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Josep Díaz
Lefteris Kirousis
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Maria Serna
Editors

Extended Abstracts Summer 2015

Strategic Behavior in Combinatorial Structures; Quantitative Finance

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Extended Abstracts Summer 2015

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Josep Díaz
Lefteris Kirousis
Maria Serna
Editors

Quantitative Finance

Luis Ortiz-Gracia
Editor

Birkhäuser

Editors
Josep Díaz
Departament de Ciències de la Computació
Universitat Politècnica de Catalunya
Barcelona, Spain

Lefteris Kirousis
Department of Mathematics
National and Kapodistrian University
Zografos, Greece

Luis Ortiz-Gracia
Department of Econometrics
University of Barcelona
Barcelona, Spain

Maria Serna
Departament de Ciències de la Computació
Universitat Politècnica de Catalunya
Barcelona, Spain

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Foreword

The *Workshop on Strategic Behavior and Phase Transitions in Random and Complex Combinatorial Structures* was held in the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona) from June 8th to 12th, 2015. This workshop was part of a research activity in CRM under the umbrella name *Algorithmic Perspectives in Economics and Physics* extended from April 7th to June 19th, 2015. Besides CRM, this research activity was funded by several Catalan organizations (Institut d' Estudis Catalans, Institució Centres de Recerca de Catalunya, Universitat Autònoma de Barcelona, and Generalitat de Catalunya) and by the Simons Institute for the Theory of Computing. The organizer committee for the program consisted of Dimitris Achlioptas (Department of Computer Science, UC Santa Cruz), Josep Díaz (Department of Computer Science, Universitat Politècnica de Catalunya), Lefteris Kirousis (Department of Mathematics, National and Kapodistrian University of Athens), and María Serna (Department of Computer Science, Universitat Politècnica de Catalunya).

The main research theme of the workshop was to explore possible ties between phase transitions on one hand, and game theory on the other. To be more specific, note that an important research area of the last decade is how atomic agents, acting locally and microscopically, lead to discontinuous macroscopic changes. This point of view has proved to be especially useful in studying the evolution of random and usually complex combinatorial objects (typically, networks) with respect to discontinuous changes in global parameters like connectivity. Naturally, there is a strategic element in the formation of a transition: the atomic agents seek "selfishly" to optimize a local microscopic parameter aiming at macroscopic changes that optimize their utility. Investigating the question of whether the connection of microscopic strategic behavior with macroscopic phase transitions is a legitimate and meaningful research objective was the scope of the workshop.

The workshop was attended by more than thirty registered participants, several of which were Ph.D. students or early career post-doctoral researchers. Because of the no-fee, open access policy that the organizers opted for, there were many more

non-registered participants. The conference followed a rather relaxed timetable that encouraged impromptu discussions and interactions.

The formal program comprised of some twenty presentations, more or less equally divided between the areas of random graphs and phase transitions on one hand, and game theory on the other. The organizers actively sought to have renowned researchers give some of the talks and at the same time to draw from the pool of early career, promising researchers to present their current work. Given the diverse background of the audience, presentations at a trans-thematic style and at a non specialized, high level were encouraged.

Barcelona, Spain — Josep Díaz
Athens, Greece — Lefteris Kirousis
Barcelona, Spain — Maria Serna
September 2015

On the Push&Pull Protocol for Rumour Spreading

Hüseyin Acan, Andrea Collevecchio, Abbas Mehrabian, and Nick Wormald

Abstract The asynchronous push&pull protocol, a randomized distributed algorithm for spreading a rumour in a graph G, is defined as follows. Independent exponential clocks of rate 1 are associated with the vertices of G, one to each vertex. Initially, one vertex of G knows the rumour. Whenever the clock of a vertex x rings, it calls a random neighbour y: if x knows the rumour and y does not, then x tells y the rumour (a push operation), and if x does not know the rumour and y knows it, y tells x the rumour (a pull operation). The average spread time of G is the expected time it takes for all vertices to know the rumour, and the guaranteed spread time of G is the smallest time t such that with probability at least $1 - 1/n$, after time t all vertices know the rumour. The synchronous variant of this protocol, in which each clock rings precisely at times $1, 2, \ldots$, has been studied extensively.

We prove the following results for any n-vertex graph: in either version, the average spread time is at most linear even if only the pull operation is used, and the guaranteed spread time is within a logarithmic factor of the average spread time, so it is $O(n \log n)$. In the asynchronous version, both the average and guaranteed spread times are $\Omega(\log n)$. We give examples of graphs illustrating that these bounds are best possible up to constant factors.

We also prove the first theoretical relationships between the guaranteed spread times in the two versions. Firstly, in all graphs the guaranteed spread time in the asynchronous version is within an $O(\log n)$ factor of that in the synchronous version, and this is tight. Next, we find examples of graphs whose asynchronous spread times are logarithmic, but the synchronous versions are polynomially large. Finally, we

H. Acan (✉) • N. Wormald
School of Mathematical Sciences, Monash University, Clayton, VIC, Australia
e-mail: huseyin.acan@monash.edu; nick.wormald@monash.edu

A. Collevecchio
School of Mathematical Sciences, Monash University, Clayton, VIC, Australia

Ca' Foscari University, Venice, Italy
e-mail: andrea.collevecchio@monash.edu

A. Mehrabian
Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada
e-mail: amehrabi@uwaterloo.ca

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show for any graph that the ratio of the synchronous spread time to the asynchronous spread time is $O(n^{2/3})$.

1 Introduction

Randomized rumour spreading is an important primitive for information dissemination in networks and has numerous applications in network science, ranging from spreading information in the WWW and Twitter to spreading viruses and diffusion of ideas in human communities. A well studied rumour spreading protocol is the *(synchronous) push&pull protocol*, introduced by Demers et al. [5] and popularized by Karp et al. [21]. Suppose that one node in a network is aware of a piece of information, the 'rumour', and wants to spread it to all nodes quickly. The protocol proceeds in rounds. In each round, every *informed* node contacts a random neighbour and sends the rumour to it ('pushes' the rumour), and every *uninformed* nodes contacts a random neighbour and gets the rumour if the neighbour knows it ('pulls' the rumour).

A point to point communication network can be modelled as an undirected graph: the nodes represent the processors and the links represent communication channels between them. Studying rumour spreading has several applications to distributed computing in such networks, of which we mention just two. The first is in broadcasting algorithms: a single processor wants to broadcast a piece of information to all other processors in the network (see [18] for a survey). There are at least four advantages to the push&pull protocol: it puts much less load on the edges than naive flooding, it is simple (each node makes a simple local decision in each round; no knowledge of the global topology is needed; no state is maintained), scalable (the protocol is independent of the size of the network: it does not grow more complex as the network grows) and robust (the protocol tolerates random node/link failures without the use of error recovery mechanisms; see [10]). A second application comes from the maintenance of databases replicated at many sites, e.g., yellow pages, name servers, or server directories. There are updates injected at various nodes, and these updates must propagate to all nodes in the network. In each round, a processor communicates with a random neighbour and they share any new information, so that eventually all copies of the database converge to the same contents; see [5] for details. Other than the aforementioned applications, rumour spreading protocols have successfully been applied in various contexts such as resource discovery [17], distributed averaging [4], data aggregation [22], and the spread of computer viruses [2].

In this paper we only consider simple, undirected and connected graphs. Given a graph and a starting vertex, the *spread time* of a certain protocol is the time it takes for the rumour to spread in the whole graph, i.e., the time difference between the moment the protocol is initiated and the moment when everyone learns the rumour. For the synchronous push&pull protocol, it turned out that the spread time is closely related to the *expansion profile* of the graph. Let $\Phi(G)$ and $\alpha(G)$ denote

the conductance and the vertex expansion of a graph G, respectively. After a series of results by various scholars, Giakkoupis [15, 16] showed the spread time is

$$O\left(\min\{\Phi(G)^{-1} \cdot \log n,\ \alpha(G)^{-1} \cdot \log^2 n\}\right).$$

This protocol has recently been used to model news propagation in social networks. Doerr et al. [6] proved an upper bound of $O(\log n)$ for the spread time on Barabási-Albert graphs, and Fountoulakis et al. [13] proved the same upper bound (up to constant factors) for the spread time on Chung-Lu random graphs.

All the above results assumed a synchronized model, i.e., all nodes take action simultaneously at discrete time steps. In many applications and certainly in real-world social networks, this assumption is not very plausible. Boyd et al. [4] proposed an asynchronous time model with a continuous time line. Each node has its own independent clock that rings at the times of a rate 1 Poisson process. (Since the time between rings is an exponential random variable, we shall call this an *exponential clock*.) The protocol now specifies for every node what to do when its own clock rings. The rumour spreading problem in the asynchronous time model has so far received less attention. Rumour spreading protocols in this model turn out to be closely related to Richardson's model for the spread of a disease [9], and to first-passage percolation [19] with edges having i.i.d. exponential weights. The main difference is that in rumour spreading protocols each vertex contacts one neighbour at a time. So, for instance, in the 'push only' protocol, the net communication rate outwards from a vertex is fixed, and hence the rate that the vertex passes the rumour to any one given neighbour is inversely proportional to its degree (the push&pull protocol is a bit more complicated). Hence, the degrees of vertices play a crucial role not seen in Richardson's model or first-passage percolation. However, on regular graphs, the asynchronous push&pull protocol, Richardson's model, and first-passage percolation are essentially the same process, assuming appropriate parameters are chosen. In this sense, Fill–Pemantle [11] and Bollobás–Kohayakawa [3] showed that a.a.s. the spread time of the asynchronous push&pull protocol is $\Theta(\log n)$ on the hypercube graph. Janson [20] and Amini et al. [1] showed the same results (up to constant factors) for the complete graph and for random regular graphs, respectively. These bounds match the same order of magnitude as in the synchronized case. Doerr et al. [8] experimentally compared the spread time in the two time models. They state that 'Our experiments show that the asynchronous model is faster on all graph classes [considered here].' However, a general relationship between the spread times of the two variants has not been proved theoretically.

Fountoulakis et al. [13] studied the asynchronous push&pull protocol on Chung-Lu random graphs with exponent between 2 and 3. For these graphs, they showed that a.a.s. after some constant time, $n - o(n)$ nodes are informed. Doerr et al. [7] showed that for the preferential attachment graph (the non-tree case), a.a.s. all but $o(n)$ vertices receive the rumour in time $O\left(\sqrt{\log n}\right)$, but to inform all vertices a.a.s., $\Theta(\log n)$ time is necessary and sufficient. Panagiotou–Speidel [23] studied

this protocol on Erdős-Renyi random graphs and proved that if the average degree is $(1 + \Omega(1)) \log n$, a.a.s. the spread time is $(1 + o(1)) \log n$.

2 Our Contribution

In this paper we answer a fundamental question about the asynchronous push&pull protocol: what are the minimum and maximum spread times on an n-vertex graph? Our proof techniques yield new results on the well studied synchronous version as well. We also compare the performances of the two protocols on the same graph, and prove the first theoretical relationships between their spread times.

We now formally define the protocols. In this paper G denotes the ground graph which is simple and connected. Its number of vertices, denoted n, is assumed to be sufficiently large.

Definition 1 (Asynchronous push&pull protocol) Suppose that an independent exponential clock of rate 1 is associated with each vertex of G. Suppose that, initially, some vertex v of G knows a piece of information, the so-called *rumour*. The rumour spreads in G as follows: whenever the clock of a vertex x rings, this vertex performs an 'action': it calls a random neighbour y; if x knows the rumour and y does not, then x tells y the rumour (a *push* operation), and if x does not know the rumour and y knows it, y tells x the rumour (a *pull* operation). Note that if both x and y know the rumour or neither of them knows it, then this action is useless. Also, vertices have no memory, hence x may call the same neighbour several consecutive times. The *spread time* of G starting from v, written $\mathrm{ST}_{\mathsf{a}}(G, v)$, is the first time that all vertices of G know the rumour. Note that this is a continuous random variable, with two sources of randomness: the Poisson processes associated with the vertices, and random neighbour-selection of the vertices. The *guaranteed spread time* of G, written $\mathrm{gst}_{\mathsf{a}}(G)$, is the smallest deterministic number t such that, for every $v \in V(G)$, we have $\mathbb{P}[\mathrm{ST}_{\mathsf{a}}(G, v) > t] \leq 1/n$. The *worst average spread time* of G, written $\mathrm{wast}_{\mathsf{a}}(G)$, is the smallest deterministic number t such that, for every $v \in V(G)$, we have $\mathbb{E}[\mathrm{ST}_{\mathsf{a}}(G, v)] \leq t$.

Definition 2 (Synchronous push&pull protocol) Initially some vertex v of G knows the rumour, which spreads in G in a round-robin manner: in each round $1, 2, \ldots$, all vertices perform actions simultaneously. That is, each vertex x calls a random neighbour y; if x knows the rumour and y does not, then x tells y the rumour (a *push* operation), and if x does not know the rumour and y knows it, y tells x the rumour (a *pull* operation). Note that this is a synchronous protocol, e.g., a vertex that receives a rumour in a certain round cannot send it on in the same round. The *spread time* of G starting from v, $\mathrm{ST}_{\mathsf{s}}(G, v)$, is the first time that all vertices of G know the rumour. Note that this is a discrete random variable, with one source of randomness: the random neighbour-selection of the vertices. The *guaranteed*

spread time of G, written $\mathrm{gst}_{\mathsf{s}}(G)$, and the *worst average spread time* of G, written $\mathrm{wast}_{\mathsf{s}}(G)$, are defined in an analogous way to the asynchronous case.

Our first main result is the following theorem.

Theorem 3 *For any n-vertex graph G, the following holds:*

(i) $(1-1/n)\,\mathrm{wast}_{\mathsf{a}}(G) \le \mathrm{gst}_{\mathsf{a}}(G) \le e\,\mathrm{wast}_{\mathsf{a}}(G)\log n$;
(ii) $\mathrm{wast}_{\mathsf{a}}(G) = \Omega(\log n)$ *and* $\mathrm{wast}_{\mathsf{a}}(G) = O(n)$;
(iii) $\mathrm{gst}_{\mathsf{a}}(G) = \Omega(\log n)$ *and* $\mathrm{gst}_{\mathsf{a}}(G) = O(n\log n)$.

Moreover, these bounds are asymptotically best possible, up to the constant factors.

Our proof of the right-hand bound in (ii) is based on the pull operation only, so this bound applies equally well to the 'pull only' protocol.

The arguments for (i) and the right hand bounds in (ii) and (iii) can easily be extended to the synchronous variant, giving the following theorem. The bound (iii) in Theorem 4 below also follows from [10, Theorem 2.1], but here we also show its tightness.

Theorem 4 *For any n-vertex graph G, the following holds:*

(i) $(1-1/n)\,\mathrm{wast}_{\mathsf{s}}(G) \le \mathrm{gst}_{\mathsf{s}}(G) \le e\,\mathrm{wast}_{\mathsf{s}}(G)\log n$;
(ii) $\mathrm{wast}_{\mathsf{s}}(G) = O(n)$;
(iii) $\mathrm{gst}_{\mathsf{s}}(G) = O(n\log n)$.

Moreover, these bounds are asymptotically best possible, up to the constant factors.

Open problem 5 Find the best possible constant factors in Theorems 3 and 4.

We next turn to studying the relationship between the asynchronous and synchronous variants on the same graph.

Theorem 6 *For any n-vertex graph G, we have*

(i) $\mathrm{gst}_{\mathsf{a}}(G) = O\left(\mathrm{gst}_{\mathsf{s}}(G)\log n\right)$; *and*
(ii) $\mathrm{wast}_{\mathsf{a}}(G) = O\left(\mathrm{wast}_{\mathsf{s}}(G)\log n\right)$.

Moreover, these bounds are best possible, up to the constant factors.

For all graphs we examined a stronger result holds, which suggests the following conjecture.

Conjecture 7 For any n-vertex graph G, we have

(i) $\mathrm{gst}_{\mathsf{a}}(G) \le \mathrm{gst}_{\mathsf{s}}(G) + O(\log n)$; and
(ii) $\mathrm{wast}_{\mathsf{a}}(G) \le \mathrm{wast}_{\mathsf{s}}(G) + O(\log n)$.

Our last main result is the following theorem, whose proof is somewhat technical, and uses couplings with the sequential rumour spreading protocol.

Theorem 8 *For any $\alpha \in [0,1)$ we have*

$$\mathrm{gst}_{\mathsf{s}}(G) \le n^{1-\alpha} + O(\mathrm{gst}_{\mathsf{a}}(G)n^{(1+\alpha)/2}). \tag{1}$$

Table 1 Summary of the known spread times of the push&pull protocols on various graph classes

Graph G	$\mathrm{wast_s}(G)$	$\mathrm{wast_a}(G)$
Path	$(4/3)n + O(1)$	$n + O(1)$
Star	2	$\log n + O(1)$
Complete	$\sim \log_3 n$ [21]	$\log n + o(1)$
General	$O\left(\Phi(G)^{-1} \cdot \log n\right)$ [15]	$\boldsymbol{O\left(\Phi(G)^{-1} \cdot \log^2 n\right)}$ **[this paper]**
General	$O\left(\alpha(G)^{-1} \cdot \log^2 n\right)$ [16]	$\boldsymbol{O\left(\alpha(G)^{-1} \cdot \log^3 n\right)}$ **[this paper]**
Hypercube	$\Theta(\log n)$ [10]	$\Theta(\log n)$ [11]
$\mathcal{G}(n,p)$ $1 < \frac{np}{\log n}$ fixed	$\Theta(\log n)$ [10]	$\sim \log n$ [23]
$\mathcal{G}(n,d)$ $2 < d$ fixed	$\Theta(\log n)$ [12]	$\sim (\log n)(d-1)/(d-2)$ [1]
Preferential attachment (Barabási–Albert)	$\Theta(\log n)$ [6]	$\Theta(\log n)$ [7]
Chung–Lu model	$\Theta(\log n)$ [13]	$\Theta(\log n)$ [13]
Random geometric graphs in $\left[0, n^{1/d}\right]^d$	$\Theta(n^{1/d}/r + \log n)$ [14]	$\boldsymbol{O\left(\log n \cdot n^{1/d}/r + \log^2 n\right)}$ **[this paper]**

Corollary 9 *We have*

$$\frac{\mathrm{gst_s}(G)}{\mathrm{gst_a}(G)} = \Omega(1/\log n) \quad \textit{and} \quad \frac{\mathrm{gst_s}(G)}{\mathrm{gst_a}(G)} = O\left(n^{2/3}\right),$$

and the left hand bound is asymptotically best possible, up to the constant factor. Moreover, there exist infinitely many graphs for which this ratio is $\Omega\left(n^{1/3}(\log n)^{-4/3}\right)$.

Open problem 10 What is the maximum possible value of the ratio $\mathrm{gst_s}(G)/\mathrm{gst_a}(G)$ for an n-vertex graph G?

A summary of known results on the spread times of the push&pull protocols on various graphs are given in Table 1.

The parameters $\mathrm{wast_s}(G)$ and $\mathrm{wast_a}(G)$ can be approximated easily using the Monte Carlo method: simulate the protocols several times, measure the spread time of each simulation, and output the average. Another open problem is to design a *deterministic* approximation algorithm for any one of $\mathrm{wast_a}(G)$, $\mathrm{gst_a}(G)$, $\mathrm{wast_s}(G)$ or $\mathrm{gst_s}(G)$.

Previous work on the asynchronous push&pull protocol has focused on special graphs. This paper is the first systematic study of this protocol on all graphs. We believe this protocol is fascinating and is quite different from its synchronous variant, in the sense that different techniques are required for analyzing it, and the spread times of the two versions can be quite different. Our work makes significant progress on better understanding of this protocol, and we hope it inspires further research on this problem.

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Random Walks That Find Perfect Objects and the Lovász Local Lemma

Dimitris Achlioptas and Fotis Iliopoulos

Abstract We give an algorithmic local lemma by establishing a sufficient condition for the uniform random walk on a directed graph to reach a sink quickly. Our work is inspired by Moser's entropic method proof of the Lovász Local Lemma (LLL) for satisfiability, and completely bypasses the Probabilistic Method formulation of the LLL. In particular, our method works when the underlying state space is entirely unstructured. Similarly to Moser's argument, the key point is that the inevitability of reaching a sink is established by bounding the entropy of the walk as a function of time.

1 Introduction

Let Ω be a (large) set of objects and let F be a collection of subsets of Ω, each subset comprising objects sharing some (negative) feature. We will refer to each subset $f \in F$ as a *flaw* and, following linguistic rather than mathematical convention, say that f is *present in* σ if $f \ni \sigma$. We will say that an object $\sigma \in \Omega$ is *flawless* (perfect) if no flaw is present in σ. For example, given a CNF formula on n variables with clauses $c_1, c_2, \ldots, c_m$, we can define a flaw for each clause c_i, comprising the subset of $\Omega = \{0,1\}^n$ violating c_i.

Given Ω and F we can often establish the *existence* of flawless objects via the Probabilistic Method. To do so, we introduce a probability measure on Ω and consider the collection of ("bad") events $\mathcal{A}$ corresponding to the flaws (one event per flaw). The existence of flawless objects is thus equivalent to the intersection of the complements of the bad events having strictly positive probability. Clearly, such

D. Achlioptas (✉)
University of California Santa Cruz, Santa Cruz, CA, USA
e-mail: optas@cs.ucsc.edu

F. Iliopoulos
University of California Berkeley, Berkeley, CA, USA
e-mail: fotis.iliopoulos@berkeley.edu

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positivity always holds if the events in $\mathcal{A}$ are independent and none of them has measure 1. One of the most powerful tools of the Probabilistic Method is the Lovász Local Lemma (LLL), asserting that such positivity also holds under a condition of limited dependence among the events in $\mathcal{A}$.

General LLL Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a set of events and let $D(i) \subseteq [m] \setminus \{i\}$ denote the set of indices of the dependency set of A_i, i.e., A_i is mutually independent of all events in $\mathcal{A} \setminus \{A_i \cup \bigcup_{j \in D(i)} A_j\}$. If there exist positive real numbers $\{\mu_i\}$ such that for all $i \in [m]$,

$$\Pr(A_i) \prod_{j \in \{i\} \cup D(i)} (1 + \mu_j) \leq \mu_i, \tag{1}$$

then the probability that none of the events in $\mathcal{A}$ occurs is at least $\prod_{i=1}^{m} 1/(1 + \mu_i) > 0$.

In a landmark work [4], Moser and Tardos made the general LLL constructive for *product* measures over explicitly presented variables. Specifically, in the variable setting of [4], each event A_i is determined by a set of variables $\mathrm{vbl}(A_i)$ so that $j \in D(i)$ if and only if $\mathrm{vbl}(A_i) \cap \mathrm{vbl}(A_j) \neq \emptyset$. Moser and Tardos proved that if (1) holds, then repeatedly selecting *any* occurring event A_i (flaw present) and resampling every variable in $\mathrm{vbl}(A_i)$ independently of all others, leads to a flawless object after a linear expected number of resamplings. Beyond the variable setting, Harris and Srinivasan in [2] algorithmized the general LLL for the uniform measure on permutations.

2 A New Framework

Inspired by the breakthrough of Moser [3], we take a more direct approach to finding flawless objects, bypassing the probabilistic formulation of the existence question. Specifically, we replace the measure on Ω by a directed graph D on Ω and we seek flawless objects by taking random walks on D. With this in mind, we refer to the elements of Ω as states. As in Moser's work [3], each state transformation (step of the walk) $\sigma \to \tau$ will be taken to *address* a flaw present at σ. Naturally, a step may eradicate other flaws beyond the one addressed but may also introduce new flaws (and, in fact, may fail to eradicate the addressed flaw). By replacing the measure with a directed graph we achieve two main effects:

(i) both the set of objects Ω and every flaw $f \subseteq \Omega$ can be entirely *amorphous*; that is, Ω does not need to have product form $\Omega = D_1 \times \cdots \times D_n$, as in Moser–Tardos [4], or any form of symmetry, as in Harris–Srinivasan [2];

(ii) the set of transformations for addressing a flaw f can differ *arbitrarily* among the different states $\sigma \in f$, allowing the actions to adapt to the "environment". This is in sharp contrast with all past algorithmic versions of the LLL, where either no or very minimal adaptivity was possible.

Concretely, for each $\sigma \in \Omega$, let $U(\sigma) = \{f \in F : \sigma \in f\}$, i.e., $U(\sigma)$ is the set of flaws present in σ. For each $\sigma \in \Omega$ and $f \in U(\sigma)$ we require a set $A(f, \sigma) \subseteq \Omega$ that must contain at least one element other than σ, which we refer to as the set of possible *actions* for addressing flaw f in state σ. To address flaw f in state σ we select uniformly at random an element $\tau \in A(f, \sigma)$ and walk to state τ, noting that possibly $\tau = \sigma \in A(f, \sigma)$. Our main point of departure is that now the set of actions for addressing a flaw f in each state σ can depend *arbitrarily* on the state, σ, itself.

We represent the set of all possible state transformations as a multi-digraph D on Ω formed as follows: for each state σ, for each flaw $f \in U(\sigma)$, for each state $\tau \in A(f, \sigma)$ place an arc $\sigma \xrightarrow{f} \tau$ in D, i.e., an arc labeled by the flaw being addressed. Thus, D may contain pairs of states σ, τ with multiple $\sigma \to \tau$ arcs, each such arc labeled by a different flaw, each such flaw f having the property that moving to τ is one of the actions for addressing f at σ, i.e., $\tau \in A(f, \sigma)$. Since we require that the set $A(f, \sigma)$ contains at least one element other than σ for every flaw in $U(\sigma)$ we see that a vertex of D is a sink if and only if it is flawless. We focus on digraphs satisfying

Atomicity D is *atomic* if for every flaw f and state τ there is *at most* one arc incoming to τ labeled by f.

The purpose of atomicity is to capture "accountability of action". In particular, note that if D is atomic, then every walk on D can be reconstructed from its final state and the sequence of labels on the arcs traversed, as atomicity allows one to trace the walk backwards unambiguously. To our pleasant surprise, in all applications we have considered so far we have found atomicity to be "a feature not a bug", serving as a very valuable *aid* in the design of flaws and actions, i.e., of algorithms.

Having defined the multi-digraph D on Ω, we will now define a digraph C on the set of flaws F, reflecting some of the structure of D.

Potential Causality For each arc $\sigma \xrightarrow{f} \tau$ in D and each flaw g present in τ, we say that f *causes* g if $g = f$ or $g \notin \sigma$. If D contains *any* arc in which f causes g we say that f *potentially causes* g.

Potential Causality Digraph The digraph $C = C(\Omega, F, D)$ of the potential causality relation, i.e., the digraph on F where $f \to g \Leftrightarrow f$ potentially causes g, is called the *potential causality digraph*. The *neighborhood* of a flaw f is $\Gamma(f) = \{g : f \to g \text{ exists in } C\}$.

In the interest of brevity, we will call C the causality digraph, instead of the potential causality digraph. It is important to note that C contains an arc $f \to g$ if there exists *even one* state transition aimed at addressing f that causes g to appear in the new state. In that sense, C is a "pessimistic" estimator of causality (or, alternatively, a lossy compression of D). This pessimism is both the strength and the weakness of our approach. On one hand, it makes it possible to extract results about algorithmic progress without tracking the state. On the other hand, it only gives good results when C remains sparse even in the presence of such stringent

arc inclusion. We feel that this tension is meaningful: maintaining the sparsity of C requires that the actions for addressing each flaw across different states are *coherent* with respect to the flaws they cause.

So far we have not discussed *which* flaw to address in each flawed state, demanding instead a non-empty set of actions $A(f,\sigma)$ for each flaw f present in a state σ. Suffice it to say that we consider algorithms which employ an *arbitrary* ordering π of F and in each flawed state σ address the greatest flaw according to π in a subset of $U(\sigma)$.

Definition 1 If π is any ordering of F, let $I_\pi : 2^F \to F$ be the function mapping each subset of F to its greatest element according to π, with $I_\pi(\emptyset) = \emptyset$. We will sometimes abuse notation and for a state $\sigma \in \Omega$, write $I_\pi(\sigma)$ for $I_\pi(U(\sigma))$ and also write I for I_π, when π is clear from context.

Definition 2 Let $D_\pi \subseteq D$ be the result of retaining for each state σ only the outgoing arcs with label $I_\pi(\sigma)$.

The next definition reflects that, since actions are selected uniformly, the *number* of actions available to address a flaw, i.e., the breadth of the "repertoire", is important.

Amenability The *amenability* of a flaw f is

$$A_f = \min_{\sigma \in f} |A(f,\sigma)| \ . \tag{2}$$

The amenability of a flaw f will be used to bound from below the amount of randomness consumed every time f is addressed. (The minimum in (2) is often inoperative with $|A(f,\sigma)|$ being the same for all $\sigma \in f$.)

3 Statement of Results

Our simplest result, stated below, concerns the case where, after choosing a single fixed permutation π of the flaws, in each flawed state σ the algorithm addresses the greatest flaw present in σ according to π, i.e., the algorithm is the uniform random walk on D_π.

Theorem 3 *If for every flaw $f \in F$,*

$$\sum_{g \in \Gamma(f)} \frac{1}{A_g} < \frac{1}{\mathrm{e}} \ ,$$

then for any ordering π of F and any $\sigma_1 \in \Omega$, the uniform random walk on D_π starting at σ_1 reaches a sink within $(\log_2 |\Omega| + |U(\sigma_1)| + s)/\delta$ steps with probability at least $1 - 2^{-s}$, where $\delta = 1 - \max_{f \in F} \sum_{g \in \Gamma(f)} \frac{\mathrm{e}}{A_g}$.

Remark 4 In applications, typically, $\delta = \Theta(1)$.

Theorem 3 has the following three features worth discussing.

Arbitrary initial state: the fact that σ_1 can be arbitrary means that any foothold on Ω suffices to apply the theorem, without needing to be able to sample from Ω according to some measure. While sampling from Ω has generally not been an issue in existing applications of the LLL, this has only been true precisely because the sets and the measures considered have been highly structured.

Arbitrary number of flaws: the running time depends only on the number of flaws present in the initial state, $|U(\sigma_1)|$, not on the total number of flaws $|F|$. This has an implication analogous to the result of Hauepler–Saha–Srinivasan [1] on core events: even when $|F|$ is very large, e.g., super-polynomial in the problem's encoding length, we can still get an efficient algorithm if we can show that $|U(\sigma_1)|$ is small, e.g., by proving that in every state only polynomially many flaws may be present. This feature provides great flexibility in the design of flaws.

Cutoff phenomenon: the bound on the running-time is sharper than a typical high probability bound, being instead akin to a mixing time cutoff bound, wherein the distance to the stationary distribution drops from near 1 to near 0 in a very small number of steps past a critical point. In our setting, the walk first makes $(\log_2 |\Omega| + |U(\sigma_1)|)/\delta$ steps without any guarantee of progress, but from that point on every single step has constant probability of being the last step. While, pragmatically, a high probability bound would be just as useful, the fact that our bound naturally takes this form suggests a potential deeper connection with the theory of Markov chains.

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Logit Dynamics with Concurrent Updates for Local Interaction Games

Vincenzo Auletta, Diodato Ferraioli, Francesco Pasquale, Paolo Penna, and Giuseppe Persiano

Abstract Game Theory is the main tool used to model the behavior of agents that are guided by their own objective in contexts where their gains depend also on the choices made by neighboring agents. Game theoretic approaches have been often proposed for modeling phenomena in a complex social network, such as the formation of the social network itself. We are interested in the *dynamics* that govern such phenomena. In this paper, we study a specific class of randomized update rules called the *logit choice function* which can be coupled with different selection rules so to give different dynamics. We study how the logit choice function behave in an extreme case of concurrency.

1 Introduction

In the last decade, we have observed an increasing interest in understanding phenomena occurring in complex systems consisting of a large number of simple networked components that operate autonomously guided by their own objectives and influenced by the behavior of the neighbors. Even though (online) social networks are a primary example of such systems, other remarkable typical instances can be found in Economics (e.g., markets), Physics (e.g., Ising model and spin systems) and Biology (e.g., evolution of life). A common feature of these systems is that the behavior of each component depends only on the interactions with a limited number of other components (its neighbors) and these interactions are usually very simple.

V. Auletta (✉) • D. Ferraioli • G. Persiano
Università di Salerno, Fisciano SA, Italy
e-mail: auletta@unisa.it; dferraioli@unisa.it; giuper@gmail.com

F. Pasquale
"Sapienza" Università di Roma, Roma, Italy
e-mail: pasquale@mat.uniroma2.it

P. Penna
Autonomous Researcher, Roma, Italy
e-mail: paolo.penna@inf.ethz.ch

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Game Theory is the main tool used to model the behavior of agents that are guided by their own objective in contexts where their gains depend also on the choices made by neighboring agents. Game theoretic approaches have been often proposed for modeling phenomena in a complex social network, such as the formation of the social network itself [2, 6, 10–12, 15, 21], the formation of opinions [8, 16, 22] and the spread of innovation [25, 27, 28]. Many of these models are based on *local interaction games* [26], where agents are represented as vertices on a *social graph* and the relationship between two agents is represented by a simple two-player game played on the edge joining the corresponding vertices.

We are interested in the *dynamics* that govern such phenomena and several dynamics have been studied in the literature like, for example, the best response dynamics [18], the logit dynamics [9], fictitious play [17] or no-regret dynamics [20]. Any such dynamics can be seen as made of two components:

(i) *selection rule:* by which the set of players that update their state (strategy) is determined;
(ii) *update rule:* by which the selected players update their strategy.

For example, the classical best response dynamics compose the *best response* update rule with a selection rule that selects one player at the time. In the best response update rule, the selected player picks the strategy that, given the current strategies of the other players, guarantees the highest utility. The Cournot dynamics [13], instead, combines the best response update rule with the selection rule that select all players. Other dynamics in which all players concurrently update their strategy are fictitious play [17] and the no-regret dynamics [20].

In this paper, we study a specific class of randomized update rules called the *logit choice function* [9, 24, 30], which is a type of noisy best response that models in a clean and tractable way the limited knowledge (or bounded rationality) of the players in terms of a parameter β called *inverse noise*. In similar models studied in Physics, β is the inverse of the temperature. Intuitively, a low value of β (that is, high temperature) models a noisy scenario in which players choose their strategies "nearly at random"; a high value of β (that is, low temperature) models a scenario with little noise in which players pick the strategies yielding higher payoffs with higher probability.

The logit choice function can be coupled with different selection rules so to give different dynamics. For example, in the *logit* dynamics [9], at every time step a single player is selected uniformly at random and the selected player updates her strategy according to the logit choice function. The remaining players are not allowed to revise their strategies in this time step. One of the appealing features of the logit dynamics is that it naturally describes an ergodic Markov chain. This means that the underlying Markov chain admits a *unique stationary distribution* which we take as solution concept. This distribution describes the long-run behavior of the system (whose states appear more frequently over a long run). The interplay between the noise and the underlying game naturally determines the system behavior: (i) as the noise becomes "very large" the equilibrium point is

"approximately" the uniform distribution; (ii) as the noise vanishes the stationary distribution concentrates on so called stochastically stable states [29] which, for certain classes of games, correspond to pure Nash equilibria [1, 9].

While the logit choice function is a very natural behavioral model for approximately rational agents, the specific selection rule selecting one single player per time step avoids any form of concurrency. Therefore a natural question arises:

What happens if *concurrent* updates are allowed?

For example, it is easy to construct games for which the best response converges to a Nash equilibrium when only one player is selected at each step and does not converge to any state when more players are chosen to concurrently update their strategies.

In this paper we study how the logit choice function behave in an extreme case of concurrency. Specifically, we couple this update rule with a selection rule by which *all* players update their strategies at every time step. We call such dynamics *all-logit*, as opposed to the classical (*one-*)logit dynamics, in which only one player at a time is allowed to move. Roughly speaking, the all-logit are to the one-logit what the Cournot dynamics are to the best response dynamics.

2 Our Contributions

We study the all-logit dynamics for local interaction games [14, 25, 26]. Here, players are vertices of a graph, called the *social graph*, and each edge is a two-player (exact) potential game. We remark that games played on different edges by a player may be different but, nonetheless, they have the same strategy set for the player. Each player picks one strategy that is used for all of her edges and the payoff is a (weighted) sum of the payoffs obtained from each game. This class of games includes coordination games on a network [14] that have been used to model the spread of innovation and of new technology in social networks [27, 28], and the Ising model [23], a model for magnetism. In particular, we study the all-logit dynamics on local interaction games for every possible value of the inverse noise β and we are interested on properties of the original one-logit dynamics that are preserved by the all-logit.

As a warm-up, we discuss two classical two-player games (these are trivial local interaction games played on a graph with two vertices and one edge): the coordination game and the prisoner's dilemma. Even though for both games the stationary distribution of the one-logit and of the all-logit are quite different, we identify three similarities. First, for both games, both Markov chains are reversible. Moreover, for both games, the expected number of players playing a certain strategy at the stationarity of the all-logit is exactly the same as if the expectation was taken on the stationary distribution of the one-logit. Finally, for these games the mixing time is asymptotically the same regardless of the selection rule. In this paper we will show that none of these findings is accidental.

We first study the *reversibility* of the all-logit dynamics, an important property of stochastic processes that is useful also to obtain explicit formulas for the stationary distribution. We *characterize* the class of games for which the all-logit dynamics (that is, the Markov chain resulting from the all-logit dynamics) are reversible and it turns out that this class coincides with the class of local interaction games. This implies that the all-logit dynamics of all two-player potential games are reversible; whereas not all potential games have reversible all-logit dynamics. This is to be compared with the well-known result saying that one-logit dynamics of every potential game are reversible with respect to the Gibbs measure; see [9]. One of the tools we develop for our characterization yields a closed formula for the stationary distribution of reversible all-logit dynamics.

Then, we focus on the *observables* of local interaction games. An observable is a function of the strategy profile (that is the sequence of strategies adopted by the players) and we are interested in its expected values at stationarity for both the one-logit and the all-logit. A prominent example of observable is the difference Diff between the number of players adopting two given strategies in a game. In a local interaction game modeling the spread of innovation on a social network this observable counts the difference between the number of adopters of the new and old technology, whereas in the Ising model it is the magnetic field of a magnet.

We show that there exists a class of observables whose expectation at stationarity of the all-logit is the same as the expectation at stationarity of the one-logit as long as the social network underlying the local interaction game is bipartite (and thus trivially for all two-player games). This class of observables includes the Diff observable. It is interesting to note that the Ising game has been mainly studied for bipartite graphs (e.g., the two-dimensional and the three-dimensional lattice). This implies that, for the Ising model, the all-logit dynamics are compatible with the observations and it is arguably more natural than the one-logit (that postulate that at any given time step only one particle updates its status and that the update strategy is instantaneously propagated). We extend this result by showing that, for general graphs, the extent at which the expectations of these observables differ can be upper and lower bounded by a function of β and of the distance of the social graph from a bipartite graph.

Finally, we give the first bounds on the mixing time of the all-logit. We start by giving a *general* upper bound on the mixing time of the all-logit in terms of the *cumulative utility* of the game. We then look at two specific classes of games: graphical coordination games and games with a dominant profile. For *graphical coordination games*, we prove an upper bound to the mixing time that exponentially depends on β. Note that in [4], the authors prove that the one-logit also takes an amount of time exponential in β for converging to the stationary distribution. For *games with a dominant profile*, we instead prove that the mixing time can be bounded by a function independent from β. Thus, also for these games the mixing time of the all-logit has the same behavior of the one-logit mixing time.

3 Related Works on Logit Dynamics

The all-logit dynamics for strategic games have been studied in Alos-Ferrer–Netzer [1]. Specifically, the authors study the logit-choice function combined with general selection rules (including the selection rule of the all-logit) and investigate conditions for which a state is *stochastically stable*. A stochastically stable state is a state that has non-zero probability as β goes to infinity; see [29]. We focus instead on a specific selection rule used by several remarkable dynamics considered in Game Theory (Cournot, fictitious play, and no-regret) and consider the whole range of values for β.

The one-logit dynamics have been actively studied starting from Blume [9] which shows that, for 2×2 coordination games, the risk dominant equilibria (see [19]) are stochastically stable. Much work has been devoted to the study of the one-logit for local interaction games with the aim of modeling and understanding the spread of innovation in a social network [14, 28]. A general upper bound on the mixing time of the one-logit dynamics for this class of games is given by Berger–Kenyon–Mossel–Peres [7]. Montanari–Saberi [25], instead, studied the hitting time of the highest potential configuration and related this quantity to a connectivity property of the underlying network. Asadpour–Saberi [3] considered the same problem for congestion games. The mixing time and the metastability of the one-logit dynamics for strategic games have been studied in [4, 5].

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The Set Chromatic Number of Random Graphs

Andrzej Dudek, Dieter Mitsche, and Paweł Prałat

Abstract We study the set chromatic number of a random graph $\mathcal{G}(n,p)$ for a wide range of $p = p(n)$. We show that the set chromatic number, as a function of p, forms an intriguing zigzag shape.

1 Introduction

A *proper colouring* of a graph is a labeling of its vertices with colours such that no two vertices sharing the same edge have the same colour. A colouring using at most k colours is called a *proper k-colouring*. The smallest number of colours needed to colour a graph G is called its *chromatic number*, and it is denoted by $\chi(G)$.

In this note we are concerned with another notion of colouring, first introduced by Chartrand–Okamoto–Rasmussen–Zhang [1]. For a given (not necessarily proper) k-colouring $c\colon V \to [k]$ of the vertex set of $G = (V,E)$, let

$$C(v) = \{c(u) : uv \in E\}$$

be the *neighbourhood colour set* of a vertex v. (In this paper, $[k] := \{1,2,\ldots,k\}$.) The colouring c is a *set colouring* if $C(u) \neq C(v)$ for every pair of adjacent vertices in G. The minimum number of colours, k, required for such a colouring is the *set chromatic number* $\chi_s(G)$ of G. One can show that

$$\log_2 \chi(G) + 1 \leq \chi_s(G) \leq \chi(G). \tag{1}$$

A. Dudek (✉)
Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA
e-mail: andrzej.dudek@wmich.edu

D. Mitsche
Laboratoire J-A Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108, Nice cedex 02, France
e-mail: dmitsche@unice.fr

P. Prałat
Department of Mathematics, Ryerson University, Toronto, ON, Canada
e-mail: pralat@ryerson.ca

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Indeed, the upper bound is trivial, since any proper colouring c is also a set colouring: for any edge uv, $N(u)$, the neighbourhood of u, contains $c(v)$ whereas $N(v)$ does not. On the other hand, suppose that there is a set colouring using at most k colours. Since there are at most 2^k possible neighbourhood colour sets, one can assign a unique colour to each set obtaining a proper colouring using at most 2^k colours. We get that $\chi(G) \leq 2^{\chi_s(G)}$, or equivalently, $\chi_s(G) \geq \log_2 \chi(G)$. With slightly more work, one can improve this lower bound by 1 (see [5]), which is tight (see [2]).

Let us recall a classic model of random graphs that we study in this paper. The *binomial random graph* $\mathcal{G}(n,p)$ is the random graph G with vertex set $[n]$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded). We also use the notations $f \ll g$ for $f = o(g)$ and $f \gg g$ for $g = o(f)$. We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough. We often write $\mathcal{G}(n,p)$ when we mean a graph drawn from the distribution $\mathcal{G}(n,p)$. For simplicity, we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$ (that is, when $f(n) = (1 + o(1))g(n)$). Finally, we use *lg* to denote logarithms with base 2 and *log* to denote natural logarithms.

Before we state the main result of this paper, we need a few definitions that we will keep using throughout the whole paper. For a given $p = p(n)$ satisfying

$$p \geq \frac{4}{\log 2} \cdot \frac{(\log n)(\log\log n)}{n} \qquad \text{and} \qquad p \leq 1 - \varepsilon$$

for some $\varepsilon > 0$, let

$$s = s(p) = \min\left\{[(1-p)^\ell]^2 + [1-(1-p)^\ell]^2 : \ell \in \mathbb{N}\right\},$$

and let ℓ_0 be a value of ℓ achieving the minimum (ℓ_0 can be assigned arbitrarily if there are at least two such values). We will show in Sect. 2 that

$$\ell_0 \in \left\{\left\lfloor \frac{\log(1/2)}{\log(1-p)} \right\rfloor, \left\lceil \frac{\log(1/2)}{\log(1-p)} \right\rceil\right\}, \tag{2}$$

and that

$$\frac{1}{2} \leq s(p) \leq \frac{1+p^2}{2}. \tag{3}$$

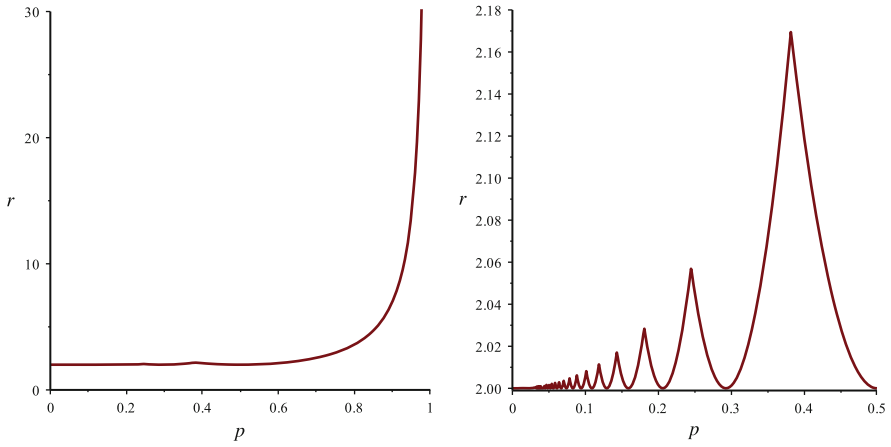

Fig. 1 The function $r = r(p)$ for $p \in (0, 1)$ and $p \in (0, 1/2]$, respectively

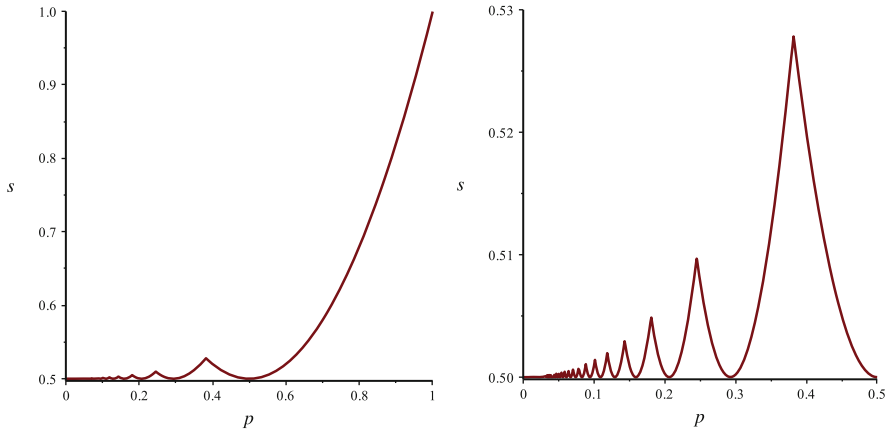

Fig. 2 The function $s = s(p)$ for $p \in (0, 1)$ and $p \in (0, 1/2]$, respectively

If p is a constant, then $r = r(p)$ is defined such that $n^2 s^{r \lg n} = 1$, that is,

$$r = r(p) = \frac{2}{\lg(1/s)}. \tag{4}$$

Observe that r tends to infinity as $p \to 1$ and undergoes a "zigzag" behaviour as a function of p (see Fig. 1). The reason for such a behaviour is, of course, that the function s is not monotone (see Fig. 2). Furthermore, observe that for each $p = 1 - (1/2)^{1/k}$, where k is a positive integer, $\ell_0 = k$, $s = 1/2$, and $r = 2$.

Now we state the main result of the paper.

Theorem 1 *Suppose that $p = p(n)$ is such that*

$$p \gg (\log n)^2(\log\log n)^2/n \qquad \text{and} \qquad p \le 1-\varepsilon,$$

for some $\varepsilon \in (0,1)$. Let $G \in \mathcal{G}(n,p)$. Then, the following holds a.a.s.:

(i) *if p is a constant, then*

$$\chi_s(G) \sim r \lg n;$$

(ii) *if $p = o(1)$ and $np = n^{\alpha+o(1)}$ for some $\alpha \in (0,1]$, then*

$$(2\alpha + o(1)) \lg n \le \chi_s(G) \le (1+\alpha+o(1)) \lg n;$$

(iii) *if $np = n^{o(1)}$, then*

$$2(\lg(np) - \lg\log n - \lg\log(np)) \le \chi_s(G) \le (1+o(1)) \lg n.$$

Note that the result is asymptotically tight for dense graphs (that is, for $np = n^{1-o(1)}$; see part (i) and part (ii) for $\alpha = 1$). For sparser graphs (part (ii) for $\alpha \in (0,1)$) the ratio between the upper and the lower bound is a constant that gets large for α small. On the other hand, the trivial lower bound of $\lg \chi(G)$ (see (1)) gives us the following: a.a.s.

$$\chi_s(\mathcal{G}(n,p)) \ge \lg \chi(\mathcal{G}(n,p)) \sim \lg\left(\frac{pn}{2\log(pn)}\right) \sim \alpha \lg n,$$

provided that $pn \to \infty$ as $n \to \infty$, and $p = o(1)$; $\chi_s(\mathcal{G}(n,p)) \ge \lg \chi(\mathcal{G}(n,p)) = \Omega(1)$ otherwise (see [3, 4]). So the lower bound we prove is by a multiplicative factor of $2 + o(1)$ larger than the trivial one, provided that $\log(np)/\log\log n \to \infty$. If $np = \log^{C+o(1)} n$ for some $C \in [2,\infty)$, then our bound is by a factor of $2(C-1)/C + o(1)$ better than the trivial one. This seemingly small improvement is important to obtain the asymptotic behaviour in the case $\alpha = 1$ and, in particular, to obtain the zig-zag for constant p.

Due to space reasons, we give only the proof of the upper bound in Sect. 2. Let us also mention that, in fact, the two bounds proved below are slightly stronger. In particular, the upper bound holds for $pn \ge (2/\log 2)(\log n)(\log\log n)$, the point where the trivial bound of $\chi(\mathcal{G}(n,p))$ becomes stronger.

2 Upper Bound

We start by proving (2) and (3). Since

$$[(1-p)^\ell]^2 + [1-(1-p)^\ell]^2 = 2\left[(1-p)^\ell - 1/2\right]^2 + 1/2,$$

it follows that $s \geq 1/2$, and consequently (2) also holds. Now, let

$$\ell = \left\lceil \frac{\log(1/2)}{\log(1-p)} \right\rceil = \frac{\log(1/2)}{\log(1-p)} + \delta,$$

where $0 \leq \delta < 1$. Observe that

$$s(p) \leq [(1-p)^\ell]^2 + [1-(1-p)^\ell]^2 = \frac{[1-(1-p)^\delta]^2 + 1}{2} \leq \frac{[1-(1-p)]^2 + 1}{2} = \frac{p^2+1}{2},$$

implying the upper bound in (3).

We keep the definition of function $r = r(p)$ for constant p introduced above (see (4)). We extend it here for sparser graphs as follows: suppose that p tends to zero as $n \to \infty$, and that $np = n^{\alpha+o(1)}$ for some $\alpha \in [0,1]$. Then, we define $r = r(p)$ such that $n^2 p s^{r \lg n} = 1$, that is, $r = r(p) \sim 1 + \alpha$, since it follows from (3) that $s \sim 1/2$.

The upper bound in Theorem 1 follows immediately from the following lemma.

Lemma 2 *Suppose that $p = p(n)$ is such that*

$$p \geq \frac{2}{\log 2} \cdot \frac{(\log n)(\log\log n)}{n} \qquad \text{and} \qquad p \leq 1-\varepsilon,$$

for some fixed $\varepsilon \in (0,1)$. Let $G \in \mathcal{G}(n,p)$. Then, a.a.s. $\chi_s(G) \leq (r+o(1)) \lg n$.

Before we move to the proof, let us note that the lower bound for p is not necessary, and the result can be extended to sparser graphs. The reason it is introduced here is that, for sparser graphs, the trivial upper bound of $\chi(G)$ is stronger; note that a.a.s.

$$\chi_s(\mathcal{G}(n,p)) \leq \chi(\mathcal{G}(n,p)) \sim \frac{pn}{2\log(pn)},$$

provided that $pn \to \infty$ as $n \to \infty$, and $p = o(1)$; $\chi_s(G) \leq \chi(G) = O(1)$ otherwise.

Proof of Lemma 2 The proof is straightforward. Let $\omega = \omega(n) = o(\log n)$ be any function tending to infinity with n (slowly enough). Before exposing the edges of the (random) graph G, we partition (arbitrarily) the vertex set into $r \lg n + \omega$ sets, each consisting of ℓ_0 *important* vertices, and one remaining set of vertices, these being not important. (For expressions such as $r \lg n + \omega$ that clearly have to be an

integer, we round up or down but do not specify which: the choice of which does not affect the argument.) Note that

$$(r \lg n + \omega)\ell_0 = O(\log n/p) = O(n/\log\log n) = o(n),$$

and so there are enough vertices to perform this operation. All vertices in a given set receive the same colour, and hence the total number of colours is equal to $(r + o(1)) \lg n$.

For a given pair of vertices, x, y, we need to estimate from above the probability $p(x, y)$ that they have the same neighbourhood colour sets. We do it by considering sets of important vertices that neither x nor y belong to. Let U be the set of (important) vertices of the same colour, and let $\ell_0 = |U|$. Then, either both x and y are not connected to any vertex from U, yielding the contribution $[(1-p)^{\ell_0}]^2$ to the probability $p(x, y)$, or both x and y are connected to at least one vertex from U, giving the contribution $[1-(1-p)^{\ell_0}]^2$. Thus,

$$p(x,y) \le \left([(1-p)^{\ell_0}]^2 + [1-(1-p)^{\ell_0}]^2\right)^{r\lg n+\omega-2} = s^{r\lg n+\omega-2}.$$

Hence, the expected number of pairs of adjacent vertices that are *not* distinguished by their neighbourhood colour sets is at most

$$\binom{n}{2} p s^{r\lg n+\omega-2} \sim n^2 p s^{r\lg n} \cdot \frac{s^{\omega-2}}{2} = \frac{s^{\omega-2}}{2},$$

where the last equality follows from the definition of r. Finally, by (3), we get that $s(p) \le (p^2+1)/2 \le ((1-\varepsilon)^2+1)/2 < 1$ and so $s^{\omega-2}/2$ tends to zero as $n \to \infty$. Hence, the lemma follows by Markov's inequality. □

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Carpooling in Social Networks

Amos Fiat, Anna R. Karlin, Elias Koutsoupias, Claire Mathieu, and Rotem Zach

Abstract We consider the online carpool fairness problem of Fagin–Williams (IBM J Res Dev 27(2):133–139, 1983), where an online algorithm is presented with a sequence of pairs drawn from a group of n potential drivers. The online algorithm must select one driver from each pair, with the objective of partitioning the driving burden as fairly as possible for all drivers. The *unfairness* of an online algorithm is a measure of the worst-case deviation between the number of times a person has driven and the number of times they would have driven if life was completely fair.

We consider the version of the problem in which drivers only carpool with their neighbors in a given social network graph; this is a generalization of the original problem, which corresponds to the social network of the complete graph. We show that, for graphs of degree d, the unfairness of deterministic algorithms against adversarial sequences is exactly $d/2$. For randomized algorithms, we show that static algorithms, a natural class of online algorithms, have unfairness $\tilde{\Theta}(\sqrt{d})$. For random sequences on stars and in bounded-genus graphs, we give a deterministic algorithm with logarithmic unfairness. Interestingly, restricting the random sequences to sparse social network graphs increases the unfairness of the natural greedy algorithm. In particular, for the line social network, this algorithm has expected unfairness $\Omega(\log^{1/3} n)$, whereas for the clique social network its expected unfairness is $O(\log \log n)$; see Ajtai–Aspnes–Naor–Rabani–Schulman–Waarts (J Algorithm 29(2):306–357, 1998).

A. Fiat (✉) • R. Zach
School of Computer Science, Tel Aviv University, Tel Aviv, Israel
e-mail: fiat@tau.ac.il; rotemz@gmail.com

A.R. Karlin
Department of Computer Science & Engineering, University of Washington, Seattle, WA, USA
e-mail: karlin@cs.washington.edu

E. Koutsoupias
Department of Computer Science, University of Oxford, Oxford, UK
e-mail: elias@cs.ox.ac.uk

C. Mathieu
Département d'Informatique, École Normale Supérieure, Paris, France
e-mail: cmathieu@di.ens.fr

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1 Introduction

In multiple experimental studies involving hundreds of graduate students, Loewenstein–Thompson–Bazerman [8] gave evidence that individuals are strongly averse to outcomes where they are at a disadvantage relative to others. Moreover, albeit significantly less so, the grad students were also averse to outcomes where they have a relative advantage in payoff. Fehr–Schmidt [6] coined the phrase *inequity aversion* to describe this phenomena. Festinger [7] had much earlier introduced the concept of cognitive dissonance, and inequity aversion is modelled as a special case thereof. Supposedly, inequity aversion may lead individuals to make significant changes, including stopping interpersonal relationships where inequities arise.

The *carpool* problem, introduced by Fagin–Williams [5] is a stylized mathematical model in which one can study questions related to minimizing inequity. As described in [5], *"suppose that n people, tired of spending their time and money in gasoline lines, decide to form a carpool. We present a scheduling algorithm for determining which person should drive on any given day. We want a scheduling algorithm that will be perceived as fair by all the members."* A priori, it seems that fairness should not be hard to achieve, but—unfortunately—precise answers as to what extent one can avoid inequity have been sought over two decades with seemingly little progress.[1]

Formally, each day t, a set of people $S_t \subset \{1, \ldots, n\}$ form a carpool. The goal is to choose who drives, so that on all days t, the overall driving burden to date has been partitioned fairly: Let $f_i(t)$ be driver i's fair share of the driving on day t, which is $1/|S_t|$ for each $i \in S_t$ and 0 otherwise. Define $F_i(t)$ to be driver i's fair share of the driving on all days up to day t, that is $F_i(t) = \sum_{\tau \le t} f_i(\tau)$, and let $D_i(t)$ be the number of times i has actually driven out of the first t days. For a particular sequence $\{S_t\}_{t=1}^{T}$, and algorithm for deciding who drives, we define

$$\text{the unfairness on day } t \;=\; \max_{\text{driver } i} |D_i(t) - F_i(t)|.$$

A carpool algorithm decides which person in S_t drives on day t; the *maximum unfairness* of the algorithm is

$$\max_{T \ge 1} \max_{\{S_t\}_{t=1}^{T}} \{\text{unfairness on day } T\}\,.$$

The offline version of the problem, when $\{S_t\}_{t=1}^{T}$ is known in advance, is easy: there is an algorithm that guarantees maximum unfairness of 1 (see, e.g., [10].)

[1] We remark that this notion of equity is not that from interactions between Tom and Jerry, both are (approximately) equally well off. The notion here is global, taking all their interactions into account. In total, Tom and Jerry should be approximately equal in payoff.

Ajtai–Aspnes–Naor–Rabani–Schulman–Waarts [1] studied the online version of problem, in which the algorithm must select a driver on day t, based only on the history up to time t. They obtained a number of extremely interesting results. First, they showed that, up to losing a factor of 2, one may assume that all the sets S_t consist of two persons. Thus, one can think of the process as a sequence of edge additions,[2] say $S_t = (i, j)$ at time t, to a multigraph on $\{1, \ldots, n\}$ (the people), with the algorithmic decision being one of choosing the orientation of the edge (towards the driver for that carpool). The goal then is to minimize[3]

$$\max_{\text{vertex } i} |\text{indegree}(i) - \text{outdegree}(i)|.$$

Ajtai et al. obtained results for two different online settings: when the requests (carpools) are selected at random, and when the request sequence is selected by an oblivious adversary that knows the algorithm, but not the outcome of any random choices the algorithm makes.

The first algorithm they considered was *Global Greedy*: on request (i, j), the driver among i and j with minimum unfairness drives; in case of a tie, the choice is arbitrary. For a uniformly random request sequence, they showed that for each t, *Global Greedy* has expected unfairness on that day of $O(\log \log n)$.

For the adversarial case, Ajtai et al. showed that every deterministic algorithm has unfairness $\lfloor n/2 \rfloor$. They also showed that this is tight: *Global Greedy* has unfairness at most $n/2$ for every request sequence. They were able to obtain a better upper bound[4] using *Randomized Local Greedy*: this algorithm considers each pair of drivers separately, and alternates which one drives each time they form a carpool. The only randomness is in the uniformly random choice of which of the two drives the very first time they carpool. They showed that *Randomized Local Greedy* has maximum unfairness equal to $\Theta(\sqrt{n \log n})$. Finally, they proved that every randomized algorithm has maximum unfairness equal to $\Omega\left((\log n)^{1/3}\right)$.

While closing this large gap between upper and lower bounds for the randomized online carpool problem remains a fascinating open problem, in this paper, we take the carpool problem in a different direction: we study it in the context of social networks.

[2] We will call these edge additions *requests*.

[3] Note that $\text{indegree}(i) - \text{outdegree}(i) = 2(D_i(t) - F_i(t))$. Dropping the factor of $1/2$ in defining the unfairness of a driver simplifies the discussion slightly.

[4] *Randomized Global Greedy*, the version of *Global Greedy* in which ties are broken at random, is conjectured to be much better, perhaps even polylog(n).

2 New Results

We study the carpool problem in the setting where the involved people belong to a social network G, and every request (carpool) is a pair of people that are connected in the social network, i.e., an edge of G. In this context, the work of [1, 5] can be seen as studying the special case where the social network is a clique.

We prove the following results for request sequences restricted to edges of a social network G with n vertices, and of maximum degree d.

2.1 Deterministic Algorithms, Adversarial Requests

We show that for every deterministic algorithm there exists a request sequence on G resulting in unfairness of at least $\lfloor d/2 \rfloor$. This is tight: we give a deterministic algorithm that never generates unfairness greater than $d/2$.

What is most interesting about this result is that, in contrast to the case where the graph is complete, the optimal deterministic algorithm is *not* the *Global Greedy* algorithm. In fact, we show that for every connected G (irrespective of its maximum degree), there is a request sequence on which *Global Greedy* has worst-case unfairness $\geq \lfloor n/2 \rfloor$. Thus, *Global Greedy* can be a factor $\Omega(n)$ worse than the optimal deterministic algorithm (e.g., when the graph has constant degree).

2.2 Random Requests

Our second set of results concerns random requests: we show that if the sequence of requests is generated by choosing edges of G uniformly at random, then the removal of edges from the graph can increase the unfairness for the *Global Greedy* algorithm: when G is a path, *Global Greedy* has expected unfairness at least $\Omega((\log n/\log\log n)^{1/3})$. This stands in contrast to the $O(\log\log n)$ upper bound of Ajtai et al. when the graph G is a clique.

For a social network G of bounded genus (e.g., planar graphs, the torus, etc.), we give a different deterministic algorithm with expected maximum unfairness $O(\log n)$.[5]

[5]The unfairness of *Global Greedy* itself is an open question when we restrict to random requests in a social network.

2.3 Randomized Algorithms

The results of Ajtai et al. show that *Randomized Local Greedy* gives maximum expected unfairness of $O(\sqrt{d \log n})$. One can view this algorithm as maintaining an invariant probability distribution over unfairness configurations: for each t, regardless of the history of requests, each edge is oriented uniformly at random. In this sense, it is a *static algorithm*. Static algorithms form a very natural class of randomized online algorithms. Intuitively, they render an adversary powerless to construct a bad request sequence: every request sequence will perform the same against such an algorithm.

One can therefore ask: what is the best randomized static algorithm? We prove that every randomized static algorithm has unfairness $\Omega(\sqrt{d})$, and therefore, *Randomized Local Greedy* is essentially optimal among static algorithms.

2.4 Other Related Work

Another problem that can model fairness issues is Tijdeman's *chairman assigment problem* [10], where a chairman has to be appointed by a community of unequal groups. An axiomatic approach to the problem and its relationship to the Shapley value of a game was given in [9]. Generalizations of the carpool problem appear in [2–4].

3 Open Questions

The outstanding open questions that follow immediately from this work are:

(i) Is there any randomized algorithm with unfairness $o(\sqrt{n})$ on the star?
(ii) Does *Randomized Global Greedy* have $o(n)$ unfairness on the star or on the line?

At this point we have no non-trivial upper bound on the star. The best algorithm we know is *Randomized Local Greedy*, which achieves $\sqrt{n}$ unfairness.

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Who to Trust for Truthful Facility Location?

Dimitris Fotakis, Christos Tzamos, and Emmanouil Zampetakis

Abstract We consider approximate mechanisms without money and with *selective verification* for k-Facility Location problems. We show how a deterministic greedy mechanism and a randomized proportional mechanism become truthful with selective verification.

1 Introduction

Suppose that we want to place a facility on the line based on the preferred locations of n strategic agents. Each agent aims to minimize the distance of her preferred location to the facility and may misreport her location, if it finds it profitable. Our objective is to minimize the maximum distance of any agent to the facility and we insist that the facility allocation should be *truthful*, i.e., no agent can improve her distance by misreporting her location. The optimal solution is to place the facility at the average of the two extreme locations. However, if we cannot incentivize truthfulness through monetary transfers (e.g., due to ethical or practical issues; see, e.g., [5] for some examples), the optimal solution is not truthful. That is, the leftmost agent can declare a location further on the left and move the facility closer to her preferred location. The fact that in this simple setting, the optimal solution is not truthful was part of the motivation for the research agenda of *approximate mechanism design without money*, introduced in Procaccia–Tennenholtz [5]. They proved that the best deterministic (resp., randomized) truthful mechanism achieves an approximation ratio of 2 (resp., $3/2$) for this problem.

However, we observe that the optimal solution can be implemented truthfully if we inspect the declared locations of the two extreme agents and verify that they coincide with their preferred locations (e.g., for their home address, we may visit them or ask them for a certificate). Inspection of the two extreme locations takes

D. Fotakis (✉)
National Technical University of Athens, 157 80, Athens, Greece
e-mail: fotakis@cs.ntua.gr

C. Tzamos • E. Zampetakis
Massachusetts Institute of Technology, 02139, Cambridge, MA, USA
e-mail: tzamos@mit.edu; mzampet@mit.edu

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place before we place the facility. If both agents are truthful, we place the facility at their average. Otherwise, we ignore any false declarations and proceed with the remaining agents. This simple modification of the optimal facility allocation is truthful, because non-extreme agents do not affect the facility allocation, while the two extreme agents cannot change the facility location, due to the verification step.

Motivated by this observation, we investigate the power of *selective verification* in approximate mechanism design without money for k-Facility Location problems, whose approximability by mechanisms without money has received considerable attention; see, e.g., [2, 4, 5] and the references therein. We show that two well-known algorithms, namely Greedy [6, Sec. 2.2] and Proportional [4], become truthful if we verify the k agents allocated the facilities. Greedy is 2-approximate for the objective of maximum cost and Proportional is $\Theta(\ln k)$-approximate for the objective of social cost. To quantify the improvement due to the use of verification, we note that [3] shows that there are no deterministic truthful mechanisms (without verification) that place $k \geq 2$ facilities in tree metrics and achieve a bounded (in terms of n and k) approximation ratio.

2 Preliminaries and the Model

We consider a metric space (M, d), where M is a finite set and $d: M \times M \to \mathbb{R}_{\geq 0}$ is a distance function, which is non-negative, symmetric, and satisfies the triangle inequality. For each $x \in M$ and $C \subseteq M$, we let $d(x, C) \equiv \min_{y \in C} d(x, y)$.

Let N be a set of n strategic agents. Each agent $i \in N$ has a preferred location $x_i \in M$. A location profile is a tuple $\boldsymbol{x} = (x_1, \ldots, x_n)$ consisting of the agent locations. A (possibly randomized) *facility allocation algorithm* $f: M^n \to \Delta(M^k)$ maps each location profile to a probability distribution over sets $C \subseteq M$ of k facilities.

The cost of any agent i is the expected distance of her location x_i to the nearest facility in $f(\boldsymbol{x})$, i.e., $\text{cost}[x_i, f(\boldsymbol{x})] = \mathbb{E}_{C \sim f(\boldsymbol{x})}[d(x_i, C)]$, which agent i seeks to minimize.

A facility allocation algorithm f satisfies *voluntary participation* (or simply, *participation*) if for any agent i and any location profile $\boldsymbol{x}$, $\text{cost}[x_i, f(\boldsymbol{x})] \leq \text{cost}[x_i, f(\boldsymbol{x}_{-i})]$, i.e., i's cost does not increase if she participates in f.

2.1 Mechanisms with Selective Verification

A *mechanism with selective verification* F takes as input a reported location profile $\boldsymbol{y}$ and has *oracle access* to a binary *verification vector* $\boldsymbol{s} \in \{0, 1\}^n$, with $s_i = 1$ if agent i has truthfully reported her preferred location, i.e., if $y_i = x_i$, and $s_i = 0$ otherwise. Actually, we assume that F verifies an agent i through a *verification oracle* ver

that, on input i, returns $\text{ver}(i) = s_i$. So, a (possibly randomized) mechanism with verification can be regarded as a function $F: M^n \times \{0,1\}^n \to \Delta(M^k)$. We denote $V(\boldsymbol{y}) \subseteq N$ the set of agents verified by F on input $\boldsymbol{y}$.

Our approach is to start from a facility allocation algorithm f and to devise a mechanism F that motivates truthful reporting by selective verification. Given an allocation f, a mechanism with verification F is an *extension* of f if, for all location profiles $\boldsymbol{x}$, $F(\boldsymbol{x}, \mathbf{1}) = f(\boldsymbol{x})$. For the converse, given a mechanism F, we say that F *induces* an allocation f if for all $\boldsymbol{x}, f(\boldsymbol{x}) = F(\boldsymbol{x}, \mathbf{1})$. A mechanism F has an algorithmic property (e.g., participation, approximation ratio) if and only if the induced algorithm f has this property.

A facility location mechanism F is *truthful*, if for any agent i, for any location pair x_i and y_i and for all reported locations $\boldsymbol{y}_{-i}$ and verification vectors $\boldsymbol{s}_{-i}$,

$$\text{cost}[x_i, F((\boldsymbol{y}_{-i}, x_i), (\boldsymbol{s}_{-i}, 1))] \leq \text{cost}[x_i, F((\boldsymbol{y}_{-i}, y_i), (\boldsymbol{s}_{-i}, 0))].$$

Namely, no matter the reported locations of the other agents and whether they report truthfully or not, the expected cost of agent i is minimized if she reports truthfully.

A remarkable property of mechanisms with selective verification is *robustness*, namely that they ignore the locations of misreporting agents and their outcome depends on the locations of truthful agents only. A mechanism F is *robust* if for all reported locations $\boldsymbol{y}$ and verification vectors $\boldsymbol{s}$, $F(\boldsymbol{y}, \boldsymbol{s}) = F(\boldsymbol{y}_{T(\boldsymbol{s})}, (1, \ldots, 1))$, where $T(\boldsymbol{s})$ is the set of truthful agents in $\boldsymbol{y}$. If F is randomized, the equality is with respect to the probability distribution of F.

We can show that robustness and participation imply truthfulness (the converse may not be true, since a truthful mechanism with verification does not need to be robust).

Lemma 1 *If a mechanism with selective verification F is robust and satisfies participation, then F is truthful.*

Focusing on truthful mechanisms, where the agents do not have an incentive to misreport, we bound the amount of verification when the agents are truthful. Given a mechanism with verification F, its *worst-case verification* is $\text{Ver}(F) \equiv \max_{\boldsymbol{x} \in M^n} |V(\boldsymbol{x})|$, i.e., the maximum number of agents verified by F in any truthful location profile.

2.2 The Greedy Mechanism for k-Facility Location

In this section, we focus on minimizing $\max_{i \in N}\{d(x_i, F(\boldsymbol{x}, \mathbf{1}))\}$, i.e, the maximum distance of any agent to the nearest facility of the mechanism (under truthful reporting). Mechanism 1 below (or G, for brevity) is a truthful and robust extension of the Greedy algorithm; see, e.g., [6, Sec. 2.2]. Greedy first allocates a facility to an arbitrary agent. As long as $|C| < k$, the next facility is allocated to the agent i maximizing $d(y_i, C)$. We extend Greedy to a truthful mechanism with selective

verification G by verifying the reported location y_i of every agent i allocated a facility. If i has reported truthfully, we place a facility at y_i and proceed with the next facility. Otherwise, we exclude y_i from the allocation and proceed with the next agent.

Mechanism 1 The Greedy Mechanism $G(\boldsymbol{y}, \boldsymbol{s})$ for k-Facility Location

pick the first agent $i \in \boldsymbol{y}$
if $\text{ver}(i) = 1$ **then** $C \leftarrow \{y_i\}$
else return $G(\boldsymbol{y}_{-i}, \boldsymbol{s}_{-i})$
while $|C| < k$ **do**
  $i \leftarrow \arg\max_{i \in \boldsymbol{y}} d(y_i, C)$
  if $\text{ver}(i) = 1$ **then** $C \leftarrow C \cup \{y_i\}$
  else $\boldsymbol{y} \leftarrow \boldsymbol{y}_{-i}$
return C

Theorem 2 *The Greedy mechanism with verification for k-Facility Location is truthful and robust, verifies k agents and is* 2*-approximate for the maximum cost.*

Proof Clearly, $\text{Ver}(G) = k$, since if all agents are truthful, G verifies only the agents allocated a facility. Moreover, the approximation ratio of Greedy for the maximum cost is 2; see, e.g., [6, Theorem 2.3].

By Lemma 1, it suffices to show that Greedy is robust and satisfies participation. As for the latter, we fix an agent i with true location x_i and a location profile $\boldsymbol{y}_{-i}$. Then, if x_i gets a facility in $(\boldsymbol{y}_{-i}, x_i)$, i's cost is 0. Otherwise, excluding x_i does not affect the execution of Greedy. Hence, i's cost is the same in $(\boldsymbol{y}_{-i}, x_i)$ and in $\boldsymbol{y}_{-i}$.

Greedy is robust because any misreporting agent i does not get a facility, either because she is not picked by the mechanism or because she is caught lying by ver. Thus, if $x_i \neq y_i$, i's presence in $G(\boldsymbol{y}, (\boldsymbol{s}_{-i}, 0))$ does not affect the outcome and $G(\boldsymbol{y}, (\boldsymbol{s}_{-i}, 0)) = G(\boldsymbol{y}_{-i}, \boldsymbol{s}_{-i})$. Applying this inductively, we can eliminate all misreporting agents and conclude that $G(\boldsymbol{y}, \boldsymbol{s}) = G(\boldsymbol{y}_{T(\boldsymbol{s})}, (1, \ldots, 1))$. Hence Greedy is robust. □

2.3 The Proportional Mechanism for k-Facility Location

We next focus on minimizing $\sum_{i=1}^{n} d(x_i, F(\boldsymbol{x}, \boldsymbol{1}))$, i.e, the total distance of the agents to the nearest facility of the mechanism (under truthful reporting). Mechanism 2 is a truthful and robust extension of the Proportional allocation [4]. Proportional is essentially a randomized version of Greedy, where the next facility is allocated to agent i with probability proportional to $d(y_i, C)$. As in Sect. 2.2, we extend Proportional to a truthful mechanism with selective verification by verifying the reported location y_i of every agent i allocated a facility.

We can show that:

Mechanism 2 The Proportional Mechanism $P(\boldsymbol{y}, \boldsymbol{s})$ for k-Facility Location

pick an agent $i \in \boldsymbol{y}$ uniformly at random
if $\text{ver}(i) = 1$ **then** $C \leftarrow \{y_i\}$
else return $P(\boldsymbol{y}_{-i}, \boldsymbol{s}_{-i})$
while $|C| < k$ **do**
  pick an agent $i \in \boldsymbol{y}$ with probability $d(y_i, C) / \sum_{j \in \boldsymbol{y}} d(y_j, C)$
  if $\text{ver}(i) = 1$ **then** $C \leftarrow C \cup \{y_i\}$
  else $\boldsymbol{y} \leftarrow \boldsymbol{y}_{-i}$
return C

Theorem 3 *The Proportional mechanism with verification for k-Facility Location is truthful and robust, verifies k agents and is* $\Theta(\ln k)$*-approximate for the social cost.*

The approximation ratio of Proportional is shown in [1, Theorem 5.1]. Moreover, $\text{Ver}(P) = k$, since if all agents report truthfully, the mechanism verifies only the agents in C. The proof that Mechanism 2 is robust and satisfies participation is a generalization of the proof of Theorem 2.

Both Greedy and Proportional have good approximation guarantees and motivate truthfulness by verifying only k agents. Interestingly, the proof that the mechanisms are truthful and robust only assumes that the cost of each agent i is minimized if i gets a facility at her true preferred location. Hence, the mechanisms remain robust and truthful for even if the agents' cost is any function of $d(t_i, C)$ (as long as it is minimized at 0).

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Metric and Spectral Properties of Dense Inhomogeneous Random Graphs

Nicolas Fraiman and Dieter Mitsche

Abstract We study metric and spectral properties of dense inhomogeneous random graphs. We generalize results known for the Erdös–Renyi model. In our case an edge (i, j) is present with probability $\kappa(X_i, X_j)p$, where $\kappa \geq 0$ is a fixed kernel and X_i are independent variables from a general distribution on a separable metric space.

1 Introduction

In this work we study metric and spectral properties of inhomogeneous random graphs, where edges are present independently but with unequal edge occupation probabilities. We obtain results about the diameter, typical distances and eigenvalues for the dense case in this random graph model, under weak assumptions.

A discrete version of this model was introduced in Söderberg [15]. The sparse case (when the number of edges is linear in the number n of vertices) was studied in detail in the paper Bollobás–Janson–Riordan [3]. Among other things, they give an asymptotic formula for the diameter of the giant component when it exists. Connectivity at the intermediate case was analyzed in Devroye–Fraiman [7]. The dense case (when the number of edges is quadratic in n) is closely related with the theory of graph limits started in Lovász–Szegedy [11] and further studied in depth by Borgs, Chayes, Lovász, Sós and Vesztergombi [4, 5] among others. For a thorough introduction to the subject of graph limits see the book Lovász [10].

The diameter of random graphs has been studied widely. In particular, for the Erdős–Rényi model $G(n, p)$, Bollobás [2] generalized the results of Klee–Larman [9] characterizing the case of constant diameter. Later, Chung–Lu [6] proved concentration results in various different ranges. More recently, Riordan–

N. Fraiman (✉)
Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, 19104, Philadelphia, PA, USA
e-mail: nfraiman@math.upenn.edu

D. Mitsche
Laboratoire J-A Dieudonné, Université Nice Sophia-Antipolis, Parc Valrose, 06108, Nice Cedex 02, France
e-mail: dmitsche@unice.fr

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Wormald [14] completed the program to give precise asymptotics for the sparse case.

The critical window for $G(n,p)$ is much harder to analyze. Nachmias–Peres [13] obtained the order of the diameter. Addario-Berry–Broutin–Goldschmidt [1] proved Gromov–Hausdorff convergence of the rescaled connected components to a sequence of continuous compact metric spaces. In particular, the rescaled diameter converges in distribution to an absolutely continuous random variable with finite mean.

The spectral gap of $G(n,p)$ was studied in Füredi–Komlós [8], and the convergence of the empirical spectral distribution to Wigner's semicircle law [16] follows from the Stieltjes' transform approach developed in Marchenko–Pastur [12].

2 Framework

Consider a separable metric space $\mathcal{S}$ and a Borel probability measure μ on $\mathcal{S}$. Let $X_1,\dots,X_n$ be μ-distributed independent random variables on $\mathcal{S}$. Let $\kappa\colon \mathcal{S}\times\mathcal{S}\to\mathbb{R}$ be a non-negative symmetric integrable kernel, $\kappa\geq 0$ and $\kappa\in L^1(\mathcal{S}\times\mathcal{S},\mu\otimes\mu)$.

Definition 1 The *inhomogeneous random graph* with kernel κ and window p is the random graph $G(n,\kappa,p)=(V,E)$, where the vertex set is $V=\{1,\dots,n\}$ and we connect each pair of vertices $i,j\in V$ independently with probability $p_{ij}=\min\{1,\kappa(X_i,X_j)p\}$.

The asymptotic expansions for distances in the graph $G(n,\kappa,p)$ are obtained by looking at upper and lower partition graphs, which are discrete approximations of κ. The formal definitions are given below. Given two subsets $\mathcal{A},\mathcal{B}\subset\mathcal{S}$ let

$$\kappa_\ell(\mathcal{A},\mathcal{B})=\operatorname{ess\,inf}\{\kappa(x,y):x\in\mathcal{A},y\in\mathcal{B}\},$$
$$\kappa_u(\mathcal{A},\mathcal{B})=\operatorname{ess\,sup}\{\kappa(x,y):x\in\mathcal{A},y\in\mathcal{B}\}.$$

We say that $\mathbb{A}=\{\mathcal{A}_1,\dots,\mathcal{A}_m\}$ is an *admissible partition* of $\mathcal{S}$ if

$$\mu(\mathcal{S}\setminus\cup_{i=1}^m\mathcal{A}_i)=0\quad\text{and}\quad\|\mathbb{A}\|_\mu:=\min\{\mu(\mathcal{A}):\mathcal{A}\in\mathbb{A}\}>0,$$

i.e., it covers $\mathcal{S}$ and contains no zero measure sets.

Definition 2 Given a kernel κ and an admissible partition $\mathbb{A}=\{\mathcal{A}_1,\dots,\mathcal{A}_m\}$ of $\mathcal{S}$, the *lower partition graph* $P_\ell(\mathbb{A})$ (resp., *upper partition graph* $P_u(\mathbb{A})$) induced by $\mathbb{A}$ is the graph with vertex set $\mathbb{A}$ and where $(\mathcal{A}_i,\mathcal{A}_j)$ is an edge if $\kappa_\ell(\mathcal{A}_i,\mathcal{A}_j)>0$ (resp., $\kappa_u(\mathcal{A}_i,\mathcal{A}_j)>0$).

For two vertices $u,v\in V$ belonging to the same connected component, denote by $d(u,v)$ the graph distance between u and v, that is, the number of edges on a shortest path between them. For a connected graph G, let $\operatorname{diam}G=\max_{u,v}d(u,v)$.

We say that a partition $\mathbb{A}$ is a *refinement* of $\mathbb{B}$, denoted by $\mathbb{A} \prec \mathbb{B}$, if for every $\mathcal{A} \in \mathbb{A}$ there exists $\mathcal{B} \in \mathbb{B}$ such that $\mathcal{A} \subset \mathcal{B}$. Note that in this case, $\mathcal{B}_i = \cup_{p=1}^{m_i} \mathcal{A}_p^{(i)}$ μ-almost everywhere. Moreover, it holds that $\operatorname{diam} P_u(\mathbb{A}) \geq \operatorname{diam} P_u(\mathbb{B})$ and $\operatorname{diam} P_\ell(\mathbb{A}) \leq \operatorname{diam} P_\ell(\mathbb{B})$. By these properties it makes sense to consider the following constants.

Definition 3 The *lower and upper partition diameters* are given by

$$\Delta_\ell := \inf_{\mathbb{A}} \operatorname{diam} P_\ell(\mathbb{A}) \qquad \text{and} \qquad \Delta_u := \sup_{\mathbb{A}} \operatorname{diam} P_u(\mathbb{A}).$$

Finally, we want to avoid trivial cases where Δ_ℓ or Δ_u are infinite because of an structural obstruction for connectivity given by κ.

Definition 4 A kernel κ on $(\mathcal{S}, \mu)$ is *reducible* if there exists a set $\mathcal{A} \subset \mathcal{S}$ with $0 < \mu(\mathcal{A}) < 1$ such that $\kappa = 0$ almost everywhere on $\mathcal{A} \times (\mathcal{S} \setminus \mathcal{A})$. Otherwise, κ is *irreducible*.

If κ is reducible then the whole graph $G(n, \kappa, p)$ is disconnected since almost surely there are no edges between vertices of type in $\mathcal{A}$ and $\mathcal{S} \setminus \mathcal{A}$. Since we want to work with connected graphs, we restrict our attention to the irreducible case.

3 Our Results

The results we prove are a generalization of previously known results for the Erdős–Renyi model obtained for the constant kernel $\kappa = 1$. We say that a sequence of events holds *with high probability*, if it holds with probability tending to 1 as $n \to \infty$. We define the *expansion factor* $\Phi := \log n / \log np$. This quantity is about the diameter of $G(n, p)$, as first shown in Bollobás [2]. Our main result about the diameter is the following.

Theorem 5 *Let κ be irreducible and continuous $(\mu \otimes \mu)$-almost everywhere. Then,*

(i) *if $\Phi < \Delta_u$ then* $\operatorname{diam} G(n, \kappa, p) = \Delta_u$ *with high probability.*
(ii) *if $\Phi > \Delta_\ell$ then* $\operatorname{diam} G(n, \kappa, p) = \Phi$ *with high probability.*

The proof relies on concentration inequalities which are inductively used to show that for all i, the i-th neighborhoods of vertices have good expansion properties. We also show that the upper and lower partition diameters cannot be very different.

Lemma 6 *If $\Delta_\ell < \infty$ then $\Delta_u \leq \Delta_\ell \leq \Delta_u + 2$.*

Let λ_i be the i-th eigenvalue (with multiplicity) of the adjacency matrix of $G(n,\kappa,p)$. The *empirical spectral distribution* is

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sigma\sqrt{np}},$$

where $\sigma = \int_{\mathcal{S}}\int_{\mathcal{S}} \kappa(x,y)(1-\kappa(x,y))d\mu(x)d\mu(y)$. Interlacing and the method of moments yield convergence to the semicircle law.

Theorem 7 *Let κ be bounded and continuous $(\mu\otimes\mu)$-almost everywhere. Then,*

$$\nu_n \xrightarrow{w} \mathbf{1}_{|x|\le 2}\frac{\sqrt{4-x^2}}{2\pi}\,dx.$$

Moreover, the eigenvalues normalized by np converge to the spectrum of T_κ, the $L^2(\mathcal{S},\mu)$ integral operator given by κ defined as

$$T_\kappa\psi(x) = \int_{\mathcal{S}} \kappa(x,y)\psi(y)\,d\mu(y).$$

Theorem 8 *Let κ be bounded and continuous $(\mu\otimes\mu)$-almost everywhere. Then, for all eigenvalues of $G(n,\kappa,p)$,*

$$\frac{\lambda_i}{np} \longrightarrow \lambda_i(T_\kappa).$$

Unlike in the empirical spectral distribution under this scaling, most of the normalized spectrum tends to the zero eigenvalue. The proof is based on the trace method and ideas for graph limits from Borgs–Chayes–Lovász–Sós–Vesztergombi [5]. A key ingredient is the following lemma. Let C_k be a cycle on k vertices. The *homomorphism densities* are

$$t(C_k,G) = \frac{|\operatorname{Hom}(C_k,G)|}{(np)^k} \quad \text{and} \quad t(C_k,\kappa) = \int\cdots\int_{\mathcal{S}}\prod_{i=1}^{k}\kappa(x_i,x_{i+1})\,d\mu(x_1)\cdots d\mu(x_k).$$

Lemma 9 *Let κ be bounded, $\varepsilon>0$ and $G_n = G(n,\kappa,p)$. Then,*

$$\mathbf{P}\left(|t(C_k,G_n)-t(C_k,\kappa)|>\varepsilon\right) \le 2\exp\left(-np\frac{\varepsilon^2}{8k^2}\right).$$

4 Ongoing Work

The special case of rank 1 kernels, where $\kappa(x, y) = \psi(x)\psi(y)$, is particularly interesting. In this case we have $\Lambda(\kappa) = \left\{\|\psi\|_2^2, 0\right\}$ because $T_\kappa\psi(x) = \|\psi\|_2^2\psi(x)$, and $T_\kappa f(x) = 0$, for all $f \perp \psi$.

Conjecture 10 Let κ be a rank 1, continuous $(\mu \otimes \mu)$-almost everywhere kernel. Then $\lambda_2 = O(\sqrt{np})$.

Theorem 8 implies that $\lambda_2 = o(np)$. We are currently working on extending the method from Füredi–Komlós [8] to bound the Rayleigh quotient characterizing the second eigenvalue.

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On-Line List Colouring of Random Graphs

Alan Frieze, Dieter Mitsche, Xavier Pérez-Giménez, and Paweł Prałat

Abstract The on-line list colouring of binomial random graphs $\mathscr{G}(n,p)$ is studied. We show that the on-line choice number of $\mathscr{G}(n,p)$ is asymptotically almost surely asymptotic to the chromatic number of $\mathscr{G}(n,p)$, provided that the average degree $d = p(n-1)$ tends to infinity faster than $(\log\log n)^{1/3}(\log n)^2 n^{2/3}$. For sparser graphs, we are slightly less successful; we show that if $d \geq (\log n)^{2+\varepsilon}$ for some $\varepsilon > 0$, then the on-line choice number is larger than the chromatic number by at most a multiplicative factor of C, where $C \in [2,4]$, depending on the range of d. Also, for $d = O(1)$, the on-line choice number is, by at most, a multiplicative constant factor larger than the chromatic number.

1 Introduction

The combinatorial game we study here is played by two players, named Mr. Paint and Mrs. Correct, and is played on a finite, undirected graph in which each vertex has assigned a non-negative number representing the number of erasers at the particular vertex. We assume for simplicity that this number is initially the same for each vertex. In each round, first Mr. Paint selects a subset of the vertices and paints them all the same colour; he cannot use this colour in subsequent rounds. Mrs. Correct then has to erase the colour from some of the vertices in order to prevent adjacent vertices having the same colour. Whenever the colour at a vertex is erased, the number of erasers at that vertex decreases by 1, but naturally, Mrs. Correct cannot

A. Frieze (✉)
Department of Mathematical Sciences, Carnegie Mellon University, 5000 Forbes Av., 15213, Pittsburgh, PA, USA
e-mail: alan@random.math.cmu.edu

D. Mitsche
Laboratoire J-A Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108, Nice cedex 02, France
e-mail: dmitsche@unice.fr

X. Pérez-Giménez • P. Prałat
Department of Mathematics, Ryerson University, Toronto, ON, Canada
e-mail: xperez@ryerson.ca; pralat@ryerson.ca

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erase the colour if she has no erasers left at that vertex. Vertices whose colours have not been erased can be considered as being permanently coloured and can be removed from the game. The game has two possible endings: (i) all vertices have been permanently coloured, in which case Mrs. Correct wins, or (ii) at some point of the game, Mr. Paint presents two adjacent vertices u and v and neither u nor v has any eraser left, in which case Mr. Paint wins. If, regardless of which sets she gets presented, there is a strategy for Mrs. Correct to win the game having initially $k-1$ erasers at each vertex, we say that the graph is *k-paintable*. The smallest k for which the graph is k-paintable is called the *paintability* number of G, and denoted by $\chi_P(G)$. Note that this parameter is indeed well defined: for any graph on n vertices, $n-1$ erasers at each vertex always guarantee Mrs. Correct to win, as she can always choose one vertex from a set presented to her and erase colours on the remaining ones. This problem is also known as the *on-line list colouring* and the corresponding graph parameter is also called the *on-line choice number of* G; see, for example, [12] and below for the relation to the (off-line) list colouring.

Let us recall a classic model of random graphs that we study in this paper. The *binomial random graph* $\mathcal{G}(n,p)$ is defined as the probability space $(\Omega, \mathcal{F}, \Pr)$, where Ω is the set of all graphs with vertex set $\{1,2,\ldots,n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$,

$$\Pr(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

This space may be viewed as the set of outcomes of $\binom{n}{2}$ independent coin flips, one for each pair (u,v) of vertices, where the probability of success (that is, adding edge uv) is p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

All asymptotics throughout are as $n \to \infty$ (we emphasize that the notations $o(\cdot)$ and $O(\cdot)$ refer to functions of n, not necessarily positive, whose growth is bounded; whereas $\Theta(\cdot)$ and $\Omega(\cdot)$ always refer to positive functions). We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. We often write $\mathcal{G}(n,p)$ when we mean a graph drawn from the distribution $\mathcal{G}(n,p)$. Finally, for simplicity, we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$ (that is, when $f(n) = (1+o(1))g(n)$).

Now, we will briefly mention the relation to other known graph parameters. A *proper colouring* of a graph is a labelling of its vertices with colours such that no two adjacent vertices have the same colour. A colouring using at most k colours is called a (proper) *k-colouring*. The smallest number of colours needed to colour a graph G is called its *chromatic number*, and it is denoted by $\chi(G)$. Let L_k be an arbitrary function that assigns to each vertex of G a list of k colours. We say that G is *L_k-list-colourable* if there exists a proper colouring of the vertices such that every vertex is coloured with a colour from its own list. A graph is *k-choosable*, if for every such function L_k, G is L_k-list-colourable. The minimum k for which a graph is k-choosable is called the *list chromatic number*, or the *choice number*, and denoted by $\chi_L(G)$. Since the choices for L_k contain the special case where each vertex is

assigned the list of colours $\{1, 2, \ldots, k\}$, it is clear that a k-choosable graph has also a k-colouring and so, $\chi(G) \leq \chi_L(G)$. It is also known that if a graph is k-paintable, then it is also k-choosable [10], that is, $\chi_L(G) \leq \chi_P(G)$. Indeed, if there exists a function L_k so that G is not L_k-list-colourable, then Mr. Paint can easily win by fixing some permutation of all colours present in L_k and presenting at the i-th step all vertices containing the i-th colour of the permutation on their lists (unless the vertex was already removed before). Finally, it was shown in [12] that the paintability of a graph G on n vertices is at most $\chi(G) \log n + 1$. (All logarithms in this paper are natural logarithms.) Combining all inequalities, we get the following:

$$\chi(G) \leq \chi_L(G) \leq \chi_P(G) \leq \chi(G) \log n + 1. \tag{1}$$

It follows from the well-known results of Bollobás [4], Łuczak [8] (see also McDiarmid [9]) that the chromatic number of $\mathscr{G}(n, p)$ a.a.s. satisfies

$$\chi(\mathscr{G}(n, p)) \sim \frac{\log(1/(1-p))n}{2\log(np)}, \tag{2}$$

for $np \to \infty$ and p bounded away from 1. The study of the choice number of $\mathscr{G}(n, p)$ was initiated in [1], where Alon proved that a.a.s., the choice number of $\mathscr{G}(n, 1/2)$ is $o(n)$. Kahn then showed (see [2]) that a.a.s. the choice number of $\mathscr{G}(n, 1/2)$ equals $(1 + o(1))\chi_{\mathscr{G}(n,1/2)}$. In [5], Krivelevich showed that this holds for $p \gg n^{-1/4}$, and Krivelevich–Sudakov–Vu–Wormald [6] improved this to $p \gg n^{-1/3}$. On the other hand, Alon–Krivelevich–Sudakov [3] and Vu [11] showed that, for any value of p satisfying $2 < np \leq n/2$, the choice number is $\Theta(np/\log(np))$. Later, Krivelevich–Vu [7] generalized this to hypergraphs; they also improved the leading constants and showed that the choice number for $C \leq np \leq 0.9n$ (where C is a sufficiently large constant) is at most a multiplicative factor of $2 + o(1)$ away from the chromatic number, the best known factor for $p \leq n^{-1/3}$. Our results below (see Theorems 1, 2 and 3) show that, even for the on-line case, for a wide range of p, we can asymptotically match the best known constants of the off-line case. Moreover, if $np \geq \log^{\omega} n$ (for some function $\omega = \omega(n)$ tending to infinity as $n \to \infty$), then we get the same multiplicative factor of $2 + o(1)$.

Our main results are the following theorems. The first one deals with dense random graphs.

Theorem 1 *Let $\varepsilon > 0$ be any constant, and suppose that*

$$(\log\log n)^{1/3}(\log n)^2 n^{-1/3} \ll p \leq 1 - \varepsilon.$$

Let $G \in \mathcal{G}(n, p)$. Then, a.a.s.,

$$\chi_P(G) \sim \frac{n}{2\log_b(np)} \sim \chi(G),$$

where $b = 1/(1-p)$.

Note that if $p = o(1)$, then

$$\frac{n}{2\log_b(np)} = \frac{n\log(1/(1-p))}{2\log(np)} \sim \frac{np}{2\log(np)} = \Theta\left(\frac{np}{\log(np)}\right).$$

For constant p it is not true that $\log(1/(1-p)) \sim p$ but the order is preserved, provided $p \le 1-\varepsilon$ for some $\varepsilon > 0$.

For sparser graphs we are less successful in determining the asymptotic behaviour of $\chi_P(\mathcal{G}(n,p))$. Nevertheless, we can prove the following two theorems determining the order of the graph parameter we study.

Theorem 2 *Let $\varepsilon > 0$ be any constant, and suppose that*

$$\frac{(\log n)^{2+\varepsilon}}{n} \le p = O((\log\log n)^{1/3}(\log n)^2 n^{-1/3}).$$

Let $G \in \mathcal{G}(n,p)$. Then, a.a.s.,

$$\chi_P(G) = \Theta\left(\frac{np}{\log(np)}\right) = \Theta(\chi(G)).$$

Moreover, if $np = (\log n)^{C+o(1)}$, a.a.s.

$$\chi(G) \le \chi_P(G) \le (1+o(1)) \begin{cases} 2\chi(G) & \text{if } C \to \infty \\ \frac{2C}{C-2}\chi(G) & \text{if } C \in [4,\infty) \\ 4\chi(G) & \text{if } C \in (2,4). \end{cases}$$

Finally, for very sparse graphs we have the following.

Theorem 3 *Let $G \in \mathcal{G}(n,p)$ with $p = O(1/n)$. Then, a.a.s., $\chi_P(G) = \Theta(1) = \Theta(\chi(G))$.*

Due to space reasons, we give here only the proof of Theorem 3.

2 Proof of Theorem 3

Before we move to the proof of Theorem 3, let us state the following simple observation.

Lemma 4 *Let G be a graph whose components are all trees or unicyclic graphs. Then Mrs. Correct has a winning strategy using 1 eraser at each vertex for tree components and using 2 erasers at each vertex for unicyclic components.*

Proof Since we may consider different components separately, we may assume that G is connected. First assume that G is a tree. For a contradiction, suppose that 1 eraser at each vertex is not enough for Mrs.Correct to win on some tree, and consider a smallest such tree T. Clearly, $|T| \geq 2$, and let us consider a leaf ℓ of T. By minimality of T, Mrs.Correct has a winning strategy on $T \setminus \ell$. Then, she extends the strategy to T as follows. The first time she is presented vertex ℓ, she considers the optimal strategy when playing on the restriction of the set to $T \setminus \ell$. If the set yielded by that strategy does not contain the neighbour of ℓ in T, she follows this strategy and simply adds ℓ to the set, and plays for the rest of the game on $T \setminus \ell$. On the other hand, if the set does contain the neighbour of ℓ, then she also follows the strategy on $T \setminus \ell$, but uses the eraser for ℓ. Since the only neighbour of ℓ cannot appear later on, she can continue with her optimal strategy on $T \setminus \ell$ and simply adds ℓ the second time she is presented this vertex. We get a contradiction, and the proof of the first part is finished.

Similarly, suppose now that G is unicyclic. As before, for a contradiction, suppose that 2 erasers at each vertex are not enough for G, and consider a smallest unicyclic graph U with this property. Clearly, $|U| \geq 3$. If U contains a leaf ℓ, then, as before, she plays optimally using two erasers on $U \setminus \ell$ and adapts her strategy exactly as before. If U does not contain a leaf, then U is a cycle. The first time she is presented a set, she picks one vertex and uses erasers for all other vertices. The rest of the game is played on a tree (in fact, a path), and she has still 1 eraser at each vertex at her disposal. By the first part of the lemma, she has a winning strategy and the proof is finished. □

We come now back to the proof of Theorem 3. It is well known that if $p < 0.99/n$, then a.a.s. G contains only trees and unicyclic components, and Mrs.Correct can win using at most 2 erasers by Lemma 4. Assume then that $p \leq c/n$ for some (perhaps large) constant $c \geq 0.99$ (assume w.l.o.g. that $c \in \mathbb{N}$). If G contains a component with at least two cycles, the component must contain a subgraph which either consists of two cycles connected by a path (or sharing a vertex), or is a cycle with a "diagonal" path. Let us call such structures *complex*. Clearly, on k vertices one can construct at most $k^2k!$ complex structures. Note that the degree of any vertex given by the existence of the complex structure is at most 4. We will show that a.a.s. there is no complex structure with the property that each vertex of this structure has a degree in the whole graph of at least $100c^2$. Note that, given a complex structure, in order for each vertex of this structure to have degree in the whole graph at least $100c^2$, it must have either at least $47c^2$ incident edges towards vertices outside the subgraph on which the complex structure is built, or it has at least $47c^2$ additional incident edges inside the subgraph on which the complex structure is built. Therefore, either half of the vertices of the complex structure have at least $47c^2$ incident edges outside, or there are at least $11c^2k$ additional edges inside the complex structure, and no information about these edges has been revealed so far.

Thus, the expected number of complex structures in which each vertex has degree at least $100c^2$ in the whole graph is at most

$$\sum_{k=4}^{n}\binom{n}{k}k^2k!p^{k+1}\left(\binom{k}{\lfloor k/2\rfloor}\left(\binom{n}{47c^2}p^{47c^2}\right)^{\lfloor k/2\rfloor}+\binom{\binom{k}{2}}{11c^2k}p^{11c^2k}\right)$$

$$\leq\sum_{k=4}^{n}\frac{k^2}{n}c^{k+1}2\left(\frac{e}{22c}\right)^{11c^2k}=O\left(\sum_{k=4}^{n}\frac{k^2}{n}c^{k+1}c^{-2k}\right)$$

$$=O\left(\sum_{k=4}^{n}\frac{k^2}{n}c^{-k}\right)=O\left(\int_{x=0}^{\infty}\frac{x^2}{n}c^{-x}dx\right)=O(1/n),\qquad(3)$$

and therefore, by Markov's inequality, the subgraph induced by vertices of degree at least $100c^2$ a.a.s. consists of components that are either trees or unicyclic components. Since we aim for a statement that holds a.a.s., we may assume that this is the case.

We may assume that each time Mrs. Correct selects a maximal independent set. Hence, the number of erasers needed to be placed at vertex v is at most $\deg(v)$ (if v uses one of its erasers at some point of the game, it must be the case that at least one of its neighbours belongs to the maximal independent set). As a result, vertices of degree at most $100c^2$ require only a constant number of erasers. Call the set of such vertices L, and let $H = V(G) \setminus L$. By (3), the graph induced by the vertices in H a.a.s. consists of components that are either trees or unicyclic components. Mrs. Correct plays as follows: whenever she is presented a set U, she plays optimally on the restriction of U to H (on which, by Lemma 4, she uses at most 2 erasers), and then she extends an independent set found there to any maximal independent set in U. This strategy uses in total at most $100c^2 = O(1)$ erasers for each vertex in L. In this way, clearly, $O(1) = \Theta(\chi(G))$ colours are used.

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Approximation Algorithms for Computing Maximin Share Allocations

Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi

Abstract We study the problem of computing maximin share guarantees, a recently introduced fairness notion. Given a set of n agents and a set of goods, the maximin share of a single agent is the best that she can guarantee to herself, if she would be allowed to partition the goods in any way she prefers, into n bundles, and then receive her least desirable bundle. The objective then in our problem is to find a partition, so that each agent is guaranteed her maximin share. Our main result is a $2/3$-approximation, that runs in polynomial time for any number of agents, improving upon recent results in the literature.

1 Introduction

We study a fair division problem in the context of allocating indivisible goods. Fair division has attracted the attention of various scientific disciplines, including among others, mathematics, economics, and political science. Ever since the first attempt for a formal treatment, Steinhaus–Banach–Knaster [7], many interesting and challenging questions have emerged. Over the past decades, a vast literature has developed, see e.g., [2, 6], and several notions of fairness have been suggested. The area gradually gained popularity in computer science as well, as most of the questions are inherently algorithmic.

The objective in fair division problems is to allocate a set of resources to a set of n agents in a way that leaves every agent satisfied. In the continuous case, the available resources are typically represented by the interval $[0, 1]$, whereas in the discrete case, we have a set of distinct, indivisible goods. The preferences of each agent are represented by a valuation function, which is usually an additive function on the

G. Amanatidis (✉) • E. Markakis
Department of Informatics, Athens University of Economics and Business, Athens, Greece
e-mail: gamana@aueb.gr; markakis@aueb.gr

A. Nikzad • A. Saberi
Department of Management Science and Engineering, Stanford University, Stanford, CA, USA
e-mail: nikzad@stanford.edu; saberi@stanford.edu

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set of goods (a probability distribution in the continuous case). Given such a setup, many solution concepts have been proposed as to what constitutes a fair solution. Some of the standard ones include *proportionality, envy-freeness, equitability* and several variants of them.

All the above solutions can be attained in the case of divisible goods. However, in the presence of indivisible goods we cannot have any such guarantees; in fact in most cases we cannot even guarantee reasonable approximations. Instead, we focus on a concept recently introduced by Budish [3], that can be seen as a relaxation of proportionality. The rationale is as follows: suppose that an agent, say agent i, is asked to partition the goods into n bundles and then the rest of the agents make a choice before i. In the worst case, agent i will be left with her least valuable bundle. Hence, a risk-averse agent would choose a partition that maximizes the minimum value of a bundle in the partition. This value is called the maximin share of agent i. The objective then is to find an allocation where every person receives at least her maximin share. Even for this notion, existence is not guaranteed under indivisible goods. But, it is possible to have constant factor approximations, as has been recently shown in Procaccia–Wang [5].

Our main result is a $(2/3 - \varepsilon)$-approximation algorithm, for any $\varepsilon > 0$, running in polynomial time for any number of agents. That is, the algorithm produces an allocation where every agent receives a bundle worth at least $2/3-\varepsilon$ of her maximin share. Our result improves upon the $2/3$-approximation of Procaccia–Wang [5], which runs in polynomial time only for a constant number of agents. To achieve this, we redesign certain parts of the algorithm in [5], arguing about the existence of appropriate, carefully constructed matchings in a bipartite graph representation of the problem. Moreover, motivated by the apparent difficulty in finding impossibility results on the approximability of the problem, we undertake a probabilistic analysis. Our analysis shows that in randomly generated instances, with high probability there always exists a maximin share allocation. This result may be seen as a justification of the experimental evidence reported in [1, 5], which show that maximin share allocations exist in most cases.

Finally, we study two special cases, motivated by previous works. The first one is the case of $n = 3$ agents. This is an interesting turning point on the approximability of the problem; for $n = 2$, there always exist maximin share allocations, but adding a third agent makes the problem significantly more complex, and the best known ratio is $3/4$ by [5]. We provide an algorithm with an approximation guarantee of $6/7$, by examining more deeply the set of allowed matchings that we can use to satisfy the agents. The second case is the setting where all item values belong to $\{0, 1, 2\}$. This is an extension of the $\{0, 1\}$ setting studied in Bouveret–Lemaître [1], and we show that there always exists a maximin share allocation, for any number of agents.

The notion we study here was introduced in Budish [3] for ordinal utilities (where agents have rankings over alternatives), building on concepts by Moulin [4]. The work Bouveret–Lemaître [1] defined the notion for cardinal utilities, in the form that we study it here, and also provided many important insights as well as experimental

evidence. Finally, in [5], a $2/3$ approximation is provided along with constructions showing instances where no maximin share allocation exists even for $n = 3$. The extensive experimentation by Bouveret and Lemaître in [1] reveals that it has been challenging to produce lower bounds, i.e., instances where no α-approximation of a maximin share allocation exists, even for α very close to 1.

2 Definitions and Notation

Let $N = \{1, \ldots, n\}$ be a set of n agents and $M = \{1, \ldots, m\}$ be a set of indivisible goods. For any $k \in \mathbb{N}$, we will also be using $[k]$ to denote the set $\{1, \ldots, k\}$. We assume each agent has an additive valuation function $v_i(\cdot)$, so that for every $S \subseteq M$, $v_i(S) = \sum_{j \in S} v_i(\{j\})$. For $j \in M$, we will use v_{ij} for $v_i(\{j\})$.

Given any subset $S \subseteq M$, an allocation of S to the n agents is a partition $T = (T_1, \ldots, T_n)$, where $T_i \cap T_j = \emptyset$ and $\bigcup T_i = S$. Let $\Pi_n(S)$ be the set of all partitions of a set S into n bundles.

The notions below were originally defined in Budish [3] and later on in Bouveret–Lemaître [1], in the same setting that we study here.

Definition 1 Given a set of n agents, and any set $S \subseteq M$, the n-maximin share of an agent i with respect to S is

$$\boldsymbol{\mu}_i(n, S) = \max_{T \in \Pi_n(S)} \min_{T_j \in T} v_i(T_j).$$

When $S = M$, we refer to $\boldsymbol{\mu}_i(n, M)$ as the maximin share of agent i. The solution concept defined in Budish [3] asks for a partition giving each agent her maximin share.

Definition 2 Given a set of agents N, and a set of goods M, a partition $T = (T_1, \ldots, T_n) \in \Pi_n(M)$ is called a *maximin share allocation* if, for every $i \in N$, $v_i(T_i) \geq \boldsymbol{\mu}_i(n, M)$.

As shown in Procaccia–Wang [5], maximin share allocations do not always exist. Hence, our focus is on approximation algorithms, i.e., on algorithms that produce a partition where each agent i receives a bundle worth (according to v_i) at least $\rho \cdot \boldsymbol{\mu}_i(n, M)$, for some $\rho \leq 1$.

3 Main Results

Our first main result is the following theorem, establishing a polynomial time algorithm for achieving a $2/3$-approximation to the maximin share of each agent.

Theorem 3 *Let N be a set of n agents with additive valuation functions, and let M be a set of goods. For any $\varepsilon > 0$, there exists a polynomial time algorithm, producing an allocation $(S_1, \ldots, S_n)$, such that for all $i \in N$:*

$$v_i(S_i) \geq \left(\frac{2}{3} - \varepsilon\right) \boldsymbol{\mu}_i(n, M) .$$

Our result is based on the algorithm from Procaccia–Wang [5], which also guarantees to each agent a $2/3$-approximation; however, it runs in polynomial time only for a constant number of agents. Here, we identify the source of exponentiality and take a different approach regarding certain parts of their algorithm, in order to turn this into a polynomial time algorithm for any number of agents, with an approximation ratio of $\frac{2}{3} - \varepsilon$, for any $\varepsilon > 0$. We refer the reader to the full version of our work for a detailed analysis of our algorithm.

Our second set of results is motivated by the extensive experimentation in [1] (and also by [5]), which showed that in all generated instances there always existed a maximin share allocation. Motivated by these experimental observations, and by the lack of impossibility results, we present a probabilistic analysis, showing that indeed we expect that in most cases there exist allocations where every agent receives her maximin share. In particular, we analyze a simple greedy round-robin algorithm on instances where each v_{ij} is drawn from the uniform distribution on $[0, 1]$. We establish the following:

Theorem 4 *Let $N = [n]$ be a set of agents and $M = [m]$ be a set of goods, and assume that the v_{ij}s are i.i.d. random variables that follow $U[0, 1]$. Then, for $m \geq 2n$ and large enough n, there exists an algorithm that allocates to each agent i a set of goods of total value at least $(\sum_{j=1}^{m} v_{ij})/n$ with probability $1 - o(1)$. Moreover, the $o(1)$ term is $O(1/n)$ when $m > 2n$ and $O(\log n/n)$ when $m = 2n$.*

The proof is based on several tools, such as Hoeffding's and Chebyshev's inequalities, and a very careful estimation of the probabilities involved when $m < 2.5n$.

We now state a similar result for any m. For this, we had to use a modification of the round-robin algorithm that we used before. While $m < 2n$, the algorithm picks any agent uniformly at random and gives her only her "best" item (phase 1). When the number of available items becomes two times the number of active agents, the algorithm proceeds as usual (phase 2).

Theorem 5 *Let $N = [n]$ be a set of agents, $M = [m]$ be a set of goods, and assume that the v_{ij}s are i.i.d. random variables that follow $U[0, 1]$. Then, for any m and large enough n, there exists an algorithm that allocates to each agent i a set of items of total value at least $\boldsymbol{\mu}_i(n, M)$ with probability $1 - o(1)$, where the $o(1)$ term is $O(1/n)$ when $m > 2n$ and $O(\log n/n)$ when $m \leq 2n$.*

Remark 1 The implicit constants in the probability bounds of Theorems 4 and 5 depend heavily on n and m. Our analysis gives good bounds for the case $2n \leq m < 2.5n$ without requiring very large values for n. For example, if $m = 2.4n$ an appropriate adjustment of our bounds gives a $o(1)$ term less than $1/1.2n$, for $n \geq 41$. When we switch from the detailed analysis of the $2n \leq m < 2.5n$ case to the sloppier general treatment for $m \geq 2.5n$, there is definitely some loss, e.g., for $m = 4n$, we get that the $o(1)$ term is less than $25/n$ for $n \geq 50$. This is corrected relatively quickly as m grows, e.g., for $m = 10n$ the $o(1)$ term can be made less than $3/n$ for $n \geq 8$.

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An Alternate Proof of the Algorithmic Lovász Local Lemma

Ioannis Giotis, Lefteris Kirousis, Kostas I. Psaromiligkos, and Dimitrios M. Thilikos

Abstract The algorithm for Lovász Local Lemma by Moser and Tardos gives a constructive way to prove the existence of combinatorial objects satisfying a system of constraints. We present an alternative probabilistic analysis of the algorithm that does not involve reconstructing the history of the algorithm. We apply our technique to improve the best known upper bound to acyclic chromatic index.

1 Introduction

The Lovász Local Lemma (LLL) first appeared in 1975 in the paper Erdős–Lovász [3]. The original proof, however, was non-constructive. Up until the seminal works Moser [8] and Moser–Tardos [9], and despite important work on the question of supplying a constructive proof by Beck [2], Alon [1], Srinivasan [11] and others, the question remained on the existence of an algorithmic process.

The constructive approaches of Moser [8] and Moser–Tardos [9] showed how to algorithmically produce a satisfying assignment to a Constraint Satisfaction Problem (CSP), given that the variables are independent. The randomized algorithms in both papers look for an unsatisfied constraint and resample its variables, until no unsatisfied constraint is left. In Moser [8], the search is local, in the sense that

I. Giotis (✉) • L. Kirousis
Department of Mathematics, National & Kapodistrian University of Athens and Computer Technology Institute & Press "Diophantus", Athens, Greece

Computer Technology Institute & Press "Diophantus", Patras, Greece
e-mail: igiotis@cs.upc.edu; lkirousis@math.uoa.gr

K.I. Psaromiligkos
Department of Mathematics, National & Kapodistrian University of Athens, Athens, Greece
e-mail: kostaspsa@gmail.com

D.M. Thilikos
Department of Mathematics, National & Kapodistrian University of Athens, Athens, Greece

Computer Technology Institute & Press "Diophantus", Patras, Greece

AlGCo project-team, CNRS, LIRMM, Montpellier, France
e-mail: sedthilk@thilikos.info

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the algorithm looks for unsatisfied constraints in the neighborhood of the current one, whereas in Moser–Tardos [9] there is no locality restriction for the search.[1] In both papers the expected time of steps for such an algorithm to halt is shown to be linear in the number of variables, given that the degree of the dependency graph of the constraints does not exceed a certain constant fraction of the inverse of the probability of a constraint not to be satisfied. The proofs depend on supplying a witness structure which encodes the history of the random choices of the algorithm. For an elegant presentation of the Moser and Tardos' proof [9] see Spencer [10]. Another proof approach (closer to the spirit of the proof in [8]) depends on the fact that the size of the witness structure cannot be less than the entropy of the sequence of random choices made by the algorithm. See [13] for a short exposition of the basic idea of this information theoretic argument known as "*entropy compression argument*". For a recent survey, see Szegedy [12].

2 Algorithmic Lovász Local Lemma

Let X_i, $i = 1, \ldots, l$, be mutually independent random variables on a common probability space, taking values in the sets D_i, $i = 1, \ldots, l$, respectively.

Let E_j, $j = 1, \ldots, m$, be a sequence of events, each depending on a sequence of the random variables X_i. The sequence of variables that an event E_j depends on is called the *scope* of E_j and is denoted by e^j. Events are assumed to be ordered as in the sequence (i.e., according to their index). The events E_j are considered "undesirable", i.e., the objective is to design a randomized algorithm that will return an assignment α to the variables X_i for which none of the events E_j hold.

We say that two events *overlap*, and write $E_j \sim E_k$, if $e^j \cap e^k \neq \emptyset$. The reflexive and symmetric binary relation $\sim$ defines a graph with all vertices looped (but no multiple edges) called the *dependency graph* of the events.

For $j = 1, \ldots, m$, let N_j be the neighborhood of the event E_j in the dependency graph, i.e., $N_j = \{E_k \mid E_k \sim E_j\}$ (observe that $E_j \in N_j$).

Let Δ be the maximum of the cardinalities $|N_j|$ (i.e., the max degree of the dependency graph counting the loop as contributing 1 to the degree) and let p be the max of the probabilities $\Pr[E_j]$.

Theorem 1 (Lovász Local Lemma) *If $ep\Delta \leq 1$, then $Pr[\overline{E_1} \wedge \overline{E_2} \wedge \cdots \wedge \overline{E_m}] > 0$, i.e., there exists an assignment to the variables X_i for which none of the events E_i hold (by e we denote the base of the natural logarithm, i.e., $e = 2.71828\ldots$).*

We work with Moser's original algorithm (Fig. 1) in [8] and we use the Principle of Deferred Decisions of Knuth [6] to show that resampling produces an assignment

[1] The algorithm in [8] was presented and analyzed only for the SAT problem, whereas the one in [9] was presented and analyzed in a general setting; the generalization of the algorithm in [8] for arbitrary events is straightforward; however, the analysis in [8] does not immediately generalize.

M-ALGORITHM

1: Sample the variables X_i, $i = 1, \ldots, l$, and let α be the resulting assignment of values to them.
2: **while** there exist an event that occurs under the current assignment, let E_j be the least indexed such event **do**
3: RESAMPLE(E_j)
4: Output current assignment α.

RESAMPLE (E_j)

1: Resample the variables in the scope e^j.
2: **while** some $E_k \in N_j$ occurs under the current assignment, let E_k be the least indexed such event **do**
3: RESAMPLE(E_k)

Fig. 1 Moser's algorithm

with the same distribution as the original one (conditional that the event that initiated this particular resampling is satisfied).

Thus we manage to express the probability that at least n steps are needed until the algorithm halts by a simple recurrence relation. This recurrence relation is then asymptotically analyzed by classical tools of the Analysis of Algorithms. Given that the degree of the dependency graph does not exceed a certain constant fraction of the inverse probability of an undesirable event to occur, it turns out that the probability that the algorithm lasts for n steps is exponentially small in n, after a cutoff point (a result made known to us by Achlioptas and Iliopoulos—private communication, 2013—but through a proof based on the entropic method). We believe that this direct and completely elementary probabilistic proof avoids some of the intricacies of the entropy compression argument. It also completely unveils, we think, the extreme elegancy of Moser's algorithm.

Theorem 2 *Assuming p and Δ are constants such that $\left(1 + \frac{1}{\Delta - 1}\right)^{\Delta - 1} p\Delta < 1$ (and, therefore, $ep\Delta \leq 1$), there exists an integer N, which depends linearly on m, and a constant $c \in (0, 1)$, depending on p and Δ, such that if $n/\log n \geq N$ then the probability that* M-ALGORITHM *executes more than n calls of* RESAMPLE *is $< c^n$.*

The integer N in the above theorem is referred to as a *cut-off* point because it marks the onset of subexponential probability for the number of steps (calls of RESAMPLE) that M-ALGORITHM takes before it stops. Clearly, when the algorithm stops we have found an assignment such that none of the events occurs. And since this happens with probability close to 1 for large enough n, Theorem 1 indeed easily follows.

We used our alternative approach to a particular coloring problem, namely the acyclic edge coloring. The entropy compression method had been used in Esperet–Parreau [4] to find bounds for the acyclic chromatic index, the least number of colors needed to properly color the edges of a graph so that no bichromatic cycle exists.

ColAlg

1: Color all edges following their ordering and choosing at each step a color uniformly at random among those that retain 4-acyclicity.
2: **while** some bichromatic cycle exists, let C be the least such and **do**
3: Recolor(C)
4: Output current coloring.

Recolor (C)

1: Recolor the edges of C (following their ordering) choosing at each step uniformly at random a color among those that retain 4-acyclicity.
2: **while** some bicromatic cycle C' sharing an edge with C exists, let C' be the least such **do**
3: Recolor(C')

Fig. 2 The coloring algorithm

Our approach leads to a direct and simple probabilistic analysis giving the upper bound of $\lceil 3.74(\Delta - 1)\rceil + 1$ for the acyclic chromatic index, improving the bound of $4(\Delta - 1)$ given by Esperet and Parreau. We also improve their bounds for graphs with bounded girth.

Theorem 3 *Assuming Δ, the maximum degree of the graph G, is constant, and given the availability of at least $3.74(\Delta - 1) + 1$ colors, there exists a coloring of the edges such that no two adjacent ones have the same color and no cycle is colored with only two colors. Furthermore, such a coloring can be found by our algorithm (see Fig. 2) almost surely.*

A proof of the *asymmetric* Lovász Local Lemma by the same technique has been recently given by Giotis–Kirousis–Psaromiligkos–Thilikos [5]. Also recently, Harvey and Vondrák gave a proof of the lopsided asymmetric Lovász Local Lemma by a similar technique (the events in that work are over an arbitrary probability space, as also in Kolipaka–Szegedy [7]).

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Learning Game-Theoretic Equilibria Via Query Protocols

Paul W. Goldberg

Abstract *Query complexity* is a very widespread and recurring theme in the analysis of algorithms and computational complexity. Algorithms are assumed to have access to their input data via certain stylised *queries*, which impose a constraint on the way an algorithm can behave. In the context of computing equilibria of games, this is a relatively recent line of work, which we review here. The talk mostly focuses on the paper Fearnley et al. (Learning equilibria of games via payoff queries. In: Proceedings of the 14th ACM-EC, pp 397–414, 2013).

1 Background and Generalities

Algorithmic Game Theory (AGT) has introduced the ideas of viewing games and auctions through the lenses of *computational complexity*, and *price of anarchy*; these are very prominent themes in AGT. Here, we try to give *query complexity* more recognition as a research theme. The following discussion is done in more detail in the survey Goldberg ("Queries and equilibrium learning: a survey", unpublished, 2015), currently a work in progress.

There are many reasons for studying query complexity, some of which are specific to game-theoretic contexts, and others which have been articulated in earlier works; query complexity is a widespread theme in algorithms and computational complexity. We note the following:

(i) There may be some function on which we wish to avoid imposing any kind of constraint (computational or syntactic), other than certain constraints (such as monotonicity) that would not, on their own, allow the function to have a concise representation. As a result, the size of the "input data" may be exponentially large, when expressed as a function of parameters such as the number of players, or the number of items being sold/traded. In that situation it is standard to assume that the data is presented as a "black-box

P.W. Goldberg (✉)
Department of Computer Science and Man Institute of Quantitative Finance, University of Oxford, Oxford, UK
e-mail: paul.goldberg@cs.ox.ac.uk

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oracle" (for example, in the context of cake-cutting). In this kind of setting, any polynomial-time algorithm implicitly gives polynomial query complexity. Note, however, that most of this work is mainly focused on computational complexity rather than query complexity.

(ii) When the entity being queried is an agent, or player, it may be considered onerous to have to answer many queries (this consideration has been noted in various previous works in the context of matchings; also in the context of auctions). Then it is important to elicit information efficiently, and avoid asking for information that will not be used. A related point, is that a full revelation of bidders' preferences may elicit more information than is needed to allocate an item efficiently. Bidders may prefer to be economical with the truth,[1] due to concerns about privacy, or due to concerns about their valuations being exploited against them, in subsequent auctions or negotiations.

(iii) It is sometimes a good model for how data is made available to an algorithm.

(iv) It imposes a benign constraint on how the algorithm may interact with the data, but with that constraint, one can obtain upper/lower bounds that are sometimes quite detailed and informative, serving as a criterion to distinguish between alternative solution concepts.[2]

(v) General understanding of algorithms being presented: for example, [10, 14] show how to compute exact CE of multiplayer games; it is useful to observe that the information obtained by these algorithms is the payoffs obtained by the players in response to mixed-strategy profiles constructed by the algorithm.

(vi) It has a strong relationship with *communication complexity*, which in turn has been studied in a game-theoretic context; e.g., the "bisection auction" is advocated as having low communication complexity.

Comparing computational complexity against query complexity, it is often easier to obtain lower bounds for query complexity. Indeed, query complexity may be used as a criterion to separate, or distinguish, the difficulty of alternative problems, in situations where computational complexity fails to do so.

2 Details on Payoff Queries

Our main focus is on equilibrium computation in the *payoff query* model. This has been studied in the papers [2, 3, 6–9, 12], but the earliest reference (that we know of) to the idea is the blog [13]. It has been noted in [4] that [14] can be

[1] In a more literal sense than the common usage of this phrase.

[2] For example, separation of randomized and deterministic computation [1, 11] in the context of computing boolean functions. Ambainis et al., [1], note that "the advantage of query complexity is that we can often prove tight lower bounds and have provable separations between different computational models. This is in contrast to the Turing machine world where lower bounds and separations between complexity classes often have to rely on unproven assumptions".

seen as a payoff-query algorithm, but the focus of [14] is on queries to mixed-strategy profiles rather than pure ones. That would appear to be a technical-looking distinction, but [14] requires *exact* payoffs for mixed-strategy queries to be obtained, and random sampling only obtains (additively) approximate ones. We focus on Fearnley–Gairing–Goldberg–Savani [6], which is the first published paper to study the pure-strategy payoff-query model. Besides to motivations mentioned above, the motivation for this paper came from experimental work (mainly by Wellman and co-authors) on *empirical game-theoretic analysis*. The following is an overview of the results of [6].

We study a variety of different settings. We first consider bimatrix games. Our first result is a lower bound for computing an exact Nash equilibrium: we show that computing an exact Nash equilibrium in a $k \times k$ bimatrix game has payoff query complexity k^2, even for zero-sum games. In other words, we have to query every pure strategy profile.

We then turn our attention to approximate Nash equilibria, where we obtain some more positive results. With the standard assumption that all payoffs lie in the range $[0, 1]$, we show that, when $2 \leq i \leq k-1$, the payoff query complexity of computing a $(1 - 1/i)$-approximate Nash equilibrium is at most $2k - i + 1$ and at least $k - i + 1$. We also observe that, when $\epsilon \geq 1 - 1/k$, no payoff queries are needed at all, because an ϵ-Nash equilibrium is achieved when both players mix uniformly over their pure strategies.

The query complexity of computing an approximate Nash equilibrium when $\epsilon < 1/2$ appears to be a challenging problem, and we provide an initial lower bound in this direction: we show that the payoff query complexity of finding a ϵ-approximate Nash equilibrium for $\epsilon = \mathcal{O}(1/\log k)$ is $\Omega(k \cdot \log k)$. This gives an interesting contrast with the $\epsilon \geq 1/2$ case. Whereas we can always compute a $(1/2)$-approximate with $2k - 1$ payoff queries, there exists a constant $\epsilon < 1/2$ for which this is not the case.

Having studied payoff query complexity in bimatrix games, it is then natural to look for improved payoff query complexity results in the context of "structured" games. In particular, we are interested in *concisely represented* games, where the payoff query complexity may be much smaller than the number of pure strategy profiles. As an initial result in this direction, we consider graphical games where we show that, for graphical games with constant degree d, a Nash equilibrium can be found with a polynomial number of payoff-queries. This algorithm works by discovering every payoff in the game, however unlike bimatrix games, this can be done without querying every pure strategy profile.

Finally, we focus on two different models of congestion games. We consider the case of *parallel links*, where the game has an origin and destination vertex, and m parallel links between them. We show both lower and upper bounds for this setting. If n denotes the number of players, then we obtain a $\log(n) + m$ payoff query lower

bound, which applies to both query models. We obtain an upper bound of

$$\mathcal{O}\left(\log(n)\cdot\frac{\log^2(m)}{\log\log(m)}+m\right)$$

normal-queries. Note that there are $n\cdot m$ different payoffs in a parallel links game, and so our upper bound implies that you do not need to discover the entire payoff function in order to solve a parallel links game.

Subsequently, we consider the more general case of symmetric network congestion games on directed acyclic graphs. We show that if the game has m edges and n players, then we can find a Nash equilibrium using $m\cdot n$ payoff queries. The algorithm discovers every payoff in the game, but it only queries a small fraction of the pure strategy profiles.

3 Conclusions and Further Work

We first consider open questions in the setting of payoff queries, which has been the main setting for the results presented here. We then consider alternative query models.

3.1 Open Questions Concerning Payoff Queries

In the context of strategic-form games, there are a number of open problems. In Fearnley–Gairing–Goldberg–Savani [6], we show a super-linear lower bound on the payoff query complexity when ϵ is allowed to depend on k. Can we prove a super-linear lower bound for a constant ϵ? Is there a deterministic algorithm that can find an ϵ-Nash equilibrium with $\epsilon < 1/2$ without querying the entire payoff matrices? Fearnley–Savani [7] achieves $\epsilon < 1/2$ with the use of randomization, but doing so with a deterministic algorithm appears to be challenging. Finally, when $2\leq i\leq k-1$, we have shown that the payoff query complexity of finding a $(1-1/i)$-Nash equilibrium lies somewhere in the range $[k-i+1,\ 2k-i+1]$. Determining the precise payoff query complexity for this case is an open problem.

For congestion games, our lower bound of $\log n+m$ arises from a game with two parallel links and a one-player game with m links. The above noted upper bound (on the number of normal queries) is a poly-logarithmic factor off from this lower bound, with the factor depending on m. Can this factor be improved? It seems unlikely that the dependence of this factor on m can be completely removed, in which case, in order to provide tight bounds, a single lower bound construction depending simultaneously on n and m would be necessary.

For symmetric network congestion games on DAGs it is unclear whether the payoff query complexity is sub-linear in n. Non-trivial lower and upper bounds for more general settings, such as asymmetric network congestion games (DAG or not) or general (non-network) congestion games would also be interesting.

3.2 Other Query Models

We have defined a payoff query as given by a *pure* (not mixed) profile $\mathtt{s}$, since that is of main relevance to empirical game-theoretic modelling. Furthermore, if $\mathtt{s}$ was a mixed profile, it could be simulated by sampling a number of pure profiles from $\mathtt{s}$ and making the corresponding sequence of pure payoff queries. An alternative definition might require a payoff query to just report a single specified player's payoff, but that would change the query complexity by a factor of at most n.

Our main results have related to exact payoff queries, though other query models are interesting too.

A very natural type of query is a *best-response query*, where a strategy $\mathtt{s}$ is chosen, and the algorithm is told the players' best responses to $\mathtt{s}$. In general, $\mathtt{s}$ may have to be a mixed strategy; it is not hard to check that pure-strategy best response queries are insufficient; even for a two-player two-action game, knowledge of the best responses to pure profiles is not sufficient to identify an ϵ-Nash equilibrium for $\epsilon < 1/2$. *Fictitious Play* can be regarded as a query protocol that uses best-response queries (to mixed strategies) to find a Nash equilibrium in zero-sum games, and essentially a 1/2-Nash equilibrium in general-sum games. We can always synthesize a pure best-response query with $n(k-1)$ payoff queries. Hence, for questions of polynomial query complexity, payoff queries are at least as powerful as best-response queries.

Are there games where best-response queries are much more useful than payoff queries? If k is large then it is expensive to synthesize best-response queries with payoff queries. A simple algorithm of Daskalakis, Mehta and Papadimitriou finds a $1/2$-Nash equilibrium via only two best-response queries, whereas [6, Theorem 7] notes that $\mathcal{O}(k)$ payoff queries are needed.

A *noisy* payoff query outputs an observation of a random variable taking values in $[0, 1]$, whose expected value is the true payoff. Alternative versions might assume that the observed payoff is within some distance ϵ from the true payoff. Noisy query models might be more realistic, and they are suggested by the experimental papers on querying games. However, in a theoretical context, one could obtain good approximations of the expected payoffs for a profile $\mathtt{s}$, by repeated sampling.

Recently, Chen–Cheng–Tang [5] showed that for general n-player games, there is an exponential lower bound on the number of payoff queries required to compute an ϵ-Nash equilibrium, for constant ϵ. This answered a question of [12]; previous an exponential lower bound was known for the stronger solution concept of ϵ-well-supported Nash equilibrium; see [2]. These results indicate that, for large games,

some kind of structure has to be assumed for their payoff function, in order to obtain a positive result for query complexity.

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The Lower Tail: Poisson Approximation Revisited

Svante Janson and Lutz Warnke

Abstract The well-known Janson's inequality gives Poisson-like upper bounds for the lower tail probability $\mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X)$ when X is the sum of dependent indicator random variables of a special form. In joint work with Svante Janson we showed that, for large deviations, this inequality is optimal whenever X is approximately Poisson, i.e., when the dependencies are weak. For subgraph counts in random graphs, this, e.g., yields new lower tail estimates, extending earlier work (for the special case $\varepsilon = 1$) of Janson, Łuczak and Ruciński.

1 Introduction

In probabilistic combinatorics and related areas it is often important to estimate the probability that a sum X of dependent indicator random variables is small or zero (e.g., to show that few or none of a collection of events occurs). Moreover, it is frequently desirable that these probabilities are exponentially small (e.g., to make union bound arguments amenable). Here, we focus on such sharp estimates for the *lower tail* $\mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X)$, where X is of a form that is commonly used in, e.g., applications of the probabilistic method or random graph theory; see [1, 4]. More precisely, the underlying probability space is the random subset $\Gamma_{\mathbf{p}} \subseteq \Gamma$, with $|\Gamma| = N$ and $\mathbf{p} = (p_i)_{i\in\Gamma}$, where each $i \in \Gamma$ is included, independently, with probability p_i. Given a family $\big(Q(\alpha)\big)_{\alpha\in\mathcal{X}}$ of subsets of Γ we define $I_\alpha = \mathbb{1}_{\{Q(\alpha)\subseteq\Gamma_{\mathbf{p}}\}}$, so that

$$X = \sum_{\alpha\in\mathcal{X}} I_\alpha \tag{1}$$

S. Janson (✉)
Department of Mathematics, Uppsala University, PO Box 480, SE-751 06, Uppsala, Sweden
e-mail: svante.janson@math.uu.se

L. Warnke
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, CB3 0WB, Cambridge, UK
e-mail: L.Warnke@dpmms.cam.ac.uk

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counts the number of sets $Q(\alpha)$ that are entirely contained in $\Gamma_{\mathbf{p}}$. We write $\alpha \sim \beta$ if $Q(\alpha) \cap Q(\beta) \neq \emptyset$ and $\alpha \neq \beta$, which intuitively means that there are 'dependencies' between I_α and I_β. Let

$$\mu = \mathbb{E}X = \sum_{\alpha \in \mathcal{X}} \mathbb{E}I_\alpha, \qquad \Pi = \max_{\alpha \in \mathcal{X}} \mathbb{E}I_\alpha,$$

$$\Lambda = \mu + \sum_{(\alpha,\beta) \in \mathcal{X} \times \mathcal{X} : \alpha \sim \beta} \mathbb{E}I_\alpha I_\beta = (1+\delta)\mu.$$

Note that δ measures how dependent the indicators I_α are (with $\delta = 0$ in the case of independent summands), and that $\operatorname{Var} X \leqslant \Lambda$ holds. In [2] the first author proved the following lower tail analogue (often called *Janson's inequality*) of the Bernstein and Chernoff bounds for sums of independent indicators (the case $\delta = 0$): with $\varphi(x) = (1+x)\log(1+x) - x$, for all $\varepsilon \in [0,1]$ we have

$$\mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X) \leqslant \exp\bigl\{-\varphi(-\varepsilon)\mu/(1+\delta)\bigr\}. \tag{2}$$

Inequality (2) is nowadays a widely used tool in probabilistic combinatorics, which makes it important to understand how 'sharp' it is, i.e., whether the exponential rate of decay given by (2) is best possible. Whenever $\Pi < 1$, Harris' inequality gives, as noted in [3],

$$\mathbb{P}(X = 0) \geqslant \prod_{\alpha \in \mathcal{X}} (1 - \mathbb{E}I_\alpha) \geqslant \exp\bigl\{-\mu/(1-\Pi)\bigr\}. \tag{3}$$

The point is that (2) and (3) yield $\log \mathbb{P}(X=0) \sim -\mu$ whenever $\delta, \Pi \to 0$. This raises the intriguing question whether the exponent of (2) is also sharp for other choices of ε, in particular when $\varepsilon \to 0$.

1.1 Main Result

In [5] we prove that "Janson's inequality" (2) is close to best possible in many situations of interest. For example, the next theorem shows that, for large deviations, the rate of decay of (2) is optimal for *any* random variable X of type (1) that is approximately Poisson (i.e., whenever $\delta, \Pi \to 0$).

Theorem 1 *With notations as above, if* $\varepsilon \in [0,1]$, $\max\{\Pi, \mathbb{1}_{\{\varepsilon<1\}}\delta\} \leqslant 2^{-14}$ *and* $\varepsilon^2\mu \geqslant \mathbb{1}_{\{\varepsilon<1\}}$, *then*

$$\mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X) \geqslant \exp\bigl\{-(1+\xi)\varphi(-\varepsilon)\mu\bigr\}, \tag{4}$$

with $\xi = 135\max\{\Pi^{1/8}, \mathbb{1}_{\{\varepsilon<1\}}\delta^{1/8}, \mathbb{1}_{\{\varepsilon<1\}}(\varepsilon^2\mu)^{-1/4}\}$.

The condition $\varepsilon^2\mu = \Omega(1)$ is natural in the context of exponentially small probabilities since $(1+\xi)\varphi(-\varepsilon) = \Theta(\varepsilon^2)$. As discussed, our favourite range is when $\delta, \Pi \to 0$. For large deviations, i.e., when $\varepsilon^2\mu \to \infty$ holds, (2) and (4) then yield

$$\log \mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X) \sim -\varphi(-\varepsilon)\mu. \tag{5}$$

In words, Theorem 1 determines the *large deviation rate function* $\log \mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}X)$ up to second order error terms, closing a gap that was left open by the first author nearly 25 years ago [2]. In fact, in [5] we also obtain variants of (4) and (5) when $\delta = O(1)$ and Π is bounded away from one.

The proof of Theorem 1 hinges on Hölder's inequality and several estimates of the Laplace transform; see [5] for the details. In fact, Theorem 1 remains valid for the more general setup stated in Riordan–Warnke [6].

1.2 Main Example

From an applications point of view it is important to also understand the sharpness of (2) in the case $\delta = \Omega(1)$, i.e., when X is no longer close to Poisson. In [5] we thus develop bootstrapping approaches which often allow us to deal with this remaining 'strongly dependent' case. In [5] our main example is the number of small subgraphs in the binomial random graph $G_{n,p}$.

Theorem 2 *Let H be a graph with $e_H \geqslant 1$ edges. Define $\Phi_H = \Phi_H(n,p) = \min\{\mathbb{E}X_J : J \subseteq H, e_J \geqslant 1\}$. There are positive constants c, C, D and n_0, all depending only on H, such that for all $n \geqslant n_0$, $p \in [0,1)$ and $\varepsilon \in [0,1]$ satisfying $\varepsilon^2\Phi_H \geqslant \mathbb{1}_{\{\varepsilon<1\}}D$ we have*

$$\exp\bigl\{-(1-p)^{-5}C\varepsilon^2\Phi_H\bigr\} \leqslant \mathbb{P}(X_H \leqslant (1-\varepsilon)\mathbb{E}X_H) \leqslant \exp\bigl\{-c\varepsilon^2\Phi_H\bigr\}. \tag{6}$$

A key feature of Theorem 2 is that ε is *not* fixed, but may depend on n. In applications, p is typically bounded away from one, so the $\varepsilon^2\Phi_H = \Omega(1)$ condition is natural. Furthermore, in this case (6) yields

$$\log \mathbb{P}(X_H \leqslant (1-\varepsilon)\mathbb{E}X_H) = -\Theta(\varepsilon^2\Phi_H), \tag{7}$$

determining the large deviation rate function of X_H up to constant factors. For the special case $\varepsilon = 1$ this was established more than 25 years ago in Janson–Łuczak–Ruciński [3]. In [5] we moreover discuss the finer behaviour of the large deviation rate function, and also consider arithmetic progressions; see [7] for related upper tail results.

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Population Protocols for Majority in Arbitrary Networks

George B. Mertzios, Sotiris E. Nikoletseas, Christoforos L. Raptopoulos, and Paul G. Spirakis

Abstract We study the problem of determining the majority type in an arbitrary connected network, each vertex of which has initially two possible types. The vertices may have a few additional possible states and can interact in pairs only if they share an edge. Any (population) protocol is required to stabilize in the initial majority. We first present and analyze a protocol with 4 states per vertex that *always* computes the initial majority value, under any fair scheduler.This protocol is optimal, in the sense that there does not exist any population protocol that always computes majority with fewer than 4 states per vertex. However, this does not rule out the existence of a protocol with 3 states per vertex that is correct with high probability (whp). To this end, we examine an elegant and very natural majority protocol with 3 states per vertex, introduced in Angluin et al. (Distrib Comput 21:87–102, 2008), where its performance has been analyzed for the clique graph. In particular, we study the performance of this protocol in arbitrary networks, under the probabilistic scheduler. We prove that, when the two initial states are put uniformly at random on the vertices, the protocol of Angluin et al. (Distrib Comput 21:87–102, 2008) converges to the initial majority with probability higher than the probability of converging to the initial minority. In contrast, we show that the resistance of the

G.B. Mertzios (✉)
School of Engineering and Computing Sciences, Durham University, Durham, UK
e-mail: george.mertzios@durham.ac.uk

S.E. Nikoletseas
Computer Technology Institute and Press "Diophantus", Patras, Greece

Computer Science Department, University of Patras, Patras, Greece
e-mail: nikole@cti.gr

C.L. Raptopoulos
Computer Technology Institute and Press "Diophantus", Patras, Greece
e-mail: raptopox@ceid.upatras.gr

P.G. Spirakis
Computer Technology Institute and Press "Diophantus", Patras, Greece

Department of Computer Science, University of Liverpool, Liverpool, UK

Computer Science Department, University of Patras, Patras, Greece
e-mail: p.spirakis@liverpool.ac.uk

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protocol to failure when the underlying graph is a clique causes the failure of the protocol in general graphs.

This abstract paper is based on our work which appeared in the Proceedings of the 41-st International Colloquium on Automata, Languages, and Programming (ICALP) in 2014.

1 Introduction and Motivation

One of the most natural computational problems in many physical systems is to compute the *majority*, i.e., to determine accurately which type of an element of the system appears more frequently. For instance, the majority problem is encountered in various settings such as in voting [9, 10], in epidemiology and interacting particles systems [13], in diagnosis of multiprocessor systems [19], in social networks [15, 17], etc. In distributed computing, the majority problem is an important and natural special case of the central problem of reaching *consensus* within a system [5, 12], where a number of processes have to agree on any single data value (e.g., leader election [8]). In all these physical systems, some pairs of elements may interact with each other, while other pairs may not be able to interact directly. This structure of the possible pairwise interactions between elements of the system can be modeled by a network (i.e., graph), where elements and possible interactions are represented by vertices and edges, respectively.

In order to solve the majority computation problem in a network, we first need to make some assumptions on the underlying model of computation. Much research has been done under the assumption that there exists a central authority, as well as unlimited available memory and full information about the whole network; see, e.g., [6, 20]. However, in many real systems we do not have (or we do not wish to have) such a powerful computational model. The weaker the considered model of computation is (e.g., lack of central authority, partial or no information about the system, lack of memory, etc.), the more challenging the majority computation becomes.

One of the ways to study distributed systems where agents may interact in pairs and each individual agent is extremely limited (in fact, being equipped only with a finite number of possible states) is by using *population protocols* [1, 3]. Then, the complex behavior of the system emerges from the rules governing the possible pairwise interactions of the agents. Population protocols have been defined by analogy to population processes [11] in probability theory, and have already been used in various fields, such as in statistical physics, genetics, epidemiology, chemistry and biology [4].

In particular, population protocols are *scalable*, i.e., they work independently of the size n of the underlying network (called the *interaction graph*) and the value of n is not even known to the protocol. Furthermore they are *anonymous*, i.e., there is only one transition function which is common to all entities/agents: the result of an interaction of an agent u at state q_u with an agent v at state q_v is the same

regardless of the identity of u and v. The transition function of a population protocol only specifies the result of every possible interaction, without specifying *which* pairs of agents interact or *when* they are chosen to interact. Usually, it is assumed that interactions between agents happen under some kind of a *fairness* condition. For a survey about population protocols we refer to [3].

In this direction, a very natural and simple population protocol for the majority problem on the clique (i.e., the complete graph), where initially every vertex has one of two possible types (states), has been introduced and analyzed in [2]. In particular, the protocol of [2] assigns only three possible states to every agent (i.e., there is a 3×3 transition table capturing all possible interactions) and the interactions between agents are dictated by a *probabilistic scheduler* (i.e., all pairs have the same probability to interact at any step). Every vertex has an identity v, but it is unaware of the identity of any other vertex, as well as of its own identity. Although the underlying interaction graph in [2] is assumed to be a clique, the authors distinguish in their protocol the agents u and v participating in an interaction into an "initiator" and a "responder" of the interaction (when agents u and v interact, each of them becomes initiator or responder with equal probability). Their main result is that, if initially the difference between the initial majority from the initial minority in the complete graph with n vertices is $\omega(\sqrt{n}\log n)$, their protocol converges to the correct initial majority value in $O(n\log n)$ time with high probability.

Most works on population and majority dynamics so far considered only two entity types (e.g., the voter model [9], the Moran process [16]). The analysis of population dynamics with more than two types is challenging. As an example we refer to the model of [2], in which, although agents can have initially one of only two types (red and green), the protocol itself allows every agent to be in one among three different states (red, green and blank) at every subsequent time point. Even though this model is quite simple, it is very hard to be analyzed. Computing the majority with as few states as possible in the more general case, where the interaction graph has an arbitrary structure (as opposed to the complete graph that has been mainly considered so far) remained an open problem.

2 Our Contribution

In [14] we study the majority problem in an *arbitrary* underlying interaction graph G, where initially every vertex has two possible states (red and green). We consider the weakest and simplest possible model of computation. In particular, we assume the existence of no central authority and we allow every vertex of G to have only a (small) constant number of available types (or states). Although every vertex of G has a unique identity, no vertex is aware of its own identity or the identity of any other vertex. Furthermore, although only two adjacent vertices can interact, vertices of G do not even know to which other vertices they are adjacent.

2.1 Stable Computation of Majority

We initially focus on the problem of *always* computing the correct majority value in an *arbitrary* (directed or undirected) interaction graph G, regardless of how large the initial difference between the majority and the minority is. In particular, assuming that the interacting pairs of vertices are chosen by an arbitrary fair scheduler, we derive matching lower and upper bounds on the number of available states, for which there exists a population protocol that always computes the correct majority value. For the lower bound, we prove the following result.

Theorem 1 *Any population protocol that stably computes the majority function uses at least 4 states.*

The proof is based on a contradiction argument; assuming there is a population protocol computing majority using 3 states, we construct two distinct instances C_1 and C_2 that (i) have different initial majorities and (ii) there is a fair scheduler that brings both C_1 and C_2 to the same intermediate configuration (after which the protocol will have no way of determining whether it started from C_1 or C_2).

On the other hand, for the matching upper bound we provide a population protocol with 4 states per vertex, namely the *Ambassador protocol*, which always computes the correct majority value, even if initially the difference between majority and minority is 1. A different 4 state protocol for majority was independently presented in [7]. Using standard results on random walks on graphs and coupon collector arguments, we provide polynomial upper bounds on the expected time needed by our Ambassador protocol to converge, and we show that in certain cases the running time is $O(n \log n)$, i.e., the same as for the fast protocol of [2].

2.2 The Protocol of [2] on Arbitrary Graphs

In [14], we also provide a detailed analysis of the 3-state protocol of [2] on an arbitrary interaction graph G. Our first result in this direction is the following:

Theorem 2 *For any strongly connected directed graph G, if the initial assignment of individuals to the vertices of G is random, then the majority protocol described in* [2] *correctly identifies the initial majority with probability at least* $1/2$.

The proof of this relies on a well known result in extremal combinatorics (in particular, on Hall's marriage Theorem).

On the other hand, we proved the following:

Theorem 3 *There exists an infinite family* $\{G_n\}_{n\in\mathbb{N}}$ *of interaction graphs where the protocol fails with high probability, even when the initial difference between majority/minority is* $n - \Theta(\log n)$.

Theorem 4 *There exists an infinite family* $\{G'_n\}_{n\in\mathbb{N}}$ *of interaction graphs where the protocol terminates in exponential expected time.*

In particular, Theorem 4 rules out the possibility to use a Markov chain Monte-Carlo approach to approximate the probability that the protocol of [2] converges to the correct majority value.

The proofs of Theorems 3 and 4 employ an intermediate result concerning the robustness of the protocol in [2] on the clique. In particular, *for any* $\varepsilon > 0$*, if the minority has size at most* $(1/7 - \varepsilon)n$ *in the complete graph with n vertices, then the protocol of* [2] *converges to the initial minority with exponentially small probability*. This result shows that, although the performance of the protocol of [2] can drop significantly when the interaction graph G is not the complete graph, it is quite robust when G is the complete graph. After submission of our original work in [14], we became aware of the paper [18], the results of which can also be used to prove a tighter version of our result concerning the robustness of the protocol of [2] on the clique. However, the techniques used there cannot be applied to graph structures other than the clique. On the other hand, our proof technique can also be applied to graph structures that are different than (but close to being) cliques by using a non-trivial coupling argument which can be of independent interest.

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The Asymptotic Value in Finite Stochastic Games

Miquel Oliu-Barton

Abstract In 1976, Bewley and Kohlberg proved that the discounted values v_λ of finite zero-sum stochastic games have a limit, as λ tends to 0, using the Tarski–Seidenberg elimination theorem from real algebraic geometry. This was a fundamental step in the development of the theory of stochastic games. The current paper provides a new and direct proof for this result, relying on the explicit description of asymptotically optimal strategies. Both approaches can be used to obtain the existence of the uniform value using the construction from Mertens and Neyman (1981).

1 Introduction

Two-person zero-sum stochastic games were introduced by Shapley [5]. They are described by a 5-tuple $(\Omega, \mathcal{I}, \mathcal{J}, g, q)$, where Ω is a finite set of states, $\mathcal{I}$ and $\mathcal{J}$ are finite sets of actions, $g\colon \Omega \times \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ is the payoff function and $q\colon \Omega \times \mathcal{I} \times \mathcal{J} \to \Delta(\Omega)$ is the transition function. Here, for any finite set X, $\Delta(X)$ denotes the set of probability distributions over X. The functions g and q are bi-linearly extended to $\Omega \times \Delta(\mathcal{I}) \times \Delta(\mathcal{J})$.

The stochastic game with initial state $\omega \in \Omega$ and discount factor $\lambda \in (0, 1]$ is denoted by $\Gamma_\lambda(\omega)$ and is played as follows: at stage $m \geq 1$, the players choose actions $(i_m, j_m) \in \mathcal{I} \times \mathcal{J}$; their choice produces a stage payoff $g(\omega_m, i_m, j_m)$ and influences the transition. A new state ω_{m+1} is drawn according to the probability distribution $q(\cdot\,|\omega_m, i_m, j_m)$ and the triple (i_m, j_m, ω_{m+1}) is announced to the players. Player 1 (resp., 2) maximizes (resp., minimizes) $\sum_{m\geq 1} \lambda(1-\lambda)^{m-1} g(\omega_m, i_m, j_m)$. A (behaviour) strategy for player 1 (resp., 2) is a function σ (resp., τ) from the set of finite histories $\cup_{m\geq 1} \Omega \times (I \times J \times \Omega)^{m-1}$ to the set of mixed actions $\Delta(\mathcal{I})$ (resp., $\Delta(\mathcal{J})$).

M. Oliu-Barton (✉)
Ceremade, Université Paris Dauphine, Paris, France
e-mail: oliubarton@ceremade.dauphine.fr

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A pair of strategies (σ, τ) induces a unique probability distribution $\mathbb{P}^{\omega}_{\sigma\tau}$ over the set of histories. The game $\Gamma_\lambda(\omega)$ has a value denoted by $v_\lambda(\omega)$, i.e.,

$$v_\lambda(\omega) = \mathrm{val}_{(\sigma,\tau)} \mathbb{E}^{\omega}_{\sigma\tau}\left[\sum\nolimits_{m\geq 1} \lambda(1-\lambda)^{m-1} g(\omega_m, i_m, j_m)\right],$$

where $\mathbb{E}^{\omega}_{\sigma\tau}$ is the expectation with respect to $\mathbb{P}^{\omega}_{\sigma\tau}$, and $\mathrm{val}_{\sigma,\tau}$ stands for $\sup_\sigma \inf_\tau = \inf_\tau \sup_\sigma$.

The vector of values $v_\lambda = (v_\lambda(\omega))_{\omega\in\Omega}$ is the unique fixed point of the so-called Shapley operator defined as follows (see [5]):

$$\forall f \in \mathbb{R}^{\Omega}, \quad \Phi(\lambda, f)(\omega) = \tag{1}$$

$$= \mathrm{val}_{(s,t)\in\Delta(\mathcal{I})\times\Delta(\mathcal{J})} \left\{ \lambda g(\omega, s, t) + (1-\lambda) \sum_{\omega'\in\Omega} q(\omega'|\omega, s, t) f(\omega') \right\}.$$

A *stationary strategy* of player 1 is a function from Ω to $\Delta(\mathcal{I})$. Similarly, $\Delta(\mathcal{J})^{\Omega}$ is the set of stationary strategies for player 2. For any pair of stationary strategies $(x, y) \in \Delta(\mathcal{I})^{\Omega} \times \Delta(\mathcal{J})^{\Omega}$, $\gamma_\lambda(\omega, x, y)$ denotes the expected payoff induced by (x, y) in $\Gamma_\lambda(\omega)$. By stationarity, the vector $\gamma_\lambda = (\gamma_\lambda(\omega, x, y))_{\omega\in\Omega}$ satisfies $\gamma_\lambda = \lambda g + (1-\lambda) Q\gamma_\lambda$, where $g(\omega) := g(\omega, x(\omega), y(\omega))$ and $Q(\omega, \omega') := q(\omega'|\omega, x(\omega), y(\omega))$, for all $\omega, \omega' \in \Omega$. The matrix Q being stochastic, $\mathrm{Id} - (1-\lambda)Q$ is invertible and

$$\forall f \in \mathbb{R}^{\Omega}, \quad \gamma_\lambda = \lambda(\mathrm{Id} - (1-\lambda)Q)^{-1} g. \tag{2}$$

From the equality $v_\lambda = \Phi(\lambda, v_\lambda)$, one deduces that both players have optimal stationary strategies.

1.1 Convergence of the Discounted Values: An Overview

The existence of the value v_λ and of optimal stationary strategies for a fixed discount factor $\lambda > 0$ is due to Shapley [5]. This result does not rely on the observation of the past actions: it holds as far as the current state is commonly known. The existence of $\lim_{\lambda\to 0} v_\lambda$ was established by Bewley–Kohlberg [1] using deep algebraic arguments, and this was so far the only proof. The current paper provides a new and direct proof for this result, relying on the explicit description of asymptotically optimal strategies. Both approaches can be used to obtain the existence of the uniform value using the construction from Mertens–Neyman [3].

The convergence of the discounted values turns out to be specific to the finite case: Vigeral [7] provided a counter-example for stochastic games and compact action sets, and one deduces from Ziliotto's counter-examples [8] the divergence for stochastic games in which one of the action sets is countable or compact.

The Algebraic Proof

For a given $\lambda \in (0,1]$, the equality $v_\lambda = \Phi(\lambda, v_\lambda)$ reduces to finitely many polynomial inequalities in finitely many variables: the discount factor λ, the optimal stationary strategies x_λ and y_λ, and the vector of values v_λ. Taking $(\lambda, x_\lambda, y_\lambda, v_\lambda)$ as variables, it defines a semi-algebraic set in $(0,1] \times \mathbb{R}^{\Omega\times\mathcal{I}} \times \mathbb{R}^{\Omega\times\mathcal{J}} \times \mathbb{R}^{\Omega}$. From the Tarski-Seidenberg elimination theorem, it admits a semi-algebraic selection above $(0,1]$. In particular, $\lambda \mapsto v_\lambda$ is a semi-algebraic, bounded function. Consequently, it admits an expansion near 0 in Puiseux series: there exists $N \in \mathbb{N}$, $\lambda_0 > 0$ and $r_m \in \mathbb{R}^{\Omega}$ $(m \geq 0)$ such that $v_\lambda(\omega) = \sum_{m\geq 0} r_m(\omega)\lambda^{m/N}$ for all $\lambda \in (0,\lambda_0)$ and $\omega \in \Omega$. In particular, the discounted values converge. Note that the same argument yields similar expansions for the optimal stationary strategies.

A New Approach

Fix an initial state $\omega_1 \in \Omega$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of discount factors such that $\lim_{n\to\infty} v_{\lambda_n}(\omega_1) = \limsup_{\lambda\to 0} v_\lambda(\omega_1)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of stationary strategies such that x_n is optimal in $\Gamma_{\lambda_n}(\omega_1)$, for each $n \in \mathbb{N}$. Consider now a family $(x_\lambda)_{\lambda\in(0,1]}$ of stationary strategies such that, for some $c, e \in \mathbb{R}^{\Omega\times\mathcal{I}}$,

$$x^i_\lambda(\omega) \sim_{\lambda\to 0} c_{\omega,i}\lambda^{e_{\omega,i}}, \quad \forall(\omega,i) \in \Omega\times\mathcal{I}. \tag{3}$$

Here, (c,e) are parameters to be determined later on, so that $(x_\lambda)_\lambda$ is asymptotically optimal.

First, one proves that for any $\lambda \in (0,1]$, $\omega \in \Omega$ and $(x,y) \in \Delta(\mathcal{I})^\Omega \times \Delta(\mathcal{I})^\Omega$, the expected payoff $\gamma_\lambda(\omega,x,y)$ can be written as a ratio of two polynomials in the variables λ, $1-\lambda$ and $x^i(\omega)$ $(\omega \in \Omega,\ i \in \mathcal{I})$, with non-negative coefficients depending on ω and y. As a consequence, the asymptotic behaviour of $\gamma_{\lambda_n}(\omega,x_n,y)$ as n tends to ∞ (resp., $\gamma_\lambda(\omega,x_\lambda,y)$ as λ tends to 0) depends only on the initial state ω, the opponent's strategy y and on the limit of finitely many ratios involving λ_n, $1-\lambda_n$ and $(x^i_n(\omega))_{(\omega,i)\in\Omega\times\mathcal{I}}$ (resp., λ, $1-\lambda$ and $(c_{\omega,i}\lambda^{e_{\omega,i}})_{(\omega,i)\in\Omega\times\mathcal{I}}$, by (3)). Up to a subsequence (resp., directly), the ratios involving λ_n and x_n (resp., λ and x_λ) converge in $[0,\infty]$. The limits determine a finite-dimensional vector ℓ^* (resp., $\ell'(c,e)$) which summarizes all the relevant information about the asymptotic behaviour of $(x_n)_n$ (resp., $(x_\lambda)_\lambda$). The equality $\ell'(c,e) = \ell^*$ implies that $\lim_{\lambda\to 0}\gamma_\lambda(\omega,x_\lambda,y) = \lim_{n\to\infty}\gamma_{\lambda_n}(\omega,x_n,y)$ for all ω and y which, together with the optimality of x_n, yields both the asymptotic optimality of $(x_\lambda)_\lambda$ and the existence of $\lim_{\lambda\to 0} v_\lambda$. The existence of a solution to the linear system with unknowns (c,e) induced by $\ell'(c,e) = \ell^*$ is obtained using a theorem of the alternative.

Comparison Between the Two Approaches

The original approach relies on important algebraic results and is elegant and clear. Moreover it proves more than the convergence of the discounted values. Indeed, in a neighborhood of 0, the value function and the optimal strategies have an expansion in Puiseux series. In particular, v_λ is of bounded variation. This property was used in Mertens–Neyman [3] to prove the existence of the uniform value for finite zero-sum stochastic games, which is a major result in the field; see Sect. 2 for more details.

Our proof is direct, relatively short, and intends to be more accessible: the main ingredients are the theory of finite-dimensional linear systems, and of finite Markov chains. Although our approach does not say much about neither the optimal strategies nor the discounted values, we prove the existence of simple, asymptotically optimal stationary strategies: it is enough to specify, for each state and action, two non-negative numbers. More important, the construction in [3] still holds. Indeed, the bounded variation of v_λ can be replaced by the bounded variation of the function ρ_λ, defined as the payoff guaranteed by an asymptotically optimal strategy in the λ-discounted game. The existence of the uniform value is thus obtained without using algebraic tools.

2 Concluding Remarks

(1) Consider the game $(\Omega, \mathcal{I}, \mathcal{J}, g, q)$ described in Sect. 1. Player 1 *can guarantee* $\alpha \in \mathbb{R}$ in $\Gamma_\infty(\omega)$ ($\omega \in \Omega$ is the initial state) if for all $\varepsilon > 0$, there exists a strategy σ_ε and $n_\varepsilon \in \mathbb{N}$ such that

$$\mathbb{E}^{\omega}_{\sigma_\varepsilon,\tau}\left[\frac{1}{n}\sum_{m=1}^{n} g(\omega_m, i_m, j_m)\right] \geq \alpha - \varepsilon, \quad \forall n > n_\varepsilon, \ \forall \tau.$$

Similarly, player 2 *can guarantee* $\alpha \in \mathbb{R}$ in $\Gamma_\infty(\omega)$ if

$$\forall \varepsilon > 0, \ \exists \tau_\varepsilon, \ \exists n_\varepsilon \in \mathbb{N}, \quad \mathbb{E}^{\omega}_{\sigma,\tau_\varepsilon}\left[\frac{1}{n}\sum_{m=1}^{n} g(\omega_m, i_m, j_m)\right] \leq \alpha + \varepsilon, \quad \forall n > n_\varepsilon, \ \forall \sigma.$$

If both players can guarantee α, this is the *uniform value* of $\Gamma_\infty(\omega)$. The existence of the uniform value, due to Mertens–Neyman [3], relies on the following result.[1]

[1] Theorem 1 is stated in this form in [4].

Theorem 1 *Let* $f\colon (0,1) \to \mathbb{R}^{\Omega}$ *be a function such that*

(i) $\|f_\lambda - f_{\lambda'}\| \leq \int_\lambda^{\lambda'} \varphi(x)dx$, *for all* $0 < \lambda < \lambda' < 1$ *and for some* $\varphi \in L^1\big((0,1],\mathbb{R}_+\big)$;
(ii) there exists $\lambda_0 > 0$ *such that* $\Phi(\lambda, f_\lambda) \geq f_\lambda$, *for all* $\lambda \in (0,\lambda_0)$.[2]

Then, player 1 *can guarantee* $\lim_{\lambda\to 0} f_\lambda$ *in* Γ_∞.

It follows from the expansion of v_λ in Puiseux series near 0 that $\lambda \mapsto v_\lambda$ satisfies (i). Moreover, (ii) is satisfied with an equality; see [5]. Thus, by Theorem 1, both players can guarantee $\lim_{\lambda\to 0} v_\lambda$, which is then the uniform value.

A natural question is whether the existence of the uniform value can be proved using our approach, and without relying on the algebraic approach. The answer is affirmative. Indeed, Theorem 1 cannot be used simply with $f_\lambda = v_\lambda$ because our approach does not provide any information about the values v_λ, apart from their convergence. Instead, define for all $\omega \in \Omega$ and $\lambda \in (0,1]$,

$$\rho_\lambda(\omega) := \min_{y\in\mathcal{J}^\Omega} \gamma_\lambda(\omega, x_\lambda, y),$$

where $(x_\lambda)_{\lambda\in(0,1]}$ is a family of asymptotically optimal stationary strategies for player 1. We claim that ρ_λ satisfies (i) and (ii) and, moreover $\lim_{\lambda\to 0}\rho_\lambda = \lim_{\lambda\to 0} v_\lambda$. One can easily check that $\rho_\lambda \leq \Phi(\lambda,\rho_\lambda)$, for all $\lambda \in (0,1]$. This property holds for any function $\lambda \mapsto \min_{y\in\mathcal{J}^\Omega} \gamma_\lambda(\cdot,x,y)$, for any $x \in \Delta(\mathcal{I})^\Omega$. Moreover, $\gamma_\lambda(\omega_1,x_\lambda,y)$ is a rational fraction in the variables $\lambda, 1-\lambda$ and $x^i_\lambda(\omega)$ $(\omega\in\Omega,\ i\in\mathcal{I})$, for any $(\omega_1,y) \in \Omega\times\mathcal{J}^\Omega$. And $x^i_\lambda(\omega)$ is an "extended" rational fraction in λ, in the sense that we allow the exponents to be in $\mathbb{R}$, for all $\omega\in\Omega,\ i\in\mathcal{I}$. It follows that $\gamma_\lambda(\omega_1,x_\lambda,y)$ is also an extended rational fraction in λ. In particular, it is of bounded variation or, equivalently, satisfies (i). This property is clearly stable by taking the minimum *over finitely many* functions. Thus, ρ_λ satisfies (i) provided that the action set $\mathcal{J}$ of player 2 is finite.

Hence, the function ρ_λ satisfies the assumptions in Theorem 1, so that player 1 can guarantee $\lim_{\lambda\to 0}\rho_\lambda$ in Γ_∞. Yet, by the choice of $(x_\lambda)_{\lambda\in(0,1]}$, $\lim_{\lambda\to 0}\rho_\lambda = \lim_{\lambda\to 0} v_\lambda$. By symmetry, player 2 can also guarantee $\lim_{\lambda\to 0} v_\lambda$, which is then the uniform value.

(2) The existence of asymptotically optimal stationary strategies described by coefficient-exponent couples for each state and action was already obtained by Solan–Vieille [6], using the algebraic approach. Their proof goes as follows. Let $(x_\lambda)_{\lambda\in(0,1]}$ be a family of optimal stationary strategies which, by semi-algebraicity arguments, admit an expansion in Puiseux series near 0. Let $c_{\omega,i}$ and $e_{\omega,i}$ be the coefficient and exponent of the dominant term in the expansion of $x^i_\lambda(\omega)$ near 0, for $(\omega,i) \in \Omega\times\mathcal{I}$. Define a family of stationary strategies

[2] Φ is the Shapley operator, defined in (1).

as above, using (c, e), and conclude using a result from the theory of perturbed Markov chains; see [6, Theorem 2].

(3) The fact that in the linear system all the coefficients are in $\{-1, 0, 1\}$, allows us to be more precise.

Corollary 2 *Let ℓ be the vector of limits obtained from $\{(a_n^k)_n,\ k \in K\}$. There exists a natural number $N \leq |K|^{\sqrt{|K|}}$, coefficients $c_k > 0$ and exponents $e_k \in \{m/N,\ m \in \mathbb{N}\}$ $(k \in K)$ such that*

$$\lim_{\lambda \to 0} \frac{\prod_{k \in A} c_k \lambda^{e_k}}{\prod_{k \in B} c_k \lambda^{e_k}} = \ell(A, B), \quad \forall A, B \subset K.$$

Thus, there exist asymptotically optimal strategies $(x_\lambda)_\lambda$ for which $x_\lambda^i(\omega) \sim_{\lambda \to 0} e_{\omega,i} \lambda^{e_{\omega,i}}$, with $e_{\omega,i} \in \{m/N,\ m \in \mathbb{N}\}$, for all $(\omega, i) \in \Omega \times \mathcal{I}$, and for some integer $N \leq |\Omega||\mathcal{I}|^{\sqrt{|\Omega||\mathcal{I}|}}$.

(4) The finiteness of both action sets is necessary in order to obtain the convergence of the discounted values; see Ziliotto [8]. However, under some regularity conditions on g and q (such as semi-algebraicity or, more generally, definability with respect to some structure) the discounted values converge in stochastic games with finite action sets on one side; see [2].

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Almost All 5-Regular Graphs Have a 3-Flow

Paweł Prałat and Nick Wormald

Abstract Tutte conjectured in 1972 that every 4-edge connected graph has a nowhere-zero 3-flow. This is equivalent to every 5-regular 4-edge-connected graph having an edge orientation in which every out-degree is either 1 or 4. We show that this property holds asymptotically almost surely for random 5-regular graphs.

1 Introduction

A *nowhere-zero* 3*-flow* (sometimes simply called a 3-flow) in an undirected graph $G = (V, E)$ is an orientation of its edge set E together with a function f assigning a number $f(e) \in \{1, 2\}$ to every $e \in E$ such that the following is satisfied: for every vertex $v \in V$,

$$\sum_{e \in D^+(v)} f(e) - \sum_{e \in D^-(v)} f(e) = 0,$$

where $D^+(v)$ is the set of all edges oriented away from v, and $D^-(v)$ is the set of all edges oriented towards v.

A well known conjecture of Tutte from 1972 (see e.g., Bondy–Murty [4, Open Problem 48] and Jensen–Toft [6, Section 13.3]) asserts that every 4-edge-connected graph admits a nowhere-zero 3-flow. This conjecture is still open. For long, it was not even known whether or not there is a fixed k such that every k-edge connected graph has a nowhere-zero 3-flow (known as the weak 3-flow conjecture of Jaeger). Weaker versions, for $k \geq c \log_2 n$ and for n-vertex graphs, were proved by Alon–Linial–Meshulam [1] and Lai–Zhang [7]. Recently, the weak 3-flow conjecture was settled by Thomassen [11], who proved that every 8-edge-connected graph admits

P. Prałat (✉)
Department of Mathematics, Ryerson University, Toronto, ON, M5B 2K3, Canada
e-mail: pralat@ryerson.ca

N. Wormald
School of Mathematical Sciences, Monash University, Melbourne, VIC, 3800, Australia
e-mail: nicholas.wormald@monash.edu

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a nowhere-zero 3-flow. This was subsequently improved to $k = 6$ by Lovász–Thomassen–Wu–Zhang [8].

It is known (see, e.g., Seymour [10]) that a graph admits a nowhere-zero 3-flow if and only if it has a nowhere-zero flow over $\mathbb{Z}_3$, or equivalently, an edge orientation in which the difference between the out-degree and the in-degree of every vertex is divisible by 3. It has also long been known (see [4, 6]) that it is enough to prove the conjecture for 5-regular graphs. Thus, Tutte's conjecture has the following equivalent form.

Conjecture 1 (Tutte) Every 4-edge-connected 5-regular graph has an edge orientation in which every out-degree is either 1 or 4.

In this paper, we show that Tutte's conjecture holds for almost all 5-regular graphs. To state this precisely, we say that a property of a probability space indexed by n holds *a.a.s.* (asymptotically almost surely) if the probability that it holds tends to 1 as n tends to ∞ (with n restricted to being even for odd-degree regular graphs). Using the small subgraph conditioning method of Robinson–Wormald [9] (see [12]) we show the following.

Theorem 2 *A random 5-regular graph G_n on n vertices a.a.s. admits a nowhere-zero 3-flow, that is, an edge orientation in which every out-degree is either* 1 *or* 4.

Since it is well known that almost all 5-regular graphs are 5-edge-connected (see e.g., [12]), it follows that almost all 4-edge-connected 5-regular graphs have a nowhere-zero 3-flow.

Jaeger [5] generalised Conjecture 1 by conjecturing that for any integer $p \geq 1$, the edges of every $4p$ edge-connected graph can be oriented so that the difference between the out-degree and the in-degree of every vertex is divisible by $2p + 1$. Similar to Tutte's conjecture, it is known that the general case can be reduced to the $(4p + 1)$-regular case. Alon–Prałat [2] showed that the assertion of Jaeger's conjecture holds for almost all $(4p + 1)$-regular graphs, provided that p is large enough. The proof used methods quite different from the present paper.

2 Sketch of Proof of Theorem 2

The pairing model for investigating properties of random regular graphs was instigated by Bollobás [3]. This consists of dn points that are arranged in n groups (called vertices) of d each, arranged in pairs uniformly at random. The pairs induce a multigraph in the obvious way, and we refer to pairs as edges. This pairing model, called $\mathcal{P}_{n,d}$, is useful because simple graphs occur with equal probabilities, and the probability that it is simple for fixed d is bounded away from 0. Hence, to show that the random regular graph has a property a.a.s., it is enough to show that the random member of the multigraph corresponding to $\mathcal{P}_{n,d}$ a.a.s. has the same property or is non-simple. (See [12] for more information on this and other claims we make about $\mathcal{P}_{n,d}$.) We will work with orientations of (the pairs of) a pairing in $\mathcal{P}_{n,5}$ in which each

vertex has in-degree 1 or 4. We call such orientations *valid*. Given an orientation, vertices of in-degree 1 will be called *in-vertices*, and those of out-degree 1 *out-vertices*, and each point contained in an edge oriented towards an in-vertex, or away from an out-vertex, is called *special*. Moreover, a point is an *in-point* if the edge containing it is oriented towards it, and an *out-point* otherwise.

Let $Y = Y(n)$ be the number of valid orientations of a random element of $\mathcal{P}_{n,5}$. It is easy to see that

$$\mathbb{E}Y = \frac{\binom{n}{n/2}5^n(5n/2)!}{M(5n)},$$

where

$$M(s) = \frac{s!}{(s/2)!2^{s/2}}$$

is the number of perfect matchings of s points. Indeed, there are $\binom{n}{n/2}$ ways to select in-vertices (since exactly half of the vertices must be such), 5^n ways to select one special point in each vertex, which determines each point to be either in or out, $(5n/2)!$ ways to pair up the points so that each "in" is paired with an "out", and $M(5n)$ pairings in total. Using Stirling's formula $s! \sim \sqrt{2\pi s}(s/e)^s$, we get

$$\mathbb{E}Y = \frac{n!5^n(5n/2)!^2 2^{5n/2}}{(n/2)!^2(5n)!} \sim \left(\frac{25}{8}\right)^{n/2}\sqrt{5}. \tag{1}$$

Hence, on average, there are many valid orientations per pairing. To show that pairings a.a.s. have at least one valid orientation, i.e., that $\mathbb{P}(Y > 0) \sim 1$, a common method would be to estimate $\mathbb{E}Y(Y-1)$, show that it is asymptotic to $(\mathbb{E}Y)^2$, and then apply Chebyshev's inequality. Alas, this fails in the present case, but only just, as there is a constant factor discrepancy in the asymptotics. In such cases, we can try the small subgraph conditioning method [12]. Here, we again compute $\mathbb{E}Y(Y-1)$, but also some joint moments of Y with short cycle counts (and then hope for the best).

To estimate $\mathbb{E}Y(Y-1)$, consider any two orientations of the same 5-regular graph. Suppose that precisely k vertices are in-vertices in both orientations, and that, of these, precisely k_{11} have the same special point in both orientations. Since the first orientation induces $n/2$ in-vertices, exactly $n/2-k$ vertices are in-vertices in the first orientation but out-vertices in the second one. Of these, suppose that k_{10} ($k_{10} \le n/2-k$) have the two special points coinciding. Similarly, there are k vertices that are out-vertices in both orientations; suppose that k_{00} of them ($k_{00} \le k$) have the two special points coinciding. Finally, there are $n/2-k$ vertices that are out- in the first and in- in the second orientation; suppose that k_{01} of them ($k_{01} \le n/2-k$) have coinciding special points.

It turns out that there are no additional restrictions on these parameters, other than integrality and non-negativity. We define

$$I = I(n) = \left\{(k, k_{00}, k_{01}, k_{10}, k_{11}) \in \mathbb{N}_0^5 : k \le \frac{n}{2},\ \max\{k_{00}, k_{11}\} \le k,\ \max\{k_{01}, k_{10}\} \le \frac{n}{2} - k\right\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix $\mathbf{k} = (k, k_{00}, k_{01}, k_{10}, k_{11}) \in I$. By calculating the number of pairings with two given orientations corresponding to this vector, we arrive eventually at

$$\mathbb{E}Y(Y-1) = \sum_{\mathbf{k}\in I} r(\mathbf{z})g(\mathbf{z}) \exp\big(nf(\mathbf{z})\big), \tag{2}$$

where $\mathbf{z} = \mathbf{z}(\mathbf{k}) = \mathbf{k}/n$ and the various factors are defined as follows. The function r has the property that $r = O(1)$ for all $\mathbf{z}$, and $r \sim 1$ if $\mathbf{z}$ is bounded away from the boundary of

$$J := \left\{(z, z_{00}, z_{01}, z_{10}, z_{11}) \in \mathbb{R}_0^5 : z \le \frac{1}{2},\ \max\{z_{00}, z_{11}\} \le z,\ \max\{z_{01}, z_{10}\} \le \frac{1}{2} - z\right\},$$

where $\mathbb{R}_0$ is the set of non-negative reals. With $b = z + 1 + z_{00} - z_{01} - z_{10} + z_{11}$ and $h(x) = x \log x$,

$$g = \frac{1}{\sqrt{32}(\pi n)^{5/2}} \left(\frac{b(5-2b)}{z_{00}z_{01}z_{10}z_{11}(z-z_{00})(z-z_{11})(1-2z-2z_{10})(1-2z-2z_{01})}\right)^{1/2}$$

from the polynomial factors in Stirling's formula, and

$$\begin{aligned} f = {} & \big(9/4 - z_{00} - z_{01} - z_{10} - z_{11}\big)\log 4 + \log 5 - h(5) + h(5/2) + h(b) + h(5/2-b) \\ & - h(z_{00}) - h(z_{01}) - h(z_{10}) - h(z_{11}) - h(z - z_{00}) - h(z - z_{11}) - h(1/2 - z - z_{01}) \\ & - h(1/2 - z - z_{10}) \end{aligned}$$

from the rest.

Standard calculations show that $\tilde{\mathbf{z}} = (1/4, 1/20, 1/20, 1/20, 1/20)$ is the unique global maximum point of f on J and, eventually, (details omitted) $\mathbb{E}Y(Y-1) \sim \frac{25}{\sqrt{21}}(25/8)^n$. Combining this with (1), we get

$$\frac{\mathbb{E}\,Y^2}{(\mathbb{E}Y)^2} \sim \frac{5}{\sqrt{21}}. \tag{3}$$

The second main computation in the small subgraph conditioning method involves the random variable X_k ($k \ge 1$), which we define to be the number of cycles of length k in $\mathcal{P}_{n,5}$. It is known that for each $k \ge 1$, $X_1, X_2, \ldots, X_k$ are asymptotically

independent Poisson random variables with $\mathbb{E}X_k \to \lambda_k := 4^k/(2k)$. Next, let $k \geq 1$ and fix $j_1, \ldots, j_k$. After some careful counting (details omitted) we obtain

$$\frac{\mathbb{E}(Y[X_1]_{j_1} \cdots [X_k]_{j_k})}{\mathbb{E}Y} \to \prod_{i=1}^{k} \mu_i^{j_i}, \quad \text{where } \mu_k = \frac{1}{2k}\left(4^k + (-4/5)^k\right).$$

The final step of small subgraph conditioning requires us to compute

$$\delta_k = \frac{\mu_k}{\lambda_k} - 1 = \left(-\frac{1}{5}\right)^k$$

and then, using $-\log(1-x) = \sum_{k\geq 1} x^k/k$,

$$\exp\left(\sum_{k\geq 1} \lambda_k \delta_k^2\right) = \exp\left(\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\left(\frac{4}{25}\right)^k\right) = \exp\left(-\frac{1}{2}\log\left(1-\frac{4}{25}\right)\right) = \frac{5}{\sqrt{21}}.$$

The fact that this is coincides with the right hand side of (3) implies, by [12, Theorem 4.1], that $\mathbb{P}(Y > 0) \sim 1$. That is, a random multigraph in $\mathcal{P}_{n,d}$ a.a.s. has an edge orientation of the required type. Theorem 2 now follows, in view of the comments at the start of this section and in the introduction.

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Part II
Quantitative Finance

Foreword

The *Interdisciplinary Workshop on Quantitative Finance* was held from June 25th to June 26th 2015, at the Centre de Recerca Matemàtica (CRM) in Bellaterra, Barcelona, Spain. It was coordinated by Elisa Alòs (Universitat Pompeu Fabra), Joan del Castillo (Universitat Autònoma de Barcelona), José Manuel Corcuera (Universitat de Barcelona), Luis Ortiz Gracia (Centre de Recerca Matemàtica), and Josep Vives (Universitat de Barcelona).

The aim of the organizers was to bring together academic researchers and professionals from financial institutions interested in quantitative finance, to interact and discuss among themselves. The focus was on the underlying mathematical models as well as in advanced numerical solution techniques used for pricing financial contracts and risk measurement. The tools and techniques to tackle these problems covered a wide spectrum of fields, like stochastic analysis, numerical analysis, partial differential equations, statistics and econometrics.

The workshop attracted a diverse audience of more than forty participants with background in Mathematics, Physics and Economics, both from the academic and the professional sides, and all of them affiliated in Spain and Europe. The invited plenary lecture was delivered by Jan de Spiegeleer from Jabre Capital Partners (Geneve) and KU Leuven (Belgium). The workshop also included fifteen contributed talks given by academics and PhD students.

The organizers would like to thank all the participants and the authors, who contributed to the success of the workshop. We would also thank La Caixa Foundation for the financial support provided through the programme *Research on Collaborative Mathematics*, and the CRM's administrative staff for the organization of this event.

Barcelona, Spain
September 2015

Luis Ortiz-Gracia

With the support of:

On the Short-Time Behaviour of the Implied Volatility Skew for Spread Options and Applications

Elisa Alòs and Jorge A. León

Abstract By means of Malliavin calculus, we construct a modification of the classical Kirk's formula for spread option prices. This new approximation is easy to compute and increases the accuracy of Kirk's formula, specially when the correlation parameter is near to one.

A spread option is a derivative written as the difference of two underlying assets. Namely, the payoff of a call spread option with strike K with time to maturity T is

$$(S_T^1 - S_T^2 - K)_+,$$

where S^1 and S^2 denote two asset prices. It is well-known that if S^1 and S^2 are two geometric Brownian motions (that may be correlated) and $K = 0$, the corresponding option price is given by the Margrabe's fomula (see Margrabe [10]), which can be deduced from the fact that S^1/S^2 is a log-normal process. Thus, in this case, the spread option value can be expressed as the classical Black–Scholes call price with initial asset price S_0^1, where we take the strike equal to the expected value of S_T^2 and volatility equal to $\sqrt{\sigma^2 - 2\rho\sigma\sigma' + (\sigma')^2}$. Here, σ and σ' are the volatility parameters of S^1 and S^2, respectively, and ρ denotes the correlation.

In the case $K \neq 0$, the fraction $S^1/\left(S^2 + K\right)$ is not log-normal and then the arguments used in the deduction of Margrabe formula cannot be applied anymore. One proposed solution is to assume the distribution of the spread is able to be approximated by the normal distribution. This leads us to the Bachelier's method; see Shimko [12]. Even though the approximation of the spread distribution by the normal one is poor, this method can be precise in some ranges of parameters;

Elisa Alòs (✉)
Dpt. d'Economia i Empresa and Barcelona Graduate School of Economics, Universitat Pompeu Fabra, c/Ramon Trias Fargas, 25-27, Barcelona, Spain
e-mail: elisa.alos@upf.edu

Jorge A. León
Control Automático, CINVESTAV-IPN, Apartado Postal 14-740, 07000, México, D.F., Mexico
e-mail: jleon@ctrl.cinvestav.mx

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see, for example, Carmona–Durrleman [7]. A more successful method is suggested by Kirk [9], who applied the Margrabe formula, approaching $S^1/\left(S^2+K\right)$ by a log-normal random process. Nowadays, Kirk's formula is the most popular option pricing approximation expression for spread options due to its accuracy and its simplicity. Recently, some other authors have proposed new formulas and methods to estimate spread option prices. Among them, we can mention Alexander–Venkatramanan [1], Bjerksund–Stensland [5], Borovkova–Permana–Weide [6], Carmona–Durrleman [7], and Deng–Li–Zhou [8]. Numerical evidence has shown that these approaches may improve Kirk's estimates, specially for high correlated cases; see, for instance, Bjerksund–Stensland [5].

An interesting tool in the development of an accurate approximation formula is the knowledge of the properties of the corresponding implied volatility. For example, in the case of vanilla options with stochastic volatility, we know that implied volatilities exhibits smiles and skews; see Renault–Touzi [11]. Then, we have to take into account that accurate approximations should reproduce them and, moreover, we can understand why some formulas fail (for example, we expect that a constant volatility expression will fail for in or out-of-the-money options).

Up to our knowledge, there are not similar analytical studies in the case of options with random strikes (as, for example, spread options). The main goal of the work by Alòs–León [3] is to consider analytically the relation between the implied volatility and the stock price S^1 for options with random strike. By means of the Malliavin calculus techniques, we develop an extension of the Margrabe formula allowing us to find an expression for the short-time skew slope for options with random strikes (and, in particular, for spread options). This analysis of the implied volatility skew evidences that the dependence between the implied volatility and the stock price S^1 increases strongly when the correlation parameter ρ is close to 1. Hence, we can expect that Kirk's approximation reduces its accuracy for highly correlated assets due to the fact that in this approximation the volatility is constant as a function of the stock price S^1. This agrees with the numerical evidence; see, for example, Baeva [4].

Our results alow us to construct a modification of the classical Kirk's formula that is able to approximate spread short-time prices when the correlation ρ is near to 1. More precisely, we propose to modify the classical Kirk's volatility (we assume the interest rate equal to zero for the sake of simplicity):

$$a(t) := \sqrt{\sigma^2 - 2\rho\sigma\sigma'\frac{S_t^2}{S_t^2+K} + \left(\sigma'\frac{S_t^2}{S_t^2+K}\right)^2}$$

by adding the corresponding first-order term in its Taylor expansion as a function of S^1. This first-order approximation can be easily computed from our obtained results

on the short-time skew. More precisely, the improvement proposed in Alòs–León [3] is to substitute $a(t)$ in Kirk's formula by

$$\sqrt{a_t^2} + \frac{1}{2}\left(\sigma'\left(\frac{S_t^2}{S_t^2+K}\right) - \rho\sigma\right)^2 \frac{1}{\left(\sqrt{a_t^2}\right)^3}\left(\sigma'\right)^2 \frac{S_t^2 K}{S_t^2+K}\left(\ln S_t^1 - \ln S_t^2 - K\right),$$

where we assume again a zero interest rate. The numerical experiments show that this approximation clearly increases the accuracy of the classical Kirk's formula, specially when the correlation parameter ρ is near to one.

The above techniques can be applied to other random strike options, as three-assets spread options. In particular, a similar improvement can be applied to the approximation formulas introduced in Alòs–Eydeland–Laurence [2]. In this case, our numerical experiments confirm again the efficiency of the presented method.

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An Alternative to CARMA Models via Iterations of Ornstein–Uhlenbeck Processes

Argimiro Arratia, Alejandra Cabaña, and Enrique M. Cabaña

Abstract We present a new construction of continuous ARMA processes based on iterating an Ornstein–Uhlenbeck operator $\mathcal{OU}_\kappa$ that maps a random variable $y(t)$ onto $\mathcal{OU}_\kappa y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} dy(s)$. This construction resembles the procedure to build an AR(p) from an AR(1) and derives in a parsimonious model for continuous autoregression, with fewer parameters to compute than the known CARMA obtained as a solution of a system of stochastic differential equations. We show properties of this operator, give state space representation of the iterated Ornstein–Uhlenbeck process and show how to estimate the parameters of the model.

1 Introduction

The link between discrete ARMA processes and stationary processes with continuous time has been of interest for many years; see, e.g., [3, 6, 7]. Also, there is a recent upsurge of interest in continuous time models, because they can be used in presence of irregularly spaced data, and in non Gaussian processes mainly due to the fact that jumps play an important role in realistic modeling in finance and other fields of applications. One approach is via the stochastic volatility model from [2], in which the volatility process V and the log asset price G satisfy:

$$dV(t) = -\lambda V(t) + d\Lambda(t) \quad \text{and} \quad dG(t) = (\gamma + \beta V(t))dt + \sqrt{V(t)}dW(t) + \rho d\Lambda(t),$$

A. Arratia (✉)
Universitat Politècnica de Catalunya, Barcelona, Spain
BGSMath, Barcelona, Spain
e-mail: argimiro@cs.upc.edu

A. Cabaña
Universitat Autònoma de Barcelona, Bellaterra, Spain
BGSMath, Barcelona, Spain
e-mail: acabana@mat.uab.cat

E.M. Cabaña
Universidad de la República, Montevideo, Uruguay
e-mail: ecabana@ccee.edu.uy

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where $\lambda > 0$, Λ is a non-decreasing Lévy process, and W is a standard Brownian motion independent of Λ. The volatility V is a Lévy driven Ornstein–Uhlenbeck (or OU) process, or a continuous time autoregression of order 1. The autocorrelations of V decay exponentially, hence they constitute a very restrictive family.

In order to include a wider family of covariances, econometric or physical models apply frequently linear combinations (superpositions) of OU processes driven by either uncorrelated or correlated noise $\sum_{j=1}^{p} a_j \int_{-\infty}^{t} \mathrm{e}^{-\kappa_j(t-s)} d\Lambda_j(s)$ (see, e.g., [8]); or models that replace the finite linear combination by a continuous version

$$\int_{s=-\infty}^{t} \int_{\Re(\kappa)>0} \mathrm{e}^{-\kappa(t-s)} d\Lambda(s,\kappa);$$

see [3–5]. In particular, Brockwell [4, 5] proposes to define CARMA processes via a state space representation of the formal equation $a(D)Y(t) = \sigma b(D)D\Lambda(t)$, where $\sigma > 0$ is a scale parameter, D denotes differentiation with respect to t, Λ is a second-order Lévy process, a is a polynomial of order p, and b is a polynomial of order $q \leq p-1$ with coefficient $b_q = 1$. When the zeroes of the AR polynomial are all different, he obtains a representation of the CARMA as a sum of Lévy driven Ornstein–Uhlenbeck processes. Brockwell estimates the CARMA parameters by adjusting an ARMA(p,q), $q < p$, to regularly spaced data and then obtain the parameters of the CARMA whose values at the observation times have the same distribution of the fitted ARMA.

We propose in [1] a parsimonious model for continuous autoregression, with fewer parameters, able to adjust slowly decaying covariances. The model is obtained by a procedure that resembles the one that allows to build an AR(p) from an AR(1), that we summarize in the sequel.

2 Iterated Lévy Driven Ornstein–Uhlenbeck Processes

The AR(p) process $X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sigma\epsilon_t$ or $\phi(B)X_t = \sigma\epsilon_t$, where $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j = \prod_{j=1}^{p}(1 - z/\rho_j)$ has roots $\rho_j = \mathrm{e}^{\kappa_j}$, is obtained by applying the composition of the moving averages $\mathcal{MA}(1/\rho_j)$ to the noise, that is, $X_t = \sigma \prod_{j=1}^{p} \mathcal{MA}(1/\rho_j)\epsilon_t$, where $\mathcal{MA}(1/\rho)$ is the moving average that maps ϵ_t onto $\mathcal{MA}(1/\rho)\epsilon_t = \sum_{j=0}^{\infty} \frac{1}{\rho^j}\epsilon_{t-j}$.

Let us denote $\mathcal{MA}_\kappa = \mathcal{MA}(\mathrm{e}^{-\kappa})$. A continuous version of the operator $\mathcal{MA}_\kappa$ mapping ϵ_t onto $\mathcal{MA}_\kappa\epsilon_t = \sum_{l\leq t,\text{integer}} \mathrm{e}^{-\kappa(t-l)}\epsilon_l$ is $\mathcal{OU}_\kappa$ that maps $y(t)$ onto $\mathcal{OU}_\kappa y(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa(t-s)} dy(s)$ and this suggests the use of the model OU(p) with a second order Lévy process Λ,

$$x_{\boldsymbol{\kappa},\sigma}(t) = \sigma \prod_{j=1}^{p} \mathcal{OU}_{\kappa_j} \Lambda(t) \quad \text{with parameters} \quad \boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_p), \sigma. \tag{1}$$

Theorem 1 (OU(p) as a superposition of OU(1)) *The* Ornstein–Uhlenbeck process (1) *can be written as a linear combination of p processes of order 1:*

(*i*) *when the components of* $\boldsymbol{\kappa}$ *are pairwise different, the linear combination is* $x_{\boldsymbol{\kappa},\sigma} = \sum_{j=1}^{p} K_j(\boldsymbol{\kappa})\xi_{\kappa_j}$, *where* $\xi_{\kappa_j}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa_j(t-s)} d(\sigma\Lambda(s))$, *and coefficients are* $K_j(\boldsymbol{\kappa}) = 1/\prod_{\kappa_l \neq \kappa_j}(1-\kappa_l/\kappa_j)$;

(*ii*) *when* $\boldsymbol{\kappa}$ *has components* κ_h *repeated* p_h *times* ($h = 1, 2, \ldots, q$, $\sum_{h=1}^{q} p_h = p$) *the linear combination is* $x_{\boldsymbol{\kappa},\sigma} = \sum_{h=1}^{q} K_h(\boldsymbol{\kappa}) \sum_{j=0}^{p_h-1} \binom{p_h-1}{j} \xi_{\kappa_h}^{(j)}$, *where* $\xi_{\kappa_h}^{(j)}(t) = \int_{-\infty}^{t} \mathrm{e}^{-\kappa_h(t-s)} \frac{(-\kappa_h(t-s))^j}{j!} d(\sigma\Lambda(s))$.

The autocovariances of $x_{\boldsymbol{\kappa},\sigma}$ *are*

$$\gamma_{\kappa,\sigma}(t) = \sum_{h'=1}^{q} \sum_{i'=0}^{p_{h'}-1} \sum_{h''=1}^{q} \sum_{i''=0}^{p_{h''}-1} K_{h'}(\boldsymbol{\kappa}) \bar{K}_{h''}(\boldsymbol{\kappa}) \binom{p_{h'}-1}{i'} \binom{p_{h''}-1}{i''} \gamma_{\kappa_{h'},\kappa_{h''},\sigma}^{(i',i'')}(t).$$

3 A State Space Representation of the OU(p) Process

The decomposition of the OU(p) process $x_{\boldsymbol{\kappa},\sigma}(t)$ as a linear combination of simpler processes of order 1, leads to an expression of the process by means of a state space model. State space modelling provides us with a unified approach for computing the likelihood of $x_{\boldsymbol{\kappa},\sigma}(t)$ through a Kalman filter, and with a tool to show that the covariances of $x_{\boldsymbol{\kappa},\sigma}(t)$ coincide with those of an ARMA($p, p-1$) whose coefficients can be computed from $\boldsymbol{\kappa}$.

In the sequel, in order to ease notation, we consider that the components of $\boldsymbol{\kappa}$ are all different. The decomposition of $x_{\boldsymbol{\kappa},\sigma}(t) = \sum_{j=1}^{p} K_j \xi_{\kappa_j}(t)$ as a linear combination of the OU(1) processes, given by Theorem 1, with innovations $\boldsymbol{\eta}_{\boldsymbol{\kappa}}$ with components $\eta_{\kappa_j}(t) = \int_{t-1}^{t} \mathrm{e}^{-\kappa_j(t-s)} d\Lambda(s)$, provides a representation of the OU(p) process in the space of states $\xi_{\boldsymbol{\kappa}} = (\xi_{\kappa_1}, \ldots, \xi_{\kappa_p})^{\mathrm{tr}}$. The transitions in the state space are

$$\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t) = \mathrm{diag}(\mathrm{e}^{-\kappa_1}, \ldots, \mathrm{e}^{-\kappa_p})\boldsymbol{\xi}_{\boldsymbol{\kappa}}(t-1) + \boldsymbol{\eta}_{\boldsymbol{\kappa}}(t)$$

and

$$\boldsymbol{x}(t) = \boldsymbol{K}^{tr}(\boldsymbol{\kappa})\boldsymbol{\xi}(t).$$

The assumption $\mathbf{E}\Lambda(1)^2 = 1$ implies that the innovations have variance $\mathbf{Var}(\boldsymbol{\eta}_{\boldsymbol{\kappa},\tau}(t)) = ((v_{j,l}))$, where $v_{j,l} = \mathbf{E}\int_{t-1}^{t} \mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)(t-s)} ds = (1-\mathrm{e}^{-(\kappa_j+\bar{\kappa}_l)})/(\kappa_j+\bar{\kappa}_l)$.

Now apply the AR operator $\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)$ to $x_{\boldsymbol{\kappa}}$ and obtain

$$\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)x_{\boldsymbol{\kappa}}(t) = \sum_{j=1}^{p} K_j G_j(B)\eta_{\kappa_j}(t) =: \zeta(t),$$

with $G_j(z) = \prod_{l\neq j}(1-\mathrm{e}^{-\kappa_l}z) := 1 - \sum_{l=1}^{p-1} g_{j,l} z^l$.

This process has the same second-order moments as the ARMA$(p, p-1)$. When Λ is a Wiener process, it is in fact an ARMA$(p, p-1)$,

$$\prod_{j=1}^{p}(1-\mathrm{e}^{-\kappa_j}B)x_{\boldsymbol{\kappa}}(t) = \sum_{j=0}^{p-1}\theta_j\epsilon(t-j) =: \zeta'(t),$$

(ϵ is a white noise) when the covariances $c_j = \mathbf{E}\zeta(t)\bar{\zeta}(t-j)$ and $c'_j = \mathbf{E}\zeta'(t)\bar{\zeta}'(t-j)$ coincide. The covariances c'_j and c_j are given respectively by the generating functions

$$\left(\sum_{h=0}^{p-1}\theta_h z^h\right)\left(\sum_{k=0}^{p-1}\bar{\theta}_k z^{-h}\right) = \sum_{l=-p+1}^{p-1} c'_l z^l$$

and

$$J(z) := \sum_{j=1}^{p}\sum_{l=1}^{p} K_j\bar{K}_l G_j(z)\bar{G}_l(1/z)v_{j,l} = \sum_{l=-p+1}^{p-1} c_l z^l.$$

Since $J(z)$ can be computed once $\boldsymbol{\kappa}$ is known, the coefficients $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_{p-1})$ are obtained by identifying the coefficients of the polynomials $z^{p-1}(\sum_{h=0}^{p-1}\theta_h z^h)(\sum_{k=0}^{p-1}\bar{\theta}_k z^{-h})$ and $z^{p-1}J(z)$.

A state space representation and its implications on the covariances of the OU process in the general case are slightly more complicated.

4 Estimation

Though $\gamma(t)$ depends continuously on $\boldsymbol{\kappa}$, the same does not happen with each term in the expression for the covariance, because of the lack of boundedness of the coefficients of the linear combination when two different values of the components of $\boldsymbol{\kappa}$ approach each other. Since we wish to consider real processes x and the process itself and its covariance $\gamma(t)$ depend only on the unordered set of the components of $\boldsymbol{\kappa}$, we shall reparameterize the process. With the notation

$$K_{j,i} = \frac{1}{(-\kappa_j)^i\prod_{l\neq j}(1-\kappa_l/\kappa_j)},$$

(in particular, $K_{j,0}$ is the same as K_j), the processes $x_i(t) = \sum_{j=1}^{p} K_{j,i}\xi_j(t)$ and the coefficients $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$ of the polynomial $g(z) = \prod_{j=1}^{p}(1+\kappa_j z) = 1 - \sum_{j=1}^{p}\phi_j z^j$ satisfy $\sum_{i=1}^{p}\phi_i x_i(t) = x(t)$. Therefore, the new parameter $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p) \in R^p$ is adopted.

4.1 The Gaussian Case

From the observations $\{\mu + x(i) \,:\, i = 0, 1, \ldots, n\}$, we obtain the likelihood L of the vector $x = (x(1)), \ldots, x(n))$,

$$\log L(\boldsymbol{x}; \boldsymbol{\phi}, \sigma) = -\tfrac{n}{2}\log(2\pi) - \tfrac{1}{2}\log(\det(V(\boldsymbol{\phi}, \sigma)) - \tfrac{1}{2}\boldsymbol{x}^{\mathrm{tr}}(V(\boldsymbol{\phi}, \sigma))^{-1}\boldsymbol{x},$$

with $V(\boldsymbol{\phi}, \sigma)$ equal to the $n \times n$ matrix with components $V_{h,i} = \gamma(|h-i|)$, that reduce to $\gamma(0)$ at the diagonal, $\gamma(1)$ at the first sub- and super-diagonals, and so on. Obtain via numerical optimisation the MLE $\hat{\boldsymbol{\phi}}$ of $\boldsymbol{\phi}$ and $\hat{\sigma}^2$ of σ^2. The estimations $\hat{\boldsymbol{\kappa}}$ follow by solving $\prod_{j=1}^{p}(1 + \hat{\kappa}_j z) = 1 - \sum_{j=1}^{p} \hat{\phi}_j z^j$.

The parameters $\boldsymbol{\kappa}, \sigma$ determine the Gaussian likelihood of $\mathcal{OU}_{\boldsymbol{\kappa},\sigma w}$, and are estimated by the values $\hat{\boldsymbol{\kappa}}$ and $\hat{\sigma}$ that maximize that likelihood.

4.2 A Non Gaussian Example

Let us assume that the noise is given by $\Lambda(t) = \sigma w(t) + c(N(t) - \lambda t)$, where w is a standard Wiener process and N is a Poisson process with intensity λ. The family of possible noises depends on the three parameters (σ, λ, c). In this case, the characteristic exponent has a simple form,

$$\psi_{\Lambda(1)}(iu) = -\frac{\sigma^2 u^2}{2} + \lambda(\mathrm{e}^{iuc} - iuc - 1)$$

hence,

$$\psi_{\boldsymbol{\eta}}(iu) = \int_0^1 \left(-\frac{\sigma^2 u^2 g^2(s)}{2} + \lambda(\mathrm{e}^{iug(s)c} - iug(s)c - 1)\right) ds.$$

With $g_h = \int_0^1 g^h(s)ds$,

$$\psi_{\boldsymbol{\eta}}(iu) = -\frac{\sigma^2 u^2 g_2}{2} + \lambda\left(-\frac{u^2 g_2 c^2}{2} - i\frac{u^3 g_3 c^3}{6} + \frac{u^4 g_4 c^4}{24} + \cdots\right).$$

Then we propose to estimate the parameters by equating the coefficients of u^2, u^3, u^4 in $\psi_{\boldsymbol{\eta}}(iu)$ with the corresponding ones in the logarithm of the empirical characteristic function of the residuals. Assuming that the mean of the residuals $r_1, r_2, \ldots, r_n$ is zero, their empirical characteristic function is

$$\frac{1}{n}\sum_{h=1}^{n} \mathrm{e}^{iur_h} = 1 - \frac{1}{2}u^2 R_2 - \frac{1}{6}iu^3 R_3 + \frac{1}{24}u^4 R_4 + \cdots,$$

where $R_m = (\sum_{h=1}^{n} r_h^m)/n$. Then, the logarithm has the expansion

$$\log \frac{1}{n} \sum_{h=1}^{n} \mathrm{e}^{iur_h} = -\frac{1}{2}u^2 R_2 - \frac{1}{6} iu^3 R_3 + \frac{1}{24} u^4 R_4 - \frac{1}{8} u^4 R_2^2 + \cdots .$$

Consequently, the estimation equations are $(\sigma^2 + \lambda c^2) g_2 = R_2$, $\lambda c^3 g_3 = R_3$, and $\lambda c^4 g_4 = R_4 - 3R_2^2$, from which the estimators follow:

$$\tilde{c} = \frac{R_4 - 3R_2^2}{R_3} \frac{g_3}{g_4}, \qquad \tilde{\lambda} = \frac{R_3^4}{(R_4 - 3R_2^2)^3} \frac{g_4^3}{g_3^4}, \qquad \tilde{\sigma}^2 = \frac{R^2}{g_2} - \frac{R_3^2}{(R_4 - 3R_2^2)} \frac{g_4}{g_3^2}.$$

5 Conclusions

We have proposed a family of continuous time stationary processes, based on p iterations of the linear operator that maps a Lévy process onto an Ornstein–Uhlenbeck process. These operators have some nice properties, such as being commutative, and their p-compositions decompose as a linear combination of simple operators of the same kind. An OU(p) process depends on $p + 1$ parameters that can be easily estimated by either maximum likelihood (ML) or by matching correlations procedures. Matching correlation estimators provide a fair estimation of the covariances of the data, even if the model is not well specified. When sampled on equally spaced instants, the OU(p) family can be written as a discrete time state space model; i.e., a VARMA model in a space of dimension p. As a consequence, the families of OU(p) models are a parsimonious subfamily of the ARMA($p, p-1$) processes in the Gaussian case. Furthermore, the coefficients of the ARMA can be deduced from those of the corresponding OU(p). We have found time series data for which the ML-estimated OU model is able to capture a long term dependence that the ML-estimated ARMA model does not show; see [1]. This leads to recommend the inclusion of OU models as candidates to represent stationary series to the users interested in such kind of dependence.

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Euler–Poisson Schemes for Lévy Processes

Albert Ferreiro-Castilla

Abstract In this note we will contextualize the recently established Wiener–Hopf Monte Carlo (WHMC) simulation technique for Lévy processes from Kuznetsov et al. (Ann Appl Probab 21(6):2171–2190, 2011), into a more general framework allowing us to use the same technique in a larger set of problems. We will briefly show how the scheme can be used to approximate Lévy driven SDEs or how to approximate different types of path dependent quantities. In a way, the present note summarizes and connects a set of results contained in Ferreiro-Castilla et al. (Stochastic Process Appl 124(2):985–1010, 2014; J Appl Probab 52(1):129–148, 2015; J Appl Probab 53(1):262–278, 2016); therefore we intentionally leave most of the technicalities aside for this note.

1 Introduction

Let $X := (X_t)_{t\geq 0}$ be a Lévy process, i.e. a (real valued) stochastic process starting from 0 with cadlag paths (right continuous and with left limits) and stationary, independent increments. The Lévy–Khintchine formula entails that the characteristic exponent Ψ, defined as $\mathbb{E}[e^{\mathrm{i}zX_t}] = e^{-t\Psi(z)}$ for all $t \geq 0$ and $z \in \mathbb{R}$, can be expressed as

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + \mathrm{i}az + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{\mathrm{i}zx} + \mathbf{1}_{\{|x|<1\}}\mathrm{i}zx)\Pi(\mathrm{d}x)\,, \tag{1}$$

where $\sigma, a \in \mathbb{R}$ and Π is a measure on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$.

Of interest in several fields are quantities depending on the path of X, the complete path of X is numerically intractable and, ultimately, any numerical scheme can only be based on simulating the increments of the driving process. A first approach consists in computing an Euler skeleton of the path; but even for a Brownian motion, where the Euler scheme is exact, computing a simple supremum in this way leads to

A. Ferreiro-Castilla (✉)
Banc Sabadell, Sabadell, Spain
e-mail: aferreiro.c@gmail.com

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a significant bias (see Broadie–Glasserman–Kou [1]) due to the fact that the skeleton misses the excursions of the process. In this note we are interested in constructing a skeleton for the path based on the Wiener–Hopf factorization. Suppose that, for any $q>0$, $\mathbf{e}(q)$ is an exponentially distributed random variable with mean q^{-1} that is independent from X. Recall that $\overline{X}_t := \sup_{s\leq t} X_s$ and let $\underline{X}_t := \inf_{s\leq t} X_s$. The Wiener–Hopf factorization states that the random variables $\overline{X}_{\mathbf{e}(q)}$ and $\overline{X}_{\mathbf{e}(q)} - X_{\mathbf{e}(q)}$ are independent. Thanks to the so-called principle of duality, that is to say the equality in law of the pair $\{X_{(t-s)-} - X_t : 0 \leq s \leq t\}$ and $\{-X_s : 0 \leq s \leq t\}$, it follows that $\overline{X}_{\mathbf{e}(q)} - X_{\mathbf{e}(q)}$ is equal in distribution to $-\underline{X}_{\mathbf{e}(q)}$. This leads to the following factorization of characteristic functions

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_{\mathbf{e}(q)}}) = \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \overline{X}_{\mathbf{e}(q)}}) \times \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta \underline{X}_{\mathbf{e}(q)}}),$$

for all $\theta \in \mathbb{R}$, known as the Wiener–Hopf factorization. Equivalently,

$$X_{\mathbf{e}(q)} \stackrel{d}{=} S_q + I_q,$$

where S_q and I_q are independent and equal in distribution to $\overline{X}_{\mathbf{e}(q)}$ and $\underline{X}_{\mathbf{e}(q)}$, respectively. Here, we use the notation $\stackrel{d}{=}$ to mean equality in distribution.

The above factorization captures in one time step the nature of the path; i.e. knowing the supremum and the infimum one would be able to answer questions relating whether the process crossed a barrier or whether there was a big jump in the time step. Obviously, the time step is itself a random variable but, thanks to the equality

$$\sum_{i=1}^{n} \frac{1}{n}\mathbf{e}_i(1) \stackrel{a.s.}{\to} 1 \text{ as } n \uparrow \infty,$$

we can create a random partition of the interval $[0, 1]$ with a mesh size $\mathbb{E}[\frac{1}{n}\mathbf{e}_i(1)]\to 0$ as $n \uparrow \infty$. We call our method the Euler–Poisson scheme, as one can think of the random partition as given by the arrival times of a Poisson process.

For the sake of simplicity, recall that $\sum_{i=1}^{k} \frac{1}{n}\mathbf{e}_i(1)$ has the same distribution as a Gamma random variable which will be denoted by $g(k, n)$. Therefore the Euler–Poisson scheme of the process X refers to the skeleton $\{X_{\mathbf{e}_k(1/n)}\}_{k\geq 0}$ or $\{X_{g(k,n)}\}_{k\geq 0}$.

1.1 Heuristics Behind the Scheme

Let us give a brief discussion of the heuristics of the Euler–Poisson scheme and an idea why this scheme should be a particular good approach to compute path dependent quantities of Lévy processes.

It can be inferred from Doney [3] that, for all $k > 0$, the random variables

$$M_k := \sup_{g(k,n)\le t<g(k+1,n)} X_t \qquad \text{and} \qquad m_k := \inf_{g(k,n)\le t<g(k+1,n)} X_t$$

can be written as

$$M_k = S_0^{(n)} + Y_k^{(+)} \qquad \text{and} \qquad m_k = I_0^{(n)} + Y_k^{(-)},$$

respectively, where $\{Y_k^{(+)}\}_{k\ge 0}$ and $\{Y_k^{(-)}\}_{k\ge 0}$ are random walks with the same distribution as $\{X_{g(k,n)}\}_{k\ge 0}$, and independent from $S_0^{(n)}$ and $I_0^{(n)}$, respectively. Since it is clear that $m_k \le X_t \le M_k$ for $g(k,n) \le t < g(k+1,n)$, the derivations in Doney [3] assert that it is possible to 'stochastically' bound the path of X from above and below by two random walks which are equal in distribution to the skeleton proposed in this paper, but with different random starting points.

There is yet another heuristic justification to support the skeleton $\{X_{g(k,n/t)}\}_{k\ge 0}$ as a good random walk approximation of the Lévy process to compute pathwise quantities. The Feynman–Kac representation identifies conditional expectations of functionals of the solution of a Stochastic Differential Equation (SDE) as solutions of a certain Partial Integro Differential Equation (PIDE). We claim that, in some sense, the solution of a Lévy driven SDE sampled over a random grid generated by the arrival times of a Poisson process is equivalent to performing a discretization in time by the method of lines to the associated Feynman–Kac equation. We are not the first to point out this relationship. It was the basis of Carr [2], where an approximation for American options of finite maturity is obtained by randomizing the time horizon by an Erlang distribution. Matache–Nitsche–Schwab [9] also pointed out, informally, the relation between a deterministic discretization in time of a Feynman–Kac PIDE and its probabilistic counterpart. Details on this relationship can be found in Ferreiro-Castilla–Kyprianou–Scheichl [6].

1.2 Feasibility of the Scheme

It is clear from the preceding section that the Euler–Poisson method is of practical interest only if samples from the distribution of $X_{\mathbf{e}(q)}$ are available. In general, there is no reason why the latter distribution is easier to handle than the distribution of X_1 itself, needed to set up a plain Euler scheme for a Lévy process. Nevertheless, recent developments in Wiener–Hopf theory for one dimensional Lévy processes have provided a rich enough variety of examples for which the necessary distributional sampling can be performed and thus the Euler–Poisson scheme may lead to simpler numerical techniques. This family of processes are called *meromorphic Lévy processes*; see Kuznetsov et al. [7, 8]. For the class of meromorphic Lévy processes, the Wiener–Hopf factors are explicit and hence we can efficiently sample from the distribution of $X_{\mathbf{e}(q)}$ through its factorization. One of the main advantages

of the Euler–Poisson algorithm is its robustness with respect to the jump structure; see, for example, Ferreiro-Castilla–Kyprianou–Scheichl–Suryanarayana [4]. Recall that many numerical algorithms to approximate the path of a Lévy process depend on the structure of the jump component of (1).

2 Applications

Let us now give a taste of what you can do with the Euler–Poisson scheme. For a comprehensive exposition of the scheme and a range of applications we refer the reader to the papers mentioned in the abstract. Assume then that we have an algorithm to perform the Wiener–Hopf factorization of a Lévy process, then we can keep track of pairs of the type $\{(X_{g(k,n/t)}, \overline{X}_{g(k,n/t)})\}_{k\geq 0}$ or $\{(X_{g(k,n/t)}, \underline{X}_{g(k,n/t)})\}_{k\geq 0}$. It is worth noting that the theory does not allow to keep track of the exact supremum and the infimum of the process simultaneously.

2.1 Computing the First Passage Time

We will give an example of how to use the scheme to compute the first passage time over a level $u > 0$, i.e., $\tau_u := \inf\{t > 0 \mid X_t > u\}$. Assume you can compute the Wiener–Hopf factorization for X and keep track of the running supremum, i.e., the sequence $\{\overline{X}_{g(k,n/t)}\}_{k\geq 0}$, then

$$k_X^{(n)} := \inf\{k \in \{0, \ldots, n\} \mid \overline{X}_{g(k,n/t)} > u\}$$

is an approximation of τ_u. Indeed,

Theorem 1 (Ferreiro-Castilla–van Schaik, [5]) *Using the same notation as above, we have*

$$\mathbb{E}\left[\left(\frac{t}{n}(k_X^{(n)} \wedge n) - \tau_u \wedge t\right)^2\right] \leq \frac{2t^2}{n}\,.$$

2.2 Approximation of Lévy Driven SDEs

Since we have now an skeleton for the path of a Lévy process, it is natural to investigate its behaviour in approximating Lévy driven SDEs. Let Y be the strong solution of

$$Y_t = y_0 + \int_0^t a(Y_{s-})\mathrm{dX_s}, \tag{2}$$

for $t \in [0, T]$. The Euler–Poisson scheme is then given by the discrete Markov chain $\widetilde{Y} := \{\widetilde{Y}_{t_i}\}_{i \geq 0}$ defined recursively by

$$\widetilde{Y}_{t_i} := \widetilde{Y}_{t_{i-1}} + a(\widetilde{Y}_{t_{i-1}})\Delta X_{\mathbf{e}_i(n/T)},$$

for $i \geq 1$ and $\widetilde{Y}_0 := y_0$, where $\Delta X_{\mathbf{e}_i} := X_{\mathbf{e}_i(n/T)} - X_{\mathbf{e}_{i-1}(n/T)} \stackrel{d}{=} X_{\mathbf{e}(n/T)}$ and $t_i := \sum_{j=0}^{i} \mathbf{e}_j(n/T)$. We claim that $\widetilde{Y}_{t_n}$ is an approximation of Y_T. Indeed,

Theorem 2 (Ferreiro-Castilla–Kyprianou–Scheichl, [6]) *Let X have second finite moments, and $a(x)$ in (2) be a Lipschitz function with constant K. Then*

$$\mathbb{E}[|Y_T - \widetilde{Y}_{t_n}|^2] \leq \widetilde{K} n^{-1/2},$$

where $\widetilde{K}$ is a constant depending only on K and T.

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On Time-Consistent Portfolios with Time-Inconsistent Preferences

Jesús Marín-Solano

Abstract Time-consistent equilibria are studied for intertemporal problems with general discount functions. An investment consumption model with life insurance is analyzed. Equilibrium strategies are derived for two classes of discount functions: time-distance discounting and heterogeneous discounting.

1 General Discount Functions

Let $d(s,t)$ be the discount function representing the valuation at time t by a decision-maker (the t-agent) of utilities obtained at time s, for $s > t$. The objective of the t-agent is

$$\max_{\{u(s)\}} E\left[\int_t^T d(s,t)L(X(s),u(s),s)\,ds + d(T,t)F(X(T))\right],$$

where the state variables evolve according to the diffusion equations

$$dX(s) = f(X(s),u(s),s)ds + \sigma(X(s),u(s),s)\,dW(s) \quad \text{and} \quad X(t) = x_t.$$

The standard HJB equation

$$-V_s^P = \max_{\{u\}}\left\{d(s,0)L + V_x^P \cdot f + \frac{1}{2}tr\left(\sigma\sigma' V_{xx}^P\right)\right\}$$

provides the solution to the problem above according to the time-preferences of the agent at time $t = 0$. This is the precommitment solution, followed by the decision-maker if she has the ability to commit herself in the future to the initial optimal decision rule. If the discount function takes the form $d(s,t) = d_1(s)d_2(t)$, this is in fact the optimal solution for all t-agents. This is the case of exponential discounting,

J. Marín-Solano (✉)
Departament de Matemàtica Econòmica, Financera i Actuarial, Universitat de Barcelona, Barcelona, Spain
e-mail: jmarin@ub.edu

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i.e., $d(s,t) = e^{-r(s-t)}$, or the more general discount function $d(s,t) = e^{-\int_t^s r(\tau)\,d\tau}$. However, for other forms of the general discount functions, the Bellman Optimality Principle is not satisfied and, if agent cannot precommit her future behavior at initial time, the solution becomes time-inconsistent. If the t-agent is naive, in the sense that she is not aware of her changing preferences, at time t she computes her future decision rule $u^t(s)$, for $s \geq t$, according to the discount function $d(s,t)$, by solving the standard HJB equation. But this decision rule will be followed just at time $s = t$, so she will be highly time inconsistent. On the contrary, if the agent is sophisticated, in the sense that she knows which will be her future decision rules, so she will maximize the current payoffs constrained to her future behavior. Instead of an optimization problem, the agent is looking for subgame perfect equilibria in a sequential game with a continuum of players, the different t-agents. In a continuous time setting, three main approaches have been introduced in the literature; see Karp [5], Ekeland–Pirvu [4], and Yong [8]. We briefly review the approach in Ekeland–Pirvu [4]. If $u^*(s) = \phi(s, X(s))$ is an equilibrium rule, then the corresponding value function is $V^S(x,t) = E\left[\int_t^T d(s,t)L(X(s), \phi(X(s),s), s)\,ds + d(T,t)F(X(T),T)\right]$, where $dX(s) = f(X(s), u(s), s)ds + \sigma(X(s), u(s), s)dW(s)$, $X(t) = x_t$. For $\epsilon > 0$ we consider the "variations"

$$u_\epsilon(s) = \begin{cases} v(s) & \text{if } s \in [t, t+\epsilon]\,, \\ \phi(X,s) & \text{if } \quad s > t+\epsilon\,. \end{cases}$$

If the t-agent can precommit her decision rule along the period $[t, t+\epsilon]$, the value function for u_ϵ is

$$V_\epsilon^S(x,t) = E\left[\max_{\{v(s),\, s\in[t,t+\epsilon]\}} \left\{ \int_t^{t+\epsilon} d(s,t)L(X(s), v(s), s)\,ds \right.\right.$$

$$\left.\left. + \int_{t+\epsilon}^T d(s,t)L(X(s), \phi(X(s),s), s)\,ds + d(T,t)F(X(T),T) \right\}\right].$$

Definition 1 Let V_ϵ^S be differentiable in ϵ in a neighbourhood of $\epsilon = 0$. A decision rule $u^*(s) = \phi(s, X(s))$ is an equilibrium if $\lim_{\epsilon\to 0^+} \left[V^S(x,t) - V_\epsilon^S(x,t)\right]/\epsilon \geq 0$.

Theorem 2 *If the value function is of class $C^{2,1}$, then it is given by*

$$V^S(x,t) = E\left[\int_t^T d(s,t)L(X(s), \phi(X(s),s), s)\,ds + d(T,t)F(X(T),T)\right],$$

where the equilibrium rule is $u^* = argmax_{\{u\}}\left[L(x,u,t) + V_x^S \cdot f(x,u,t) + \frac{1}{2}tr\left(\sigma\sigma' V_{xx}^S\right)\right]$.

Note that, unless $d(s,t) = d_1(s)d_2(t)$, when we differentiate the value function, we do not obtain a second order partial differential equation.

2 An Investment-Consumption Model with Life Insurance

Following Pliska–Ye [7], consider a decision maker with a working life of T years and an initial wealth of $W(t_0) = W_0 > 0$, who earns money at a rate $i(t)$. T is a fixed planning horizon that can be interpreted as the retirement time of the wage earner. Let $\tau \in [t_0, \infty)$ be a random variable that models the wage earner's time of death, and let $F(u)$ and $F'(u) = f(u)$ be the distribution and density functions of τ, respectively. At each time $t \in [t_0, \min\{T, \tau\}]$ the agent must decide how to allocate her money between consumption, investment and the purchase of life insurance. If the agent dies before reaching age T ($\tau < T$), a bequest utility depending on the life insurance purchased by the agent plus the accumulated wealth is considered. Otherwise, if the agent reaches age T, a plain final utility is introduced. The agent can invest a fraction $w(t)$ of her wealth at time t, $W(t)$, in a risky asset whose price follows a geometric Brownian motion process $dP(t) = \mu P(t)dt + \sigma P(t)dz(t)$. The remaining fraction $(1-w(t))$ is invested in a risk-free asset. The agent can contract a life insurance. By denoting the insurance premium per Euro of coverage for age t by $p(t)$ and by $Q(t)$ the amount of insurance purchased, the insurance premium at time t will be $p(t)Q(t)$. Finally, the investor can allocate an amount of $c(t)$ to consumption. Assuming full rationality (and time-consistency) of the decision maker, the problem consists in finding the equilibrium rule to

$$E\Bigg[\int_t^{T\wedge\tau} d(s,t)U(c(s))ds + d(\tau,t)B(Z(\tau))\mathbf{1}_{\{\tau\le T\}} + \\ + d(T,t)L(W(T))\mathbf{1}_{\{\tau>T\}} \mid W(t) = W\ ,\ \tau > t\Bigg],$$

$$dW(t) = [w(t)(\mu-\rho)W(t) + \rho W(t) + i(t) - c(t) - p(t)Q(t)]dt + w(t)\sigma W(t)dz(t),$$

where $T \wedge \tau = \min\{T, \tau\}$; $\mathbf{1}_A$ is the indicator function of event A; $U(c)$ is the utility function from consumption; $B(Z)$, $Z = W + Q$, is the bequest function in case of death before the retirement date T and $L(W(T))$ is the bequest function under the assumption of being alive at time T. Functions $U(\cdot)$, $B(\cdot)$ and $L(\cdot)$ are strictly concave.

2.1 Time Distance Discounting

Let $d(s,t) = \theta(s-t)$ with discount rate $r(z) = -\frac{\theta'(z)}{\theta(z)}$. The objective functional becomes

$$E\Big[\int_t^T \{(1-F_t(s))\theta(s-t)U(c(s)) + f_t(s)\theta(s-t)B(Z(s))\}ds +$$

$$+(1-F_t(T))\theta(T-t)L(W(T)) \mid W(t)=W\Big],$$

where $F_t(\tau)$ and $f_t(\tau)$ are the conditional distribution and conditional density of the time of death. The precommitment and naive solutions can be obtained by using the standard techniques. To find time-consistent equilibria, by differentiating the value function we have

$$r(T-t)V^S + K - V_t^S = \max_{\{c,w,Q\}} \big\{S(t)U(c) + f(t)B(Z) + \big[w(t)(\mu-\rho)W$$

$$+\rho W + i(t) - c(t) - p(t)Q(t)\big]V_W^S + \frac{1}{2}w(t)^2\sigma^2W^2V_{WW}^S\big\},$$

and

$$K(W,t) = E\Big[\int_t^T \theta(s-t)\big[r(s-t) - r(T-t)\big]\big[(1-F(s))U(c^*(s)) +$$

$$+f(s)B(Z^*(s))\big]ds \mid W(t)=W\Big].$$

This is a partial integro-differential equation that, in general, must be solved numerically; see Ekeland–Mbodji–Pirvu [3]. However, by differentiating $K(W,t)$ with respect to t, it is possible, for some particular discount functions, to simplify the problem. Take, for example, $\theta(\tau) = (1-d\tau)$, $t < 1/d$. Then $K(W,t)$ solves the PDE

$$-\frac{d}{1-d(T-t)}K - K_t = -\frac{d^2}{1-d(T-t)}(T-t)\left[S(t)U(c^*(t)) + f(t)B(Z^*(t))\right] +$$

$$+K_W\big[w^*(t)(\mu-\rho)W + \rho W + i(t) - c^*(t) - p(t)Q^*(t)\big] + \frac{1}{2}w^*(t)^2\sigma^2W^2K_{WW}.$$

For the case of CARA utility functions, namely $U(c) = -e^{-\gamma c}/\gamma$, $B(Z) = -ae^{-\gamma Z}$, and $L(W(T)) = -be^{\gamma W(T)}$, with $\gamma > 0$, by guessing $V(W,s) = -ae^{-\gamma(\alpha(s)+\beta(s)W)}$ and $K = A(t)e^{-\gamma(\alpha(t)+\beta(t)W)}$, equilibrium rules are derived by solving a system of two ordinary differential equations; see Marín-Solano–Navas–Roch [6]. Proceeding similarly, equilibria can be obtained for power utility functions.

2.2 Heterogeneous Discounting

Next, we consider the family of discount functions $d(s,t) = \exp(-\delta(s-t))$, for $s < T$, and $d(T,t) = \exp(-\rho(s-t))$. Time-consistent equilibria can be obtained by first maximizing the Hamiltonian function

$$L(x,u,t) + V_x(x,t)f(x,u,t) + \frac{1}{2}tr\left(\sigma(x,u,t)\cdot\sigma'(x,u,t)\cdot V_{xx}(x,t)\right),$$

and solving later on a system of two coupled PDEs. In our problem with a life insurance, the objective is to "maximize"

$$E\left[\int_t^T \left(S(s)e^{-\delta(s-t)}U(c_s) + f(s)e^{-\rho(s-t)}B(Z(s),s)\right)ds + S(T)e^{-\rho(T-t)}L(W(T)) \mid \mathcal{F}_t\right].$$

Time-consistent equilibria can be found by transforming the problem from the Lagrange form to the Mayer form (de Paz–Marín-Solano–Navas–Roch [2]), or interpreting it as a kind of two-player cooperative differential game with asymmetric players (de Paz–Marín-Solano–Navas [1]). By using either of these two methods, explicit solutions can be found for log-utilities. In the case of power and exponential utilities, equilibria are obtained by solving a system of two first order ordinary differential equations. In the equilibria, under heterogeneous discounting a change in the consumption pattern compared with the standard case appears: consumption decreases near the retirement date. On the contrary, as an effect of the constancy of drift and volatilities, investment strategies remain unchanged, both with time distance and heterogeneous discounting.

3 Concluding Remarks

The general problem with arbitrary discount functions admits a natural extension. We can consider the problem in which there are N agents who aim to *maximize* their joint payoffs. Let $d_i(s,t)$, L_i and $\lambda_i \geq 0$ be the discount function, utility function and weight in the whole coalition of agent i, respectively. Weights can be constant or not. The objective of the group at time t is described by

$$\sum_{i=1}^{N} \lambda_i(x_t,t)E\left[\int_t^T d_i(s,t)L_i(X(s),u(s),s)\,ds + d_i(T,t)F_i(X(T))\right].$$

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A Generic Decomposition Formula for Pricing Vanilla Options Under Stochastic Volatility Models

Raúl Merino and Josep Vives

Abstract We obtain a decomposition of the call option price for a very general stochastic volatility diffusion model, extending a previous decomposition formula for the Heston model. We realize that a new term arises when the stock price does not follow an exponential model. The techniques used for this purpose are non-anticipative. In particular, we also see that equivalent results can be obtained by using Functional Itô Calculus. Using the same generalizing ideas, we also extend to non-exponential models the alternative call option price decomposition formula written in terms of the Malliavin derivative of the volatility process. Finally, we give a general expression for the derivative of the implied volatility under both the anticipative and the non-anticipative cases.

1 Introduction

Stochastic Volatility models are a natural extension of the Black–Scholes model in order to manage the skew and the smile observed in real data. It is well known that in these models the average of future volatilities is a relevant quantity; see, for example, [8, Chapter 2]. Unfortunately, adding a stochastic volatility structure makes pricing and calibration more complicated, as closed formulas do not always exist. Moreover, even when these formulas exist, like for the Heston model, in general, they do not allow a fast calibration of the parameters.

During last years, different developments for finding approximations to the closed-form option pricing formulas have been published. Malliavin techniques are naturally used to solve this problem in [1] as the average future volatility is an anticipative quantity. Otherwise, a non-anticipative method to obtain an

R. Merino (✉)
Investment Control Department, Vida Caixa S.A., Juan Gris 20-26, 08014, Barcelona (Catalunya), Spain
e-mail: raul.merino85@gmail.com

J. Vives
Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007, Barcelona (Catalunya), Spain
e-mail: josep.vives@ub.edu

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approximation of the pricing formula is developed for the Heston model in [2]. The method is based on the use of the adapted projection of the average future volatility. As a result, the model allows obtaining a decomposition of the call option price in terms of such future volatility.

In the present paper we generalize [2] to general stochastic volatility diffusion models. Similarly, following the same kind of ideas, we extend the expansion based on Malliavin calculus obtained in [1]. This is important because Heston model is not the unique stochastic volatility model currently used in practice, and some of them, like SABR model, are not of exponential type. For a general discussion about stochastic volatility models in practice, see [9].

The main ideas developed in this paper are the following:

- a generic call option price decomposition is found without having to specify the volatility structure;
- a new term emerges when the stock option prices does not follow an exponential model, as for example in the SABR case;
- the Feynman–Kac formula is a key element in the decomposition: it allows to express the new terms that emerges under the new framework (i.e. stochastic volatility) as corrections of the Black–Scholes formula;
- the decomposition found using Functional Itô calculus appears to be the same as the decomposition obtained through our technique;
- a general expression of the derivative of the implied volatility, both for non-anticipative and anticipative cases.

2 Framework

Let $S = \{S(t), t \in [0,T]\}$ be a positive price process under a market chosen risk neutral probability that follows the model

$$dS(t) = \mu(t, S(t))dt + \theta(t, S(t), \sigma(t)) \left(\rho dW(t) + \sqrt{1-\rho^2} dB(t)\right), \tag{1}$$

where W and B are independent Brownian motions, $\rho \in (-1,1)$, $\mu: [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, $\theta: [0,T] \times \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\sigma(t)$ is a positive square-integrable process adapted to the filtration of W. We assume on μ and σ sufficient conditions to ensure the existence and uniqueness of the solution of equation (1). Notice that we do not assume any concrete volatility structure. Thus, our decompositions can be adapted to many different models. In particular, we cover models as Black–Scholes, CEV, Heston and SABR.

We will denote by $BS(t, S, \sigma)$ the price of a plain vanilla European call option under the classical Black–Scholes model with constant volatility σ, current stock

price S, time to maturity $\tau = T - t$, strike price K and interest rate r. In this case,

$$BS(t, S, \sigma) = S\Phi(d_+) - Ke^{-r\tau}\Phi(d_-),$$

where $\Phi(\cdot)$ denotes the cumulative probability function of the standard normal law, and

$$d_\pm = \frac{\ln(S/K) + (r \pm \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

We use the notation $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$, where $\{\mathcal{F}_t, t \geq 0\}$ is the natural filtration of S. In our setting, the call option price is given by $V(t) = e^{-r\tau}\mathbb{E}_t[(S(T) - K)^+]$. We will also use the following definitions for $y \geq 0$:

$$G(t, S(t), y) := S^2(t)\partial_S^2 BS(t, S(t), y),\ H(t, S(t), y) := S(t)\partial_S G(t, S(t), y),$$
$$K(t, S(t), y) := S^2(t)\partial_S^2 G(t, S(t), y) \text{ and } L(t, S(t), y) := \frac{\theta(t, S(t), y)}{S(t)}.$$

3 Decomposition Formulas

On one hand we extend the decomposition formula obtained in [2] to a generic stochastic volatility diffusion process. We note that the new formula can be extended without having to specify the underlying volatility process, obtaining a more flexible decomposition formula. When the stock price does not follow an exponential process, a new term emerges. The formula proved in [2] becomes a particular case.

It is well known that if the stochastic volatility process is independent from the price process, then the pricing formula of a plain vanilla European call is given by

$$V(t) = \mathbb{E}_t[BS(t, S(t), \bar{\sigma}(t))],$$

where $\bar{\sigma}^2(t)$ is the so called average future variance and it is defined by

$$\bar{\sigma}^2(t) := \frac{1}{T-t}\int_t^T \sigma^2(s)ds.$$

Naturally, $\bar{\sigma}(t)$ is called the *average future volatility*; see [8, Pag. 51].

The idea used in [2] consists in considering the adapted projection of the average future variance

$$v^2(t) := \mathbb{E}_t(\bar{\sigma}^2(t)) = \frac{1}{T-t}\int_t^T \mathbb{E}_t[\sigma^2(s)]ds$$

to obtain a decomposition of $V(t)$ in terms of $v(t)$. This idea switches an anticipative problem related with the anticipative process $\bar{\sigma}(t)$ into a non-anticipative one with the adapted process $v(t)$. We apply the same technique to our generic stochastic differential equation (1) and the result is the following.

Theorem 1 *For all $t \in [0, T)$ we have*

$$\begin{aligned} V(t) &= BS(t, S(t), v(t)) \\ &+ \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u, S(u), v(u)) \left(L^2(u, S(u), \sigma(u)) - \sigma^2(u)\right) du\right] \\ &+ \frac{1}{8}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u, S(u), v(u)) d[M, M](u)\right] \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} L(u, S(u), \sigma(u)) H(u, S(u), v(u)) d[W, M](u)\right], \end{aligned}$$

where $M(t) := \int_0^T \mathbb{E}_t\left[\sigma^2(s)\right] ds = \int_0^t \sigma^2(s) ds + (T-t)v(t)^2$.

Remark 2 We have extended the decomposition formula in [2] to the generic SDE (1). When we apply Itô calculus, we realize that Feynman–Kac formula absorbs some of the terms that emerge. It is important to note that this technique works for any payoff or any diffusion model satisfying Feynman–Kac formula.

Remark 3 Note that when $\theta(t, S(T), \sigma(t)) = \sigma(t)S(t)$, that is, the stock price follows an exponential type process, we recuperate exactly the formula proved in [2].

Remark 4 The decomposition problem is an anticipative path-dependent problem. But, as we have seen using a smart choice of the volatility process into the Black–Scholes formula, we can convert it into a non-anticipative one. It is natural to wonder whether the functional Itô calculus, developed in [3–6] brings some new insights into the problem. We have proved that an analogous decomposition can be obtained and this decomposition coincides with the previous one. So, we found an equivalence of the ideas developed by [3–6] and [2] in this decomposition problem. Note that both formulas come from different points of view; the ideas under [3–6] are based on a functional extension of the ideas in [7], while the main idea of [2] is to change a process by its expectation.

On other hand we can use the same ideas to extend the call option price decomposition obtained under the anticipative framework in [1]. This decomposition formula is written in terms of the Malliavin derivative; see [10] for a general reference on Malliavin calculus. Note also that this decomposition formula has one term less than the formula in the non anticipative setup. In this case, we have the following result.

Theorem 5 *For all $t \in [0, T)$, we have*

$$V(t) = \mathbb{E}_t \left[BS(t, S(t), \bar{\sigma}(t)) \right] + \frac{1}{2}\mathbb{E}_t \left[\int_t^T e^{-ru} G(u, S(u), \bar{\sigma}(u)) \left(L^2(u, S(u), \sigma(u)) - \sigma^2(u) \right) du \right] + \frac{\rho}{2}\mathbb{E}_t \left[\int_t^T e^{-r(u-t)} L(u, S(u), \sigma(u)) H(u, S(u), \bar{\sigma}_u) \left(\int_u^T D_u^W \sigma^2(r) dr \right) du \right].$$

Remark 6 As it is expected, a new term emerges in relation with the formula in [1], that covers only the exponential type models. In particular, when $\theta(t, S(T), \sigma(t)) = \sigma(t)S(t)$ we recuperate results from [1].

Remark 7 Note that, when $v(t)$ is a deterministic function, we have that all decomposition formulas are equal.

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A Highly Efficient Pricing Method for European-Style Options Based on Shannon Wavelets

Luis Ortiz-Gracia and Cornelis W. Oosterlee

Abstract In the search for robust, accurate and highly efficient financial option valuation techniques, we present here the SWIFT method (Shannon Wavelets Inverse Fourier Technique), based on Shannon wavelets. SWIFT comes with control over approximation errors made by means of sharp quantitative error bounds. The nature of the local Shannon wavelets basis enables us to adaptively determine the proper size of the computational interval. Numerical experiments on European-style options confirm the bounds, robustness and efficiency.

1 Introduction

European options are financial derivatives, governed by the solution of an integral, the so-called discounted expectation of a final condition, i.e., the pay-off function. A strain of literature dealing with highly efficient pricing of these contracts already exists, where the computation often takes place in Fourier space. For the computation of the expectation we require knowledge about the probability density function governing the stochastic asset price process, which is typically not available for relevant price processes. Methods based on quadrature and the Fast Fourier Transform (FFT) [1, 6, 7], methods based on Fourier cosine expansions [4, 12] and methods based on wavelets [8, 9] have therefore been developed because for relevant *log-asset price processes* the characteristic function appears to be available. The characteristic function is defined as the Fourier transform of the density function. In this paper, we will explore the potential of Shannon wavelets [2] for

L. Ortiz-Gracia (✉)
Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C, 08193 Bellaterra, Barcelona, Spain
e-mail: lortiz@crm.cat

C.W. Oosterlee
CWI – Centrum Wiskunde & Informatica, NL-1090 GB, Amsterdam, The Netherlands

Delft Institute of Applied Mathematics, Delft University of Technology, 2628 CD, Delft, The Netherlands
e-mail: C.W.Oosterlee@cwi.nl

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the valuation of European-style options, which is also based on the availability of the characteristic function. We will call the resulting numerical wavelets technique *"SWIFT"* (Shannon Wavelet Inverse Fourier Technique). Further details on the method can be found in [10].

The pricing of European options in computational finance is governed by the numerical solution of partial differential, or partial integro-differential, equations. The corresponding solution, being the option value at time t, can also be found by means of the Feynman–Kac formula as a discounted expectation of the option value at final time $t = T$, the so-called pay-off function. Here, we consider this *risk-neutral option valuation formula*,

$$v(x,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left(v(y,T)|x\right) = e^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\,dy, \tag{1}$$

where v denotes the option value, T is the maturity, t the initial date, $\mathbb{E}^{\mathbb{Q}}$ the expectation operator under the risk-neutral measure $\mathbb{Q}$, x and y are state variables at time t and T, respectively, $f(y|x)$ is the probability density of y given x, and r is the deterministic risk-neutral interest rate.

Whereas f is typically not known, the characteristic function of the log-asset price is often available (sometimes in closed-form), as the Fourier transform of f. We represent the option values as functions of the scaled log-asset prices, and denote these prices by $x = \ln(S_t/K)$ and $y = \ln(S_T/K)$, with S_t the underlying price at time t and K the strike price.

The pay-off $v(y,T)$ for European options in log-asset space is then given by

$$v(y,T) = \left[\alpha \cdot K\left(e^{y} - 1\right)\right]^{+}, \tag{2}$$

with $\alpha = 1$ for a call, and $\alpha = -1$ for a put.

2 SWIFT

The strategy to follow to determine the price of the option consists of approximating the density function f in (1) by means of a *finite combination of Shannon scaling functions* and recovering the coefficients of the approximation from its Fourier transform.

Let us consider the probability density function f in (1) and its Fourier transform,

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-iwy} f(y|x)\,dy. \tag{3}$$

Following the wavelets theory from [3], the function f can be approximated at a level of resolution m, i.e.,

$$f(y|x) \approx \mathcal{P}_m f(y|x) = \sum_{k\in\mathbb{Z}} c_{m,k}(x)\phi_{m,k}(y), \tag{4}$$

where $\mathcal{P}_m f$ converges to f in $L^2(\mathbb{R})$, that is, $\|f - \mathcal{P}_m f\|_2 \to 0$, when $m \to +\infty$, and where $\phi_{m,k}(y) = 2^{m/2}\phi(2^m y - k)$, $\phi(y) = \text{sinc}(y)$, $c_{m,k}(x) = \langle f, \phi_{m,k}\rangle$, and $< f, g >= \int_{\mathbb{R}} f(y|x)\overline{g(y)}\,dy$ denotes the inner product in $L^2(\mathbb{R})$ (the bar denoting complex conjugation).

Lemma 1 in [10] guarantees that the infinite series in (4) is well-approximated by a finite summation without loss of considerable density mass,

$$\mathcal{P}_m f(y|x) \approx f_m(y|x) := \sum_{k=k_1}^{k_2} c_{m,k}(x)\phi_{m,k}(y|x), \tag{5}$$

for certain accurately chosen values k_1 and k_2.

2.1 Density Coefficients

We compute the coefficients in expression (5) by considering

$$c_{m,k}(x) = \langle f, \phi_{m,k}\rangle = \int_{\mathbb{R}} f(y|x)\overline{\phi}_{m,k}(y)\,dy = 2^{m/2}\int_{\mathbb{R}} f(y|x)\phi(2^m y - k)\,dy. \tag{6}$$

Using the classical *Vieta formula* [5], the cardinal sinus can be expressed as an infinite product, i.e.,

$$\text{sinc}(t) = \prod_{j=1}^{+\infty} \cos\left(\frac{\pi t}{2^j}\right). \tag{7}$$

If we truncate this infinite product to a finite product with J factors, then, thanks to the cosine product-to-sum identity [11], we have

$$\prod_{j=1}^{J} \cos\left(\frac{\pi t}{2^j}\right) = \frac{1}{2^{J-1}}\sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi t\right). \tag{8}$$

By (7) and (8), the sinc function can thus be approximated as

$$\operatorname{sinc}(t) \approx \operatorname{sinc}^*(t) := \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi t\right). \tag{9}$$

If we replace the function ϕ in (6) by its approximation (9) then,

$$c_{m,k}(x) \approx c^*_{m,k}(x) := \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_{\mathbb{R}} f(y|x) \cos\left(\frac{2j-1}{2^J}\pi(2^m y - k)\right) dy.$$

Taking into account that $\Re\left(\hat{f}(w)\right) = \int_{\mathbb{R}} f(y|x)\cos(wy)\,dy$ in expression (3) (where $\Re(z)$ denotes the real part of z), and observing that

$$\hat{f}(w)e^{ik\pi\frac{2j-1}{2^J}} = \int_{\mathbb{R}} e^{-i\left(wy-\frac{k\pi(2j-1)}{2^J}\right)} f(y|x)\,dy,$$

we end up with the following expression for computing the density coefficients,

$$c_{m,k}(x) \approx c^*_{m,k}(x) = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re\left[\hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J}\right) e^{\frac{ik\pi(2j-1)}{2^J}}\right].$$

2.2 Pay-off Coefficients

The pay-off functions for European call or put options have been given in equation (2). We truncate the infinite integration range in (1) to a finite domain $\mathcal{I}_m = [\frac{k_1}{2^m}, \frac{k_2}{2^m}]$, which gives,

$$v(x,t) = e^{-r(T-t)} \int_{\mathbb{R}} v(y,T) f(y|x)\,dy \approx v_1(x,t) = e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f(y|x)\,dy.$$

If we now replace f by its approximation f_m, we find

$$\begin{aligned} v_1(x,t) &= e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f(y|x)\,dy \approx v_2(x,t) = e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f_m(y|x)\,dy \\ &= e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{m,k}(x) \cdot V^{\alpha}_{m,k}, \end{aligned}$$

with the pay-off coefficients $V^{\alpha}_{m,k} := \int_{\mathcal{I}_k} v(y,T)\phi_{m,k}(y)\,dy$. Then, let us define, $\bar{k}_1 := \max(k_1, 0)$. The pay-off coefficients for a European call option are computed

as follows,

$$V_{m,k}^{1} \approx V_{m,k}^{1,*} := \begin{cases} \frac{K2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[I_1\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right) - I_2\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right) \right], & \text{if } k_2 > 0, \\ 0, & \text{if } k_2 \leq 0, \end{cases}$$

where

$$I_1(a,b) = \frac{C_j 2^m}{1 + (C_j 2^m)^2} \Bigg[e^b \sin(C_j(2^m b - k)) - e^a \sin(C_j(2^m a - k)) + \frac{1}{C_j 2^m} \left(e^b \cos(C_j(2^m b - k)) - e^a \cos(C_j(2^m a - k)) \right) \Bigg],$$

and

$$I_2(a,b) = \frac{1}{C_j 2^m} \left(\sin(C_j(2^m b - k)) - \sin(C_j(2^m a - k)) \right), \quad C_j = \frac{2j-1}{2^J}\pi.$$

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A New Pricing Measure in the Barndorff-Nielsen–Shephard Model for Commodity Markets

Salvador Ortiz-Latorre

Abstract For a commodity spot price dynamics given by an Ornstein–Uhlenbeck process with Barndorff-Nielsen–Shephard stochastic volatility, we price forward contracts using a new class of pricing measures, extending the classical Esscher transform, that simultaneously allow for change of level and speed in the mean reversion of both the price and the volatility.

1 Introduction

Benth and Ortiz-Latorre [3] analysed a structure preserving class of pricing measures for Ornstein–Uhlenbeck (OU) processes with applications to forward pricing in electricity markets. In particular, they considered multi-factor OU models driven by Lévy processes having positive jumps (so-called subordinators) or Brownian motions for the spot price dynamics, and analysed the risk premium when the level and speed of mean reversion in these factor processes were changed.

In this work we present the results obtained in [4], where we continue this study for OU processes driven by Brownian motion, but with a stochastic volatility perturbing the driving noise. The stochastic volatility process is modelled again as an OU process, but driven by a subordinator. This class of stochastic volatility models were first introduced by Barndorff-Nielsen–Shephard [1] for equity prices, and later analysed by Benth [2] in commodity markets. The class of pricing measures we study here allows for a simultaneous change of speed and level of mean reversion for both the (logarithmic) spot price and the stochastic volatility process. The mean reversion level can be flexibly shifted up or down, while the speed of mean reversion can be slowed down. It significantly extends the classical Esscher transform, see Gerber–Shiu [8], which only allows for changes in the level of mean reversion.

The affine structure of the model can be exploited to reduce the forward pricing to solving a system of Riccati equations by resorting to the theory of Kallsen–

S. Ortiz-Latorre (✉)
Department of Mathematics, University of Oslo, Blindern N-0316, P.O. Box 1053, Oslo, Norway
e-mail: salvadoo@math.uio.no

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Muhle-Karbe [10]. The forward price becomes a function of both the spot and the volatility, and has a deterministic asymptotic dynamics when we are far from maturity. By careful analysis of the associated system of Riccati equations, we can study how the implied risk premium of our class of measure changes as a function of its parameters. The risk premium is defined as the difference between the forward price and the predicted spot price at maturity, and it is a notion of great importance in commodity markets since it measures the price for entering a forward hedge position in the commodity (see, e.g., Geman [7] for more details). In particular, under rather mild assumptions on the parameters, we can show that the risk premium may change sign stochastically, and may be positive for short times to maturity and negative when maturity is farther out in time. This is a profile of the risk premium that one may expect in energy markets based on both economical and empirical findings, which cannot be obtained by using the Esscher transform.

2 Mathematical Model

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ is a filtered probability space satisfying the usual hyphotesis, where $T > 0$ is a fixed finite time horizon. On this probability space there are defined W, a standard Wiener process, and L, a pure jump Lévy subordinator with finite expectation, that is, a finite variation Lévy process with jumps supported on the positive axis and $\mathbb{E}[L(1)] < \infty$ or, equivalently, a Lévy process with the following Lévy–Itô representation $L(t) = \int_0^t \int_0^\infty z N^L(ds, dz), t \in [0, T]$, where $N^L(ds, dz)$ is a Poisson random measure with Lévy measure ℓ satisfying $\int_0^\infty z\ell(dz) < \infty$. We shall suppose that W and L are independent of each other. As we are going to consider an Esscher change of measure and geometric spot price models, we introduce an assumption on the existence of exponential moments for L.

Assumption 1 Suppose that

$$\Theta_L \triangleq \sup\{\theta \in \mathbb{R}_+ : \mathbb{E}_P[e^{\theta L(1)}] < \infty\}, \tag{1}$$

is a constant strictly greater than zero, which may be ∞.

In $(-\infty, \Theta_L)$ the cumulant (or log moment generating) function $\kappa_L(\theta) \triangleq \log \mathbb{E}_P[e^{\theta L(1)}]$ is well defined and analytic. As $0 \in (-\infty, \Theta_L)$, L has moments of all orders. Also, $\kappa_L(\theta)$ is convex, which yields that $\kappa_L''(\theta) \geq 0$ and, hence, that $\kappa_L'(\theta)$ is non decreasing. Finally, as a consequence of $L \geq 0, a.s.$, we have that $\kappa_L'(\theta)$ is non negative. Moreover,

$$\kappa_L(\theta) = \int_0^\infty (e^{\theta z} - 1)\ell(dz) < \infty \qquad \text{and} \qquad \kappa_L^{(n)}(\theta) = \int_0^\infty z^n e^{\theta z} \ell(dz) < \infty$$

for $n \in \mathbb{N}$, where $\kappa_L^{(n)}(\theta)$ denotes the n-th derivative of $\kappa_L(\theta)$.

Consider the OU-type processes

$$X(t) = X(0) - \alpha \int_0^t X(s)ds + \int_0^t \sigma(s)dW(s), \tag{2}$$

$$\sigma^2(t) = \sigma^2(0) + \int_0^t (\kappa_L'(0) - \rho\sigma^2(s))ds + \int_0^t \int_0^\infty z\tilde{N}^L(ds, dz), \tag{3}$$

where $t \in [0, T], \alpha, \rho > 0, X(0) \in \mathbb{R}, \sigma^2(0) > 0$ and $\tilde{N}^L(ds, dz) \triangleq N^L(ds, dz) - ds\, \ell(dz)$ is the compensated version of $N^L(ds, dz)$.

3 The Change of Measure

We will consider a parametrized family of measure changes that will allow us to simultaneously modify the speed and the level of mean reversion in equations (2) and (3). The density processes of these measure changes are given by

$$\left.\frac{dQ_{\bar{\theta},\bar{\beta}}}{dP}\right|_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t), \quad t \in [0, T], \tag{4}$$

where

$$\tilde{G}_{\theta_1,\beta_1}(t) \triangleq \int_0^t \sigma^{-1}(s)\left(\theta_1 + \alpha\beta_1 X(s)\right) dW(s), \quad t \in [0, T],$$

$$\tilde{H}_{\theta_2,\beta_2}(t) \triangleq \int_0^t \int_0^\infty \left(e^{\theta_2 z}\left(1 + \frac{\rho\beta_2}{\kappa_L''(\theta_2)} z\sigma^2(s-)\right) - 1\right)\tilde{N}^L(ds, dz), \quad t \in [0, T],$$

and $\mathcal{E}(\cdot)$ denotes the stochastic exponential, see Protter [11, Chapter II, Section 8]. The parameters $\bar{\beta} \triangleq (\beta_1, \beta_2)$ and $\bar{\theta} \triangleq (\theta_1, \theta_2)$ will take values on the sets $[0, 1]^2$ and $\bar{D}_L \triangleq \mathbb{R} \times D_L$, respectively, where $D_L \triangleq (-\infty, \Theta_L/2)$ and Θ_L is given by equation (1). Note that when $\beta_1 = \beta_2 = 0$ we get the Esscher transform; see Benth–Sgarra [6]. The following theorem ensure that we have a bona fide change of probability measure.

Theorem 2 *Let $\theta \in \bar{D}_L, \bar{\beta} \in [0, 1]^2$. Then $\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2}) = \{\mathcal{E}(\tilde{G}_{\theta_1,\beta_1} + \tilde{H}_{\theta_2,\beta_2})(t)\}_{t\in[0,T]}$ is a strictly positive true martingale under P.*

Then, by Girsanov's theorem for semimartingales (Theorems 1 and 3 in Shiryaev [12]), under $Q_{\bar{\theta},\bar{\beta}}$, the processes $X(t)$ and $\sigma^2(t)$ become

$$X(t) = X(0) + \int_0^t (\theta_1 - \alpha(1 - \beta_1)X(s))ds + \sigma(t)W_{Q_{\bar{\theta},\bar{\beta}}}(t),$$

$$\sigma^2(t) = \sigma^2(0) + \int_0^t \left(\kappa_L'(\theta_2) - \rho(1 - \beta_2)\sigma^2(s)\right) ds + \int_0^t \int_0^\infty z\tilde{N}^L_{Q_{\bar{\theta},\bar{\beta}}}(ds, dz),$$

where $t \in [0,T]$, $W_{Q_{\bar{\theta},\bar{\beta}}}$ is a $Q_{\bar{\theta},\bar{\beta}}$-standard Wiener process and the $Q_{\bar{\theta},\bar{\beta}}$-compensator measure of σ^2 (and L) is

$$v^{\sigma^2}_{Q_{\bar{\theta},\bar{\beta}}}(dt,dz) = v^{L}_{Q_{\bar{\theta},\bar{\beta}}}(dt,dz) = H_{\theta_2,\beta_2}(t,z)\ell(dz)dt.$$

Remark 3 Under $Q_{\bar{\theta},\bar{\beta}}$, σ^2 still satisfies the Langevin equation with different parameters, that is, the measure change preserves the structure of the equations for σ^2. However, the process L is not a Lévy process under $Q_{\bar{\theta},\bar{\beta}}$, but it remains an affine semimartingale; see [9]. The equation for X is the same under the new measure but with different parameters.

4 Geometric Spot Model

For modeling the commodity spot price S we consider a geometric model, that is,

$$S(t) = \Lambda_g(t)\exp(X(t)), \quad t \in [0,T^*], \tag{5}$$

where $T^* > 0$ is a fixed time horizon. The process Λ_g is assumed to be deterministic and it accounts for the seasonalities observed in the spot prices. Using a risk-neutral pricing argument (see Benth–Šaltytė-Benth–Koekebakker [5]), under the assumption of deterministic interest rates, the forward price at time $0 \le t$, with time of delivery T with $t \le T < T^*$, is given by $F_Q(t,T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t]$. The risk premium for forward prices with a fixed delivery time is defined by the following expression

$$R^F_Q(t,T) \triangleq \mathbb{E}_Q[S(T)|\mathcal{F}_t] - \mathbb{E}_P[S(T)|\mathcal{F}_t].$$

Here, Q is any probability measure equivalent to the historical measure P (not necessarily a martingale measure) and $\mathcal{F}_t$ is the market information up to time t. In what follows we will use the probability measure $Q = Q_{\bar{\theta},\bar{\beta}}$ in order to compute the forward prices and the risk premium. The following is a sufficient condition to ensure that $S(T) \in L^1(P)$.

Assumption 4 We assume that $\alpha, \rho > 0$ and that, for some $\delta > 0$, Θ_L satisfies

$$\frac{1}{2\rho}\left(\frac{\rho}{2\alpha}\right)^{\frac{1}{1-\frac{\rho}{2\alpha}}} \le \Theta_L - \delta.$$

Based on an affine tranform formula for affine semimartingales, see Kallsen–Muhle-Karbe [10], we can provide semi-explicit expressions for $\mathbb{E}_Q[S(T)|\mathcal{F}_t]$.

Theorem 5 *Let $\bar{\beta} = (\beta_1,\beta_2) \in [0,1]^2$, $\bar{\theta} = (\theta_1,\theta_2) \in \bar{D}_L$ and $T > 0$. Assume the existence of functions $\Psi_i^{\bar{\theta},\bar{\beta}} \in C^1([0,T];\mathbb{R})$, $i = 0,1,2$, satisfying the generalised*

Riccati differential equation

$$\begin{aligned}
\tfrac{d}{dt}\Psi_0^{\bar\theta,\bar\beta}(t) &= \theta_1\Psi_2^{\bar\theta,\bar\beta}(t) + \kappa_L(\Psi_1^{\bar\theta,\bar\beta}(t)+\theta_2) - \kappa_L(\theta_2),\\
\tfrac{d}{dt}\Psi_1^{\bar\theta,\bar\beta}(t) &= -\rho\Psi_1^{\bar\theta,\bar\beta}(t) + \tfrac{(\Psi_2^{\bar\theta,\bar\beta}(t))^2}{2} + \tfrac{\rho\beta_2}{\kappa_L''(\theta_2)}(\kappa_L'(\Psi_1^{\bar\theta,\bar\beta}(t)+\theta_2) - \kappa_L'(\theta_2)),\\
\tfrac{d}{dt}\Psi_2^{\bar\theta,\bar\beta}(t) &= -\alpha(1-\beta_1)\Psi_2^{\bar\theta,\bar\beta}(t),
\end{aligned} \tag{6}$$

with initial conditions $\Psi_0^{\bar\theta,\bar\beta}(0) = \Psi_1^{\bar\theta,\bar\beta}(0) = 0, \Psi_2^{\bar\theta,\bar\beta}(0) = 1$ *and the integrability condition* $\sup_{t\in[0,T]}\kappa_L''(\theta_2 + \Psi_1^{\bar\theta,\bar\beta}(t)) < \infty$. *Then,*

$$\mathbb{E}_Q[\exp(X(T))|\mathcal{F}_t] = \exp\left(\Psi_0^{\bar\theta,\bar\beta}(T-t) + \Psi_1^{\bar\theta,\bar\beta}(T-t)\sigma^2(t) + \Psi_2^{\bar\theta,\bar\beta}(T-t)X(t)\right), \tag{7}$$

and

$$\begin{aligned}
R_Q^F(t,T) = \mathbb{E}_P[S(T)|\mathcal{F}_t]\Bigg\{\exp\Bigg(&\Psi_0^{\bar\theta,\bar\beta}(T-t) - \int_0^{T-t}\kappa_L\left(e^{-\rho s}\frac{1-e^{-(2\alpha-\rho)s}}{2(2\alpha-\rho)}\right)ds\\
&+\left(\Psi_1^{\bar\theta,\bar\beta}(T-t) - e^{-\rho(T-t)}\frac{1-e^{-(2\alpha-\rho)(T-t)}}{2(2\alpha-\rho)}\right)\sigma^2(t)\\
&+\left(\Psi_2^{\bar\theta,\bar\beta}(T-t) - e^{-\alpha(T-t)}\right)X(t)\Bigg) - 1\Bigg\},
\end{aligned} \tag{8}$$

for $t \in [0,T]$.

The applicability of Theorem 5 is quite limited as it is stated. This is due to the fact that it is very difficult to see a priori if there exist functions $\Psi_i^{\bar\theta,\bar\beta}, i = 0,1,2$ belonging to $C^1([0,T];\mathbb{R}^2)$ and satisfying equation (6). One has to study existence and uniqueness of solutions of equation (6) and the possibility of extending the solution to arbitrary large $T > 0$. The idea is to provide sufficient conditions on $(\bar\theta,\bar\beta)$ ensuring that $\Psi_i^{\bar\theta,\bar\beta}(t)$, $i = 0,1,2$, remain bounded and do not explode in finite time. We have the following result:

Theorem 6 *Let* $\Lambda^{\theta,\beta,a}\colon[0,\Theta_L-\theta) \to \mathbb{R}$ *be the function defined by*

$$\Lambda^{\theta,\beta,a}(u) = -\rho u + a + \frac{\rho\beta}{\kappa_L''(\theta)}(\kappa_L'(u+\theta) - \kappa_L'(\theta)),$$

where $a \geq 0$ *and* $(\theta,\beta) \in D_L \times (0,1)$, *and consider the set*

$$\mathcal{D}_b(a) = \{(\theta,\beta) \in D_L \times (0,1) : \exists u \in [0,\Theta_L-\theta) \quad s.t. \quad \Lambda^{\theta,\beta,a}(u) \leq 0\}.$$

If $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ *and* $(\theta_1, \beta_1) \in \mathbb{R} \times [0, 1)$ *then* $\Psi_0^{\bar{\theta},\bar{\beta}}(t)$, $\Psi_1^{\bar{\theta},\bar{\beta}}(t)$ *and* $\Psi_2^{\bar{\theta},\bar{\beta}}(t)$ *are in* $C^1([0, T]; \mathbb{R})$ *for any* $T > 0$. *Moreover,*

$$\Psi_0^{\bar{\theta},\bar{\beta}}(t) \longrightarrow \frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \{\kappa_L(\Psi_1^{\bar{\theta},\bar{\beta}}(t) + \theta_2) - \kappa_L(\theta_2)\} ds, \quad t \to \infty,$$

$$(\Psi_1^{\bar{\theta},\bar{\beta}}(t), \Psi_2^{\bar{\theta},\bar{\beta}}(t)) \longrightarrow (0, 0), \quad t \to \infty,$$

and

$$t^{-1} \log \left\| (\Psi_1^{\bar{\theta},\bar{\beta}}(t), \Psi_2^{\bar{\theta},\bar{\beta}}(t)) \right\| \to \gamma, \quad t \to \infty,$$

where $\gamma = -\alpha(1-\beta_1)$ *or* $\gamma = -\rho(1-\beta_2)$.

An immediate consequence of Theorem 6 is that the forward price will be equal to the seasonal function $\Lambda_g(T)$ in the long end; that is, when $(\theta_2, \beta_2) \in \mathcal{D}_b(1/2)$ and $(\theta_1, \beta_1) \in \mathbb{R} \times [0, 1)$, it holds that

$$\lim_{T\to\infty} \frac{F_Q(t,T)}{\Lambda_g(T)} = \exp\left(\frac{\theta_1}{\alpha(1-\beta_1)} + \int_0^\infty \{\kappa_L(\Psi_1^{\bar{\theta},\bar{\beta}}(s) + \theta_2) - \kappa_L(\theta_2)\} \, ds \right).$$

In [4] we provide a comprehensive qualitatively analysis of the possible risk premium profiles that can be obtained using our change of measure. In particular, we show that it is possible to generate risk profiles with positive values in the short end of the forward curve and negative values in the long end. Moreover, we show that the sign of the risk premium can change stochastically. These two features cannot be obtained by using the Esscher transform.

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