Clusterization of Correlation Functions

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Abstract Using the Zhu recursion formulas for correlation functions for vertex operator algebras, we introduce a cluster algebra structure over a non-commutative set of variables.

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1 Introduction

The deep theory of cluster algebras [5] is connected to many different areas of mathematics. In particular, it has intersections with the theory of Riemann surfaces, the moduli spaces of local systems, higher Teichmüller theory, stability structures, Donaldson–Thomas invariants, dilogarithm identities, and many others, [1–4, 7–9]. Several applications of cluster algebras in conformal field theory are known [3, 9]. Non-trivial but natural definition of seeds and mutations this notion allows to apply this kind of relations in various algebraic configurations. In some sense cluster algebras unify alternative ways of description of previously known structures.

The rich theory of vertex operator algebras which constitute an algebraic language of the conformal field theory are also known. Being a natural generalization for Lie algebras, vertex algebras represent a version of Fourier analysis with non-commutative modes. The expansion of vertex operators in terms of modes allows us to operate in an algebraic manner with analytic structures associated with powers of formal parameters attached to modes. This serves as a tool relating complicated algebraic relations vertex operator algebra modes with descriptions of algebraic-geometry objects.

Since both cluster algebras and vertex algebras represent two classes of quite universal algebraic instrumentation, one would be naturally interested in possible

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connections between these two machineries. In this note we would like to sketch a way to relate cluster algebras [5] with vertex operator algebras [6]. We formulate definition of a vertex operator cluster algebra which possesses a structure similar to an ordinary cluster algebra. The seeds are defined over non-commutative variables, coordinates around marked points, and matrix elements of a number of vertex operators. In [6] it was proven that one can describe a vertex operator algebra by the set of all its correlation functions.

1.1 Cluster and Vertex Operator Algebras

Let $\mathbb{P}$ be an abelian group with binary operation $\oplus$. Let $\mathbb{ZP}$ be the group ring of $\mathbb{P}$ and let $\mathbb{QP}(x_1, \ldots, x_n)$ be the field of rational functions in n variables with coefficients in $\mathbb{QP}$. A seed is a triple $(\mathbf{x}, \mathbf{y}, B)$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a basis of $\mathbb{QP}\,(x_1, \ldots, x_n)$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, is an n-tuple of elements $y_i \in \mathbb{P}$, and B is a skew-symmetrizable matrix. Given a seed $(\mathbf{x}, \mathbf{y}, B)$ its mutation $\mu_k(\mathbf{x}, \mathbf{y}, B)$ in direction k is a new seed $(\mathbf{x}', \mathbf{y}', B')$ defined as follows. Let $[x]_+ = max(x, 0)$. Then we have $B' = (b'_{ij})$ with $b'_{ij} = b_{ij}$ for $i = k$ or $j = k$, and $b'_{ij} = b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+$, otherwise. For new coefficients $\mathbf{y}' = \left(y'_1, \ldots, y'_n\right)$, with $y'_j = y_k^{-1}$ if $j = k$, $y'_j = y_j y_k^{[b_{kj}]+} (y_k \oplus 1)^{-b_{kj}}$ if $j \neq k$, and $\mathbf{x} = \{x_1, \ldots, x_n\}$, where $x'_k = \left(y_k \prod_{i=1}^{n} x_i^{[b_{ik}]+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]+} \right) ((y_k \oplus 1)x_k)^{-1}$. Mutations are involutions, i.e., $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}, \mathbf{y}, B)$.

A vertex operator algebra (VOA) [6] is determined by a quadruple $(V, Y, \mathbf{1}, \omega)$, where is a linear space endowed with a $\mathbb{Z}$-grading with $V = \bigoplus_{r\in\mathbb{Z}} V_r$ with $\dim V_r < Y\infty$. The state $\mathbf{1} \in V_0$, $\mathbf{1} \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ for formal variable z so that for any vector $u \in V$ we have a vertex operator $Y(u, z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1}$. The linear operators (modes) $u(n) : V \to V$ satisfy creativity $Y(u, z)\mathbf{1} = u + O(z)$, and lower truncation $u(n)v = 0$, conditions for each u, $v \in V$ and $n \gg 0$. For the conformal vector ω one has $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, where $L(n)$ satisfies the Virasoro algebra for some central charge C: $[\,L(m), L(n)\,] = (m - n)L(m + n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}\mathrm{Id}_V$, where Id_V is identity operator on V. Each vertex operator satisfies the translation property $Y(L(-1)u, z) = \partial_z Y(u, z)$. The Virasoro operator $L(0)$ provides the $\mathbb{Z}$-grading with $L(0)u = ru$ for $u \in V_r$, $r \in \mathbb{Z}$. Finally, the vertex operators satisfy the Jacobi identity which we omit here. These axioms imply locality, $(z_1 - z_2)^N Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^N Y(v, z_2)Y(u, z_1)$, skew-symmetry, $Y(u, z)v = e^{zL(-1)} Y(v, -z)u$, associativity $(z_0 + z_2)^N Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w$, and commutativity $u(k)Y(v, z) - Y(v, z)u(k) = \sum_{j\geq 0} \binom{k}{j} Y(u(j)v, z) z^{k-j}$, conditions for u, v, $w \in V$ and integers $N \gg 0$. For $v = \mathbf{1}$ one has $Y(\mathbf{1}, z) = \mathrm{Id}_V$. Note also that modes of homogeneous states are graded operators on V, i.e., for $v \in V_k$, $v(n) : V_m \to$

$V_{m+k-n-1}$. In particular, let us define the zero mode $o(v)$ of a state of weight $wt(v) = k$, i.e., $v \in V_k$, as $o(v) = v(wt(v) - 1)$, extending to V additively.

1.2 Correlation Functions of Genus Zero and One Riemann Surfaces

We define the restricted dual space of V by Frenkel [6]. Let V be a vertex operator algebra. $V' = \bigoplus_{n \geq 0} V_n^*$, where V_n^* is the dual space of linear functionals on the finite dimensional space V_n. Let $\langle .,. \rangle$ denote the canonical pairing between V' and V. Define matrix elements for $v' \in V'$, $v \in V$ and n vertex operators $Y(v_1, z_1)$, …, $Y(v_n, z_n)$ by $\langle v', Y(v_1, z_1) \ldots Y(v_n, z_n) v \rangle$. Choosing $v = \mathbf{1}$ and $v' = \mathbf{1}'$ we obtain the n-point correlation function on the sphere: $F_V^{(0)}(v_1, z_1; \ldots; v_n, z_n) = \langle \mathbf{1}', Y(v_1, z_1) \ldots Y(v_n, z_n) \mathbf{1} \rangle$. Here the upper index of $F^{(0)}$ stands for the genus. For $u \in V_n$,

$$u(k) : V_m \to V_{m+n-k-1}. \tag{1}$$

Hence it follows that for $v' \in V'_{m'}$, $v \in V_m$, and $u \in V_n$ we obtain a monomial $\langle v', Y(u, z) v \rangle = C^u_{v'v} z^{m'-m-n}$, where $C^u_{v'v} = \langle v', u(m + n - m' - 1) v \rangle$. Recall now the following formal expansion: for variable x, y we adopt the convention that $(z_1 + z_2)^m = \sum_{n \geq 0} \binom{m}{n} z_1^{m-n} z_2^n$, i.e., for $m < 0$ we formally expand in the second parameter z_2. Using the vertex commutator property, i.e., $[u(m), Y(v, z)] = \sum_{i \geq 0} \binom{m}{i} Y(u(i)v, z) z^{m-i}$, one can also derive [10] a recursive relationship. In [10] we find a recurrent formula expressing an $n + 1$-point matrix element on the sphere as a finite sum of n-point matrix elements [10, Lemma 2.2.1]. For $v_1, \ldots, v_n \in V$, and a homogeneous $v \in V$, we find

$$\begin{aligned}
&\langle v', Y(v_1, z_1) \ldots Y(v_n, z_n) v \rangle \\
&= \sum_{r=2}^{n} \sum_{m \geq 0} f_{wt(v_1),m}(z_1, z_r) \cdot \langle v', Y(v_2, z_2) \ldots Y(v_1(m)\, v_r, z_r) \ldots Y(v_n, z_n) v \rangle \\
&\quad + \langle v', o(v_1)\, Y(v_2, z_2) \ldots Y(v_n, z_n) v \rangle,
\end{aligned} \tag{2}$$

where $f_{wt(v_1),m}(z_1, z_r)$ is a rational function defined by $f_{n,m}(z, w) = \frac{z^{-n}}{m!} \left(\frac{d}{dw}\right)^m \frac{w^n}{z-w}$. $\iota_{z,w} f_{n,m}(z, w) = \sum_{j \in \mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1}$. In order to consider modular-invariance of n-point functions at genus one, Zhu introduced [10] a second "square-bracket" VOA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ associated with a given VOA $(V, Y(,), \mathbf{1}, \omega)$. The new square bracket vertex operators are $Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1)$, with $q_z = e^z$, while the new conformal vector is $\tilde{\omega} = \omega - \frac{c}{24} \mathbf{1}$. For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and

$m \geq 0$, $v[m] = m! \sum\limits_{i \geq m} c(wt(v), i, m) v(i)$, where $\sum\limits_{m=0}^{i} c(wt(v), i, m) x^m = \binom{wt(v)-1+x}{i}$. For $v_1, \ldots, v_n \in V$ the genus one n-point function [10] has the form

$$F_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = Tr_V \left(Y(q_1^{L(0)} v_1, q_1) \ldots Y(q_n^{L(0)} v_n, q_n) \; q^{L(0)-C/24} \right),$$

for $q = e^{2\pi i \tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parameter. Then the genus one Zhu recursion formula is given by the following [10]. For any $v, v_1, \ldots, v_n \in V$ we find for an $n+1$-point function

$$\begin{aligned} &F_V^{(1)}(v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\ &= \sum_{r=1}^{n} \sum_{m \geq 0} P_{m+1}(z - z_r, \tau) \cdot F_V^{(1)}(v_1, z_1; \ldots; v[m] v_r, z_r; \ldots; v_n, z_n; \tau) \\ &\quad + F_V^{(1)}\left(o(v); v_1, z_1; \ldots; v_n, z_n; \tau\right), \end{aligned} \tag{3}$$

$F_V^{(1)}\left(o(v); v_1, z_1; \ldots; v_n, z_n; \tau\right) = Tr_V \left(o(v) \; Y(q_1^{L(0)} v_1, q_1) \ldots Y(q_n^{L(0)} v_n, q_n) \; q^{L(0)-C/24} \right)$. In this theorem $P_m(z, \tau)$ denote higher Weierstrass functions defined by $P_m(z, \tau) = \frac{(-1)^m}{(m-1)!} \sum\limits_{n \in \mathbb{Z}_{\neq 0}} \frac{n^{m-1} q_z^n}{1-q^n}$.

2 Cluster Structure for a Vertex Operator Algebra Correlation Functions

Fix a vertex operator algebra V. Choose n-marked points p_i, $i = 1, \ldots, n$ on a compact Riemann surface. In the vicinity of each marked point p_i define a local coordinate z_i with zero at p_i. Consider n-tuples $\mathbf{v} \equiv \{v_1, \ldots, v_n\}$, of arbitrary states $v_i \in V$, and local corresponding vertex operators $\mathbf{Y}(\mathbf{v}, \mathbf{z}) \equiv \{Y(v_1, z_1), \ldots, Y(v_n, z_n)\}$, with coordinates $\mathbf{z} \equiv \{z_1, \ldots, z_n\}$ around p_i, $i = 1, \ldots, n$. We define a *vertex operator cluster algebra seed*

$$(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})), \tag{4}$$

where $F_n(\mathbf{v}, \mathbf{z}) \equiv F_n(v_1, z_1; \ldots; v_n, z_n)$ is an n-point correlation function (matrix element for the sphere case) for n states v_i. Now, define the mutation $\mu_k(v, m, z)$:

$$\left(\mathbf{v}', \mathbf{Y}(\mathbf{v}', \mathbf{z}), F_n'(\mathbf{v}', \mathbf{z})\right) = \mu_k(v, m, z) \, (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})), \tag{5}$$

of the seed (4) in direction $k \in 1, \ldots, n$ for $v \in V$, according to the Zhu reduction formula for corresponding Riemann surface genus, e.g., for the sphere as in (2), for the torus as in (3), etc. Namely, for $\mathbf{v}$, we define $\mathbf{v}'$ as the mutation of $\mathbf{v}$ in direction $k \in 1, \ldots, n$ as

$$\mathbf{v}' = \mu_k(v, m, z)\mathbf{v} = (v_1, \ldots, v(m)v_k \ldots, v_n), \tag{6}$$

for some $m \geq 0$. Note that due to the lower truncation property we get a finite number of terms as a result of the action of $v(m)$ on v_r. For the n-tuple of vertex operators we define

$$\mathbf{Y}(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z)\mathbf{Y}(\mathbf{v}, \mathbf{z}) = (Y(v_1, z_1), \ldots, Y(v(m)v_k, z_k), \ldots, Y(v_n, z_n)) \,. \tag{7}$$

The mutation

$$F_n'(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z)F_n(\mathbf{v}, \mathbf{z}), \tag{8}$$

is defined by summing over mutations in all possible directions with auxiliary functions $f(\text{wt } \ \text{v}, m, k, z)$, $k \in 1, \ldots, n$ and all $m \geq 0$:

$$\begin{aligned} F_n'(\mathbf{v}', \mathbf{z}) &= \mu_k(v, m, z)F_n(v_1, z_1; \ldots; v_n, z_n) \\ &= \sum_{k=1}^{n} \sum_{m \geq 0} f(\text{wt } \ \text{v}, m, k, z)F_n(v_1, z_1; \ldots; v(m)v_k, z_k; \ldots; v_n, z_n) + \widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}), \end{aligned} \tag{9}$$

where $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z})$ denote higher terms in the Zhu reduction formula for a specific genus of a Riemann surfaces used in the consideration. In particular, for the genus zero case we have $f(\text{wt } \ \text{v}, m, k, z) = f_{\text{v},m}(z, z_k)$ for some $m \geq 0$, $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(0)}(o(v); \mathbf{v}, \mathbf{z}) = \langle \mathbf{1}', o(v) \quad Y(v_1, z_1) \ldots Y(v_n, z_n)\mathbf{1} \rangle$, while for the genus one Riemann surface we take and $f(\text{wt } v, m, k, z) = P_{m+1}(z - z_k; \tau)$ given by $P_m(z, \tau)$, $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(1)}(o(v); \mathbf{v}, \mathbf{z}) = Tr_V\,(o(v)Y(v_1, z_1) \ldots Y(v_n, z_n))$. The mutation $\mu_k(v, m, z)$ defined by (6)–(9) is an involution, i.e.,

$$\mu_k(v, m, z)\mu_k(v, m, z)\,(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})) = (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z}))\,,$$

subject a few conditions. As the first condition, one can take $v(m)v(m)v_k = v_k$, $k = 1, \ldots, n$ for the actions (6)–(7). The simplest case, in particular, for $v \in V_k$, for some specific $k = 1, \ldots, n$, when $k - m - 1 = 0$, then $v(m) = o(v) \equiv v(\text{wt } v - 1)$. Then due to the property (1), $v(m)v(m) : V_p \longrightarrow V_p$. Note that when we sum in (9) over mutations in all possible directions $k \in 1, \ldots, n$ and all $m \geq 0$, we obtain a correlation function (matrix element for the sphere) of rank $n + 1$ (see (2) and (3)) with extra $v \in V$ inserted at a point p with corresponding local coordinate z:

$$F_{n+1}^{(g)}(v, z; v_1, z_1; \ldots; v_n, z_n; \tau)$$

$$= \sum_{k=1}^{n} \sum_{m \geq 0} f(\mathrm{wt}\ v, m, k, z) \cdot F_n^{(g)}(v_1, z_1; \ldots; v(m)v_k \ldots; v_n, z_n; \tau) + \widetilde{F}_n^{(g)}(v, z; \mathbf{v}, \mathbf{z}).$$

When we reduce $F_n^{(g)}(v_1, z_1; \ldots; v(m)v_k \ldots; v_n, z_n)$ in (9) to the partition function $F_0^{(g)}$ (i.e., the zero point function) according to the Zhu reduction formulas ((2) or (3)), we obtain multiple action of modes $\prod_{m \geq 0} v_r(m)$ on various v_k as well as products of $f(\mathrm{wt}\ v_r, m_r, r, z_r)$ functions as a result of action on z_k.

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