Properties of Abelian Groups Determined by Their Endomorphism Ring

Ulrich Albrecht

Abstract The goal of this paper is to give a survey of how endomorphism rings can be used to study the behavior of modules. While the first part considers modules over arbitrary rings, the second half focuses mainly on the case of torsion-free Abelian groups. Although there are many applications of endomorphism rings to the theory of mixed Abelian groups, a comprehensive discussion of this subject is beyond the framework of a survey article. In particular, we only present core results, and provide an extensive literature list for those who want to get deeper into the subject.

Keywords Abelian Groups • Endomorphism Rings • Flatness • Adjoint Functors

1 Introduction

This paper has been motivated to a large part by Rüdiger's seminal work on endomorphism algebras. Since his contributions to this subject are discussed in another paper in this volume, it is our goal to highlight the connections between Rüdiger's work and the many ways endomorphism rings are used in Abelian Group Theory.

Traditionally, the goal of Abelian Group Theory has been to describe as large classes of Abelian groups as possible in terms of meaningful numerical invariants. The first major class of groups characterized in this way were the countable p-groups [50]. Ulm's work directly lead to the discussion of the totally projective p-groups as the largest class of torsion groups which are determined by their Ulm-Kaplansky invariants [32]. Baer published a similarly important result for torsion-free groups in 1937 [27]. He showed that the subgroups of the rational numbers are determined up to isomorphism by their types. Moreover, he showed that their rank 1 summands completely determine the completely decomposable groups.

U. Albrecht (✉)
Department of Mathematics, Auburn University, Auburn, AL 36849, USA
e-mail: albreuf@mail.auburn.edu

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Unfortunately, the hopes of the 1940s that it is possible to develop a comprehensive description of the structure of torsion-free groups (of finite rank) were disappointed by a series of examples which Jonsson gave in the 1950s [46, 47].

Example 1.1 ([46]) There exists a torsion-free Abelian group G such that $G = A \oplus B = C \oplus D$ where A, B, C, and D are indecomposable groups with $r_0(A) = 1$, $r_0(B) = 3$, and $r_0(C) = r_0(D) = 2$.

Furthermore, even the pure subgroups of completely decomposable groups cannot be classified in any meaningful way [39]. These and many other results, which cannot be mentioned within the framework of this survey, clearly indicate that a variety of approaches are needed to understand the behavior of torsion-free Abelian groups (of finite rank) better. Although numerical invariants, albeit in a more general form, have been found for large classes of Butler groups, even subgroups of finite index of completely decomposable groups have a structure which is too complex to describe comprehensively in this way [39].

One way to overcome the previously mentioned difficulties is to consider methods and tools from other areas of Mathematics. Rüdiger was one of the pioneers using tools from set-theory and infinite combinatorics to construct large classes of Abelian groups with prescribed properties. We follow a similar approach, but focus on applications of non-commutative ring-theory to Abelian groups instead, an approach initiated by Arnold in the 1970s [24]. Rüdiger's realization theorems for endomorphism ring clearly play a central role in this as is shown in Sect. 3.

Studying Abelian groups via their endomorphism rings takes a point of view which is radically different from the traditional approach. Instead of developing a structure theory, it views an Abelian group A as an object that is best studied by looking at its interaction with other objects. This approach is philosophically related to the one taken in modern Physics where objects like elementary particles are studied through their interaction with other particles. To study this interaction, methods from homological algebra and ring-theory are employed. To facilitate this type of investigation, one usually relies on an adjoint pair of functors between the category of Abelian groups and the category of right modules over the endomorphism ring of A. Section 2 looks at these functors, and introduces some of the basic concepts.

Applications are discussed in Sects. 4 and 5. We give several examples of A-solvable Abelian groups which will answer questions concerning the size and generality of this class. Given the constraints of a survey paper, many interesting topics have to be omitted. Since it is the goal to relate our discussions to Rüdiger's work, we concentrate mostly on torsion-free groups of arbitrary rank. In particular, the discussion of quasi-properties of torsion-free groups of finite rank as well as properties of mixed groups which are described in terms of endomorphism rings have to be omitted in spite of the large amount of literature related to these topics.

2 Adjoint Functors

The interaction of a right R-module A with other R-modules is often described by the functors $\mathrm{Hom}_R(A,-)$ and $\mathrm{Hom}_R(-,A)$. Each of these functors actually carries a structure which is richer than that of an Abelian group, namely that of a right, respectively left, module over the endomorphism ring $E = End_R(A)$ of A. These module structures are induced by A since the latter can be viewed as an E-R-bimodule. Although many properties of a module, e.g., its direct sum decompositions, can be described in terms of its endomorphism ring, the classical theory of Abelian groups makes very little use of the information which can be obtained from this ring. This is quite surprising in view of the Baer-Kaplansky Theorem which states that two Abelian p-groups are isomorphic exactly if they have isomorphic endomorphism rings [28, 48]. The situation is quite different in the torsion-free case. Rüdiger's work shows that there exist proper classes of non-isomorphic torsion-free groups with isomorphic endomorphism rings [41].

Our discussion concentrates on the covariant functor $H_A(-) = \mathrm{Hom}_R(A,-)$ between the categories $\mathcal{M}_R$ of right R-modules and $\mathcal{M}_E$ of right E-modules. It forms one component of the adjoint pair (H_A, T_A) of functors between $\mathcal{M}_R$ and $\mathcal{M}_E$ where T_A is defined by $T_A(X) = X \otimes_E A$ for all right E-modules X. Associated with this adjoint pair are natural transformations $\theta_M : T_A H_A(M) \to M$ for $M \in \mathcal{M}_R$ and $\Phi_X : X \to H_A T_A(X)$ for $X \in \mathcal{M}_E$ defined by $\theta_M(\alpha \otimes a) = \alpha(a)$ and $[\Phi_X(x)](a) = x \otimes a$. The image of θ_M is called *the A-socle of M*, and is denoted by $S_A(M)$.

The idea to consider the category $\mathcal{M}_E$ to investigate properties of a torsion-free Abelian group A originated in two papers which appeared in 1975. Arnold and Lady showed in [25] that H_A and T_A induce an equivalence between the category of A-projective modules of finite A-rank and the category of finitely generated projective right E-modules. Here a right R-module P is *A-projective (of finite A-rank)* if it is a direct summand of a (finite) direct sum of copies of A. Arnold and Murley removed the finiteness conditions in [26] in case that A is a self-small module where a right R-module A is *self-small* if, for every index-set I and every $\alpha \in H_A(\oplus_I A)$, there is a finite subset I' of I such that $\alpha(A) \subseteq \oplus_{I'} A$. Every torsion-free Abelian group of finite rank is self-small, and so is every R-module with a countable endomorphism ring [26]. In contrast to slenderness, which arises in the discussion of the contravariant functor $\mathrm{Hom}_R(-,A)$, self-smallness is not affected by the existence of large cardinals.

A right R-module M is *(finitely, respectively κ-) A-generated* if it is an epimorphic image of a module of the form $\oplus_I A$ for some index-set I (with $|I| < \infty$, respectively $|I| < \kappa$). Since T_A is right exact, all R-modules of the form $T_A(X)$ with $X \in \mathcal{M}_E$ are A-generated, and it is easy to see that M is A-generated if and only if $S_A(M) = M$. We say that a right R-module M has an *A-projective resolution* if we can find an exact sequence

$$\dots P_{n+1} \xrightarrow{\alpha_{n+1}} P_n \xrightarrow{\alpha_n} \dots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

in which each P_n is A-projective.

Proposition 2.1 ([2]) *Let A be a self-small right R-module. A right R-module M has an A-projective resolution if and only if* $M \cong T_A(X)$ *for some right E-module X.*

However, not every A-generated module has an A-projective resolution, nor is the class of modules described by the last result closed with respect to direct summands as the following example shows:

Example 2.2 Let A be an Abelian group which fits into a non-splitting exact sequence $0 \to \mathbb{Z} \to A \to \mathbb{Q} \to 0$. Then, $E(A) = \mathbb{Z}$ by Arnold [24, Section 3]. If G is a torsion-free Abelian group of finite rank, then $r_0(T_A(G)) = 2r_0(G)$ since $r_0(A) = 2$. In particular, every finite rank group with an A-projective resolution has to have even rank by Proposition 2.1. Therefore, $\mathbb{Q}$ does not have an A-projective resolution although it is a direct summand of $T_A(\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ which has an A-projective resolution by Proposition 2.1.

The reason for the difficulties illustrated by the last example is that the module A need not be projective with respect to the sequences

$$0 \to \ker\ \alpha_n \to P_n \xrightarrow{\alpha_n} im\ \alpha_n \to 0$$

induced by an A-projective resolution of an R-module M. Adopting a standard notion from Abelian Group Theory, we say that an exact sequence

$$0 \to U \to V \to W \to 0$$

of right R-modules is *A-balanced* if the induced sequence

$$0 \to H_A(U) \to H_A(V) \to H_A(W) \to 0$$

is exact. The R-module M has an *A-balanced A-projective resolution* if it admits an A-projective resolution

$$\dots P_{n+1} \xrightarrow{\alpha_n} P_n \xrightarrow{\alpha_{n-1}} \dots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

for which the induced sequences $0 \to \ker\ \alpha_n \to P_{n+1} \xrightarrow{\alpha_n} im\ \alpha_n \to 0$ are A-balanced for all $n < \omega$.

Proposition 2.3 ([2]) *Let A be a self-small Abelian group. If a module M has an A-balanced A-projective resolution, then* θ_M *is an isomorphism.*

Although this is a survey article, a brief proof of this result is included since it nicely illustrates the use of the adjointness of (H_A, T_A) without having to deal with the complexities of some of the later results:

Proof An A-balanced A-projective resolution of M induces an A-balanced exact sequence

$$0 \to U \to P_0 \to M \to 0$$

in which U is A-generated as an image of P_1. Applying the functors H_A and T_A successively induces the commutative diagram

$$\begin{array}{ccccccccc} & & T_AH_A(U) & \longrightarrow & T_AH_A(P_0) & \longrightarrow & T_AH_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr \downarrow \theta_{P_0} & & \downarrow \theta_M & & \\ 0 & \longrightarrow & U & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which θ_U is onto, and θ_{P_0} is an isomorphism by Arnold and Murley [26]. The Snake-Lemma yields that θ_M is an isomorphism.

A right R-module M is *A-solvable* if θ_M is an isomorphism; and the class of A-solvable right R-modules is denoted by $\mathscr{C}_A$. By Arnold and Lady [25] and Arnold and Murley [26], A-projective modules are A-solvable if A is self-small. Arnold and Murley also showed in [26] that every locally A-projective module M is A-solvable if the endomorphism ring of A is discrete in the finite topology. Here M is *locally A-projective (locally A-free)* if every finite subset of M is contained in an A-projective direct summand (isomorphic to A^n for some $n < \omega$) of M. The endomorphism ring E of a module A is *discrete in the finite topology* if there is a finitely generated E-submodule of A with $\mathrm{Hom}_R(A/U, A) = 0$. By Arnold and Murley [26], A is self-small if its endomorphism ring is discrete in the finite topology.

An A-projective resolution of an A-solvable module need not be A-balanced without additional conditions on A. For instance, if $A = \mathbb{Q} \oplus \mathbb{Z}$, then all groups are A-generated; and there exists an exact sequence $0 \to U \to \oplus_\omega A \to A \to 0$ with respect to which A is not projective. Before continuing our discussion, we want to remind the reader that a left R-module A is *faithful* if $X \otimes_R A \neq 0$ for all non-zero finitely generated right R-modules X. It is *fully faithful* if this holds for all right R-modules. Faithfully flat modules are fully faithful.

Theorem 2.4 ([2] and [3]) *The following conditions are equivalent for a right R-module M:*

(a) A is fully faithful as a left R-module.
(b) A right R-module M admits an A-balanced exact sequence

$$0 \to U \to \oplus_I A \to M \to 0$$

with $S_A(U) = U$ if and only if $M \in \mathscr{C}_A$.
(c) An exact sequence $0 \to U \xrightarrow{\alpha} M \to P \to 0$ splits if $\alpha(U) + S_A(M) = M$ and P is A-projective.

Baer had shown in [27] that every subgroup A of $\mathbb{Q}$ satisfies condition (c) for $P = A$. Hence, condition (c) is often referred to as *Baer's Lemma*. Arnold and Lady established the equivalence of (a) and (c) in [25] if A is a torsion-free Abelian group of finite rank and $M = A$. Unfortunately, neither Baer's nor their arguments carry over to the case that P is an arbitrary A-projective group.

It is well known from category theory that the kernel of a map $A^n \to A$ need not be A-generated unless A is flat as a module over its endomorphism ring (Ulmer's Theorem [51]). In particular, if $\alpha \in \mathrm{Hom}_R(M, N)$ for A-solvable modules M and N, then neither *ker* α nor *im* α need to be A-solvable. Therefore, we call a class $\mathscr{C}$ of A-generated groups *A-closed* if

(i) $\mathscr{C}$ is closed with respect to finite direct sums and A-generated submodules, and
(ii) ker $\alpha \in \mathscr{C}$ for all $M, N \in \mathscr{C}$ and all $\alpha \in \mathrm{Hom}_R(M, N)$.

Theorem 2.5 ([7]) *The following are equivalent or a self-small right R-module A:*

(a) A is flat as a module over its endomorphism ring.
(b) There exists an A-closed class $\mathscr{C}$ containing A.
(c) $\mathscr{C}_A$ is the largest A-closed class containing the A-projective modules.

However, there exit R-modules A which are flat as modules over their endomorphism ring, but not faithful. For instance, the group $A = \mathbb{Q} \oplus \mathbb{Z}$ has this property. Therefore, A may not be projective with respect to exact sequences in $\mathscr{C}_A$ even if the latter is A-closed. Since the existence of sequences in $\mathscr{C}_A$ which are not A-balanced makes it difficult to develop a comprehensive homological algebra for $\mathscr{C}_A$, we call an A-closed class $\mathscr{C}$ *A-balanced* if every exact sequence $0 \to B \to C \to M \to 0$ with $B, C, M \in \mathscr{C}$ is A-balanced.

Theorem 2.6 ([7]) *The following are equivalent for a self-small right R-module A:*

(a) A is faithfully flat as a left E-module.
(b) There exists an A-balanced, A-closed class containing all of the A-projective modules.
(c) $\mathscr{C}_A$ is the largest A-balanced, A-closed class containing all of the A-projective modules.
(d) A right R-module has an A-balanced A-projective resolution if and only if it is A-solvable.

In particular, A-balanced A-projective resolutions of an A-solvable module M induce derived functors $Bext^n_A(-,-)$ on $\mathscr{C}_A$ such that

$$Bext^n_A(M, N) \cong \mathrm{Ext}^n_E(H_A(M), H_A(N))$$

for all A-solvable modules M and N [9].

Finally, the concept of A-solvable modules carries over naturally to the quasi-category of Abelian groups. Unfortunately, the discussion of quasi-concepts is beyond the framework of this survey.

3 Realization Theorems

The results of the last section raise the question whether it is possible to construct self-small modules A such that

(a) A is flat (or, e.g., faithfully flat or projective) when viewed as an E-module, and
(b) the endomorphism ring of A can be prescribed to belong to a specific class of rings, e.g., principal ideal domains, hereditary rings, or polynomial rings?

[38, Chapter 111] is dedicated to this question, and [38, Problem 84] particularly asks for criteria for certain types of rings to be endomorphism rings. However, the following example shows that one has to be somewhat careful when combining properties of the module ${}_EA$ in (a) with ring-theoretic properties of E in (b):

Example 3.1 Suppose that A is a right R-module such that E is a principal ideal domain. Then, A is an indecomposable Abelian group since E does not contain any non-trivial idempotents. If ${}_EA$ were projective as a left E-module, then ${}_EA \cong \oplus_I E$ for some index-set I which is only possible if $|I| = 1$. Thus, ${}_EA \cong E$. For instance, all Murley groups A have a principal ideal domain as an endomorphism ring [24]. Here a torsion-free groups A of finite rank is a *Murley-group* if $dim \mathbb{Z}/p\mathbb{Z} A/pA \leq 1$ for all primes p.

Fortunately, module-theoretic properties like faithfulness and flatness are not nearly as restrictive as projectivity. To see this, we are going to look at some of the standard construction methods of modules with a prescribed endomorphism ring. Although most of them have their origin in Abelian Group Theory, they actually hold for substantially more general classes of rings. For instance, Rüdiger and the author extended the construction of E-algebras to a non-commutative setting in [15]. The methods used in this extension can also be applied to the realization theorems for endomorphism rings in [31] and [34]. As in the commutative setting, some restrictions on R are necessary to avoid immediate counterexamples.

An element c of a ring R is *regular* if $cr = 0$ or $rc = 0$ implies $r = 0$. For any ring R, let

$$C(R) = \{s \in R \mid rs = sr \text{ for all } r \in R\}$$

denote *the center of* R. Clearly, $C(R)$ is a subring of R and $1_R \in C(R)$. As in [41], we consider a countable, multiplicatively closed subset $\mathbb{X} \subseteq C(R)$ of regular central elements of R which contains precisely one unit of R, the identity 1_R. The notions of $\mathbb{X}$-density, $\mathbb{X}$-purity, $\mathbb{X}$-torsion-freeness, and $\mathbb{X}$-cotorsion-freeness carry over literally from the commutative setting [41]. In particular, $\widehat{R}$ denotes the $\mathbb{X}$-completion of R.

Theorem 3.2 *Let S be an extension ring of R which is $\mathbb{X}$-cotorsion-free and torsion-free as a $C(R)$-module. If $\kappa^+ \leq \mu \leq \lambda$ are cardinals such that $|S| = \kappa$, μ is regular, and $\lambda^\kappa = \lambda^{\aleph_0}$, then there exists an $\mathbb{X}$-cotorsion-free right R-module A such that*

(a) $End_R(A) = S$*, and*
(b) *every countably generated* S*-submodule of* ${}_SA$ *is contained in a free* S*-submodule.*

In particular, the endomorphism ring of A is discrete in the finite topology, and A is flat as a left E-module. Moreover, if S is countable, then A is also faithful.

Proof Since the description of the actual construction of A is beyond the framework of this survey, the interested reader is referred to [15] to identify which modifications need to be made to the proof of [41, Theorem 12.3.4] in order to obtain A. In particular, Rüdiger had pointed out during the writing of [15] that the module A can be constructed in [41, Theorem 12.3.4] in such a way that it contains a family $\mathscr{F}$ of countably generated free submodules with the following properties:

(a) Every countable subset of A is contained in an element of $\mathscr{F}$.
(b) $\Sigma_{n<\omega} F_n \in \mathscr{F}$ for all families $\{F_n \mid n < \omega\} \subseteq \mathscr{F}$.

Clearly, the existence of $\mathscr{F}$ guarantees that A is flat as an S-module. To see that the endomorphism ring of A is discrete in the finite topology, observe that A is constructed as an $\mathbb{X}$-dense submodule of the $\mathbb{X}$-completion of a free S-module. Therefore, we can find a left S-module monomorphism $\alpha : S \to A$. Consider $\beta \in S = End_R(A)$ with $0 = \beta(\alpha(1_A))$. Since α is S-linear, we have $\beta(\alpha(1_A)) = \alpha(\beta 1_A)) = \alpha(\beta)$. Thus, $\beta = 0$ since α is one-to-one.

To see that A is faithful if S is countable, let I be a maximal right ideal of S with $IA = A$, and select $F_0 \in \mathscr{F}$. There is a countable S-submodule Y_0 of A such that $F_0 \subseteq IY_0$. Select $F_1 \in \mathscr{F}$ with $IY_0, Y_0 \subseteq F_1$. Continuing inductively, we obtain an ascending chain $\{F_n \in \mathscr{F} \mid n < \omega\}$ such that $F_n \subseteq IF_{n+1} \subseteq F_{n+1}$ for all $n < \omega$. Hence, $F' = \cup_{n<\omega} F_n$ is a free submodule of A such that $IF' = F'$. However, this is only possible if $I = E$.

The countability condition in the last result can be removed under **V = L** by adapting the arguments of [34] to the non-commutative setting:

Corollary 3.3 (ZFC + $\diamondsuit_\kappa$) *Let S be an extension ring of R which is* $\mathbb{X}$*-cotorsion-free and torsion-free as a* $C(R)$*-module. If* κ *is a regular uncountable cardinal such that* $|S| < \kappa$*, then there exists an* $\mathbb{X}$*-cotorsion-free right R-module A with the following properties:*

(a) $End_R(A) = S$*, and*
(b) *Every* κ*-generated* S*-submodule of* ${}_SA$ *is contained in a free* S*-submodule.*

In particular, the endomorphism ring of A is discrete in the finite topology, and A is faithfully flat as left E-module [7].

We want to point out that Faticoni used a Black Box construction similar to the one in Theorem 3.2 to construct an Abelian group A which is faithful, but not fully faithful as a module over its endomorphism ring [36].

Theorem 3.2 and Corollary 3.3 can be used to construct large class of A-solvable groups which are not A-projective:

Theorem 3.4 ([7]) *Let A be a cotorsion-free self-small Abelian group which is faithfully flat as a module over its endomorphism ring.*

(a) If A is countable, then there exist a proper class of A-solvable groups with endomorphism ring E^{op}*.*
(b) (ZFC + V = L) There exist a proper class of A-solvable groups with endomorphism ring E^{op}*.*

Proof We use either Theorem 3.2 or Corollary 3.3 to obtain a proper class of Abelian groups G with $End(G) = E^{op}$. Then, G^{op} is a right E-module and $T_A(G^{op})$ is A-solvable. An application of the Adjoint-Functor-Theorem completes the proof.

However, there are several question arising from the last results:

Problem 3.5 Can the Black Box be used directly to construct arbitrarily large classes of A-solvable groups in case A is countable instead of using E^{op}?

In [37], Franzen and Rüdiger used the Black Box to obtain modules over commutative rings R with prescribed endomorphism rings which contain a module of the form $\oplus_I B$ as a dense and pure submodule where B is a cotorsion-free faithful R-module. Combining this construction with the arguments from [15] should yield the desired result by replacing the free modules in the definition of the family $\mathscr{F}$ by B-projective modules.

Problem 3.6 Show directly that large classes of A-solvable groups exist assuming $\mathbf{V} = \mathbf{L}$ instead of using E^{op}.

In addition to the previously mentioned realization theorems, there are also the classical results by Zassenhaus and Corner from the 1960s, each of which will also produce Abelian groups which are faithfully flat as modules over their endomorphism ring:

Theorem 3.7 *Let R be a countable ring whose additive group is torsion-free and reduced.*

(a) There exists a countable Abelian group A with $E(A) = R$ *[30].*
(b) If $r_0(R) = n$*, then A can be chosen to have rank* $2n$ *[30].*
(c) If R^+ *is a free group of rank n, then A can be chosen to have rank n too [53].*

In either case, A has an endomorphism ring which is discrete in the finite topology, and is faithfully flat as an E-module [7].

Finally, we want to remark that the contra-variant functor $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ induces a duality between the direct summands of cartesian powers of A and projective left E-modules if A is a slender Abelian group. This duality was initially discussed by Huber and Warfield in [45] in case that A is a torsion-free group of finite rank, while the author considered the general case in [4]. Again, Rüdiger's realization theorems provide us with large classes of slender groups with a prescribed endomorphism ring.

4 Torsion-Free Abelian Groups

We now turn our discussion to Abelian groups, although many of our results will carry over to a more general setting, e.g., to modules over Dedekind domains. [38, Problem 84] asks to find criteria for certain types of rings to be endomorphism rings, but does specify what form these criteria should take, e.g., whether or not they are to be numerical invariants or properties describing the interaction of a group with a certain type of endomorphism ring with other groups. In the following, we interpret this problem to have two parts, namely

(a) How are ring-theoretic properties of the endomorphism ring of an Abelian group A reflected in the structure and the homological properties of A?
(b) How are structural and homological properties of an Abelian group A reflected in ring-theoretic properties of its endomorphism ring?

If A is fully faithful as an E-module, then H_A and T_A induce a one-to-one and onto correspondence between the right ideals of E and the A-generated subgroups of A. Because of this, it is frequently possible to address these questions for properties of a ring, which are definable in terms of ideals and submodules of projective modules. On the other hand, properties like commutativity, or more generally those given by polynomial identities, are virtually impossible to describe as can, for instance, be seen in [23] which looks at Abelian groups with commutative endomorphism rings.

We begin our discussion by investigating the connection between ring-theoretic properties of A and some of the fundamental properties of homogeneous completely decomposable groups which Baer considered in his 1937 paper [27]. For instance, if G is a subgroup of a homogeneous completely decomposable group of type τ and $G = G(\tau)$, then G is homogeneous completely decomposable.

Theorem 4.1 ([1] and [25]) *The following conditions are equivalent for a self-small torsion-free Abelian group A:*

(a) A is faithfully flat as an E-module and E is right hereditary.
(b) (i) A satisfies the conclusions of Baer's Lemma (see Theorem 2.4).
(ii) A-generated subgroups of A-projective groups are A-projective.

Arnold and Lady had investigated the conditions in (b) in the case that A is a torsion-free group of finite rank [25]. However, their arguments do not carry over to the general case. Furthermore, condition (b.ii) alone need not imply that E is right hereditary as was shown in [11].

Rings satisfying chain conditions are of particular interest in ring-theory, and they are often considered in conjunction with the requirement that the ring is right or left non-singular [43, 49]. However, when describing groups whose endomorphism ring satisfies chain conditions, we need to be aware of several facts that make it difficult to describe these groups in terms of numerical invariants:

- The endomorphism ring of torsion-free groups of finite rank has finite right and left Goldie-dimension.

- A semi-prime subring of a finite dimensional $\mathbb{Q}$-algebra is right and left Noetherian [24, Chapter 9].
- Descending chain conditions on right or left ideals are usually too restrictive to yield interesting classes of groups [38, Theorem 11.3].
- Standard group-theoretic concepts like types and purity have only limited bearings on ring-theoretic properties of an endomorphism ring unless we restrict our discussion to the finite rank case [6].

To avoid immediate restrictions on the rank of E, we turn to the notion of non-singularity introduced by Goodearl and Stenstrom [43, 49]. Taking this approach, the author was able to give a description of the Abelian groups A with a right and left Noetherian, hereditary endomorphism ring in [1, Theorem 5.1]. Since these groups have many of the homological properties usually associated with rank 1 groups, they are called *generalized rank* 1 *groups*, and play an important role in the theory of A-solvable groups. An important class of generalized rank 1 groups are the finitely faithful S-groups, which consists of all finite rank torsion-free groups A such that $r_p(E) = [r_p(A)]^2$ for all primes p [42]. Goeters showed that each finally faithful S-group has a hereditary endomorphism ring [42]. Hence, $Bext^n_A(G,H) \cong \mathrm{Ext}^n_E(H_A(G), H_A(H)) = 0$ for all A-solvable groups G and H and all $n > 1$. Moreover, we can describe how $Bext^1_A(G,A)$ is embedded into $Ext_{\mathbb{Z}}(G,A)$ in this case.

Proposition 4.2 ([14]) *If A is a finitely faithful A-group, then the group*

$$\mathrm{Ext}_{\mathbb{Z}}(G,A)/Bext^1_A(G,A)$$

is torsion-free and divisible for all torsion-free A-solvable groups G.

For a right R-module M, *the singular submodule of M* is

$$Z(M) = \{x \in M | xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$$

which takes the place of the torsion submodule in the general setting. The module M is called *non-singular* if $Z(M) = 0$, and *singular* if $M = Z(M)$. The ring R is *right non-singular* if it is non-singular as a right R-module. A ring is *a right p.p.-ring* if all principal right ideals are projective. Right p.p.-ring play an important role in the theory of non-singular rings and modules, e.g., see [22, 29, 33, 44], and [18]. Finally, a submodule U of an R-module M is *$\mathcal{S}$-closed* if M/U is non-singular.

However, the endomorphism ring E of a non-singular module M over a non-commutative non-singular ring may behave quite different from that of a torsion-free module over an integral domain. For instance, R need not be a subring of E, and M may not be non-singular over its endomorphism ring as Rüdiger and the author showed in [16].

Problem 4.3 Abelian groups whose endomorphism ring is a right p.p.-ring were described in [6]. Is it possible to give a description of the torsion-free Abelian groups with a right non-singular endomorphism ring?

However, finitely generated non-singular modules over a non-singular ring need not be submodules of free modules in contrast to the situation in the case of integral domains. Rings having this properties are called *strongly right non-singular*, and include the semi-prime right and left Goldie-rings. In particular, a ring is right and left strongly non-singular if it is a semi-prime subring of a finite dimensional $\mathbb{Q}$-algebra. For Abelian groups A with a strongly non-singular endomorphism ring, it is possible to define more meaningful notions of torsion-freeness and purity for the class of A-generated groups.

An A-generated group G is *A-torsion-free* if every finitely A-generated subgroup U is isomorphic to a subgroup of an A-projective group (which need not be a subgroup of G). An A-generated subgroup U of an A-torsion-free group G is *A-pure* if $(U + P)/U$ is A-torsion-free for all finitely A-generated subgroups P of G. We want to emphasize that A-pure subgroups need not be A-balanced. Using the concept of the $\mathscr{S}$-closure of a submodule of a non-singular module it is also possible to introduce the notion of the A-closure of an A-generated subgroup of a A-torsion-free group.

Theorem 4.4 ([6]) *Let A be a self-small torsion-free Abelian group which is E-flat such that E is a right strongly non-singular ring.*

(a) A group G is A-torsion-free if and only if G is A-solvable and $H_A(G)$ is non-singular. In particular, A-generated subgroups and direct sums of A-torsion-free groups are A-torsion-free.
(b) An A-generated subgroup U of an A-torsion-free group G is A-pure if and only if $H_A(G)/H_A(U)$ is non-singular.

Problem 4.5 Define the notions of A-torsion-freeness and A-purity in case that E is not a strongly non-singular ring.

C. Walker called a subgroup U of an Abelian group G *A^*-pure* if it is a direct summand of all subgroups H of G which contain U and have the property that H/U is an image of A [52]. It is *P_A^*-pure* if it is a direct summand of all subgroups H of G which contain U and have the property that H/U is finitely A-generated.

Theorem 4.6 ([6]) *The following conditions are equivalent for an Abelian group A which is E-flat and has a right strongly non-singular endomorphism ring:*

(a) E is a right p.p.- (right semi-hereditary) ring.
(b) If $\alpha \in E$ ($\alpha \in E(A^n)$ for some $n < \omega$), then $\ker \alpha$ is a direct summand.
(c) An A-generated subgroup U of an A-torsion-free group is A_*-pure (P_A^*-pure) if and only if it is A-pure.

Furthermore, the question arises how A-purity and the standard notion of purity are related.

Theorem 4.7 ([6]) *The following conditions are equivalent for a self-small E-flat Abelian group A with a strongly right non-singular endomorphism ring:*

(a) If G is a torsion-free A-solvable group, then G is A-torsion-free, and every pure A-generated subgroup of G is A-pure in G.
(b) A/U is torsion for all A-generated subgroups U of A with $\mathrm{Hom}_{\mathbb{Z}}(A/U, A) = 0$.

The last condition is, for instance, satisfied if $\mathbb{Q}E$ is a semi-simple Artinian ring.

Problem 4.8 Do the last two results remain true if E is not strongly non-singular?

We conclude this section by looking at locally A-projective groups and their A-pure subgroups. In particular, we obtain a version of Pontryagin's criterion for A-solvable groups:

Theorem 4.9 ([8]) *Let A be an E-flat Abelian group with a right strongly non-singular, right semi-hereditary endomorphism ring which is discrete in the finite topology.*

(a) A-pure subgroups of locally A-projective subgroups are A-projective.
(b) An A-pure subgroup of a locally A-projective group is A-projective if it is an epimorphic image of $\oplus_\omega A$.
(c) A countably A-generated A-torsion-free group G is A-projective if every finitely A-generated subgroup of G is contained in a finitely A-generated A-pure subgroup of G.

In particular, $S_A(A^I)$ is locally A-free if E is left Noetherian [20]. Surprisingly, the converse holds too:

Corollary 4.10 ([20]) *Let A be a slender Abelian group of non-measurable cardinality whose endomorphism ring is discrete in the finite topology. If* $S_A(A^I)$ *is locally A-free for all index-sets of non-measurable cardinality, then E is left Noetherian.*

Problem 4.11 Can the various Black Box methods used in [41] to construct separable Abelian groups be adapted to obtain large classes of locally A-projective groups?

In view of Corollary 4.10, some additional ring-theoretic restrictions on E may be necessary.

5 Applications

We want to remind the reader that the class $\mathscr{C}_A$ of A-solvable groups consists of all Abelian groups G for which the evaluation map $\theta_G : T_A H_A(G) \to G$ is an isomorphism. When looking at $\mathscr{C}_A$, the question immediately arises which groups, in addition to A-projective groups, belong to $\mathscr{C}_A$? Arguing as in the proof of Theorem 3.2 or Corollary 3.3, it is easy to see that $\mathscr{C}_A$ contains the κ-A-projective groups whenever $\kappa > |A|$ is a regular cardinal and A is faithfully flat as an E-module [7]. Here, an A-generated group G is *κ-A-projective* if every

κ-A-generated subgroup U of G can be embedded into an A-projective subgroup of G. If $|A| < \kappa$ and E is right hereditary, then this is equivalent to the condition that all A-generated subgroups U with $|U| < \kappa$ are A-projective. However, $\mathscr{C}_A$ may contain cotorsion groups even if A is cotorsion-free.

Proposition 5.1 *(a) If A is subgroup of $\mathbb{Q}$ of type τ, then all torsion-free groups G with $G = G(\tau)$ are A-solvable, and so is $\mathbb{Z}/p\mathbb{Z}$ for all primes p with $A \neq pA$. However, $\mathscr{C}_A$ need not be closed under direct sums unless A has idempotent type [5].*

(b) If A is a generalized rank 1 group, then $\mathbb{Q}$ is A-solvable if and only if A is homogeneous completely decomposable [5].

We now turn to the case that $r_0(A) > 1$, and focus on the following questions raised by the last example:

- Can we find indecomposable generalized rank 1 groups A such that all A-solvable groups are (co-) torsion-free? Which indecomposable generalized rank 1 groups other than subgroups of $\mathbb{Q}$ admit torsion A-solvable groups?
- Can we find cotorsion-free indecomposable generalized rank 1 groups other than subgroups of $\mathbb{Q}$ such that all A-generated reduced torsion-free groups are A-solvable?

The first of these is answered by

Theorem 5.2 ([5]) *The following are a equivalent for a generalized rank 1-group A and a prime p with $A/pA \neq 0$.*

(a) Every bounded p-group is A-solvable.
(b) $[r_p(A)]^2 = r_p(E) < \infty$.

In particular, A a torsion-free Abelian group of finite rank is a finitely faithful S-group if and only if it is fully faithful as an E-module and $\mathbb{Z}/p\mathbb{Z}$ is A-solvable for all primes p with $A \neq pA$. On the other hand, Corner's realization theorem in Theorem 3.7b always produces a torsion-free group A of finite rank with $r_p(A) = r_p(E)$ [7]. Thus, if A is a group of rank 4 with $E \cong \mathbb{Z} + i\mathbb{Z}$ which is constructed in this way, then the elements of $\mathscr{C}_A$ are torsion-free and reduced.

Surprisingly, the question whether there exits A-solvable torsion groups also is closely related to categorical properties of $\mathscr{C}_A$. However, since $\mathscr{C}_A$ is not an Abelian category unless A is a subgroup of $\mathbb{Q}$ of idempotent type [5], we investigate when the category of A-solvable groups is pre-Abelian.

Theorem 5.3 ([5]) *The following conditions are equivalent for an indecomposable generalized rank 1-group A with $r_0(A) > 1$:*

(a) $\mathscr{C}_A$ is a pre-Abelian category which does not contain J_p for any prime p.
(b) If $r_p(A) < \infty$ for some prime p with $A \neq pA$, then $[r_p(A)]^2 \neq r_p(E)$.
(c) The elements of $\mathscr{C}_A$ are cotorsion-free.

The realization theorems discussed in Sect. 3 allow the construction of large classes of groups such that $\mathscr{C}_A$ is pre-Abelian:

Example 5.4 Let A be an indecomposable generalized rank 1 such that $\mathbb{Q}E$ is semi-simple and $r_p(A) \geq 2^{\aleph_0}$ for all primes p with $A \neq pA$. One of the Göbel's realization theorems guarantees that there exist proper classes of Abelian groups satisfying these conditions. Since $r_p(A) \geq 2^{\aleph_0}$, there is a subgroup U of A such that $A/U \cong J_p$. If $J_p \in \mathscr{C}_A$, then U is A-generated since A is a generalized rank 1 group. By the results of Sect. 4, U is a direct summand of A which is not possible since A is indecomposable. Hence, $\mathscr{C}_A$ is pre-Abelian by the last result.

Furthermore, since the realization theorems produce proper classes of groups with isomorphic endomorphism rings, the question arises which categorical properties are shared by Abelian groups A and B with isomorphic, or more generally Morita-equivalent, endomorphism rings. Surprisingly, the categories $\mathscr{C}_A$ and $\mathscr{C}_B$ need not be equivalent:

Example 5.5 Let A be a subgroup of $\mathbb{Q}$ with $E(A) \cong \mathbb{Z}$ whose type is not idempotent. By Albrecht [5], the category $\mathscr{C}_A$ is not pre-Abelian. On the other hand, we can use one of the Rüdiger's construction methods to obtain a group B with $E(B) \cong \mathbb{Z}$ such that $r_p(A) \geq 2^{\aleph_0}$ for all primes p. Arguing as in Example 5.4, we obtain that $\mathscr{C}_B$ is pre-Abelian. Clearly, $\mathscr{C}_A$ and $\mathscr{C}_B$ are not equivalent.

On the other hand, the categories of locally A-projective and locally B-projective groups are equivalent if A and B are Abelian groups with $End(A) \cong End(B)$ whose endomorphism rings are discrete in the finite topology. Every locally A-projective be longs to the class $\mathscr{T}L_A$ of *A-torsion-less groups* which consists of the A-generated subgroups of cartesian powers of A.

Theorem 5.6 *Let A and B be torsion-free Abelian groups which are faithfully flat as modules over their endomorphism rings, and whose endomorphism rings are discrete in the finite topology. If* $End(A)$ *is left Noetherian, and* $End(B)$ *is Morita-equivalent to* $End(A)$*, then the categories* $\mathscr{T}L_A$ *and* $\mathscr{T}L_A$ *are equivalent.*

Proof Since being Noetherian is a Morita-invariant property, $End(B)$ is left Noetherian too. Moreover, Morita-equivalence preserves torsion-less modules. Because $End(A)$ is left Noetherian, $S_A(A^I)$ is A-solvable [20]. Therefore, $\mathscr{T}L_A$ is equivalent to the category of torsion-less right $End(A)$-modules since A is faithfully flat as an $End(A)$-module. Because a similar result holds for $\mathscr{T}L_B$, the theorem follows.

The author investigated Abelian groups with Morita-equivalent endomorphism rings in [10] showing that any equivalence of $\mathscr{C}_A$ and $\mathscr{C}_B$ is induced by a Morita-equivalence between $End(A)$ and $End(B)$. This and Example 5.5 give rise to

Problem 5.7 Let A and B be Abelian groups with Morita equivalent endomorphism ring. Identify (the largest) subclasses $\mathscr{C}_1 \subseteq \mathscr{C}_A$ and $\mathscr{C}_2 \subseteq \mathscr{C}_B$ for which the Morita-equivalence between $End(A)$ and $End(B)$ induces an equivalence between $\mathscr{C}_1$ and $\mathscr{C}_2$.

We now turn to the question when $\mathscr{C}_A$ is pre-Abelian if it contains J_p for some prime p.

Theorem 5.8 ([5]) *The following conditions are equivalent for an indecomposable generalized rank* 1*-group* A *with* $r_0(A) > 1$ *for which* $P(A) = \{p \mid J_p \in \mathscr{C}_A\}$ *is not empty:*

(a) $\mathscr{C}_A$ *is a pre-Abelian category.*
(b) (i) There exists an exact sequence $0 \to Ea \to A \to G \to 0$ *such that* $G = tG \oplus [\oplus_I \mathbb{Q}]$ *for some index-set* I *and* $G[p] = 0$ *for all* $p \in P(A)$.
(ii) If $r_p(A) < \infty$ *for some prime* p *with* $A \neq pA$ *and* $[r_p(A)]^2 = r_p(E)$*, then* $p \in P(A)$.

To see that $P(A)$ may be not empty, we consider the class of irreducible Murley groups. A group A is *irreducible* if it does not have any proper, non-zero pure fully invariant subgroups. A torsion-free group A is a *Murley group* if $r_p(A) \leq 1$ for all primes p. A homogeneous Murley group is indecomposable; and irreducible Murley groups are homogeneous [24, Chapter 15].

Theorem 5.9 ([17]) *If* A *is an irreducible Murley group, then every reduced* A*-generated torsion-free group* G *is* A*-solvable. In particular,* $P(A) \neq \emptyset$ *in this case.*

Problem 5.10 Is an indecomposable finitely faithful S-groups A for which J_p is A-solvable for all primes p with $A \neq pA$ an irreducible Murley group?

Problem 5.11 Can we describe the structure of the A-solvable groups if A is a Murley group?

A particular interesting class of $\aleph_1$-A-projective groups are the A-coseparable groups. Here, an A-generated group G is said to be *A-coseparable* if it is $\aleph_1$-A-projective and every subgroup U of G such that G/U is finitely A-presented contains a direct summand V of G such that G/V is A-projective of finite A-rank. In particular, every A-projective group is A-coseparable, and it is undecidable in ZFC if there exist A-coseparable groups which are not A-projective [35].

Theorem 5.12 ([14]) *Let* A *be a torsion-free finitely faithful* S*-group. A reduced torsion-free* A*-generated group* G *such that* $\operatorname{Ext}(G, A)$ *is torsion-free is locally* A*-projective.*

An Abelian group B is said to be *finitely projective with respect to* A if it is projective with respect to all sequences $0 \to U \to A^n \to G \to 0$ with $S_A(U) = U$.

Theorem 5.13 ([14]) *Let* A *be a torsion-free finitely faithful* S*-group. Then, the following are equivalent for a torsion-free reduced* A*-generated group* G*:*

(a) $\operatorname{Ext}(G, A)$ *is torsion-free.*
(b) G *is finitely* A*-projective.*
(c) G *is* A*-coseparable.*
(d) G *is* A*-coseparable and locally* A*-projective.*

Similarly, an A-generated group G is said to be $\aleph_1$-*A-coseparable* if it is $\aleph_1$-A-projective and every A-generated subgroup U of G such that G/U is countable contains a direct summand V of G such that G/V is countable.

Theorem 5.14 [12, Theorem 3.3]) *Let A be a self-small countable torsion-free generalized rank 1 group. A group G is $\aleph_1$-A-coseparable if and only if G is A-solvable, and every exact sequence*

$$0 \to P \to X \to G \to 0$$

such that P is a direct summand of $\oplus_\omega A$ and X is A-generated splits.

We conclude this paper with an application of endomorphism rings to mixed Abelian groups. While a detailed discussion of this interesting topic is beyond the framework of this survey, we want to mention that self-small mixed groups A such that $r_0(A/tA)$ naturally arise in various problems concerning mixed groups. For instance, Rüdiger and the author discussed cellular covers of Abelian groups in 2014. Here, a *cellular covering sequence* for an Abelian group A is an exact sequence $0 \to K \to G \xrightarrow{\gamma} A \to 0$ for which the induced map

$$\gamma_* : \mathrm{Hom}_{\mathbb{Z}}(G, G) \to \mathrm{Hom}_{\mathbb{Z}}(G, A)$$

is an isomorphism. Every group A admits a cellular covering sequence

$$0 \to 0 \to A \xrightarrow{\gamma} A \to 0$$

with γ an automorphism of A, *called a trivial cellular cover*. In this discussion, Rüdiger asked whether there exist (large classes of) honest, i.e., non-splitting, mixed groups without any non-trivial covering sequences. This question was answered positively in [13]. In the following, tA denotes the torsion subgroup of A, and A_p its p-torsion subgroup.

Theorem 5.15 *(a) No self-small Abelian group A such that A/tA is a divisible group of finite rank has a non-trivial cellular cover.*

(b) Let A be a mixed Abelian group of finite torsion-free rank such that A_p is finite for all primes p. If A/pA is finite for all primes p with $A_p \neq 0$ and $A = pA$ for all primes p with $A_p = 0$, then A has no non-trivial covering sequence $0 \to K \to G \to A \to 0$ with $tE(G) \cong tE(A)$.

(c) There exist honest self-small mixed groups A_1 and A_2 of torsion-free rank $n \geq 2$ with $tA_1 \cong tA_2$ and $E(A_1) \cong E(A_2)$ such that A_1 admits a non-trivial cellular cover $0 \to K \to G \to A_1 \to 0$ with $E(G) \cong E(A_1)$, while A_2 admits no non-trivial cellular covering sequences at all.

Problem 5.16 In [40], Rüdiger and Laszlo Fuchs showed that a subgroups of $\mathbb{Q}$ has a non-trivial cellular cover if and only if it does not have idempotent type. Is it possible to determine which self-small mixed groups A with $r_0(A) = 1$ have a non-trivial cellular cover?

We did not discuss self-small mixed groups A such that $r_0(A/tA)$ is finite in this paper mostly because we were mainly focused on topics that are closely related to Rüdiger's work. The endomorphism rings of these groups were investigated in a series of papers by the B. Wickless, S. Breaz and the author, e.g., see [21] and [19].

Problem 5.17 Let A be a self-small mixed group such that $r_0(A/tA)$ is finite. When is $\mathscr{C}_A$ pre-Abelian?

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