{C}}$ over $\mathbb{P}T^*M$ via the fiberwise Hopf map.*

1. *If $X \subset M$ is a complex analytic subvariety then there exists a closed integral current $\mathbb{P}\,\mathbf{nc}(X)$ of dimension $2n-2$ such that $\mathbf{nc}(X) = \mathbb{P}\,\mathbf{nc}(X) \times_{\mathscr{P}_{\mathscr{C}}} S^1$.*
2. *There exist closed differential forms $\gamma_i \in \Omega^{2n-2i-2}(\mathbb{P}T^*M), i = 0, \ldots, n-1$, such that for any compact complex subvariety $X \subset M$ the closed currents $c_i(X) = \pi_*(\mathbb{P}\,\mathbf{nc}(X) \llcorner \gamma_i), i = 0, \ldots, n-1$ represent the Chern-Schwartz-MacPherson homology classes of X.*

Remark 10.45 In fact the projectivized conormal cycle $\mathbf{nc}(X)$ may be expressed as a weighted sum of ordinary conormal spaces of the (smooth) strata of X. The conormal cycle approach gives rise to a practical method of computing the weights (cf. [8]).

Together with the classical relations between the Chern classes and the Stiefel-Whitney classes of the tangent bundles of smooth complex manifolds, these parallel constructions imply the following.

Corollary 10.46 ([25]) *If X is a (singular) complex analytic subvariety of a complex manifold, then the Sullivan Stiefel-Whitney classes of X coincide with the mod 2 reductions of its Schwartz-MacPherson Chern classes.*

The Sullivan Stiefel-Whitney classes and the Schwartz-MacPherson Chern classes are both characterized by a functoriality property with respect to the category of *constructible functions*, viz. functions that may be expressed as locally finite sums and differences of characteristic functions of subanalytic sets (resp. complex analytic subvarieties)—the relation (10.15) (finite additivity of conormal cycles) implies that the conormal cycle operation may be extended uniquely to a linear map from constructible functions to Legendrian cycles. The key to the proofs of Theorems 10.43 and 10.44 is to establish this functoriality with respect to general maps f. In each case this is accomplished by examining the conormal cycle of a geometric object associated to f, viz. the mapping cylinder in the first case and the graph in the second.

10.6.2 Weyl's Theorem

The main theme of the present chapter is that the formalism of the normal cycle is well suited to meeting Federer's criteria for success in finding a second order geometric measure theory, and moreover extends to an impressively wide range of singular objects. However, these criteria are not completely definitive, as indicated by the following classical yet still astounding theorem of Hermann Weyl.

Theorem 10.47 (Weyl, [42]) *If $M \subset \mathbb{R}^n$ is a smooth submanifold then appropriate multiples of the tube coefficients are Riemannian invariants. In particular they are unchanged under a change in the isometric immersion into Euclidean space.*

In other words, when applied to smooth submanifolds $M \subset \mathbb{R}^n$ the primary invariants of classical integral geometry are actually characteristics of the inner metric structure of M. To what extent does this remain true if M is replaced by objects that are merely integral geometrically regular? Although our understanding of this question remains rudimentary, we have enough evidence to make the following.

Conjecture 10.48 Let $X \subset$ be a connected compact set admitting a normal cycle.

1. Any two points of X may be joined by a rectifiable curve $\gamma \subset X$.
2. Endow X with the metric given by $d(p, q)$ = the infimal length of all curves $\subset X$ joining p, q. Then the signed measures

$$\Phi_i^X := \pi_*(\mathbf{nc}(X) \llcorner \mu_i), \qquad i = 0, \ldots, n-1.$$

may be computed solely from this metric.

This is not known even in the relatively simple case of a convex hypersurface in $\mathbb{R}^n$, $n \geq 4$ (see Sect. 10.6.3 below for the $n = 3$ case). However, it is not hard to confirm it in some other basic cases:

Theorem 10.49

1. *The conjecture holds if $X \subset \mathbb{R}^n$ is a compact convex set.*
2. *(Bröcker-Kuppe [10]) If $X, Y \subset \mathbb{R}^n$ are closed subanalytic sets, and $g : X \to Y$ is a subanalytic homeomorphism such that* $\text{length}(g(\gamma)) = \text{length}(\gamma)$ *for every rectifiable curve $\gamma \subset X$, then*

$$g_* \Phi_i^X = \Phi_i^Y, \qquad i = 0, \ldots, n-1.$$

Proof (Sketch)

(1) The boundary of X is topologically distinguished. From the discussion above it is not hard to see that the intrinsically defined function

$$f(r) := \int_X \chi(B(x,r) \cap \partial X)\, dx$$

is a polynomial of degree n in $r > 0$. The i-th coefficient is (up to scale) equal to $(-1)^{n-i-1}$ times the i-th intrinsic volume of X.

(2) For simplicity we confine the discussion to the globalizations V_i of the curvature measures Φ_i. We proceed by induction on the dimension d of X.

Let $\mathscr{S}$ be a nice (Whitney b is enough) stratification of X such that $\{f(S) : S \in \mathscr{S}\}$ is again a nice stratification of Y. Put $S = S^d$ for top-dimensional stratum of $\mathscr{S}$

and $T := X \setminus S$. The length-preserving condition implies that $f|_S$ is a Riemannian isometry with its image.

Let $f \geq 0$ be any subanalytic function with $f^{-1}(0) = T$, and put $X_r := f^{-1}[r,\infty), T_r := f^{-1}[0,r], Q_r := f^{-1}(r)$. It is known that $\mathbf{nc}(T) = \lim_{r\downarrow 0} \mathbf{nc}(T_r)$, and therefore

$$V_i(X) = V_i(X_r) + V_i(T_r) - V_i(Q_r) \to \lim_{r\downarrow 0} V_i(X_r) + V_i(T) - \lim_{r\downarrow 0} V_i(Q_r)$$

as $r \downarrow 0$. Since the X_r, Q_r are smooth, the first and third terms are intrinsically defined, and so is $\mu_i(T)$ by the inductive hypothesis. □

Bernig [7] has given an intrinsic geometric expression for the scalar curvature measure $\Phi_{d-2}(X)$ for subanalytic X.

Part 1 of Conjecture 10.48 suggests the following scenario. We might naively attempt to approximate the normal cycle of a connected $X \subset \mathbb{R}^n$ by those of smooth hypersurfaces $X_1, X_2, \ldots \to X$ in the Hausdorff metric, under the assumption that the masses of the $\mathbf{nc}(X_i)$ are uniformly bounded. In any case, if such an approximation exists then X might be viewed as at least marginally igregular, and so we expect that the first part of Conjecture 10.48 holds. The most direct approach to proving this proceeds by way of the following.

Conjecture 10.50 There exists a universal constant C such that the intrinsic diameter of a smooth compact hypersurface $M \subset \mathbb{R}^n$ is no greater than $C \cdot \text{mass}(\mathbf{nc}(M))$.

This may be restated in more common terminology by replacing mass($\mathbf{nc}(M)$) by the equivalent (up to scale)

$$\int_M \sum_{j=0}^{n-1} \sum_{i_1<\cdots<i_j} |k_{i_1}\cdots k_{i_j}|\, d\mathcal{H}^{n-1}$$

where the k_i are the principal curvatures. This is true for surfaces, i.e. $n = 3$ (cf. [23, 41]).

10.6.3 Surfaces

In other words, Weyl's Theorem 10.47 suggests that the ostensibly extrinsic notion of integral geometric regularity also suggests a species of generalized Riemannian geometry tame enough to admit precise quantitative curvature measures, as opposed for example to the curvature bounds that are the foundation of the theory of what are now called Alexandrov spaces. Since the spaces belonging to this domain may not even be C^1, the absence of an analytic framework in which to compute these curvatures is an obvious obstacle.

However, such a framework does exist in case the objects in question are topological surfaces: Alexandrov's theory of *manifolds of bounded curvature* (MBC). The underlying idea relies on the special nature of the Gauss-Bonnet theorem in dimension 2. One considers all possible geodesic triangulations $\mathscr{T}$ of a given a compact metric surface Σ. Assuming for simplicity that the interior angles of the constituent triangles $T \in \mathscr{T}$ are well-defined, one asserts the Ansatz that the integral of the Gauss curvature measure Φ_0^Σ over the interior of T is equal to the angle deficit. An Alexandrov MBC is then defined to be a surface Σ for which the sum of all absolute angle deficits/surpluses over all the triangles $T \in \mathscr{T}$ is bounded by an absolute constant C, independent of $\mathscr{T}$, which then serves as an upper bound for the total absolute Gauss curvature $\int_\Sigma |K|$. Alexandrov showed that under these conditions Σ admits a well-defined signed measure Φ_0^Σ that fulfills the Ansatz.

Theorem 10.51 (Alexandrov) *A convex surface $\Sigma \subset \mathbb{R}^3$ is an MBC. Its Alexandrov Gauss curvature measure agrees with the Gauss curvature measure constructed from its normal cycle.*

Alexandrov's Theorem 10.51 may be generalized as follows.

Theorem 10.52 ([22]) *Suppose $\Sigma \subset \mathbb{R}^3$ is a surface that may be expressed locally as the graphs of Lipschitz functions $g_i : \mathbb{R}^2 \supset U \to \mathbb{R}$, such that each g_i is a strongly approximable* MA *function. Then Σ, as a length space, is an Alexandrov MBC, with Alexandrov Gauss curvature measure equal to the Gauss curvature measure arising from the normal cycle.*

The local graph condition may be weakened slightly. The proof proceeds by showing that the induced metrics on the domain $\mathbb{R}^2$ of the functions g_i converge. These metrics are of course smooth, with Gauss curvature induced by the curvature of the graphs, converging weakly to a signed measure obtained by contracting the normal cycle of Σ with the area form of S^2. Thus the conditions of Alexandrov's general Theorem 10.51 are fulfilled.

Pogorelov made the amazing claim that a C^1 surface in $\mathbb{R}^3$ is an MBC if its Gauss map has finite mapping area (cf. [4]). This hypothesis is weaker than finiteness of the area of its normal cycle. However, a fundamental heuristic states that normal cycle finiteness is equivalent to the statement that finite Gauss mapping area is finite for generic smooth perturbations: if the L^1 norm of the second fundamental form of Σ is infinite, then we may choose some direction v in space so that the integral of the absolute value of the second fundamental form, applied to the projection of v onto Σ, is infinite. Applying a small quadratic perturbation of space along the direction v, the resulting perturbed surface now has infinite total absolute Gauss curvature.

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Chapter 11
Valuations and Boolean Models

Julia Schulte and Wolfgang Weil

Abstract Valuations, as additive functionals, allow various applications in Stochastic Geometry, yielding mean value formulas for specific random closed sets and processes of convex or polyconvex particles. In particular, valuations are especially adapted to Boolean models, the latter being the union sets of Poisson particle processes. In this chapter, we collect mean value formulas for scalar- and tensor-valued valuations applied to Boolean models under quite general invariance assumptions.

11.1 Introduction

Hadwiger's characterization theorem for the intrinsic volumes (see Theorem 1.23) has important applications in integral geometry. Besides a kinematic formula for arbitrary continuous valuations on $\mathscr{K}^n$, the celebrated principal kinematic formula was proved by Hadwiger using his characterization result. In its general form, the principal kinematic formula for the intrinsic volumes V_j reads

$$\int_{G_n} V_j(K \cap gM)\mu(dg) = \sum_{k=j}^{n} c_{n,j,k} V_k(K) V_{n+j-k}(M), \tag{11.1}$$

for convex bodies $K, M \in \mathscr{K}^n$, $j = 0, \dots, n$, and with given constants $c_{n,j,k} \geq 0$. In 1959, Federer proved a local version of (11.1), for curvature measures, a notion he invented on the larger class of sets with positive reach. For both results, the global formula (11.1) and its local analog for curvature measures, a more direct proof was given in [20] by splitting the integration over the motion group G_n into

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a translation integral and a subsequent integration over the rotation group. This approach led to translative integral formulas for intrinsic volumes and curvature measures, introducing certain expressions of mixed type.

The need for translative integral formulas arose with the development of Stochastic Geometry in the 1970s by Matheron and Miles (we refer here and in the following to the book [21], for more details and specific references). In particular, in two important papers by Miles and Davy in 1976, the problem was discussed how geometric mean values for particles in a randomly overlapping system can be estimated from measurements at the union set. The formulas, which they proved, modeled the particle system by a stationary and isotropic Poisson process of convex bodies (a random countable subset of $\mathscr{K}^n$ with rigid motion invariant distribution) and then used the principal kinematic formula. As a surprising result, in two and three dimensions, it was possible to estimate the mean number of particles (per unit volume) in an overlapping particle system by measuring the specific area, boundary length and Euler characteristic of the union set in a bounded planar sampling window (respectively, the volume, surface area, integral mean curvature and Euler characteristic in the spatial situation). In addition, also mean particle quantities were obtained (mean area and boundary length, respectively, mean volume, mean surface area and mean integral mean curvature). Such overlapping particle systems occurred frequently in microscopic investigations and became more and more important for techniques in image analysis. There, the random set model, given as the union of a Poisson particle process, the *Boolean model*, was not only used for systems with given real particles but also for spatially homogeneous random structures without that there were particles in the background. Then, the mean particle characteristics served as important parameters to find an appropriate distribution for a fictive particle process to adjust a Boolean model to the given structure.

For such applications, the assumption of stationarity (spatial homogeneity) was mostly acceptable, but isotropy (rotation invariance) was often not fulfilled. This initiated the study of non-isotropic Boolean models, for which translative integral formulas were needed. In general dimensions, and for the intrinsic volumes, a corresponding system of formulas for stationary Boolean models was presented in [24]. A further important step was made in [1] by showing that the translative integral formulas for intrinsic volumes, in their local form for curvature measures, even produced mean value results in the non-stationary case. In the subsequent years, many related integral-geometric results were obtained for mixed volumes, support functions, area measures, and applied to particle processes and Boolean models. Recently, translative integral formulas for general valuations and local versions for measure-valued valuations became available and corresponding mean value formulas for Boolean models were established in [26, 27]. As we shall see, some of these results can also be applied to tensor valuations.

In the following survey, we describe the interrelations between valuations, translative integral formulas and Boolean models and give appropriate references. We mostly concentrate on the stationary situation. After discussing the general results, we collect various examples in Sect. 11.6. Formulas for tensor valuations are given in Sect. 11.7. The final Sect. 11.8 describes shortly extensions to non-stationary structures and also gives an outlook on the use of harmonic intrinsic volumes in the directional analysis of non-isotropic Boolean models, a development which was started in [6]. In two and three dimensions measurements of the specific harmonic intrinsic volumes allow the estimation of the mean number of particles per unit volume and of the mean harmonic intrinsic volumes (which include in particular the usual intrinsic volumes). Thus, these recent results are a natural extension of the formulas by Miles and Davy from 1976 to the non-isotropic situation.

11.2 Basic Definitions and Background Information

We consider the space $\mathscr{K}^n$ of convex bodies in $\mathbb{R}^n, n \geq 2$, supplied with the Hausdorff metric $\delta(\cdot, \cdot)$ and the dense subset $\mathscr{P}^n$ of convex polytopes. We refer to [19], for notions from Convex Geometry which are used in this chapter. In the following, we study *valuations* on $\mathscr{K}^n$ or $\mathscr{P}^n$. These are mappings $\varphi : \mathscr{K}^n \to \mathscr{X}$ (or $\varphi : \mathscr{P}^n \to \mathscr{X}$), which are additive in the sense that

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M),$$

whenever K, M and $K \cup M$ lie in $\mathscr{K}^n$ (respectively, in $\mathscr{P}^n$). Here $\mathscr{X}$ is a commutative (topological) semigroup, but we concentrate on the situations where $\mathscr{X} = \mathbb{R}$ (real valuations), $\mathscr{X} = \mathscr{M}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ (measure valuations; here $\mathscr{M}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is the space of finite signed Borel measures on $\mathbb{R}^n \times \mathbb{S}^{n-1}$), and $\mathscr{X} = \mathscr{T}^n$ the space of tensors in $\mathbb{R}^n$ (tensor valuations).

11.2.1 Real Valuations

Concerning real valuations, we mostly concentrate on the class Val of translation invariant continuous valuations, in the following. The standard examples of valuations in Val are the *intrinsic volumes* $V_m(K), m = 0, \dots, n$, for $K \in \mathscr{K}^n$. They are, in addition, invariant under rotations. McMullen [14, 16] has shown that every valuation $\varphi \in$ Val admits a (unique) decomposition

$$\varphi = \sum_{j=0}^{n} \varphi_j \tag{11.2}$$

into j-homogeneous valuations φ_j (which are again translation invariant and continuous). Here, φ_0 is a constant and Hadwiger [4] has proved that $\varphi_n = c_n V_n$. For $m = 1, \dots, n-1$, the vector space Val_m of m-homogeneous valuations is infinite-dimensional. In particular, McMullen [15] has shown that $\varphi \in \mathrm{Val}_{n-1}$ if and only if

$$\varphi(K) = \int_{\mathbb{S}^{n-1}} f(u) S_{n-1}(K, du), \qquad K \in \mathscr{K}^n, \tag{11.3}$$

for some continuous function $f = f_\varphi$ on $\mathbb{S}^{n-1}$ which is uniquely determined, up to a linear function (see Theorem 1.25). Here, $S_{n-1}(K, \cdot)$ is the area measure of K (see Sect. 1.3).

We also recall from Theorem 1.18 that, for a polytope $P \in \mathscr{P}^n$ and $\varphi_j \in \mathrm{Val}_j, j = 1, \dots, n-1$, we have

$$\varphi_j(P) = \sum_{F \in \mathscr{F}_j(P)} f_j(n(P,F))V_j(F), \tag{11.4}$$

where the summation is over all j-dimensional faces of P, $n(P,F)$ is the set of all unit vectors which are normals of P at relative interior points of F ($n(P,F)$ is a spherical polytope of dimension $n-j-1$) and f_j is a simple valuation on the spherical polytopes of dimension $\leq n-j-1$.

11.2.2 Measure Valuations

Concerning measure valuations, we mention the *support measures* $\Lambda_j(K, \cdot), j = 0, \dots, n-1$, which are finite measures on $\mathbb{R}^n \times \mathbb{S}^{n-1}$, continuous with respect to the weak convergence of measures and translation covariant in the sense that

$$\Lambda_j(K, A \times B) = \Lambda_j(K + x, (A + x) \times B)$$

for Borel sets $A \subset \mathbb{R}^n, B \subset \mathbb{S}^{n-1}$, and all $x \in \mathbb{R}^n$. They are also rotation covariant,

$$\Lambda_j(K, A \times B) = \Lambda_j(\vartheta K, \vartheta(A \times B)), \qquad \vartheta \in \mathrm{SO}_n.$$

If $\varphi : \mathscr{K}^n \to \mathscr{M}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ is a continuous, translation covariant measure-valued functional, which is locally determined in the spatial component in the sense of Theorem 1.29, Condition (d), then there is a decomposition similar to (11.2), which follows from Theorem 3.1 in [12]. Namely,

$$\varphi = \sum_{j=0}^{n} \varphi_j \tag{11.5}$$

with j-homogeneous measure-valued functionals $\varphi_j : \mathscr{K}^n \to \mathscr{M}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ (which are again translation covariant, continuous and locally determined in the spatial component). Here, homogeneity means that

$$\varphi(\alpha K, (\alpha A) \times B) = \alpha^j \varphi(K, A \times B) \tag{11.6}$$

for Borel sets $A \subset \mathbb{R}^n, B \subset \mathbb{S}^{n-1}$, and all $\alpha \geq 0$. This decomposition does not require additivity of φ, but if φ is a valuation then the homogeneous components φ_j are also valuations, $j = 0, \ldots, n$.

The support measures give rise to two further series of measures, the *curvature measures* $C_0(K, \cdot), \ldots, C_{n-1}(K, \cdot)$ and the *area measures* $S_0(K, \cdot), \ldots, S_{n-1}(K, \cdot)$ of K. The former are (up to some constant) the projections of the support measures onto the first component and the latter are (up to the same constant) the projections onto the second component. For different normalizations of curvature and area measures, see Sect. 1.3.

11.2.3 Tensor Valuations

For $j = 0, \ldots, n-1$ and $r, s \in \mathbb{N}_0$, the basic tensor valuations, the *Minkowski tensors* $\Phi_j^{r,s}(K)$ (of rank $r + s$), arise as (tensor) integrals of the support measures,

$$\Phi_j^{r,s}(K) = c_{n-j}^{r,s} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} x^r u^s \Lambda_j(K, d(x, u))$$

where $c_k^{r,s} := \frac{1}{r!s!} \frac{\omega_k}{\omega_{k+s}}$ for $k \in \{1, \ldots, n\}$. Here ω_i is the surface area of the i-dimensional unit ball (see (1.14)). The Minkowski tensors with $r = 0$ are translation invariant; for $r > 0$ they have a special covariance property with respect to translations (see Sect. 2.2).

11.3 The Basic Equation for Boolean Models

The *Boolean model* is a random closed set $Z \subset \mathbb{R}^n$ which arises in a special way, namely as the union of sets (called *grains*) from a Poisson process Y. Usually, the grains are assumed to be compact or even compact and convex. More general random sets Z can be considered if Y is an arbitrary point process on the class $\mathscr{C}^n$ of nonempty compact sets in $\mathbb{R}^n$ or on the subclass of convex bodies $\mathscr{K}^n$. In particular, if Z and Y are *stationary*, that is, have a distribution invariant under translations, the random set Z can be interpreted as a *germ-grain model*,

$$Z := \bigcup_{i=0}^{\infty} (x_i + Z_i),$$

where points (*germs*) $x_1, x_2, \ldots$ are distributed in $\mathbb{R}^n$ according to a stationary point process X and then random compact (or compact, convex) sets Z_i (the grains) are attached to the germs in a suitable way. We shall describe this construction in the next subsection, but will concentrate on the Poissonian case and convex sets, that is to Boolean models, where the grains are convex and independent from each other and from the underlying germ process X. These strong independence properties together with the fact that the realizations of Z are locally polyconvex allow to apply valuations φ to Z and to express the expected value $\mathbf{E}\varphi(Z \cap K_0)$ in a bounded sampling window K_0 by the characteristic parameters of X and the Z_i. This will be explained in the second subsection. The effective further investigation of Boolean models then requires formulas from Translative Integral Geometry, as they will be provided in Sect. 11.4. Background material on random sets, point processes and the integral geometric results as well as further material on Boolean models can be found in [21] and we refer to this book for all details which are not explained in the following.

11.3.1 Boolean Models

Since we will only consider stationary Boolean models Z with convex grains throughout the following, we start with a stationary Poisson process in $\mathbb{R}^n$. A stationary *point process* X in $\mathbb{R}^n$ is a (simple) random counting measure

$$X := \sum_{i=1}^{\infty} \delta_{\xi_i},$$

where δ_x denotes the Dirac measure in $x \in \mathbb{R}^n$ and where the ξ_i are distinct random points in $\mathbb{R}^n$. We also assume that X is locally finite meaning that (almost surely) each $C \in \mathscr{C}^n$ contains only finitely many points ξ_i from X. Here, in the description, we already made use of the fact that such a point process can be represented in an alternative way as a locally finite (random) closed set

$$X = \{\xi_1, \xi_2, \dots\} \subset \mathbb{R}^n.$$

To make these definitions precise, we need an underlying probability space $(\Omega, \mathscr{A}, \mathbf{P})$ and σ-algebras on the class $\mathscr{F}^n$ of closed sets in $\mathbb{R}^n$, respectively on the class N of counting measures in $\mathbb{R}^n$. For details we refer to [21] but mention that the former is chosen as the Borel σ-algebra $\mathscr{B}(\mathscr{F}^n)$ of the hit-or-miss topology on $\mathscr{F}^n$ and the latter, $\mathscr{N}$, is generated by the evaluation (or counting) maps

$$\Phi_A : \eta \mapsto \eta(A), \qquad A \in \mathscr{B}(\mathbb{R}^n).$$

The *stationarity*, which we assume in addition, means that $X + t$ has the same distribution as X for all translations $t \in \mathbb{R}^n$. X is *isotropic* if the distribution of X is invariant under rotations. (Here translations and rotations act in the natural way on counting measures, respectively on closed sets.)

The (stationary) point process X is a *Poisson process* if $X(A)$ has a Poisson distribution for all bounded Borel sets $A \subset \mathbb{R}^n$,

$$\mathbf{P}(X(A) = k) = \mathrm{e}^{-\gamma\,\lambda_n(A)}\,\frac{(\gamma\,\lambda_n(A))^k}{k!}, \qquad k = 0, 1, 2, \dots$$

Here $\gamma = \mathbf{E}X([0,1]^n)$ is the *intensity* of the Poisson process. It describes the mean number of points of X per unit volume. Because of the stationarity we have $\mathbf{E}X(A) = \gamma\,\lambda_n(A)$, for all $A \in \mathscr{B}(\mathbb{R}^n)$. As a consequence of the Poisson property, the random variables $X(A_1), \dots, X(A_m)$ are (stochastically) independent if the Borel sets $A_1, \dots, A_m$ are disjoint. More generally, in this case, also the restrictions $X \llcorner A_1, \dots, X \llcorner A_m$ are independent random measures. The Poisson process X is uniquely determined (in distribution) by the parameter γ. Since the Lebesgue measure λ_n is rotation invariant, X is isotropic.

Now assume that X is a stationary Poisson process with intensity $\gamma > 0$, enumerated (in a measurable way) as $X = \{\xi_1, \xi_2, \dots\}$. Let $\mathbf{Q}$ be a probability measure on $\mathscr{K}^n$ (supplied with the Borel σ-algebra with respect to the Hausdorff metric) and let $Z_1, Z_2, \dots,$ be a sequence of independent random convex bodies with distribution $\mathbf{Q}$ (and independent of the Poisson process X). Then

$$Z := \bigcup_{i=1}^{\infty} (\xi_i + Z_i)$$

is a stationary random set, a *Boolean model*. Some additional assumptions are helpful. First, we require that

$$\int_{\mathscr{K}^n} V_n(K + B^n)\,\mathbf{Q}(dK) < \infty, \tag{11.7}$$

since then Z is a closed set (and moreover $Z \cap K$ is polyconvex for each $K \in \mathscr{K}^n$). Second, we assume that $\mathbf{Q}$ is concentrated on the *centered* convex bodies $\mathscr{K}_c^n$ (the class of bodies $K \in \mathscr{K}^n$ with center of the circumsphere at the origin). The effect of this condition is that $\mathbf{Q}$ is uniquely determined by Z and that Z is isotropic if and only if $\mathbf{Q}$ is invariant under rotations.

For the following it is often convenient to use the particle process $Y = \{\xi_1 + Z_1, \xi_2 + Z_2, \dots\}$. This is a point process on the locally compact space $\mathscr{K}^n$ (that is, a (simple) random counting measure on $\mathscr{K}^n$ or, equivalently, a locally finite random closed subset of $\mathscr{K}^n$). The process Y also has the Poisson property, that means, the random number $Y(A)$ of particles from Y in a Borel set $A \subset \mathscr{K}^n$ has a Poisson distribution. Later we will use this for the sets

$$\mathscr{K}_C := \{K \in \mathscr{K}^n : K \cap C \neq \emptyset\}, \qquad C \in \mathscr{C}^n.$$

Proposition 11.1 *For $A \in \mathcal{B}(\mathcal{K}^n)$, we have*

$$\mathbf{P}(Y(A) = k) = \mathrm{e}^{-\Theta(A)} \frac{(\Theta(A))^k}{k!}, \qquad k = 0, 1, \ldots,$$

where Θ is the image measure (on $\mathcal{K}^n$) of $\gamma\,\lambda_n \otimes \mathbf{Q}$ under the mapping Φ : $\mathbb{R}^n \times \mathcal{K}_c^n \to \mathcal{K}^n$, $(x, K) \mapsto x + K$.

Proof By the extension theorem of measure theory, and since Φ is a homeomorphism, it is sufficient to prove the result for sets $A = \Phi(B \times C)$, $B \in \mathcal{B}(\mathbb{R}^n)$, $C \in \mathcal{B}(\mathcal{K}_c^n)$. In this case, the independence properties of Z yield

$$\begin{aligned}
\mathbf{P}(Y(A) = k) &= \mathbf{P}\Big(\sum_{i=1}^{\infty} \mathbf{1}\{\xi_i \in B, Z_i \in C\} = k\Big) \\
&= \sum_{j=k}^{\infty} \mathbf{P}(X(B) = j) \binom{j}{k} \mathbf{Q}(C)^k (1 - \mathbf{Q}(C))^{j-k} \\
&= \mathrm{e}^{-\gamma\,\lambda_n(B)} \mathbf{Q}(C)^k \sum_{j=k}^{\infty} \binom{j}{k} (1 - \mathbf{Q}(C))^{j-k} \frac{(\gamma\,\lambda_n(B))^j}{j!} \\
&= \mathrm{e}^{-\gamma\,\lambda_n(B)} \mathbf{Q}(C)^k \frac{(\gamma\,\lambda_n(B))^k}{k!} \sum_{i=0}^{\infty} (1 - \mathbf{Q}(C))^i \frac{(\gamma\,\lambda_n(B))^i}{i!} \\
&= \mathrm{e}^{-\gamma\,\lambda_n(B)} \frac{(\gamma\,\lambda_n(B)\mathbf{Q}(C))^k}{k!} \mathrm{e}^{\gamma\,\lambda_n(B) - \gamma\,\lambda_n(B)\mathbf{Q}(C)} \\
&= \mathrm{e}^{-\Theta(A)} \frac{(\Theta(A))^k}{k!}.
\end{aligned}$$

Here $\mathbf{1}\{\cdot\}$ denotes the indicator function of the event $\{\cdot\}$. □

We emphasize the fact that the (stationary) Boolean model Z is uniquely determined (in distribution) by the two quantities γ (a constant which we always assume to be > 0) and $\mathbf{Q}$ (a probability measure on $\mathcal{K}_c^n$). Thus, in order to fit a Boolean model to given data (in form of closed sets in a window K_0, say), one has to determine (more precisely, to estimate) γ and $\mathbf{Q}$ from the data, that is from observations of realizations $Z(\omega) \cap K_0$ of Z in K_0.

11.3.2 Additive Functionals

Concerning the estimation problem described above, let us assume that we observe $\varphi(Z(\omega) \cap K_0)$ for some geometric functional φ in the window K_0. The mean value $\mathbf{E}\varphi(Z \cap K_0)$ is then the quantity which can be estimated unbiasedly by $\varphi(Z(\omega) \cap K_0)$.

It is natural to assume that the window is convex, hence $K_0 \in \mathscr{K}^n$. Then $Z \cap K_0$ is polyconvex a.s. Therefore it is another natural assumption that φ is additive, hence a valuation. In order to have a smooth behavior with respect to approximations (at least on $\mathscr{K}^n$), we also assume that φ is continuous on $\mathscr{K}^n$. Hence, we consider now $\mathbf{E}\varphi(Z \cap K_0)$, for $K_0 \in \mathscr{K}^n$ and a continuous real valuation φ on $\mathscr{K}^n$.

The following result from [28] (see also [21, Theorem 9.1.2]), expresses the mean value $\mathbf{E}\varphi(Z \cap K_0)$ in terms of γ and $\mathbf{Q}$. Notice that we do no require that φ is translation invariant.

Theorem 11.2 *Let Z be a stationary Boolean model in $\mathbb{R}^n$ with convex grains, let $K_0 \in \mathscr{K}^n$ be a sampling window and φ a continuous real valuation on $\mathscr{K}^n$. Then $\mathbf{E}|\varphi(Z \cap K_0)| < \infty$ and*

$$\mathbf{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k \int_{\mathscr{K}_c^n} \cdots \int_{\mathscr{K}_c^n} \Phi(K_0, K_1, \ldots, K_k)\, \mathbf{Q}(dK_1) \cdots \mathbf{Q}(dK_k)$$

with

$$\Phi(K_0, K_1, \ldots, K_k) := \int_{(\mathbb{R}^n)^k} \varphi(K_0 \cap (K_1 + x_1) \cap \cdots \cap (K_k + x_k))\, \lambda_n^k(d(x_1, \ldots, x_k)) \,. \tag{11.8}$$

Proof We sketch the proof since it sheds some light on the role of the Poisson assumption underlying the Boolean model. To simplify the formulas, we use the particle process $Y = \{\xi_1 + Z_1, \xi_2 + Z_2, \ldots\}$.

Almost surely the window K_0 is hit by only finitely many grains $M_1, \ldots, M_\nu \in Y$ (here ν is a random variable). The additivity of φ implies

$$\begin{aligned}
\varphi(Z \cap K_0) &= \varphi\Big(\bigcup_{i=1}^{\nu} M_i \cap K_0\Big) \\
&= \sum_{k=1}^{\nu} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le \nu} \varphi(K_0 \cap M_{i_1} \cap \cdots \cap M_{i_k}) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{(N_1, \ldots, N_k) \in Y_{\neq}^k} \varphi(K_0 \cap N_1 \cap \cdots \cap N_k).
\end{aligned} \tag{11.9}$$

Here, $Y_{\neq}^k$ denotes the set of all k-tuples of pairwise distinct bodies in Y and we could extend the summation to infinity since $\varphi(\emptyset) = 0$.

The continuity of φ on $\mathcal{K}^n$ implies $|\varphi(M)| \le c(K_0)$ for all $M \in \mathcal{K}^n, M \subset K_0$. Hence,

$$|\varphi(Z \cap K_0)| \le \sum_{k=1}^{\nu} \binom{\nu}{k} c(K_0) \le 2^\nu c(K_0).$$

From Proposition 11.1 we get

$$\begin{aligned}\mathbf{E}2^\nu &= \sum_{k=0}^{\infty} 2^k \mathbf{P}(Y(\mathcal{K}_{K_0}) = k)\\ &= \mathrm{e}^{-\Theta(\mathcal{K}_{K_0})} \sum_{k=0}^{\infty} \frac{(2\Theta(\mathcal{K}_{K_0}))^k}{k!}\\ &= \mathrm{e}^{\Theta(\mathcal{K}_{K_0})} < \infty\end{aligned}$$

since

$$\begin{aligned}\Theta(\mathcal{K}_{K_0}) &= \gamma \int_{\mathcal{K}_c^n} \int_{\mathbb{R}^n} \mathbf{1}\{(x+K) \cap K_0 \neq \emptyset\}\, \lambda_n(dx)\mathbf{Q}(dK)\\ &= \gamma \int_{\mathcal{K}_c^n} V_n(K + (-K_0))\, \mathbf{Q}(dK)\\ &\le \gamma \max\{r(K_0), 1\}^n \int_{\mathcal{K}_c^n} V_n(K + B^n)\, \mathbf{Q}(dK) < \infty,\end{aligned}$$

due to condition (11.7) (here $r(K_0)$ is the circumradius of K_0). Hence we obtain $\mathbf{E}|\varphi(Z \cap K_0)| < \infty$.

This integrability property allows to use the dominated convergence theorem for $\mathbf{E}\varphi(Z \cap K_0)$, where $\varphi(Z \cap K_0)$ is expressed by formula (11.9), and interchange expectation and summation. We get

$$\mathbf{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \mathbf{E} \sum_{(N_1,\dots,N_k) \in Y_{\neq}^k} \varphi(K_0 \cap N_1 \cap \dots \cap N_k).$$

Now we use the Campell theorem for point processes [21, Theorem 3.1.2] (applied to the special point process $Y_{\neq}^k$ on $(\mathcal{K}_c^n)^k$) and the fact that the intensity measure of $Y^k_{\neq}$, for a Poisson process Y, is the product measure Θ^k of Θ (see [21, Corollary 3.2.4]). We obtain

$$\mathbf{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathscr{K}^n_c} \cdots \int_{\mathscr{K}^n_c} \varphi(K_0 \cap N_1 \cap \cdots \cap N_k)\, \Theta(dN_1) \cdots \Theta(dN_k). \qquad (11.10)$$

Inserting the special form of Θ now yields the result. □

Remarks

(1) There is a simple and obvious generalization of the last theorem to Boolean models with polyconvex grains if the integrability condition (11.7) is modified appropriately (the number of convex bodies which constitute the typical polyconvex grain should be limited). There is also another, less obvious generalization to non-stationary Boolean models. This requires to consider a general Poisson process Y on $\mathscr{K}^n$ where the intensity measure Θ can have a more general form (this induces that the underlying Poisson process X in $\mathbb{R}^n$ is also not stationary anymore). Formula (11.10) then still holds, provided Θ is *translation regular*. We will explain this and give more results in Sect. 11.8.

(2) Due to the stationarity and the independence properties of the Poisson process Y, the grains $M_1, M_2, \ldots \in Y$ are almost surely in general relative position. This implies that geometric functionals φ on $\mathscr{K}^n$ or $\mathscr{P}^n$ can have an additive extension to the polyconvex set $Y \cap K_0$, although they are not valuations. Examples are the local functionals on $\mathscr{P}^n$ considered in [26]. For them, Theorem 11.2 still holds for Boolean models with polytopal grains. A j-homogeneous local functional $\varphi_j(P), P \in \mathscr{P}^n$, of interest is the total content of the j-dimensional skeleton (the union of the j-dimensional faces) of P,

$$\varphi_j(P) := \sum_{F \in \mathscr{F}_j(P)} V_j(F).$$

For $j = 0, \ldots, n-2$, this functional is not additive on $\mathscr{K}^n$. Since we concentrate on valuations in this chapter, we will not discuss general local functionals further and refer to [26] for information (but observe the remarks on local extensions of valuations in Sect. 11.4.1).

11.4 Integral Geometry for Valuations

In order to simplify the expectation formula in Theorem 11.2, it would be helpful to have a more explicit expression for the (iterated) translative integral in (11.8). This is obtained in the following subsection. In the second subsection, we discuss kinematic formulas.

11.4.1 The Translative Formula

Some results on translative integrals in dimension 2 and 3 are due to Blaschke, Berwald and Varga in 1937. A first general translative integral formula for intrinsic volumes in $\mathbb{R}^n$ and their local counterparts, the curvature measures, was obtained in [20] and the iterated version was proved in [24] (see the Notes for Sect. 6.4 in [21], for further references, variations and extensions).

The k-fold iterated translative integral formula for the intrinsic volume V_j involved mixed functionals which were denoted by $V^{(j)}_{m_1,\dots,m_k}$ (where the parameters satisfy $m_1 + \cdots + m_k = (k-1)n + j$), a notation which was subsequently used also for various related results on support measures and other local functionals. In the sequel, we use a special notation, which was introduced in [6] to simplify the resulting formulas. First, we observe that the exponent (j) in such mixed expressions can be determined from $j = m_1 + \cdots + m_k - (k-1)n$ and is therefore redundant. Then, we introduce a multi-index $\mathbf{m} = (m_1, \dots, m_k)$ from the class

$$\mathrm{mix}(j,k) := \{\mathbf{m} = (m_1, \dots, m_k) \in \{j, \dots, n\}^k : m_1 + \cdots + m_k = (k-1)n + j\},$$

for $j \in \{0, \dots, n\}$ and $k \in \mathbb{N}$, and abbreviate the mixed functional $\varphi_{m_1,\dots,m_k}$ by $\varphi_{\mathbf{m}}$. For $\mathbf{m} \in \mathrm{mix}(j,k)$, we also write $|\mathbf{m}| := k$.

The following theorem was obtained in [27], based on a corresponding result for polytopes in [26]. We now assume that the functional φ is translation invariant.

Theorem 11.3 *For $\varphi \in \mathrm{Val}$, let φ_j be its j-homogeneous part, $j = 0, \dots, n$, with $\varphi_n = c_n V_n$. Then, for $k \geq 2$, there exist mixed functionals $\varphi_{\mathbf{m}}$, $\mathbf{m} \in \mathrm{mix}(j,k)$, on $(\mathcal{K}^n)^k$ such that for convex bodies $K_1, \dots, K_k \in \mathcal{K}^n$,*

$$\int_{(\mathbb{R}^n)^{k-1}} \varphi_j(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k))\, \lambda_n^{k-1}(d(x_2, \dots, x_k)) = \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} \varphi_{\mathbf{m}}(K_1, \dots, K_k). \tag{11.11}$$

For $(m_1, \dots, m_k) \in \mathrm{mix}(j,k)$ the mapping $(K_1, \dots, K_k) \mapsto \varphi_{m_1,\dots,m_k}(K_1, \dots, K_k)$ is symmetric (w.r.t. permutations of the indices $1, \dots, k$), it is homogeneous of degree m_i in K_i and it is a valuation in Val *in each of its variables K_i.*

Proof Again, we give only a sketch of the proof and refer to [26, 27] for details. In particular, we omit the discussion of the necessary measurability properties. Also, we concentrate on the case $k = 2$, the general case follows then by iteration. On the other hand, we give a more general proof using measures, since this will give us also a local version of the theorem, as explained in the Remark given below.

The main idea is to consider polytopes first and then extend the result to arbitrary convex bodies by approximation. Since the restriction of φ_j to $\mathcal{P}^n$ is a continuous, translation invariant valuation, homogeneous of degree j, we can use (11.4) to decompose $\varphi_j(P), P \in \mathscr{P}^n$, as

$$\varphi_j(P) = \sum_{F \in \mathscr{F}_j(P)} f_j(n(P,F))V_j(F).$$

We define a measure $\Phi_j(P, \cdot)$ on $\mathbb{R}^n$ by

$$\Phi_j(P, \cdot) := \sum_{F \in \mathscr{F}_j(P)} f_j(n(P,F))\lambda_F.$$

Here, λ_F denotes the restriction to F of the j-dimensional Lebesgue measure in the affine space generated by F. Then, $P \mapsto \Phi_j(P, \cdot)$ is a measure valuation, homogeneous of degree j (in the sense of (11.6)) and translation covariant.

The following arguments are similar to those used in the proofs of Theorems 5.2.2 and 6.4.1 in [21]. Since f_j can be written as a difference of two positive functions $f_j = f_j^+ - f_j^-$, we may assume $f_j \geq 0$ (the additivity of f_j, which can get lost in this decomposition, does not play a role in the following arguments).

Let $P, Q \in \mathscr{P}^n$, $A, B \in \mathscr{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then,

$$\begin{aligned}
&\Phi_j(P \cap (Q + x), A \cap (B + x)) \\
&\qquad = \sum_{F' \in \mathscr{F}_j(P \cap (Q+x))} f_j(n(P \cap (Q + x), F'))\lambda_{F'}(A \cap (B + x)).
\end{aligned}$$

For λ_n-almost all x, the face F' is the intersection $F' = F \cap (G + x)$ of some m-face F of P with the translate of a $(n + j - m)$-face G of Q, $m \in \{j, \dots, n\}$ (such that F and $G + x$ meet in relative interior points). The normal cone of $F \cap (G + x)$ does not depend on the choice of x, let $n(P, Q; F, G)$ be its intersection with $\mathbb{S}^{n-1}$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi_j(P \cap (Q + x), A \cap (B + x))\, \lambda_n(dx) \\
&= \sum_{m=j}^{n} \sum_{F \in \mathscr{F}_m(P)} \sum_{G \in \mathscr{F}_{n+j-m}(Q)} f_j(n(P, Q; F, G)) \int_{\mathbb{R}^n} \lambda_{F \cap (G+x)}(A \cap (B + x))\, \lambda_n(dx).
\end{aligned}$$

In [21, pp. 185–186] it is shown that

$$\int_{\mathbb{R}^n} \lambda_{F \cap (G+x)}(A \cap (B + x))\, \lambda_n(dx) = [F, G]\lambda_F(A)\lambda_G(B),$$

where $[F, G]$ denotes the determinant between F and G (see [21, p. 183]). Hence, if we define a measure $\Phi_{m,n+j-m}(P, Q; \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\Phi_{m,n+j-m}(P, Q; \cdot) := \sum_{F \in \mathscr{F}_m(P)} \sum_{G \in \mathscr{F}_{n+j-m}(Q)} f_j(n(P, Q; F, G))[F, G]\lambda_F \otimes \lambda_G, \tag{11.12}$$

we arrive at

$$\int_{\mathbb{R}^n} \Phi_j(P \cap (Q + x), A \cap (B + x))\, \lambda_n(dx) = \sum_{m=j}^{n} \Phi_{m,n+j-m}(P, Q; A \times B). \tag{11.13}$$

Now we consider the total measures $\Phi_j(P \cap (Q + y), \mathbb{R}^n)$ (which equals our valuation $\varphi_j(P \cap (Q + y)))$ and $\varphi_{m,n+j-m}(P, Q) := \Phi_{m,n+j-m}(P, Q; \mathbb{R}^n \times \mathbb{R}^n)$. Then (11.13) implies

$$\int_{\mathbb{R}^n} \varphi_j(P \cap (Q + x))\, \lambda_n(dx) = \sum_{m=j}^{n} \varphi_{m,n+j-m}(P, Q). \tag{11.14}$$

We remark that

$$\varphi_{m,n+j-m}(P, Q) = \sum_{F \in \mathscr{F}_m(P)} \sum_{G \in \mathscr{F}_{n+j-m}(Q)} f_j(n(P, Q; F, G))[F, G]V_j(F)V_{n+j-m}(G) \tag{11.15}$$

and therefore

$$\varphi_{m,n+j-m}(rP, sQ) = r^m s^{n+j-m} \varphi_{m,n+j-m}(P, Q) \tag{11.16}$$

for $r, s > 0$.

We define a functional J on $\mathscr{K}^n \times \mathscr{K}^n$ by

$$J(K, M) := \int_{\mathbb{R}^n} \varphi_j(K \cap (M + x))\, \lambda_n(dx).$$

Let $K_i \to K, M_i \to M$ be convergent sequences. Since φ_j is continuous, there is a constant $c(K + B^n)$ such that

$$|\varphi_j(K_i \cap (M_i + x))| \le c(K + B^n)\mathbf{1}_{(K+B^n)-(M+B^n)}(x)$$

for all large enough i. For λ_n-almost all x the integrand $\varphi_j(K_i \cap (M_i + x))$ converges to $\varphi_j(K \cap (M + x))$ (namely, for all x for which K and $M + x$ do not touch each other). Hence the dominated convergence theorem implies $J(K_i, M_i) \to J(K, M)$, thus J is continuous.

Choosing polytopes $P_i \to K, Q_i \to M$, we obtain $J(rP_i, sQ_i) \to J(rK, sM)$ for all $r, s > 0$. Since

$$J(rP_i, sQ_i) = \sum_{m=j}^{n} r^m s^{n+j-m} \varphi_{m,n+j-m}(P_i, Q_i),$$

the coefficients $\varphi_{m,n+j-m}(P_i, Q_i)$ of this polynomial have to converge, and we denote the limits by $\varphi_{m,n+j-m}(K, M), m = j, \ldots, n$. Hence,

$$J(rK, sM) = \sum_{m=j}^{n} r^m s^{n+j-m} \varphi_{m,n+j-m}(K, M)$$

and, putting $r = s = 1$, we get (11.11) (for $k = 2$).

It remains to prove the properties of the mixed functionals $\varphi_{m,n+j-m}$. The symmetry and the homogeneity property follow for polytopes from (11.15) and (11.16), and for arbitrary bodies by approximation. The valuation property follows from (11.11) if one takes into account the additivity properties of the integrand on the left hand side and compares these with the different homogeneity properties of the summands on the right hand side. □

Remarks

(1) Theorem 11.3 also holds for measure valuations. More precisely, let $\Phi : \mathcal{K}^n \to \mathcal{M}(\mathbb{R}^n)$ be a continuous, translation covariant and locally determined valuation and let Φ_j be its j-homogeneous part, $j = 0, \ldots, n$ (see (11.5)). Then, for $k \geq 2$, there exist mixed measure-valued functionals $\Phi_{\mathbf{m}}$, $\mathbf{m} \in \text{mix}(j, k)$, on $(\mathcal{K}^n)^k$ such that

$$\begin{aligned}\int_{(\mathbb{R}^n)^{k-1}} & \Phi_j(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k), A_1 \cap (A_2 + x_2) \cap \cdots \cap (A_k + x_k)) \\ & \times \lambda_n^{k-1}(d(x_2, \ldots, x_k)) \\ = & \sum_{\mathbf{m} \in \text{mix}(j,k)} \Phi_{\mathbf{m}}(K_1, \ldots, K_k; A_1 \times \cdots \times A_k) \end{aligned} \tag{11.17}$$

for $K_1, \ldots, K_k \in \mathcal{K}^n$ and $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^n)$.

For $(m_1, \ldots, m_k) \in \text{mix}(j, k)$ the mapping $(K_1, \ldots, K_k) \mapsto \Phi_{m_1,\ldots,m_k}(K_1, \ldots, K_k; A_1 \times \cdots \times A_k)$ is symmetric (w.r.t. permutations of the indices $1, \ldots, k$), it is homogeneous of degree m_i in K_i and A_i, and it is a continuous, translation covariant and locally determined measure valuation in each of its variables K_i.

The proof follows the same lines as in the case of Theorem 11.3 by starting with the case of polytopes. For $k = 2$, we then arrive again at (11.13). In order to extend this expansion to arbitrary bodies $K, M \in \mathcal{K}^n$, we use the fact that (11.13) is equivalent to

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x,x-y)\Phi_j(P\cap(Q+y),dx)\,\lambda_n(dy)$$
$$=\sum_{m=j}^{n}\int_{(\mathbb{R}^n)^2} g(x,y)\Phi_{m,n+j-m}(P,Q;d(x,y)) \tag{11.18}$$

for each continuous function g on $(\mathbb{R}^n)^2$. Again, the dominated convergence theorem shows that the integral on the left, for arbitrary bodies K, M, is a continuous functional $J(g,K,M)$. The homogeneity properties are then used again to show that, for $P_i \to K, Q_i \to M$, each of the integrals

$$\int_{(\mathbb{R}^n)^2} g(x,y)\Phi_{m,n+j-m}(P_i,Q_i;d(x,y))$$

on the right hand side converges. Therefore, the measures $\Phi_{m,n+j-m}(P_i,Q_i;\cdot)$ converge weakly and the limit measures $\Phi_{m,n+j-m}(K,M;\cdot)$ satisfy (11.17) (for $k=2$).

(2) Of course, for a measure valuation Φ_j as above, the total measure $\varphi_j(K) = \Phi_j(K,\mathbb{R}^n)$ satisfies Theorem 11.3 with mixed functionals which are given by the total measures

$$\varphi_{\mathbf{m}}(K_1,\dots,K_k) := \Phi_{\mathbf{m}}(K_1,\dots,K_k;\mathbb{R}^n\times\cdots\times\mathbb{R}^n), \qquad \mathbf{m}\in \operatorname{mix}(j,k).$$

We then say that the measure valuation Φ_j is a *local extension* of the scalar valuation φ_j (more generally $\Phi := \sum_{j=0}^n \Phi_j$ is a local extension of $\varphi := \sum_{j=0}^n \varphi_j$). Thus, if a valuation $\varphi \in$ Val has such a local extension, then the iterated translative formula holds in a global as well as a local version. This fact is of importance if expectation formulas for non-stationary Boolean models are considered. It is an open question, whether each valuation $\varphi \in$ Val has a local extension. Local extensions, if they exist, are not unique (corresponding examples are given in [26, 27]). It is another open problem to describe all local extensions of a valuation φ.

(3) Due to the summation condition $m_1+\cdots+m_k = (k-1)n+j$, only finitely many different mixed functionals arise in the iterated translative formula (11.11). Namely, the mixed functionals $\varphi_{m_1,\dots,m_k}$ (as well as the local versions $\Phi_{m_1,\dots,m_k}$) split if one of the parameters m_i equals n. In fact, if we consider (11.12) for $m=n$, then $F=P$ (we may assume that P is full dimensional) and $G\in\mathscr{F}_j(Q)$. Then $n(P,Q;F,G) = n(Q,G)$ and $[F,G]=1$, hence

$$\Phi_{n,j}(P,Q;\cdot) = \lambda_P\otimes\Big(\sum_{G\in\mathscr{F}_j(Q)} f_j(n(Q,G))\lambda_G\Big)$$
$$=\lambda_P\otimes\Phi_j(Q,\cdot).$$

This extends to arbitrary bodies K, M (if φ_j has a local extension Φ_j) and to the total measures $\varphi_{n,j}(K, M)$. More generally, for $k \geq 2$ and $(m_1, \dots, m_{k-1}, n) \in \text{mix}(j, k)$ we have

$$\varphi_{m_1,\dots,m_{k-1},n}(K_1, \dots, K_{k-1}, K_k) = \varphi_{m_1,\dots,m_{k-1}}(K_1, \dots, K_{k-1}) V_n(K_k)$$

and, if a local extension exists,

$$\Phi_{m_1,\dots,m_{k-1},n}(K_1, \dots, K_{k-1}, K_k; \cdot) = \Phi_{m_1,\dots,m_{k-1}}(K_1, \dots, K_{k-1}; \cdot) \otimes \lambda_{K_k}.$$

Because of the symmetry, the case $m_k = n$ also implies corresponding decompositions if $m_i = n, i \in \{1, \dots, k-1\}$.

11.4.2 Kinematic Formulas

The Boolean model Z is isotropic if and only if the grain distribution $\mathbf{Q}$ is rotation invariant. If this is the case, the translative integrals in (11.8) can be replaced by an integration over the group G_n of rigid motions (with invariant measure μ), hence Theorem 11.2 holds with

$$\Phi(K_0, K_1, \dots, K_k) := \int_{(G_n)^k} \varphi(K_0 \cap g_1 K_1 \cap \dots \cap g_k K_k)\, \mu^k(d(g_1, \dots, g_k))\,. \tag{11.19}$$

Here, Hadwiger's general integral-geometric theorem (see [21, Theorem 5.1.2]) shows that

$$\int_{G_n} \varphi(K \cap gM)\, \mu(dg) = \sum_{k=0}^{n} \varphi_{n-k}(K) V_k(M), \tag{11.20}$$

for $K, M \in \mathcal{K}^n$, where the coefficients $\varphi_{n-k}(K)$ are given by the Crofton-type integrals

$$\varphi_{n-k}(K) := \int_{A(n,k)} \varphi(K \cap E)\, \mu_k(dE) \tag{11.21}$$

over the space $A(n, k)$ of affine k-dimensional flats in $\mathbb{R}^n$ with invariant measure μ_k. (11.20) follows from an application of Hadwiger's characterization theorem (Theorem 1.23) and holds for continuous valuations, even without the assumption of translation invariance. The φ_{n-k} are then also continuous valuations. If $\varphi \in \text{Val}_j$, then $\varphi_{n-k} \in \text{Val}_{n+j-k}$. If $\varphi \in \text{Val}$ has a local extension $\Phi \geq 0$, then a direct proof of (11.20) is possible, based on the translative integral formula (11.13) for polytopes and the representation (11.12) (see [27] for details).

Formula (11.20) can be easily iterated and yields

$$\int_{(G_n)^k} \varphi(K_0 \cap g_1 K_1 \cap \cdots \cap g_k K_k)\, \mu^k(d(g_1, \ldots, g_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(0,k+1)} c^n_{n-m_0} \varphi_{m_0}(K_0) \prod_{i=1}^{k} c^{m_i}_n V_{m_i}(K_i), \tag{11.22}$$

with $\mathbf{m} = (m_0, \ldots, m_k)$ and constants defined by

$$c^r_s := \frac{r!\kappa_r}{s!\kappa_s}, \tag{11.23}$$

see [21, Theorem 5.1.4]. Here κ_i is the volume of the i-dimensional unit ball (see (1.14)).

For $\varphi = V_j$, the integral (11.21) can be solved by the Crofton formula and we get

$$\varphi_{n-k}(K) = c^k_j c^{n+j-k}_n V_{n+j-k}(K)$$

for $k \geq j$ (and $\varphi_{n-k}(K) = 0$ otherwise), which yields the principal kinematic formula

$$\int_{G_n} V_j(K \cap gM)\, \mu(dg) = \sum_{k=j}^{n} c^k_j c^{n+j-k}_n V_k(K) V_{n+j-k}(M) \tag{11.24}$$

and the iterated version

$$\int_{(G_n)^k} V_j(K_0 \cap g_1 K_1 \cap \cdots \cap g_k K_k)\, \mu^k(d(g_1, \ldots, g_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k+1)} c^n_j \prod_{i=0}^{k} c^{m_i}_n V_{m_i}(K_i). \tag{11.25}$$

11.5 Mean Values for Valuations

Combining Theorems 11.2 and 11.3, we obtain the following expectation formula for a stationary Boolean model Z and $\varphi \in \mathrm{Val}_j$,

$$\mathbf{E}\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \gamma^k$$
$$\sum_{\mathbf{m} \in \mathrm{mix}(j,k+1)} \int_{\mathcal{K}^n_c} \cdots \int_{\mathcal{K}^n_c} \varphi_{\mathbf{m}}(K_0, K_1, \ldots, K_k)\, \mathbf{Q}(dK_1) \cdots \mathbf{Q}(dK_k). \tag{11.26}$$

Here, it is important to observe that the right hand side can be simplified due to the decomposition property which we mentioned above. The resulting formulas are presented in [27]. They depend on the shape and size of the window K_0. We can get simpler results if we eliminate the effect of the window by a suitable limit procedure, namely by normalizing with $V_n(K_0)$ (here we assume $V_n(K_0) > 0$) and then letting K_0 grow to $\mathbb{R}^n$ (for simplicity, we consider $rK_0, r \to \infty$). Then, on the right hand side all summands with multi-index $\mathbf{m} = (m_0, \dots, m_k)$ and $m_0 < n$ will vanish asymptotically. If we define for $\mathbf{m} \in \mathrm{mix}(j,k)$ the *density* (mean value) $\overline{\varphi}_{\mathbf{m}}(Y, \dots, Y)$ of the mixed valuation $\varphi_{\mathbf{m}}$ for the (Poisson) particle process Y by

$$\overline{\varphi}_{\mathbf{m}}(Y, \dots, Y) := \gamma^k \int_{\mathcal{K}_c^n} \cdots \int_{\mathcal{K}_c^n} \varphi_{\mathbf{m}}(K_1, \dots, K_k)\, \mathbf{Q}(dK_1) \cdots \mathbf{Q}(dK_k),$$

the right hand side thus reads

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} \overline{\varphi}_{\mathbf{m}}(Y, \dots, Y).$$

This also indicates that the corresponding limit

$$\lim_{r \to \infty} \frac{1}{V_n(rK_0)} \mathbf{E}\varphi(Z \cap rK_0) \tag{11.27}$$

on the left hand side exists. We will discuss this in the following subsection. The second subsection then contains the central result, the explicit expectation formula for valuations and Boolean models. The third subsection shortly discusses the isotropic case.

11.5.1 Densities for Valuations and Random Sets

The following result is Theorem 9.2.1 in [21] in a slightly less general form. It shows that the limit in (11.27) exists for valuations $\varphi \in \mathrm{Val}$ (additively extended to the convex ring $\mathscr{R}^n$) and stationary random sets Z with values in the *extended convex ring*

$$\mathscr{S}^n := \{F \subset \mathbb{R}^n : F \cap rB^n \in \mathscr{R}^n \text{ for all } r > 0\},$$

satisfying the condition

$$\mathbf{E} 2^{N(Z \cap B^d)} < \infty. \tag{11.28}$$

Here, $N(A)$, for a set $A \in \mathscr{R}^n$, is the minimal number k of convex bodies $K_1, \ldots, K_k \in \mathscr{K}^n$ such that $A = \bigcup_{i=1}^k K_i$.

The class $\mathscr{S}^n$ consists of countable unions of convex bodies (locally polyconvex sets) and is supplied with the Borel σ-algebra generated by the Hausdorff metric on $\mathscr{R}^n$ (which is the same σ-algebra as the one generated by the *hit-or-miss* topology, see [21, Sect. 2.4]). For a stationary Boolean model Z with convex or polyconvex grains, we have $Z(\omega) \in \mathscr{S}^n$ and (11.28) is satisfied.

Theorem 11.4 *Let Z be a stationary random set with values in $\mathscr{S}^n$, satisfying* (11.28). *Let $\varphi \in \mathrm{Val}$ and $K \in \mathscr{K}^n$ with $V_n(K) > 0$. Then the limit*

$$\overline{\varphi}(Z) := \lim_{r\to\infty} \frac{1}{V_n(rK)} \mathbf{E}\varphi(Z \cap rK)$$

exists and is independent of K.

For the proof, a functional $\phi \in \mathrm{Val}$ is defined by

$$\phi(K) := \mathbf{E}\varphi(Z \cap K), \qquad K \in \mathscr{K}^n.$$

Then, Theorem 1.12 is used to obtain an additive extension of ϕ to ro-polyhedra (see Sect. 1.2). Since $\mathbb{R}^n$ allows a lattice decomposition into half-open unit cubes $C_0^d, C_1^d, \ldots$, one can show directly that

$$\lim_{r\to\infty} \frac{\phi(rW)}{V_n(rW)} = \phi(C_0^d).$$

This argument shows slightly more, namely that

$$\overline{\varphi}(Z) = \mathbf{E}\varphi(Z \cap C_0^d).$$

We call $\overline{\varphi}(Z)$ the *φ-density* (or *specific φ-value*) of Z. We can estimate the φ-density by the φ-value of Z on the unit cube C^d minus the value $\varphi(Z \cap \partial^+ C^d)$ on the upper right boundary $\partial^+ C^d$ of C^d (observe that $\partial^+ C^d \in \mathscr{R}^n$).

We get, in particular, the existence of the densities $\overline{\varphi}_{m,n+j-m}(Z, K)$ for the mixed functionals $\varphi_{m,n+j-m}$ and $K \in \mathscr{K}^n$.

The following result is a nice application of our translative formula (11.11) and generalizes Theorem 9.4.1 in [21].

Theorem 11.5 *Let Z be a stationary random set with values in $\mathscr{S}^n$, satisfying* (11.28). *Let $\varphi_j \in \mathrm{Val}_j$ and $K \in \mathscr{K}^n$. Then*

$$\mathbf{E}\varphi_j(Z \cap K) = \sum_{m=j}^{n} \overline{\varphi}_{m,n+j-m}(Z, K).$$

Proof We can follow the proof of Theorem 9.4.1 in [21]. We use the stationarity of Z and the translation invariance of φ_j to get

$$\mathbf{E}\int_{\mathbb{R}^n} \varphi_j(Z \cap K \cap (rB^n + x))\lambda_n(dx) = \mathbf{E}\int_{\mathbb{R}^n} \varphi_j(Z \cap (K + x) \cap rB^n)\lambda_n(dx).$$

Now we apply the translative formula to both sides and obtain

$$\sum_{m=j}^{n} \mathbf{E}\varphi_{m,n-m+j}(Z \cap K, rB^n) = \sum_{m=j}^{n} \mathbf{E}\varphi_{m,n-m+j}(Z \cap rB^n, K).$$

We divide both sides by $V_n(rB^n)$ and let $r \to \infty$. On the left hand side the homogeneity properties of the mixed functionals induce that only the summand for $m = j$ remains (and yields $\mathbf{E}\varphi_j(Z \cap K)$). Each summand on the right hand side converges to the corresponding density. □

The summand on the right hand side for $m = j$ is $\overline{\varphi}_j(Z)V_n(K)$. Theorem 11.5 thus gives the error (or bias) if the mean value $\overline{\varphi}_j(Z)$ is estimated by the values of φ_j for realisations $Z(\omega)$ in a window K. As mentioned above, an unbiased estimator of $\overline{\varphi}_j(Z)$ is given by $\varphi_j(Z(\omega) \cap C^d) - \varphi_j(Z(\omega) \cap \partial^+ C^d)$.

11.5.2 The Mean Value Formula for Boolean Models

We now come back to Boolean models Z and continue with the formula

$$\overline{\varphi}_j(Z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{\mathbf{m}\in\mathrm{mix}(j,k)} \overline{\varphi}_{\mathbf{m}}(Y, \dots, Y),$$

for $\varphi_j \in \mathrm{Val}_j$, $j \in \{0, \dots, n\}$, which we have developed. To simplify this formula further, we use again the decomposition property. For $j = n$, we have only one summand

$$\overline{\varphi}_{n,\dots,n}(Y, \dots, Y) = \overline{\varphi}_n(Y)\overline{V}_n(Y)^{n-1} = c_n\overline{V}_n(Y)^n,$$

which gives us

$$\begin{aligned}\overline{\varphi}_n(Z) &= c_n\overline{V}_n(Z)\\ &= c_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\overline{V}_n(Y)^k\\ &= c_n\left(1 - \mathrm{e}^{-\overline{V}_n(Y)}\right).\end{aligned}$$

For $j < n$ and $\mathbf{m} = (m_1, \dots, m_k) \in \operatorname{mix}(j,k)$, let s be the number of indices which are smaller than n. By symmetry we can assume $m_{s+1} = \cdots = m_k = n$. Then

$$\overline{\varphi}_{m_1,\dots,m_s,n,\dots,n}(Y,\dots,Y) = \overline{\varphi}_{m_1,\dots,m_s}(Y,\dots,Y)\overline{V}_n(Y)^{k-s},$$

where $s \in \{1,\dots,n-j\}$ and $m_i \in \{j,\dots,n-1\}$, for $i = 1,\dots,s$. Introducing the notation

$$\operatorname{mix}(j) := \{(m_1,\dots,m_s) \in \{j,\dots,n-1\}^s \cap \operatorname{mix}(j,s) : 1 \le s \le n-j\},$$

we obtain

$$\sum_{\mathbf{m}\in\operatorname{mix}(j,k)} \overline{\varphi}_m(Y,\dots,Y) = \sum_{s=1}^{(n-j)\wedge k} \binom{k}{s} \sum_{\substack{\mathbf{m}\in\operatorname{mix}(j)\\ |\mathbf{m}|=s}} \overline{\varphi}_m(Y,\dots,Y)\overline{V}_n(Y)^{k-s},$$

where $(n-j)\wedge k$ denotes the minimum of $n-j$ and k. This implies

$$\begin{aligned}
\overline{\varphi}_j(Z) &= \sum_{k=1}^{\infty}(-1)^{k-1} \sum_{s=1}^{(n-j)\wedge k} \frac{1}{(k-s)!}\overline{V}_n(Y)^{k-s} \sum_{\substack{\mathbf{m}\in\operatorname{mix}(j)\\ |\mathbf{m}|=s}} \frac{1}{s!}\overline{\varphi}_{\mathbf{m}}(Y,\dots,Y)\\
&= \sum_{s=1}^{n-j}\sum_{r=0}^{\infty} \frac{(-1)^{r+s-1}}{r!}\overline{V}_n(Y)^r \sum_{\substack{\mathbf{m}\in\operatorname{mix}(j)\\ |\mathbf{m}|=s}} \frac{1}{s!}\overline{\varphi}_{\mathbf{m}}(Y,\dots,Y)\\
&= \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m}\in\operatorname{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!}\overline{\varphi}_{\mathbf{m}}(Y,\dots,Y).
\end{aligned}$$

For $j = n-1$, the sum in the formula above reduces to $\overline{\varphi}_{n-1}(Y)$. Hence we have obtained the following result.

Theorem 11.6 *Let Z be a stationary Boolean model with convex grains and let $\varphi_j \in \mathrm{Val}_j$. Then,*

$$\overline{\varphi}_n(Z) = c_n\left(1 - \mathrm{e}^{-\overline{V}_n(Y)}\right),$$
$$\overline{\varphi}_{n-1}(Z) = \mathrm{e}^{-\overline{V}_n(Y)}\overline{\varphi}_{n-1}(Y),$$

and

$$\overline{\varphi}_j(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m}\in\operatorname{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!}\overline{\varphi}_{\mathbf{m}}(Y,\dots,Y), \tag{11.29}$$

for $j = 0,\dots,n-2$.

In (11.29), the multi-index $\mathbf{m} \in \mathrm{mix}(j)$ with $|\mathbf{m}| = 1$ is $\mathbf{m} = (j)$, it yields the summand $\overline{\varphi}_j(Y)$. The remaining summands have multi-indices $\mathbf{m} = (m_1, \dots, m_s)$ with $s > 1$ and $m_i \in \{j+1, \dots, n-1\}$, due to the definition of $\mathrm{mix}(j)$ and the summation rule in $\mathrm{mix}(j, s)$.

11.5.3 The Isotropic Case

If the Boolean model Z is stationary and isotropic, we can obtain a result analogous to Theorem 11.6 by using the iterated kinematic formula (11.22). Equivalently, we can use the rotation invariance of $\mathbf{Q}$ to show that the mean value $\overline{\varphi}_{\mathbf{m}}(Y, \dots, Y)$ in (11.29) satisfies

$$\overline{\varphi}_{\mathbf{m}}(Y, \dots, Y) = c_{\mathbf{m}}(\varphi_j) \prod_{i=1}^{s} \overline{V}_{m_i}(Y),$$

for $\mathbf{m} = (m_1, \dots, m_s)$ with constants $c_{\mathbf{m}}(\varphi_j)$ depending on φ_j.

In the case $\varphi_j = V_j$, we have

$$c_{\mathbf{m}}(V_j) = c_j^n \prod_{i=1}^{s} c_n^{m_i}.$$

Using this in Theorem 11.6, we see that, in the isotropic case, all densities $\overline{V}_j(Z)$ can be expressed by the densities $\overline{V}_j(Y)$ and we obtain the famous Miles formulas

$$\overline{V}_n(Z) = 1 - \mathrm{e}^{-\overline{V}_n(Y)},$$

$$\overline{V}_{n-1}(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \overline{V}_{n-1}(Y),$$

and

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m} \in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} c_j^n \prod_{i=1}^{s} c_n^{m_i} \overline{V}_{m_i}(Y),$$

for $j = 0, \dots, n-2$.

These density formulas can be inverted successively from top to bottom. Since, for convex grains, the mean value $\overline{V}_0(Y)$ equals the intensity γ, we obtain in this way an expression for γ in terms of the densities $\overline{V}_j(Z)$ of the Boolean model Z.

11.6 Special Cases

We now discuss which formulas arise from Theorems 11.3 and 11.6, if special valuations φ are considered. More details on these examples can be found in [27].

11.6.1 Mixed Volumes

As a first case, we consider the *mixed volume* $\varphi(K) = V(K[j], M_{j+1}, \ldots, M_n)$, for fixed bodies $M_{j+1}, \ldots, M_n \in \mathscr{K}^n$. It follows from the properties of the intrinsic volume V_j (which corresponds to the case $M_{j+1} = \cdots = M_n = B^n$) that φ is in Val_j. Formula (11.11) thus gives

$$\int_{(\mathbb{R}^n)^{k-1}} V(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k)[j], M_{j+1}, \ldots, M_n)\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} V_{\mathbf{m}}(K_1, \ldots, K_k; M_{j+1}, \ldots, M_n), \tag{11.30}$$

with mixed functionals $V_{\mathbf{m}}(K_1, \ldots, K_k; M_{j+1}, \ldots, M_n)$. The special case $M_{j+1} = \cdots = M_n = B^n$ yields the iterated translative formula for intrinsic volumes,

$$\int_{(\mathbb{R}^n)^{k-1}} V_j(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k))\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} V_{\mathbf{m}}(K_1, \ldots, K_k). \tag{11.31}$$

(Note, however, that $V_j(K)$ and $V(K[j], B^n, \ldots, B^n)$ differ by a constant. Therefore, $V_{\mathbf{m}}(K_1, \ldots, K_k)$ and $V_{\mathbf{m}}(K_1, \ldots, K_k; B^n, \ldots, B^n)$ also differ by the same constant.)

We also remark that $K \mapsto V(K[j], M_{j+1}, \ldots, M_{n-1}, B^n)$, for strictly convex bodies $M_{j+1}, \ldots, M_{n-1}$, has a local extension given, up to a constant, by the mixed curvature measure $C(K[j], M_{j+1}, \ldots, M_{n-1}; (\cdot) \times M_{j+1} \times \cdots \times M_{n-1})$ introduced and studied in [12] (see also [8] and [9]). This implies a corresponding local integral formula coming from (11.17) which we do not copy here. Instead, we emphasize the special case $M_{j+1} = \cdots = M_{n-1} = B^n$, where we have a multiple of the j-th order curvature measure $C_j(K, \cdot)$, $j = 0, \ldots, n$, and where we obtain the iterated translative formula

$$\int_{(\mathbb{R}^n)^{k-1}} C_j(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k), A_1 \cap (A_2 + x_2)$$
$$\cap \cdots \cap (A_k + x_k))\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} C_{\mathbf{m}}(K_1, \ldots, K_k; A_1 \times \cdots \times A_k), \tag{11.32}$$

with mixed measures, which are different from the mixed curvature measures mentioned above. The different nature of these measures of mixed type can be most easily seen from the homogeneity properties of the total measures. For example, the mixed volume $V(K_1[j_1], \dots, K_k[j_k])$ has total degree of homogeneity $j_1 + \dots + j_k = n$. In contrast to this, the mixed functional $V_{m_1,\dots,m_k}(K_1, \dots, K_k)$, for $\mathbf{m} = (m_1, \dots, m_k) \in \mathrm{mix}(j, k)$, has total degree of homogeneity $m_1 + \dots + m_k = (k-1)n + j$. There is a special case where the two series of functionals meet, for $k = 2$ and $j = 0$ (where we have the translative formula for the Euler characteristic). Here,

$$V_{m,n-m}(K, M) = \binom{n}{m} V(K[m], -M[n-m]),$$

for $m = 0, \dots, n$.

Theorem 11.6 implies mean value formulas for mixed volumes and Boolean models. We only state the result for the intrinsic volumes, which is Theorem 9.1.5 in [21] and reads

$$\overline{V}_n(Z) = 1 - \mathrm{e}^{-\overline{V}_n(Y)},$$

$$\overline{V}_{n-1}(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \overline{V}_{n-1}(Y),$$

and

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m} \in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{V}_{\mathbf{m}}(Y, \dots, Y), \tag{11.33}$$

for $j = 0, \dots, n-2$.

If Z is isotropic, then the density of the mixed functional $V_{m_1,\dots,m_s}$ splits,

$$\overline{V}_{m_1,\dots,m_s}(Y, \dots, Y) = c_j^n \prod_{i=1}^{s} c_n^{m_i} \overline{V}_{m_i}(Y), \tag{11.34}$$

with constants c_i^k defined in (11.23).

This is Theorem 9.1.4 in [21] (with corrected constants).

11.6.2 Support Functions

As a next case, we consider the *(centered) support function* $\varphi(K) = h^*(K, \cdot)$. This is a translation invariant, continuous and additive functional, which is homogeneous of degree 1, with values in the Banach space of centered continuous functions on $\mathbb{S}^{n-1}$. To fit this case into our framework, we may apply the results for scalar valuations point-wise, that is, for $h^*(K,u), u \in \mathbb{S}^{n-1}$. The iterated translative formula then reads

$$\int_{(\mathbb{R}^n)^{k-1}} h^*(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k), \cdot)\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(1,k)} h^*_{\mathbf{m}}(K_1, \ldots, K_k, \cdot), \tag{11.35}$$

with mixed support functions $h^*_{m_1,\ldots,m_k}(K_1, \ldots, K_k, \cdot)$, $(m_1, \ldots, m_k) \in \mathrm{mix}(1,k)$. This integral formula was studied in [25] and [2]. In the latter paper, it was also shown that, for $k = 2$, the mixed function $h^*_{m,n+1-m}(K_1, K_2, \cdot)$ is indeed a support function. For general k this was shown, with a different proof, by Schneider [18].

The formula for Boolean models Z reads

$$\overline{h}^*(Z, \cdot) = \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m} \in \mathrm{mix}(1)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{h}^*_{\mathbf{m}}(Y, \ldots, Y, \cdot).$$

Again, there is a local extension of $K \mapsto h^*(K,u)$ given by the mixed measure $\phi_{1,n-1}(K, u^+; \cdot)$ where u^+ is the closed half-space with outer normal u (see [2, 25]). The corresponding iterated translative formula for this mixed measure is a consequence of Theorem 11.3, but it also follows from the general results in [21, Sect. 6.4].

11.6.3 Area Measures

Next, we consider the *area measure* map $S_j : K \mapsto S_j(K, \cdot)$. It is a translation invariant, additive and measure-valued functional which is continuous with respect to the weak topology of measures. To fit these measure-valued notions into our results, we cannot consider them point-wise, for a given Borel set, since this would not yield a continuous valuation. However, we can apply our results to the integral

$$\varphi_j^f(K) := \int_{\mathbb{S}^{n-1}} f(u) S_j(K, du)$$

with a continuous function f on $\mathbb{S}^{n-1}$. Namely, φ_j^f is an element of Val and fulfills, by Theorem 11.3, the iterated translative formula

$$\int_{(\mathbb{R}^n)^{k-1}} \varphi_j^f\big(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k)\big)\lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m}\in \mathrm{mix}(j,k)} \varphi_{\mathbf{m}}^f(K_1, \ldots, K_k)$$

with unique mixed functionals $\varphi^f_{m_1,\ldots,m_k}$, $(m_1, \ldots, m_k) \in \mathrm{mix}(j,k)$. Let $C(\mathbb{S}^{n-1})$ denote the space of continuous functions on the sphere. For $\mathbf{m} \in \mathrm{mix}(j,k)$ the mapping $f \mapsto \varphi_{\mathbf{m}}^f(K_1, \ldots, K_k)$ is a continuous linear functional on $C(\mathbb{S}^{n-1})$. The Riesz representation theorem therefore implies the existence of a unique finite (signed) measure $S_{\mathbf{m}}(K_1, \ldots, K_k; \cdot)$ on $\mathbb{S}^{n-1}$ with

$$\varphi_{\mathbf{m}}^f(K_1, \ldots, K_k) = \int_{\mathbb{S}^{n-1}} f(u) S_{\mathbf{m}}(K_1, \ldots, K_k; du),$$

which we call the mixed measure of area type. Therefore, we obtain a translative formula for area measures reading

$$\int_{(\mathbb{R}^n)^{k-1}} S_j\big(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k), \cdot\big)\lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m}\in \mathrm{mix}(j,k)} S_{\mathbf{m}}(K_1, \ldots, K_k; \cdot).$$

Since area measures have centroid 0, the same is true for the mixed measures. We emphasize the difference between the mixed measures of area type, which arise in the translative formula, and the mixed area measures $S(K_1, \ldots, K_{n-1}, \cdot)$ which are defined as a coefficient in the multilinear expansion of $S_{n-1}(\alpha_1 K_1 + \cdots + \alpha_{n-1} K_{n-1}, \cdot)$, $\alpha_i \geq 0$. Both types of measures are measures on the unit sphere but they depend on different numbers of bodies and have different homogeneity properties.

The translative formula for area measures was originally obtained in [8].

We remark that the mixed area-type measures for $j = 0$ are trivial. Since $S_0(K, \cdot) = V_0(K)\sigma(\cdot)$, where σ is the spherical Lebesgue measure, we have by the uniqueness of the mixed measures

$$S_{\mathbf{m}}(K_1, \ldots, K_k; \cdot) = V_{\mathbf{m}}(K_1, \ldots, K_k)\, \sigma(\cdot), \qquad \mathbf{m} \in \mathrm{mix}(k, 0), k \geq 2.$$

The translative formulas imply formulas for Boolean models (see [6, Corollary 4.1.4]) of the form

$$\overline{S}_j(Z,\,\cdot) = \mathrm{e}^{-\overline{V}_n(X)} \sum_{\mathbf{m}\in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{S}_{\mathbf{m}}(Y,\ldots,Y;\,\cdot). \tag{11.36}$$

This result follows now also from Theorem 11.6 using the functional analytic approach described above.

If Z is an isotropic Boolean model, the measure $\overline{S}_j(Z,\,\cdot)$ is rotation invariant. Since the spherical Lebesgue measure σ is up to normalization the unique measure on $\mathbb{S}^{n-1}$ with this property, we have

$$\overline{S}_j(Z,\,\cdot) = \frac{n\kappa_{n-j}}{\omega_n\binom{n}{j}}\,\overline{V}_j(Z)\,\sigma$$

and the formula (11.36) is equivalent to the corresponding result for the specific intrinsic volume $\overline{V}_j(Z)$ (this also implies a rotation formula for mixed area-type measures).

The local extension of the valuation $K \mapsto S_j(K;C), C \subset \mathbb{S}^{n-1}$, is (up to a constant) given by $K \mapsto \Lambda_j(K;\,\cdot\,\times C)$, where $\Lambda_j(K,\dot{)}$ is the support measure of K introduced in Sect. 11.2.2. For the support measures a local translative integral formula similar to (11.32) holds which was originally shown in [8].

11.6.4 Flag Measures

Now, we use the functional analytic approach just described in a similar situation for *flag measures* of convex bodies. A general reference for flag measures is the overview article [10]. The flag measures we consider in the following are a version of the translation invariant flag area measures which are considered in [3] and related to the flag area measures in [10] via [3, (2.1)] and a renormalization. We first describe the underlying notions concerning flag manifolds. Recall that $G(n,j)$ denotes the Grassmannian of j-dimensional subspaces (which we supply with the invariant probability measure ν_j) and define corresponding flag manifolds by

$$F(n,j) := \{(u,L) : L \in G(n,j), u \in L \cap \mathbb{S}^{n-1}\}$$

and

$$F^{\perp}(n,j) := \{(u,L) : L \in G(n,j), u \in L^{\perp} \cap \mathbb{S}^{n-1}\}.$$

Both flag manifolds carry natural topologies (and invariant Borel probability measures) and $F(n,n-j)$ and $F^{\perp}(n,j)$ are homeomorphic via the orthogonality map $\rho : (u, L) \mapsto (u, L^\perp)$. We define a flag measure $\psi_j(K, \cdot)$ as a projection mean of area measures,

$$\psi_j(K, A) := \int_{G(n,j+1)} \int_{\mathbb{S}^{n-1} \cap L} \mathbf{1}\{(u, L^\perp \vee u) \in A\} S'_j(K|L, du) \nu_{j+1}(dL) \tag{11.37}$$

for a Borel set $A \subset F(n, n-j)$, where $L^\perp \vee u$ is the subspace generated by $L^\perp$ and the unit vector u and where the prime indicates the area measure calculated in the subspace L (for the necessary measurability properties needed here and in the following, we refer to [5]). Using the homeomorphism ρ, we can replace $\psi_j(K, \cdot)$ by a measure $\psi_j^\perp(K, \cdot)$ on $F^\perp(n, j)$ given by

$$\psi_j^\perp(K, A) := \int_{G(n,j+1)} \int_{\mathbb{S}^{n-1} \cap L} \mathbf{1}\{(u, L \cap u^\perp) \in A\} S'_j(K|L, du) \nu_{j+1}(dL). \tag{11.38}$$

These two (equivalent) versions of the same flag measure are motivated by the fact that their images under the map $(u, L) \mapsto u$ are in both cases the j-th order area measure $S_j(K, \cdot)$. Both measures, $\psi_j(K, \cdot)$ and $\psi_j^\perp(K, \cdot)$ have a local version $\lambda_j(K, \cdot)$, respectively $\lambda_j^\perp(K, \cdot)$, which is obtained by replacing in (11.37) and (11.38) the area measure $S'_j(K|L, \cdot)$ by a multiple of the support measure $\Lambda'_j(K|L, \cdot)$ (see [10, Theorem 4]). In the following, we concentrate on $\psi_j(K, \cdot)$, formulas for the other representation $\psi_j^\perp(K, \cdot)$ follow in a similar way.

The measure $\psi_j(K, \cdot)$ is centered in the first component,

$$\int_{F(n,n-j)} u \psi_j(K, d(u, L)) = 0,$$

as follows from the corresponding property of area measures. Let $C(F(n, n-j))$ be the Banach space of continuous functions on $F(n, n-j)$ and choose $f \in C(F(n, n-j))$. Then,

$$\varphi_j^f : K \mapsto \int_{F(n,n-j)} f(u, L) \psi_j(K, d(u, L))$$

is in Val_j. Consequently, we obtain the iterated translative formula

$$\int_{(\mathbb{R}^n)^{k-1}} \varphi_j^f(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k))\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} \varphi_{\mathbf{m}}^f(K_1, \ldots, K_k), \tag{11.39}$$

with mixed functionals $\varphi_{\mathbf{m}}^f(K_1, \ldots, K_k)$, $\mathbf{m} \in \mathrm{mix}(j, k)$. For fixed bodies $K_1, \ldots, K_k$, the left hand side is a continuous linear functional on $C(F(n, n-j))$ if we let f vary.

Namely,

$$f \mapsto \varphi_j^f(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k))$$

is continuous and linear, for each $x_1, \ldots, x_k$, and this carries over to the integral. Replacing $K_1, \ldots, K_k$ by $\alpha_1 K_1, \ldots, \alpha_k K_k, \alpha_i > 0$, we use the homogeneity properties of $\varphi^f_{m_1,\ldots,m_k}$ to see that the right side is a polynomial in $\alpha_1, \ldots, \alpha_k$. This shows that the coefficients $\varphi^f_{m_1,\ldots,m_k}(K_1, \ldots, K_k)$ of this polynomial must be continuous linear functionals on $C(F(n, n-j))$, too. By the Riesz representation theorem we obtain unique finite (signed) measures $\psi_{m_1,\ldots,m_k}(K_1, \ldots, K_k; \cdot)$ on $F(n, n-j)$ such that

$$\varphi^f_{m_1,\ldots,m_k}(K_1, \ldots, K_k) = \int_{F(n,n-j))} f(u, L)\psi_{m_1,\ldots,m_k}(K_1, \ldots, K_k; d(u, L))$$

for all $f \in C(F(n, n-j))$. We call them the *mixed flag measures*. They are again centered in the first component.

Hence we obtain the iterated translative formula for flag measures,

$$\int_{(\mathbb{R}^n)^{k-1}} \psi_j(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k), \cdot)\, \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \text{mix}(j,k)} \psi_{\mathbf{m}}(K_1, \ldots, K_k; \cdot). \qquad (11.40)$$

A mean value formula for flag measures of Boolean models Z follows from Theorem 11.6. Since it looks very similar to (11.36) (but is in fact a generalization), we do not copy it here.

11.7 Tensor Valuations and Boolean Models

Finally, we consider the *Minkowski tensors* $K \mapsto \Phi_j^{r,s}(K)$ which are the central objects of various chapters of this volume. They are defined as integrals with respect to the support measures. Therefore, Sect. 11.6.3 implies the iterated translative formula

$$\int_{(\mathbb{R}^n)^{k-1}} \Phi_j^{r,s}(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k))\lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \text{mix}(j,k)} \Phi_{\mathbf{m}}^{r,s}(K_1, \ldots, K_k), \qquad (11.41)$$

with mixed tensor valuations $(K_1, \dots, K_k) \mapsto \Phi_{\mathbf{m}}^{r,s}(K_1, \dots, K_k)$, $\mathbf{m} \in \operatorname{mix}(j,k)$. The mixed tensor valuations $\Phi_{m_1,\dots,m_k}^{r,s}$ are homogeneous of order $m_1 + r$ with respect to the first argument K_1 and homogeneous of order m_i with respect to K_i for $i \geq 2$. For $r = 0$ the Minkowski tensors are translation invariant. In this case their coordinates are elements of Val and as an alternative to the above approach via support measures, Theorem 11.3 can be applied directly.

It is convenient to define *local Minkowski tensors* as the tensor-valued signed measures on $\mathbb{R}^n$ given by

$$\Phi_j^{r,s}(K,A) := c_{n-j}^{r,s} \int_{A \times \mathbb{S}^{n-1}} x^r u^s \Lambda_j(K, d(x,u))$$

for Borel sets $A \subset \mathbb{R}^n$. In generalization of (11.41), they fulfill the translative formula

$$\begin{aligned} \int_{(\mathbb{R}^n)^{k-1}} & \Phi_j^{r,s}\big(K_1 \cap (K_2 + x_2) \cap \dots \cap (K_k + x_k), A_1 \cap (A_2 + x_2) \\ & \cap \dots \cap (A_k + x_k)\big)\, \lambda_n^{k-1}(d(x_2, \dots, x_k)) \\ & = \sum_{\mathbf{m} \in \operatorname{mix}(j,k)} \Phi_{\mathbf{m}}^{r,s}(K_1, \dots, K_k; A_1 \times \dots \times A_k) \end{aligned} \tag{11.42}$$

with mixed local Minkowski tensors $\Phi_{\mathbf{m}}^{r,s}(K; \cdot)$, $\mathbf{m} \in \operatorname{mix}(j,k)$. For $r = 0$, the mapping $K \mapsto \Phi_j^{0,s}(K, \cdot)$ is a local extension of the Minkowski tensor $K \mapsto \Phi_j^{0,s}(K)$.

For the translation invariant Minkowski tensors we then obtain density formulas for Boolean models reading

$$\overline{\Phi}_j^{0,s}(Z) = \mathrm{e}^{-\overline{V}_n(Y)} \sum_{\mathbf{m} \in \operatorname{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{\Phi}_{\mathbf{m}}^{0,s}(Y, \dots, Y)$$

for $j = 0, \dots, n-1$. The case $j = 0$ is special here, since

$$\Phi_0^{0,s}(K) = \frac{2}{s!\,\omega_{s+1}} \mathbf{1}\{s \in 2\mathbb{N}_0\} Q^{\frac{s}{2}} V_0(K)$$

and

$$\Phi_{\mathbf{m}}^{0,s}(K_1, \dots, K_k) = \frac{2}{s!\,\omega_{s+1}} \mathbf{1}\{s \in 2\mathbb{N}_0\} Q^{\frac{s}{2}} V_{\mathbf{m}}(K_1, \dots, K_k)$$

for $\mathbf{m} = (m_1, \ldots, m_k) \in \text{mix}(0)$. Thus we get

$$\overline{\Phi}_0^{0,s}(Z) = \frac{2}{s!\omega_{s+1}} \mathbf{1}\{s \in 2\mathbb{N}_0\} Q^{\frac{s}{2}} \overline{V}_0(Z),$$

where $\overline{V}_0(Z)$ can be expressed by mixed densities of Y using (11.33).

If Z is an isotropic Boolean model, we have more generally

$$\overline{\Phi}_j^{0,s}(Z) = \alpha_{n,j,s} \mathbf{1}\{s \in 2\mathbb{N}_0\} Q^{\frac{s}{2}} \overline{V}_j(Z)$$

for $j = 0, \ldots, n-1$, where

$$\alpha_{n,j,s} := \frac{2}{s!} \frac{\omega_{n-j}\, \omega_{s+n}}{\omega_n\, \omega_{n-j+s}\, \omega_{s+1}}.$$

The translative formula (11.41) and the density formulas for Minkowski tensors are contained in [7]. In [7] also mean value formulas for general Minkowski tensors $\Phi_j^{r,s}$ are obtained expressing

$$\mathbf{E}\Phi_j^{r,s}(Z \cap K_0)$$

for $K_0 \in \mathcal{K}^n$ by mixed expressions of K_0 and Y. They result by combining Theorem 11.2 and (11.41) (here it is important that Theorem 11.2 does not require translation invariance of φ). For $r > 0$ the tensors $\Phi_j^{r,s}$ need not be translation invariant (but see [7, p. 35 or p. 62], for exceptions) and densities of the Boolean model Z in the sense of Theorem 11.4 are not defined. Therefore, in this case the mean value results do not lead to density formulas.

The usefulness of the Minkowski tensors for the study of special non-isotropic Boolean models is illustrated in [7] and [22] by parametric examples.

11.8 Concluding Remarks and Outlook

As we have mentioned before, the methods and results for the use of valuations with stationary Boolean models Z can be extended to non-stationary Z under mild regularity assumptions. We describe this situation in the following, but leave out many details for which we refer to the literature.

We recall from Proposition 11.1 the definition of the measure Θ and the Poisson property of Y, which shows that Θ is a translation invariant measure on $\mathcal{K}^n$, which satisfies

$$\mathbf{E}Y = \Theta.$$

Since therefore $\Theta(A)$, for a Borel set $A \subset \mathcal{K}^n$, describes the mean number of particles from Y which fall into A, Θ is called the *intensity measure* of Y. For the discussion of the non-stationary case, we allow Poisson particle processes Y on $\mathcal{K}^n$, where the intensity measure $\Theta := \mathbf{E}Y$ on $\mathcal{K}^n$ is no longer translation invariant, but is absolutely continuous with respect to a translation invariant measure. We call such a measure *translation regular*. It then follows that

$$\Theta(A) = \int_{\mathcal{K}_c^n} \int_{\mathbb{R}^n} \mathbf{1}_A(K+x)\eta(K,x)\lambda_n(dx)\mathbf{Q}(dK)$$

for some probability measure $\mathbf{Q}$ on $\mathcal{K}_c^n$ and a measurable function $\eta \geq 0$ on $\mathcal{K}_c^n \times \mathbb{R}^n$ (see [21, (11.1)]). In general, $\mathbf{Q}$ and η are not uniquely determined by Θ, but they are if η does not depend on K, hence can be considered as a function on $\mathbb{R}^n$ alone. We will assume this throughout the following and refer to [21, Sect. 11.1] and [27] for the more general situation. Then, η is called the *intensity function* and $\mathbf{Q}$ the *distribution of the typical grain* of the Poisson particle process Y. The interpretation is similar to the stationary case. Points in space are distributed according to the intensity function η (by a Poisson process X_0 in $\mathbb{R}^n$ with intensity measure $\int \eta d\lambda_n$). Then convex bodies are attached to the points independently and with distribution $\mathbf{Q}$.

Let now $\varphi \in \mathrm{Val}$ have a local extension Φ. Then $\Phi(Z,\cdot)$ is a signed Radon measure (defined on bounded Borel sets of $\mathbb{R}^n$) which is absolutely continuous to the Lebesgue measure λ_n. We denote its (almost everywhere existing) density by $\overline{\varphi}(Z,\cdot)$ (this is a measurable function on $\mathbb{R}^n$). Then we have, as a generalization of Theorem 11.2,

$$\overline{\varphi}(Z,z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\mathcal{K}_c^n} \cdots \int_{\mathcal{K}_c^n} \int_{(\mathbb{R}^n)^k} \eta(z-x_1)\cdots\eta(z-x_k)$$
$$\times \Phi_{(k)}(K_1,\dots,K_k; d(x_1,\dots,x_k))\, \mathbf{Q}(dK_1)\cdots\mathbf{Q}(dK_k)$$

where the measure $\Phi_{(k)}(K_1,\dots,K_k;\cdot)$ is given by

$$\begin{aligned}\Phi_{(k)}&(K_1,\dots,K_k; A_1\times\cdots\times A_k)\\ &:= \int_{(\mathbb{R}^n)^{k-1}} \Phi\big(K_1\cap(K_2+x_2)\cap\cdots\cap(K_k+x_k),\\ &\qquad\qquad A_1\cap(A_2+x_2)\cap\cdots\cap(A_k+x_k)\big)\\ &\quad\times \lambda_n^{k-1}(d(x_2,\dots,x_k)),\end{aligned} \tag{11.43}$$

for Borel sets $A_1,\dots,A_k \subset \mathbb{R}^n$. It is remarkable that, in this non-stationary situation, still an iterated translative integral shows up. Using (11.17), we can now proceed as in Sect. 11.5.2 and obtain

$$\overline{\varphi}_n(Z,z) = c_n\big(1 - e^{-\overline{V}_n(Y,z)}\big),$$
$$\overline{\varphi}_{n-1}(Z,z) = e^{-\overline{V}_n(Y,z)}\overline{\varphi}_{n-1}(Y,z),$$

and

$$\overline{\varphi}_j(Z,z) = e^{-\overline{V}_n(Y,z)} \sum_{\mathbf{m}\in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{\varphi}_{\mathbf{m}}(Y,\dots,Y,z,\dots,z), \tag{11.44}$$

for $j = 0,\dots,n-2$ and λ_n-almost all $z \in \mathbb{R}^n$. Here, the mean values for Y are defined by

$$\overline{V}_n(Y,z) := \int_{\mathcal{K}_c^n}\int_{\mathbb{R}^n} \eta(z-x)\lambda_n(dx)\mathbf{Q}(dK)$$

and

$$\overline{\varphi}_{\mathbf{m}}(Y,\dots,Y,z,\dots,z) := \int_{\mathcal{K}_c^n}\cdots\int_{\mathcal{K}_c^n}\int_{(\mathbb{R}^n)^k} \eta(z-x_1)\cdots\eta(z-x_k)$$
$$\times\, \Phi_{\mathbf{m}}(K_1,\dots,K_k; d((x_1,\dots,x_k))\mathbf{Q}(dK_1)\cdots\mathbf{Q}(dK_k),$$

see [27, Theorem 6.2].

Specializing to the examples discussed in Sect. 11.6, we obtain from (11.44) formulas for various geometric mean values for general Boolean models. In particular, for the translation invariant local Minkowski tensors we obtain the formulas

$$\overline{\Phi}_{n-1}^{0,s}(Z,z) = e^{-\overline{V}_n(Y,z)}\overline{\Phi}_{n-1}^{0,s}(Y,z),$$

and

$$\overline{\Phi}_j^{0,s}(Z,z) = e^{-\overline{V}_n(Y,z)} \sum_{\mathbf{m}\in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{\Phi}_{\mathbf{m}}^{0,s}(Y,\dots,Y,z,\dots,z), \tag{11.45}$$

for $j = 0,\dots,n-2$, $s \in \mathbb{N}_0$ and λ_n-almost all $z \in \mathbb{R}^n$.

We conclude this article with an outlook on the recent development of applying harmonic intrinsic volumes in the study of stationary non-isotropic Boolean models. Harmonic intrinsic volumes are integrals of spherical polynomials with respect to the area measures $S_j(K;\cdot)$, $j \in \{0,\dots,n-1\}$. Let $\mathbf{S}_l$ denote the space of spherical harmonics (i.e. homogeneous spherical polynomials p with $\Delta p = 0$) of degree l and let $D(n,l)$ be the dimension of $\mathbf{S}_l$. Let $Y_{l,1},\dots,Y_{l,D(n,l)}$ be an orthonormal basis of $\mathbf{S}_l$ with respect to the L^2-scalar product with the measure $\omega_n^{-1}\sigma$. Then, *harmonic intrinsic volumes* are defined by

$$V_j^{l,p}(K) := c_{n,j} \int\limits_{S^{n-1}} Y_{l,p}(u) S_j(K, du),$$

where

$$c_{n,j} := \binom{n}{j} \frac{1}{n\kappa_{n-j}}.$$

The harmonic intrinsic volumes $V_j^{l,p}$ are elements of Val. Furthermore it holds

$$V_j^{0,1} = V_j,$$

i.e. the usual intrinsic volumes are contained in the collection of harmonic intrinsic volumes. They fulfill an interesting rotation formula

$$\int\limits_{\mathrm{SO}_d} V_j^{l,p}(\vartheta K)\nu(d\vartheta) = \begin{cases} V_j(K), & (l,p) = (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Since the harmonic intrinsic volumes are integrals with respect to the area measures, Sect. 11.6.3 implies the iterated translative formula,

$$\int\limits_{(\mathbb{R}^n)^{k-1}} V_j^{l,p}(K_1 \cap (K_2 + x_2) \cap \cdots \cap (K_k + x_k)) \lambda_n^{k-1}(d(x_2, \ldots, x_k))$$
$$= \sum_{\mathbf{m} \in \mathrm{mix}(j,k)} V_{\mathbf{m}}^{l,p}(K_1, \ldots, K_k).$$

Consequently, also density formulas for Boolean models are obtained reading

$$\overline{V}_j^{l,p}(Z) = \mathrm{e}^{-\overline{V}_n(X)} \sum_{\mathbf{m} \in \mathrm{mix}(j)} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}|!} \overline{V}_{\mathbf{m}}^{l,p}(Y, \ldots, Y).$$

If Z is an isotropic Boolean model, we have

$$\overline{V}_j^{l,p}(Z) = \begin{cases} \overline{V}_j(Z), & (l,p) = (0,1), \\ 0, & \text{otherwise,} \end{cases}$$

a property which already indicates that the harmonic intrinsic volumes are particularly useful for non-isotropic Boolean models. This turns out to be true if we consider a Boolean model where the grain distribution $\mathbf{Q}$ is *rotation regular*, i.e. satisfies

$$\mathbf{Q}(A) = \int_{\mathcal{K}_c^n} \int_{\mathrm{SO}_n} \mathbf{1}_A(\vartheta K)\eta(K,\vartheta)\nu(d\vartheta)\tilde{\mathbf{Q}}(dK)$$

for some rotation invariant probability measure $\tilde{\mathbf{Q}}$ on $\mathcal{K}_c^n$ and a measurable function $\eta \geq 0$ on $\mathcal{K}_c^n \times \mathrm{SO}_n$. The measure $\tilde{\mathbf{Q}}$ is unique and the function η is unique $\tilde{\mathbf{Q}} \otimes \nu$-everywhere under the additional assumptions

$$\int\limits_{\mathrm{SO}_n} \eta(K,\vartheta)\nu(d\vartheta) = 1 \quad \text{and} \quad \eta(\sigma K,\vartheta) = \eta(K,\vartheta\sigma),$$

for $K \in \mathcal{K}_c^n, \vartheta, \sigma \in \mathrm{SO}_n$. It was recently shown in [6] that in two and three dimensions, for a stationary Boolean model with rotation regular grain distribution, the intensity can be expressed as a series of products of the densities $\overline{V}_j^{l,p}(Z)$ of the harmonic intrinsic volumes. For the proofs and the definition of the constants in the following two theorems we refer to [6].

Theorem 11.7 *In two dimensions, the intensity γ has the series representation*

$$\gamma = \rho\overline{V}_0(Z) + \rho^2 \sum_{l,m=0}^{\infty} \sum_{p=1}^{D(2,l)} \sum_{q=1}^{D(2,m)} c_{l,m}^{p,q}\, \overline{V}_1^{l,p}(Z)\, \overline{V}_1^{m,q}(Z)$$

with some constants $c_{l,m}^{p,q} \in \mathbb{R}$ and

$$\rho := \frac{1}{1-\overline{V}_2(Z)}.$$

Theorem 11.8 *In three dimensions, the intensity γ has the series representation*

$$\gamma = \rho\overline{V}_0(Z) + \rho^2 \sum_{l,m=0}^{\infty} \sum_{p=1}^{D(3,l)} \sum_{q=1}^{D(3,m)} d_{l,m}^{p,q}\overline{V}_1^{l,p}(Z)\overline{V}_2^{m,q}(Z)$$
$$+ \rho^3 \sum_{l,m,o=0}^{\infty} \sum_{p=1}^{D(3,l)} \sum_{q=1}^{D(3,m)} \sum_{s=1}^{D(3,o)} e_{l,m,o}^{p,q,s}\, \overline{V}_2^{l,p}(Z)\overline{V}_2^{m,q}(Z)\overline{V}_2^{o,s}(Z)$$

with some constants $d_{l,m}^{p,q}, e_{l,m,o}^{p,q,s} \in \mathbb{R}$ and

$$\rho := \frac{1}{1-\overline{V}_3(Z)}.$$

These representations of the intensity can be seen as a generalization of the results by Miles and Davy for isotropic Boolean models from 1976 which were mentioned in the introduction and described in Sect. 11.5.3. The article [23] will also contain applications of the series representation in Theorem 11.7 to specific examples of Boolean models.

Harmonic intrinsic volumes are real-valued functionals but they are closely related to tensor-valued functionals. For the corresponding Minkowski tensors, see [11, 17] and [13].

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Chapter 12
Second Order Analysis of Geometric Functionals of Boolean Models

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Abstract This chapter presents asymptotic covariance formulae and central limit theorems for geometric functionals, including volume, surface area, and all Minkowski functionals and translation invariant Minkowski tensors as prominent examples, of stationary Boolean models. Special focus is put on the anisotropic case. In the (anisotropic) example of aligned rectangles, we provide explicit analytic formulae and compare them with simulation results. We discuss which information about the grain distribution second moments add to the mean values.

12.1 Introduction

In this chapter we study a large class of functionals of the Boolean model, a fundamental benchmark model of stochastic geometry [6, 33, 37] and continuum percolation [9, 30]. It has many applications in materials science [42], physics [1, 38], and astronomy [17, 26], as well as, for the measurement of biometrical data [28] or the estimation of percolation thresholds [27, 29]. Intuitively speaking, a Boolean model is a collection of overlapping random grains, scattered in space in a purely random manner. This random object is defined as follows. Let $X = \{X_1, X_2, \ldots\}$ be a stationary Poisson process of intensity γ in $\mathbb{R}^n$, that is, a countable collection of random points in $\mathbb{R}^n$ such that the numbers of points in disjoint sets are independent and the number of points in each set follows a Poisson distribution whose parameter is γ times the Lebesgue measure of the set. Let $(Z_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random convex bodies (nonempty compact convex subsets of $\mathbb{R}^n$), independent of X. The Boolean model Z is the random closed set [37]

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defined by

$$Z := \bigcup_{i \in \mathbb{N}} (Z_i + X_i),$$

where $Z_i + X_i := \{z + X_i : z \in Z_i\}$; see also Chap. 11. An example is the spherical Boolean model, where the Z_i are balls with random radii centred at the origin and $Z_i + X_i$ is the corresponding ball centred at X_i.

In this chapter we study geometric functionals of the Boolean model Z. Prominent examples of such functionals are the intrinsic volumes (Minkowski functionals) and Minkowski tensors, which are efficient shape descriptors that have been successfully applied to a variety of physical systems [40]. In [41], the different approaches and notations in the physics and mathematics literature are compared. We are interested in the second order properties of the random variables obtained by applying geometric functionals to the restriction $Z \cap W$ of the Boolean model to a convex observation window $W \subset \mathbb{R}^n$. For a stationary and isotropic Boolean model, Miles [31] and Davy [7] obtained explicit formulae expressing the mean values of the Minkowski functionals in terms of the intensity and geometric mean values of the typical grain (see also [6, 37]). For mean value formulae for more general functionals of Boolean models we refer to Chap. 11. We shall discuss here formulae for asymptotic covariances as well as multivariate central limit theorems for an increasing observation window. Much of the presented theory is taken from [16]. However, some results are new. In particular this is the first publication providing explicit covariance formulae involving the Euler characteristic of planar non-isotropic Boolean models. Our methods are based on the Fock space representation of Poisson functionals from [20] and the Stein-Malliavin approach to their normal approximation [21, 22, 34]. A completely different treatment of second moments of curvature measures of an isotropic Boolean model with an interesting application to morphological thermodynamics was presented in [25]. There, two different scenarios are considered: first, a Poisson distributed number of grain centres in the observation window (Poisson process), and second, a fixed number of grains (binomial process). In statistical physics, these two choices are called the grand canonical and the canonical ensemble. The second moments of geometric quantities show a similar behaviour as thermodynamical quantities in statistical physics [24, 25]. For the isotropic examples of overlapping discs or spheres, the covariances of the Minkowski functionals are also discussed in [4] or [17], respectively.

This chapter is organized in the following way. After introducing Boolean models and geometric functionals in Sect. 12.2, Sect. 12.3 is devoted to the covariance structure of geometric functionals of Boolean models. First, we present general covariance formulae. Then, we concentrate on planar Boolean models. Univariate and multivariate central limit theorems for geometric functionals of Boolean models are discussed in Sect. 12.4. In Sect. 12.5, we explicitly compute the covariance formulae for a special Boolean model of aligned rectangles. In the final Sect. 12.6, we present and discuss simulation results for Boolean models with rectangles and compare them with our theoretical findings. The agreement is excellent.

Let us finish this introduction with an informal summary of our results for applied scientists. We calculate for certain models of disordered systems of overlapping grains the second moments of a quite general class of robust shape descriptors, which include as well-known examples volume, surface area, Euler characteristic, and, more generally, all Minkowski functionals and tensors. Our results apply to general anisotropic grain distributions, see Theorems 12.4 and 12.5. The anisotropic case of aligned (planar) rectangles is discussed in great detail; see Sect. 12.5 and Fig. 12.2. It is interesting to note that the asymptotic formulae for the infinite volume system are actually exact for finite systems with periodic boundary conditions; see Sect. 12.3.3. The central limit theorem for the geometric functionals (see Theorems 12.8 and 12.10) ascertains that in the limit of infinite system size the probability distributions of the normalized geometric functionals are normal distributions. If the structure of a given sample is reasonably well described by the (joint) cumulative probability distributions of the geometric functionals, it is possible to construct tests of certain model hypotheses for random heterogeneous media based on the asymptotic normality and our explicit covariance formulae. We discuss the behaviour of the second moments (e.g., how they differ for various models) and probability distributions in finite systems for specific examples, such as isotropically oriented rectangles or rectangles that are aligned with the coordinate axes (but still distributed randomly in space). In the latter case, the formulae for the asymptotic covariances take a very explicit form (see Fig. 12.2). Moreover, these formulae allow for an exact analysis of the dependence of asymptotic covariances on the grains and in particular of their scaling behaviour. Furthermore, they serve as a benchmark for our general formulae. Indeed, the analytic results for aligned rectangles are in excellent agreement with Monte Carlo simulations, see Figs. 12.3 and 12.5.

12.2 Preliminaries

In the introduction, we have defined a Boolean model, as in Sect. 11.3.1, in terms of a stationary Poisson process in $\mathbb{R}^n$ which is independently marked with random convex bodies. In this chapter we use an equivalent description based on a Poisson process in the space $\mathscr{K}^n$ of convex bodies in $\mathbb{R}^n$, $n \geq 1$. For our purposes this representation is more convenient.

We equip $\mathscr{K}^n$ with its Borel σ-field $\mathscr{B}(\mathscr{K}^n)$ with respect to the Hausdorff metric. We call a measure Θ on $\mathscr{K}^n$ *locally finite* if

$$\Theta(\{K \in \mathscr{K}^n : K \cap C \neq \emptyset\}) < \infty, \qquad C \in \mathscr{C}^n,$$

where $\mathscr{C}^n$ is the space of compact subsets of $\mathbb{R}^n$. Let $\mathbf{N}$ be the space of all locally finite counting measures on $\mathscr{K}^n$ and let it be equipped with the smallest σ-field $\mathscr{N}$

such that all maps $\nu \mapsto \nu(A)$, $A \in \mathscr{B}(\mathscr{K}^n)$, from $\mathbf{N}$ to $\mathbb{N} \cup \{0, \infty\}$ are measurable. Each element $\nu \in \mathbf{N}$ has a representation

$$\nu = \sum_{i=1}^{N} \delta_{K_i}, \qquad K_1, K_2, \ldots \in \mathscr{K}^n,\ N \in \mathbb{N} \cup \{0, \infty\},$$

where δ_K stands for the Dirac measure concentrated at $K \in \mathscr{K}^n$. Because of this representation one can think of ν as a countable collection of convex bodies (or grains).

Throughout this chapter all random objects are defined on a fixed (sufficiently rich) probability space $(\Omega, \mathscr{A}, \mathbf{P})$. We call a random element η in $\mathbf{N}$ a Poisson process with a locally finite intensity measure Θ if

(i) $\eta(A_1), \ldots, \eta(A_m)$ are independent for disjoint sets $A_1, \ldots, A_m \in \mathscr{B}(\mathscr{K}^n)$,
(ii) $\eta(A)$ follows a Poisson distribution with parameter $\Theta(A)$ for $A \in \mathscr{B}(\mathscr{K}^n)$, i.e.

$$\mathbf{P}(\eta(A) = k) = \frac{\Theta(A)^k}{k!} e^{-\Theta(A)}, \qquad k \in \mathbb{N} \cup \{0\}.$$

The second property explains the name. Since $\Theta(A) = \mathbf{E}\eta(A)$ for any $A \in \mathscr{B}(\mathscr{K}^n)$, Θ is called intensity measure of η. The Poisson process η is called stationary if it is invariant under the shifts $K \mapsto K + x := \{y + x : y \in K\}$ for all $x \in \mathbb{R}^n$. This means that the distribution of η does not change under simultaneous translations of its grains. The stationarity of the Poisson process η is equivalent to the translation invariance of the intensity measure Θ.

In the following we always assume that η is a stationary Poisson process in $\mathscr{K}^n$ with a locally finite intensity measure Θ such that $\Theta(\mathscr{K}^n) > 0$. It follows from [37, Theorem 4.1.1] that the intensity measure Θ has the representation

$$\Theta(\,\cdot\,) = \gamma \iint \mathbf{1}\{K + x \in \,\cdot\,\}\, dx\, \mathbf{Q}(dK),$$

where $\gamma \in (0, \infty)$ is an intensity parameter and $\mathbf{Q}$ is a probability measure on $\mathscr{K}^n$ such that

$$\int V_n(K + C)\, \mathbf{Q}(dK) < \infty, \qquad C \in \mathscr{C}^n. \tag{12.1}$$

Without loss of generality we can assume in the following that $\mathbf{Q}$ is concentrated on convex bodies for which the origin is the centre of the circumscribed ball. A random convex body Z_0 distributed according to the probability measure $\mathbf{Q}$ is called typical grain. It follows from Steiner's formula that (12.1) is equivalent to

$$v_i := \mathbf{E}V_i(Z_0) < \infty, \qquad i = 0, \ldots, n,$$

where $V_0, \dots, V_n$ stand for the intrinsic volumes. Later we shall require that some higher moments of the intrinsic volumes exist. When studying covariances we have to assume that

$$\mathbf{E}V_i(Z_0)^2 < \infty, \qquad i = 0, \dots, n. \tag{12.2}$$

For some results we need the stronger assumption that

$$\mathbf{E}V_i(Z_0)^3 < \infty, \qquad i = 0, \dots, n. \tag{12.3}$$

The Boolean model Z based on the Poisson process η is the union of all grains of the Poisson process η, that is

$$Z := \bigcup_{K \in \eta} K.$$

This is a random closed set (see [37, Chap. 2] for an introduction to random closed sets), whose distribution is completely determined by the intensity γ and the distribution $\mathbf{Q}$ of the typical grain Z_0. The stationarity of the Poisson process η implies the stationarity of the Boolean model Z, that is, the distribution of Z is invariant under translations. Throughout this chapter we investigate the stationary Boolean model Z within compact convex observation windows. For a convex body $W \in \mathscr{K}^n$ the number of convex bodies of η that intersect W is almost surely finite so that the random closed set $Z \cap W$ belongs almost surely to the convex ring $\mathscr{R}^n$, which is the set of all unions of finitely many convex bodies and the empty set. Most results in this chapter are for the asymptotic regime that the observation window is increased. More precisely, we shall assume that the inradius of the observation window goes to infinity.

To study the behaviour of the intersection of the Boolean model with the observation window W, we consider functionals of $Z \cap W$ with specific properties. We say that a functional $\psi : \mathscr{R}^n \to \mathbb{R}$ is

(i) additive (or a valuation), if $\psi(\emptyset) = 0$, and $\psi(A \cup B) = \psi(A) + \psi(B) - \psi(A \cap B)$ for all $A, B \in \mathscr{R}^n$;
(ii) locally bounded, if

$$M(\psi) := \sup\{|\psi(K + x)| : x \in \mathbb{R}^n, K \in \mathscr{K}^n \text{ with } K \subset [0,1]^n\} < \infty;$$

(iii) translation invariant, if $\psi(A + x) = \psi(A)$, for any $A \in \mathscr{R}^n$ and any $x \in \mathbb{R}^n$.

A measurable functional $\psi : \mathscr{R}^n \to \mathbb{R}$ with all three properties is called geometric. In this case property (ii) can be simplified using the translation invariance (iii). Fundamental examples of geometric functionals are the intrinsic volumes $V_0, \dots, V_n$, where V_n is the volume, V_{n-1} is half the surface area (if the set is the closure of its interior) and V_0 is the Euler characteristic (see also Sect. 1.2).

More general geometric functionals are of the form

$$V_{g,i}(A) := \Psi_i(A; g) := \int g(u)\, \Psi_i(A; du), \qquad A \in \mathscr{R}^n, \tag{12.4}$$

where $\Psi_i(A; \cdot) := \Lambda_i(A; \mathbb{R}^n \times \cdot)$, $i \in \{0, \dots, n\}$, is the (additive extension of the) i-th area measure of A (a signed measure on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$), and $g : \mathbb{S}^{n-1} \to \mathbb{R}$ is measurable and bounded. If $g \equiv 1$, then $V_{g,i} = V_i$. We refer to Sect. 1.3 and [36, p. 216] for more detail on the support measures Λ_i. An example for geometric functionals of the form (12.4) are the so-called harmonic intrinsic volumes, which are used in [12] to give a representation of the intensity γ of non-isotropic Boolean models (see also Sect. 11.8).

The next class of geometric functionals we consider are the components of translation invariant Minkowski tensors (see Chaps. 1 and 2 for a more detailed introduction to tensor valuations). Let us denote by $\mathbb{T}^s$ the space of s-dimensional tensors in $\mathbb{R}^n$. Let $(e_1, \dots, e_n)$ denote the standard basis of $\mathbb{R}^n$. Then, for $u \in \mathbb{R}^n$ and $s \in \mathbb{N}$, the s-dimensional tensor u^s is given by its coordinates

$$(u^s)_{i_1,\dots,i_s} = \prod_{j=1}^{s} u_{i_j}, \qquad i_1, \dots, i_s \in \{1, \dots, n\},$$

with respect to the tensor basis $e_{i_1} \otimes \cdots \otimes e_{i_s}$, $i_1, \dots, i_s \in \{1, \dots, n\}$. See Chap. 2 for a description in terms of a basis of the vector space $\mathbb{T}^s$ of symmetric tensors.

Now the Minkowski tensors $\Phi_m^{0,s} : \mathscr{R}^n \to \mathbb{T}^s$, $s \in \mathbb{N}$, $m \in \{0, \dots, n-1\}$, are given by

$$\Phi_m^{0,s}(A) = \frac{1}{s!} \frac{\omega_{d-m}}{\omega_{d-m+s}} \int u^s\, \Psi_m(A; du),$$

where $\omega_i := i\kappa_i$ with κ_i being the volume of the unit ball in $\mathbb{R}^i$. Each component of $\Phi_m^{0,s}$ is obviously measurable, additive and translation invariant. For any $i_1, \dots, i_r \in \{1, \dots, n\}$ and $u \in \mathbb{S}^{n-1}$ we have $|(u^r)_{i_1,\dots,i_r}| \le 1$ so that

$$|(\Phi_m^{0,s}(K))_{i_1,\dots,i_r}| \le \frac{1}{s!} \frac{\omega_{d-m}}{\omega_{d-m+s}} \int 1\, \Psi_m(K; du) = \frac{1}{s!} \frac{\omega_{d-m}}{\omega_{d-m+s}} V_m(K)$$

for $K \in \mathscr{K}^n$. This shows that the components are also locally bounded.

12.3 Covariance Structure

We first consider general covariance formulae for geometric functionals of Boolean models in any dimension n. Then, we concentrate on planar Boolean models and derive explicit integral formulae for the asymptotic covariances of intrinsic volumes.

12.3.1 General Covariance Formulae

In this subsection we consider the asymptotic covariance of two geometric functionals of the Boolean model Z within an observation window W as the inradius of W is increased. This means that we consider sequences of convex bodies $(W_i)_{i\in\mathbb{N}}$ such that $r(W_i) \to \infty$ as $i \to \infty$, where $r(K)$ stands for the inradius of a convex body $K \in \mathcal{K}^n$. We denote this asymptotic regime by $r(W) \to \infty$ in the sequel.

In order to present a formula for the asymptotic covariance of two geometric functionals of a Boolean model Z we have to introduce some notation. For a geometric functional $\psi : \mathcal{R}^n \to \mathbb{R}$ the integrability assumption (12.1) implies for any $A \in \mathcal{R}^n$ that $\mathbf{E}|\psi(Z \cap A)| < \infty$; see [16]. Hence we can define $\psi^* : \mathcal{R}^n \to \mathbb{R}$ by

$$\psi^*(A) = \mathbf{E}\psi(Z \cap A) - \psi(A), \qquad A \in \mathcal{R}^n.$$

The functional ψ^* is again geometric, see [16, Eq. (3.11)]. The mapping $\psi \mapsto \psi^*$ is a key operation for the second order analysis of the Boolean model. The following proposition provides explicit formulae in some important examples. To state these (and other formulae) we need the measure $\overline{\Psi}_{n-1}(\,\cdot\,) := \mathbf{E}\Psi_{n-1}(Z_0;\,\cdot\,)$. For a bounded measurable function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ we use the notation

$$\overline{\Psi}_{n-1}(g) := \int g(u)\,\overline{\Psi}_{n-1}(du) = \mathbf{E}\int g(u)\,\Psi_{n-1}(Z_0; du).$$

The *volume fraction* of Z is defined by $p := \mathbf{E}V_n(Z \cap [0,1]^n)$ and can be expressed in the form

$$p = 1 - e^{-\gamma v_n}. \tag{12.5}$$

Proposition 12.1 *Let $g : \mathbb{S}^{n-1} \to \mathbb{R}$ be bounded and measurable. Then*

$$V_n^* = -(1-p)V_n, \tag{12.6}$$

$$V_{g,n-1}^* = -(1-p)V_{g,n-1} + (1-p)\gamma\overline{\Psi}_{n-1}(g)V_n. \tag{12.7}$$

Proof Formula (12.6) follows from an easy calculation; see [16]. For $j \in \{0,\ldots,n-1\}$ and $K_0 \in \mathcal{K}$ we obtain from Theorem 9.1.2 in [37] that

$$\mathbf{E}V_{g,j}(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int \Psi_j(K_0 \cap \ldots \cap K_k; g)\,\Theta^k(d(K_1,\ldots,K_k)).$$

Using a result in [14, Sects. 3.2–3.4] or [13, Theorem 3.1] (for $g \equiv 1$ see also [37, p. 390]), we obtain that

$$\mathbf{E}V_{g,j}(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\gamma^k}{k!} \tag{12.8}$$

$$\times \sum_{\substack{m_0,\dots,m_k=j \\ m_0+\dots+m_k=kn+j}}^{n} \int V^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k;g)\,\mathbf{Q}^k(d(K_1,\dots,K_k)),$$

where

$$V^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k;\cdot) := \Lambda^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k;(\mathbb{R}^n)^{k+1}\times\cdot)$$

are finite Borel measures on $\mathbb{S}^{n-1}$, the mixed area measures of order j; see [13–15]. As usual, we abbreviate $V^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k) := V^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k;\mathbb{S}^{n-1})$.

Consider (12.8) for $j = n-1$. In the summation on the right-hand side we have $m_i = n-1$ for exactly one $i \in \{0,\dots,k\}$ and $m_r = n$ for $r \neq i$. Using the decomposability

$$V^{(n-1)}_{n-1,n,\dots,n}(K_0,\dots,K_k;g) = \Psi_{n-1}(K_0;g)V_n(K_1)\cdots V_n(K_k) \tag{12.9}$$

and the symmetry properties of the mixed area measures (see [13, 14]) we hence obtain that

$$\begin{aligned}
&\mathbf{E}V_{g,n-1}(Z \cap K_0) \\
&= \Psi_{n-1}(K_0;g)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\gamma^k}{k!}v_n^k + V_n(K_0)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\gamma^k}{k!}kv_n^{k-1}\overline{\Psi}_{n-1}(g) \\
&= (1-e^{-\gamma v_n})\Psi_{g,n-1}(K_0) + \gamma\overline{\Psi}_{n-1}(g)e^{-\gamma v_n}V_n(K_0).
\end{aligned}$$

Inserting here (12.5) yields formula (12.7). □

For two geometric functionals ψ, ϕ, we define the inner product

$$\varrho(\psi,\phi) := \sum_{i=1}^{\infty}\frac{\gamma}{i!}\int_{\mathcal{K}^n}\int_{(\mathcal{K}^n)^{i-1}} \psi(K_1\cap\dots\cap K_i) \tag{12.10}$$

$$\times\,\phi(K_1\cap\dots\cap K_i)\,\Theta^{i-1}(d(K_2,\dots,K_i))\,\mathbf{Q}(dK_1),$$

whenever this infinite series is well defined. The importance of this operation for the covariance analysis of the Boolean model is due to (12.17) below. In Proposition 12.2 below and in Sect. 12.3.2 we shall see that (12.10) can be computed in some specific examples.

We need to introduce further notation. The *mean covariogram* of the typical grain Z_0 is

$$C_n(x) = \mathbf{E}V_n(Z_0 \cap (Z_0 + x)), \qquad x \in \mathbb{R}^n.$$

For a measurable and bounded function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ we define

$$C_{n-1}(x; g) = \mathbf{E} \int \mathbf{1}\{y \in Z_0^\circ + x\} g(u)\, \Lambda_{n-1}(Z_0; d(y,u)), \qquad x \in \mathbb{R}^n, \tag{12.11}$$

where A° denotes the interior of A. Moreover, we use the mixed moment measures

$$N_{n-1,n}(\cdot) = \mathbf{E} \iint \mathbf{1}\{(y,u,z) \in \cdot\}\mathbf{1}\{z \in Z_0\}\, \Lambda_{n-1}(Z_0; d(y,u))\, dz$$

and

$$N_{n-1,n-1}(\cdot) = \mathbf{E} \iint \mathbf{1}\{(y,u,z,v) \in \cdot\}\, \Lambda_{n-1}(Z_0; d(y,u))\, \Lambda_{n-1}(Z_0; d(z,v)).$$

Proposition 12.2 *Let $g, h : \mathbb{S}^{n-1} \to \mathbb{R}$ be bounded and measurable. Then*

$$\varrho(V_n, V_n) = \int \left(e^{\gamma C_n(x)} - 1\right) dx, \tag{12.12}$$

$$\varrho(V_{g,n-1}, V_n) = \gamma \int g(u) e^{\gamma C_n(y-z)}\, N_{n-1,n}(d(y,u,z)), \tag{12.13}$$

$$\varrho(V_0, V_n) = (1-p)^{-1} - 1. \tag{12.14}$$

If, additionally, $\mathbf{P}(V_n(Z_0) > 0) = 1$*, then*

$$\begin{aligned}\varrho(V_{g,n-1}, V_{h,n-1}) = \gamma^2 \int & e^{\gamma C_n(y-z)} C_{n-1}(y-z; g) h(v)\, N_{n-1,n}(d(z,v,y)) \\ & + \gamma \int e^{\gamma C_n(y-z)} g(u) h(v)\, N_{n-1,n-1}(d(y,u,z,v)),\end{aligned} \tag{12.15}$$

$$\varrho(V_0, V_{g,n-1}) = \gamma(1-p)^{-1}\overline{\Psi}_{n-1}(g). \tag{12.16}$$

Proof Formulae (12.12) and (12.14) are implied by [16, Theorem 5.2]. The formulae (12.13) and (12.16) can be derived as in the proof of the latter theorem; cf. the computation of $\varrho_{d-1,d}$ and of $\varrho_{0,d}$ in [16].

As in the computation of $\varrho_{i,j}$ in [16] (for $i = j = n-1$) we obtain that

$$\varrho(V_{g,n-1}, V_{h,n-1}) = A_0 + A_1,$$

where

$$A_0 := \gamma^2 \iiiint e^{\gamma C_n(y-z)} \mathbf{1}\{y \in K_2^\circ, z \in K_1^\circ\} g(u)h(v) \times \Lambda_{n-1}(K_1; d(y,u))\, \Lambda_{n-1}(K_2; d(z,v))\, \Theta(dK_1)\, \mathbf{Q}(dK_2)$$

and

$$A_1 := \gamma \iiint e^{\gamma C_n(y-z)} g(u)h(v)\, \Lambda_{n-1}(K; d(y,u))\, \Lambda_{n-1}(K; d(z,v))\, \mathbf{Q}(dK).$$

An easy calculation based on the covariance property of Λ_{n-1} shows that

$$A_0 = \gamma^2 \int e^{\gamma C_n(x-z)} C_{n-1}(x-z; g)h(v)\, N_{n-1,n}(d(z,v,x)).$$

As the number A_1 can be expressed directly as an integral with respect to $N_{n-1,n-1}$, (12.15) follows. □

The following theorem establishes the existence of asymptotic covariances for general geometric functionals. Moreover, formula (12.17) provides a tool for their computation.

Theorem 12.3 *Assume that* (12.2) *is satisfied and let* ψ *and* ϕ *be geometric functionals. Then the limit*

$$\sigma(\psi,\phi) := \lim_{r(W)\to\infty} \frac{\mathbf{cov}(\psi(Z\cap W), \phi(Z\cap W))}{V_n(W)}$$

exists and is given by

$$\sigma(\psi,\phi) = \varrho(\psi^*,\phi^*). \tag{12.17}$$

If (12.3) *holds, there is a constant* c_Θ*, depending only on* Θ*, such that, for* $W \in \mathcal{K}^n$ *with* $r(W) \geq 1$*,*

$$\left| \frac{\mathbf{cov}(\psi(Z\cap W), \phi(Z\cap W))}{V_n(W)} - \sigma(\psi,\phi) \right| \leq \frac{c_\Theta M(\psi)M(\phi)}{r(W)}. \tag{12.18}$$

Theorem 12.3 is taken from [16, Theorem 3.1]. Its proof is involved and depends on the Fock space representation [20] and several non-trivial integral-geometric inequalities for geometric functionals. The inequality (12.18) allows us to control the error if we approximate the exact covariance for a given observation window by the asymptotic covariance. By evaluating the left-hand side of (12.18) for the volume one obtains a lower bound of order $1/r(W)$ (see [16, Proposition 3.8]), which shows that the rate on the right-hand side of (12.18) is optimal, in general.

Using Propositions 12.1 and 12.2 in formula (12.17), we obtain the following result for the asymptotic covariances involving volume and surface content.

Theorem 12.4 *Assume that* (12.2) *holds and let* $g, h : \mathbb{S}^{n-1} \to \mathbb{R}$ *be measurable and bounded. Then,*

$$\sigma(V_n, V_n) = (1-p)^2 \int (e^{\gamma C_n(x)} - 1)\, dx,$$

$$\begin{aligned}\sigma(V_{g,n-1}, V_n) = & -(1-p)^2 \gamma \overline{\Psi}_{n-1}(g) \int (e^{\gamma C_n(x)} - 1)\, dx \\ & + (1-p)^2 \gamma \int g(u) e^{\gamma C_n(x-y)}\, N_{n-1,n}(d(x,u,y)).\end{aligned}$$

If, in addition, $\mathbf{P}(V_n(Z_0) > 0) = 1$*, then*

$$\begin{aligned}\sigma(V_{g,n-1}, V_{h,n-1}) = & (1-p)^2 \gamma^2 \overline{\Psi}_{n-1}(g) \overline{\Psi}_{n-1}(h) \int (e^{\gamma C_n(x)} - 1)\, dx \\ & + (1-p)^2 \gamma^2 \int e^{\gamma C_n(x-y)} h(u) C_{n-1}(x-y; g)\, N_{n-1,n}(d(y,u,x)) \\ & - (1-p)^2 \gamma^2 \int e^{\gamma C_n(x-y)} \big(g(u) \overline{\Psi}_{n-1}(h) \\ & \qquad + h(u) \overline{\Psi}_{n-1}(g)\big)\, N_{n-1,n}(d(y,u,x)) \\ & + (1-p)^2 \gamma \int e^{\gamma C_n(x-y)} g(u) h(v)\, N_{n-1,n-1}(d(x,u,y,v)).\end{aligned}$$

In the case $h = g \equiv 1$ the formula for $\sigma(V_{g,n-1}, V_{h,n-1})$ simplifies to [16, Corollary 6.2], that is

$$\begin{aligned}\sigma(V_{n-1}, V_{n-1}) = & (1-p)^2 \gamma^2 v_{n-1}^2 \int (e^{\gamma C_n(x)} - 1)\, dx \\ & + (1-p)^2 \gamma^2 \int e^{\gamma C_n(x-y)} (C_{n-1}(x-y) - 2v_1)\, N_{n-1,n}(d(y,u,x)) \\ & + (1-p)^2 \gamma \int e^{\gamma C_n(x-y)}\, N_{n-1,n-1}(d(x,u,y,v)), \end{aligned} \tag{12.19}$$

where $C_{n-1}(x) := C_{n-1}(x; 1)$ is defined by (12.11) with $g \equiv 1$.

In the planar case (treated in Sect. 12.3.2) we will complement Theorem 12.4 with the asymptotic covariances involving the Euler characteristic. Integral representations of asymptotic covariances of intrinsic volumes in general dimensions (with respect to some special curvature based measures) can be found in [16, Sects. 5 and 6].

Theorem 12.3 establishes the existence of an asymptotic covariance matrix $\Sigma = (\sigma(\psi_i, \psi_j))_{i,j=1,\ldots,m}$ for geometric functionals $\psi_1, \ldots, \psi_m$. It is natural to ask whether this matrix is positive definite. The next result (see [16, Theorem 4.1]) gives sufficient, but presumably not necessary conditions for positive definiteness.

Theorem 12.5 *Let (12.2) be satisfied and assume that* $\mathbf{P}(V_n(Z_0) > 0) > 0$. *Let* $\psi_0, \ldots, \psi_n$ *be geometric functionals such that, for* $i \in \{0, \ldots, n\}$, ψ_i *is homogeneous of degree i (that is,* $\psi_i(\lambda K) = \lambda^i \psi_i(K)$ *for* $\lambda > 0$*) and satisfies*

$$|\psi_i(K)| \geq \tilde{\beta}(\psi_i) r(K)^i, \qquad K \in \mathscr{K}^n,$$

with a constant $\tilde{\beta}(\psi_i)$ *only depending on* ψ_i. *Then* $\Sigma = (\sigma(\psi_i, \psi_j))_{i,j=0,\ldots,n}$ *is positive definite.*

Since the intrinsic volumes satisfy the assumptions of Theorem 12.5, we obtain the following corollary.

Corollary 12.6 *Let (12.2) be satisfied and assume that the typical grain has nonempty interior with positive probability. Then the matrix* $\Sigma = (\sigma(V_i, V_j))_{i,j=0,\ldots,n}$ *is positive definite.*

12.3.2 Covariance Formulae for Planar Boolean Models

In this section we consider the Boolean model in the planar case $n = 2$. For measurable and bounded $g : \mathbb{S}^1 \to \mathbb{R}$ we consider the additive and measurable functional

$$V_{g,1}(K) := \Psi_1(K; g) := \int g(u)\, \Psi_1(K; du), \qquad K \in \mathscr{R}^n,$$

see (12.4). We will compute the asymptotic covariances between V_0 and the vector $(V_0, V_{g,1}, V_2)$.

We define a function $\bar{h} : \mathbb{S}^1 \to \mathbb{R}$ by

$$\bar{h}(u) := \int h(K^*, u)\, \mathbf{Q}(dK), \qquad u \in \mathbb{S}^1,$$

where $K^* := -K$ and $h(K^*, \cdot)$ is the support function of K^*. Indeed, if K is a convex body containing the origin, then the basic properties of V_1 together with the definition of the support function easily imply that $0 \leq h(K^*, u) \leq cV_1(K^*) = cV_1(K)$ for a constant $c > 0$ that does only depend on the dimension. Therefore dominated convergence implies that $\bar{h}$ is continuous and in particular bounded. We also define

$$v_{1,1} := \overline{\Psi}_1(\bar{h}) = \int \bar{h}(u)\, \overline{\Psi}_1(du) = \iiint h(K^*, u)\, \overline{\Psi}_1(L; du)\, \mathbf{Q}(dK)\, \mathbf{Q}(dL).$$

Theorem 12.7 *Assume that* (12.2) *and* $\mathbf{P}(V_2(Z_0) > 0) = 1$ *hold and let* $g : \mathbb{S}^1 \to \mathbb{R}$ *be measurable and bounded. Then*

$$\sigma(V_0, V_2) = p(1-p) - (1-p)^2\gamma(1-\gamma v_{1,1}) \int \left(e^{\gamma C_2(x)} - 1\right) dx$$
$$- 2(1-p)^2\gamma^2 \int \bar{h}(u)e^{\gamma C_2(y-z)}\, N_{1,2}(d(y,u,z)), \qquad (12.20)$$

$$\sigma(V_0, V_{g,1}) = (1-p)^2\gamma\overline{\Psi}_1(g) + (1-p)^2\gamma^2\overline{\Psi}_1(g)(1-\gamma v_{1,1}) \int \left(e^{\gamma C_2(x)} - 1\right) dx$$
$$+ (1-p)^2 \int \big(\chi'(y-z)g(u)$$
$$+ 2\gamma^3\overline{\Psi}_1(g)e^{\gamma C_2(y-z)}\bar{h}(u)\big)\, N_{1,2}(d(z,u,y))$$
$$- 2(1-p)^2\gamma^2 \int e^{\gamma C_2(y-z)}\bar{h}(u)g(v)\, N_{1,1}(d(y,u,z,v)), \qquad (12.21)$$

$$\sigma(V_0, V_0) = (1-2p)(1-p)\gamma + (1-p)(2p-3)v_{1,1}\gamma^2$$
$$+ (1-p)^2\gamma^2(1-\gamma v_{1,1})^2 \int \left(e^{\gamma C_2(x)} - 1\right) dx$$
$$+ (1-p)^2 \int \bar{h}(u)\chi''(y-z)\, N_{1,2}(d(z,u,y))$$
$$+ 4(1-p)^2\gamma^3 \int e^{\gamma C_2(y-z)}\bar{h}(u)\bar{h}(v)\, N_{1,1}(d(y,u,z,v)), \qquad (12.22)$$

where

$$\chi'(x) := e^{\gamma C_2(x)}\left(\gamma^3(v_{1,1} - 2C_1(x;\bar{h})) - \gamma^2\right), \qquad x \in \mathbb{R}^2,$$
$$\chi''(x) := e^{\gamma C_2(x)}\left(4\gamma^4(C_1(x;\bar{h}) - v_{1,1}) + 4\gamma^3\right), \qquad x \in \mathbb{R}^2.$$

Proof We wish to apply (12.17). In view of Proposition 12.1 we need to determine V_0^*. To do so we consider (12.8) for $j = 0$ and $g \equiv 1$. For the summation we distinguish four cases. In the first two cases we have $m_i = 0$ for exactly one $i \in \{0, \ldots, k\}$ and either $m_0 = 0$ or $m_0 = 2$. In the third and fourth case we have $m_i = m_r = 1$ for exactly two $i, r \in \{0, \ldots, k\}$ and either $m_0 = 0$ or $m_0 = 1$. Accordingly we can write

$$\mathbf{E}V_0(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\gamma^k}{k!}(a_{1,k} + a_{2,k} + a_{3,k} + a_{4,k}).$$

The decomposability property (12.9) (for $n = 2$) and the symmetry of mixed functionals imply that

$$a_{1,k} = V_0(K_0)v_2^k, \qquad a_{2,k} = V_2(K_0)kv_2^{k-1}.$$

To treat $a_{3,k}$ and $a_{4,k}$ we use the decomposability property

$$V^{(0)}_{1,1,2\dots,2}(K_0,\dots,K_k) = V^{(0)}_{1,1}(K_0,K_1)V_2(K_2)\cdots V_2(K_k)$$

and again the symmetry of mixed functionals (see [14]) to obtain that

$$a_{3,k} = kv_2^{k-1}\int V^{(0)}_{1,1}(K_0,K)\,\mathbf{Q}(dK),$$
$$a_{4,k} = \frac{k(k-1)}{2}v_2^{k-2}V_2(K_0)\int V^{(0)}_{1,1}(K,L)\,\mathbf{Q}^2(d(K,L)).$$

It follows that

$$V_0^*(K_0) = -(1-p)V_0(K_0) + (1-p)\gamma\int V^{(0)}_{1,1}(K_0,K)\,\mathbf{Q}(dK)$$
$$+(1-p)V_2(K_0)\Big(\gamma - \frac{\gamma^2}{2}\int V^{(0)}_{1,1}(K,L)\,\mathbf{Q}^2\big(d(K,L)\big)\Big)$$

or

$$V_0^*(K_0) = -(1-p)V_0(K_0) + (1-p)\gamma\overline{V}_{1,1}(K_0)$$
$$+(1-p)\Big(\gamma - \frac{\gamma^2}{2}w_{1,1}\Big)V_2(K_0), \tag{12.23}$$

where

$$\overline{V}_{1,1}(K_0) := \int V^{(0)}_{1,1}(K_0,K)\,\mathbf{Q}(dK),$$
$$w_{1,1} := \int V^{(0)}_{1,1}(K,L)\,\mathbf{Q}^2(d(K,L)).$$

Using (12.17) together with (12.23) and Proposition 12.1, we obtain the following intermediate formulae for the asymptotic covariances:

$$\sigma(V_0,V_2) = (1-p)^2\varrho(V_0,V_2) - (1-p)^2\gamma\varrho(\overline{V}_{1,1},V_2)$$
$$-(1-p)^2\Big(\gamma - \frac{\gamma^2}{2}w_{1,1}\Big)\varrho(V_2,V_2), \tag{12.24}$$

$$\begin{aligned}\sigma(V_0, V_{g,1}) = & -(1-p)^2\gamma\overline{\Psi}_1(g)\varrho(V_0, V_2) + (1-p)^2\gamma^2\overline{\Psi}_1(g)\varrho(\overline{V}_{1,1}, V_2)\\ & + (1-p)^2\overline{\Psi}_1(g)\Big(\gamma^2 - \frac{\gamma^3}{2}w_{1,1}\Big)\varrho(V_2, V_2)\\ & + (1-p)^2\varrho(V_0, V_{g,1}) - (1-p)^2\gamma\varrho(\overline{V}_{1,1}, V_{g,1})\\ & - (1-p)^2\Big(\gamma - \frac{\gamma^2}{2}w_{1,1}\Big)\varrho(V_2, V_{g,1}), \end{aligned} \tag{12.25}$$

and

$$\begin{aligned}\sigma(V_0, V_0) = & (1-p)^2\varrho(V_0, V_0) - 2(1-p)^2\gamma\varrho(V_0, \overline{V}_{1,1})\\ & + (1-p)^2\gamma^2\varrho(\overline{V}_{1,1}, \overline{V}_{1,1}) - 2(1-p)^2\Big(\gamma - \frac{\gamma^2}{2}w_{1,1}\Big)\varrho(V_0, V_2)\\ & + 2(1-p)^2\Big(\gamma^2 - \frac{\gamma^3}{2}w_{1,1}\Big)\varrho(\overline{V}_{1,1}, V_2)\\ & + (1-p)^2\Big(\gamma - \frac{\gamma^2}{2}w_{1,1}\Big)^2\varrho(V_2, V_2). \end{aligned} \tag{12.26}$$

At this stage we can use the formula

$$V_{1,1}^{(0)}(K, L) = 2\int h(L^*, u)\, \Psi_1(K; du), \qquad K, L \in \mathscr{K},$$

(which follows from (6.25) and (14.21) in [37] along with $S_1 = 2\Psi_1$) implying that

$$\overline{V}_{1,1}(K) = 2\int \bar{h}(u)\, \Psi_1(K; du) = 2V_{\bar{h},1}(K), \tag{12.27}$$

$$w_{1,1} = \int \overline{V}_{1,1}(K)\, \mathbf{Q}(dK) = 2\overline{\Psi}_1(\bar{h}) = 2v_{1,1}. \tag{12.28}$$

Theorem 5.2 in [16] shows that

$$\varrho(V_0, V_0) = e^{\gamma v_2}\Big(\gamma + \frac{\gamma^2 v_{1,1}'}{2}\Big),$$

where

$$v_{1,1}' := \int \Phi_0(K_1 \cap (K_2 + x); \partial K_1 \cap (\partial K_2 + x))\, dx\, \mathbf{Q}^2(d(K_1, K_2)).$$

It follows from [37, Theorem 6.4.1] (together with the decomposability property and the fact that the boundary of a convex body has vanishing volume) that

$$\int \Phi_0(K_1 \cap (K_2 + x); \partial K_1 \cap (\partial K_2 + x))\, dx = \Phi_{1,1}^{(0)}(K_1, K_2; \partial K_1 \times \partial K_2),$$

where $\Phi^{(0)}_{1,1}(K_1, K_2; \cdot)$ is a mixed measure. Since $\Phi^{(0)}_{1,1}(K_1, K_2; \cdot)$ is concentrated on $\partial K_1 \times \partial K_2$ by [37, Theorem 6.4.1 (b)], we have

$$\Phi^{(0)}_{1,1}(K_1, K_2; \partial K_1 \times \partial K_2) = \Phi^{(0)}_{1,1}(K_1, K_2; \mathbb{R}^2 \times \mathbb{R}^2) = V^{(0)}_{1,1}(K_1, K_2).$$

Therefore $v'_{1,1} = w_{1,1} = 2v_{1,1}$ and

$$\varrho(V_0, V_0) = (1-p)^{-1}(\gamma + \gamma^2 v_{1,1}). \tag{12.29}$$

Now we can insert (12.27) and (12.28) as well as (12.29) and the formulae of Proposition 12.2 into (12.24)–(12.26) to obtain the assertions. From (12.24) we get

$$\begin{aligned}\sigma(V_0, V_2) = (1-p)^2\varrho(V_0, V_2) - (1-p)^2\gamma 2\varrho(V_{\bar{h},1}, V_2)\\ - (1-p)^2(\gamma - \gamma^2 v_{1,1})\varrho(V_2, V_2)\end{aligned}$$

so that (12.20) follows from (12.14), (12.13) and (12.12).

Next we deduce from (12.25) that

$$\begin{aligned}\sigma(V_0, V_{g,1}) = & -p(1-p)\gamma\overline{\Psi}_1(g)\\ & + 2(1-p)^2\gamma^3\overline{\Psi}_1(g)\int \bar{h}(u)e^{\gamma C_2(y-z)}\, N_{1,2}(d(y,u,z))\\ & + (1-p)^2\overline{\Psi}_1(g)(\gamma^2 - \gamma^3 v_{1,1})\int \left(e^{\gamma C_2(x)} - 1\right) dx\\ & + (1-p)^2\gamma e^{\gamma v_2}\overline{\Psi}_1(g)\\ & - 2(1-p)^2\gamma^3\int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})g(v)\, N_{1,2}(d(z,v,y))\\ & - 2(1-p)^2\gamma^2\int e^{\gamma C_2(y-z)}\bar{h}(u)g(v)\, N_{1,1}(d(y,u,z,v))\\ & - (1-p)^2\Big(\gamma^2 - \gamma^3 v_{1,1}\Big)\int g(u)e^{\gamma C_2(y-z)}\, N_{1,2}(d(y,u,z)).\end{aligned}$$

Equation (12.21) follows upon some simplification and rearrangement.

From (12.26) we obtain that

$$\begin{aligned}\sigma(V_0, V_0) = & (1-p)\big(\gamma + \gamma^2 v_{1,1}\big) - 4(1-p)\gamma^2 v_{1,1}\\ & + 4(1-p)^2\gamma^4\int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\bar{h}(v)\, N_{1,2}(d(z,v,y))\\ & + 4(1-p)^2\gamma^3\int e^{\gamma C_2(y-z)}\bar{h}(u)\bar{h}(v)\, N_{1,1}(d(y,u,z,v))\end{aligned}$$

$$
\begin{aligned}
&- 2(1-p)^2\big(\gamma - \gamma^2 v_{1,1}\big)((1-p)^{-1} - 1) \\
&+ 4(1-p)^2(\gamma^3 - \gamma^4 v_{1,1}) \int \bar{h}(u) e^{\gamma C_2(y-z)}\, N_{1,2}(d(y,u,z)) \\
&+ (1-p)^2\big(\gamma - \gamma^2 v_{1,1}\big)^2 \int \big(e^{\gamma C_2(x)} - 1\big)\, dx.
\end{aligned}
$$

Equation (12.22) now follows from an easy calculation. □

In the isotropic case, $\bar{h} := \bar{h}(u)$ does not depend on $u \in \mathbb{S}^1$. By [37, (14.21)], we have for $L \in \mathscr{K}^n$

$$
V_1(L) = \int h(L,u)\, \Psi_1(B^2; du) = \tfrac{1}{2} \int h(L,u)\, \mathscr{H}^1(du),
$$

so that

$$
v_1 = \tfrac{1}{2} \iint h(L,u)\, \mathscr{H}^1(du)\, \mathbf{Q}(dL) = \pi \bar{h}.
$$

Further

$$
v_{1,1} = \iint \bar{h}(u)\ \Psi_1(K; du)\, \mathbf{Q}(dK) = \bar{h} v_1.
$$

Hence

$$
\bar{h} = \frac{v_1}{\pi}, \qquad v_{1,1} = \frac{v_1^2}{\pi}. \tag{12.30}
$$

Inserting (12.30) into (12.20)–(12.22) yields Corollary 6.3 in [16].

12.3.3 The Boolean Model on the Torus

One obtains the n-dimensional (unit) torus T^n by identifying opposite sides of the boundary of $[-1/2, 1/2]^n$. As in $\mathbb{R}^n$ one can consider a translation invariant Poisson process of grains on the torus (with intensity measure Θ, grain distribution $\mathbf{Q}$ and intensity γ) and consider the resulting Boolean model Z_{T^n}. The Boolean model on the torus T^n can be constructed in the following way from a random closed set in $\mathbb{R}^n$ (see Fig. 12.1). We start with a homogeneous Poisson process in $[-1/2, 1/2]^n$ and put around each point an independent copy of the typical grain. For each grain, we also place all translates by vectors $v \in \mathbb{Z}^n$ and take the union of all resulting grains. Finally, we restrict this random closed set to $[-1/2, 1/2]^n$ and identify opposite boundaries. This setting is also denoted as periodic boundary conditions.

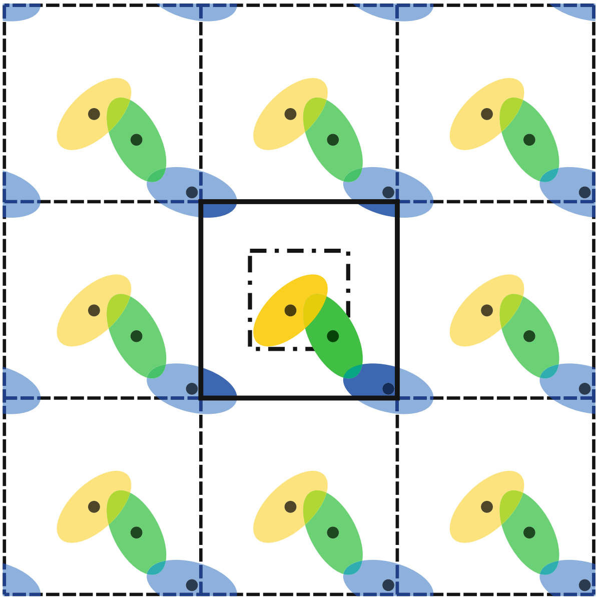

Fig. 12.1 The Boolean model with periodic boundary conditions: we consider grains with centers in $[-1/2, 1/2]^n$ (*square with solid line*) and all their translations by $\mathbb{Z}^n$ valued vectors (in *squares with dashed lines*). The Boolean model with periodic boundary conditions is obtained by taking the union of all the grains and restricting to the square with the *solid line*

For a geometric functional $\psi : \mathscr{R}^n \to \mathbb{R}$ we can define $\psi(Z_{T^n})$ in the following way. For a set $K \subset T^n$ whose embedding $K_{\mathbb{R}^n} = \{x \in \mathbb{R}^n : x \in K\}$ into $\mathbb{R}^n$ is a convex body and is contained in $(-1/2, 1/2)^n$ we put $\psi(K) = \psi(K_{\mathbb{R}^n})$. By further requiring that ψ is translation-invariant and additive on T^n, this gives us $\psi(Z_{T^n})$.

By computing the Fock space representation of $\psi(Z_{T^n})$ and $\phi(Z_{T^n})$ for geometric functionals $\psi, \phi : \mathscr{R}^n \to \mathbb{R}$ as in [16, Sect. 3] for a Boolean model in $\mathbb{R}^n$, one obtains that

$$\begin{aligned}
&\mathbf{cov}(\psi(Z_{T^n}), \phi(Z_{T^n})) \\
&\quad = \sum_{n=1}^{\infty} \frac{\gamma}{n!} \iint \big(\mathbf{E}\psi(Z_{T^n} \cap K_1 \cap \ldots \cap K_n) - \psi(K_1 \cap \ldots \cap K_n)\big) \\
&\qquad \times \big(\mathbf{E}\phi(Z_{T^n} \cap K_1 \cap \ldots \cap K_n) - \phi(K_1 \cap \ldots \cap K_n)\big) \\
&\qquad \times \Theta^{n-1}\big(d(K_2, \ldots, K_n)\big)\, \mathbf{Q}(dK_1).
\end{aligned}$$

Now let us assume that the grain distribution $\mathbf{Q}$ is such that the typical grain Z_0 is almost surely contained in $[-1/4, 1/4]^n$, which is depicted by the dot-dash line in Fig. 12.1. In this case the intersection of two grains is always convex and the intersections on the right-hand side of the covariance formula are the same as for a Boolean model in $\mathbb{R}^n$ with grain distribution $\mathbf{Q}$ and intensity γ. Thus, it follows from the above definition of ψ and ϕ of a subset of the torus whose embedding into $\mathbb{R}^n$ is a convex body and a subset of $(-1/2, 1/2)^n$ and the additivity that

$$\mathbf{cov}(\psi(Z_{T^n}), \phi(Z_{T^n})) = \varrho(\psi^*, \phi^*).$$

In other words, if the typical grain is sufficiently bounded, the exact covariances for the Boolean model on the torus coincide with the asymptotic covariances for the corresponding Boolean model in $\mathbb{R}^n$. This provides a way to compute estimates for the asymptotic covariances via simulations on the torus.

12.4 Central Limit Theorems

In this section we consider the asymptotic behaviour of the distributions of geometric functionals or of vectors of geometric functionals for growing observation window. Recall that a sequence of m-dimensional random vectors $(Y_i)_{i\in\mathbb{N}}$ converges in distribution to an m-dimensional random vector Y if

$$\lim_{i\to\infty} \mathbf{P}(Y_i \le x) = \mathbf{P}(Y \le x)$$

for all $x \in \mathbb{R}^m$ for which $y \mapsto \mathbf{P}(Y \le y)$ is continuous at x. (Here the relation $\le$ is to be understood componentwise.) In this case we write $Y_i \xrightarrow{d} Y$ (as $i \to \infty$). We are not only interested in the convergence in distribution but also in error bounds. In order to measure the distance between the distributions of two m-dimensional random vectors Z_1, Z_2, we use the d_3-metric which is given by

$$d_3(Z_1, Z_2) = \sup_{h\in\mathscr{H}_m} |\mathbf{E}h(Z_1) - \mathbf{E}h(Z_2)|,$$

where $\mathscr{H}_m$ is the set of all C^3-functions $h : \mathbb{R}^m \to \mathbb{R}$ such that the absolute values of the second and the third partial derivatives are bounded by one. For two random variables Z_1, Z_2 we consider the Wasserstein distance

$$d_W(Z_1, Z_2) = \sup_{h\in\mathrm{Lip}(1)} |\mathbf{E}h(Z_1) - \mathbf{E}h(Z_2)|,$$

where Lip(1) is the set of all functions $h : \mathbb{R} \to \mathbb{R}$ whose Lipschitz constant is at most one. Note that convergence in the d_3-distance or in the Wasserstein distance implies convergence in distribution.

For the quantitative bounds we assume that there is a constant $\varepsilon \in (0, 1]$ such that

$$\mathbf{E}V_i(Z_0)^{3+\varepsilon} < \infty, \qquad i = 0, \dots, n. \tag{12.31}$$

We begin with a multivariate central limit theorem for a vector of geometric functionals.

Theorem 12.8 *Assume that* (12.2) *is satisfied, let* $\Psi := (\psi_1, \dots, \psi_m)$ *for geometric functionals* $\psi_1, \dots, \psi_m$*, and let* N_Σ *be an m-dimensional centred Gaussian random vector with covariance matrix* $\Sigma = (\sigma(\psi_i, \psi_j))_{i,j=1,\dots,m}$. *Then*

$$\frac{1}{\sqrt{V_n(W)}}\big(\Psi(Z \cap W) - \mathbf{E}\Psi(Z \cap W)\big) \xrightarrow{d} N_\Sigma \quad as \quad r(W) \to \infty.$$

If (12.31) *holds, there is a constant* $C_{\psi_1,\dots,\psi_m}$ *depending on* $\psi_1, \dots, \psi_m$*,* Θ *and* ε *such that*

$$d_3\Big(\frac{1}{\sqrt{V_n(W)}}\big(\Psi(Z \cap W) - \mathbf{E}\Psi(Z \cap W)\big), N_\Sigma\Big) \leq \frac{C_{\psi_1,\dots\psi_m}}{r(W)^{\min\{\varepsilon n/2,1\}}}$$

for $W \in \mathcal{K}^n$ *with* $r(W) \geq 1$.

This result was proved in [16, Theorem 9.1] by using the Stein-Malliavin method and a truncation argument.

As tensors can be interpreted as vectors, we can define convergence of tensor valued random elements and their d_3-distance via convergence and d_3-distance for random vectors. Since the components of $\Phi_m^{0,s}$ are geometric functionals, Theorem 12.8 can be applied to the translation invariant Minkowski tensors.

Corollary 12.9 *Assume that* (12.2) *holds, let* $s \in \mathbb{N}$ *and* $m \in \{0, \dots, n-1\}$*, and let* $N_m^{0,s}$ *be a random element in* $\mathbb{T}^s$ *such that each component is a centred Gaussian random variable and*

$$\mathbf{cov}\left((N)_{i_1,\dots,i_s}, (N)_{j_1,\dots,j_s}\right) = \sigma\left((\Phi_m^{0,s})_{i_1,\dots,i_s}, (\Phi_m^{0,s})_{j_1,\dots,j_s}\right)$$

for $i_1, \dots, i_s, j_s, \dots, j_s \in \{1, \dots, n\}$. *Then*

$$\frac{1}{\sqrt{V_n(W)}}\big(\Phi_m^{0,s}(Z \cap W) - \mathbf{E}\Phi_m^{0,s}(Z \cap W)\big) \xrightarrow{d} N_m^{0,s} \qquad as\ r(W) \to \infty.$$

If (12.31) *holds, there is a constant* $C_{s,m}$ *depending on s, m,* Θ *and* ε *such that*

$$d_3\Big(\frac{1}{\sqrt{V_n(W)}}(\Phi_m^{0,s}(Z \cap W) - \mathbf{E}\Phi_m^{0,s}(Z \cap W)), N_m^{0,s}\Big) \leq \frac{C_{s,m}}{r(W)^{\min\{\varepsilon n/2,1\}}}$$

for $W \in \mathcal{K}^n$ *with* $r(W) \geq 1$.

In the multivariate case we assume translation invariance of the geometric functionals in order to ensure the existence of an asymptotic covariance matrix. In the univariate case this is not required since one can standardize be dividing by the standard deviation. For this reason, we can drop the assumption of translation invariance in the following univariate central limit theorem, which is taken from [16, Theorem 9.3].

Theorem 12.10 *Let* (12.2) *be satisfied, let* $\psi : \mathcal{R}^n \to \mathbb{R}$ *be measurable, additive and locally bounded. Assume that there are constants* $r_0 \geq 1$ *and* $\sigma_0 > 0$ *such that*

$$\frac{\mathbf{var}\, \psi(Z \cap W)}{V_n(W)} \geq \sigma_0 \tag{12.32}$$

for $W \in \mathcal{K}^n$ *with* $r(W) \geq r_0$, *and let* N *be a standard Gaussian random variable. Then*

$$\frac{\psi(Z \cap W) - \mathbf{E}\psi(Z \cap W)}{\sqrt{\mathbf{var}\, \psi(Z \cap W)}} \xrightarrow{d} N \qquad \text{as } r(W) \to \infty.$$

If, additionally, (12.31) *is satisfied, there is a constant* c_ψ *depending on* ψ, Θ, r_0, σ_0, *and* ε *such that*

$$d_W\left(\frac{\psi(Z \cap W) - \mathbf{E}\psi(Z \cap W)}{\sqrt{\mathbf{var}\, \psi(Z \cap W)}}, N\right) \leq \frac{c_\psi}{V_n(W)^{\min\{\varepsilon/2, 1/2\}}}$$

for $W \in \mathcal{K}^n$ *with* $r(W) \geq r_0$.

Theorem 12.5 shows that (12.32) holds for a geometric functional ψ if the typical grain has a nonempty interior with positive probability and if there exist an integer $i \in \{0, \ldots, d\}$ and a constant $c > 0$ such that ψ is homogeneous of degree i and $\psi(K) \geq c \cdot r(K)^i$ for all $K \in \mathcal{K}^n$.

The results presented in this section generalize previous findings in [2, 3, 10, 11, 23, 32, 35], which only deal with volume, surface area or closely related functionals.

12.5 Boolean Model of Aligned Rectangles

In this section we assume that $n = 2$ and that the typical grain Z_0 is a deterministic rectangle of the form

$$K := \left[-\frac{a}{2}e_1, \frac{a}{2}e_1\right] + \left[-\frac{b}{2}e_2, \frac{b}{2}e_2\right] = \left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]$$

for some fixed $a, b > 0$, where $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Then $v_2 = ab$ and $v_1 = a + b$.

12.5.1 Asymptotic Variance $\sigma(V_2, V_2)$

For any $x = (x_1, x_2) \in \mathbb{R}^2$ we have

$$C_2(x) = V_2(K \cap (K + x)) = \mathbf{1}\{|x_1| \le a, |x_2| \le b\}(a - |x_1|)(b - |x_2|).$$

A change of variables and a symmetry argument imply that

$$\int \left(e^{\gamma C_2(x)} - 1\right) dx = 4v_2 H(\gamma v_2), \tag{12.33}$$

where the function $H : [0, \infty) \to [0, \infty)$ is defined by

$$H(r) := \int_0^1 \int_0^1 \left(e^{rst} - 1\right) ds\, dt = \sum_{k=1}^{\infty} \frac{r^k}{k!(k+1)^2}, \qquad r \ge 0. \tag{12.34}$$

Hence we obtain from Theorem 12.4 that

$$\sigma(V_2, V_2) = 4(1-p)^2 v_2 H(\gamma v_2), \tag{12.35}$$

where we recall that $p = 1 - e^{-\gamma v_2}$. The variance is visualized in Fig. 12.2a.

12.5.2 Asymptotic Covariance $\sigma(V_1, V_2)$

At this stage it is convenient to complement the definition (12.34) with the following easy to check formulae:

$$\int_0^1 \int_0^1 e^{rst} s\, ds\, dt = \frac{1}{r^2} e^r - \frac{1}{r^2} - \frac{1}{r}, \tag{12.36}$$

$$\int_0^1 \int_0^1 e^{rst} s^2\, ds\, dt = \frac{1}{r^2} e^r - \frac{1}{r^3} e^r + \frac{1}{r^3} - \frac{1}{2r}, \tag{12.37}$$

$$\int_0^1 \int_0^1 e^{rst} st\, ds\, dt = \frac{1}{r^2} e^r - \frac{1}{r^2} - \frac{1}{r} H(r) - \frac{1}{r}. \tag{12.38}$$

A consequence is

$$\int_0^1 \int_0^1 e^{rst}(st + s^2)\, ds\, dt = \frac{2}{r^2} e^r - \frac{1}{r^3} e^r + \frac{1}{r^3} - \frac{1}{r^2} - \frac{3}{2r} - \frac{1}{r} H(r). \tag{12.39}$$

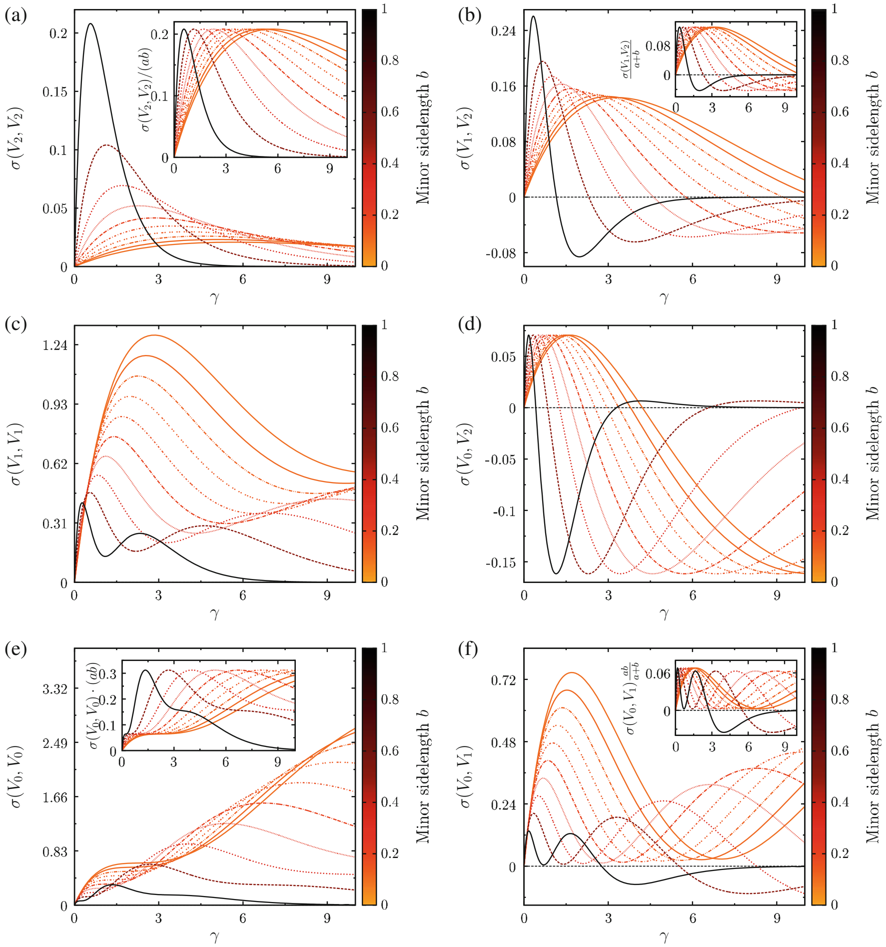

Fig. 12.2 Asymptotic covariances $\sigma(V_i, V_j)$ as a function of the intensity γ for Boolean models of aligned rectangles with side lengths a, b and varying aspect ratio b/a; we choose $a = 1$, hence $b \in (0, 1]$; see (12.35), (12.43), (12.45), (12.53), (12.56), and (12.59). The *insets* show covariances that are rescaled by ab or $a + b$ so that they only depend on γv_2 but not on the aspect ratio, which also holds for $\sigma(V_0, V_2)$ in (**d**) without rescaling

Now we use Theorem 12.4 (for $n = 2$ and $g \equiv 1$) to compute $\sigma(V_1, V_2)$. For any measurable and even functions $f : \mathbb{R}^2 \to [0, \infty)$ and $\tilde{f} : \mathbb{S}^1 \to [0, \infty)$ we have

$$\int f(y - z)\tilde{f}(u)\, N_{1,2}(d(y, u, z)) = a_1^+ + a_1^- + a_2^+ + a_2^-,$$

where

$$a_i^{\pm} := \frac{1}{2} \iint \mathbf{1}\{y \in A_i^{\pm}, z \in K\} f(y-z)\tilde{f}(e_i)\, \mathscr{H}^1(dy)\, dz, \qquad i \in \{1, 2\},$$

and $A_1^{\pm} := \{(x_1, x_2) \in K : x_1 = \pm a/2\}$, $A_2^{\pm} := \{(x_1, x_2) \in K : x_2 = \pm b/2\}$. By Fubini's theorem and a change of variables

$$a_1^{\pm} = \frac{\tilde{f}(e_1)}{2} \iint \mathbf{1}\{y \in A_1^{\pm}, y + z \in K\} f(z)\, \mathscr{H}^1(dy)\, dz.$$

For any $z = (z_1, z_2) \in K$ with $-a \le z_1 \le 0$ and $z_2 \ge 0$ we have

$$\int \mathbf{1}\{y \in A_1, y + z \in K\}\, \mathscr{H}^1(dy) = \mathscr{H}^1([-b/2, b/2 - z_2]) = b - z_2 = b - |z_2|.$$

For $-a \le z_1 \le 0$ and $z_2 \le 0$ this integral takes the same value. Since the set of all z with $z_1 \notin [-a, 0]$ or $|z_2| > b$ does not contribute to a^+ while the set of all z with $z_1 \notin [0, a]$ or $|z_2| > b$ does not contribute to a^- it follows that

$$a_1^+ + a_1^- = \frac{\tilde{f}(e_1)}{2} \int \mathbf{1}\{|z_1| \le a, |z_2| \le b\} f(z_1, z_2)(b - |z_2|)\, d(z_1, z_2).$$

Using a similar result for $b_1^+ + b_1^-$ gives

$$\begin{aligned} &\int f(y-z)\tilde{f}(u)\, N_{1,2}(d(y, u, z)) \\ &\quad = \frac{\tilde{f}(e_1)}{2} \int \mathbf{1}\{|z_1| \le a, |z_2| \le b\} f(z_1, z_2)(b - |z_2|)\, d(z_1, z_2) \\ &\qquad + \frac{\tilde{f}(e_2)}{2} \int \mathbf{1}\{|z_1| \le a, |z_2| \le b\} f(z_1, z_2)(a - |z_1|)\, d(z_1, z_2). \end{aligned} \tag{12.40}$$

Inserting here $f(z) := e^{\gamma C_2(z)}$ and using a change of variables gives

$$\begin{aligned} &\int e^{\gamma C_2(y-z)} \tilde{f}(u)\, N_{1,2}(d(y, u, z)) \\ &\quad = 2ab^2 \tilde{f}(e_1) \int_0^1 \int_0^1 e^{\gamma a b y_1 y_2} y_2\, dy_1\, dy_2 + 2a^2 b \tilde{f}(e_2) \int_0^1 \int_0^1 e^{\gamma a b y_1 y_2} y_1\, dy_1\, dy_2. \end{aligned}$$

From (12.36) we obtain that

$$\int e^{\gamma C_2(y-z)}\tilde{f}(u)\, N_{1,2}(d(y,u,z)) = \tilde{f}(e_1)\left(\frac{2}{\gamma^2 a}e^{\gamma ab} - \frac{2}{\gamma^2 a} - \frac{2b}{\gamma}\right) + \tilde{f}(e_2)\left(\frac{2}{\gamma^2 b}e^{\gamma ab} - \frac{2}{\gamma^2 b} - \frac{2a}{\gamma}\right). \tag{12.41}$$

In the case $\tilde{f}(e_1) = \tilde{f}(e_2) = 1$ this yields

$$\int e^{\gamma C_2(x-y)}\, N_{1,2}(d(x,u,y)) = 2v_1\left(\frac{1}{\gamma^2 v_2}e^{\gamma v_2} - \frac{1}{\gamma^2 v_2} - \frac{1}{\gamma}\right). \tag{12.42}$$

Inserting this result together with (12.33) into the formula of Theorem 12.4 yields

$$\sigma(V_1,V_2) = 2(1-p)^2 v_1\left[\frac{1}{\gamma v_2}(e^{\gamma v_2} - 1) - 1 - 2\gamma v_2 H(\gamma v_2)\right], \tag{12.43}$$

which is visualized in Fig. 12.2b.

12.5.3 Asymptotic Covariance $\sigma(V_0, V_2)$

Next we use (12.20) to compute $\sigma(V_0,V_2)$, starting with the observation

$$h(K,-e_1) = h(K,e_1) = \tfrac{a}{2}, \qquad h(K,-e_2) = h(K,e_2) = \tfrac{b}{2}.$$

Therefore we obtain from (12.41)

$$\int \bar{h}(u)e^{\gamma C_2(y-z)}\, N_{1,2}(d(y,u,z)) = \frac{2}{\gamma^2}e^{\gamma v_2} - \frac{2}{\gamma^2} - \frac{2v_2}{\gamma}. \tag{12.44}$$

To evaluate

$$v_{1,1} = \int h(K,u)\, \Psi_1(K;du)$$

we split the integration according to $u \in \{-e_1, e_1, -e_2, e_2\}$. As all four integrals yield the same value $ab/4$, we get $v_{1,1} = v_2$. Summarizing, we obtain from (12.20)

$$\sigma(V_0,V_2) = p(1-p) - 4(1-p)^2\gamma v_2(1 - 1\gamma v_2)H(\gamma v_2) - 4(1-p)^2\big((1-p)^{-1} - 1 - \gamma v_2\big),$$

that is

$$\sigma(V_0, V_2) = (1-p)\,[2(1-p)\gamma v_2 - 3p - (1-p)\gamma v_2(4-2\gamma v_2)H(\gamma v_2)]\,. \tag{12.45}$$

Figure 12.2d visualizes this asymptotic covariance.

12.5.4 Asymptotic Variance $\sigma(V_1, V_1)$

Next we turn to $\sigma(V_1, V_1)$ as given by (12.19) for $n = 2$. Some of our calculations will also be required to compute $\sigma(V_0, V_1)$ and $\sigma(V_0, V_0)$. We have

$$C_1(x;\bar{h}) = \frac{a}{4}\int \mathbf{1}\{y - x \in K^\circ, y \in A_1^+ \cup A_1^-\}\,\mathscr{H}^1(dy) + \frac{b}{4}\int \mathbf{1}\{y - x \in K^\circ, y \in A_2^+ \cup A_2^-\}\,\mathscr{H}^1(dy)$$

and a straightforward calculation (left to the reader) yields

$$C_1(x;\bar{h}) = \mathbf{1}\{|x_1| \le a, |x_2| \le b\}\Big(\frac{a}{4}(b - |x_2|) + \frac{b}{4}(a - |x_1|)\Big) \tag{12.46}$$

as well as

$$C_1(x) = \tfrac{1}{2}\mathbf{1}\{|x_1| \le a, |x_2| \le b\}((a - |x_1|) + (b - |x_2|)).$$

From $C_1(x;\bar{h}) = C_1(-x;\bar{h})$ (see (12.46)) and (12.40) (with $f(x) := e^{\gamma C_2(x)}$ and $\tilde{f} \equiv 1$) it follows that

$$\int e^{\gamma C_2(y-z)} C_1(y - z;\bar{h})\, N_{1,2}(d(z,u,y)) = J_1 + J_2,$$

where

$$J_1 := \frac{a}{8}\int_{[-a,a]\times[-b,b]} e^{\gamma(a-|z_1|)(b-|z_2|)}((a - |z_1|) + (b - |z_2|))(b - |z_2|)\, d(z_1, z_2)$$

and J_2 is defined similarly. We have

$$\begin{aligned}
J_1 &= \frac{a}{2}\int \mathbf{1}\{0 \le z_1 \le a, 0 \le z_2 \le b\} e^{\gamma z_1 z_2}(z_1+z_2)z_2\, d(z_1,z_2) \\
&= \frac{a^2b^2}{2}\int_0^1\int_0^1 e^{\gamma abst}(as+bt)t\, ds\, dt \\
&= \frac{a^3b^2}{2}\int_0^1\int_0^1 e^{\gamma abst} st\, ds\, dt + \frac{a^2b^3}{2}\int_0^1\int_0^1 e^{\gamma abst} t^2\, ds\, dt.
\end{aligned}$$

Together with the analogous formula for J_2 this yields

$$\int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\, N_{1,2}(d(z,u,y)) = \frac{v_1 v_2^2}{2}\int_0^1\int_0^1 e^{\gamma v_2 st}(st+t^2)\, ds\, dt.$$

Now we can use (12.39) with $r = \gamma v_2$ to obtain

$$\begin{aligned}
&\int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\, N_{1,2}(d(z,u,y)) \\
&\quad = \frac{v_1}{\gamma^2} e^{\gamma v_2} - \frac{v_1}{2\gamma^3 v_2} e^{\gamma v_2} + \frac{v_1}{2\gamma^3 v_2} - \frac{v_1}{2\gamma^2} - \frac{3 v_1 v_2}{4\gamma} - \frac{v_1 v_2}{2\gamma} H(\gamma v_2).
\end{aligned} \tag{12.47}$$

Similarly,

$$\begin{aligned}
&\int e^{\gamma C_2(y-z)} C_1(y-z)\, N_{1,2}(d(z,u,y)) \\
&\quad = \int \mathbf{1}\{0 \le z_1 \le a, 0 \le z_2 \le b\} e^{\gamma z_1 z_2}(z_1+z_2)^2\, d(z_1,z_2) \\
&\quad = ab\int_0^1\int_0^1 e^{\gamma abst}(as+bt)^2\, ds\, dt \\
&\quad = ab(a^2+b^2)\int_0^1\int_0^1 e^{\gamma abst} s^2\, ds\, dt + 2a^2b^2\int_0^1\int_0^1 e^{\gamma abst} st\, ds\, dt.
\end{aligned}$$

It follows from (12.37) and (12.38) that the latter sum equals

$$\begin{aligned}
&ab(a^2+b^2)\Big(\frac{1}{\gamma^2 a^2 b^2} e^{\gamma v_2} - \frac{1}{\gamma^3 a^3 b^3} e^{\gamma v_2} + \frac{1}{\gamma^3 a^3 b^3} - \frac{1}{2\gamma ab}\Big) \\
&\quad + 2a^2b^2\Big(\frac{1}{\gamma^2 a^2 b^2} e^{\gamma ab} - \frac{1}{\gamma^2 a^2 b^2} - \frac{1}{\gamma ab} H(\gamma ab) - \frac{1}{\gamma ab}\Big).
\end{aligned}$$

Therefore

$$\gamma^2 \int e^{\gamma C_2(y-z)} C_1(y-z)\, N_{1,2}(d(z,u,y))$$
$$= (a^2+b^2)\Big(\frac{1}{v_2}e^{\gamma v_2} - \frac{1}{\gamma v_2^2}e^{\gamma v_2} + \frac{1}{\gamma v_2^2} - \frac{\gamma}{2}\Big)$$
$$+ 2e^{\gamma v_2} - 2 - 2v_2\gamma(H(\gamma v_2)+1).$$

To proceed, we need to compute the integrals

$$I_1^{++} := \int \mathbf{1}\{y \in A_1^+, z \in A_1^+\} e^{\gamma C_2(y-z)}\, \mathscr{H}^1(dy)\, \mathscr{H}^1(dz), \tag{12.48}$$

$$I_1^{+-} := \int \mathbf{1}\{y \in A_1^+, z \in A_1^-\} e^{\gamma C_2(y-z)}\, \mathscr{H}^1(dy)\, \mathscr{H}^1(dz),$$

$$I_{1,2}^{+} := \int \mathbf{1}\{y \in A_1^+, z \in A_2^+\} e^{\gamma C_2(y-z)}\, \mathscr{H}^1(dy)\, \mathscr{H}^1(dz) \tag{12.49}$$

as well as I_2^{++} (resp. I_2^{+-}), arising from I_1^{++} (resp. I_1^{+-}) by replacing (A_1^+, A_1^+) (resp. (A_1^+, A_1^-)) with (A_2^+, A_2^+) (resp. (A_2^+, A_2^-)). A straightforward calculation gives

$$I_1^{++} = \frac{2b}{\gamma a}e^{\gamma ab} - \frac{2}{\gamma^2 a^2}e^{\gamma ab} + \frac{2}{\gamma^2 a^2}, \quad I_2^{++} = \frac{2a}{\gamma b}e^{\gamma ab} - \frac{2}{\gamma^2 b^2}e^{\gamma ab} + \frac{2}{\gamma^2 b^2}, \tag{12.50}$$

$$I_1^{+-} = b^2, \qquad I_2^{+-} = a^2, \tag{12.51}$$

$$I_{1,2}^{+} = ab(H(\gamma ab)+1). \tag{12.52}$$

We prove here (12.50). The proof of (12.51) and (12.52) is even simpler. By the parametrisation $y = (a/2, s)$ with $s \in [-b/2, b/2]$ for $y \in A_1^+$ and $z = (a/2, t)$ with $t \in [-b/2, b/2]$ for $z \in A_2^+$ we get

$$I_1^{++} = \int_{-b/2}^{b/2}\int_{-b/2}^{b/2} e^{\gamma a(b-|s-t|)}\, ds\, dt.$$

Splitting the domain of integration into $s < t$ and $s \geq t$ yields

$$I_1^{++} = 2e^{\gamma ab}\int_{-b/2}^{b/2} e^{\gamma at}\int_t^{b/2} e^{-\gamma as}\, ds\, dt$$
$$= \frac{2}{\gamma a}e^{\gamma ab}\int_{-b/2}^{b/2}\big(1 - e^{-\gamma ab/2}e^{\gamma at}\big)\, dt.$$

Continuing this calculation gives

$$I_1^{++} = \frac{2b}{\gamma a} e^{\gamma ab} - \frac{2}{\gamma^2 a^2} e^{\gamma ab/2} \big(e^{\gamma ab/2} - e^{-\gamma ab/2} \big)$$

and hence the first identity in (12.50). The second follows by symmetry.

By symmetry arguments we have

$$\int e^{\gamma C_2(x-y)} N_{1,1}(d(x,u,y,v)) = 2\tfrac{1}{4} I_1^{++} + 2\tfrac{1}{4} I_2^{++} + 2\tfrac{1}{4} I_1^{+-} + 2\tfrac{1}{4} I_2^{+-} + 8\tfrac{1}{4} I_{1,2}^{+},$$

so that (12.50)–(12.52) yield

$$\int e^{\gamma C_2(x-y)} N_{1,1}(d(x,u,y,v)) = \frac{b}{\gamma a} e^{\gamma ab} - \frac{1}{\gamma^2 a^2} e^{\gamma ab} + \frac{1}{\gamma^2 a^2} + \frac{a}{\gamma b} e^{\gamma ab} - \frac{1}{\gamma^2 b^2} e^{\gamma ab} + \frac{1}{\gamma^2 b^2} + \frac{a^2+b^2}{2} + 2ab(H(\gamma ab)+1).$$

Therefore,

$$\gamma \int e^{\gamma C_2(x-y)} N_{1,1}(d(x,u,y,v)) = \frac{a^2+b^2}{v_2} e^{\gamma v_2} - \frac{a^2+b^2}{\gamma v_2^2} e^{\gamma v_2} + \frac{a^2+b^2}{\gamma v_2^2} + \gamma \frac{a^2+b^2}{2} + 2\gamma v_2 (H(\gamma v_2)+1),$$

so that

$$\begin{aligned} &\gamma^2 \int e^{\gamma C_2(y-z)} C_1(y-z) N_{1,2}(d(z,u,y)) + \gamma \int e^{\gamma C_2(x-y)} N_{1,1}(d(x,u,y,v)) \\ &= (a^2+b^2)\Big(\frac{1}{v_2} e^{\gamma v_2} - \frac{1}{\gamma v_2^2} e^{\gamma v_2} + \frac{1}{\gamma v_2^2} - \frac{\gamma}{2} \Big) + 2e^{\gamma v_2} - 2 \\ &\quad + \frac{a^2+b^2}{v_2} e^{\gamma v_2} - \frac{a^2+b^2}{\gamma v_2^2} e^{\gamma v_2} + \frac{a^2+b^2}{\gamma v_2^2} + \gamma \frac{a^2+b^2}{2} \\ &= 2(a^2+b^2)\Big(\frac{1}{v_2} e^{\gamma v_2} - \frac{1}{\gamma v_2^2} e^{\gamma v_2} + \frac{1}{\gamma v_2^2} \Big) + 2e^{\gamma v_2} - 2. \end{aligned}$$

Now we can conclude from (12.19), (12.33) and (12.42) that

$$\begin{aligned} \sigma(V_1, V_1) &= 4(1-p)^2 \gamma^2 v_1^2 v_2 H(\gamma v_2) - 4(1-p)^2 v_1^2 \gamma^2 \Big(\frac{1}{\gamma^2 v_2} e^{\gamma v_2} - \frac{1}{\gamma^2 v_2} - \frac{1}{\gamma} \Big) \\ &\quad + (1-p)^2 2(a^2+b^2) \Big(\frac{1}{v_2} e^{\gamma v_2} - \frac{1}{\gamma v_2^2} e^{\gamma v_2} + \frac{1}{\gamma v_2^2} \Big) \\ &\quad + (1-p)^2 (2e^{\gamma v_2} - 2), \end{aligned}$$

that is

$$\sigma(V_1,V_1) = (1-p)\Big[2p + 4(1-p)\gamma^2 v_1^2 v_2 H(\gamma v_2) \\ - 4\gamma^2 v_1^2\Big(\frac{p}{\gamma^2 v_2} - \frac{1-p}{\gamma}\Big) + 2(a^2+b^2)\Big(\frac{1}{v_2} - \frac{p}{\gamma v_2^2}\Big)\Big], \tag{12.53}$$

which is shown in Fig. 12.2c.

12.5.5 Asymptotic Covariance $\sigma(V_0, V_1)$

We now turn to $\sigma(V_0, V_1)$. It follows from (12.47) that

$$- 2\gamma^3 \int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\, N_{1,2}(d(z,u,y)) \\ = -2\gamma v_1 e^{\gamma v_2} + \frac{v_1}{v_2} e^{\gamma v_2} - \frac{v_1}{v_2} + \gamma v_1 + \frac{3\gamma^2 v_1 v_2}{2} + \gamma^2 v_1 v_2 H(\gamma v_2).$$

Furthermore, from (12.42) and $v_{1,1} = v_2$ we get

$$(\gamma^3 v_{1,1} - \gamma^2) \int e^{\gamma C_2(y-z)}\, N_{1,2}(d(z,u,y)) \\ = 2v_1\gamma^3 v_2\Big(\frac{1}{\gamma^2 v_2} e^{\gamma v_2} - \frac{1}{\gamma^2 v_2} - \frac{1}{\gamma}\Big) - 2v_1\gamma^2\Big(\frac{1}{\gamma^2 v_2} e^{\gamma v_2} - \frac{1}{\gamma^2 v_2} - \frac{1}{\gamma}\Big) \\ = 2\gamma v_1 e^{\gamma v_2} - 2\gamma v_1 - 2\gamma^2 v_1 v_2 - \frac{2v_1}{v_2} e^{\gamma v_2} + \frac{2v_1}{v_2} + 2\gamma v_1.$$

From (12.44) and $\overline{\Psi}(1) = v_1$ we deduce that

$$2\gamma^3 \overline{\Psi}_1(1) \int e^{\gamma C_2(y-z)} \bar{h}(u)\, N_{1,2}(d(z,u,y)) = 2\gamma^3 v_1\Big(\frac{2}{\gamma^2} e^{\gamma v_2} - \frac{2}{\gamma^2} - \frac{2v_2}{\gamma}\Big) \\ = 4\gamma v_1 e^{\gamma v_2} - 4\gamma v_1 - 4\gamma^2 v_1 v_2.$$

Summarizing the previous formulae we arrive at

$$(1-p)^2 \int \big(\chi'(y-z) + 2\gamma^3 \overline{\Psi}_1(1) e^{\gamma C_2(y-z)} \bar{h}(u)\big)\, N_{1,2}(d(z,u,y)) \\ = (1-p)\gamma v_1 \left[1 + \left(3 - \frac{1}{\gamma v_2}\right) p + (1-p)\gamma v_2 \left(H(\gamma v_2) - \frac{9}{2}\right)\right]. \tag{12.54}$$

Next we consider

$$I := \int e^{\gamma C_2(y-z)} \bar{h}(u)\, N_{1,1}(d(y,u,z,v)).$$

Then $I = I_1 + I_2$, where

$$I_1 := \frac{a}{8} \int \mathbf{1}\{y \in A_1^+ \cup A_1^-\} e^{\gamma C_2(y-z)}\, \mathcal{H}^1(dy)\, \mathcal{H}^1(dz),$$
$$I_2 := \frac{b}{8} \int \mathbf{1}\{y \in A_2^+ \cup A_2^-\} e^{\gamma C_2(y-z)}\, \mathcal{H}^1(dy)\, \mathcal{H}^1(dz).$$

By symmetry,

$$I_1 = 2\frac{a}{8} I_1^{++} + 2\frac{a}{8} I_1^{+-} + 4\frac{a}{8} I_{1,2}^{+}, \qquad I_2 = 2\frac{b}{8} I_2^{++} + 2\frac{b}{8} I_2^{+-} + 4\frac{a}{8} I_{1,2}^{+},$$

where the occurring integrals have been defined by (12.48)–(12.49). The formulae (12.50)–(12.52) yield

$$I = \Big(\frac{a}{2\gamma} + \frac{b}{2\gamma}\Big) e^{\gamma ab} - \Big(\frac{1}{2\gamma^2 a} + \frac{1}{2\gamma^2 b}\Big) e^{\gamma ab} + \frac{1}{2\gamma^2 a} + \frac{1}{2\gamma^2 b}$$
$$+ \frac{ab^2}{4} + \frac{a^2 b}{4} + \Big(\frac{a^2 b}{2} + \frac{ab^2}{2}\Big)(H(\gamma ab) + 1),$$

that is

$$I = \frac{v_1}{2\gamma} e^{\gamma v_2} - \frac{v_1}{2\gamma^2 v_2} e^{\gamma v_2} + \frac{v_1}{2\gamma^2 v_2} + \frac{3 v_1 v_2}{4} + \frac{v_1 v_2}{2} H(\gamma v_2).$$

It follows that

$$- 2(1-p)^2 \gamma^2 \int e^{\gamma C_2(y-z)} \bar{h}(u)\, N_{1,1}(d(y,u,z,v))$$
$$= (1-p)\gamma v_1 \left[\frac{p}{\gamma v_2} - 1 - \big(\tfrac{3}{2} + H(\gamma v_2)\big)(1-p)\gamma v_2 \right]. \qquad (12.55)$$

Now we conclude from (12.21) and (12.33) that

$$\sigma(V_0, V_1) = (1-p)\gamma v_1 \left[1 - p + 4(1-p)\gamma v_2 (1 - \gamma v_2) H(\gamma v_2)\right] + c_{1,2} + c_{1,1},$$

where $c_{1,2}$ is given by the right-hand side of (12.54) and $c_{1,1}$ is given by the right-hand side of (12.55). Thus, we derive

$$\sigma(V_0, V_1) = (1-p)\gamma v_1[1 + 2p + (1-p)\gamma v_2(4(1-\gamma v_2)H(\gamma v_2) - 6)]. \tag{12.56}$$

The asymptotic covariance is plotted in Fig. 12.2f.

12.5.6 Asymptotic Variance $\sigma(V_0, V_0)$

Finally, we determine $\sigma(V_0, V_0)$, as given by (12.22). From (12.40) and (12.46) we get

$$\begin{aligned}
&\int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\bar{h}(u)\, N_{1,2}(d(z,u,y)) \\
&\quad = \frac{a}{4}\int \mathbf{1}\{|z_1| \le a, |z_2| \le b\} e^{\gamma |z_1||z_2|}|z_2|\Big(\frac{a}{4}|z_2| + \frac{b}{4}|z_1|\Big)\, d(z_1, z_2) \\
&\qquad + \frac{b}{4}\int \mathbf{1}\{|z_1| \le a, |z_2| \le b\} e^{\gamma |z_1||z_2|}|z_1|\Big(\frac{a}{4}|z_2| + \frac{b}{4}|z_1|\Big)\, d(z_1, z_2) \\
&\quad = \frac{a}{4}\int_0^b \int_0^a e^{\gamma z_1 z_2} z_2(az_2 + bz_1)\, dz_1\, dz_2 + \frac{b}{4}\int_0^b \int_0^a e^{\gamma z_1 z_2} z_1(az_2 + bz_1)\, dz_1\, dz_2 \\
&\quad = \frac{a^2b^2}{4}\int_0^1 \int_0^1 e^{\gamma abst} t(abt + abs)\, ds\, dt + \frac{a^2b^2}{4}\int_0^1 \int_0^1 e^{\gamma abst} s(abt + abs)\, ds\, dt \\
&\quad = \frac{a^3b^3}{2}\int_0^1 \int_0^1 e^{\gamma abst}(s^2 + st)\, ds\, dt.
\end{aligned}$$

Using now (12.39), we obtain

$$\begin{aligned}
&4\gamma^4 \int e^{\gamma C_2(y-z)} C_1(y-z;\bar{h})\bar{h}(u)\, N_{1,2}(d(z,u,y)) \\
&\quad = 4\gamma^2 v_2 e^{\gamma v_2} - 2\gamma e^{\gamma v_2} + 2\gamma - 2\gamma^2 v_2 - 3\gamma^3 v_2^2 - 2\gamma^3 v_2^2 H(\gamma v_2).
\end{aligned}$$

From (12.44) we see that

$$\begin{aligned}
&(-4\gamma^4 v_2 + 4\gamma^3)\int e^{\gamma C_2(y-z)}\bar{h}(u)\, N_{1,2}(d(z,u,y)) \\
&\quad = (-4\gamma^4 v_2 + 4\gamma^3)\Big(\frac{2}{\gamma^2}e^{\gamma v_2} - \frac{2}{\gamma^2} - \frac{2v_2}{\gamma}\Big) \\
&\quad = -8\gamma^2 v_2 e^{\gamma v_2} + 8\gamma^2 v_2 + 8\gamma^3 v_2^2 + 8\gamma e^{\gamma v_2} - 8\gamma - 8\gamma^2 v_2.
\end{aligned}$$

Summarizing, we have

$$(1-p)^2 \int \bar{h}(u)\chi''(y-z)\, N_{1,2}(d(z,u,y))$$
$$= (1-p)\gamma[6p - 4\gamma v_2 + (1-p)\gamma v_2(5\gamma v_2 - 2H(\gamma v_2)\gamma v_2 - 2)]. \qquad (12.57)$$

Next we note that

$$\int e^{\gamma C_2(y-z)}\bar{h}(u)\bar{h}(v)\, N_{1,1}(d(y,u,z,v))$$
$$= 2\frac{a^2}{16}I_1^{++} + 2\frac{a^2}{16}I_1^{+-} + 2\frac{b^2}{16}I_2^{++} + 2\frac{b^2}{16}I_2^{+-} + 8\frac{ab}{16}I_{1,2}^{+},$$

where $I_1^{++}, I_1^{+-}, I_{1,2}^{+}, I_2^{++}, I_2^{+-}$ have been defined by (12.48)–(12.49). The formulae (12.50)–(12.52) give

$$\int e^{\gamma C_2(y-z)}\bar{h}(u)\bar{h}(v)\, N_{1,1}(d(y,u,z,v))$$
$$= \frac{a^2}{8}\Big(\frac{2b}{\gamma a}e^{\gamma ab} - \frac{2}{\gamma^2 a^2}e^{\gamma ab} + \frac{2}{\gamma^2 a^2}\Big) + \frac{a^2b^2}{8}$$
$$+ \frac{b^2}{8}\Big(\frac{2a}{\gamma b}e^{\gamma ab} - \frac{2}{\gamma^2 b^2}e^{\gamma ab} + \frac{2}{\gamma^2 b^2}\Big)$$
$$+ \frac{a^2b^2}{8} + \frac{a^2b^2}{2}(H(\gamma ab)+1)$$
$$= \frac{v_2}{2\gamma}e^{\gamma v_2} - \frac{1}{2\gamma^2}e^{\gamma v_2} + \frac{1}{2\gamma^2} + \frac{v_2^2}{4} + \frac{v_2^2}{2}H(\gamma v_2) + \frac{v_2^2}{2}.$$

Therefore

$$4(1-p)^2\gamma^3 \int e^{\gamma C_2(y-z)}\bar{h}(u)\bar{h}(v)\, N_{1,1}(d(y,u,z,v))$$
$$= (1-p)\gamma\left[(2H(\gamma v_2)+3)(1-p)(\gamma v_2)^2 + 2\gamma v_2 - 2p\right]. \qquad (12.58)$$

Now we conclude from (12.22) and (12.33) that

$$\sigma(V_0,V_0) = (1-p)\gamma\big[1 - 2p + (2p-3)\gamma v_2$$
$$+ 4(1-p)(1-\gamma v_2)^2\gamma v_2 H(\gamma v_2)\big] + d_{1,2} + d_{1,1},$$

where $d_{1,2}$ is given by the right-hand side of (12.57) and $d_{1,1}$ is given by the right-hand side of (12.58). Thus, we finally derive

$$\sigma(V_0, V_0) = (1-p)\gamma\big[1 + 2p + (4p-7)\gamma v_2 \\ + 4(1-p)\gamma v_2\big(2\gamma v_2 + (1-\gamma v_2)^2 H(\gamma v_2)\big)\big], \qquad (12.59)$$

which is plotted in Fig. 12.2e.

12.5.7 Invariance Properties

The reader might have noticed that the asymptotic covariances $\sigma(V_i, V_j)$ (with the exception of $\sigma(V_1, V_1)$) depend on the parameters γ, v_1, and v_2 in a specific way. In order to explain these invariance properties, let $Z_{a,b,\gamma}$ denote the Boolean model with grains $K = [0, a] \times [0, b]$ and intensity γ. By applying to each rectangle of the underlying Poisson process the linear transformation $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (ax_1, bx_2)$, one obtains the distributional identity

$$Z_{a,b,\gamma} \overset{d}{=} T_{a,b} Z_{1,1,ab\gamma}.$$

Together with the fact that $V_i(T_{a,b}A) = (ab)^{i/2} V_i(A)$, for $A \in \mathscr{R}^2$ and $i \in \{0, 2\}$, we see that

$$\begin{aligned}
\sigma(V_i, V_j) &= \lim_{r(W)\to\infty} \frac{\mathbf{cov}(V_i(Z_{a,b,\gamma} \cap W), V_j(Z_{a,b,\gamma} \cap W))}{V_2(W)} \\
&= \lim_{r(W)\to\infty} \frac{\mathbf{cov}(V_i(T_{a,b}Z_{1,1,ab\gamma} \cap W), V_j(T_{a,b}Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \\
&= (ab)^{i/2+j/2-1} \lim_{r(W)\to\infty} \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap T_{a,b}^{-1}W), V_j(Z_{1,1,ab\gamma} \cap T_{a,b}^{-1}W))}{V_2(T_{a,b}^{-1}W)} \\
&= v_2^{i/2+j/2-1} \lim_{r(W)\to\infty} \frac{\mathbf{cov}(V_i(Z_{1,1,\gamma v_2} \cap W), V_j(Z_{1,1,\gamma v_2} \cap W))}{V_2(W)},
\end{aligned}$$

for $i, j \in \{0, 2\}$. This shows that for all Boolean models of deterministic rectangles with fixed γv_2, the asymptotic covariances between volume and Euler characteristic are a power of v_2 times a constant depending on γv_2.

Next we investigate the invariance properties of $\sigma(V_0, V_1)$ and $\sigma(V_1, V_2)$. For $i \in \{1, 2\}$ we define

$$V_{1,e_i}(A) := \int \mathbf{1}\{u = \pm e_i\}\, \Psi_1(A; du), \qquad A \in \mathscr{R}^2,$$

which are again geometric functionals. If W is a rectangle with sides in the directions e_1 and e_2, which we can assume in the following, we have that

$$V_1(Z_{a,b,\gamma} \cap W) = V_{1,e_1}(Z_{a,b,\gamma} \cap W) + V_{1,e_2}(Z_{a,b,\gamma} \cap W).$$

By the same arguments as in the previous computation we obtain that, for $i \in \{0, 2\}$,

$$\begin{aligned}
\sigma(V_i, V_1) &= \lim_{r(W)\to\infty} \frac{\mathbf{cov}(V_i(Z_{a,b,\gamma} \cap W), V_1(Z_{a,b,\gamma} \cap W))}{V_2(W)} \\
&= \lim_{r(W)\to\infty} \Bigg\{ \frac{\mathbf{cov}(V_i(T_{a,b}Z_{1,1,ab\gamma} \cap W), V_{1,e_1}(T_{a,b}Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \\
&\qquad + \frac{\mathbf{cov}(V_i(T_{a,b}Z_{1,1,ab\gamma} \cap W), V_{1,e_2}(T_{a,b}Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \Bigg\} \\
&= \lim_{r(W)\to\infty} \Bigg\{ (ab)^{i/2-1} b \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap W), V_{1,e_1}(Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \\
&\qquad + (ab)^{i/2-1} a \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap W), V_{1,e_2}(Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \Bigg\}.
\end{aligned}$$

Using that the asymptotic covariances between V_i and V_{1,e_1} and between V_i and V_{1,e_2} are the same for the Boolean model $Z_{1,1,ab\gamma}$ due to symmetry, we conclude that

$$\begin{aligned}
\sigma(V_i, V_1) &= \lim_{r(W)\to\infty} (ab)^{i/2-1}(a+b) \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap W), V_{1,e_1}(Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \\
&= \lim_{r(W)\to\infty} \Bigg\{ (ab)^{i/2-1} \frac{(a+b)}{2} \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap W), V_{1,e_1}(Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \\
&\qquad + (ab)^{i/2-1} \frac{(a+b)}{2} \frac{\mathbf{cov}(V_i(Z_{1,1,ab\gamma} \cap W), V_{1,e_2}(Z_{1,1,ab\gamma} \cap W))}{V_2(W)} \Bigg\} \\
&= \lim_{r(W)\to\infty} v_2^{i/2-1} \frac{v_1}{2} \frac{\mathbf{cov}(V_i(Z_{1,1,\gamma v_2} \cap W), V_1(Z_{1,1,\gamma v_2} \cap W))}{V_2(W)}.
\end{aligned}$$

Thus, the asymptotic covariance between volume and surface area is v_1 times a constant depending on γv_2, while the covariance between Euler characteristic and surface area is $v_1 v_2^{-1}$ times a constant depending on γv_2.

Figure 12.2 summarizes the results of this section visually. It shows the asymptotic covariances $\sigma(V_i, V_j)$ as a function of the intensity γ for Boolean models of aligned rectangles for a variety of aspect ratios b/a.

12.6 Simulations of Boolean Models with Isotropic or Aligned Rectangles

Planar Boolean models with either squares or rectangles with aspect ratio $1/2$ as grains are simulated in a finite observation window. We study the variances and covariances of the intrinsic volumes as well as their relative frequency histograms weighted by the size of each bin. We compare the simulation results for aligned rectangles to the analytic formulae for the covariances in the previous Sect. 12.5. Moreover, we simulate rectangles with a uniform (isotropic) orientation distribution and find, e.g., for $\sigma(V_0, V_1)$ a qualitatively different behaviour.

Tensor valuation densities and the density of the Euler characteristic $\overline{V}_0$ of anisotropic Boolean models are studied in [13]. The same simulation procedure is applied here with even better statistics for reliable estimates of the second moments and the histograms of the intrinsic volumes.

The grain centres are random points, uniformly distributed within the simulation box. The union of the rectangles is computed using the Computational Geometry Algorithms Library (CGAL) [5]. The program PAPAYA then calculates the Minkowski functionals of the Boolean model [39].

For aligned rectangles, the covariances $\sigma(V_2, V_2)$, $\sigma(V_0, V_0)$, and $\sigma(V_0, V_2)$ as well as the rescaled covariances $\sigma(V_1, V_2)/(2a + 2b)$ and $\sigma(V_0, V_1)/(2a + 2b)$ are only functions of v_2 and γv_2, as shown in the previous Sect. 12.5. In other words, if the unit of area is chosen to be the area of a single grain $v_2 = a \cdot b = 1$ (so that the area of the typical grain does not depend on the aspect ratio), the rescaled covariances are independent of the aspect ratio. Therefore, we define in the following the unit of length by the square root of the area of a single grain.

Parts of this section are taken from the PhD thesis of one of the authors [18].

12.6.1 Variances and Covariances

The first moments of area or perimeter of a Boolean model are rather insensitive to the grain distribution. Indeed, if the unit of area is chosen to be the mean area of a single grain, the density of the area, i.e., the occupied area fraction, of the Boolean model is only a function of the intensity. Moreover, if the density of the perimeter in the asymptotic limit is divided by the mean perimeter of a single grain, it is also independent of the grain distribution [37, Theorem 9.1.4].

Does the same hold for the second moments? Is there a qualitatively different behaviour in the variances and covariances depending on whether the orientation distribution of the grains is isotropic or anisotropic? Which covariances or variances are invariant under affine transformations of the grain distributions?

Depending on the computational costs, for each different set of parameters we perform between $M_s = 21{,}000$ and 600,000 simulations of Boolean models with rectangles: at varying intensities γ, with aspect ratio 1 or $1/2$, and for rectangles either aligned w.r.t. the observation window or with an isotropic orientation distribution. The simulation box is a square with side length $L = 4a$, where periodic boundary conditions are applied. The number of grains within the simulation box is a random number and follows a Poisson distribution with mean $\gamma \cdot L^2$. To estimate the covariances, we simulate more than 5,800,000 samples of Boolean models including about 54,000,000 rectangles in total.

Because of the periodic boundary conditions, the covariances of this system coincide with the asymptotic covariances for the infinite volume system from Sect. 12.5, as we have pointed out in Sect. 12.3.3.

For each sample $m \in \{1, \ldots M_s\}$ of a Boolean model, we determine the intrinsic volumes $V_i^{(m)}$ $(i = 0, 1, 2)$. The sample covariance then provides an estimate for the covariance between the Minkowski functionals:

$$s(V_i, V_j) := \frac{1}{M_s - 1} \sum_{m=1}^{M_s} (V_i^{(m)} - \langle V_i \rangle)(V_j^{(m)} - \langle V_j \rangle)$$

using the sample mean $\langle V_i \rangle := \frac{1}{M_s} \sum_{m=1}^{M_s} V_i^{(m)}$ as an estimator for the expectation. In accordance with the definition of the asymptotic covariances in Theorem 12.3, the sample covariance is then divided by the size L^2 of the observation window. We finally use bootstrapping (with 1000 bootstrap samples) to estimate the mean and the error of the estimators.

Figure 12.3 shows the simulation results for the variances and covariances of the intrinsic volumes for an isotropic orientation distribution of the grains as well as for aligned rectangles. In the latter case, the simulation results are compared to the analytic results in (12.35), (12.43), (12.45), (12.53), (12.56), and (12.59). They are in excellent agreement.

The variances and covariances of the Minkowski functionals of overlapping rectangles exhibit a complex behaviour as functions of the intensity γ similar to the Boolean model with discs in [16, Sect. 7]. The variances of area and Euler characteristic apparently have one maximum and no other extrema. The variance of the perimeter has a global maximum and (at least) one local minimum and one local maximum. As expected, the three Minkowski functionals are positively correlated at low intensities γ, but at higher intensities there are also regimes where the area is anti-correlated to the Euler characteristic or the perimeter.

The covariance $\sigma(V_0, V_1)$ between the perimeter and the Euler characteristic shows a qualitatively different behaviour for rectangles with an isotropic orientation distribution when compared to aligned rectangles. There is a regime in the intensity γ (around the first local minimum) for which the rectangles with an isotropic orientation distribution are anti-correlated, while the aligned grains are positively correlated. In the same regime, also the discs in [16] are positively correlated like the aligned rectangles and in contrast to the rectangles with an isotropic orientation distribution. This is probably related to the fact that rotated rectangles can more easily form clusters with holes than aligned rectangles or discs. The zero-crossing of the expectation of the Euler characteristic χ for the rectangles with an isotropic

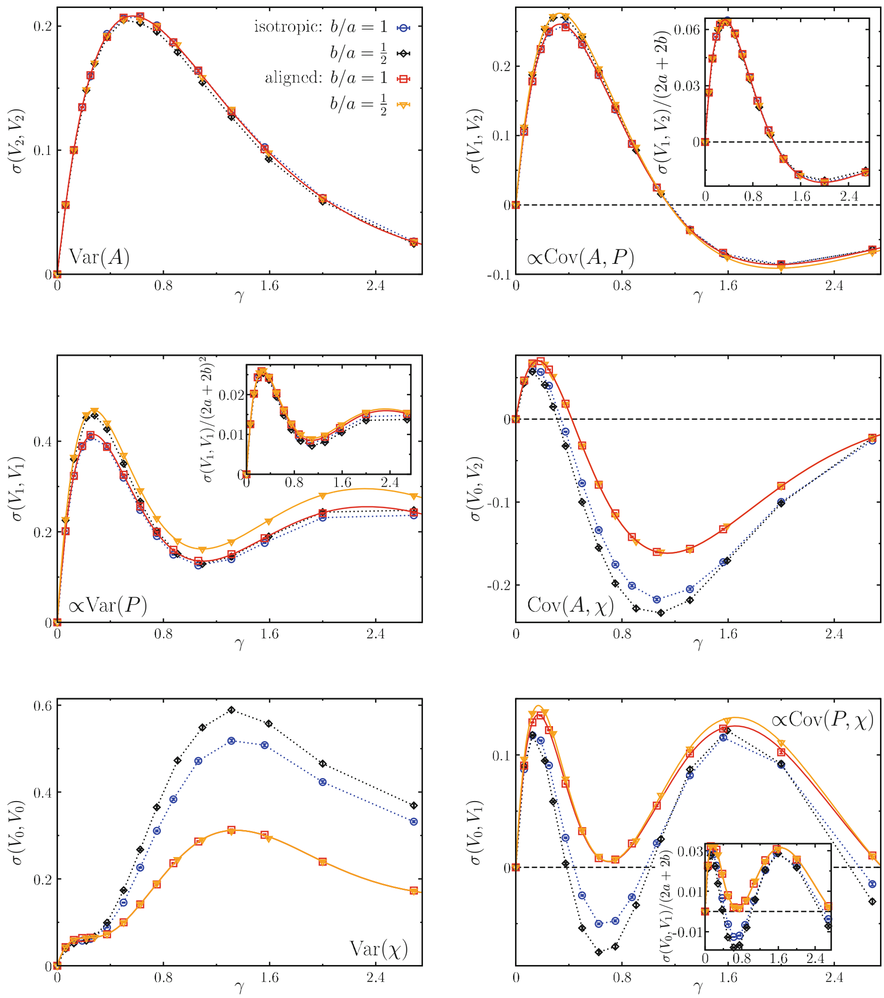

Fig. 12.3 Variances and covariances of the intrinsic volumes V_2 (area A), V_1 (proportional to perimeter P), and V_0 (Euler characteristic χ) of Boolean models as a function of the intensity γ. Depicted are both numerical estimates in finite observation windows with periodic boundary conditions (marks with *dotted lines* as guides to the eye) and analytic curves of the covariances (*solid lines*), see (12.35), (12.43), (12.45), (12.53), (12.56), and (12.59). Four different Boolean models are simulated: both for squares ($b/a = 1$) and rectangles ($b/a = 1/2$) either an isotropic orientation distribution is used or the grains are aligned with the x-direction. In the *insets*, the covariances and the variance of the perimeter of the Boolean model are rescaled by the perimeter of a single grain. In contrast to Fig. 12.2, the unit of area is the size of a single grain, that is $v_2 = ab = 1$

orientation distribution is within this regime. For the aligned rectangles, the zero-crossing of the mean value of χ is at the end of this regime, see [13].

The question remains whether or not the variances and covariances of area or rescaled perimeter of the Boolean model are independent of the grain distribution like the first moments of these functionals. Equations (12.35) and (12.43) show that at least for aligned rectangles the variance $\sigma(V_2, V_2)$ as well as the covariance $\sigma(V_1, V_2)$ divided by the perimeter of a single grain $(2a+2b)$ are indeed independent of the aspect ratio. The simulation results from Fig. 12.3 might suggest that this could also be valid for the isotropic orientation distributions. However, the variance $\sigma(V_2, V_2)$ and the rescaled covariance $\sigma(V_1, V_2)/(2a + 2b)$ do depend on the grain distribution, although only weakly for the models studied here. To show this, we evaluate (12.35) and (12.43) numerically and compare the covariances to those of the Boolean model with discs from [16]. Figure 12.4 shows that there is a weak but significant difference in the analytic curves of $\sigma(V_2, V_2)$ and $\sigma(V_1, V_2)$ for the two different models. The variance of the perimeter depends more clearly on the grain distribution. Even if it is rescaled by the perimeter of a single grain and even for aligned grains, the variance distinctly depends on the aspect ratio of the rectangles (except for small intensities γ). So, in contrast to the first moments of the area and rescaled perimeter of the Boolean model, the second moments in general depend on the grain distribution, e.g., the orientation distribution, even if this dependence may be weak. As expected, also the variance $\sigma(V_0, V_0)$ of the Euler characteristic as well as the covariances $\sigma(V_0, V_1)$ and $\sigma(V_0, V_2)$ depend on the grain distribution, see Fig. 12.4.

12.6.2 Central Limit Theorem

We also determine the histograms of the intrinsic volumes in a finite observation window, where the histograms are weighted by the total number of samples and the bin width. The histograms are then compared to the density of a standard normal distribution in order to numerically validate the central limit theorems in Sect. 12.4. The information content of a histogram is up to the binning almost equivalent to the empirical distribution function, but in plotting it is more convenient to compare histograms and densities.

The histograms resemble probability density functions. However, the intrinsic volumes of the considered Boolean model do not have probability density functions. Indeed, with positive probability, there is no overlap between the grains and there are no intersections with the boundary so that some multiples of the intrinsic volumes of the fixed grain have positive probability.

In this subsection, we simulate larger systems than in the previous Sect. 12.6.1. For a simulation box with side length $L = 20a$, we perform for each different set of parameters between $M_s = 5000$ and 150,000 simulations of Boolean models with rectangles at varying intensities γ. Like in Sect. 12.6.1, the rectangles have aspect ratio 1 or $1/2$, and they are either aligned w.r.t. the observation window or their

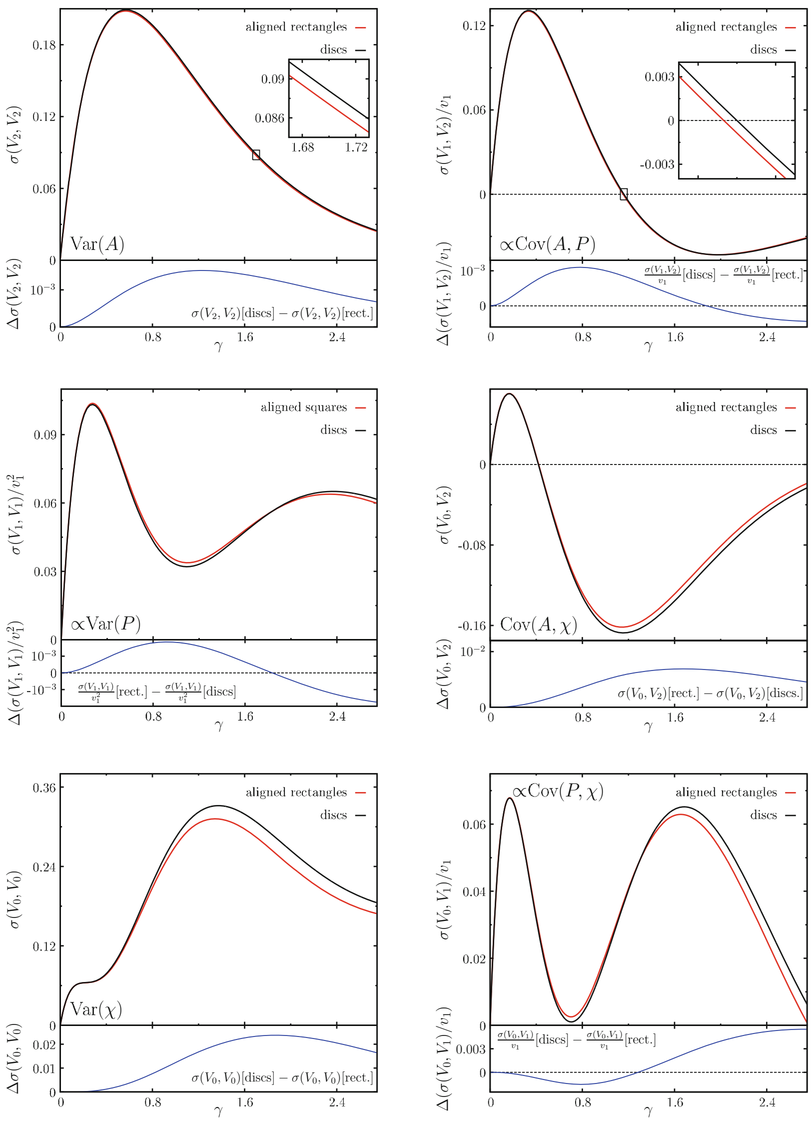

Fig. 12.4 Variances and covariances: $\sigma(V_2, V_2)$, the variance of the area; $\sigma(V_1, V_2)/v_1$, proportional to the covariance of area and perimeter; v_1 is half of the perimeter of a single grain; $\sigma(V_1, V_1)/v_1^2$, proportional to the variance of the perimeter; $\sigma(V_0, V_2)$, the covariance of area and Euler characteristic; $\sigma(V_0, V_0)$, the variance of the Euler characteristic; $\sigma(V_0, V_1)/v_1$, proportional to the covariance of perimeter and Euler characteristic. They are shown both for Boolean models with aligned rectangles, see (12.35), (12.43), (12.45), (12.53), (12.56), and (12.59), and for overlapping discs, see [16]. Note that except for $\sigma(V_1, V_1)/v_1^2$ the curves for the rectangles are independent of the aspect ratio of the rectangle, see also Figs. 12.2 and 12.3. The *insets* in the figures at the top are close-up views which show that the covariances differ slightly for Boolean models with rectangles or with discs. Below each subfigure, the differences of the covariances for Boolean models with rectangles or discs are plotted

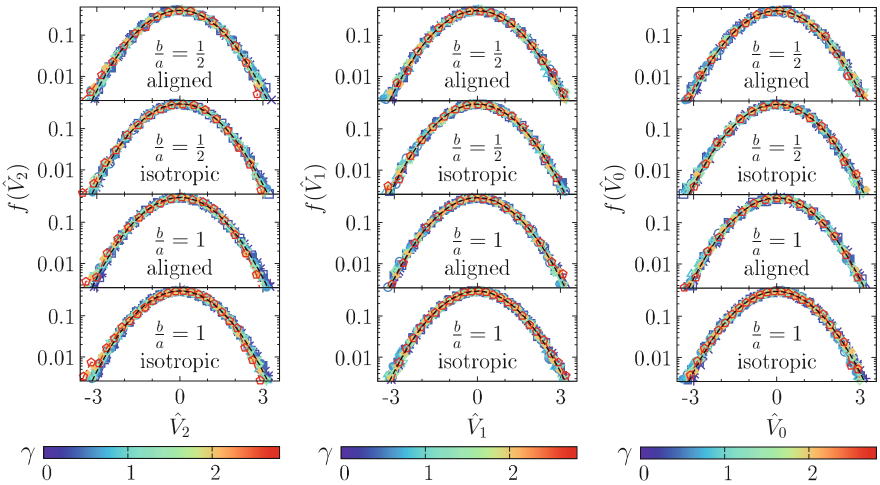

Fig. 12.5 Histograms f of the normalized intrinsic volumes $\hat{V}_i$ (see (12.60)) of Boolean models of rectangles with different aspect ratios b/a for different intensities γ; for both aligned rectangles and rectangles with an isotropic orientation distribution. For all of these different models, the rescaled distributions are already for the relatively small system size $L = 20a$ in very good agreement with a normal distribution (*dashed black line*)

orientation is isotropically distributed. To produce the histograms, we simulate more than 1,400,000 samples of Boolean models including about 350,000,000 rectangles in total.

We normalize the intrinsic volumes V_i, i.e., we subtract the estimated mean values $\langle V_i \rangle$ of the intrinsic volumes and divide by $\sqrt{s(V_i, V_i)}$:

$$\hat{V}_i := \frac{V_i - \langle V_i \rangle}{\sqrt{s(V_i, V_i)}}. \tag{12.60}$$

Figure 12.5 plots the histograms f of the normalized intrinsic volumes of Boolean models with different aspect ratios b/a for either aligned rectangles or rectangles with an isotropic orientation distribution and for varying intensities γ.

These histograms are in good agreement with the density function of a normal distribution for all intrinsic volumes, for all intensities, and for all of the simulated models (despite the relatively small simulation box). In other words, even in small observation windows, the probability distributions of the intrinsic volumes of Boolean models can be well approximated by Gaussian distributions.

As we have mentioned in the introduction, the central limit theorems for the geometric functionals (see Theorems 12.8 and 12.10) and the exact formulae for the second moments (see Theorems 12.4 and 12.5) can be used for hypothesis testing of models of random heterogeneous media. A hypothesis test could, e.g., use the intrinsic volumes to decide whether or not a random two-phase medium can be modeled by overlapping grains. The joint probability distribution of the Minkowski functionals allows for a characterization of the shape by several geometrical functionals and hence for a construction of tests using their full covariance structure. For a different random field (with a Poisson distributed number of counts in a binned gamma-ray sky map) such a sensitive morphometric data analysis has already been developed [8, 19]. The same concepts could be applied to the Boolean model.

In Fig. 12.5, there are only small deviations from a normal distribution relative to the error bars. So, the systematic deviations, e.g., due to the finite observation window size, seem to be small. In order to determine these deviations, a very high numerical accuracy is needed. We simulate $3 \cdot 10^6$ samples of two Boolean models for rectangles with aspect ratio $1/2$ that are either aligned or follow an isotropic orientation distribution. Here, we apply minus sampling boundary conditions, i.e., we consider all grains with centres in a slightly larger simulation box $[-\sqrt{a^2+b^2}/2, L+\sqrt{a^2+b^2}/2]^2$, but the observation window is still the original square $(0,L)^2$ with $L = 20a$. Contributions caused by the boundary are here neglected as it is often done in physics. The expected number of grains in the simulation box is adjusted accordingly and follows a Poisson distribution with mean $\gamma \cdot (L+\sqrt{a^2+b^2})^2$. To minimize the computational costs, a relatively low intensity γ is chosen for these simulations. It corresponds to an expected occupied area fraction $\phi = 1/15$. The resulting histograms are plotted in Fig. 12.6. As expected, for the small system size the large number of samples reveals deviations from the normal distribution that are significantly larger than the error bars.

For each underlying Boolean model, the histograms of the normalized intrinsic volumes coincide within error bars. This is not surprising because at the low intensity chosen here the intrinsic volumes are strongly correlated. (The correlation coefficients are larger than 0.9.)

For different Boolean models (isotropic orientation distribution or aligned grains), the histograms of the non-rescaled Minkowski functionals differ slightly but distinctly already for the relatively small intensity studied here, see the inset of Fig. 12.6. In contrast to this, the histograms of the normalized intrinsic volumes collapse for the different Boolean models within the error bars to a single curve, which can be well approximated by a standardized Poisson distribution. This can be explained by the strong correlation between the intrinsic volumes and the number of grains N_h hitting the observation window for each Boolean model. The latter follows a Poisson distribution with parameter $\mathbf{E}[N_h] = \gamma \cdot \mathbf{E}[V_2([0;L]^2 + Z_0)]$. (The correlation coefficients are larger than 0.85.) There is only a small relative difference between the parameters $\mathbf{E}[N_h]$ for the different considered Boolean models, because the observation window is large when compared to the typical grain Z_0. Therefore, the corresponding Poisson distributions are very close after standardization (dashed green line in Fig. 12.6) and coincide with the histograms of the normalized intrinsic volumes.

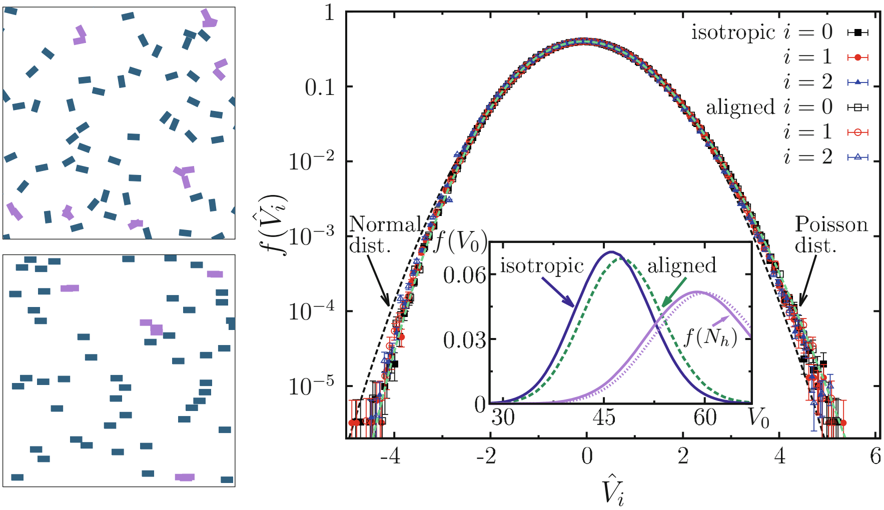

Fig. 12.6 Histograms f of the normalized intrinsic volumes $\hat{V}_i$ (see (12.60)) of Boolean models with either aligned rectangles or rectangles with an isotropic orientation distribution. The aspect ratio of the rectangles is $1/2$. The *dashed black line* depicts the density of a normal distribution. The *dashed green line* represents a standardized Poisson distribution. On the *left hand side*, two samples of the Boolean models with either an isotropic orientation distribution or aligned rectangles are shown; clusters of rectangles are *colored purple*. The *inset* shows the histograms of the non-rescaled Euler characteristic V_0 in the isotropic and aligned case as well as the corresponding probability mass functions of the number of grains N_h hitting the observation window with mean values 59.4 and 60.5, respectively

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Chapter 13
Cell Shape Analysis of Random Tessellations Based on Minkowski Tensors

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Abstract To which degree are shape indices of individual cells of a tessellation characteristic for the stochastic process that generates them? Within the context of stochastic geometry and the physics of disordered materials, this corresponds to the question of relationships between different stochastic processes and models. In the context of applied image analysis of structured synthetic and biological materials, this question is central to the problem of inferring information about the formation process from spatial measurements of the resulting random structure. This chapter addresses this question by a theory-based simulation study of cell shape indices derived from tensor-valued intrinsic volumes, or Minkowski tensors, for a variety of common tessellation models. We focus on the relationship between two indices: (1) the dimensionless ratio $\langle V\rangle^2/\langle A\rangle^3$ of empirical average cell volumes to areas, and (2) the degree of cell elongation quantified by the eigenvalue ratio $\langle \beta_1^{0,2}\rangle$ of the interface Minkowski tensors $W_1^{0,2}$. Simulation data for these quantities, as well as for distributions thereof and for correlations of cell shape and cell volume, are presented for Voronoi mosaics of the Poisson point process, determinantal and permanental point processes, Gibbs hard-core processes of spheres, and random sequential

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absorption processes as well as for Laguerre tessellations of configurations of polydisperse spheres, STIT-tessellations, and Poisson hyperplane tessellations. These data are complemented by experimental 3D image data of mechanically stable ellipsoid configurations, area-minimising liquid foam models, and mechanically stable crystalline sphere configurations. We find that, not surprisingly, the indices $\langle V\rangle^2/\langle A\rangle^3$ and $\langle\beta_1^{0,2}\rangle$ are not sufficient to unambiguously identify the generating process even amongst this limited set of processes. However, we identify significant differences of these shape indices between many of the tessellation models listed above. Therefore, given a realization of a tessellation (e.g., an experimental image), these shape indices are able to narrow the choice of possible generating processes, providing a powerful tool which can be further strengthened by considering density-resolved volume-shape correlations.

13.1 Shape Descriptors for Random Cells

In 1966, Mark Kac [39] posed the now famous question "Can one hear the shape of a drum"? This question referred to the uniqueness of the spectrum of the Helmholtz equation, i.e., the eigenmodes—perceptible as acoustic waves to the ear—with respect to different shapes of the Dirichlet boundary conditions. In general, the answer to Mark Kac's question is "No" as examples of distinct drum shapes exist that give the same spectrum of eigenmodes. Nevertheless, while not providing a unique characterization of the shape of the drum, the eigenmode spectrum contains substantial information about the shape of the drum. Given a specific eigenmode spectrum, many drum shapes can be excluded as the possible origin of the sound. While short of being a unique determinant, several aspects and properties of the shape of the drum can be inferred from an observed eigenmode spectrum. For example, Mark Kac showed how to some degree we can "hear" the connectivity of the drum. In fact, Mark Kac derived a relationship between the eigenmode spectrum of a drum and its shape quantified by Minkowski functionals or intrinsic volumes [39]. This family of integral geometric measures are also at the heart of this chapter.

Here, we address a similar question for stochastic spatial tessellation models, "Can one 'see' the stochastic process that generates a disordered structure (by considering only geometric characteristics of individual cells)?". This question refers to the uniqueness of average or distributional properties of geometric characteristics of a random tessellation—those that can be 'seen'—with respect to different stochastic processes that underly the formation of the tessellation. However, the answer to this question bears close analogies to the answer to Kac's question: the answer to our question is also "no", since there are at least two distinct tessellation models that agree in any single cell property. An example of such a pair is the Poisson hyperplane tessellation [16] and the STIT tessellation (STable with respect to ITeration) [76], see Fig. 13.1 and Sect. 13.2.4; a measurement of single cell properties can never uniquely infer which of these two models has generated

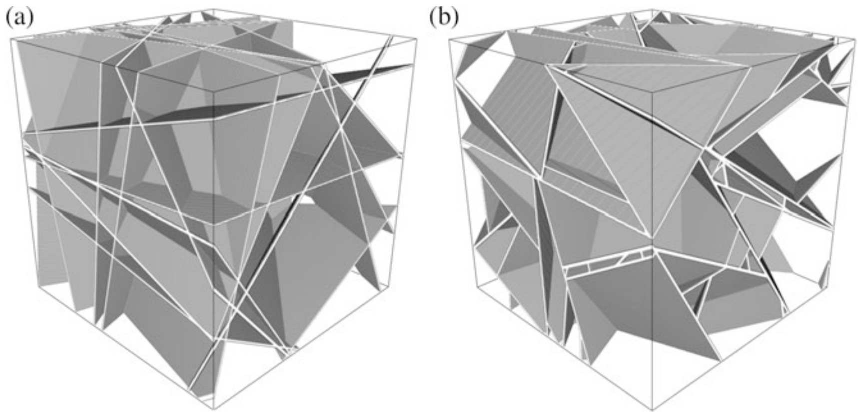

Fig. 13.1 Two random tessellations that have identical single cell properties but a different global arrangement: a Poisson hyperplane tessellation (**a**), see Sect. 13.2.4.1, and a STIT tessellation (**b**), see Sect. 13.2.4.2

the tessellation. On the other hand, like the eigenmode spectrum in Kac's question, the distributions and averages of single-cell properties of the tessellation are strongly dependent on the underlying stochastic process. In reverse, their measurement can be used to discriminate between possible underlying stochastic processes.

We investigate a variety of random tessellations that are stationary, that is, statistically homogeneous. We address the question which of their properties can be captured by geometric shape indices of the typical cell. That is to say, what information is contained in a local structure characterization.

There is a plethora of very different tessellation models that are important across many disciplines, from mathematics, physics, chemistry and biology to computer, life, and social sciences. Very different types of random or disordered tessellations appear ubiquitously in nature in very different systems, e.g., metal alloys, foams, biological tissues, and geological formations. Moreover, they are not only used to model cellular spatial systems but also for point pattern analysis or local optimization. For an overview see, e.g., [16, 71, 73, 79]. In our simulation study, we focus on three-dimensional systems, but the definitions and concepts can be applied or generalized to arbitrary dimensions.

Random tessellations can vary both in their geometric construction principle [19] and in the underlying stochastic process [16]. Although based on the same point pattern, different cells can be constructed according to varying protocols, but the same construction principle can also be applied to different random processes. We analyze a variety of important mathematical models and physical systems. We especially compare random tessellations with the same construction principles but different underlying stochastic processes. How are the shape indices of their typical cells related to each other? How do they differ from one another?

A special emphasis is on the characterization of anisotropy or elongation of the cell. Even in a statistically isotropic ensemble, a single cell usually exhibits a non-uniform orientation of its normal vectors. Similar to the method described in [96], we quantify the latter geometric anisotropy by Minkowski tensors.

An overview of the different construction principles and underlying stochastic processes that are analyzed here is given in Sect. 13.2. Here, we also relate the different nomenclatures that are commonly used in various fields of research. Moreover, we here provide the simulation details.

In Sect. 13.3, we present the here used shape indices and discuss how the Minkowski tensors serve as robust and sensitive measures of anisotropy.

In Sect. 13.4, we estimate the expectations of anisotropy indices of a typical cell in these models and physical systems, from which a "map of anisotropy" is constructed that to some extent classifies and relates the different tessellations. In Sect. 13.5, we extend the analysis and estimate the full probability density functions of both Minkowski functionals and anisotropy parameters. There, we find that the shape characterization based only on a single descriptor can distinguish such different tessellations as Voronoi or hyperplane tessellations. However, the probability density functions of the normalized Minkowski functionals for the Voronoi tessellations of (random) point processes can hardly distinguish quite different physical systems.

Therefore, we introduce in Sect. 13.6 a more sensitive structure characterization: we determine the mean anisotropy index as a function of the cell volume, i.e., conditional on the cell size. Loosely speaking, we distinguish the shape of small from that of large cells. For the different physical systems which are indistinguishable w.r.t. the former structure based on a single index, we detect a qualitatively different behavior with the more sensitive analysis based on two different shape descriptors. The latter can clearly distinguish the different Voronoi tessellations. Moreover, for the Poisson hyperplane tessellations with an isotropic orientation distribution compared to those with cuboidal cells, we find that while small cells in the latter tessellations tend to be more anisotropic, this trend changes for the large cells that are on average more isotropic. Section 13.7 contains the conclusion.

13.2 Construction of Random Tessellations

In its most general form, a tessellation is a collection of subsets of $\mathbb{R}^n$ (cells) with pairwise disjoint interiors whose union is $\mathbb{R}^n$. In such a generality the cells need not be connected, let alone convex. A rigorous introduction to the mathematical theory of random tessellations with convex cells is given in [90]; see also [16]. Some basic properties of more general (stationary) random tessellations (partitions) are discussed in [51].

The mathematical properties of a random tessellation are the result of a sometimes subtle interplay of the geometric construction principle and the underlying stochastic process. Section 13.2.1 describes some basic geometric construction principles and Sects. 13.2.2–13.2.5 some stochastic processes driving random tessellations. Figure 13.2 visualizes some of these examples.

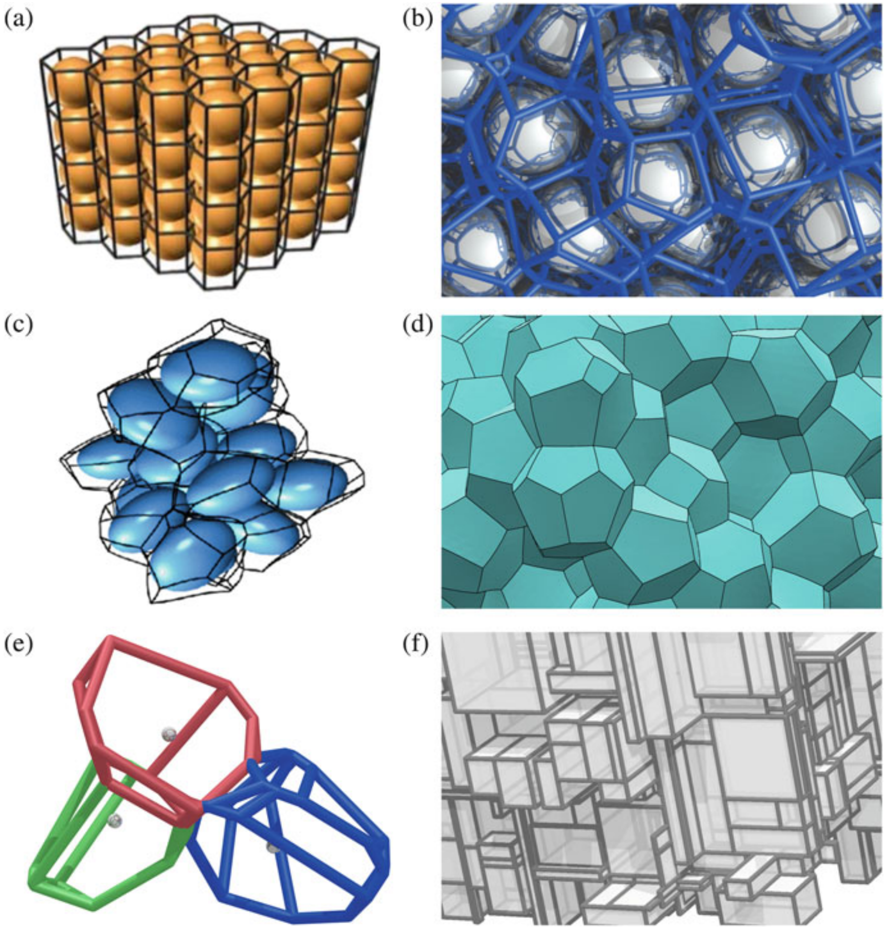

Fig. 13.2 Examples of tessellations: (**a**) a crystalline packing of hard spheres and the corresponding Voronoi cells of the sphere centers (reproduced from [97], with the permission of AIP Publishing), (**b**) Voronoi diagram of the sphere centers of an equilibrium hard-sphere fluid (using sphere configurations from Steven Atkinson), (**c**) an experimental random packing of ellipsoids and some corresponding Set Voronoi cells (reprinted with permission from [87]. Copyright 2015 by the American Physical Society), (**d**) random monodisperse foam in the dry limit (data by Andy Kraynik), (**e**) three cells out of a Poisson-Voronoi tessellation, and (**f**) cells of a STIT tessellation

13.2.1 Geometric Construction Principles

A Voronoi tessellation (also known as Voronoi diagram) is constructed from a given finite or locally finite subset of points (centers). The (Voronoi) cell of a center is the set of points in $\mathbb{R}^n$ that are closer to this center than to any other center. The cells are convex polytopes (finite intersections of half-spaces) which are bounded if the convex hull of the points equals $\mathbb{R}^n$; for examples, see Fig. 13.2a, b. Voronoi tessellations can be generalized in several ways [79, Chap. 3]. For instance one might replace the set of centers by a set of particles of equal or different shape. This leads to the Set Voronoi diagrams. In such a tessellation, a cell of a particle is the set of points in $\mathbb{R}^n$ that are closer to the surface of this particle than to any other particle. Instead of the distance to the center of a particle, like in the standard Voronoi tessellation, the distance to the surface of a particle is used. This definition results in curved facets, see Fig. 13.2c; for details of the algorithm applied here, see [84]. Another rich source for more general models are distances other than the Euclidean one. A *Laguerre tessellation* (also called power diagram), for instance, is based on a weighted power distance, where each cell has its individual weight. Such tessellations do have convex cells. However, not every center has a non-empty cell. An example is a system of impenetrable polydisperse spheres, i.e., spheres with different volumes, where the radius R of a sphere is the weight for the corresponding cell. Instead of the Euclidean distance r between a point outside of a sphere and a sphere center, the Laguerre tessellation then uses the weighted power distance $r^2 - R^2$. Voronoi tessellations are widely used in such diverse fields as, e.g., astronomy [26], wireless networks [6, 7], archeology, biology, chemistry, computational geometry, geology, or marketing; see [16, 79, 108].

Tessellations very different from Voronoi diagrams are determined by a (locally finite) collection of (intersecting) hyperplanes. The interiors of the cells are the connected components of the complement of the union of these hyperplanes. Hence any such hyperplane is the union of some $(n-1)$-dimensional facets. The planar case of this hyperplane tessellation is a line tessellation. The vertices of a non-degenerate tessellation all have degree 2^n [90].

A third class of tessellations results from iterative cell divisions; see [19]. One starts with a (bounded) convex window which is divided by a (random) hyperplane. Then the cell division process is continued independently on both resulting cells. In contrast to Voronoi and hyperplane tessellations, this cracking pattern algorithm does not produce face-to-face tessellations, compare Figs. 13.1b and 13.2f. In a face-to-face tessellation, the intersection of two cells is either empty or a face of both. The properties of the tessellation depend on the rules for randomly selecting the cells to be split and the (random) choice of the dividing hyperplanes. One important example, the STIT-tessellation [76], is discussed in Sect. 13.2.4.2, see Fig. 13.2f.

13.2.2 Point Processes

This section considers the completely random Poisson point process, repulsive determinantal point processes, clustering permanental processes, and patterns with a minimal distance between the points constructed by random sequential addition [34, 38].

A point process Φ (on $\mathbb{R}^n$) is a random collection of points which is locally finite, that is, the points are not allowed to accumulate in bounded sets. The number of points in a (Borel) set $B \subset \mathbb{R}^n$ is denoted by $\Phi(B)$. Here, point processes are always assumed to be stationary, that is, the distribution of Φ does not change under simultaneous shifts of all points. The mean (or expected) number $\gamma = \mathbf{E}[\Phi([0, 1]^n)]$ of points falling into the unit cube $[0, 1]^n$ is called intensity of Φ.

In most cases the distribution of Φ can be described by the correlation functions. This is a countable family of functions $g_k: \mathbb{R}^n \to [0, \infty)$, $k \in \mathbb{N}$, satisfying

$$\mathbf{E}[\Phi(B_1) \cdots \Phi(B_k)] = \gamma^k \int_{B_1 \times \cdots \times B_k} g_k(x_1, \ldots, x_k)\, d(x_1, \ldots, x_k),$$

for all $k \in \mathbb{N}$ and all pairwise disjoint Borel sets $B_1, \ldots, B_k \subset \mathbb{R}^n$. Note that the mathematical expectation operator $\mathbf{E}[\cdot]$ corresponds to the ensemble average in physics literature.

For a finite point pattern, $\gamma^k \cdot g_k(x_1, \ldots, x_k) d(x_1, \ldots, x_k)$ can be interpreted as the probability to find a point in each of the infinitesimally small neighborhoods of the positions $x_1, \ldots, x_k$.

13.2.2.1 The Poisson Process

Intuitively speaking, a Poisson point process (PPP) is a completely independent point process, where the points are randomly placed in space uniformly distributed, e.g., see [20, 52].

The (stationary) Poisson process is the most fundamental example of a point process. In this case, the correlation functions do not depend on the locations and are simply given by

$$g_k(x_1, \ldots, x_k) = 1.$$

For a Poisson process Φ with intensity $\gamma > 0$, the number of points in pairwise disjoint sets are stochastically independent. Moreover, the number of points in a set B has a Poisson distribution, that is

$$\mathbf{P}(\Phi(B) = m) = \frac{\gamma^m V_n(B)^m}{m!} e^{-\gamma V_n(B)}, \qquad m = 0, 1, \ldots,$$

where $V_n(B)$ denotes the volume (Lebesgue measure) of B.

The PPP can be interpreted as discrete white noise (appearing, e.g., as random noise in a detector), and the configurations are equivalent to the grand-canonical version of the ideal gas model. In the latter case, the intensity is equivalent to the fugacity $z := \mathrm{e}^{\mu/(k_B T)}$ of the ideal gas (if the unit of length is defined by the thermal de Broglie wavelength), where μ is the chemical potential, T the temperature, and k_B the Boltzmann constant.

In this chapter we shall be concerned with the Poisson Voronoi tessellation, that is with the Voronoi tessellation generated by the points of a Poisson process, see Fig. 13.2e. Many probabilistic properties of this benchmark model of stochastic geometry are now well understood; see [16, 71, 90] for first order properties and [53] for (asymptotic) second order properties.

To simulate a Poisson point pattern with intensity γ, the random number of points within a finite simulation box is drawn from a Poisson distribution with parameter $\gamma \cdot V_{\text{box}}$, where V_{box} is the volume of the simulation box. The points are then uniformly distributed within the simulation box. Data in Figs. 13.4, 13.5, 13.8a, and 13.9 is based on simulations of about 1000 patterns, each with on average 2000 points.

13.2.2.2 Determinantal Point Processes

Determinantal point processes (DPP) were introduced in [58] (see also [14]) to model the behavior of fermions in quantum mechanics. They are also used to describe transmitters in wireless networks [21]. Determinantal processes model "soft" repulsive particles in the sense that although it is unlikely, particles can get arbitrarily close to each other. (This is in contrast to the hard-sphere systems discussed below.)

The mathematical definition of a DPP is based on a kernel $K\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is assumed to generate a self-adjoint, non-negative and locally trace-class integral operator on $L^2(\mathbb{R}^n)$ with eigenvalues in the interval [0, 1], see [102]. Then the correlation functions are given by the determinants

$$g_k(x_1, \ldots, x_k) = \gamma^{-k} \det(K(x_i, x_j))_{1 \le i,j \le k}.$$

We refer to [34] for a nice survey of the probabilistic properties of a DPP. For our present studies we have simulated the DPP with a software package provided by Ege Rubak [55] written for SPATSTAT [8], which is a package for the statistics software R. Data in Figures 13.4, 13.5, 13.8a, and 13.9 is based on simulations of 100 point patterns with on average 2000 points (using a power exponential spectral model with $\alpha = 0.12$ and $\nu = 10$, as explained in [55]).

13.2.2.3 Permanental Point Processes

Permanental point processes are the attractive counterpart of determinantal processes and can be used to model bosons in quantum mechanics [58]. The definition of a permanental point process is again based on a kernel $K\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is assumed to generate a self-adjoint, non-negative and locally trace-class integral operator on $L^2(\mathbb{R}^n)$. Then the correlation functions are given by

$$g_k(x_1, \ldots, x_k) = \gamma^{-k} \operatorname{per}(K(x_i, x_j))_{1 \le i,j \le k},$$

where the permanent per$A$ of a $k \times k$-matrix $A = (a_{i,j})$ is given by

$$\operatorname{per} A := \sum_{\sigma \in \Sigma_k} \prod_{i=1}^{k} a_{i,\sigma(i)}.$$

Here Σ_k denotes the group of permutations σ of $\{1, \ldots, k\}$.

The mathematical properties of the permanental process are analyzed in [34, 62], see also [52]. A remarkable feature of such a process is that it is doubly stochastic Poisson, i.e., an inhomogeneous PPP with a random intensity function. If the latter is stationary, so is the resulting process. To explain this we let $(Y_1(x))_{x \in \mathbb{R}^n}$ and $(Y_2(x))_{x \in \mathbb{R}^n}$ denote two independent centered Gaussian random fields with the same covariance function

$$\mathbf{E}[Y_1(x)Y_1(y)] = \mathbf{E}[Y_2(x)Y_2(y)] = \frac{K(x,y)}{2}, \qquad x, y \in \mathbb{R}^n.$$

Let $Z := (Z(x))_{x \in \mathbb{R}^n}$, where $Z(x) := Y_1^2(x) + Y_2^2(x)$. Then the permanental process with this kernel K has the same distribution as a point process Φ with the following two properties. Given Z, the number of points in pairwise disjoint sets are conditional independent, while the number of points in a set B has a conditional Poisson distribution with parameter $\int_B Z(x)\,dx$, that is,

$$\mathbf{P}(\Phi(B) = m \mid Z) = \frac{\left(\int_B Z(x)\,dx\right)^m}{m!} e^{-\int_B Z(x)\,dx}, \qquad m = 0, 1, \ldots.$$

As pointed out in [62], this doubly-stochastic construction allows for a fast simulation of permanental point processes. First, the two Gaussian random fields Y_1 and Y_2 are simulated in an observation window W. Then, we simulate an inhomogeneous Poisson point process with intensity function $Z(x) := Y_1^2(x) + Y_2^2(x)$. Therefore, we simulate a homogeneous PPP Φ' (see Sect. 13.2.2.1) with intensity $\gamma_0 \geq \max_{x \in W} Z(x)$ but only accept a point $y \in \Phi'$ with probability $Z(y)/\gamma_0$, which is called independent Poisson thinning [72].

Because of the vast number of possible Gaussian random fields, there is a great wealth of permanental point processes that produce quite different point patterns. Here, we restrict ourselves to a class of Gaussian random fields that is especially important in physics: the *Gaussian random wave model* (GRWM), see [22] and references therein. Intuitively speaking, the GRWM is defined as a superposition of plane waves with random phases and orientations of the wave vector but a constant absolute value of the wave vector; with an increasing number of random plane waves, the random field converges to a Gaussian random field. More precisely, for each run we add $N_w = 100$ plane waves and use

$$f(x) = a_w \cdot \sqrt{\frac{2}{N_w}} \cdot \sum_{i=1}^{N_w} \cos(k_i \cdot x + \eta_i), \tag{13.1}$$

as a very good approximation of a Gaussian random field. The random phases η_i are uniformly distributed in $[0, 2\pi)$.

An anisotropic GRWM can be constructed by choosing a non-isotropic orientation distribution of the wave vectors k_i. Instead of a uniform distribution of the orientation of k_i on the unit sphere, the orientation can, for example, be restricted to a spherical cap with opening angle ω, which is the angle between the edge and the center of the spherical cap (with the origin as the vertex). Both arbitrarily anisotropic and perfectly isotropic models can be obtained by varying $\omega \in (0, \pi/2]$.

For the present study, we have simulated GRWMs in $W = [0; L)^3$ with $L = 25$, $|k_i| = 10/L$, $a_w = 1/2$, and $2\omega/\pi = 0.01, 0.03, \ldots, 0.09, 0.1, 0.3, \ldots, 0.9, 1$. For each value of ω, we analyze 100 point patterns; each contains on average about 8000 points, but the number of points in the single samples vary between 500 and 40,000.

13.2.2.4 Random Sequential Addition

An intuitive explanation of "random sequential addition" (RSA), also called "random sequential adsorption" or "simple sequential inhibition", is to subsequently place spheres uniformly distributed into space. However, a sphere is only accepted if there is no overlap with spheres that have already been accepted [108]. The point process is then formed by the centers of the accepted spheres. This notion of RSA is nearly identical to the (in mathematics well-known) Matérn III process [75, 105]. There is only a difference in finite observation windows. In the latter process, the intensity is fixed and the global packing fraction can (slightly) fluctuate, that is, the fraction of the volume occupied by the particles. However, in the RSA process studied here the global packing fraction is fixed.

The RSA process can be generalized by introducing "ghost" particles, i.e., particles that have been rejected but for a finite time still block later arriving spheres [109]. In the limit that such a rejected sphere blocks all later arriving spheres, these ghost packings correspond to the Matérn II process [107]. The latter is exactly solvable in that all correlation functions are explicitly known across all dimensions [109]. In contrast to the Matérn I and II processes (see e.g. [16, 44, 104, 107]), the RSA model does not seem to be amenable to a first and second order analysis. A likelihood based statistical inference, however, is possible; see [35].

Sometimes, RSA is only referring to the saturation limit, i.e., a configuration where no additional sphere can be accepted. The volume fraction in the saturation limit in the spherical spatial case is about $0.384131(\pm 2 \cdot 10^{-6})$ [112]. In this chapter, RSA refers to a system for which the global packing fraction is chosen to be some value below this limit. Spheres are added to the ensemble only until this global packing fraction is reached. In the dilute limit, i.e., vanishing global packing fraction, an overlap with a previously accepted sphere gets increasingly unlikely. Thus, the structure of the RSA sphere configurations becomes similar to a Poisson point process.

In chemistry and physics, RSA is used to model the irreversible adsorption or adhesion of, for example, proteins or cells at solid interfaces. If such a particle is (randomly) attached to the surface, it can no longer move or leave the interface, and it prevents other particles from being absorbed in its neighborhood [24, 106].

For Sect. 13.4, we simulate for each occupied volume fraction ($\phi = 0.03, 0.06, \ldots, 0.36$) 100 samples of 2000 spheres. For Sects. 13.5 and 13.6, we simulate 1000 patterns where each again consists of 2000 spheres. A description of the simulation procedure (which follows straightforwardly from the definition of the process) can, e.g., be found in [108, Sect. 12.3, p. 280].

13.2.3 Systems of Hard Particles

We also analyze systems of hard, impenetrable particles and the resulting Voronoi tessellations: simulations of equilibrium hard spheres, a variety of crystalline arrangements of spheres, and experimentally packed spheres and ellipsoids.

13.2.3.1 Equilibrium Hard Spheres

If hard spheres are not motionless but allowed to move, the system can equilibrate, so that the point process defined via snapshots of the sphere centers becomes statistically invariant over time, see Fig. 13.2b. It can then serve as a simple model for liquids. In mathematics, this model is often called a hard-sphere Gibbs process [16, 20, 81]. This is a point process Φ satisfying the integral equation

$$\mathbf{E}\Big[\sum_{x\in\Phi} f(x,\Phi)\Big] = b\cdot\mathbf{E}\Big[\int f(x,\Phi\cup\{x\})e^{-E(x,\Phi)}\,dx\Big]$$

for all (measurable) functions f of x and Φ. In physics, it corresponds to a grand canonical ensemble of impenetrable spheres that are in thermodynamic equilibrium with a reservoir that allows for an exchange of energy and particles. The number $-\log b$ can be interpreted as chemical activity. The intensity γ is an increasing function of b, see [16, p. 189]. $E(x,\Phi)$ is some fixed positive parameter if the sphere around x does not intersect the spheres around the points of Φ; otherwise $E(x,\Phi) := \infty$. Here, all spheres have the same radius. Replacing in the above integral equation $E(x,\Phi)$ by a more general (energy) function, yields Gibbs processes as characterized in [78]. The usual definition of a Gibbs process proceeds with specifying the conditional finite window distributions given the configuration in the complement and then using the Dobrushin-Lanford-Ruelle (DLR) equation, see [16, 83].

Like for the RSA process, the equilibrium hard-sphere fluid becomes, in the dilute limit, similar to a Poisson point process. It can then be called a hard-sphere gas. It was shown in [61] that the packing fraction of equilibrium hard spheres tends, as $b \to \infty$, monotonically to the closest packing density

$\phi_{\max} = \pi/\sqrt{18} \approx 0.74$ [30]. With increasing packing fraction, there is a disorder-to-order phase transition [1, 111] and at a maximum global packing fraction the spheres form a face-centered-cubic crystal (or one of its stacking variants).

This work analyzes equilibrium configurations of hardcore spheres without gravity in the isotropic fluid phase obtained by Monte Carlo simulations. The data is taken from [41]. The results are averaged over 10 systems per global packing fraction with up to 16,384 particles. Dense systems contain only 4000 particles due to computational costs.

13.2.3.2 Crystalline Sphere Packings

We also determine the geometric anisotropy of cells in deterministic, perfectly ordered tessellations. More specifically, we analyze Voronoi cells of crystal lattices that are formed by the centers of hard spheres in mechanically stable packings [18, 113]. We compare the random packings of hard spheres to crystalline packings of hard spheres that are locally jammed, which means no sphere can be moved while all other spheres are kept fix. (In three dimensions, each sphere touches at least four other spheres of which not all are on the same hemisphere.) Moreover, each sphere is connected to any other sphere via a chain of contacts. The data includes a great range of Bravais lattices, which in three-dimensions are lattices that are generated by discrete translations of three independent vectors, therefore all lattice sites are equivalent. The most isotropic unit cells (w.r.t. the distribution of the normal vectors on the cell boundary) are in our analysis those of the simple cubic, body-centered, and face-centered cubic packing as well as the hexagonal close-packed arrangement and the hexagonal AAA stacking, see Fig. 13.2a.

The data for the Minkowski functionals and tensors of the Voronoi cells for these sphere packings are provided by Richard Schielein, for more details see [97].

13.2.3.3 Jammed Ellipsoids and Spheres

Jammed packings of hard particles need not to be perfectly ordered like the crystalline sphere packings. There are also mechanically stable packings that are disordered. The local structure is more similar to an equilibrium liquid like in Sect. 13.2.3.1 but the particles are jammed.

Moreover, the simple model of hard spheres can be extended to non-spherical particles, namely ellipsoids. However, in contrast to the equilibrium hard-sphere fluid, we here analyze experiments with non-equilibrium jammed packings. This work uses oblate ellipsoids (e.g. two equally long and one small half-axis) created by 3D printing with aspect ratios $\alpha = 0.40, 0.60, 0.80, 1.00$, where the aspect ratio is defined as the ratio of the smallest to the largest length of a semi-axis. For each aspect ratio at least ten packings of 5000 ellipsoids with different global packing fraction are analyzed. The particles are randomly packed into a cylindrical container by different preparation protocols to get an initial loose packing. They can then be compactified by tapping to get a variety of different packing fractions [87]. A 3D image is gained by imaging the packing with X-ray tomography. The positions and orientations of the ellipsoids are detected from the grayscale image of the tomographic reconstruction, for more details see [85]. To reduce boundary effects, the outer particles are removed for the analysis leaving approximately 800 particles in the core region.

The Voronoi diagram of the ellipsoid centers is not a useful tessellation of the void space for non-spherical particles like ellipsoids. The particles are anisotropic, and facets of the Voronoi cell could cut the particles. Therefore, we construct the Set Voronoi diagram of the jammed ellipsoid packings, see Fig. 13.2c and [84], as described in Sect. 13.2.1.

13.2.4 Tessellations Constructed by Hyperplanes

Sections 13.2.2 and 13.2.3 discuss particle processes and their Voronoi tessellations or generalizations thereof. Quite different tessellations can be constructed by a collection of intersecting hyperplanes. In Sect. 13.1, Fig. 13.1 compares realizations of two such tessellations, which have the same distribution of the typical cell, i.e., the same single cell properties, but an obviously different global structure.

13.2.4.1 Poisson Hyperplane Tessellations

Loosely speaking, a Poisson process Φ of hyperplanes can be defined by replacing the points of a PPP by hyperplanes [90]. This means that Φ is a countable collection of random hyperplanes with the following two properties. First, the random number of hyperplanes with a prescribed property follows a Poisson distribution. Second, given a finite number of mutually exclusive properties, the random numbers of hyperplanes with these properties are stochastically independent. Such a process is stationary if its distribution does not change under simultaneous translations of the hyperplanes. In this case, the distribution of Φ is determined by an intensity parameter (the cumulative surface area of hyperplanes in the unit volume) and a directional orientation distribution $\mathbb{Q}$, an (even) probability measure on the unit sphere. If $\mathbb{Q}$ is uniform, then the tessellation is statistically isotropic. In general, $\mathbb{Q}$ can be used to model preferred directions.

Here, we analyze Poisson hyperplane tessellations (PHP) with two different orientation distributions of the hyperplanes. They are either isotropically distributed, see Fig. 13.1a, or only three directions are allowed that are orthogonal to each other, and the probability for each is $1/3$. Thus, the single cells form cuboids.

We simulate these tessellations with unit intensity. For the isotropic tessellations, we use software written by Felix Ballani. We analyze $2 \cdot 10^7$ cells in the isotropic tessellation and 10^7 cuboidal cells in the tessellation with three allowed directions.

13.2.4.2 STIT Tessellations

STIT (STable with respect to ITeration) tessellations [16, 76] are the result of an iterative cell division process. The original cell (e.g., the simulation box) has an exponential random lifetime whose parameter (in the isotropic case) is proportional to its first intrinsic volume, i.e., the $(n-1)$-st Minkowski functional. At the end of a lifetime, the cell is divided by a random hyperplane. The two new cells again have exponential lifetimes whose parameters are determined as above and which are assumed to be independent. Stopping this process after some fixed deterministic positive time yields the STIT tessellation. Different STIT models can be constructed from different orientation distributions of the random hyperplanes, compare Figs. 13.1b and 13.2f. Potential applications of the STIT tessellations are approximations of 'hierarchical' crack or fracture structures [76].

Although being globally very different from Poisson hyperplane tessellations, the typical cell of a STIT tessellation has the same distribution as that of the corresponding Poisson hyperplane tessellations; see [76]. Therefore, also the joint distribution of all Minkowski functionals and tensors of a typical cell is the same in a STIT or in a Poisson hyperplane tessellation (with the same intensity and orientation distribution of the hyperplane directions). For further properties of STIT-tessellations we refer to [67, 93, 94].

13.2.5 Random Dry Foam and Laguerre Tessellations

Foams like dry soap froth in the limit of a vanishing liquid content [47], that is with infinitely thin soap films, are important examples of tessellations which minimize surface area [32]. The foam that is analyzed here, see Fig. 13.2d, is a realistic model for monodisperse dry soap froth, where all cells have the same volume [49].

Interestingly, for random foams, the basic stochastic and geometric building blocks are closely connected [48]. In this sense, it can be seen as a hybrid model of different construction principles and stochastic processes.

The simulation starts from the Voronoi diagram of a random hard-sphere packing, derived by the Lubachevsky-Stillinger packing algorithm [57]. Then, the soap froth is equilibrated by Kenneth Brakke's SURFACE EVOLVER [15]. The surface area is minimized, and the mechanical forces balanced [25, 49, 50]. The data is provided by Andy Kraynik.

We also analyze random Laguerre tessellations with varying volume distributions, which are intended as a mathematical model similar to polydisperse foam structures, where the volume of the cells can vary strongly [54].

The data is taken from [82]. First, a random ensemble of hard spheres is simulated where the volume of the spheres follows a log-normal distribution with a coefficient of variation (CV) between 0.2 and 2.0. The packing fraction, i.e., the volume fraction that is occupied by the spheres, is 60%. For each chosen parameter, we simulate five samples of sphere packings in the unit cube (with periodic boundary conditions) each containing 10,000 spheres. Using the radii of the hard spheres as weights, the Laguerre tessellation is constructed as described in Sect. 13.2.1.

13.3 Minkowski Tensors and Anisotropy Indices

Intrinsic volumes are in physical literature commonly referred to as "Minkowski functionals" and the tensor valuations as "Minkowski tensors". (The only difference to the mathematical literature is in the normalization.) They are efficient numerical tools, which have been successfully applied to a variety of biological [9, 13] and physical systems [45, 65, 66] on all length scales from nuclear physics [99, 100], over condensed and soft matter [31, 42, 92, 110], to astronomy and cosmology [17, 23, 27, 28, 43, 89] as well as to pattern analysis [12, 60, 64, 91]. They allow for a versatile morphometric analysis of random spatial structures on very different length scales [45].

The Minkowski tensors allow for a comprehensive [2, 3] and systematic approach to quantify various aspects of structural anisotropy [96]. A local analysis based on the anisotropy of a single cell quantified by the Minkowski tensors, e.g., allows to detect phase transitions and the onset of crystallinity in jammed packings [40–42, 69].

Free software to calculate Minkowski functionals and tensors of both voxelized and triangulated data, PAPAYA and KARAMBOLA (for two and three dimensions, respectively), is available at:

www.theorie1.physik.fau.de/research/software.html

A comprehensive introduction to the Minkowski tensors as anisotropy indices and exemplary applications can be found in [95, 96]. A comparison of physical and mathematical notation is provided in [98].

13.3.1 Integral Geometric Definition

To allow easy comparison of the results presented here with the existing physical literature, we shortly recall the definitions of these quantities with the relevant notational adaptations; see Sect. 1.3 in Chap. 1 for details.

Let K be a convex body in $\mathbb{R}^n$, that is, a compact convex subset of $\mathbb{R}^n$. The parallel body $K + \rho B^n$, $\rho \geq 0$, consists of all points in $\mathbb{R}^n$ with Euclidean distance at most ρ from K. The volume $V_n(K + \rho B^n)$ of the parallel body $K + \rho B^n$ can be expressed by the following version of Steiner's formula:

$$V_n(K + \rho B^n) = W_0(K) + \frac{1}{n} \sum_{\nu=1}^{n} \rho^\nu \binom{n}{\nu} W_\nu(K).$$

A comparison with the intrinsic volumes in Eq. (1.16) shows that $V_n = W_0$ is the volume and that

$$V_\nu = \frac{1}{n\kappa_{n-\nu}} \binom{n}{\nu} W_{n-\nu} \qquad \text{for } \nu \leq n-1.$$

A similar reindexing and rescaling applies to the Minkowski tensors defined in Chap. 2. For $r, s \in \mathbb{N}_0$ we define following [63, 90]

$$W_0^{r,0}(K) = \int_K x^r \mathrm{d}x,$$

and

$$W_\nu^{r,s}(K) = \frac{n\kappa_\nu}{\binom{n}{\nu}} \int_{\mathbb{R}^n \times S^{n-1}} x^r u^s \Lambda_{n-\nu}(K; \mathrm{d}(x,u)) \qquad \text{for } n \geq \nu \geq 1,$$

where x^r and u^s are symmetric tensor products, and $x^r u^s$ is the symmetric tensor product of x^r and u^s, and $\Lambda_\nu(K;\,\cdot\,)$ is the ν-th support measure of K; see Sect. 1.3 in Chap. 1. Comparing this with (2.4), (2.6), and (2.7), we see that

$$\Phi_n^{r,0}(K) = \frac{1}{r!} W_0^{r,0}(K),$$

and

$$\Phi_\nu^{r,s}(K) = \frac{1}{r!s!\omega_{n-\nu+s}} \binom{n-1}{\nu} W_{n-\nu}^{r,s}(K) \qquad \text{for } n \geq \nu \geq 1.$$

In this work, we concentrate on the translation invariant Minkowski tensors $W_\nu^{0,2}$. For a sufficiently smooth three-dimensional compact convex set K, these tensors of rank two can be represented by symmetric matrices

$$W_1^{0,2} = \int_{\partial K} \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix} \mathrm{d}A, \tag{13.2}$$

and

$$W_2^{0,2} = \int_{\partial K} \frac{\kappa_1 + \kappa_2}{2} \cdot \begin{pmatrix} n_x^2 & n_x n_y \\ n_x n_y & n_y^2 \end{pmatrix} \mathrm{d}A, \tag{13.3}$$

where $\mathbf{n} = (n_x, n_y)$ describes the normal vector at the boundary of K; κ_1 and κ_2 are the principle curvatures on ∂K, and $(\kappa_1 + \kappa_2)/2$ is the mean curvature.

13.3.2 Geometrical Interpretation

In three dimensions, the Minkowski functionals are proportional to either the volume, the surface area, the integrated mean curvature, or the Euler characteristic. The latter is a topological constant, which measures in a certain way connectivity. For a single cell without holes, the Euler characteristic is trivially equal to unity.

The Minkowski functionals of a domain K can be expressed by integrals over K or over its boundary ∂K. These scalar measures are naturally generalized to the Minkowski tensors by including an integral over the tensor products of the position vector $\mathbf{r}$ and the surface normal vector $\mathbf{n}$. In other words, they are the moment tensors of the position or normal vectors.

The tensors using the position vector are closely related to tensors of inertia where the mass is located in the region of integration and possibly weighted by the curvature. $W_0^{2,0}$ contains the information of the tensor of inertia of the solid object, $W_1^{2,0}$ of a hollow object where the mass is located in the shell. For the example of polytopes in three dimensions, $W_2^{2,0}$ and $W_3^{2,0}$ are related to the tensor of inertia if the mass is distributed on the edges or vertices but weighted with the opening angles. The tensor $W_1^{0,2}$ is proportional to the moment or covariance tensor of the distribution of normal vectors, in other words, of the orientation of the surface; $W_2^{0,2}$ is proportional to the according moment tensor weighted by the curvature distribution. In contrast to the tensors that are related to the tensors of inertia, the moment tensors of the normal distributions are translation invariant.

The Minkowski tensors allow for a systematic analysis of anisotropy w.r.t. different geometrical aspects, like the distribution of volume or of the orientation of the surface. While a domain K might be perfectly isotropic w.r.t. one of these properties, it can be strongly anisotropic w.r.t. another property. For example, a homogeneous random two-phase medium, like a stationary Boolean model, has an isotropic distribution of the volume; hence, the tensors $W_\nu^{2,0}$ are isotropic [33, 96]. However, it can be strongly anisotropic w.r.t. the distribution of the normal vectors on the interface, which is detected by $W_1^{0,2}$ [33, 96].

Different normalizations are used in different fields of research. In this chapter, we use for the Minkowski functionals W_ν and tensors $W_\nu^{r,s}$ a normalization such that W_0 is the volume and W_1 is the surface area. The normalizations of the Minkowski tensors are then defined accordingly [45].

Here, the tensor $W_3^{0,2}$ is not of interest because it is proportional to the unit tensor times the Euler characteristic (see detailed discussion in the conclusion section of [68]). The latter is, as mentioned above, for a compact, convex cell always equal to one.

13.3.3 Shape Indices

Even in a statistically isotropic random tessellation, a typical cell is usually geometrically anisotropic. There is no globally preferred orientation. However, given a specific realization of a single cell, the distribution of its normal vectors will be most likely non-uniform. This geometric anisotropy of each single cell is here characterized by the Minkowski tensors. Each tensor $W_\nu^{0,2}$ contains both information about the preferred direction and the amplitude or degree of the anisotropy. The latter can conveniently be described by a scalar anisotropy index: the ratio of the smallest to the largest eigenvalue $\beta_\nu^{0,2}$ of the Minkowski tensor $W_\nu^{0,2}$ [95, 96]. It describes the "degree of anisotropy" that is captured by the Minkowski tensor $W_\nu^{0,2}$. (In physics literature, such shape indices are sometimes referred to as "shape measures." However, this does not refer to a measure in the mathematical sense.) For example, if λ_1 and λ_3 are the smallest and largest eigenvalues of $W_1^{0,2}$, respectively, the anisotropy index is given by:

$$\beta_1^{0,2} := \frac{\lambda_1}{\lambda_3}. \tag{13.4}$$

Smaller values indicate stronger anisotropy. For a ball, $\beta_1^{0,2} = 1$. A cube also appears perfectly isotropic to a second rank tensor. Therefore, $\beta_1^{0,2} = 1$ for a cube. For a cuboid, the index $\beta_1^{0,2}$ is equal to the ratio of the surface areas of the smallest and largest faces.

Alternative scalar anisotropy indices q_r can also be derived for Minkowski tensors of arbitrary rank r [40, 69]. The index q_2, for example, is equivalent to a weighted bond orientational order parameter [69]. For two dimensions, see [45].

Given a sample of M cells, we determine for each cell C_m (with $m \in \{1, \ldots M\}$) the Minkowski functionals $W_\nu(C_m)$ and the anisotropy indices $\beta_\nu^{0,2}(C_m)$. We then estimate the corresponding mean values for the typical cell by the sample means:

$$\begin{aligned} \langle W_\nu \rangle &:= \frac{1}{M} \sum_{m=1}^{M} W_\nu(C_m), \\ \langle \beta_\nu^{0,2} \rangle &:= \frac{1}{M} \sum_{m=1}^{M} \beta_\nu^{0,2}(C_m). \end{aligned} \tag{13.5}$$

Put differently, the shape index W_ν (or $\beta_\nu^{0,2}$) is evaluated for each cell separately, and the arithmetic average is the estimator for the expectation. These estimators can be justified by the ergodicity properties of the underlying tessellations. Similarly, the sample variance over all cells provides an estimate for the variance:

$$s_{W_\nu}^2 := \frac{1}{M-1} \sum_{m=1}^{M} (W_\nu(C_m) - \langle W_\nu \rangle)^2, \tag{13.6}$$

$$s_{\beta_\nu^{0,2}}^2 := \frac{1}{M-1} \sum_{m=1}^{M} (\beta_\nu^{0,2}(C_m) - \langle \beta_\nu^{0,2} \rangle)^2. \tag{13.7}$$

Based on the sample mean and sample variance, we define normalized Minkowski functionals and normalized anisotropy indices:

$$\hat{W}_\nu := \frac{W_\nu - \langle W_\nu \rangle}{s_{W_\nu}}, \tag{13.8}$$

$$\hat{\beta}_\nu^{0,2} := \frac{\beta_\nu^{0,2} - \langle \beta_\nu^{0,2} \rangle}{s_{\beta_\nu^{0,2}}}. \tag{13.9}$$

For these normalized or rescaled shape indices, we also determine estimated probability density functions (EPDFs) f, that means empirical histograms weighted by the total number of samples and the bin width. (A bin is a range of values for which the frequency of occurrence is determined.) In other words, the EPDF is a relative frequency histogram weighted by the size of each bin.

In Sect. 13.4, we use another shape index

$$Q := 36\pi \cdot \langle V \rangle^2 / \langle A \rangle^3, \tag{13.10}$$

where $V(= W_0)$ is the volume of the cell, $A(\propto W_1)$ its surface area, and $\langle . \rangle$ denotes the sample mean as defined in Eq. (13.5). The mean volume is rescaled by the mean surface area, so that the resulting index has no unit. The prefactor 36π is chosen such that the upper bound (which is given by the perfectly isotropic sphere) is unity.

Q can be interpreted as an isoperimetric ratio of the empirical average cell volume and area. However, the ratio of mean values is in general not equal to the mean value of the corresponding ratio of volume and surface area. Therefore, Q can be quite different from the mean isoperimetric ratio (or quotient) of the typical cell. For example, we will discuss in the following Sect. 13.4 how in strongly polydisperse tessellations, that is, in tessellations with strong fluctuations in the cell volume, a few large cells can strongly influence the mean volume and surface area and thus Q.

For a single body K, the isoperimetric ratio is defined as $Q_s(K) := 36\pi \cdot V(K)^2/A(K)^3$, where $V(K)$ and $A(K)$ are the volume or surface area of K. It characterizes to some extent the deviation from a spherical shape: only for a sphere $Q_s = 1$, for all other bodies $Q_s < 1$. The isoperimetric quotient is, e.g., used to describe equilibrium phases of packings of hard convex polyhedra. It can be

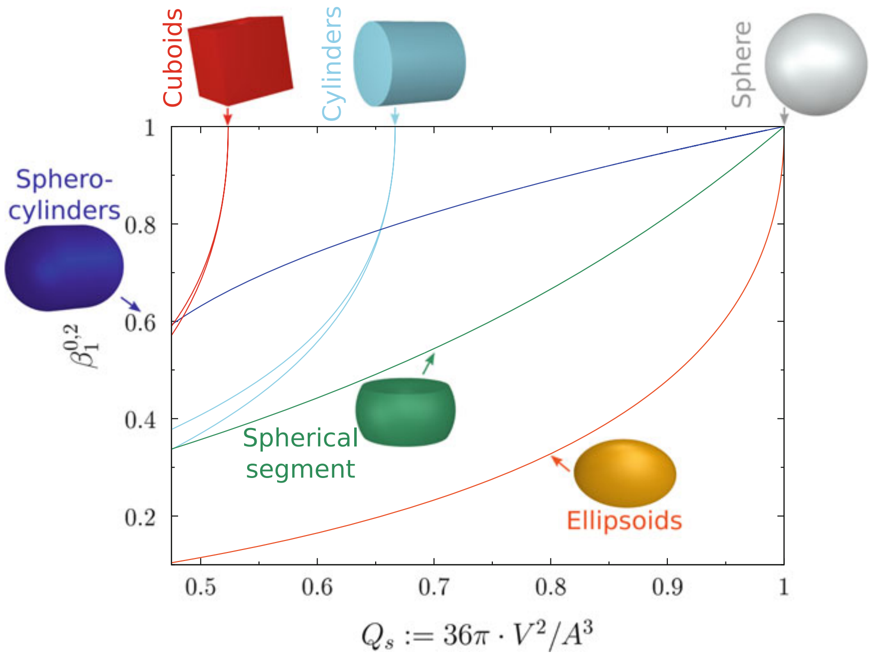

Fig. 13.3 Anisotropy indices for exemplary single bodies. The curves show the anisotropy indices for a family of objects with different aspect ratios

correlated to the coordination number in a map of phases [59] similar to our map of average anisotropies in the following Sect. 13.4.

In Fig. 13.3, the anisotropy index $\beta_1^{0,2}$ from Eq. (13.4) is compared to the isoperimetric ratio Q_s for some exemplary single bodies. Only for a sphere, both $\beta_1^{0,2} = 1$ and $Q_s = 1$. For a cube, $\beta_1^{0,2} = 1$ but $Q_s = \pi/6 \approx 0.52$. The two (red) lines for cuboids in Fig. 13.3 correspond to either an elongation or a contraction of one of the sidelengths of a cube. Similarly, the two (cyan) lines show the anisotropy indices of cylinders with a varying ratio of their heights to their diameters, which can be larger or smaller than unity. If the ratio is exactly equal to one, $\beta_1^{0,2} = 1$. If spherical caps are added to the cylinder, it is called a sphero-cylinder. In the limit of a vanishing ratio of height to diameter, it becomes a sphere. The spherical segments in Fig. 13.3 are defined by cutting a sphere with two parallel hyperplanes. More precisely, they are the intersections of the unit sphere with $[-1, 1]\times[-1, 1]\times[-h, h]$. The (green) line represents different values of this height h. Finally, Fig. 13.3 plots the anisotropy indices of prolate ellipsoids (where the two smaller principal axes are equally long) with different aspect ratios (orange line).

13.4 Averages and Map of Mean Anisotropy Indices

First, we compare for the tessellations discussed in Sect. 13.2 the mean values of the Minkowski functionals and tensors of the typical cells.

These averages allow us to construct a "map of anisotropy" [45], see Fig. 13.4. Each tessellation is represented by its mean anisotropy index $\langle\beta_1^{0,2}\rangle$, see Eq. (13.4), and the parameter Q from Eq. (13.10). The map of anisotropy provides a kind of classification of the tessellation models based on their anisotropy, which can, e.g., help to choose the appropriate model for applications.

It also reveals how some of the models can be related to each other. For example, the systems of equilibrium hard spheres at different global packing fractions connect, in the two limits of vanishing or maximal packing fraction, the uncorrelated Poisson point process (+) and the perfectly ordered close packing on a lattice (gray dots). The anisotropy indices for the equilibrium hard spheres are represented by two solid (black) lines. The curve separates into an ordered and a disordered branch because of the disorder-to-order phase transition, see Sect. 13.2.3.1. (For clarity of presentation, we connect single data points close to each other by a solid line.)

At the densest global packing fraction $\phi_{\max} = \pi/\sqrt{18} \approx 0.74$ [30], the spheres form a face-centered-cubic crystal or one of its stacking variants. Due to

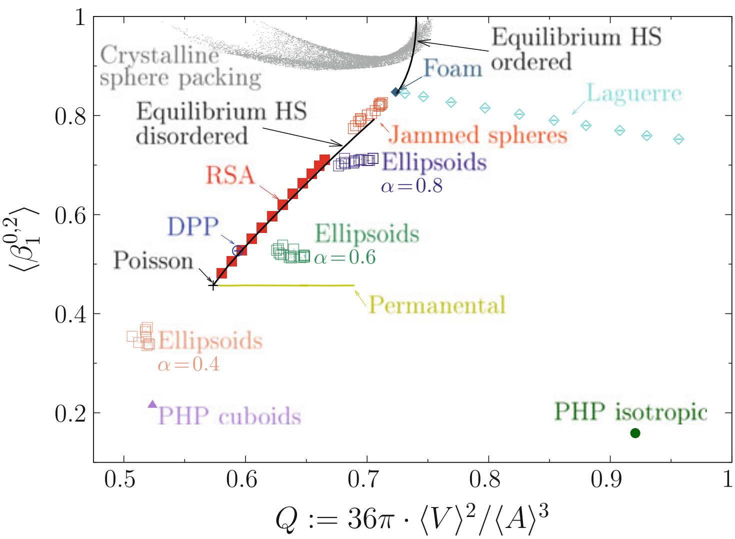

Fig. 13.4 Map of local anisotropy for various tessellations ranging from Poisson and equilibrium hard sphere (HS) Voronoi tessellations, over PHP tessellations to cells in crystalline sphere packings. For each tessellation, the mean anisotropy index $\langle\beta_1^{0,2}\rangle$, see Eq. (13.4), and the parameter Q from Eq. (13.10) are shown. For the jammed ellipsoid packings, experimental data is shown for different aspect ratios α. The map of anisotropy reveals structural differences between the models, but also how they are related to each other

the symmetry of these crystals, the single cells are perfectly isotropic w.r.t. the anisotropy index $\beta_1^{0,2}$ of the second-rank Minkowski tensor. The cells in the other crystalline sphere packings (gray dots) are, as expected, also either isotropic or only slightly anisotropic w.r.t. $\beta_1^{0,2}$.

For the RSA process (■), different points in the figure correspond to different global packing fractions. The Poisson point process (+) is quite irregular, and its Voronoi cells are therefore rather anisotropic. In the dilute limit the RSA process, like the equilibrium hard spheres, have a structure similar to a Poisson point process. The same holds for any dilute system of hard particles where the distances between the particles compared to the size of the single particles diverge. In such a case, collisions and thus interactions between the particles can be neglected, and the Voronoi diagram is well approximated by a Poisson Voronoi tessellation. However, the denser the hard-particle systems get, the more correlated and ordered they become. Therefore, the typical cells become more isotropic w.r.t. both $\beta_1^{0,2}$ and the ratio Q. Put differently, an increasing packing fraction decreases the anisotropy in the hard-particle systems ($\langle\beta_1^{0,2}\rangle$ gets closer to unity).

In Fig. 13.4, the mean anisotropy indices of the DPP (∘) are similar to that of an RSA pattern (■). However, the mean values for the packings of jammed ellipsoids (□) are distinctly different. For ellipsoids with an aspect ratio $\alpha = 0.4$, the corresponding Set Voronoi cells are on average more anisotropic than Poisson Voronoi cells.

The different points for the Laguerre tessellations (⋄) correspond to different polydispersities of the underlying hard sphere packings, that is different degrees of variation in the sphere volumes. A stronger polydispersity decreases $\langle\beta_1^{0,2}\rangle$, i.e., the typical cell is more anisotropic. At the same time, however, the ratio Q increases, for which we can provide a heuristic argument. The estimate $\langle\beta_1^{0,2}\rangle$ of the mean value for a typical cell is dominated by a vast number of small anisotropic cells. The same would apply to the mean isoperimetric ratio $\langle Q_s\rangle$ of a typical cell. However, Q is based on the ratio of mean volume and surface area. These are more strongly influenced by large cells while small cells contribute values close to zero. In the Laguerre tessellations studied here, large cells are on average more isotropic. Therefore, the ratio Q increases with increasing polydispersity. A comparison of $\langle\beta_1^{0,2}\rangle$ to Q, which is the ratio of mean values, can thus distinguish more systems than a comparison to $\langle Q_s\rangle$, which is the mean isoperimetric ratio of the typical cell.

The PHP tessellations are also strongly polydisperse, as we will show in the following Sect. 13.5. The range of observed cell volumes covers several orders of magnitude. Similar to the polydisperse Laguerre tessellations, the isotropic PHP tessellations (•) exhibit a small anisotropy index $\langle\beta_1^{0,2}\rangle$ but a large ratio Q. This does not hold for the PHP tessellations with cuboids as cells (▲), where both $\langle\beta_1^{0,2}\rangle$ and Q are small. For an isotropic distribution of the hyperplanes, the shape of a large typical cell is, with high probability, close to a that of ball [36]. However, the most isotropic shape of a cuboid is the cube. We indeed find for the PHP tessellations with cuboids as cells a value of Q that is within error bars equivalent to Q_s of a cube, see Fig. 13.3.

The average anisotropy index $\langle\beta_1^{0,2}\rangle$ of a typical cell in a statistically isotropic PHP tessellation is more anisotropic than the average anisotropy of a typical cell in a PHP tessellation with cuboids as cells. As expected, the anisotropy indices of the typical cell differ for different orientation distributions of the hyperplanes. However, they are independent of the intensity of the hyperplanes, because the tessellation model itself and the two shape indices are scale free. This follows for Q directly from the mean value formulas in [77]. The intensity only defines a unit of length that can be chosen arbitrarily.

The data for the permanental point process is represented by the solid (yellow) line which connects the single data points. Because a single realization of a permanental point process is here the outcome of an inhomogeneous PPP, the pattern can locally be very similar to a homogeneous PPP. Therefore, $\langle\beta_1^{0,2}\rangle$ of the typical Voronoi cell is for the models that we have simulated similar to $\langle\beta_1^{0,2}\rangle$ of a typical Poisson Voronoi cell. However, with increasingly anisotropic underlying Gaussian random fields, the index Q becomes larger (possibly because of a stronger polydispersity of the cells like for the Laguerre tessellations).

For the monodisperse foam (◆), the jammed sphere packings (□), and the Laguerre tessellations in the monodisperse limit (◇), the anisotropy takes very similar values. The tessellations are all related in that they are at least based on the Voronoi tessellation of a rather dense and disordered packing of hard spheres, see Sect. 13.2. (Note that an equilibrium hard-sphere liquid at the same global packing fraction is significantly more regular than these disordered packings.) However, there are also distinctive geometric differences between these systems, which this coarse analysis cannot capture. For example, the faces of a foam cell are curved [48].

Moreover, the anisotropy of the Voronoi tessellations for the RSA and equilibrium hard-sphere systems as well as for the DPP, are at least for the range of parameters chosen here very similar. All three point processes have in common that they are purely repulsive.

13.5 Shape Distribution Functions

Going beyond the average, we consider the full probability density function of the Minkowski functionals of the typical cell as well as of the anisotropy index.

From the data (described in Sect. 13.2), we determine the EPDFs of the normalized Minkowski functionals and of the normalized anisotropy indices, which are derived from the Minkowski tensors. Figures 13.5 and 13.6 show the resulting curves.

How sensitive is this description of the qualitative behavior of several features of the local structure? Figs. 13.5 and 13.6 clearly show that the probability density functions of the Minkowski functionals are distinctly qualitatively different for the Voronoi or hyperplane tessellations. At least for the here simulated models of PHP (or equivalently STIT) tessellations, the probability density function of the volume appears to be monotonically decreasing at least for a large range of

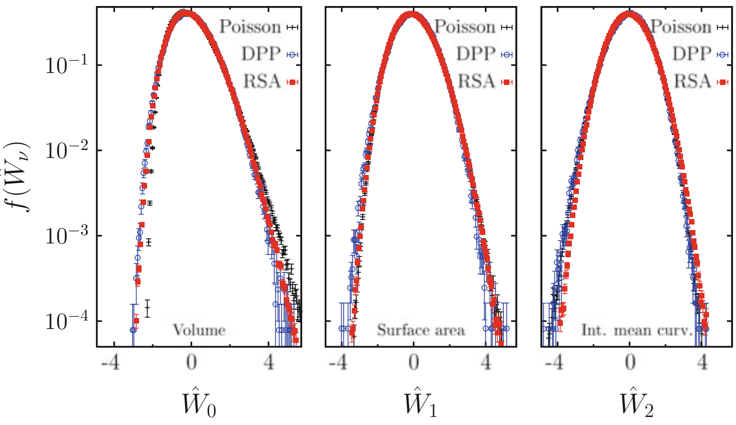

Fig. 13.5 Estimated probability density functions f of the normalized Minkowski functionals $\hat{W}_\nu$, see Eq. (13.8), for cells in Voronoi tessellations of three different point processes: Poisson point process (see Sect. 13.2.2.1), DPP (see Sect. 13.2.2.2), and RSA (see Sect. 13.2.2.4)

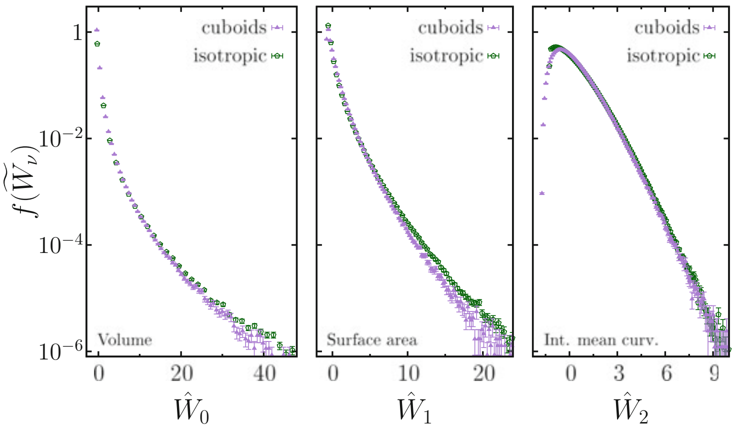

Fig. 13.6 Estimated probability density functions f of the normalized Minkowski functionals $\hat{W}_\nu$, see Eq. (13.8), for cells in PHP tessellations with different orientation distributions of the hyperplanes (see Sect. 13.2.4.1)

volumes. In contrast to this, the distributions of the Voronoi cell volumes show a clear maximum close to the mean values.

In the latter case, the distributions are for several point processes well-known to be in good agreement with (generalized) Gamma distributions [4, 5, 41, 56, 80, 101]. However, statistically significant deviations have been found for jammed particle packings, see, e.g., [46, 88].

The occurrence of Gamma distributions in tessellations driven by Poisson processes of points or flats was observed in the seminal mathematical work [70, 74]. Even for Poisson point processes on very general spaces the intensity measure of certain random sets is conditionally Gamma-distributed [114]. A systematic and unifying explanation of these phenomena in a Euclidean setting was given in [11]; see also [10]. One of the results in [11, 74] ascertains (for $n = 3$) that the distribution of the $(n-2)$-nd Minkowski functional of the typical cell of a statistically isotropic Poisson hyperplane tessellation is conditionally Gamma-distributed given the number m of neighbors. The shape parameter of this Gamma distribution is just $m - n$. Hence, the unconditional distribution is a mixture of Gamma distributions (with the same scale parameter) with respect to the distribution of the number of neighbors of the typical cell.

In a Poisson hyperplane tessellation, the number of neighbors of a cell coincides with the number of $(n-1)$-dimensional facets. Note that this is in general not the case in a STIT tessellation. (However, the distribution of the number of $(n-1)$-facets of the typical cell and the distributions of the Minkowski functionals are the same as for the corresponding PHP tessellation.)

According to [11], in both cases the probability density function of the $(n-2)$-nd Minkowski functional of the typical cell can be expressed by the probability mass function p of the number of $(n-1)$-dimensional facets:

$$f(W_{n-1}) = \frac{1}{2\kappa_{n-1}} \sum_{m=4}^{\infty} p(m) \cdot g(m-n, \gamma; \frac{W_{n-1}}{2\kappa_{n-1}}), \tag{13.11}$$

where $g(\alpha, \beta; x) = \beta^{\alpha} x^{\alpha-1} \mathrm{e}^{-x\beta} / \Gamma(\alpha)$ is the probability density function of the Gamma distribution. This is demonstrated in Fig. 13.7 (for $n = 3$), where the mixture of Gamma distributions given in Eq. (13.11) (the solid line in Fig. 13.7b) is in very good agreement with the EPDF of W_2. In a three-dimensional stationary random hyperplane tessellation (with finite intensity), the average number of facets of the typical cell is $n_{3,2} = 6$ [90, Eq. (10.35), p. 484], which is in agreement with the sample mean $\langle m \rangle = 5.9994$ (where the standard error of the mean is ≈ 0.0004).

Equation (13.11) shows that the information content of the number of facets and of the Minkowski functional W_{n-1} is somehow related. However, the latter is an additive, continuous measure where small changes in the positions and orientations of the hyperplanes correspond to only small changes in the Minkowski functionals. This is in contrast to the number of facets which is a topological index, which can change distinctly and discontinuously for small variations of the positions and orientations of the hyperplanes. Moreover, in experimental observations, the resolution of small facets might not be possible. Therefore, the Minkowski functionals are more robust structure characteristics, which are also suitable for an analysis of noisy data sets.

The above described phenomenon extends to typical faces of lower dimensions [11]. For other intrinsic volumes, there seems to be no mathematical argument supporting the occurrence of mixed Gamma distributions.

For the Poisson Voronoi tessellation, the situation is more complicated. There is no obvious reason for the typical cell to have conditionally Gamma-distributed Minkowski functionals. There is, however, the concept of the (typical) Voronoi

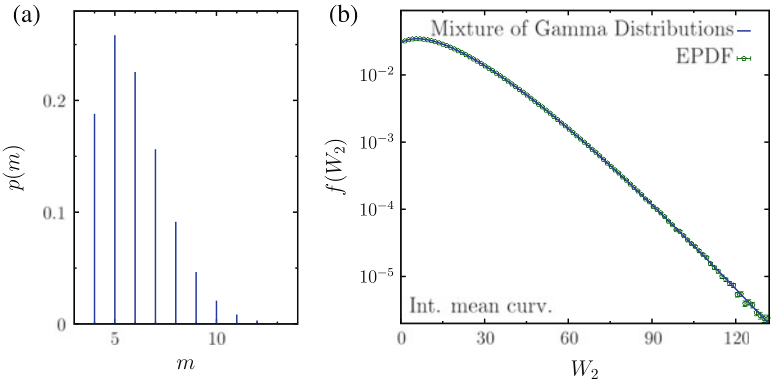

Fig. 13.7 Relation between the distribution of the number of faces and the Minkowski functional W_2 in an isotropic PHP tessellation: (**a**) the empirical probability mass function p of the number m of two-dimensional facets; (**b**) the EPDF $f(W_2)$ of the integrated mean curvature of a cell (marks) is in very good agreement with the mixture of Gamma distributions given in Eq. (13.11) and using the empirical probability mass function p in (**a**)

flower, whose geometry is closely connected to the typical cell. Given the number m of neighbors of the typical cell, the volume of this flower has a Gamma distribution with shape parameter m [74]. Again this can be extended to flowers of typical faces of lower dimensions [10, 11]. There are some reasons to believe that similar results hold for other Minkowski functionals.

The agreement of the EPDFs of the normalized Minkowski functionals of Voronoi cells for very different point processes reveals some limitations of this univariate qualitative shape descriptors. For physically different systems like the relatively long-ranged DPP, the uncorrelated PPP, or a non-equilibrium RSA system with only short-ranged interactions, the EPDFs are at least very similar. For the systems studied here, there are small but statistically significant deviations only for the volume distribution of the Poisson Voronoi cells and for the distribution of the integrated mean curvature for the cells of an RSA process. On the one hand, this reveals an interesting similarity in the local structure of Voronoi diagrams of random point processes. On the other hand, this agreement for very different systems indicates that the EPDF of the normalized Minkowski functionals is not sensitive enough to detect the structural differences between these systems. Note that the global structure differs distinctly for the four systems. Such differences in the global structure of the Poisson point process, the equilibrium hard-sphere liquid, or a non-equilibrium jammed packing of hard spheres is, for example, discussed in detail in [46].

Also the EPDFs of the anisotropy indices show a qualitatively different behavior for both PHP tessellations (and thus for the corresponding STIT tessellations) and the Voronoi tessellations, see Fig. 13.8. However, the EPDFs of both the anisotropy indices $\beta_1^{0,2}$ and $\beta_2^{0,2}$ appear to be qualitatively indistinguishable for the three different Voronoi tessellations of a Poisson point process, the RSA process, or

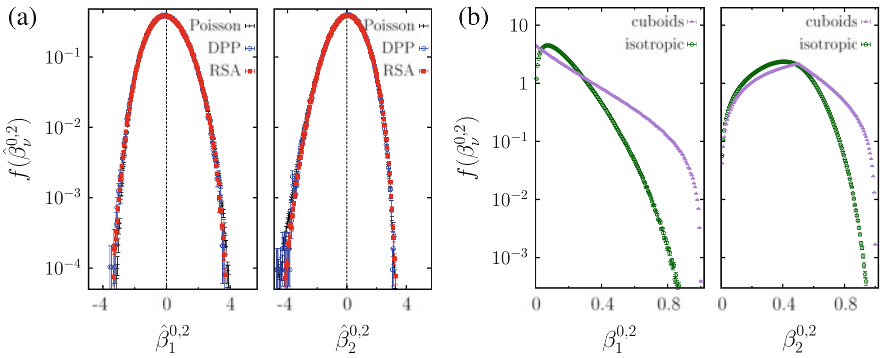

Fig. 13.8 Estimated probability density functions f (**a**) of the normalized Minkowski tensor anisotropy indices $\hat{\beta}_\nu^{0,2}$, see Eq. (13.9), for cells in Voronoi tessellations and (**b**) of the anisotropy indices $\beta_\nu^{0,2}$ for cells in PHP tessellations

the DPP (like for the Minkowski functionals). In an analysis based on a single characteristic, we do not find a qualitatively different behavior in the local structure of these three models. Depending on the chosen parameters of the models there can be distinct quantitative differences, see Fig. 13.4. However, for some sets of parameters also these quantitative differences nearly vanish.

13.6 Local Joint Characterization via Volume and Anisotropy

For a more sensitive characterization of the local structure, we need to take the relations between different characteristics into account. By doing so, we reveal a qualitatively different behavior of local structural characteristics even for systems where the single characteristics exhibited a qualitatively indistinguishable EPDF in the previous Sect. 13.5. Moreover, this sensitive joint local analysis of different characteristics allows for intuitive geometric insights.

Following the analysis of [86], we perform a conditional analysis based on the cell volume and consider the shape of small or large cells separately. We quantify the anisotropy of the cells by the Minkowski tensor $W_1^{0,2}$ conditional on their volume. In other words, we estimate the conditional expectation $\langle\beta_1^{0,2}\rangle_V$ of the anisotropy index $\beta_1^{0,2}$ as a function of the cell volume V. More precisely, the condition is on the cell volume being in an interval $[V - \Delta V; V + \Delta V)$, i.e., we bin the cell volumes and then estimate $\langle\beta_1^{0,2}\rangle_V$ separately for each bin.

13.6.1 Distinguishing Local Structures

Figure 13.9 shows the resulting curves for the Voronoi tessellations of the PPP, RSA process, and DPP. In contrast to the shape description by a single index in Sect. 13.5, the new analysis combining two shape indices can qualitatively distinguish the different point processes.

The Poisson points are non-interacting. The hard spheres in the RSA process are rigid, i.e., perfectly repulsive at contact. The points in the DPP can be interpreted as "soft" repulsive particles. Although unlikely, the particles can get arbitrarily close to each other, which is in contrast to the RSA process.

The cells in the Poisson Voronoi tessellation are on average most anisotropic for all cell volumes. The mean anisotropy index conditional on the cell volume is also (at least for a large range of cell volumes) smaller than for the Voronoi cells of the other systems. The anisotropy index increases monotonically with increasing cell volumes, i.e., larger cells are on average more isotropic than smaller ones. It is well known that the shape of a typical large cell in the Poisson Voronoi tessellation converges to a sphere in the limit of arbitrarily large cell volume [37]. Therefore, also the anisotropy parameter $\langle\beta_1^{0,2}\rangle_V$ must approach unity, i.e., perfect isotropy.

As expected, the RSA process of hard-spheres at different global packing fractions $\phi = 0.1$ and $\phi = 0.2$ have qualitatively similar curves in Fig. 13.9. Because a hard-sphere system at finite packing fraction is more ordered than a Poisson point process, it can be expected that a typical cell is more isotropic in the hard-sphere system. This corresponds to an increase in the mean anisotropy index. For large cell volumes, the anisotropy index also increases as a function of the cell

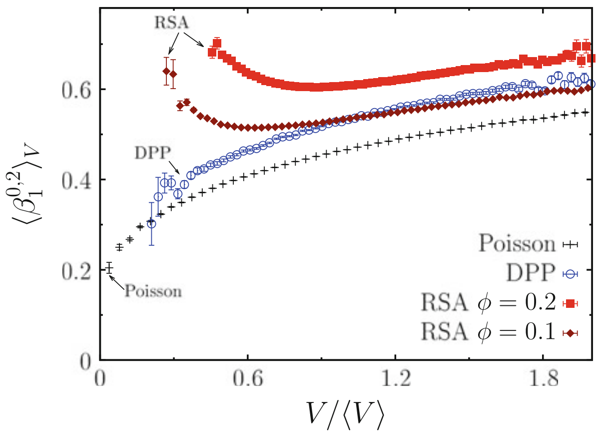

Fig. 13.9 Anisotropy as a function of cell volume for Voronoi tessellations of a Poisson point process, a DPP, or RSA hard-sphere process with volume fractions $\phi = 0.1$ or 0.2. The combined analysis based on volume and anisotropy reveals the qualitatively different behavior of the different models

volume. However, in contrast to the uncorrelated Poisson point process, the curves for the RSA process exhibit a minimum.

For small cells, the trend reverses and the smaller the cell, the larger the anisotropy index gets. The smallest possible Voronoi cell for a hard sphere packing is well-known to be a regular dodecahedron where the central sphere touches all faces. This is the dual to an icosahedral arrangement of the neighboring spheres, which is the locally densest possible configuration with a maximum of 12 contacting neighbors. (The volume of the smallest Voronoi cell for hard spheres divided by the mean Voronoi volume is given by $V/\langle V\rangle = 6/(\pi\sqrt{5/2+11/10\sqrt{5}})\cdot\phi \approx 0.1729\cdot\phi$ and rather small for the here chosen global packing fraction ϕ.) In this limit, the anisotropy index converges to unity (because a regular dodecahedron appears perfectly isotropic w.r.t. $\beta_\nu^{0,2}$). Therefore, there is a minimum in the anisotropy index as a function of the cell volume. The curve is qualitatively different from the corresponding curve for a Poisson Voronoi tessellation.

Moreover, it is also qualitatively different from the DPP. Interestingly, the repulsive particles in the DPP show an intermediate behavior between the hard spheres and the non-interacting Poisson points. For large cells, the anisotropy of the Voronoi cells of the DPP is comparable to those of the RSA process. It is more regular than a PPP. However, in contrast to the hard-sphere system, small cells get on average more anisotropic ($\langle\beta_1^{0,2}\rangle$ decreases). These small cells are more similar to those in the irregular PPP. A heuristic explanation of this behavior is that it is unlikely but possible that two points get close to each other. However, it is then very unlikely that also a third point is located nearby. Therefore, the two corresponding cells would, in this case, tend to be elongated because they are strongly restricted in the direction of the nearest neighbors.

Be reminded that the typical cells in a dilute hard-sphere gas or a weakly interacting DPP are very similar to those of a PPP, i.e., the mean values are very close to each other, see Fig. 13.4. Moreover, the EPDFs of the Minkowski functionals or of the anisotropy indices are qualitatively nearly indistinguishable, see Figs. 13.5 and 13.8a.

Analyzed by single characteristics alone, the local structure seemed at least qualitatively the same for the three different point processes. However, the qualitative behavior of the local structural characteristics is actually not the same. The rescaled univariate EPDFs were only not sensitive enough to find this qualitatively different behavior. The characterization based on both the volume and the anisotropy can clearly and qualitatively distinguish the local structure of the Voronoi tessellations for the different generating processes (PPP, DPP and RSA).

13.6.2 Equilibrium and Non-equilibrium Hard Particles

Using this improved local shape analysis, Fig. 13.10 compares the local structure of the equilibrium hard-sphere liquids (a) and the non-equilibrium jammed ellipsoids (b).

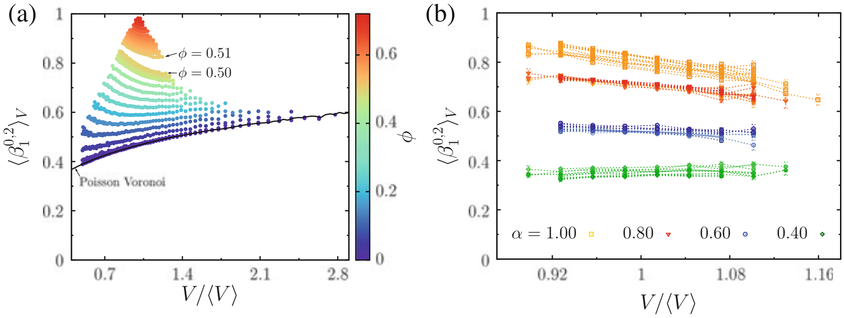

Fig. 13.10 Anisotropy as a function of cell volume for hard-particle systems: (**a**) equilibrium hard-sphere liquids from the dilute limit (i.e., vanishing packing fraction) to nearly crystalline structures; the color scale indicates the varying packing fraction ϕ; the *black line* corresponds to the Poisson point process (see also Fig. 13.9); the gap at roughly $\phi = 0.5$ results from the freezing transition; (**b**) experimental data of random jammed ellipsoids for different aspect ratios α (indicated by *different colors*); the different curves with the same colors represent different experimental realizations with varying global packing fraction ($\phi = 0.54 - 0.68$) [86]

In Fig. 13.10a, the curve for the Poisson point process from Fig. 13.9 is extended to larger cell volumes. As mentioned above, it must converge to unity. However, the simulation reveals that it converges rather slowly. Only the extremely dilute hard-sphere systems are numerically difficult to distinguish from the PPP (because of the vanishing number of close neighbors and thus of small Voronoi cells). Already for equilibrium liquids at relatively small global packing fractions, the anisotropy as a function of the volume deviates distinctly from the uncorrelated PPP.

With increasing packing fraction, the range of observed Voronoi cell volumes shrinks because the configurations become more regular and thus the fluctuations in the Voronoi volume decrease. In the limit of maximal packing fraction, only a single value of the volume is possible (which corresponds to the dual of the unit cell).

At a global packing fraction of about $\phi = 0.5$, a gap in the curves of the anisotropy index is observed, see Fig. 13.10a. This is related to the solid-liquid hard-sphere phase transition. Our samples are initially prepared in a crystalline state before equilibration. Therefore, the transition occurs at the lower end of coexistence regime which is for an equilibrium hard-sphere system between $\phi \approx 0.494$ and 0.545 [41].

In contrast to the globally loose fluids where the system behaves like the PPP, the anisotropy index in globally dense systems decreases monotonically as a function the cell volume. This shows that in dense hard-sphere liquids the locally dense configurations are more ordered than the looser ones, and thus more isotropic.

For the jammed ellipsoid packings, the monotony of this functions changes with the aspect ratio of the ellipsoids. Like in dense equilibrium hard-sphere systems, smaller cells are more isotropic. For very oblate ellipsoids with $\alpha = 0.60$ or 0.40, the anisotropy in Fig. 13.10b appears to be rather independent of the cell volume.

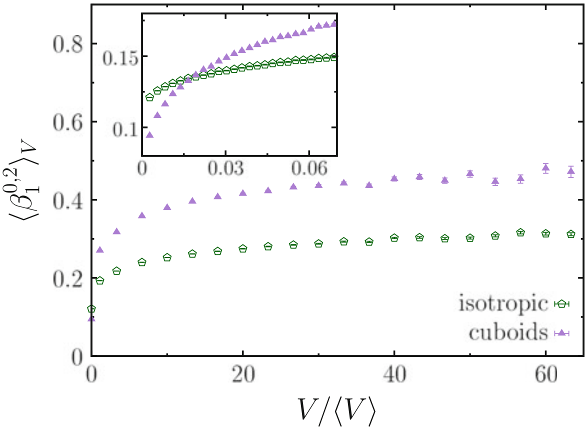

Fig. 13.11 Anisotropy as a function of cell volume for PHP tessellations with either an isotropic orientation distribution of the hyperplanes or with three allowed directions, i.e., all cells are cuboids. For most volumes V, the cells in the isotropic tessellation are on average more anisotropic than the cuboid shaped cells. However, the *inset* shows that this changes for very small cells, for which the cells in the isotropic system are more isotropic than the cuboid-shaped cells

13.6.3 Poisson Hyperplane Tessellations

Figure 13.11 displays the results for the PHP tessellations with either an isotropic orientation distribution of the hyperplanes or with three allowed directions, i.e., all cells are cuboids.

The main plot shows the anisotropy as a function of the cell volume for large cells, which get exceedingly unlikely with increasing cell volume, see Fig. 13.6. These large cells are on average more isotropic than a typical cell in the tessellation, and the cuboid-shaped cells are more isotropic than the cells in the statistically isotropic tessellation (i.e., larger values of $\langle \beta_1^{0,2} \rangle_V$). However, the inset shows that this order reverses for small cells, where the cells in the statistically isotropic system are more isotropic than the cuboid shaped cells.

13.7 Conclusions

Random or disordered tessellations appear in very different physical, chemical, or biological systems as well as in life sciences. Their complex structure calls for advanced mathematical tools that can quantify their geometry.

The Minkowski functionals and tensors allow for a robust yet concise characterization of the shape of single cells. They are powerful tools to narrow the choice of possible underlying stochastic processes.

Here, we have applied, in a theory-based simulation study, such an analysis to a variety of important and common tessellations, see Sect. 13.2.

- The "map of anisotropy" from Sect. 13.4 analyzes the relationship between the dimensionless ratio $\langle V\rangle^2/\langle A\rangle^3$ of average cell volumes to average cell areas and the degree of cell elongation quantified by the eigenvalue ratio $\langle\beta_1^{0,2}\rangle$ of the interfacial Minkowski tensor $W_1^{0,2}$. It provides an overview of the various tessellations considered here. It can highlight relations between different point processes but also reveals some structural differences.
- The probability density functions of single local characteristics in Sect. 13.5 can also clearly distinguish two different types of tessellations such as the PHP and Voronoi tessellations. However, the rescaled probability density functions for Voronoi cells from different stochastic processes can be qualitatively similar. On the one hand, this agreement reveals interesting relations between the models. On the other hand, it does not imply that the local structure, i.e., the shape distribution of single cells, is the same for these physically quite different point processes.
- To detect qualitative differences in the local structure, we combine different characteristics. More precisely, we determine the mean anisotropy index as a function of the cell volume, see Sect. 13.6. Thus, we find a qualitatively different behavior, for example, for determinantal point processes and equilibrium hard spheres. We also use this analysis for additional insights into the Poisson hyperplane or STIT tessellations, e.g., discussing the different anisotropy for small or large cells. The numerical tools which we apply here are efficient and can be easily used for a detailed structure analysis of any tessellation of interest.

We have thus demonstrated how the Minkowski functionals and tensors can serve as sensitive and robust local shape descriptors. Given a simple object like a single cell and starting with simple and efficient shape indices like volume and surface area (following the rule of parsimony), the straightforward generalization to Minkowski functionals and tensors allows for a comprehensive shape analysis. Each additive, continuous, and motion invariant or covariant tensor is essentially a linear combination of Minkowski tensors [2, 3, 29]. Moreover, these geometrical shape descriptors are more robust than so-called "topological measures," like the number of faces, vertices, or edges. Such topological quantities are sensitive to noise in that a small change can strongly alter the topology of the cell. For example, whether or not a small additional face is resolved can lead to faces with a very different number of vertices.

Prominent examples are also the so-called bond-orientational order parameters, which are standard tools in condensed matter physics to characterize particle arrangements [103]. They are based on the definition of a neighborhood, where different choices can even lead to qualitatively different behavior of the bond-orientational order parameters. Moreover, because of the discrete nature of neighborhood, the bond-orientational order parameters can change discontinuously for infinitesimally small changes in the particle positions. This can be avoided by a morphometric approach that assigns weights to the neighbors. These weights lead to robust measures that are continuous in the particle coordinates and that are equivalent to the Minkowski tensors presented here.

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Chapter 14
Stereological Estimation of Mean Particle Volume Tensors in $\mathbb{R}^3$ from Vertical Sections

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Abstract In this chapter, we discuss stereological estimation of mean particle volume tensors in $\mathbb{R}^3$ from vertical sections. We consider a particle process of compact particles that can be represented as a stationary marked point process. Under the assumption that the particle distribution is invariant under rotations around a fixed axis, called the vertical axis, we show how the mean particle volume tensors can be estimated consistently (in a probabilistic sense) from observations in vertical sections through a sample of particles. In a simulation study, the new estimator has a superior behaviour compared to an earlier estimator based on observations in several optical planes.

14.1 Introduction

Volume tensors, or more generally Minkowski tensors, have been used with success for shape and orientation description of spatial structures in materials science, see Chap. 13 or [2, 5, 11, 12]. An early example from the biosciences is given in [3].

Information about shape and orientation from tensors can fairly easily be determined if a 3D voxel image of the spatial structure under study is available. However, for biostructures like cells it is even in conventional microscopy difficult to construct such voxel images. For such cases, local stereological methods of estimating volume tensors from observations in planar sections have been developed in [9] and [14]. A particular focus has been on methods of obtaining information on shape and orientation for particle populations.

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In this chapter, we give an introduction to these methods and also present a new estimator that has great potential use in optical microscopy.

14.2 The Particle Model

Let X be a particle process of compact particles in $\mathbb{R}^3$. We assume that the process can be represented as a stationary marked point process

$$\{[x(K); K - x(K)] : K \in X\}.$$

Here, $x(K) \in K$ is a reference point associated to the particle $K \in X$ and the mark $K - x(K)$ is the particle translated such that its reference point is at the origin o. We let $\mathbf{K}_0$ be a random compact set with distribution equal to the particle mark distribution $\mathbb{Q}$, say. The random set $\mathbf{K}_0$ may be regarded as a randomly chosen particle or a typical particle with o as its reference point. The intensity of the marked point process, that is the mean number of reference points per unit volume, is denoted by λ. For a detailed description of stationary particle processes and the definition of the mark distribution, see [10, Chap. 3].

Our aim is to estimate the mean particle volume tensors $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$ where, as in the previous chapters, the volume tensor of rank $r \in \mathbb{N}_0$ of a compact set K is given by

$$\Phi_3^{r,0}(K) := \frac{1}{r!} \int_K x^r \,\mathrm{d}x. \tag{14.1}$$

Recall that for $x = (x_1, x_2, x_3)$, x^r is the rank r tensor that can be identified with an array of elements of the form

$$(x^r)_{i_1 i_2 i_3} := x_1^{i_1} x_2^{i_2} x_3^{i_3} \qquad \text{for } i_1, i_2, i_3 \in \{0, \dots, r\} \text{ with } \sum_{j=1}^{3} i_j = r.$$

In contrast to Chap. 2, we thus identify the r-linear mapping $\Phi_3^{r,0}$ with its coefficients with respect to an arbitrarily chosen basis. The integration in (14.1) is to be understood elementwise.

The estimation will be based on a sample of particles, collected as those particles with reference point in a full-dimensional compact sampling window W,

$$\{K \in X : x(K) \in W\}. \tag{14.2}$$

For an illustration of the sampling procedure, see Fig. 14.1 (left).

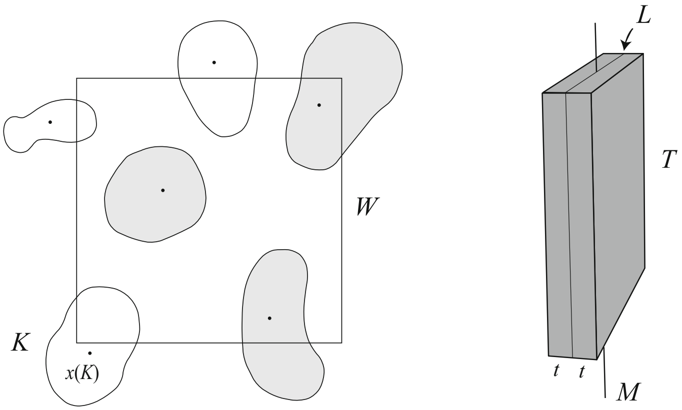

Fig. 14.1 *Left:* a particle K is sampled if its reference point $x(K)$ belongs to W. Sampled particles are shown hatched. *Right:* a vertical slice T of thickness $2t$. The central plane L contains the vertical axis M

Due to the stationarity of the particle process X, we have for any $\mathbb{Q}$-integrable function f on compact subsets of $\mathbb{R}^3$

$$\mathbb{E} \sum_{K \in X,\, x(K) \in W} f(K - x(K)) = \lambda V_3(W) \mathbb{E} f(\mathbf{K}_0),$$

where V_3 denotes volume. If we let $N(W)$ be the number of sampled particles, it follows that

$$\frac{\mathbb{E} \sum_{K \in X,\, x(K) \in W} f(K - x(K))}{\mathbb{E} N(W)} = \mathbb{E} f(\mathbf{K}_0). \tag{14.3}$$

In particular if f in (14.3) equals the elements of $\Phi_3^{r,0}$, we find that

$$\frac{\sum_{K \in X,\, x(K) \in W} \Phi_3^{r,0}(K - x(K))}{N(W)} \tag{14.4}$$

is a ratio-unbiased estimator of $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$. For the proof of consistency of estimators, the following theorem will be useful, see [4, Corollary 12.2.V].

Theorem 14.1 *Suppose that the particle process X is a stationary ergodic marked point process. Let $\{W_n\}$ be an increasing sequence of convex bodies in $\mathbb{R}^3$ such that*

$$r(W_n) \to \infty \qquad \text{as } n \to \infty,$$

where

$$r(W) := \sup\{r \geq 0 : W \text{ contains a ball of radius } r\}.$$

Let f be a measurable function on compact subsets of $\mathbb{R}^3$ *which is integrable with respect to the mark distribution* $\mathbb{Q}$*. Then, almost surely,*

$$\frac{1}{V_3(W_n)} \sum_{K \in X, x(K) \in W_n} f(K - x(K)) \to \lambda \mathbb{E} f(\mathbf{K}_0),$$

as $n \to \infty$.

Using Theorem 14.1 with f equal to the elements of $\Phi_3^{r,0}$ and $f \equiv 1$, it is seen that the estimator (14.4) is consistent when X is ergodic and the increasing sequence $\{W_n\}$ of bounded convex windows satisfies $r(W_n) \to \infty$ as $n \to \infty$.

The estimator (14.4) requires that the volume tensor $\Phi_3^{r,0}$ can be determined on the sampled particles. If we do not have direct access to the particles in 3D, we need to develop stereological methods of estimating the volume tensors of the sampled particles from planar sections.

Stereological estimators of volume tensors based on observations in vertical slices have been derived in [9] and [14]. In a model-based setting, these estimators are valid under the *restricted isotropy assumption* where the distribution of the typical particle $\mathbf{K}_0$ is invariant under rotations around a line M in the Grassmannian $G(3, 1)$ of one-dimensional linear subspaces in $\mathbb{R}^3$. The line M is called the vertical axis, but may indeed be an arbitrary fixed line through the origin.

To be more specific, let $T := L + tB^3$ be a vertical slice. Here, L is a plane through the origin, containing M, and tB^3 is a ball centred at o and with radius t, see Fig. 14.1 (right). Let

$$\widehat{\Phi}_3^{r,0}(K) := \frac{1}{r!} \int_{K \cap T} x^r G(t^2/\|p_{M^\perp}(x)\|^2)^{-1} \, \mathrm{d}x, \tag{14.5}$$

where G is the distribution function of the Beta distribution with parameters $(1/2, 1/2)$ and $p_{M^\perp}$ is the orthogonal projection on $M^\perp$. Then, cf. [14, Sect. 3 and Appendix A (online supporting information)],

$$\mathbb{E}\widehat{\Phi}_3^{r,0}(\mathbf{K}_0) = \mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0),$$

and, combining this identity with (14.3),

$$\frac{\sum_{K \in X, x(K) \in W} \widehat{\Phi}_3^{r,0}(K - x(K))}{N(W)} \tag{14.6}$$

is a ratio-unbiased (and consistent) estimator of $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$. The consistency holds under the assumptions of Theorem 14.1. This estimator will be called *the slice estimator* in the following.

Note that under a restricted isotropy assumption, the mean particle volume tensors $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$ do not vary freely. For $\mathbb{E}\Phi_3^{1,0}(\mathbf{K}_0)$ and $\mathbb{E}\Phi_3^{2,0}(\mathbf{K}_0)$, we have

$$\mathbb{E}\Phi_3^{1,0}(\mathbf{K}_0) \in M, \tag{14.7}$$

and

$$\mathbb{E}\Phi_3^{2,0}(\mathbf{K}_0) - \frac{(\mathbb{E}\Phi_3^{1,0}(\mathbf{K}_0))^2}{2\mathbb{E}\Phi_3^{0,0}(\mathbf{K}_0)} = B \begin{pmatrix} \eta & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} B^T, \tag{14.8}$$

where B is any orthogonal matrix with first column spanning M [14, p. 819]. The slice estimator may be adjusted such that constraints of this type are fulfilled [14, pp. 821–822].

14.3 Stereological Estimation from Vertical Sections

In this section, we will show that, under the restricted isotropy assumption, $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$ can be estimated from observations only in the central plane L of the slice T. To the best of our knowledge, this estimator has not been described before.

To show this claim, we assume for simplicity that the vertical axis M is the z-axis and use cylindrical coordinates to obtain

$$\begin{aligned} \mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0) &= \frac{1}{r!}\mathbb{E}\int_{\mathbf{K}_0} x^r \, \mathrm{d}x \\ &= \frac{1}{r!}\int_{z=-\infty}^{\infty}\int_{u=0}^{\infty}\int_{\theta=0}^{2\pi} P\big((u\cos\theta, u\sin\theta, z) \in \mathbf{K}_0\big) \\ &\qquad \times (u\cos\theta, u\sin\theta, z)^r u \, \mathrm{d}\theta \, \mathrm{d}u \, \mathrm{d}z. \end{aligned}$$

Using restricted isotropy, we get

$$\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0) = \frac{1}{r!}\int_{z=-\infty}^{\infty}\int_{u=0}^{\infty} P((u,0,z) \in \mathbf{K}_0) f_r(u,z) \, \mathrm{d}u \, \mathrm{d}z, \tag{14.9}$$

where $f_r(u,z)$ is the rank r tensor

$$f_r(u,z) := \int_{\theta=0}^{2\pi} (u\cos\theta, u\sin\theta, z)^r u \, \mathrm{d}\theta$$

for $u > 0$ and $z \in \mathbb{R}$. The elements of the tensor $f_r(u, z)$, $u > 0$, $z \in \mathbb{R}$, are given by

$$\begin{aligned} f_r(u,z)_{i_1 i_2 i_3} &= \int_0^{2\pi} u(u\cos\theta)^{i_1}(u\sin\theta)^{i_2} z^{i_3}\, \mathrm{d}\theta \\ &= u^{i_1+i_2+1} z^{i_3} \int_0^{2\pi} (\cos\theta)^{i_1}(\sin\theta)^{i_2}\, \mathrm{d}\theta \\ &= c_{i_1 i_2} u^{i_1+i_2+1} z^{i_3}, \end{aligned}$$

say, for $i_1, i_2, i_3 \in \{0, \ldots, r\}$ with $\sum_{j=1}^{3} i_j = r$, where

$$c_{i_1 i_2} = \begin{cases} 2\dfrac{\omega_{i_1+i_2+2}}{\omega_{i_1+i_2+1}} \dfrac{\binom{(i_1+i_2)/2}{i_1/2}}{\binom{i_1+i_2}{i_1}}, & \text{for } i_1, i_2 \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, as in previous chapters, ω_i is the surface area of the unit sphere in $\mathbb{R}^i$. It follows that

$$\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)_{i_1 i_2 i_3} = (r+1) c_{i_1 i_2} \mathbb{E}\big[\Phi_{2,L}^{r+1,0}(\mathbf{K}_0 \cap L_+)_{i_1+i_2+1, i_3}\big], \tag{14.10}$$

where

$$\begin{aligned} L &:= \{(u, 0, z) : u, z \in \mathbb{R}\}, \\ L_+ &:= \{(u, 0, z) : u > 0, z \in \mathbb{R}\}, \end{aligned}$$

and $\Phi_{2,L}^{r+1,0}(\mathbf{K}_0 \cap L_+)$ is the rank $r+1$ volume tensor of $\mathbf{K}_0 \cap L_+$, considered as a subset of L; see Chap. 5 for a precise definition of this intrinsic version of the volume tensor. Alternatively, one can use the larger set $\mathbf{K}_0 \cap L$ and obtain

$$\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)_{i_1 i_2 i_3} = (r+1)\frac{c_{i_1 i_2}}{2} \mathbb{E}\big[\Phi_{2,L}^{r+1,0}(\mathbf{K}_0 \cap L)_{i_1+i_2+1, i_3}\big].$$

If, for a compact set K, we let $\widetilde{\Phi}_3^{r,0}(K)$ be the rank r tensor with

$$\widetilde{\Phi}_3^{r,0}(K)_{i_1 i_2 i_3} := (r+1)\frac{c_{i_1 i_2}}{2} \Phi_{2,L}^{r+1,0}(K \cap L)_{i_1+i_2+1, i_3},$$

we find

$$\mathbb{E}\widetilde{\Phi}_3^{r,0}(\mathbf{K}_0) = \mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0),$$

and

$$\frac{\sum_{K \in X, x(K) \in W} \widetilde{\Phi}_3^{r,0}(K - x(K))}{N(W)} \tag{14.11}$$

is a ratio-unbiased (consistent) estimator of $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$. The consistency holds under the assumptions of Theorem 14.1. This estimator will be called *the section estimator*.

The section estimator is much simpler to implement in microscopy than the slice estimator and, furthermore, it has technical advantages. For instance, the estimator is not sensitive to shrinkage in the direction perpendicular to the slice. Both estimators rely on restricted isotropy, which must be assured in applications. Note, however, that $\mathbf{K}_0$ need not be a body of revolution around the vertical axis, but only its distribution must be invariant under rotations, fixing this axis, see also Fig. 14.3 below.

14.4 The Lévy Particle Model

We have compared by simulation the statistical behaviour of the section estimator and the slice estimator under a flexible Lévy particle model [1, 7, 8].

Under such a model, the random particle $\mathbf{K}_0$ is star-shaped with respect to a point $c_0 \in \mathbb{R}^3$ and distributed as $c_0 + \mathbf{Z}$, where $\mathbf{Z}$ is modelled as a random deformation of a fixed particle Z_0, say, which is star-shaped relative to the origin o. The random set $\mathbf{Z}$ is also star-shaped with respect to o and therefore uniquely determined by its radial function $\mathbf{R} : \mathbb{S}^2 \to [0, \infty)$ relative to o. (Recall that $\mathbf{R}(u)$ is the distance from o to the boundary of $\mathbf{Z}$ in direction $u \in \mathbb{S}^2$.) In the model, the radial function $\mathbf{R}$ is given by

$$\mathbf{R}(u) := r(u)\mathbf{X}(u), \qquad u \in \mathbb{S}^2,$$

where $r : \mathbb{S}^2 \to [0, \infty)$ is the radial function of the fixed particle Z_0 and $\mathbf{X} : \mathbb{S}^2 \to [0, \infty)$ is an isotropic non-negative Lévy-based stochastic process on $\mathbb{S}^2$ of the form

$$\mathbf{X}(u) := \int_{\mathbb{S}^2} k(u, v)\mathbf{Y}(\mathrm{d}v).$$

Here, k is chosen as the von Mises-Fisher kernel [6] and $\mathbf{Y}$ is a Gamma Lévy basis. The parameters of the stochastic process $\mathbf{X}$ are chosen such that $\mathbb{E}V_3(\mathbf{K}_0) = V_3(Z_0)$. This ensures that $\mathbf{Z}$ is a random deformation of Z_0. For more details, see [14, Sect. 6].

The set-up is illustrated in Fig. 14.2. We choose $x(\mathbf{K}_0) = o$ as the reference point for $\mathbf{K}_0$. If $c_0 \neq o$, the reference points of the particles in the resulting particle process may be non-centrally placed in the particles, as illustrated on the profile to

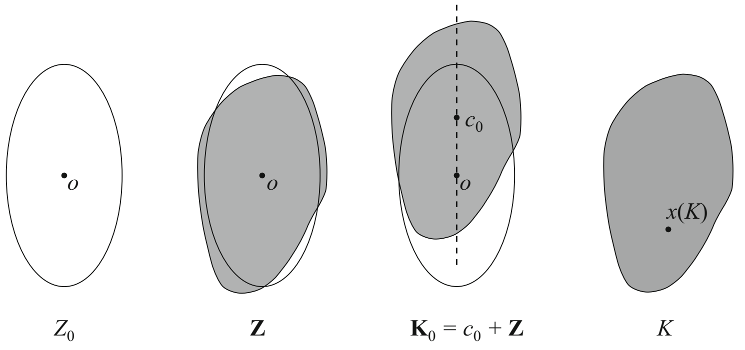

Fig. 14.2 2D illustration relating to the particle model, used in the simulation study. The typical particle $\mathbf{K}_0$ is distributed as $c_0 + \mathbf{Z}$ where $c_0 \in \mathbb{R}^3$ and $\mathbf{Z}$ is a random deformation of the ellipse Z_0. If $c_0 \neq o$, the reference points of the particles in the resulting particle process may be non-centrally placed in the particles, as illustrated on the profile to the right

the right in Fig. 14.2. The restricted isotropy assumption is fulfilled if c_0 belongs to the vertical axis M and Z_0 is a solid of revolution around M.

14.5 The Simulation Study

In this section, we compare by simulation the statistical behaviour of the slice estimator and the section estimator. We focus on the quality of the estimators of $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$ for $r = 0, 1, 2$.

We use a Lévy particle model, fulfilling the restricted isotropy assumption. The fixed particle Z_0 is chosen as a prolate ellipsoid with its longest axis parallel to the vertical axis. The mean particle volume tensors $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$, $r = 0, 1, 2$, determine the model parameters $v := \mathbb{E}\Phi_3^{0,0}(\mathbf{K}_0) = \mathbb{E}V_3(\mathbf{K}_0)$, $c_0 \in M$ and the lengths $a > b$ of the semi-axes of the ellipsoid Z_0. In the simulation study, we use the volume tensors to estimate this set of natural model parameters. Since $c_0 \in M$, $c_0 = ze$, where e spans M, so the focus is here on estimating z. In Fig. 14.3, five replicated simulations of $\mathbf{K}_0$ are shown from the actual model used in the simulation study together with the ellipsoid Z_0 (left).

For a sample of n particles $\mathbf{K}_{01}, \ldots, \mathbf{K}_{0n}$, we have determined for $r = 0, 1, 2$

$$\frac{1}{n}\sum_{i=1}^{n} \widehat{\Phi}_3^{r,0}(\mathbf{K}_{0i}) \qquad \text{(the slice estimator)} \tag{14.12}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \widetilde{\Phi}_3^{r,0}(\mathbf{K}_{0i}) \qquad \text{(the section estimator).} \tag{14.13}$$

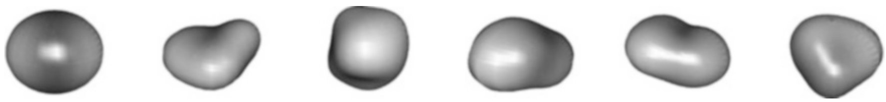

Fig. 14.3 Particles simulated under the Lévy particle model as random deformations of a prolate ellipsoid. The ellipsoid is shown to the left, followed by five random deformations

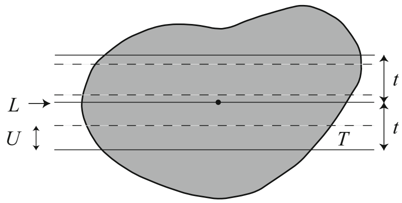

Fig. 14.4 2D illustration of the subsampling of a slice T of thickness $2t$. The slice is subsampled by three equidistant planes (shown as stippled lines) with distance $2t/3$ between neighbour planes. The position of the lower plane is determined by U which is uniform random in the interval $[0, 2t/3)$

In principle, the slice estimator (14.5) requires measurements in the whole slice T which typically covers the central part of the particle, as illustrated in Fig. 14.4. By focusing on the central part of the particle, overprojection at the peripheral parts of the particle is avoided in optical microscopy. For further details, see [13]. In practice (and in the simulations), the slice is subsampled by a systematic set of parallel planes. We used three equidistant planes in T, as also shown in Fig. 14.4. Each plane was subsampled by a systematic set of lines that was alternately parallel and perpendicular to the vertical axis. The distance between lines in a plane was chosen such that on the average two lines hit the particle in each plane. For more details, see [9, Fig. 2].

For the section estimator, $K_{0i} \cap L$ was subsampled by a systematic set of parallel lines in L, perpendicular to M. Again, the distance between lines was chosen such that on the average two lines hit the particle. With this set-up, the amount of work involved for the slice estimator is approximately three times that of the section estimator.

The simulation results for the slice estimator and the section estimator are shown in Tables 14.1 and 14.2, respectively, for the case of n=10, 20, 50 and 100 particles. For a sample of n particles $\mathbf{K}_{01}, \dots, \mathbf{K}_{0n}$, the estimators of v are

$$\widehat{v} := \frac{1}{n}\sum_{i=1}^{n} \widehat{\Phi}_3^{0,0}(\mathbf{K}_{0i}), \qquad \widetilde{v} := \frac{1}{n}\sum_{i=1}^{n} \widetilde{\Phi}_3^{0,0}(\mathbf{K}_{0i}), \tag{14.14}$$

depending on whether the slice estimator or the section estimator is used. Likewise, the estimators of z become

Table 14.1 For the slice estimator, we show the mean (and CV) of the estimated mean particle volume v, displacement z and semi-axis lengths $a > b$ of the prolate ellipsoid Z_0, determined from estimated mean volume tensors based on n simulated particles in $500{,}000/n$ simulations. The true parameter values are $v = 606.553$, $z = -0.073$, $a = 5.857$ and $b = 4.972$. The parameter values resemble the ones obtained in concrete analyses of microscopy data from the human brain cortex [14, p. 827]. The ellipsoid Z_0 is shown to the left in Fig. 14.3 followed by five random particles from the Lévy particle model, used in the simulation study. Estimation is done under the assumption of restricted isotropy

n	10	20	50	100
v	606.860 (0.151)	606.860 (0.095)	606.860 (0.067)	606.860 (0.047)
z	−0.073 (6.162)	−0.074 (4.021)	−0.074 (2.867)	−0.074 (2.034)
a	5.821 (0.082)	5.841 (0.054)	5.848 (0.039)	5.852 (0.028)
b	4.981 (0.068)	4.977 (0.044)	4.976 (0.031)	4.975 (0.022)

Table 14.2 For the section estimator, we show the mean (and CV) of the estimated mean particle volume v, displacement z and semi-axis lengths $a > b$ of the prolate ellipsoid Z_0, determined from estimated mean volume tensors based on n simulated particles in $500{,}000/n$ simulations. The true parameter values are given in the legend to Table 14.1

n	10	20	50	100
v	606.333 (0.152)	606.333 (0.096)	606.333 (0.068)	606.333 (0.048)
z	−0.069 (7.057)	−0.069 (4.560)	−0.069 (3.258)	−0.069 (2.337)
a	5.797 (0.098)	5.832 (0.064)	5.844 (0.047)	5.850 (0.033)
b	4.992 (0.070)	4.981 (0.044)	4.976 (0.032)	4.974 (0.022)

$$\widehat{z} := \frac{\frac{1}{n}\sum_{i=1}^{n} \widehat{\Phi}_3^{1,0}(\mathbf{K}_{0i})}{\frac{1}{n}\sum_{i=1}^{n} \widehat{\Phi}_3^{0,0}(\mathbf{K}_{0i})} \cdot e, \qquad \widetilde{z} := \frac{\frac{1}{n}\sum_{i=1}^{n} \widetilde{\Phi}_3^{1,0}(\mathbf{K}_{0i})}{\frac{1}{n}\sum_{i=1}^{n} \widetilde{\Phi}_3^{0,0}(\mathbf{K}_{0i})} \cdot e, \tag{14.15}$$

where $x \cdot e$ denotes the usual inner product of $x \in \mathbb{R}^3$ with the unit vector e that spans the vertical axis M and was used in the definition of z. The estimators of the semi-axis lengths a and b of the ellipsoid Z_0 are non-linear functions of the estimators of mean particle tensors of rank 0,1 and 2.

A total of 500,000 particles was simulated. These particles are used in Tables 14.1 and 14.2 to produce $500{,}000/n$ samples of n particles. Since the same 500,000 particles are used for all n and the estimated mean particle volume is a simple average, according to (14.14), the mean of the estimated mean particle volume v in Tables 14.1 and 14.2 does not depend on n. The mean of the estimated displacement z is also virtually constant which shows that for the model used in the simulation study the bias of the estimators of z in (14.15) is negligible, also for as small n as 10.

Tables 14.1 and 14.2 show that both the slice estimator and the section estimator provide estimators of the mean particle volume v and the semi-axis lengths a and b of the prolate ellipsoid Z_0 with CVs less than 10 % if 20 or more particles are sampled while it is needed to sample more than 100 particles if the very small displacement z is to be discovered. Comparing the section estimator with n particles to the slice estimator with $n/3$ particles (same amount of work), the section estimator is superior.

14.6 Non-parametric Inference

In the simulation study, we used the estimators of mean particle volume tensors $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$, $r = 0, 1, 2$, to estimate the parameters in the simulated Lévy particle model. In cases where the particle model is not a suitable description of the particle population under consideration, we may still use the mean particle volume tensors to obtain information about particle size, position, shape and orientation. Here, $\mathbb{E}\Phi_3^{0,0}(\mathbf{K}_0) = \mathbb{E}V_3(\mathbf{K}_0)$ is, of course, a size parameter (mean particle volume) while $\bar{c} := \mathbb{E}\Phi_3^{1,0}(\mathbf{K}_0)/\mathbb{E}\Phi_3^{0,0}(\mathbf{K}_0)$ contains information about the deviation of the centre of gravity from the reference point of the typical particle. Likewise, we can use $\mathbb{E}\Phi_3^{r,0}(\mathbf{K}_0)$, $r = 0, 1, 2$, to construct an approximating ellipsoid $\bar{c} + \bar{e}$, say, that contains information about particle shape and orientation of the typical particle. Here, $\bar{e}$ is a centred ellipsoid, called the *Miles ellipsoid*. It can be constructed from a spectral decomposition of

$$\mathbb{E}\Phi_3^{2,0}(\mathbf{K}_0) - \frac{(\mathbb{E}\Phi_3^{1,0}(\mathbf{K}_0))^2}{2\mathbb{E}\Phi_3^{0,0}(\mathbf{K}_0)}.$$

For more details, see [14, Sects. 4.2 and 4.3].

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Chapter 15
Valuations in Image Analysis

Anne Marie Svane

Abstract When intrinsic volumes and Minkowski tensors of a real world structure are computed, this is often based on a digital image. The digitization causes some estimation problems due to the anisotropic nature of the digital grid. Even the most natural and frequently used algorithms based on counting the local pixel/voxel configurations are often biased. In this chapter, we survey the known results on convergence of these local algorithms with a focus on estimation of intrinsic volumes. Moreover, we present some of the latest attempts to define convergent algorithms.

15.1 Introduction

Let $X \subseteq \mathbb{R}^n$ be a geometric object. If X is sufficiently well-behaved, we can gain information about its geometry by computing its intrinsic volumes $V_0(X), \ldots, V_n(X)$. These include such important characteristics as volume V_n, surface area $2V_{n-1}$, integrated mean curvature $2\pi(n-1)^{-1}V_{n-2}$, and Euler characteristic V_0. See [18] for the general definition of intrinsic volumes when X is convex. All intrinsic volumes are rotation and translation invariant. Non-invariant properties, such as position, orientation, and elongation, are captured by the Minkowski tensors. The r-th Minkowski volume tensor for $r \geq 0$ is an element of the space $\mathbb{T}^r$ of symmetric r-tensors on $\mathbb{R}^n$ and is given by

$$\Phi_n^{r,0}(X) := \frac{1}{r!}\int_X x^r dx,$$

where x^r is the r-fold tensor product of x. Moreover, for $r \geq 0$ and $s > 0$, we define $\Phi_n^{r,s}(X) := 0$. For $r, s \geq 0$ and $0 \leq m \leq n-1$, there is a Minkowski tensor

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$\Phi_m^{r,s}(X) \in \mathbb{T}^{r+s}$ defined by

$$\Phi_m^{r,s}(X) := \frac{1}{r!s!} \frac{\omega_{n-m}}{\omega_{n-m+s}} \int_{\Sigma^n} x^r u^s \Lambda_m(X; d(x,u)). \tag{15.1}$$

Here $\Sigma^n := \mathbb{R}^n \times S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$, and ω_n is the surface area of S^{n-1}. Moreover, $\Lambda_m(X; \cdot)$ is the m-th support measure on Σ^n, see [18] when X is convex and [28] for more general set classes. The integrand $x^r u^s$ is the symmetric tensor product of r copies of x and s copies of u. Intrinsic volumes are special cases of the Minkowski tensors since $\Phi_m^{0,0}(X) = V_m(X)$. More information about Minkowski tensors of convex sets can be found in [18, Sect. 5.4].

As explained in [19, 20] Minkowski tensors are useful tools for physicists to characterize geometric properties of a material. The tensors are often computed based on a digital image, for instance from a microscope or a scanner. This causes several problems. Not only are such images often blurred and noisy, the digitization itself may also introduce a bias. The latter is the topic of the present chapter.

A digital image is divided into pixels or voxels and the object is observed inside each. The pixel (voxel) midpoints form a lattice. Gaining information about the underlying object can thus be considered a stereological type of problem. In stereology, the object is usually only known inside an affine plane. Unbiased estimators for intrinsic volumes and, more generally, Minkowski tensors can be obtained by randomizing the rotation and translation of the intersection plane. In image analysis, we observe the object along a lattice rather than a plane. If the object is randomly translated and rotated with respect to the observation lattice, it is sometimes possible to find algorithms that are unbiased when the resolution goes to infinity. The assumption that the lattice is randomly rotated is not always realistic. This causes a rotation bias in many digital algorithms.

Another problem is that, while the boundary of the object is still visible on lower dimensional planes, a lattice will most likely not hit any boundary points. The boundary can in principle behave wildly between the lattice points. To avoid this, some regularity of the boundary must normally be assumed. In grey-scale images, the object boundary is represented by a blurred zone around the true boundary. As we shall see, this makes it easier to gain information about the boundary.

A third problem is that the data amount is often large. Therefore, fast algorithms are required. The focus of this chapter will therefore be on the development of algorithms with low computation time.

We are going to review some of the mathematical results on digital estimators for intrinsic volumes and Minkowski tensors. In Sect. 15.2, we consider the ideal situation where there is no noise or blurring. The emphasis will be on the so-called local algorithms, which are the most frequently applied ones, but some global methods will also be discussed. In Sect. 15.3, we consider the situation where the digital image is blurred. Noisy images will not be treated.

15.2 Digital Algorithms for Black-and-White Images

In this section, we consider the ideal situation where the digital image is sharp and noise-free. A mathematical model for such an image is given in Sect. 15.2.1. Local algorithms are defined and discussed in Sects. 15.2.2–15.2.6. Some other types of algorithms are described in Sect. 15.2.7.

15.2.1 Black-and-White Images

Let $X \subseteq \mathbb{R}^n$ be the object that we are trying to observe. A digital image is divided into pixels (voxels). If there is no noise or blurring, we can measure exactly whether or not each pixel midpoint belongs to X. A pixel (voxel) is colored black if its midpoint lies in X and white otherwise. This is illustrated in Fig. 15.1.

If we let $\mathbb{L}$ denote the lattice formed by the pixel (voxel) midpoints, then the information contained in a black-and-white image corresponds to the set $X \cap \mathbb{L}$ of black pixel midpoints. We will assume throughout that X is compact and topologically regular, i.e., X is the closure of its own interior. This ensures that X does not have any lower dimensional parts that we are not able to see in the image.

Clearly, there is not enough information in $X \cap \mathbb{L}$ to determine the Minkowski tensors. But going to a higher resolution will give us more information about X. This corresponds to scaling $\mathbb{L}$ by a small factor $a > 0$, resulting in the image $X \cap a\mathbb{L}$.

In most applications, $\mathbb{L}$ is the standard lattice $\mathbb{Z}^n$ or a rotation and translation of this. However, other cases can occur, for instance the hexagonal lattice in 2D [14]. For this reason, we let $\mathbb{L}$ be arbitrary.

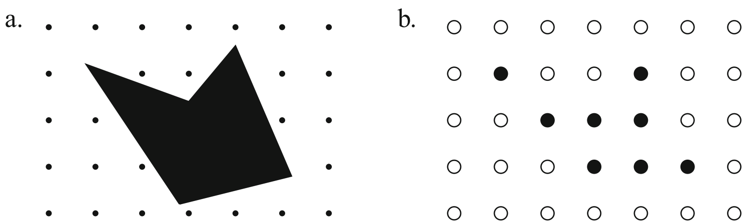

Fig. 15.1 Example of a black-and-white image. Figure (**a**) shows the object together with the grid of pixel midpoints. Figure (**b**) shows the resulting digital image

15.2.2 Local Algorithms for Intrinsic Volumes

The most popular type of algorithms for estimating intrinsic volumes and Minkowski tensors is the class of so-called local algorithms. The reason for the name is that the algorithm only depends on what the image looks like locally.

The intuition behind is the additivity of Minkowski tensors: By the inclusion-exclusion formula, they can be computed as a sum of contributions from each $k \times \cdots \times k$ lattice cell depending only on the intersection of X with that cell. Since the only thing we know about X in each cell is the configuration of black and white points, we estimate the contribution from each cell by a so-called weight depending only on the configuration. The Minkowski tensor is then estimated by counting the number of occurrences of each possible $k \times \cdots \times k$ configuration of black and white points in the image and taking a weighted sum of configuration counts.

Local algorithms are a popular choice in applications since they only require reading through the image once. Hence they are very fast in the sense that their computation times are linear in the number of pixels. They do, however, become computationally involved for $n > 2$ and $k > 2$ [16]. Geometric intuition can give an idea about how to choose the weights.

To give a precise definition of local algorithms, we first introduce some notation: Suppose the lattice is given by $\mathbb{L} := A(\mathbb{Z}^n + c)$ where $A \in \mathrm{Gl}(n)$ and $c \in [0,1)^n$. The fundamental lattice k-cell of $\mathbb{L}$ is $C_0^k := A([0,k)^n)$. The volume of C_0^1 is denoted $c_{\mathbb{L}}$. The set of lattice points in C_0^k is denoted by $C_{0,0}^k := C_0^k \cap \mathbb{L}$. The translation of $C_{0,0}^k$ by $z \in \mathbb{R}^n$ is denoted by $C_{z,0}^k := C_{0,0}^k + z$.

A $k \times \cdots \times k$ configuration is a partition of $C_{0,0}^k$ into two disjoint sets B (black points) and W (white points). We denote the 2^{k^n} possible $k \times \cdots \times k$ configurations by (B_l, W_l) for $l = 1, \ldots, 2^{k^n}$. For instance, up to rotation there are six possible 2×2 configurations in 2D. These are shown in Fig. 15.2.

Let $N_l(X \cap a\mathbb{L})$ be the number of occurrences of the configuration (B_l, W_l) in the image, i.e.

$$N_l(X \cap a\mathbb{L}) := \sum_{z \in a\mathbb{L}} \mathbf{1}_{\{z + aB_l \subseteq X, (z + aW_l) \cap X = \emptyset\}}.$$

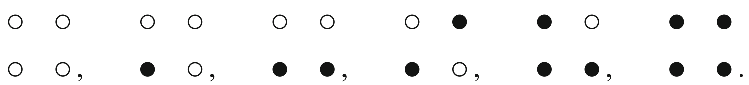

Fig. 15.2 The possible 2×2 configurations in 2D

We estimate $V_m(X)$ by a weighted sum of these configuration counts:

Definition 15.1 A local algorithm for the intrinsic volume V_m is an algorithm of the form

$$\hat{V}_m(X) := a^m \sum_{l=1}^{2^{k^n}} w_l^{(m)} N_l(X \cap a\mathbb{L}), \tag{15.2}$$

where $w_l^{(m)}$ can be arbitrary real numbers, referred to as the weights.

Many natural approaches to defining digital estimators result in a local algorithm. The most simple one, see [13, Sect. 2.3.1], is based on approximating X by the union $\hat{X}$ of all $2 \times \cdots \times 2$ lattice cells with midpoint in X, i.e.,

$$\hat{X} := \bigcup_{z \in X \cap a\mathbb{L}} (z - p_{\mathbb{L}} + aC_0^1), \tag{15.3}$$

where $p_{\mathbb{L}} := A(\frac{1}{2}, \ldots, \frac{1}{2})$ is the midpoint of the fundamental cell. Then $V_m(\hat{X})$ can be used as an estimate for $V_m(X)$. The intrinsic volumes of $\hat{X}$ can be computed by a local algorithm. One can realize this by applying the inclusion-exclusion formula to $\hat{X}$. Local algorithms based on different reconstructions of X have also been considered, see e.g. [13, 15].

Other approaches are inspired by integral geometry. These take as a starting point the Steiner formula as in [12] or a discretization of the Crofton formula as in [14, 15]. The results are again local algorithms.

15.2.3 *Convergence of Local Algorithms in the Design-Based Setting*

To evaluate the quality of an algorithm, it can be tested in a design based setting. This means that the object X is considered as deterministic, whereas the lattice is randomized. In this section, we consider what we will call a stationary lattice, that is, a lattice of the form $\mathbb{L} = \mathbb{L}_0 + c$, where $\mathbb{L}_0$ is a fixed lattice and $c \in C_0^1$ is a uniform random translation vector. This means that the lattice is randomly translated with respect to the underlying object which seems like a natural assumption in applications. It is natural to require a local algorithm to be unbiased in this setting, at least asymptotically when the resolution tends to infinity, i.e., $\lim_{a \to 0} \mathbf{E}\hat{V}_m(X) = V_m(X)$. Alternatively, one could consider the exact error, see Remark 15.7.

There is a simple estimator for the volume of X that is unbiased even in finite resolution:

Theorem 15.2 *The volume estimator that counts the number of black lattice points and multiplies by the volume of each* $2 \times \cdots \times 2$ *lattice cell is a local algorithm with* $k = 1$ *given by*

$$\hat{V}_n(X) := a^n c_{\mathbb{L}} \sum_{z \in a\mathbb{L}} \mathbf{1}_{\{z \in X \cap a\mathbb{L}\}}.$$

This algorithm is unbiased in the design based setting, i.e.,

$$\mathbf{E}\hat{V}_n(X) = V_n(X).$$

Note that the algorithm in Theorem 15.2 is actually computing the volume of the approximating set $\hat{X}$ defined in (15.3).

In 2D and 3D, the Euler characteristic $V_0(X)$ can also be estimated by $V_0(\hat{X})$. This requires the following smoothness condition on the boundary:

Definition 15.3 A set $X \subseteq \mathbb{R}^n$ is called r-regular if for every boundary point $x \in \operatorname{bd} X$, there are two balls of radius r containing x whose interiors are completely contained in X and $\mathbb{R}^n \setminus X$, respectively.

The following theorem is proved in 2D in [17] and in 3D in [3]:

Theorem 15.4 *Let* X *be an* r*-regular subset of* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Then for* a *sufficiently small compared to* r*,*

$$V_0(\hat{X}) = V_0(X).$$

Unfortunately, the estimator $V_m(\hat{X})$ is not unbiased for $1 \le m \le n-1$, not even when $a \to 0$. This is part of a more general phenomenon. Even when the underlying set is a convex polytope, the following was proved in [23], see also [8] when $k = 2$:

Theorem 15.5 *There exists no estimator of the form* (15.2) *for* V_m *with* $m < n$ *that is asymptotically unbiased for all compact convex polytopes with non-empty interior.*

In fact, there is a measure ν on the set $\mathscr{P}_0^n$ of compact convex polytopes with non-empty interior, such that any local algorithm for V_m, $m < n$, is biased on a set of polytopes with positive ν-measure. See [23] for details.

Moreover, for $0 \le m \le n-2$, it can be shown that the worst case bias

$$\sup_{X \in \mathscr{P}_0^n} \left| \frac{\lim_{a \to 0} \mathbf{E}\hat{V}_m(X) - V_m(X)}{V_m(X)} \right|$$

is always at least 100%. For surface area in 3D, i.e. when $m = 2$ and $n = 3$, one can do a bit better. It was shown in [29] that the best possible algorithm has a worst case bias of 4%. The authors give an explicit algorithm that minimizes the bias.

Theorem 15.4 showed that under certain smoothness assumptions on the boundary, estimation of the Euler characteristic is possible. This is not the case in general. The following was proved in [23]:

Theorem 15.6 *Let $m > 0$ and $m = n - 1$ or $m = n - 2$. There exists no estimator for V_m of the form (15.2) that is asymptotically unbiased for all r-regular sets.*

To prove these theorems, it is necessary to study the mean $\mathbf{E}\hat{V}_m(X)$, which is a linear combination of the mean configuration counts $\mathbf{E}N_l(X \cap a\mathbb{L})$. A simple computation shows that

$$\begin{aligned}\mathbf{E}N_l(X \cap a\mathbb{L}) &= \mathbf{E}\Big(\sum_{z \in a\mathbb{L}} \mathbf{1}_{\{z+aB_l \subseteq X, (z+aW_l)\cap X=\emptyset\}}\Big)\\ &= \sum_{z\in a\mathbb{L}_0} \int_{C_0^1} \mathbf{1}_{\{z+ac+aB_l\subseteq X,(z+ac+aW_l)\cap X=\emptyset\}} dc\\ &= a^{-n}\int_{\mathbb{R}^n} \mathbf{1}_{\{z+aB_l\subseteq X,(z+aW_l)\cap X=\emptyset\}} dz. \end{aligned} \tag{15.4}$$

If X is r-regular and f is a function supported within distance r from $\operatorname{bd} X$, then the generalized Steiner formula of Hug et al. [6] yields the following formula for the integral

$$\int_{\mathbb{R}^n} f(x)dx = \sum_{i=0}^{n-1} \int_{\operatorname{bd} X} \int_{-r}^{r} t^i f(x + tu(x)) dt \mu_i(dx),$$

where the μ_i are signed measures on $\operatorname{bd} X$ and $u(x)$ is the outward pointing normal vector at $x \in \operatorname{bd} X$.

If B_l and W_l are both non-empty, then all points z satisfying $z + aB_l \subseteq X$ and $(z + aW_l) \cap X = \emptyset$ lie at distance at most $ak\sqrt{n}$ from the boundary, so the generalized Steiner formula can be applied to $\mathbf{1}_{\{z+aB_l\subseteq X,(z+aW_l)\cap X=\emptyset\}}$ when a is small. Thus, to determine the asymptotic behavior of (15.4), one must study the function $\mathbf{1}_{\{z+aB_l\subseteq X,(z+aW_l)\cap X=\emptyset\}}$ along the normal lines $x + tu(x)$. This idea first appeared in [10] and was extended in [25].

Remark 15.7 One could also consider convergence when the lattice is fixed and require that $\lim_{a\to 0} \hat{V}_m(X) = V_m(X)$. This property is known as multigrid convergence. It was shown in [23] that if a local algorithm is not asymptotically unbiased, then it can also not be multigrid convergent. Thus, the above results on non-existence of asymptotically unbiased algorithms translate to results on non-existence of multigrid convergent algorithms.

15.2.4 Local Algorithms for Minkowski Tensors

Local algorithms for Minkowski 2-tensors have been suggested in [19, 20]. Since these are position dependent, the weights in Definition 15.1 will generally have to depend on position, i.e. we must consider algorithms of the form

$$\sum_l \sum_{z\in a\mathbb{L}} w_l(z,a)\mathbf{1}_{\{z+aB_l\subseteq X,(z+aW_l)\cap X=\emptyset\}}, \tag{15.5}$$

where $w_l : \mathbb{R}^n \times (0,\infty) \to \mathbb{T}^{r+s}$ depends on position and resolution.

Estimation of volume tensors is easy, since a Riemann sum

$$\hat{\Phi}_n^{r,0}(X) := a^n c_{\mathbb{L}} \frac{1}{r!} \sum_{z\in X\cap a\mathbb{L}} z^r$$

yields an unbiased local algorithm.

There are no convergence results in the literature about local algorithms for other Minkowski tensors than intrinsic volumes, but asymptotic formulas for the mean of a local algorithm of the form (15.5) could easily be derived by computations similar to [25] and [23]. Apart from the most trivial tensors, asymptotically unbiased local estimators for Minkowski tensors are not expected to exist.

15.2.5 Isotropic Design

One could also consider a version of the design based setting where the lattice is both randomly translated and rotated. That is, we consider the lattice $\mathbb{L} = R(\mathbb{L}_0 + c)$ where the translation vector $c \in C_0^1$ and the rotation $R \in SO(n)$ are both uniform random and mutually independent. We say that $\mathbb{L}$ is stationary isotropic. In this setting, asymptotically unbiased estimators do exist [25]:

Theorem 15.8 *If X is r-regular and $\mathbb{L}$ is stationary isotropic, then there exist local algorithms for V_m with $m = n, n-1, n-2$ that are asymptotically unbiased, i.e.*

$$\lim_{a\to 0} \mathbf{E}\hat{V}_m(X) = V_m(X).$$

Explicit asymptotically unbiased algorithms are given in [25].

If the algorithms are applied to a stationary isotropic Boolean model with a fixed lattice, a similar result seems to hold: There exists asymptotically unbiased estimators for V_n, V_{n-1}, and V_{n-2}. At least, this has been shown in both 2D [22] and 3D [5]. Again, isotropy is essential.

These results suggest that it is the lack of isotropy of $\mathbb{L}$ that causes the bias in the results of Sect. 15.2.3.

15.2.6 Variance of the Local Volume Estimator

We consider again the volume estimator given by lattice point counting

$$\hat{V}_n(X) = a^n c_{\mathbb{L}} \sum_{z \in a\mathbb{L}} \mathbf{1}_X(z)$$

from Theorem 15.2. While this has the correct mean in the design based setting, determining the exact error $|\hat{V}_n(X) - V_n(X)|$ is a classical and difficult topic. Even when X is the unit ball centered at the origin, the optimal bound on the error when $a \to 0$ is unknown. This is known as the Gauss circle problem.

Instead, we will consider the variance in the design based setting where $\mathbb{L}$ is stationary and isotropic. To study the variance, we first consider $\mathbf{E}(\hat{V}_n(X)^2)$:

$$\begin{aligned}
\mathbf{E}(\hat{V}_n(X)^2) &= a^{2n} c_{\mathbb{L}}^2 \mathbf{E}\Big(\sum_{z \in a\mathbb{L}} \mathbf{1}_X(z)\Big)^2 \\
&= a^{2n} c_{\mathbb{L}}^2 \int_{SO(n)} \int_{C_0^1} \Big(\sum_{z_1 \in aR\mathbb{L}_0} \sum_{z_2 \in aR(\mathbb{L}_0 + c)} \mathbf{1}_X(z_2)\mathbf{1}_X(z_2 + z_1)\Big) dc dR \\
&= a^n c_{\mathbb{L}} \int_{SO(n)} \Big(\sum_{z_1 \in \mathbb{L}_0} \int_{\mathbb{R}^n} \mathbf{1}_X(Rz_2)\mathbf{1}_X(R(z_2 + az_1)) dz_2\Big) dR \\
&= a^n c_{\mathbb{L}} \int_{SO(n)} \Big(\sum_{z_1 \in \mathbb{L}_0} \mathbf{1}_X * (\mathbf{1}_X)^-(-aRz_1)\Big) dR,
\end{aligned}$$

where g^- denotes the function $x \mapsto g(-x)$. The Poisson summation formula [21, VII, Corollary 2.6] yields:

$$\begin{aligned}
a^n c_{\mathbb{L}} &\int_{SO(n)} \Big(\sum_{z \in \mathbb{L}_0} \mathbf{1}_X * (\mathbf{1}_X)^-(-aRz)\Big) dR \\
&= \sum_{\xi \in \mathbb{L}_0^*} \int_{SO(n)} \mathscr{F}(\mathbf{1}_X * (\mathbf{1}_X)^-)(a^{-1}R\xi) dR \\
&= \sum_{\xi \in \mathbb{L}_0^*} \int_{SO(n)} |\mathscr{F}(\mathbf{1}_X)(a^{-1}R\xi)|^2 dR \\
&= \omega_n^{-1} \sum_{\xi \in \mathbb{L}_0^*} \int_{S^{n-1}} \big|\mathscr{F}(\mathbf{1}_X)(a^{-1}|\xi|u)\big|^2 du.
\end{aligned}$$

Here $\mathscr{F}$ denotes the Fourier transform and $\mathbb{L}_0^*$ is the so-called dual lattice of $\mathbb{L}_0$. We have used the fact that the Fourier transform of a convolution is a product of Fourier transforms and that $\mathscr{F}(g^-)$ is the complex conjugate of $\mathscr{F}(g)$. Recalling that

$$\mathscr{F}(\mathbf{1}_X)(0) = \int_{\mathbb{R}^n} \mathbf{1}_X(x)dx = \mathbf{E}(\hat{V}_n(X)),$$

we find

$$\begin{aligned}\mathbf{var}(\hat{V}_n(X)) &= \mathbf{E}(\hat{V}_n(X)^2) - \mathbf{E}(\hat{V}_n(X))^2 \\ &= \omega_n^{-1} \sum_{\xi \in \mathbb{L}_0^* \setminus \{0\}} \int_{S^{n-1}} \left|\mathscr{F}(\mathbf{1}_X)(a^{-1}|\xi|u)\right|^2 du. \end{aligned} \tag{15.6}$$

It was shown in [1] that if X is a smooth manifold, then for $a^{-1}|\xi|$ sufficiently large,

$$\int_{S^{n-1}} \left|\mathscr{F}(\mathbf{1}_X)(a^{-1}|\xi|u)\right|^2 du \leq C(X)a^{d+1}|\xi|^{-d-1},$$

where $C(X) > 0$ is a constant depending on X. It follows that:

Theorem 15.9 *If X is a smooth manifold, then for a sufficiently small*

$$\mathbf{var}(\hat{V}_n(X)) \leq a^{d+1}\omega_n^{-1}C(X) \sum_{\xi \in \mathbb{L}_0^* \setminus \{0\}} |\xi|^{-d-1}.$$

Getting a precise formula for the variance is not possible. When X is smooth and convex with nowhere vanishing Gauss curvature, there are formulas for the Fourier coefficients. These show that each term $\int_{S^{n-1}} |\mathscr{F}(\mathbf{1}_X)(a^{-1}|\xi|u)|^2 du$ oscillates between 0 and $8V_{n-1}(X)a^{d+1}|\xi|^{-d-1}$, see [4]. It is therefore hard to determine the sum (15.6).

If the underlying set is a random set $\mathbf{X}$, it is sometimes possible to obtain precise formulas for the asymptotic variance. Under suitable conditions on $\mathbf{X}$, it is shown in [11] that

$$\lim_{a \to 0} a^{-d-1}\,\mathbf{var}(\hat{V}_n(\mathbf{X})) = 4\omega_n^{-1}\mathbf{E}V_{n-1}(\mathbf{X}) \sum_{\xi \in \mathbb{L}_0^*} |\xi|^{-d-1}.$$

15.2.7 Other Types of Algorithms

Despite the negative convergence results, local algorithms are still being used because of their low computation time. But there are also various attempts in the literature to define algorithms that take the global structure of the image into account without loosing too much speed.

In [9], a semi-local algorithm is suggested for estimation of Euler characteristic. Let $\mathscr{K}_0^n$ be the class of compact convex sets with non-empty interior in $\mathbb{R}^n$. Assume that X is known to be a finite union of planar sets from $\mathscr{K}_0^2$ satisfying mild conditions on their intersections [9, Definition 1]. Generally, $\hat{X}$ does not have the same Euler characteristic as X, but it is shown in [9] that after throwing away certain connected components of $\hat{X}$ in a systematic way, it does. The computation time of this algorithm is also linear in the number of pixels.

There are also convergent algorithms for the remaining $\Phi_m^{r,s}$ when $X \in \mathscr{K}_0^n$. The convex hull of $X \cap a\mathbb{L}$, $\mathrm{conv}(X \cap a\mathbb{L})$, converges to X in the Hausdorff metric when $a \to 0$, see [7]. The Minkowski tensors are continuous with respect to the Hausdorff metric, so

$$\lim_{a \to 0} \Phi_m^{r,s}(\mathrm{conv}(X \cap a\mathbb{L})) = \Phi_m^{r,s}(X).$$

Hence $\Phi_m^{r,s}(\mathrm{conv}(X \cap a\mathbb{L}))$ can be taken as an estimate for $\Phi_m^{r,s}(X)$. The optimal computation time for the convex hull of a set of N points is $\mathrm{O}(N \log N + N^{\lfloor n/2 \rfloor})$, see [2].

The method of convex hulls does obviously not generalize to non-convex sets. Another approach [7] is based on computing the Voronoi cells of $X \cap a\mathbb{L}$. The optimal computation time for the Voronoi cells of N points is almost as good as for the convex hull, namely $\mathrm{O}(N \log N + N^{\lceil n/2 \rceil})$, see [2]. This algorithm applies to all sets of positive reach:

Definition 15.10 Let $X \subseteq \mathbb{R}^n$ and $R \geq 0$. Then $X^R := \{x \in \mathbb{R}^n \mid d(x, X) \leq R\}$ denotes the parallel set of X. The reach of X, $\mathrm{Reach}(X)$, is the supremum over all $R \geq 0$ for which every point in X^R has a unique nearest point in X. If $\mathrm{Reach}(X) > 0$, then we say that X has positive reach.

The idea is to define the Voronoi tensor of a set $Y \subseteq \mathbb{R}^n$ for each pair $r, s \geq 0$ by

$$\mathscr{V}_R^{r,s}(Y) := \int_{Y^R} p_Y(x)^r (x - p_Y(x))^s dx \in (\mathbb{R}^n)^{\otimes(r+s)}.$$

Here $p_Y(x)$ denotes the point in Y closest to x. This is well-defined for almost all x.

If X has positive reach and $R < \mathrm{Reach}(X)$, then the Voronoi tensors satisfy the following Steiner formula:

$$\mathscr{V}_R^{r,s}(X) = r!s! \sum_{j=0}^{n} \kappa_{j+s} R^{j+s} \Phi_{n-j}^{r,s}(X).$$

This follows from an application of the generalized Steiner formula in [6].

If the Voronoi tensors are known for $n+1$ distinct values $R_0, \ldots, R_n$ of R, then we get $n+1$ equations:

$$\begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(X) \\ \vdots \\ \mathcal{V}_{R_n}^{r,s}(X) \end{pmatrix} = r!s! \begin{pmatrix} \kappa_s R_0^s & \ldots & \kappa_{s+n} R_0^{s+n} \\ \vdots & & \vdots \\ \kappa_s R_n^s & \ldots & \kappa_{s+n} R_n^{s+n} \end{pmatrix} \begin{pmatrix} \Phi_n^{r,s}(X) \\ \vdots \\ \Phi_0^{r,s}(X) \end{pmatrix}$$

The matrix is invertible since it is a product of a diagonal matrix and a Vandermonde matrix, so the system can be solved for the Minkowski tensors.

The Voronoi tensors of the set $X \cap a\mathbb{L}$ can be computed from the image. If we take this as an estimate for the Voronoi tensors of X, we obtain the following estimators for the Minkowski tensors:

$$\begin{pmatrix} \hat{\Phi}_n^{r,s}(X) \\ \vdots \\ \hat{\Phi}_0^{r,s}(X) \end{pmatrix} = \frac{1}{r!s!} \begin{pmatrix} \kappa_s R_0^s & \ldots & \kappa_{s+n} R_0^{s+n} \\ \vdots & & \vdots \\ \kappa_s R_n^s & \ldots & \kappa_{s+n} R_n^{s+n} \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{V}_{R_0}^{r,s}(X \cap a\mathbb{L}) \\ \vdots \\ \mathcal{V}_{R_n}^{r,s}(X \cap a\mathbb{L}) \end{pmatrix} \tag{15.7}$$

An asymptotic convergence result for these estimators was proved in [7]. Note that the result holds for any translation of the lattice, so we do not need to assume randomization of the lattice.

Theorem 15.11 *Suppose X is a topologically regular set of positive reach and $R <$ Reach(X). Then*

$$\lim_{a \to 0} \mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \mathcal{V}_R^{r,s}(X).$$

By linearity in (15.7), we obtain

$$\lim_{a \to 0} \hat{\Phi}_m^{r,s}(X) = \Phi_m^{r,s}(X).$$

The Voronoi tensors of $X \cap a\mathbb{L}$ have a simple expression in terms of the Voronoi cells

$$V_x := \{y \in \mathbb{R}^n \mid \forall z \in (X \cap a\mathbb{L}) \setminus \{x\} : \|x - y\| < \|z - y\|\}.$$

Namely,

$$\mathcal{V}_R^{r,s}(X \cap a\mathbb{L}) = \sum_{x \in X \cap a\mathbb{L}} x^r \int_{V_x \cap B_x(R)} (y - x)^s dy,$$

where $B_x(R)$ is the ball around x of radius R. Thus, in order to compute the estimator, one needs to compute the Voronoi cells of $X \cap a\mathbb{L}$ and do an integral over each of these. This is more computationally involved than the local algorithms, but there exist relatively fast algorithms to compute Voronoi cells.

15.3 Grey-Scale Images

The black-and-white model for digital images introduced in Sect. 15.2.2 is often too idealized for real world images. Due to limitations of the measuring device, the light from a single point will be spread out. We are thus unable to measure precisely whether or not a point lies in X. Instead, we measure a light intensity. Associating a grey tone to each intensity, this results in a grey-scale image where we see a blurred zone around the boundary of the object.

Blurring may seem like an obstacle to the estimation of intrinsic volumes. The simplest way to deal with it is to use thresholding, i.e. to choose a threshold value β and convert all pixels with grey-value larger than β to black and all other pixels to white. The algorithms for black-and-white images may then be applied to the thresholded image. One would expect this to introduce an extra bias. At the same time, a lot of information is thrown away when an image is thresholded. We shall see below that algorithms based directly on the grey-values perform much better.

15.3.1 Models for Grey-Scale Images

Let $\rho : \mathbb{R}^n \to [0, \infty)$ be the point spread function (PSF) that describes how the light originating from a point at the origin is spread out over $\mathbb{R}^n$. The intensity $\theta^X(x)$ that can be measured at $x \in \mathbb{R}^n$ is then an integral of the contributions from all points in X:

$$\theta^X(x) := \int_X \rho(x - y)dy.$$

In other words, θ^X is the convolution of $\mathbf{1}_X$ with ρ. We have assumed that the PSF is independent of the position of the point. Moreover, we assume that ρ is bounded, continuous, and that $\int_{\mathbb{R}^n} \rho(x)dx = 1$. Since the results below only deal with rotation invariant PSF's, we will assume throughout that $\rho(x) = \rho(|x|)$, i.e. the light received from a point depends only on the distance to the point. More general PSF's have been considered in [24].

In applications, the PSF is often modeled by the Gaussian $\rho(x) = (2\pi)^{-d/2}e^{-x^2/2}$, which satisfies all the above assumptions.

In a digital grey-scale image, we measure the intensity θ^X at the midpoint of each pixel. That is, the information we have is

$$\theta^X_{|\mathbb{L}} : \mathbb{L} \to [0, 1].$$

We also consider the following transformation of ρ:

$$\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\varepsilon^{-1}x).$$

Small values of ε correspond to little blurring, meaning that the grey-values are concentrated close to the boundary of X. The intensity function corresponding to ρ_ε will be denoted by θ_ε^X.

15.3.2 Local Algorithms for Grey-Scale Images

Local algorithms for grey-scale images are algorithms based on the local $k \times \cdots \times k$ configurations of grey-values in the image. A $k \times \cdots \times k$ configuration of grey-values is an element of

$$[0,1]^{C_{0,0}^k} = \{\{\theta_s\}_{s\in C_{0,0}^k} \mid \theta_s \in [0,1]\}.$$

We denote the configuration $\{\theta_\varepsilon^X(x)\}_{x\in z+aC_{0,0}^k}$ of grey-values observed on $z + aC_{0,0}^k$ by $\theta_\varepsilon^X(z,a,k)$. To each configuration we associate a weight. We can think of this as a function $f : [0,1]^{C_{0,0}^k} \to \mathbb{R}$.

Definition 15.12 A local algorithm for V_m is an algorithm of the form

$$\hat{V}_m(X) = a^n\varepsilon^{m-n}\sum_{z\in a\mathbb{L}} f(\theta_\varepsilon^X(z,a,k)),$$

where $f : [0,1]^{C_{0,0}^k} \to \mathbb{R}$ is a measurable function called the weight function.

The factor a^n compensates for the growing number of terms in the sum when $a \to 0$. The factor ε^{m-n} ensures the right degree of homogeneity.

15.3.3 Convergence of Grey-Scale Local Algorithms

We again test the convergence of the algorithms in the design based setting with a stationary lattice.

We restrict ourselves to estimators with $k = 1$. Thus, the weight function is a function $f : [0,1] \to \mathbb{R}$ and $\hat{V}_m$ takes the following simple form

$$\hat{V}_m(X) = a^n\varepsilon^{m-n}\sum_{z\in a\mathbb{L}} f(\theta_\varepsilon^X(z)). \tag{15.8}$$

The asymptotic behavior of estimators based on larger $k \times \cdots \times k$ configurations is studied in [24], but the results are harder to interpret in this case, so we omit them here.

The mean of an estimator of the form (15.8) is again given by a simple formula:

$$\begin{aligned}\mathbf{E}\hat{V}_m(X) &= a^n \varepsilon^{m-n} \mathbf{E}\Big(\sum_{z \in a\mathbb{L}} f(\theta_\varepsilon^X(z)) \Big) \\ &= a^n \varepsilon^{m-n} \int_{C_0^1} \Big(\sum_{z \in a\mathbb{L}_0} f(\theta_\varepsilon^X(z + ac)) \Big) dc \qquad (15.9) \\ &= \varepsilon^{m-n} c_{\mathbb{L}}^{-1} \int_{\mathbb{R}^n} f \circ \theta_\varepsilon^X(z) dz.\end{aligned}$$

Note that this is independent of the resolution a^{-1}. Instead, we consider the convergence of $\hat{V}_m(X)$ when $\varepsilon \to 0$, i.e. when the blurring becomes small. To determine the asymptotic behavior of (15.9) when $\varepsilon \to 0$, we introduce a function $\theta : \mathbb{R} \to [0, 1]$ that will appear in the results below. This is given by

$$\theta(t) := \int_{\mathbb{R}^n} \mathbf{1}_{\{\langle x,u \rangle \leq 0\}} \rho(tu - x) dx,$$

where $u \in S^{n-1}$ is a unit vector. By rotation invariance of ρ, θ is independent of u.

The map $t \mapsto \theta(t)$ is the intensity function of a halfspace perpendicular to u measured at a point of signed distance t from the boundary of the halfspace. If we zoom in on the boundary of a sufficiently smooth set, it will look almost like a halfspace. Therefore, the blurred image will locally look almost like a blurred halfspace when ε is small. This is the intuitive reason why θ shows up in the limit $\varepsilon \to 0$.

The theorem is stated under the assumption that X is a gentle set. This is a mild smoothness condition ensuring that almost every boundary point has a well-defined tangent space. It is satisfied by all finite unions elements from $\mathscr{K}_0^n$ and all r-regular sets. See [10] for the precise definition.

We can now state the following convergence result for surface area estimators:

Theorem 15.13 *Let X be a gentle set. Suppose $f : [0, 1] \to \mathbb{R}$ is continuously differentiable on the interval $[\beta, \omega] \subseteq (0, 1)$ and that f is zero outside $[\beta, \omega]$. Then*

$$\lim_{\varepsilon \to 0} \mathbf{E}\hat{V}_{n-1}(X) = c_1(f, \rho) V_{n-1}(X),$$

where

$$c_1(f, \rho) := 2c_{\mathbb{L}}^{-1} \int_{\mathbb{R}} f \circ \theta(t) dt.$$

If $f > 0$ *on* (β, ω), *then* $c_1(f, \rho) \neq 0$. *In this case,*

$$c_1(f, \rho)^{-1} \hat{V}_{n-1}(X)$$

is an asymptotically unbiased estimator for $V_{n-1}(X)$.

Similarly, there is a result for estimation of integrated mean curvature:

Theorem 15.14 *Let X be an r-regular set and assume that ρ has compact support. Suppose* $f : [0, 1] \to \mathbb{R}$ *is continuously differentiable on* $[\beta, 1-\beta] \subset (0, 1)$ *and zero outside* $[\beta, 1-\beta]$. *If* $f(t) = -f(1-t)$, *then*

$$\lim_{\varepsilon \to 0} \mathbf{E}\hat{V}_{n-2}(X) = c_2(f, \rho) V_{n-2}(X).$$

15.3.4 Some Examples

To illustrate the results, we can look at some simple examples of local algorithms for grey-scale images. The simplest algorithm for surface area is of the form (15.8) with $f = \mathbf{1}_{[\beta, 1-\beta]}$, i.e.

$$\hat{V}_{n-1}(X) := a^n \varepsilon^{-1} \sum_{z \in a\mathbb{L}} \mathbf{1}_{[\beta, 1-\beta]}(\theta_\varepsilon^X(z)).$$

Up to a factor $a^n \varepsilon^{-1}$, this is the number of pixels with grey-value in the interval $[\beta, 1-\beta]$. According to Theorem 15.2, the mean of this estimator is ε^{-1} times the volume of the band around bd X with grey-values in $[\beta, 1-\beta]$. Intuitively, the volume of this a band should be proportional to $\varepsilon V_{n-1}(X)$. Indeed, Theorem 15.13 shows that

$$\lim_{a \to 0} \mathbf{E}\hat{V}_{n-1}(X) = c_1(\beta, \rho) V_{n-1}(X),$$

where

$$c_1(\beta, \rho) = 4 c_{\mathbb{L}}^{-1} \theta^{-1}(\beta).$$

When ρ is the standard Gaussian PSF, θ is the distribution function of a standard normal distribution on $\mathbb{R}$, so $c_1(\beta, \rho)$ can be computed directly. If the PSF is unknown, it may be necessary to determine $c_1(\beta, \rho)$ experimentally.

This algorithm is extremely simple. It only requires thresholding at two different levels and computing the difference in the number of black lattice points. However, other algorithms for surface area could also be worth considering. For instance, it might be relevant to choose a function that puts more emphasis on grey-values close to $1/2$ since these are expected to lie close to bd X.

For estimating V_{n-2}, one can consider the weight function $f = \mathbf{1}_{[\beta,1/2]} - \mathbf{1}_{[1/2,1-\beta]}$. The resulting algorithm is given by counting the number of grey-values between β and $1/2$ and subtracting the number of grey-values between $1/2$ and $1-\beta$. For suitable β, the constant $c_2(f,\rho)$ in Theorem 15.14 will be non-zero and hence we can divide by it to get an estimator for V_{n-2}.

15.3.5 Variance of Local Algorithms for Grey-Scale Images

The above convergence results hold in any resolution a. This may seem a bit counterintuitive. If ε is small, which is necessary to obtain good precision, the grey-values in the interval $[\beta, \omega]$ are concentrated in a very narrow band around $\operatorname{bd} X$. If, at the same time, the resolution is low, then it is likely that the lattice does not intersect this band. Thus we expect to see large deviations from the mean. This is captured by the variance, as shown in [26]:

Theorem 15.15 *Suppose that X is a topologically regular set and its boundary is a smooth $(n-1)$-dimensional manifold where $n > 1$ and that ρ and f are smooth functions. Let $\mathbb{L}$ be a stationary isotropic lattice. Then there is a constant $C(X,\rho,f) > 0$ such that for all a and ε sufficiently small,*

$$\mathbf{var}(\hat{V}_{n-1}(X)) \leq C(X,\rho,f)a^n\varepsilon^{-1}.$$

The interesting thing here is that the effect of the resolution on the variance is much larger than the effect of ε. In particular, if a and ε tend to 0 at the same rate, the variance will also go to zero. So to obtain small variance, it is more important to have high resolution than little blurring.

A computation similar to the one in Sect. 15.2.6 with $\varepsilon^{-1}f \circ \theta_\varepsilon^X$ replacing $c_{\mathbb{L}}\mathbf{1}_X$ shows that

$$\mathbf{var}(\hat{V}_{n-1}(X)) = \varepsilon^{-2}\omega_n^{-1}c_{\mathbb{L}}^{-2} \sum_{z\in\mathbb{L}_0^*\setminus\{0\}} \int_{S^{n-1}} \left|\mathscr{F}(f \circ \theta_\varepsilon^X)(a^{-1}|z|u)\right|^2 du.$$

As in the case of volume estimators, the variance can be studied by considering the Fourier coefficients in this sum. This is the approach in [26].

15.3.6 Minkowski Tensors from Grey-Scale Images

As in the black-and-white case, local estimators for Minkowski tensors would have to take the position of each configuration into account. That is, we must consider estimators of the form

$$a^n \varepsilon^{-m} \sum_{z \in a\mathbb{L}} f(z, \theta_\varepsilon^X(z, a, k)),$$

where $f : \mathbb{R}^n \times [0,1]^{C_{0,0}^k} \to \mathbb{T}^{r+s}$. Such estimators were studied in [27] with the purpose of defining estimators for Minkowski tensors. Under the assumption that $a = \varepsilon$ and X is r-regular, it was shown that all Minkowski tensors of the form $\Phi_{n-1}^{r,s}(X)$ and $\Phi_{n-2}^{r,0}(X)$ can be estimated by an asymptotically unbiased algorithm of this form. This requires that the point spread function is known.

We will not show this in general here but just outline the idea for surface tensors in 2D. We assume that $\mathbb{L}$ is the standard lattice $\mathbb{Z}^2$ spanned by the basis vectors e_1 and e_2. Consider the estimator

$$\hat{\Phi}_1^{r,s}(X) := a \sum_{z \in a\mathbb{Z}^2} z^r f(\theta_a^X(z, a, 2)).$$

Comparing with the definition (15.1), we expect that f should be an estimate for u^s.

The idea is to estimate the normal direction u by the direction in which the grey-values decrease fastest. Given a 2×2 configuration, we can look at how fast the grey-values change in the vertical and horizontal direction to get an idea about the normal direction, see Fig. 15.3.

More precisely, consider a boundary point x with normal vector u and suppose that a is small. Then, in a neighborhood around x, the image will look almost like a blurred halfspace with normal vector u. In particular, if y lies in this neighborhood, then the grey-value at y will be approximately $\theta(\langle y - x, u \rangle)$ since $\theta(t)$ is the grey-value of a point at signed distance t to the boundary of the halfspace.

If the whole 2×2 lattice block $z + aC_{0,0}^2 = \{z, z + ae_1, z + ae_2, z + ae_1 + ae_2\}$ lies in this neighborhood of x, then

$$\theta_a^X(z) \approx \theta(\langle z - x, u \rangle),$$
$$\theta_a^X(z + ae_1) \approx \theta(\langle z + ae_1 - x, u \rangle),$$
$$\theta_a^X(z + ae_2) \approx \theta(\langle z + ae_2 - x, u \rangle).$$

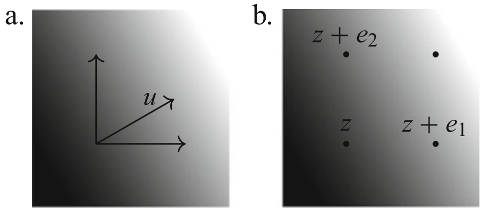

Fig. 15.3 (**a**) A *blurred halfspace* with normal u. The normal vector is determined by how fast the grey-values change in the horizontal and vertical direction. (**b**) A configuration of grey-values. To determine u one can look at how the grey-vales change from z to $z + e_1$ and from z to $z + e_2$

It follows that

$$\theta^{-1}(\theta_a^X(z + ae_1)) - \theta^{-1}(\theta_a^X(z)) \approx \langle ae_1, u\rangle = au_1,$$
$$\theta^{-1}(\theta_a^X(z + ae_2)) - \theta^{-1}(\theta_a^X(z)) \approx \langle ae_2, u\rangle = au_2,$$

where $u = (u_1, u_2)$. We may use the left hand side to estimate u and estimate u^s from this. Note that this requires that θ, which is determined by the point spread function, is known. It is shown in [27] that this algorithm is asymptotically unbiased.

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3. Manuscripts should be submitted either online at www.editorialmanager.com/lnm to Springer's mathematics editorial in Heidelberg, or electronically to one of the series editors. Authors should be aware that incomplete or insufficiently close-to-final manuscripts almost always result in longer refereeing times and nevertheless unclear referees' recommendations, making further refereeing of a final draft necessary. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter, a bibliography and several sample chapters. Parallel submission of a manuscript to another publisher while under consideration for LNM is not acceptable and can lead to rejection.

4. In general, **monographs** will be sent out to at least 2 external referees for evaluation.

   A final decision to publish can be made only on the basis of the complete manuscript, however a refereeing process leading to a preliminary decision can be based on a pre-final or incomplete manuscript.

   Volume Editors of **multi-author works** are expected to arrange for the refereeing, to the usual scientific standards, of the individual contributions. If the resulting reports can be forwarded to the LNM Editorial Board, this is very helpful. If no reports are forwarded or if other questions remain unclear in respect of homogeneity etc, the series editors may wish to consult external referees for an overall evaluation of the volume.

5. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include

   – a table of contents;
   – an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   – a subject index: as a rule this is genuinely helpful for the reader.
   – For evaluation purposes, manuscripts should be submitted as pdf files.

6. Careful preparation of the manuscripts will help keep production time short besides ensuring satisfactory appearance of the finished book in print and online. After acceptance of the manuscript authors will be asked to prepare the final LaTeX source files (see LaTeX templates online: https://www.springer.com/gb/authors-editors/book-authors-editors/manuscriptpreparation/5636) plus the corresponding pdf- or zipped ps-file. The LaTeX source files are essential for producing the full-text online version of the book, see http://link.springer.com/bookseries/304 for the existing online volumes of LNM). The technical production of a Lecture Notes volume takes approximately 12 weeks. Additional instructions, if necessary, are available on request from lnm@springer.com.

7. Authors receive a total of 30 free copies of their volume and free access to their book on SpringerLink, but no royalties. They are entitled to a discount of 33.3 % on the price of Springer books purchased for their personal use, if ordering directly from Springer.

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