Manfred Droste · László Fuchs
Brendan Goldsmith · Lutz Strüngmann
Editors

Groups, Modules, and Model Theory - Surveys and Recent Developments

In Memory of Rüdiger Göbel

Springer

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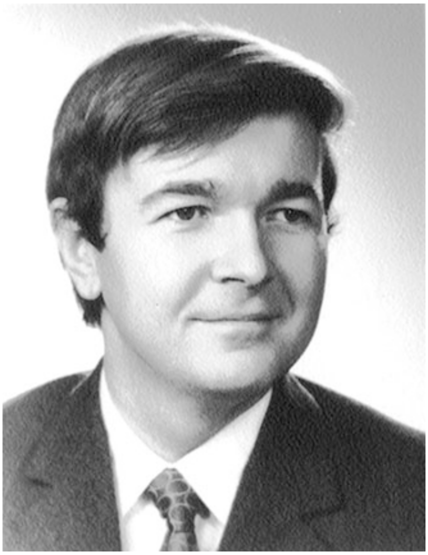

A youthful Rüdiger

To the memory of Rüdiger Göbel – a beloved friend and distinguished colleague who inspired generations of young mathematicians

Rüdiger in more recent years

Preface

When Rüdiger Göbel died in July 2014, his many friends and colleagues from around the world felt it appropriate to acknowledge his contributions to a wide-ranging area of group theory and model theory by organising a conference in his honour. Out of this developed the conference New Pathways Between Group Theory and Model Theory which took place in Mülheim an der Ruhr, Germany, during February 1–4, 2016. The enthusiastic response to this conference led directly to this volume; we are most thankful to all the participants of that conference who helped to make it an unforgettable event that Rüdiger would certainly have enjoyed. However, the material appearing in this book is not of the usual conference proceedings type: the editors have tried to present a balanced mix of survey papers, which will enable expert and non-expert alike to get a good overview of developments across a range of areas of group, module and model theory, along with research papers presenting some of the most recent developments in these same areas. Every effort has been made to make these research papers easily accessible in their introductory sections. We would hope that the material is of interest to both beginning graduate students and experienced researchers alike. The topics covered are, inevitably, just a cross section of the vast areas of group, module and model theory, but they reflect in a strong way the areas in which Rüdiger Göbel contributed so much. The book is divided into two sections, surveys and recent research developments, with each section containing material from all the areas of the title.

Finally, we would like to express our sincere thanks to the colleagues who contributed papers so enthusiastically, to the many experts who acted as referees for all the papers, to the professional staff at Springer and in particular to Dimana Tzvetkova, for their help in producing a volume which we hope is an appropriate commemoration of our late friend Rüdiger Göbel. A special word of thanks goes

to Katrin Leistner for her invaluable help during the organisation of the conference and also during the preparation of this memorial volume. We would also like to acknowledge the help given by Gabor Braun and Daniel Herden at various stages of the organisation of both the conference and this volume.

Leipzig, Germany — Manfred Droste
New Orleans, LA, USA — László Fuchs
Dublin, Ireland — Brendan Goldsmith
Mannheim, Germany — Lutz Strüngmann
November 2016

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Part II Research Articles

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Rüdiger Göbel—An Appreciation

Rüdiger Göbel, who died on July 28, 2014, was one of the leading algebraists of his generation. He was the only child of Gotthard and Ruth Göbel and was born in Fürstenwalde, Germany (later in the German Democratic Republic), on December 27, 1940, during World War II. His family eventually fled the German Democratic Republic settling in what was then West Germany. Rüdiger went on to study mathematics and physics at the Johann Wolfgang Goethe University in Frankfurt am Main, receiving his doctorate in 1967 under the supervision of Reinhold Baer; he was Baer's last doctoral student, and the two remained good friends for the rest of Baer's life even though they still maintained the very formal means of address: Rüdiger often joked that despite his new position as a professor, he was still Herr Göbel, while Baer remained Herr Professor Baer.

Rüdiger's academic career began with positions in Würzburg, Germany, and Austin, Texas, working in the area of physics and particularly in relativity theory. His habilitation followed in 1974 with the title 'General Relativity Theory and Group Theory'. He moved in the same year to the University of Essen as a professor of applied mathematics, eventually changing to pure mathematics; he remained in the University of Essen (or Duisburg-Essen as it became) for the rest of his career.

On his way to the university in Frankfurt on December 3, 1964, he met a fellow student (of English and history) Heidi Drexler, and they married in 1969 and have one daughter Ines, who also studied mathematics. Rüdiger often commented that that trip to the university was his 'lucky day'; he and Heidi had an enduring, warm and loving relationship which was so evident to the numerous visitors who enjoyed the hospitality of Rüdiger, Heidi and Ines (and James, the dog) in the Göbel family home; the wonderful aroma of freshly brewed FortMason tea at breakfast is never to be forgotten. (Incidentally, this tea was the drink he carried in his flask at conferences, seminars and colloquium talks and jokingly referred to as his 'whiskey'.) Rüdiger and Heidi had another bond: they were co-authors in 1978 of a paper on an old English riddle in the Exeter book—see [202].

At the outset, let us stress that it is impossible in a few pages to give a detailed overview of the many research contributions made by Rüdiger Göbel, and it will be for a later generation to assess his impact on the world of algebra. His publication list numbers 211 papers, and of these, three could be viewed as historical, four are unpublished manuscripts and one (with Heidi) non-mathematical; in addition, there are two important research books with Jan Trlifaj [1, 2] and 11 books, mainly conference proceedings, where he was a coeditor, and we know of six further works which he had listed prior to his death as 'in preparation'—see the publication section at the end of this Appreciation.

An obvious feature of Rüdiger's research output is the number of co-authors, some 53 in total, but perhaps more surprising is the number of co-authors with whom he wrote multiple papers and the duration of these collaborations. Four co-authors had more than 10 joint papers with him, and these collaborations endured for more than 25 years: Saharon Shelah co-authored 35 papers between 1985 and 2014; Manfred Dugas co-authored 28 papers from 1979 to 2007; Manfred Droste co-authored 18 papers in the 34-year period 1979–2013 and Brendan Goldsmith co-authored 11 papers between 1984 and 2010. Many others had collaborations resulting in more than 5 joint papers. Rüdiger always enjoyed this joint approach to working on a problem and often expressed the view that 'it's fun working together'; in fact, just 23 of his listed papers are singly authored. He was a generous co-author, quick to share ideas but always demanding in terms of getting the best results possible. His ability to move from one topic to another was impressive. His research work can be split crudely into a number of categories; the definition of such categories is, of course, somewhat arbitrary, and many other divisions are possible; nor could the categories ever be regarded as disjoint. We look at each of these briefly:

(1) Physics

Rüdiger published just four papers in the area of physics [194, 203, 204, 208], and these relate mainly to topological issues arising in general relativity theory. Despite his comparatively small output in physics, Rüdiger was very proud of his work in this area, particularly as it had drawn praise from no less a figure than Stephen Hawking.

(2) Informatics

Rüdiger also worked in an area that can be broadly described as informatics with Manfred Droste—see, for example, [130, 136, 147, 148, 149, 151, 152]. These papers are mainly concerned with domain theory, the mathematics underpinning denotational semantics of programming languages and the application of model theoretic methods to systematically construct universal Scott domains. Broadly, similar ideas influenced the later probabilistic approach to the classical Ulm-Zippin theorem on reduced p-groups [16].

(3) Non-commutative Group Theory

Rüdiger's earliest works in algebra were, not surprisingly as a student of Baer, in the area of group theory, specifically on products of groups. Interestingly, his earlier paper 'On stout and slender groups' [206] would indirectly lead him into the study of cotorsion-free Abelian groups, an area where he made significant contributions—see (4) below. His later work on non-commutative groups, mainly with Braun, Droste, Dugas and Paras, dealt largely with the construction of groups with prescribed automorphism groups—see, for example, [63, 66, 68, 79, 82, 91, 97, 105, 111, 114, 122, 137, 143]. He also retained a long-term interest in infinite permutation groups and related matters—see, for example, [7, 54, 76, 192, 199]. In many of these works, one can see the influence of his work in Abelian groups. Rüdiger, like his close friend Lászlό Fuchs, had a strong belief that Abelian group theory provides a powerful source of ideas and techniques that can be applied in other areas of algebra, an approach he outlined in [115]. The validity of this belief can be seen in his approach to the works just listed and also to papers such as [32, 86, 89, 109, 123, 162, 166] which are actually outside of the realm of pure group theory; incidentally, Rüdiger was very proud of the $25 prize for solving C.U. Jensen's problem in [162] and for many years had the cheque from Jensen pinned to the wall in his office.

(4) Abelian Group Theory

For many, the name of Rüdiger Göbel is synonymous with Abelian group theory. A detailed evaluation of his contributions to this area alone would require many pages. A characteristic feature of his many contributions in this area was his pioneering use of techniques from set theory and infinite combinatorics; these techniques, in growing levels of sophistication, can be traced throughout his work. At the risk of omitting some of his significant research outputs—and it is important to stress that the selection made below reflects the tastes and knowledge of the editors; it is not intended to suggest that papers not specifically mentioned below are in any way of lesser value than those mentioned—we shall consider the following broad areas:

(a) Products, Slenderness and the Baer-Specker Group

As mentioned above, Rüdiger was initially interested in products of noncommutative groups, but by the late 1970s, his interests had turned to questions relating to the notion of slenderness and its interconnections with the Baer-Specker group, $\mathbb{Z}^{\aleph_0}$ and its higher analogues, $\mathbb{Z}^{\kappa}$, for arbitrary cardinals κ. Working independently with Manfred Dugas and Burkhard Wald, he established many interesting results in this area—see, for example, [189, 190, 191, 193, 196, 197, 198, 200]; his paper with Wald [196] was, for many Abelian group theorists, their first taste of Martin's axiom used by an algebraist rather than a logician. In some sense, this paper was a prelude to Rüdiger's importation into Abelian group theory of many of the ideas arising in set theory and infinite combinatorics, which, as noted above, became one of his characteristics. The Baer-Specker group and the wonderful complexity of its set of subgroups were a topic of constant interest to Rüdiger, and he had many subsequent works in this area—see, for example, [24, 25, 31, 84, 85, 100, 103, 107, 119].

(b) Endomorphism Rings

In the early 1960s, the theory of Abelian groups had been thrown into a certain amount of disarray by the dramatic results, (A), of the late A.L.S. Corner on the realisation of rings as endomorphism rings of torsion-free Abelian groups. Soon after arriving in Essen, Rüdiger embarked on a serious study of this area, visiting Corner in Oxford on several occasions to discuss the topic. Probably during this period, he learned of Corner's unpublished conjecture that every cotorsion-free ring is an endomorphism ring, and the proof of this conjecture in conjunction with Manfred Dugas became one of his most significant early achievements; the proof in [187] worked in $V = L$, while [185] exploited a combinatorial principle due to Shelah to settle the question in ZFC. At around this time, Rüdiger was introduced to the notion of *inessential endomorphisms*; this was a very general concept developed by Corner in an unpublished paper presented at the 1967 Montpellier Conference and later refined by Goldsmith in his doctoral thesis. This notion turned out to be the key to realising rings as 'almost' the endomorphism rings of groups possessing many projections that cannot be suppressed. Starting with [181], Rüdiger and various co-authors turned the notion into a key concept in the theory of realisations—see, for example, [135, 139, 141, 150, 155, 161, 173, 177]—and in the process, the abbreviation 'Iness' was transformed to 'Ines', the name of Rüdiger's daughter! The paper [177] became an important introduction to the whole area and was the origin of many subsequent far-reaching works by Rüdiger, often in conjunction with Shelah, involving more sophisticated combinatorial arguments—his books with Jan Trlifaj [1, 2] are an excellent source of more up-to-date information on these developments. It seems fitting that Corner's original work, (B), introducing the notion of inessential now appears in this volume.

(c) Modules with Distinguished Submodules

Rüdiger also had a keen interest in the area of representation theory that arose from an early work, (C), of Sheila Brenner and Michael Butler on what might be loosely described as 'vector spaces with distinguished subspaces'. Corner's subsequent generalisation of that work, (D), involving five distinguished submodules, intrigued Rüdiger, and he produced several interesting papers on the topic, showing that four submodules suffice and investigating the situation when a lesser number of submodules is used—see [103, 110, 116, 131, 132, 140, 153, 157, 163, 167, 168, 169].

(d) E-Rings

The notion of an E-ring was introduced by Phill Schultz, (E), in 1973 and was a topic which, with subsequent generalisations, interested Rüdiger for a long time; in fact, his second-last published paper [2] was on this topic, and we are aware that he was considering a further paper in the same area at the time of his death.

(e) Cellular Covers

The notion of a cellular cover arises from homotopy theory and is dual to the notion of localisation investigated by Rüdiger and Shelah in [77] in the context of simple groups—see also [72]. Rüdiger began working on this topic in 2007 with Farjoun and Segev, [42], and retained an interest in it right up to the time of his death: a paper on cellular covers of h-divisible modules is listed as being 'in preparation'—see [3] in the appropriate section below. See also the papers [11, 13, 22, 27, 36].

(f) Other Topics in Module Theory

As noted at the beginning of this Appreciation, it is not really possible to give a complete survey of Rüdiger's work in just a few pages. His interests were extensive, and in addition to the topics mentioned in the preceding subsections, he produced important papers in a wide range of other areas including Butler groups [9, 108], entropy [12], torsion and mixed modules [15, 67, 138, 145, 164, 171, 180, 188], Crawley modules [47, 49], group algebras [34, 65], cotorsion theories [28, 34, 83, 93], cotilting modules [54, 88, 92] and measure theoretic algebra [70, 95, 124, 125].

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(B) A.L.S. Corner, *A class of pure subgroups of the Specker group,* this Volume.

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(D) A.L.S. Corner, *Endomorphism algebras of large modules with distinguished submodules,* J. Algebra, **11** (1969) pp.155–185.

(E) P. Schultz, *The endomorphism ring of the additive group of a ring,* J. Austral. Math. Soc., **15** (1973) pp.60–69.

Personal Comments

Manfred Droste:

Rüdiger Göbel was a great teacher as well as a supportive colleague. He had so many interesting and difficult research works but was always so modest. In spite of the demands of this research work, he also had time for personal talks. We will always remember his enthusiasm and personal friendship.

László Fuchs:

It was quite an experience to work with Rüdiger. I admired his huge knowledge, his quick responses to difficult questions and his good judgement in selecting important features. We lost a prominent mathematician, a good friend. He will be sorely missed.

Brendan Goldsmith:

When Rüdiger Göbel died, the world of mathematics lost an important member of its community, but his colleagues lost more than this: a generous friend always with a word of encouragement and a smile.

Ní bheidh a leithéid arís ann!

Lutz Strüngmann:

Rüdiger has been my mentor for 25 years. I took linear algebra courses from him when I was a freshman, and ever since then, he has become not only a colleague, but a true friend. I will never forget his passion for mathematics, his unique way of teaching and his warm friendship.

Rüdiger, du warst ein großartiger Lehrer!

Students of Rüdiger Göbel

Rüdiger's students were very important to him, and he was always generous in sharing his ideas and motivating his students. For many years, the students' office in the Mathematics Department at the University of Duisburg-Essen has been a stimulating and comfortable one for both students and visitors alike, in no small measure due to the influence of Simone Wallutis and her three 'Jungs'. Rüdiger was very proud of this and devoted time and energy to ensuring that this tradition remained alive.

The students whom he formally supervised for doctorates are listed below, but it is important to say that others regarded themselves as being his 'informal' students and were active participants in his research seminars at various times; Ulrich Albrecht (professor at Auburn University, Alabama) and Berthold Franzen (professor at Technische Hochschule Mittelhessen, Gießen, Germany) certainly fall into this category.

Burkhard Wald,	1979	
Manfred Droste,	1982	Professor at Universität Leipzig, Germany, Institut für Informatik
Claudia Böttinger,	1990	
Klaus Kowalski,	1990	
Simone Wallutis, (née Pabst)	1994	
Anja Elter,	1996	
Lutz Strüngmann,	1998	Professor at Hochschule Mannheim, Germany, Fakultät für Informatik
Georg Hennecke,	1999	
Ansgar Opdenhövel,	1999	
Gábor Braun,	2003	
Daniel Herden,	2005	Professor at Baylor University, Texas, Department of Mathematics
Sebastian Pokutta,	2005	Professor at Georgia Institute of Technology, Atlanta, ISyE, ARC and ML@GT
Nicole Hülsmann,	2006	
Héctor Gabriel Salazar Pedroza,	2012	
Montakarn Petapirak,	2014	
Katrin Leistner,	2015	
Christian Müller,	open	

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Part I
Survey Articles

In this chapter we present a collection of survey articles and introductory articles on various topics within the theory of groups, modules and models.

Properties of Abelian Groups Determined by Their Endomorphism Ring

Ulrich Albrecht

Abstract The goal of this paper is to give a survey of how endomorphism rings can be used to study the behavior of modules. While the first part considers modules over arbitrary rings, the second half focuses mainly on the case of torsion-free Abelian groups. Although there are many applications of endomorphism rings to the theory of mixed Abelian groups, a comprehensive discussion of this subject is beyond the framework of a survey article. In particular, we only present core results, and provide an extensive literature list for those who want to get deeper into the subject.

Keywords Abelian Groups • Endomorphism Rings • Flatness • Adjoint Functors

1 Introduction

This paper has been motivated to a large part by Rüdiger's seminal work on endomorphism algebras. Since his contributions to this subject are discussed in another paper in this volume, it is our goal to highlight the connections between Rüdiger's work and the many ways endomorphism rings are used in Abelian Group Theory.

Traditionally, the goal of Abelian Group Theory has been to describe as large classes of Abelian groups as possible in terms of meaningful numerical invariants. The first major class of groups characterized in this way were the countable p-groups [50]. Ulm's work directly lead to the discussion of the totally projective p-groups as the largest class of torsion groups which are determined by their Ulm-Kaplansky invariants [32]. Baer published a similarly important result for torsion-free groups in 1937 [27]. He showed that the subgroups of the rational numbers are determined up to isomorphism by their types. Moreover, he showed that their rank 1 summands completely determine the completely decomposable groups.

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Unfortunately, the hopes of the 1940s that it is possible to develop a comprehensive description of the structure of torsion-free groups (of finite rank) were disappointed by a series of examples which Jonsson gave in the 1950s [46, 47].

Example 1.1 ([46]) There exists a torsion-free Abelian group G such that $G = A \oplus B = C \oplus D$ where A, B, C, and D are indecomposable groups with $r_0(A) = 1$, $r_0(B) = 3$, and $r_0(C) = r_0(D) = 2$.

Furthermore, even the pure subgroups of completely decomposable groups cannot be classified in any meaningful way [39]. These and many other results, which cannot be mentioned within the framework of this survey, clearly indicate that a variety of approaches are needed to understand the behavior of torsion-free Abelian groups (of finite rank) better. Although numerical invariants, albeit in a more general form, have been found for large classes of Butler groups, even subgroups of finite index of completely decomposable groups have a structure which is too complex to describe comprehensively in this way [39].

One way to overcome the previously mentioned difficulties is to consider methods and tools from other areas of Mathematics. Rüdiger was one of the pioneers using tools from set-theory and infinite combinatorics to construct large classes of Abelian groups with prescribed properties. We follow a similar approach, but focus on applications of non-commutative ring-theory to Abelian groups instead, an approach initiated by Arnold in the 1970s [24]. Rüdiger's realization theorems for endomorphism ring clearly play a central role in this as is shown in Sect. 3.

Studying Abelian groups via their endomorphism rings takes a point of view which is radically different from the traditional approach. Instead of developing a structure theory, it views an Abelian group A as an object that is best studied by looking at its interaction with other objects. This approach is philosophically related to the one taken in modern Physics where objects like elementary particles are studied through their interaction with other particles. To study this interaction, methods from homological algebra and ring-theory are employed. To facilitate this type of investigation, one usually relies on an adjoint pair of functors between the category of Abelian groups and the category of right modules over the endomorphism ring of A. Section 2 looks at these functors, and introduces some of the basic concepts.

Applications are discussed in Sects. 4 and 5. We give several examples of A-solvable Abelian groups which will answer questions concerning the size and generality of this class. Given the constraints of a survey paper, many interesting topics have to be omitted. Since it is the goal to relate our discussions to Rüdiger's work, we concentrate mostly on torsion-free groups of arbitrary rank. In particular, the discussion of quasi-properties of torsion-free groups of finite rank as well as properties of mixed groups which are described in terms of endomorphism rings have to be omitted in spite of the large amount of literature related to these topics.

2 Adjoint Functors

The interaction of a right R-module A with other R-modules is often described by the functors $\mathrm{Hom}_R(A, -)$ and $\mathrm{Hom}_R(-, A)$. Each of these functors actually carries a structure which is richer than that of an Abelian group, namely that of a right, respectively left, module over the endomorphism ring $E = End_R(A)$ of A. These module structures are induced by A since the latter can be viewed as an E-R-bimodule. Although many properties of a module, e.g., its direct sum decompositions, can be described in terms of its endomorphism ring, the classical theory of Abelian groups makes very little use of the information which can be obtained from this ring. This is quite surprising in view of the Baer-Kaplansky Theorem which states that two Abelian p-groups are isomorphic exactly if they have isomorphic endomorphism rings [28, 48]. The situation is quite different in the torsion-free case. Rüdiger's work shows that there exist proper classes of non-isomorphic torsion-free groups with isomorphic endomorphism rings [41].

Our discussion concentrates on the covariant functor $H_A(-) = \mathrm{Hom}_R(A, -)$ between the categories $\mathscr{M}_R$ of right R-modules and $\mathscr{M}_E$ of right E-modules. It forms one component of the adjoint pair (H_A, T_A) of functors between $\mathscr{M}_R$ and $\mathscr{M}_E$ where T_A is defined by $T_A(X) = X \otimes_E A$ for all right E-modules X. Associated with this adjoint pair are natural transformations $\theta_M : T_A H_A(M) \to M$ for $M \in \mathscr{M}_R$ and $\Phi_X : X \to H_A T_A(X)$ for $X \in \mathscr{M}_E$ defined by $\theta_M(\alpha \otimes a) = \alpha(a)$ and $[\Phi_X(x)](a) = x \otimes a$. The image of θ_M is called *the A-socle of M*, and is denoted by $S_A(M)$.

The idea to consider the category $\mathscr{M}_E$ to investigate properties of a torsion-free Abelian group A originated in two papers which appeared in 1975. Arnold and Lady showed in [25] that H_A and T_A induce an equivalence between the category of A-projective modules of finite A-rank and the category of finitely generated projective right E-modules. Here a right R-module P is *A-projective (of finite A-rank)* if it is a direct summand of a (finite) direct sum of copies of A. Arnold and Murley removed the finiteness conditions in [26] in case that A is a self-small module where a right R-module A is *self-small* if, for every index-set I and every $\alpha \in H_A(\oplus_I A)$, there is a finite subset I' of I such that $\alpha(A) \subseteq \oplus_{I'} A$. Every torsion-free Abelian group of finite rank is self-small, and so is every R-module with a countable endomorphism ring [26]. In contrast to slenderness, which arises in the discussion of the contravariant functor $\mathrm{Hom}_R(-, A)$, self-smallness is not affected by the existence of large cardinals.

A right R-module M is *(finitely, respectively κ-) A-generated* if it is an epimorphic image of a module of the form $\oplus_I A$ for some index-set I (with $|I| < \infty$, respectively $|I| < \kappa$). Since T_A is right exact, all R-modules of the form $T_A(X)$ with $X \in \mathscr{M}_E$ are A-generated, and it is easy to see that M is A-generated if and only if $S_A(M) = M$. We say that a right R-module M has an *A-projective resolution* if we can find an exact sequence

$$\ldots P_{n+1} \xrightarrow{\alpha_{n+1}} P_n \xrightarrow{\alpha_n} \ldots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

in which each P_n is A-projective.

Proposition 2.1 ([2]) *Let A be a self-small right R-module. A right R-module M has an A-projective resolution if and only if $M \cong T_A(X)$ for some right E-module X.*

However, not every A-generated module has an A-projective resolution, nor is the class of modules described by the last result closed with respect to direct summands as the following example shows:

Example 2.2 Let A be an Abelian group which fits into a non-splitting exact sequence $0 \to \mathbb{Z} \to A \to \mathbb{Q} \to 0$. Then, $E(A) = \mathbb{Z}$ by Arnold [24, Section 3]. If G is a torsion-free Abelian group of finite rank, then $r_0(T_A(G)) = 2r_0(G)$ since $r_0(A) = 2$. In particular, every finite rank group with an A-projective resolution has to have even rank by Proposition 2.1. Therefore, $\mathbb{Q}$ does not have an A-projective resolution although it is a direct summand of $T_A(\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ which has an A-projective resolution by Proposition 2.1.

The reason for the difficulties illustrated by the last example is that the module A need not be projective with respect to the sequences

$$0 \to \ker\ \alpha_n \to P_n \xrightarrow{\alpha_n} im\ \alpha_n \to 0$$

induced by an A-projective resolution of an R-module M. Adopting a standard notion from Abelian Group Theory, we say that an exact sequence

$$0 \to U \to V \to W \to 0$$

of right R-modules is *A-balanced* if the induced sequence

$$0 \to H_A(U) \to H_A(V) \to H_A(W) \to 0$$

is exact. The R-module M has an *A-balanced A-projective resolution* if it admits an A-projective resolution

$$\ldots P_{n+1} \xrightarrow{\alpha_n} P_n \xrightarrow{\alpha_{n-1}} \ldots \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

for which the induced sequences $0 \to \ker\ \alpha_n \to P_{n+1} \xrightarrow{\alpha_n} im\ \alpha_n \to 0$ are A-balanced for all $n < \omega$.

Proposition 2.3 ([2]) *Let A be a self-small Abelian group. If a module M has an A-balanced A-projective resolution, then θ_M is an isomorphism.*

Although this is a survey article, a brief proof of this result is included since it nicely illustrates the use of the adjointness of (H_A, T_A) without having to deal with the complexities of some of the later results:

Proof An A-balanced A-projective resolution of M induces an A-balanced exact sequence

$$0 \to U \to P_0 \to M \to 0$$

in which U is A-generated as an image of P_1. Applying the functors H_A and T_A successively induces the commutative diagram

$$\begin{array}{ccccccccc} & & T_AH_A(U) & \longrightarrow & T_AH_A(P_0) & \longrightarrow & T_AH_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_U & & \wr\downarrow \theta_{P_0} & & \downarrow \theta_M & & \\ 0 & \longrightarrow & U & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which θ_U is onto, and θ_{P_0} is an isomorphism by Arnold and Murley [26]. The Snake-Lemma yields that θ_M is an isomorphism.

A right R-module M is *A-solvable* if θ_M is an isomorphism; and the class of A-solvable right R-modules is denoted by $\mathscr{C}_A$. By Arnold and Lady [25] and Arnold and Murley [26], A-projective modules are A-solvable if A is self-small. Arnold and Murley also showed in [26] that every locally A-projective module M is A-solvable if the endomorphism ring of A is discrete in the finite topology. Here M is *locally A-projective (locally A-free)* if every finite subset of M is contained in an A-projective direct summand (isomorphic to A^n for some $n < \omega$) of M. The endomorphism ring E of a module A is *discrete in the finite topology* if there is a finitely generated E-submodule of A with $\mathrm{Hom}_R(A/U, A) = 0$. By Arnold and Murley [26], A is self-small if its endomorphism ring is discrete in the finite topology.

An A-projective resolution of an A-solvable module need not be A-balanced without additional conditions on A. For instance, if $A = \mathbb{Q} \oplus \mathbb{Z}$, then all groups are A-generated; and there exists an exact sequence $0 \to U \to \oplus_\omega A \to A \to 0$ with respect to which A is not projective. Before continuing our discussion, we want to remind the reader that a left R-module A is *faithful* if $X \otimes_R A \neq 0$ for all non-zero finitely generated right R-modules X. It is *fully faithful* if this holds for all right R-modules. Faithfully flat modules are fully faithful.

Theorem 2.4 ([2] and [3]) *The following conditions are equivalent for a right R-module M:*

(a) A is fully faithful as a left R-module.
(b) A right R-module M admits an A-balanced exact sequence

$$0 \to U \to \oplus_I A \to M \to 0$$

with $S_A(U) = U$ if and only if $M \in \mathscr{C}_A$.
(c) An exact sequence $0 \to U \xrightarrow{\alpha} M \to P \to 0$ splits if $\alpha(U) + S_A(M) = M$ and P is A-projective.

Baer had shown in [27] that every subgroup A of $\mathbb{Q}$ satisfies condition (c) for $P = A$. Hence, condition (c) is often referred to as *Baer's Lemma*. Arnold and Lady established the equivalence of (a) and (c) in [25] if A is a torsion-free Abelian group of finite rank and $M = A$. Unfortunately, neither Baer's nor their arguments carry over to the case that P is an arbitrary A-projective group.

It is well known from category theory that the kernel of a map $A^n \to A$ need not be A-generated unless A is flat as a module over its endomorphism ring (Ulmer's Theorem [51]). In particular, if $\alpha \in \mathrm{Hom}_R(M, N)$ for A-solvable modules M and N, then neither *ker* α nor *im* α need to be A-solvable. Therefore, we call a class $\mathscr{C}$ of A-generated groups *A-closed* if

(i) $\mathscr{C}$ is closed with respect to finite direct sums and A-generated submodules, and
(ii) $\ker\ \alpha \in \mathscr{C}$ for all $M, N \in \mathscr{C}$ and all $\alpha \in \mathrm{Hom}_R(M, N)$.

Theorem 2.5 ([7]) *The following are equivalent or a self-small right R-module A:*

(a) A is flat as a module over its endomorphism ring.
(b) There exists an A-closed class $\mathscr{C}$ containing A.
(c) $\mathscr{C}_A$ is the largest A-closed class containing the A-projective modules.

However, there exit R-modules A which are flat as modules over their endomorphism ring, but not faithful. For instance, the group $A = \mathbb{Q} \oplus \mathbb{Z}$ has this property. Therefore, A may not be projective with respect to exact sequences in $\mathscr{C}_A$ even if the latter is A-closed. Since the existence of sequences in $\mathscr{C}_A$ which are not A-balanced makes it difficult to develop a comprehensive homological algebra for $\mathscr{C}_A$, we call an A-closed class $\mathscr{C}$ *A-balanced* if every exact sequence $0 \to B \to C \to M \to 0$ with $B, C, M \in \mathscr{C}$ is A-balanced.

Theorem 2.6 ([7]) *The following are equivalent for a self-small right R-module A:*

(a) A is faithfully flat as a left E-module.
(b) There exists an A-balanced, A-closed class containing all of the A-projective modules.
(c) $\mathscr{C}_A$ is the largest A-balanced, A-closed class containing all of the A-projective modules.
(d) A right R-module has an A-balanced A-projective resolution if and only if it is A-solvable.

In particular, A-balanced A-projective resolutions of an A-solvable module M induce derived functors $Bext^n_A(-,-)$ on $\mathscr{C}_A$ such that

$$Bext^n_A(M, N) \cong \mathrm{Ext}^n_E(H_A(M), H_A(N))$$

for all A-solvable modules M and N [9].

Finally, the concept of A-solvable modules carries over naturally to the quasi-category of Abelian groups. Unfortunately, the discussion of quasi-concepts is beyond the framework of this survey.

3 Realization Theorems

The results of the last section raise the question whether it is possible to construct self-small modules A such that

(a) A is flat (or, e.g., faithfully flat or projective) when viewed as an E-module, and
(b) the endomorphism ring of A can be prescribed to belong to a specific class of rings, e.g., principal ideal domains, hereditary rings, or polynomial rings?

[38, Chapter 111] is dedicated to this question, and [38, Problem 84] particularly asks for criteria for certain types of rings to be endomorphism rings. However, the following example shows that one has to be somewhat careful when combining properties of the module ${}_EA$ in (a) with ring-theoretic properties of E in (b):

Example 3.1 Suppose that A is a right R-module such that E is a principal ideal domain. Then, A is an indecomposable Abelian group since E does not contain any non-trivial idempotents. If ${}_EA$ were projective as a left E-module, then ${}_EA \cong \oplus_I E$ for some index-set I which is only possible if $|I| = 1$. Thus, ${}_EA \cong E$. For instance, all Murley groups A have a principal ideal domain as an endomorphism ring [24]. Here a torsion-free groups A of finite rank is a *Murley-group* if $dim \mathbb{Z}/p\mathbb{Z} A/pA \leq 1$ for all primes p.

Fortunately, module-theoretic properties like faithfulness and flatness are not nearly as restrictive as projectivity. To see this, we are going to look at some of the standard construction methods of modules with a prescribed endomorphism ring. Although most of them have their origin in Abelian Group Theory, they actually hold for substantially more general classes of rings. For instance, Rüdiger and the author extended the construction of E-algebras to a non-commutative setting in [15]. The methods used in this extension can also be applied to the realization theorems for endomorphism rings in [31] and [34]. As in the commutative setting, some restrictions on R are necessary to avoid immediate counterexamples.

An element c of a ring R is *regular* if $cr = 0$ or $rc = 0$ implies $r = 0$. For any ring R, let

$$C(R) = \{s \in R \mid rs = sr \text{ for all } r \in R\}$$

denote *the center of* R. Clearly, $C(R)$ is a subring of R and $1_R \in C(R)$. As in [41], we consider a countable, multiplicatively closed subset $\mathbb{X} \subseteq C(R)$ of regular central elements of R which contains precisely one unit of R, the identity 1_R. The notions of $\mathbb{X}$-density, $\mathbb{X}$-purity, $\mathbb{X}$-torsion-freeness, and $\mathbb{X}$-cotorsion-freeness carry over literally from the commutative setting [41]. In particular, $\widehat{R}$ denotes the $\mathbb{X}$-completion of R.

Theorem 3.2 *Let S be an extension ring of R which is $\mathbb{X}$-cotorsion-free and torsion-free as a $C(R)$-module. If $\kappa^+ \leq \mu \leq \lambda$ are cardinals such that $|S| = \kappa$, μ is regular, and $\lambda^\kappa = \lambda^{\aleph_0}$, then there exists an $\mathbb{X}$-cotorsion-free right R-module A such that*

(a) $End_R(A) = S$, and
(b) every countably generated S-submodule of ${}_SA$ is contained in a free S-submodule.

In particular, the endomorphism ring of A is discrete in the finite topology, and A is flat as a left E-module. Moreover, if S is countable, then A is also faithful.

Proof Since the description of the actual construction of A is beyond the framework of this survey, the interested reader is referred to [15] to identify which modifications need to be made to the proof of [41, Theorem 12.3.4] in order to obtain A. In particular, Rüdiger had pointed out during the writing of [15] that the module A can be constructed in [41, Theorem 12.3.4] in such a way that it contains a family $\mathscr{F}$ of countably generated free submodules with the following properties:

(a) Every countable subset of A is contained in an element of $\mathscr{F}$.
(b) $\Sigma_{n<\omega} F_n \in \mathscr{F}$ for all families $\{F_n \mid n < \omega\} \subseteq \mathscr{F}$.

Clearly, the existence of $\mathscr{F}$ guarantees that A is flat as an S-module. To see that the endomorphism ring of A is discrete in the finite topology, observe that A is constructed as an $\mathbb{X}$-dense submodule of the $\mathbb{X}$-completion of a free S-module. Therefore, we can find a left S-module monomorphism $\alpha : S \to A$. Consider $\beta \in S = End_R(A)$ with $0 = \beta(\alpha(1_A))$. Since α is S-linear, we have $\beta(\alpha(1_A)) = \alpha(\beta 1_A)) = \alpha(\beta)$. Thus, $\beta = 0$ since α is one-to-one.

To see that A is faithful if S is countable, let I be a maximal right ideal of S with $IA = A$, and select $F_0 \in \mathscr{F}$. There is a countable S-submodule Y_0 of A such that $F_0 \subseteq IY_0$. Select $F_1 \in \mathscr{F}$ with $IY_0, Y_0 \subseteq F_1$. Continuing inductively, we obtain an ascending chain $\{F_n \in \mathscr{F} \mid n < \omega\}$ such that $F_n \subseteq IF_{n+1} \subseteq F_{n+1}$ for all $n < \omega$. Hence, $F' = \cup_{n<\omega} F_n$ is a free submodule of A such that $IF' = F'$. However, this is only possible if $I = E$.

The countability condition in the last result can be removed under **V = L** by adapting the arguments of [34] to the non-commutative setting:

Corollary 3.3 (ZFC + $\diamondsuit_\kappa$) *Let S be an extension ring of R which is $\mathbb{X}$-cotorsion-free and torsion-free as a $C(R)$-module. If κ is a regular uncountable cardinal such that $|S| < \kappa$, then there exists an $\mathbb{X}$-cotorsion-free right R-module A with the following properties:*

(a) $End_R(A) = S$, and
(b) Every κ-generated S-submodule of ${}_SA$ is contained in a free S-submodule.

In particular, the endomorphism ring of A is discrete in the finite topology, and A is faithfully flat as left E-module [7].

We want to point out that Faticoni used a Black Box construction similar to the one in Theorem 3.2 to construct an Abelian group A which is faithful, but not fully faithful as a module over its endomorphism ring [36].

Theorem 3.2 and Corollary 3.3 can be used to construct large class of A-solvable groups which are not A-projective:

Theorem 3.4 ([7]) *Let A be a cotorsion-free self-small Abelian group which is faithfully flat as a module over its endomorphism ring.*

(a) If A is countable, then there exist a proper class of A-solvable groups with endomorphism ring E^{op}.
(b) (ZFC + V = L) There exist a proper class of A-solvable groups with endomorphism ring E^{op}.

Proof We use either Theorem 3.2 or Corollary 3.3 to obtain a proper class of Abelian groups G with $End(G) = E^{op}$. Then, G^{op} is a right E-module and $T_A(G^{op})$ is A-solvable. An application of the Adjoint-Functor-Theorem completes the proof.

However, there are several question arising from the last results:

Problem 3.5 Can the Black Box be used directly to construct arbitrarily large classes of A-solvable groups in case A is countable instead of using E^{op}?

In [37], Franzen and Rüdiger used the Black Box to obtain modules over commutative rings R with prescribed endomorphism rings which contain a module of the form $\oplus_I B$ as a dense and pure submodule where B is a cotorsion-free faithful R-module. Combining this construction with the arguments from [15] should yield the desired result by replacing the free modules in the definition of the family $\mathscr{F}$ by B-projective modules.

Problem 3.6 Show directly that large classes of A-solvable groups exist assuming $\mathbf{V} = \mathbf{L}$ instead of using E^{op}.

In addition to the previously mentioned realization theorems, there are also the classical results by Zassenhaus and Corner from the 1960s, each of which will also produce Abelian groups which are faithfully flat as modules over their endomorphism ring:

Theorem 3.7 *Let R be a countable ring whose additive group is torsion-free and reduced.*

(a) There exists a countable Abelian group A with $E(A) = R$ [30].
(b) If $r_0(R) = n$, then A can be chosen to have rank $2n$ [30].
(c) If R^+ is a free group of rank n, then A can be chosen to have rank n too [53].

In either case, A has an endomorphism ring which is discrete in the finite topology, and is faithfully flat as an E-module [7].

Finally, we want to remark that the contra-variant functor $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ induces a duality between the direct summands of cartesian powers of A and projective left E-modules if A is a slender Abelian group. This duality was initially discussed by Huber and Warfield in [45] in case that A is a torsion-free group of finite rank, while the author considered the general case in [4]. Again, Rüdiger's realization theorems provide us with large classes of slender groups with a prescribed endomorphism ring.

4 Torsion-Free Abelian Groups

We now turn our discussion to Abelian groups, although many of our results will carry over to a more general setting, e.g., to modules over Dedekind domains. [38, Problem 84] asks to find criteria for certain types of rings to be endomorphism rings, but does specify what form these criteria should take, e.g., whether or not they are to be numerical invariants or properties describing the interaction of a group with a certain type of endomorphism ring with other groups. In the following, we interpret this problem to have two parts, namely

(a) How are ring-theoretic properties of the endomorphism ring of an Abelian group A reflected in the structure and the homological properties of A?
(b) How are structural and homological properties of an Abelian group A reflected in ring-theoretic properties of its endomorphism ring?

If A is fully faithful as an E-module, then H_A and T_A induce a one-to-one and onto correspondence between the right ideals of E and the A-generated subgroups of A. Because of this, it is frequently possible to address these questions for properties of a ring, which are definable in terms of ideals and submodules of projective modules. On the other hand, properties like commutativity, or more generally those given by polynomial identities, are virtually impossible to describe as can, for instance, be seen in [23] which looks at Abelian groups with commutative endomorphism rings.

We begin our discussion by investigating the connection between ring-theoretic properties of A and some of the fundamental properties of homogeneous completely decomposable groups which Baer considered in his 1937 paper [27]. For instance, if G is a subgroup of a homogeneous completely decomposable group of type τ and $G = G(\tau)$, then G is homogeneous completely decomposable.

Theorem 4.1 ([1] and [25]) *The following conditions are equivalent for a self-small torsion-free Abelian group A:*

(a) A is faithfully flat as an E-module and E is right hereditary.
(b) (i) A satisfies the conclusions of Baer's Lemma (see Theorem 2.4).
(ii) A-generated subgroups of A-projective groups are A-projective.

Arnold and Lady had investigated the conditions in (b) in the case that A is a torsion-free group of finite rank [25]. However, their arguments do not carry over to the general case. Furthermore, condition (b.ii) alone need not imply that E is right hereditary as was shown in [11].

Rings satisfying chain conditions are of particular interest in ring-theory, and they are often considered in conjunction with the requirement that the ring is right or left non-singular [43, 49]. However, when describing groups whose endomorphism ring satisfies chain conditions, we need to be aware of several facts that make it difficult to describe these groups in terms of numerical invariants:

- The endomorphism ring of torsion-free groups of finite rank has finite right and left Goldie-dimension.

- A semi-prime subring of a finite dimensional $\mathbb{Q}$-algebra is right and left Noetherian [24, Chapter 9].
- Descending chain conditions on right or left ideals are usually too restrictive to yield interesting classes of groups [38, Theorem 11.3].
- Standard group-theoretic concepts like types and purity have only limited bearings on ring-theoretic properties of an endomorphism ring unless we restrict our discussion to the finite rank case [6].

To avoid immediate restrictions on the rank of E, we turn to the notion of non-singularity introduced by Goodearl and Stenstrom [43, 49]. Taking this approach, the author was able to give a description of the Abelian groups A with a right and left Noetherian, hereditary endomorphism ring in [1, Theorem 5.1]. Since these groups have many of the homological properties usually associated with rank 1 groups, they are called *generalized rank* 1 *groups*, and play an important role in the theory of A-solvable groups. An important class of generalized rank 1 groups are the finitely faithful S-groups, which consists of all finite rank torsion-free groups A such that $r_p(E) = [r_p(A)]^2$ for all primes p [42]. Goeters showed that each finally faithful S-group has a hereditary endomorphism ring [42]. Hence, $Bext^n_A(G,H) \cong \mathrm{Ext}^n_E(H_A(G), H_A(H)) = 0$ for all A-solvable groups G and H and all $n > 1$. Moreover, we can describe how $Bext^1_A(G,A)$ is embedded into $Ext_{\mathbb{Z}}(G,A)$ in this case.

Proposition 4.2 ([14]) *If A is a finitely faithful A-group, then the group*

$$\mathrm{Ext}_{\mathbb{Z}}(G,A)/Bext^1_A(G,A)$$

is torsion-free and divisible for all torsion-free A-solvable groups G.

For a right R-module M, *the singular submodule of M* is

$$Z(M) = \{x \in M | xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$$

which takes the place of the torsion submodule in the general setting. The module M is called *non-singular* if $Z(M) = 0$, and *singular* if $M = Z(M)$. The ring R is *right non-singular* if it is non-singular as a right R-module. A ring is *a right p.p.-ring* if all principal right ideals are projective. Right p.p.-ring play an important role in the theory of non-singular rings and modules, e.g., see [22, 29, 33, 44], and [18]. Finally, a submodule U of an R-module M is *$\mathscr{S}$-closed* if M/U is non-singular.

However, the endomorphism ring E of a non-singular module M over a non-commutative non-singular ring may behave quite different from that of a torsion-free module over an integral domain. For instance, R need not be a subring of E, and M may not be non-singular over its endomorphism ring as Rüdiger and the author showed in [16].

Problem 4.3 Abelian groups whose endomorphism ring is a right p.p.-ring were described in [6]. Is it possible to give a description of the torsion-free Abelian groups with a right non-singular endomorphism ring?

However, finitely generated non-singular modules over a non-singular ring need not be submodules of free modules in contrast to the situation in the case of integral domains. Rings having this properties are called *strongly right non-singular*, and include the semi-prime right and left Goldie-rings. In particular, a ring is right and left strongly non-singular if it is a semi-prime subring of a finite dimensional $\mathbb{Q}$-algebra. For Abelian groups A with a strongly non-singular endomorphism ring, it is possible to define more meaningful notions of torsion-freeness and purity for the class of A-generated groups.

An A-generated group G is *A-torsion-free* if every finitely A-generated subgroup U is isomorphic to a subgroup of an A-projective group (which need not be a subgroup of G). An A-generated subgroup U of an A-torsion-free group G is *A-pure* if $(U + P)/U$ is A-torsion-free for all finitely A-generated subgroups P of G. We want to emphasize that A-pure subgroups need not be A-balanced. Using the concept of the $\mathscr{S}$-closure of a submodule of a non-singular module it is also possible to introduce the notion of the A-closure of an A-generated subgroup of a A-torsion-free group.

Theorem 4.4 ([6]) *Let A be a self-small torsion-free Abelian group which is E-flat such that E is a right strongly non-singular ring.*

(a) A group G is A-torsion-free if and only if G is A-solvable and $H_A(G)$ is non-singular. In particular, A-generated subgroups and direct sums of A-torsion-free groups are A-torsion-free.
(b) An A-generated subgroup U of an A-torsion-free group G is A-pure if and only if $H_A(G)/H_A(U)$ is non-singular.

Problem 4.5 Define the notions of A-torsion-freeness and A-purity in case that E is not a strongly non-singular ring.

C. Walker called a subgroup U of an Abelian group G *A^*-pure* if it is a direct summand of all subgroups H of G which contain U and have the property that H/U is an image of A [52]. It is *P^*_A-pure* if it is a direct summand of all subgroups H of G which contain U and have the property that H/U is finitely A-generated.

Theorem 4.6 ([6]) *The following conditions are equivalent for an Abelian group A which is E-flat and has a right strongly non-singular endomorphism ring:*

(a) E is a right p.p.- (right semi-hereditary) ring.
(b) If $\alpha \in E$ ($\alpha \in E(A^n)$ for some $n < \omega$), then ker α *is a direct summand.*
(c) An A-generated subgroup U of an A-torsion-free group is A_*-pure (P^*_A-pure) if and only if it is A-pure.

Furthermore, the question arises how A-purity and the standard notion of purity are related.

Theorem 4.7 ([6]) *The following conditions are equivalent for a self-small E-flat Abelian group A with a strongly right non-singular endomorphism ring:*

(a) If G is a torsion-free A-solvable group, then G is A-torsion-free, and every pure A-generated subgroup of G is A-pure in G.
(b) A/U is torsion for all A-generated subgroups U of A with $\mathrm{Hom}_{\mathbb{Z}}(A/U, A) = 0$.

The last condition is, for instance, satisfied if $\mathbb{Q}E$ is a semi-simple Artinian ring.

Problem 4.8 Do the last two results remain true if E is not strongly non-singular?

We conclude this section by looking at locally A-projective groups and their A-pure subgroups. In particular, we obtain a version of Pontryagin's criterion for A-solvable groups:

Theorem 4.9 ([8]) *Let A be an E-flat Abelian group with a right strongly non-singular, right semi-hereditary endomorphism ring which is discrete in the finite topology.*

(a) A-pure subgroups of locally A-projective subgroups are A-projective.
(b) An A-pure subgroup of a locally A-projective group is A-projective if it is an epimorphic image of $\oplus_\omega A$.
(c) A countably A-generated A-torsion-free group G is A-projective if every finitely A-generated subgroup of G is contained in a finitely A-generated A-pure subgroup of G.

In particular, $S_A(A^I)$ is locally A-free if E is left Noetherian [20]. Surprisingly, the converse holds too:

Corollary 4.10 ([20]) *Let A be a slender Abelian group of non-measurable cardinality whose endomorphism ring is discrete in the finite topology. If* $S_A(A^I)$ *is locally A-free for all index-sets of non-measurable cardinality, then E is left Noetherian.*

Problem 4.11 Can the various Black Box methods used in [41] to construct separable Abelian groups be adapted to obtain large classes of locally A-projective groups?

In view of Corollary 4.10, some additional ring-theoretic restrictions on E may be necessary.

5 Applications

We want to remind the reader that the class $\mathscr{C}_A$ of A-solvable groups consists of all Abelian groups G for which the evaluation map $\theta_G : T_A H_A(G) \to G$ is an isomorphism. When looking at $\mathscr{C}_A$, the question immediately arises which groups, in addition to A-projective groups, belong to $\mathscr{C}_A$? Arguing as in the proof of Theorem 3.2 or Corollary 3.3, it is easy to see that $\mathscr{C}_A$ contains the κ-A-projective groups whenever $\kappa > |A|$ is a regular cardinal and A is faithfully flat as an E-module [7]. Here, an A-generated group G is *κ-A-projective* if every

κ-A-generated subgroup U of G can be embedded into an A-projective subgroup of G. If $|A| < \kappa$ and E is right hereditary, then this is equivalent to the condition that all A-generated subgroups U with $|U| < \kappa$ are A-projective. However, $\mathscr{C}_A$ may contain cotorsion groups even if A is cotorsion-free.

Proposition 5.1 *(a) If A is subgroup of $\mathbb{Q}$ of type τ, then all torsion-free groups G with $G = G(\tau)$ are A-solvable, and so is $\mathbb{Z}/p\mathbb{Z}$ for all primes p with $A \neq pA$. However, $\mathscr{C}_A$ need not be closed under direct sums unless A has idempotent type [5].*
(b) If A is a generalized rank 1 group, then $\mathbb{Q}$ is A-solvable if and only if A is homogeneous completely decomposable [5].

We now turn to the case that $r_0(A) > 1$, and focus on the following questions raised by the last example:

- Can we find indecomposable generalized rank 1 groups A such that all A-solvable groups are (co-) torsion-free? Which indecomposable generalized rank 1 groups other than subgroups of $\mathbb{Q}$ admit torsion A-solvable groups?
- Can we find cotorsion-free indecomposable generalized rank 1 groups other than subgroups of $\mathbb{Q}$ such that all A-generated reduced torsion-free groups are A-solvable?

The first of these is answered by

Theorem 5.2 ([5]) *The following are a equivalent for a generalized rank 1-group A and a prime p with $A/pA \neq 0$.*

(a) Every bounded p-group is A-solvable.
(b) $[r_p(A)]^2 = r_p(E) < \infty$.

In particular, A a torsion-free Abelian group of finite rank is a finitely faithful S-group if and only if it is fully faithful as an E-module and $\mathbb{Z}/p\mathbb{Z}$ is A-solvable for all primes p with $A \neq pA$. On the other hand, Corner's realization theorem in Theorem 3.7b always produces a torsion-free group A of finite rank with $r_p(A) = r_p(E)$ [7]. Thus, if A is a group of rank 4 with $E \cong \mathbb{Z} + i\mathbb{Z}$ which is constructed in this way, then the elements of $\mathscr{C}_A$ are torsion-free and reduced.

Surprisingly, the question whether there exits A-solvable torsion groups also is closely related to categorical properties of $\mathscr{C}_A$. However, since $\mathscr{C}_A$ is not an Abelian category unless A is a subgroup of $\mathbb{Q}$ of idempotent type [5], we investigate when the category of A-solvable groups is pre-Abelian.

Theorem 5.3 ([5]) *The following conditions are equivalent for an indecomposable generalized rank 1-group A with $r_0(A) > 1$:*

(a) $\mathscr{C}_A$ is a pre-Abelian category which does not contain J_p for any prime p.
(b) If $r_p(A) < \infty$ for some prime p with $A \neq pA$, then $[r_p(A)]^2 \neq r_p(E)$.
(c) The elements of $\mathscr{C}_A$ are cotorsion-free.

The realization theorems discussed in Sect. 3 allow the construction of large classes of groups such that $\mathscr{C}_A$ is pre-Abelian:

Example 5.4 Let A be an indecomposable generalized rank 1 such that $\mathbb{Q}E$ is semi-simple and $r_p(A) \geq 2^{\aleph_0}$ for all primes p with $A \neq pA$. One of the Göbel's realization theorems guarantees that there exist proper classes of Abelian groups satisfying these conditions. Since $r_p(A) \geq 2^{\aleph_0}$, there is a subgroup U of A such that $A/U \cong J_p$. If $J_p \in \mathcal{C}_A$, then U is A-generated since A is a generalized rank 1 group. By the results of Sect. 4, U is a direct summand of A which is not possible since A is indecomposable. Hence, $\mathcal{C}_A$ is pre-Abelian by the last result.

Furthermore, since the realization theorems produce proper classes of groups with isomorphic endomorphism rings, the question arises which categorical properties are shared by Abelian groups A and B with isomorphic, or more generally Morita-equivalent, endomorphism rings. Surprisingly, the categories $\mathcal{C}_A$ and $\mathcal{C}_B$ need not be equivalent:

Example 5.5 Let A be a subgroup of $\mathbb{Q}$ with $E(A) \cong \mathbb{Z}$ whose type is not idempotent. By Albrecht [5], the category $\mathcal{C}_A$ is not pre-Abelian. On the other hand, we can use one of the Rüdiger's construction methods to obtain a group B with $E(B) \cong \mathbb{Z}$ such that $r_p(A) \geq 2^{\aleph_0}$ for all primes p. Arguing as in Example 5.4, we obtain that $\mathcal{C}_B$ is pre-Abelian. Clearly, $\mathcal{C}_A$ and $\mathcal{C}_B$ are not equivalent.

On the other hand, the categories of locally A-projective and locally B-projective groups are equivalent if A and B are Abelian groups with $End(A) \cong End(B)$ whose endomorphism rings are discrete in the finite topology. Every locally A-projective be longs to the class $\mathcal{T}L_A$ of *A-torsion-less groups* which consists of the A-generated subgroups of cartesian powers of A.

Theorem 5.6 *Let A and B be torsion-free Abelian groups which are faithfully flat as modules over their endomorphism rings, and whose endomorphism rings are discrete in the finite topology. If $End(A)$ is left Noetherian, and $End(B)$ is Morita-equivalent to $End(A)$, then the categories $\mathcal{T}L_A$ and $\mathcal{T}L_A$ are equivalent.*

Proof Since being Noetherian is a Morita-invariant property, $End(B)$ is left Noetherian too. Moreover, Morita-equivalence preserves torsion-less modules. Because $End(A)$ is left Noetherian, $S_A(A^I)$ is A-solvable [20]. Therefore, $\mathcal{T}L_A$ is equivalent to the category of torsion-less right $End(A)$-modules since A is faithfully flat as an $End(A)$-module. Because a similar result holds for $\mathcal{T}L_B$, the theorem follows.

The author investigated Abelian groups with Morita-equivalent endomorphism rings in [10] showing that any equivalence of $\mathcal{C}_A$ and $\mathcal{C}_B$ is induced by a Morita-equivalence between $End(A)$ and $End(B)$. This and Example 5.5 give rise to

Problem 5.7 Let A and B be Abelian groups with Morita equivalent endomorphism ring. Identify (the largest) subclasses $\mathcal{C}_1 \subseteq \mathcal{C}_A$ and $\mathcal{C}_2 \subseteq \mathcal{C}_B$ for which the Morita-equivalence between $End(A)$ and $End(B)$ induces an equivalence between $\mathcal{C}_1$ and $\mathcal{C}_2$.

We now turn to the question when $\mathcal{C}_A$ is pre-Abelian if it contains J_p for some prime p.

Theorem 5.8 ([5]) *The following conditions are equivalent for an indecomposable generalized rank* 1*-group* A *with* $r_0(A) > 1$ *for which* $P(A) = \{p \mid J_p \in \mathcal{C}_A\}$ *is not empty:*

(a) $\mathcal{C}_A$ *is a pre-Abelian category.*
(b) *(i) There exists an exact sequence* $0 \to Ea \to A \to G \to 0$ *such that* $G = tG \oplus [\oplus_I \mathbb{Q}]$ *for some index-set* I *and* $G[p] = 0$ *for all* $p \in P(A)$*.*
(ii) If $r_p(A) < \infty$ *for some prime* p *with* $A \neq pA$ *and* $[r_p(A)]^2 = r_p(E)$*, then* $p \in P(A)$*.*

To see that $P(A)$ may be not empty, we consider the class of irreducible Murley groups. A group A is *irreducible* if it does not have any proper, non-zero pure fully invariant subgroups. A torsion-free group A is a *Murley group* if $r_p(A) \leq 1$ for all primes p. A homogeneous Murley group is indecomposable; and irreducible Murley groups are homogeneous [24, Chapter 15].

Theorem 5.9 ([17]) *If* A *is an irreducible Murley group, then every reduced* A*-generated torsion-free group* G *is* A*-solvable. In particular,* $P(A) \neq \emptyset$ *in this case.*

Problem 5.10 Is an indecomposable finitely faithful S-groups A for which J_p is A-solvable for all primes p with $A \neq pA$ an irreducible Murley group?

Problem 5.11 Can we describe the structure of the A-solvable groups if A is a Murley group?

A particular interesting class of $\aleph_1$-A-projective groups are the A-coseparable groups. Here, an A-generated group G is said to be *A-coseparable* if it is $\aleph_1$-A-projective and every subgroup U of G such that G/U is finitely A-presented contains a direct summand V of G such that G/V is A-projective of finite A-rank. In particular, every A-projective group is A-coseparable, and it is undecidable in ZFC if there exist A-coseparable groups which are not A-projective [35].

Theorem 5.12 ([14]) *Let* A *be a torsion-free finitely faithful* S*-group. A reduced torsion-free* A*-generated group* G *such that* $\mathrm{Ext}(G, A)$ *is torsion-free is locally* A*-projective.*

An Abelian group B is said to be *finitely projective with respect to* A if it is projective with respect to all sequences $0 \to U \to A^n \to G \to 0$ with $S_A(U) = U$.

Theorem 5.13 ([14]) *Let* A *be a torsion-free finitely faithful* S*-group. Then, the following are equivalent for a torsion-free reduced* A*-generated group* G*:*

(a) $\mathrm{Ext}(G, A)$ *is torsion-free.*
(b) G *is finitely* A*-projective.*
(c) G *is* A*-coseparable.*
(d) G *is* A*-coseparable and locally* A*-projective.*

Similarly, an A-generated group G is said to be $\aleph_1$-*A-coseparable* if it is $\aleph_1$-A-projective and every A-generated subgroup U of G such that G/U is countable contains a direct summand V of G such that G/V is countable.

Theorem 5.14 [12, Theorem 3.3]) *Let A be a self-small countable torsion-free generalized rank 1 group. A group G is $\aleph_1$-A-coseparable if and only if G is A-solvable, and every exact sequence*

$$0 \to P \to X \to G \to 0$$

such that P is a direct summand of $\oplus_\omega A$ and X is A-generated splits.

We conclude this paper with an application of endomorphism rings to mixed Abelian groups. While a detailed discussion of this interesting topic is beyond the framework of this survey, we want to mention that self-small mixed groups A such that $r_0(A/tA)$ naturally arise in various problems concerning mixed groups. For instance, Rüdiger and the author discussed cellular covers of Abelian groups in 2014. Here, a *cellular covering sequence* for an Abelian group A is an exact sequence $0 \to K \to G \xrightarrow{\gamma} A \to 0$ for which the induced map

$$\gamma_* : \mathrm{Hom}_{\mathbb{Z}}(G, G) \to \mathrm{Hom}_{\mathbb{Z}}(G, A)$$

is an isomorphism. Every group A admits a cellular covering sequence

$$0 \to 0 \to A \xrightarrow{\gamma} A \to 0$$

with γ an automorphism of A, *called a trivial cellular cover*. In this discussion, Rüdiger asked whether there exist (large classes of) honest, i.e., non-splitting, mixed groups without any non-trivial covering sequences. This question was answered positively in [13]. In the following, tA denotes the torsion subgroup of A, and A_p its p-torsion subgroup.

Theorem 5.15 *(a) No self-small Abelian group A such that A/tA is a divisible group of finite rank has a non-trivial cellular cover.*

(b) Let A be a mixed Abelian group of finite torsion-free rank such that A_p is finite for all primes p. If A/pA is finite for all primes p with $A_p \neq 0$ and $A = pA$ for all primes p with $A_p = 0$, then A has no non-trivial covering sequence $0 \to K \to G \to A \to 0$ with $tE(G) \cong tE(A)$.

(c) There exist honest self-small mixed groups A_1 and A_2 of torsion-free rank $n \geq 2$ with $tA_1 \cong tA_2$ and $E(A_1) \cong E(A_2)$ such that A_1 admits a non-trivial cellular cover $0 \to K \to G \to A_1 \to 0$ with $E(G) \cong E(A_1)$, while A_2 admits no non-trivial cellular covering sequences at all.

Problem 5.16 In [40], Rüdiger and Laszlo Fuchs showed that a subgroups of $\mathbb{Q}$ has a non-trivial cellular cover if and only if it does not have idempotent type. Is it possible to determine which self-small mixed groups A with $r_0(A) = 1$ have a non-trivial cellular cover?

We did not discuss self-small mixed groups A such that $r_0(A/tA)$ is finite in this paper mostly because we were mainly focused on topics that are closely related to Rüdiger's work. The endomorphism rings of these groups were investigated in a series of papers by the B. Wickless, S. Breaz and the author, e.g., see [21] and [19].

Problem 5.17 Let A be a self-small mixed group such that $r_0(A/tA)$ is finite. When is $\mathscr{C}_A$ pre-Abelian?

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The Zero-Divisor Graph of a Commutative Semigroup: A Survey

David F. Anderson and Ayman Badawi

Abstract Let S be a (multiplicative) commutative semigroup with 0. Associate to S a (simple) graph $G(S)$ with vertices the nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this survey article, we collect some properties of the zero-divisor graph $G(S)$.

Keywords Zero-divisor graph • Semigroup • Poset • Lattice • Semi-lattice • Annihilator graph

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1 Introduction

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be its set of zero-divisors. Over the past several years, there has been considerable attention in the literature to associating graphs with commutative rings (and other algebraic structures) and studying the interplay between their corresponding ring-theoretic and graph-theoretic properties; for recent survey articles, see [13, 17, 18, 29, 56, 58], and [61]. For example, as in [11], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [23], who let all the elements of R be vertices and was mainly interested in colorings (also see [7]). The zero-divisor

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graph of a commutative ring R has been studied extensively by many authors. For other types of graphs associated to a commutative ring, see [2–4, 8–10, 16, 19–21, 24, 43, 55, 57, 59, 63, 67], and [73].

The concept of zero-divisor graph of a commutative ring in the sense of Anderson-Livingston as in [11] was extended to the zero-divisor graph of a commutative semigroup by DeMeyer, McKenzie, and Schneider in [33]. Let S be a (multiplicative) commutative semigroup with 0 (i.e., $0x = 0$ for every $x \in S$), and let $Z(S) = \{x \in S \mid xy = 0 \text{ for some } 0 \neq y \in S\}$ be the set of zero-divisors of S. As in [33], the *zero-divisor graph* of S is the (simple) graph $G(S)$ with vertices $Z(S) \setminus \{0\}$, the set of nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph of a commutative semigroup with 0 has also been studied by many authors, for example, see [8, 9, 15, 30, 32–38, 41, 44, 46, 49, 51–54, 68, 69, 71], and [74–81].

The purpose of this survey article is to collect some properties of the zero-divisor graph of a commutative semigroup with 0. Our aim is to give the flavor of the subject, but not be exhaustive. In Sect. 2, we give several examples of zero-divisor graphs of semigroups. In Sect. 3, we give some properties of $G(S)$ and investigate which graphs can be realized as $G(S)$ for some commutative semigroup S with 0. In Sect. 4, we continue the investigation of which graphs can be realized as $G(S)$ and are particularly interested in the number (up to isomorphism) of such semigroups S. Finally, in Sect. 5, we briefly give some more results and references for further reading. An extensive bibliography is included.

Throughout, G will be a simple graph with $V(G)$ its set of vertices, i.e., G is undirected with no multiple edges or loops. We say that G is *connected* if there is a path between any two distinct vertices of G. For vertices x and y of G, define $\mathrm{d}(x, y)$ to be the length of a shortest path from x to y ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles).

A graph G is *complete* if any two distinct vertices of G are adjacent. The complete graph with n vertices will be denoted by K_n (we allow n to be an infinite cardinal number). A *complete bipartite graph* is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices of G are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call G a *star graph*. We denote the complete bipartite graph by $K_{m,n}$, where $|A| = m$ and $|B| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is a $K_{1,n}$.

Let H be a subgraph of a graph G. Then H is an *induced subgraph* of G if every edge in G with endpoints in H is also an edge in H, and G is a *refinement* of H if $V(H) = V(G)$. For a vertex x of a graph G, let $N(x)$ be the set of vertices in G that are adjacent to x and $\overline{N(x)} = N(x) \cup \{x\}$. A vertex x of G is called an *end* if there is only one vertex adjacent to x (i.e., if $|N(x)| = 1$). The *core* of G is the largest subgraph of G in which every edge is the edge of a cycle in G. Also, recall that a *component*, say C, of a graph G is a connected induced subgraph of G such that $a - b$ is not an edge of G for every vertex a of C and every vertex b of $G \setminus C$. It is known that every graph is a union of disjoint components.

Let S be a (multiplicative) commutative semigroup with 0. A $\emptyset \neq I \subseteq S$ is an *ideal* of S if $xI \subseteq I$ for every $x \in S$. A proper ideal I of S is a *prime ideal* if $xy \in I$ for $x, y \in S$ implies $x \in I$ or $y \in I$. An $x \in S$ has *finite order* if $\{ x^n \mid n \geq 1 \}$ is finite. Recall that S is *nilpotent* (resp., *nil*) if $S^n = \{0\}$ for some integer $n \geq 1$ (resp., for every $x \in S, x^n = 0$ for some integer $n = n(x) \geq 1$). Thus, a nilpotent semigroup is a nil semigroup, and a finite nil semigroup is a nilpotent semigroup. If every element of S is a zero-divisor (i.e., $Z(S) = S$), then we call S a *zero-divisor semigroup*. Note that we can usually assume that a commutative semigroup S with 0 is a zero-divisor semigroup since $Z(S)$ is an (prime) ideal of S and $G(S) = G(Z(S))$. Clearly, a nonzero nil semigroup, and hence a nonzero nilpotent semigroup, is a zero-divisor semigroup.

A general reference for graph theory is [26], and a general reference for semigroups is [42]. Other definitions will be given as needed.

2 Examples of Zero-Divisor Graphs

Let S be a (multiplicative) commutative semigroup with 0. Associate to S a (simple) graph $G(S)$ with vertices the nonzero zero-divisors of S, and two distinct vertices x and y are adjacent if and only if $xy = 0$. Note that $G(S)$ is the empty graph if and only if $S = \{0\}$ or $Z(S) = \{0\}$ (i.e., $\{0\}$ is a prime semigroup ideal of S). To avoid any trivialities, we will implicitly assume that $G(S)$ is not the empty graph.

In this section, we give several specific examples of "zero-divisor" graphs that have appeared in the literature and show that they are all the zero-divisor graph $G(S)$ for some commutative semigroup S with 0. This illustrates the power of this unifying concept and explains why these "zero-divisor" graphs all share common properties related to diameter and girth.

Example 2.1 Let R be a commutative ring with $1 \neq 0$.

1. The "usual" zero-divisor graph $\Gamma(R)$ defined in [11] has vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. Thus, $\Gamma(R) = G(S)$, where $S = R$ considered as a multiplicative semigroup.
2. Let I be an ideal of R. As in [63], the *ideal-based zero-divisor graph* of R with respect to I is the (simple) graph $\Gamma_I(R)$ with vertices $\{ x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I \}$, and distinct vertices x and y are adjacent if and only if $xy \in I$. Thus, $\Gamma_I(R) = G(S)$, where $S = R/I$ is the Rees semigroup of (the multiplicative semigroup) R with respect to I (i.e., the ideal I collapses to 0). In particular, $\Gamma_{\{0\}}(R) = \Gamma(R)$.
3. Define an (congruence) equivalence relation $\sim$ on R by $x \sim y \Leftrightarrow \mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, and let $R_E = \{ [x] \mid x \in R \}$ be the commutative monoid of (congruence) equivalence classes under the induced multiplication $[x][y] = [xy]$. Note that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$; so $[x] \subseteq Z(R)^*$ for every $x \in R \setminus ([0] \cup [1])$. The *compressed zero-divisor graph* of R is the (simple) graph $\Gamma_E(R)$ with vertices $R_E \setminus \{[0], [1]\}$, and distinct vertices $[x]$ and $[y]$ are adjacent if and only if

$[x][y] = [0]$, if and only if $xy = 0$. Thus, $\Gamma_E(R) = G(R_E)$. This zero-divisor graph was first defined (using different notation) in [57] and has been studied in [8, 9, 29], and [67]. The semigroup analog has been studied in [35] and [38].

4. Let $\sim$ be a multiplicative congruence relation on R (i.e., $x \sim y \Rightarrow xz \sim yz$ for $x, y, z \in R$). As in [10], the *congruence-based zero-divisor graph* of R with respect to $\sim$ is the (simple) graph $\Gamma_\sim(R)$ with vertices $Z(R/\sim) \setminus \{[0]_\sim\}$, and distinct vertices $[x]_\sim$ and $[y]_\sim$ are adjacent if and only if $[xy]_\sim = [0]_\sim$, if and only if $xy \sim 0$. Thus, $\Gamma_\sim(R) = G(R/\sim)$, where $R/\sim = \{\,[x]_\sim \mid x \in R\,\}$ is the commutative monoid of congruence classes under the induced multiplication $[x]_\sim[y]_\sim = [xy]_\sim$. The congruence-based zero-divisor graph includes the three above zero-divisor graphs as special cases.
5. Let S be the semigroup of ideals of R under the usual ideal multiplication. As in [24], $\mathbb{AG}(R) = G(S)$ is called the *annihilating-ideal graph* of R (this zero-divisor graph was first defined in [73]). Similarly, as in [32], define the *annihilating-ideal graph* of a commutative semigroup S with 0 to be $\mathbb{AG}(S) = G(T)$, where T is the semigroup of (semigroup) ideals of S under the usual multiplication of (semigroup) ideals.
6. Let $(S, \wedge)$ be a meet semilattice with least element 0. As in [60], the *zero-divisor graph* of S is the (simple) graph $\Gamma(S)$ with vertices $Z(S) \setminus \{0\} = \{\,0 \neq x \in S \mid x \wedge y = 0 \text{ for some } 0 \neq y \in S\,\}$, and distinct vertices x and y are adjacent if and only if $x \wedge y = 0$. Recall that S becomes a commutative (Boolean) semigroup S' with 0 under the multiplication $xy = x \wedge y$; so $\Gamma(S) = G(S')$. Similar zero-divisor graphs have been defined for posets and lattices [see [39, 40, 46, 48, 53], and Theorem 4.1(1)].

However, not all "zero-divisor" graphs can be realized as $G(S)$ for a suitable commutative semigroup S with 0. For example, as in [19], the *annihilator graph* of a commutative ring R with $1 \neq 0$ is the (simple) graph $AG(R)$ with vertices $Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(x) \cup \mathrm{ann}_R(y) \neq \mathrm{ann}_R(xy)$. Then $\Gamma(R)$ is a subgraph of $AG(R)$, and may be a proper subgraph (e.g., $\Gamma(\mathbb{Z}_8) = K_{1,2}$, while $AG(\mathbb{Z}_8) = K_3$). Thus, $AG(R)$ need not be a $G(S)$. Similarly, as in [1], one can also define the *annihilator graph* $AG(S)$ of a commutative semigroup S with 0. The zero-divisor graph $G(S)$ is a subgraph of $AG(S)$.

As in [81], for a commutative semigroup S with 0, let $\overline{G}(S)$ be the (simple) graph with vertices $Z(S) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xSy = \{0\}$. Then $G(S)$ is a subgraph of $\overline{G}(S)$, and may be a proper subgraph (e.g., if $S = \{0, 2, 4, 6\} \subseteq \mathbb{Z}_8$, then $G(S) = K_{1,2}$, while $\overline{G}(S) = K_3$).

3 Some Properties of the Zero-Divisor Graph $G(S)$

In this section, we give some properties of the zero-divisor graph $G(S)$ of a commutative semigroup S with 0 and are particularly interested in which graphs can be realized as $G(S)$ for some commutative semigroup S with 0. We start with some basic properties of $G(S)$. Parts (1)–(3) of Theorem 3.1 were first proved for $\Gamma(R)$ (cf. [11, 12, 31], and [57]).

Theorem 3.1 *Let S be a commutative semigroup with* 0.

1. ([33, Theorem 1.2]) *$G(S)$ is connected with diam$(G(S)) \in \{0, 1, 2, 3\}$.*
2. ([33, Theorem 1.3]) *If $G(S)$ does not contain a cycle, then $G(S)$ is a connected subgraph of two star graphs whose centers are connected by a single edge.*
3. ([33, Theorem 1.5]) *If $G(S)$ contains a cycle, then the core of $G(S)$ is a union of triangles and squares, and any vertex not in the core of $G(S)$ is an end. In particular, gr$(G(S)) \in \{3, 4, \infty\}$.*
4. ([30, Theorem 1(4)]) *For every pair x, y of distinct nonadjacent vertices of $G(S)$, there is a vertex z of $G(S)$ with $N(x) \cup N(y) \subseteq \overline{N(z)}$.*

Remark 3.2 (1) In Theorem 3.1(4), it is easily shown that $N(x) \cup N(y) \subsetneq \overline{N(z)}$ (for any such z), and either case $z \in N(x) \cup N(y)$ or $z \notin N(x) \cup N(y)$ may occur. Moreover, we can always choose $z = xy$, but there may be other choices for z.

(2) In [53], a (simple) connected graph which satisfies condition (4) of Theorem 3.1 is called a *compact graph*. In [53, Theorem 3.1], it was shown that a simple graph G is the zero-divisor graph of a poset if and only if G is a compact graph.

For small graphs, conditions (1), (3), and (4) of Theorem 3.1 actually characterize zero-divisor graphs.

Theorem 3.3 ([30, Theorem 2]) *Let G be a (simple) graph with $|V(G)| \leq 5$ satisfying conditions (1), (3), and (4) of Theorem 3.1. Then $G \cong G(S)$ for some commutative semigroup S with* 0.

([30, Example 2]). In view of Theorem 3.3, Fig. 1 is a graph with six vertices which satisfies conditions (1), (3), and (4) of Theorem 3.1, but G is not the zero-divisor graph of any commutative semigroup with 0. (Also, see [35, Fig. 2, p. 3372].)

The next theorem gives several classes of graphs which can be realized as the zero-divisor graph of a commutative semigroup with 0. As to be expected, many more graphs can be realized as $G(S)$ for a commutative semigroup S with 0 than as

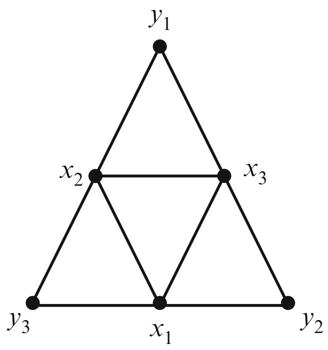


Fig. 1 A graph with six vertices which satisfies conditions (1), (3), and (4) of Theorem 3.1, but G is not the zero-divisor graph of any commutative semigroup with 0

$\Gamma(R)$ for a commutative ring R with $1 \neq 0$ (cf. [11, 12, 64], and [65]). For example, K_n and $K_{1,n}$ (for an integer $n \geq 1$) can be realized as a $G(S)$ for every $n \geq 1$, but can be realized as a $\Gamma(R)$ if and only if $n + 1$ is a prime power [11, Theorem 2.10 and p. 439].

Theorem 3.4 ([30, Theorem 3]) *The following graphs are the zero-divisor graph of some commutative semigroup with* 0.

1. *A complete graph or a complete graph together with an end.*
2. *A complete bipartite graph or a complete bipartite graph together with an end.*
3. *A refinement of a star graph.*
4. *A graph which has at least one end and diameter* ≤ 2.
5. ([33, Theorem 1.3(2)]) *A graph which is the union of two star graphs whose centers are connected by a single edge.*

([30, Example 3]). By (3) and (5) of Theorem 3.4, the refinement of a star graph and the union of two star graphs whose centers are connected by an edge are each the zero-divisor graph of a commutative semigroup with 0. The graph in Fig. 2 is also a refinement of the union of two star graphs with centers at vertex a and vertex b. However, it is not the zero-divisor graph of any commutative semigroup with 0. The vertices a and f do not satisfy condition (4) of Theorem 3.1 since vertex a is adjacent to d and vertex f is adjacent to c, but there is no vertex adjacent to both c and d.

The following theorem gives necessary and sufficient conditions on the semigroup S for $G(S)$ to be a refinement of a star graph (cf. [11, Theorem 2.5] for commutative rings).

Theorem 3.5 ([79, Theorem 1.1]) *Let S be commutative semigroup with* 0 *and* $Z(S) \neq \{0\}$. *Then $G(S)$ is a refinement of a star graph if and only if either $Z(S)$ is an annihilator ideal (and hence a prime ideal) of S or* $Z(S) = A \cup B$, *where* $A \cong (\mathbb{Z}_2, \cdot)$, $A \cap B = \{0\}$, *and A, B are ideals of S.*

For a vertex c of a graph G, let G_c^* be the induced subgraph of G with vertices $V(G_c^*) = V(G) \setminus \{\, u \in V(G) \mid u = c$ or u is an end vertex adjacent to $c\,\}$.

Theorem 3.6 ([79, Theorem 2.3]) *Let S be a set with a commutative binary operation and a zero element* 0 *such that* $S = \{0\} \cup \{c\} \cup T \cup S_1$ *is the disjoint union of four nonempty subsets. Assume further that* $Z(S) = S$, *whose zero-divisor*

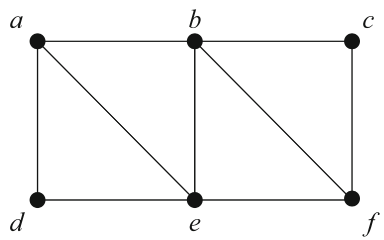


Fig. 2 A graph which is a refinement of the union of two star graphs with centers at vertex a and vertex b. However, it is not the zero-divisor graph of any commutative semigroup with 0

graph $G(S)$ is a refinement of a star graph with center c such that $S_1 = V(G_c^*)$ and G_c^* has at least two components. Then the following statements are equivalent.

1. *S is a commutative zero-divisor semigroup (i.e., the binary operation is associative).*
2. *$S_1^2 = \{0, c\}$, $T^2 \subseteq \{0, c\}$, $c^2 = 0$, and $ts_1 = c$ for every $t \in T$ and $s_1 \in S_1$.*
3. *$S^2 = \{0, c\}$ and $S^3 = \{0\}$.*

Recall that a vertex x of a graph G has *degree* m, denoted by $\deg(x) = m$, if $|N(x)| = m$. For an integer $k \geq 1$, let G_k be the induced subgraph of G with vertices $V(G_k) = \{x \in V(G) \mid \deg(x) \geq k\}$. For a commutative semigroup S with 0 and an integer $k \geq 1$, let $I_k = \{x \in V(G) \mid \deg(x) \geq k\} \cup \{0\}$. Results in the next two theorems from [30] were stated for nilpotent semigroups, but their proofs show that they hold for nil semigroups (i.e, every element is nilpotent).

Theorem 3.7 *Let S be a commutative semigroup with 0.*

1. ([30, Theorem 4]) *I_k is a descending chain of ideals in S.*
2. ([30, Corollary 1]) *The core of $G(S)$ together with $\{0\}$ is an ideal of S whose zero-divisor graph is the core of $G(S)$.*
3. ([30, Corollary 2]) *If S is a nil semigroup, then $G(S)_k = G(I_k)$ for every integer $k \geq 1$.*
4. ([30, Corollary 3]) *Let G be a graph and assume that G_k is not the zero-divisor graph of any commutative semigroup with 0 for some integer $k \geq 1$. Then G is not the zero-divisor graph of any commutative nil semigroup.*
5. ([30, Corollary 4]) *Let G be a graph which is equal to its core, but is not the zero-divisor graph of any commutative semigroup with 0, and let H be the graph obtained from G by adding ends to G. Then H is not the zero-divisor graph of any commutative semigroup with 0.*

Sharper results hold when S is a nil semigroup. A well-known special case is for $\Gamma(R)$ when $Z(R) = \mathrm{nil}(R)$ (e.g., when R is a finite local ring).

Theorem 3.8 *Let S be a commutative semigroup with 0.*

1. ([30, Theorem 5]) *Assume that S is a nil semigroup. Then*
   a. *$diam(G(S)) \in \{0, 1, 2\}$.*
   b. *Every edge in the core of $G(S)$ is the edge of a triangle in $G(S)$. In particular, $gr(G(S)) \in \{3, \infty\}$.*
2. ([30, Corollary 5]) *If every element of S has finite order and some edge in the core of $G(S)$ is the edge of a square, but not a triangle, then S contains a nonzero idempotent element.*

In [35], the authors gave several criteria for a graph G to be a zero-divisor graph in terms of the number of edges of G and adding or removing edges from a given zero-divisor graph $G(S)$. In the next theorem, we are removing edges from K_n (which has $n(n-1)/2$ edges).

Theorem 3.9 ([35, Theorem 2.5(1)]) *Let G be a connected graph with n vertices and $n(n-1)/2-p$ edges. Then G is the zero-divisor graph of a commutative semigroup with* 0 *if* $0 \le p \le \lceil n/2 \rceil + 1$ *(i.e., if G has at least $n(n-1)/2 - \lceil n/2 \rceil - 1$ edges).*

Theorem 3.10 ([35, Theorem 3.22]) *Let $G = G(S)$ be a zero-divisor graph with cycles for a commutative semigroup S with* 0 .

1. *If a is an end adjacent to x in G, then adding another end adjacent to x results in a zero-divisor graph.*
2. *Removing an end from G results in a zero-divisor graph.*

For a commutative semigroup S with 0, let $G^{\bullet}(S)$ be the (simple) graph with vertices the nonzero zero-divisors of S, and distinct vertices x and y are adjacent if and only if $xy \neq 0$ [in [36], 0 was allowed to be a vertex of $G^{\bullet}(S)$]. As in [36], a graph G is called *admissible* if $G \cong G^{\bullet}(S)$ for some commutative zero-divisor semigroup S. In [36], the authors study $G(S)$ by studying $G^{\bullet}(S)$.

Theorem 3.11 ([36, Theorem 2]) *Given a connected graph G, let G' be the graph obtained by the following procedure: For every edge $a - b$ in G, add a vertex $c_{a,b}$ and edges $a - c_{a,b}$, $b - c_{a,b}$. Then G' is connected and admissible.*

For a graph G, let $\overline{G}$ be the *complement graph* of G (i.e., $V(\overline{G}) = V(G)$ and $a-b$ is an edge in $\overline{G}$ if and only if $a-b$ is not an edge in G for every two distinct vertices a, b of G). Thus, $G^{\bullet}(S) = \overline{G(S)}$. The next theorem gives some necessary conditions on $\overline{G}$ for G to be admissible.

Theorem 3.12 ([36, Theorem 4], cf. Theorem 3.1) *Let G be an admissible graph.*

1. *$\overline{G}$ has at most one nontrivial component, i.e., with more than one vertex.*
2. *For every connected pair $a, b \in V(\overline{G})$, $d(a, b) \le 3$.*
3. *The induced cycles in $\overline{G}$ are either* 3*-cycles or* 4*-cycles.*
4. *For every pair a, b of distinct nonadjacent vertices of $\overline{G}$, there is a vertex c of $\overline{G}$ such that $N(a) \cup N(b) \subseteq \overline{N(c)}$.*

Let G be a simple connected graph, and let $S \subseteq V(G)$. Then a vertex x of G is said to *bound* S if for every $y \in N(x)$, we have $\mathrm{d}(y,t) \le 1$ for every $t \in S$. The set of boundary vertices of S is denoted by $B_G(S)$. A set $S \subseteq V(G)$ is said to be *bounded* if $B_G(S) \neq \emptyset$; otherwise, S is said to be *unbounded* (see [36]).

Theorem 3.13 ([36, Theorem 3 and Corollary (p. 1490)]) *Let G be an admissible graph and $a, b \in V(G)$, not necessarily distinct. Then $ab \in B_G(\{a,b\}) \cup \{0\}$. In particular, if $a - b$ is an edge of G, then $B_G(\{a\}) \neq \emptyset$ and $B_G(\{a,b\}) \neq \emptyset$.*

The following theorem gives some connections between elements in an admissible graph.

Theorem 3.14 *Let G be an admissible graph. Then*

1. ([36, Lemma 1]) *If $a - b$ is an edge of G, then $d(ab, a) \leq 2$ and $d(ab, b) \leq 2$.*
2. ([36, Proposition 5]) *For every $a \in V(G)$, $a^2 \in B_G(\{a\}) \cup \{0\}$.*
3. ([36, Proposition 6]) *If $a - b$ is an edge of G and $a^2 = b^2 = 0$, then a and b are adjacent to a common vertex of G.*
4. ([36, Proposition 7]) *If $a - b$ is an edge of G and $a^2 = 0$, then $ab \notin \overline{N(a)}$.*
5. ([36, Proposition 8]) *If $a - b$ is an edge of G and $a^2 = a$, then $ab \in \overline{N(a)}$.*
6. ([36, Corollary (p. 1495)]) *If $a - b$ is an edge of G such that $a^2 = 0$ and $b^2 = b$, then $ab \in N(b) \setminus N(a)$.*

4 The Number of Zero-Divisor Semigroups

Not only is it of interest to know which graphs can be realized as $G(S)$ for some commutative semigroup S with 0, but more precisely, what are the choices for such semigroups S? The case for commutative semigroups S with 0 and $G(S)$ is somewhat different than for commutative rings R with $1 \neq 0$ and $\Gamma(R)$. It is well known that $|R| \leq |Z(R)|^2$ when $Z(R) \neq \{0\}$; so (up to isomorphism) there are only finitely many commutative rings with $1 \neq 0$ that have a given (nonempty) finite zero-divisor graph. However, for semigroups, one can always adjoin units; so if there is a commutative semigroup S with 0 and $G \cong G(S)$, then for every cardinal number $n \geq |S|$, there is a commutative semigroup $S(n)$ with 0 [and $Z(S(n)) = Z(S)$] such that $G \cong G(S(n))$ and $|S(n)| = n$. Thus, to determine which commutative semigroups with 0 realize a given graph G, we will restrict our attention to commutative zero-divisor semigroups [i.e., $S = Z(S)$].

While it is usually not true that $G(S) \cong G(T)$ implies that $S \cong T$ for commutative semigroups S and T with 0, we can get better results when we restrict to certain classes of zero-divisor semigroups. We first consider the case when S is reduced (i.e., $x^n = 0$ implies $x = 0$). The zero-divisor graph of reduced commutative semigroups with 0 has been studied in [8, 9, 38, 46], and [53]. The next theorem shows that this case reduces to Boolean semigroups (i.e., $x^2 = x$ for every element). Call a monoid S with 0 a *zero-divisor monoid* if $S \setminus \{1\} = Z(S)$. Special cases of the next theorem have been proved in [53, Theorem 4.3] for (1) and [51, Theorem 4.2] for (2).

Theorem 4.1 1. ([46, Corollary 1.2]) *The following statements are equivalent for a graph G with at least two vertices.*

   a. *$G \cong G(S)$ for some reduced commutative semigroup S with 0.*
   b. *$G \cong G(S)$ for some commutative Boolean semigroup S with 0.*
   c. *$G \cong G(S)$ for some meet semilattice S.*

2. ([8, Theorem 2.1]) *Let S and T be commutative Boolean zero-divisor monoids. Then $G(S) \cong G(T)$ if and only if $S \cong T$.*

We next give several classes of graphs for which one can determine all possible commutative zero-divisor semigroups with a given graph. However, we will be content to just give the number (up to isomorphism) of such semigroups rather than list them all explicitly.

In [76], the authors gave recursive formulas for the number (up to isomorphism) of commutative zero-divisor semigroups whose zero-divisor graphs are either K_n or $K_n + 1$ (a K_n with an end adjoined) and compute these numbers up to $n = 10$. For example, there are (up to isomorphism) 139 commutative zero-divisor semigroups with zero-divisor graph K_{10} and 7, 101 with zero-divisor graph $K_{10} + 1$. We give the explicit formula for K_n; let $p(m, r)$ be the number of partitions $x_1 + \cdots + x_r = m$ of the integer m with $x_1 \geq x_2 \geq \cdots \geq x_r \geq 1$.

Theorem 4.2 ([76, Theorem 2.2]) *For every integer $n \geq 1$, there are (up to isomorphism)*

$$1 + \sum_{r}^{n} = 1 \sum_{e=0}^{n-r} p(n-e, r)$$

commutative zero-divisor semigroups whose zero-divisor graph is K_n.

We next consider star graphs.

Theorem 4.3 ([71]) *Let $n \geq 1$ be an integer and $f(n)$ be the number (up to isomorphism) of commutative semigroups with n elements. Then there are (up to isomorphism) $n+2+2f(n-1)+2f(n)$ commutative zero-divisor semigroups whose zero-divisor graph is $K_{1,n}$.*

Theorem 4.4 ([79, Theorem 2.13])

1. *If S is a nilpotent commutative semigroup with $G(S)$ a star graph, then $S^4 = \{0\}$.*
2. *For every cardinal number $n \geq 2$, there is a unique (up to isomorphism) nilpotent commutative semigroup $S(n)$ such that $G(S(n)) = K_{1,n}$ and $S(n)^3 \neq \{0\}$.*

Theorem 4.5 *Let n be an integer.*

1. ([79, Theorem 3.6]) *For every $n \geq 2$, there are (up to isomorphism) $n + 2$ nilpotent commutative semigroups with 0 whose zero-divisor graph is the star graph K_1, n.*
2. ([69, Theorem 2.1]) *There are (up to isomorphism) 12 commutative zero-divisor semigroups whose zero-divisor graph is the star graph $K_1, 2$.*
3. ([69, Theorem 2.2]) *There is (up to isomorphism) a unique commutative zero-divisor semigroup whose zero-divisor graph is the path graph P_4: $a - b - c - d$.*
4. ([69, Theorem 2.5]) *There are (up to isomorphism) 35 commutative zero-divisor semigroups whose zero-divisor graph is the graph $K_1, 3$.*
5. ([69, Theorem 2.7]) *There are (up to isomorphism) 31 commutative zero-divisor semigroups whose zero-divisor graph is the graph in Fig. 3.*

By [79, p. 339], the number of commutative zero-divisor semigroups whose zero-divisor graph is K_2 (resp., K_3, K_4, and $K_3 + 1$) is 4 (resp., 7, 12, and 22). Combining this with Theorem 4.5 gives all commutative zero-divisor semigroups whose zero-divisor graph has at most four vertices.

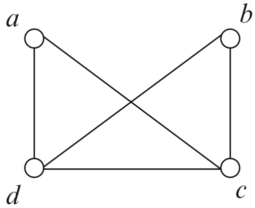


Fig. 3 There are (up to isomorphism) 31 commutative zero-divisor semigroups with this zero-divisor graph

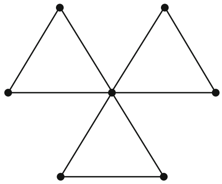


Fig. 4 F_3, i.e., a friendship graph with $|I| = 3$

Recall that a graph G is a *friendship graph* if G is graph-isomorphic to $(\cup_I K_2) + K_1$, for some set I; this graph is denoted by $F_{|I|}$. For example, Fig. 4 is a friendship graph with $|I| = 3$. We call G a *fan-shaped graph* if G is graph-isomorphic to $P_n \cup \{c\}$, where P_n is the path graph on n vertices and c is adjacent to every vertex of P_n, and denote this graph by F'_n.

Theorem 4.6 ([79, Lemma 3.1]) *For every integer $n \geq 2$, there are (up to isomorphism) $\frac{(n+1)(n+2)}{2}$ commutative zero-divisor semigroups whose zero-divisor graph is the friendship graph F_n.*

Theorem 4.7 ([79, Theorem 3.2]) *Let G be the friendship graph F_n together with m end vertices adjacent to its center, where $n \geq 2$, $m \geq 0$. Then there are (up to isomorphism) $\frac{(n+1)(n+2)(m+1)}{2}$ commutative zero-divisor semigroups whose zero-divisor graph is the graph G.*

The number of fan-shaped graphs F'_n for $n \geq 6$ is a special case of the next theorem (let $T = \emptyset$, so the number is $g(n)$). For $n = 2$ (resp., 3, 4, and 5), the number (up to isomorphism) of commutative zero-divisor semigroups whose zero-divisor graph is F'_n is 4 (resp., 12, 47, and 26) (see [68] for $n = 4$ and [80, Theorem 3.1] for $n = 5$).

Theorem 4.8 ([79, Theorem 3.5]) *For every integer $n \geq 6$ and any finite set T, let $G = (P_n \cup T) + c$ be the graph with $G^*_c = P_n$, where P_n is the path graph with n vertices. Then there are (up to isomorphism) $(|T| + 1)g(n)$ commutative zero-divisor semigroups whose zero-divisor graph is the graph G, where*

$$g(n) = \begin{cases} \frac{1}{2}(2^n + 2^{\frac{n}{2}}) & \text{if } n \text{ is even} \\ \frac{1}{2}(2^n + 2^{\frac{n+1}{2}}) & \text{if } n \text{ is odd.} \end{cases}$$

The next two theorems from [77] concern the complete graph K_n with an end adjoined to some vertices of K_n.

Theorem 4.9 *Let n be an integer.*

1. ([77, Theorem 2.1]) *For $n \geq 4$, there is (up to isomorphism) a unique commutative zero-divisor semigroup whose zero-divisor graph is the graph K_n together with two end vertices.*
2. ([77, Theorem 2.2]) *For $n \geq 4$, there is no commutative semigroup with 0 whose zero-divisor graph is the graph K_n together with three end vertices.*
3. ([77, Proposition 3.1]) *There are (up to isomorphism) 20 commutative zero-divisor semigroups whose zero-divisor is the graph K_3 together with an end vertex.*

Theorem 4.10 ([77, Theorem 3.2]) *For integers n and k with $1 \leq k \leq n$, let $M_{n,k} = K_n \cup \{x_1, \ldots, x_k\}$ be the complete graph K_n with vertices $\{a_1, \ldots, a_n\}$ together with k end vertices $\{x_1, \ldots, x_k\}$, where a_i is adjacent to x_i for every $1 \leq i \leq k$.*

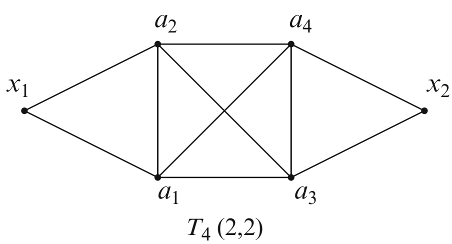
1. *For every integer $n \geq 4$, there is a unique commutative zero-divisor semigroup whose zero-divisor graph is either $M_{3,3}$ or $M_{n,2}$.*
2. ([30, Theorem 3(1)]) *For every integer $n \geq 1$, there are multiple commutative zero-divisor semigroups whose zero-divisor graph is either $M_{n,0}$ (i.e., K_n) or $M_{n,1}$.*
3. *For every integer $n \geq 4$ and $k \geq 3$, there is no commutative zero-divisor semigroup whose zero-divisor graph is $M_{n,k}$.*
4. *There are (up to isomorphism) three commutative zero-divisor semigroups whose zero-divisor graph is $M_{3,2}$.*

For an integer $n \geq 4$, let $T_n(2,2) = K_n \cup \{x_1, x_2\}$ be the complete graph K_n with vertices $M_n = \{a_1, \ldots, a_n\}$ together with the edges: $x_1 - a_1$, $x_1 - a_2$, $x_2 - a_3$, and $x_2 - a_4$. For example, Fig. 5 is the graph $T_4(2,2)$. The following two theorems from [41] give the number (up to isomorphism) of commutative zero-divisor semigroups with zero-divisor graph $T_n(2,2)$ for every integer $n \geq 4$.

Theorem 4.11 *1.* ([41, Lemma 2.1]) *There is no commutative zero-divisor semigroup whose zero-divisor graph is $T_4(2,2)$.*
2. ([41, Theorem 2.2]) *There are (up to isomorphism) 18 commutative zero-divisor semigroups whose zero-divisor graph is $T_5(2,2)$.*

Theorem 4.12 ([41, Theorem 2.3]) *Let $n \geq 6$ be an integer and $M_n(2,2) = \{a_1, \ldots, a_n\} \cup \{0, x_1, x_2\}$. Then $M_n(2,2)$ is a commutative zero-divisor semigroup whose zero-divisor graph is $T_n(2,2)$ if and only if the following conditions hold.*

Fig. 5 The graph $T_4(2,2)$



1. *$a_ix_1 = 0$ ($i = 1, 2$), $a_jx_2 = 0$ ($j = 3, 4$), $x_i^2 = x_i$ ($i = 1, 2$), $a_ia_j = 0$ for every $i \neq j$, $a_i^2 \in \{0, a_1, a_2\}$ ($i = 1, 2$), and $a_j^2 \in \{0, a_3, a_4\}$ ($j = 3, 4$).*
2. *$x_1x_2 \in \{a_5, \ldots, a_n\}$. If $x_1x_2 = a_t$, then $a_tx_i = a_t$ ($i = 1, 2$), $a_t^2 = a_t$ and $a_r^2 = 0$ for every $r \geq 5$ and $r \neq t$.*
3. *$a_rx_1 \in \{a_3, a_4\}$ for every $r \neq 1, 2, t$. If $a_rx_1 = a_3(a_4)$ for $r \neq 3(4)$, then $a_3x_1 = a_3$ ($a_4x_1 = a_4$) and $a_2^3 = 0$ ($a_2^4 = 0$)). In particular, if $a_4x_1 = a_3$ ($a_3x_1 = a_4$), then $a_4^2 = 0$ ($a_3^2 = 0$).*
4. *$a_rx_2 \in \{a_1, a_2\}$ for every $r \neq 3, 4, t$. If $a_rx_2 = a_1(a_2)$ for $r \neq 1(2)$, then $a_1x_2 = a_1$ ($a_2x_2 = a_2$) and $a_1^2 = 0$ ($a_2^2 = 0$). In particular, if $a_2x_1 = a_1$ ($a_1x_2 = a_2$), then $a_2^2 = 0$ ($a_1^2 = 0$).*

   Moreover, if P_n is the number (up to isomorphism) of commutative zero-divisor semigroups with zero-divisor graph $T_n(2, 2)$, then

$$P_n = \begin{cases} \frac{1}{48}(n^3 - 6n^2 + 89n + 204) & \text{if } n = 4m + 1 \\ \frac{1}{48}(n^3 + n^2 + 64n - 12) & \text{if } n = 4m + 2 \\ \frac{1}{48}(n^3 - 3n^2 + 71n + 219) & \text{if } n = 4m + 3 \\ \frac{1}{48}(n^3 - 6n^2 + 80n + 144) & \text{if } n = 4m. \end{cases}$$

5 Other Results

We conclude this survey article by referencing a few other results on zero-divisor graphs. Many topics related to associating graphs to algebraic systems have been left untouched; the interested reader may consult the seven survey articles mentioned in the introduction, unreferenced papers in the bibliography, and MathSciNet for many more relevant articles.

Remark 5.1 Some more results.

1. In [27, 62], and [74], the authors studied directed zero-divisor graphs of a noncommutative semigroup with 0.
2. It was shown in [51] that a graph G with more than two vertices has a unique corresponding commutative zero-divisor semigroup if G is a zero-divisor graph of some Boolean ring.
3. In [9], the authors determined the monoids R_E for which $\Gamma_E(R) = G(R_E)$ is a star graph.
4. For other types of graphs associated to semigroups, see, for example, [1, 5, 6, 25, 32], and [81].
5. The authors in [35, 52], and [54] studied commutative zero-divisor semigroups whose zero-divisor graphs are complete r-partite graphs.
6. In [70], the authors determined the number (up to isomorphism) of commutative rings and semigroups whose zero-divisor graphs are regular polyhedra.

7. The authors in [78] studied sub-semigroups determined by the zero-divisor graph.
8. The authors in [15] studied minimal paths in commutating graphs of semigroups.
9. For graphs associated to groups, see, for example, [14, 22], and [50].
10. For graphs of posets, lattices, semilattices, or Boolean monoids, see, for example, [8, 39, 40, 44, 46, 48], and [53].
11. The authors in [27] (resp., [28]) studied the zero-divisor graph (resp., annihilator graph) of near rings.
12. The author in [47] studied the zero-divisor graph of a groupoid.
13. In [45, 66], and [72], the authors gave algorithms for determining if a given graph can be realized as the zero-divisor graph of a commutative ring with $1 \neq 0$.
14. In [33, 40, 44, 53, 54], and [60], the authors studied colorings of commutative semigroups with 0.

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A Remak-Krull-Schmidt Class of Torsion-Free Abelian Groups

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Abstract The class of almost completely decomposable groups with a critical typeset of type $(1, 5)$ and a homocyclic regulator quotient of exponent p^3 is shown to be of bounded representation type, i.e., in particular, a Remak-Krull-Schmidt class of torsion-free abelian groups. There are precisely 20 near-isomorphism classes of indecomposables all of rank 7, 8, 9.

Keywords Remak-Krull-Schmidt • Almost completely decomposable group • Indecomposable • Bounded representation type

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1 Introduction

The basis theorem for finite abelian groups completely classifies these groups. There is a countable set of directly indecomposable finite abelian groups (the cyclic groups of prime power order) and every finite abelian group is up to isomorphism uniquely the direct sum of indecomposable groups. This result served as a model for the study of infinite abelian groups. It soon turned out that not even primary abelian groups are direct sums of cyclic subgroups. Torsion-free groups of finite rank are trivially the direct sums of indecomposable subgroups but here one problem is the hopeless abundance of indecomposable groups of finite rank and the other problem is "pathological decompositions", i.e., indecomposable decompositions that are not unique up to isomorphism. The most striking result in this direction is due to A.L.S. Corner: *Given integers* $n \geq k \geq 1$, *there exists a torsion-free group* X *of rank* n *such that for any partition* $n = r_1 + \cdots + r_k$, *there is a decomposition of* X *into a direct sum of* k *indecomposable subgroups of ranks* $r_1, \ldots, r_k$ *respectively.* E.g., for $n = 10, k = 2, n = 1 + 9 = 2 + 8 = \cdots = 5 + 5$ and Corner's result says that there is a group G that has indecomposable decompositions $G = G_1 \oplus G_2$ such that the ranks of the summands are 1 and 9, 2 and 8, ..., 5 and 5. Kaplansky called the theory of torsion-free abelian groups a strange subject that largely consists of a collection of examples, in particular, examples of pathological decompositions, proving all wrong that one might hope for. Thus to obtain results one has to consider subclasses of torsion-free groups. A first such class is the class of *completely decomposable groups*, direct sums of rank-one groups. This is a Remak-Krull-Schmidt class, i.e., it has unique decompositions with indecomposable summands, namely rank-one subgroups. This was settled in 1940 by Reinhold Baer. It may be considered a warning sign that there are $2^{\aleph_0}$ non-isomorphic rank-one groups (necessarily indecomposable). A sophisticated, yet amenable class is the class of *almost completely decomposable groups*, e.g. Jonsson in the 1950s and possibly Baer. However, first studied in depth by Lady[9]. These are finite extensions of completely decomposable groups of finite rank. Even in the case of almost completely decomposable groups there are pathological decompositions. In fact, Corner's examples were almost completely decomposable groups. For almost completely decomposable groups a weakening of isomorphism, also due to Lady, called "near-isomorphism", proved to be essential. Arnold [1] showed that *nearly isomorphic groups have the same decomposition properties.* If one is indecomposable, then so is the other. If one is the direct sum of two subgroups, then so is the other with summands that are nearly isomorphic to the summands of the one. This means that nothing is lost by way of decompositions if instead of isomorphism one works with the coarser notion of near-isomorphism. We finally arrive at a Remak-Krull-Schmidt Category as follows. Every almost completely decomposable group G contains by definition a completely decomposable subgroup A of finite index. If the index $[G : A]$ is least possible among indices of completely decomposable subgroups, then A is called a *regulating subgroup* and $[G : A]$ is the *regulating index* of G. Given a prime p, the category of all almost completely decomposable groups whose regulating index is a p-power with near-isomorphism as equivalence relation is a Remak-Krull-Schmidt category. This is the Faticoni-Schultz Theorem [8]: *The "indecomposable" decompositions of an almost completely decomposable group G with p-power regulating index are unique up to near-isomorphism.* In such a Remak-Krull-Schmidt category a classification up to near-isomorphism is achieved as soon as the indecomposable groups in the class are found. As was shown in [2] most of these classes contain indecomposable groups of arbitrarily large rank in which case it is hopeless to try to describe all near-isomorphism classes of indecomposable groups. This leaves some special classes that may have a finite number of near-isomorphism classes of indecomposable groups. The class considered in this paper is shown to be such a class and the indecomposables are explicitly determined.

Any torsion-free abelian group G is an additive subgroup of a $\mathbb{Q}$-vector space V. The $\mathbb{Q}$-subspace of V generated by G is denoted by $\mathbb{Q}\,G$ and $\dim(\mathbb{Q}\,G)$ is the *rank* of G. A torsion-free abelian group R of finite rank is *completely decomposable* if R is the direct sum of rank-1 groups. A *type* is an isomorphism class $[X]$ of a rank-one group X and $\tau = [X]$ is the type of X. The set of all types is partially ordered where $[X] \leq [Y]$ if there is a non-zero homomorphism $X \to Y$. Given a completely decomposable group R, we get a decomposition $R = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ where R_ρ is obtained by combining the rank-1 summands of type ρ of R into a summand $R_\rho (\neq 0)$. The set $\mathrm{T}_{\mathrm{cr}}(R)$ is the *critical typeset of* R, e.g. [1] or [11].

An almost completely decomposable group G contains a well-understood fully invariant completely decomposable subgroup of finite index, the *regulator* $\mathrm{R}(G)$, [7]. In fact, the regulator is the intersection of all regulating subgroups. The critical typeset of G is the critical typeset of $\mathrm{R}(G)$, $\mathrm{T}_{\mathrm{cr}}(G) = \mathrm{T}_{\mathrm{cr}}(R)$. If the critical typeset is an inverted forest, then there is a unique regulating subgroup that equals the regulator. This is the case for the class studied in this paper.

A type τ is *p-locally free* if $pX \neq X$ for any rank-1 subgroup X of G with $[X] = \tau$. Given a finite poset S of p-locally free types, an almost completely decomposable group G is an *(S, p^k)-group* if $S = \mathrm{T}_{\mathrm{cr}}(G)$ and the exponent of the *regulator quotient* $G/\,\mathrm{R}(G)$ is p^k, i.e., $\exp(G/\,\mathrm{R}(G)) = p^k$. In the survey article [6] we used a more general definition of (S, p^k)-groups, namely $\mathrm{T}_{\mathrm{cr}}(G) \subset S$ and $p^k G \subset \mathrm{R}(G)$, so that the class (S, p^k) is closed under direct summands. Our approach here is motivated by obtaining a complete list of indecomposables. Two (S, p^k)-groups G and H are *nearly isomorphic* c.f. Lady[10] if there is an integer n relatively prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n$ and $gf = n$. Consequently, a classification of all indecomposable (S, p^k)-groups up to near isomorphism results in a classification of all (S, p^k)-groups up to near isomorphism. Hence, for almost completely decomposable groups G with $G/\,\mathrm{R}(G)$ p-primary, the main question is to determine the near-isomorphism classes of indecomposable (S, p^k)-groups.

There is an interesting connection between almost completely decomposable groups and representations of finite partially ordered sets.

Let G be an almost completely decomposable group with regulator $R = \bigoplus_{\rho \in S} R_\rho$, critical typeset S, and regulator quotient G/R that is a finite abelian group of exponent p^k. We assume that the critical types are p-locally free. Set

$R(\tau) = \bigoplus_{\tau \leq \rho \in \mathrm{T}_{\mathrm{cr}}(A)} R_\rho$. Let $*$ be an element defined to be incomparable to any element in S. Then

$$U_G = \left(\frac{R}{p^kR}, \frac{R(\tau)+p^kR}{p^kR}, \frac{p^kG}{p^kR} \mid \tau \in S^{\mathrm{opp}} \cup \{*\}\right)$$

is a representation of the poset $S^{\mathrm{opp}} \cup \{*\}$ in the category of $\mathbb{Z}/p^k\mathbb{Z}$-modules where $\tau \mapsto \frac{R(\tau)+p^kR}{p^kR}$ and $* \mapsto \frac{p^kG}{p^kR}$. Two representations U_G and U_H are isomorphic (as representations) if and only if G and H are nearly isomorphic, and U_G is indecomposable if and only if G is indecomposable. The terms "bounded representation type" and "unbounded representation type" stem from the theory of representation of posets. Details are in [6] that also contains a complete survey of the known and open problems in the subject.

In the representation U_G the term $\frac{R}{p^kR}$ is a free $\mathbb{Z}/p^k\mathbb{Z}$-module. Choosing a basis of $\frac{R}{p^kR}$ and a basis of the finite p-group p^kG/p^kR the crucial module p^kG/p^kR can be encoded by a "representing matrix" with coefficients in $\mathbb{Z}/p^k\mathbb{Z}$. In this way representation problems are turned into matrix problems.

Similarly, by choosing a (suitable) basis of the finite p-group G/R and expressing its elements in terms of a (suitable) basis of R one encodes the group G by an integral matrix, its *coordinate matrix*. In this matrix the entries are determined only modulo p^k, and therefore the matrix may be considered to be a matrix with coefficients in $\mathbb{Z}/p^k\mathbb{Z}$. Doing so makes the coordinate matrix identical with the representing matrix.

We denote by $(1,n)$ the poset $\{\tau_0, \tau_1 < \cdots < \tau_n\}$ where τ_0 is incomparable to any one of the other elements. In this paper we study homocyclic $((1,5), p^3)$-groups, where $G \in ((1,5), p^3)$ is *homocyclic* if $G/\mathrm{R}(G)$ is a direct sum of cyclic groups all of the same order, $p^3 = \exp(G/\mathrm{R}(G))$. We present a complete catalogue of near-isomorphism types of indecomposable homocyclic groups in $((1,5), p^3)$. There are precisely 20 near-isomorphism classes, and all have rank 7, 8, 9. The proof includes finding a normal form for coordinate matrices of $((1,5), p^3)$-groups, see Sect. 3.

For example, there are infinitely many indecomposable $((1,5), p^3)$-groups when the regulator quotient is not required to be homocyclic [6] but there are only 20 if the regulator quotient is homocyclic (this paper).

Open Problems In [6] there is a list of nine classes of groups that are not known to have finitely many near-isomorphism classes of indecomposables or not, and the solution of those few problems would complete our theory of groups with an inverted forest as a critical typeset. Meanwhile three of those open problems have been solved, including the class dealt with in the present paper, c.f. [12, 13].

A few classes play a key role among the classes of groups with an inverted forest as a critical typeset. These "boundary classes" are such that classes that are in a (suggestive) sense "below" such a boundary class have finitely many indecomposables up to near-isomorphism and those "above" do not.

To emphasize: Our theory of almost completely decomposable groups with an inverted forest as a critical typeset is complete if and only if we know whether the remaining boundary classes have finitely many indecomposables or not. Moreover, it seems that in each of the remaining six problematic boundary classes the difficulties aggregate. If the following six classes are shown to be bounded or not, then our theory is complete.

1. $((1,2),p^5),((1,4),p^3),((2,3),p^2),((2,4),p^2)$ if the regulator quotient is not required to be homocyclic;
2. $((1,3),p^5),((2,4),p^2)$ if the regulator quotient is homocyclic.

2 Matrices

We deal with integer matrices. A *line* of a matrix is a row or a column. *Transformations* of matrices are successive applications of elementary transformations. Matrices are simplified by making entries equal to 0. While annihilating an entry, other entries that were originally zero may become nonzero; such entries are called *fill-ins* and must be removed, i.e., the original 0 must be restored. There is a fixed exponent p^k and entries may be changed modulo p^k, in particular $p^h = 0$ if $h \geq k$. A *unit* in our context is an integer that is relatively prime to p. An integer matrix $A = [a_{i,j}]$ is called *p-reduced (modulo p^k)* if

1. there is at most one 1 in a line and all other entries are in $p\,\mathbb{Z}$,
2. if an entry 1 of A is at the position (i_s, j_s), then $a_{i_s,j} = 0$ for all $j > j_s$ and $a_{i,j_s} = 0$ for all $i < i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\,\mathbb{Z}$ for all $j < j_s$ and all $i > i_s$.

Thus in a p-reduced matrix, the entries left of and below an entry 1 are in $p\,\mathbb{Z}$.

Lemma 2.1 (cf. [5, Lemma 1]) *Let A be an integer matrix.*

1. *The matrix A can be transformed into a p-reduced matrix by elementary row transformations upward and elementary column transformations to the right, i.e., interchange of lines is not used.*
2. *If in addition row transformations down are allowed, then the matrix A can be transformed into a p-reduced matrix where all entries are* 0 *below a* 1.

3 Homocyclic $((1,n),p^k)$-Groups and Coordinate Matrices

The following terminology is used in this paper. Details, equivalent formulations, and confirmation of assertions can be found in [1] or [11]. For a general treatment of (S,p^k)-groups see [6].

Let G be an almost completely decomposable group. The isomorphism types of the regulator $\mathrm{R}(G)$ and the regulator quotient $G/\mathrm{R}(G)$ are near-isomorphism invariants of G. In particular, the rank r of the regulator quotient is an invariant of G. Given a prime p, G is *p-reduced* if the localization $G_{(p)}$ of G at p is a free $\mathbb{Z}_{(p)}$-module, or, equivalently, if each type $\tau \in \mathrm{T_{cr}}(G)$ is *p-locally free*, i.e., $pX \neq X$

for any rank-1 subgroup X of G with $[X] = \tau$. ($[X]$ denotes the isomorphism class of X.) An almost completely decomposable group without summands of rank 1 is called *clipped*.

A coordinate matrix of G is obtained by means of bases of R and G/R. Write $R = S_1 x_1 \oplus \cdots \oplus S_m x_m$ with $x_i \in R$, $S_i = \{s \in \mathbb{Q} : s x_i \in R\}$, and $p^{-1} \notin S_i$. In this case, $\{x_1, \ldots, x_m\}$ is called a *p-basis* of R.

A matrix $M = [m_{i,j}]$ is a *coordinate matrix* of G modulo R if M is integral, there is a basis $(\gamma_1, \ldots, \gamma_r)$ of G/R, there are representatives $g_i \in G$ of γ_i, and there is a p-basis $\{x_1, \ldots, x_m\}$ of R such that

$$g_i = p^{-k_i}(\textstyle\sum_{j=1}^{m} m_{i,j} x_j) \quad \text{where} \quad \langle \gamma_i \rangle \cong \mathbb{Z}_{p^{k_i}}, \quad 1 \le k_i \le k = \exp(G/R).$$

A coordinate matrix M of G is of size $r \times m$ and coordinate matrices that are congruent modulo p^k describe equal groups. Since $(\gamma_1, \ldots, \gamma_r)$ is a basis of G/R, a coordinate matrix M of size $r \times m$ has (p-)rank r, i.e., the r rows of M are linearly independent modulo p^k. Each column of a coordinate matrix corresponds to a type. So we speak of the type of a column and of τ-columns of M. So there corresponds the sequence $(\tau_1, \tau_2, \ldots, \tau_m)$ of the column types to the coordinate M. The number $r_\tau(G)$ of τ-columns of M is called the *τ-homogeneous rank of* G. This is a near-isomorphism invariant of G.

A matrix M is said to be *decomposable* if there are permutation matrices X, Y, such that $XMY = M_1 \oplus M_2$. There are the special cases $XMY = [M_1 \,|\, 0]$ and $XMY = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$. The following lemma is a well-known fact and we include the simple argument for the convenience of the reader.

Lemma 3.1 (cf. [6, Lemma 3.1]) *The almost completely decomposable group G is decomposable if and only if there exists a decomposable coordinate matrix of G.*

In particular, if G has a decomposable coordinate matrix M, i.e., $XMY = M_1 \oplus M_2$ with permutation matrices X, Y, then $G = G_1 \oplus G_2$ where G_i has the coordinate matrix M_i.

Clearly, a 0*-column of M displays a summand of rank* 1*, i.e., G is not clipped.*

Proof The coordinate matrix is obtained by means of a p-basis B of $R = \mathrm{R}(G)$. Each column of M corresponds to a basis element and the columns of the M_i determine a partition $B = B_1 \cup B_2$ of the p-basis and there is a corresponding direct decomposition $R = R_1 \oplus R_2$. It is easy to see that $G = G_1 \oplus G_2$ where $G_i = \langle R_i \rangle_*$, the purification of R_i in G.

Henceforth let G be a homocyclic $((1, n), p^k)$-group of rank m with regulator $R = \mathrm{R}(G)$ and critical typeset $\mathrm{T}_{\mathrm{cr}}(G) = \{\tau_0, \tau_1 < \cdots < \tau_n\}$, where $\mathrm{T}_{\mathrm{cr}}(G)$ is a poset of p-locally free types, and G/R is a homocyclic group of rank r and of exponent p^k, and we write $M = \left[M_{\tau_0} \,\|\, M_{\tau_1} \,\middle|\, \cdots \,\middle|\, M_{\tau_n}\right]$ where M_{τ_i} contains all τ_i-columns.

We call transformations of rows and of columns of a coordinate matrix of G *allowed* if the transformed coordinate matrix is the coordinate matrix of a near-isomorphic group. In particular, transformations are allowed if they are due to changes of the two bases involved. The following transformations are allowed in our case (see [6, p. 11], [3, Theorem 12])

(a), (b) Add an integer multiple of a row of M to any other row of M (this is because our groups are homocyclic);
(c) multiply a row of M by a unit modulo p^k;
(d) interchange any two rows of M;
(e) for $j \geq i$, add an integer multiple of a column of M_{τ_i} to a column of M_{τ_j};
(f) multiply a column of M by a unit modulo p^k;
(g) interchange any two columns of M_{τ_i}.

If the coordinate matrix M is formed with respect to the regulator R, then the submatrices of M formed by all τ_0-columns and the rest matrix both have rank equal to the rank r of the regulator quotient. Conversely, if the coordinate matrix M is formed with respect to a completely decomposable subgroup R of finite index and M satisfies the stated rank conditions, then R is the regulator ([11, Theorem 8.1.10], [3, Lemma 13]). These rank conditions are called the *Regulator Criterion*.

For clipped groups the τ_0-columns of a coordinate matrix always can be transformed to the identity matrix without any change of the rest, because of the Regulator Criterion. By Lemma 2.1 the part $\left[M_{\tau_1} \mid \cdots \mid M_{\tau_n}\right]$ of a coordinate matrix M can be changed into p-reduced form, cf. Proposition 4.1.

4 Standard Coordinate Matrices

We establish a standard form for coordinate matrices of homocyclic $((1,n),p^k)$-groups. If $A = [A_{i,j}]$ is a block matrix, then we denote by $A_{*,j}$ and by $A_{i,*}$ the jth block column and the ith-block row of A, respectively. Integer entries that are prime to p are called units.

Our main technique is forming Smith Normal Forms and variations thereof. Two matrices A, B are said to be *equivalent* if there are invertible matrices X, Y such that $B = XAY$. It is well known that every integral matrix is equivalent to a matrix in *Smith Normal Form*, $\operatorname{diag}(a_{1,1}, \ldots, a_{k,k}, 0, \ldots, 0)$ where $a_{i,i}$ divides $a_{i+1,i+1}$. Here we consider integer (coordinate) matrices and deal with them modulo p^k, because in our setting matrices that are congruent modulo p^k describe the same group. So we have the *(modified) Smith Normal Form*

$$\begin{bmatrix} I & & & & \\ & pI & & & \\ & & \ddots & & \\ & & & p^{k-1}I & \\ & & & & 0 \end{bmatrix}$$

where the empty space indicates 0-blocks and I stands for identity matrices of various sizes. We often call $\begin{bmatrix} p^hI & 0 \\ 0 & p^{h+1}I \end{bmatrix}$ the *(partial) Smith Normal Form*. In our case $p^3 = 0$, and the possible (partial) Smith Normal Forms are $\begin{bmatrix} I & 0 \\ 0 & pX \end{bmatrix}, \begin{bmatrix} pI & 0 \\ 0 & p^2X \end{bmatrix}, \begin{bmatrix} p^2I & 0 \\ 0 & 0 \end{bmatrix}$.

An integer matrix is said to be *reduced* if it is either 0 or of the form $p^l I$, l some nonnegative integer. A block matrix $A = [A_{i,j}]$ with blocks $A_{i,j}$ is said to be *completely reduced* if all blocks are reduced. Let $[A_{i,j}]$ be an integer block matrix. A single block $A_{i,j}$ is either reduced or it is called a *placeholder*.

Let $[A_1 | A_2 | \ldots | A_r]$ be a sequence of blocks in a block row of a block matrix M. The situation occurs often where arbitrary row transformations in the block row can be applied, arbitrary column transformations can be applied in each A_i-column and all column transformations to the right can be done. Suppose that all Smith Normal Forms of the A_i are of the form $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then we form the Smith Normal Form of A_1. Annihilate, assuming that this is possible, with $I \subset A_1$ in all $A_2, A_3, \ldots$. This splits $A_2 = \left[\begin{smallmatrix} 0 \\ X_2 \end{smallmatrix}\right]$. Actually $\forall\, i : A_i = \left[\begin{smallmatrix} 0 \\ X_i \end{smallmatrix}\right]$. Then form the Smith Normal Form of $X_2 = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and annihilate with $I \subset X_2$ in $A_3, A_4, \ldots$. Continue so through the sequence of the A_i's. We obtain the so-called *iterated Smith Normal Form* of the sequence $[A_1 | A_2 | \ldots]$ *starting with* A_1. The iterated Smith Normal Form looks like:

$$[A_1 | A_2 | A_3 | \cdots] = \left[\begin{array}{cc|cc|cc|c} I & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & I & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & I & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{array}\right].$$

There is an obvious variant for columns instead of rows and the start is from below. In general, if we form the iterated Smith Normal Form we tacitly assure that the already obtained "reduced blocks" of the whole coordinate matrix can be reestablished.

Changing the block matrix $[A_{i,j}]$ by a collection W of allowed transformations a block matrix $[A^W_{i,j}]$ is obtained. Let i_0, j_0 be fixed and let A_{i_0,j_0} be a 0-block. The 0-block A_{i_0,j_0} is called *restorable* if there is always a collection U of allowed transformations leaving all blocks $A^W_{i,j}$ unchanged if $(i,j) \neq (i_0, j_0)$ and changing the block $A^W_{i_0,j_0}$ back to 0. In particular, if W caused fill-ins in the 0-block A_{i_0,j_0}, then these fill-ins can be removed by U without additional change of the other blocks.

We improve the notation of [5, Proposition 2] and, for the convenience of the reader, we give an adapted proof.

Proposition 4.1 *Let n be a natural number and let p be a prime and $(1,n) = (\tau_0, \tau_1 < \cdots < \tau_n)$. A homocyclic $((1,n), p^k)$-group without summands of rank ≤ 3 has a coordinate matrix of the form*

$$[I \| pA \| I] = \left[\begin{array}{c||ccccc||ccccc} & pA_{2,1} & 0 & 0 & \cdots & 0 & I(\tau_2) & 0 & \cdots & \cdots & 0 \\ & pA_{3,1} & pA_{3,2} & 0 & \cdots & 0 & 0 & I(\tau_3) & 0 & \cdots & 0 \\ I & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots \\ & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \\ & pA_{n,1} & pA_{n,2} & \cdots & \cdots & pA_{n,n-1} & 0 & 0 & \cdots & \cdots & I(\tau_n) \\ \hline \tau_0 & \tau_1 & \tau_2 & \tau_3 & \cdots & \tau_{n-1} & \tau_2 & \tau_3 & \cdots & \cdots & \tau_n \end{array}\right] \equiv \left[\begin{array}{c||ccccccccc} & pA_{2,1} & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ & pA_{3,1} & 0 & pA_{3,2} & I & \cdots & \cdots & 0 & 0 \\ I & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ & pA_{n,1} & 0 & pA_{n,2} & 0 & \cdots & \cdots & pA_{n,n-1} & I \\ \hline \tau_0 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \cdots & \cdots & \tau_{n-1} & \tau_n \end{array}\right] \tag{1}$$

The lower triangular block matrix $[A_{i,j}]$ ($[pA_{i,j}]$ is stripped to $[A_{i,j}]$) is p-reduced. For the block $pA_{n,1}$ there is a matrix D such that $pA_{n,1} = p^2 D$. The blocks $pA_{i,j}$ are of the form

$$pA_{i,j} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & pI & 0 \\ 0 & 0 & p^2 A'_{i,j} \end{array}\right], \textit{ where lines may be absent.}$$

In particular, if G is a homocyclic $((1,5), p^3)$-group, then the standard coordinate matrix has the form

$$\left[\begin{array}{cccc||cccc||ccc||cc|c}
0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_2 \\
0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\hline\hline
0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & B_1 & B_2 & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & \tau_3 \\
0 & 0 & B_3 & B_4 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & \\
\hline\hline
pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & C_1 & C_2 & C_3 & 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & C_4 & C_5 & C_6 & 0 & 0 & F_1 & F_2 & 0 & pI & 0 & 0 & 0 & \tau_4 \\
0 & C_7 & C_8 & C_9 & 0 & 0 & F_3 & F_4 & 0 & 0 & H & 0 & 0 & \\
\hline\hline
D_1 & D_2 & D_3 & D_4 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
D_5 & D_6 & D_7 & D_8 & 0 & G_1 & G_2 & G_3 & pI & 0 & 0 & 0 & 0 & \tau_5 \\
D_9 & D_{10} & D_{11} & D_{12} & 0 & G_4 & G_5 & G_6 & 0 & J_1 & J_2 & pI & 0 & \\
D_{13} & D_{14} & D_{15} & D_{16} & 0 & G_7 & G_8 & G_9 & 0 & J_3 & J_4 & 0 & K & \\
\hline
\multicolumn{4}{c||}{\tau_1} & \multicolumn{4}{c||}{\tau_2} & \multicolumn{3}{c||}{\tau_3} & \multicolumn{2}{c|}{\tau_4} &
\end{array}\right] \tag{2}$$

where, saving space, we write the coordinate matrix differently. We omit the identity matrix in front formed by the τ_0-columns, and τ_i in the last column indicates the location of $I(\tau_i)$. matrix in the back. All placeholders, like C_9, have entries all in $p^2\,\mathbb{Z}$.

Proof Let G be given by a coordinate matrix M where the columns are ordered as their types. As G is clipped, the τ_0-columns form a square matrix N that by the Regulator Criterion is invertible. Hence N can be transformed by column transformations alone to the identity matrix without changing the rest, cf. [3, Proposition 4]. So $M = [I \mid M']$ and we disregard the leading identity matrix and call M' the coordinate matrix. The regulator quotient is homocyclic. This allows arbitrary row transformations and Lemma 2.1(2) applies. So this coordinate matrix M' can be transformed to a p-reduced matrix, and by the Regulator Criterion M' contains columns forming a permutation matrix of size r, where r is the number of rows of M'.

We move the columns of the included permutation matrix to the right, keeping the order of the types, and rearrange this matrix by a row permutation of the full coordinate matrix to I. As the coordinate matrix M' is p-reduced we obtain the complete coordinate matrix in the form $[I \mid M'] = [I_0 \mid pA \mid I_1]$. The identity matrix I_1 contains all the remaining units in M'. Note that the columns of pA and of I_1 are ordered as their types, respectively. Since each column of the identity matrix I_1 allows to annihilate with its entry 1 in all columns of type greater or equal we obtain the lower triangular form of pA.

By Lemma 2.1(1) the (stripped) part A can be transformed to a p-reduced matrix. The induced row transformations of the identity matrix I_0 can be compensated by column transformations alone. The induced row transformations (upward) of the identity matrices I_0, I_1 can be undone by column transformations alone, respectively, and give a block structure due to the types. This and the ordering of the columns of pA define a block structure on pA as shown in (1).

A τ_n-column in pA is 0, as A is p-reduced. So there cannot be a τ_n-column if G is clipped. By Lemma 3.1 a τ_1-column in I_1 displays a summand of rank 2, hence there is no such column. As A is p-reduced, $A_{i,j} = 0$ if $j \geq i$. Thus we get the claimed block matrix for $[I_0 \mid pA \mid I_1]$. A $p \in pA_{n,1}$ allows to annihilate in its whole row and in its whole column displaying a summand of rank 3, by Lemma 3.1. So there is a matrix D such that $pA_{n,1} = p^2 D$.

Since arbitrary column transformations are allowed in the first block column $A_{*,1}$ and since arbitrary row operations are allowed in each block row $A_{i,*}$ we may form the iterated Smith Normal Form of the first block column $A_{*,1}$, starting with $A_{n-1,1}$. So we already obtained the first block column of the coordinate matrix (1). Then we annihilate with all $p \in pA_{*,1}$ horizontally to the right in the rows of pA.

Arbitrary column transformations are allowed in the second block column $A_{*,2}$. If we leave the 0-rows unchanged that are due to the p's in the first block column, then arbitrary row operations are allowed in the remaining rows of each block row $A_{i,*}$. So excluding the rows that we leave unchanged we may form the iterated Smith Normal Form of the remaining rows of the second block column $A_{*,2}$, starting with $A_{n,2}$. Again we annihilate with the p's in the second block column horizontally to the right in the rows of pA. So we obtain the second block column of the coordinate matrix (1). We continue to treat all block columns to the right successively and we get the claimed coordinate matrix. This procedure automatically transforms the blocks $pA_{i,j}$ as indicated.

A coordinate matrix as in Proposition 4.1 is called *standard*. The block format of a standard coordinate matrix of G and the number of entries p in each block $pA_{i,j}$ of pA are near-isomorphism invariants of G.

Proposition 4.2 (cf. [5, Proposition 5]) *Let G be a homocyclic $((1, n), p^k)$-group with the standard coordinate matrix. Then the size of the $I(\tau_i)$'s, the size of the blocks $A_{i,j}$ and the numbers of entries p in a block $pA_{i,j}$ are near-isomorphism invariants of G for all i, j.*

5 Indecomposable Groups in the Class of Homocyclic $((1,5),p^3)$-Groups

$(1,5)$-groups are of rank ≥ 6 because the critical typeset consists of 6 types. Indecomposable $(1,5)$-groups have a regulator quotient of rank ≥ 2. Thus coordinate matrices of indecomposable $(1,5)$-groups must have at least two rows. This means that τ_0 contributes at least two ranks and the remaining five critical types τ_i must contribute at least one rank each, so the rank of an indecomposable $(1,5)$-group must be ≥ 7.

We denote groups by their *scheme*. With placeholders, like G_6, we refer to the coordinate matrix (2) or the Basic Template (3) below. Recall that all entries in a placeholder matrix are in $p^2\,\mathbb{Z}$. For instance,

$$\left[\begin{array}{cc||ccccc} 1 & 0 & p & 0 & 1 & 0 & 0 \\ 0 & 1 & p^2 & p^2 & 0 & p & 1 \\ \hline \tau_0 & \tau_0 & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \end{array}\right] = \left[\begin{array}{cc||ccc||cc} 1 & 0 & p & 0 & 0 & 1 & 0 \\ 0 & 1 & p^2 & p^2 & p & 0 & 1 \\ \hline \tau_0 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_5 \end{array}\right] \text{ with scheme } \left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & G_6 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]^1_7 = [7.1]$$

denotes a group of rank 7 by its coordinate matrix and its *scheme* where 7 indicates the rank and 1 is a running number. There are five groups of rank 7, so [7.1] till [7.5] denote all groups of rank 7 in the list below by their schemes.

List of Indecomposable Homocyclic $((1,5),p^3)$-Groups

(rank 7)
$$\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & G_6 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]^1_7, \left[\begin{array}{ccc|c} C_6 & 0 & p & \tau_4 \\ D_4 & p & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^2_7, \left[\begin{array}{ccc|c} B_2 & p & 0 & \tau_3 \\ 0 & G_5 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]^3_7, \left[\begin{array}{ccc|c} C_9 & 0 & 0 & \tau_4 \\ D_8 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^4_7, \left[\begin{array}{ccc|c} C_3 & p & 0 & \tau_4 \\ 0 & G_7 & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^5_7,$$

(rank 8)
$$\left[\begin{array}{cc|c} 0 & p & \tau_3 \\ p & 0 & \tau_4 \\ D_{13} & G_8 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]^1_8, \left[\begin{array}{cc|c} p & 0 & \tau_2 \\ C_8 & 0 & \tau_4 \\ D_7 & p & \tau_5 \\ \hline \tau_1 & \tau_3 & \end{array}\right]^2_8, \left[\begin{array}{cc|c} p & 0 & \tau_3 \\ C_7 & 0 & \tau_4 \\ D_{14} & G_9 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]^3_8, \left[\begin{array}{cc|c} p & 0 & \tau_2 \\ C_5 & p & \tau_4 \\ 0 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_3 & \end{array}\right]^4_8, \left[\begin{array}{cc|c} B_4 & 0 & \tau_3 \\ C_3 & p & \tau_4 \\ 0 & G_7 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]^5_8,$$

(rank 9)
$$\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & 0 & p & \tau_5 \\ D_{14} & G_9 & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]^1_9, \left[\begin{array}{ccc|c} C_6 & 0 & p & \tau_4 \\ D_4 & p & 0 & \tau_5 \\ 0 & 0 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^2_9, \left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ C_7 & 0 & 0 & \tau_4 \\ D_{10} & G_6 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]^3_9, \left[\begin{array}{ccc|c} 0 & B_2 & p & \tau_3 \\ p & 0 & 0 & \tau_4 \\ D_{13} & 0 & G_8 & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \end{array}\right]^4_9, \left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ 0 & C_9 & 0 & \tau_4 \\ D_7 & D_8 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_3 & \end{array}\right]^5_9,$$

$$\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ C_7 & F_4 & 0 & \tau_4 \\ 0 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^6_9, \left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ C_8 & 0 & 0 & \tau_4 \\ D_7 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^7_9, \left[\begin{array}{ccc|c} B_4 & 0 & 0 & \tau_3 \\ C_3 & p & 0 & \tau_4 \\ 0 & G_7 & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^8_9, \left[\begin{array}{ccc|c} B_2 & 0 & p & \tau_3 \\ C_3 & p & 0 & \tau_4 \\ 0 & 0 & G_8 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \end{array}\right]^9_9, \left[\begin{array}{ccc|c} 0 & p & 0 & \tau_4 \\ C_6 & 0 & p & \tau_4 \\ 0 & G_7 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]^{10}_9,$$

Proposition 5.1 *The homocyclic* $((1,5),p^3)$*-groups in the list above are indecomposable and pairwise not near-isomorphic.*

Proof By Proposition 4.2 the groups in the list are pairwise not near-isomorphic. We illustrate the details with examples. First of all near-isomorphic groups have equal ranks so we only need to consider groups in the subsets of equal rank. Secondly, the types in the bottom row and the last column of the schemes have to coincide for near-isomorphic group (Proposition 4.2) because these encode the sizes of the I_{τ_i} and the blocks $A_{i,j}$. This says by itself that the group [9.1] is not near-isomorphic with any other group of rank 9 and hence of any other group in the list. On the other hand, this does not say that [9.6] and [9.8] are not near-isomorphic. But [9.6] has

$A_{1,3} = [p]$ while [9.8] has $A_{1,3} = [p^2]$, so the number of entries p is different in the block and therefore [9.6] and [9.8] are not near-isomorphic.

To show that the groups in the list are indecomposable we utilize the connection with representations. We restrict ourselves to the situation at hand and let $S := (\tau_0, \tau_1 < \cdots < \tau_n)$ where the τ_i are p-free types.

Let $G \in (S, p^k)$ with regulator $R := \mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(G)} R_\rho$ and suppose that $\mathrm{T}_{\mathrm{cr}}(G) \subset S$ and $p^k G \subset R$. Define $\overline{} : R \to R/p^k R : \overline{x} = x + p^k R$, so $\overline{R} = R/p^k R$. The *(anti)-representation* U_G *of* G is given by $U_G = (\overline{R}, \overline{R(\sigma)}, \overline{p^k G} : \sigma \in S)$. Let G' be another group of (S, p^k) with regulator R' and representation $U_{G'}$. A homomorphism $f : U_G \to U'_G$ is a $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$-module homomorphism $f : \overline{R} \to \overline{R'}$ such that $\forall, \sigma \in S : f(\overline{R(\sigma)}) \subset \overline{R'(\sigma)}$ and $f(\overline{p^k G}) \subset \overline{p^k G'}$. The following facts are well-known and can be found in [1, 11] and [4].

- $\overline{R}$ is a finite free $\mathbb{Z}_{p^k}$-module.
- In general, $\overline{p^k G}$ is a finite $\mathbb{Z}_{p^k}$-module, and in the homocyclic case it is free.
- $\overline{p^k G} \cong G/R$.
- G is nearly isomorphic with G' if and only if U_G and $U_{G'}$ are isomorphic.
- G is indecomposable if and only if End U_G contains no idempotents other than 0 and 1.

For simplicity, we assume in the following that our groups are homocyclic. We associate with a representation U_G a $\mathbb{Z}_{p^k}$-matrix that encodes $\overline{p^k G}$. Let $\{x_1, \ldots, x_m\}$ be a p-basis of R and let $\{g_1, \ldots, g_r\}$ be a basis of $\overline{p^k G}$. Then $\{\overline{x_1}, \ldots, \overline{x_m}\}$ is a basis of the free module $\overline{R}$. Expressing the generators g_i in terms of the basis $\{\overline{x_1}, \ldots, \overline{x_m}\}$ we obtain the *representing matrix* of G. This matrix (in the homocyclic case) is identical with the earlier (integral) coordinate matrix except that instead of "working modulo p^k", the entries are considered elements of $\mathbb{Z}_{p^k}$.

Exemplarily we show that the group G of rank 9 with scheme [9.10] is indecomposable. Its representing matrix is

$$M = \left[\begin{array}{ccc||c|c|c|cc|c} 1 & 0 & 0 & 0 & p & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & p^2 & 0 & p & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & p^2 & p^2 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & p & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & p^2 & 0 & p & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & p^2 & p^2 & 0 & 0 & 1 \end{bmatrix}$$

The representing matrix comes with basis of $\overline{R}$ and we view the elements of $\overline{R}$ as coordinate vectors with respect to this basis. In particular, $\overline{p^3 G}$ is simply the row space of M. To show that G is indecomposable we show that any idempotent endomorphism of $U_G = (\overline{R}, \overline{R(\sigma)}, \overline{p^3 G} : \sigma \in \{\tau_0, \tau_1, \ldots, \tau_5\})$ is either the zero map 0 or the identity 1. In terms of coordinates an endomorphism of $\overline{R}$ is a Z_{p^3}-matrix that acts by right multiplication on the elements $(x_{01}, x_{02}, x_{02}, x_1, x_2, x_3, x_{41}, x_{42}, x_5)$ of $\overline{R}$. The requirement that $f(\overline{R(\sigma)}) \subset \overline{R(\sigma)}$ implies that f, as a matrix, is of the form

$$f = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ 0 & 0 & 0 & 0 & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ 0 & 0 & 0 & 0 & 0 & d_{11} & d_{12} & d_{13} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{11} & e_{12} & e_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{21} & e_{22} & e_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{11} \end{bmatrix}.$$

The additional requirement that $f(\overline{p^3G}) \subset \overline{p^3G}$, i.e., that the row space of M is invariant under right multiplication by f, has a very handy description due to the fact that M has the right inverse $M^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In fact, [4, Theorem 4.3], $f(\overline{p^3G}) \subset \overline{p^3G}$ if and only if $Mf = MfM^*M$. Using a computer algebra program we find that (in our example) $Mf = MfM^*M$ if and only if

$$\begin{bmatrix} e_{13} - a_{13} + pc_{15} \\ b_{16}p^2 + d_{14}p + e_{23} - a_{23} \\ h_{11} - a_{33} + p^2c_{15} + p^2d_{14} \end{bmatrix} = 0, \quad \begin{bmatrix} -p^2a_{12} \\ p^2b_{11} - p^2a_{22} \\ -p^2a_{32} \end{bmatrix} = 0, \quad \begin{bmatrix} pc_{11} - pa_{11} - p^2a_{13} \\ p^2b_{12} - p^2a_{23} - pa_{21} \\ p^2c_{11} - p^2a_{33} - pa_{31} \end{bmatrix} = 0,$$

$$\begin{bmatrix} pc_{12} - pa_{12} - p^2a_{13} \\ pd_{11} - pa_{22} - p^2a_{23} + p^2b_{13} \\ p^2c_{12} - p^2a_{33} - pa_{32} + p^2d_{11} \end{bmatrix} = 0, \quad \begin{bmatrix} e_{11} - a_{11} + pc_{13} \\ b_{14}p^2 + d_{12}p + e_{21} - a_{21} \\ p^2c_{13} - a_{31} + p^2d_{12} \end{bmatrix} = 0, \quad \begin{bmatrix} e_{12} - a_{12} + pc_{14} \\ b_{15}p^2 + d_{13}p + e_{22} - a_{22} \\ p^2c_{14} - a_{32} + p^2d_{13} \end{bmatrix} = 0.$$

It follows immediately that $p^2a_{12} = 0$, $p^2a_{21} = 0$, $pa_{31} = 0$, $pa_{32} = 0$. Furthermore, we get immediately that $h_{11} \equiv a_{33} \mod p$, $b_{11} \equiv a_{22} \mod p$, $c_{11} \equiv a_{11} \mod p$, $d_{11} \equiv a_{22} \mod p$, $e_{11} \equiv a_{11} \mod p$, $e_{22} \equiv a_{22} \mod p$. Using these results we find in two steps that $c_{11} \equiv a_{33} \mod p$, $p^2c_{12} = p^2a_{12} = 0$, $p^2e_{12} = p^2a_{12} = 0$, and finally, in three steps that $a_{33} \equiv d_{11} \mod p$. We obtain that $\alpha := a_{11} \equiv a_{22} \equiv a_{33} \equiv b_{11} \equiv c_{11} \equiv d_{11} \equiv e_{11} \equiv e_{22} \equiv h_{11}$. In particular, we get $\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \equiv \begin{bmatrix} \alpha & 0 \\ e_{21} & \alpha \end{bmatrix} \mod p$ and $\begin{bmatrix} \alpha & 0 \\ e_{21} & \alpha \end{bmatrix}^2 = \begin{bmatrix} \alpha & 0 \\ e_{21} & \alpha \end{bmatrix}$. It now follows that $\alpha \in \{0, 1\}$ and $e_{21} \equiv 0 \mod p$. For the full matrix f we have

$$f^2 = f \equiv \begin{bmatrix} \alpha & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ 0 & 0 & 0 & 0 & \alpha & c_{12} & c_{13} & c_{14} & c_{15} \\ 0 & 0 & 0 & 0 & 0 & \alpha & d_{12} & d_{13} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & e_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & e_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix} \mod p$$

with $\alpha \in \{0, 1\}$. Finally, from $f^2 = f$ and either $f = pf'$ or $f = 1 + pf'$ it follows that $f \in \{0, 1\}$ and that G is indecomposable.

All other indecomposability proofs are similar, somewhat lengthy, but straightforward.

6 The Class of Homocyclic $((1,5),p^3)$-Groups is Bounded

Theorem 6.1 *There are precisely the* 20 *near-isomorphism types of homocyclic* $((1,5),p^3)$*-groups of rank* 7*,* 8 *and* 9 *in the list above that are indecomposable.*

Proof Let G be an indecomposable $((1,5),p^3)$-group with coordinate matrix in standard form, cf. (2). Recall that an indecomposable $(1,5)$-group has rank ≥ 7. Thus summands of rank ≤ 6 are contradictory.

Moreover, if the coordinate matrix displays a summand, then this summand is near-isomorphic to G. If this summand occurs in the indecomposables list, then it can be omitted without loss of generality due to the tacit assumption that the starting group is not the one that appeared. The starting group is then the last indecomposable group that appears. In particular the following transformations are allowed. With the blocks pI in the columns with number $1,2,3,6,7,10$ we annihilate D_1, D_5, D_9, C_1, C_4, D_2, D_6, B_1, C_2, D_3, G_1, G_4, F_1, G_2 and J_1, respectively. The fill-ins in the identity matrix to the right can be deleted by the respective pI to the left. The 0-blocks obtained this way always can be restored and we indicate this by leaving such blocks empty meaning that fill-ins in those blocks can be ignored.

Now we show that some additional blocks can be seen to be 0. A $p^2 \in D_{16}$ allows to annihilate in its complete row and then in its complete column. This leads to a summand of rank 3. Thus $D_{16} = 0$. In turn a $p^2 \in D_{15}$ allows to annihilate in its complete row and then in its column, except for the pI above. This leads to a summand of rank 5. Thus $D_{15} = 0$. Further, as $D_{16} = 0$, a $p^2 \in D_{12}$ allows to annihilate in its row, except of in pI to the right, and then in its complete column. This leads to a summand of rank 4. Thus $D_{12} = 0$. A $p^2 \in D_{11}$ allows to annihilate in its row except of in pI to the right and then in its column except of in pI above This leads to a summand of rank 6. Thus $D_{11} = 0$.

Moreover, we show that the A-row is not present. We form the iterated Smith Normal Form of $\begin{bmatrix} D_4 \\ D_8 \end{bmatrix}$. Above the 0-columns of $\begin{bmatrix} D_4 \\ D_8 \end{bmatrix}$ we form the iterated Smith Normal Form of this part of $\begin{bmatrix} C_3 \\ C_6 \\ C_9 \end{bmatrix}$. Above the 0-columns of this part of $\begin{bmatrix} C_3 \\ C_6 \\ C_9 \end{bmatrix}$ we form the iterated Smith Normal Form of $\begin{bmatrix} B_2 \\ B_4 \end{bmatrix}$. Now we annihilate with all of the just obtained p^2I's in A. First with those of B, then with those of C and lastly with those of D. So all nonzero entries of A are above 0-columns and they cause summands of rank 3. Hence $A = 0$ and the A-row is not present.

By similar arguments, switching from rows to columns, we conclude that the K-column is not present. For this we form the iterated Smith Normal Form of $[D_{13} \mid D_{14}]$ starting with D_{14}. We form the iterated Smith Normal Form of the part of $[G_7 \mid G_8 \mid G_9]$ that continues the 0-rows of $[D_{13} \mid D_{14}]$, starting with G_9. Then we form the iterated Smith Normal Form of the part of $[J_3 \mid J_4]$ that continues the 0-rows of $[D_{13} \mid D_{14} \mid G_7 \mid G_8 \mid G_9]$, starting with J_4. Now we annihilate in K with the just obtained p^2I's. First with those of J, then with those of G and lastly with those of D. So all nonzero entries of K are to the right of 0-rows and they cause summands of rank 3. Hence $K = 0$ and the K-column is not present. So we get

Basic Template

$$\left[\begin{array}{cccc||cccc||ccc||c||c}
\square & \square & pI & \square & \multicolumn{4}{c||}{\square} & \multicolumn{3}{c||}{\square} & \square & \tau_2 \\
\hline\hline
\square & pI & \square & \square & \square & \square & \square & \square & & & & & \\
\square & \square & \square & B_2 & \square & \square & pI & \square & & \square & & \square & \tau_3 \\
\square & \square & B_3 & B_4 & \square & \square & \square & E & & & & & \\
\hline\hline
pI & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & & \\
\square & \square & \square & C_3 & \square & pI & \square & \square & \square & \square & \square & \square & \tau_4 \\
\square & \square & C_5 & C_6 & \square & \square & \square & F_2 & \square & pI & \square & & \\
\square & C_7 & C_8 & C_9 & \square & \square & F_3 & F_4 & \square & \square & H & & \\
\hline\hline
\square & \square & \square & D_4 & pI & \square & \square & \square & \square & \square & \square & \square & \\
\square & \square & D_7 & D_8 & \square & \square & \square & G_3 & pI & \square & \square & \square & \tau_5 \\
\square & D_{10} & 0 & 0 & \square & \square & G_5 & G_6 & \square & \square & J_2 & pI & \\
D_{13} & D_{14} & 0 & 0 & \square & G_7 & G_8 & G_9 & \square & J_3 & J_4 & \square & \\
\hline\hline
\multicolumn{4}{c||}{\tau_1} & \multicolumn{4}{c||}{\tau_2} & \multicolumn{3}{c||}{\tau_3} & \tau_4 &
\end{array}\right] \tag{3}$$

Note that the meanings of placeholders, for instance F_4, changes in the various matrices. In the coordinate matrix displayed next the name F_4 denotes a part of the F_4 in (3). Doing so avoids a proliferation of indices.

Completely Reduced Forms of $\left[\begin{smallmatrix} C_5 & C_6 \\ C_8 & C_9 \end{smallmatrix}\right]$ and of $\left[\begin{smallmatrix} G_5 & G_6 \\ G_8 & G_9 \end{smallmatrix}\right]$

The completely reduced forms of $\left[\begin{smallmatrix} C_5 & C_6 \\ C_8 & C_9 \end{smallmatrix}\right]$ and of $\left[\begin{smallmatrix} G_5 & G_6 \\ G_8 & G_9 \end{smallmatrix}\right]$ can be produced independently although both processes split the F-blocks in various ways.

Annihilations in the adjoint blocks C_3, C_7 and G_3, G_7, respectively, also can be done independently and, in particular, without any effect to the block pI where the C_3-row and the G_7-column are crossing. We get

$$\left[\begin{array}{c|c|ccc|ccc||c|c|ccc|ccc||c|ccc|c||c||c}
 & & pI & & & & & & & & & & & & & & & & & & & & \tau_2 \\
\hline\hline
 & pI & & & & & & & & & & & & & & & & & & & & & \\
\hline
 & & & & & B_2^1 & 0 & 0 & & & pI & & & & & & & & & & & & \\
 & & & & & B_2^2 & 0 & 0 & & & & pI & & & & & & & & & & & \tau_3 \\
 & & & & & B_2^3 & 0 & 0 & & & & & pI & & & & & & & & & & \\
\hline
 & & B_3 & 0 & 0 & B_4 & 0 & 0 & & & & & & E & 0 & 0 & & & & & & & \\
\hline\hline
pI & & & & & & & & & & & & & & & & & & & & & & \\
\hline
 & & & & & C_3 & 0 & 0 & & pI & & & & & & & & & & & & & \\
\hline
 & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & F_2 & 0 & 0 & & pI & & & & & \\
 & & 0 & p^2I & 0 & 0 & 0 & 0 & & & & & & 0 & 0 & 0 & & & pI & & & & \tau_4 \\
 & & 0 & 0 & 0 & 0 & p^2I & 0 & & & & & & 0 & 0 & 0 & & & & pI & & & \\
\hline
 & C_7 & 0 & 0 & 0 & 0 & 0 & 0 & & & F_3 & 0 & 0 & F_4 & 0 & 0 & & & & & H & & \\
 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & 0 & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & 0 & & \\
\hline\hline
 & & & & & D_4^1 & D_4^2 & D_4^3 & pI & & & & & & & & & & & & & & \\
\hline
 & & D_7^1 & D_7^2 & D_7^3 & D_8^1 & D_8^2 & D_8^3 & & & & & & G_3 & 0 & 0 & pI & & & & & & \\
\hline
 & D_{10}^1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & & & & & J_2 & & \\
 & D_{10}^2 & & 0 & & & 0 & & & & 0 & p^2I & 0 & 0 & 0 & 0 & & & & & 0 & pI & \tau_5 \\
 & D_{10}^3 & & & & & & & & & 0 & 0 & 0 & 0 & p^2I & 0 & & & & & 0 & & \\
\hline
D_{13}^1 & D_{14}^1 & & & & & & & & G_7 & 0 & 0 & 0 & 0 & 0 & 0 & & J_3^1 & J_3^2 & J_3^3 & J_4 & & \\
D_{13}^2 & D_{14}^2 & & 0 & & & 0 & & & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & & \\
D_{13}^3 & D_{14}^3 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & p^2I & & 0 & 0 & 0 & 0 & & \\
\hline\hline
\multicolumn{8}{c||}{\tau_1} & \multicolumn{8}{c||}{\tau_2} & \multicolumn{5}{c||}{\tau_3} & \tau_4 &
\end{array}\right]$$

Note that empty blocks can be kept to be 0 and the explicitly denoted 0-blocks are created by forming Smith Normal Forms or if nonzero entries would create low rank summands.

By successively forming Smith Normal Forms and assuming that the group G is not one of the indecomposables list, we show in three steps that several blocks are 0. This causes certain block lines of the coordinate matrix to be absent.

Blocks $D_8^1, D_8^2, D_{14}^1, D_{14}^2, D_4^1, D_{13}^1, D_4^3, D_{13}^3, D_7^1, D_{10}^1, D_7^2, D_{10}^2$ are 0
We show that those blocks are 0 in the indicated sequence. A $p^2 \in D_8^1$ allows to annihilate in its complete column and then in its complete row except of p to the right, causing a summand of rank 4. Thus $D_8^1 = 0$.

In turn a $p^2 \in D_8^2$ allows to annihilate in D_7^2, D_7^3, D_8^3. We annihilate first in G_3, D_7^1 and then in D_4^2 and in p^2I above. This displays a summand of rank 4. Thus $D_8^2 = 0$.

A $p^2 \in D_{14}^1$ allows to annihilate in its complete row and then in its complete column except of p above, causing a summand of rank 5. Thus $D_{14}^1 = 0$.

In turn a $p^2 \in D_{14}^2$ allows to annihilate in $D_{10}^2, D_{10}^3, D_{14}^3$. We annihilate first in C_7, D_{10}^1 and then in D_{13}^2 and in p^2I to the right. This displays a summand of rank 5. Thus $D_{14}^2 = 0$.

A $p^2 \in D_4^1$ allows to annihilate in D_4^2, D_4^3, C_3, B_2. So there is no 0-column in B_4 above such a $p^2 \in D_4^1$, to avoid a summand of rank 4. We form the Smith Normal Form of D_4^1 and do all the possible annihilations with this $p^2I \subset D_4^1$. Then we form the Smith Normal Form of that part X of B_4 above the 0-columns of D_4^1. Further we annihilate with $p^2I \subset X$ in B_4 and in turn we form the Smith Normal Form of the part Y of B_4 above $p^2I \subset D_4^1$. Since a $p^2 \in Y$, i.e., above a $p^2 \in D_4^1$, allows to annihilate in B_3, E, we get a summand of rank 6. Hence $D_4^1 = 0$.

A $p^2 \in D_{13}^1$ allows to annihilate in $D_{13}^2, D_{13}^3, G_7, J_3$. So there is no 0-row in J_4 to the right of such a $p^2 \in D_{13}^1$, to avoid a summand of rank 5. We form the Smith Normal Form of D_{13}^1, and do all the possible annihilations with this $p^2I \subset D_{13}^1$. Then we form the Smith Normal Form of that part X of J_4 to the right of the 0-rows of D_{13}^1. Further we annihilate with $p^2I \subset X$ in J_4 and in turn we form the Smith Normal Form of the part Y of J_4 to the right of $p^2I \subset D_{13}^1$. Since a $p^2 \in Y$, i.e., to the right of a $p^2 \in D_{13}^1$, allows to annihilate in J_2, H, we get a summand of rank 5. Hence $D_{13}^1 = 0$.

A $p^2 \in D_4^2$ allows to annihilate in D_4^3 and a $p^2 \in D_8^3$ allows to annihilate in D_4^3, hence a $p^2 \in D_4^3$ leads to a summand of rank 6. Thus $D_4^3 = 0$.

A $p^2 \in D_{13}^2$ allows to annihilate in D_{13}^3 and a $p^2 \in D_{14}^3$ allows to annihilate in D_{13}^3, hence a $p^2 \in D_{13}^3$ leads to a summand of rank 5. Thus $D_{13}^3 = 0$.

An entry $p^2 \in D_7^1$ allows to annihilate in D_7^2, D_7^3, G_3 and after that it allows to annihilate in B_3. Fill-ins above D_8^3 in the B_3-row can be removed by $p^2I \subset C_9$ below. We form the Smith Normal Form of D_7^1, then we form the Smith Normal Form of the part $X \subset D_8^3$ to the right of the 0-rows of D_7^1. We annihilate with $p^2 I \subset X$ in D_8^3 and form the Smith Normal Form of the rest of D_8^3. After that we annihilate with $p^2 I \subset D_7^1$ in D_7^2 and in G_3. A row with $p^2 \in D_7^1$ does not continue to a 0-row in D_8^3 to avoid a summand of rank 6. Hence this $p^2 \in D_7^1$ displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ 0 & C_9 & 0 & \tau_4 \\ D_7 & D_8 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_3 & \end{array}\right]$, [9.5]. Thus $D_7^1 = 0$.

An entry $p^2 \in D_{10}^1$ allows to annihilate in D_{10}^2, D_{10}^3, C_7 and after that it allows to annihilate in J_2. Fill-ins to the right of D_{14}^3 in the J_2-column can be removed by $p^2 I \subset G_9$ to the left. We form the Smith Normal Form of D_{10}^1, then we form the Smith Normal Form of the part $X \subset D_{14}^3$ below the 0-columns of D_{10}^1. We annihilate with $p^2 I \subset X$ in D_{14}^3 and form the Smith Normal Form of the rest of D_{14}^3. After that we annihilate with $p^2 I \subset D_{10}^1$ in D_{10}^2 and in C_7. A column with $p^2 \in D_{10}^1$ does not continue to a 0-column in D_{14}^3 to avoid a summand of rank 5. Hence this $p^2 \in D_{10}^1$ displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & 0 & p & \tau_5 \\ D_{14} & G_9 & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]$, [9.1]. Thus $D_{10}^1 = 0$.

We form the Smith Normal Form of D_7^2, then we form the Smith Normal Form of the part $X \subset D_8^3$ to the right of the 0-rows of D_7^2. We annihilate with $p^2 I \subset X$ in D_8^3 and form the Smith Normal Form of the rest $Y \subset D_8^3$. There is no 0-row in Y, because $p^2 I \subset D_7^2$ allows to annihilate in D_7^3, G_3 and in $p^2 I$ above, creating summands of rank 6. So the Smith Normal Form of Y is $[p^2 I \mid 0]$. Fill-ins above D_8^3 in the C_6-row can be removed by $p^2 I \subset C_9$ below. Hence a $p^2 \in D_7^2$ displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ 0 & C_9 & 0 & \tau_4 \\ D_7 & D_8 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_3 & \end{array}\right]$, again [9.5]. Thus $D_7^2 = 0$.

We form the Smith Normal Form of D_{10}^2, then we form the Smith Normal Form of the part $X \subset D_{14}^3$ below of the 0-columns of D_{10}^2. We annihilate with $p^2 I \subset X$ in D_{14}^3 and form the Smith Normal Form of the rest $Y \subset D_{14}^3$. There is no 0-column in Y, because $p^2 I \subset D_{14}^2$ allows to annihilate in D_{14}^3, C_7 and in $p^2 I$ to the right, creating summands of rank 5. So the Smith Normal Form of Y is $\left[\begin{smallmatrix} p^2 I \\ 0 \end{smallmatrix}\right]$. Fill-ins to the right of D_{14}^3 in the G_8-column can be removed by $p^2 I \subset G_9$ to the right. Hence a $p^2 \in D_{10}^2$ displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & 0 & p & \tau_5 \\ D_{14} & G_9 & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]$, again [9.1]. Thus $D_{10}^2 = 0$.

We include those 0-blocks and obtain the new coordinate matrix

		pI										τ_2
	pI											
			B_2^1 0 0			pI						
			B_2^2 0 0			pI						τ_3
			B_2^3 0 0			pI						
		B_3 0 0	B_4 0 0				E 0 0					
pI												
			C_3 0 0		pI							
		0 0 0	0 0 0				F_2 0 0		pI			
		0 p^2I 0	0 0 0				0 0 0		pI			τ_4
		0 0 0	0 p^2I 0				0 0 0		pI			
	C_7	0 0 0	0 0 0			F_3 0 0	F_4 0 0			H		
	0	0 0 p^2I	0 0 0			0 0 0	0 0 0			0		
	0	0 0 0	0 0 p^2I			0 0 0	0 0 0			0		
			0 D_4^2 0	pI								
		0 0 D_7^3	0 0 D_8^3				G_3 0 0	pI				
	0					0 0 0	0 0 0			J_2		
	0	0	0			0 p^2I 0	0 0 0			0	pI	τ_5
	D_{10}^3					0 0 0	0 p^2I 0			0		
0	0				G_7	0 0 0	0 0 0		J_3^1 J_3^2 J_3^3	J_4		
D_{13}^2	0	0	0		0	0 0 p^2I	0 0 0		0 0 0	0		
0	D_{14}^3				0	0 0 0	0 0 p^2I		0 0 0	0		
		τ_1				τ_2			τ_3		τ_4	

Blocks $J_2, B_3, H, E, J_3^2, B_2^2, F_2, J_3^1, F_3, B_2^1$ are 0

We form the iterated Smith Normal Form of $\begin{bmatrix} J_2 \\ J_4 \end{bmatrix}$, starting with J_4, and annihilate with $p^2I \subset J_2$ in H. This displays summands of rank 4. Thus $J_2 = 0$.

We form the iterated Smith Normal Form of $[B_3 | B_4]$, starting with B_4, and annihilate with $p^2I \subset B_3$ in E. This displays summands of rank 5. Thus $B_3 = 0$.

We form the Smith Normal Forms of C_7 and of that part $X \subset F_4$ that continues the 0-rows of C_7. Then we annihilate with $p^2I \subset C_7$ in F_3 and with $p^2I \subset X$ we annihilate in F_4, F_3. We form the Smith Normal Form of the part $Z \subset F_3$ that continues the 0-rows of $[C_7 | F_4]$.

Further we form the Smith Normal Form of J_4. With all p^2I's in C_7, Z, J_4, X we annihilate in H. There remains a rest $Y \subset H$ with lines that are 0 outside of H. Forming the Smith Normal Form of Y summands of rank $2, 3$ occur. So we get $H = 0$.

We form the Smith Normal Forms of G_3 and of that part $X \subset F_4$ that continues the 0-columns of G_3. Then we annihilate with $p^2I \subset G_3$ in F_2 and with $p^2I \subset X$ we annihilate in F_4, F_2. We form the Smith Normal Form of the rest $Z \subset F_2$. Further we form the Smith Normal Form of B_4. With all p^2I's in G_3, Z, B_4, X we annihilate in E. There remains a rest $Y \subset E$ with lines that are 0 outside of E. Forming the Smith Normal Form of Y summands of rank $2, 3$ occur. So we get $E = 0$.

We form the Smith Normal Form of G_7 and we form the Smith Normal Form of the part $X \subset J_4$ that continues the 0-rows of G_7. With $p^2I \subset X$ we annihilate in J_4 and in J_3. Then we form the iterated Smith Normal Form of the part Y of $[J_3^1 | J_3^2 | J_3^3]$ that continues the 0-rows of $[G_7 | J_4]$, starting with J_3^3. Further we annihilate with

$p^2I \subset G_7$ in $[J_3^1|J_3^2]$. Hence non-zero rows of J_3^1 continue to 0-rows outside of J_3^1 and non-zero rows of J_3^2 continue to 0-rows outside of J_3^2. Thus an entry $p^2 \in J_3^2$ displays a summand of rank 8 with scheme $\left[\begin{array}{ccc|c} p & 0 & & \tau_2 \\ C_5 & p & & \tau_4 \\ 0 & J_3 & & \tau_5 \\ \hline \tau_1 & \tau_3 & & \end{array}\right]$, [8.4]. Thus $J_3^2 = 0$.

We form the Smith Normal Form of C_3 and we form the Smith Normal Form of the part $X \subset B_4$ that continues the 0-columns of C_3. With $p^2I \subset X$ we annihilate in B_4 and in B_2. Then we form the iterated Smith Normal Form of the part Y of $\left[\begin{smallmatrix} B_2^1 \\ B_2^2 \\ B_2^3 \end{smallmatrix}\right]$ that continues the 0-columns of $\left[\begin{smallmatrix} B_4 \\ C_3 \end{smallmatrix}\right]$, starting with B_2^3. Further we annihilate with $p^2I \subset C_3$ in $\left[\begin{smallmatrix} B_2^1 \\ B_2^2 \end{smallmatrix}\right]$. Hence non-zero columns of B_2^1 continue to 0-columns outside of B_2^1 and non-zero columns of B_2^2 continue to 0-columns outside of B_2^2. Thus an entry $p^2 \in B_2^2$ displays a summand of rank 7 with scheme $\left[\begin{array}{ccc|c} B_2 & p & 0 & \tau_3 \\ 0 & G_5 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]$, [7.3]. Thus $B_2^2 = 0$.

We form the Smith Normal Form of G_3 and we form the Smith Normal Form of the part $X \subset F_4$ that continues the 0-columns of G_3. Then we annihilate with $p^2I \subset X$ in F_4. After that we annihilate with $p^2I \subset G_3$ and with $p^2I \subset X$ in F_2. The part $Y \subset F_2$ that continues the 0-columns of $\left[\begin{smallmatrix} F_4 \\ G_3 \end{smallmatrix}\right]$ and the block J_3^1 are connected. Note that non-zero columns of Y (and so of F_2) are 0 outside of Y. Up to now we did not change the completely reduced form of that part of $[J_3^1|J_3^2|J_3^3]$ that continues the 0-rows of $[G_7|J_4]$, obtained proving $J_3^2 = 0$. The non-zero rows of J_3^1 are of the form $[p^2I|0]$. The columns with $p^2 \in J_3^1$ are 0 except of the entry p above, and all other entries in such a row are 0. The form $J_3^1 = [p^2I|0]$ splits $Y = \left[\begin{smallmatrix} Y_1 \\ Y_2 \end{smallmatrix}\right]$, where Y_2 is connected with the 0-columns of J_3^1. An entry $p^2 \in Y_1$ allows to annihilate in Y_1, Y_2, where the intermediate block pI between Y and J_3^1 can be reestablished without changing J_3^1 because all fill-ins in pI can be removed by entries of pI that are above a 0-column of J_3^1. So an entry $p^2 \in Y_1$ leads to a summand of rank 6, thus $Y_1 = 0$. We form the Smith Normal Form of Y_2. The Smith Normal Form of J_3^1 can be reestablished. An entry $p^2 \in Y_1$ displays a summand of rank 4. Thus $Y = 0$, hence $F_2 = 0$. But then an entry $p^2 \in J_3^1$ displays a summand of rank 5. Thus $J_3^1 = 0$.

We form the Smith Normal Form of C_7 and we form the Smith Normal Form of the part $X \subset F_4$ that continues the 0-rows of C_7. Then we annihilate with $p^2I \subset X$ in F_4. After that we annihilate with $p^2I \subset C_7$ and with $p^2I \subset X$ in F_3. The part $Y \subset F_3$ that continues the 0-rows of $[C_7|F_4]$ and the block B_2^1 are connected. Note that non-zero rows of Y (and so of F_3) are 0 outside of Y. Up to now we did not change the completely reduced form of that part of $\left[\begin{smallmatrix} B_2^1 \\ B_2^2 \\ B_2^3 \end{smallmatrix}\right]$ that continues the 0-columns of $[C_3|B_4]$, obtained proving $B_2^2 = 0$. The non-zero columns of B_2^1 are of the form $\left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$. The rows with $p^2 \in B_2^1$ are 0 except of the entry p to the right, and all other entries in such a column are 0. The form $B_2^1 = \left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$ splits $Y = [Y_1|Y_2]$, where Y_2 is connected to the 0-rows of B_2^1.

An entry $p^2 \in Y_1$ allows to annihilate in Y_1, Y_2, where the intermediate block pI between Y and B_2^1 can be reestablished without changing B_2^1 because all fill-ins in pI can be removed by entries of pI that are to the right of a 0-row of B_2^1. So an entry $p^2 \in Y_1$ leads to a summand of rank 6, thus $Y_1 = 0$. We form the Smith Normal Form of Y_2. The Smith Normal Form of B_2^1 can be reestablished. An entry $p^2 \in Y_2$ displays a summand of rank 5. Thus $Y = 0$, hence $F_3 = 0$. But then an entry $p^2 \in B_2^1$ displays a summand of rank 4. Thus $B_2^1 = 0$.

By the discussion above, that certain blocks are 0, we get that some block lines are not present. The B_3-, the J_3^1- and the J_3^2-block column are not present. The B_2^1-, the B_2^2- and the J_2-row are not present. Further, if there are blocks pI or p^2I in a block line that is no more present, also the crossing block line through this pI or p^2I is not present. Moreover, we include $E = 0$ and $H = 0$ and obtain the new coordinate matrix

$$
\left[\begin{array}{cccccc|cccccc|ccc|c|c}
 & & pI & & & & & & & & & & & & & & \tau_2 \\
\hline\hline
 & pI & & & & & & & & & & & & & & & \\
 & & & B_2^3 & 0 & 0 & & & pI & & & & & & & & \tau_3 \\
 & & 0 & B_4 & 0 & 0 & & & & & 0 & & & & & & \\
\hline\hline
pI & & & & & & & & & & & & & & & & \\
 & & & C_3 & 0 & 0 & & pI & & & & & & & & & \\
 & & 0 & 0 & p^2I & 0 & & & & & 0 & & & pI & & & \\
 & C_7 & 0 & 0 & 0 & 0 & & & & F_4 & 0 & 0 & & & & & \tau_4 \\
 & 0 & p^2I & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & & & 0 & & \\
 & 0 & 0 & 0 & 0 & p^2I & & & & 0 & 0 & 0 & & & & & \\
\hline\hline
 & & & 0 & D_4^2 & 0 & pI & & & & & & & & & & \\
 & & D_7^3 & 0 & 0 & D_8^3 & & & & G_3 & 0 & 0 & pI & & & & \\
 & D_{10}^3 & & & & & & & 0 & 0 & p^2I & 0 & & & 0 & pI & \tau_5 \\
0 & 0 & & & & & & G_7 & 0 & 0 & 0 & 0 & & J_3^3 & J_4 & & \\
D_{13}^2 & 0 & 0 & & 0 & & & 0 & p^2I & 0 & 0 & 0 & & 0 & 0 & & \\
0 & D_{14}^3 & & & & & & 0 & 0 & 0 & 0 & p^2I & & 0 & 0 & & \\
\hline\hline
 & & & \tau_1 & & & & & \tau_2 & & & & & \tau_3 & & \tau_4 &
\end{array}\right]
$$

This 16×16 matrix is decomposable displaying two 8×8 matrices, M_1 and M_2. We have to discuss both parts separately.

$$
M_1 = \left[\begin{array}{ccc|ccc|c|c|c}
 & pI & & & & & & & \tau_2 \\
\hline\hline
pI & & & & & & & & \tau_3 \\
\hline\hline
C_7 & 0 & 0 & F_4 & 0 & 0 & & & \\
0 & p^2I & 0 & 0 & 0 & 0 & & & \tau_4 \\
0 & 0 & p^2I & 0 & 0 & 0 & & & \\
\hline\hline
 & D_7^3 & D_8^3 & G_3 & 0 & 0 & pI & & \\
D_{10}^3 & 0 & 0 & 0 & p^2I & 0 & & pI & \tau_5 \\
D_{14}^3 & 0 & 0 & 0 & 0 & p^2I & & & \\
\hline\hline
 & \tau_1 & & & \tau_2 & & \tau_3 & \tau_4 &
\end{array}\right], \quad
M_2 = \left[\begin{array}{ccc|ccc|cc|c}
 & B_2^3 & 0 & & & pI & & & \tau_3 \\
 & B_4 & 0 & & & & & & \\
\hline\hline
pI & & & & & & & & \\
 & C_3 & 0 & & pI & & & & \tau_4 \\
 & 0 & p^2I & & & & pI & & \\
\hline\hline
 & 0 & D_4^2 & pI & & & & & \\
0 & 0 & 0 & & G_7 & 0 & J_3^3 & J_4 & \tau_5 \\
D_{13}^2 & 0 & 0 & & 0 & p^2I & 0 & 0 & \\
\hline\hline
 & \tau_1 & & & \tau_2 & & \tau_3 & &
\end{array}\right]
$$

Discussion of M_1

The blocks D_{10}^3, D_{14}^3 have no 0-row and a $p^2 \in D_{14}^3$ allows to annihilate in D_{10}^3, so the completely reduced form of $\left[\begin{array}{c} D_{10}^3 \\ \hline D_{14}^3 \end{array}\right]$ is $\left[\begin{array}{ccc} 0 & p^2I & 0 \\ \hline p^2I & 0 & 0 \end{array}\right]$. A 0-column in C_7 leads to summands of rank 3, or rank 6 or, if there is a $p^2 \in D_{10}^3$ in this column, then a summand of rank 7 is displayed with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ D_{10} & G_6 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]$, [7.1]. Thus C_7 has no 0-columns and we get the completely reduced form of

$$\left[\begin{array}{c} C_7 \\ \hline D_{10}^3 \\ \hline D_{14}^3 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & p^2I \\ 0 & p^2I & 0 \\ p^2I & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & p^2I & 0 \\ \hline p^2I & 0 & 0 \end{array}\right].$$

The blocks D_7^3, D_8^3 have no 0-column and a $p^2 \in D_8^3$ allows to annihilate in D_7^3, so the completely reduced form of $[D_7^3 | D_8^3]$ is $\left[\begin{array}{c|c} 0 & p^2I \\ p^2I & 0 \\ 0 & 0 \end{array}\right]$. A 0-row in G_3 leads to summands of rank 3, or rank 6 or, if there is a $p^2 \in D_7^3$ in this row, then a summand of rank 8 is displayed with scheme $\left[\begin{array}{cc|c} p & 0 & \tau_2 \\ C_8 & 0 & \tau_4 \\ D_7 & p & \tau_5 \\ \hline \tau_1 & \tau_3 & \end{array}\right]$, [8.2]. Thus G_3 has no 0-rows and we get the completely reduced form of

$$[D_7^3 | D_8^3 | G_3] = \left[\begin{array}{c|c|cccc} 0 & p^2I & 0 & 0 & p^2I & 0 \\ p^2I & 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 & 0 \end{array}\right].$$

We include these completely reduced forms and obtain the new coordinate matrix

$$M_1 = \left[\begin{array}{ccc|c|c||cccccc||c||c||c}
 & & & pI & & & & & & & & & & \tau_2 \\
\hline\hline
\multicolumn{3}{c|}{pI} & & & & & & & & & & & \tau_3 \\
\hline\hline
0 & 0 & p^2I & 0 & 0 & F_4^1 & F_4^2 & F_4^3 & F_4^4 & 0 & 0 & & & \\
0 & p^2I & 0 & 0 & 0 & F_4^5 & F_4^6 & F_4^7 & F_4^8 & 0 & 0 & & & \\
p^2I & 0 & 0 & 0 & 0 & F_4^9 & F_4^{10} & F_4^{11} & F_4^{12} & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & F_4^{13} & F_4^{14} & F_4^{15} & F_4^{16} & 0 & 0 & & & \tau_4 \\
0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
\hline\hline
 & & & 0 & p^2I & 0 & 0 & p^2I & 0 & 0 & 0 & & & \\
 & & & p^2I & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & pI & & \\
 & & & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & & & \tau_5 \\
\cline{1-13}
0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I & 0 & & pI & \\
\cline{1-13}
p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I & & & \\
\hline\hline
\multicolumn{5}{c||}{\tau_1} & \multicolumn{6}{c||}{\tau_2} & \tau_3 & \tau_4 &
\end{array}\right]$$

Discussion of F_4

In F_4 annihilation is allowed upward and to the left. All 0-lines of F_4 immediately lead to summands. Obviously $F_4^{16} = 0$.

Discussion of 0-Lines of F_4

A 0-row in the F_4^{13}-row leads to a summand of rank 2. A 0-row in the F_4^{9}-row leads to a summand of rank 8 with scheme $\left[\begin{array}{cc|c} p & 0 & \tau_3 \\ C_7 & 0 & \tau_4 \\ D_{14} & G_9 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]$, [8.3]. A 0-row in the F_4^{5}-row leads to a summand of rank 8 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ C_7 & 0 & 0 & \tau_4 \\ D_{10} & G_6 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_4 & \end{array}\right]$, [9.3]. A 0-row in the F_4^{1}-row leads to a summand of rank 5.

A 0-column in the F_4^{16}-block column leads to a summand of rank 1. A 0-column in the F_4^{15}-block column leads to a summand of rank 7 with scheme $\left[\begin{array}{ccc|c} C_9 & 0 & 0 & \tau_4 \\ D_8 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [7.4]. A 0-column in the F_4^{14}-block column leads to a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ C_8 & 0 & 0 & \tau_4 \\ D_7 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.7]. A 0-column in the F_4^{13}-block column leads to a summand of rank 4. Thus, altogether, the whole block F_4 has no 0-line.

Discussion of the Blocks $F_4^{15}, F_4^{12}, F_4^{14}, F_4^{8}, F_4^{4}, F_4^{13}$

A $p^2 \in F_4^{15}$ leads to a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} 0 & F_4 & 0 & \tau_4 \\ C_9 & 0 & 0 & \tau_4 \\ D_8 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 3 and 6.

A $p^2 \in F_4^{12}$ leads to a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ C_7 & F_4 & 0 & \tau_4 \\ D_{13} & 0 & G_9 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \end{array}\right]$. This group decomposes into summands of rank 3 and 6.

A $p^2 \in F_4^{14}$ leads to a summand of rank 11 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_2 \\ 0 & F_4 & 0 & \tau_4 \\ C_8 & 0 & 0 & \tau_4 \\ D_7 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 3 and 8.

A $p^2 \in F_4^{8}$ leads to a summand of rank 10 with scheme $\left[\begin{array}{cccc|c} p & 0 & 0 & 0 & \tau_3 \\ C_7 & F_4 & 0 & 0 & \tau_4 \\ D_{10} & 0 & G_6 & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \tau_4 & \end{array}\right]$. This group decomposes into summands of rank 3 and 7.

Entries $p^2 \in F_4^{4}$ and $p^2 \in F_4^{13}$ lead to summands of rank 5 and 6, respectively. Thus the F_4^{16}-row and the F_4^{16}-column are not present.

Discussion of the Blocks $F_4^{11}, F_4^{10}, F_4^{9}, F_4^{7}, F_4^{3}$

A $p^2 \in F_4^{11}$ leads to a summand of rank 15 with scheme $\left[\begin{array}{ccccc|c} p & 0 & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & 0 & \tau_4 \\ 0 & C_9 & 0 & 0 & 0 & \tau_4 \\ 0 & D_8 & 0 & 0 & p & \tau_5 \\ D_{14} & 0 & 0 & G_9 & 0 & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes, one summand of rank 3 and two summands of rank 6.

A $p^2 \in F_4^{10}$ leads to a summand of rank 17 with scheme $\left[\begin{array}{ccccc|c} 0 & p & 0 & 0 & 0 & \tau_2 \\ p & 0 & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & 0 & \tau_4 \\ 0 & C_8 & 0 & 0 & 0 & \tau_4 \\ 0 & D_7 & G_3 & 0 & 0 & \tau_5 \\ D_{14} & 0 & 0 & G_9 & 0 & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 3, 6 and 8.

A $p^2 \in F_4^9$ leads to a summand of rank 12 with scheme $\left[\begin{array}{cccc|c} p & 0 & 0 & 0 & \tau_3 \\ C_7 & F_4 & 0 & 0 & \tau_4 \\ 0 & G_3 & 0 & p & \tau_5 \\ D_{10} & 0 & G_6 & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 6.

A $p^2 \in F_4^7$ leads to a summand of rank 16 with scheme $\left[\begin{array}{cccccc|c} p & 0 & 0 & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & 0 & 0 & \tau_4 \\ 0 & C_9 & 0 & 0 & 0 & 0 & \tau_4 \\ 0 & D_8 & G_3 & 0 & p & 0 & \tau_5 \\ D_{10} & 0 & 0 & G_6 & 0 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \tau_4 & \end{array}\right]$. This group decomposes into summands of rank 3, 6 and 7.

A $p^2 \in F_4^3$ leads to a summand of rank 12 with scheme $\left[\begin{array}{cccc|c} p & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & \tau_4 \\ 0 & C_9 & 0 & 0 & \tau_4 \\ 0 & D_8 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 6. Thus the F_4^{11}-block row and the F_4^{11}-block column are not present.

Discussion of the Blocks $F_4^6, F_4^5, F_4^2, F_4^1$

A $p^2 \in F_4^6$ leads to a summand of rank 18 with scheme $\left[\begin{array}{cccccc|c} 0 & p & 0 & 0 & 0 & 0 & \tau_2 \\ p & 0 & 0 & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & 0 & 0 & \tau_4 \\ 0 & C_8 & 0 & 0 & 0 & 0 & \tau_4 \\ 0 & D_7 & G_3 & 0 & p & 0 & \tau_5 \\ D_{10} & 0 & 0 & G_6 & 0 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \tau_4 & \end{array}\right]$. This group decomposes into summands of rank 3, 7 and 8.

A $p^2 \in F_4^5$ leads to a summand of rank 13 with scheme $\left[\begin{array}{ccccc|c} p & 0 & 0 & 0 & 0 & \tau_3 \\ C_7 & F_4 & 0 & 0 & 0 & \tau_4 \\ 0 & G_3 & 0 & p & 0 & \tau_5 \\ D_{10} & 0 & G_6 & 0 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \tau_3 & \tau_4 & \end{array}\right]$. This group decomposes into summands of rank 6 and 7.

A $p^2 \in F_4^2$ leads to a summand of rank 14 with scheme $\left[\begin{array}{cccc|c} 0 & p & 0 & 0 & \tau_2 \\ p & 0 & 0 & 0 & \tau_3 \\ C_7 & 0 & F_4 & 0 & \tau_4 \\ 0 & C_8 & 0 & 0 & \tau_4 \\ 0 & D_7 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 6 and 8.

A $p^2 \in F_4^1$ leads to a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} p & 0 & 0 & \tau_3 \\ C_7^3 & F_4 & 0 & \tau_4 \\ 0 & G_3 & p & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.6]. Thus $F_4^1 = 0$.

So we finally got the contradiction $F_4 = 0$. Hence we obtained all indecomposable groups that come with the coordinate matrix M_1.

Discussion of M_2

G_7 and C_3 are connected, and the Smith Normal Form $G_7 = \left[\begin{array}{c|c} p^2 I & 0 \\ 0 & 0 \end{array}\right]$ splits $C_3 = \left[\begin{array}{c} C_3^1 \\ \hline C_3^2 \end{array}\right]$ where C_3^2 is connected with the 0-columns of G_7. We form the Smith Normal Form of $C_3^1 = \left[\begin{array}{cc} p^2 I & 0 \\ 0 & 0 \end{array}\right]$ and annihilate with $p^2 I \subset C_3^1$ in C_3^2. Note that C_3^2 has no 0-row to avoid a summand of rank 3. Thus the Smith Normal Form of the rest $X \subset C_3^2$ is $[p^2 I | 0]$. Doing this G_7 does not change and we obtain in completely reduced form

$$\left[\begin{array}{c|c} C_3 & pI \\ \hline 0 & G_7 \end{array}\right] = \left[\begin{array}{ccc|ccc} p^2I & 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ 0 & p^2I & 0 & 0 & 0 & pI \\ \hline 0 & 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & 0 & p^2I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We show

(1) *G_7 has no 0-row and $[J_3^3 | J_4]$ has no 0-column,*

(2) *C_3 has no 0-column and $\left[\begin{smallmatrix} B_2^3 \\ B_4 \end{smallmatrix}\right]$ has no 0-row.*

Ad (1): We form the Smith Normal Form of G_7 and we form the Smith Normal Form of the part $X \subset J_4$ that continues the 0-rows of G_7. Then we annihilate with $p^2I \subset X$ in J_4 and in J_3^3. A $p^2 \in X$ displays a summand of rank 3, thus $X = 0$.

The blocks D_4^2, J_3^3 are connected. The Smith Normal Form of D_4^2 is $[p^2I | 0]$ because D_4^2 has no 0-rows. The Smith Normal Forms of D_4^2, G_7, J_4 split the block $J_3^3 = \left[\begin{smallmatrix} Z_1 & Y_1 \\ Z_0 & Y_0 \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} Y_1 \\ Y_0 \end{smallmatrix}\right]$ is connected with the 0-columns of D_4^2 and $[Z_0 | Y_0]$ continues the 0-rows of $[G_7 | J_4]$. We form the Smith Normal Form of Y_0 and annihilate with $p^2I \subset Y_0$ in Y_1 and in Z_0. There is no change in the blocks D_4^2, G_7. The entries $p^2 \in Y_0$ display summands of rank 6, thus $Y_0 = 0$.

Now we form the Smith Normal Form of Z_0 and annihilate with $p^2I \subset Z_0$ in Z_1. There is no change in G_7 and $p^2I \subset D_4^2$ can be reestablished by row transformations. So a $p^2 \in Z_0$ displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} C_6 & 0 & p & \tau_4 \\ D_4 & p & 0 & \tau_5 \\ 0 & 0 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.2]. Thus $Z_0 = 0$. Hence $[Z_0 | Y_0] = 0$ displaying summands of rank 2. Thus the $[Z_0 | Y_0]$-block row is not present, i.e., $[G_7 | J_4]$ has no 0-row. Moreover, if a $p^2 \in J_4$ continues to a 0-row of G_7, then a summand of rank 3 is displayed. Thus G_7 has no 0-row.

A 0-column of Y_1 displays a summand of rank 4, and a 0-column of Z_1 displays a summand of rank 7 with scheme $\left[\begin{array}{ccc|c} C_6 & 0 & p & \tau_4 \\ D_4 & p & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [7.2]. Thus J_3^3 has no 0-column. Clearly J_4 has no 0-column. This shows (1).

Ad (2): We form the Smith Normal Form of C_3 and we form the Smith Normal Form of the part $X \subset B_4$ that continues the 0-columns of C_3. Then we annihilate with $p^2I \subset X$ in B_4 and in B_2^3. A $p^2 \in X$ displays a summand of rank 3, thus $X = 0$.

The blocks D_{13}^2, B_2^3 are connected. The Smith Normal Form of D_{13}^2 is $\left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$ because D_{13}^2 has no 0-columns. The Smith Normal Forms of D_{13}^2, C_3, B_4 split the block $B_2^3 = \left[\begin{smallmatrix} Y_1 & Z_1 \\ Y_0 & Z_0 \end{smallmatrix}\right]$ where $[Y_0 | Z_0]$ is connected to the 0-rows of D_{13}^2 and $\left[\begin{smallmatrix} Z_1 \\ Z_0 \end{smallmatrix}\right]$ continues the 0-columns of $\left[\begin{smallmatrix} B_4 \\ C_3 \end{smallmatrix}\right]$. We form the Smith Normal Form of Z_0 and annihilate with $p^2I \subset Z_0$ in Z_1 and in Y_0. There is no change in the blocks D_{13}^2, C_3. The entries $p^2 \in Z_0$ display summands of rank 6, thus $Z_0 = 0$.

Now we form the Smith Normal Form of Z_1 and annihilate with $p^2I \subset Z_1$ in Y_1. There is no change in C_3 and $p^2I \subset D_{13}^2$ can be reestablished by column transformations. So a $p^2 \in Z_1$ displays a summand of rank 9 with scheme

$\left[\begin{array}{ccc|c} 0 & B_2 & p & \tau_3 \\ p & 0 & 0 & \tau_4 \\ D_{13} & 0 & G_8 & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \end{array}\right]$, [9.4]. Thus $Z_1 = 0$. Hence $\left[\begin{smallmatrix} Z_1 \\ Z_0 \end{smallmatrix}\right] = 0$ displaying summands of rank 1. Thus the $\left[\begin{smallmatrix} Z_1 \\ Z_0 \end{smallmatrix}\right]$-block column is not present, i.e., $\left[\begin{smallmatrix} B_4 \\ C_3 \end{smallmatrix}\right]$ has no 0-column. Moreover, if a $p^2 \in B_4$ is above a 0-column of C_3, then a summand of rank 3 is displayed. Hence C_3 has no 0-column.

A 0-row of Y_1 displays a summand of rank 8 with scheme $\left[\begin{array}{cc|c} 0 & p & \tau_3 \\ p & 0 & \tau_4 \\ D_{13} & G_8 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]$, [8.1]. Thus B_2^3 has no 0-row. Clearly B_4 has no 0-row. This shows (2).

So we obtain the completely reduced form

$$\left[\begin{array}{c|c} C_3 & pI \\ \hline 0 & G_7 \end{array}\right] = \left[\begin{array}{cc|ccc} p^2I & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & p^2I & 0 & 0 & pI \\ \hline 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & p^2I & 0 \end{array}\right].$$

Next we show

(3) $[J_3^3|J_4]$ *has no* 0*-line*,

(4) $\left[\begin{smallmatrix} B_2^3 \\ B_4 \end{smallmatrix}\right]$ *has no* 0 *-line*.

We know already that $[J_3^3|J_4]$ has no 0-column and that $\left[\begin{smallmatrix} B_2^3 \\ B_4 \end{smallmatrix}\right]$ has no 0-row.

Ad (3): We show that $[J_3^3|J_4]$ has no 0-row. We form the Smith Normal Form of that part $X \subset B_4$ that continues the columns of the lower $p^2I \subset C_3$ and annihilate with $p^2I \subset X$ in B_4, B_2^3.

There is a rest $Y \subset B_4$ that continues the columns of the upper $p^2I \subset C_3$. We form the Smith Normal Form of Y and annihilate with $p^2I \subset Y$ in B_2^3.

A p^2 in the upper $p^2I \subset C_3$ allows to annihilate in B_2^3. We show how to remove the occurring fill-ins because this is quite lengthy. Annihilating with p^2 in B_2^3 creates a first fill-in to the left of pI in the B_2^3-block row. This first fill-in is in $p\,\mathbb{Z}$. We annihilate this fill-in with p to the left. This creates a second fill-in below the upper $p^2I \subset G_7$. This second fill-in is in $p^2\,\mathbb{Z}$. We annihilate this second fill-in with a $p^2 \in G_7$ above. This creates a third fill-in to the right of p^2I in the D_{13}^2-block row below J. This third fill-in is in $p^2\,\mathbb{Z}$ and we annihilate it using p^2I to the right. The created fourth fill-in in the J-block column and the B-block row can be removed by the identity matrix to the right with τ_3-columns. Thus the non-zero columns of $\left[\begin{smallmatrix} B_2^3 \\ B_4 \end{smallmatrix}\right]$ that continue the upper $p^2I \subset C_3$ contain only one non-zero entry $p^2 \in B_4$, respectively.

A $p^2 \in G_7$ that continues to a 0-row of $[J_3^3|J_4]$ displays a summand of rank 5 or of rank 8, depending on if its column is connected with a 0-row of C_3 or not. The summand of rank 8 has the scheme $\left[\begin{array}{cc|c} B_4 & 0 & \tau_3 \\ C_3 & p & \tau_4 \\ 0 & G_7 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]$, [8.5]. Thus $[J_3^3|J_4]$ has no 0-row. This shows (3).

Ad (4): We show that $\begin{bmatrix} B_2^3 \\ B_4 \end{bmatrix}$ has no 0-column. We form the Smith Normal Form of that part $X \subset J_4$ that continues the rows of the lower $p^2 I \subset G_7$ and annihilate with $p^2 I \subset X$ in J_4, J_3^3.

There is a rest $Y \subset J_4$ that continues the rows of the upper $p^2 I \subset G_7$. We form the Smith Normal Form of Y and annihilate with $p^2 I \subset Y$ in J_3^3.

A p^2 in the upper $p^2 I \subset G_7$ allows to annihilate in J_3^3. We show how to remove the occurring fill-ins because this is quite lengthy. Annihilating with p^2 in J_3^3 creates a first fill-in above pI in the J_3^3-block column. This first fill-in is in $p\,\mathbb{Z}$. We annihilate this fill-in with p below. This creates a second fill-in to the right of the upper $p^2 I \subset C_3$. This second fill-in is in $p^2\,\mathbb{Z}$. We annihilate this second fill-in with a $p^2 \in C_3$ to the left. This creates a third fill-in above $p^2 I$ in the D_4^2-block column to the right of B. This third fill-in is in $p^2\,\mathbb{Z}$ and we annihilate it using $p^2 I$ below. The created fourth fill-in in the J-block column and the B-block row can be removed by the identity matrix to the right with τ_3-columns. Thus the non-zero rows of $[J_3^3 | J_4]$ that continue the upper $p^2 I \subset G_7$ contain only one non-zero entry $p^2 \in J_4$, respectively.

A $p^2 \in C_3$ that continues to a 0-column of $\begin{bmatrix} B_2^3 \\ B_4 \end{bmatrix}$ displays a summand of rank 4 or of rank 9, depending on if its column is connected with a 0-column of G_7 or not. The summand of rank 9 has the scheme $\left[\begin{array}{ccc|c} B_4 & 0 & 0 & \tau_3 \\ C_3 & p & 0 & \tau_4 \\ 0 & G_7 & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.8]. Thus $\begin{bmatrix} B_2^3 \\ B_4 \end{bmatrix}$ has no 0-column. This shows (4).

Blocks $G_7 = p^2 I, C_3 = p^2 I, J_3^3 = 0, B_2^3 = 0, D_4^2 = 0, D_{13}^2 = 0$

We show $G_7 = p^2 I$. The Smith Normal Form of G_7 is $[p^2 I | 0]$. Let N denote the 0-part of G_7. Let X denote the lower block $p^2 I \subset C_3$. Then N is connected with X. We form the Smith Normal Form of the part $Y \subset B_4$ above X and annihilate with $p^2 I \subset Y$ in B_4, B_2^3. An entry $p^2 \in Y$ displays a summand of rank 6. Thus $Y = 0$. We form the Smith Normal Form of the part $Z \subset B_2^3$ above X and annihilate with $p^2 I \subset Z$ in B_2^3. The Smith Normal Form of Z is $p^2 I$ because B_2^3 has no 0-row and because a 0-column of Z displays a summand of rank 4. So the 0-part N of G_7 is connected with D_{13}^2. We form The Smith Normal Form of D_{13}^2, i.e., $\begin{bmatrix} p^2 I \\ 0 \end{bmatrix}$, because D_{13}^2 has no 0-column. All intermediate blocks between D_{13}^2 and N can be reestablished. A 0-row of D_{13}^2 displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} B_2 & 0 & p & \tau_3 \\ C_3 & p & 0 & \tau_4 \\ 0 & 0 & G_8 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \end{array}\right]$, [9.9]. We get $D_{13}^2 = p^2 I$, and this displays summands of rank 12 with scheme $\left[\begin{array}{cccc|c} 0 & B_2 & 0 & p & \tau_3 \\ p & 0 & 0 & p & \tau_4 \\ 0 & C_3 & p & 0 & \tau_4 \\ D_{13} & 0 & 0 & G_8 & \tau_5 \\ \hline \tau_1 & \tau_1 & \tau_2 & \tau_2 & \end{array}\right]$. This group decomposes into summands of rank 6. Finally the 0-part N of G_7 is not present, i.e., $G_7 = p^2 I$.

We show $C_3 = p^2 I$. The Smith Normal Form of C_3 is $\begin{bmatrix} p^2 I \\ 0 \end{bmatrix}$. Let N denote the 0-part of C_3. Let X denote the part $p^2 I \subset G_7$ that is connected with N. We form the Smith Normal Form of the part $Y \subset J_4$ to the right of X and annihilate with $p^2 I \subset Y$ in J_4, J_3^3. An entry $p^2 \in Y$ displays a summand of rank 6. Thus $Y = 0$.

We form the Smith Normal Form of the part $Z \subset J_3^3$ to the right of X and annihilate with $p^2I \subset Z$ in J_3^3. Doing this G_7 can be reestablished. The Smith Normal Form of Z is p^2I because J_3^3 has no 0-row to avoid a summand of rank 6 and because J_3^3 has no 0-column. So the 0-part N of C_3 is connected with D_4^2. We form The Smith Normal Form of D_4^2, i.e., $[p^2I|0]$, because D_4^2 has no 0-row. All intermediate blocks between D_4^2 and N can be reestablished. A 0-column of D_4^2 displays a summand of rank 9 with scheme $\left[\begin{array}{ccc|c} 0 & p & 0 & \tau_4 \\ C_6 & 0 & p & \tau_4 \\ 0 & G_7 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.10]. Thus $D_4^2 = p^2I$, and this displays summands of rank 12 with scheme $\left[\begin{array}{cccc|c} 0 & 0 & p & 0 & \tau_4 \\ C_6 & 0 & 0 & p & \tau_4 \\ D_4 & p & 0 & 0 & \tau_5 \\ 0 & 0 & G_7 & J_3 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_2 & \tau_3 & \end{array}\right]$. This group decomposes into summands of rank 5 and 7. Finally the 0-part N of C_3 is not present, i.e., $C_3 = p^2I$.

We show $J_3^3 = 0$. Since $G_7 = p^2I$ we may annihilate J_3^3. This creates fill-ins in the J_3^3-block column that are in $p\,\mathbb{Z}$. We annihilate these fill-ins by pI below. This creates fill-ins to the right of C_3 that can be deleted by C_3. Again there are fill-ins to the right of B_2^3, B_4 that can be removed by p^2I below. Hence $J_3^3 = 0$.

We show $B_2^3 = 0$. Since $C_3 = p^2I$ we may annihilate B_2^3. This creates fill-ins in the B_2^3-block row that are in $p\,\mathbb{Z}$. We annihilate these fill-ins by pI to the right. This creates fill-ins below G_7 that can be deleted by G_7. Again there are fill-ins below J_3^3, J_4 that can be removed by p^2I to the left. Hence $B_2^3 = 0$.

We form the Smith Normal Form of D_4^2, i.e, $[p^2I|0]$. The 0-columns of D_4^2 display summands of rank 4, because $J_3^3 = 0$. The non-zero columns of D_4^2 display summands of rank 7 with scheme $\left[\begin{array}{ccc|c} C_6 & 0 & p & \tau_4 \\ D_4 & p & 0 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [7.2]. Thus $D_4^2 = 0$.

We form the Smith Normal Form of D_{13}^2, i.e, $\left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$. The 0-rows of D_{13}^2 display summands of rank 5, because $B_2^3 = 0$. The non-zero rows of D_{13}^2 display summands of rank 8 with scheme $\left[\begin{array}{cc|c} 0 & p & \tau_3 \\ p & 0 & \tau_4 \\ D_{13} & G_8 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]$, [8.1]. Thus $D_{13}^2 = 0$.

By the discussion above, that certain blocks are 0, we get that some block lines are not present. The D_4^2-, the D_{13}^2-block lines, the J_3^2-block column and the B_2^3-block column are not present. Further, if there are blocks pI or p^2I in a block line that is no more present, also the crossing block line through this pI or p^2I is not present. Moreover, we include $C_3 = p^2I$ and $G_7 = p^2I$ and obtain the new coordinate matrix

$$\left[\begin{array}{c|c|c|c} B_4 & & & \tau_3 \\ \hline p^2I & pI & & \tau_4 \\ \hline 0 & p^2I & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$$

We form the Smith Normal Form of $J_4 = \left[\begin{smallmatrix} p^2I \\ 0 \end{smallmatrix}\right]$. Since J_4 and B_4 are connected, the block J_4 splits $B_4 = [B_4^1|B_4^2]$ where B_4^2 is connected with the 0-rows of J_4. We form the Smith Normal Form of B_41 and annihilate with $p^2I \subset B_4^1$ in B_4^2. After that we form the Smith Normal Form of the rest of B_4^2. The intermediate blocks between B_4 and J_4 can be reestablished without changing the Smith Normal

Form of J_4, because B_4^2 is connected with the 0-rows of J_4. Eventually we obtain a completely reduced matrix and four cases

If a 0-column of B_4 is connected with a 0-row of J_4, then a summand of rank 6 is obtained.

If a 0-column of B_4 is connected with a non-zero row of J_4, then a summand of rank 7 is obtained with scheme $\left[\begin{array}{ccc|c} C_3 & p & 0 & \tau_4 \\ 0 & G_7 & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [7.5]. Thus B_4 has no 0-column.

If a non-zero column of B_4 is connected with a 0-row of J_4, then a summand of rank 8 is obtained with scheme $\left[\begin{array}{cc|c} B_4 & 0 & \tau_3 \\ C_3 & p & \tau_4 \\ 0 & G_7 & \tau_5 \\ \hline \tau_1 & \tau_2 & \end{array}\right]$, [8.5].

Thus a non-zero column of B_4 is connected with a non-zero row of J_4, and then a summand of rank 9 is obtained with scheme $\left[\begin{array}{ccc|c} B_4 & 0 & 0 & \tau_3 \\ C_3 & p & 0 & \tau_4 \\ 0 & G_7 & J_4 & \tau_5 \\ \hline \tau_1 & \tau_2 & \tau_3 & \end{array}\right]$, [9.8].

We obtained all groups in the list above, finishing the proof of the theorem.

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Rigid $\aleph_1$-Free Abelian Groups with Prescribed Factors and Their Role in the Theory of Cellular Covers

Gábor Braun and Lutz Strüngmann

Abstract In Rodríguez and Strüngmann (J. Algebra Appl. 14, 2016) Rodríguez and the second author gave a new method to construct cellular exact sequences of abelian groups with prescribed torsion-free kernels and co-kernels. In particular, the method was applied to the class of $\aleph_1$-free abelian groups in order to complement results from Rodríguez–Strüngmann (Mediterr. J. Math. 6:139–150, 2010) and Göbel–Rodríguez–Strüngmann (Fundam. Math. 217:211–231, 2012). However, $\aleph_1$-free abelian groups G with trivial dual but $\mathrm{Hom}(G, R) \neq \{0\}$ for all rational groups $R \subseteq \mathbb{Q}$ not isomorphic to $\mathbb{Z}$ had to be excluded. Here we give two constructions of such groups, e.g., using Shelah's Black Box prediction principle.

Keywords Cellular cover • $\aleph_1$-Free group • Black Box

Mathematical Subject Classification (2010): Primary: 20K20, 20K30; Secondary: 16S50, 16W20

1 Introduction

The theory of cellular covers, in particular of cellular covers of (abelian) groups, has been under intensive investigation over the last decade. Especially, cellular covers of torsion-free abelian groups are of interesting complexity and variety. This paper contributes another piece to the global picture of cellular covers.

Originating from homotopy theory by the work of Dror Farjoun [4] the theory of cellular covers merged into algebra. In general, a *cellular cover* is a group (module) homomorphism $c: G \to M$ such that composition with c induces an

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isomorphism of sets between $\mathrm{Hom}(G,G)$ and $\mathrm{Hom}(G,M)$. Cellular covers are the algebraic counterpart of cellular approximations of topological spaces in the sense of J.H.C. Whitehead, or the more general cellularization maps extensively studied in homotopy theory in the 1990s by, e.g., Bousfield [1]. As for the dual case, namely for *localizations*, there is sometimes a good interplay between cellularization of spaces and cellularization of groups and modules. This motivated a careful study of the algebraic setting. For applications in homological algebra, see, e.g., Rodríguez–Scherer [14] or [15] and topological applications can be found in, e.g., Farjoun [4], Shoham [18] and the references cited there. A good reference for the basic facts of cellular covers is the book by Göbel and Trlifaj [12]. See Sect. 2 for a brief summary on cellular covers of and [10] for a reference on abelian groups.

Here we are considering abelian groups only, but definitions naturally extend for other categories. Recall that a homomorphism $\pi\colon G \to H$ of abelian groups is a *cellular cover* over H, if every homomorphism $\varphi\colon G \to H$ lifts to a unique endomorphism $\widetilde{\varphi}$ of G such that $\pi\widetilde{\varphi} = \varphi$. (Note that maps are written on the left.) If additionally the mapping $\pi\colon G \to H$ is an epimorphism we say that

$$0 \to K \to G \xrightarrow{\pi} H \to 0 \tag{1}$$

is a *cellular exact sequence* and K is the *cellular kernel* of π. Obviously, for the sequence (1) to be cellular exact we necessarily have $\mathrm{Hom}(G,K) = \{0\}$. A very useful criterion to prove cellular exactness of a sequence (1) was given in [6] stating that (1) is cellular exact if

(i) $\mathrm{End}(H) = \mathbb{Z}$;
(ii) K is a fully invariant subgroup of G;
(iii) $\mathrm{Hom}(K,H) = \{0\} = \mathrm{Hom}(G,K)$.

In this case also the endomorphism ring of G will be $\mathbb{Z}$. In principle using this criterion, Göbel, Rodríguez and the second author gave in [13] and [16] constructions of cellular exact sequences of modules (over a commutative ring of size less than the continuum) with prescribed co-kernel. However, in order to prove the cellular exactness of the sequences it was assumed that the co-kernel H did not have any non-trivial homomorphisms into $\aleph_0$-free modules. The reason for this was the criterion above, i.e., the $\aleph_0$-freeness of the constructed module G in the cellular exact sequence. Recall that an R-module is $\aleph_0$*-free* if all its finite rank submodules are free. For abelian groups, applying Pontryagin's Criterion [9, IV.2.3], this is equivalent to saying that the group is $\aleph_1$-free, i.e., all its countable subgroups are free. Hence the problematic case is when H itself is $\aleph_1$-free. As a detour, an alternative construction for abelian groups was given in [17] and the following was proved in Gödel's constructible universe.

Theorem 1.1 ([17, Theorem 3.8]) *Assume* $(V = L)$. *Let* H *be an* $\aleph_1$*-free abelian group such that* $\mathrm{End}(H) = \mathbb{Z}$. *Assume that there is a countable torsion-free abelian group* K *with* $\mathrm{End}(K) \subseteq \mathbb{Q}$ *and* $\mathrm{Hom}(H,K) = \{0\} = \mathrm{Hom}(K,H)$. *Then there is a cellular exact sequence*

$$0 \to K \to G \to H \to 0$$

such that $\mathrm{End}(G) = \mathbb{Z}$.

The set-theoretic assumption of $(V = L)$ was needed because the construction of G is based on the fact that $\mathrm{Ext}(H, K)$ is non-torsion in this case. The authors remarked the following in [17]: Let H be an $\aleph_1$-free abelian group with $\mathrm{End}(H) = \mathbb{Z}$ and $R \subseteq \mathbb{Q}$ be any subgroup of the rational numbers such that $R \not\cong \mathbb{Z}$. Then automatically $\mathrm{End}(R) \subseteq \mathbb{Q}$ and $\mathrm{Hom}(R, H) = \{0\}$. Thus the only remaining assumption in Theorem 1.1 is $\mathrm{Hom}(H, R) = \{0\}$.

Corollary 1.2 (V=L) *Let H be an $\aleph_1$-free abelian group with* $\mathrm{End}(H) = \mathbb{Z}$ *and let* $\mathbb{Z} \not\cong R \subseteq \mathbb{Q}$ *such that* $\mathrm{Hom}(H, R) = \{0\}$. *Then there is a cellular exact sequence*

$$0 \to R \to G \to H \to 0$$

such that $\mathrm{End}(G) = \mathbb{Z}$.

The following open problem, which asks for the existence of $\aleph_1$-free groups H violating the assumption in Corollary 1.2, was stated in [17]:

Problem 1.3 Given any $\aleph_1$-free group H with $\mathrm{End}(H) = \mathbb{Z}$, is there a subgroup $\mathbb{Z} \not\cong R \subseteq \mathbb{Q}$ such that $\mathrm{Hom}(H, R) = \{0\}$ or at least a countable torsion-free group K such that $\mathrm{End}(K) = \mathbb{Z}$ and $\mathrm{Hom}(H, K) = \{0\} = \mathrm{Hom}(K, H)$?

In this paper we will give two constructions of counterexamples H, showing that the answer to Problem 1.3 is negative. The first construction (Theorem 4.1 in Sect. 4) is an easy push-out construction reusing known pathological abelian groups, at the price of only achieving $\mathrm{Hom}(H, \mathbb{Z}) = \{0\}$ instead of $\mathrm{End}(H) = \mathbb{Z}$. The second construction (Theorem 5.6 and Corollary 5.7 in Sect. 5.2) is a variant of the well-known Black Box construction of abelian groups, and thus more involved.

2 Overview of Cellular Covers

In this section we provide a short overview of classification of cellular covers.

As mentioned above, reversing arrows cellular covering can be seen as dual to localization (see, e.g., Dugas [7] and the literature cited therein). The general goal is to completely classify up to isomorphism all possible cellular exact sequences with fixed kernel or co-kernel, respectively. For groups, the study of cellular covers was initiated by Rodríguez–Scherer in [14] and more systematically by Farjoun–Göbel–Segev in [5] and extended in Chachólski, Dror Farjoun, Göbel, Segev [3] and Dror Farjoun–Göbel–Segev–Shelah [6]. Basic results on cellular covers of groups have been established in Dror Farjoun–Göbel–Segev [5], while Chachólski–Dror Farjoun–Göbel–Segev [3] deals with the case of divisible abelian groups, and provides a complete characterization of all possible surjective covers of such groups (see also [11]): Given a divisible abelian group D, any cellular cover of D is of the form $G = D_0 \oplus \bigoplus_{p\in X} D_p \oplus \mathrm{Hom}(\mathbb{Q}, \bigoplus_{p\in Y} D_p) \oplus H$ where D_0, D_p are the torsion-free and torsion parts of D respectively, X and Y are disjoint sets of primes and H is some subgroup of a direct product of the form $\prod_{p\in Y} D[p^{k_p}]$.

In general, cellular exact covers induce covers of the reduced and divisible parts which implies that the general case can be split into the direct sum of the divisible and reduced cases. Naturally, most efforts have thus been made to investigate the reduced case. Again, the reduced case splits into the torsion and torsion-free case since for reduced mixed groups every cellular covering map already induces an isomorphism between the torsion subgroups involved (see Fuchs–Göbel [11]). However, from Buckner–Dugas [2] and Dugas [8] it follows that the only cellular covers of reduced torsion groups are the trivial ones. Therefore, the reduced torsion-free case is of most interest.

Buckner–Dugas [2] and Dugas [8] investigated the kernels of cellular covering maps under the name of 'co-local' subgroups proving that these kernels are always torsion-free and reduced. Moreover, every cotorsion-free abelian group appears as the kernel of a cellular covering map. Hence, for torsion-free groups, the situation is far more complex. This was also evidenced by Fuchs–Göbel [11] who showed that the collections of non-equivalent cellular covers of certain torsion-free groups of rank one form proper classes complementing a result by Dror Farjoun–Göbel–Segev–Shelah [6] showing that the kernels of cellular covering maps for some fixed torsion-free group may be arbitrarily large. However, this cannot happen whenever the kernel is a free abelian group. In fact, Rodríguez and the second author proved in [16] that no subgroup of $\mathbb{Q}$ admits cellular covers with free kernel. In contrast they showed in [16] that every cotorsion-free abelian group of finite rank is the kernel of some cellular exact sequence with co-kernel of rank two.

Realizing groups as the co-kernel of cellular exact sequences is even more delicate. In [13] Göbel–Rodríguez and the second author gave a realization theorem for certain cotorsion-free abelian groups with $\mathbb{Z}$ as endomorphism ring as the co-kernel of cellular exact sequences. However, the main assumption on those groups H was that they do not allow non-trivial homomorphisms into any $\aleph_1$-free abelian group. The construction was based on Shelah's Black Box principle and therefore the constructed groups G in a sequence like (1) were $\aleph_1$-free. The assumption on H thus was needed to ensure that there are no homomorphisms from H into G in the end. This case was then attacked in [17] where a simple construction principle for cellular covering maps was given that can be applied to $\aleph_1$-free groups, as mentioned above.

3 Preliminaries

Recall that a subgroup A of an abelian group G is *pure* if for all $x \in A$ which is divisible by an integer n in G there is an $y \in A$ with $x = ny$. For a prime p, the subgroup A is *p-pure* if this holds for all p-power n.

Given a torsion-free group G and a subgroup $A \subseteq G$, let A_{p*} denote the p-purification of the subgroup A, i.e., the smallest p-pure subgroup of G containing A:

$$A_{p*} := \{x \in G \mid \exists k \colon p^k x \in A\}.$$

Similarly, let A_* denote the purification of A:

$$A_* := \{x \in G \mid \exists n \colon nx \in A\}.$$

4 Rigid $\aleph_1$-Free Groups with Rational Groups as Prescribed Factor

Here we prove that there is indeed an $\aleph_1$-free group H such that $\mathrm{Hom}(H, R) \neq \{0\}$ for any rational group $R \subseteq \mathbb{Q}$ with $R \not\cong \mathbb{Z}$ but $\mathrm{Hom}(H, \mathbb{Z}) = \{0\}$. This is a weaker version of what is required in Problem 1.3 since we do not achieve that $\mathrm{End}(G) = \mathbb{Z}$ but only get a trivial dual of G. Hence Theorem 1.1 does not cover all $\aleph_1$-free groups. The construction is based on a push out and does not involve any set-theoretic tools but is purely algebraic. In contrast, we will extend Theorem 4.1 below to a more general setting using Shelah's (strong) Black Box prediction principle, which is valid in ZFC.

Theorem 4.1 *There exists an $\aleph_1$-free abelian group H of size $2^{\aleph_0}$ such that* $\mathrm{Hom}(H, \mathbb{Z}) = \{0\}$ *and* $\mathrm{Hom}(H, R) \neq \{0\}$ *for all rational groups $R \subseteq \mathbb{Q}$ with $R \not\cong \mathbb{Z}$.*

Proof Fix a prime p and let $T = \{R \subseteq \mathbb{Q} : \frac{1}{p} \notin R, R \not\cong \mathbb{Z}\}$ and put $C = \bigoplus_{R \in T} R$ be the direct sum of all rational groups $R \subseteq \mathbb{Q}$ which are p-reduced and not isomorphic to $\mathbb{Z}$. We choose a short exact sequence

$$0 \to K \to F \to C \to 0$$

with free abelian groups K and F. Clearly, F must be of rank $2^{\aleph_0}$. By standard constructions, see, e.g., [12, Corollary 12.3.42, Theorem 12.3.4 with Construction 12.3.6], we may extend the kernel K to an $\aleph_1$-free abelian group L such that $\mathrm{End}(L) = \mathbb{Z}$ and the quotient L/K is isomorphic to a direct sum $A = \bigoplus \mathbb{Z}[1/p]$ of copies of the rational ring $\mathbb{Z}[1/p] = \{\frac{1}{p^n} : n \in \mathbb{N}\}$. Thus we have another short exact sequence

$$0 \to K \to L \to A \to 0$$

We now construct the push out H of L and F over K.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & \circlearrowright & \downarrow & & \Vert & & \\
0 & \longrightarrow & L & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Obviously, since A and C are epimorphic images of H, there are non-trivial homomorphisms from H into all rational groups $R \subseteq \mathbb{Q}$ which are not isomorphic to $\mathbb{Z}$. Note that a rational group is either p-reduced or contains $\mathbb{Z}[1/p]$. Thus $\mathrm{Hom}(H,R) \neq \{0\}$ for all $R \subseteq \mathbb{Q}$ with $R \ncong \mathbb{Z}$. Moreover, since $\mathrm{End}(L) = \mathbb{Z}$ we conclude that $\mathrm{Hom}(L,\mathbb{Z}) = \{0\}$. Hence any $\varphi \in \mathrm{Hom}(H,\mathbb{Z})$ restricted to L must be trivial, so φ induces a map $\tilde{\varphi}: H/L = C \to \mathbb{Z}$ which must be trivial as well since $\mathrm{Hom}(R,\mathbb{Z}) = \{0\}$ for all $R \in T$. Therefore $\mathrm{Hom}(H,\mathbb{Z}) = \{0\}$.

It remains to prove that H is $\aleph_1$-free. By the push-out property we may assume that $H = F + L$ and that $F \cap L = K$. Thus let $H_0 \subseteq H$ be a finite rank pure subgroup of H. We have to prove that H_0 is free. Choose maximally independent elements $h_1, \ldots, h_n \in H_0$ such that $H_0 = \langle h_1, \ldots, h_n \rangle_* \subseteq H$. Since $H = F + L$ we can choose finitely generated subgroups $F_0 \subseteq F$ and $L_0 \subseteq L$ such that $\{h_1, \ldots, h_n\} \subseteq F_0 + L_0$ and $(F_0 + L)/L$ is p-pure in C as well as $(L_0 + F)/F$ is q-pure in A for all $q \neq p$. Note that such a choice is possible since C is p-reduced and A is q-reduced for all $q \neq p$. Choose a finitely generated pure subgroup K_0 of K such that $L_0 \cap K \subseteq K_0$ and $F_0 \cap K \subseteq K_0$. We finally put

$$L_1 = \langle L_0, K_0 \rangle_* \subseteq L \qquad \text{and} \qquad F_1 = \langle F_0, K_0 \rangle_* \subseteq F$$

Then L_1 is a finitely generated pure subgroup of L and F_1 is a finitely generated pure subgroup of F. Note that L is $\aleph_1$-free. Moreover, we have

$$L_1 \cap K = K_0 \qquad \text{and} \qquad F_1 \cap K = K_0$$

as well as the fact that $(F_1 + L)/L$ is still p-pure in C and $(L_1 + F)/F$ is q-pure in A for all $q \neq p$. We claim that $F_1 + L_1$ is pure in H, thus $F_1 + L_1$ is a finitely generated pure (hence free) subgroup of H containing H_0, which will finish the proof.

We first show that $F_1 + L_1$ is q-pure for all $q \neq p$. Let $x \in H$ and assume that $mx \in F_1 + L_1$ with $\gcd(m,p) = 1$. Then $mx + F \in (L_1 + F)/F$. Since the latter is

q-pure and q-torsion-free for all $q \neq p$ we conclude that $x + F \in (L_1 + F)/F$. Thus there is $l \in L_1$ such that $x - l \in F$. Hence

$$m(x - l) \in (F_1 + L_1) \cap F = F_1 + (L_1 \cap F) = F_1 + (L_1 \cap K) = F_1 + K_0 = F_1$$

However, F_1 is pure, so $x - l \in F_1$ and so $x \in L_1 + F_1$.

Similarly, it follows that $F_1 + L_1$ is also p-pure in H, and thus $F_1 + L_1$ is a pure subgroup of H, as claimed. □

5 Rigid $\aleph_1$-Free Groups with Large Prescribed Factor

In this section we construct counterexamples to Problem 1.3 using Shelah's Black Box principle. First we recall the Black Box, and then we proceed with the construction.

5.1 Black Box

Here we recall the version of Black Box used in our construction. As a preliminary, we recap support arguments, a key tool for verifying linear independence in an abelian group, e.g., to show that the rank of a block diagonal matrix is the sum of the rank of the blocks. We shall use support arguments for the p-adic closure $\widehat{B}$ of a free group $B = \bigoplus_{i \in I} \mathbb{Z} e_i$ with a fixed basis $\{e_i\}_{i \in I}$. Recall that every element of $\widehat{B}$ can be uniquely written as infinite sums $x = \sum_{i \in I} n_i e_i$ where the $n_i \in J_p$ are p-adic integers, such that for every $k > 0$ all but finitely many of them is divisible by p^k. The *support* of x is the set of coordinates with non-zero coefficient: $\operatorname{supp} \sum_i n_i e_i = \{i : n_i \neq 0\}$. The elements of B have finite support, but there are elements of $\widehat{B}$ with infinite support if I is infinite, e.g., $\operatorname{supp} \sum_{n=0}^{\infty} p^n e_{i_n} = \{i_0, i_1, \ldots, i_n, \ldots\}$ where the i_n are pairwise distinct elements of I.

For a subset $X \subseteq I$ of indices, let B_X denote the subgroup of B of all elements with support in X, i.e., B_X is generated by the e_i with $i \in X$. The set I will be a set of ordinals below. To measure the size of X, recall that the norm of a set X of ordinals is the smallest ordinal strictly upper bounding X, i.e., $\|X\| := \sup_{\alpha \in X} \alpha + 1$. The norm of a function $\phi \colon B_X \to \widehat{B}$ is the norm of its domain: $\|\phi\| := \|X\|$.

Now we are ready to state the Black Box.

Proposition 5.1 *([12, Strong Black Box 9.2.2, Corollary 9.2.7]) Let p be a prime number, and λ be a cardinal with $\lambda^{\aleph_0} = \lambda$. Let E be a stationary set of λ^+ consisting only of ordinals with cofinality ω. Let $B := \bigoplus_{\alpha < \lambda^+} \mathbb{Z} e_\alpha$, and $B_X := \bigoplus_{\alpha \in X} \mathbb{Z} e_\alpha$ for any set X of ordinals. Then there is a family $\{\phi_\alpha \colon B_{X_\alpha} \to \widehat{B_{X_\alpha}}\}_{\alpha < \lambda^+}$ such that*

(i) $\|\phi_\alpha\| \in E$
(ii) $\|\phi_\alpha\| \leq \|\phi_\beta\|$ *and* $\|X_\alpha \cap X_\beta\| < \|X_\beta\|$ *for* $\alpha < \beta < \lambda^+$
(iii) $\|\phi_\alpha(x)\| < \|X_\alpha\|$ *for all* $x \in B_{X_\alpha}$

(iv) PREDICTION: *For any homomorphism* $\psi: B \to \widehat{B}$ *and for any subset* $X \subseteq \lambda^+$ *with* $|X| \le \aleph_0$*, the following set is stationary on* λ^+*:*

$$\{\alpha \in E \mid \exists \beta < \lambda^+ : \|\phi_\beta\| = \alpha, \phi_\beta \subseteq \psi, X \subseteq X_\beta\}.$$

Remark 5.2 Condition (iii) is not present in the original theorem; it has been added for the sole purpose of simplifying applications by omitting traps, which are actually not useful. The condition can be easily arranged by simply omitting traps not satisfying this condition. With the omission of traps, Condition (iv) still holds, as the set of ordinals α with $\|\psi(x)\| < \alpha$ for all $x \in B$ with $\|x\| < \alpha$ form a closed unbounded set.

For applications of the Black Box, some auxiliary results are needed. We shall often use the next folklore lemma:

Lemma 5.3 *Let* $\phi: H \to G$ *be a homomorphism into a torsion-free abelian group* G *from a subgroup* H *of* G*. If* $\phi(b) \in \mathbb{Z}b$ *for all* $b \in H$*, then* ϕ *is multiplication by an integer* n*, i.e.,* $\phi(b) = nb$ *for all* $b \in H$*.*

The main step of the construction, getting rid of a single unwanted endomorphism, is summarized as follows.

Lemma 5.4 ([12, Step Lemma 12.3.5 Simplified]) *Let* $P := \bigoplus_{i \in I} \mathbb{Z}e_i$ *be an infinite rank free group, and* $\phi: P \to \widehat{P}$ *be a homomorphism from* P *to its p-adic completion, and* $b \in P$ *a pure element with* $b\phi \notin \mathbb{Z}b$*. Let* $i_0, i_1, \ldots, i_n, \ldots$ *be countable many pairwise different indices from* I*, such that* $\operatorname{supp}\phi(x)$ *contains only finitely many elements from the sequence for all* $x \in P$*. Then there is a p-adic integer* $\pi \in J_p$ *such that the element* $y := \sum_{n=0}^{\infty} p^n e_{i_n} + \pi b \in \widehat{P}$ *satisfies* $y\phi \notin \langle P, y\rangle_{p*}$*.*

Remark 5.5 Actually, [12, Step Lemma 12.3.5] states only the existence of a suitable b, but from its proof it is clear that any pure element $b \in P$ will do. Note also that the lemma states that y is either $\sum_{n=0}^{\infty} p^n e_{i_n} + \pi b$ or $\sum_{n=0}^{\infty} p^n e_{i_n}$ but the latter case is included in the former one as the special case $\pi = 0$.

5.2 Rigid $\aleph_1$-Free Groups with Large Prescribed Factors

Here we state and prove our main theorem on the existence of rigid $\aleph_1$-free groups with prescribed factors.

Theorem 5.6 *Let* H *be a torsion-free abelian group such that* $\operatorname{Hom}(H, M) = \{0\}$ *for all* $\aleph_1$*-free abelian groups* M*. Let* λ *be a cardinal with* $\lambda = \lambda^{\aleph_0}$ *and* $|H| \le \lambda$*. Then there is an* $\aleph_1$*-free abelian group* G *of cardinality* λ^+ *such that* $\operatorname{End} G = \mathbb{Z}$ *and* H *is a factor of* G*.*

Proof We start by applying Proposition 5.1 for the λ given in the theorem and an arbitrary prime p and stationary set E. Let $F_G := \bigoplus_{\alpha<\lambda} \mathbb{Z}e_\alpha$ be the free subgroup of B generated by the first λ basis elements. As $|H| \leq \lambda$, the group H can be written as a factor $H = F_G/F_K$ of F_G by a free subgroup F_K of F_G of rank λ. Then also $H = B/F$ for $F := F_K \oplus \bigoplus_{\lambda<\alpha\leq\lambda^+} \mathbb{Z}e_\alpha$. Note that as H is torsion-free, F is a pure subgroup of B.

For $\beta \leq \lambda^+$ with $\|X_\beta\| > \lambda$, we inductively choose p-pure elements $y_\beta \in \widehat{B}$ satisfying

(i) For a sequence $\gamma(\beta,0), \gamma(\beta,1), \dots, \gamma(\beta,n), \dots \in X_\beta$ of indices, a p-adic integer $\pi_\beta \in J_p$ and a $b_\beta \in F \cap B_{X_\beta}$ we have

$$y_\beta = \sum_{n=0}^{\infty} p^n e_{\gamma(\beta,n)} + \pi_\beta b_\beta,$$

and the support $\operatorname{supp} b_\beta$ of b_β does not contain any of the $\gamma(\beta,n)$;

(ii) $y_\beta \in \widehat{F}$;

(iii) $\|y_\beta\| = \|X_\beta\|$, and $\operatorname{supp} y_\beta$ has order type ω;

(iv) $y_\beta\phi_\beta \notin \langle B_{X_\beta}, y_\beta \rangle_{p*}$ unless ϕ_β is multiplication by an integer on $F \cap B_{X_\beta}$.

The existence of the y_β easily follows by induction from Lemma 5.4, as we explain now.

We choose b_β with Condition (iv) in mind, so that it is a witness for ϕ_β not being a multiplication by an integer if that is the case. More precisely, if ϕ_β is a multiplication by an integer on $F \cap B_{X_\beta}$, then we can choose b_β arbitrarily, e.g., $b_\beta := 0$. Otherwise we choose $b_\beta \in F \cap B_{X_\beta}$ pure with $b_\beta\phi_\beta \notin \mathbb{Z}b_\beta$, which is possible by Lemma 5.3.

Recall that every $\|X_\beta\| \in E$ has cofinality ω, hence there is a strictly increasing sequence $\lambda < \gamma(\beta,0) < \gamma(\beta,1) < \cdots < \gamma(\beta,n) < \dots$ of ordinals, with $\sup_n \gamma(\beta,n) = \|X_\beta\|$. As the support of b_β is finite, we can assume that none of the $\gamma(\beta,n)$ lie in the support of b_β.

If $b_\beta\phi_\beta \notin \mathbb{Z}b_\beta$, then we choose $\pi_\beta \in J_p$ such that $y_\beta\phi_\beta \notin \langle B_{X_\beta}, y_\beta \rangle_{p*}$. Such a π_β exists by Lemma 5.4. Otherwise we choose π_β arbitrarily, e.g., $\pi_\beta := 0$. Condition (iv) is satisfied in both cases. Conditions (i), (ii) and (iii) hold by construction.

Finally, we define our desired group G, which will satisfy the claims of the theorem, together with a kernel $K \subseteq G$ with the intent $H = G/K$. Both G and K are defined as subgroups of $\widehat{B}$:

$$G := \langle B, y_\beta : \beta \in E \rangle_{p*},$$

$$K := \langle F, y_\beta : \beta \in E \rangle_{p*}.$$

Note that the p-purification in the definition of K and G can be computed 'by hand': the key observation is that B and F are pure subgroups, while the y_β are p-divisible modulo F, so we define p-pure elements $y_\beta^{(k)}$ approximating 'y_β/p^k modulo F'.

To find these new elements, we first approximate the p-adic integers π_β by writing them as infinite sums

$$\pi_\beta = \sum_{n=0}^{\infty} m_{\beta,n} p^n, \quad m_{\beta,n} \in \mathbb{Z}.$$

Now the $y_\beta^{(k)}$ are defined as 'approximations for y_β/p^k' with the fractional part removed:

$$y_\beta^{(k)} := \sum_{n=0}^{\infty} p^n e_{\gamma(\beta,n+k)} + \left(\sum_{n=0}^{\infty} m_{\beta,n+k} p^n\right) b_\beta.$$

Note that $y_\beta^{(0)} = y_\beta$. In particular,

$$y_\beta^{(k)} - p y_\beta^{(k+1)} = e_{\gamma(\beta,k)} + m_{\beta,k} b_\beta =: x_{\beta,k} \in F \cap B_{X_\beta}. \tag{2}$$

Thus

$$G := \left\langle F_G, e_\alpha, y_\beta^{(k)} : \lambda < \alpha, \beta \in E, k \in \mathbb{N} \right\rangle,$$
$$K := \left\langle F_K, e_\alpha, y_\beta^{(k)} : \lambda < \alpha, \beta \in E, k \in \mathbb{N} \right\rangle.$$

Having determined the generators of G, we now determine the relations between the generators of G. Actually, we claim that all relations between the generators are generated by the basic relations (2), i.e., G has the following presentation in terms of generators and relations:

$$G = \left\langle F_G, e_\alpha, y_\beta^{(k)} : \lambda < \alpha, \beta \in E, k \in \mathbb{N} \;\middle|\; y_\beta^{(k)} = x_{\beta,k} + p y_\beta^{(k+1)} \right\rangle.$$

In particular, this shows $H = G/K$.

The proof is a well-known standard support argument, which we reproduce here for completeness. Given a linear relation

$$\sum_j m_j e_j + \sum_i n_i y_{\beta_i}^{(k_i)} = 0 \tag{3}$$

between the generators, it can be reduced to the case where all the β_i are different using the relations from (2). Now we compare the support of the summands. As $y_{\beta_i} - p^{k_i} \in B$, the supports of y_{β_i} and $y_{\beta_i}^{(k_i)}$ differ only on finitely many elements (i.e., the symmetric difference $\operatorname{supp} y_{\beta_i} \triangle \operatorname{supp} y_{\beta_i}^{(k_i)}$ is finite). As $\operatorname{supp} y_{\beta_i}$ is infinite, we have that $\operatorname{supp} y_{\beta_i} \cap \operatorname{supp} y_{\beta_i}^{(k_i)}$ is infinite.

However, for $i \neq j$, we have that $\operatorname{supp} y_{\beta_i} \cap \operatorname{supp} y_{\beta_j}^{(k_j)}$ is finite. It follows that we must have $n_i = 0$ as otherwise $\operatorname{supp} y_{\beta_i}$ would intersect the support of the left-hand side of (3) in an infinite set. Thus the linear relation (3) collapses to $\sum_j m_j e_j = 0$, and as the e_j are linearly independent, we must have $m_j = 0$.

A standard extension of this support argument shows that G is $\aleph_1$-free, i.e., every finite subset $Y \subseteq G$ is contained in a finitely generated pure subgroup of G (applying Pontryagin's Criterion [9, IV.2.3]). As Y is finite, it is contained in a subgroup generated by finitely many generators e_{α_j}, $y_{\beta_i}^{(k_i)}$ with $j \in J$, $i \in I$. The main trick for achieving purity is to adjust the generators to have essentially pairwise disjoint support. To this end, first we choose integers $l_i \geq k_i$ so that $\gamma(\beta_i, l_i)$ is not contained in the support of the e_{α_j} and the $y_{\beta_{i'}}$ for $i' \neq i$. Let $J' := J \cup \{\gamma(i, n) : i \in I, m \leq l_i\}$, and consider the subgroup A generated by the finitely many elements e_{α_j}, $y_{\beta_i}^{(l_i)}$ for $j \in J'$, $i \in I$. The choice of J' ensures that A contains the e_{α_j} for $j \in J$, and the $y_{\beta_i}^{(k_i)}$ for $i \in I$. In particular, A contains Y. Moreover, the only generator of A where $e_{\gamma(\beta_i, l_i)}$ has non-zero coefficient is $y_{\beta_i}^{(l_i)}$, where it has coefficient 1.

To prove that A is a pure subgroup of G, consider an arbitrary element $a = \sum_{j \in J'} n_j e_{\alpha_j} + \sum_{i \in I} m_i y_{\beta_i}^{(l_i)}$ of A divisible by an integer N in G, i.e.,

$$\sum_{j \in J'} n_j e_{\alpha_j} + \sum_{i \in I} m_i y_{\beta_i}^{(l_i)} = N \left(\sum_{j \in J''} n'_j e_{\alpha_j} + \sum_{i \in I} m'_i y_{\beta_i}^{(m_i)} \right)$$

for a finite index set J'', and integers n'_j, m'_i. (Here we use linear independence of the e_α, y_β for ruling out the occurrence of y_β with $\beta \neq \beta_i$ on the right-hand side.) We will show that a is already divisible by N in A, i.e., all the n_j and m_i are divisible by N. Comparing the coordinates $e_{\gamma(\beta_i, l_i)}$ on both sides, the coefficient on the left-hand side is m_i, while the one on the right-hand side is an integer divisible by N, hence m_i is divisible by N. It follows that $\sum_{j \in J'} n_j e_{\alpha_j}$ is already divisible by N in G, i.e., using linear independence of the e_α and y_β again,

$$\sum_{j \in J'} n_j e_{\alpha_j} = N \left(\sum_{j \in \tilde{J}} \tilde{n}_j e_{\alpha_j} \right)$$

for some $\tilde{J}$ and integers $\tilde{n}_j$. It follows that the n_j are divisible by N, too, as claimed. This finishes the proof of pureness of A, and hence that G is $\aleph_1$-free.

Finally, we prove that $\operatorname{End} G = \mathbb{Z}$. Let ψ be any endomorphism of G, and $x \in F$. By Proposition 5.1(iv), there is a $\beta \in E$ with $\beta > \lambda$, $\phi_\beta \subseteq \psi$ and $\operatorname{supp} x \subseteq X_\beta$. In particular, $y_\beta \in \widehat{B_{X_\beta}}$ and $y_\beta \psi = y_\beta \phi_\beta \in G \cap \widehat{B_{X_\beta}}$. Now $y_\beta \psi_\beta = b + \sum_{i=1}^{k} m_k y_{\beta_k}^{(n_k)}$ for some $b \in B$, $\beta_k < \lambda^+$ and integers m_k, n_k. As $\operatorname{supp} y_\beta \psi_\beta \subseteq X_\beta$, by a support argument the expression of $y_\beta \psi_\beta$ reduces to $y_\beta \psi_\beta = b + m y_\beta^{(n)}$ with $b \in B_{X_\beta}$, i.e., $y_\beta \phi_\beta \in \left\langle B_{X_\beta}, y_\beta \right\rangle_{p*}$. It follows that ϕ_β is a multiplication by an integer, i.e., $x\psi = x\phi_\beta = n_x x$ for some $n_x \in \mathbb{Z}$. Therefore ψ is a multiplication by an integer n on the group F by Lemma 5.3.

As $y_\beta \in \widehat{F}$ for all β, the group F is dense in K in the p-adic topology. It follows that ψ is multiplication by n on K. Hence $\psi - n$ factors through the natural projection q to $G/K = H$: i.e., $\psi - n = fq$ for some $f \in \mathrm{Hom}(H, G)$. As G is $\aleph_1$-free, $\mathrm{Hom}(H, G) = \{0\}$ by assumption, leading to $f = 0$ and $\psi = n$. □

As a corollary we obtain a negative answer to Problem 1.3. Recall that by Stein's lemma any countable abelian group C is a direct sum $C = C' \oplus F$ where F is free abelian and C' has trivial dual, i.e., $\mathrm{Hom}(C', \mathbb{Z}) = \{0\}$. Clearly, since C' is countable, any homomorphism from C' into an $\aleph_1$-free group must be zero.

Corollary 5.7 *Let* $H = \bigoplus\{C : C$ *torsion-free,* $\mathrm{Hom}(C, \mathbb{Z}) = \{0\}$ *and* $|C| = \aleph_0\}$. *Then there are arbitrarily large* $\aleph_1$*-free abelian groups* G *with* $\mathrm{End}(G) = \mathbb{Z}$ *and* H *is a factor of* G. *In particular,* $\mathrm{Hom}(G, C) \neq \{0\}$ *for all torsion-free countable abelian groups* C *with* $\mathrm{Hom}(C, \mathbb{Z}) = \{0\}$.

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Definable Valuations Induced by Definable Subgroups

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Abstract In his paper Definable Valuations (1994) Koenigsmann shows that every field that admits a t-henselian topology is either real closed or separably closed or admits a definable valuation inducing the t-henselian topology. To show this Koenigsmann investigates valuation rings induced by certain (definable) subgroups of the field. The aim of this paper, based on the author's PhD thesis (Dupont, PhD thesis, University of Konstanz, 2015), is to look at the methods used in Koenigsmann (Definable Valuations, 1994) in greater detail and Koenigsmann (Definable Henselian Valuations, J. Symb. Log. 80(01):85–99, 2015).

Keywords Valuations • Definable valuations • q-Henselian valued fields • t-Henselian topologies

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1 Introduction

In this paper we will show that any non-real closed, non-separably closed field K, which admits a t-henselian topology, admits a non-trivial definable valuation (see Theorem 6.19). Our main tool will be to construct valuation rings using subgroups of K. More precisely we will treat simultaneously additive subgroups of K and multiplicative subgroups of $K^{\times}$.

This paper arose as follows. Motivated by recent considerations on definable valuations under model theoretic assumptions the author reconsidered in her PhD thesis, [6], an unpublished preprint of Koenigsmann, see [16]. This paper is mainly a revised version of the preprint. In Proposition 6.14, using [12], we will give an

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alternative proof for one case in [16, Theorem 3.1] for which the original proof was incorrect. Corollary 6.16 provides the crucial idea for the model theoretic investigation, which will be pursued in a forthcoming paper, see [7].

The research on definable valuations has been very active lately. Recent works include [1] and [10] on the complexity of the formulas defining valuations. In [12] conditions are given under which a definable valuation is henselian. Further [4, 13] and [11] deal with uniformly definable valuation rings. As well [14] and [15] on dp-minimal fields include sections on definable valuations.

The paper is organized as follows.

We will start with some preliminaries on fractional ideals on valued fields, topologies induced by valuations and absolute values and discrete valuations, that we will refer to later on.

In Sect. 2, for every additive or multiplicative subgroup of a field K we will define the valuation ring $\mathcal{O}_G$ and prove some of its basic properties.

In Sect. 3 we will give criteria under which $\mathcal{O}_G$ is non-trivial.

In Sect. 4 we will examine under which criteria $\mathcal{O}_G$ is definable.

In Sect. 5 we will bring together the results of the previous two sections for the group of qth powers $(K^{\times})^q$ for $q \neq \mathrm{char}(K)$ and for the Artin-Schreier group $K^{(p)}$ for $p = \mathrm{char}(K)$. That way in Theorem 6.17 we will show that (under additional assumptions) if K admits a non-trivial q-henselian valuation for some prime q, then it admits a non-trivial definable valuation. From this we will finally establish Theorem 6.19 on t-henselian fields as announced at the beginning of the Introduction.

Notation: *In this paper K will always denote a field and $\mathcal{O}$ a valuation ring on K with $\mathcal{M}$ its maximal ideal. By $\varrho : K \longrightarrow \mathcal{O}/\mathcal{M} =: \overline{K}$ we denote the residue homomorphism. By v we will denote a valuation on K and by $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ the valuation ring induced by v with maximal ideal $\mathcal{M}_v$. A valuation will be called discrete, if its value group contains a minimal positive element. Without loss of generality, we shall assume that $\mathbb{Z}$ is a convex subgroup of the value group and hence* 1 *is the minimal positive element.*

Some of the following definitions and theorems will be slightly different for additive and multiplicative subgroups. Often we will write the differences for multiplicative subgroups in square brackets "[...]" if there is no danger of misunderstanding. If we say G is a subgroup of K, this can mean either a subgroup of the additive group $(K, +)$ or the multiplicative group $(K^{\times}, \cdot)$, unless explicitly otherwise noted. We will say G is a proper subgroup of K if $G \subsetneq K$ [resp. $G \subsetneq K^{\times}$].

2 Preliminaries

The following can be shown by simple calculation.

Remark 2.1 Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ be a valuation. Let $\{0\} \subsetneq \mathcal{A} \subsetneq K$.

(a) $\mathscr{A}$ is a fractional ideal of $\mathscr{O}_v$ if and only if for every $x \in K$, if there exists $a \in \mathscr{A}$ such that $v(x) \geq v(a)$, then $x \in \mathscr{A}$.
(b) The fractional ideals of $\mathscr{O}_v$ are linearly ordered, i.e. if $\mathscr{A}_1$ and $\mathscr{A}_2$ are fractional ideals of $\mathscr{O}_v$, then $\mathscr{A}_1 \subseteq \mathscr{A}_2$ or $\mathscr{A}_2 \subseteq \mathscr{A}_1$.
(c) Let $\mathscr{A} \subsetneq \mathscr{O}_v$. $\mathscr{A}$ is a prime ideal of $\mathscr{O}_v$ if and only if for every $x \in \mathscr{O}_v$, if there exists $a \in \mathscr{A}$ and an $n \in \mathbb{N}$ with $n \cdot v(x) \geq v(a)$, we have $x \in \mathscr{A}$.

Lemma 2.2 *Let $\mathscr{O}_2 \subsetneq \mathscr{O}_1$ be two valuation rings on K with maximal ideals $\mathscr{M}_1$ and $\mathscr{M}_2$. Let $\mathscr{A}$ be an $\mathscr{O}_2$-ideal with $\sqrt{\mathscr{A}} = \mathscr{M}_2$. Then $\mathscr{M}_1 \subsetneq \mathscr{A}$.*

Proof Suppose $\mathscr{A} \subseteq \mathscr{M}_1$. Then $\mathscr{M}_2 = \sqrt{\mathscr{A}} \subseteq \mathscr{M}_1$. But this contradicts $\mathscr{O}_2 \subsetneq \mathscr{O}_1$. Hence $\mathscr{M}_1 \subsetneq \mathscr{A}$ by Remark 2.1 (b). □

Lemma 2.3 *Let $\mathscr{O}$ be a valuation ring and $\mathscr{A}$ an $\mathscr{O}$-ideal. Then $(1 + \mathscr{A})$ is a multiplicative subgroup of $\mathscr{O}^\times$.*

Proof It is clear that $1 + \mathscr{A} \subseteq \mathscr{O}^\times$. Let $a \in \mathscr{A}$. Then $1 + a \in \mathscr{O}^\times$. Hence $(1 + a) - 1 \in \mathscr{O}^\times$. Therefore $a \cdot (1 + a) - 1 \in \mathscr{A}$ and hence $(1 + a) - 1 = 1 + a \cdot (1 + a) - 1 \in 1 + \mathscr{A}$. Further for $a, b \in \mathscr{A}$ we have $(1 + a) \cdot (1 + b) = 1 + a + b + a \cdot b \in 1 + \mathscr{A}$. □

Lemma 2.4 *Let v_1, v_2 be independent valuations on K. Let $\mathscr{A}_1$ be a non-trivial $\mathscr{O}_{v_1}$-ideal and $\mathscr{A}_2$ a non-trivial $\mathscr{O}_{v_2}$-ideal. Then $K = \mathscr{A}_1 + \mathscr{A}_2$ and $K^\times = (1 + \mathscr{A}_1) \cdot (1 + \mathscr{A}_2)$.*

Proof Let $b_1, b_2 \in K$ and $c_1, c_2 \in K^\times$. From the Approximation Theorem (see [9, Theorem 2.4.1]) follows with Remark 2.1 $(b_1 + c_1 \cdot \mathscr{A}_1) \cap (b_2 + c_2 \cdot \mathscr{A}_2) \neq \emptyset$.

Let $x \in K$. With $b_1 = x$, $c_1 = -1$, $b_2 = 0$ and $c_2 = 1$ follows that there exist $a_1 \in \mathscr{A}_1$ and $a_2 \in \mathscr{A}_2$ such that $x - a_1 = a_2$. Thus $x = a_1 + a_2 \in \mathscr{A}_1 + \mathscr{A}_2$. Therefore $K = \mathscr{A}_1 + \mathscr{A}_2$.

Now let $x \in K^\times$. Then with $b_1 = c_1 = x$ and $b_2 = c_2 = 1$ follows that there exist $a_1 \in \mathscr{A}_1$ and $a_2 \in \mathscr{A}_2$ such that $x + x \cdot a_1 = 1 + a_2$. We have $x = (1 + a_1)^{-1} \cdot (1 + a_2) \in (1 + \mathscr{A}_1)^{-1} \cdot (1 + \mathscr{A}_2) = (1 + \mathscr{A}_1) \cdot (1 + \mathscr{A}_2)$ by Lemma 2.3. Hence $K^\times = (1 + \mathscr{A}_1) \cdot (1 + \mathscr{A}_2)$. □

Lemma 2.5 *Let $\mathscr{O}_1$ and $\mathscr{O}_2$ be two non-comparable valuation rings on a field K. Let $\mathscr{O}$ be the finest common coarsening of $\mathscr{O}_1$ and $\mathscr{O}_2$ and $\mathscr{M}$ the maximal ideal of $\mathscr{O}$. Let $\mathscr{A}_1$ be an $\mathscr{O}_1$-ideal with $\mathscr{M} \subsetneq \mathscr{A}_1$ and $\mathscr{A}_2$ an $\mathscr{O}_2$-ideal with $\mathscr{M} \subsetneq \mathscr{A}_2$. Then $\mathscr{O} = \mathscr{A}_1 + \mathscr{A}_2$ and $\mathscr{O}^\times = (1 + \mathscr{A}_1) \cdot (1 + \mathscr{A}_2)$.*

Proof Apply Lemma 2.4 to the valuation rings $\overline{\mathscr{O}_1}$ and $\overline{\mathscr{O}_2}$ induced by $\mathscr{O}_1$ and $\mathscr{O}_2$ on $\overline{K} = \mathscr{O}/\mathscr{M}$. □

Lemma 2.6 *Let $\mathscr{A}$ be an $\mathscr{O}$-ideal.*

(a) Let $x \in K^\times$ such that $x^{-1} \notin \mathscr{A}$. Then for every $0 \neq a \in \mathscr{A}$ we have $\left(x - a^{-1}\right) - 1 \in \mathscr{A}$.
(b) The multiplicative group generated by the non-zero elements of $\mathscr{A}$ is $K^\times$.

Proof (a) Let $0 \neq a \in \mathscr{A}$. Let $x \in K^\times$ with $x^{-1} \notin \mathscr{A}$. Let v be a valuation with $\mathscr{O} = \mathscr{O}_v$. By Remark 2.1 follows $v\left(x^{-1}\right) < v(a)$ and therefore $v(x) > v\left(a^{-1}\right)$. Hence $v\left(x - a^{-1}\right) = v\left(a^{-1}\right)$ and therefore $v\left(\left(x - a^{-1}\right) - 1\right) = v(a)$. Again by Remark 2.1 follows $\left(x - a^{-1}\right) - 1 \in \mathscr{A}$.

(b) Let $0 \neq x \in \mathscr{O}$. Let $0 \neq a \in \mathscr{A}$. Then $a \cdot x \in \mathscr{A}$. Therefore $x = a^{-1} \cdot a \cdot x$ is contained in the multiplicative group generated by the non-zero elements of $\mathscr{A}$.

For $x \notin \mathscr{O}$ we have $x^{-1} \in \mathscr{O}$. Therefore as shown above x^{-1} and hence as well x is contained in the multiplicative group generated by the non-zero elements of $\mathscr{A}$. □

Lemma 2.7 *Let K be a field and $\mathscr{N} \subseteq \mathscr{P}(K)$ such that*

(V1) $\bigcap \mathscr{N} := \bigcap_{U \in \mathscr{N}} U = \{0\}$ *and* $\{0\} \notin \mathscr{N}$;
(V2) $\forall\, U, V \in \mathscr{N}\ \exists\, W \in \mathscr{N}\ W \subseteq U \cap V$;
(V3) $\forall\, U \in \mathscr{N}\ \exists\, V \in \mathscr{N}\ V - V \subseteq U$;
(V4) $\forall\, U \in \mathscr{N}\ \forall\, x, y \in K\ \exists\, V \in \mathscr{N}\ (x + V) \cdot (y + V) \subseteq x \cdot y + U$;
(V5) $\forall\, U \in \mathscr{N}\ \forall\, x \in K^\times\ \exists\, V \in \mathscr{N}\ (x + V)^{-1} \subseteq x^{-1} + U$;
(V6) $\forall\, U \in \mathscr{N}\ \exists\, V \in \mathscr{N}\ \forall\, x, y \in K\ x \cdot y \in V \longrightarrow x \in U \vee y \in U$.

Then $\mathscr{T}_{\mathscr{N}} := \{U \subseteq K \mid \forall\, x \in U\ \exists\, V \in \mathscr{N}\ x + V \subseteq U\}$ *is a Topology on K.*
$\mathscr{N}$ *is a basis of zero neighbourhoods of* $\mathscr{T}_{\mathscr{N}}$.

Definition 2.8 A topology such that (V 1) to (V 6) hold for the set of neighbourhoods of zero is called V-topology.

Remark 2.9 By Prestel and Ziegler [20, Theorem 1.1] (V 1) to (V 6) hold for the set of neighbourhoods of zero if and only if they hold for any basis of the neighbourhoods of zero.

The following was first shown in [8]. A proof can be found in [9, Appendix B].

Theorem 2.10 *A topology is a V-topology if and only if it is induced by a non-trivial valuation or by a non-trivial absolute value.*

A detailed proof of the following claim can be found in [6, Claim 3.8]. As it is very technical and of not much interest for the rest of the paper, we will only give a brief idea of the proof here.

Proposition 2.11 *Let K be a field and $|\,.\,|$ an archimedean absolute value on K.*

(a) Let G be an additive subgroup of K. If G is open with respect to $|\,.\,|$, then $G = K$.
(b) Let G be a multiplicative subgroup of $K^\times$. If G is open with respect to $|\,.\,|$, then either $G = K^\times$ or $G \cup \{0\}$ is an ordering on K.

Proof (idea) As any field which admits an archimedean absolute value embeds into $\mathbb{R}$ or $\mathbb{C}$, we can assume without loss of generality $K \subseteq \mathbb{R}$ or $K \subseteq \mathbb{C}$.

If G is open, it contains an open neighbourhood U of 0 [resp. 1]. As G is closed under addition [resp. multiplication] for any $g \in G\ g + U$ [resp. $g \cdot U$] is still contained in G. By recursively approximating all the elements of K [resp. $K^\times$ or $K^{>0}$ if K is an ordered field with $g > 0$ for all $g \in G$], we show the claim. □

The following lemma is well known. A proof can be found, for example, in [6, Claim A.43]

Lemma 2.12 *Let $v : K \twoheadrightarrow \Gamma \cup \{\infty\}$ be a discrete valuation on K.*

(a) Let $x \in K$. Then $x \cdot \mathcal{O}_v = \mathcal{M}_v$ if and only if $v(x) = 1$. In particular there exists $x \in K$ with $x \cdot \mathcal{O}_v = \mathcal{M}_v$.
(b) Let $x \in K^\times$ such that $v(x) = 1$. Then for every $y \in K^\times$ with $v(y) \in \mathbb{Z}$ there exists $z \in \mathcal{O}_v^\times$ such that $y = x^{v(y)} \cdot z$.

Proposition 2.13 *Let $\mathcal{O}$ be a non-trivial valuation ring on a field K.*

(a) If $\widetilde{\mathcal{O}}$ is a maximal non-trivial coarsening of $\mathcal{O}$, then $\widetilde{\mathcal{O}}$ has rank-1.
(b) If there exists no maximal non-trivial coarsening of $\mathcal{O}$, then the non-zero prime ideals of $\mathcal{O}$ form a basis of the neighbourhoods of zero of the topology $\mathcal{T}_{\mathcal{O}}$.

Proposition 2.13 is a shortened version of [9, Proposition 2.3.5].

3 The Valuation Ring $\mathcal{O}_G$ Induced by a Subgroup G

In this section for every (additive or multiplicative) subgroup G of a field, we want to define a valuation ring $\mathcal{O}_G$. For this valuation we will first define when a valuation is coarsely compatible with a subgroup. We will define $\mathcal{O}_G$ as the intersection over all valuation rings that are coarsely compatible with G. Before we will come to the definition we will prove some lemmas that we will need to show that with this definition $\mathcal{O}_G$ is a valuation ring. We will conclude the section with defining three cases that will reappear in the subsequent sections.

Definition 3.1 Let G be a subgroup of K.

(a) $\mathcal{O}$ is *compatible* with G if and only if $\mathcal{M} \subseteq G$ [resp. $1 + \mathcal{M} \subseteq G$].
(b) $\mathcal{O}$ is *weakly compatible* with G if and only if there exists an $\mathcal{O}$-ideal $\mathcal{A}$ with $\sqrt{\mathcal{A}} = \mathcal{M}$ such that $\mathcal{A} \subseteq G$ [resp. $1 + \mathcal{A} \subseteq G$].
(c) $\mathcal{O}$ is *coarsely compatible* with G if and only if v is weakly compatible with G and there is no proper coarsening $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ such that $\widetilde{\mathcal{O}}^\times \subseteq G$.

Let v be a valuation on K. We call v *compatible* (respectively *weakly compatible*, *coarsely compatible*) with G if and only if $\mathcal{O}_v$ is compatible (respectively weakly compatible, coarsely compatible) with G.

We omit "with G" whenever the context is clear.

Remark 3.2 If $\mathcal{O}^\times \subseteq G$, then $\mathcal{O}$ is compatible. Further if G is an additive group, then $\mathcal{O} \subseteq G$.

Proof If G is an additive group, $-1 \in \mathcal{O}^\times \subseteq G$ and hence $\mathcal{M} = 1 + \mathcal{M} - 1 \subseteq \mathcal{O}^\times - 1 \subseteq G$. Hence $\mathcal{O}$ is compatible and $\mathcal{O} \subseteq G$.

If G is a multiplicative subgroup, then $1 + \mathcal{M} \subseteq \mathcal{O}^\times \subseteq G$ and hence $\mathcal{O}$ is compatible. □

Lemma 3.3 *Let char* $(\mathcal{O}_v/\mathcal{M}_v) = q$*. Let* G *be a subgroup of* K*. Let* v *be weakly compatible. Then there exists* $n \in \mathbb{N}$ *such that* $q^n \cdot \mathcal{M}_v \subseteq G$ [resp. $1 + q^n \cdot \mathcal{M}_v \subseteq G$].

Proof Let $\mathcal{A}$ be an $\mathcal{O}_v$-ideal with $\mathcal{A} \subseteq G$ [resp. $1 + \mathcal{A} \subseteq G$] and $\sqrt{\mathcal{A}} = \mathcal{M}_v$. As $q \in \mathcal{M}_v$ there exists $n \in \mathbb{N}$ such that $q^n \in \mathcal{A}$. Let $x \in q^n \cdot \mathcal{M}_v$. Then $v(x) > v(q^n)$ and therefore by Remark 2.1 (a) $x \in \mathcal{A}$. Hence $q^n \cdot \mathcal{M}_v \subseteq G$ [resp. $1 + q^n \cdot \mathcal{M}_v \subseteq 1 + \mathcal{A} \subseteq G$]. □

Lemma 3.4 *Let* G *be a subgroup of a field* K*. Then any two coarsely compatible valuation rings are comparable.*

Proof Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two weakly compatible valuation rings on K. For $i = 1, 2$ let $\mathcal{M}_i$ be the maximal ideal of $\mathcal{O}_i$ and $\mathcal{A}_i$ $\mathcal{O}_i$-ideals with $\mathcal{A}_i \subseteq G$ [resp. $1 + \mathcal{A}_i \subseteq G$] and $\sqrt{\mathcal{A}_i} = \mathcal{M}_i$. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are not comparable. Let $\mathcal{O}$ be the finest common coarsening of $\mathcal{O}_1$ and $\mathcal{O}_2$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}$. From Lemma 2.2 follows that $\mathcal{M} \subsetneq \mathcal{A}_1$ and $\mathcal{M} \subsetneq \mathcal{A}_2$. By Lemma 2.5 we have $\mathcal{O}^\times \subseteq \mathcal{A}_1 + \mathcal{A}_2 \subseteq G$ [resp. $\mathcal{O}^\times = (1 + \mathcal{A}_1) \cdot (1 + \mathcal{A}_2) \subseteq G$]. Hence by definition $\mathcal{O}_1$ and $\mathcal{O}_2$ are not coarsely compatible. □

Set $\mathcal{O}_G := \bigcap \{\mathcal{O} \mid \mathcal{O}$ coarsely compatible with $G\}$.

Theorem 3.5 *(a)* $\mathcal{O}_G$ *is a valuation ring on* K*.*
(b) $\mathcal{O}_G$ *is coarsely compatible.*

Proof (a) This follows from Lemma 3.4.

(b) Let $\mathcal{C} := \{\mathcal{O} \mid \mathcal{O}$ coarsely compatible with $G\}$. For every $\mathcal{O} \in \mathcal{C}$ let $\mathcal{M}_\mathcal{O}$ be the maximal ideal of $\mathcal{O}$ and let $\mathcal{A}_\mathcal{O}$ be an $\mathcal{O}$-ideal with $\sqrt{\mathcal{A}_\mathcal{O}} = \mathcal{M}_\mathcal{O}$ and $\mathcal{A}_\mathcal{O} \subseteq G$ [resp. $1 + \mathcal{A}_\mathcal{O} \subseteq G$]. Define $\mathcal{A}_G := \bigcup \{\mathcal{A}_\mathcal{O} \mid \mathcal{O} \in \mathcal{C}\}$. Let $\mathcal{M}_G$ be the maximal ideal of $\mathcal{O}_G$.

Let $a, b \in \mathcal{A}_G$ and $x \in \mathcal{O}_G$. There exist $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{C}$ such that $a \in \mathcal{A}_{\mathcal{O}_1} =: \mathcal{A}_1$ and $b \in \mathcal{A}_{\mathcal{O}_2} =: \mathcal{A}_2$. By Lemma 3.4 let without loss of generality $\mathcal{O}_1 \subseteq \mathcal{O}_2$. Then $\mathcal{A}_2 \subseteq \mathcal{A}_1$ and therefore $a, b \in \mathcal{A}_1$. As $\mathcal{A}_1$ is an ideal $a + b \in \mathcal{A}_1 \subseteq \mathcal{A}_G$. Further $x \in \mathcal{O}_G$ and therefore $x \in \mathcal{O}_1$. Therefore $x \cdot a \in \mathcal{A}_1 \subseteq \mathcal{A}_G$. For every valuation $\mathcal{O} \in \mathcal{C}$ $\mathcal{A}_\mathcal{O} \subseteq \mathcal{M}_\mathcal{O} \subseteq \mathcal{M}_G$. Hence $\mathcal{A}_G \subseteq \mathcal{M}_G$ and thus $\sqrt{\mathcal{A}_G} \subseteq \mathcal{M}_G$.

On the other hand, let $x \in \mathcal{M}_G$. It is easy to see that there exists $\mathcal{O} \in \mathcal{C}$ such that $x \in \mathcal{M}_\mathcal{O} = \sqrt{\mathcal{A}_\mathcal{O}}$. Therefore there exists an $n \in \mathbb{N}$ such that $x^n \in \mathcal{A}_\mathcal{O} \subseteq \mathcal{A}_G$ and hence $x \in \sqrt{\mathcal{A}_G}$. Therefore $\mathcal{M}_G \subseteq \sqrt{\mathcal{A}_G}$. As $\mathcal{A}_\mathcal{O} \subseteq G$ [resp. $1 + \mathcal{A}_\mathcal{O} \subseteq G$] for every $\mathcal{O} \in \mathcal{C}$ we have $\mathcal{A}_G \subseteq G$ [resp. $1 + \mathcal{A}_G \subseteq G$]. Hence $\mathcal{O}_G$ is weakly compatible.

Assume $\mathcal{O}_G$ is not coarsely compatible. Let $\mathcal{O}$ be a valuation ring such that $\mathcal{O}_G \subsetneq \mathcal{O}$ and $\mathcal{O}^\times \subseteq G$. Without loss of generality let $\mathcal{O}$ be coarsely compatible. Let $x \in \mathcal{O} \setminus \mathcal{O}_G$. Then there exists a valuation ring $\widetilde{\mathcal{O}} \in \mathcal{C}$ with $x \notin \widetilde{\mathcal{O}}$. By Lemma 3.4 $\widetilde{\mathcal{O}}$ and $\mathcal{O}$ are comparable. As $x \in \mathcal{O} \setminus \widetilde{\mathcal{O}}$ we have $\widetilde{\mathcal{O}} \subsetneq \mathcal{O}$. But this contradicts $\widetilde{\mathcal{O}}$ coarsely compatible. This shows that $\mathcal{O}_G$ is coarsely compatible. □

Definition 3.6 We call $\mathcal{O}_G$ the valuation ring induced by G.

In the whole paper let $\mathcal{M}_G$ denote the maximal ideal of $\mathcal{O}_G$ and let v_G be a valuation with $\mathcal{O}_{v_G} = \mathcal{O}_G$.

Theorem 3.7 *For any subgroup G of a field K one of the following cases holds:*

group case There is a valuation ring $\mathcal{O}$ with $\mathcal{O}^\times \subseteq G$.
In this case $\mathcal{O}_G$ is the only coarsely compatible valuation ring with this property. All weakly compatible valuations are compatible.

weak case There exists a weakly compatible valuation ring which is not compatible.
In this case $\mathcal{O}_G$ is the only valuation ring with this property.

residue case All weakly compatible valuations are compatible and there is no valuation ring $\mathcal{O}$ with $\mathcal{O}^\times \subseteq G$.
In this case $\mathcal{O}_G$ is the finest compatible valuation ring.

Proof *group case* Let $\mathcal{O}$ be a valuation ring with $\mathcal{O}^\times \subseteq G$. Let $\widetilde{\mathcal{O}} := \bigcup\{\mathcal{O} \mid \mathcal{O}$valuation ring suchthat $\mathcal{O}^\times \subseteq G\}$. Let $x, y \in \widetilde{\mathcal{O}}$. Then there exist $\mathcal{O}_1, \mathcal{O}_2 \in \{\mathcal{O} \mid \mathcal{O}^\times \subseteq G\}$ such that $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are comparable $x + y, x \cdot y \in \widetilde{\mathcal{O}}$ is clear. Otherwise let $\mathcal{O}$ be the finest common coarsening of $\mathcal{O}_1$ and $\mathcal{O}_2$. By Lemma 2.5, $\mathcal{O} = \mathcal{M}_1 + \mathcal{M}_2 \subseteq G$ [resp. $\mathcal{O}^\times = (1 + \mathcal{M}_1) \cdot (1 + \mathcal{M}_2) \subseteq G$]. As $x, y \in \mathcal{O}$ we have $x + y, x \cdot y \in \mathcal{O}$ and therefore $x + y, x \cdot y \in \widetilde{\mathcal{O}}$. Further if $x \in \widetilde{\mathcal{O}}$ then $x \in \mathcal{O}$ for some valuation ring $\mathcal{O}$ such that $\mathcal{O}^\times \subseteq G$. Hence $-x \in \mathcal{O} \subseteq \widetilde{\mathcal{O}}$. Hence $\widetilde{\mathcal{O}}$ is a ring. By assumption it is clear that $\widetilde{\mathcal{O}}$ is a valuation ring.
Now let $x \in \widetilde{\mathcal{O}}^\times$. As above we can find a valuation ring $\mathcal{O}$ such that $x, x^{-1} \in \mathcal{O}$ and $\mathcal{O}^\times \subseteq G$. Hence $\widetilde{\mathcal{O}}^\times \subseteq G$. Further by definition, $\widetilde{\mathcal{O}}$ is coarsely compatible. Hence $\mathcal{O}_G \subseteq \widetilde{\mathcal{O}}$. As $\mathcal{O}_G$ is by Theorem 3.5 (b) coarsely compatible, it follows that $\mathcal{O}_G = \widetilde{\mathcal{O}}$. In particular $\mathcal{O}_G$ is compatible.
By Lemma 3.4 follows that there can be at most one coarsely compatible valuation ring $\mathcal{O}$ with $\mathcal{O}^\times \subseteq G$.
Let $\mathcal{O}$ be weakly compatible.
If $\mathcal{O}^\times \subseteq G$ then $\mathcal{O}$ is compatible.
If $\mathcal{O}^\times \nsubseteq G$ we have $\mathcal{O}_G \subsetneq \mathcal{O}$. Hence $\mathcal{M} \subseteq \mathcal{M}_G$ and therefore $\mathcal{O}$ is compatible.

weak case Let $\mathcal{O}$ be weakly compatible but not compatible.
By the group case $\mathcal{O}^\times \nsubseteq G$. Hence $\mathcal{O}$ is coarsely compatible and therefore $\mathcal{O}_G \subseteq \mathcal{O}$. From Lemma 2.2 follows $\mathcal{O}_G = \mathcal{O}$ as otherwise $\mathcal{O}$ would be compatible.

residue case $\mathcal{O}_G$ is the finest coarsely compatible valuation ring. By assumption in the residue case the coarsely compatible valuation rings are exactly the compatible valuation rings. □

In the group case the $\mathcal{O}_G^\times$, and in the additive case even $\mathcal{O}_G$, is contained in the subgroup. In the residue case G induces a proper subgroup on the residue field $\mathcal{O}_G/\mathcal{M}_G$. In Sect. 6, when proving the definability of $\mathcal{O}_G$ under certain conditions, in the residue case for part of the proof we will be working in the residue field. The name weak case does not need any further motivation.

4 Criteria for the Non-Triviality of $\mathcal{O}_G$

In the whole section let $G \subseteq K$ [resp. $G \subseteq K^\times$] be a subgroup of K.

The valuation ring $\mathcal{O}_G$, that we have defined in the last section, is not necessarily non-trivial. In this section we will give criteria under which $\mathcal{O}_G$ is non-trivial. In particular we will show that we can express the non-triviality of $\mathcal{O}_G$ in a suitable first order language.

Lemma 4.1 *$\mathcal{O}_G$ is non-trivial if and only if $G \neq K$* [resp. $G \neq K^\times$] *and there exists a non-trivial weakly compatible valuation.*

Proof Assume that $G \neq K$ [resp. $G \neq K^\times$] and $\mathcal{O}$ is a non-trivial weakly compatible valuation ring.

If we are in the group case we have $\mathcal{O}_G \subseteq G \subsetneq K$ $\left[\text{resp. } \mathcal{O}_G^\times \subseteq G \subsetneq K^\times\right]$ and therefore $\mathcal{O}_G$ non-trivial.

If we are in the weak case $\mathcal{M}_G \nsubseteq G$ [resp. $1 + \mathcal{M}_G \nsubseteq G$]. Hence $\mathcal{M}_G \neq \{0\}$ and thus $\mathcal{O}_G$ is non-trivial.

In the residue case we have $\mathcal{O}_G \subseteq \mathcal{O} \subsetneq K$ and hence $\mathcal{O}_G$ is non-trivial.

Conversely assume $\mathcal{O}_G \subsetneq K$ is non-trivial. Then $\mathcal{O}_G$ is a non-trivial weakly compatible valuation ring.

Further suppose $G = K$ [resp. $G = K^\times$]. For the trivial valuation $\mathcal{O}_{tr} = K$ we have $\mathcal{O}_{tr}^\times \subseteq G$. Therefore no non-trivial valuation can be coarsely compatible. □

Definition 4.2 We denote the coarsest topology for which G is open and for which Möbius transformations [resp. linear transformations] are continuous, by $\mathcal{T}_G$. We call $\mathcal{T}_G$ the topology induced by G.

Theorem 4.3 *Let*

$$\mathcal{S}_G := \left\{ \left\{ \frac{a \cdot x + b}{c \cdot x + d} \;\middle|\; x \in G, c \cdot x \neq -d \right\} \;\middle|\; a, b, c, d \in K, a \cdot d - b \cdot c \neq 0 \right\}$$
$$\left[\text{resp. } \mathcal{S}_G := \{a \cdot G + b \mid a \in K^\times, b \in K\}\right].$$

Then $\mathcal{S}_G$ is a subbase of this topology.

Proof As $G \in \mathcal{S}_G$ it is open in the topology induced by $\mathcal{S}_G$.

The inverse functions and compositions of a Möbius transformation [resp. linear transformations] are again a Möbius transformation [resp. linear transformations]. Hence Möbius transformations [resp. linear transformation] are continuous in the topology induced by $\mathcal{S}_G$.

On the other hand, every Möbius transformation [resp. linear transformation] is the inverse function of a Möbius transformation [resp. linear transformation] and therefore every element of $\mathcal{S}_G$ is the preimage of G under a Möbius transformation [resp. linear transformation]. Hence there can be no coarser topology for which G is open and for which Möbius transformations [resp. linear transformations] are continuous. □

We will denote the topology induced by a valuation v by $\mathscr{T}_v$ and the topology induced by a valuation ring $\mathscr{O}$ by $\mathscr{T}_{\mathscr{O}}$. We will examine the relation between $\mathscr{T}_G$ and $\mathscr{T}_{\mathscr{O}_G}$.

Claim 4.1 *Let v be weakly compatible. Then G is open with respect to the topology $\mathscr{T}_v$.*

Proof Let $\mathscr{A}$ be an $\mathscr{O}_v$-ideal with $\mathscr{A} \subseteq G$ [resp. $1 + \mathscr{A} \subseteq G$] and $\sqrt{\mathscr{A}} = \mathscr{M}$. Let $a \in \mathscr{A}$. Then by Remark 2.1 $\mathscr{A}' := \{x \in K \mid v(x) > v(a)\}$ is an open subset of $\mathscr{A}$.

If G is an additive subgroup of K, then for every $x \in G$ as well $x + \mathscr{A}'$ is open in $\mathscr{T}_v$. As $x + \mathscr{A}' \subseteq x + \mathscr{A} \subseteq x + G \subseteq G$ for all $x \in G$ and $0 \in \mathscr{A}'$, we have $G = \bigcup_{x \in G} (x + \mathscr{A}')$.

If G is a multiplicative subgroup of $K^\times$, $g \cdot (1 + \mathscr{A}') \subseteq g \cdot (1 + \mathscr{A}) \subseteq G$ for all $g \in G$. As $1 \in 1 + \mathscr{A}'$ this implies $G = \bigcup\limits_{g \in G} g \cdot \left(1 + \mathscr{A}'\right)$.

$\mathscr{A}'$ is open in $\mathscr{T}_v$ and therefore, as $\mathscr{T}_v$ is a field topology, G is open. □

Proposition 4.4 *Assume $\mathscr{O}$ is weakly compatible.*

(a) Let $G \subseteq K$ be an additive subgroup. Then $\mathscr{S}_G$ is a basis of $\mathscr{T}_{\mathscr{O}}$.
(b) Let $G \subseteq K^\times$ be a multiplicative subgroup. Then
$\{(a_1 \cdot G + b_1) \cap (a_2 \cdot G + b_2) \mid a_1, a_2 \in K^\times, b_1, b_2 \in K\}$ is a basis of $\mathscr{T}_{\mathscr{O}}$.

Proof First note that $\{\alpha \cdot \mathscr{M}_G + \beta \mid \alpha \in K^\times, \beta \in K\}$ is a basis of $\mathscr{T}_{v_G}$.

(a) Let $a, b, c, d \in K$ such that $a \cdot d - b \cdot c \neq 0$. As $G \in \mathscr{T}_{\mathscr{O}}$ by Claim 4.1, $G \setminus \left\{-\frac{d}{c}\right\} \in \mathscr{T}_{\mathscr{O}}$. As field operations are continuous in $\mathscr{T}_{\mathscr{O}}$ $\left\{\frac{a \cdot x + b}{c \cdot x + d} \mid x \in G, x \neq -\frac{d}{c}\right\} \in \mathscr{T}_{\mathscr{O}}$. Hence $\mathscr{S}_G \subseteq \mathscr{T}_{\mathscr{O}}$ and therefore $\mathscr{T}_G \subseteq \mathscr{T}_{\mathscr{O}}$.

To prove $\mathscr{T}_{\mathscr{O}} \subseteq \mathscr{T}_G$ let $\mathscr{A}$ be an $\mathscr{O}_G$-ideal with $\mathscr{A} \subseteq G$ and $\sqrt{\mathscr{A}} = \mathscr{M}_G$. We can choose $d \in K \setminus G$ with $d^{-1} \in \mathscr{A}$ as follows. Choose $\widetilde{d} \in K \setminus G$. If $0 \neq \widetilde{d}^{-1} \in \mathscr{A}$, set $d := \widetilde{d}$. If $0 \neq \widetilde{d}^{-1} \notin \mathscr{A}$, choose $0 \neq e \in \mathscr{A}$. By Lemma 2.6 (a) $0 \neq \left(\widetilde{d} - e^{-1}\right)^{-1} \in \mathscr{A}$.

If $e^{-1} \notin G$, set $d := e^{-1}$.

If $e^{-1} \in G$, we have $\widetilde{d} - e^{-1} \notin G$. In this case set $d := \widetilde{d} - e^{-1}$.

Let $0 \neq \widetilde{a}, \widetilde{b} \in \mathscr{A}$. Let $a := d^{-1} \cdot \widetilde{a}$, $b := a \cdot \widetilde{b}$ and $U := \left\{\frac{a \cdot x + b}{x + d} \mid x \in G\right\}$.

We have $a \cdot d - b = a \cdot \left(d - \widetilde{b}\right) \neq 0$. Hence $a \cdot d - b \neq 0$. Further $x \neq -\frac{d}{1}$ for all $x \in G$. Therefore $U \in \mathscr{S}_G$. Note that $v_G(d) < 0$ and $v_G\left(\widetilde{b}\right) > 0$. Let $x \in G$.

Let us first assume $v_G(x) < v_G(d)$. Then $v_G(x + d) = v_G(x)$. Further $v_G(a \cdot x) < v_G(a) < v_G(a) + v_G\left(\widetilde{b}\right) = v_G(b)$. Therefore $v_G(a \cdot x + b) = v_G(a) + v_G(x)$. Hence $v_G\left(\frac{a \cdot x + b}{x + d}\right) = v_G(a) > 0$.

Now assume $v_G(x) \geq v_G(d)$. Then $v_G(a \cdot x) \geq v_G(a) + v_G(d) = v_G(\widetilde{a})$.

As $\widetilde{a} \in \mathscr{A}$ by Remark 2.1 (a) we have $a \cdot x \in \mathscr{A}$ and therefore $a \cdot x + b \in \mathscr{A}$. As $x + d \notin G$ we have $x + d \notin \mathscr{A} \subseteq G$. Again by Remark 2.1 (a) follows $v_G(a \cdot x + b) > v_G(x + d)$ and therefore $v_G\left(\frac{a \cdot x + b}{x + d}\right) > 0$. Hence $\frac{a \cdot x + b}{x + d} \in \mathscr{M}_G$.

That shows $U \subseteq \mathscr{M}_G$. For $\alpha \in K^\times$ and $\beta \in K$ we have $\alpha \cdot U + \beta \subseteq \alpha \cdot \mathscr{M}_G + \beta$ and $\alpha \cdot \mathscr{U} + \beta \in \mathscr{S}_G$.

(b) Let $n \in \mathbb{N}, a_1, \ldots, a_n \in K^\times$ and $b_1, \ldots, b_n \in K$. By Claim 4.1 $G \in \mathscr{T}_\mathscr{O}$. As field operations are continuous in $\mathscr{T}_\mathscr{O}$ and $\mathscr{T}_\mathscr{O}$ is a topology, $\bigcap_{i=1}^n (a_i \cdot G + b_i) \in \mathscr{T}_\mathscr{O}$. Hence $\mathscr{S}_G \subseteq \mathscr{T}_\mathscr{O}$ and therefore $\mathscr{T}_G \subseteq \mathscr{T}_\mathscr{O}$.

To show $\mathscr{T}_\mathscr{O} \subseteq \mathscr{T}_G$ let $\mathscr{A}$ be an $\mathscr{O}_G$-ideal with $1 + \mathscr{A} \subseteq G$ and $\sqrt{\mathscr{A}} = \mathscr{M}_G$.

Suppose $c \in K^\times$ and $\mathscr{A} \subseteq c \cdot G \cup \{0\}$. Then for all $0 \neq a \in \mathscr{A}\,\{0\}$ there exists $x \in G$ with $a = c \cdot x$. As $\mathscr{A}$ is an ideal we have $0 \neq (c \cdot x)^2 \in \mathscr{A}$. Hence $(c \cdot x)^2 \in c \cdot G$ and therefore $c \cdot x^2 \in G$. Hence $c \in G$ as $x^{-2} \in G$. Therefore $c \cdot G \subseteq G$. Hence G contains all non-zero elements of $\mathscr{A}$ and hence the group generated by them. But by Lemma 2.6 (b) this contradicts $G \neq K^\times$.

Therefore there exist $c, d \in K^\times$ with $\mathscr{A} \cap c \cdot G \neq \emptyset$, $\mathscr{A} \cap d \cdot G \neq \emptyset$ and $c \cdot G \cap d \cdot G = \emptyset$.

Let $a \in \mathscr{A} \cap c \cdot G$ and $b \in \mathscr{A} \cap d \cdot G$. Suppose $(a - c \cdot G) \cap (b - d \cdot G) \nsubseteq \mathscr{M}_G$. Let $x \in ((a - c \cdot G) \cap (b - d \cdot G)) \setminus \mathscr{M}_G$. Then there exist $g_1, g_2 \in G$ with $x = a - c \cdot g_1 = b - d \cdot g_2$. As $x^{-1} \in \mathscr{O}_G$ we have $a \cdot x^{-1} \in \mathscr{A}$ and $b \cdot x^{-1} \in \mathscr{A}$. Therefore $-c \cdot g_1 = x - a = x \cdot \left(1 - a \cdot x^{-1}\right) \in x \cdot (1 + \mathscr{A}) \subseteq x \cdot G$ and $-d \cdot g_2 = x - b = x \cdot \left(1 - b \cdot x^{-1}\right) \in x \cdot (1 + \mathscr{A}) \subseteq x \cdot G$. Hence there exist $h_1, h_2 \in G$ with $-c \cdot g_1 = x \cdot h_1$ and $-d \cdot g_2 = x \cdot h_2$. We have $g_1 \cdot h_1^{-1} \in G$ and $g_2 \cdot h_2^{-1} \in G$ and therefore $-x = c \cdot g_1 \cdot h_1^{-1} \in c \cdot G$ and $-x = d \cdot g_2 \cdot h_2^{-1} \in d \cdot G$. Hence $-x \in c \cdot G \cap d \cdot G$ but this contradicts $c \cdot G \cap d \cdot G = \emptyset$.

Therefore $U := (-c \cdot G + a) \cap (-d \cdot G + b) \subseteq \mathscr{M}_G$ and $U \in \mathscr{S}_G$.

For $\alpha \in K^\times$ and $\beta \in K$ we have $\alpha \cdot U + \beta \subseteq \alpha \cdot \mathscr{M}_G + \beta$ and $\alpha \cdot \mathscr{U} + \beta \in \mathscr{S}_G$. □

Lemma 4.5 *Let* $G \subsetneq K$ [resp. $G \subsetneq K^\times$]. *Then* $\mathscr{T}_\mathscr{O} = \mathscr{T}_G$ *if and only if there exists a non-trivial weakly compatible coarsening* $\mathscr{O}'$ *of* $\mathscr{O}$. *In this case* $\mathscr{B}_G := \mathscr{S}_G$ $\left[\text{resp. } B_G := \left\{ (a \cdot G + b) \cap (c \cdot G + d) \mid a, c \in K^\times, b, d \in K \right\}\right]$ *is a basis of* $\mathscr{T}_G$.

Proof Let us first assume $\mathscr{T}_\mathscr{O} = \mathscr{T}_G$. As G is open in $\mathscr{T}_G = \mathscr{T}_\mathscr{O}$ and the $\mathscr{O}$-ideals form a basis of neighbourhoods of zero of $\mathscr{T}_\mathscr{O}$, there exists an $\mathscr{O}$-ideal $\mathscr{A} \neq \{0\}$ such that $\mathscr{A} \subseteq G$ [resp. $1 + \mathscr{A} \subseteq G$]. $\mathscr{O}' := \mathscr{O}_{\sqrt{\mathscr{A}}} \supseteq \mathscr{O}$ is a valuation ring with maximal ideal $\mathscr{M}' = \sqrt{\mathscr{A}}$ and $\mathscr{O} \subseteq \mathscr{O}'$. Hence $\mathscr{O}'$ is weakly compatible.

Now assume $\mathscr{O}' \supseteq \mathscr{O}$ is weakly compatible. By Proposition 4.4 $\mathscr{B}_G$ is basis of $\mathscr{T}_{\mathscr{O}'}$ and hence $\mathscr{T}_{\mathscr{O}'} = \mathscr{T}_G$. As $\mathscr{O}'$ and $\mathscr{O}$ are dependent $\mathscr{T}_\mathscr{O} = \mathscr{T}_{\mathscr{O}'} = \mathscr{T}_G$ (see [9, Theorem 2.3.4]). □

Theorem 4.6 *Let K be a field with a proper additive subgroup G or with a proper multiplicative subgroup G such that* $G \cup \{0\}$ *is not an ordering. Then there is a non-trivial weakly compatible valuation ring if and only if* $\mathscr{T}_G$ *is a V-topology.*

Proof Let $\mathscr{O}$ be a weakly compatible valuation ring. Then by Lemma 4.5 $\mathscr{T}_\mathscr{O} = \mathscr{T}_G$ and therefore by Theorem 2.10 $\mathscr{T}_G$ is a V-topology.

On the other hand, let $\mathscr{T}_G$ be a V-topology. As G is open with respect to $\mathscr{T}_G$ by Proposition 2.11 $\mathscr{T}_G$ cannot be induced by an archimedean absolute value. Hence by Theorem 2.10 $\mathscr{T}_G$ is induced by a valuation ring $\mathscr{O}$. By Lemma 4.5 there exists a non-trivial weakly compatible coarsening of $\mathscr{O}$. □

Corollary 4.7 *Let $G \subsetneq K$ be a proper additive subgroup of K. [Resp. let $G \subsetneq K^\times$ be a proper multiplicative subgroup of K such that $G \cup \{0\}$ is not an ordering on K.] The following are equivalent*

(i) $\mathscr{O}_G$ *is non-trivial.*
(ii) There exists a non-trivial weakly compatible valuation ring $\mathscr{O}$ *on* K.
(iii) $\mathscr{T}_G$ *is a V-topology.*
(iv) $\mathscr{B}_G$ *is a basis of a V-topology.*

This follows at once by Lemma 4.1, Theorem 4.6 and Lemma 4.5.

Lemma 4.8 *Let $G \subsetneq K$ be a proper additive subgroup of K. [Resp. let $G \subsetneq K^\times$ be a proper multiplicative subgroup of K such that $G \cup \{0\}$ is not an ordering on K.] Let $\mathscr{L}_G := \{+, -, \cdot ; 0, 1; \underline{G}\}$, where G is a unary relation symbol. Then any of the equivalent assertions is an elementary property in $\mathscr{L}_G$.*

Proof We can express in $\mathscr{L}_G$, that the axioms (V 1) to (V 6) hold for $\mathscr{B}_G$ and hence by Remark 2.9 that $\mathscr{B}_G$ is a basis of a V-topology. □

5 Criteria for the Definability of $\mathscr{O}_G$

Let $\mathscr{L}$ always denote a language and $\mathscr{L}(K)$ the extension of the language $\mathscr{L}$ by adding a constant for every element of K.

Definition 5.1 (a) We call $\mathscr{O}$ $\mathscr{L}$*-definable* (with parameters) or *definable in* $\mathscr{L}$, if there exists an $\mathscr{L}(K)$-formula $\varphi(x)$ such that $\mathscr{O} = \{x \in K \mid \varphi(x)\}$. We say φ defines $\mathscr{O}$.
(b) We call v $\mathscr{L}$*-definable* if $\mathscr{O}_v$ is $\mathscr{L}$-definable.
(c) We call $\mathscr{O}$ (resp. v) $\mathscr{L}$*-$\emptyset$-definable* or *parameterfree* $\mathscr{L}$*-definable*, if the formula φ above, is an $\mathscr{L}$-formula.
(d) We call $\mathscr{O}$ (respectively v) *definable* if it is $\mathscr{L}_{\text{ring}}$-definable.

In some of the theorems in Sect. 5 we need assumptions that might only be fulfilled in a finite field extension of K but not in K itself. With the following theorem we will still obtain a definable valuation on K.

Proposition 5.2 *Let L/K be a finite field extension. If $\mathscr{O}$ is a non-trivial definable valuation ring on L, then $\mathscr{O} \cap K$ is a non-trivial definable valuation ring on K.*

Proof As L/K is algebraic, if $\mathscr{O}$ is non-trivial, then $\mathscr{O} \cap K$ is also non-trivial.
As L/K is finite, L is interpretable in K and hence $\mathscr{O} \cap K$ is definable. □

Note that if $\mathscr{O}$ in the proposition above is parameter-free definable, it does not follow that $\mathscr{O} \cap K$ is parameter-free definable in K.

Example 5.3 For every prime number $q \in \mathbb{N}$ the q-adic valuation is definable in the q-adic numbers $\mathbb{Q}_q$. The valuation ring is $\mathscr{O}_q = \left\{x \in \mathbb{Q}_q \mid \exists y\, y^2 - y = q \cdot x^2\right\}$.

This follows from [2].

We now want to explore under which conditions $\mathcal{O}_G$ is definable in $\mathcal{L}_G := \{0, 1; +, -, \cdot; \underline{G}\}$. We will first look at the group case, then at the weak case and at last at the residue case.

The proofs all follow the same pattern. Let $\mathcal{L}' := \mathrm{L}_G(\underline{\mathcal{O}})$, the language $\mathcal{L}_G$ extended by a unary relation symbol. We will show that under certain assumptions for $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$ we have $\mathcal{O}' = \mathcal{O}_{G'}$. Hence for every $(K', G') \equiv (K, G)$ there exists at most one $\mathcal{O}'$ such that $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$ and therefore $\underline{\mathcal{O}}$ is implicitly defined in $\mathrm{Th}(K, G, \mathcal{O}_G)$. By Beth's Theorem (see, for example, [18, Theorem 9.3]) $\underline{\mathcal{O}}$ is explicitly defined in $\mathrm{Th}(K, G, \mathcal{O}_G)$. Hence there exists an $\mathcal{L}_G$-formula φ such that $Th(K, G, \mathcal{O}_G) \vdash \forall x\, \varphi(x) \leftrightarrow \underline{\mathcal{O}}(x)$ and hence $\mathcal{O}$ is $\mathcal{L}_G$-definable.

We will further prove that the assumptions for the existence of an $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$ that we give, are not only sufficient but also necessary. For this we will use the following easy observation.

Remark 5.4 Let $\mathcal{L}_G := \{0, 1; +, -, \cdot; \underline{G}\}$ and $\mathcal{L}' = \mathrm{L}_G(\underline{\mathcal{O}})$. If there exists $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$ such that $\mathcal{O}' \neq \mathcal{O}_{G'}$, then there exists no $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$.

For the proof of Theorem 5.6 we will need the following lemma.

Lemma 5.5 *(a) Let $G \subsetneq K$ be an additive subgroup of K such that the group case holds. Let $\mathcal{O}_G$ be discrete. Let $x_0 \in K$ such that $\mathcal{M}_G = x_0 \cdot \mathcal{O}_G$. Then there exists $n \in \mathbb{N}$ such that $x_0^{-n} \cdot \mathcal{O}_G \subseteq G$ and $x_0^{-(n+1)} \cdot \mathcal{O}_G \nsubseteq G$.*

(b) Let $G \subsetneq K^\times$ be a multiplicative subgroup of K such that the group case holds. Let $x \in \mathcal{M}_G$. Then $\mathcal{M}_G \setminus x \cdot \mathcal{M}_G \nsubseteq G$.

Proof (a) As we are in the group case by Theorem 3.7 $\mathcal{O}_G \subseteq G$ and therefore for all $y \in K \setminus G$ we have $v_G\left(y^{-1}\right) > 0$. Assume for all $y \in K \setminus G$ we have $v_G\left(y^{-1}\right) > n$ for all $n \in \mathbb{N}$. Let $\mathfrak{p} := \{z \in K \mid v_G(z) > n \text{for all } n \in \mathbb{N}\} \neq \emptyset$. By Remark 2.1 (c) $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_G$ and hence $\mathcal{O}_\mathfrak{p} := (\mathcal{O}_G)_\mathfrak{p}$ is a valuation ring on K with $\mathcal{O}_G \subsetneq \mathcal{O}_\mathfrak{p}$. Let $z \in \mathcal{O}_\mathfrak{p}$. Then there exist $a, b \in \mathcal{O}_G$ with $b \notin \mathfrak{p}$ and $z = a \cdot b^{-1}$. As $b \notin \mathfrak{p}$ there exists $n \in \mathbb{N}$ with $v_G(b) \leq n$. We have $v_G\left(z^{-1}\right) = v_G(b) - v_G(a) \leq n - v_G(a) \leq n$. Hence by assumption $z \in G$. Hence $\mathcal{O}_G \subsetneq \mathcal{O}_\mathfrak{p} \subseteq G$. This contradicts Theorem 3.5 (b). Choose $y \in K \setminus G$ such that $v_G(y^{-1}) > 0$ is minimal. Then $v_G(y^{-1}) \in \mathbb{N}$. By Lemma 2.12 $v_G(x_0) = 1$ and there exists $a \in \mathcal{O}_G^\times$ such that $y^{-1} = x_0^{n+1} \cdot a$. Hence $G \not\ni y = x_0^{-(n+1)} \cdot a^{-1} \in x_0^{-(n+1)} \cdot \mathcal{O}_G$. Hence $G \nsupseteq x_0^{-(n+1)} \cdot \mathcal{O}_G$.

Assume $z \in \left(x_0^{-n} \cdot \mathcal{O}_G\right) \setminus G$. Then $z = x_0^{-n} \cdot b$ for some $b \in \mathcal{O}_G$. As $v_G(z) = v_G\left(x_0^{-n}\right) + v_G(b) \geq -n$ we have $v_G\left(z^{-1}\right) \leq n < n + 1 = v_G(y)$. But this contradicts the minimality of $v_G\left(y^{-1}\right)$.

Hence we have found $n \in \mathbb{N}$ with $x_0^{-n} \cdot \mathcal{O}_G \subseteq G$ and $x_0^{-(n+1)} \cdot \mathcal{O}_G \nsubseteq G$.

(b) Assume there exists $x_0 \in \mathcal{M}_G$ such that $\mathcal{M}_G \setminus x_0 \cdot \mathcal{M}_G \subseteq G$. Let $\mathfrak{p} := \{y \in K \mid v_G(y) > n \cdot v_G(x_0) \text{ for all } n \in \mathbb{N}\}$. By Remark 2.1 (c) $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_G$ and therefore $(\mathcal{O}_G)_\mathfrak{p} =: \mathcal{O}_\mathfrak{p}$ is a coarsening of $\mathcal{O}_G$. As $x_0^{-1} \in \mathcal{O}_\mathfrak{p} \setminus \mathcal{O}_G$ we have $\mathcal{O}_\mathfrak{p} \supsetneq \mathcal{O}_G$. Let $\langle \mathcal{M}_G \setminus x_0 \cdot \mathcal{M}_G \rangle$ denote the smallest multiplicative subgroup of $K^\times$ which contains $\mathcal{M}_G \setminus x_0 \cdot \mathcal{M}_G$. As $\mathcal{M}_G \setminus x_0 \cdot \mathcal{M}_G \subseteq G$ we have $\langle \mathcal{M}_G \setminus x_0 \cdot \mathcal{M}_G \rangle \subseteq G$. Let $y \in K$ such that $0 < v_G(y) \leq m \cdot v_G(x_0)$ and

$(m-1) \cdot v_G(x_0) < v_G(y)$ for some $m \in \mathbb{N}$. Then $y \cdot x_0^{-(m-1)} \in \mathscr{M}_G$. Further $v_G\left(y \cdot x_0^{-(m-1)}\right) \leq m \cdot v_G(x_0) - (m-1) \cdot v_G(x_0) = v_G(x_0)$. Thus $y \cdot x_0^{-(m-1)} \in \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G$. As $x_0 \in \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G$ and therefore $x_0^{m-1} \in \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$ it follows that $y = x_0^{m-1} \cdot y \cdot x_0^{-(m-1)} \in \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$.

Now let $y \in \mathscr{O}_{\mathfrak{p}}^{\times} \setminus \mathscr{O}_G^{\times}$. Then $y \notin \mathfrak{p}$ and $y^{-1} \notin \mathfrak{p}$. Hence there exist $n_1, n_2 \in \mathbb{N}$ such that $v_G(y) \leq n_1 \cdot v_G(x_0)$ and $v_G\left(y^{-1}\right) \leq n_2 \cdot v_G(x_0)$. As $y \notin \mathscr{O}_G^{\times}$ by assumption, we have $v_G(y) \neq 0$. If $v_G(y) > 0$, then $y \in \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$ as shown above. If $v_G(y) < 0$, then $y^{-1} \in \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$ and hence $y \in \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$. Therefore $\mathscr{O}_{\mathfrak{p}}^{\times} \setminus \mathscr{O}_G^{\times} \subseteq \langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle$. As $\mathscr{O}_G^{\times} \subseteq G$ and $\langle \mathscr{M}_G \setminus x_0 \cdot \mathscr{M}_G \rangle \subseteq G$ we have $\mathscr{O}_{\mathfrak{p}}^{\times} \subseteq G$. But this contradicts 3.5(b). □

Theorem 5.6 *(a) Let G be an additive subgroup of K such that the group case holds. Then there exists an $\mathscr{L}_G$-formula φ such that φ defines $\mathscr{O}_{G'}$ for all $(K', G') \equiv (K, G)$ if and only if $\mathscr{O}_G$ is discrete or $x^{-1} \cdot \mathscr{O}_G \nsubseteq G$ for all $x \in \mathscr{M}_G$.*

(b) Let $G \subsetneq K^{\times}$ be a multiplicative subgroup of K such that the group case holds. Then there exists an $\mathscr{L}_G$-formula φ such that φ defines $\mathscr{O}_{G'}$ for all $(K', G') \equiv (K, G)$.

Proof (a) Let $(K', G', \mathscr{O}') \equiv (K, G, \mathscr{O}_G)$ be an $\mathscr{L}'$-structure. Let $\mathscr{M}'$ denote the maximal ideal of $\mathscr{O}'$. As $\mathscr{O}_G \subseteq G$, we have $\mathscr{O}' \subseteq G'$. Hence we are in the group case and therefore by Theorem 3.7 we have $\mathscr{O}_{G'} \subseteq G'$ and $\mathscr{O}' \subseteq \mathscr{O}_{G'}$.

Let us first assume that $\mathscr{O}_G$ is discrete. By Lemma 2.12 there exists $x_0 \in K$ such that $\mathscr{M}_G = x_0 \cdot \mathscr{O}_G$. By Lemma 5.5 (a) there exists $n \in \mathbb{N}$ such that $x_0^{-n} \cdot \mathscr{O}_G \subseteq G$ and $x_0^{-(n+1)} \cdot \mathscr{O}_G \nsubseteq G$. As $(K', G', \mathscr{O}') \equiv (K, G, \mathscr{O}_G)$ there exists $x' \in K'$ such that $x' \cdot \mathscr{O}' = \mathscr{M}'$, $(x')^{-n} \cdot \mathscr{O}' \subseteq G'$ and $(x')^{-(n+1)} \nsubseteq G'$. Assume $x' \notin \mathscr{M}_{G'}$. Then $(x')^{-1} \in \mathscr{O}_{G'}$ and thus $(x')^{-(n+1)} \cdot \mathscr{O}' \subseteq (x')^{-(n+1)} \cdot \mathscr{O}_{G'} \subseteq \mathscr{O}_{G'} \subseteq G'$. But this contradicts the choice of x'. Hence $x' \in \mathscr{M}_{G'}$ and therefore $x' \cdot \mathscr{O}_{G'} \subseteq \mathscr{M}_{G'}$. Thus $\mathscr{M}' = x' \cdot \mathscr{O}' \subseteq x' \cdot \mathscr{O}_{G'} \subseteq \mathscr{M}_{G'}$ and therefore $\mathscr{O}_{G'} \subseteq \mathscr{O}'$. Altogether follows $\mathscr{O}_{G'} = \mathscr{O}'$.

Now assume $x^{-1} \cdot \mathscr{O}_G \nsubseteq G$ for all $x \in \mathscr{M}_G$. Assume $\mathscr{O}' \subsetneq \mathscr{O}_{G'}$. Let $x \in \mathscr{M}' \setminus \mathscr{M}_{G'}$. Then $x^{-1} \in \mathscr{O}_{G'}^{\times}$ and therefore $x^{-1} \cdot \mathscr{O}' \subseteq x^{-1} \cdot \mathscr{O}_{G'} = \mathscr{O}_{G'} \subseteq G'$. But as $(K', G', \mathscr{O}') \equiv (K, G, \mathscr{O}_G)$ this is a contradiction. Therefore $\mathscr{O}' = \mathscr{O}_{G'}$.

Hence in both cases by Beth's Theorem there exists an $\mathscr{L}_G$-formula φ such that φ defines $\mathscr{O}_{G'}$ for all $(K', G') \equiv (K, G)$.

Finally assume $x \in \mathscr{M}_G$ such that $x^{-1} \cdot \mathscr{O}_G \subseteq G$ and $\mathscr{O}_G$ is not discrete. Then for every $n \in \mathbb{N}$ there exists $y_n \in \mathscr{M}_G \setminus \{0\}$ such that $v_G(x) \geq n \cdot v_G(y_n) \geq k \cdot v_G(y_n)$ for all $k \leq n$. For all $a \in \mathscr{O}_G$ we have $x \cdot a \cdot y_n^{-k} \in \mathscr{O}_G$ and therefore $y_n^{-k} \cdot a \in x^{-1} \cdot \mathscr{O}_G$. Thus $y_n^{-k} \cdot \mathscr{O}_G \subseteq x^{-1} \cdot \mathscr{O}_G \subseteq G$ for all $k \leq n$. Hence $\Phi(y) = \{y \in \mathscr{M}_G \wedge 0 \neq y \wedge y^{-n} \cdot \mathscr{O}_G \subseteq G \mid n \in \mathbb{N}\}$ is a finitely satisfiable type. Thus there exists an elementary extension $(K', G', \mathscr{O}')$ of $(K, G, \mathscr{O}_G)$ and $y' \in K'$ such that y' realizes $\Phi(y)$. Let $\mathscr{O}'' = \bigcup_{n=0}^{\infty} (y')^{-n} \cdot \mathscr{O}'$. As $(y')^{-n} \cdot \mathscr{O}' \subseteq G'$ for every $n \in \mathbb{N}$, we have $\mathscr{O}'' \subseteq G'$. Further $\mathscr{O}' \subseteq \mathscr{O}''$. As $y' \in \mathscr{M}'$ we have $(y')^{-1} \notin \mathscr{O}'$ but $(y')^{-1} \in (y')^{-1} \cdot \mathscr{O}' \subseteq \mathscr{O}''$ and therefore $\mathscr{O}' \subsetneq \mathscr{O}'' \subseteq G'$. Thus $\mathscr{O}' \neq \mathscr{O}_{G'}$. Hence by Remark 5.4 there exists no $\mathscr{L}_G$-formula φ such that φ defines $\mathscr{O}_{G'}$ for all $(K', G') \equiv (K, G)$.

(b) Let $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$. As $\mathcal{O}_G^\times \subseteq G$ we have $(\mathcal{O}')^\times \subseteq G'$. By Theorem 3.7 we have $\mathcal{O}' \subseteq \mathcal{O}_{G'}$ and $\mathcal{O}_{G'}^\times \subseteq G'$. Assume $\mathcal{O}' \subsetneq \mathcal{O}_{G'}$. Let $x \in \mathcal{M}' \setminus \mathcal{M}_{G'}$. As $x \in \mathcal{O}_{G'}^\times$ we have $x \cdot \mathcal{M}_{G'} = \mathcal{M}_{G'}$. Therefore $\mathcal{M}' \setminus x \cdot \mathcal{M}' \subseteq \mathcal{M}' \setminus x \cdot \mathcal{M}_{G'} = \mathcal{M}' \setminus \mathcal{M}_{G'} \subseteq \mathcal{O}_{G'}^\times \subseteq G'$. Hence there exists $x \in \mathcal{M}'$ such that $\mathcal{M}' \setminus x \cdot \mathcal{M}' \subseteq G'$. But as by Lemma 5.5 (b) $\mathcal{M}_G \setminus x \cdot \mathcal{M}_G \nsubseteq G$, this contradicts $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$.

Therefore $\mathcal{O}' = \mathcal{O}_{G'}$ and hence by Beth's Theorem there exists an $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$. □

Theorem 5.7 *Let* $G \subsetneq K$ [resp. $G \subsetneq K^\times$] *be a subgroup of* K *such that the weak case holds. Then there exists an* $\mathcal{L}_G$*-formula* φ *such that* φ *defines* $\mathcal{O}_{G'}$ *for all* $(K', G') \equiv (K, G)$ *if and only if* $\mathcal{O}_G$ *is discrete.*

Proof Let us first assume that $\mathcal{O}_G$ is discrete. Let $\mathcal{A}$ be an $\mathcal{O}$-ideal with $\mathcal{A} \subseteq G$ [resp. $1 + \mathcal{A} \subseteq G$] and $\mathcal{M}_G = \sqrt{\mathcal{A}}$. Let $x_0 \in \mathcal{M}_G$ with $v_G(x_0) = 1$. Let $a \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $x_0^k = a$. Then $v_G(a) = k \in \mathbb{N}$. Choose $y_0 \in \mathcal{M}_G \setminus G$ [resp. $y_0 \in \mathcal{M}_G \setminus G - 1$] such that $v_G(y_0)$ is maximal. Such a y_0 exists as $v_G(\mathcal{M}_G \setminus G)$ [resp. $v_G(\mathcal{M}_G \setminus G - 1)$] is bounded by $v_G(a)$ by Remark 2.1 and v_G is discrete. As $y_0 \notin G \supseteq \mathcal{A}$ [resp. $y_0 \notin G - 1 \supseteq \mathcal{A}$] by Remark 2.1 we have $0 < v_G(y_0) < v_G(a) = k$. Hence $v_G(y_0) \in \mathbb{N}$. From Lemma 2.12 follows that there exists $n \in \mathbb{N}$ and $b \in \mathcal{O}_G^\times$ such that $y_0 = x_0^n \cdot b$. Hence $y_0 \in x_0^n \cdot \mathcal{O}_G \setminus G$ $\left[\text{resp. } y_0 \in x_0^n \cdot \mathcal{O}_G \setminus (G-1)\right]$ and therefore $G \nsupseteq x_0^n \cdot \mathcal{O}_G$ $\left[\text{resp. } G - 1 \nsupseteq x_0^n \cdot \mathcal{O}_G\right]$. Assume there exists $z \in x_0^{n+1} \cdot \mathcal{O}_G \setminus G$ $\left[\text{resp. } z \in x_0^{n+1} \cdot \mathcal{O}_G \setminus G - 1\right]$. Let $z_0 \in \mathcal{O}_G$ such that $z = x_0^{n+1} \cdot z_0$. We have $v_G(z) \geq n + 1 > v_G(y_0)$. But this contradicts the maximality of $v_G(y_0)$. Hence $x_0^{n+1} \cdot \mathcal{O}_G \subseteq G$ $\left[\text{resp. } 1 + x_0^{n+1} \cdot \mathcal{O}_G \subseteq G\right]$.

Now let $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$. Let $\mathcal{M}'$ be the maximal ideal of $\mathcal{O}'$. As $\mathcal{O}_G$ is not compatible with G, $\mathcal{O}'$ is not compatible with G'. Further there exists $x' \in K'$ such that $x' \cdot \mathcal{O}' = \mathcal{M}'$, $(x')^n \cdot \mathcal{O}' \nsubseteq G'$ and $(x')^{n+1} \cdot \mathcal{O}' \subseteq G'$ $\left[\text{resp. } x' \cdot \mathcal{O}' = \mathcal{M}', 1 + (x')^n \cdot \mathcal{O}' \nsubseteq G' \text{ and } 1 + (x')^{n+1} \cdot \mathcal{O}' \subseteq G'\right]$. Let v' be a valuation with $\mathcal{O}_{v'} = \mathcal{O}'$. Let $\mathcal{A} := \left\{a \in K \;\middle|\; v'(a) > v'\left((x')^{n+1}\right)\right\}$. $\mathcal{A}$ is an $\mathcal{O}'$-ideal with $\mathcal{A} \subseteq (x')^{n+1} \cdot \mathcal{O}' \subseteq G'$ $\left[\text{resp. } 1 + \mathcal{A} \subseteq 1 + (x')^{n+1} \cdot \mathcal{O}' \subseteq G'\right]$. Further for every $z \in \mathcal{M}'$ there exists $a \in \mathcal{O}'$ such that $z = x' \cdot a$. We have $v'\left(z^{n+2}\right) = v'(x')^{n+1} + v'(x') + v'\left(a^{n+2}\right) > v'\left((x')^{n+1}\right)$ and hence $z \in \sqrt{\mathcal{A}}$. Therefore $\sqrt{\mathcal{A}} = \mathcal{M}$ and thus $\mathcal{O}'$ is weakly compatible with G'. By Theorem 3.7 $\mathcal{O}' = \mathcal{O}_{G'}$. Hence by Beth's Theorem if $\mathcal{O}_G$ is discrete there exists an $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$.

Now assume $\mathcal{O}_G$ is not discrete. Let $x_0 \in \mathcal{M}_G \setminus G$ [resp. $x_0 \in \mathcal{M}_G \setminus G - 1$]. Then $x_0 \cdot \mathcal{O}_G \nsubseteq G$ [resp. $x_0 \cdot \mathcal{O}_G \nsubseteq G - 1$]. As $\mathcal{O}_G$ is not discrete, for every $n \in \mathbb{N}$ there exists $y \in \mathcal{M}_G \setminus \{0\}$ such that $v_G(x_0) \geq n \cdot v_G(y) \geq k \cdot v_G(y)$ for all $k \leq n$. For $a \in \mathcal{O}_G$ we have $x_0 \cdot a \cdot y^{-k} \in \mathcal{O}_G$. Therefore $x_0 \cdot a \in y^k \cdot \mathcal{O}_G$. Hence $y^k \cdot \mathcal{O}_G \supseteq x_0 \cdot \mathcal{O}_G \nsubseteq G$ $\left[\text{resp. } y^k \cdot \mathcal{O}_G \supseteq x_0 \cdot \mathcal{O}_G \nsubseteq G - 1\right]$ for all $k \geq n$. Let $z \in y^n \cdot \mathcal{O}_G \setminus G$ [resp. $z \in y^n \cdot \mathcal{O}_G \setminus (G-1)$]. As $y \in \mathcal{O}_G$ we have $y^n \cdot \mathcal{O}_G \subseteq y^k \cdot \mathcal{O}_G$ and therefore $z \in y^k \cdot \mathcal{O}_G$ for every $k \leq n$. Thus there exists $z \in \bigcap_{k=1}^{n} y^k \cdot \mathcal{O}_G = y^n \cdot \mathcal{O}_G$ with $z \notin G$. Therefore $\Phi\,(y,z) = \{y \in \mathcal{M}_G \wedge 0 \neq y \wedge z \in y^n \cdot \mathcal{O}_G \wedge z \notin G \mid n \in \mathbb{N}\}$ [resp. $\Phi\,(y,z) = \{y \in \mathcal{M}_G \wedge 0 \neq y \wedge z \in y^n \cdot \mathcal{O}_G \wedge z \notin G - 1 \mid n \in \mathbb{N}\}$] is a finitely satisfiable type. Hence there exist an elementary extension $(K', G', \mathcal{O}')$ and y', $z' \in K'$ such that (y', z') realizes $\Phi\,(y,z)$. Let $\mathfrak{p} = \bigcap_{n=1}^{\infty} (y')^n \cdot \mathcal{O}'$. Let $a, b \in \mathfrak{p}$. Then for all $n \in \mathbb{N}$ there exist a_n, $b_n \in \mathcal{O}'$ such that $a = (y')^n \cdot a_n$ and $b = (y')^n \cdot b_n$. We have $a + b = (y')^n \cdot (a_n + b_n) \in (y')^n \cdot \mathcal{O}'$. Hence $a + b \in \mathfrak{p}$. Let $c \in \mathcal{O}'$. For every $n \in \mathbb{N}$ we have $c \cdot a = c \cdot (y')^n \cdot a_n \in (y')^n \cdot \mathcal{O}'$. Hence $c \cdot a \in \mathfrak{p}$. Now let $a, b \in \mathcal{O}'$ with $a \cdot b \in \mathfrak{p}$. Assume $a \notin \mathfrak{p}$. Then there exists $n_0 \in \mathbb{N}$ such that $a \notin (y')^{n_0} \cdot \mathcal{O}'$. Hence $v_G\left(a \cdot (y')^{-n_0}\right) < 0$. Let $m \in \mathbb{N}$. We have $a \cdot b \in (y')^{n_0+m} \cdot \mathcal{O}'$ and thus $0 \leq v_G\,(a \cdot (y')^{-n_0}) + v_G\,(b \cdot (y')^{-m})$. Hence we have $v_G\,(b \cdot (y')^{-m}) > 0$ and therefore $b \in (y')^m \cdot \mathcal{O}'$. Thus $b \in \mathfrak{p}$. Hence $\mathfrak{p}$ is an $\mathcal{O}'$-prime ideal. As $z' \in \mathfrak{p}$ we have $\mathfrak{p} \nsubseteq G'$ [resp. $\mathfrak{p} \nsubseteq G' - 1$]. As $(y')^n \cdot \mathcal{O}' \subseteq \mathcal{M}'$ for all $n \in \mathbb{N}$ we have $\mathfrak{p} \subseteq \mathcal{M}'$. As $(y')^{-1} \notin \mathcal{O}'$ we have $y' \notin (y')^2 \cdot \mathcal{O}'$. Hence $\mathfrak{p} \subsetneq \mathcal{M}'$. By Remark 2.1 for every ideal $\mathcal{A} \subseteq G'$ [resp. $\mathcal{A} \subseteq G' - 1$] we have $\mathcal{A} \subseteq \mathfrak{p}$ and therefore a $\sqrt{\mathcal{A}} \subseteq \mathfrak{p} \subsetneq \mathcal{M}'$. Hence $\mathcal{O}'$ is not coarsely compatible with G'. In particular $\mathcal{O}' \neq \mathcal{O}_{G'}$. By Remark 5.4 there exists no $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$. □

For every subgroup G of K, $\overline{G} := \varrho(G)$ is a subgroup of the residue field $\overline{K}$. We will show the following lemma.

Lemma 5.8 *Let G be a subgroup of K such that the group case or the residue case holds.*

(a) Let $G \subseteq K$ be an additive subgroup of K. Let $x \in K$. Then $\overline{x} \in \overline{G}$ if and only if $x \in G$.
(b) Let $G \subseteq K$ be a multiplicative subgroup of K. Let $x \in \mathcal{O}_G^{\times}$. Then $\overline{x} \in \overline{G}$ if and only if $x \in G$.

Proof (a) Let $\overline{x} \in \overline{G}$. Then there exists $y \in G$ with $\overline{y} = \overline{x}$ hence $x = y + \alpha$ for some $\alpha \in \mathcal{M}_G \subseteq G$. As $\alpha, y \in G$, we have $x = y + \alpha \in G$.

The other direction is clear.

(b) Let $x \in \mathcal{O}_G^{\times}$. Assume $\overline{x} \in \overline{G}$. Then there exists $y \in G$ with $\overline{y} = \overline{x}$ hence $x = y + \alpha$ for some $\alpha \in \mathcal{M}_G$. Let v_G be a valuation with $\mathcal{O}_G = \mathcal{O}_{v_G}$. We have $v_G\,(y) = \min\{v_G(x), v_G(\alpha)\} = 0$ and therefore $y \in \mathcal{O}_G^{\times}$. Hence y–$1 \in \mathcal{O}_G$ and therefore $\alpha \cdot y^{-1} \in \mathcal{M}_G$. As $1 + \mathcal{M}_G \subseteq G$ $1 + \alpha \cdot y^{-1}, y \in G$. Therefore $x = y \cdot \left(1 + \alpha \cdot y^{-1}\right) \in G$.

The other direction is again clear. □

Theorem 5.9 *Let $G \subseteq K$ be a subgroup of a field such that the residue case holds. Then there exists an $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$ if and only if G is additive or G is multiplicative and $\overline{G} \cup \{0\}$ is no ordering.*

Proof Let us first assume G is additive or G is multiplicative and $\overline{G} \cup \{0\}$ is no ordering. Assume $\mathcal{O}^*$ is a non-trivial valuation ring on $\overline{K}$ which is weakly compatible with $\overline{G}$. Let $\widetilde{\mathcal{O}} := \varrho^{-1}\,(\mathcal{O}^*)$. As $\mathcal{O}^*$ is non-trivial, $\widetilde{\mathcal{O}}$ is a valuation ring on K with $\widetilde{\mathcal{O}} \subsetneq \mathcal{O}_G$. Let $\mathcal{M}^*$ denote the maximal ideal of $\mathcal{O}^*$ and $\widetilde{\mathcal{M}}$ the maximal ideal of $\widetilde{\mathcal{O}}$. Let $\mathcal{A}$ be an $\mathcal{O}^*$-ideal such that $\sqrt{\mathcal{A}} = \mathcal{M}^*$ and $\mathcal{A} \subseteq \overline{G}$ $\left[\text{resp. } 1 + \mathcal{A} \subseteq \overline{G}\right]$. Then $\varrho^{-1}(\mathcal{A})$ is an $\widetilde{\mathcal{O}}$-ideal with $\sqrt{\varrho^{-1}(\mathcal{A})} = \widetilde{\mathcal{M}}$. With Lemma 5.8 $\varrho^{-1}(\mathcal{A}) \subseteq \varrho^{-1}\left(\overline{G}\right) = G$ $\left[\text{resp. } 1 + \varrho^{-1}(\mathcal{A}) \subseteq \varrho^{-1}(1) + \varrho^{-1}(\mathcal{A}) = \varrho^{-1}(1 + \mathcal{A}) \subseteq \varrho^{-1}\left(\overline{G}\right) = G\right]$. Therefore $\widetilde{\mathcal{O}}$ is a weakly compatible refinement of $\mathcal{O}_G$. As we are in the residue case by Theorem 3.7 this is a contradiction. Hence there exists no non-trivial valuation ring on $\overline{K}$ which is weakly compatible with $\overline{G}$.

Now let $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$. $\mathcal{O}'$ is coarsely compatible with G' and hence $\mathcal{O}_{G'} \subseteq \mathcal{O}'$. Assume $\mathcal{O}_{G'} \subsetneq \mathcal{O}'$. Let ϱ' denote the residue homomorphism $\varrho' : \mathcal{O}' \longrightarrow \mathcal{O}'/\mathcal{M}'$. Then $\varrho'(\mathcal{O}_{G'})$ is a non-trivial valuation ring on $\overline{K'} := \mathcal{O}'/\mathcal{M}'$. We have $\varrho'(\mathcal{M}_{G'}) \subseteq \varrho'(G') = \overline{G'}$ $\left[\text{resp. } 1 + \varrho'(\mathcal{M}_{G'}) = \varrho'(1 + \mathcal{M}_{G'}) \subseteq \varrho'(G') = \overline{G'}\right]$. Therefore $\overline{\mathcal{O}}_{G'}$ is a non-trivial valuation ring on $\overline{K'}$ which is weakly compatible with $\overline{G'}$. But this contradicts $(K', G', \mathcal{O}') \equiv (K, G, \mathcal{O}_G)$ by Corollary 4.8.

Now assume G is a multiplicative subgroup of $K^\times$ and $\overline{G} \cup \{0\}$ an ordering on the residue field $\overline{K}$ of $(K, \mathcal{O}_G)$. Assume $\overline{G} \cup \{0\}$ is not archimedean. Then the valuation ring $\mathcal{O}^* := \left\{x \in \overline{K} \mid \text{there exists } a \in \mathbb{Z}\; a - x \in \overline{G},\; a + x \in \overline{G}\right\}$ on $\overline{K}$ is non-trivial (compare [9, page 36]). Let $\varrho : \mathcal{O}_G \longrightarrow \overline{K}$ denote the residue homomorphism. Then $\varrho^{-1}(\mathcal{O}^*) := \widetilde{\mathcal{O}}$ is a valuation ring on K with $\mathcal{O}_G \supsetneq \widetilde{\mathcal{O}}$. Denote by $\mathcal{M}^*$ the maximal ideal of $\mathcal{O}^*$ and by $\widetilde{\mathcal{M}}$ the maximal ideal of $\widetilde{\mathcal{O}}$.

$\mathcal{O}^*$ is $\left(\overline{G} \cup \{0\}\right)$-convex. Hence $1 + \mathcal{M}^* \subseteq \overline{G}$ (see, for example, [9, Proposition 2.2.4]). As by Lemma 5.8 $\varrho^{-1}\left(\overline{G}\right) = G$ $1 + \widetilde{\mathcal{M}} \subseteq \varrho^{-1}(1) + \varrho^{-1}(\mathcal{M}^*) = \varrho^{-1}(1 + \mathcal{M}^*) \subseteq \varrho^{-1}\left(\overline{G}\right) = G$. Hence $\widetilde{\mathcal{O}} \subsetneq \mathcal{O}_G$ is a coarsely compatible valuation ring on K. This is a contradiction. Therefore $\overline{G} \cup \{0\}$ must be an archimedean ordering. Let $\Phi(y) := \left\{y \in \overline{K} \wedge n - y \notin \overline{G} \mid n \in \mathbb{N}\right\}$. For every $n \in \mathbb{N}$ there exists $y \in K$ such that $n - y \notin \overline{G}$ and therefore $k - y \notin \overline{G}$ for all $k \leq n$. Therefore $\Phi(y)$ is a finitely satisfiable type. Hence there exists an elementary extension $(K', G', \mathcal{O}')$ and $y' \in K'$ such that y' realizes $\Phi(y)$. $\overline{G'} \cup \{0\}$ is a non-archimedean order on $\overline{K'}$ as $y' > n$ for all $n \in \mathbb{N}$. As above from $\overline{G'} \cup \{0\}$ non-archimedean follows that there exists a valuation ring $\widetilde{\mathcal{O}} \subsetneq \mathcal{O}'$ which is compatible with G'. As we have $\mathcal{O}_G^\times \not\subseteq G$ we have $(\mathcal{O}')^\times \not\subseteq G'$. Hence $\mathcal{O}'$ has a proper refinement which is coarsely compatible with G' and hence $\mathcal{O}' \neq \mathcal{O}_{G'}$. By Remark 5.4 there exists no $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$. □

The following table summarizes Theorem 5.6, Theorem 5.7 and Theorem 5.9.

Theorem 5.10 *Let $G \subsetneq K$ [resp. $G \subsetneq K^\times$] be a subgroup of K.*

Then there exists an $\mathcal{L}_G$-formula φ such that φ defines $\mathcal{O}_{G'}$ for all $(K', G') \equiv (K, G)$ if and only if

	$G\subseteq K$ additive	$G\subseteq K^\times$ multiplicative
Group case	Iff either $\mathcal{O}_G$ is discrete or for all $x \in \mathcal{M}_G$ $x^{-1}\cdot\mathcal{O}_G \nsubseteq G$	Always
Weak case	If and only if $\mathcal{O}_G$ is discrete	
Residue case	Always	Iff $\overline{G}\cup\{0\}$ is no ordering

6 $\mathcal{O}_G$ for Groups of Prime Powers and the Artin Schreier Group

In this section we want to apply the results from the previous sections to the Artin-Schreier group $G = K^{(p)}$ for $p = \mathrm{char}(K) > 0$ and the group of prime powers $G = (K^\times)^q$ for $q \neq \mathrm{char}(K)$ prime. As these groups are $\mathcal{L}_{\mathrm{ring}}$-$\emptyset$-definable, any $\mathcal{L}_G$-$\emptyset$-definable valuation will be $\mathcal{L}_{\mathrm{ring}}$-$\emptyset$-definable.

We will start with a lemma that shows that for these groups the weak case can only occur if $G = (K^\times)^q$ for $q = \mathrm{char}\left(\overline{K}\right)$.

Lemma 6.1 *Let $\mathcal{O}$ be a valuation ring on a field K.*

- *Let G be an additive subgroup of K and $K^{(p)} \subseteq G$ for $p := char\,(K) > 0$* ***or***
- *let G be a multiplicative subgroup such that there exists $n \in \mathbb{N}$ with $(K^\times)^n \subseteq G$ and $\gcd\big(n, char\left(\overline{K}\right)\big) = 1$ if $char\left(\overline{K}\right) \neq 0$.*

Then v is compatible if and only if it is weakly compatible.

Proof It is clear that if $\mathcal{O}$ is compatible, then it is weakly compatible.

Assume $\mathcal{O}$ is weakly compatible but not compatible. Let $\mathcal{A}$ be an $\mathcal{O}$-ideal with $\sqrt{\mathcal{A}} = \mathcal{M}$ and $\mathcal{A} \subseteq G$ [resp. $1+\mathcal{A} \subseteq G$]. By Remark 2.1 we can choose $\mathcal{A}$ maximal with $\mathcal{A} \subseteq G$ [resp. $1+\mathcal{A} \subseteq G$]. Let $a \in \mathcal{M}\setminus\mathcal{A}$. Let $k \in \mathbb{N}$ with $a^k \notin \mathcal{A}$ and $a^{k+1} \in \mathcal{A}$. Define the $\mathcal{O}$-ideal $\mathcal{B} := a^k\cdot\mathcal{O}$. As $a^k \in \mathrm{B}\setminus\mathcal{A}$ we have $\mathcal{B} \nsubseteq \mathcal{A}$ and hence by Remark 2.1 $\mathcal{A} \subsetneq \mathrm{B}$. Let $x \in \mathrm{B}^2$. Then there exists $y \in \mathcal{O}$ with $x = \left(a^k\cdot y\right)^2$. As $a^{k-1}\cdot y^2 \in \mathcal{O}$ and $a^{k+1} \in \mathcal{A}$, we have $x \in \mathcal{A}$. Hence $\mathcal{B}^2 \subseteq \mathcal{A}$.

Let us first show that if G is an additive subgroup of K then $\mathcal{B} \subseteq G$. Let $b \in \mathrm{B}$. As $p = \mathrm{char}(K) \geq 2$, we have $b^{p-2} \in \mathcal{O}$. Further $b^2 \in \mathcal{A}$. Therefore $b^p \in \mathcal{A}$. As $(-b)^p + b \in K^{(p)} \subseteq G$, therefore $(-b)^p + b \pm b^p \in G$. Thus $\mathcal{B} \subseteq G$.

Now assume that G is a multiplicative subgroup. We will show $1+\mathrm{B} \subseteq G$.

Let $b \in \mathrm{B}$. Then

$$G \ni \left(\frac{b}{n}+1\right)^n = 1 + b + \binom{n}{2}\cdot\left(\frac{1}{n}\right)^2\cdot b^2 + b\cdot\left(\sum_{i=0}^{n-3}\binom{n}{i+3}\cdot\left(\frac{1}{n}\right)^{i+3}\cdot b^i\right)\cdot b^2.$$

As $\gcd\big(n, char\left(\overline{K}\right)\big) = 1$ we have $n \in \mathcal{O}^\times$. Furthermore $b \in \mathcal{O}$ and for all $i, j \in \mathbb{N}$ with $i \leq j$ we have $\binom{i}{j} \in \mathbb{N} \subseteq \mathcal{O}$. Hence $\sum_{i=0}^{n-3}\binom{n}{i+3}\cdot\left(\frac{1}{n}\right)^{i+3}\cdot b^i \in \mathcal{O}$ and $\binom{n}{2}\cdot\left(\frac{1}{n}\right)^2 \in \mathcal{O}$. As $\mathcal{B}^2$ is an $\mathcal{O}$-ideal, from this follows $\binom{n}{2}\cdot\left(\frac{1}{n}\right)^2\cdot b^2 \in \mathrm{B}^2 \subseteq \mathcal{A}$

and $\left(\sum_{i=0}^{n-3}\binom{n}{i+3}\cdot\left(\frac{1}{n}\right)^{i+3}b^i\right)\cdot b^2\in \mathrm{B}^2\subseteq\mathscr{A}$. Therefore $\left(\frac{b}{n}+1\right)^n\in 1+b+\mathscr{A}+b\cdot\mathscr{A}=(1+b)\cdot(1+\mathscr{A})$. By Lemma 2.3 $(1+\mathscr{A})^{-1}=1+\mathscr{A}$. Hence $1+b\in(K^\times)^q\cdot(1+\mathscr{A})^{-1}\subseteq G\cdot(1+\mathscr{A})^{-1}=G\cdot(1+\mathscr{A})\subseteq G$. Hence $1+\mathrm{B}\subseteq G$.

Therefore $\mathscr{B}$ is an $\mathscr{O}$-ideal with $\mathscr{B}\subseteq G$ [resp. $1+\mathrm{B}\subseteq G$] and $\mathscr{A}\subsetneq\mathrm{B}$. But this contradicts the choice of $\mathscr{A}$. ☐

Theorem 6.2 *Let K be a field with char $(K)=p>0$ and $G:=K^{(p)}$. Then $\mathscr{O}_G$ is $\emptyset$-definable.*

Proof As the case $\mathscr{O}_G=K$ is trivial we can assume $\mathscr{O}_G\neq K$ and hence as well $G\neq K$.

By Lemma 6.1 there exists no valuation which is weakly compatible but not compatible. Hence by the definition of the cases (Theorem 3.7), we are not in the weak case.

If we are in the residue case $\mathscr{O}_G$ is $\emptyset$-definable by Theorem 5.10.

Now assume we are in the group case. Suppose for a contradiction, there exists an $x_0\in\mathscr{M}_G$ such that $x_0^{-1}\cdot\mathscr{O}_G\subseteq G$. Then $x_0^{-1}\cdot\mathscr{O}_G$ is a fractional $\mathscr{O}_G$-ideal and therefore there exists a maximal fractional $\mathscr{O}_G$-ideal $\mathscr{A}$ with $\mathscr{A}\subseteq G$. We have $\mathscr{O}_G\subsetneq x_0^{-1}\cdot\mathscr{O}_G\subseteq\mathscr{A}$. Let $\mathscr{A}_\alpha:=\{x\in K\mid v_G(x)\geq\alpha\cdot v_G(y)\text{ for some }y\in\mathscr{A}\}$. Let $x\in\mathscr{A}$. If $v_G(x)\geq 0=\alpha\cdot v_G(1)$, then $x\in\mathscr{A}_\alpha$. If $v_G(x)<0$, then $v_G(x)>\alpha\cdot v_G(x)$ and therefore $x\in\mathscr{A}_\alpha$. Hence $\mathscr{A}\subseteq\mathscr{A}_\alpha$. Assume for all $x_1\in\mathscr{A}\setminus\mathscr{O}$ there exists $x_2\in\mathscr{A}$ such that $\left(1+p^{-1}\right)\cdot v_G(x_1)\geq v_G(x_2)$. Define $\mathfrak{p}:=\{x\in K\mid -v_G(x)<v_G(a)\text{for all }a\in\mathscr{A}\}$. Let $a\in\mathscr{A}\setminus\mathscr{O}\neq\emptyset$. Then for all $x\in\mathfrak{p}$ $v_G(x)>-v_G(a)>0$ and hence $x\in\mathscr{M}$. As further $a^{-1}\in\mathscr{M}\setminus\mathfrak{p}$ we have $\mathfrak{p}\subsetneq\mathscr{M}$. Let $x,y\in\mathfrak{p}$. Then $-v_G(x+y)\leq-\max\{v_G(x),v_G(y)\}<v_G(a)$ for all $a\in\mathscr{A}$. Hence $x+y\in\mathfrak{p}$. Let $x\in\mathfrak{p}$ and $k\in\mathscr{O}$. For all $a\in\mathscr{A}$ we have $k\cdot a\in\mathscr{A}$. Hence $-v_G(x)>v_G(k\cdot a)$ and therefore $v_G(a)<-v_G(k\cdot x)$. Thus $k\cdot x\in\mathfrak{p}$. Let $x,y\in\mathscr{O}\setminus\mathfrak{p}$. Let $a,b\in\mathscr{A}$ such that $-v_G(x)\geq v_G(a)$ and $-v_G(y)\geq v_G(b)$.

If $a\in\mathscr{O}$ or $b\in\mathscr{O}$ we have $a\cdot b\in\mathscr{A}$. As $-v_G(x\cdot y)\geq v_G(a)+v_G(b)$ we have $x\cdot y\notin\mathfrak{p}$.

If $a,b\in\mathscr{A}\setminus\mathscr{O}$ let $a_0\in\{a,b\}$ such that $v_G(a_0)=\min\{v_G(a),v_G(b)\}\in\mathscr{A}\setminus\mathscr{O}$. By assumption there exists $a_1\in\mathscr{A}$ such that $0>\left(1+p^{-1}\right)\cdot v_G(a_0)\geq v_G(a_1)$. Recursively for all $n\geq 0$ we can define $a_{n+1}\in\mathscr{A}\setminus\mathscr{O}$ with $\left(1+p^{-1}\right)\cdot v_G(a_n)\geq v_G(a_{n+1})$. We then get $\left(1+p^{-1}\right)^n\cdot v_G(a_0)\geq\left(1+p^{-1}\right)^{n-1}\cdot v_G(a_1)\geq\ldots\geq v_G(a_n)$. As $\left(1+p^{-1}\right)^n\longrightarrow\infty$ for $n\to\infty$, for some $n\in\mathbb{N}$ we have $\left(1+p^{-1}\right)^n\geq 2$ and thus $2\cdot v_G(a_0)\geq\left(1+p^{-1}\right)^n\cdot v_G(a_0)\geq v_G(a_n)$. Hence $-v_G(x\cdot y)\geq v_G(a)+v_G(b)\geq v_G(a_n)$. As $a_n\in\mathscr{A}$ from this follows $x\cdot y\notin\mathfrak{p}$.

Altogether we see that for all $x,y\in\mathscr{O}$ if $x\cdot y\in\mathfrak{p}$ then $x\in\mathfrak{p}$ or $y\in\mathfrak{p}$.

Hence $\mathfrak{p}$ is a prime ideal. Therefore $\mathscr{O}_\mathfrak{p}$ is a proper coarsening of $\mathscr{O}$.

Let $x\cdot y^{-1}\in\mathscr{O}_\mathfrak{p}$. As $y\notin\mathfrak{p}$ there exists $a\in\mathscr{A}$ such that $v_G(a)\leq-v_G(y)$. We therefore have $v_G(x\cdot y^{-1})\geq v_G(x)+v_G(a)\geq v_G(a)$. Hence by Remark 2.1 (a) we have $x\cdot y^{-1}\in\mathscr{A}$. Thus $\mathscr{O}_G\subsetneq\mathscr{O}_\mathfrak{p}\subseteq\mathscr{A}\subseteq G$. But this contradicts the definition of $\mathscr{O}_G$. Hence for some $x_0\in\mathscr{A}\setminus\mathscr{O}_G\neq\emptyset$ we have $\left(1+p^{-1}\right)\cdot v_G(x_0)<v_G(x)$ for all $x\in\mathscr{A}$. As $\mathscr{A}\subseteq G$, there exists $y_0\in K$ such that $x_0=y_0^p-y_0$. We have

$0 > v_G(x_0) = v_G(y_0^p - y_0) = p \cdot v_G(y_0)$. Therefore $v_G(y_0) = p^{-1} \cdot v_G(x_0)$ and hence $v_G(x_0 \cdot y_0) = \left(1 + p^{-1}\right) \cdot v_G(x_0)$. As $\left(1 + p^{-1}\right) \cdot v_G(x_0) < v_G(x)$ for all $x \in \mathscr{A}$, from this follows $x_0 \cdot y_0 \notin \mathscr{A}$. As $\left(1 + p^{-1}\right) \leq \alpha$ and $v_G(x_0) < 0$, we have $v_G(x_0 \cdot y_0) \geq \alpha \cdot v_G(x_0)$ and hence $x_0 \cdot y_0 \in \mathscr{A}_\alpha \setminus \mathscr{A}$. This shows $\mathscr{A} \subsetneq \mathscr{A}_\alpha$.

Let $x \in \mathscr{A}_\alpha \setminus \mathscr{A}$. Then there exists $y \in \mathscr{A}$ such that

$$v_G(x) > \alpha \cdot v_G(y). \tag{1}$$

As $\mathscr{O}_G \subseteq \mathscr{A}$ we have $v_G(x) < 0$. Hence $v_G(y) < 0$. As $x \notin \mathscr{A}$ by Remark 2.1 $v_G(x) < v_G(y)$ and therefore $v_G\left(x \cdot y^{-1}\right) < 0$. Further $v_G\left(x \cdot y^{-1}\right) > \alpha \cdot v_G(y) - v_G(y) > v_G(y)$ as $\alpha - 1 \in (0, 1)$ and $v_G(y) < 0$. Hence

$$0 > v_G\left(x \cdot y^{-1}\right) > v_G(y). \tag{2}$$

Again by Remark 2.1 we get $x \cdot y^{-1} \in \mathscr{A} \setminus \mathscr{O}_G$. As $\mathscr{A} \subseteq G$ there exists $a \in K$ such that $x \cdot y^{-1} = a^p - a$. As $0 < v_G\left(x \cdot y^{-1}\right)$ we have $v_G(a) < 0$ and hence

$$v_G\left(x \cdot y^{-1}\right) = v_G(a^p). \tag{3}$$

Therefore $x \cdot (y \cdot a^p)^{-1} \in \mathscr{O}_G^\times \subseteq \mathscr{A}$. As $y \in \mathscr{A}$ we have $x \cdot a^{-p} \in \mathscr{A} \subseteq G$. Hence there exists $b \in K$ such that $x \cdot a^{-p} = b^p - b$. Hence

$$x = a^p \cdot b^p - a^p \cdot b = (a \cdot b)^p - a \cdot b + a \cdot b - a^p \cdot b. \tag{4}$$

We have $(a \cdot b)^p - a \cdot b \in G$. Further

$$\begin{aligned} \min\{p \cdot v_G(b),\ v_G(b)\} &= v_G(b^p - b) \\ &= v_G(x) - v_G(a^p) \\ &\overset{(3)}{=} v_G(x) - v_G(x \cdot y^{-1}) \\ &= v_G(y) \\ &< 0. \end{aligned} \tag{5}$$

Hence

$$p \cdot v_G(b) = \min\{p \cdot v_G(b), v_G(b)\} = v_G(y). \tag{6}$$

Further as $1 < \alpha \leq 2 - p^{-1}$

$$\begin{aligned} v_G(a^p \cdot b) &\overset{(3)}{=} v_G\left(x \cdot y^{-1}\right) + v_G(b) \\ &\overset{(5)}{=} v_G(x) - v_G(y) + p^{-1} \cdot v_G(y) \end{aligned}$$

$$\overset{(1)}{>} \alpha \cdot v_G(y) - v_G(y) + p^{-1} \cdot v_G(y)$$
$$= \left(\alpha - 1 + p^{-1}\right) \cdot v_G(y)$$
$$\overset{(2)}{\geq} \left(2 - p^{-1} - 1 - p^{-1}\right) \cdot v_G(y)$$
$$= v_G(y).$$

Therefore as $y \in \mathscr{A}$ again with Remark 2.1 follows $a^p \cdot b \in \mathscr{A} \subseteq G$. As $p > 1$ and $v_G(a) < 0$ we get $v_G(a \cdot b) > v_G(a^p \cdot b)$. Therefore with Remark 2.1 follows $a \cdot b \in \mathscr{A} \subseteq G$. As G is closed under addition $x \overset{(4)}{=} (a \cdot b)^p - a \cdot b + a \cdot b - a^p \cdot b \in G$.

Hence $\mathscr{A}_\alpha \setminus \mathscr{A} \subseteq G$ and therefore $\mathscr{A} \subseteq \mathscr{A}_\alpha \subseteq G$. But this contradicts the maximality of $\mathscr{A}$ and therefore for all $x \in \mathscr{M}_G$ we have $x^{-1} \cdot \mathscr{O}_G \nsubseteq G$. Hence by Theorem 5.10 $\mathscr{O}_G$ is $\emptyset$-definable. □

Proposition 6.3 *Let $q \in \mathbb{N}$ be prime. Let K be a field with $char(K) \neq q$ and the qth-root of unity $\zeta_q \in K$. Let $G := (K^\times)^q$. Assume we are in the group case or we are in the residue case and $\overline{G} \cup \{0\}$ is no ordering on K. Then $\mathscr{O}_G$ is $\emptyset$-definable. In particular $\mathscr{O}_G$ is $\emptyset$-definable if $char(K) > 0$.*

Proof The case $\mathscr{O}_G = K$ is clear.

If $\mathscr{O}_G \neq K^\times$ we have $G \neq K^\times$ and hence the claim follows by Theorem 5.10.

Further by Lemma 6.1 and Theorem 3.7 if $q \neq \mathrm{char}(K) > 0$ the weak case cannot occur. □

Proposition 6.4 *Let K be a field with $char(K) = 0$ and $\zeta_2 \in K$. Let K not be euclidean, i.e. K^2 is not a positive cone. Let $G := (K^\times)^2$. Let $\overline{G} \cup \{0\}$ be an ordering on $\overline{K}$. Then $G \cup (-G)$ is a multiplicative subgroup of $K^\times$ and $\mathscr{O}_{G\cup(-G)}$ is $\emptyset$-definable. Further if $\mathscr{O}_G$ is non-trivial then it induces the same topology as $\mathscr{O}_{G\cup(-G)}$.*

Proof As G is a multiplicative subgroup of K it is easy to see that $G \cup (-G)$ is a multiplicative subgroup of K as well. As $\overline{K}$ is real, K is real as well (see [9, Corollary 2.2.6]). Suppose $K^\times = G \cup (-G)$. Then $K = K^2 \cup \left(-K^2\right)$. It is clear that $K^2 \cdot K^2 \subseteq K^2$, $K^2 \subseteq K^2$ and $-1 \notin \sum K^2$. Suppose $K^2 + K^2 \nsubseteq K^2$. Hence there exist $x, y \in K$ such that $x^2 + y^2 \notin K^2$. By assumption $K = K^2 \cup \left(-K^2\right)$ and therefore $x^2 + y^2 \in -K^2$. Thus $x^2 \cdot \left(-\left(x^2 + y^2\right)\right)^{-1}, y^2 \cdot \left(-\left(x^2 + y^2\right)\right)^{-1} \in K^2$ and hence $-1 = x^2 \cdot \left(-\left(x^2 + y^2\right)\right)^{-1} + y^2 \cdot \left(-\left(x^2 + y^2\right)\right)^{-1} \in \sum K^2$. But this is a contradiction to K real. From this follows that K^2 is a positive cone. As we assumed that K is not euclidean this is a contradiction and hence $K^2 \cup \left(-K^2\right) \neq K$. Thus $G \cup (-G) \neq K^\times$. By Lemma 6.1 we are not in the weak case. Let $x \in \mathscr{O}^\times_{G\cup(-G)}$. If $x \notin G$ by Lemma 5.8 (a) follows $\overline{x} \notin \overline{G}$ and therefore $\overline{-x} \in \overline{G}$. Again by Lemma 5.8 (a) $-x \in G$ and thus $x \in -G$. Hence $\mathscr{O}^\times_{G\cup(-G)} \subseteq G \cup (-G)$. Therefore we are in the group case and $\mathscr{O}_{G\cup(-G)}$ is $\emptyset$-definable by Theorem 5.10.

Assume $\mathscr{O}_G$ is non-trivial. As $1 + \mathscr{M}_G \subseteq G \subseteq G \cup (-G)$, $\mathscr{O}_G$ is compatible with $G \cup (-G)$. Therefore by Lemma 4.1 $\mathscr{O}_{G\cup(-G)}$ is non-trivial. As $\mathscr{O}_{G\cup(-G)}$ is as well compatible with $G \cup (-G)$, $\mathscr{O}_{G\cup(-G)}$ induces the same topology as $\mathscr{O}_G$. □

We will generalize the notion of henselianity slightly and define when a valued field is called q-henselian for a certain prime q. We denote by $K\langle q\rangle$ the compositum of all finite Galois extensions of q-power degree. $(K, \mathcal{O})$ is *q-henselian* if $\mathcal{O}$ extends uniquely $K\langle q\rangle$.

Proposition 6.5 *Let (K, v) be a valued field, let q be prime and if $q \neq char(K)$ assume $\zeta_q \in K$.*

(a) If $char\left(\overline{K}\right) \neq q$, then v is q-henselian if and only if $1 + \mathcal{M}_v \subseteq (K^\times)^q$.
(b) If $char(K) = q$, then v is q-henselian if and only if $\mathcal{M}_v \subseteq K^{(q)}$.
(c) If $char(K) = 0$, $char\left(\overline{K}\right) = q$ and v is a rank-1-valuation, then v is q-henselian if and only if $1 + q^n \cdot \mathcal{M}_v \subseteq (K^\times)^q$ for some $n \in \mathbb{N}$. In this case $1 + q^n \cdot \mathcal{M}_v \subseteq (K^\times)^q$ for every $n \geq 2$.

Proposition 6.5 is essentially [17, Proposition 1.4], assertion (6.5) is slightly adjusted as in [17] this is only shown for $n = 2$. As the proof works the same way, we will not repeat it here (for details see [6, Proposition 5.10]). The original proof Assertion (6.5) has a gap. For a corrected proof see [3].

Proposition 6.6 *Let (K, v) be a valued field, let q be prime such that v is q-henselian.*

(a) Let $char(K) = p = q$ and $G := K^{(p)}$. Then v is compatible.
(b) Let $char\left(\overline{K}\right) \neq q$, $\zeta_q \in K$ and $G := (K^\times)^q$. Then v is compatible.
(c) Let $char(K) = 0$, $char\left(\overline{K}\right) = q$, $\zeta_q \in K$ and $G := (K^\times)^q$. Then $1 + q^2 \cdot \mathcal{M}_v \subseteq G$. If further v is a rank-1 valuation, then v is weakly compatible.

Proof Assertion (a) and (b) and the first part of (c) follow at once from Proposition 6.5

Now assume char $(K) = 0$, char $\left(\overline{K}\right) = q$, $\zeta_q \in K$, $G := (K^\times)^q$ and v is of rank-1. Let $\mathcal{A} = q^2 \cdot \mathcal{M}_v = \left\{x \in K \mid v(x) > v\left(q^2\right)\right\}$. $\mathcal{A}$ is an $\mathcal{O}_v$-ideal. As v is of rank-1, Γ is archimedean. Hence for every $x \in \mathcal{M}_v$ there exists $n \in \mathbb{N}$ with $v(x^n) > v\left(q^2\right)$ and thus $x^n \in \mathcal{A}$. Therefore $\sqrt{\mathcal{A}} = \mathcal{M}_v$. As $1 + \mathcal{A} \subseteq \mathcal{M}_v$, it follows that v is weakly compatible. □

Proposition 6.7 *Let K be a valued field and let $p = char(K) > 0$. Let $G := K^{(p)}$. Then $\mathcal{O}_G$ is p-henselian.*

Proof By Lemma 6.1 $\mathcal{O}_G$ is compatible. Hence $\mathcal{M}_G \subseteq G = K^{(p)}$. By Proposition 6.5 (6.5) $\mathcal{O}_G$ is p-henselian. □

Proposition 6.8 *Let K be a valued field, let $q \neq char(K)$ be prime and $\zeta_q \in K$. Let $G := (K^\times)^q$.*

(a) If $char(\mathcal{O}_G/\mathcal{M}_G) \neq q$, then $\mathcal{O}_G$ is q-henselian
(b) If $char(K) = 0$ and $char(\mathcal{O}_G/\mathcal{M}_G) = q$, then $\mathcal{O}_G$ has a non-trivial q-henselian coarsening.

Proof (a) By Lemma 6.1 $\mathcal{O}_G$ is compatible. Hence $1 + \mathcal{M}_G \subseteq G = (K^\times)^q$. By Proposition 6.5 (6.5) $\mathcal{O}_G$ is p-henselian.

(b) By Proposition 2.13, either there exists a maximal non-trivial coarsening of $\mathcal{O}_G$ or the non-zero prime ideals of $\mathcal{O}_G$ form a basis of the neighbourhoods of zero of the topology $\mathcal{T}_{\mathcal{O}}$.

Let us first assume $\widetilde{\mathcal{O}}$ is a maximal non-trivial coarsening of $\mathcal{O}_G$. Then $\widetilde{\mathcal{O}}$ has rank-1. Let $\widetilde{\mathcal{M}}$ denote the maximal ideal of $\widetilde{\mathcal{O}}$. As $\mathcal{O}_G$ is coarsely compatible, so is $\widetilde{\mathcal{O}}$ and hence by Lemma 3.3 there exists $n \in \mathbb{N}$ with $1 + q^n \cdot \widetilde{\mathcal{M}} \subseteq G = (K^\times)^q$. If $\operatorname{char}\left(\widetilde{\mathcal{O}}/\widetilde{\mathcal{M}}\right) = q$ then by Proposition 6.5 (6.5) $\widetilde{\mathcal{O}}$ is q-henselian. If $\operatorname{char}\left(\widetilde{\mathcal{O}}/\widetilde{\mathcal{M}}\right) = 0$ then $1 + \widetilde{\mathcal{M}} = 1 + q^n \cdot \widetilde{\mathcal{M}} \subseteq (K^\times)^q$ and hence by Proposition 6.5 (6.5) $\widetilde{\mathcal{O}}$ is q-henselian.

Now assume the non-zero prime ideals of $\mathcal{O}_G$ form a basis of the neighbourhoods of zero of $\mathcal{T}_{\mathcal{O}}$. Then there exists an $\mathcal{O}_G$-prime ideal $\mathfrak{p} \neq \{0\}$ such that $q \notin \mathfrak{p}$. $\widetilde{\mathcal{O}} := (\mathcal{O}_G)_{\mathfrak{p}}$ is a proper coarsening of $\mathcal{O}_G$ with maximal ideal $\widetilde{\mathcal{M}} := \mathfrak{p} \subsetneq \mathcal{M}_G$. As $1 + \widetilde{\mathcal{M}} \subseteq G = (K^\times)^q$ and $\operatorname{char}\left(\widetilde{\mathcal{O}}/\widetilde{\mathcal{M}}\right) = 0$ by Proposition 6.5 (6.5) $\widetilde{\mathcal{O}}$ is q-henselian. □

Similar as the canonical henselian valuation (see [9, Section 4.4]) we can define the canonical q-henselian valuation. (See [13, Section 2] for details.) A field K is hereby called q-closed if it has no proper finite Galois extensions of q-power degree.

Lemma 6.9 *Let q be prime. Let K be a field which is not q-closed. We divide the class of q-henselian valuations into two subclasses,*

$$H_1^q(K) := \left\{ v \mid v textisaqhenselianvaluationand\, \overline{K}_v \neq \overline{K}_v\langle q\rangle \right\}$$

and

$$H_2^q(K) := \left\{ v \mid v \textit{ is a } q \textit{ henselian valuation and } \overline{K}_v = \overline{K}_v\langle q\rangle \right\}.$$

If $H_2^q(K) \neq \emptyset$, then there exists a unique coarsest valuation $v_K^q \in H_2^q(K)$.
Otherwise there exists a unique finest valuation $v_K^q \in H_1^q(K)$.

Definition 6.10 We call v_K^q the canonical q-henselian valuation.

Remark 6.11 Note that v_K^q is the trivial valuation if and only if K admits no non-trivial q-henselian valuation or $K = K\langle q\rangle$.

Theorem 6.12 *Let K be a field which is not q-closed. Let* char $(K) \neq q$, *$\zeta_q \in K$ and if $q = 2$ assume the residue field of the canonical henselian valuation $\mathcal{O}_{v_K^q}/\mathcal{M}_{v_K^q}$ is not euclidean. Then v_K^q is $\emptyset$-definable.*

Theorem 6.12 is a simplified version of [13, Main Theorem 3.1] omitting some details we will not need.

Proposition 6.13 *Let $K \neq K\langle 2\rangle$ and assume $\mathcal{O}_{v_K^q}/\mathcal{M}_{v_K^q}$ is euclidean. Then the coarsest 2-henselian valuation ${v_K^2}^*$ on K which has a euclidean residue field is $\emptyset$-definable.*

Proposition 6.13 is [13, Observation 3.2 (a)].

The following proposition is in particular interesting in the weak case, where $\mathcal{O}_G$ in general is not definable.

Proposition 6.14 *Let $q \in \mathbb{N}$ be prime. Let K be a field with char $(K) = 0$ and $\zeta_q \in K$. Let $G := (K^\times)^q$. Assume that we are in the weak case. Then K admits a q-henselian $\emptyset$-definable valuation which induces the same topology as $\mathcal{O}_G$.*

Proof The case $\mathcal{O}_G = K$ is clear. Hence assume $\mathcal{O}_G \neq K$ and therefore $G \neq K^\times$. As we are in the weak case, by Lemma 6.1 char$(\mathcal{O}_G/\mathcal{M}_G) = q$. By Proposition 6.8 $\mathcal{O}_G$ has a non-trivial q-henselian coarsening. By Theorem 6.12 and Proposition 6.13 either v_K^q or v_K^{2*} is $\emptyset$-definable, non-trivial and induces the same topology as $\mathcal{O}_G$.

By Lemma 6.1 the weak case can only occur if char $(\mathcal{O}_G/\mathcal{M}_G) = q$. □

Theorem 6.15 *Let K be a field.*

- *Let char$(K) = q$ and $G := K^{(q)}$* ***or***
- *let* char $(K) \neq q$*, $\zeta_q \in K$, $G := (K^\times)^q$ and if $q = 2$ assume K is not euclidean.*

Assume $\mathcal{O}_G$ is non-trivial. Then K admits a non-trivial $\emptyset$-definable valuation inducing the same topology as $\mathcal{O}_G$.

Proof If char$(K) = q$ let $G := K^{(q)}$. By Theorem 6.2 $\mathcal{O}_G$ is $\emptyset$-definable.

If char $(K) \neq q$, $\zeta_q \in K$ by Proposition 6.3, Proposition 6.4 and Proposition 6.14 there exists a $\emptyset$-definable valuation which induces the same topology as $\mathcal{O}_G$. □

Corollary 6.16 *Let K be a field.*

- *Let char$(K) = q$ and $G := K^{(q)}$* ***or***
- *let* char $(K) \neq q$*, $\zeta_q \in K$, $G := (K^\times)^q$ and if $q = 2$ assume K is not euclidean.*

Assume that for $\mathcal{N} = \{U \in \mathcal{T}_G \mid 0 \in U\}$ the following holds:

(V1) $\bigcap \mathcal{N} := \bigcap_{U \in \mathcal{N}} U = \{0\}$ *and* $\{0\} \notin \mathcal{N}$;
(V2) $\forall\, U, V \in \mathcal{N}\ \exists\, W \in \mathcal{N}\ W \subseteq U \cap V$;
(V3) $\forall\, U \in \mathcal{N}\ \exists\, V \in \mathcal{N}\ V - V \subseteq U$;
(V4) $\forall\, U \in \mathcal{N}\ \forall\, x, y \in K\ \exists\, V \in \mathcal{N}\ (x + V) \cdot (y + V) \subseteq x \cdot y + U$;
(V5) $\forall\, U \in \mathcal{N}\ \forall\, x \in K^\times\ \exists\, V \in \mathcal{N}\ (x + V)^{-1} \subseteq x^{-1} + U$;
(V6) $\forall\, U \in \mathcal{N}\ \exists\, V \in \mathcal{N}\ \forall\, x, y \in K\ x \cdot y \in V \longrightarrow x \in U \vee y \in U$.

Then K admits a non-trivial $\emptyset$-definable valuation.

This follows at once by Theorem 6.15 and Corollary 4.7.

Theorem 6.17 *Let K be a field which is not q-closed.*

- *Let char$(K) = q$* ***or***
- *let* char $(K) \neq q$*, $\zeta_q \in K$ and if $q = 2$ assume K is not euclidean.*

Assume K admits a non-trivial q-henselian valuation v. Then K admits a non-trivial $\emptyset$-definable valuation which induces the same topology as v.

Proof As K is not q-closed $G \neq K$ [resp. $G \neq K^\times$].

If char$(K) = q$ let $G := K^{(q)}$, otherwise let $G := (K^\times)^q$.

If $\operatorname{char}(K) = q$ or $\operatorname{char}(\overline{K}) \neq q$, then v is weakly compatible by Proposition 6.6. Hence by Lemma 4.1 $\mathcal{O}_G$ is non-trivial.

If $\operatorname{char}(K) = 0$ and $\operatorname{char}(\overline{K}) = q$ by Proposition 2.13 either there exists a maximal non-trivial coarsening of $\mathcal{O}_v$ or the non-zero prime ideals of $\mathcal{O}$ form a basis of the neighbourhoods of zero of $\mathcal{T}_{\mathcal{O}}$.

If $\widetilde{\mathcal{O}}$ is a maximal non-trivial coarsening of $\mathcal{O}_v$, then $\widetilde{\mathcal{O}}$ has rank-1. As a coarsening of a q-henselian valuation ring, $\widetilde{\mathcal{O}}$ is q-henselian and hence by Proposition 6.6 (c) $\widetilde{\mathcal{O}}$ is weakly compatible. Again by Lemma 4.1 $\mathcal{O}_G$ is non-trivial.

If the non-zero prime ideals of $\mathcal{O}$ form a basis of the neighbourhoods of zero of $\mathcal{T}_{\mathcal{O}}$, there exists an $\mathcal{O}_v$-prime ideal $\mathfrak{p} \neq \{0\}$ such that $q \notin \mathfrak{p}$. $\widetilde{\mathcal{O}} := (\mathcal{O}_v)_{\mathfrak{p}}$ is a proper coarsening of $\mathcal{O}_v$ with maximal ideal $\widetilde{\mathcal{M}} := \mathfrak{p}$. As a coarsening of a q-henselian valuation ring, $\widetilde{\mathcal{O}}$ is q-henselian and hence by Proposition 6.6 (b) compatible. Again by Lemma 4.1 $\mathcal{O}_G$ is non-trivial.

By Theorem 6.15 K admits a non-trivial $\emptyset$-definable valuation inducing the same topology as $\mathcal{O}_G$ and hence as v. As $\mathcal{O}_v$ and $\mathcal{O}_G$ are both weakly compatible, v induces the same topology as $\mathcal{O}_G$ (see [9, Theorem 2.3.4]). □

Remark 6.18 Under the assumptions of Theorem 6.17:

(a) There exists a non-trivial q-henselian definable valuation inducing the same topology as v.
(b) If $q = \operatorname{char}(K)$, v induces the same topology as $\mathcal{O}_G$ for $G = K^{(q)}$.
(c) If $q \neq \operatorname{char}(K)$, v induces the same topology as $\mathcal{O}_G$ for $G = (K^{\times})^q$.

Proof (a) If $q = \operatorname{char}(K)$ or if $q \neq 2$ and $q \neq \operatorname{char}(\mathcal{O}_G/\mathcal{M}_G)$, the definable valuation in Theorem 6.17 is $\mathcal{O}_G$ and, by Proposition 6.7 or Proposition 6.8 (a), $\mathcal{O}_G$ is q-henselian. If $q = 2 \neq \operatorname{char}(\mathcal{O}_G/\mathcal{M}_G)$ as well by Proposition 6.8 (a), $\mathcal{O}_G$ is q-henselian. Therefore the q-henselian definable valuation we obtain by Theorem 6.12 or Proposition 6.13 is non-trivial. If $q \neq \operatorname{char}(\mathcal{O}_G/\mathcal{M}_G)$ with the same proof as for the weak case in Proposition 6.14, we obtain a q-henselian definable valuation in all cases.

(b) By Proposition 6.7 $\mathcal{O}_G$ is q-henselian. As K is not q-closed any two q-henselian topologies are dependent and therefore v induces the same topology as $\mathcal{O}_G$.

(c) By Proposition 6.8 (a) some coarsening $\widetilde{\mathcal{O}}$ of $\mathcal{O}_G$ is q-henselian. As K is not q-closed any two q-henselian topologies are dependent and therefore v induces the same topology as $\widetilde{\mathcal{O}}$ and hence as $\mathcal{O}_G$. □

A field with a V-topology is called t-henselian if it is locally equivalent to a field with a topology induced by a henselian valuation. For details see [20]. In particular any field with a topology induced by a henselian valuation is t-henselian. The converse is not true. An example was indicated in [20, Page 338], details are given in [5, Konstruktion 5.3.5].

Theorem 6.19 *Let $(K, \mathcal{T})$ be a t-henselian field. There exists a definable valuation on K which induces the topology $\mathcal{T}$ if and only if K is neither real closed nor separably closed.*

Proof In archimedean ordered real closed fields for every definable set either the set itself or the complement is bounded by a natural number, which cannot be true for a non-trivial valuation ring. As the theory of real closed fields is complete, from this follows already that no real closed field admits a definable valuation. If a field admits a non-trivial definable valuation we can construct a formula with the strong order property. Hence the field is not simple and therefore in particular not separably closed. For more details see [6, 6.58–6.61] and [21, Section 8.2].

If $(K, \mathscr{T})$ is a not real closed and not separably closed t-henselian field, it is locally equivalent, and hence elementary equivalent, to a field $\widetilde{K}$ with a topology induced by a henselian valuation v.

$\widetilde{K}$ is as well not real closed and not separably closed, hence there exists a prime q and a field L such that $L/\widetilde{K}$ is a finite separable extension, $L \neq L\langle q\rangle$, and if $q \neq \text{char}(L)$ then $\zeta_q \in L$. Let w be the unique extension of v to L. w is henselian and hence q-henselian.

If $q = \text{char}(L)$ or $q \neq \text{char}(L)$ and $q \neq 2$ or L is not euclidean, then by Theorem 6.17 there exists a definable valuation $\widetilde{w}$ on L which induces the same topology as w.

Now assume $q = 2 \neq \text{char}(L)$ and L is euclidean. As L is not real closed and euclidean, there exists a polynomial $f \in L[X]$ of odd degree such that f has no roots in L (see, for example, [19, Theorem 1.2.10 (Artin,Schreier)]). Without loss of generality let f be irreducible. Let $x \in L^{\text{sep}}$ be a root of f. We have $[L(x) : L] = \deg(f)$ positive and odd. Hence there exists $\widetilde{q} \neq 2$ prime such that $\widetilde{q} \mid [L(x) : L]$. Now we can find a field extension $\widetilde{L}$ such that we can prove as above for $\widetilde{L}$ and $\widetilde{q}$ that there is a definable valuation $\widetilde{w}$ on $\widetilde{L}$.

By Proposition 5.2 $\widetilde{w}|_K$ is a definable valuation on $\widetilde{K}$. As $\widetilde{w}$ induces the same topology on L as w it is easy to see that $\widetilde{w}|_{\widetilde{K}}$ and $w|_{\widetilde{K}}$ induce the same topology on $\widetilde{K}$.

As K and $\widetilde{K}$ are elementary equivalent, there exists a definable valuation v_0 on K. As $\left(\widetilde{K}, \mathscr{T}_v\right)$ and $(K, \mathscr{T})$ are locally equivalent, follows $\mathscr{T}_{v_0} = \mathscr{T}$. □

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Groups of Automorphisms of Totally Ordered Sets: Techniques, Model Theory and Applications to Decision Problems

A.M.W. Glass

In memory of Rüdiger Göbel, the mathematician and the man.

Abstract This is a survey of topics likely to be of interest to algebraists in general. It has been written accordingly.

Keywords ℓ-permutation groups • Model theory • Word problems • Decision problems

Mathematical Subject Classification (2010): 06F15, 20B07, 05C05, 03C60

1 Introduction and Motivation

Let $(\Omega, \leq)$ be a totally ordered set. Then the group $\mathrm{Aut}(\Omega, \leq)$ has a natural lattice order defined on it given by

$$\alpha(f \vee g) := \max\{\alpha f, \alpha g\}, \quad \alpha(f \wedge g) := \min\{\alpha f, \alpha g\} \quad (\alpha \in \Omega,\ f, g \in \mathrm{Aut}(\Omega, \leq)).$$

Moreover,

$$h(f \vee g)k = hfk \vee hgk \quad \text{and} \quad h(f \wedge g)k = hfk \wedge hgk$$

for all $f, g, h, k \in \mathrm{Aut}(\Omega, \leq)$, making $\mathrm{Aut}(\Omega, \leq)$ into a lattice-ordered group. W. Charles Holland proved an analogue of Cayley's Theorem: Every lattice-ordered group can be embedded (as a group and lattice) in some $\mathrm{Aut}(\Omega, \leq)$. This gives the essential tool for studying the model theory of lattice-ordered groups and constructing lattice-ordered groups with undecidable properties in this richer language. One can show that for each real number $r > 0$, there is a two-generator abelian totally ordered group $D(r)$ that can be embedded (as a group and lattice) in a finitely presented lattice-ordered group if and only if r is a recursive real number. Alternatively, one can study groups with a total order that is preserved by

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multiplication on the right. These, too, can be embedded in some $\mathrm{Aut}(\Omega, \leq)$. The theory of these right-orderable groups mirrors that of combinatorial group theory despite the absence of the amalgamation property. Moreover, one can construct a finitely presented totally orderable group with insoluble word problem. All of this is outlined in the article below which has been written for a general algebraist. After the introductory background, I provide a brief survey of some recent developments in the theory of groups of automorphisms of totally ordered sets. Such a survey is overdue since all books on the topic were written in the last century and the most recent surveys [6] and [9] are over 5 years old. I have chosen topics of a model-theoretic or decision-theoretic character (as appropriate for the Proceedings in which this survey appears), including some results of the last few months.

2 Introductory Examples

The symmetric group on a set Ω, $Sym(\Omega) = \mathrm{Aut}(\Omega, =)$, has been thoroughly studied. A next step is to study $\mathrm{Aut}(\Omega, \leq)$ and its subgroups where $\leq$ is a total order on an infinite set Ω. To help appreciate the context of this survey, I'll begin with some important examples to help focus our thinking.

Example 2.1 Probably the most familiar example of such a subgroup is the affine group A of order-preserving affine maps of a totally ordered field such as the rationals $\mathbb{Q}$ or reals $\mathbb{R}$. That is, all maps: $f_{a,b} : \alpha \mapsto a\alpha + b$ where a, b belong to the field and $a > 0$; i.e., $\alpha f_{a,b} = a\alpha + b$ $(\alpha \in \Omega)$. Now $f_{a,b} = m_a t_b$ where $m_a : \alpha \mapsto a\alpha$ and $t_b : \alpha \mapsto \alpha + b$. Indeed, $m_a^{-1} t_b m_a = t_{ab}$ so the subgroup $T := \{t_b \mid b \in \Omega\}$ is normalised by the subgroup $M := \{m_a \mid a \in \Omega, a > 0\}$. In the special case that $\Omega = \mathbb{R}$, the maps $e_p : \alpha \mapsto \alpha^p$ belong to $\mathrm{Aut}(\mathbb{R}, \leq)$ where p is an odd positive integer and one takes the unique real pth root to get e_p^{-1}. Then the subgroup $P := \langle e_p \mid p = 2q + 1, q \in \mathbb{Z}, q > 0\rangle$ in turn normalises M $(e_p^{-1} m_a e_p = m_{a^p})$. It can be shown that the subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$ generated by T, M, P is the free product of $A = MT$ and MP with M amalgamated ([2] or [15]) thus providing a plethora of explicitly defined free groups of rank 2 (and hence of free subgroups of rank $\aleph_0$) in $\mathrm{Aut}(\mathbb{R}, \leq)$. This generalised the result of White [45] where $b = 1$ and p is a fixed odd prime.

Example 2.2 Each of the elements in the examples above are order-preserving differentiable functions from $\mathbb{R}$ onto $\mathbb{R}$. Indeed, the set of all order-preserving differentiable functions from $\mathbb{R}$ onto $\mathbb{R}$ under composition forms a subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$.

Example 2.3 There is a weak form of the Ehrenfeucht-Mostowski Theorem that is related to the topic. If T is a countable first-order theory having infinite models, then for any infinite totally ordered set $(\Omega, \leq)$, there is a model $\mathscr{M}$ of T such that $\mathrm{Aut}(\Omega, \leq)$ can be embedded in $\mathrm{Aut}(\mathscr{M})$ [12, Sect. 3.3].

Notation *Throughout this survey,* $\leq$ *will always denote a total order on an infinite set* Ω*. I will write* $\mathrm{supp}(g) := \{\alpha \in \Omega \mid \alpha g \neq \alpha\}$ *for the support of* $g \in Aut(\Omega, \leq)$*,* f^g *as shorthand for* $g^{-1}fg$ *(the conjugate of* f *by* g*) and* $[f, g]$ *for* $f^{-1}f^g = f^{-1}g^{-1}fg$*, the commutator of* f *and* g *(*$f, g \in \mathrm{Aut}(\Omega, \leq)$*).*

3 Ordering

Let $f, g \in \mathrm{Aut}(\Omega, \leq)$. Write $f \leq g$ if $\alpha f \leq \alpha g$ for all $\alpha \in \Omega$. This is the pointwise ordering on $\mathrm{Aut}(\Omega, \leq)$. This partial ordering is actually a lattice with least upper bound and greatest lower bound given by

$$\alpha(f_1 \vee f_2) := \max\{\alpha f_1, \alpha f_2\} \quad \text{and} \quad \alpha(f_1 \wedge f_2) := \min\{\alpha f_1, \alpha f_2\} \quad (\alpha \in \Omega).$$

Furthermore, for all $f, g, h, k \in \mathrm{Aut}(\Omega, \leq)$, we have

$$f(g \vee h)k = fgk \vee fhk \quad \text{and} \quad f(g \wedge h)k = fgk \wedge fhk;$$

that is, $\mathrm{Aut}(\Omega, \leq)$ is a *lattice-ordered group* or ℓ-group for short. If G is a subgroup of $\mathrm{Aut}(\Omega, \leq)$ and $g_1 \vee g_2, g_1 \wedge g_2 \in G$ for all $g_1, g_2 \in G$, then G is called a *sublattice subgroup* of $\mathrm{Aut}(\Omega, \leq)$, or ℓ*-subgroup* for short, and (G, Ω) is called an ℓ*-permutation group*. An ℓ-group in which the order is total is called an o-*group*. For example, the subgroup T of $\mathrm{Aut}(\Omega, \leq)$ from Example 2.1 is an ℓ-subgroup that is an o-group. An *orderable group* is a group that can be totally ordered to become an o-group.

In Example 2.1 with $\Omega = \mathbb{R}$, the affine group A is not a sublattice subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$ as $m_2 \vee 1 \notin A$ (where 1 denotes the identity permutation) since $m_2 \vee 1$ is not even differentiable at 0. The ℓ-subgroup L of $\mathrm{Aut}(\mathbb{R}, \leq)$ generated by A comprises all (finite) piecewise linear permutations of $\mathbb{R}$ that are order-preserving. Consider the ℓ-subgroup C of elements $g \in L$ of bounded support; that is, $(\exists \beta, \gamma \in \mathbb{R})\, \mathrm{supp}(g) \subseteq (\beta, \gamma)$. Then $(C, \mathbb{R})$ is a simple group [13]. Indeed, this example is an o-group under the order $f <_r g$ if the rightmost non-identity slope of gf^{-1} exceeds 1, but it is not an o-group under the inherited pointwise ordering from $\mathrm{Aut}(\mathbb{R}, \leq)$.

Example 2.2 is not an ℓ-subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$.

There is another way to partially order $\mathrm{Aut}(\Omega, \leq)$. Let $\preceq$ be any well order on Ω and define $f \prec g$ if $\alpha_0 f < \alpha_0 g$ where α_0 is the least element of $\mathrm{supp}(gf^{-1})$ under $\preceq$ (distinct $f, g \in \mathrm{Aut}(\Omega, \leq)$). This is a total order on $\mathrm{Aut}(\Omega, \leq)$ with $fh \preceq gh$ for all $f, g, h \in \mathrm{Aut}(\Omega, \leq)$ satisfying $f \preceq g$. Such a total order is called a *right order* on $\mathrm{Aut}(\Omega, \leq)$. Every subgroup of $\mathrm{Aut}(\Omega, \leq)$ inherits this right order. Conversely, if $(G, \preceq)$ is any right-ordered group, then $(G, \preceq)$ can be embedded in $\mathrm{Aut}(G, \preceq)$ via Cayley's right regular representation of G.

A deeper result that I will use is due to Holland [29].

Theorem 3.1 *Every lattice-ordered group can be ℓ-embedded (as a group and lattice) in* $\mathrm{Aut}(\Omega, \leq)$ *for some infinite totally ordered set* $(\Omega, \leq)$.

By Holland's Theorem, every ℓ-group is right orderable. The converse is false: $\langle x, y \mid x^y = x^{-1} \rangle$ is right orderable but cannot be made into an ℓ-group. However, every right-orderable group $(G, \preceq)$ can be embedded in a group, $\mathrm{Aut}(G, \preceq)$, which can be made into an ℓ-group.

4 Multiple Transitivity

Definition 4.1 Let G be a subgroup of $\mathrm{Aut}(\Omega, \leq)$ and m be a positive integer. (G, Ω) is said to be o-*m transitive* if for every $\alpha_1 < \cdots < \alpha_m$ and $\beta_1 < \cdots < \beta_m$ there is $g \in G$ such that $\alpha_i g = \beta_i$ $(i = 1, \ldots, m)$.

Since any line is determined by two points lying on it, the affine group in Example 2.1 is sharply o-2 transitive; that is, for all $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, there is a *unique* $g \in A$ such that $\alpha_i g = \beta_i$ $(i = 1, 2)$; it is not o-3 transitive. In contrast, $\mathrm{Aut}(\mathbb{R}, \leq)$ is o-m transitive for all m belonging to $\mathbb{Z}_+$, the set of all strictly positive integers. Indeed, matters are propitious if one considers ℓ-permutation groups.

Lemma 4.2 *Let* (G, Ω) *be an o-2 transitive ℓ-permutation group. Then* (G, Ω) *is o-m transitive for all* $m \in \mathbb{Z}_+$.

Proof By induction. Assume that (G, Ω) has been shown to be o-k transitive. Let $\alpha_1 < \cdots < \alpha_{k+1}$ and $\beta_1 < \cdots < \beta_{k+1}$ in Ω. By hypothesis, there is $f \in G$ with $\alpha_i f = \beta_i$ $(i = 1, \ldots, k)$. If $\alpha_{k+1} f > \beta_{k+1}$, by o-2 transitivity there is $g \in G$ such that $\alpha_1 g = \beta_k$ and $\alpha_{k+1} g = \beta_{k+1}$. Then $\alpha_i (f \wedge g) = \beta_i$ $(i = 1, \ldots, k+1)$. On the other hand, if $\alpha_{k+1} f \leq \beta_{k+1}$, there is $g \in G$ such that $\alpha_k g = \beta_1$ and $\alpha_{k+1} g = \beta_{k+1}$. Now $\alpha_i (f \vee g) = \beta_i$ $(i = 1, \ldots, k+1)$ and the lemma is proved. □

We can apply Lemma 4.2 to o-2 transitive $\mathrm{Aut}(\Omega, \leq)$ to recognise the lattice (to within duality) in the language of groups ([36] or [31]).

Lemma 4.3 *Let* $\mathrm{Aut}(\Omega, \leq)$ *be o-2 transitive and* $1 \neq p \in \mathrm{Aut}(\Omega, \leq)$. *Then* $p > 1$ *or* $p < 1$ *if and only if*

$$(\exists f, g \neq 1)(\forall h)([f, p^{-h} g p^h] = 1).$$

Proof If $p > 1$ and $\alpha \in \mathrm{supp}(p)$, let $f, g \in \mathrm{Aut}(\Omega, \leq)$ each have a single bounded interval of support with $\mathrm{supp}(f) \subseteq (\alpha, \alpha p)$ and $\mathrm{supp}(g) \subseteq (\alpha p, \alpha p^2)$. Then $\mathrm{supp}(f) < \mathrm{supp}(g)$. For all $h \in \mathrm{Aut}(\Omega, \leq)$ we have $p^h > 1$; so $\mathrm{supp}(f) < \alpha p < \mathrm{supp}(p^{-h} g p^h)$ and the condition holds. Similarly, the condition holds if $p < 1$. But if $p \not< 1$, let $\alpha \in \Omega$ with $\alpha < \alpha p$. If $f, g \in \mathrm{Aut}(\Omega, \leq)$ and there are $\beta \in \mathrm{supp}(f)$ and $\gamma \in \mathrm{supp}(g)$ with $\beta f > \beta > \gamma g > \gamma$, then we can conjugate p (by h, say) so that $\gamma g p^h > \gamma g > \beta$ and $[f, p^{-h} g p^h] \neq 1$. So there are no f, g with $\mathrm{supp}(f)$ containing points exceeding elements of $\mathrm{supp}(g)$ and satisfying the condition. Similarly, if $p \not> 1$, the only possibilities for f, g satisfying the condition are for $\mathrm{supp}(f) > \mathrm{supp}(g)$ and $p < 1$. □

For any ℓ-group G, let $G_+ = \{g \in G \mid g > 1\}$, the set of strictly positive elements of G. So the lemma gives a sentence that determines G_+ or G_+^{-1}. The result is enough to deduce that the lattice operation (or its dual) can be determined from the group (see *op. cit.* or [18, Sect. 1.10]).

There is an ℓ-embedding (preserves both the group and lattice operations) of the free lattice-ordered group on $\aleph_0$ generators into $\mathrm{Aut}(\mathbb{Q}, \leq)$ such that the image is o-2 transitive (and so o-m transitive for all $m \in \mathbb{Z}_+$). The same is true for the free lattice-ordered group on n generators for any $n \in \mathbb{Z}$ with $n \geq 2$ [18, Theorem 6.7], and the ℓ-group free product of a non-empty finite set of non-trivial countable ℓ-groups [20, Theorem 8.F].

5 Conjugacy

I'll begin with a simple example of conjugacy. Call $g \in \mathrm{Aut}(\Omega, \leq)_+$ a *bump* if for all $\alpha, \beta \in \mathrm{supp}(g)$, there is $m \in \mathbb{Z}_+$ such that $\alpha < \beta g^m$; that is, g has a single interval of support.

Lemma 5.1 *If $f, g \in \mathrm{Aut}(\mathbb{R}, \leq)_+$ are bumps of bounded support, then f and g are conjugate in* $\mathrm{Aut}(\mathbb{R}, \leq)$.

Proof Let $\alpha \in \mathrm{supp}(f)$ and $\beta \in \mathrm{supp}(g)$. Let $h_0 : [\alpha, \alpha f] \longrightarrow [\beta, \beta g]$ be an order-preserving bijection. For each $n \in \mathbb{Z}$, let $h_n = f^{-n} h_0 g^n$. It is an order-preserving bijection between $[\alpha f^n, \alpha f^{n+1}]$ and $[\beta g^n, \beta g^{n+1}]$. Then $h^* := \bigcup\{h_n \mid n \in \mathbb{Z}\}$ is an order-preserving bijection from $\mathrm{supp}(f)$ to $\mathrm{supp}(g)$. If h is any order-bijection from $\mathbb{R}$ to all of $\mathbb{R}$ extending h^*, then $h \in \mathrm{Aut}(\mathbb{R}, \leq)$ and $f^h = g$ by construction. □

Remark 5.2 Note that h_0 was an arbitrary bijection between $[\alpha, \alpha f]$ and $[\beta, \beta g]$. This will be used later.

Remark 5.3 If I replace $\mathbb{R}$ by $\mathbb{Q}$, the conclusion fails for if $f, g \in \mathrm{Aut}(\mathbb{Q}, \leq)$ are bumps with $\mathrm{supp}(f) = (0, 1)$ and $\mathrm{supp}(g) = (0, \sqrt{2})$, then $f^h = g$ implies that $1h = \sqrt{2}$ which is impossible if $h \in \mathrm{Aut}(\mathbb{Q}, \leq)$.

The idea of the proof of Lemma 5.1 is very important and gives much stronger results. I'll begin with two, the first of which allows "simultaneous conjugacy" and is used to obtain undecidability results [23] (also see the next section).

Lemma 5.4 *Let $\beta, \gamma, \delta \in \mathbb{R}$ and $a, b, c, d \in$* $\mathrm{Aut}(\mathbb{R}, \leq)$ *be bumps with $\beta < \mathrm{supp}(a) < \beta b$, $\gamma < \mathrm{supp}(b) < \gamma c$ and $\delta < \mathrm{supp}(c) < \delta d$. Then there is* $h \in \mathrm{Aut}(\mathbb{R}, \leq)$ *such that $a^h = a,\ b^h = c$ and $d^h = d$.*

Proof Note that $\delta < \gamma < \beta < \beta b < \gamma c < \delta d$ since $\beta \in \mathrm{supp}(b),\ \gamma \in \mathrm{supp}(c),\ \delta \in \mathrm{supp}(d)$. Let h_0 be any order-preserving bijection between $[\beta, \beta b]$ and $[\gamma, \gamma c]$ that is the identity on $\mathrm{supp}(a)$. As in the proof of Lemma 5.1, one can define $\{h_n \mid n \in \mathbb{Z}\}$ so that their union h^* is an order-preserving bijection between $\mathrm{supp}(b)$ and $\mathrm{supp}(c)$ with $a^{h^*} = a$ and $b^{h^*} = c$. Extend h^* to an order-preserving permutation $\bar{h}_0$ of $[\delta, \delta d]$ and extend $\bar{h}_0$ to an element $h \in \mathrm{Aut}(\mathbb{R}, \leq)$ as in the proof of Lemma 5.1 with $f = g = d$. By construction, $a^h = a,\ b^h = c$ and $d^h = d$. □

Corollary 5.5 *If $G = \mathrm{Aut}(\Omega, \leq)$ is o-2 transitive, then any element of G is a commutator and G is divisible.*

Proof Since $\mathrm{supp}(f^2) = \mathrm{supp}(f)$, the proof of Lemma 5.1 applies, interval by interval, with f^2 in place of g. In this case, each end point of a bump of f is an end point of the corresponding bump of f^2 and conversely. So the definitions of the $\{h_n \mid n \in \mathbb{Z}\}$ on each interval of support can be sewn together to get $h \in G$ with $f^h = f^2$. That is, $f = [f, h]$. Similarly, if $m \in \mathbb{Z}_+$ I can use f^m in place of g to get $h \in G$ with $f^h = f^m$. Then $f^{h^{-1}}$ is an mth root of f. □

If $f, g \in \mathrm{Aut}(\Omega, \leq)_+$ with f a bump and $f \wedge gf^{-1} = 1$, then f is a bump of g; i.e., g agrees with f on $\mathrm{supp}(f)$. So g is not itself a bump unless $f = g$. Conversely, $g \in \mathrm{Aut}(\Omega, \leq)_+$ is a bump if

$$(\forall f \geq 1)((f \wedge gf^{-1} = 1) \longrightarrow (f = 1 \text{ or } f = g)).$$

Thus one can recognise a "bump" in $\mathrm{Aut}(\Omega, \leq)$ in the first-order language of ℓ-groups. Write *bump*(g) for this formula. Therefore one cannot conjugate a bump in $\mathrm{Aut}(\Omega, \leq)$ to a strictly positive element of $\mathrm{Aut}(\Omega, \leq)$ that has more than one bump. Note that any conjugate of a strictly positive (strictly negative) element of an ℓ-group is strictly positive (strictly negative), and any conjugate of an element incomparable to the identity is also incomparable to the identity. By a tour-de-force permutation proof using a transfinite sequence of extensions of the orbit Wreath product, Pierce [38] proved the following result (see also [18, pp. 193–205]) where I write ℓ-embedding for a group and lattice embedding.

Theorem 5.6 *Every ℓ-group can be ℓ-embedded in one in which any two strictly positive elements are conjugate (and so any two strictly negative elements are conjugate).*

The method of proof was finally extended to include the set of elements unrelated to the identity (see [4]).

Theorem 5.7 *Every ℓ-group can be ℓ-embedded in one with exactly* 4 *conjugacy classes.*

The proofs give two important corollaries.

Corollary 5.8 *Let $\langle h_i \rangle$ be a cyclic subgroup of a lattice-ordered group G_i ($i = 1, 2$). Then there is a lattice-ordered group L and ℓ-embeddings $\tau_i : G_i \longrightarrow L$ ($i = 1, 2$) such that $h_1\tau_1 = h_2\tau_2$ provided that h_1, h_2 are both strictly positive, both strictly negative or both unrelated to the identity.*

Corollary 5.9 *Every right-orderable group can be embedded in one in which all non-identity elements are conjugate.*

Note that if $\mathrm{Aut}(\Omega, \leq)$ is o-2 transitive and $f, g > 1$, then $\mathrm{supp}(f) < \mathrm{supp}(g)$ if and only if

$$f \mathscr{L} g := (f > 1 \;\&\; g > 1) \;\&\; (\forall h > 1)(f \wedge g^h = 1)$$

(cf. Lemma 4.3). Moreover, if $\bar{\alpha} < \bar{\beta}$ in $\bar{\Omega}$, the Dedekind completion of Ω, then there is $g \in \mathrm{Aut}(\Omega, \leq)$ such that $\bar{\alpha} = \inf(\mathrm{supp}(g)) < \sup(\mathrm{supp}(g)) = \bar{\beta}$. If we let $Adj(f, g)$ be the formula

$$f \mathscr{L} g \ \& \ \neg(\exists k > 1)(f \mathscr{L} k \ \& \ k \mathscr{L} g),$$

then $(\Omega, \leq)$ is Dedekind complete if and only if

$$(\forall g \neq 1)((bump(g) \ \& \ (\exists f > 1)(g^f \wedge g = 1)) \longrightarrow (\exists h > 1)Adj(g, g^h));$$

it has countable coterminality if and only if there is a bump $g \in \mathrm{Aut}(\Omega, \leq)$ with $(\forall f)(f \wedge g = 1 \longrightarrow f = 1)$. Using these facts one can obtain the following results in [27].

Theorem 5.10 *Let* $\mathrm{Aut}(\Omega, \leq)$ *be o-2 transitive satisfying the same group-theoretic sentences as* $\mathrm{Aut}(\mathbb{R}, \leq)$. *Then* $(\Omega, \leq) \cong (\mathbb{R}, \leq)$ *as ordered sets.*

Theorem 5.11 *Let* $\mathrm{Aut}(\Omega, \leq)$ *be o-2 transitive satisfying the same group-theoretic sentences as* $\mathrm{Aut}(\mathbb{Q}, \leq)$. *Then* $(\Omega, \leq) \cong (\mathbb{Q}, \leq)$ *or* $(\Omega, \leq) \cong (\mathbb{R} \setminus \mathbb{Q}, \leq)$ *as ordered sets.*

For an analogue for quotients of $\mathrm{Aut}(\mathbb{R}, \leq)$, etc., see [17].

Droste [16] has shown that there are continuum many doubly homogeneous chains $(\Omega, \leqslant)$ whose automorphism groups $\mathrm{Aut}(\Omega, \leq)$ are pairwise elementarily inequivalent as groups (and also as lattices with identity element). This, the maximum possible, shows the complexity of these studies.

For further attractive aspects of conjugacy, see [42].

6 Applications to Decision Problems for Lattice-Ordered Groups

In this section I'll provide a brief summary of some results on decision problems whose proofs rely on the ideas of the previous section; a fuller account can be found in the cited papers.

Groups with undecidable word problem are usually constructed using the amalgamation property and its equivalent, the Higman-Neumann-Neumann construction (that uses conjugation). Unfortunately, the class of lattice-ordered groups fails even a weak form of the amalgamation property [38] or [18, Theorem 10.C]:

Proposition 6.1 *There are* ℓ*-groups* C, G_1, G_2 *with* ℓ*-embeddings* $\sigma_i : C \longrightarrow G_i$ $(i = 1, 2)$ *such that there is no* ℓ*-group* L *with* ℓ*-embeddings* $\tau_i : G_i \longrightarrow L$ $(i = 1, 2)$ *such that* $c\sigma_1\tau_1 = c\sigma_2\tau_2$ *for all* $c \in C$.

So one cannot use the tricks from combinatorial group theory to obtain decidability and undecidability results for ℓ-groups. What is necessary is to write down a finite set of generators and defining relations of an ℓ-group, i.e., *a finite presentation*, and prove that certain relations are consequences of these relations and others, critically, are not. The relations in any presentation are equalities between ℓ-group words.

We use the idea in Lemma 5.4. We prove that for each recursive function $f : \mathbb{N} \longrightarrow \mathbb{N}$ one can construct a *finitely* presented ℓ-group $G(f)$ with elements a, b, c, d, a_f among the generators of $G(f)$ such that, *inter alia*, the infinite set of relations

$$a^{a_f} = a, \quad a^{c^m a_f} = a^{b^{f(m)} c^m}, \quad d^{a_f} = d \quad (m \in \mathbb{N}) \quad (**)$$

hold in $G(f)$. This is done step by step on the way that f is formed beginning with a basic finitely presented ℓ-group G. If $f(m) = 0$ for all $m \in \mathbb{N}$, these relations are trivially satisfied if a_f is the identity of G. For f the successor function the construction is somewhat more complicated. If f is the composition of g and h and $G(g)$ and $G(h)$ have already been constructed, then an appropriate $G(f)$ can be constructed using a_f and the generators of $G(g)$ and $G(h)$ with a finite set of extra relations that ensure that $a^{c^m a_f} = a^{b^{h(g(m))} c^m}$. A more complicated finite set of generators and relations is needed to construct $G(f)$ if finitely presented ℓ-groups $G(g), G(h), G(u), G(v)$ have already been constructed and f is obtained from g, h, u, v by "Julia Robinson" Induction (see [23]). In this way one can prove that for each recursive function $f : \mathbb{N} \longrightarrow \mathbb{N}$ there is a finitely presented ℓ-group $G(f)$ satisfying the relations $(**)$.

Now let $X \subseteq \mathbb{N}$ be a recursively enumerable set that is not recursive and f be any recursive function with image X. Let h be the characteristic function of $\mathbb{N} \setminus X$; that is,

$$h(m) = 0 \text{ if } m \in X \quad \text{and} \quad h(m) = 1 \text{ if } m \in \mathbb{N} \setminus X.$$

Caution: h is not recursive.

We next write down the analogous generators and equations for the composition of f and h that were used above for the composition of recursive functions requiring that the composition is the zero function. This ℓ-group L is finitely presented and can be shown to have the property that

$$a^{c^m a_h} = a^{c^m} \quad \text{for all } m \in X.$$

Since h is not recursive, for all we know at this stage it would be quite possible for $a^{c^m a_h} = a^{c^m}$ for all $m \in \mathbb{N}$.

For groups, this obstacle was overcome using spelling in HNN-extensions to obtain non-equality if $m \in \mathbb{N} \setminus X$ (see, e.g., [11]) but this is invalid for ℓ-groups in general by Proposition 6.1. So instead one constructs elements of $\mathrm{Aut}(\mathbb{R}, \leq)$ which interpret each generator of G and of each $G(f)$ (f recursive) so that all the defining relations of these ℓ-groups hold with this interpretation in $\mathrm{Aut}(\mathbb{R}, \leq)$ with $a^{c^m a_f} = a^{b^{f(m)} c^m}$ for all $m \in \mathbb{N}$. Indeed, one can write down an interpretation of each *function*

$f : \mathbb{N} \longrightarrow \mathbb{N}$ so that these equations hold in $\mathrm{Aut}(\mathbb{R}, \leq)$ for all $m \in \mathbb{N}$. This ensures that there is an ℓ-homomorphism of L into $\mathrm{Aut}(\mathbb{R}, \leq)$ and that if $m \in \mathbb{N} \setminus X$, then $a^{c^m a_h} = a^{bc^m} \neq a^{c^m}$ in $\mathrm{Aut}(\mathbb{R}, \leq)$. Thus $a^{c^m a_h} \neq a^{c^m}$ in the pre-image L if $m \in \mathbb{N} \setminus X$. Consequently, the finitely presented ℓ-group L has insoluble word problem.

Theorem 6.2 *There is a finitely presented lattice-ordered group G with insoluble group word problem. Indeed, such a lattice-ordered group G can be found with two generators and a single defining relation.*

The above proof is loosely modelled on a permutation proof of a corresponding theorem for groups [37] but has interesting necessary differences.

In 1961, Graham Higman proved the remarkable result that every finitely generated group that can be defined by a recursively enumerable set of relations can be embedded in a finitely presented group, that is in a finitely generated group with a finite set of defining relations [28]. Proofs rely heavily on the amalgamation property for groups and spellings in Higman-Neumann-Neumann groups. No such method is available for ℓ-groups. The analogue for ℓ-groups was first obtained for ℓ-embedding abelian finitely generated recursively enumerably defined ℓ-groups in finitely presented ℓ-groups in [24] using continued fractions and techniques from [23].

Theorem 6.3 *Every finitely generated abelian lattice-ordered group that can be defined by a recursively enumerable set of relations can be ℓ-embedded in a finitely presented lattice-ordered group.*

This result is enough to establish a classification of recursive real numbers. For any real number $r \in \mathbb{R}_+$, let $r = r_0.r_1r_2r_3\ldots$ where $r_0 \in \mathbb{Z}_+ \cup \{0\}$, each $r_n \in \{0, \ldots, 9\}$ $(n \in \mathbb{Z}_+)$ and the expression for r does not end in all 9s. Let $\varrho_n := r_0r_1\ldots r_n$ and $D(r)$ be the abelian o-group on two generators x, y with the recursive set of defining relations

$$\varrho_n x \leq 10^n y < d_n x \quad (n \in \mathbb{Z}_+),$$

where $d_n := \varrho_n + 1$. So, by Theorem 6.3,

Corollary 6.4 *$D(r)$ can be ℓ-embedded in a finitely presented lattice-ordered group if and only if r is a recursive real number.*

That is, the recursive real numbers are precisely those that are algebraic in the language of lattice-ordered groups (using the group and lattice operations); i.e., the recursive real numbers are those that can be recovered from finitely many sentences of the (not necessarily abelian) language of lattice-ordered groups.

The full analogue of Higman's Embedding Theorem actually holds for ℓ-groups [21]. As with the undecidability of the word problem, the proof uses elements of the ℓ-permutation group $\mathrm{Aut}(\mathbb{R}, \leq)$ as witnesses of non-equalities.

Theorem 6.5 *Every finitely generated lattice-ordered group that can be defined by a recursively enumerable set of relations can be ℓ-embedded in a finitely presented lattice-ordered group.*

Using the Higman Embedding Theorem, Boone and Higman [10] gave a beautiful connection between the logic and algebra of the situation. They proved that a finitely generated group has the logical property of having soluble word problem if and only if it can be embedded in a simple group that can be embedded in a finitely presented group (a purely algebraic property). A strengthening of the result was proved by Richard Thompson [41] using permutation groups. The natural analogue of [10] holds for ℓ-groups using ideas in [23] and [21]. The proof is also intricate—see [22].

Theorem 6.6 *A finitely generated lattice-ordered group has soluble word problem if and only if it can be ℓ-embedded in an ℓ-simple lattice-ordered group that can be ℓ-embedded in a finitely presented lattice-ordered group.*

We can further deduce that many problems about finitely presented ℓ-groups are undecidable using ℓ-permutation groups [19]. The key is the following full analogue of Rabin's Lemma [39] (which he proved for groups using Higman-Neumann-Neumann extensions).

Lemma 6.7 *Let G be a finitely presented ℓ-group. Then for each ℓ-group word w in the generators of G, there is a uniform explicit construction of a finitely presented ℓ-group $G(w)$ such that*

(1) $G(w) = \{1\}$ *if* $w = 1$ *in* G; *and*
(2) G *can be ℓ-embedded in* $G(w)$ *if* $w \neq 1$ *in* G.

We can apply the lemma to prove that many problems are undecidable. For example, the triviality problem, the abelian problem, the isomorphism problem: If we could determine if an arbitrary finitely presented ℓ-group were trivial, then taking G to be a finitely presented ℓ-group with insoluble word problem and w a word in G, we would have $G(w)$ is trivial if and only if $w = 1$ in G. Since there is no algorithm to determine whether or not an arbitrary word w in the generators of G is the identity in G (Theorem 6.2), the undecidability of the triviality problem, abelian problem (G is abelian) and isomorphism problem (to the trivial ℓ-group) all follow.

The natural analogue to the group theory result on conjugacy in finitely presented ℓ-groups remains open.

Question 6.8 Is there a finitely presented lattice-ordered group with soluble word problem but insoluble conjugacy problem?

7 Structure Theory

In this section, I provide the facts that will be needed to understand the model theory of ℓ-permutation groups. Throughout it, (G, Ω) will be a transitive ℓ-permutation group.

A *convex congruence* $\mathscr{C}$ on Ω is a G-congruence with all $\mathscr{C}$-classes convex; these classes are called *o-blocks*. By transitivity, each o-block Δ is a class of a unique convex congruence (the classes are $\{\Delta g \mid g \in G\}$). Denote this convex congruence by $\kappa(\Delta)$. Of course, for $\mathrm{Aut}(\mathbb{R}, \leq)$ or $\mathrm{Aut}(\mathbb{Q}, \leq)$ the only convex congruences are the universal and trivial convex congruences.

The set of convex congruences $\mathfrak{K}^*$ of (G, Ω) is totally ordered by inclusion [18, Theorem 3.A]. So if Δ_1, Δ_2 are o-blocks of possibly different convex congruences and $\Delta_1 \cap \Delta_2 \neq \emptyset$ then $\Delta_1 \subseteq \Delta_2$ or $\Delta_2 \subseteq \Delta_1$. If $\mathscr{C}$ and $\mathscr{D}$ are convex congruences, $\mathscr{C} \subset \mathscr{D}$ and there is no convex congruence strictly between $\mathscr{C}$ and $\mathscr{D}$, then we say that $\mathscr{D}$ *covers* $\mathscr{C}$ and $\mathscr{C}$ is *covered* by $\mathscr{D}$.

Let $\mathfrak{K} := \{K \in \mathfrak{K}^* \mid K \text{ covers some } K' \in \mathfrak{K}^*\}$.

To get some insight into the definitions, consider the following example.

Example 7.1 Let $\mathbb{Z}_-$ be the set of negative integers and for each $n \in \mathbb{Z}_-$, let $(R_n, \leq_n)$ be a copy of $(\mathbb{R}, \leq)$. Let $\Omega := \prod_{n=-1}^{-\infty} R_n$. Put a total order on Ω as follows. Let $\alpha, \beta \in \Omega$ be distinct. Define $\alpha < \beta$ if $\alpha_m <_m \beta_m$ where $m := \max\{n \in \mathbb{Z}_- \mid \alpha_n \neq \beta_n\}$. Then $\mathrm{Aut}(\Omega, \leq)$ is transitive and has convex congruences $\mathfrak{K} := \{K_n \mid n \in \mathbb{Z}_-\}$ where $\alpha\, K_n\, \beta$ if $\alpha_m = \beta_m$ for all $m > n$. Note that $\gamma K_{n-1} = \{\delta \in \Omega \mid \gamma_m = \delta_m \text{ for all } m \geq n\}$ for all $\gamma \in \Omega$ and $\mathfrak{K} = \mathfrak{K}^* \setminus \mathscr{E}$, where $\mathscr{E}$ is the trivial convex congruence.

In general, $\mathfrak{K}^*$ is generated by $\mathfrak{K}$ in the following sense.

Proposition 7.2 ([18, Theorem 3D]) *For any transitive ℓ-permutation group (G, Ω), the following holds. Every convex congruence of (G, Ω) other than the trivial convex congruence $\mathscr{E}$ is the union of all covering convex congruences of (G, Ω) that are contained in it, and every convex congruence of (G, Ω) other than the universal convex congruence $\mathscr{U}$ is also the intersection of all covered convex congruences of (G, Ω) containing it.*

In Example 7.1, the universal congruence $\mathscr{U}$ equals $K_{-1} \in \mathfrak{K}$ but $\mathscr{E} \in \mathfrak{K}^* \setminus \mathfrak{K}$. Let $\lambda_n = 5$ if $n \geq -10$ and $\lambda_n = 1$ otherwise, and $\mu_n = 5$ if $n \geq -7$ and $\mu_n = 0$ otherwise. Then $\lambda\, K_{-8}\, \mu$ but λ and μ belong to distinct o-blocks of K_n if $n < -8$.

Let $\alpha, \beta \in \Omega$ be distinct. Then both the union $U(\alpha, \beta)$ of all convex congruences $\mathscr{C}$ for which α, β lie in distinct o-blocks and the intersection $V(\alpha, \beta)$ of all convex congruences $\mathscr{C}$ for which α, β lie in the same o-block are convex congruences. So $U(\alpha, \beta)$ is covered by $V(\alpha, \beta)$. (In the above example, $V(\lambda, \mu) = K_{-8}$ and $U(\lambda, \mu) = K_{-9}$.) Hence

$$\mathfrak{K} = \{V(\alpha, \beta) \mid \alpha, \beta \in \Omega, \alpha \neq \beta\} \subseteq \mathfrak{K}^*.$$

Now $\mathfrak{K}$ inherits the total order (inclusion) from $\mathfrak{K}^*$ and is called the *spine* of (G, Ω). For all $\alpha, \beta \in \Omega$ we have $\beta = \alpha g$ for some $g \in G$ by transitivity. Therefore $\mathfrak{K}$ can also be described as follows. Fix $\alpha \in \Omega$. Then

$$\mathfrak{K} = \{V(\alpha, \alpha g) \mid g \in G, \alpha g \neq \alpha\}.$$

Write T for the set of o-blocks of elements of $\mathfrak{K}$. If $\Delta \in T$, then $\kappa(\Delta) \in \mathfrak{K}$; so κ restricts to a surjective map from T to $\mathfrak{K}$. For each $\mathscr{C} \in \mathfrak{K}$, write $\pi(\mathscr{C})$ for both the convex congruence covered by $\mathscr{C}$ and its set of o-blocks; the latter inherits a total order from Ω. If Δ is a $\mathscr{C}$-class, let $\pi(\Delta)$ be the set of all $\pi(\mathscr{C})$-classes contained in Δ. In the above example, $\mu K_{-3} = \{\alpha \in \Omega \mid \alpha_{-1} = \alpha_{-2} = 5\}$ and $\pi(\mu K_{-3}) = \{\nu K_{-4} \mid \nu_{-1} = \nu_{-2} = 5\}$.

Define the stabiliser $\mathrm{st}(\Delta)$ and rigid stabiliser $\mathrm{rst}(\Delta)$ of an o-block Δ of a convex congruence by:

$$\mathrm{st}(\Delta) := \{g \in G \mid \Delta g = \Delta\}, \quad \text{and} \quad \mathrm{rst}(\Delta) := \{g \in G \mid \mathrm{supp}(g) \subseteq \Delta\}.$$

So $\mathrm{st}(\Delta)$ and $\mathrm{rst}(\Delta)$ are convex ℓ-subgroups of G and $\mathrm{rst}(\Delta) \subseteq \mathrm{st}(\Delta)$.

Let $\Delta \in T$. Each $g \in \mathrm{st}(\Delta)$ induces an action g_Δ on the ordered set $\pi(\Delta)$ given by

$$\Gamma g_\Delta = \Gamma g \quad (\Gamma \in \pi(\Delta),\ g \in \mathrm{st}(\Delta)).$$

Let

$$G(\Delta) := \{g_\Delta \mid g \in \mathrm{st}(\Delta)\}.$$

In our example, $G(\mu K_{-3}) \cong \mathrm{Aut}(R_4, \leq_4) \cong \mathrm{Aut}(\mathbb{R}, \leq)$.

Note that $(G(\Delta), \pi(\Delta))$ is transitive and o-primitive. Furthermore, if $K \in \mathfrak{K}$ and Δ, Δ' are both K-classes, then $(G(\Delta), \pi(\Delta))$ and $(G(\Delta'), \pi(\Delta'))$ are isomorphic, the isomorphism being induced by conjugation by any $f \in G$ with $\Delta f = \Delta'$ since $(\Gamma f)(f^{-1} g f) = (\Gamma g) f$ for all $g \in \mathrm{rst}(\Delta)$, $\Gamma \in \pi(\Delta)$. It is customary to write (G_K, Ω_K) for any of these ℓ-permutation groups; they are independent of Δ to within ℓ-permutation isomorphism.

Let $g \in G$ and Λ be a union of g-invariant convex subsets of Ω. Write $\mathrm{dep}(g, \Lambda)$ for the element of $\mathrm{Aut}(\Omega, \leqslant)$ that agrees with g on Λ and with the identity elsewhere; thus

$$\alpha\, \mathrm{dep}(g, \Lambda) = \begin{cases} \alpha g & \text{if } \alpha \in \Lambda \\ \alpha & \text{if } \alpha \notin \Lambda. \end{cases}$$

Of course, in general the automorphism $\mathrm{dep}(g, \Lambda)$ need not belong to G. Now (G, Ω) is said to be *depressible to covering o-blocks* or *abundant* if for each $\Delta \in T$, and each $g \in \mathrm{st}(\Delta)$, we have $\mathrm{dep}(g, \Delta) \in G$. So if (G, Ω) is abundant and $\Delta \in T$, then for each $g \in \mathrm{st}(\Delta)$, the elements g and $\mathrm{dep}(g, \Delta) \in \mathrm{rst}(\Delta)$ induce the same action on $\pi(\Delta)$: $\Gamma \mathrm{dep}(g, \Delta)_\Delta = \Gamma g_\Delta$ for all $\Gamma \in \pi(\Delta)$. Clearly, $(\mathrm{Aut}(\Omega, \leqslant), \Omega)$ is abundant.

It is easy to see that

Lemma 7.3 *Suppose that* (G, Ω) *is abundant. Let* $\Delta \in T$ *and* $\Gamma \in \pi(\Delta)$. *If* Δ *is not a minimal element of* T, *then* $\mathrm{rst}(\Gamma) \neq 1$.

Let (G, Ω) be a transitive ℓ-permutation group, $g \in G$ and $\alpha \in \mathrm{supp}(g)$. I will denote the supporting interval of g containing α by $\Lambda(\alpha, g) := \{\beta \in \Omega \mid (\exists m, n \in \mathbb{Z})(\alpha g^n < \beta < \alpha g^m)\}$. Then (G, Ω) is said to be *depressible* if $dep(g, \Lambda(\alpha, g)) \in G$ for all $g \in G$ and $\alpha \in \mathrm{supp}(g)$ (c.f. bump in Sect. 4).

McCleary provided a classification of all o-primitive transitive ℓ-permutation groups (McCleary's Trichotomy Theorem, [35], or [18, Theorem 4A] or [20, Theorem 7E]). In the special case that the transitive o-primitive ℓ-permutation group is depressible, this reduces to a dichotomy.

Proposition 7.4 (S.H. McCleary) *Let* (G, Ω) *be a transitive depressible* ℓ*-permutation group. Then* (G, Ω) *is* o*-primitive if and only if*

(i) $(\Omega, \leqslant)$ *is order-isomorphic to a subgroup of the reals and the action of G is the right regular representation on* Ω; *or*
(ii) (G, Ω) *is* o-2 *transitive.*

Remark 7.5 The o-primitive ℓ-permutation groups arising in Proposition 7.4(i) are abelian o-groups, whereas those in (ii) have trivial centre and are o-*m* transitive for all $m \in \mathbb{Z}_+$ (Lemma 4.2).

8 Model Theory

Gurevich and Holland showed how to recognise $(\mathbb{R}, \leq)$ and essentially $(\mathbb{Q}, \leq)$ from their automorphism groups in the language of ℓ-groups from among o-2 transitive $\mathrm{Aut}(\Omega, \leq)$. I stated this as Theorems 5.10 and 5.11 in Sect. 4 and outlined the proof. Recently, John Wilson and I showed how to achieve this under the much weaker hypothesis that $\mathrm{Aut}(\Omega, \leq)$ is merely transitive [25] (version 1). I spoke on this at the Conference and now give a summary of what I said incorporating some subsequent simplifications we have managed.

For ease of presentation and for the rest of this section, let (G, Ω) be a depressible, abundant transitive ℓ-permutation group such that $\mathfrak{K}$ either has no least element or the least o-primitive component is o-2 transitive.

For each $\Delta \in T$, let

$$Q_\Delta := \{h \in \mathrm{rst}(\Delta) \mid h_\Delta \neq 1\}.$$

Equivalently, we have $Q_\Delta = \{h \in \mathrm{rst}(\Delta) \mid (\exists \alpha \in \Delta)(V(\alpha, \alpha h) = \kappa(\Delta))\} \neq \emptyset$. For $h \in Q_\Delta$, let

$$X_h := \{[h^{-1}, h^g] \mid g \in G\}, \quad \text{and} \quad W_h = \bigcup \{X_{h^g} \mid g \in G,\ [X_h, X_{h^g}] \neq 1\},$$

where $[A, B] = \{[a, b] \mid a \in A,\ b \in B\}$.

Write C_G^2 as shorthand for $\mathrm{C}_G\mathrm{C}_G$. The key result to date in [25] is

Proposition 8.1 *For every $\Delta \in T$ and $h \in Q_\Delta$,*

$$\mathrm{C}_G^2(W_h) = \mathrm{rst}(\Delta).$$

Thus if $\beta \in \Delta$ and $h' \in \mathrm{rst}(\Delta)$ with $V(\beta, \beta h') = \kappa(\Delta)$, then

$$\mathrm{C}_G^2(W_h) = \mathrm{C}_G^2(W_{h'}).$$

In particular,

$$\mathrm{C}_G^2(W_h) = \mathrm{C}_G^2(W_{h'}) \quad \textit{for all} \;\; h, h' \in Q_\Delta.$$

To prove this, we need a lemma.

Lemma 8.2 *Let $\Delta \in T$ and $h \in Q_\Delta$. Then*

$$W_h = \bigcup \{X_{h^g} \mid g \in \mathrm{st}(\Delta)\},$$

and $\mathrm{C}_G(W_h)$ is the pointwise stabiliser of Δ.

The first statement of Lemma 8.2 is proved by reducing to o-primitive ℓ-permutation groups and this in turn is proved in the o-2 transitive case by establishing that for any $g, h \in G$ with $h \neq 1$ and $\mathrm{supp}(h) \cap \mathrm{supp}(h^g) = \emptyset$, there are $f, k \in G$ such that $[h^{-1}, h^f][h^{-g}, h^{gk}] \neq [h^{-g}, h^{gk}][h^{-1}, h^f]$. Our proof of the existence of f, k uses o-8 transitivity which we have seen follows from the o-2 transitivity of an ℓ-permutation group (G, Ω) (Lemma 4.2).

Since (G, Ω) is abundant and depressible, for each $\Delta \in T$, there is $h \in Q_\Delta$ with *bump*(h) and any element $h \in G_+$ that is a bump must belong to Q_Δ for some $\Delta \in T$. Since X_h is first-order definable, so is W_h by Proposition 8.1. I can therefore immediately obtain a first-order formula $\gamma(h, x)$ that holds in G if and only if h is a bump and $x \in \mathrm{C}_G^2(W_h) = \mathrm{rst}(\Delta)$ where Δ is the o-block of $V(\delta h, \delta)$ containing $\delta \in \mathrm{supp}(h)$; and also derive a formula $\vartheta(h'h)$ which holds if and only if h, h' are bumps and $\mathrm{C}_G^2(W_{h'}) \subset \mathrm{C}_G^2(W_h)$ and a formula $\chi(h', h)$ that is the conjunction of $\vartheta(h', h)$ and the universal formula: $(\forall$ bump $h'')\, \neg(\vartheta(h', h'')$ & $\vartheta(h'', h))$. Since $\mathrm{C}_G^2(W_h) = \mathrm{rst}(\Delta)$, it is therefore straightforward to determine the first-order theory of the totally ordered spine $\mathfrak{K}_G$ of G from the first-order language of the ℓ-group G. Thus

Proposition 8.3 *Let (G, Ω) and (H, Λ) be transitive, depressible ℓ-permutation groups that are abundant but not locally abelian. If G and H satisfy the same first-order sentences in the language of lattice-ordered groups, then $\mathfrak{K}_G$ and $\mathfrak{K}_H$ satisfy the same first-order sentences in the language of totally ordered sets. Moreover, if the first-order theory of G is decidable, then so is the first-order theory of $\mathfrak{K}_G$.*

Hence if (G, Ω) is a depressible o-2 transitive ℓ-permutation group and (H, Λ) is a transitive, depressible, abundant ℓ-permutation group with H and G satisfying the same sentences in the language of ℓ-groups, then $\mathfrak{K}_H$ has only one element since $\mathfrak{K}_G$ does, whence (H, Λ) is o-primitive (and so o-2 transitive by Proposition 7.4). From this and Theorems 5.10 and 5.11, we can immediately deduce

Theorem 8.4 *If* $\mathrm{Aut}(\Omega, \leq)$ *is transitive and satisfies the same* ℓ*-group-theoretic sentences as* $\mathrm{Aut}(\mathbb{R}, \leq)$*, then* $(\Omega, \leq) \cong (\mathbb{R}, \leq)$ *as ordered sets.*

and

Theorem 8.5 *If* $\mathrm{Aut}(\Omega, \leq)$ *is transitive and satisfies the same* ℓ*-group-theoretic sentences as* $\mathrm{Aut}(\mathbb{Q}, \leq)$*, then* $(\Omega, \leq) \cong (\mathbb{Q}, \leq)$ *or* $(\Omega, \leq) \cong (\mathbb{R} \setminus \mathbb{Q}, \leq)$ *as ordered sets.*

But we can prove more. We want to translate ℓ-group formulae from $G(\Delta)$ to equivalent ones in G. We need to set up conditions on (G, Ω) that make this possible. I will do this here under the restriction that every element of $\mathfrak{K}$ has a predecessor in $\mathfrak{K}$. That is, if $K \in \mathfrak{K}$ covers $K' \in \mathfrak{K}^*$, then $K' \in \mathfrak{K}$. This can be expressed in our first-order language by Proposition 8.3: $(\forall \text{ bump } h)(\exists \text{ bump } h')\chi(h', h)$.

Let $h \in Q_\Delta$ be a bump and $\bar{x} = \{x_1, \ldots, x_n\}$ be a finite set of variables. First replace $u(\bar{x}) = v(\bar{x})$ by $u(\bar{x})v(\bar{x})^{-1} = 1$ and $u(\bar{x}) \neq v(\bar{x})$ by $u(\bar{x})v(\bar{x})^{-1} \neq 1$. Next replace

$$t(\bar{x}) = 1 \quad \text{by} \quad (\exists \text{ bump } h')(\chi(h', h) \quad \& \quad \gamma(h', t(\bar{x}))),$$

and

$$t(\bar{x}) \neq 1 \quad \text{by} \quad (\exists \text{ bump } h')(\chi(h', h) \ \& \ \neg\gamma(h', t(\bar{x}))).$$

For any formula $\rho(\bar{x})$ free in $\bar{x}$, let $\rho_h^*(\bar{x})$ be the result of replacing each basic subformula of ρ as above. For a formula

$$\sigma(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) :\equiv \exists x_j \rho(\bar{x}),$$

let

$$\sigma_{j,h}^*(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) :\equiv \exists x_j(\gamma(h, x_j) \quad \& \quad \rho_h^*(\bar{x})),$$

and for $\sigma(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) :\equiv \forall x_j \rho(\bar{x})$, let

$$\sigma_{j,h}^*(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) :\equiv \forall x_j(\gamma(h, x_j) \to \rho_h^*(\bar{x})).$$

For example, if $\alpha \in \Omega$ and $h \in H_\Delta$ with $\kappa(\Delta) = V(\alpha, \alpha h)$, then we can express that $G(\Delta)$ is abelian by modifying the sentence $\sigma :\equiv (\forall f, g)[f, g] = 1$. In this case, σ_h^* is the sentence

$$(\forall f, g)([\gamma(h, f) \ \& \ \gamma(h, g)] \to (\exists \text{ bump } h')(\chi(h', h) \ \& \ \gamma(h', [f, g])),$$

and $G(\Delta) \models \sigma$ if and only if $G \models \sigma_h^*$.

We say that a transitive ℓ-permutation group (G, Ω) is *locally abelian* if $\mathfrak{K}$ has a minimal element with the corresponding o-primitive component abelian. By the above, we obtain the following result.

Proposition 8.6 *Let (G, Ω) be a transitive depressible abundant ℓ-permutation group that is not locally abelian. If every element of $\mathfrak{K}_G$ has a predecessor, then for any bump $h \in Q_\Delta$ and sentence σ of the language of ℓ-groups,*

$$G(\Delta) \models \sigma \quad \text{if and only if} \quad G \models \sigma_h^*.$$

Definition 8.7 A *coloured chain* is a totally ordered set $(C, \leqslant)$ together with a countable set $\{P_i \mid i \in I\}$ of unary predicates called *colours* such that for all $c \in C$ the set $\{i \mid P_i(c)\}$ is non-empty. One says that c has colour i if $P_i(c)$ holds. (Elements of C are allowed to have more than one colour.)

Let Σ be the set of all sentences in the first-order language of ℓ-groups. The coloured chain $\mathfrak{C}_G$ associated with an ℓ-permutation group (G, Ω) is the totally ordered set $\mathfrak{K}_G$ together with the set $\{P_\sigma \mid \sigma \in \Sigma\}$, such that $P_\sigma(K)$ if and only if $G_K \models \sigma$ (for $K \in \mathfrak{K}$, $\sigma \in \Sigma$) ; cf. the first-order theory of coloured chains associated with abelian o-groups [26].

We note that if $\rho, \tau \in \Sigma$ and $F \models \rho \longrightarrow \tau$ for every ℓ-group F, then $P_\rho(c) \longrightarrow P_\tau(c)$ is an axiom for the theory of coloured chains of ℓ-groups.

In [25] (version 1), we prove a stronger version of the following theorem (which is not quite as much as I claimed in my conference lecture).

Theorem 8.8 *Let (G, Ω) and (H, Λ) be transitive depressible ℓ-permutation groups that are abundant but not locally abelian. If $G \equiv H$, then the totally ordered spine $\mathfrak{K}_G$ of G is elementarily equivalent to the totally ordered spine $\mathfrak{K}_H$ of H and if every element of $\mathfrak{K}_G$ has a predecessor, then $\mathfrak{C}_G \equiv \mathfrak{C}_H$.*

This applies to many examples, in particular to many wreath and Wreath products of o-primitive ℓ-permutation groups (dependent on finiteness conditions or otherwise of the set of supporting intervals) for those familiar with these concepts (see [18] or [30] for definitions).

Corollary 8.9 *Let (G, Ω) and (H, Λ) be transitive abundant ℓ-permutation groups that are depressible but not locally abelian with $\mathfrak{K}_G = \{K_n : n \in \mathbb{Z}_-\}$ where $K_n > K_{n-1}$ for all $n \in \mathbb{Z}_-$. If $G \equiv H$, then $\mathfrak{K}_H = \{K'_n : n \in \mathbb{Z}_-\} \cup \mathfrak{K}'$ for some totally ordered set $\mathfrak{K}'$ with $K'_n > K'_{n-1} > K'$ for all $n \in \mathbb{Z}_-$ and $K' \in \mathfrak{K}'$ and $G_{K_n} \equiv H_{K'_n}$ for all $n \in \mathbb{Z}_-$.*

In particular, Corollary 8.9 applies when (G, Ω) is given by Example 7.1. In this case, the resulting o-primitive components of (H, Λ) satisfy the same first-order sentences as $\mathrm{Aut}(\mathbb{R}, \leq)$; if $H = \mathrm{Aut}(\Lambda, \leq)$, then all o-primitive components of (H, Λ) are isomorphic to $\mathrm{Aut}(\mathbb{R}, \leq)$.

Corollary 8.10 *Let (G, Ω) and (H, Λ) be transitive, depressible ℓ-permutation groups that are abundant but not locally abelian. If $G \equiv H$ and the totally ordered* *spine* $\mathfrak{K}_G$ *of* G *is* $\{K_1, \ldots, K_r\}$ *where* $K_1 < \cdots < K_r$*, then the totally ordered spine* $\mathfrak{K}_H$ *of* H *is* $\{K_1', \ldots, K_r'\}$ *where* $K_1' < \cdots < K_r'$ *and* $G_{K_i} \equiv H_{K_i'}$ *(*$i = 1, \ldots, r$*).*

This corollary even holds when (G, Ω) is locally abelian.

The converse to Theorem 8.8 fails.

Recent improvements of [25] (version 1) are being written up and incorporated to form [25] (version 2). They involve working only in the group language. The strengthening of Theorems 8.4 and 8.5 to satisfying the same group-theoretic sentences can be found in A. M. W. Glass, J. S. Wilson, Recognizing the real line, arxiv:1701.07235.

9 Amalgamation for Right-Orderable Groups

In Sect. 5 we obtained undecidability results for lattice-ordered groups. In the next two sections of this survey we consider undecidability for order*able* and right-order*able* groups. The context is generators and relations in *group* words. For this, Higman-Neumann-Neumann constructions can be used and we will need to show that each resulting group can be made into an o-group or a right-ordered group.

The class of orderable groups is closed under isomorphisms, subgroups and ultraproducts. It is therefore definable in the first-order theory of groups. The same is true for the class of right-orderable groups. This is in sharp contrast to the class of lattice-orderable groups; there are groups G, H satisfying exactly the same sentences in group theory with only G being lattice orderable [44].

To begin, consider the amalgamation of just the trivial subgroup; i.e., the free product. In 1949, A. A. Vinogradov proved [43]

Proposition 9.1 *The free product of orderable groups is orderable.*

In 1990, George Bergman provided other special cases when amalgamation is possible for orderable groups and also for right-orderable groups, [3]. The analogue of Vinogradov's result easily follows from Bergman's work (see, e.g., [32], Corollary 6.1.3).

Proposition 9.2 *If* $\{G_i \mid i \in I\}$ *is a family of right-ordered groups, then their free product* $*_{i \in I} G_i$ *is right orderable with a right order extending that on each* G_i*.*

Although the classes of orderable and right-orderable groups fail to be closed under free products with amalgamation in general, amalgamation is possible in the class of right-orderable groups when the amalgamated subgroup is cyclic by the same proof as that for Corollary 5.8; for details, see [4]. In contrast to the classes of orderable groups and ℓ-groups, it has been possible to give necessary and sufficient conditions for amalgamation in the class of right-orderable groups [5]. The necessary and sufficient conditions are complicated; they involve the existence of normal families of right orders on the constituent groups that are compatible under the isomorphisms between the amalgamated subgroups. The amalgamation of cyclic subgroups for ℓ-groups and right-orderable (or right-ordered) groups required a permutation tour-de-force (see Corollaries 5.8 and 5.9 at the end of Sect. 5). The proof that the conditions are necessary and sufficient for amalgamation for right-orderable groups also requires a very delicate and different inductive permutation proof; I do not know how to prove it without using automorphism groups of totally ordered sets. For details, see [5, Theorem A].

Natural analogues hold for HNN-extensions. The ideas and conditions also extend to graphs of groups [14].

Although the necessary and sufficient conditions are complicated, the results easily imply all previously known cases. To give a flavour of what can be deduced, here are three consequences that I'll use in the next section; the first two are easy to state. As is standard, I will write $*_{i\in I}G_i\,(H_i \overset{\varphi_i}{\cong} H)$ for the group free product of $\{G_i \mid i \in I\}$ with the subgroups H_i amalgamated via φ_i $(i \in I)$.

Corollary 9.3 *Let H_i be a subgroup of a right-ordered group G_i such that any right order on H_i is extendable to a right order on G_i $(i \in I)$. Suppose that each H_i is isomorphic to a group H. Then $L := *_{i\in I}G_i\,(H_i \overset{\varphi_i}{\cong} H)$ is right orderable for any amalgamating isomorphisms φ_i $(i \in I)$. Moreover, the initial right orders on G_i $(i \in I)$ extend to a right order on L if $\varphi_i\varphi_j^{-1}$ preserves the induced orders on H_i and H_j for all $i, j \in I$.*

Corollary 9.4 *Let G be a right-ordered group with normal convex subgroup N. Let $\varphi : H_1 \cong H_2$ be an order-preserving isomorphism between subgroups H_1 and H_2. Assume that $H_1 \cap N = H_2 \cap N = \{1\}$ and the induced isomorphism $\bar{\varphi} : H_1N/N \cong H_2N/N$ is the identity. Then $K = \langle G, t \mid h_1^t = h_1\varphi\ (h_1 \in H_1)\rangle$ is right orderable with an order extending that of G.*

Corollary 9.4 is a special case of the third corollary which is only needed to establish that $B(T)_3$ in the next section is right orderable.

Corollary 9.5 *Let H_1 and H_2 be isomorphic subgroups of a right-ordered group G; say $\varphi : H_1 \cong H_2$. Let N be a convex normal subgroup of G such that $(N \cap H_1)\varphi = N \cap H_2$. Let $\bar{G} := G/N$ and $\bar{H}_i := H_iN/N$ $(i = 1, 2)$. Define $\bar{\varphi} : \bar{H}_1 \cong \bar{H}_2$ by $\bar{h}_1\bar{\varphi} := (h_1\varphi)N$, where $\bar{h}_1 := h_1N \in \bar{H}_1$. If $K_1 := \langle \bar{G}, \bar{t} \mid \bar{h}_1^{\bar{t}} = \bar{h}_1\bar{\varphi}\ (\bar{h}_1 \in \bar{H}_1)\rangle$ and $K_2 := \langle G, t \mid h_1^t = h_1\varphi\ (h_1 \in H_1 \cap N)\rangle$ are right orderable, then $K := \langle G, t \mid h_1^t = h_1\varphi\ (h_1 \in H_1)\rangle$ is right orderable with a right order extending that on G.*

I'll close this section with the analogue of a result about countable groups. Using Wreath products, one can actually replace *three* by *two* (cf. [18], Theorem 10.A) but I want to demonstrate a further application of our necessary and sufficient conditions for right orderability. The result suffices for proving Corollary 10.6 and Theorem 10.7.

Corollary 9.6 (cf. [34], Corollary IV.3.1) *Every countable right-ordered group C can be embedded in a three-generator right-ordered group which is defined by a finite or recursively enumerable set of relations if C is.*

10 Applications to Decision Problems for Right Orderable Groups

I'll now apply these corollaries to derive undecidability results for right-orderable groups. Unlike the case for lattice-ordered groups where the group and lattice operations were allowed in forming words (see Sect. 6), the words in this context are just group words.

If T is any Turing machine, there is a recursively enumerable set of semigroup words (numbers) $E := E(T)$ in the alphabet of symbols $\{a_1, \ldots, a_m\}$ associated with T. Let $\gamma(T)$ be the finitely presented semigroup associated with T (see [40], Chap. 12). It has generators $s_1, \ldots, s_M$ (symbols) and $q_0, \ldots, q_N$ (states), and defining relations $\Sigma_i = \Gamma_i$ ($i = 1, \ldots, I$), where each Σ_i and Γ_i is special; i.e., has the form wq_jw' where w, w' are semigroup words in $\{a_1, \ldots, a_m\} \subseteq \{s_1, \ldots, s_M\}$ and $j \in \{0, \ldots, N\}$. One can build a finitely presented group $B(T)$ associated with $\gamma(T)$. This was done originally by P. S. Novikov and independently by W. W. Boone, then simplified by J.L. Britton in [11]. Britton's finitely presented group $B(T)$ is a homomorphic image of Boone's original group. It is built from the free group F_2 on 2 generators x, y by a succession of *HNN*-extensions each of which is finitely presented over its predecessor and involves the generators of $\gamma(T)$ *inter alia*, subject to a finite set of relations. Now

Lemma 10.1 *The free group on x, y is orderable (and hence right orderable).*

Proof Each quotient $\gamma_n(F_2)/\gamma_{n+1}(F_2)$ in the lower central series is torsion-free abelian and so orderable. Define $f < g$ in F_2 if and only if $gf^{-1} \in \gamma_n(F_2) \setminus \gamma_{n+1}(F_2)$ for some (necessarily unique) n and $gf^{-1} > 1$ in the order on $\gamma_n(F_2)/\gamma_{n+1}(F_2)$. □

Each of the successive HNN-extensions in forming $B(T)$ can be shown to successively satisfy the hypotheses for right orderability in the requisite corollaries of the previous section. Thus

Theorem 10.2 *Let T be a Turing machine and $B(T)$ be the Britton group associated with T. Then $B(T)$ is right orderable.*

Remark *There are distinct $a, b \in B(T)$ having the same square. In any o-group, $a = b$ whenever $a^2 = b^2$ (since $a < b$ implies that $a^2 < ab < b^2$). Hence $B(T)$ cannot be made into an o-group for any Turing machine T.*

By taking T to be any Turing machine yielding a recursively enumerable but non-recursive set $E(T)$, we obtain a right-orderable finitely presented group with insoluble word problem. Thus

Corollary 10.3 (cf. [23]) *There is a right-orderable finitely presented group with insoluble word problem.*

Theorem 10.4 (cf. [21]) *A finitely generated group can be embedded in a right-orderable finitely presented group if and only if it is right orderable and can be defined by a recursively enumerable set of relations.*

Outline of Proof It is immediate that any finitely generated subgroup of a finitely presented right-orderable group must be right orderable and definable by a recursively enumerable set of relations.

To prove the converse, follow Aanderaa's proof of the Higman Embedding Theorem in [1]. (A slight oversight in the proof of Lemma 8 of [1] can easily be remedied.)

Let U be any finitely generated right-orderable group that is defined by a recursively enumerable set of relations. By increasing the set of generators and relations, I may assume that each of these relations is a semigroup word in the generators. Let the resulting set of generators for U be $\{u_1, \ldots, u_m\}$. Let $a_1, \ldots, a_m$ be formal symbols. For w any word in $a_1, \ldots, a_m$, let w_u be the word obtained from w by replacing each a_i by u_i $(i = 1, \ldots, m)$. Let T be the Turing machine that enumerates the set of all words w in $a_1, \ldots, a_m$ such that $w_u = 1$ in U. So $E = E(T)$ and

$$U := \langle u_1, \ldots, u_m \mid w_u = 1 \ (w \in E)\rangle.$$

Let $B(T)$ be the Britton group of the previous proof with the right order so constructed. So $a_1, \ldots, a_m \in \{s_1, \ldots, s_M\}$. $B(T)$ also contains k_0, t_0 with $t_0 > 1$ and all powers of t_0 less than k_0 in the right order on $B(T)$. Now take a sequence of three HNN-extensions (Britton extensions) $B(T)_2$ embedded in $B(T)_3$ embedded in $B(T)_4$ embedded in $B(T)_5$ starting with

$$B(T)_2 := U * B(T).$$

Specifically,

$$B(T)_3 := \langle B(T)_2, c_1, \ldots, c_m \mid u_j^{c_i} = u_j,\ a_j^{c_i} = a_j,\ k_0^{c_i} = k_0 u_i^{-1} \ \ (i, j \in \{1, \ldots, m\})\rangle,$$

$$B(T)_4 := \langle B(T)_3, d \mid (a_j c_j)^d = a_j,\ k_0^d = k_0 \ \ (j = 1, \ldots, m)\rangle,$$

and

$$B(T)_5 := \langle B(T)_4, p \mid a_j^p = a_j,\ k_0^p = k_0,\ t_0^p = t_0 d \ \ (j = 1, \ldots, m)\rangle.$$

Now $B(T)_2$ is right orderable by Proposition 9.2 with an order extending that of $B(T)$. It can be shown that the constructed right-order on $B(T)_2$ extends to a right order on $B(T)_3$ using Corollary 9.5. This is the more difficult step in the proof. Corollary 9.4 can be used to prove that successively $B(T)_4$ and then $B(T)_5$ are right orderable. Aanderaa's very clever proof relies on showing that (like the Higman Rope Trick—see [34], Lemma IV.7.6.) the relations defining U are not necessary in $B(T)_5$ and so $B(T)_5$ can actually be finitely presented. Since U embeds in $B(T)_5$, Higman's Embedding Theorem follows for right-orderable groups.

The proof of Theorem 10.4 actually gives (cf. Theorem 6.5 in Sect. 6)

Corollary 10.5 *A finitely generated right-ordered group can be embedded (as a right-ordered group) in some finitely presented right-ordered group if and only if it can be defined by a recursively enumerable set of relations.*

We obtain the standard consequence of the Higman Embedding Theorem by taking the free product of all finitely presented right-ordered groups (to within isomorphism), embedding this countable recursively generated and related right-ordered countable group in a three-generator recursively defined right-ordered group (Corollary 9.6), and then applying Theorem 10.4. This gives a universal finitely presented right-orderable group.

Corollary 10.6 *There is a finitely presented right-ordered group in which every finitely presented right-ordered group can be embedded (as a right-ordered group).*

The natural analogue of the Boone-Higman Theorem also follows for right-orderable groups by the same proof (also cf. Theorem 6.6).

Theorem 10.7 (cf. [10]) *A finitely generated right-orderable group has soluble word problem if and only if it can be embedded in a simple group which can be embedded in a finitely presented right-orderable group.*

Again one can milk the proof and obtain

Corollary 10.8 *A finitely generated right-ordered group has soluble word problem if and only if it can be embedded in a simple right-ordered group which can be embedded in a finitely presented right-ordered group (all the embeddings preserving order).*

Being right orderable is a Markov property for finitely presented groups (see [34], Sect. IV.4): isomorphic finitely presented groups are either both right orderable or neither is, there is a right-orderable finitely presented group (e.g., $\langle x \mid x = 1\rangle$) and there is a finitely presented group (e.g., $\langle y \mid y^2 = 1\rangle$) that cannot be embedded in any right-orderable group. Therefore, there is no algorithm to determine if an arbitrary finitely presented group is right orderable or not ([34], Theorem IV.4.1).

The question arises whether one could solve the isomorphism problem for right-orderable finitely presented groups, assuming that one were provided with an oracle which would tell us (truthfully) that the finitely presented groups we are considering are right orderable. Unfortunately, the construction given by Rabin (see the proof of Theorem IV.4.1. in [34]) does not fit the hypotheses of our corollaries, so this remains open.

The existence of a finitely presented right-orderable group with insoluble word problem (Corollary 10.3) can be used to construct a stronger result [7].

Theorem 10.9 *There is a finitely presented orderable group with insoluble word problem.*

11 Sketch of the Proof of Theorem 10.9

I now provide a sketch of Vasily Bludov's beautiful idea to prove Theorem 10.9. Take a finitely presented *right*-orderable group with insoluble word problem, e.g., Britton's group $B(T)$. Write it as the quotient of the free group F on generators $x_1, \ldots, x_m$, say, by the normal subgroup N generated by the finite set of relations $u_1, \ldots, u_n$. So F/N is a finitely presented right-orderable group with insoluble word problem. Let G_0 be a semidirect product of F by a free group on $2m$ generators that normalises N. Take a Higman-Neumann-Neumann-extension G_1 of G_0 with stable letter t that fixes each element of N. Let T_0 be the normal closure of $\langle t \rangle$ by G_0 (equivalently, in G_1). We use Higman-Neumann-Neumann-extensions to add $2m$ *endomorphisms* of G_1 that fix each element of G_0 but map t appropriately. This is our group G which can be finitely presented. We derive the infinite set of relations $t^{-1}ut = u$ for all $u \in N$ from the *finite* set of defining relations of G which include $t^{-1}u_it = u_i$ $(i = 1, \ldots, n)$. Since t commutes (in G) with $x \in F$ if and only if $x \in N$, and F/N has insoluble word problem, G will provide the requisite group in Theorem 10.9 once we establish that G is an orderable group. To achieve this, we use the right order on F/N to give a right order $\prec$ on G_0 and thence an order on the generators of the free group T_0 by $t^{f_1} < t^{f_2}$ if and only if $f_1 \prec f_2$. We will give an ordering of basic commutators in a free group and derive a G_1-invariant order on T, the normal closure of $\langle t \rangle$ in G, using groups on each of which the endomorphisms become *automorphisms*. This ordering of T is also invariant under these automorphisms. We will realise G as an extension of T by an orderable group. Hence G will be orderable.

Specifically, G_0 is generated by $x_1, \ldots, x_m,\ b_1, \ldots, b_{2m}$ and has defining relations

$$x_i^{b_j} = x_i^{x_j}, \qquad x_i^{b_{m+j}} = x_i^{x_j^{-1}} \qquad (i, j = 1, \ldots, m). \tag{1}$$

By (1), $x_j^{b_j} = x_j$ $(j = 1, \ldots, m)$. This immediately gives

$$x_i^{b_j^{-1}} = x_i^{x_j^{-1}} \qquad (i, j = 1, \ldots, m). \tag{1'}$$

So G_0 is a semidirect product of the free group $F(\bar{x})$ by the free group $F(\bar{b})$ with free generators $b_1, \ldots, b_{2m}$, and N is normalised by $F(\bar{b})$ in G_0.

Let G_1 be generated by $x_1, \ldots, x_m,\ b_1, \ldots, b_{2m}$, and the extra generator t and have defining relations (1) and

$$[t, u_j^g] = 1 \qquad (j = 1, \ldots, n;\ \ g \in F(\bar{x})). \tag{2}$$

So G_1 is a Higman-Neumann-Neumann-extension of G_0. Now G_0 (and hence $F(\bar{x})$ and $F(\bar{b})$) can be embedded in G_1 in the natural way. We will regard N, $F(\bar{x})$ and $F(\bar{b})$ as subgroups of G_1. By Britton's Lemma,

Lemma 11.1 *If $w \in F(\bar{x})$, then $[t, w] = 1$ in G_1 if and only if $w \in N$.*

For $i = 1, \ldots, 2m$, let A_i be the subgroup of G_1 generated by $G_0 \cup \{[b_i, t]\}$. Since $N^{b_i} = N$, we have

$$u^{[b_i,t]} = u \quad \text{if and only if} \quad u \in N.$$

Hence $G_1 \cong A_i$. We can therefore define the requisite group G as the sequence of Higman-Neumann-Neumann-extensions starting from G_1 and obtained by successively adjoining stable letters $y_1, \ldots, y_{2m}$ subject to

$$[x_i, y_j] = 1 \qquad (i = 1, \ldots, m; \;\; j = 1, \ldots, 2m), \tag{3}$$

$$[b_i, y_j] = 1 \qquad (i, j = 1, \ldots, 2m), \tag{4}$$

$$t^{y_i} = [b_i, t] \qquad (i = 1, \ldots, 2m). \tag{5}$$

We will regard G_1 as a subgroup of G in the natural way.

We now show that the infinite set of relations (2) can be deduced from these three finite sets of relations, (1) and the finite subset (6) of (2) where

$$[t, u_j] = 1 \qquad (j = 1, \ldots, n). \tag{6}$$

Lemma 11.2 *The group G can be finitely presented. It is generated by the $5m + 1$ elements*

$$x_1, \ldots, x_m, \;\; b_1, \ldots, b_{2m}, \;\; t, \;\; y_1, \ldots, y_{2m},$$

and can be defined by the $8m^2 + 2m + n$ relations (1)–(6).

Proof We derive (2) from the remaining relations. Conjugating $[t, u_j] = 1$ by y_i and then by b_i^{-1} (and by y_{i+m} and then by b_{i+m}^{-1}) and using (5), (3) and (1), we obtain $[t, u_j^{x_i^{\pm 1}}] = 1$ in G ($j = 1, \ldots, n$; $i = 1, \ldots, m$). An easy induction now gives that $[t, u_j^{w(\bar{x})}] = 1$ in G for all $j \in \{1, \ldots, n\}$ and $w(\bar{x}) \in F(\bar{x})$. Hence (2) follows and the lemma is proved. □

Also note that since G was formed from G_1 by successively adding $y_1, \ldots, y_{2m}$ as stable letters, by Britton's Lemma

Lemma 11.3 *$\langle y_1, \ldots, y_{2m} \rangle$ is a free subgroup of G of rank $2m$.*

We will write $F(\bar{y})$ for the free group with free generators $y_1, \ldots, y_{2m}$ and identify it with the subgroup of G given by the lemma.

Let $w \in F(\bar{x})$. Since $[t, w] = 1$ in G if and only if $w \in N$, it follows that G has insoluble word problem.

We must now show that G is an orderable group. This is technical. Let T_0 be the normal subgroup of G_0 generated by t. This is also the normal subgroup of G_1 generated by t. It can be shown that T_0 is the free group on t^{vh} ($v \in F(b_1, \ldots, b_{2m})$, $h \in F/N$). In Lemma 10.1 we proved that a free group can be made into an o-group. One can give a special central ordering of each abelian quotient $\gamma_k(T_0)/\gamma_{k+1}(T_0)$ and thereby an order on T_0 that is preserved by G_1 (see [7]). But $y_1, \ldots, y_{2m}$ are only endomorphisms of T_0. Let T_0^* be the topological completion of T_0 under the interval topology. Then T_0^* is an o-group with a total order extending that of T_0 such that the endomorphisms $y_1, \ldots, y_{2m}$ of T_0 extend to order-preserving *automorphisms* of T_0^*. From this and the given defining relations, one can obtain the following result.

Lemma 11.4 *T has generators*

$$t^{\alpha(\bar{y})v(\bar{b})h(\bar{x})}, \tag{7}$$

where $h(\bar{x}) \in H$, $v(\bar{b}) \in F(\bar{b})$ and $\alpha(\bar{y})$ is either empty or, for some $i \in \{1, \ldots, 2m\}$, $\alpha(\bar{y})$ is a non-trivial element of $F(\bar{y})$ that begins with y_i^{-1} and $v(\bar{b})$ does not begin with $b_i^{\pm 1}$.

Using this lemma, an intricate argument can be provided to show that T is a G-invariant o-group under the induced ordering on T. This uses linear algebra in torsion-free abelian quotient groups and Nielsen's method to lift to the non-abelian case. Since G/T is clearly an o-group under the natural ordering, Theorem 10.9 follows. □

12 A Model-Theoretic Consequence

Corollary 10.3 can also be combined with the Ehrenfeucht-Mostowski Theorem (see Example 2.3 in Sect. 2) to prove the following purely model-theoretic result [8].

Theorem 12.1 *If T is a first-order theory having infinite models, then T has a model $\mathscr{M}$ whose automorphism group has undecidable universal theory (in the language of groups).*

Finally, we mention that S. Lemieux has proved that Novikov's finitely presented groups (which have soluble word problem and insoluble conjugacy problem) are right orderable [33].

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Algebraic Entropies for Abelian Groups with Applications to the Structure of Their Endomorphism Rings: A Survey

Brendan Goldsmith and Luigi Salce

Abstract The algebraic entropies most frequently used for endomorphisms of Abelian groups are illustrated, their properties and mutual relationships are discussed, and several applications to endomorphism rings, both of torsion and torsion-free Abelian groups, are presented.

Keywords Abelian groups • Modules • Endomorphism rings • Algebraic entropy • Length functions

Mathematical Subject Classification (2010): Primary: 20K30; Secondary: 20K10, 20K15, 16D10

1 Introduction

In dynamical systems, entropy is a notion that measures the rate of increase in dynamical complexity as the system evolves with time. Well-known entropies in mathematics are the measure-theoretic entropy for probability spaces introduced by Kolmogorov in 1958 [42] and Sinai in 1959 [61], and the topological entropy for continuous endomaps of compact spaces, first defined by Adler-Konheim-McAndrew in 1965 [1]. The iteration of an endomorphism of an Abelian group also generates a discrete-time dynamical system. The complexity of such a system may be measured in ways that are clearly analogous to the more classical concepts of entropy mentioned above; the resulting concepts are generally referred to as

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algebraic entropies. In this survey paper we will present the algebraic entropies most frequently used for endomorphisms of Abelian groups; the differing notions of algebraic entropy that arise are, of course, related to the differing algebraic properties of the underlying groups.

Algebraic entropies may be regarded as a further step in a long-standing approach to the study of algebraic structures endowed with an endomorphism. This approach started with vector spaces and linear transformations, was extended to modules over commutative rings, and then to categories of modules over arbitrary rings R and their endomorphisms (see, for instance, Sect. 12 in [40] and Chap. VII, Sect. 5 of [3]). We refer to Sect. 2.1 below for more information on this subject. Algebraic entropies provide the right tools to measure the complexity of Abelian groups viewed as $\mathbb{Z}[X]$-modules, and more generally of R-modules viewed as $R[X]$-modules, by means of the additional structure produced by an endomorphism.

Algebraic entropies can be defined also for other algebraic structures, e.g.,

1. endomorphisms of modules over arbitrary rings R, once a length function on Mod(R) is available (see Salce-Zanardo [58], Salce-Vámos-Virili [60] and Salce-Virili [56]);
2. self-maps of finite length of local Noetherian rings (see Majidi Zolbanin-Miasnikov-Szpiro [47]);
3. group actions on $R[\Gamma]$-modules of amenable and sofic groups Γ (see Li-Liang [44, 45] and Virili [67] and the references therein);
4. rational maps, i.e., homogeneous polynomials acting on homogeneous coordinates of the projective completion of an affine space (see Bellon-Viallet [5]);
5. product MV-algebras and MV-algebras (see Petrovicova [52] and Riecan [54]);
6. entropy for non-commutative groups has been studied in [14] and has been used to establish connections with the classical notion of growth developed by Milnor in the context of Geometric group theory [16].

We note at the outset that the word 'group' will mean an additively written Abelian group unless specified to the contrary.

The development of the theory of entropies in an algebraic setting started in 1969 with a sketched definition of the algebraic entropy *ent* for endomorphisms of Abelian groups; this appeared at the end of the paper [1] where the topological entropy was introduced. The precise definition and the first basic properties of *ent* were given by Weiss in 1975 [70], where also a connection with the topological entropy via the Pontryagin-Van Kampen duality was established. The relationship between a variation of *ent*, denoted by h, for automorphisms of discrete countable groups and the topological entropy of their duals was proved by Peters in 1979 [51].

We list below, in chronological order of their appearance in the literature, the five algebraic entropies we are going to illustrate:

– the algebraic entropy *ent*, the first algebraic entropy sketched in [1]; research on algebraic entropy has flourished since the appearance of the paper [19] in 2009, where *ent* was thoroughly investigated, and many new directions and applications have subsequently emerged; *ent* works non-trivially only on torsion groups;

- the rank-entropy ent_{rk} introduced by Salce-Zanardo in 2009 [58], which works non-trivially only on torsion-free groups; Rüdiger Göbel and the second author investigated this entropy in 2012 [32], obtaining interesting applications to endomorphism rings of torsion-free groups, that will be illustrated at the end of Sect. 4;
- the adjoint entropy ent^* investigated by Dikranjan-Giordano Bruno-Salce in 2010 [20] and Goldsmith-Gong in 2012 [34], which is related to *ent* by means of the Pontryagin-Van Kampen duality;
- the intrinsic entropy $\widetilde{ent}$, introduced by Dikranjan-Giordano Bruno-Salce-Virili in 2015 [25], which probably offers the most attractive tool for algebraists in the dynamical study of endomorphisms of arbitrary groups;
- the entropy denoted by h, still sometimes called algebraic entropy since it coincides with *ent* on torsion groups (but referred to also as Peters entropy), which takes non-trivial values also outside of torsion groups; this entropy is of more combinatorial character, and was recently deeply investigated by Dikranjan-Giordano Bruno in [18]; its interest arises in part due to its remarkable connection with a famous open question in algebraic number theory, namely, Lehmer's problem.

The aim of this paper is to provide an up-to-date general overview of the theory of the above algebraic entropies.

One of our main goals is to show how algebraic entropies help in understanding the structure of the endomorphism rings of Abelian groups, in the case of p-groups as well as in the case of torsion-free groups. In particular, classical results and examples of endomorphism rings will be revisited with the new tool of the suitable algebraic entropies, with special emphasis on celebrated theorems obtained in the 1960s by Pierce and Corner.

We will not discuss the connections with the topological entropy for topological groups, even if its interactions with the algebraic entropies are very strong, as mentioned above and as many "Bridge Theorems" testify (see [15, 17]). For readers interested in this aspect of the subject, we refer to the survey papers by Dikranjan-Giordano Bruno [14] and Dikranjan-Sanchis-Virili [21], and to the paper by Virili [66], who studies the algebraic entropy of continuous endomorphisms of locally compact groups, appropriately modifying Peters's definition.

The present survey has unavoidable overlaps with parts of these papers, but here we focus more on the role of length functions and on applications to the structure of endomorphism rings of discrete groups. In order to eliminate overlaps, we will omit proofs, or even outlines of proofs, of most results concerning the algebraic entropy h, which was already discussed in [14, Sect. 5] and in [21, Sects. 3.1, 3.2]. Needless to say, the choice of the results presented in this survey reflects our own taste on the subject. We dedicated this work to the memory of our late friend Rüdiger Göbel who influenced both authors with his own interest in the subject. His insight, as reflected in [32], provided a stimulus to further ongoing work in the area of algebraic entropy.

2 Preliminaries

2.1 The $R[X]$-Module Associated with an Endomorphism of an R-Module

It is an old and classical point of view to look at a vector space V over a field K endowed with a K-linear map ϕ as a $K[X]$-module, and to denote it by V_ϕ. When V is finite dimensional, the decomposition of the finitely generated $K[X]$-module V_ϕ as a direct sum of cyclic modules gives rise to the canonical rational form of the matrix associated with ϕ. We refer to Chap. 10, Sect. 5 in [28], or to [69] for a detailed description of this matter. Also Kaplansky, in his “Little Red Book” [40], devotes Sect. 12 to this subject, in order to apply the theorems on Abelian p-groups developed in the previous sections (Ulm’s theorem included) to vector spaces endowed with a K-linear map, viewed as modules over the PID $K[X]$.

This point of view, extended in Chap. VII, Sect. 5 of [3] to modules over commutative rings, and in [60] to modules over arbitrary unitary rings R, is the most fruitful when one works with the algebraic entropies of endomorphisms of modules. So in this preliminary section we describe it, considering the category of $R[X]$-modules and recalling the basic results proved in [60]; no particular complication arises at this stage in considering general R-modules instead of $\mathbb{Z}$-modules.

Let M be a left module over the ring R, and let $\phi : M \to M$ be an endomorphism. The map $R[X] \times M \to M$ defined by $(f(X), x) \mapsto f(\phi)(x)$ makes M a left $R[X]$-module, denoted by M_ϕ. Conversely, if M_X is an $R[X]$-module, multiplication by X induces an R-endomorphism ϕ of M_R, i.e., M_X viewed as an R-module, and $M_X = (M_R)_\phi$.

If ϕ_X denotes the $R[X]$-endomorphism of $R[X] \otimes_R M$ induced by ϕ, we have the exact sequence of $R[X]$-modules

$$0 \to R[X] \otimes_R M \stackrel{\Psi}{\to} R[X] \otimes_R M \stackrel{\Phi}{\to} M_\phi \to 0$$

where Ψ and Φ are defined by setting $\Psi = X - \phi_X$ and $\Phi(f(X) \otimes x) = f(\phi)(x)$. Note that Ψ is injective, since

$$\Psi(\sum_k (X^k \otimes x_k)) = \sum_k X^k \otimes (x_{k-1} - \phi(x_k))$$

hence, if $\sum_k (X^k \otimes x_k) \in Ker(\Psi)$, then $x_{k-1} = \phi(x_k)$ for all k, and this implies that the x_k are all 0, since almost all x_k are 0 (see also Proposition 2.2 in [55]).

The properties relating the structures as R-module and as $R[X]$-module of $M \in \text{Mod}(R)$ endowed with the endomorphism ϕ are listed below.

(1) A homomorphism of $R[X]$-modules $\alpha : M_\phi \to N_\psi$ is a homomorphism $\alpha : M \to N$ of R-modules such that $\alpha \circ \phi = \psi \circ \alpha$;

(2) M_ϕ is isomorphic to N_ψ if and only if there is an isomorphism $\alpha : M \to N$ of R-modules such that $\phi = \alpha^{-1} \circ \psi \circ \alpha$, that is, ϕ and ψ are conjugated under α;
(3) a submodule N of the R-module M endowed with the endomorphism ϕ is an $R[X]$-submodule if and only if it is ϕ-invariant; if this is the case, this submodule is denoted by N_ϕ, and ϕ induces an endomorphism $\bar{\phi}$ of the factor $R[X]$-module M_ϕ/N_ϕ;

The Bernoulli shifts are a fundamental tool in the study of the various algebraic entropies. Given an R-module M, the *right Bernoulli shift* for M is the endomorphism $\beta : \bigoplus_{\mathbb{N}} M \to \bigoplus_{\mathbb{N}} M$ defined by setting

$$\beta(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots),$$

where the $x_i \in M$ are almost all 0. The *left Bernoulli shift* is the endomorphism $\lambda : \bigoplus_{\mathbb{N}} M \to \bigoplus_{\mathbb{N}} M$ defined by setting $\lambda(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$.

The $R[X]$-module $R[X] \otimes_R M$ is isomorphic to $(\bigoplus_{\mathbb{N}} M)_\beta$ via the isomorphism which sends $(a_0 + a_1X + \cdots + a_nX^n) \otimes x$ into $(a_0x, a_1x, \cdots, a_nx, 0, 0, \cdots)$ for $a_i \in R$ and $x \in M$. The converse isomorphism sends $(x_0, x_1, x_2, \cdots)$ into $\sum_{n\geq 0} X^n \otimes x_n$.

Thus the exact sequence $0 \to R[X] \otimes_R M \overset{\Psi}{\to} R[X] \otimes_R M \overset{\Phi}{\to} M_\phi \to 0$ can be viewed as

$$0 \to (\bigoplus_{\mathbb{N}} M)_\beta \overset{\Psi}{\to} (\bigoplus_{\mathbb{N}} M)_\beta \overset{\Phi}{\to} M_\phi \to 0$$

with $\Phi((x_0, x_1, x_2, \cdots)) = \sum_{n\geq 0} \phi^n(x_n)$ and $\Psi(0, \cdots, 0, x, 0, \cdots) = (0, \cdots, 0, -\phi(x), x, 0, \cdots)$, where, if x takes the k-th place in $(0, \cdots, 0, x, 0, \cdots)$, it takes the $(k+1)$-st place in its image $(0, \cdots, 0, -\phi(x), x, 0, \cdots)$.

2.2 Cyclic $\mathbb{Z}[X]$-Modules

We will apply the above approach to the ring of integers $\mathbb{Z}$ and the category Mod($\mathbb{Z}$) of Abelian groups. In this context it is relevant to know the prime ideals of the domain $\mathbb{Z}[X]$, which is a 2-dimensional Noetherian ring. Besides the null ideal (0), the prime ideals of $\mathbb{Z}[X]$ can be distinguished by being maximal or (non-zero) minimal.

The **maximal ideals** are 2-generated of the form $(p, f(X))$, where p is a prime integer and $f(X)$ is a polynomial of $\mathbb{Z}[X]$ which is irreducible modulo p (hence, in particular, irreducible in $\mathbb{Z}[X]$). Notice that the cyclic $\mathbb{Z}[X]$-module $\mathbb{Z}[X]/(p, f(X))$ is isomorphic to the finite ring $F_p[X]/(f_p(X))$, where F_p is the Galois field with p elements and $f_p(X)$ is the reduction mod p of $f(X)$.

The **minimal prime ideals** of $\mathbb{Z}[X]$ are principal, either of the form (p), for p a prime integer, or of the form $(f(X))$, for $f(X)$ an irreducible polynomial of $\mathbb{Z}[X]$ (hence, in particular, primitive). In the first case we have $\mathbb{Z}[X]/(p) \cong F_p[X]$, in the latter case $\mathbb{Z}[X]/(f(X))$ is a torsion-free group of rank equal to the degree of $f(X)$.

The four types of cyclic $\mathbb{Z}[X]$-modules with prime ideal annihilators described above will play a central role in the theory of the algebraic entropies for Abelian groups.

2.3 Trajectories and Partial Trajectories

Let $\phi : M \to M$ be an endomorphism of a left module M over an arbitrary ring R. Let F be a submodule of M. The minimal ϕ-invariant submodule of M containing F is

$$T(\phi, F) = F + \phi F + \phi^2 F + \cdots$$

which is called the *ϕ-trajectory of F*. The submodule $T(\phi, F)$ can be viewed as the $R[X]$-submodule of M_ϕ generated by F. If we stop the infinite sum at the nth summand, we obtain what is called the *nth partial ϕ-trajectory*:

$$T_n(\phi, F) = F + \phi F + \cdots + \phi^{n-1} F.$$

Note that, if F is finite, then $T_n(\phi, F)$ is also finite for any n, and that the finitely generated $R[X]$-submodules of M_ϕ are exactly the ϕ-trajectories of finitely generated R-submodules of M. Similarly, if R is a commutative integral domain and F is a submodule of finite rank, then also $T_n(\phi, F)$ has finite rank. Less usual, but crucial for our purposes, is to consider when F is not a submodule, but just a *finite subset*. In this case $T_n(\phi, F)$ is also a finite subset (the sum of two finite subsets consists of the elements obtained by summing one element in the first summand and one element in the second summand in all possible ways). The only operation needed in this case is the addition, and not the multiplication by scalars in R, so the fact that M is an R-module is redundant, and only the Abelian group structure of M is needed. From now on, we will focus on endomorphisms ϕ of Abelian groups G.

If F is a subgroup of G, of particular interest is when $F = \mathbb{Z}x$ is cyclic, in which case we call $T(\phi, \mathbb{Z}x)$ the *cyclic ϕ-trajectory* generated by x and we denote it simply by $T(\phi, x)$. Cyclic trajectories are strongly related to the finite topology on the endomorphism ring $\mathrm{End}(G)$ of an Abelian group G. Recall that the finite topology on $\mathrm{End}(G)$, denoted by fin_G, has as basis of neighbourhoods of zero the left ideals $K_F = \{\alpha \in \mathrm{End}(G) | \alpha(F) = 0\}$, ranging F over the finite subsets of G. The topological ring $(\mathrm{End}(G), fin_G)$ is Hausdorff and complete—see, for example, [27, Theorem 107.1]. When $F = \{g\}$ is a singleton, we write simply K_g. Obviously $\mathrm{End}(G)/K_g$ is isomorphic through the evaluation map at g to the orbit $O_g = \{\phi(g) : \phi \in \mathrm{End}(G)\}$. The next lemma, which will be used later on, provides a connection between cyclic ϕ-trajectories and the subring of $\mathrm{End}(G)$ generated by ϕ.

Lemma 2.1 *Let $\phi : G \to G$ be an endomorphism of the group G and $\mathbb{Z}[\phi]$ the subring of* $\mathrm{End}(G)$ *generated by ϕ. Then:*

(1) *$T(\phi, g)$ is isomorphic as $\mathbb{Z}[X]$-module to $\mathbb{Z}[\phi]/(\mathbb{Z}[\phi] \cap K_g)$, for all $g \in G$;*
(2) *if $F = \{g_i | 1 \leq i \leq n\}$ is a finite subset of G, there is a monomorphism of $\mathbb{Z}[X]$-modules $\epsilon : \mathbb{Z}[\phi]/(\mathbb{Z}[\phi] \cap K_F) \to \bigoplus_{1\leq i\leq n} T(\phi, g_i)$.*

The isomorphism in (1) is the restriction of the evaluation map at g; the embedding in (2) is the diagonal map of the isomorphisms arising from (1) for each of the g_i's.

2.4 Invariants and Length Functions

Given an arbitrary ring R, an *invariant* on Mod(R) is a function $i : \text{Mod}(R) \to \mathbb{R}^* = \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying the two conditions: $i(0) = 0$ and $i(M) = i(N)$ when $M \cong N$. In the following, we will use an invariant i to define algebraic entropies, but two additional properties are needed for i:

(i) $N \leq M$ implies $i(M) \geq i(M/N)$;
(ii) $N_1, N_2 \leq M$ implies $i(N_1 + N_2) \leq i(N_1) + i(N_2)$.

An invariant i is *subadditive* if it satisfies (i) and (ii); it is *faithful* if $i(M) = 0$ implies $M = 0$; it is *discrete* if the set of its values in $\mathbb{R}^*$ is order isomorphic to $\mathbb{N}$.

The invariants we are going to investigate in this paper, introduced by Northcott-Reufel [50] and investigated also by Vámos [63, 64], satisfy much stronger conditions.

Definition 2.2 Let R be a ring. A length function on Mod(R) is an invariant L : Mod(R) $\to \mathbb{R}^*$ satisfying the two conditions:

(A) L is additive on short exact sequences, that is, given an exact sequence: $0 \to A \to B \to C \to 0$ in Mod(R), the equality $L(B) = L(A) + L(C)$ holds;
(U) L is upper continuous, that is, for every R-module M, $L(M) = \sup_F L(F)$, where F ranges over the set of the finitely generated submodules of M.

Note that an additive invariant L satisfies conditions (i) and (ii), and also:

(iii) $N \leq M$ implies $L(N) \leq L(M)$.

A useful characterization of upper continuity was given by Vámos in [64], who proved that an invariant L satisfying (iii) is upper continuous if and only if $L(M) = \sup_i L(M_i)$ whenever $M = \cup_i M_i$, where $\{M_i\}_i$ is a directed system of submodules of M. This characterization will be used to prove that the entropies investigated below are upper continuous invariants.

The three most popular examples of length functions and invariants on Mod($\mathbb{Z}$) are given below.

Example 2.3 1) Let $R = \mathbb{Z}$. Then a length function L : Mod($\mathbb{Z}$) $\to \mathbb{R}^*$ is the rank-function $rk_{\mathbb{Z}}(-)$, that can be defined, for any Abelian group G, as $rk_{\mathbb{Z}}(G) = dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$. The rank-function $rk_{\mathbb{Z}}(-)$ is not faithful, since $rk_{\mathbb{Z}}(G) = 0$ for any torsion group G.

2) Another length function $L : \mathrm{Mod}(\mathbb{Z}) \to \mathbb{R}^*$ is defined by setting $L(G) = \log |G|$, where it is understood that, if G is infinite, then $\log |G| = \infty$. The additive property of $\log |-|$ follows by Lagrange's theorem. Obviously $\log |-|$ is faithful and it differs from the classical length function just by the factors $\log p$ on each elementary group $\mathbb{Z}/p\mathbb{Z}$.
3) A faithful invariant on $\mathrm{Mod}(\mathbb{Z})$ which is subadditive but not additive is *gen*, defined by setting $gen(G)$ equal to the minimum number of generators of G, if G is finitely generated, or $gen(G) = \infty$ otherwise. In this case *gen* is upper continuous and satisfies condition (iii).

Some basic results for length functions on commutative rings proved in [50] and [64] are collected in the next proposition; the interested reader may also wish to consult [72], where length functions over arbitrary valuation domains have been investigated.

Proposition 2.4 *Let R be a commutative ring and $L : \mathrm{Mod}(R) \to \mathbb{R}^*$ a length function. Then:*

(1) *if $P_1 < P_2$ are prime ideals of R with $L(R/P_2) > 0$, then $L(R/P_1) = \infty$;*
(2) *if R is an integral domain and $\infty > L(R) = r > 0$ ($r \in \mathbb{R}$), then L is essentially the rank-function, in the sense that $L = r \cdot rk(-)$;*
(3) *if R is Noetherian and F is a finitely generated R-module, then F has a finite chain of submodules:*

$$0 = F_0 < F_1 < F_2 < \cdots < F_{n-1} < F_n = F$$

such that $F_i/F_{i-1} \cong R/P_i$ for all $1 \leq i \leq n$, where the P_i's are prime ideals; consequently, $L(F) = \sum_{1 \leq i \leq n} L(R/P_i)$.

The important consequence of the preceding proposition is that, given a commutative Noetherian ring R, like $\mathbb{Z}$ or $\mathbb{Z}[X]$, a length function L on $\mathrm{Mod}(R)$ is determined by the values it takes on the cyclic modules $L(R/P)$, where P ranges in the prime spectrum $Spec(R)$. In fact, these values determine the values of $L(F)$ for every finitely generated module F, by (3), and these values determine the value $L(M)$ of any module M, by upper continuity.

Example 2.5 1) Let $R = \mathbb{Z}$. Then a length function $L : \mathrm{Mod}(\mathbb{Z}) \to \mathbb{R}^*$ is determined by the values $L(\mathbb{Z}/p\mathbb{Z})$ (p a prime integer) and $L(\mathbb{Z})$. If $\infty > L(\mathbb{Z}/p\mathbb{Z}) > 0$ for all p, then necessarily $L(\mathbb{Z}) = \infty$ by Proposition 2.4 (1), and L coincides (up to multiplication by a positive real) with the classical length function (which is equivalent to the function $\log | - |$). If $L(\mathbb{Z}/p\mathbb{Z}) = 0$ for all p and $\infty > L(\mathbb{Z}) > 0$, then L coincides (up to multiplication by a positive real) with the function $rk_{\mathbb{Z}}(-)$.
2) Let $R = \mathbb{Z}[X]$. A length function $L : \mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ is determined by the values on the cyclic modules $\mathbb{Z}[X]/(p, f(X))$, $\mathbb{Z}[X]/(p)$, $\mathbb{Z}[X]/(f(X))$ and $\mathbb{Z}[X]$ (see Sect. 2.2). If $\infty > L(\mathbb{Z}[X]) > 0$, then L essentially coincides with the function $rk_{\mathbb{Z}[X]}(-)$, and $L(R/P) = 0$ for all non-zero prime ideals.

Let R be an arbitrary ring and $L : \mathrm{Mod}(R) \to \mathbb{R}^*$ a length function. Following [65], given an R-module M we call its fully invariant submodule $z_L(M) = \{x \in M | L(Rx) = 0\}$ the *L-singular submodule* of M. This notion is useful only if L is not faithful, since $z_L(M) = 0$ for all M when L is faithful. The module M is called *L-singular* if $z_L(M) = M$ and the class of the L-singular modules is denoted by Ker_L. Clearly $M \in Ker_L$ if and only if $L(M) = 0$.

Setting $\mathscr{F}_L = \{M \in \mathrm{Mod}(R) | L(Rx) > 0 \text{ for all } 0 \neq x \in M\}$, the pair $(Ker_L, \mathscr{F}_L)$ is a hereditary torsion theory and z_L is its associated idempotent radical. One of the main concerns in this paper will be the investigation of the L-singular submodules $z_L(G_\phi)$ of the $\mathbb{Z}[X]$-modules G_ϕ, and of the L-singular modules of the torsion class Ker_L, when L coincides with one of the algebraic entropies we are going to introduce in the next section.

3 Algebraic Entropies for Abelian Groups

3.1 Definition of the Algebraic Entropies

We start defining the three entropies *ent*, ent_{rk} and h, which use in their definition the partial trajectories. The difference between *ent* and h is that the former considers trajectories of finite subgroups, while the latter of finite subsets; they share the length function $log| - |$ to evaluate the size of the partial trajectories. The entropy ent_{rk} considers trajectories of subgroups of finite rank, and it makes use of the length function $rk_{\mathbb{Z}}(-)$ to evaluate the size of the partial trajectories.

The definition of the three entropies relies on the existence of certain limits of sequences of non-negative real numbers, which is ensured by the following well-known lemma due to Fekete [26]. Recall that a sequence of non-negative real numbers $\{a_n\}_n$ is *subadditive* if $a_{m+n} \leq a_m + a_n$ for all m, n.

Lemma 3.1 (Fekete) *Let $\{a_n\}_n$ be a subadditive sequence of non-negative real numbers. Then the limit* $\lim_{n\to\infty} a_n/n$ *exists finite and it coincides with* $\inf_n\{a_n/n\}$.

Now let $\phi : G \to G$ be an endomorphism of the Abelian group G. Let F be a finite subgroup of G; then the partial ϕ-trajectories of F form the ascending chain of finite subgroups

$$F = T_1(\phi, F) \leq T_2(\phi, F) \leq \cdots \leq T_n(\phi, F) \leq \cdots$$

from which we derive the sequence of non-negative real numbers $\{\log |T_n(\phi, F)|\}_n$. From the equality

$$T_{m+n}(\phi, F) = T_m(\phi, F) + \phi^m T_n(\phi, F)$$

and from the fact that $\phi^m T_n(\phi, F)$ is an epic image of $T_n(\phi, F)$, we get that

$$\log |T_{m+n}(\phi, F)| \leq \log |T_m(\phi, F)| + \log |T_m(\phi, F)$$

for all m, n, that is, the sequence $\{\log |T_n(\phi, F)|\}_n$ is subadditive, so we can apply Fekete's lemma.

If we replace the finite subgroup F by a finite subset S of G (resp., by a subgroup of finite rank H), we get the subadditive sequence of non-negative real numbers $\{\log |T_n(\phi, S)|\}_n$ (resp., $\{rk(T_n(\phi, H))\}_n$). We can now collect the definitions of the three entropies considered up to now.

Definition 3.2 (1) The algebraic entropy of ϕ with respect to the finite subgroup F is the limit

$$ent(\phi, F) = \lim_{n\to\infty} \log |T_n(\phi, F)|/n.$$

The algebraic entropy of ϕ is $ent(\phi) = \sup_F ent(\phi, F)$, the sup taken over the set of the finite subgroups of G.

(2) The rank-entropy of ϕ with respect to the subgroup of finite rank H of G is the limit

$$ent_{rk}(\phi, H) = \lim_{n\to\infty} rk(T_n(\phi, H))/n.$$

The rank-entropy of ϕ is $ent_{rk}(\phi) = \sup_H ent_{rk}(\phi, H)$, the sup taken over the set of subgroups of finite rank of G.

(3) The algebraic entropy h of ϕ with respect to the finite subset S of G is the limit

$$h(\phi, S) = \lim_{n\to\infty} \log |T_n(\phi, S)|/n.$$

The algebraic entropy h of ϕ is $h(\phi) = \sup_S h(\phi, S)$, the sup taken over the set of the finite subsets of G.

A first comment is immediately in order. Since finite non-zero subgroups of G are contained in its torsion part $t(G)$, it follows that $ent(\phi) = ent(\phi \restriction t(G))$; so the algebraic entropy ent is useful for torsion groups only and it vanishes for all endomorphisms of torsion-free groups.

Dually, since for an arbitrary group G we have $rk(G) = rk(G/t(G))$, it follows that $ent_{rk}(\phi) = ent_{rk}(\bar{\phi})$, where $\bar{\phi} : G/t(G) \to G/t(G)$ is the map induced by ϕ; so the rank-entropy ent_{rk} is useful for torsion-free groups only and it vanishes for all endomorphisms of torsion groups.

The definition of the intrinsic entropy $\widetilde{ent}(\phi)$ is based on a characterization of the algebraic entropy $ent(\phi)$, which makes its computation limit-free. This characterization relies on the observation that, given a subgroup F of G, and setting, for the sake of simplicity, $T_n = T_n(\phi, F)$, one has two surjective homomorphisms

$$T_n/T_{n-1} \to T_n/(T_{n-1} + (T_n \cap Ker\phi)) \cong \phi^n F/(\phi T_{n-1} \cap \phi^n F) \to$$
$$\to \phi^n F/(T_n \cap \phi^n F) \cong T_{n+1}/T_n.$$

So we get for every $n > 1$ an epimorphism $T_n/T_{n-1} \to T_{n+1}/T_n$. This implies that, if F is finite, then the decreasing sequence $\{\log |T_n/T_{n-1}|\}_{n>1}$ is stationary, and similarly, if F has finite rank, then the sequence $\{rk(T_n/T_{n-1})\}_{n>1}$ is stationary. From these facts, an easy calculation (for details we refer to [58, Proposition 1.10]) gives the following

Proposition 3.3 *Let $\phi : G \to G$ be an endomorphism of the Abelian group G.*

(1) *If F is a finite subgroup of G, then $ent(\phi, F) = \log |T_{n+1}(\phi, F)/T_n(\phi, F)|$ for all n large enough.*
(2) *If H is a subgroup of finite rank of G, then $ent_{rk}(\phi, H) = rk(T_{n+1}(\phi, H)/T_n(\phi, H))$ for all n large enough.*

As a byproduct of Proposition 3.3 we derive an easy way to compute the algebraic entropy and the rank-entropy of Bernoulli shifts.

Corollary 3.4 *Let $\beta : \bigoplus_{\mathbb{N}} G \to \bigoplus_{\mathbb{N}} G$ be the right Bernoulli shift for the group G. Then*

(1) $ent(\beta) = \log |t(G)|$;
(2) $ent_{rk}(\beta) = rk(G)$.

Proof We just sketch the proof. Let F be a finite subgroup of $\bigoplus_{\mathbb{N}} G$. Then $F \leq K = \bigoplus_{i \leq n} F'$ for F' a finite subgroup of $t(G)$. A basic property of the entropy ensures that $ent(\beta, F) \leq ent(\beta, K)$, and clearly for each $n \geq 1$ we have $ent(\beta, K) = \log |T_{n+1}(\beta, K)/T_n(\beta, K)| = \log |F'|$. Taking suprema, from this fact it is not difficult to prove (1). Replacing F and F' by subgroups of finite rank of G, from the similar equality $rk(T_{n+1}(\beta, K)/T_n(\beta, K)) = rk(F')$ one derives (2). □

A consequence of Proposition 3.3 (1) and of the fact that T_{n+1}/T_n is a quotient of T_2/T_1 for all n, is that, in order to have the finiteness of $ent(\phi, F)$, we do not need F finite, but rather $T_2(\phi, F)/T_1(\phi, F) = (F + \phi F)/F$ finite will suffice. Thus we are led to the following:

Definition 3.5 Let $\phi : G \to G$ be an endomorphism of the Abelian group G. A subgroup H of G is ϕ-inert if $(H + \phi H)/H$ is finite.

An inductive argument shows that, if H is ϕ-inert in G, then $T_n(\phi, H)/H$ is finite for all n. Finite subgroups, as well as subgroups of finite index, and fully invariant subgroups are examples of subgroups which are ϕ-inert for all endomorphisms ϕ. For more details of this and the related concept of fully inert subgroups, see [22, 24, 37] and [38].

We can now give the definition of intrinsic entropy.

Definition 3.6 Let $\phi : G \to G$ be an endomorphism of the Abelian group G. The intrinsic entropy of ϕ with respect to the ϕ-inert subgroup H is the limit

$$\widetilde{ent}(\phi, H) = \lim_{n\to\infty} \log |T_n(\phi, H)/H|/n.$$

The intrinsic entropy of ϕ is $\widetilde{ent}(\phi) = \sup_H \widetilde{ent}(\phi, H)$, the sup taken over the set of the ϕ-inert subgroups of G.

An argument parallel to that used in the proof of Proposition 3.3 gives the next result, which makes also the computation of $\widetilde{ent}(\phi)$ limit-free and shows that the algebraic entropy and the intrinsic entropy may be computed in the same way, with the only difference that in the first case we consider finite subgroups, while in the latter case we consider the larger family of ϕ-inert subgroups.

Proposition 3.7 *Let $\phi : G \to G$ be an endomorphism of the Abelian group G. If H is a ϕ-inert subgroup of G, then $\widetilde{ent}(\phi, H) = \log|T_{n+1}(\phi, H)/T_n(\phi, H)|$ for all n large enough.*

The first question arising from the above definition is: can we compare the intrinsic entropy of an endomorphism with the algebraic entropy *ent* and with the entropy h? The answer is given by the following proposition.

Proposition 3.8 *Let $\phi : G \to G$ be an endomorphism of the Abelian group G. Then*

(1) $ent(\phi) \leq \widetilde{ent}(\phi) \leq h(\phi)$;
(2) if G is a torsion group, then the two inequalities are indeed equalities.

Proof (1) Since the family of finite subgroups of G is contained in the family of ϕ-inert subgroups, and since *ent* and $\widetilde{ent}$ may be computed in the same way, as Proposition 3.3 and Proposition 3.7 show, the inequality $ent(\phi) \leq \widetilde{ent}(\phi)$ holds. To prove the latter inequality, let H be a ϕ-inert subgroup of G. Then $(H + \phi H)/H$ is finite, hence there exists a finite subset S of G such that $H + \phi H = H + \phi S$; an easy inductive argument shows that $T_n(\phi, H) = H + T_n(\phi, S)$ for all n, hence $|T_n(\phi, H)/H| = |(H + T_n(\phi, S))/H| \leq |T_n(\phi, S)|$. Taking logarithms, dividing by n and passing to the limit we get $\widetilde{ent}(\phi, H) \leq \lim_{n\to\infty} \log|T_n(\phi, S)|/n \leq h(\phi)$, consequently $\widetilde{ent}(\phi) \leq h(\phi)$.

(2) If G is torsion, every finite subset of G is contained in a finite subgroup, from which it immediately follows that $ent(\phi) = h(\phi)$. □

We will see that also the inequality $ent_{rk}(\phi) \leq \widetilde{ent}(\phi)$ holds. Instead than giving a direct proof, we prefer to postpone it after the Addition Theorem will be available.

In analogy with Proposition 3.3, Proposition 3.7 can be used to compute the intrinsic entropy of Bernoulli shifts. As the case of torsion groups is covered by Corollary 3.4, since in that case the intrinsic entropy coincides with the algebraic entropy, by Proposition 3.8, we consider only the case of torsion-free groups.

Corollary 3.9 *Let $\beta : \bigoplus_{\mathbb{N}} G \to \bigoplus_{\mathbb{N}} G$ be the right Bernoulli shift for the torsion-free group G. Then $\widetilde{ent}(\beta) = \infty$, and consequently $h(\beta) = \infty$.*

Proof One can easily restrict to the case $G = \mathbb{Z}$; so consider the right shift on $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_n$, with $\mathbb{Z}_n = \mathbb{Z}$ for each n. Fix an integer $k > 1$ and set $H_k = \mathbb{Z}_0 \oplus (\bigoplus_{n\geq 1} k\mathbb{Z}_n)$. For each $n \geq 1$ we have

$$T_n(\beta, H_k) = \mathbb{Z}_0 \oplus \cdots \oplus \mathbb{Z}_{n-1} \oplus k\mathbb{Z}_n \oplus k\mathbb{Z}_{n+1} \oplus \cdots ,$$

therefore $|T_n(\beta, H_k)/H_k| = |(\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_{n-1})/(k\mathbb{Z}_1 \oplus \cdots \oplus k\mathbb{Z}_{n-1})| = k^{n-1}$. This shows that the subgroup H_k is ϕ-inert in G. Now

$$\widetilde{ent}(\beta, H_k) = \lim_{n\to\infty} \log |T_n(\beta, H_k)/H_k|/n = \lim_{n\to\infty} \log(k^{n-1})/n = \log k.$$

Consequently $\widetilde{ent}(\beta) \geq \sup_{H_k} \widetilde{ent}(\beta, H_k) = \sup_k \log k = \infty$. □

Basic properties satisfied by the four algebraic entropies defined in this section are listed below; their proofs are straightforward. Denoting by *Ent* an arbitrary entropy among $ent, ent_{rk}, \widetilde{ent}, h$, and $\phi : G \to G$, $\psi : K \to K$ endomorphisms of groups, the following hold:

(i) if ϕ and ψ are conjugated endomorphisms of the isomorphic groups G, K(i.e. there exists an isomorphism $\theta : G \to K$ such that $\theta \circ \phi = \psi \circ \theta$), then $Ent(\phi) = Ent(\psi)$;
(ii) $Ent(\phi^k) = k \cdot Ent(\phi)$ for every $k \geq 1$;
(iii) $Ent(\phi \oplus \psi) = Ent(\phi) + Ent(\psi)$;
(iv) if H is a ϕ-invariant subgroup of G, then $Ent(\phi) \geq Ent(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the induced map;
(v) if H is a ϕ-invariant subgroup of G, then $Ent(\phi) \geq Ent(\phi \restriction H)$;
(vi) if G is the union of a directed system of ϕ-invariant subgroups G_i $(i \in I)$, then $Ent(\phi) = \sup_i Ent(\phi \restriction G_i)$.

Some of the above properties are used to prove the Addition Theorem (see next section), and, except (ii) and (vi), they are obvious consequences of it.

3.2 The Addition Theorem

Let *Ent* denote any function among $ent_{rk}, \widetilde{ent}$ and h. We look at *Ent* as a function

$$Ent : \mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$$

defined by setting $Ent(G_\phi) = Ent(\phi)$. In a similar way we look at *ent* as the function

$$ent : \mathrm{Tor}(\mathbb{Z}[X]) \to \mathbb{R}^*$$

defined by $ent(G_\phi) = ent(\phi)$, where $\mathrm{Tor}(\mathbb{Z}[X])$ consists of those $\mathbb{Z}[X]$-modules G_ϕ such that G is a torsion group.

Note that property (i) in the preceding section ensures that the functions $Ent : \mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ and $ent : \mathrm{Tor}(\mathbb{Z}[X]) \to \mathbb{R}^*$ are invariants. Furthermore, properties (iii) and (iv) ensure that *Ent* and *ent* are subadditive invariants, property (v) ensures that they satisfy property (iii) in Sect. 2.4, and property (vi) that they are upper continuous invariants.

Definition 3.10 We say that the function *Ent* (respectively, *ent*) satisfies the Addition Theorem (AT, for short) if it is additive on $\mathrm{Mod}(\mathbb{Z}[X])$ (respectively, on $\mathrm{Tor}(\mathbb{Z}[X])$).

Recalling property (III) in Sect. 2.1, this amounts to say that, if H is a ϕ-invariant subgroup of the group G endowed with the endomorphism ϕ, then $Ent(\phi) = Ent(\phi \restriction H) + Ent(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the map induced by ϕ (similarly for *ent* with G torsion).

Notice that we cannot hope that *ent* satisfies AT on the whole category $\mathrm{Mod}(\mathbb{Z}[X])$. In fact, let $\beta : \bigoplus_{\mathbb{N}} \mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z}$ be the right Bernoulli shift; consider the induced map $\bar{\beta} : \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, where $n > 1$ is a fixed integer. From Corollary 3.4 we know that $ent(\bar{\beta}) = \log n$, while $ent(\beta) = 0$, since $\bigoplus_{\mathbb{N}} \mathbb{Z}$ is torsion-free; therefore, *ent* is not additive on the exact sequence of $\mathbb{Z}[X]$-modules

$$0 \to (\bigoplus_{\mathbb{N}} n\mathbb{Z})_{\beta} \to (\bigoplus_{\mathbb{N}} \mathbb{Z})_{\beta} \to (\bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z})_{\bar{\beta}} \to 0.$$

The main achievements of the whole theory of the algebraic entropies, one for each different entropy, are collected below in a single statement.

Theorem 3.11 (Addition Theorem) *The functions* $ent_{rk}, \widetilde{ent}$ *and* h *satisfy AT on* $\mathrm{Mod}(\mathbb{Z}[X])$, *and the function ent satisfies AT on* $\mathrm{Tor}(\mathbb{Z}[X])$.

The first proof of AT, given for the entropy ent in [19], was by induction on the values of $ent(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the map induced by ϕ on the factor group G/H of G modulo the ϕ-invariant subgroup H of G. This approach is possible since *ent* is a discrete invariant. The proof of AT for the four entropies always requires hard work, and there is no room in a survey like this to give full details of them. But we think that it is useful for the interested reader to have an idea of the tools and of the techniques used to prove AT; therefore we will outline:

A) the adaptation to *ent* of the proof of AT given in a recent paper [56] for entropies induced by arbitrary length functions. This result covers, with the due modifications, also AT for the rank-entropy ent_{rk}. However, it is worthwhile remarking that in [58], using the properties of the rank-entropy, it is proved that $ent_{rk}(G_\phi) = rk_{\mathbb{Z}[X]}(G_\phi)$. From this equality AT for ent_{rk} follows as an immediate corollary;
B) the proof of AT for $\widetilde{ent}$ given in [25], with an improvement at a certain step given in [30].

We do not offer even a sketch of the more complicated proof of AT given in [18] for the entropy h. It is worthwhile recalling that this proof was available some years before the submission for publication of [18], and offered a basis and inspiration for the proof of AT for $\widetilde{ent}$ in [25].

Outline A The proof that, given an endomorphism $\phi : G \to G$ of a torsion group G and a ϕ-invariant subgroup H of G, then $ent(\phi) = ent(\phi \restriction H) + ent(\bar{\phi})$, is achieved via the following steps:

A.1. Reduction to an injective endomorphism passing from ϕ to $\bar{\phi}$, where $\bar{\phi}: G/K_\infty \to G/K_\infty$ is the map induced by ϕ on the quotient of G modulo the hyperkernel $K_\infty = \cup_n Ker\phi^n$, showing that $ent(\phi) = ent(\bar{\phi})$.

A.2. Reduction to an automorphism by means of the functor $- \otimes_{\mathbb{Z}[X]} \mathbb{Z}[X^{\pm 1}]$ (localization at $\mathbb{Z}[X^{\pm 1}]$); setting $N_\psi = G_\phi \otimes_{\mathbb{Z}[X]} \mathbb{Z}[X^{\pm 1}]$, one can prove that $ent(\phi) = ent(\psi)$, using the fact that N_ψ is isomorphic to the direct limit of the directed system: $G_\phi \stackrel{\phi}{\to} G_\phi \stackrel{\phi}{\to} \cdots \stackrel{\phi}{\to} G_\phi \stackrel{\phi}{\to} \cdots$.

A.3. Proof of AT in case ϕ is an automorphism, using the fact that, in this case, one can avoid the limit calculation, since it is possible to prove that

$$ent(\phi) = \sup_N \{\log |N/\phi^{-1}N|\},$$

where the sup is taken over the set of ϕ^{-1}-invariant subgroups N of G having the property that $N/\phi^{-1}N$ is finite.

The proof outlined above is completely different from the proof of AT given in [19] for *ent* and from that in [60] for algebraic entropies induced by *discrete* length functions. However, non-discrete length functions appear as soon as the Noetherian condition on the ground ring is no longer assumed [73], and in this non-discrete case the new proof in [56] is needed. We recall that a limit-free computation of the entropy *ent* for endomorphisms of p-groups was found also in [12].

Outline B The proof that, given an endomorphism $\phi : G \to G$ of an arbitrary group G and a ϕ-invariant subgroup H of G, then $\widetilde{ent}(\phi) = \widetilde{ent}(\phi \restriction H) + \widetilde{ent}(\bar{\phi})$, is achieved via the following steps.

B.1. Reduction from the general case to the case when $G = T(\phi, F)$, for F a finitely generated subgroup of G. This reduction follows from the general fact, holding over a Noetherian ring like $\mathbb{Z}[X]$, that the additivity of an upper continuous invariant can be tested looking only at finitely generated modules. Of course, this needs the preliminary proof that the invariant $\widetilde{ent}$ is upper continuous. This fact is proved using basic simple properties of $\widetilde{ent}$ and Vámos's characterization of upper continuity mentioned after Definition 2.2.

B.2. Proof of AT in case $G = T(\phi, F)$ in two different cases, according to whether G has finite or infinite rank. In the crucial case that G has finite rank, one can consider separately the torsion-free and the torsion case, via a thorough inspection of the structure of G. The torsion-free case is treated using next steps B.3 and B.4; the torsion case uses AT for *ent*.

B.3. If $\phi : G \to G$ is an endomorphism of a torsion-free group G, and $\psi : D(G) \to D(G)$ is the unique extension of ϕ to its divisible hull, then $\widetilde{ent}(\phi) = \widetilde{ent}(\psi)$.

B.4. Proof of AT in case $G = T(\phi, F)$ is torsion-free of finite rank and H is a ϕ-invariant pure subgroup; by means of B.3 one can pass to an endomorphism ψ of $\mathbb{Q}^n$ and a ψ-invariant $\mathbb{Q}$-vector subspace K, and show that $\widetilde{ent}(\psi) = \widetilde{ent}(\psi \restriction K) + \widetilde{ent}(\bar{\psi})$ (where $\bar{\psi} : \mathbb{Q}^n/K \to \mathbb{Q}^n/K$ is the induced map).

This step for $\mathbb{Q}$-vector spaces was proved in [25] with the aid of the Intrinsic Yuzvinski Formula (see next Sect. 3.3), and in a direct simpler way in [30].

Remark 3.12 The proof of AT for the algebraic entropy h given in [18] requires at a certain step ([18, Proposition 3.12]) the Algebraic Yuzvinski Formula (AYF for short; see next section). This step is analogous to step B.4 above for h. It would be interesting to have a proof of this step for the algebraic entropy h similar to that given in [30] for $\widetilde{ent}$.

We already mentioned in step B.1 above that $\widetilde{ent}$ is an upper continuous invariant on $\mathrm{Mod}(\mathbb{Z}[X])$. Observe that ent is upper continuous by definition on $\mathrm{Tor}(\mathbb{Z}[X])$. Furthermore, the upper continuity on arbitrary $\mathbb{Z}[X]$-modules G_ϕ of ent_{rk} depends on the fact that, given a subgroup H of finite rank of G, there exists a finitely generated subgroup F of H such that $rk(H) = rk(F)$. Finally, the proof that h is upper continuous makes use, as in case of $\widetilde{ent}$, of Vámos's characterization of upper continuity mentioned after Definition 2.2, and of basic properties of h. So we have the following

Proposition 3.13 *The entropies $ent_{rk}, \widetilde{ent}$ and h are upper continuous, therefore they are length functions on* $\mathrm{Mod}(\mathbb{Z}[X])$. *The algebraic entropy ent is upper continuous, therefore it is a length function on* $\mathrm{Tor}(\mathbb{Z}[X])$.

The four entropies considered in Proposition 3.13 are not faithful invariants. The investigation of the consequences of this fact will be made in Sect. 3.5.

Looking at Sect. 2.2 on cyclic $\mathbb{Z}[X]$-modules, the advantage of having at disposal Proposition 3.13 is that $ent_{rk}, \widetilde{ent}$ and h are completely determined by the values they assume on the cyclic $\mathbb{Z}[X]$-modules $\mathbb{Z}[X]/(p, f(X))$ ($f(X)$ irreducible mod p), $\mathbb{Z}[X]/(p)$ (p prime integer), $\mathbb{Z}[X]/(f(X))$ ($f(X)$ irreducible) and $\mathbb{Z}[X]$; also, their comparison follows easily from the comparison of their values on these cyclic modules. Concerning the algebraic entropy ent, it is enough to consider the values it assumes on the $\mathbb{Z}[X]$-modules which are torsion groups, namely, $\mathbb{Z}[X]/(p, f(X))$ and $\mathbb{Z}[X]/(p)$.

The following table shows the different values on these cyclic $\mathbb{Z}[X]$-modules of the four entropies. In the fourth row, where $\log s$ and $\sum_{|\lambda_i|>1} \log |\lambda_i|$ appear, $s \geq 1$ denotes the leading coefficient of the irreducible polynomial $f(X) \in \mathbb{Z}[X]$, and the λ_i denote its (complex) eigenvalues.

TABLE OF THE VALUES OF THE 4 ENTROPIES ON THE CYCLIC MODULES $\mathbb{Z}[X]/P$

	$\mathbf{ent_{rk}}$	$\mathbf{ent}$	$\widetilde{ent}$	$\mathbf{h}$
$\mathbb{Z}[\mathbf{X}]/(\mathbf{p}, \mathbf{f}(\mathbf{X}))$	0	0	0	0
$\mathbb{Z}[\mathbf{X}]/(\mathbf{p})$	0	$\log p$	$\log p$	$\log p$
$\mathbb{Z}[\mathbf{X}]/(\mathbf{f}(\mathbf{X}))$	0	0	$\log s$	$\log s + \sum_{\lvert\lambda_i\rvert>1} \log \lvert\lambda_i\rvert$
$\mathbb{Z}[\mathbf{X}]$	1	0	∞	∞

The comparison between the four entropies, included the inequality $ent_{rk} \leq \widetilde{ent}$, can be easily deduced from the preceding table. We remark that some computation in the above table is obvious, as the 0's in the first row, due to the fact that $\mathbb{Z}[X]/(p, f(X)) \cong F_p[X]/(f_p(X))$ is a finite group. Some other computation is easy and has already been made, as the value $\log p$ in the second row [since $\mathbb{Z}[X]/(p) \cong F_p[X]$—see Corollary 3.4 (1), and Proposition 3.8 (2)], or the value $ent_{rk}(\mathbb{Z}[X]) = 1$ [see Corollary 3.4 (2)]. Some other computation is less easy, such as $\widetilde{ent}(\mathbb{Z}[X]) = \infty = h(\mathbb{Z}[X])$ proved in Corollary 3.9. Finally, the computations of the values different from 0 in the third row are challenging, and are the subject of the next Sect. 3.3.

3.3 Algebraic Yuzvinski Formulas and Uniqueness Theorems

Let $f(X) \in \mathbb{Z}[X]$ be a primitive polynomial of degree $n \geq 1$, not necessarily irreducible, with leading coefficient $s \geq 1$ and (complex) eigenvalues $\lambda_1, \cdots, \lambda_n$ (counted with their multiplicities). The (additive) *Mahler measure* of $f(X)$ is

$$m(f(X)) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|.$$

The Mahler measure was first considered by Lehmer in 1933 [43] and later on defined independently by Mahler in 1962 [46]. The famous Lehmer's Problem in number theory asks whether the infimum of the positive Mahler measures of monic integral polynomials is strictly positive. The paper [39] reports that Mossinghoff et al. verified by computer that the number $\log \lambda$, where $\lambda = 1.17628\ldots..$ is the Lehmer number, i.e., the largest real root of the palindromic polynomial

$$f_\lambda(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

is the smallest positive Mahler measure for all monic integral polynomials of degree up to 40. We refer to [39, 49] and [62] for more information on this subject.

The Algebraic Yuzvinski Formula, proved recently by Giordano Bruno-Virili in [29], relates the algebraic entropy h of an endomorphism of a finite dimensional $\mathbb{Q}$-vector space to the Mahler measure of its characteristic polynomial over $\mathbb{Z}$, which is nothing other than the (monic) characteristic polynomial over $\mathbb{Q}$ multiplied by the minimal common multiple s of the denominators of its rational coefficients.

Theorem 3.14 (Algebraic Yuzvinski Formula-AYF) *Let $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ be a linear transformation. Then $h(\phi) = m(f(X))$, where $f(X)$ is the characteristic polynomial of ϕ over $\mathbb{Z}$.*

It is worth mentioning that a first step in demonstrating the Algebraic Yuzvinski Formula was done proving the case zero of AYF in [23], with arguments exclusively of linear algebra. Indeed, Corollary 1.4 in [23] shows that $h(\phi) = 0$ if and only if $m(f(X)) = 0$.

The Intrinsic Algebraic Yuzvinski Formula, proved in [25], is a simplified variation of the Algebraic Yuzvinski Formula, related to the intrinsic algebraic entropy $\widetilde{ent}$ rather than to the algebraic entropy h.

Theorem 3.15 (Intrinsic Algebraic Yuzvinski Formula-IAYF) *Let $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$ be a linear transformation. Then $\widetilde{ent}(\phi) = \log s$, where s is the leading coefficient of $f(X)$, the characteristic polynomial of ϕ over $\mathbb{Z}$.*

It is not difficult to see that AYF and IAYF are equivalent to the equalities $h(\mathbb{Z}[X]/(f(X)) = m(f(X))$ and $\widetilde{ent}(\mathbb{Z}[X]/(f(X)) = \log s$, respectively, which are the values different from 0 in the third row of the table in the preceding Sect. 3.2.

The source of AYF is the nice formula obtained by Yuzvinski in 1968 [71], which states that the topological entropy of a continuous endomorphism $\psi : \hat{\mathbb{Q}}^n \to \hat{\mathbb{Q}}^n$ ($\hat{\mathbb{Q}}$ is the Pontryagin dual of $\mathbb{Q}$) equals the Mahler measure of the characteristic polynomial over $\mathbb{Z}$ of the dual endomorphism $\hat{\psi} : \mathbb{Q}^n \to \mathbb{Q}^n$. We refer to the papers by Giordano Bruno-Virili [29] and [30] for detailed descriptions, comparison and comments on these theorems.

It is outside of our objectives to enter into the very complicated proof of the Algebraic Yuzvinski Formula concerning the algebraic entropy h; the interested reader can consult directly the papers [29] and [30]. In this survey we confine ourselves to sketch the proof of the Intrinsic Algebraic Yuzvinski Formula concerning the intrinsic entropy $\widetilde{ent}$, which explains in some sense the somewhat mysterious term $\log s$ appearing in AYF. We will not follow the original proof in [25], but the simpler and direct proof provided recently in [30].

Outline of the Proof of IAYF

(i) The core of the proof is the case of $f(X)$ irreducible. Under this hypothesis, for any $0 \neq x \in \mathbb{Q}^n$, the nth partial ϕ-trajectory $F = T_n(\phi, x)$ has rank n, and a simple argument shows that $\widetilde{ent}(\phi) = \widetilde{ent}(\phi, F)$. Since $f(\phi)(x) = 0$, we get $s\phi^n(x) \in F$, hence $(F + \phi F)/F$ is a quotient of $\mathbb{Z}/s\mathbb{Z}$ and consequently $\widetilde{ent}(\phi, F) \leq \log |(F + \phi F)/F| \leq \log s$. To prove the converse inequality, one first needs to prove that every polynomial $p(X) \in \mathbb{Z}[X]$ such that $p(\phi)(x) = 0$ is a multiple in $\mathbb{Z}[X]$ of $f(X)$. Then, assuming by way of contradiction that

$|T_{k+1}(\phi, F)/T_k(\phi, F)| = t < s$ for some k, one obtains a primitive polynomial $p(X) \in \mathbb{Z}[X]$ with leading coefficient t such that $p(\phi)(x) = 0$. So, from the fact that $f(X)$ divides $p(X)$, one derives that s divides t, a contradiction. Henceforth, $|T_{k+1}(\phi, F)/T_k(\phi, F)| = s$ for all k and from Proposition 3.7 one concludes that $\widetilde{ent}(\phi) = \log s$.

(ii) The passage from the irreducible case to the general case when $f(X)$ is just primitive makes use of a typical argument of finite dimensional vector spaces. The $\mathbb{Q}$-vector space $\mathbb{Q}^n = V$ is the union of a finite chain of ϕ-invariant subspaces $0 = V_0 < V_1 < \cdots < V_k < V_{k+1} = V$ such that each map $\phi_i : V_{i+1} \to V_i$ induced by ϕ has characteristic polynomial $f_i(X)$ over $\mathbb{Z}$ which is irreducible. Let s_i be the leading coefficient of $f_i(X)$ for each i. Then $f(X) = \prod_{i=0}^{k} f_i(X)$ and $s = \prod_{i=0}^{k} s_i$, so from step B.4 in Outline B in Sect. 3.2 we deduce from (i) that $\widetilde{ent}(\phi) = \sum_{i=1}^{k+1} \widetilde{ent}(\phi_i) = \sum_{i=0}^{k} \log s_i = \log s$.

Recall that from Proposition 2.4 we deduced that a length function $L : \mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ is determined by the values it takes on the cyclic modules $\mathbb{Z}[X]/P$, where P ranges over the prime spectrum $Spec(\mathbb{Z}[X])$. From this fact we can derive uniqueness results for the various entropies considered up to now.

The simplest case is when $L = ent_{rk}$, because of the result by Northcott-Reufel [50] mentioned earlier, which states that a length function L on a domain R such that $L(R) = 1$ necessarily coincides with the function $rk(-)$. When $R = \mathbb{Z}[X]$, the fact that $ent_{rk}(\mathbb{Z}[X]) = 1$ gives the following

Theorem 3.16 *The rank entropy ent_{rk} is the unique length function L : $\mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ such that $L(\mathbb{Z}[X]) = 1$; consequently $ent_{rk} = rk_{\mathbb{Z}[X]}$.*

Note that $ent_{rk}(\mathbb{Z}[X]) = 1$ implies that $ent_{rk}(\mathbb{Z}[X]/P) = 0$ for all non-zero prime ideals of $\mathbb{Z}[X]$, by Proposition 2.4 (1). Also the case of the algebraic entropy *ent* is very simple, since only its values on the torsion groups $\mathbb{Z}[X]/(p) \cong F_p[X]$ (p a prime integer) must be checked.

Theorem 3.17 *The entropy ent is the unique length function $L : \mathrm{Tor}(\mathbb{Z}[X]) \to \mathbb{R}^*$ such that $L(\mathbb{Z}[X]/(p)) = \log p$ for every prime integer p.*

Finally, the uniqueness results for the intrinsic entropy $\widetilde{ent}$ and the algebraic entropy h take care of the fact that the two entropies coincide with *ent* on torsion groups, and of AYF and IAYF.

Theorem 3.18 *The intrinsic entropy $\widetilde{ent}$ is the unique length function L : $\mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ such that $L(\mathbb{Z}[X]/(p)) = \log p$ for every prime integer p and $L(\mathbb{Z}[X]/(f(X))) = \log s$ for every primitive polynomial $f(X)$ with leading coefficient s.*

Theorem 3.19 *The algebraic entropy h is the unique length function L : $\mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ such that $L(\mathbb{Z}[X]/(p)) = \log p$ for every prime integer p*

and $L(\mathbb{Z}[X]/(f(X)) = \log s + \sum_{|\lambda_i|>1} \log|\lambda_i|$ *for every primitive polynomial* $f(X)$ *with leading coefficient* s *and eigenvalues* λ_i.

3.4 Adjoint Algebraic Entropy, Hopficity and Co-Hopficity

It is well known in measure-theoretic approaches to entropy that, in certain circumstances, measure-preserving transformations having zero entropy are necessarily invertible—see, for example, Chap. IV of the notes by Ward [68]. An obvious algebraic analogue of this result asks if monomorphisms having zero algebraic entropy are necessarily invertible. The global version of this question: *"if G is a group all whose endomorphisms have zero algebraic entropy, is every monomorphism necessarily invertible?"*, is of interest since groups with this type of invertibility property have long been studied under the name of co-Hopfian groups. Recall that a group is said to be *co-Hopfian* if every monomorphism is an automorphism. In the case of p-groups, the following connection was observed in [19] in a Note after Proposition 2.9:

Proposition 3.20 *If* G *is a* p*-group and* $ent(\phi) = 0$ *for all* $\phi \in \mathrm{End}(G)$*, then* G *is co-Hopfian.*

The proof of Proposition 3.20 is an immediate consequence of the fact that $ent(\phi) = 0$ if and only if ϕ is strongly recurrent, that is, for every $x \in G$ there exists an integer $n > 0$ such that $\phi^n(x) = x$, and this fact follows immediately from the equivalence of (1) and (2) in Theorem 3.38 in the next section.

Proposition 3.20 motivates the following definition, of general interest also for the other entropies.

Definition 3.21 Let $Ent \in \{ent, ent_{rk}, \widetilde{ent}, h\}$ be one of the entropies discussed above. Given a group G, the global entropy of G is the supremum of $Ent(\phi)$, where ϕ ranges over $\mathrm{End}(G)$, in symbols:

$$gl.Ent(G) = \sup\{Ent(\phi) \,:\, \phi \in \mathrm{End}(G)\}.$$

From the basic result (ii) at the end of Sect. 3.1, which says that $Ent(\phi^k) = k \cdot Ent(\phi)$ for every $k \geq 1$, it immediately follows that the global entropy of every group G is either 0 or ∞. Using this terminology, Proposition 3.20 says that for a p-group G, if $gl.ent(G) = 0$ then G is co-Hopfian.

There is, of course, a notion weakly dual to co-Hopficity: a group G is said to be *Hopfian* if every surjective endomorphism of G is an automorphism. (For further discussion of these notions, particularly in the context of Abelian groups, see, for example, [35] and the references therein.)

In trying to connect the notion of Hopficity to the vanishing of some type of entropy, a 'weakly dual' notion of entropy was used; this entropy was introduced in [20] and investigated in [59] and [34]. In this 'weakly dual' notion, one replaces finite subgroups by subgroups of finite index and trajectories utilize inverse images of an endomorphism rather than images. The resulting notion is called *adjoint entropy*—the reason for the choice of name will become apparent below.

Specifically let G be a group and N a finite index subgroup of G, ϕ an endomorphism of G, and for every fixed positive natural number n set

$$C_n(\phi, N) = N \cap \phi^{-1}N \cap \phi^{-2}N \cap \cdots \cap \phi^{-(n-1)}N.$$

It is pointed out in [20, 34] that $C_n(\phi, N)$ is a finite index subgroup in G. The subgroup $C_n(\phi, N)$ is called the *nth partial co-trajectory* of N with respect to ϕ, and the subgroup $C(\phi, N) = \bigcap_{n\geq 0}\phi^{-n}N$ is called the *co-trajectory* of N with respect to ϕ.

Denote $\log |G/C_n(\phi, N)|$ by $I_n(\phi, N)$, then the following limit exists as shown in [20, 34]:

$$I(\phi, N) = \lim_{n\to\infty} \frac{I_n(\phi, N)}{n}.$$

Let $\mathscr{N}(G)$ denote the family of all finite index subgroups of G. Then the *adjoint entropy* of ϕ is defined as

$$ent^*(\phi) = \sup\{I(\phi, N) \mid N \in \mathscr{N}(G)\}.$$

A word is in order about the choice of name: if G is a group, $\phi \in \mathrm{End}(G)$ and $\hat{G}$ is the Pontryagin dual of G (the continuous characters of G), $\hat{\phi}$ denotes the Pontryagin adjoint of ϕ (so $\hat{\phi}(\chi) = \chi \circ \phi$ for all $\chi \in \hat{G}$). Then the following main result was shown in [20].

Theorem 3.22 *If $\phi : G \to G$ is an endomorphism of a group G, then $ent^*(\phi) = ent(\hat{\phi})$.*

Notice that, unlike the algebraic entropy which works non-trivially only for torsion groups, the adjoint entropy is of interest in the category of all reduced groups; since a divisible group has no subgroups of finite index other than the whole group itself, adjoint entropy is trivial for divisible groups. We shall see shortly that a non-reduced group has adjoint entropy equal to that of its reduced summand.

The adjoint entropy has many interesting properties, but it is worth pointing out immediately that it fails to satisfy the fundamental property of the other entropies discussed in this survey: the Addition Theorem fails for adjoint entropy. Indeed, adjoint entropy does not even satisfy the weaker monotonicity property: suppose $B = \bigoplus_{n\geq 1} \mathbb{Z}(p^n)$ is a standard p-group and λ is the left Bernoulli shift on B, then it is easy to show that $ent^*(\lambda) = \infty$. However, if D denotes a divisible hull of B, the endomorphism λ of B extends to an endomorphism ψ of D. However, $ent^*(\psi) = 0$ as D is divisible, while $ent^*(\psi \restriction B) = ent^*(\lambda) = \infty$.

When H is a ϕ-invariant subgroup of a group G and $ent^*(H) = 0$, we do, however, get a weak analogue of the Addition Theorem:

Proposition 3.23 *Let ϕ be an endomorphism of a group G, H a ϕ-invariant subgroup of G, and $\overline{\phi} : G/H \to G/H$ the induced endomorphism. If $ent^*(\phi \restriction H) = 0$, or H is of finite index in G, then $ent^*(\phi) = ent^*(\overline{\phi})$.*

For a proof see [20, Lemma 4.10] and [34, Proposition 2.9]. Notice that an immediate consequence is that if $G = D \oplus R$, where D is divisible and R is reduced, then for any endomorphism ϕ of G, we have $ent^*(\phi) = ent^*(\overline{\phi})$, so that the adjoint entropy for a non-reduced group is the same as the adjoint entropy for the reduced part.

The situation improves, however, if one works with pure subgroups. In particular, the following holds:

Proposition 3.24 *Let G be a group and $\phi \in \mathrm{End}(G)$. If H is a ϕ-invariant pure subgroup of G and $\bar{\phi}$ is the induced endomorphism on G/H, then*

(i) $ent^*(\phi) \geq ent^*(\phi \restriction H)$;
(ii) $ent^*(\phi) \geq ent^*(\bar{\phi})$;
(iii) if $ent^*(\phi \restriction H) = 0$, then also $ent^*(\phi) = ent^*(\bar{\phi})$.

The proof of Proposition 3.24 is straightforward, if a little computational; see, for example, [20, Lemma 4.8], [59, Proposition 1.2] or [34, Proposition 2.9]. The significance of pure subgroups in this context is, in part, explained by the following fact observed in the last two of the preceding references.

Proposition 3.25 *If H is a pure subgroup of a group G, then there is an injection from $\mathcal{N}(H)$ into $\mathcal{N}(G)$. Moreover, if H is also dense in the natural topology on G, then the injection is actually a bijection.*

The proof of Proposition 3.25 follows easily from the observation that if H is pure in G and M is a finite index subgroup of H with $nH \leq M$, then G/nH splits as $H/nH \oplus X/nH$ for some X. The injection is then given by $M \mapsto M + X$. When H is also dense, the mapping $M + X \mapsto (M + X) \cap H$ is the required inverse.

The import of Proposition 3.25 is that if H is a pure dense ϕ-invariant subgroup of G, then $ent^*(\phi) = ent^*(\phi \restriction H)$. If G is a reduced p-group, then a basic subgroup B of G is pure and dense but not in general ϕ-invariant for an arbitrary endomorphism ϕ of G. However, by utilizing an old result of Szele—a basic subgroup of a p-group is always an endomorphic image of the group—one can establish the following interesting result:

Theorem 3.26 *A reduced p-group G satisfies $gl.ent^*(G) = 0$ if, and only if, it is finite.*

Details of the proof may be found in [59, Theorem 2.6] or [34, Corollary 2.24].

Recall that the first Ulm subgroup of a group G is $U(G) = \bigcap_{n \geq 1} nG$ (see [27, Sect. 6]); it is well known that $U(G)$ is the intersection of all the finite index subgroups of G. The following observation—see, for example, [34, Lemma 2.19]—is key to establishing a link between adjoint entropy and Hopficity.

Lemma 3.27 *Suppose that an epimorphism ϕ of a group G has zero adjoint entropy, then for any finite index subgroup N of G, the kernel of ϕ is contained in N.*

The precise connection is given by the following:

Theorem 3.28 *If G is a reduced group such that $gl.ent^*(G) = 0$ and $U(G)$ is Hopfian, then G is Hopfian. In particular, if G is reduced torsion-free such that $gl.ent^*(G) = 0$, then G is Hopfian.*

Proof Suppose that ϕ is an epimorphism of G. Then ϕ has zero adjoint algebraic entropy and so, by Lemma 3.27, letting N range over the finite index subgroups of G, we have $Ker\phi \subseteq \bigcap N = \bigcap nG = U(G)$. Since ϕ is epic, $G \cong G/Ker\phi$ and so $U(G) \cong U(G/Ker\phi)$. However, $Ker\phi \subseteq U(G)$ and as U is a radical, we have $U(G/Ker\phi) = U(G)/Ker\phi$. Thus, $U(G) \cong U(G)/Ker\phi$. If $Ker\phi \neq 0$, then $U(G)$ would have a proper isomorphic quotient, contrary to $U(G)$ being Hopfian. So we conclude $Ker\phi = 0$ and G is Hopfian as required.

The final observation follows immediately from the fact that a reduced torsion-free group has trivial first Ulm subgroup. □

We hasten to remark that the converse of Theorem 3.28 is not true: Corner [9] gives an example of a Hopfian group which has an endomorphism of infinite adjoint entropy—see [34, Sect. 3].

The observant reader may have noticed that we have only exhibited groups having endomorphisms with zero or infinite adjoint entropy. This is no coincidence since we have the following striking result ([20, Theorem 7.6]) establishing a dichotomy for the adjoint entropy, which has no counterpart in algebraic entropy.

Theorem 3.29 *If G is a group and $\phi \in \mathrm{End}(G)$, then either $ent^*(\phi) = 0$ or $ent^*(\phi) = \infty$.*

The proof of Theorem 3.29 relies on the fact that, if $ent^*(\phi) > 0$, then there exists a prime p such that $ent^*(\phi_p) > 0$, where $\phi_p : G/pG \to G/pG$ is the induced map. Then the core of the proof is contained in the next remarkable result of independent interest on vector spaces.

Theorem 3.30 *Let $\psi : V \to V$ be a linear transformation of the vector space V over the finite field F_p. Then $ent^*(\psi) < \infty$ if and only if $ent^*(\psi) = 0$, if and only if ψ is algebraic over F_p (i.e., V_ψ is a bounded $F_p[X]$-module).*

The reader interested in a more extensive treatment of the rank-entropy of linear transformations of vector spaces should consult [31].

Theorem 3.28 motivates the study of the reduced torsion-free and mixed groups G such that $gl.ent^*(G) = 0$. The classification of torsion-free groups with this property is essentially an impossible task since the groups exist in such abundance. A good method of generating interesting examples is via the so-called Realization Theorems. These theorems have their origin in Corner's famous result [6]: every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free group (see Sect. 4.3 for more details). There have been numerous generalizations of this result, for a good survey of results see the book by Göbel and Trlijaj [33]. Typical results obtained in this way are the following which may be found in [59] and [34] (recall that a group is said to be *superdecomposable* if every nonzero direct summand admits a non-trivial decomposition as a direct sum).

Example 3.31 (i) There exist arbitrary large indecomposable torsion-free groups G such that $gl.ent^*(G) = 0$ and arbitrary large indecomposable groups with endomorphisms of adjoint entropy ∞.

(ii) There exist countable superdecomposable torsion-free groups G such that $gl.ent^*(G) = 0$ and countable superdecomposable torsion-free groups with endomorphisms of adjoint entropy ∞.

As far as mixed groups are concerned, results are sparse unless we impose the restriction that the reduced mixed group G have countable torsion-free rank. This restriction enables the use of a result of Corner which is analogous to Szele's theorem on basic subgroups being endomorphic images. Details may be found in [34, Sect. 4]; we restrict ourselves to one result:

Theorem 3.32 *Let G be a reduced mixed group of countable torsion-free rank having torsion subgroup a p-group T. Then $gl.ent^*(G) = 0$ if, and only if, $G = T \oplus X$, where T is finite and X is a reduced countable torsion-free group with $gl.ent^*(X) = 0$.*

There is one other special class of mixed groups that is easily handled from the viewpoint of adjoint entropy. Recall that a group G is *cotorsion* if it satisfies $\mathrm{Ext}(\mathbb{Q}, G) = 0$ – details of such groups may be found in [27, Sect. 54]. The following may be found in [59, Theorem 3.4] or [34, Proposition 4.4] (recall that the natural topology on a group G has as basis of neighbourhoods of 0 the subgroups nG, $n \neq 0$).

Theorem 3.33 *A reduced cotorsion group G has $gl.ent^*(G) = 0$ if, and only if, it has the form $G = \prod_p G_p$ where, for each prime p, G_p is a finitely generated p-adic module, equivalently, if G is compact in the natural topology.*

3.5 Ent-Singular Submodules and Ent-Singular Modules

Let R be an arbitrary ring and $L : \mathrm{Mod}(R) \to \mathbb{R}^*$ a length function. Given an R-module M, at the end of Sect. 2.4 we considered its fully invariant submodule $z_L(M) = \{x \in M | L(Rx) = 0\}$, called the L-singular submodule of M. The first goal in this section is to describe the L-singular submodule of a given $\mathbb{Z}[X]$-module G_ϕ, when the length function L is an algebraic entropy, that is, when $L = Ent$ for some $Ent \in \{ent_{rk}, \widetilde{ent}, h\}$, or $L = ent$ and G is a torsion group. Our second goal is to characterize the Ent-singular modules, that is, the $\mathbb{Z}[X]$-modules G_ϕ such that $Ent(\phi) = 0$, and the ent-singular modules, that is, those $G_\phi \in \mathrm{Tor}(\mathbb{Z}[X])$ such that $ent(\phi) = 0$.

Inspired by the topological Pinsker factor of a topological flow (X, ϕ) (see [2] and [41]), Dikranjan-Giordano Bruno defined in [13] the *Pinsker subgroup* of an algebraic flow (G, ϕ), where $\phi : G \to G$ is an endomorphism of an Abelian group. In our notation and terminology, given a $\mathbb{Z}[X]$-module G_ϕ, the Pinsker subgroup of G_ϕ is the h-singular submodule of G_ϕ, that is,

$$z_h(G_\phi) = \{x \in G \mid h(\phi \restriction T(\phi, x)) = 0\}.$$

This is the greatest $\mathbb{Z}[X]$-submodule H_ϕ of G_ϕ such that $h(\phi \restriction H) = 0$.

One characterization of $z_h(G_\phi)$ given in [13] needs the following notions. Define by induction on $n \geq 0$ an increasing chain of ϕ-invariant subgroups of G as follows:

$$P_0(G) = 0\;;\; P_{n+1}(G) = \{x \in G \mid \phi^r(x) - x \in P_n(G),\ \text{for some } r \geq 1\}$$

and let $P_\infty(G_\phi) = \cup_n P_n(G)$. Then define by induction on $n \geq 0$ another increasing chain of ϕ-invariant subgroups of G as follows:

$$Q_0(G) = 0\;;\; Q_{n+1}(G) = \{x \in G \mid \phi^r(x) - \phi^s(x) \in Q_n(G),\ \text{for some } r > s \geq 0\}$$

and let $Q_\infty(G_\phi) = \cup_n Q_n(G)$. The subgroup $Q_1(G) = \{x \in G \mid \phi^r(x) = \phi^s(x)$ for some $r > s \geq 0\}$ is the set of *quasi-periodic* points of G.

It is not difficult to prove that

$$P_\infty(G_\phi) = \{x \in G \mid (\phi^{n_1} - 1)\cdots(\phi^{n_k} - 1)(x) = 0;\ \text{for some } k \geq 1,\ n_i \geq 1\}$$

$$Q_\infty(G_\phi)=\{x \in G|\phi^n \prod_1^k(\phi^{n_i} - 1)(x)=0 \text{ or } \phi^n(x)=0;\ \text{for some } k \geq 1,\ n \geq 0,\ n_i\geq 1\}.$$

Then, recalling that $Ker_\infty(\phi) = \cup_n Ker(\phi^n)$ denotes the hyperkernel of ϕ, the following characterization of the h-singular submodule of G_ϕ is proved in [13].

Theorem 3.34 *Let $\phi : G \to G$ be an endomorphism of an Abelian group G. Then the Pinsker subgroup $z_h(G_\phi)$ of G coincides with $Q_\infty(G_\phi) = P_\infty(G_\phi) \oplus Ker_\infty(\phi)$.*

Replacing the algebraic entropy h by the intrinsic entropy $\widetilde{ent}$, we may consider in a similar way the $\widetilde{ent}$-singular submodule of G_ϕ, which is the greatest $\mathbb{Z}[X]$-submodule H_ϕ of G_ϕ such that $\widetilde{ent}(\phi \restriction H) = 0$. This subgroup, denoted by $z_{\widetilde{ent}}(G_\phi)$, is called *intrinsic Pinsker subgroup* of G_ϕ in [36]. From the definition of L-singular submodule, applied to $L = \widetilde{ent}$, we get

$$z_{\widetilde{ent}}(G_\phi) = \{x \in G \mid \widetilde{ent}(\phi \restriction T(\phi, x)) = 0\}.$$

We have the following characterizations of the $\widetilde{ent}$-singular submodule of G_ϕ (see [36]).

Theorem 3.35 *Let $\phi : G \to G$ be an endomorphism of an Abelian group G. Then the $\widetilde{ent}$-singular submodule of G_ϕ satisfies:* $z_{\widetilde{ent}}(G_\phi) = \{x \in G \mid f(\phi)(x) = 0,$ for some $f(X)$ monic $\in \mathbb{Z}[X]\} = \{x \in G \mid T(\phi, x) = T_n(\phi, x),$ for some $n \geq 1\}$.

The subgroup

$$t_\phi(G) = \{x \in G \mid T(\phi, x) = T_n(\phi, x) \text{ for some } n \geq 1\}$$

was defined for torsion groups G in [19]. It coincides with $z_{ent}(G_\phi)$. It follows from the above characterization of $z_{\widetilde{ent}}(G_\phi)$ and from [14] that

$$t_\phi(G) = t(G) \cap z_h(G_\phi) = t(G) \cap z_{\widetilde{ent}}(G_\phi).$$

When G is an arbitrary group, we can also define $z_{ent_{rk}}(G_\phi)$, namely, the ent_{rk}-singular submodule of G_ϕ, which coincides with the torsion part of the $\mathbb{Z}[X]$-module G_ϕ, that is,

$$z_{ent_{rk}}(G_\phi) = \{x \in G \mid f(\phi)(x) = 0 \text{ for some } 0 \neq f(X) \in \mathbb{Z}[X]\}.$$

Since we have the inequalities of length functions: $ent_{rk} \leq \widetilde{ent} \leq h$, for the corresponding radicals the following inequalities hold: $z_h \leq z_{\widetilde{ent}} \leq z_{ent_{rk}}$. So for the three *Ent*-singular submodules of G_ϕ we have $z_h(G_\phi) \leq z_{\widetilde{ent}}(G_\phi) \leq z_{ent_{rk}}(G_\phi)$. These inclusions may be strict, as the next simple examples show.

Example 3.36 Let $G = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1$ be the free group of rank 2, and let $\phi : G \to G$ be the endomorphism defined by setting:

$$\phi(e_0) = e_1 \ , \ \phi(e_1) = e_0 + e_1.$$

The characteristic polynomial of ϕ over $\mathbb{Z}$ is $f(X) = X^2 - X - 1$, so the eigenvalues of ϕ are $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$. It follows that $\phi^2(x) = \phi(x) + x$ for all $x \in G$, consequently $T(\phi, x) = T_2(\phi, x)$ and so $z_{\widetilde{ent}}(G_\phi) = G_\phi = z_{ent_{rk}}(G_\phi)$. On the other hand, ϕ is injective, hence $Ker_\infty(\phi) = 0$, and an easy check gives $P_n(G) = 0$ for all $n \geq 0$, consequently $z_h(G_\phi) = 0$ by Theorem 3.34. Note that from the AYF and the IAYF we get that $h(\phi) = \log((1 + \sqrt{5})/2)$ and $\widetilde{ent}(\phi) = 0$.

Example 3.37 Let $G = \mathbb{Q}e_0 \oplus \mathbb{Q}e_1$ be the $\mathbb{Q}$-vector space of dimension 2, and let $\phi : G \to G$ be the endomorphism defined by setting:

$$\phi(e_0) = e_1 \ , \ \phi(e_1) = e_0 + 3/2\, e_1.$$

The characteristic polynomial of ϕ over $\mathbb{Z}$ is $f(X) = 2X^2 - 3X - 2$, so the eigenvalues of ϕ are 2 and $-1/2$. It follows that $2\phi^2(x) = 3\phi(x) + 2x$ for all $x \in G$, consequently $|T_{n+1}(\phi, x)/T_n(\phi, x)| = 2$ for all n. From the characterization above of $z_{\widetilde{ent}}$ we obtain that $z_{\widetilde{ent}}(G_\phi) = 0 = z_h(G_\phi)$, while $z_{ent_{rk}}(G_\phi) = G_\phi$, since G_ϕ is a torsion $\mathbb{Z}[X]$-module. Note that from the AYF and the IAYF we get, respectively, that $h(\phi) = \log 2 + \log 2 = \log 4$ and $\widetilde{ent}(\phi) = \log 2$.

We pass now to consider the *Ent*-singular modules, where *Ent* denotes one of the three entropies in the set $\{ent_{rk}, \widetilde{ent}, h\}$, that is, the $\mathbb{Z}[X]$-modules G_ϕ such that $Ent(G_\phi) = 0$; these modules form the hereditary torsion class Ker_{Ent}. The following inclusions for the three torsion classes are clear, in view of the inequalities $ent_{rk} \leq \widetilde{ent} \leq h$:

$$Ker_h \subseteq Ker_{\widetilde{ent}} \subseteq K_{ent_{rk}}.$$

We will also consider the *ent*-singular modules G_ϕ, assuming that G is a p-group, since *ent* is a length function only on the subcategory $\mathrm{Tor}(\mathbb{Z}[X])$ (it is well known that a torsion group decomposes into its p-primary components, which are fully invariant subgroups, so the investigation of the endomorphisms of a torsion group can be reduced to the case of p-groups).

The next result is Proposition 2.4 in [19].

Theorem 3.38 *Let $\phi : G \to G$ be an endomorphism of a p-group G. The following are equivalent:*

(1) *G_ϕ is ent-singular (i.e., $ent(\phi) = 0$);*
(2) *the ϕ-trajectory $T(\phi, x)$ of each element $x \in G$ is finite;*
(3) *ϕ is point-wise integral, that is, for every $x \in G$ there exists a monic polynomial $f(X) \in \mathbb{Z}[X]$ such that $f(\phi)(x) = 0$;*
(4) $G = Q_1(G) = \{x \in G \mid \phi^r(x) - \phi^s(x) = 0$ for some $r > s \geq 0\}$.

In point (3) of the preceding theorem one may replace the monic polynomial $f(X) \in \mathbb{Z}[X]$ with a monic polynomial of $J_p[X]$, where J_p denotes the ring of the p-adic integers. Recall that the elements in $Q_1(G)$ are called *quasi-periodic*.

Although we shall not make use of it, it seems appropriate to quote from [20, Corollary 7.7] a similar result for the adjoint entropy ent^*.

Theorem 3.39 *Let $\phi : G \to G$ be an endomorphism of the reduced group G and let $\phi_p : G/pG \to G/pG$ be the endomorphism induced by ϕ for every prime p. The following are equivalent:*

(1) *G_ϕ is ent^*-singular (i.e., $ent^*(\phi) = 0$);*
(2) *for every prime p there exists a monic polynomial $f_p \in \mathbb{Z}[X]$ such that $f_p(\phi)(G) \leq pG$;*
(3) *$ent^*(\phi_p) = 0$ for every prime p;*
(4) *ϕ_p is algebraic (i.e., $(G/pG)_{\phi_p}$ is a bounded $F_p[X]$-module for every prime p);*
(5) *ϕ_p is quasi-periodic for every prime p (i.e., $\phi_p^r = \phi_p^s$ for some $r > s \geq 0$).*

The next result is Theorem 3.6 in [57].

Theorem 3.40 *Let $\phi : G \to G$ be an endomorphism of a group G. The following are equivalent:*

(1) *G_ϕ is ent_{rk}-singular (i.e., $ent_{rk}(\phi) = 0$);*
(2) *the rank of the ϕ-trajectory $T(\phi, x)$ of each element $x \in G$ is finite;*

(3) *ϕ is point-wise algebraic, that is, for every $x \in G$ there exists a polynomial $f(X) \in \mathbb{Z}[X]$ such that $f(\phi)(x) = 0$;*
(4) *G is the union of a well-ordered ascending chain of ϕ-invariant pure subgroups:*

$$t(G) = G_0 < G_1 < \cdots < G_\sigma < \cdots < \bigcup G_\sigma = G$$

such that $rk(G_{\sigma+1}/G_\sigma)$ is finite for all σ.

The next result is contained in Proposition 3.3 and Corollary 5.11 in [25].

Theorem 3.41 *Let $\phi : G \to G$ be an endomorphism of a group G. The following are equivalent:*

(1) *G_ϕ is $\widetilde{ent}$-singular (i.e., $\widetilde{ent}(\phi) = 0$);*
(2) *for every ϕ-inert subgroup H of G, $T(\phi, H)/H$ is finite;*
(3) *for each element $x \in G$, there exists $n \geq 1$ such that $T(\phi, x) = T_n(\phi, x)$, i.e., ϕ is point-wise integral;*
(4) *$ent(t(G)_\phi) = 0$ and G is the union of a well-ordered ascending chain of ϕ-invariant pure subgroups:*

$$t(G) = G_0 < G_1 < \cdots < G_\sigma < \cdots < \bigcup G_\sigma = G$$

such that $rk(G_{\sigma+1}/G_\sigma)$ is finite for all σ and the characteristic polynomial over $\mathbb{Z}$ of each induced map $\phi_\sigma : D(G_{\sigma+1}/G_\sigma) \to D(G_{\sigma+1}/G_\sigma)$ is monic.

In the preceding theorem $D(G_{\sigma+1}/G_\sigma)$ denotes the divisible hull of the torsion-free group of finite rank $G_{\sigma+1}/G_\sigma$ and ϕ_σ its endomorphism which is the unique extension of the endomorphism of $G_{\sigma+1}/G_\sigma$ induced by ϕ. The next result is an immediate consequence of Theorem 3.34.

Theorem 3.42 *Let $\phi : G \to G$ be an endomorphism of a group G. The following are equivalent:*

(1) *G_ϕ is h-singular (i.e., $h(\phi) = 0$);*
(2) *for each element $x \in G$ there exists $n \geq 0$ such that either $\phi^n(x) = 0$, or $\phi^n \prod_1^k (\phi^{n_i} - 1)(x) = 0$ for certain integers $n_i \geq 1$.*

4 Applications to the Structure of Endomorphism Rings

4.1 The Three Faces of Algebraic Entropies

We can look at each entropy $Ent \in \{ent, ent_{rk}, ent^*, \widetilde{ent}, h\}$ discussed above from three different points of view, which reveal the three different faces of algebraic entropies.

According to the first point of view, each entropy Ent defines a function Ent_G : $\mathrm{End}(G) \to \mathbb{R}^*$, which sends the endomorphism ϕ to $Ent(\phi)$. This point of view was applied, for instance, in [57], where the "kernel" $ent_0(G) = \{\phi \in \mathrm{End}(G) : ent(\phi) = 0\}$ of the map $ent_G : \mathrm{End}(G) \to \mathbb{R}^*$ was investigated. It was proved, *inter alia*, that, if G is a p-group and $E_s(G)$ is the two-sided ideal of $\mathrm{End}(G)$ consisting of the small endomorphisms, then the subset $ent_0(G)$ is a subring of $\mathrm{End}(G)$, provided the factor ring $\mathrm{End}(G)/E_s(G)$ is commutative (see Sect. 4.2 for definitions and more details).

This point of view gives rise naturally to the following question.

Question 4.1 If two groups G and H have isomorphic endomorphism rings, is there a relationship between the two maps $Ent_G : \mathrm{End}(G) \to \mathbb{R}^*$ and Ent_H : $\mathrm{End}(H) \to \mathbb{R}^*$?

If the two groups G and H are p-groups and the considered algebraic entropy is ent, the Baer-Kaplansky theorem (see Theorem 108.1 in [27]) gives the answer to the above question. This theorem states that every isomorphism $\Phi : \mathrm{End}(G) \to \mathrm{End}(H)$ between the two endomorphism rings is induced by an isomorphism α : $G \to H$ of the two groups. This implies that two corresponding endomorphisms $\phi \in \mathrm{End}(G)$ and $\Phi(\phi) \in \mathrm{End}(H)$ are conjugated under α, that is, the image $\Phi(\phi)$ is given by $\Phi(\phi) = \alpha \circ \phi \circ \alpha^{-1}$. As conjugated endomorphisms have the same algebraic entropy, it follows that $ent(\phi) = ent(\Phi(\phi))$ for every $\phi \in \mathrm{End}(G)$, equivalently, $ent_G = ent_H \circ \Phi$. We will see that the situation is completely different when the entropies $ent_{rk}, \widetilde{ent}, h$ are considered in general.

The second point of view considers the global entropy function $gl.Ent$: $\mathrm{Mod}(\mathbb{Z}) \to \{0, \infty\}$, sending a group G into $gl.Ent(G)$ (we have seen that these maps take only the two values 0 and ∞). This point of view gave rise to several results of Sect. 3.4. As another application of this approach, in [19] it was proved that for each $n \geq 1$ there exist p-groups G of length ωn such that $gl.ent(G) = 0$ (see Theorem 4.7 in Sect. 4.2 for more details).

The third point of view of the algebraic entropies, which produces most relevant consequences and was developed in Sect. 3.2, considers the category of $\mathbb{Z}[X]$-modules. Accordingly, the algebraic entropy can be viewed as a map Ent : $\mathrm{Mod}(\mathbb{Z}[X]) \to \mathbb{R}^*$ sending the $\mathbb{Z}[X]$-module G_ϕ to $Ent(\phi)$. We have seen that in this way one can express nicely the facts that the entropies are invariants, that they are upper continuous (except ent^*), that $ent_{rk}, \widetilde{ent}, h$ are length functions on $\mathrm{Mod}(\mathbb{Z}[X])$ (and ent on $\mathrm{Tor}(\mathbb{Z}[X])$). Furthermore, in this setting the notions of Ent-singular submodule and of Ent-singular modules come up, and these will be a main tool in the investigation of endomorphism rings in Sect. 4.3.

4.2 Endomorphism Rings of p-Groups

If $\phi : G \to G$ is an endomorphism of a p-group G, we have seen that $ent(\phi) = \widetilde{ent}(\phi) = h(\phi)$, and that $ent_{rk}(\phi) = 0$. Therefore applications of algebraic entropies to endomorphism of p-groups will concern only the entropy *ent*.

We recall some basic facts concerning endomorphisms of p-groups G. The centre of $\mathrm{End}(G)$ is $\mathbb{Z}/p^k\mathbb{Z} \cdot 1$ (where 1 denotes the identity map of G) if G is bounded of exponent k, otherwise it is $J_p \cdot 1$.

Pierce [53] called an endomorphism ϕ *small* if for every positive integer k there exists an integer $n \geq 0$ such that $\phi(p^nG[p^k]) = 0$ (as usual, $p^nG[p^k] = p^nG \cap G[p^k]$). Obvious examples of small endomorphisms are the bounded endomorphisms and the projections on cyclic summands. Pierce proved that the small endomorphisms form a two-sided ideal of $\mathrm{End}(G)$, denoted by $E_s(G)$, and that $\mathrm{End}(G)$ is a direct sum of $E_s(G)$ by a torsion-free J_p-module A, which is the completion of a free J_p-module.

Recall Corner's notion [8] that $\mathrm{End}(G)$ is a split extension of a J_p-algebra A by the two-sided ideal $E_s(G)$: this means that A is a subring of $\mathrm{End}(G)$ and there exists a ring homomorphism $\mathrm{End}(G) \to A$ that is the identity map on A, with kernel $E_s(G)$. So we get the direct group decomposition:

$$\mathrm{End}(G) = A \oplus E_s(G).$$

Motivated by these results, Corner proved in [8] and [9] several beautiful realization theorems, stating that, under suitable conditions, there exists $2^{2^{\aleph_0}}$ non-isomorphic p-groups G, such that there is a ring split extension $\mathrm{End}(G) \cong A \oplus E_s(G)$. It is worthwhile remarking that in Corner's realization theorems the multiplicative structure of the factor ring $\mathrm{End}(G)/E_s(G) \cong A$ plays no role, while it is the crucial factor in computing the global entropy $gl.ent(G)$, as the next results in this section will show.

In the applications of the algebraic entropy *ent* a relevant role is played by the semi-standard groups. Recall that Corner [8] called a reduced p-group G *semi-standard* if its Ulm-Kaplansky invariants of finite index are all finite, that is, for all integers $n \geq 0$:

$$\alpha_n(G) = \dim_{F_p}(p^nG[p]/p^{n+1}G[p]) < \aleph_0.$$

The relevance of semi-standard groups in connection with *ent* is clear from the following:

Lemma 4.2 *If G is a reduced p-group which is not semi-standard, then there exist endomorphisms of G of positive algebraic entropy.*

Proof If $\alpha_n(G)$ is infinite, then G contains a summand $B \cong \bigoplus_{\mathbb{N}} \mathbb{Z}/p^{n+1}\mathbb{Z}$, so the right Bernoulli shift on B extends to an endomorphism ϕ of G such that $ent(\phi) = \log(p^{n+1})$. □

The following proposition collects results, proved in [19], that are samples of the role of semi-standard groups; recall that a p-group is *essentially finitely indecomposable* if it does not admit direct summands which are infinite direct sums of cyclic groups.

Proposition 4.3 (1) *A reduced p-group G such that $gl.ent(G) = 0$ is necessarily semi-standard and essentially finitely indecomposable; in particular, $|G| \leq 2^{\aleph_0}$.*
(2) *If G is semi-standard and θ is a small endomorphism, then θ is point-wise integral, hence $ent(\theta) = 0$.*
(3) *If G is unbounded semi-standard, then all the endomorphisms of the subring $J_p \cdot 1 \oplus E_s(G)$ are point-wise integral, hence have zero entropy.*

An example of a semi-standard group G such that $\mathrm{End}(G) = J_p \cdot 1 \oplus E_s(G)$ was first constructed by Pierce [53]. Megibben proved [48] that quasi-complete groups G (defined by the property that the closure in the p-adic topology of a pure subgroup is still pure) such that $\bar{G}/G \cong \mathbb{Z}(p^\infty)$ ($\bar{G}$ is the torsion-completion of G) also satisfy $\mathrm{End}(G) = J_p \cdot 1 \oplus E_s(G)$. Hence for all these groups we have that $gl.ent(G) = 0$.

Notice that the converse of Proposition 4.3 (1) is not true. In fact, using the Corner's realization result proved in [8, Theorem 2.1], it is possible to construct a p-group G (with all its Ulm-Kaplansky invariants equal to 1 and essentially indecomposable, i.e., if $G = G_1 \oplus G_2$, then one of the two summands is finite) admitting an endomorphism of infinite algebraic entropy (see [19, Theorem 4.4]). This property of having an endomorphism of infinite algebraic entropy is shared by semi-standard groups belonging to many important classes of p-groups, such as totally projective, $p^{\omega+1}$-projective and torsion-complete groups (see [19, Theorem 4.5]).

The above discussion shows that a crucial question is whether, besides the quasi-complete groups studied by Megibben mentioned above, which form a relatively small class, there exist other reduced semi-standard p-groups G such that $gl.ent(G) = 0$.

Bearing in mind the restriction on the cardinality of these groups imposed by Proposition 4.3 (1), we first look for a classification of countable reduced p-groups G with $gl.ent(G) = 0$: such a group is necessarily finite. This follows from [19] (Proposition 4.1 and Theorem 4.4, for the bounded and unbounded cases, respectively).

If we allow groups of cardinality the continuum, then examples of groups G with $gl.ent(G) = 0$ exist in such abundance that no reasonable classification seems possible—see [19, Sect. 5] and [57, Sect. 3]. Nevertheless, we can offer nice sufficient conditions in order that $gl.ent(G) = 0$ when $|G| = 2^{\aleph_0}$.

Theorem 4.4 *If G is an unbounded semi-standard p-group such that the ring $\mathrm{End}(G)/E_s(G)$ is integral over J_p, then $gl.ent(G) = 0$.*

The hypothesis of Theorem 4.4 is certainly satisfied if the J_p-module $A \cong \mathrm{End}(G)/E_s(G)$ has finite rank. If we look for J_p-algebras A of infinite rank, another one of Corner's realization theorem proved in [8] helps us. This theorem states that if the J_p-algebra A has a countable chain of left ideals $A \geq A_1 \geq A_2 \geq \cdots$ such that

A_i/A_{i+1} is a free J_p-module of finite rank for all i, and if $pA = \cap_i(pA + A_i)$, then there exists a separable semi-standard p-group G such that $\mathrm{End}(G) = A \oplus E_s(G)$.

Using this theorem of Corner, it is possible to construct a torsion-free J_p-algebra A, the completion of a free J_p-module of countable rank, which is integral over J_p and satisfies the conditions of the Corner's theorem mentioned above. This J_p-algebra is the Nagata idealization of J_p by the completion of a free J_p-module of countable rank (see [19, Example 5.12]). Thus the p-group G constructed in this way satisfies $gl.ent(G) = 0$.

Surprisingly enough, if we consider reduced p-groups G of size $\aleph_1$ and $gl.ent(G) = 0$, we find ourselves in set-theoretic trouble. Indeed we have the following:

Theorem 4.5 *The following statements are consistent with ZFC:*

(i) *there is a reduced p-group G of cardinality $\aleph_1$ with $gl.ent(G) = 0$;*
(ii) *every reduced p-group G of cardinality $< 2^{\aleph_0}$ with $gl.ent(G) = 0$ is finite.*

This result follows from a recent work of Braun and Strüngmann [4] on Hopfian and co-Hopfian groups. For part (i) they worked in the forcing extension of the universe by $Fin(\omega_1, 2)$, but for our purposes it suffices to assume the Continuum Hypothesis (CH): as we have seen, the example of Pierce of a standard reduced p-group G with $\mathrm{End}(G) = J_p 1 \oplus E_s(G)$ is a group of size $2^{\aleph_0} = \aleph_1$ and satisfies $gl.ent(G) = 0$. Thus (i) holds in ZFC + (CH).

For (ii) we exploit Martin's Axiom (MA), specifically we work in ZFC + (MA) + ¬(CH). It follows from [4, Theorem 1.1(2)] that an infinite reduced p-group of size strictly less than the continuum cannot be co-Hopfian. Since a reduced p-group with zero global entropy is necessarily co-Hopfian by Proposition 3.20, in this model of set theory, no infinite reduced p-group of cardinality less than the continuum can have zero global entropy, i.e., (ii) holds.

Since both ZFC + (CH) and ZFC + (MA) + ¬(CH) are consistent with ZFC, it follows that the existence of a reduced p-group G of size $\aleph_1$ and of zero global entropy, $gl.ent(G) = 0$ is undecidable.

Another multiplicative property of the factor ring $\mathrm{End}(G)/E_s(G)$ is crucial for a different purpose, namely, its commutativity (see [57, Theorem 3.6]). Recall that $ent_0(G) = \{\phi \in \mathrm{End}(G) : ent(\phi) = 0\}$ denotes the "kernel" of the map $ent_G : \mathrm{End}(G) \to \mathbb{R}^*$.

Theorem 4.6 *If G is an unbounded semi-standard p-group such that the ring* $\mathrm{End}(G)/E_s(G)$ *is commutative, then $ent_0(G)$ is a subring of* $\mathrm{End}(G)$.

It is easy to see that the hypothesis that G is semi-standard in Theorem 4.6 cannot be omitted, and to provide examples of groups satisfying the hypotheses of Theorem 4.6 such that $ent_0(G)$ is a proper subring of $\mathrm{End}(G)$. Note that $ent_0(G) \geq p\mathrm{End}(G)$, since, given any $\phi \in \mathrm{End}(G)$, $p\phi$ annihilates the socle $G[p]$, from which it easily follows that $ent(p\phi) = 0$. Thus the factor group $\mathrm{End}(G)/ent_0(G)$ is an F_p-vector space, whose dimension measures how far is G from having $gl.ent(G) = 0$.

Theorem 4.6 can be improved, replacing the ideal $E_s(G)$ by the two-sided ideal of the socle-finite endomorphisms. Recall that the endomorphism ϕ is called *socle-finite* if $\phi(G[p])$ is finite. The socle-finite endomorphisms form the two-sided ideal of $\mathrm{End}(G)$ denoted by $E_{sf}(G)$, which obviously contains $p\mathrm{End}(G)$. It is not difficult to prove that G is semi-standard exactly when $E_{sf}(G) \geq E_s(G)$ (see [57, proposition 4.1]), so the commutativity of the factor ring $\mathrm{End}(G)/E_{sf}(G)$ is a weaker assumption than the commutativity of $\mathrm{End}(G)/E_s(G)$, ensuring that $ent_0(G)$ is a subring of $\mathrm{End}(G)$.

The restricted cardinality of a reduced p-group G having global algebraic entropy $gl.ent(G) = 0$ automatically restricts the possible length of the group to be the continuum also. Exhibiting groups G having global algebraic entropy $gl.ent(G) = 0$ of length up to the continuum seems to be difficult and the best results to date are in [19, Theorem 5.18] where the following is established; recall that an endomorphism $\phi : G \to G$ is *thin* if, for every positive integer k, there is an integer $n \geq 0$ such that $\phi((p^n G)[p^k]) \leq p^\omega G$.

Theorem 4.7 *Given an ordinal $\lambda < \omega^2$, there exists a family of $2^{2^{\aleph_0}}$ p-groups G, each of length λ and with $gl.ent(G) = 0$, such that there are only thin homomorphisms between the different members of the family.*

The proof of Theorem 4.7 relies on the fact that, if G is a semi-standard p-group of length $< \omega^2$ such that $\mathrm{End}(G/p^\omega G) = J_p \oplus E_s(G/p^\omega G)$, then $gl.ent(G) = 0$; this fact shows that for such a group, in order to get zero global algebraic entropy, it does not matter what the subgroup $p^\omega G$ is, since only the quotient $G/p^\omega G$ is relevant. So one can reduce to the consideration of separable groups and apply results proved by Corner [9] connecting properties of G to properties of $G/p^\omega G$.

4.3 Endomorphism Rings of Torsion-Free Groups

This section is devoted to surveying the results obtained in the two papers [32] and [36] on algebraic entropies applied to endomorphism rings of torsion-free groups. The problem considered in both papers is to give a satisfactory answer to Question 4.1, for the entropy ent_{rk} in [32], and for the entropy $\widetilde{ent}$ in [36].

The difference between the two papers, besides the fact that they apply the two different entropies ent_{rk} and $\widetilde{ent}$, is two-fold: with respect to the realization results they use, and with respect to the rings they realize as endomorphism rings via these results.

In [32], a powerful realization result contained in the 1985 Corner-Göbel paper [11] is used, that provides constructions of groups of large cardinalities, and a complete topological ring is realized that has no primitive idempotents; hence the groups which have endomorphism ring isomorphic to it are superdecomposable.

In [36], the original Corner theorems of the 1960s on endomorphism rings of countable torsion-free groups are used, and the realized rings are the integral polynomial ring $\mathbb{Z}[X]$ and the power series ring $\mathbb{Z}[[X]]$; these rings are integral domains, hence the groups which have endomorphism ring isomorphic to them are indecomposable.

We start illustrating first the results in [36], since its simpler approach gives considerable advantages when applying algebraic entropies.

The following result proved by Corner in [7] characterizes the rings which are endomorphism rings of countable reduced torsion-free groups; recall that fin_G denotes the finite topology on the endomorphism ring of a group G. It is well known that this topology makes $\mathrm{End}(G)$ a complete topological ring - see, for example, [27, Theorem 107.1].

Theorem 4.8 (Corner 1967) *A topological ring (A, τ) is isomorphic to the topological ring $(\mathrm{End}(G), fin_G)$, for G a countable reduced torsion-free group, if, and only if, (A, τ) is complete and Hausdorff and τ admits a basis of neighbourhoods of 0 consisting of a countable descending chain of left ideals $N_1 \geq N_2 \geq \cdots$ such that A/N_k is a countable reduced torsion-free group for all k.*

When the topology τ is the discrete one, then the ring A itself is countable reduced torsion-free, so we obtain the first realization result, already mentioned earlier, proved by Corner in [6].

Theorem 4.9 (Corner 1963) *Every countable reduced torsion-free ring is isomorphic to the endomorphism ring of a countable reduced torsion-free group.*

If we look now at some countable reduced torsion-free rings A, which are to be realized as endomorphism rings via Theorem 4.9, the easiest possible candidate is $A = \mathbb{Z}$. It is well known that there exist torsion-free groups G of arbitrary cardinality such that $\mathrm{End}(G) \cong \mathbb{Z}$. But all these groups provide trivial examples with respect to the entropical behaviour, since their subgroups are fully invariant, hence their global entropies are trivially zero, whatever entropy among $ent_{rk}, \widetilde{ent}$ one chooses.

A more interesting example, which provides non-trivial outcomes, is $A = \mathbb{Z}[X]$. In the following proposition the properties satisfied by any group G such that $\mathrm{End}(G) \cong \mathbb{Z}[X]$ are collected. We identify $\mathrm{End}(G)$ with $\mathbb{Z}[X]$ and we denote by ω the distinguished endomorphism of G induced by the multiplication by the indeterminate X.

Proposition 4.10 *Let G be a group such that $\mathrm{End}(G) = \mathbb{Z}[X]$. Then:*

(1) *G is indecomposable, torsion-free and reduced;*
(2) *the finite topology of $\mathrm{End}(G)$ is discrete;*
(3) *there exist elements $g \in G$ such that, for every endomorphism ϕ of G different from the multiplication by an integer, $T(\phi, g)_\phi \cong (\bigoplus_{\mathbb{N}} \mathbb{Z})_\beta$, hence $\widetilde{\mathrm{ent}}(T(\phi, x)_\phi) = \infty$;*
(4) *if there exists $x \in G$ such that $\mathrm{rk}_{\mathbb{Z}}(T(\omega, x)) = n < \aleph_0$, then:*

    (4.1) $T_n(\omega, x) = \bigoplus_{0 \leq i \leq n-1} \omega^i x\mathbb{Z}$
    (4.2) *there exists a minimal positive integer s such that $s^k \omega^{n+k-1} x \in T_n(\omega, x)$ for all $k \geq 1$*
    (4.3) $\widetilde{\mathrm{ent}}(T(\omega, x)_\omega) = \log s$.

It is possible that the case considered in point (4) above does not occur, as Theorem 4.12 below shows. Of course, more can be said when the countable reduced torsion-free group G such that $\mathrm{End}(G) \cong \mathbb{Z}[X]$ is constructed in a specific way, e.g., by means of Theorem 4.9. But this theorem is not the only one available for constructing G. In fact, Corner proved also the following result in [10, Example 1] explicitly for the ring $\mathbb{Z}[X]$. The relevance of this theorem is due to the fact that, if R is a countable reduced torsion-free integral domain, then any group realizing R as its endomorphism ring via Theorem 4.9 is Hopfian.

Theorem 4.11 (Corner 1965) *The ring $\mathbb{Z}[X]$ is isomorphic to the endomorphism ring $\mathrm{End}(H)$ of a torsion-free non-Hopfian group H of countable rank.*

The next theorem compares the properties of two groups G and H with endomorphism ring isomorphic to $\mathbb{Z}[X]$ constructed by means of Theorem 4.9 and Theorem 4.11, respectively.

Theorem 4.12 (1) *A group G such that $\mathrm{End}(G) \cong \mathbb{Z}[X]$, constructed by means of Theorem 4.9, is a torsion-free $\mathbb{Z}[X]$-module of countable rank, so it contains a free $\mathbb{Z}[X]$-module $F \cong \bigoplus_{\aleph_0} \mathbb{Z}[X]$, and G/F is a torsion group. Furthermore, the $\widetilde{\mathrm{ent}}$-singular submodule of G_ω is 0.*

(2) *The non-Hopfian group H such that $\mathrm{End}(H) \cong \mathbb{Z}[X]$, constructed by means of Theorem 4.11, has rank 2 as $\mathbb{Z}[X]$-module. The $\widetilde{\mathrm{ent}}$-singular submodule of H_ω is isomorphic to $\bigoplus_{\mathbb{N}} \mathbb{Z}(r^\infty)$ (r a prime integer) endowed by the left Bernoulli shift.*

The reader interested in a more detailed description of the structure as $\mathbb{Z}[X]$-module of the non-Hopfian group H above can consult [36, Proposition 3.6].

In order to apply Theorem 4.8, a natural ring to be considered is the integral power series $\mathbb{Z}[[X]]$ with the X-adic topology χ. However, we can consider $\mathbb{Z}[[X]]$ endowed also with the discrete topology δ, in view of the following theorem proved in [36], whose key idea goes back to Corner.

Theorem 4.13 *Let (A, τ) be a topological ring satisfying the conditions of Theorem 4.8. Then there exists a reduced torsion-free group H of cardinality $2^{\aleph_0}$ such that $(\mathrm{End}(H), \mathit{fin}_H) \cong (A, \delta)$.*

The next theorem compares the properties of two groups G and H with endomorphism ring isomorphic to $\mathbb{Z}[[X]]$ constructed by means of Theorem 4.8 and Theorem 4.13, respectively.

Theorem 4.14 (1) *A group G such that $(\mathrm{End}(G), \mathit{fin}_G) \cong (\mathbb{Z}[[X]], \chi)$, constructed by means of Theorem 4.8, is countable and all its orbits $\mathbb{Z}[[\omega]]g$ ($g \in G$) equal the trajectories $T(\omega, g) = \mathbb{Z}[\omega]g$. Furthermore, $gl.\widetilde{\mathrm{ent}}(G) = 0$, so, in particular, G_ω coincides with its $\widetilde{\mathrm{ent}}$-singular submodule.*

(2) *The group H such that $(\mathrm{End}(H), \mathit{fin}_H) \cong (\mathbb{Z}[[X]], \delta)$, constructed by means of Theorem 4.13, has rank $2^{\aleph_0}$ as $\mathbb{Z}[x]$-module, it contains elements g such that $T(\omega, g)_\omega \cong (\bigoplus_{\mathbb{N}} \mathbb{Z})_\beta$, so $\widetilde{\mathrm{ent}}(\omega) = \infty$ and the $\widetilde{\mathrm{ent}}$-singular submodule of H_ω is properly contained in H.*

The reader interested in more information on the structure of the groups G and H as $\mathbb{Z}[X]$-modules should consult [36]. A consequence of the above results is the following:

Corollary 4.15 *There exist reduced torsion-free groups G, H such that G is countable, H has rank $2^{\aleph_0}$ as $\mathbb{Z}[X]$-module,* $\mathrm{End}(G) \cong \mathbb{Z}[[X]] \cong \mathrm{End}(H)$*, but $gl.\widetilde{ent}(G) = 0$, while $gl.\widetilde{ent}(H) = \infty$.*

The above results show that, given any isomorphism $\Phi : \mathrm{End}(H) \to \mathrm{End}(G)$, $\widetilde{ent}(\omega) = \infty$ and $\widetilde{ent}(\Phi(\omega)) = 0$; so the entropical behaviour of a ring isomorphic to the endomorphism ring of a torsion-free group depends on the group and not on the ring itself. Thus, in this case, the answer to Question 4.1 is completely different from the torsion case.

We pass now to illustrate the results in [32], starting with the description of the complete topological ring $\hat{A}$ realized in two different ways as endomorphism ring of torsion-free groups.

Let Λ denote the countable set of the finite subsets of $\mathbb{N}$. Let $A = \bigoplus_{\lambda\in\Lambda} \mathbb{Z}e_\lambda$ be the free group generated by the symbols e_λ. Define a multiplication on the generators of A by setting:

$$e_\lambda \cdot e_\mu = e_{\lambda\cup\mu}$$

and extending it by linearity on A. Note that A is commutative and $e_\lambda \cdot e_\emptyset = e_\lambda = e_\lambda \cdot e_\lambda$, so $e_\emptyset = 1_A$ and every generator is idempotent. This implies that every element of A is the root of an integral polynomial. For every $\lambda \in \Lambda$, let

$$N_\lambda = \langle e_\mu \mid \mu \not\subseteq \lambda \rangle \, , \; A_\lambda = \langle e_\mu \mid \mu \subseteq \lambda \rangle$$

be generated as subgroups. The subgroup N_λ is an ideal of A and A_λ is a finitely generated subgroup (even subring), hence free of finite rank; furthermore, it is easily seen that A is a split extension of A_λ by N_λ. Since $\cap_{\lambda\in\Lambda} N_\lambda = 0$, A is a Hausdorff topological ring with the topology τ having the N_λ as neighbourhoods of 0. Let $\hat{A}$ denote the completion of A with respect to the topology τ, and $\hat{N}_\lambda$ the closure of N_λ in it. Since $N_\lambda \cap A_\lambda = 0$, $\hat{A}$ is a split extension of its discrete subring A_λ by the ideal $\hat{N}_\lambda$. In particular, $\hat{A}/\hat{N}_\lambda \cong A_\lambda$ is free of finite rank.

The ring $\hat{A}$ can be embedded into $\prod_\lambda \hat{A}/\hat{N}_\lambda \cong \prod_\lambda A_\lambda$. Since $\prod_\lambda A_\lambda$ is isomorphic to the Baer-Specker group $\prod_{\mathbb{N}} \mathbb{Z}$, $\hat{A}$ is $\aleph_1$-free, by [27, Theorem 19.2], hence cotorsion-free (i.e., torsion-free, reduced and with no subgroups isomorphic to J_p for all p). This condition is necessary, in view of the results in [11], to realize $\hat{A}$ as endomorphism ring.

The two noteworthy properties of the ring $\hat{A}$ are that it has no primitive idempotents, hence the groups which have endomorphism ring isomorphic to $\hat{A}$ are superdecomposable, i.e., they have no indecomposable non-zero direct summands, and that it contains a family of $2^{\aleph_0} = |\hat{A}|$ algebraically independent elements over $\mathbb{Z}$. With this ring $\hat{A}$ at ones disposal, and using the powerful realization Theorem 6.3

of [11], the main result proved in [32] is the following theorem, in which $\hat{B}_M$ and $\hat{B}_H$ denote completions in the p-adic topology for a fixed prime p, and $\subseteq_*$ denotes pure embeddings.

Theorem 4.16 *Let $(\hat{A}, \tau)$ be the topological ring constructed above and let λ, μ be two infinite cardinals such that $\lambda = \lambda^{\aleph_0}$, $\mu = \mu^{\aleph_0}$. Then:*

(i) there are two $\aleph_1$-free abelian groups G, H of cardinality $|G| = \lambda$, $|H| = \mu$, with $(\hat{A}, \tau) \cong (\mathrm{End}(G), fin_G)$ and (denoting by δ the discrete topology) $(\hat{A}, \delta) \cong (\mathrm{End}(H), fin_H)$;
(ii) if $B_M = \bigoplus_{i<\lambda} e_i(\bigoplus_{\alpha\in\Lambda} \hat{A}/\hat{N}_\alpha)$ and $B_H = \bigoplus_{i<\mu} e_i\hat{A}$, then $B_M \subseteq M \subseteq_* \hat{B}_M$ and $B_H \subseteq H \subseteq_* \hat{B}_H$. Moreover $|M \setminus B_M| = \lambda$ and $|H \setminus B_H| = \mu$;
(iii) for all $a \in \hat{A}, g \in M$ there is $0 \neq f(x) \in \mathbb{Z}[x]$ with $f(a)(g) = 0$;
(iv) if $a \in \hat{A}$ is a transcendental element and $0 \neq f(x) \in \mathbb{Z}[x]$, there exist suitable elements $g \in H$ (called "generating elements"), such that $f(a)(g) \neq 0$. There are $2^{\aleph_0} = |\hat{A}|$ such elements $a \in \hat{A}$ and $\mu = |H|$ such elements $g \in H$.

A consequence of Theorem 4.16, similar to Corollary 4.15, with ent_{rk} replacing $\widetilde{ent}$, is the following:

Corollary 4.17 *There exist two reduced $\aleph_1$-free groups G and H with $\mathrm{End}(G) \cong A \cong \mathrm{End}(H)$, such that $gl.ent_{rk}(G) = 0$ and $gl.ent_{rk}(H) = \infty$.*

The fact that $gl.ent_{rk}(G) = 0$ is a direct consequence of Theorem 4.16 (iii), which shows that every endomorphism of G is point-wise algebraic (see Theorem 3.40). Theorem 4.16 (iv). says that $ent_{rk}(a) > 0$ for all transcendental elements $a \in \hat{A}$, when viewed as endomorphisms of H, hence $gl.ent_{rk}(H) = \infty$ follows.

A much more powerful result than Theorem 4.16 (iv) is proved in [32, Corollary 4.9]; it states that, for every "generating element" $g \in H$, there exist two families $\{a_\lambda\}_{\lambda\in 2^{\aleph_0}}$ and $\{b_\mu\}_{\mu\in 2^{\aleph_0}}$ of distinct elements of $\hat{A}$, such the orbit $\hat{A}g$ contains the direct sum of fully invariant cyclic trajectories $T_\lambda = \bigoplus_\mu T(a_\lambda, b_\mu(g))$, and this happens for $2^{\aleph_0}$ transcendental elements a_λ.

5 Concluding Remarks

Although notions of algebraic entropy have been around for some 50 years, serious investigation of them seems to have been sparked by the appearance of [19] in 2009. The growth of interest in the subject can be witnessed by the fact that almost 50% of the references below post-date the appearance of the first preprint of [19]. Thus, in some sense, the subject is comparatively new but its natural interaction with many classical algebraic notions such as Kaplansky's approach to Abelian group theory via the notion of $\mathbb{Z}[X]$-modules or Corner's approach to realizing rings as endomorphism rings, along with clear similarities to wider-established notions in physics, communications theory and computing, give it the feel of a central topic in modern algebra. This, of course, presents the survey writer with a key problem:

many areas of interest are still emerging and the deep connections with topological entropy make the setting of key open problems a difficult task. Faced with this and the confines of space, we have decided, at this stage, not to try to lay out specific problems for future study, even though it is clear that many such problems arise naturally from our discussions above. We have no doubt that with the current level of interest in algebraic entropy, such paths for future study will emerge in the not too distant future.

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On Subsets and Subgroups Defined by Commutators and Some Related Questions

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Abstract The aim of this paper is to survey some results concerning the following subsets of a group G: subsets consisting of commutators, subsets defined by commutators identities, subsets defined by autocommutators. We are interested in conditions under which each of these sets forms a subgroup of G. In particular we continue the investigation started by L.C. Kappe and R.F. Morse on the subset of all commutators in G. Furthermore, we present some recent results concerning the subset of autocommutators in a group and under what conditions these subsets form a subgroup.

Keywords Commutators • Commutator subgroup • Abelian groups

Mathematical Subject Classification (2010): Primary 20K30; Secondary 20F12, 20F28, 20F45, 20D45, 20E36

1 Introduction

The aim of this paper is to present some results concerning the following three topics related to commutators in groups:

- subsets consisting of commutators;
- subsets defined by commutator identities;
- subsets defined by autocommutators.

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We are interested in conditions under which each of these subsets forms a subgroup.

These topics have drawn considerable attention in recent years. This survey should be of interest to group theorists who are experts in the described areas of research, as well as to those who wish only to take a glimpse at the topic.

In Sect. 2 we address these questions for the subset of commutators. We can be brief here, since a comprehensive survey by R.F. Morse and the first author [30] on this topic appeared in 2007. We will focus here on new developments in recent years, such as the Ore Conjecture.

Subsets defined by commutator identities and conditions when such a set forms a subgroup are discussed in Sect. 3. We do not claim that this is a complete survey of all known results on this topic. This has to wait for a future publication.

Finally, in Sects. 4–6 we present some results concerning the subset of autocommutators in a group and under what conditions these subsets form a subgroup. This is a fairly new topic which is also of interest for abelian groups. In addition to some recent results, we also include some work in progress, by the authors of this survey, in particular concerning answers to this question in infinite abelian groups.

2 Commutators and the Commutator Subgroup

Let G be a group, x, y elements of G. The **commutator** of x and y is the element

$$[x,y] := x^{-1}y^{-1}xy = x^{-1}x^{y}.$$

We will first give a brief historic overview of the problem on when the set of commutators forms a subgroup. For further details we refer to [30]. Our focus then will be on the Ore Conjecture, which was completely solved only in 2010 after the appearance of [30] in 2007.

The fundamental concept of the commutator was introduced by R. Dedekind in 1880, while he was interested in extending group characters from abelian to nonabelian groups. But only in 1896 it appeared in a paper by F.G. Frobenius [15]. There he wrote that the element F such that $BA = ABF$ was called "commutator of A and B" by Dedekind. He said that Dedekind had proved the following:

The conjugate of a commutator is again a commutator, therefore the **commutator subgroup** *generated by the commutators of a group is a normal subgroup of the group*, and also that *any normal subgroup with abelian quotient contains the commutator subgroup*; and *the commutator subgroup is trivial if and only if the group is abelian.*

These results were first published by G.A. Miller in [40] in 1896, where he spoke about *"the operation* $sts^{-1}t^{-1}$".

The word *commutator*, again attributed to Dedekind, was used by Miller only in a paper [41] of 1898, where he expanded the basic properties of the commutator subgroup and introduced the derived series of a group; he also showed that the derived series is finite and ends with 1 if and only if the group is solvable. The year after, in [42], Miller investigated commutators, in particular when the product of two commutators is again a commutator. He proved that every element in the alternating group $\mathscr{A}_n$ on n letters, $n \geq 5$, is a commutator, and that the same happens for every element in the commutator subgroup of the holomorph of a cyclic group C_n of order n. In particular, the commutator subgroup of the holomorph is C_n, if n is odd, and the subgroup of index 2 in C_n, if n is even.

The first textbook in which commutators and the commutator subgroup were introduced is H. Weber's 1899 *Lehrbuch der Algebra* [60], the last important textbook on algebra published in the nineteenth century. In this book there appears for the first time explicitly the question:

Is the set of all commutators a subgroup, i.e. does the commutator subgroup consist entirely of commutators?

Weber stated that *the set of commutators is not necessarily a subgroup*, but no example was provided.

The first example of a group in which the set of commutators is not equal to the commutator subgroup appeared in 1902, in a paper by W.B. Fite [14], a student of Weber, where the term "metabelian" in the title is used to denote nilpotent groups of class ≤ 2.

Fite constructed an example of a group G of order 1024, attributed to Miller, and also provided a homomorphic image H of order 256 of G which is again an example. The group $H = \langle g_1, g_2, g_3, g_4 \rangle$ is a subgroup of S_{16}, where

$$g_1 = (1,3)(5,7)(9,11), g_2 = (1,2)(3,4)(13,15),$$
$$g_3 = (5,6)(7,8)(13,14)(15,16), g_4 = (9,10)(11,12).$$

Commutators appear also in a paper by W. Burnside in 1903 [7], where characters are used to obtain a criterion for when an element of the commutator subgroup is the product of two or more commutators.

The first occurrence of the commutator notation probably is in a paper by F.W. Levi and B.L. van der Waerden [36] in 1933, where the commutator of two group elements i, j is denoted by

$$(i,j) = iji^{-1}j^{-1}.$$

The first book in which this notation is used is the *Lehrbuch der Gruppentheorie* by H.J. Zassenhaus [61] in 1937, where the famous paper by P. Hall [23] on groups of prime power order is quoted for definitions, notation, and results.

Now, let G be a group and denote with

$$K(G) := \{[g,h] \,|\, g,h \in G\},$$

the set of commutators of G. Then the commutator subgroup G' of G, *i.e*. the derived subgroup of G, is

$$G' =< K(G) > .$$

Some natural questions arise:

When is $G' = K(G)$?

Which is the minimal order of a counterexample?

Only in 1977, in his PhD Thesis [21], R.M. Guralnick gave the answer to the last question.

He proved that there are exactly *two* nonisomorphic groups G of order *96* such that $K(G) \neq G'$. In both cases G' is nonabelian of order 32 and $|K(G)| = 29$. The groups are:

$$G = H \rtimes < y >, \text{ where } H =< a > \times < b > \times < i, j >, a^2 = b^2 = y^3 = 1,$$

$$< i, j > \simeq Q_8, a^y = b, b^y = ab, i^y = j, j^y = ij;$$

$$G = H \rtimes < y >, \text{ where } H = N \rtimes < c >, N =< a > \times < b >,$$

$$a^2 = b^4 = c^4 = 1, a^c = a, b^c = ab, y^3 = 1, a^y = c^2b^2, b^y = cba, c^y = ba.$$

Much more information about all of these questions is contained in [30]. The final section there is devoted to the so-called *Ore Conjecture*, made by O. Ore in 1951 in [46], where finite groups with every element in the commutator subgroup a commutator are investigated. Ore rediscovered Miller's result [42] (see also the paper [28] by N. Ito) that every element in the alternating group $\mathscr{A}_n$ on n letters, $n \geq 5$, is a commutator and conjectured:

Every element in a non-abelian finite simple group is a commutator.

Much work on this conjecture has been done over the years. Important contributions are due to R.C. Thompson [55–57], K'en-ch'eng Ts'eng and Ch'eng-hao Hsü [58], K'en-ch'eng Ts'eng and Chiung-sheng Li [59], Gow [18, 19], Neubüser et al. [45], Bonten [6], Blau [5], Ellers and Gordeev [13], as well as Shalev [53]. But in 2007 the conjecture was still open for some of the finite simple groups of Lie type over small fields, and the survey [30] could furnish only a precise account of which cases were settled and which were still open at that time. Finally, in 2010, the conjecture has been completely solved by Liebeck et al. [37]. Using character theory, induction on the dimension, and certain computer calculations with very deep arguments, they have been able to settle all the cases.

3 Subsets Defined by Commutator Identities

In this section we study kind of a dual problem to the one of the preceding section. The most familiar example here is the center of a group G as follows:

$$Z(G) = \{a \in G \mid [a, g] = 1, \forall g \in G\}.$$

We can generalize this concept by starting from a word $w = w(x_1, \ldots, x_t)$. One says that this word is a *law* in G if $w(g_1, \ldots, g_t) = 1$ for every t-tuple $(g_1, \ldots, g_t)$ of elements of G. For instance, $w = [x_1, x_2]$ is a law for every abelian group.

For any word $w = w(x_1, \ldots, x_t)$, we can look at the following subsets in a group G:

$$W_i(G) = \{a_i \in G \mid w(g_1, \cdots, g_{i-1}, a_i, g_{i+1} \cdots g_t) = 1, \forall g_j \in G,\ j \neq i\}.$$

In case $w = [x, y]$, we have $W_1(G) = W_2(G) = Z(G)$, the center of G. As we will see, $W_i(G)$ in general need not be a subgroup. So the question arises under which conditions do we have that the subset $W_i(G)$ forms a subgroup of G for a given word $w = w(x_1, \ldots, x_t)$. We should mention here that $W_i(G)$ is always a normal set in G and hence $\langle W_i(G) \rangle$ is a normal subgroup.

The simple commutator word of weight n is defined recursively as

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$$

with $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$. Consider $w(x_1, \cdots, x_{n+1}) = [x_1, \cdots, x_{n+1}]$. Then it is easy to show that $W_i(G)$ is always a subgroup and $W_1(G) = Z_n(G)$, the n-th term of the upper central series.

The focus of investigations on the sets $W_i(G)$ has been in context with the n-Engel word $w_n(x, y) = [x,_n y]$, recursively defined as

$$[x,_n y] = [[x,_{n-1} y], y], n \geq 2,$$

and

$$[x,_1 y] = [x, y].$$

We mention here that this question has been considered in context with other words, e.g. the n-commutator word $w(x, y) = (xy)^n y^{-n} x^{-n}$, $n \geq 2$ an integer (see [4] and [31]).

An element a of a group G is called a *right n-Engel element* of G if it belongs to the set

$$R_n(G) = \{b \in G \mid [b,_n g] = 1, \forall g \in G\},$$

and a *left n-Engel element* of G if it belongs to the set

$$L_n(G) = \{b \in G \mid [g,_n b] = 1, \forall g \in G\}.$$

Obviously $R_1(G) = Z(G) = L_1(G)$.

A celebrated result of 1961 by W.P. Kappe [34] ensures that $R_2(G)$ is always a subgroup. In 1970, however, I.D. Macdonald in [38] gave an example that shows that right 3-Engel elements do not form a subgroup in general. This result was generalized in 1999 by W. Nickel [44]. For each $n \geq 3$ he constructed a group with a right n-element a where neither a^{-1} nor a^2 is a right n-Engel element. Therefore $R_n(G)$ need not be a subgroup, for $n \geq 3$.

The same is true for the set $L_2(G)$. The left 2-Engel elements of a group do not need to form a subgroup in general. For instance, the standard wreath product of a group of order 2 with an elementary abelian group of order 4 is generated by left 2-Engel elements but does not consist of such elements. Furthermore, for $n > 2$ the first author gave examples of metabelian groups in which $L_n(G)$ does not form a subgroup (see [29]).

In contrast to that P.M. Ratchford and the first author showed in [32] that $R_n(G)$ is a subgroup of G whenever G is metabelian or center-by-metabelian with certain extra conditions attached. In [2] A. Abdollahi and H. Khosravi proved that the set of right 4-Engel elements of a group G is a subgroup for locally nilpotent groups G without elements of orders 2, 3, or 5. See also the survey [1] by Abdollahi.

An element a of a group G is called a *right Engel element* of G if for each $g \in G$ there is an integer $n = n(a, g) \geq 0$ such that $[a,_n g] = 1$, a *left Engel element* if $[g,_n a] = 1$. Now set

$$\bar{R}(G) = \bigcup_{n \geq 2} R_n(G) \text{ and } \bar{L}(G) = \bigcup_{n \geq 2} L_n(G),$$

and consider also the following sets:

$$R(G) = \{a \in G | a \text{ a right Engel element of } G\}$$

and

$$L(G) = \{a \in G | a \text{ a left Engel element of } G\}.$$

In 1966, T.A. Peng [47] generalized results previously obtained by B.I. Plotkin [49], R. Baer [3] and E. Schenkman [51] and proved that if G satisfies the maximal condition on abelian subgroups, then the previous four subsets are actually subgroups. Precisely he proved that $R(G)$ and $\bar{R}(G)$ coincide with the hypercenter of G, and $L(G)$ and $\bar{L}(G)$ coincide with the Fitting subgroup of G.

K.W. Gruenberg in [20] showed that in any soluble group G the subsets $\bar{R}(G)$, $L(G)$ and $\bar{L}(G)$ are subgroups, in particular the latter coincides with the Baer radical of G. See also [50] for many results on this topic.

In the paper [48], Peng defined a group G to be an E-group if, for every $x \in G$, the set of all $y \in G$ such that $[y,_n x] = 1$ for some positive integer n (depending on x and y) is a subgroup. He studied finite soluble E-groups.

H. Heineken in [22] continued the study of E-groups, and showed in particular that the class of finite E-groups is a formation.

A generalization of the word $w(x, y) = [x, y]$ has been considered by W.P. Kappe in 2003 [35]. He started from the word

$$w(x, y, x_1, \ldots, x_n) = [x, y, x_1, \ldots, x_n, y]$$

and, for any $n \geqslant 1$, studied the set

$$B_n(G) = \{b \in G \mid [b, g, a_1, \ldots, a_n, g] = 1, \forall g, a_1, \ldots, a_n \in G\}.$$

He proved that $B_n(G)$ is always a subgroup of G.

4 The Set of Autocommutators and the Autocommutator Subgroup

Let G be a group, $g \in G$ and $\varphi \in Aut(G)$. The **autocommutator** of g and φ is the element

$$[g, \varphi] = g^{-1} g^{\varphi}.$$

Obviously, the autocommutator $[g, \varphi]$ is the commutator of g and φ in the holomorph of G. We denote by

$$K^{\star}(G) = \{[g, \varphi] \mid g \in G, \varphi \in Aut(G)\}$$

the set of all autocommutators of G and, following P.V. Hegarty [26], we write

$$G^{\star} = \langle K^{\star}(G) \rangle,$$

where $G^{\star}$ is called the autocommutator subgroup of G.

At “Groups in Galway 2003” D. MacHale brought the following problem to the attention of the first author:

Is $G^{\star}$ always equal to $K^{\star}(G)$?

He added that there might be even a finite abelian counterexample.

D. Garrison, D. Yull and the first author showed in 2006 that the answer to this conjecture is negative. In fact they proved the following result:

Theorem 4.1 ([17]) *Let G be a finite abelian group. Then the set of autocommutators always forms a subgroup.*

MacHale also suggested that the two groups of order 96 given by Guralnick [21] as the minimal counterexamples to the conjecture $G' = K(G)$ might also be minimal counterexamples to the conjecture $G^{\star} = K^{\star}(G)$. In this case the answer is also negative. In fact, in [17] the following result is proved.

Theorem 4.2 ([17]) *There exists a finite nilpotent group of class 2 and order* 64 *in which the set of autocommutators does not form a subgroup, namely*

$$G =< a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, [a, b] = [a, c] =$$
$$[a, d] = [b, c] = [b, d] = [c, d] = e^2, [a, e] = [b, e] = [c, e] = [d, e] = 1 > .$$

In fact, G is unique of that order and for all groups of order less than 64 *the set of autocommutators forms a subgroup.*

Many of these results were obtained with the help of GAP (see [54]). It was also proved there that in the two groups of order 96 given by Guralnick in [21] as minimal counterexamples for $K(G) \neq G'$ we have $K^\star(G) = G^\star$. This result was also obtained with the help of GAP.

In [17], the authors were able to give a complete description of $K^\star(G)$, if G is a finite abelian group. In fact, they proved the following theorem.

Theorem 4.3 ([17]) *Let G be a finite abelian group*

$$G = B \times O,$$

where O is of odd order, and B is a 2*-group. Then we have:*

(i) If either $B = 1$ *or* $B = \langle b_1 \rangle \times \langle b_2 \rangle \times H$ *with* $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, *then* $K^\star(G) = G^\star = G$.
(ii) If $B = \langle b_1 \rangle \times H$, *with* $|b_1| = 2^n$, $expH \leq 2^{n-1}$, *then* $K^\star(G) = G_{2^{n-1}} \times O$, *where* $G_{2^{n-1}} = \{x \in G \mid x^{2^{n-1}} = 1\}$.

In any case, $K^\star(G)$ *is a subgroup of G.*

Notice that Theorem 4.3 implies immediately the following corollary, a result obtained by C. Çis, M. Çis and G. Silberberg:

Theorem 4.4 ([8]) *Every finite abelian group is the autocommutator subgroup of some finite abelian group.*

Similar as in the case of the commutator subgroup G', there exist non-abelian finite groups that are not the autocommutator subgroup of any group. For example, M. Deaconescu and G.L. Walls showed in [11] that this is the case for the symmetric group S_3. In the same paper they also classified all finite groups G such that $G^\star$ is infinite cyclic or cyclic of prime order. There are only three groups G such that $G^\star$ is infinite cyclic, the group $\mathbb{Z}$ of the integers, $\mathbb{Z} \times \mathbb{Z}_2$ and D_∞, the infinite dihedral group. Only $G \simeq \mathbb{Z}_4$ has the property $K^\star(G) \simeq \mathbb{Z}_2$. However, if p is odd, the equation $G^\star \simeq \mathbb{Z}_p$ has the solution: $G \simeq \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_2, T$ or $T \times \mathbb{Z}_2$, where T is a partial holomorph of $\mathbb{Z}_p$ containing $\mathbb{Z}_p$. Furthermore, Hegarty showed in [27] that for any finite group H there are only finitely many finite groups G such that $G^\star \simeq H$.

The group S_3 is a complete group. More generally, M. Naghshineh, M. Farrokhi D.G., and M.R.R. Moghaddam showed in [43] that if H is a finite complete group and $G^\star \simeq H$, then H is perfect and $G \simeq H$ or $G \simeq H \times \mathbb{Z}_2$. In the same paper the authors studied the groups G such that $G^\star \simeq H$ for other finite groups H and conjectured that there is no finite group G such that $G^\star$ is the finite dihedral group of order $2n$.

In the paper [26] Hegarty also defined the absolute center of a group G as

$$Z^\star(G) = \{g \mid [g, \varphi] = 1, \forall \varphi \in Aut(G)\}.$$

He proved the analogue of the classical result of I. Schur [52] that $G/Z(G)$ finite implies G' finite. In fact he showed that if $G/Z^\star(G)$ is finite, then $G^\star$ is finite and $Aut(G)$ is finite.

Schur's theorem was extended by A. Mann in [39], who showed that if $G/Z(G)$ is locally finite of exponent n, then G' is locally finite of n-bounded exponent. H. Dietrich and P. Moravec in [12] proved the analogue result for $G/Z^\star(G)$ and $G^\star$. They showed that if $G/Z^\star(G)$ is locally finite of exponent n, then $G^\star$ is locally finite of n-bounded exponent.

5 The Set of Autocommutators and the Autocommutator Subgroup in Infinite Abelian Groups

In this section we report some recent results on the autocommutators in infinite abelian groups, obtained by the three authors of this survey in [33], and by Dietrich and Moravec in [12]. Since we will deal with abelian groups, we will use additive notation for the operation of G.

Hence, if $g \in G$ and $\varphi \in Aut(G)$, we will write the **autocommutator** of g and φ as

$$[g, \varphi] := -g + g^{\varphi}.$$

It is an easy exercise to prove the following two propositions.

Proposition 5.1 *Let G be an abelian torsion group without elements of even order. Then*

$$K^\star(G) = G^\star = G.$$

Proposition 5.2 *In any abelian group G, we have $2G \subseteq K^\star(G)$.*

First, notice that in infinite abelian groups the autocommutators do not always form a subgroup, as shown by the following example.

Example 5.3 ([33]) Let $G = \langle a\rangle \oplus \langle c\rangle$, where $\langle a\rangle$ is infinite cyclic and $|c| = 2$. Then $K^\star(G)$ is not a subgroup of G.

Proof Let $\varphi \in Aut(G)$, then $\varphi(c) = c$, and $\varphi(a) = \gamma a + \delta c$, where $\gamma \in \{1, -1\}$ and $\delta \in \{0, 1\}$. Therefore we have: $Aut(G) = \{1, \varphi_1, \varphi_2, \varphi_3\}$, where $1 = id_G$, $\varphi_1(a) = -a, \varphi_1(c) = c, \varphi_2(a) = a + c, \varphi_2(c) = c, \varphi_3(a) = -a + c, \varphi_3(c) = c$.
For any $g = \alpha a + \beta c \in G$, where $\alpha \in \mathbb{Z}$ and $\beta \in \{0, 1\}$, we have

$$-g + g^{\varphi_1} = (-\alpha)a + (-\beta)c + (-\alpha)a + \beta c =, (-2\alpha)a;$$
$$-g + g^{\varphi_2} = (-\alpha)a + (-\beta)c + \alpha a + \alpha c + \beta c = \alpha c;$$
$$-g + g^{\varphi_3} = (-\alpha)a + (-\beta)c + -\alpha a + \alpha c + \beta c = (-2\alpha)a + \alpha c.$$

In particular, $2a \in K^\star(G)$, $2a + c \in K^\star(G)$, but $4a + c \neq (-2\gamma)a + \gamma c$, for any integer γ. □

More generally, for finitely generated abelian groups we have the following result.

Theorem 5.4 ([33]) *Let G be a finitely generated abelian group. Set*

$$G = \langle a_1 \rangle \oplus \cdots \oplus \langle a_s \rangle \oplus B \oplus O,$$

where $a_1, \cdots, a_s$ are aperiodic, O is a finite group of odd order, B is a finite 2-group. Then we have:

(i) If $s > 1$, then $K^\star(G) = G^\star = G$.
(ii) If $s = 0$ and either $B = 0$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$ with $|b_1| = |b_2| = 2^n$, and $\exp H \leq 2^n$ then $K^\star(G) = G^\star = 2(\langle a_1 \rangle) \oplus B \oplus O$. Then $K^\star(G)$ is a subgroup of G.
(iii) If $s = 0$ and $B = \langle b_1 \rangle \oplus H$ with $|b_1| = 2^n$ and $\exp H \leq 2^{n-1}$, then $K^\star(G)$ is not a subgroup of G.
(iv) If $s = 0$ and either $B = 0$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$ with $|b_1| = |b_2| = 2^n$ and $\exp H \leq 2^n$, then $K^\star(G) = G^\star = G$. If $s = 0$ and $B = \langle b_1 \rangle \oplus H$ with $|b_1| = 2^n$ and $\exp H \leq 2^{n-1}$, then $K^\star(G) = G_{2^{n-1}} \oplus O$ where $G_{2^{n-1}} = \{x \in G \mid 2^{n-1} x = 0\}$.

In any case, if $s = 0$, then $K^\star(G)$ is a subgroup of G.

In order to study the autocommutators in infinite abelian groups, we start from the case G periodic. In this case the autocommutators do form a subgroup, in fact we have the following result.

Theorem 5.5 ([33]) *Let G be a periodic abelian group.*

Set $G = O \oplus D \oplus R$, where D is a divisible 2-group, R is a reduced 2-group and every element of O has odd order.

Then

$$K^\star(G) = O \oplus D \oplus K^\star(R),$$

where

(i) $K^\star(R) = R$ if R is of infinite exponent;
(ii) $K^\star(R) = R$ if R is of finite exponent 2^n, and $R = \langle a \rangle \oplus \langle b \rangle \oplus H$, with $|a| = |b| = 2^n$;
(iii) $K^\star(R) = R_{2^{n-1}}$ if R is of finite exponent 2^n, and $R = \langle a \rangle \oplus H$, with $|a| = 2^n$ and $\exp H = 2^{n-1}$.

In particular, $K^\star(G)$ is a subgroup of G.

In the mixed case, generalizing our previous example, it is easy to construct many other examples of mixed abelian groups G in which $K^\star(G)$ is not a subgroup. In fact, we have:

Theorem 5.6 ([33]) *Let T be a periodic abelian group with $K^{\star}(T) \subset T$ and consider the group $G = T \oplus \langle a \rangle$, where $\langle a \rangle$ is an infinite cyclic group. Then $K^{\star}(G)$ is not a subgroup.*

In the group G of the previous theorem the torsion subgroup $T(G) = T$ is contained in $K^{\star}(G)$, but $K^{\star}(T) \subset T$. Thus it is not true that $T \cap K^{\star}(G) \subseteq K^{\star}(T)$. Surprisingly, the reverse inclusion holds. In fact we have the following result.

Theorem 5.7 ([33]) *Let G be a mixed abelian group and write $T = T(G)$ for the torsion subgroup of G. Then*

$$K^{\star}(T) \subseteq K^{\star}(G).$$

The study of the automorphism group of a torsion-free abelian group is usually very complicated.

In the following section we restrict our investigation to torsion-free abelian groups with finite automorphism group.

6 Autocommutators and the Autocommutator Subgroup in Torsion-Free Groups with Finite Automorphism Group

In this section we study autocommutators in a torsion-free group G with $Aut(G)$ finite. Notice that in this case G is abelian. In fact, we have $G/Z(G) \simeq Inn(G) \subseteq Aut(G)$, thus $G/Z(G)$ is finite. Then G' is finite by Schur's theorem. Thus G' is trivial, since G is torsion-free.

Torsion-free abelian groups with finite automorphism group have been studied by A. de Vries and A.B. de Miranda in [10], as well as by J.T. Hallet and K.A. Hirsch in [24] and [25], and recently by A.L.S. Corner in [9].

The final description of the automorphism group of these groups refers to six groups, which are called **primordial groups**. They are the basic building blocks for the groups under consideration. Besides the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6$, the following groups are the primordial groups:

(1) the quaternion group Q_8 of order 8;
(2) the dicyclic group $DC_{12} =< a, b | a^3 = b^2 = (ab)^2 >$ of order 12;
(3) the binary tetrahedral group $BT_{24} =< a, b | a^3 = b^3 = (ab)^2 >$ of order 24.

We have the following:

Theorem 6.1 ([9]) *A finite group H is the automorphism group of a torsion-free abelian group if and only if*

(i) it is a subdirect product of primordial groups, and

(ii) if H has no cyclic direct factor of order 2, then G contains an element whose centralizer is a 2*-group and* $\{\alpha \in H \mid \alpha^3 = 1\}$ *is a direct product of groups whose 3-Sylow are trivial or cyclic of order 3.*

Next we will give a brief survey on the results obtained in [33] concerning autocommutators and the autocommutator subgroup in torsion-free abelian groups with finite automorphism group. We start with the following useful remark.

Let $\varphi \in Aut(G)$, then the map

$$\theta_\varphi : x \in G \longmapsto -x + x^\varphi \in G$$

is a homomorphism of G. Hence $Im\theta_\varphi$ is a subgroup of G, contained in $K^\star(G)$. Moreover,

$$K^\star(G) = \bigcup_{\varphi \in Aut(G)} Im\theta_\varphi.$$

We start from the case $Aut(G)$ primordial and first consider the case that $Aut(G)$ is cyclic.

Theorem 6.2 ([33]) *Let G be a group with cyclic automorphism group. Then* $K^\star(G)$ *is a subgroup of G.*

Another positive result is the following.

Theorem 6.3 ([33]) *Let G be a group with* $Aut(G) \simeq BT_{24}$*. Then* $K^\star(G)$ *is a subgroup of G.*

The case $Aut(G) \simeq Q_8$ is more complicated. In fact, we have the following result.

Theorem 6.4 ([33]) *Let G be a group with* $Aut(G) \simeq Q_8 = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 \rangle$ *and let* $F = Im\theta_\alpha \cap Im\theta_\beta \cap Im\theta_{\alpha\beta}$.

Then $K^\star(G)$ *is a subgroup of G if and only if either* $K^\star(G) = Im\theta_\varphi$ *for some* $\varphi \in Aut(G)$ *or* $G^\star/F \simeq V_4$.

The group A in Example 129.6 of [16] has $Aut(A) \simeq Q_8$ and $K^\star(A)$ is a subgroup of A. In fact, we have: $A/2A = \langle a + 2A \rangle \oplus \langle b + 2A \rangle \oplus \langle c + 2A \rangle \oplus \langle d + 2A \rangle$ and $A/F = \{F, a+b+F = c+d+F, a+c+F = b+d+F, b+c+F = a+d+F\}$.

We mention here that it is possible to construct a group G with $Aut(G) \simeq Q_8$, $G/2G$ of rank 8 such that $K^\star(G)$ is not a subgroup of G.

Concerning the primordial group DC_{12}, we notice that the group A in Example 129.7 of [16] has $Aut(A) \simeq DC_{12}$ and $K^\star(A)$ is a subgroup of A. So far we have not been able to find a torsion-free abelian group with $Aut(G) \simeq DC_{12}$ and $K^\star(G) \neq G^\star$.

In conclusion, we notice that de Vries and de Miranda [10] as well as Hallett and Hirsch [24] constructed many examples of abelian groups G, indecomposable or not, of rank ≥ 2 with $Aut(G) \simeq V_4$. In their examples we have $K^\star(G) = 2G$. Thus $K^\star(G)$ is always a subgroup of G. But we have been able to show the following.

Theorem 6.5 ([33]) *There exists a torsion-free abelian group G of rank 2 such that* $Aut(G) \simeq V_4$ *and* $K^\star(G)$ *is not a subgroup of G.*

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Recent Progress in Module Approximations

Jan Trlifaj

To the memory of my dear friend and coauthor Rüdiger Göbel.

Abstract We present two recent developments in the approximation theory of modules. The first one investigates boundaries of this theory, namely the classes naturally occurring in homological algebra, but not providing for approximations (e.g., the class of all flat Mittag-Leffler modules). We introduce the key tools for their study which involve set-theoretic methods combined with (infinite dimensional) tilting theory. The second development concerns tilting classes, their structure over commutative rings, and the recent generalization to silting modules and classes.

Keywords Approximations of modules • Infinite dimensional tilting theory • Set-theoretic homological algebra • Silting classes and modules

Mathematical Subject Classification (2010): 16DXX, 16E30, 18G25, 13D07, 03E75

1 Introduction

Since the solution of the Flat Cover Conjecture [9], a number of classes $\mathcal{C}$ of (right R-) modules were shown to be deconstructible, that is, each of their modules expressible as a transfinite extension of small modules from $\mathcal{C}$. The deconstructibility implies existence of $\mathcal{C}$-precovers, hence makes $\mathcal{C}$ fit in the machinery of relative homological algebra [15].

Though deconstructible classes may appear ubiquitous, some important non-precovering classes of modules have gradually emerged. First, an extension of ZFC was constructed in [13] such that the class of all Whitehead groups is not

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precovering. In ZFC, the class of all flat Mittag-Leffler (= $\aleph_1$-projective) modules has recently been shown to be precovering, if only if R is a right perfect ring, [28].

The first part of this survey studies these boundaries of the approximation theory in more detail. We introduce tree modules, a key tool used to prove non-existence of approximations, and more in general, non-existence of factorizations of maps. The construction of tree modules goes back to [12] and [17], but it is now available in much broader contexts: the flat Mittag-Leffler (tree) modules are just the zero dimensional instances, for $T = R$ and $n = 0$, of locally T-free (tree) modules, where T is any n-tilting module. The phenomenon of non-precovering occurs for locally T-free modules, if and only if T is not $\sum$-pure split. In particular, the phenomenon can be traced even to the setting of finite dimensional hereditary algebras: it occurs when R is of infinite representation type and T is the Lukas tilting module [29]. We will also see that the same tools apply to the (non-tilting) setting of very flat modules. The latter modules have recently been introduced in algebraic geometry [25].

In the second part of the survey, we present recent results on the structure of tilting classes which directly continue the research presented in [18, Vol. 1]. We start with a natural extension of the classification of tilting classes over commutative noetherian rings to the general commutative setting [2]. Then we deal with a recent generalization originating in representation theory: the theory of silting modules and classes [1, 5], and pursue the analogies with the tilting setting: e.g., the finite type result for silting classes [23], and the classification of silting classes over commutative rings [2]. We finish by presenting Saorín's problem on 1-tilting modules from [24], and its recent solution in [8].

2 Preliminaries

2.1 Module Approximations

In order to present the new developments, we need to recall briefly the relevant basic notions and facts from the approximation theory of modules. For more details, we refer the reader to [18, Part II].

For an (associative, but not necessarily commutative) ring R with unit, we denote by Mod-R the category of all (unitary right R-) modules. Moreover, given an infinite cardinal κ and a class of modules $\mathcal{C}$, we will use the notation $\mathcal{C}^{<\kappa}$ and $\mathcal{C}^{\leq\kappa}$ to denote the subclass of $\mathcal{C}$ consisting of all less than κ-presented modules, and at most κ-presented modules, respectively.

The notation mod–R stands for the category of all *strongly finitely presented* modules, i.e, the modules possessing a projective resolution consisting of finitely generated projective modules. Note that if R is right coherent, then mod–R = $(\text{Mod–}R)^{<\omega}$ is the category of all finitely presented modules.

Definition 2.1 Let $\mathcal{C}$ be a class of modules. A module M is said to be *$\mathcal{C}$-filtered* (or a *transfinite extension* of the modules in $\mathcal{C}$), provided there exists an increasing chain $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ of submodules of M with the following properties: $M_0 = 0$,

$M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \le \sigma$, $M_{\alpha+1}/M_\alpha \cong C_\alpha$ for some $C_\alpha \in \mathcal{C}$ for each $\alpha < \sigma$, and $M_\sigma = M$.

The chain $\mathcal{M}$ is called a *$\mathcal{C}$-filtration* of the module M of length σ. If σ is finite, then M is said to be finitely $\mathcal{C}$-filtered. The class of all $\mathcal{C}$-filtered modules will be denoted by Filt($\mathcal{C}$). We will say that $\mathcal{C}$ is *closed under transfinite extensions* provided that $\mathcal{C} = \text{Filt}(\mathcal{C})$.

For example, if $\mathcal{C}$ is the class of all simple modules, then Filt($\mathcal{C}$) is the class of all semiartinian modules, and finitely $\mathcal{C}$-filtered modules coincide with the modules of finite length.

Given a class of (infinitely generated) modules $\mathcal{C}$ and $M \in \mathcal{C}$, it is rarely possible to decompose M into a direct sum of small, or even indecomposable, modules from $\mathcal{C}$. Deconstructibility is much more feasible:

Definition 2.2 Let $\mathcal{C}$ be a class of modules and κ an infinite cardinal. Then $\mathcal{C}$ is κ-*deconstructible* provided that $\mathcal{C} = \text{Filt}(\mathcal{C}^{<\kappa})$. The class $\mathcal{C}$ is called *deconstructible*, if $\mathcal{C}$ is κ-deconstructible for some infinite cardinal κ.

Example 2.3 If R has cardinality $\le \kappa$ where κ is an infinite cardinal, then the class $\mathcal{F}_n$ of all modules of flat dimension at most n is κ^+-deconstructible.

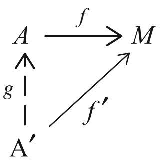
The class of all projective modules $\mathcal{P}_0$ is $\aleph_1$-deconstructible, because each projective module is a direct sum of countably generated projective modules by a classic theorem of Kaplansky.

Let $n \ge 0$ be finite, and κ be an infinite cardinal. If each right ideal of R is $\le \kappa$-generated, then the class $\mathcal{P}_n$ of all modules of projective dimension at most n is κ^+-deconstructible. As recently proved in [29, §3], the latter fact can substantially be generalized:

Theorem 2.4 *Assume that κ is an infinite cardinal such that each right ideal of R is $\le \kappa$-generated. Let $n \ge 0$ be finite, and $\mathcal{C}$ be any κ^+-deconstructible class of modules. Then the class of all modules possessing a $\mathcal{C}$-resolution of length $\le n$ is also κ^+-deconstructible.*

Right and left approximations of modules were introduced by Auslander, Reiten and Smalø in the setting of finitely generated modules over artin algebras. Independently, Enochs and Xu studied them in the general setting of Mod–R, but they used the terminology of precovers and preenvelopes, respectively. Since we will primarily be interested in the general setting, we prefer the latter terminology (following [15] and [18]):

Definition 2.5 (i) A class of modules $\mathcal{A}$ is *precovering* if for each module M there is $f \in \text{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \text{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:



The map f is called an $\mathcal{A}$-*precover* of M (or a *right* $\mathcal{A}$-*approximation* of M).

(ii) An $\mathcal{A}$-precover is *special* in case it is surjective, and its kernel K satisfies $\operatorname{Ext}^1_R(A, K) = 0$ for each $A \in \mathcal{A}$.

(iii) Let $\mathcal{A}$ be precovering. Assume that in the setting of (i), if $f' = f$ then each factorization g is an automorphism. Then f is called an $\mathcal{A}$-*cover* of M. The class $\mathcal{A}$ is *covering* in case each module has an $\mathcal{A}$-cover.

We note that each covering class containing $\mathcal{P}_0$ and closed under extensions is necessarily special precovering. The class $\mathcal{P}_0$ is easily seen to be precovering, while $\mathcal{F}_0$ is covering by Bican et al. [9]. Recall that by the classic Bass' Theorem P, $\mathcal{P}_0$ is covering, iff $\mathcal{P}_0 = \mathcal{F}_0$, i.e., iff R is a right perfect ring.

Dually, we define *(special) preenveloping* and *enveloping* classes of modules. For example, $\mathcal{I}_0$, the class of all injective modules, is an enveloping class.

Precovering classes are ubiquitous because of the following basic facts due to Enochs and Šťovíček:

Theorem 2.6 *Let $\mathcal{S}$ be a set of modules and $\mathcal{C} = Filt(\mathcal{S})$. Then $\mathcal{C}$ is precovering. Moreover, if $\mathcal{C}$ is closed under direct limits, then $\mathcal{C}$ is covering.*

The first claim of Theorem 2.6 implies that each deconstructible class closed under transfinite extensions is precovering.

The converse of the second claim, namely whether each covering class of modules is necessarily closed under direct limits, is still open—this is the *Enochs Problem*. (For a case where the Enochs Problem has recently been solved in the positive, see Sect. 3.3 below.)

Example 2.7 The classes $\mathcal{P}_n$ ($n < \omega$) for any ring R, as well as $\mathcal{GP}$, the class of all Gorenstein projective modules over an Iwanaga–Gorenstein ring R, are special precovering. The classes $\mathcal{F}_n$ ($n < \omega$) over any ring, and $\mathcal{GF}$ of all Gorenstein flat modules over an Iwanaga–Gorenstein ring R, are covering. The classes $\mathcal{I}_n$ ($n < \omega$) of all modules of injective dimension $\leq n$ for any ring R (resp. $\mathcal{GI}$ for R Iwanaga–Gorenstein) are special preenveloping (resp. enveloping).

Precovering classes $\mathcal{C}$, and preenveloping classes $\mathcal{E}$, can be employed in developing relative homological algebra, where the (absolute) classes of all projective and injective modules are replaced by the (relative) classes $\mathcal{C}$ and $\mathcal{E}$, respectively, cf. [15].

Besides the formal duality between the definitions of precovering and preenveloping classes, there is also an explicit duality discovered by Salce, mediated by complete cotorsion pairs:

Definition 2.8 Let R be a ring. A pair of classes of modules $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a (hereditary) *cotorsion pair* provided that

1. $\mathcal{A} = {}^{\perp}\mathcal{B} := \{A \in \text{Mod-}R \mid \operatorname{Ext}^i_R(A, B) = 0 \text{ for all } i \geq 1 \text{ and } B \in \mathcal{B}\}$, and
2. $\mathcal{B} = \mathcal{A}^{\perp} := \{B \in \text{Mod-}R \mid \operatorname{Ext}^i_R(A, B) = 0 \text{ for all } i \geq 1 \text{ and } A \in \mathcal{A}\}$.

In this case $\mathcal{A}$ is closed under transfinite extensions. If moreover

3. For each module M, there exists an exact sequence $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$,
   then $\mathfrak{C}$ is called *complete*.

Condition 3 implies that $\mathcal{A}$ is a special precovering class. In fact, 3 is equivalent to its dual:

3'. For each module M there is an exact sequence $0 \to M \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The latter condition implies that $\mathcal{B}$ is a special preenveloping class.

Complete cotorsion pairs, and hence special precovering and special preenveloping classes, are abundant:

Theorem 2.9 *For each set of modules $\mathcal{S}$, there is a complete cotorsion pair of the form $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$ in Mod-R.*

Remark 2.10 The rich supply of complete cotorsion pairs yields a variety of ways to do relative homological algebra. This is certainly not restricted to module categories: the modern way of doing homological algebra is working with the derived category of a given Grothendieck category $\mathcal{G}$. In order to compute morphisms between two objects A and B in the derived category, it suffices to introduce a model category structure on $C(\mathcal{G})$) (= the category of unbounded chain complexes on $\mathcal{G}$). Morphisms between A and B are then computed as the $C(\mathcal{G})$-morphisms between cofibrant and fibrant replacements of A and B modulo chain homotopy. Hovey [20] has shown that compatible model category structures correspond 1–1 to certain complete cotorsion pairs in $C(\mathcal{G})$, and the latter arise naturally from complete cotorsion pairs in $\mathcal{G}$, [34]. We refer to the survey [31] for more on this link between approximation theory and homological algebra for general Grothendieck categories, notably for quasi-coherent sheaves over schemes. A further extension of (parts of) the theory to approximations over locally presentable abelian categories has recently been obtained in [26].

2.2 Tilting Theory

Next, we recall the basics of (infinite dimensional) tilting theory. For more details, we refer to [18, Part III].

For a module T, denote by Add(T) (resp. add(T)) the class of all direct summands of arbitrary (resp. finite) direct sums of copies of T.

Definition 2.11 A module T is *tilting* provided that

(T1) T has finite projective dimension.
(T2) $\mathrm{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i$ and all cardinals κ.

(T3) There exist $r < \omega$ and an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ where $T_i \in \mathrm{Add}(T)$ for each $i \leq r$.

The class $\mathcal{T}_T := T^\perp$ is the *right tilting class*, $\mathcal{A}_T := {}^\perp\mathcal{T}_T$ the *left tilting class*, and the (complete) cotorsion pair $\mathfrak{C}_T := (\mathcal{A}_T, \mathcal{T}_T)$ the *tilting cotorsion pair*, induced by T. If T has projective dimension $\leq n$, then the tilting module T is called *n-tilting*, and similarly for $\mathcal{T}_T$, $\mathcal{A}_T$, and $\mathfrak{C}_T$. If T and T' are tilting modules, then T is *equivalent* to T' in case T and T' induce the same tilting class.

If $n = 1$, then $\mathcal{T}_T$ is a torsion class in Mod–R, so there is a *tilting torsion pair* $(\mathcal{T}_T, \mathcal{F}_T)$ in Mod–R.

Tilting theory originated in the realm of finitely generated modules/representations of finite dimensional algebras, but many of its aspects extend to the general setting of possibly infinitely generated modules over arbitrary rings. Such extension is especially desired for commutative rings, because all finitely generated tilting modules over a commutative ring are projective, that is, 0-tilting.

The main focus of the classical tilting theory is on category equivalences induced by tilting modules. Here, we will need to recall only the approximation properties of the corresponding tilting classes. The first one concerns (1-) tilting torsion classes:

Proposition 2.12 *Let R be a ring and $\mathcal{T}$ be a torsion class of modules. Then $\mathcal{T}$ is a right 1-tilting class, iff $\mathcal{T}$ is special preenveloping.*

A much more complex argument is needed to prove the following characterization of general tilting classes and tilting cotorsion pairs:

Theorem 2.13 *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then $\mathfrak{C}$ is tilting, iff $\mathcal{A} \subseteq \mathcal{P}_n$ for some $n < \omega$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

Even though tilting modules are allowed to be infinitely generated, there is always a grain of finiteness preserved. A class of modules $\mathcal{T}$ is said to be *of finite type*, in case there exists $n < \omega$ and a set $\mathcal{S}$ consisting of strongly finitely presented modules of projective dimension $\leq n$ such that $\mathcal{T} = \mathcal{S}^\perp$. Then $\mathcal{T}$ is also axiomatizable, by a (possibly infinite) set of formulas of the language of the first order theory of modules. Also, $\mathcal{T}$ is *definable*, that is, $\mathcal{T}$ is closed under arbitrary direct products, direct limits, and pure submodules.

Theorem 2.13 easily implies that each class of finite type is a right tilting class. The converse is a major accomplishment of (infinite-dimensional) tilting theory:

Theorem 2.14 *Let R be a ring, T be an n-tilting module, and $(\mathcal{A}_T, \mathcal{B}_T)$ be the induced tilting cotorsion pair. Then $\mathcal{A}_T$ is $\aleph_1$-deconstructible, and $\mathcal{B}_T$ is of finite type.*

Theorem 2.14 makes it possible to classify tilting modules and classes over Dedekind domains, because finitely presented modules are classified in this case. Further tools are needed to handle the general commutative noetherian case. The main result from [4] offers the following classification:

A sequence $(P_0, \ldots, P_{n-1})$ consisting of subsets of the Zariski spectrum Spec(R) is called *characteristic* provided that $P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{n-1}$, and for each $i < n$, P_i is an upper subset of the poset (Spec(R), $\subseteq$) such that P_i contains no associated primes of $\Omega^{-i}(R)$, where $\Omega^{-i}(R)$ denotes the ith cosyzygy in the minimal injective coresolution of R.

Theorem 2.15 *Let R be a commutative noetherian ring and $n < \omega$. Then right n-tilting classes in Mod–R are parametrized by characteristic sequences: the class $\mathcal{T}$ corresponding to a characteristic sequence $(P_0, \dots, P_{n-1})$ is defined by the formula*

$$\mathcal{T} = \{M \in Mod\text{–}R \mid Tor_i^R(M, R/p) = 0 \text{ for all } i < n \text{ and } p \in P_i\}.$$

In Chap. 4, we will present recent generalizations of Theorem 2.15, both for the commutative, but not necessarily noetherian, tilting setting, and for the silting setting.

3 Boundaries of the Approximation Theory

3.1 Locally T-Free Modules

Having defined tilting modules T, we can now proceed to the locally T-free ones. We start with a slightly more general notion:

Definition 3.1 Let R be a ring. A system $\mathcal{S}$ consisting of countably presented submodules of a module M is a *dense system* provided that $\mathcal{S}$ is closed under unions of well-ordered countable ascending chains, and each countable subset of M is contained in some $N \in \mathcal{S}$.

Let $\mathcal{F}$ be a class of countably presented modules. Denote by $\mathcal{C}$ the class of all modules possessing a countable $\mathcal{F}$-filtration. A module M is *locally $\mathcal{F}$-free* provided that M contains a dense system of submodules consisting of elements of $\mathcal{C}$.

Notice that if M is countably presented, then M is locally $\mathcal{F}$-free, iff $M \in \mathcal{C}$. Also, each locally $\mathcal{F}$-free module is a directed union of the modules in $\mathcal{C}$. A less trivial fact proved in [29] is

Lemma 3.2 *Let $\mathcal{F}$ be a class of countably presented modules. Then the class of all locally $\mathcal{F}$-free modules is closed under transfinite extensions.*

Definition 3.3 Consider the particular case of Definition 3.1 when $\mathcal{F} = \mathcal{A}^{\leq\omega}$ for a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$. Then $\mathcal{C} = \mathcal{F}$, and a module is locally $\mathcal{F}$-free, iff it admits a dense system of countably presented submodules from $\mathcal{A}$.

In particular, if T is a tilting module with the induced tilting cotorsion pair $(\mathcal{A}_T, \mathcal{B}_T)$, then the locally $\mathcal{A}_T^{\leq\omega}$-free modules are simply called *locally T-free*.

Example 3.4 If $T = R$, then the locally T-free modules coincide with the $\aleph_1$-projective modules in the sense of [11]. By Herbera and Trlifaj [19], they also coincide with the *flat Mittag-Leffler modules*, that is, the modules M such that the functor $M \otimes_R -$ is exact, and for each sequence of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic. We will denote by $\mathcal{FM}$ the class of all flat Mittag-Leffler modules. Clearly $\mathcal{P}_0 \subseteq \mathcal{FM} \subseteq \mathcal{F}_0$.

Lemma 3.2 suggests the question of whether the classes of locally $\mathcal{F}$-free modules are deconstructible (if so, they are even precovering by Theorem 2.6).

This is true in the setting of Example 3.4 when R is a right perfect ring, because then all flat (Mittag-Leffler) modules are projective. However, the picture changes completely in the non-right perfect case. In [16], the following result was proved in ZFC: the class of all flat Mittag-Leffler (= $\aleph_1$-free) abelian groups is not precovering. The proof used an idea from [13], where it was proved that the assertion ‘the class of all Whitehead groups is not precovering’ is consistent with ZFC. But the latter assertion is not provable in ZFC, because it is also consistent with ZFC that all Whitehead groups are free. The ZFC result from [16] mentioned above has gradually been extended from the case of $R = \mathbb{Z}$ to all countable non-right perfect rings R, [7]. Also, it turned out that other classes of locally T-free modules exhibit non-precovering properties in ZFC. Tree modules provide a key tool for proving these properties.

3.2 Tree Modules

In order to define a tree module, we first have to introduce its basic combinatorial component—a tree.

Definition 3.5 Let κ be an infinite cardinal, and T_κ be the set of all finite sequences of ordinals less than κ, or equivalently, all maps $\tau : n \to \kappa$ with $n < \omega$. The symbol $\ell(\tau)$ will denote the *length* of τ (so $\ell(\tau) = n$ for $\tau : n \to \kappa$).

We define a partial order on T_κ by letting $\tau' \leq \tau$, iff $\ell(\tau') \leq \ell(\tau)$ and $\tau \restriction \ell(\tau') = \tau'$. This partial order gives T_κ the structure of a tree, called the *tree on* κ; the maps τ are the nodes of T_κ.

Let $\mathrm{Br}(T_\kappa)$ denote the set of all branches of T_κ. Each branch $\nu \in \mathrm{Br}(T_\kappa)$ can be identified with an ω-sequence of ordinals less than κ, so $\mathrm{Br}(T_\kappa) = \{\nu : \omega \to \kappa\}$.

Notice that card $(T_\kappa) = \kappa$, while card $(\mathrm{Br}(T_\kappa)) = \kappa^\omega$. Also, at each node τ, the tree T_κ branches to κ successive nodes, but T_κ has only short branches, of length ω.

Next, we turn to the basic algebraic component: the Bass module.

Definition 3.6 Let R be a ring and $\mathcal{F}$ be a class of countably presented modules. A module B is a *Bass module* over $\mathcal{F}$, provided that B is a countable direct limit of modules from $\mathcal{F}$. W.l.o.g., such B is the direct limit of a chain

$$F_0 \overset{f_0}{\to} F_1 \overset{f_1}{\to} \dots \overset{f_{i-1}}{\to} F_i \overset{f_i}{\to} F_{i+1} \overset{f_{i+1}}{\to} \dots \tag{1}$$

where $F_i \in \mathcal{F}$ and $f_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

Example 3.7 Consider the particular setting when $\mathcal{F} = \mathcal{P}_0^{\leq\omega}$ is the class of all countably presented projective modules. Then the Bass modules over $\mathcal{F}$ coincide with the countably presented flat modules, hence they have projective dimension at most 1.

If R is not right perfect, then the *classic Bass module* is a particular instance of the direct limit above when $F_i = R$ and f_i is the left multiplication by a_i ($i < \omega$), where $Ra_0 \supsetneq \cdots \supsetneq Ra_n \ldots a_0 \supsetneq Ra_{n+1}a_n \ldots a_o \supsetneq \ldots$ is a strictly decreasing chain of principal left ideals in R. Bass proved that in this case, the projective dimension of B equals 1. In particular, $B \in \mathcal{F}_0 \setminus \mathcal{FM}$.

Returning to the general setting, we observe that since our tree T_κ has branches of length ω, and the Bass module B is a direct limit of the system of modules $(F_i \mid i < \omega)$ indexed in ω, we can use B to decorate T_κ. The resulting tree module L will be contained in the product $P = \prod_{\tau \in T_\kappa} F_{\ell(\tau)}$ as follows:

Definition 3.8 For each $\nu \in \mathrm{Br}(T_\kappa)$, $i < \omega$, and $x \in F_i$, we define $x_{\nu i} \in P$ by

$$\pi_{\nu \restriction i}(x_{\nu i}) = x,$$

$$\pi_{\nu \restriction j}(x_{\nu i}) = f_{j-1} \ldots f_i(x) \text{ for each } i < j < \omega,$$

$$\pi_\tau(x_{\nu i}) = 0 \text{ otherwise},$$

where $\pi_\tau \in \mathrm{Hom}_R(P, F_{\ell(\tau)})$ denotes the τth projection for each $\tau \in T_\kappa$.

Let $X_{\nu i} = \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of P isomorphic to F_i. Further, let $X_\nu := \sum_{i<\omega} X_{\nu i}$, and $L = \sum_{\nu \in \mathrm{Br}(T_\kappa)} X_\nu$. L is called the *tree module* corresponding to κ and to the presentation (1) of B above.

Let $D = \bigoplus_{\tau \in T_\kappa} F_{\ell(\tau)}$. Then $D \subseteq L$, and D is κ-presented. Despite the fact that L/D is isomorphic to a large direct sum of copies of B, L inherits the property from D of being locally $\mathcal{F}$-free:

Lemma 3.9 *[29] There is an exact sequence* $0 \to D \hookrightarrow L \to B^{(Br(T_\kappa))} \to 0$. *Moreover, the module L is locally $\mathcal{F}$-free; this is witnessed by the dense system* $\mathcal{S} = \{X_C \mid X_C = \sum_{\nu \in C} X_\nu,\ \ C \text{ a countable subset of } Br(T_\kappa)\}$.

Lemma 3.9 is used in [28] to determine the crucial role of Bass modules in testing for existence of approximations:

Lemma 3.10 (Šaroch's Lemma) *Let $\mathcal{F}$ be a class of countably presented modules, and $\mathcal{L}$ the class of all locally $\mathcal{F}$-free modules. Let B be a Bass module over $\mathcal{F}$, such that B is not a direct summand in any module from $\mathcal{L}$. Then B has no $\mathcal{L}$-precover.*

Remark 3.11 The construction of tree modules above can be generalized further: the trees T_κ are replaced by ones whose branches have length λ, where λ is a regular infinite cardinal, and the decorating Bass modules by well-ordered direct limits of small modules indexed in λ. Such generalized tree modules have recently been employed in proving results on non-existence of factorizations of maps, whose applications include the solution of Auslander's Problem concerning almost split sequences in Mod–R. For more details, we refer to [27].

3.3 Locally T-Free Modules and Approximations

Let us now have a closer look at the particular case of locally T-free modules where T is a tilting module (see Definition 3.3). By Angeleri Hügel et al. [6], Bass modules play a central role in deciding further questions here:

Theorem 3.12 *Let T be a tilting modules and $\mathcal{A}_T$ be the corresponding left tilting class (so $\mathcal{A}_T = \mathrm{Filt}(\mathcal{C}_T)$ where $\mathcal{C} = \mathcal{A}_T^{\leq\omega}$ by Theorem 2.14). Let $\mathcal{L}_T$ denote the class of all locally T-free modules. Then the following are equivalent:*

1. *$\mathcal{L}_T$ is a (pre-) covering class.*
2. *All Bass modules over $\mathcal{C}_T$ are contained in $\mathcal{C}_T$.*
3. *The class $\mathcal{A}_T$ is closed under direct limits.*
4. *T is $\sum$-pure split.*

Here, a module T is *$\sum$-pure split* provided that each pure embedding $T_0 \hookrightarrow T_1$ with $T_1 \in \mathrm{Add}\,(T)$, splits. For example, any $\sum$-pure injective module is $\sum$-pure split.

Since $\mathcal{A}_T \subseteq \mathcal{L}_T \subseteq \varinjlim \mathcal{C}_T$, condition (3) above is further equivalent to $\mathcal{L}_T$ being closed under direct limits. This shows that the Enochs Problem from Theorem 2.6 has a positive solution for all left tilting classes of modules.

We note the following corollary of Theorem 3.12 for the zero-dimensional case of $T = R$ (cf. Examples 3.4 and 3.7). It may be viewed as an approximation theoretic extension of the classic Bass' Theorem P:

Corollary 3.13 *The following are equivalent for a ring R:*

1. *The class $\mathcal{FM}$ of all flat Mittag-Leffler modules is (pre-) covering.*
2. *All (classical) Bass modules over $\mathcal{P}_0^{\leq\omega}$ are projective.*
3. *$\mathcal{P}_0 = \mathcal{F}_0$ (i.e., R is a right perfect ring).*
4. *The regular module R is $\sum$-pure split.*

We note that even in the particular setting of Corollary 3.13, the regular module R satisfying 4 need not be $\sum$-pure injective, since there exist right artinian rings that are not pure-injective, [35].

Next, we present an application of Theorem 3.12 to an unexpected setting, namely to hereditary finite dimensional algebras of infinite representation type. Here, the relevant tilting module is the Lukas tilting module. In order to define it, we recall some terminology for this particular setting (for more details, we refer to [3]):

We denote by p the representative set of all indecomposable finite dimensional *preprojective* modules, i.e., the indecomposable projective modules and their τ^{-1}-shifts, where τ^{-1} is the inverse of the AR-translation, given in this setting by the formula $\tau^{-1}(M) = \mathrm{Ext}^1_R(D(R), M)$ where D is the standard duality. Dually, the set q of all indecomposable finite dimensional *preinjective* modules is defined. The remaining indecomposable finite dimensional modules are called *regular*.

The class $p^{\perp}$ is clearly of finite type, hence it is a right tilting class for a (1-) tilting module L, called the *Lukas tilting module*. In fact, a module M belongs to $p^{\perp}$, iff M has no direct summands from p.

The left tilting class induced by L is the class $\mathcal{B}$ of all *Baer modules*. The locally L-free modules are called *locally Baer modules*, their class is denoted by $\mathcal{L}$. With this notation, we have

Theorem 3.14 ([3, 6]) *Let R be a hereditary finite dimensional algebras of infinite representation type. Then*

1. $\mathcal{B} = \mathrm{Filt}(p)$.
2. *The Lukas tilting module L is countably generated, but has no finite dimensional direct summands, and it is not Σ-pure split.*
3. *$\mathcal{L}$ is not precovering (and hence not deconstructible).*

Remark 3.15 By Theorem 3.12, there exists a Bass module over $\mathcal{B}^{\leq\omega}$ which is not a Baer module. Such Bass module can be constructed as the union of a chain $P_0 \overset{f_0}{\hookrightarrow} P_1 \overset{f_1}{\hookrightarrow} \dots \overset{f_{i-1}}{\hookrightarrow} P_i \overset{f_i}{\hookrightarrow} P_{i+1} \overset{f_{i+1}}{\hookrightarrow} \dots$ such that all the P_i are direct sums of the modules from p, but the cokernels of all the f_i are direct sums of regular modules.

3.4 Very Flat and Locally Very Flat Modules

As mentioned above, the results in [6] are quite general, and apply far beyond the tilting setting. We now present one such application, concerning a new class of flat modules discovered recently in algebraic geometry. We only sketch the motivation here, leaving details to [25]:

As shown in [14], quasi-coherent sheaves over a scheme X with the structure sheaf O_X can equivalently be studied as certain ‘quasi-coherent’ representations of the quiver Q whose vertices are affine open subschemes U of X, and arrows $U \to V$ are pairs of affine open subschemes $V \subseteq U$. A *quasi-coherent representation M* assigns to each vertex an $O_X(U)$-module $M(U)$, and to each arrow $U \to V$ an $O_X(U)$-homomorphism $f_{VU} : M(U) \to M(V)$, such that

$$\mathrm{id}_{O_X(V)} \otimes f_{VU} : O_X(V) \otimes_{O_X(U)} M(U) \to O_X(V) \otimes_{O_X(U)} M(V) \cong M(V)$$

is an $O_X(V)$-isomorphism, and $f_{WV} f_{VU} = f_{WU}$ whenever $U \to V \to W$.

This representation theoretic approach to quasi-coherent sheaves has recently been dualized by Positselski in [25]:

Definition 3.16 Let X be a scheme and O_X its structure sheaf. A *contraherent cosheaf P* on X is defined by assigning

(i) to every affine open subscheme $U \subseteq X$, an $O_X(U)$-module $P(U)$, and

(ii) to each pair of affine open subschemes $V \subseteq U \subseteq X$, an $O_X(U)$-homomorphism $g_{UV} : P(V) \to P(U)$ such that

$$\mathrm{Hom}_{O_X(U)}(O_X(V), g_{UV}) : P(V) \to \mathrm{Hom}_{O_X(U)}(O_X(V), P(U))$$

is an $O_X(V)$-isomorphism, and $\mathrm{Ext}^1_{O_X(U)}(O_X(V), P(U)) = 0$.

(iii) $g_{UW} = g_{UV} g_{VW}$ for each triple of affine open subschemes $W \subseteq V \subseteq U \subseteq X$.

The condition on $P(U)$ involving vanishing of Ext^1 in (ii) is imposed because the $O_X(U)$-module $O_X(V)$ is not projective in general, so (unlike the functor $O_X(V) \otimes_{O_X(U)} -$ which is exact, because $O_X(V)$ is a flat $O_X(U)$-module), the functor $\mathrm{Hom}_{O_X(U)}(O_X(V), -)$ need not be exact. The crucial observation here is that the $O_X(U)$-module $O_X(V)$ is always very flat:

Lemma 3.17 *[25] Let $\varphi : R \to S$ be a homomorphism of commutative rings such that the induced morphism of affine schemes $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an open embedding. Then S is a very flat R-module.*

Here, a module M over a commutative ring R is *very flat* provided that $M \in {}^\perp(\mathcal{S}^\perp)$ where $\mathcal{S} = \{R[s^{-1}] \mid s \in R\}$, and $R[s^{-1}]$ denotes the localization of R at the multiplicative set $\{s^i \mid i < \omega\}$.

The class of all very flat modules is denoted $\mathcal{VF}$. It fits in the complete cotorsion pair $(\mathcal{VF}, \mathcal{CA})$ where $\mathcal{CA} = \mathcal{S}^\perp$ is the class of all *contraadjusted* modules.

It is easy to see that $\mathcal{P}_0 \subseteq \mathcal{VF} = \mathrm{Filt}(\mathcal{VF}^{\leq\omega}) \subseteq \mathcal{F}_0 \cap \mathcal{P}_1$. If we view very flat modules as analogs of projective modules, then the locally very flat modules are analogs of the flat Mittag-Leffler ones:

Definition 3.18 Let R be a commutative ring and $\mathcal{C} = \mathcal{VF}^{\leq\omega}$. Then the locally $\mathcal{C}$-free modules are called *locally very flat*. The class of all locally very flat modules is denoted by $\mathcal{LV}$.

Clearly, $\mathcal{ML} \subseteq \mathcal{LV} \subseteq \mathcal{F}_0$. Lemma 3.10 makes it possible to pursue the analogy further, at least in the case of noetherian integral domains (cf. Corollary 3.7):

Corollary 3.19 ([30]) *The following are equivalent for a noetherian integral domain R:*

1. *The class $\mathcal{LV}$ of all locally very flat modules is (pre-) covering.*
2. *All Bass modules over $\mathcal{VF}^{\leq\omega}$ are very flat.*
3. *$\mathcal{VF} = \mathcal{F}_0$.*
4. *The Zariski spectrum $\mathrm{Spec}(R)$ is finite.*

Note that Condition 3 implies that $\mathcal{LV}$ is closed under direct limits, so we have yet another particular instance where Enochs Conjecture holds.

Condition 4 implies that R has Krull dimension at most 1. Moreover, if R is a Dedekind domain, then it even implies that R is a PID. In the Dedekind domain case, there is more information available on the structure of very flat and locally very flat modules:

Theorem 3.20 ([30]) *Let R be a Dedekind domain and M a torsion-free module of rank r.*

1. *If r is finite, then M is very flat, iff there exists* $0 \neq s \in R$ *such that the localization* $M \otimes_R R[s^{-1}]$ *is a projective* $R[s^{-1}]$*-module (of rank r).*
2. *M is very flat, iff M has a* $\mathcal{T}$*-filtration (of length r), where* $\mathcal{T}$ *denotes the set of all non-zero submodules of the localizations* $R[s^{-1}]$ *with* $0 \neq s \in R$*.*
3. *M is locally very flat, iff each finite rank submodule of M is very flat.*

Notice that the statement 3 above is the analog of Potryagin's Criterion for $\aleph_1$-freeness of torsion-free abelian groups.

4 Tilting and Silting Theory

4.1 Tilting Classes Over Commutative Rings

Theorem 2.15 gives a characterization of right tilting classes over commutative noetherian rings in terms of the characteristic sequences of subsets of their Zariski spectra. It has recently been generalized to arbitrary commutative rings in [22]. In order to state this generalization, we need more terminology:

Definition 4.1 Let R be a commutative ring with the Zariski spectrum $\mathrm{Spec}(R)$.

(i) A subset $X \subseteq \mathrm{Spec}(R)$ is *Thomason* provided there is a set S consisting of finitely generated ideals of R such that $X = \bigcup_{I \in S} V(I)$ where $V(I) = \{p \in \mathrm{Spec}(R) \mid I \subseteq p\}$.

(ii) For a module M, let $\mathcal{C}_M$ denote the smallest subclass of Mod–R containing M and closed under submodules and direct limits. A prime $p \in \mathrm{Spec}(R)$ is a *vaguely associated* prime of M, in case $R/p \in \mathrm{Ass}_R(\mathcal{C}_M)$, i.e., R/p embeds in a module from $\mathcal{C}_M$.

(iii) A sequence $(P_0, \ldots, P_{n-1})$ consisting of subsets of the Zariski spectrum $\mathrm{Spec}(R)$ is called *characteristic* provided that $P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{n-1}$, and for each $i < n$, P_i is a Thomason subset of $\mathrm{Spec}(R)$ such that P_i contains no vaguely associated primes of $\Omega^{-i}(R)$.

We note that if R is noetherian, then Thomason subsets coincide with the upper subsets of $(\mathrm{Spec}(R), \subseteq)$, and vaguely associated primes coincide with the associated primes of M, [21]. So in the particular case when R is noetherian, the definition of a characteristic sequence above coincides with the one given in Sect. 2.2.

Theorem 4.2 ([22]) *Let R be a commutative ring and* $n < \omega$*. Then right n-tilting classes in* Mod–*R are parametrized by characteristic sequences: the class* $\mathcal{T}$ *corresponding to a characteristic sequence* $(P_0, \ldots, P_{n-1})$ *is defined by the formula*

$$\mathcal{T} = \{M \in \text{Mod–}R \mid \mathit{Tor}_i^R(M, R/I) = 0 \textit{ for each } i < n \textit{ and each finitely generated ideal } I \textit{ such that } V(I) \subseteq P_i\}. \tag{2}$$

In the particular case of 1-tilting classes, we obtain

Corollary 4.3 ([21]) *Let R be a commutative ring. Then right* 1*-tilting classes in* Mod–R *correspond to Thomason subsets P of* Spec(R) *such that P contains no primes vaguely associated with R. The right* 1*-tilting class corresponding to such P is* $\{M \in \text{Mod–}R \mid MI = M$ *for all finitely generated ideals I such that* $V(I) \subseteq P\}$.

In the one-dimensional case, instead of Thomason subsets, one can use the better known Gabriel filters of ideals of R in order to characterize right 1-tilting classes:

Definition 4.4 Let R be a commutative ring. A filter $\mathcal{G}$ consisting of ideals of R is a *Gabriel filter* provided that

(i) If $I \in \mathcal{G}$, then the ideal $(I : x) = \{r \in R \mid x \cdot r \in I\}$ belongs to $\mathcal{G}$ for each $x \in R$.
(ii) If J is an ideal in R such that there exists $I \in \mathcal{G}$ with $(J : x) \in \mathcal{G}$ for all $x \in I$, then $J \in \mathcal{G}$.

A Gabriel filter $\mathcal{G}$ is said to be of *finite type* in case $\mathcal{G}$ has a filter basis consisting of finitely generated ideals, and $\mathcal{G}$ is *faithful* provided that $\text{Ann}(I) = 0$ for all $I \in \mathcal{G}$.

Theorem 4.5 ([21]) *Let R be a commutative ring. Then right* 1*-tilting classes in* Mod–R *correspond to faithful Gabriel filters of finite type. Given such a filter $\mathcal{G}$, the corresponding right* 1*-tilting class is* $\{M \in \text{Mod–}R \mid MI = M$ *for all* $I \in \mathcal{G}\}$.

Notice that in the latter formulation, the characterization of right 1-tilting classes is a direct generalization of the known characterization—due to Bazzoni and Salce—for the particular case of Prüfer domains (see, e.g., [18, §14.2]).

Remark 4.6 We briefly comment on some recent developments in cotilting theory of commutative rings: recall that dually to Definition 2.11, one can define n-cotilting modules and classes, see [18, Chap. 15]. In the commutative noetherian case, the dual of Theorem 2.15 holds, whence characteristic sequences parametrize also all cotilting classes: the left n-cotilting class corresponding to the characteristic sequence $\mathcal{P} = (P_0, \ldots, P_{n-1})$ is $\mathcal{C}_{\mathcal{P}} = \{M \in \text{Mod–}R \mid \text{Ext}^i_R(R/p, M) = 0$ for all $i < n$ and $p \in P_i\}$. In fact, in this case each n-cotilting module is equivalent to an n-cotilting module which is a dual (or character module) of an n-tilting module, [4].

Recently, it was shown in [32] that among all n-cotilting modules inducing $\mathcal{C}_{\mathcal{P}}$ there is one, $M_{\mathcal{P}}$, which is minimal, that is, $M_{\mathcal{P}}$ is isomorphic a direct summand in any other cotilting module inducing $\mathcal{C}_{\mathcal{P}}$. This minimal cotilting module is unique up to isomorphism; its construction is presented in [32].

Relations between cotilting modules over commutative noetherian rings and their localization at maximal ideals have recently been studied in [33]—for example, each cotilting module C is equivalent to the direct product $\prod_{m \in \text{mSpec}(R)} C^m$ where R_m denotes the localization of R at m and $C^m = \text{Hom}_R(R_m, C) \in \text{Mod–}R_m$ the colocalization of C at m.

4.2 Silting Modules and Classes

Silting modules and classes generalize 1-tilting modules and classes. They have first appeared in the representation theory of artin algebras, cf. [1].

In the following, $\operatorname{Gen} T$ will denote the class of all homomorphic images of all direct sums of copies of the module T.

Definition 4.7 Let R be a ring and T be a module.

(i) Let $\Phi = \{\varphi_i \mid i \in I\}$ be a set of morphisms between projective modules. We denote by $\mathcal{D}_\Phi$ the class of all modules M such that $\operatorname{Hom}_R(\varphi_i, M)$ is surjective for each $i \in I$. If Φ contains only one element, $\Phi = \{\varphi\}$, we will simply use $\mathcal{D}_\varphi$ in place of $\mathcal{D}_\Phi$.

(ii) T is a *τ-rigid* module provided there exists a projective presentation

$$P_1 \overset{\varphi}{\to} P_0 \to T \to 0 \tag{3}$$

such that $\mathcal{D}_\varphi \subseteq \operatorname{Gen} T$.

(iii) T is *silting* provided that there exists a projective presentation (3) such that $\mathcal{D}_\varphi = \operatorname{Gen} T$. In this case, $\operatorname{Gen} T$ is called the *silting class* generated by T, and the map φ is said to *witness* that T is a silting module.

The peculiar terminology in (ii) comes from the case when R is an artin algebra: there, a finitely generated module is τ-rigid, iff $\operatorname{Hom}_R(T, \tau T) = 0$ where τ denotes the AR-translation, cf. [1]. Equivalently, $\operatorname{Gen} T \subseteq T^{\perp_1}$, where $T^{\perp_1} = \operatorname{Ker} \operatorname{Ext}^1_R(T, -)$.

The latter equivalence holds more in general, when R is any right perfect ring (or R is semiperfect and T is finitely generated), and also when T is a module of projective dimension ≤ 1 over any ring (in particular, when R is right hereditary). However, $\operatorname{Gen} T \subseteq T^{\perp_1}$ is a weaker condition in general than τ-rigidity, cf. [8].

Let $\operatorname{Pres} T$ ($\subseteq \operatorname{Gen} T$) denote the class of all modules M such that there exist cardinals κ and λ and an exact sequence $T^{(\kappa)} \to T^{(\lambda)} \to M \to 0$.

Definition 4.8 Let R be a ring and T be a module.

(i) T is *finendo*, provided T is finitely generated over its endomorphism ring (equivalently, $\operatorname{Gen} T$ is a preenveloping class, see [18, 13.52]).

(ii) T is *quasi-tilting* provided that $\operatorname{Pres} T = \operatorname{Gen} T \subseteq T^\perp$.

The following Lemma gives basic relations among the notions defined above (for its proof, we refer to [5]):

Lemma 4.9 *Let R be a ring and T be a module.*

1. *Each silting module is finendo, quasi-tilting, and τ-rigid.*
2. *T is finendo and quasi-tilting, iff T is a* 1*-tilting* $R/Ann(T)$*-module.*
3. *T is* 1*-tilting, iff T is faithful silting, iff T is faithful finendo and quasi-tilting.*

Example 4.10 Assume that R is an artin algebra and T is a finitely presented module. Then, as mentioned above, T is τ-rigid, iff $\operatorname{Gen} T \subseteq T^{\perp_1}$. Since T is always finendo, the notions of a silting and quasi-tilting module coincide, namely with the notion of a 1-tilting $R/\operatorname{Ann}(T)$-module.

In general, by Marks and Šťovíček [23], silting modules can equivalently be studied as 1-tilting modules, but over a different ring: the upper triangular 2×2 matrix ring $S = UT_2(R)$. Recall that Mod–S is equivalent to the *morphism category* of R, that is, the category whose objects are morphisms in Mod–R, and morphisms between two objects f and f' are pairs (g, g') of morphisms in Mod–R, where g (g') maps the domain (codomain) of f into the domain (codomain) of f', which satisfy $f'g = g'f$.

Theorem 4.11 ([23]) *Let R be a ring and T be a module with a projective presentation (3). Then φ witnesses that T is silting, if and only if $\varphi \oplus id_R$ is a 1-tilting object in the morphism category, i.e., the S-module $T' = P_1 \oplus R \oplus P_0 \oplus R$ is 1-tilting.*

Here, the S-module structure on T' is given by

$$(p_1, r_1, p_0, r_0).A = (p_1.u, r_1.u, \varphi(p_1).v + p_0.w, r_1.v + r_0.w)$$

for $A = \left(\begin{smallmatrix} u & v \\ 0 & w \end{smallmatrix}\right) \in S$.

Notice that the finite type of right 1-tilting classes (see Theorem 2.14) can be restated as follows: a class $\mathcal{D}$ is right 1-tilting, iff $\mathcal{D} = \mathcal{D}_\Phi$ for a set Φ consisting of monomorphisms between finitely generated projective modules. The following result from [23] extends this to all silting classes:

Theorem 4.12 *Let R be a ring and $\mathcal{D} \subseteq$ Mod–R. Then $\mathcal{D}$ is silting, iff $\mathcal{D} = \mathcal{D}_\Phi$ for a set Φ consisting of morphisms between finitely generated projective modules.*

It easily follows that each silting class is a preenveloping (and definable) torsion class (cf. Proposition 2.12). For left noetherian rings, also the converse holds:

Theorem 4.13 ([2]) *Let R be a left noetherian ring. Then silting classes coincide with the preenveloping definable torsion classes.*

However, Theorem 4.13 does not extend to non-noetherian rings:

Example 4.14 ([2]) Let R be a commutative local ring with an idempotent maximal ideal $m \neq 0$, and $\mathcal{D} =$ Mod–R/m be the class of all completely reducible R-modules. Then $\mathcal{D}$ is a preenveloping and definable torsion class, but it is not silting.

We finish this section by returning to the commutative noetherian setting (cf. the particular case of $n = 1$ of Theorem 2.15):

Theorem 4.15 ([2]) *Let R be a commutative noetherian ring. Then silting classes $\mathcal{D}$ in Mod–R correspond 1–1 to upper subsets P of the Zariski spectrum. For such P, $\mathcal{D}$ is defined by $\mathcal{D} = \{M \in \text{Mod–}R \mid Mp = M \text{ for all } p \in P\}$.*

The language of Gabriel filters makes it possible to extend the latter characterization further, to all commutative rings:

Theorem 4.16 ([2]) *Let R be a commutative ring. Then silting classes in* Mod–R *correspond 1–1 to (not necessarily faithful) Gabriel filters of finite type. The silting class corresponding to such filter $\mathcal{G}$ is $\{M \in \text{Mod–}R \mid MI = M \text{ for all } I \in \mathcal{G}\}$.*

4.3 Saorín's Problem

We finish our survey by considering a particular recent problem on 1-tilting modules motivated in the theory of Grothendieck categories. We only sketch the motivation here, referring the interested reader to [24] for more details.

The paper [24] deals with the torsion pairs $t = (\mathcal{T}, \mathcal{F})$ in Grothendieck categories $\mathcal{G}$, such that t is induced by a self-small 1-tilting object in $\mathcal{G}$, or equivalently, $\mathcal{T}$ is a cogenerating class in $\mathcal{G}$, and the heart $\mathcal{H}_t$ of the Happel-Reiten-Smalø t-structure on the bounded derived category $\mathcal{D}^b(\mathcal{G})$ is a module category. This is shown to imply that t is a tilting torsion pair with $\mathcal{F}$ closed under direct limits, but the converse implication fails in general. (By the Addendum to [24], the condition of $\mathcal{F}$ being closed under direct limits is equivalent to $\mathcal{H}_t$ being a Grothendieck category.)

For the particular case of $\mathcal{G} = \text{Mod–}R$, the validity of the converse implication is equivalent to a positive answer to [24, Question 5.5], stated here as

Problem 4.17 (Saorín's Problem) Let R be a ring and T a 1-tilting module. Let $(\mathcal{T}_T, \mathcal{F}_T)$ the corresponding tilting torsion pair. Are the following conditions (i) and (ii) equivalent?

(i) The class $\mathcal{F}_T$ is closed under direct limits.
(ii) T is equivalent to a finitely generated tilting module.

Clearly (ii) always implies (i), because if T is equivalent to a finitely generated tilting module T', then $\mathcal{F}_T = \operatorname{Ker}\operatorname{Hom}_R(\mathcal{T}, -) = \operatorname{Ker}\operatorname{Hom}_R(T', -)$ is closed under direct limits, since T' is even finitely presented, cf. [18, 2.7 and 13.3].

So the question is whether (i) implies (ii). It turns out that the answer depends on the ring R in case. The key point, due to Bazzoni, is that Condition (i) can be translated into a more accessible one, for any ring R.

Theorem 4.18 ([8]) *In the setting of Problem 4.17, Condition (i) is equivalent to*

(iii) *T is pure-projective, i.e., a direct summand in a direct sum of finitely presented modules.*

So restated in terms of the properties of T, Saorín's Problem just asks whether each pure-projective 1-tilting module is equivalent to a finitely generated one. On the positive side, we have

Theorem 4.19 ([8, 21]) *Assume that either R is commutative or right artinian. Then the Conditions (i)–(iii) above are equivalent.*

However, the answer may be negative even for (non-commutative) two-sided noetherian domains:

Example 4.20 [8] Let R be the universal enveloping algebra of $sl(2,\mathbb{C})$, i.e., $R = \mathbb{C}(x,y,z)/K$ where K is the ideal generated by $x - yz + zy$, $2y - xy + yz$ and $-2z - xz + zx$. Let I denote the idempotent ideal generated by $x + K$, $y + K$ and $z + K$. Then the class $\mathcal{D}_I$ of all I-divisible modules (i.e., the modules M such that $MI = M$) is a tilting class induced by a pure-projective 1-tilting module which is not equivalent to any finitely generated tilting module.

Another counter-example is provided by the Dubrovin-Puninski ring:

Example 4.21 Let R be a nearly simple uniserial domain, and S be the Dubrovin-Puninski ring, i.e., S is the endomorphism ring of (any) cyclically presented non-projective right R-module, [10]. Then S is a left and right coherent ring with a unique finitely presented simple injective left S-module M. Moreover, M has a projective resolution $0 \to S \to P \to P \xrightarrow{\pi} M \to 0$ where P is countably presented. Let P' be the kernel of π. Then $T_1 = P \oplus P'$ is a pure-projective 1-tilting left S-module which is not equivalent to any finitely presented tilting left S-module. For a classification of all the tilting (and other definable) classes of left S-modules, we refer to [8].

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Part II
Research Articles

In this chapter we present a collection of recent research articles on various topics within the theory of groups, modules and models.

A Class of Pure Subgroups of the Specker Group

A.L.S. Corner

Abstract This paper seeks to extend results realizing rings as endomorphism rings, to situations where one cannot hope to prescribe the endomorphism ring completely. The notion of an inessential endomorphism is introduced and exploited in the context of subgroups of the higher Specker groups, the product over an infinite cardinal of copies of the group of integers.

Keywords Torsion-free Abelian groups • Endomorphism rings • Slender groups • Specker groups

Mathematical Subject Classification: Primary 20K30, Secondary 20K20

1 Introduction

This paper was originally delivered by A.L.S. Corner at the Montpellier conference on Abelian groups in June 1967; see 'Liste des participants et des conferences' in [1] below. The paper was never published but as it contained the first concept of an *inessential endomorphism*, its influence in subsequent years was significant. The version below has been transcribed by Brendan Goldsmith from a typescript given to him by Tony Corner when he was a doctoral student of Corner. Some of the final pages of the typescript were missing but a number of hand-written replacements by Corner exist; the part reproduced below closely follows the earliest of these versions. Some explanations which Corner had indicated on the manuscript as being required have been included in italics.

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2 Definition and Preliminary Results

Let D be a divisible torsion group of (infinite) cardinal $m < \aleph_\iota$ (= the first strongly inaccessible cardinal). Consider any short exact sequence

$$0 \longrightarrow X_1 \longrightarrow X_0 \xrightarrow{\varepsilon_X} D \longrightarrow 0 \tag{X}$$

in which X_0, and therefore also X_1, is free Abelian of rank m; we call such a short exact sequence an **admissible** resolution of D. Then we have an induced short exact sequence

$$0 \longrightarrow X_0^* \longrightarrow X_1^* \xrightarrow{\delta_X} K \longrightarrow 0, \tag{X^*}$$

where $()^* = \mathrm{Hom}(,\mathbb{Z})$, and $K = \mathrm{Ext}(D, \mathbb{Z})$.

The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces an exact sequence

$0 = \mathrm{Hom}(D, \mathbb{Q}) \to \mathrm{Hom}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(D, \mathbb{Z}) \to \mathrm{Ext}(D, \mathbb{Q}) = 0$; therefore we may (and shall) make the identification

$$K = \mathrm{Hom}(D, \mathbb{Q}/\mathbb{Z}). \tag{1}$$

Since K is torsion-free, we may regard X_0^* as a pure subgroup of X_1^*. Note further that X_0^* and X_1^* are isomorphic copies of the 'higher' Specker group $\mathbb{Z}^m$, namely the direct product of m copies of the group $\mathbb{Z}$ of integers.

Given any pure subgroup S of K, we now define

$$G(S) = G(S, X) = S\delta_X^{-1}.$$

It is immediate that

$$X_0^* \subseteq G(S, X) \subseteq X_1^*, \tag{2}$$

where both inclusions are pure. As a pure subgroup of the homogeneous separable group X_1^*, $G(S)$ is homogeneous and separable, its type being of course the type of $\mathbb{Z}$. For future reference we note that

$$G(S, X)/X_0^* \cong S, \ \ X_1^*/G(S, X) \cong K/S. \tag{3}$$

Proposition 2.1 *(i) Up to isomorphism, $G(S, X)$ depends only on the pure subgroup S of K (and its embedding in K), and not on the admissible resolution X.*
(ii) $G(S, X) \cong G(S, X) \oplus \mathbb{Z}^m$.

The proof depends on two simple lemmas.

Lemma 2.2 *Let Y_0 be a direct summand of X_0 such that $Y_0\varepsilon_X = D$. Then Y_0 admits a direct complement U in X_0 such that $U \subseteq \mathrm{Ker}\ \varepsilon_X$.*

Proof Write $X_0 = Y_0 \oplus U'$, where U' is an arbitrary direct complement. Since U' is free, the condition on Y_0 guarantees the existence of a homomorphism $\phi : U' \to Y_0$ such that $\phi(\varepsilon_X \restriction Y_0) = \varepsilon_X \restriction U' : U' \to D$. Then the subgroup $U = U'(1 - \phi)$ is a direct complement of Y_0 contained in $\mathrm{Ker}\ \varepsilon_X$.

Lemma 2.3 *There exists a direct summand Y_0 of corank m in X_0 such that $Y_0 \varepsilon_X = D$.*

Proof Choose a free basis e_i $(i \in I)$ of X_0, and write $d_i = e_i \varepsilon_X$, so that d_i $(i \in I)$ is a generating system for D indexed by a set I of cardinal m. Note that any generating system for D remains a generating system if a single element is removed; for the quotient of D by the subgroup generated by the diminished system is both divisible and cyclic, and, as such, must vanish. Now consider the set of pairs (J, Σ), where J is a subset of I and Σ is a function that assigns to each $j \in J$ a finite subset Σ_J of the complement $I \setminus J$ such that $d_j \in \langle d_i : i \in \Sigma_J \rangle$. We order the pairs in the usual way: $(J, \Sigma) \leq (J', \Sigma')$ if and only if $J \subseteq J'$ and $\Sigma_j = \Sigma'_j$ for all $j \in J$. The ordering is visibly inductive; so Zorn's Lemma provides a maximal pair, say (J, Σ). By construction, d_i $(i \in I \setminus J)$ is a generating system for D. Consider any index $j_0 \in I \setminus J$. By our previous remark, d_i $(i \in I \setminus (J \cup \{j_0\}))$ is still a generating system for D, so there exists a finite subset Σ_{j_0} of $I \setminus (J \cup \{j_0\})$ such that $d_{j_0} \in \langle d_i : i \in \Sigma_{j_0} \rangle$. Thus I is the union of J and the finite subsets Σ_j $(j \in J)$; which implies that $|J| = |I| = m$. The subgroup Y_0 of X_0 generated by the e_i $(i \in I \setminus J)$ is then a direct summand of corank m in X_0 such that $Y_0 \varepsilon_X = D$; and the lemma is proved.

Remark 2.4 The argument just given shows that if $\{g_i : i \in I\}$ is any generating system for a group G which admits no non-zero cyclic quotient, then there is a subset J of I such that $|J| = |I|$ and $\{g_i : i \in I \setminus J\}$ is still a generating system; the converse is also true, trivially.

We are now in a position to prove Proposition 2.1.

Proof To establish (ii), it is enough to show that $G(S, X)$ has a direct summand isomorphic with $\mathbb{Z}^m$; for then (say) $G(S, X) \cong H \oplus \mathbb{Z}^m \cong H \oplus \mathbb{Z}^m \oplus \mathbb{Z}^m \cong G(S, X) \oplus \mathbb{Z}^m$. Now, by the two lemmas just proved, we may choose a direct decomposition $X_0 = Y_0 \oplus U$, where U is free Abelian of rank m, $U \subseteq \operatorname{Ker} \varepsilon_X$. Write $\varepsilon_Y = \varepsilon_X \restriction Y_0$, $Y_1 = \operatorname{Ker} \varepsilon_Y$. Then we have a commutative diagram with exact rows and split exact columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \xrightarrow{\varepsilon_Y} & \mathrm{D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \xrightarrow{\varepsilon_X} & \mathrm{D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & U & = & U & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

and this induces another such diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & Y_0^* & \longrightarrow & Y_1^* & \xrightarrow{\delta_Y} & K & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \pi} & & \| & & \\
0 & \longrightarrow & X_0^* & \longrightarrow & X_1^* & \xrightarrow{\delta_X} & K & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & U^* & = & U^* & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

In this second diagram, the kernel of $\pi : X_1^* \to Y_1^*$ is a direct summand of X_1^*, isomorphic with U^* and contained in X_0^*. Therefore Ker π is a direct summand of $G(S, X)$ and as π clearly maps $G(S, X)$ onto $G(S, Y)$, it follows that

$$G(S, X) \cong G(S, Y) \oplus U_0^*, \tag{4}$$

where (Y) is the admissible resolution $0 \to Y_1 \to Y_0 \to D \to 0$. But $U^* \cong \mathbb{Z}^m$; so we have established (ii). Applying (ii) with (Y) in place of (X), we deduce from (4) that $G(S, X) \cong G(S, Y)$, which establishes a special case of (i).

Now consider any two admissible resolutions (X), (Y) of D. Put $W_0 = X_0 \oplus Y_0$, $\varepsilon_W = (\varepsilon_X, \varepsilon_Y) : W_0 \to D$, $W_1 = \text{Ker } \varepsilon_W$. Then the short exact sequence $0 \longrightarrow W_1 \longrightarrow W_0 \xrightarrow{\varepsilon_W} D \longrightarrow 0$ is an admissible resolution (W) of D, and we may clearly identify X_1, Y_1 with the subgroups Ker $(\varepsilon_W \restriction X_0)$, Ker $(\varepsilon_W \restriction Y_0)$ of W_0. Therefore, by the special case just considered, $G(S, X) \cong G(S, W) \cong G(S, Y)$. The proposition is proved.

Proposition 2.1 justifies our use of the simplified notation $G(S)$ for $G(S, X)$ when we are interested only in the isomorphism class and not in the admissible resolution being used. In the remaining sections of the paper, homomorphisms of the $G(S)$ into slender groups play an important role. Such homomorphisms are made manageable by the following technical lemma.

Lemma 2.5 *Let Y^* be a direct summand of finite corank in X_0^* (so that the quotient $U = X_0^*/Y^*$ is free of finite rank). Then there is a direct decomposition*

$$G(S, X)/Y^* = S_0 \oplus V,$$

such that $V \cong U$ and the epimorphism $G(S, X)/Y^* \to S$ induced by δ_X maps S_0 isomorphically onto a subgroup of finite index in S.

(Note that every direct summand of X_0^* is isomorphic with a group of the form Y^* for some free group Y of rank $\leq m$.)

Proof First consider the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \xrightarrow{\varepsilon_X} & D & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 \to K^* & \longrightarrow & X_1^{**} & \longrightarrow & X_0^{**} & \longrightarrow & \mathrm{Ext}(K,\mathbb{Z}) & & \end{array} \tag{5}$$

where the bottom row is induced from (X^*) and the first two vertical rows are the evaluation maps which are known to be isomorphisms (*due to the slenderness of* $\mathbb{Z}$ *and the fact that the first* ω*-measurable cardinal, if it exists, is strongly inaccessible*). It is immediate that $K^* = 0$, and that the image of X_0^{**} in $\mathrm{Ext}(K,\mathbb{Z})$ is (canonically) isomorphic with D.

Write $C = X_1^*/Y^*$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_0^* & \longrightarrow & X_1^* & \xrightarrow{\delta_X} & K & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & U & \longrightarrow & C & \longrightarrow & K & \longrightarrow & 0 \end{array} \tag{6}$$

in which the first two vertical arrows are the canonical epimorphisms. This induces another commutative with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1^{**} & \longrightarrow & X_0^{**} & \longrightarrow & \mathrm{Ext}(K,\mathbb{Z}) \\ & & \big\uparrow & & \big\uparrow & & \big\| \\ 0 & \longrightarrow & C^* & \longrightarrow & U^* & \longrightarrow & \mathrm{Ext}(K,\mathbb{Z}) \end{array} \tag{7}$$

The top row of (7) coincides with the bottom row of (5). Therefore the image E (say) of U^* in $\mathrm{Ext}(K,\mathbb{Z})$ is a torsion group, because it lies in an isomorphic copy of D; since $U^* (\cong U)$ is finitely generated, E is even a finite group. Extracting from the bottom row of (7) the exact sequence

$$0 \to C^* \to U^* \to E \to 0, \tag{8}$$

we find that C^* is free of the same finite rank as U^*, whence the (non-canonical) isomorphisms $C^{**} \cong C^* \cong U^* \cong U$.

Let C_0 be the kernel of the evaluation map $C \to C^{**}$. Then C/C_0 is isomorphic with a subgroup of the free group C^{**}; so the canonical exact sequence $0 \to C_0 \to C \to C/C_0 \to 0$ splits. Hence $C_0^* = 0$; for otherwise we could extend a non-zero element of C_0^* to a homomorphism $C \to \mathbb{Z}$ that would not vanish on the whole of C_0, contrary to the definition of C_0. Therefore we have canonical isomorphisms $C^* \cong (C/C_0)^*$, $C^{**} \cong (C/C_0)^{**}$. It follows that we have a commutative diagram with exact rows and split exact middle column,

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & & C_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & C & \longrightarrow & K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U^{**} & \longrightarrow & C^{**} & \longrightarrow & \mathrm{Ext}(E,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the bottom row is induced from (8) and all other arrows represent inclusions, canonical epimorphisms or evaluation maps. A simple diagram-chase reveals that U, C_0 generate their direct sum in C, and that the quotient $C/(C_0 \oplus U)$ is isomorphic with $\mathrm{Ext}(E, \mathbb{Z})$. But E is finite, so $\mathrm{Ext}(E, \mathbb{Z}) \cong E$. Thus $C_0 \oplus U$ is of finite index in C.

Now the quotient $H = G(S, X)/Y^*$ is a subgroup of C containing U. Therefore the evaluation map $C \to C^{**}$ carries H onto a subgroup of the free group C^{**}; which implies that there is a direct decomposition $H = S_0 \oplus V$, where $S_0 = C_0 \cap H$, and $V \cong C^{**} \cong U$ (non-canonically). Since $C_0 \oplus U$ is of finite index in C, the subgroup $S_0 \oplus U$ $(= (C_0 \oplus U) \cap H)$ is of finite index in H. But U is the kernel of the epimorphism $H = G(S, X)/Y^* \to S$ induced by δ_X; consequently, this epimorphism carries S_0 isomorphically onto a subgroup of finite index in S. The lemma is proved.

3 Inessential Homomorphisms: A Splitting Theorem

Let $\mathscr{P}$ be a non-empty class of Abelian groups that is closed under finite direct sums, at least up to isomorphism. Given Abelian groups G, H we shall say that a homomorphism $\phi : G \to H$ is **$\mathscr{P}$-inessential** if, for some $P \in \mathscr{P}$, there exists a homomorphism $\pi : P \to H$ such that $G\phi \subseteq P\pi$. If $\phi_1, \phi_2 : G \to H$ are two $\mathscr{P}$-inessential homomorphisms, say $G\phi_i \subseteq P_i\pi_i$ where $P_i \in \mathscr{P}$, then it is clear that $G(\phi_1 - \phi_2) \subseteq (P_1 \oplus P_2)(\pi_1, \pi_2)$, so that $\phi_1 - \phi_2$ is $\mathscr{P}$-inessential; thus the set $\mathrm{Iness}_{\mathscr{P}}(G, H)$ of all $\mathscr{P}$-inessential homomorphisms $G \to H$ is a subgroup of $\mathrm{Hom}(G, H)$. An equally obvious argument shows that a composite of homomorphisms is $\mathscr{P}$-inessential whenever one of the factors is $\mathscr{P}$-inessential. In particular, $\mathrm{Iness}_{\mathscr{P}}(G) = \mathrm{Iness}_{\mathscr{P}}(G, G)$ is always an ideal of the endomorphism ring $E(G)$ of G; and, more generally, it is clear what we mean by saying that the class of all $\mathscr{P}$-inessential homomorphisms is an ideal $\mathrm{Iness}_{\mathscr{P}}$ of the additive category of all Abelian groups. Although we shall make no formal use of the fact, we observe that we may obtain an additive category $\mathrm{Ess}_{\mathscr{P}}$ by taking its objects to be (all) Abelian groups, and regarding the quotient group $\mathrm{Ess}_{\mathscr{P}}(G,H) = \mathrm{Hom}(G,H)/\mathrm{Iness}_{\mathscr{P}}(G,H)$ as the group of morphisms $G \to H$ in $\mathrm{Ess}_{\mathscr{P}}$, with the obvious rule for composition of morphisms; note that an Abelian group G is a zero object of $\mathrm{Ess}_{\mathscr{P}}$ if and only if the identity map on G is $\mathscr{P}$-inessential, that is, if and only if G is a homomorphic image (in the usual sense) of some group $P \in \mathscr{P}$.

From now on, we take $\mathscr{P}$ to be the class of all higher Specker groups $\mathbb{Z}^m$ ($m < \aleph_\iota$); as this is the only case we need, we drop the prefixed and suffixed $\mathscr{P}$ and write simply **inessential**, $\mathrm{Iness}(G,H)$, etc.

Proposition 3.1 *Let $D = M \otimes (\mathbb{Q}/\mathbb{Z})$, where M is a free Abelian group of rank $m < \aleph_\iota$. Let S be a pure subgroup of $K = \mathrm{Hom}(D,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(M,\hat{\mathbb{Z}})$, where $\hat{\mathbb{Z}} = E(\mathbb{Q}/\mathbb{Z})$, and suppose that S has the following properties:*

(i) S is slender;
(ii) for any subgroup S_0 of finite index in S, every homomorphism $S_0 \to S$ extends to an endomorphism of S;
(iii) there is an anti-homomorphism $E(S) \to E(M)$, written $\alpha \to {}_M\alpha$, such that, for each $\alpha \in E(S)$, α coincides with the restriction to S of the endomorphism $({}_M\alpha)^*$ of K induced by ${}_M\alpha$.

Then $E(G(S))$ is a split extension of $E(S)$ by $\mathrm{Iness}(G(S))$.

Proof We take advantage of Proposition 2.1 to choose a convenient admissible resolution of D. Take an admissible resolution of $\mathbb{Q}/\mathbb{Z}$,

$$0 \to Y_1 \to Y_0 \to \mathbb{Q}/\mathbb{Z} \to 0, \qquad (Y)$$

so that Y_0 and Y_1 are free Abelian of countable rank. Then

$$0 \to M \otimes Y_1 \to M \otimes Y_0 \to M \otimes (\mathbb{Q}/\mathbb{Z}) = D \to 0 \qquad (X)$$

is an admissible resolution of D, which induces the exact sequence

$$0 \to (M \otimes Y_0)^* \to (M \otimes Y_1)^* \to \mathrm{Hom}(M,\hat{\mathbb{Z}}) = K \to 0. \qquad (X^*)$$

Write $A = E(S)$. We use the anti-homomorphism of (iii) to endow M with the structure of a left A-module, by setting $\alpha x = x\,{}_M\alpha$ for $x \in M,\ \alpha \in A$; then $M \otimes Y_0$ and $M \otimes Y_1$ are canonically left A-modules, while (X^*) is canonically an exact sequence of right A-modules if in each of the terms scalar multiplication by $\alpha \in A$ is taken to be the endomorphism induced by ${}_M\alpha$. Visibly, by (iii), S is a sub-A-module of K, so its inverse image $G = G(S)$ is a sub-A-module of $(M \otimes Y_1)^*$. We write $\alpha_G = ({}_M\alpha)^* \restriction G$ ($\alpha \in A$), so that $\alpha_G \in E(G)$ is scalar multiplication by α in the A-module G, and we write $A_G = \{\alpha_G : \alpha \in A\}$ for the ring of scalar multiplications of G. Then the map $A \to A_G$ given by $\alpha \mapsto \alpha_G$ is a ring homomorphism. The proposition will be proved once we have shown that

$$E(G) = A_G \oplus \mathrm{Iness}(G), \quad A_G \cong A;$$

and for this it is clearly enough to show that (a) the kernel of the composite map $A \to A_G \subseteq E(G) \to E(G)/\mathrm{Iness}(G)$ is trivial, and (b) $E(G) = A_G + \mathrm{Iness}(G)$.

(a) Consider any $\alpha \in A$ such that α_G is inessential; we must prove that $\alpha = 0$. By hypothesis, for some $n < \aleph_\iota$, there is a homomorphism $\pi : \mathbb{Z}^n \to G$ such that $G\alpha_G \subseteq \mathbb{Z}^n\pi$. Therefore $S\alpha = G\alpha_G\delta_X \subseteq \mathbb{Z}^n\pi\delta_X$. But S is slender, so the restriction on n implies that $\mathbb{Z}^n\pi\delta_X$ is free (of finite rank). Consequently $S\alpha$ is also free. Assume for a contradiction that $S\alpha \neq 0$. Then S admits a non-zero homomorphism into $\mathbb{Z}$, which means that S must have an infinite cyclic direct summand, say $S = T \oplus \langle s \rangle$. Then the subgroup $S_0 = T \oplus \langle 2s \rangle$ is of finite index in S and it is easy to define a homomorphism $\omega : S_0 \to S$ such that $(2s)\omega = s$. Obviously ω cannot extend to an endomorphism of S. This contradicts (ii); so $\alpha = 0$, as required.

(b) Consider any endomorphism $\phi \in E(G)$. Since S is slender, the composite map

$$X_0^* \subseteq G \xrightarrow{\phi} G \xrightarrow{\delta_X} S$$

must vanish on some direct summand Y^* of finite corank in X_0^*. By Lemma 2.5, we have a direct decomposition $G/Y^* = S_0 \oplus V$, where V is free of finite rank and S_0 is isomorphically mapped by δ_X onto a subgroup of finite index in S. Hence a commutative diagram (of Abelian groups) in which $\delta_X = \bar{\delta}\delta_0$

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ {\scriptstyle \bar{\delta}}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ S_0 \oplus V & \xrightarrow[\phi_0]{} & S \\ {\scriptstyle \delta_0}\downarrow & & \\ S & & \end{array}$$

and where all the maps that have not so far been defined are canonically induced from the others. Applying (ii) and (iii) to the homomorphism $S_0\delta_0 \to S$ induced by $\phi_0 \restriction S_o$, we find that there is an $\alpha \in A$ such that $(\delta_0 \restriction S_0)\alpha = \phi_0 \restriction S_0$. Since $\alpha_G\delta_X = \delta_X\alpha$, we have $G(\phi - \alpha_G)\delta_X = G\bar{\delta}(\phi_0 - \delta_0\alpha) = V(\phi_0 - \delta_0\alpha)$. But V is free; so there exists a homomorphism $\pi : V \to G$ such that $\pi\delta_X = \phi_0 - \delta_0\alpha$. Therefore $G(\phi - \alpha_G) \subseteq (X_0^* \oplus V)(1, \pi)$; and since $X_0^* \oplus V \cong \mathbb{Z}^m$, this proves that $\phi - \alpha_G$ is inessential. Thus $E(G) = A_G + \mathrm{Iness}(G)$; and the proof of the proposition is complete.

4 A Ring-Realization Theorem

Theorem 4.1 *Let A be a ring whose additive group is free Abelian of rank* $\leq m < \aleph_\iota$. *Then there exists a pure subgroup G of* $\mathbb{Z}^m$ *whose endomorphism ring* $E(G)$ *is a split extension of A by* $\mathrm{Iness}(G)$.

Proof Let V, F be free Abelian groups of rank m and set $M = A \otimes V$. Now A may be regarded either as a left or a right A-module. We use these module structures to regard M and $A \otimes F$ as a left and a right A-module, respectively, so that $M^* = (A \otimes V)^*$ is a right A-module. We prove first that M^* has a sub-A-module isomorphic with $A \otimes F$ that is pure as a subgroup. To this end choose free bases a_i $(i \in I)$ of the additive group of A, f^λ $(\lambda \in \Lambda)$ of F and b^j_μ $(j \in I,\ \mu \in \Lambda)$ of M. Define $a^j \in A^*,\ e^\mu \in M^*$ by setting

$$a_i a^j = \delta^j_i,\ (a_i \otimes b^j_\lambda)e^\mu = \delta^j_i \delta^\mu_\lambda,$$

so that

$$(\alpha \otimes b^j_\lambda)e^\mu = \delta^\mu_\lambda(\alpha a^j) \quad (\alpha \in A).$$

Now suppose that we have elements α^λ $(\lambda \in \Lambda)$, almost all zero, and an integer q such that

$$\sum e^\mu \alpha_\mu \in qM^* = q(A \otimes V)^*.$$

Applying this homomorphism on the basis element $a_i \otimes b^j_\lambda$ of $A \otimes V$, we find that $q\mathbb{Z}$ contains the image

$$(a_i \otimes b^j_\lambda)\sum_\mu e^\mu \alpha_\mu = \sum_\mu (\alpha_\mu a_i \otimes b^j_\lambda)e^\mu = (\alpha_\lambda a_i)a^j;$$

therefore $\alpha_\lambda a_i = \sum_j (\alpha_\lambda a_i)a^j a_j \in qA$, whence $a_\lambda \in a_\lambda A \subseteq qA$. In particular, taking $q = 0$, we find that the e^μ form a free basis for a free right sub-A-module of M^*. Clearly this submodule is isomorphic with $A \otimes F$ under the isomorphism $\sum e^\lambda \alpha_\lambda \mapsto \sum \alpha_\lambda \otimes f^\lambda$. If we use this isomorphism to identify $A \otimes F$ as a submodule of M^*, then the case $q > 0$ shows that $A \otimes F$ is embedded as a pure submodule of M^*.

Reverting to the notation of Proposition 3.1, we now have pure embeddings of Abelian groups

$$A \otimes F \rightarrowtail M^* = \mathrm{Hom}(M, \mathbb{Z}) \rightarrowtail \mathrm{Hom}(M, \hat{\mathbb{Z}}) = K,$$

and the embeddings respect the right-A-module structures. Here $\hat{\mathbb{Z}} = E(\mathbb{Q}/\mathbb{Z})$ may be identified with the natural or $\mathbb{Z}$-adic completion of $\mathbb{Z}$; so K is complete in its natural topology. Consequently, K contains the natural completion of its pure subgroup $A \otimes F$,

$$\widehat{(A \otimes F)} \leq_* K.$$

Utilizing ideas from [3], one can find a finite set $\mathscr{U} = \{U_1, \ldots U_5\}$ of direct summands of $A \otimes F$ such that (i) $A \otimes F = \sum U_i$ and (ii) $E(A \otimes F;\ \mathscr{U}) = A$ (acting as right multiplication on $A \otimes F$), where the ring on the left-hand side of (ii) consists of all $\mathbb{Z}$-endomorphisms ϕ of $A \otimes F$ such that $U_i\phi \subseteq U_i$ $(1 \leq i \leq 5)$.

Following the notation of [2], choose elements $\pi_1, \ldots \pi_5$ in the subring $\mathbf{P}$ of $\hat{\mathbb{Z}}$ which are algebraically independent over $\mathbb{Z}$ and take S to be the pure subgroup of $\widehat{(A \otimes F)}$ generated by $A \otimes F, \pi_i U_i$ $(1 \leq i \leq 5)$, i.e.,

$$S = \widehat{(A \otimes F)}\ \cap\ \mathbb{Q}(A \otimes F \oplus \bigoplus_{i=1}^{5} \pi_i U_i).$$

Then $S \leq_* K$. We check the conditions (i),(ii), (iii) of Proposition 3.1.

(i) S lies in the pure subgroup of $\widehat{(A \otimes F)}$ generated by $A \otimes F, \pi_i(A \otimes F)$ $(1 \leq i \leq 5)$, which may be identified with $\Pi \otimes A \otimes F$, where Π is the pure subgroup of $\hat{\mathbb{Z}}$ generated by $1, \pi_1, \ldots \pi_5$. Thus S is a subgroup of a direct sum of copies of the countable reduced torsion-free group Π. As such, S is slender.

(iii) Obviously S is a faithful right sub-A-module of $\widehat{(A \otimes F)}$. But any $\mathbb{Z}$-module endomorphism ϕ of S extends by continuity to a $\hat{\mathbb{Z}}$-endomorphism $\hat{\phi}$ of $\hat{S} = \widehat{(A \otimes F)}$; and the choice of the π_i then easily forces $\phi \restriction A \otimes F \in E(A \otimes F;\ \mathscr{U}) = A$. Hence an identification $E(S) = A$ (acting as right multiplication on S). Since M is a left A-module, there is a natural ring anti-homomorphism $E(S) = A \to E(M_{\mathbb{Z}})$: and the way in which $A \otimes F$ was embedded as a sub-A-module of $\mathrm{Hom}(M, \hat{\mathbb{Z}}) = K$ guarantees that (iii) is satisfied.

(ii) Let S_0 be a subgroup of finite index q in S, and $\phi : S_0 \to S$ any homomorphism. Then by (iii) the composite

$$(S \xrightarrow{q} qS \hookrightarrow S_0 \xrightarrow{\phi} S) = \text{scalar multiplication by some } b \in A.$$

Choose a basis f_i $(i \in I)$ of $F_{\mathbb{Z}}$. Since I is infinite, there exist distinct $i, j \in I$ such that $1 \otimes f_i \equiv 1 \otimes f_j \mod S_0$. Then $f = f_i - f_j$ generates a free cyclic direct summand $\mathbb{Z}f$ of F, and $1 \otimes f \in S_0$. Applying $q\phi$ we obtain $b \otimes f \in (A \otimes F) \cap qS = q(A \otimes F)$, whence $b \in qA$. Say $b = qa$. Then scalar multiplication by a gives a $\mathbb{Z}$-endomorphism of S which extends ϕ, as required.

It now follows by Proposition 3.1 that $E(G(S)) = A \oplus \mathrm{Iness}(G(S))$: and the theorem is proved.

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Countable 1-Transitive Trees

Katie M. Chicot and John K. Truss

Abstract We give a survey of three pieces of work, on 2-transitive trees (Droste, Memoirs Am Math Soc 57(334) 1985), on weakly 2-transitive trees (Droste et al., Proc Lond Math Soc 58:454–494, 1989), and on lower 1-transitive linear orders (Barbina and Chicot, Towards a classification of the countable 1-transitive trees: countable lower 1-transitive linear orders. arXiv:1504.03372), all in the countable case. We lead on from these to give a complete description of all the countable 1-transitive trees. In fact the work of Barbina and Chicot was carried out as a preliminary to finding such a description. This is because the maximal chains in any 1-transitive tree are easily seen to be lower 1-transitive, but are not necessarily 1-transitive. In fact a more involved set-up has to be considered, namely a coloured version of the same situation (where 'colours' correspond to various types of ramification point), so a major part of what we do here is to describe a large class of countable *coloured* lower 1-transitive linear orders and go on to use this to complete the description of all countable 1-transitive trees. This final stage involves analyzing how the possible coloured branches can fit together, with particular attention to the possibilities for cones at ramification points.

Keywords Tree • Lower semilinear order • 1-Transitive • Cone • Ramification point • Coding tree

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1 Introduction

The main aim of this paper is to give a complete ‘description’ of all the countable 1-transitive trees (we use this expression since it is unclear whether what we provide really constitutes a ‘classification’). We also give a survey of the work leading up to this, which is taken from three main sources, [11] on 2-transitive trees, [10] on weakly 2-transitive trees, and [5] on lower 1-transitive linear orders. The reason for considering lower 1-transitive linear orders at all is that they arise naturally as branches (that is, maximal chains) of countable 1-transitive trees, which are the structures we are really aiming to describe.

In the literature, several authors have studied 2-transitive trees in a variety of contexts. For instance, they arise naturally in the analysis of Jordan groups, see [1], on their own merits, and also via B-, C-, and D-relations. See also [4], where reducts of 2-transitive trees have been studied. Bodirsky and co-authors have used 2-transitive trees in the context of constraint satisfaction problems in theoretical computer science [3], and they were used by Andreka, Givant, Nemeti [2] to show that the lattice of varieties of representable relation algebras embeds the power set of the integers, hence has a complicated structure. We also mention that there are natural countable homogeneous relational structures, obtained from these trees over a slightly expanded language, whose automorphism groups have $2^{2^{\aleph_0}}$ normal subgroups [10].

A **tree** is taken to be a partially ordered set in which any two elements have a common lower bound, and the set of all lower bounds of any one element is linearly ordered (and for non-triviality we also suppose that there are incomparable points, so it is not a chain, which is called being **proper**). These are also called **lower semilinear orders** (for instance in [1]), since they are linear going downwards. Droste initiated the study of sufficiently transitive trees in [11]. Let us say that a tree is k-**transitive** if for any two isomorphic k-element substructures, there is an automorphism taking the first to the second. He showed that in non-trivial cases, no tree can be 4-transitive, and he classified the countable 2- and 3-transitive trees. Unless stated otherwise, all the trees and linear orders we consider are countable.

Now in a tree A there are two different kinds of 2-element substructure, chains and antichains, so it is required by the definition of 2-transitivity that the automorphism group of A act transitively both on the set of 2-element chains and on the set of 2-element antichains. If we relax the condition, and only require transitivity on the former, then we arrive at the class of **weakly** 2-**transitive** trees, as defined and studied in [10]. The key and immediate difference that this makes is that there need no longer be a constant value throughout the tree of ‘ramification order’ (degree of branching; see below for the formal definition). Any ramification point is the greatest lower bound of a 2-element antichain, and so if we require transitivity on 2-element antichains, it easily follows that the ramification order is the same throughout.

To understand the definitions in the previous paragraph, it is helpful to explain the notion of ramification point more precisely. For this, and since we shall need it later anyhow, we introduce the notion of the 'completion' A^D of A (D here for 'Dedekind'). This is also called the 'Dedekind–MacNeille completion' in [14], though as this term is used in more than one sense in the literature, we just call it 'completion' here. An **ideal** is a non-empty bounded subset I of A, which is equal to the set of lower bounds of its set of upper bounds. For instance, in $\mathbb{Q}$, $I = \{q \in \mathbb{Q} : q < \pi\}$ is an ideal since its set of upper bounds is $\{q : q > \pi\}$, and the set of lower bounds of *this* is again I. Also $\{q : q \leq 4\}$ is an ideal but $\{q : q < 4\}$ is not. Then A^D is defined to be the set of all ideals partially ordered by inclusion. One can check that this is again a tree, and A may be viewed as a subset of A^D, via the embedding $a \mapsto A_{\{a\}}$ (which preserves the ordering). It is called the **completion** of A. If $A = A^D$, then A is **complete**.

If $X \subseteq A$, let $A_X = \{a \in A : (\forall x \in X) a \leq x\}$. If X is non-empty and bounded below, this is an ideal, so lies in A^D. We write A^+ for the set of all A_X for finite non-empty X. Thus $A \subseteq A^+ \subseteq A^D$. If A is infinite, then $|A| = |A^+|$, but usually $|A| < |A^D|$. If X is finite having a least member (including the case that it is a finite chain), then $A_X = A_{\{\min X\}}$. Otherwise, there are at least two minimal members of A_X, which must form an antichain, and then A_X is called a **ramification point**. As A is a tree, it is easy to see that such X may be taken to be an antichain of size 2. A ramification point may equal $A_{\{a\}}$ for some $a \in A$, in which case it is said to be of **positive type**, or it may not, in which case it lies in $A^+ \setminus A$ and is said to be of **negative type**. There are natural induced actions of $\mathrm{Aut}(A, \leq)$ on A^+ and A^D.

In a tree there is a notion of 'cone', which plays a crucial role in describing its structure. If x is a ramification point, then a **cone** at x is an equivalence class under the relation on points above x given by $y \sim z$ if for some $t, x < t < y, z$. In a tree this is easily seen to be an equivalence relation (in general partial orders it may not be). The **ramification order** of x is the number of cones at x. If x is not a ramification point, then we may say that it has ramification order 1 (so in a sense, has just one cone). Throughout, by **branch** of a tree we understand a maximal chain.

Given these definitions, we are already in a position to state Droste's main results for the 2-transitive case. He shows that in any 2-transitive tree A, either all ramification points are of positive type or all are of negative type, and the ramification order is constant. For every countable cardinal κ between 2 and $\aleph_0$ inclusive, there is a 2-transitive tree, unique up to isomorphism, in which all branches have order-type $\mathbb{Q}$, and all ramification points are of positive type, and have ramification order κ. Similarly for negative type. Every 2-transitive tree is of one of these kinds, or else has a root and all other points are immediate successors of the root. These last examples are technically 2-transitive, but are not 1-transitive, and are not of interest (for instance, their automorphism groups are symmetric groups on the set of non-root points). A proof of Droste's result is sketched in [13].

A key difference between 2-transitive and weakly 2-transitive trees is that in the latter, ramification order does not need to be constant. In fact, it can vary wildly throughout the tree, and this enables us to build $2^{\aleph_0}$ non-isomorphic countable examples. It used to be thought that the uncountability of a class of structures would render its classification futile. However, it is still quite easy to describe fairly explicitly what all the weakly 2-transitive trees are, in terms of a 'real parameter' (or an arbitrary subset of ω), and so one can regard this as good as a classification. (Another famous example of a classification of an uncountable class of structures is Cherlin's [8], where the countable homogeneous directed graphs are given, again in terms of a real parameter.)

We focus a little further on the structure of a weakly 2-transitive tree A, as part of our survey, and also because this will lead on naturally to the more complicated classification in the 1-transitive case. To keep track of the way the ramification points arise and interact, we view A^+ as 'coloured', where two points have the same colour if and only if they lie in the same orbit under $\mathrm{Aut}(A, \leq)$. In the 2-transitive case there are at most 2 orbits (ramification points and in the negative type case, the points of A), but in the weakly 2-transitive case there can be many more. First note that (assuming there is a chain of length at least 3) weak 2-transitivity at once implies that branches are densely ordered without endpoints, hence order-isomorphic to $\mathbb{Q}$; it is not however immediately clear that the same is true in the coloured case (that is, whether all branches of A^+ need be isomorphic). If X is a branch of A, X^+ will stand for the coloured linear order obtained from the branch of A^+ containing X. This need not now be dense; however, the only possibility for consecutive elements $x < y$ is that $y \in X$ and $x \notin X$, as is easy to see. If these pairs are 'collapsed' and viewed as single points, the result *is* densely ordered, and indeed is $\mathbb{Q}_C$ where C is a colour set, the 'C-coloured version of the rationals' (defined below). The point y may be described as the least member of a cone at x. As mentioned above, at any ramification point x there will be cones: some may have least elements, but not all, and the numbers arising sum to the total ramification order: both need to be taken into account in specifying what the 'colour' is at x. Any ramification point having a cone with a least member is called 'special'. Note that all special ramification points have the same colour, since they are immediate predecessors of points of A.

Giving a few more details from [10], the 'type' of a tree is specified by the following information:

(1) a list of all the ramification orders which arise at non-special ramification points (which will be numbers between 2 and ∞ inclusive),
(2) whether the points of A ramify; if so there will be no special ramification points, but the ramification order of points of A should be given,
(3) if the points of A do not ramify, whether there are any special ramification points; if so then the number of cones there with or without a least element should be given.

In [10] this information is realized as a certain triple. Given this notion, the following steps are then required: (1) any weakly 2-transitive tree has a type, (2) any two countable weakly 2-transitive trees having the same type are isomorphic, (3) any type satisfying certain stated properties is the type of some weakly 2-transitive tree. This then provides a classification of all the weakly 2-transitive trees. There are $2^{\aleph_0}$ non-isomorphic structures in the 'list', since any subset of $\{2, 3, 4, \ldots, \aleph_0\}$ may arise as the possible ramification orders (even not taking into account the issue of special ramification points), but given this, the description is concrete enough to constitute a 'classification'.

Now we move on to the 1-transitive case. Since the weaker the hypothesis on a structure, the more complicated one expects any classification of such structures to be, it is unsurprising that this presents many difficulties. An immediate problem is as follows. For weakly 2-transitive trees, it is easy to check that all the branches are order-isomorphic to the ordered rationals $\mathbb{Q}$, because one checks directly that they are densely ordered without endpoints. If we drop 2 to 1, then we would expect that the branches B should be 1-transitive. This is however not clear, since if we seek to show that $x \in B$ can be mapped to $y \in B$, all we can say is that there is an automorphism *of* A taking x to y, but there seems to be no obvious reason why it should fix B setwise. Similarly, it seems impossible to demonstrate that all the branches will be isomorphic. Thus we are inevitably led to revised definitions; the most we can say is that if $x, y \in B$, then there is an isomorphism between the *initial segments* determined by x and y, which we call being **lower** 1-**transitive**, and that if B_1 and B_2 are branches, then there is an isomorphism between initial segments of B_1 and B_2, which we call being **lower isomorphic**.

Now the countable 1-transitive linear orders are described by Morel [12]. They are $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for countable ordinals α (where we are taking restricted lexicographic powers). The class of lower 1-transitive linear orders is however vastly bigger than this. As part of our survey, we outline the main ideas of [5], where a description is given, and since a modified version is required in this paper. The principal 'base' examples of lower 1-transitive linear orders are $\mathbb{Z}$ and $\mathbb{Q}$ (which are actually 1-transitive) and ω^* (the least infinite ordinal ω under the reversed ordering) and $\dot{\mathbb{Q}}$ (the rationals with an extra point on the right). The main result of [5] is that all countable lower 1-transitive linear orders can be built up from these, by an admittedly complicated process, but generalizing the way in which $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ are built up from $\mathbb{Z}$ and $\mathbb{Q}$.

A key method of construction which is definitely new in the lower 1-transitive case involves taking '$\mathbb{Q}_n$- or $\dot{\mathbb{Q}}_n$—combinations'. Here for $2 \le n \le \aleph_0$, $\mathbb{Q}_n$ is the 'n-coloured version of the rationals', obtained by colouring $\mathbb{Q}$ by n colours (usually taken to be $0, 1, 2, \ldots, n-1$), each of which occurs densely. This is easily seen to exist and to be unique up to isomorphism; $\dot{\mathbb{Q}}_n$ is obtained by adding one extra point on the right. Since $\mathbb{Q}_n$ and $\dot{\mathbb{Q}}_n$ are coloured, they are not lower 1-transitive linear orders. However they are used in the construction of such. If $Y_0, Y_1, \ldots, Y_{n-1}$ are linear orders, then $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ is the linear order obtained from $\mathbb{Q}_n$ by replacing each point coloured i by a copy of Y_i, and similarly for $\dot{\mathbb{Q}}_n(Y_0, Y_1, \ldots, Y_n)$. Then if all the Y_i are lower isomorphic and lower 1-transitive, one can easily check that $\mathbb{Q}_n(Y_0, Y_1, \ldots, Y_{n-1})$ and $\dot{\mathbb{Q}}_n(Y_0, Y_1, \ldots, Y_n)$ are also lower 1-transitive. For instance, ω^* and $\mathbb{Z}$ are lower isomorphic, and $\mathbb{Q}_2(\omega^*, \mathbb{Z})$ is lower 1-transitive, since all its initial segments are isomorphic to $\mathbb{Q}_2(\omega^*, \mathbb{Z}) + \omega^*$.

The natural way to represent a general construction of a lower 1-transitive linear order built up from the base structures by $\mathbb{Q}_n$ or $\dot{\mathbb{Q}}_n$ combinations, and by taking 'limits' (in a similar way to passing from $\mathbb{Z}^n$ for $n \in \omega$ to $\mathbb{Z}^\omega$, formally expressed by the use of the symbol *lim*) is via 'coding trees', to keep track of how this is done. This technique was introduced in [6] and used more heavily in [7]; there the class of structures to be described was the 1-transitive coloured linear orders. This use of the word ‘tree’ is not quite the same as for the class of structures we are trying to classify, so in order to avoid confusion, we use ‘downward growing’ trees as coding trees (just reverse the ordering in all definitions about trees), which are meant to describe the way in which the linear orders are built up iteratively from singletons (following the intuition as in computer science, for instance parse trees). Such a tree will have a root (now at the top), which corresponds to the encoded structure. There will be leaves (minimal elements) which encode singletons, and at intermediate stages, each vertex has a label which tells us how the structure encoded at that vertex is determined from the encoded structures below it. Because it is important to distinguish the two uses of trees in our work, as coding trees, forming one ingredient in the classification, and also as the structures being classified, we use A for the 1-transitive tree, and T for a coding tree; also T (but not A) will be labelled, the labels telling us how the linear order is built up in stages.

A major discovery in [5] was that coding trees must necessarily be ‘levelled’, meaning that they may be partitioned into a union of maximal antichains, called ‘levels’ in such a way that the set of levels is itself linearly ordered, and if $x < y$ in the tree, then the level that x lies in is less than the one that y lies in. The sense in which coding trees then provide a classification of the class in question is that first, any coding tree does encode a uniquely determined member of the class (here a lower 1-transitive linear order), and second, any member of the class is encoded by some coding tree. The first task is the easier in the sense that, once the correct definition of coding tree is found, it is then just necessary to show that it does encode a lower 1-transitive linear order, by suitably interpreting the labels. This is not completely straightforward since the tree needn’t be well-founded or conversely well-founded, so a method of coping with limits in either direction must be found. The second task involves finding a way of recognizing inside the lower 1-transitive linear order which coding tree it came from; a key notion here is that of ‘invariant partitions’, which correspond to the levels of the resulting tree.

Providing a few more details, for technical reasons (associated with the ‘second task’) we insist that the coding tree be complete, though most of its points (irrational cuts) have no actual impact on the encoding procedure. A **coding tree** is then a complete (not necessarily countable) labelled levelled tree, with a greatest element (root), countably many minimal elements (leaves) such that every element is a leaf or above a leaf, with possible labels $\varsigma(v)$ of a non-leaf vertex $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$, or *lim*. It is also required that if $\varsigma(x) = \mathbb{Z}$ or $\mathbb{Q}$, it has one child (a vertex immediately below it which is $\geq y$ for any $y < x$), if $\varsigma(x) = \omega^*$ or $\dot{\mathbb{Q}}$, it has 2 children, if $\varsigma(x) = \mathbb{Q}_n$ it has n children, if $\dot{\mathbb{Q}}_n$, it has $n + 1$ children, and if $\varsigma(x) = lim$, it has no children, and exactly one cone. For vertices x and y on the same level, if $\varsigma(x) \neq \varsigma(y)$ then $\varsigma(x)$ and $\varsigma(y)$ are lower isomorphic. (There are two further technical conditions.) A simple example of a coding tree is given in Fig. 1, and the (labelled, and levelled) tree shown is clearly intended to encode the lower 1-transitive linear order $\mathbb{Q}_2(\omega^*, \mathbb{Z})$ mentioned above.

For a finite coding tree, an easy recursion tells us how it is meant to be interpreted. Since however, coding trees can in general be infinite, and very complicated, we have to have an alternative method of describing what they are meant to encode.

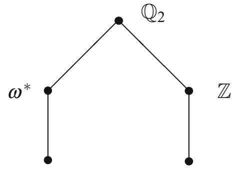

Fig. 1 Coding tree for $\mathbb{Q}_2(\omega^*, \mathbb{Z})$

For this purpose it is helpful to introduce an auxiliary notion, called an *expanded coding tree*, whose definition parallels that of coding tree in many respects, but which describes more explicitly the implementation of the intended encoding procedure. The definition is that an **expanded coding tree** is a complete labelled levelled tree, with a greatest element (root), countably many minimal elements (leaves) such that every element is a leaf or above a leaf, with possible labels $\varsigma(v)$ of a non-leaf vertex $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$, or *lim*. If $\varsigma(x) = \mathbb{Z}, \mathbb{Q}, \omega^*, \dot{\mathbb{Q}}, \mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$, its children are ordered in type $\mathbb{Z}$, $\mathbb{Q}$, etc. In the first two cases, the trees rooted at the children are all isomorphic (where isomorphisms of labelled levelled trees are required to preserve labels and levels, as well as the partial ordering), for ω^* and $\dot{\mathbb{Q}}$ the trees rooted at all the children except the greatest are isomorphic, for $\mathbb{Q}_n$, the trees rooted at the children are isomorphic if and only if they have the same colour, and for $\dot{\mathbb{Q}}_n$ the same applies except for the greatest point. If $\varsigma(x) = lim$, it has no children, and exactly one cone. For any two vertices on the same level, if one is a parent vertex, so is the other, and if one is a leaf, so is the other; if they are parent vertices, the order-types of their children are lower isomorphic. (Again there are other technical conditions.)

Any expanded coding tree gives rise to an encoded linear order as its set of leaves. For if x and y are distinct leaves, their least upper bound z is a ramification point, which has at least two cones and is not a leaf, so is labelled by one of the first six labels, and the ordering between x and y is determined by the ordering of the children of z. To say how a coding tree encodes a linear order, there is a notion of an expanded coding tree being 'associated' with a coding tree, which essentially means that the expanded coding tree is obtained from the coding tree by 'carrying out' all the instructions it requires. (See the definitions given in Sect. 3 in the coloured case.) This therefore tells us what it means for a coding tree to encode a linear order.

The main theorems proved in [5] about this situation are as follows:

Theorem 1.1 *Any coding tree encodes some linear order, and any two countable linear orders encoded by the same coding tree are isomorphic.*

Theorem 1.2 *The linear order encoded by any coding tree is countable and lower 1-transitive.*

Theorem 1.3 *Any countable lower 1-transitive linear order is encoded by some coding tree.*

Theorem 1.1 is proved by constructing an expanded coding tree $(E, \leq)$ associated with the given coding tree $(T, \leq)$. Since each member of E represents a choice of interpretation of the labels of T, we view members of E as functions on branches of T, called 'decoding functions'. In order to stop this family becoming uncountable, we require such functions to take the same ('default') value on all except finitely many points. Then the encoded linear order is taken to be the set of leaves of the resulting tree. Uniqueness of the encoded (countable) linear order is proved by back-and-forth.

In view of the previous theorem, we may refer to *the* order encoded, and to prove the properties of it stated in Theorem 1.2, we may assume that it is given by the particular decoding functions just mentioned.

By far the most complicated step is in Theorem 1.3. We have to recover the (expanded coding) tree $(E, \leq)$ from the linear order $(X, \leq)$. In fact the levels of E precisely correspond to partitions of X into convex subsets which are invariant under automorphisms and lower isomorphisms of $(X, \leq)$, and it is this property which provides a way of recovering E from X—one shows that the family of all subsets of X occurring in some invariant partition forms an expanded coding tree for $(X, \leq)$. There are quite a number of technical details to be verified along the way, in particular the fact that the resulting tree is levelled. Finally, a coding tree for X is obtained from E by 'coalescing' all the children of a vertex which will be identified in the coding tree.

Much of this material will arise again in the more complicated situation described in this paper, where (a class of) lower 1-transitive *coloured* linear orders has to be described.

2 Strategy for Classifying Countable 1-Transitive Trees

Having surveyed the work leading up to this paper, we now outline our strategy for analyzing countable 1-transitive trees. As explained when discussing the weakly 2-transitive case, we need to consider *coloured* branches, to keep track of how and where the tree ramifies. Section 3 is devoted to a description of the required class of lower 1-transitive coloured linear orders, using a modification of coding trees as explained in the (monochromatic) lower 1-transitive case. Since it turns out that not all coloured chains can arise, it suffices to classify those which actually do, which we call 'branch-coloured chains'. Even here things are considerably more complicated than in the monochromatic case. Coding trees need not be levelled (they will be 'nearly levelled', defined below), and we require three new labels for the vertices. Despite this, the main outline is as before.

We now give some more precise details. A **coloured linear order** is a triple $(X, \leq, F)$ where $(X, \leq)$ is a linear order, and F is a function from X onto a set C, called the set of 'colours'. It is **lower 1-transitive** if $\forall x, y \in X(F(x) = F(y) \rightarrow (-\infty, x] \cong (-\infty, y])$, and two coloured linear orders X and Y with the same colour set are **lower isomorphic** if there are $x \in X, y \in Y$ such that $(-\infty, x] \cong (-\infty, y]$ (where the isomorphisms must preserve the ordering *and* colours). As explained in

Sect. 1, for any branch X of a 1-transitive tree $(A, \leq)$, the corresponding coloured branch X^+ of A^+ must be lower 1-transitive, where the colours are the orbits of the action of $\mathrm{Aut}(A)$ on A^+, and any two branches are lower isomorphic. A **branch-coloured chain** is defined to be a coloured chain without maximal or minimal elements, such that one designated colour $\bar{c}$ occurs densely (meaning that for any $x < y$ there is z coloured $\bar{c}$ with $x \leq z \leq y$) and if $x < y$ are consecutive points of X then y is coloured $\bar{c}$. It can be verified (see Lemma 5.2) that any branch of a countable 1-transitive tree is a branch-coloured chain. It turns out that this *precisely* characterizes the possible (coloured) branches of countable 1-transitive trees (see Corollary 6.5). It eases the technical details of the proof (specifically of Corollary 3.25) if we also encode initial segments at points coloured $\bar{c}$, so in certain places we relax the requirements to allow this.

At this stage we just give one example of what a branch of a 1-transitive tree could be, namely $\mathbb{Q}.(1 + \mathbb{Z})$, which is $\mathbb{Q}$ 'copies' of a singleton followed by $\mathbb{Z}$. Here the singletons are coloured 'red', and are ramification points, and all the other points will be points of the tree (which we initially are assuming do not themselves ramify). This is indeed lower 1-transitive, as is easy to verify; the maximal chains of the resulting tree have order-type $\mathbb{Q}.\mathbb{Z}$, and the only (but vital) role of the red points is to tell us where the tree branches. To specify the tree, further information such as ramification order must be given. Variants on this example are $\mathbb{Q}.(\mathbb{Z} + 1)$ and $\mathbb{Q}.(1+\mathbb{Z}+1)$ which ramify immediately *above* each copy of $\mathbb{Z}$, and below *and* above each copy, respectively. In each of these examples it is also possible for the points of the tree to ramify. If we wish to emphasize the fact that the singletons are coloured, we may also write $\mathbb{Q}.(\text{red} + \mathbb{Z})$, $\mathbb{Q}.(\mathbb{Z} + \text{blue})$, and $\mathbb{Q}.(\text{red} + \mathbb{Z} + \text{blue})$ for these three (and similarly in other cases).

In modifying the classification of countable lower 1-transitive linear orders for branch-coloured chains, we have to alter the definition of 'coding tree'. First the labels allowed are as before, together with $\vee$, $\wedge$, $-$, and members of a set C of colours including $\bar{c}$. Vertices labelled $\vee$ or $\wedge$ have two children (they will correspond to concatenation), and $-$ one child (included for technical reasons, to retain the levels, see Fig. 3a for an example). To make some of the definitions run more smoothly, we may view each of $\vee$ and $\wedge$ as 2-element linear orders, coloured by distinct colours, and $-$ as a singleton linear order. The colour labels are used for labelling the leaves. Further, the tree is required to be **nearly levelled**, which means that the tree obtained by removing all leaves labelled by colours, other than $\bar{c}$, is levelled, which, to recap, means that it can be expressed as a disjoint union of maximal antichains ('levels') in such a way that for any two levels l_1 and l_2, either every member of l_1 is below some member of l_2, or the other way round. We illustrate some colour coding trees in Figs. 2 and 3, also indicating the coloured orders that they are meant to encode.

For ease we extend the function F which colours leaves, to all vertices, by saying that $F(x)$ is the set of all colours of leaves below x. An additional requirement is that for any non-leaf x, $\bar{c} \in F(x)$, and several clauses govern the use of the labels $\vee$, $\wedge$, and $-$. A key point is that the left child of any vertex labelled $\vee$ and the right child of any vertex labelled $\wedge$ are leaves not coloured $\bar{c}$, but their other children are

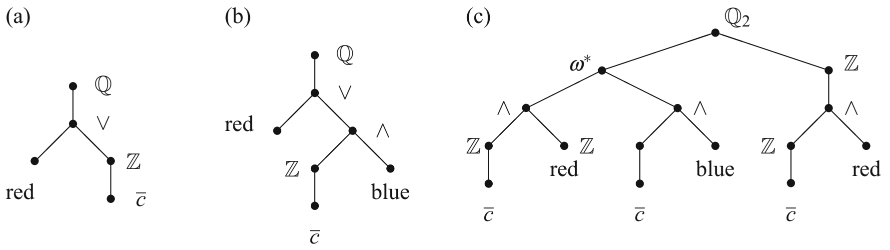

Fig. 2 Colour coding trees for (**a**) $\mathbb{Q}.(1 + \mathbb{Z})$, (**b**) $\mathbb{Q}.(1 + \mathbb{Z} + 1)$, (**c**) $\mathbb{Q}_2(\omega^*.(\mathbb{Z} + 1) + \mathbb{Z} + 1, \mathbb{Z}.(\mathbb{Z} + 1))$

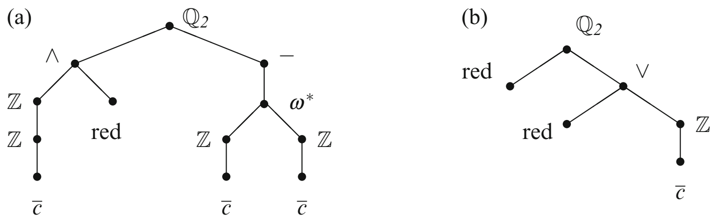

Fig. 3 Colour coding trees for (**a**) $\mathbb{Q}_2(\mathbb{Z}^2 + \text{red}, \omega^*.\mathbb{Z})$, (**b**) $\mathbb{Q}_2(\text{red}, \text{red} + \mathbb{Z})$

not leaves. Also any two leaves having the same colour are on levels that are at most one apart; if they are one apart, then the vertex on the lower level is the left child of a vertex labelled $\vee$. The reader may verify the truth of these clauses in the examples given. For the full definition, see Sect. 3.

Now that we have introduced colour coding trees, we can reproduce the material sketched at the end of Sect. 1 for (ordinary) coding trees and the way in which they encode lower 1-transitive coloured linear orders. There is again a notion of 'expanded coding tree', and a colour coding tree encodes a coloured linear order if there is an associated coloured expanded coding tree whose set of leaves, under the induced ordering, is colour- and order-isomorphic to the given coloured linear order. Our task once again is to prove the analogues of Theorems 1.1–1.3. The first two require fairly straightforward adaptations of the previous methods. We should take a little time to explain the modification of 'invariant partitions' which are now used for Theorem 1.3.

For the monochromatic case, a partition π of a linearly ordered set $(X, <)$ into convex subsets is **invariant** if for any lower isomorphism $f : (-\infty, a] \to (-\infty, b]$ of X, and any $x, y \leq a$, x and y lie in the same member of π if and only if $f(x)$ and $f(y)$ do. It is then shown that if $(X, <)$ is countable and lower 1-transitive, the members of all invariant partitions form an expanded coding tree of $(X, <)$ under a suitable labelling. For instance, in the example $\mathbb{Q}_2(\omega^*, \mathbb{Z})$ given in Sect. 1, there are just three invariant partitions, two of them trivial, into singletons, and into just one set X, and one non-trivial one, into the maximal discrete blocks, which are all ordered like ω^* or $\mathbb{Z}$. The corresponding expanded coding tree therefore has three levels (and the 'coalesced' coding tree is shown in Fig. 1).

The first difficulty in the coloured case is that the members of invariant partitions (defined in precisely the same way, with respect to colour-preserving isomorphisms) need no longer form a tree. For instance in $X = \mathbb{Z}.(1 + \mathbb{Z})$, there are invariant partitions into convex subsets of order-type $\mathbb{Z} + 1$, and $1 + \mathbb{Z}$, and the members of these overlap but are incomparable. This problem also arose in [7], and the solution adopted there was to take a maximal subtree. Here we can be more explicit, and favour one of these two partitions over the other, giving the notion of a 'restricted' invariant partition. The other complication over the monochromatic case is that the tree we get need no longer be levelled, as illustrated in the examples given in Figs. 2 and 3. Examining those examples, we can see that the trees do become levelled if we remove the leaves having non-$\bar{c}$ labels, so they are nearly levelled. The only vertices x not lying in a level are therefore leaves labelled by some colour $\neq \bar{c}$. However, even these can be 'assigned' a level, as it turns out that such leaves are children of a vertex labelled $\vee$, $\wedge$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$, whose other child or children do lie in a level; and x is also assigned to this level. This is intuitively correct (see the examples of coding trees provided) but the definition of 'levelled' is strictly speaking violated, as x may lie in more than one of the maximal antichains which form the levels, and we have chosen to assign it to the highest possible one. The rather mysterious label $-$ features in Fig. 3a; without that the tree would not even be nearly levelled. The invariant partitions corresponding to the two non-trivial levels in Fig. 2a are into copies of $1+\mathbb{Z}$, and into red singletons and copies of $\mathbb{Z}$. In the next tree they are into copies of $1 + \mathbb{Z} + 1$, into red singletons and copies of $\mathbb{Z} + 1$, and into red and blue singletons and copies of $\mathbb{Z}$, respectively. (Note that we have favoured red singletons and copies of $\mathbb{Z} + 1$ over the other possibility which would be blue singletons and copies of $1+\mathbb{Z}$, ruled out by the correct definition of 'restricted'.) One can similarly trace through the other examples. Notice from the examples that $\vee$ is used when we are concatenating a non-$\bar{c}$ colour on the left, and $\wedge$ when we are doing this on the right. In the final part of Sect. 3, we describe how colour coding trees for lower isomorphic colour lower 1-transitive linear orders are related. This is important, because this is the situation which applies to the branches of a 1-transitive tree. A surprising, but vital piece of information is that the lower isomorphism class of any branch-coloured chain is countable (without this, we would be unable to 'build' countable trees exhibiting all branch-coloured chains in some lower isomorphism class).

In Sect. 4 we describe and analyze the notion of 'cone type'. As we have seen, this is a crucial ingredient in giving the structure of a weakly 2-transitive tree. In that situation, there were two cases, 'special' and 'normal' ramification points. For the 1-transitive case one takes into account which level of the colour coding tree of a branch ramification occurs. The cone type is then a list telling us how many cones there are at a ramification point of all possible types.

Since the precise characterization of 1-transitive trees is complicated, the most significant aspects are singled out in Sect. 5, where the notion of 'structured tree' is introduced. This is a countable tree $(A, \leq)$ with a colouring function on A^+ such that the points coloured $\bar{c}$ are precisely those of A, points with equal colours have the same cone type, all coloured branches are lower isomorphic, and every final segment of the isomorphism type of any branch occurs above every point of A. These are clearly necessary conditions for A to be 1-transitive, and the strengthening required to make them also sufficient is explored in Sect. 6.

The 'classification' in the final section is carried out by means of the 'type' of a tree A, which comprises the following information:

the set Υ of isomorphism types of branches of A^+,
the colour set of A^+,
for each colour, the cone type of points having that colour.

The main theorem of Sect. 5 then says that two 1-transitive trees are isomorphic if and only if they have the same type. It is clear that any 1-transitive tree has a type, and Sect. 6 is devoted to describing which types actually arise, the tricky point being whether Υ has to be the whole of a lower isomorphism class of lower 1-transitive coloured chains, or just part of it (which can happen).

3 A Class of Lower 1-Transitive Coloured Linear Orders

The general structure of the coloured linear orders we need can be extremely complicated, and we use an auxiliary notion to help describe how they are built up, namely that of 'coding tree', as outlined for the monochromatic and coloured cases in the previous sections. For lower 1-transitive coloured linear orders, we require three extra labels, $\vee$, $\wedge$, and $-$, over the monochromatic situation and additional colours are allowed. To motivate how this comes about, we begin by analyzing the structure of a branch-coloured chain in terms of its invariant partitions, show how this leads naturally to the idea of a (colour) 'coding tree', starting with the expanded version, and then 'collapse it' to derive a coding tree.

3.1 Finding a Tree Corresponding to a Coloured Linear Order

Let $(X, \leq, F)$ be a lower 1-transitive branch-coloured chain or an initial segment of such a chain at a $\bar{c}$-coloured point. Unlike for the monochromatic case the family of members of *all* invariant partitions of $(X, \leq, F)$ need not form a tree, as explained in Sect. 2, and that is why we formulate the notion of 'restricted' invariant partition below. A further complication over the monochromatic case is that we have to replace 'levelled' by 'nearly levelled'.

The definition is that an invariant partition π of X is **restricted** if whenever $p \in \pi$ is covered in π by q (that is, has q as an immediate successor) which has a least member a, then a is coloured $\bar{c}$. For instance, in $\mathbb{Q}.(1 + \mathbb{Z} + 1)$ (see Fig. 2b), the partition into the copies of red $+ \mathbb{Z}$ and blue singletons is not restricted, since the condition is violated by taking p to be a copy of red $+ \mathbb{Z}$ (which is covered by a blue singleton). Nor is the partition into copies of $\mathbb{Z}$ and red and blue singletons. First we establish some properties of arbitrary invariant partitions, and then look in more detail at the restricted ones.

Lemma 3.1 *The parts of any invariant partition π of X are lower 1-transitive coloured linear orders which are lower isomorphic to each other (provided their colour sets intersect).*

Proof Let $x, y \in \pi$, and suppose $a \in x, b \in y$, and $F(a) = F(b)$. By lower 1-transitivity of X there is an isomorphism $\varphi : (-\infty, a] \to (-\infty, b]$. Since π is invariant, φ takes $(-\infty, a] \cap x$ to $(-\infty, b] \cap y$, so the desired isomorphism is the restriction of φ to $(-\infty, a] \cap x$, and x and y are lower isomorphic. To deduce that the parts of π are lower 1-transitive coloured linear orders, use the same argument starting with $x = y$. □

Lemma 3.2 *Any invariant partition π of a lower 1-transitive coloured linear order X is also lower 1-transitive, when it is re-coloured by saying that two members of π have the same colour if they have at least one colour in common in X.*

Proof First we show that 'have a colour in common' is an equivalence relation on π. Suppose $x < y < z$ in π, where $a_1 \in x$ and $b_1 \in y$ are both coloured c_1 and $a_2 \in y$ and $b_2 \in z$ are both coloured c_2. If $b_1 \leq a_2$, we use lower 1-transitivity of X to find $g \in \mathrm{Aut}(X)$ taking a_2 to b_2, and then as π is invariant, $g(b_1) \in z$, so x and z have a colour in common (at a_1 and $g(b_1)$). If $a_2 \leq b_1$, we instead take b_1 to a_1 and consider the image of a_2 in x. Hence the 're-colouring' F' of π is well-defined.

We show that $(\pi, <, F')$ is lower 1-transitive. Let $x, y \in \pi$ be such that $F'(x) = F'(y)$, and let $a \in x$ and $b \in y$ have the same colour under F. Let $\varphi : (-\infty, a] \to (-\infty, b]$ be an isomorphism. Since π is invariant, φ induces an isomorphism from $(-\infty, x]$ to $(-\infty, y]$. □

Lemma 3.3 *If π is an invariant partition of X and π is coloured as in Lemma 3.2, then the colour containing $\bar{c}$ is dense in the rest.*

Proof Let χ be the colour containing $\bar{c}$. Then by the density of points coloured $\bar{c}$ in X, if distinct $x, y \in \pi$ are not coloured χ there is a point of π in between coloured χ. □

Definition 3.4 The **restricted refining invariant tree** I^{RR} is the family of all subsets of X which are members of some restricted invariant partition of X, partially ordered by $\subset$.

We shall see that this is a tree (see Theorem 3.8), and by adding some extra vertices, we shall be able to turn it into an expanded coding tree for $(X, \leq)$. Note that it clearly has root X and the leaves are the singletons $\{x\}$ for $x \in X$. The next two lemmas show that it is nearly levelled.

Lemma 3.5 *If π_1 and π_2 are restricted invariant partitions, one is a refinement of the other.*

Proof If not, there must be $x < y$ and $a < b$ such that x and y lie in the same member p_1^1 of π_1 but in different members p_1^2, p_2^2 of π_2, and a and b are in the same member p_3^2 of π_2 but different members p_2^1, p_3^1 of π_1. Let us first suppose that there are points $b_1 \leq b$ and $y_1 \leq y$ coloured $\bar{c}$ which are greater than all members of p_2^1,

p_1^2, respectively. Note that since $a < b_1 \le b$, $b_1 \in p_3^2$ and similarly $y_1 \in p_1^1$. By lower 1-transitivity, there is an isomorphism θ from $(-\infty, y_1]$ to $(-\infty, b_1]$. Since θ preserves π_1 and π_2 below y, θx and b_1 are in the same member of π_1, which implies that $a < \theta x$, but different members of π_2, which implies that $\theta x < a$, which gives a contradiction.

Since the $\bar{c}$ points are dense, if there is no such b_1 then b must be the least member of p_3^1 and it is not coloured $\bar{c}$, and in addition, p_2^1 and p_3^1 must be consecutive members of π_1. This however violates the definition of π_1 restricted, so cannot arise. Therefore a suitable b_1 exists, and similarly, so does y_1. □

Lemma 3.6 *If π_1, π_2 are restricted invariant partitions, π_1 a proper refinement of π_2, and $p \in \pi_1 \cap \pi_2$, then p is a singleton not coloured by $\bar{c}$.*

Proof Suppose for a contradiction that $p \in \pi_1 \cap \pi_2$ which is either not a singleton or is a singleton coloured by $\bar{c}$. In each case there is $x \in p$ coloured by $\bar{c}$. Since π_1 is a proper refinement of π_2, there are $p_1 \in \pi_1$ and $p_2 \in \pi_2$ such that $p_1 \subset p_2$. If p_1 has a member y coloured $\bar{c}$, there is an isomorphism θ from $(-\infty, x]$ to $(-\infty, y]$. Since π_1 and π_2 are invariant, $p_1 \cap (-\infty, y] = p_2 \cap (-\infty, y]$, so p_1 and p_2 agree 'on the left'. Since $p_1 \subset p_2$ and π_1 is restricted, there is $p_1' \neq p_1$ in π_1 having a member coloured $\bar{c}$ such that $p_1' \subseteq p_2$. The same proof shows that p_1' and p_2 also agree on the left, contrary to p_1 and p_1' disjoint.

If however p_1 has no member coloured $\bar{c}$, then it must be a singleton $\{b\}$ say. If π_1 has a member p_1' to the left of b with a point coloured $\bar{c}$, we may apply the same argument to p_1' instead. Otherwise b must be the least member of p_2, and since $p_1 \subset p_2$, there is $b_1 > b$ in p_2 coloured $\bar{c}$. Now map b_1 to x by a lower isomorphism, and this must take p_2 to p on the left, but it also takes p_2 on the left to at least two members of π_1, giving a contradiction. □

The tree I^R we actually require as an expanded coding tree for X is obtained from I^{RR} by adding some more points. This entails that *some* invariant partitions which are not restricted are allowed, but in the process, certain points need to be duplicated. For instance, in $\mathbb{Q}_2(\mathbb{Z}^2 + 1, \omega^*.\mathbb{Z})$ (see Fig. 3a), there are invariant partitions into copies of $\mathbb{Z}^2 + 1$ and $\omega^*.\mathbb{Z}$ (restricted), and into copies of $\mathbb{Z}^2$ and $\omega^*.\mathbb{Z}$ and red singletons (not restricted); since the copies of $\omega^*.\mathbb{Z}$ occur in both, we have to 'count' them twice. Notice that if p lies in the invariant partition π, then, by invariance, for any other $q \in \pi$ which has a colour in common with p, p has a least element under the induced ordering if and only if q does (and then they have the same colour), but it is possible for one to have a greatest element but not the other; this situation precisely corresponds to the introduction of the $-$ label.

Definition 3.7 If π is a restricted invariant partition, then it is a $\{b\}$-partition if for some $p \in \pi$, $\max p$ exists but $\min p$ does not. The **refining invariant tree** I^R, still partially ordered by $\subset$ (except that some vertices may be duplicated), is obtained from I^{RR} by adding as new sets all $\{\max p\}$ and $p \setminus \{\max p\}$ for $p \in \pi$ where π is a $\{b\}$-partition (and if $\max q$ is not defined for $q \in \pi$, then we include q twice, once in π and once for the new partition).

This tells us what the vertices and levels are of I^R are, and we can now prove the following basic result.

Theorem 3.8 *The refining invariant tree for X is a complete nearly levelled tree.*

Proof First we see that I^{RR} is a tree. For let $x, y, z \in I^{RR}$ with $x \subset y, z$. Thus $y \cap z \neq \varnothing$. Let $y \in \pi_1$, $z \in \pi_2$, where π_i are restricted invariant partitions. Since by Lemma 3.5 one of these refines the other, and $y \cap z \supseteq x$, $y \subseteq z$ or $z \subseteq y$. So $\{t \in I^{RR} : x \subseteq t\}$ is linearly ordered as required. To see that it stays a tree when we add in the additional vertices as in Definition 3.7, let $x \subset y, z$ lie in I^R. We may suppose that x is a singleton, so lies in I^{RR}. If $y, z \in I^{RR}$, then they are comparable as already shown. If $y, z \notin I^{RR}$, then $y = p \setminus \{\max p\}$, $z = q \setminus \{\max q\}$ for some $p, q \in I^{RR}$ whose maxima exist. Suppose that $p \subseteq q$. Then we see that $y \subseteq z$ by treating the two cases $\max p = \max q$ and $\max p \neq \max q$. Otherwise assume that $y \in I^R \setminus I^{RR}$ but $z \in I^{RR}$. Hence $y = p \setminus \{\max p\}$ for $p \in I^{RR}$ for which $\max p$ exists. Now $p \subseteq z$ or $z \subset p$. In the former case, $y \subset z$, and in the latter, if $z \not\subseteq y$ then $\max p \in z$ giving $p = z$, a contradiction.

The fact that I^R is nearly levelled follows from Lemma 3.6, noting that in passing from I^{RR} to I^R, we have taken care to retain levels, apart from leaves not coloured by $\bar{c}$. The root is X itself.

The fact that I^R is complete is verified as in [5]. The key point is that the infimum of a descending sequence of restricted invariant partitions is also a restricted invariant partition, and this is not altered by the addition of extra levels in passing from I^{RR} to I^R. □

We wish to use I^R to 'encode' $(X, \leq, F)$, and for this purpose, we have to assign labels so that it contains information about how $(X, \leq, F)$ can be recovered, and also linearly order the children of each parent vertex. The ordering is immediate from the fact that the set of children of each parent vertex is a subset of a level, which is a partition of X into convex subsets, so receives an induced linear ordering. The assignment of labels is motivated by the examples given in Figs. 2 and 3. A **descendant** of x is any $y \leq x$.

First consider the 'exceptional' duplicated points (written q in the definition). Each such appears in two levels of I^R, and they are labelled $-$ at their occurrence in the higher level. The other labels are assigned as follows:

Any leaf x is coloured by the singleton $\{F(x)\}$.

If p is not a leaf, then

if $\max p$ and $\min p$ both exist, p is labelled $\vee$, and $p \setminus \{\min p\}$ is labelled $\wedge$,
if $\max p$ exists but not $\min p$, then p is labelled $\wedge$, and
if $\min p$ exists but not $\max p$, then p is labelled $\vee$,
and if x is a parent vertex not of these forms then x is labelled by the (coloured) order type of its children, where two children have the same 'colour' if their sets of descendants are isomorphic,
if x has descendants but no children then x is labelled lim.

3.2 Coding Trees

Now that we have shown how labelled trees arise naturally in analyzing the structure of a branch-coloured chain, we can now start 'from the other end', and define more formally what sort of trees we actually require. We first introduce some more definitions which are needed.

Definition 3.9 The **left forest** of a vertex x labelled $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}_n$, ω^*, $\dot{\mathbb{Q}}$, $\dot{\mathbb{Q}}_n$, $\vee$ is the forest (disjoint union of trees) consisting of the descendants of all children of x except for the rightmost (when it exists). The left forests of vertices labelled $\wedge$ or $-$ are empty.

An **ordered forest** is a forest together with a linear ordering $\lhd$ of its maximal elements; two ordered forests are **lower isomorphic** if for any maximal element z of one there is a maximal element t of the other such that there is a $\lhd$-preserving isomorphism $(-\infty, z] \to (-\infty, t]$ induced by an isomorphism of the forests below these points.

Definition 3.10 The **middle forest** of a vertex labelled $\vee$, $\wedge$, $-$ is the forest (tree actually) consisting of the descendants of its right, left, only child, respectively. The middle forests of other vertices are empty.

Note that the middle forest is not usually in the 'middle'. Even though it can appear on the left it must be distinguished from the left forest. The encodings of middle forests do often occur between two coloured points in the resulting linear order.

Definition 3.11 A **right descendant** y of a vertex x is a descendant of x such that for any consecutive z, t such that $y \le z < t \le x$, z is a right child of t (which in particular means that t can only be labelled by ω^*, $\dot{\mathbb{Q}}$, $\dot{\mathbb{Q}}_n$, $\vee$, or $\wedge$).

Definition 3.12 A **colour coding tree** is a sextuple $(T, \leqslant, \lhd, \varsigma, F, \ll)$ such that:

1. $(T, \leqslant)$ is a nearly levelled tree with a greatest element (its root), $\lhd$ partially orders the children of each parent, and $\ll$ linearly orders the levels,
2. T is complete,
3. every vertex is a leaf or is above a leaf, the leaves are labelled by F with singleton labels, (so this is a 'colouring' function), and we extend F to all vertices x by letting $F(x) = \bigcup\{F(y) : y \text{ a leaf below } x\}$,
4. if x is not a leaf, then $\bar{c} \in F(x)$,
5. the non-leaf vertices are labelled by ς, taking values in $\{\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$ $(2 \leqslant n \leqslant \aleph_0)$, $\vee$, $\wedge$, $-$, $\lim\}$,
6. • if $\varsigma(x) = \mathbb{Z}$, $\mathbb{Q}$, or $-$ then x has one cone, having a greatest member (child),
   • if $\varsigma(x) = \omega^*$, $\dot{\mathbb{Q}}$, $\vee$ or $\wedge$ then x has two cones, each having greatest elements (children), linearly ordered by $\lhd$,
   • if $\varsigma(x) = \mathbb{Q}_n$ then x has n cones, each having greatest elements (children), indexed by n, which are pairwise $\lhd$-incomparable,

- if $\varsigma(x) = \dot{\mathbb{Q}}_n$ then x has $n+1$ cones, each having greatest elements (children), one of which is $\lhd$-greatest, and the others are indexed by n and pairwise $\lhd$-incomparable,
- if $\varsigma(x) = \lim$ then x has just one cone, no children, and is not a leaf,

7. if non-leaf vertices x and y are in the same level, then $\varsigma(x) = \varsigma(y)$, or one is $\wedge$ and the other $-$, or $\varsigma(x), \varsigma(y)$ are lower isomorphic coloured linear orders such that for each i, if z and t are the ith left children of x and y, respectively, under the indexing, then $F(z) = F(t)$,
8. the left child of any vertex labelled $\vee$ and the right child of any vertex labelled $\wedge$ are leaves not coloured $\bar{c}$, but their other children are not leaves,
9. any two leaves having the same colour are in levels that are at most one apart; if they are one apart, then the vertex on the lower level is the left child of a vertex labelled $\vee$,
10. at each given level of T the left forests from non-leaf vertices at that level are isomorphic; in addition if x and y are on the same level and $\varsigma(x) = \varsigma(y) = \wedge$ and $F(x) = F(y)$, then the middle forests are also isomorphic,
11. the root is not labelled $\wedge$, $\vee$, or $-$; the children of a vertex labelled $\wedge$ or $-$ are not labelled $\wedge$, $\vee$, or $-$; the children of a vertex labelled $\vee$ are not labelled $\vee$; if $\varsigma(x) = -$ then, for some vertex y level with x, $\varsigma(y) = \wedge$; in addition if $\varsigma(x) = \vee$ then the parent of x (if it exists) is labelled $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$,
12. if $\varsigma(x) = \wedge$ and y is the left child of x, then y has no right descendant which is a leaf,
13. if $\varsigma(x) = \omega^*$, $\dot{\mathbb{Q}}$ or $\dot{\mathbb{Q}}_n$ and y' is the right child of x, then $\bar{c} \in F(y')$, and there is a left child y of x such that $\bar{c} \in F(y)$,
14. there are countably many leaves.

Note that we usually write $F(x) = c$ rather than $F(x) = \{c\}$ in cases where this is a singleton. As explained earlier, to define what it means for a coding tree to encode a coloured linear order, we require the intermediate notion of 'expanded coding tree'. Five examples of colour coding trees were given in Figs. 2 and 3. Figure 3b illustrates clause 9 of the definition.

Definition 3.13 An **expanded coding tree** is a sextuple $(E, \leqslant, \lhd, \varsigma, F, \ll)$ such that:

1. $(E, \leqslant)$ is a nearly levelled tree with a greatest element (the root), $\lhd$ linearly orders the children of each parent, and $\ll$ linearly orders the levels,
2. E is complete,
3. every vertex is a leaf or above a leaf, the leaves are labelled by F with singleton labels, and F is extended to all vertices x by letting $F(x) = \bigcup\{F(y) : y \text{ a leaf below } x\}$,
4. if x is not a leaf, then $\bar{c} \in F(x)$,
5. the non-leaf vertices are labelled by ς, where $\varsigma(x)$ lies in $\{\mathbb{Z}, \omega^*, \mathbb{Q}, \dot{\mathbb{Q}}, \mathbb{Q}_n, \dot{\mathbb{Q}}_n (2 \leqslant n \leqslant \aleph_0), \vee, \wedge, -, \lim\}$,
6. for any parent vertex x of the tree, its cones all have greatest elements (children) which are indexed by the members of $\varsigma(x)$ so that if $\varsigma(x)$ is a coloured linear ordering, $\lhd$ corresponds to the ordering of $\varsigma(x)$, and one of the following holds:

- $\varsigma(x) = \mathbb{Z}, \mathbb{Q}, -$, and the trees below x are all isomorphic,
- $\varsigma(x) = \omega^*, \dot{\mathbb{Q}}$ and the left trees below x are all isomorphic,
- $\varsigma(x) = \mathbb{Q}_n$ and the trees below any two children of x having the same colour are isomorphic,
- $\varsigma(x) = \dot{\mathbb{Q}}_n$ and the trees below any two left children of x having the same colour are isomorphic,
- $\varsigma(x) = \vee$ or $\wedge$ and x has just 2 children,

7. if non-leaf vertices x and y are in the same level, then $\varsigma(x) = \varsigma(y)$, or one is $\wedge$ and the other $-$, or $\varsigma(x)$, $\varsigma(y)$ are lower isomorphic coloured linear orders,
8. the left child of any vertex labelled $\vee$ and the right child of any vertex labelled $\wedge$ are leaves not coloured $\bar{c}$, but their other children are not leaves,
9. any two leaves having the same colour are in levels that are at most one apart; if they are one apart then the vertex on the lower level is the left child of a vertex labelled $\vee$,
10. at each given level of E the ordered left forests from non-leaf vertices at that level are lower isomorphic; in addition if x and y are on the same level and $\varsigma(x) = \varsigma(y) = \wedge$ and $F(x) = F(y)$, then the middle forests are also isomorphic,
11. the root is not labelled $\wedge$, $\vee$, or $-$, the children of a vertex labelled $\wedge$ or $-$ are not labelled $\vee$, $\wedge$, or $-$; the children of a vertex labelled $\vee$ are not labelled $\vee$; if $\varsigma(x) = -$ then for some vertex y level with x, $\varsigma(y) = \wedge$; if $\varsigma(x) = \vee$ then the parent of x (if it exists) is labelled $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$,
12. if $\varsigma(x) = \wedge$ and y is the left child of x, then y has no right descendant which is a leaf,
13. if $\varsigma(x) = \omega^*$, $\dot{\mathbb{Q}}$ or $\dot{\mathbb{Q}}_n$ and y' is the right child of x, then $\bar{c} \in F(y')$, and there is a left child y of x such that $\bar{c} \in F(y)$,
14. there are countably many leaves.

Theorem 3.14 *The labelled refining invariant tree I^R for X is an expanded coding tree.*

Proof It was shown in Theorem 3.8 that I^R is a complete nearly levelled tree. We verify the remaining properties.

3. Every vertex has a leaf below it, as the trivial partition into singletons is clearly invariant (and restricted). The rest of this clause follows from the definitions.
4. By the density of the points coloured $\bar{c}$, if x is not a leaf, then $\bar{c} \in F(x)$.
5. follows from the way the labels were assigned.

Now we check that the labels have been correctly assigned. First consider parent vertices p, with child q. If p is labelled $\vee$, then $\min p$ exists, and clauses 6, 8, and 11 are satisfied (except that we still must verify the last part of 11). If p is labelled $\wedge$, then $\max p$ exists, and the same argument applies. So now suppose that p is not labelled $\vee$ or $\wedge$, in which case for clause 6 we must show that the label is given by the 'coloured' order type of its children, which is one of $\mathbb{Z}, \omega^*, \mathbb{Q}, \dot{\mathbb{Q}}, \mathbb{Q}_n, \dot{\mathbb{Q}}_n$ (for $2 \leqslant n \leqslant \aleph_0$).

Then p (or $p \cup \{b\}$ where $b > p$, if $p \notin I^{RR}$) lies in a restricted invariant partition π, whose members are all parent vertices, by Lemma 3.1, and a child q of p lies in an invariant partition π'. It follows from lower 1-transitivity that all members of π' contained in p are children of p. Let F' be the colouring of π' given by Lemma 3.2. Let $\sim$ be defined on π' by $x \sim y$ if x and y are contained in the same member of π, and there are just finitely many points between x and y, and the F' values of any two points between them have non-empty intersection. As in the proof of Lemma 3.2 this is an equivalence relation, which clearly has convex classes and is invariant. By Lemma 3.1 the classes are all themselves lower 1-transitive and lower isomorphic, and if non-trivial can clearly only be isomorphic to $\mathbb{Z}$ or ω^* (the only other option is that they are distinctly coloured singletons, in which case p is labelled $\vee$ or $\wedge$, already covered). In this case we get the corresponding label for p.

In other cases, the parts of π' must be dense within p. We shall show that then p is a $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$ combination of its set Z of children. If all the left children are isomorphic, then $Z = \mathbb{Q}$, or $\dot{\mathbb{Q}}$ if the right child exists. If not all the left children are isomorphic then we shall show that $Z = \mathbb{Q}_n$ (or $\dot{\mathbb{Q}}_n$ if p has a right child) where the set Γ of (colour, order-)isomorphism types of the left children of p is of size n (which may be $\aleph_0$). Suppose, for a contradiction, that p is not the $\mathbb{Q}_n$ mixture of its children, in which case the members of Γ occur densely in p but there are two of them such that not all other members of Γ occur between them. Let γ be a member of Γ which does not occur between all pairs, and let us define $\sim'$ on π' by $y \sim' z$ if $y = z$, or if no point of $[y, z]$ (or $[z, y]$ if $z < y$) has isomorphism type γ. This provides a restricted invariant partition of X into convex pieces refining π, and is a proper refinement not equal to π', contrary to π and π' being on consecutive levels.

7. Suppose that x and y are non-leaf level vertices. By the definition of I^R from I^{RR}, x is labelled $\vee$ if and only if y is, and x is labelled $\wedge$ or $-$ if and only if y is. Otherwise, by Lemma 3.1, and since they are not leaves, x and y are lower isomorphic linear orders, and it follows from this that $\varsigma(x)$ lies in $\{\mathbb{Z}, \omega^*, \mathbb{Q}, \dot{\mathbb{Q}}, \mathbb{Q}_n, \dot{\mathbb{Q}}_n\}$ if and only if $\varsigma(y)$ does, and then $\varsigma(x)$ and $\varsigma(y)$ are lower isomorphic. If none of the above apply, then $\varsigma(x)$ and $\varsigma(y)$ must both equal lim.
8. follows from the definition at the stage when $\vee$ and $\wedge$ are assigned as labels.
9. Let x and y be leaves coloured $c \neq \bar{c}$, and let π_1, π_2 be the suprema of the sets of restricted invariant partitions containing $\{x\}$, $\{y\}$, respectively. By Lemma 3.5 we may suppose that π_2 refines π_1. Let p be the member of π_1 containing y. Since x and y have the same colour, there is an isomorphism $\theta : (-\infty, x] \to (-\infty, y]$, and since π_1 is invariant, y is the least member of p. If $p = \{y\}$, then $\pi_1 = \pi_2$ and x and y are on the same level. Otherwise, by definition, p is labelled $\vee$, and y is the left child of p on the next level down, given by π_2.
10. Let x and y be non-leaf level vertices. First suppose that $\varsigma(x)$ and $\varsigma(y)$ are lower isomorphic linear orders. Then x and y lie in an invariant partition π (the level in question), so by Lemma 3.1 they are lower isomorphic, and this isomorphism induces a lower isomorphism between the ordered left forests of x and y. If $\varsigma(x)$ and $\varsigma(y)$ are not coloured linear orders, but are equal, then we may similarly appeal to Lemma 3.2. Otherwise, one is $\wedge$ and the other is $-$ in which case

the left forests are empty by definition, so are vacuously isomorphic. For the middle forests the only case requiring verification is this final case, but then by construction, the children of the $\wedge$ and $-$ vertices are lower isomorphic.

11. Most clauses here are immediate from the definition. Note that the fact that the root is not labelled by $\wedge$, $\vee$, or $-$ follows since X has no greatest or least element. We concentrate on the final statement. Consider a vertex x labelled $\vee$, having a parent y. We just have to rule out the possibilities that y is labelled $\mathbb{Z}$ or ω^*. From the definition, we see that the invariant partition π that x lies in is restricted, and $\min x$ exists. Hence every member of π has a minimum, and if $\varsigma(y) = \mathbb{Z}$ or ω^*, then members of π (except the greatest, if it exists) are covered in π by a set having a least member not coloured by $\bar{c}$, contrary to π restricted.
12. If y has a right descendant which is a leaf, then X has consecutive elements corresponding to it, and the right child of x. Since X is a branch-coloured chain, the right child of x is coloured $\bar{c}$, contrary to clause 8.
13. Let $\varsigma(x) = \omega^*$, $\dot{\mathbb{Q}}$ or $\dot{\mathbb{Q}}_n$ and let y, y' be left and right children of x (so that y' is uniquely determined, but not y). We show that $\bar{c} \in F(y')$ and y may be chosen so that $\bar{c} \in F(y)$. The result for $F(y)$ follows from the density of the points coloured $\bar{c}$ (allowing suitable choice of y—this is only necessary in the $\dot{\mathbb{Q}}_n$ case, for the others, any y will serve).

    Suppose, for a contradiction, that $\bar{c} \notin F(y')$. Then y' is a leaf not coloured $\bar{c}$, in other words, it is a singleton. Let π be the invariant partition containing x. Since $y' = \max x$, π is restricted, and by definition of the labelling, x is labelled $\vee$ or $\wedge$, contrary to assumption.
14. Since X is countable, I^R has countably many leaves.

□

3.3 Decoding a Coding Tree

The above material leads on naturally to the definition of 'encodes'.

Definition 3.15 Let $(T, \leqslant, \lhd, \varsigma, F, \ll)$ be a coding tree, and $(E, \leqslant, \lhd, \varsigma, F, \ll)$ be an expanded coding tree. We say that E is **associated** with T if there is a function ϕ from E to T which takes the root r of E to the root of T, each leaf of E to some leaf of T, and

(i) $t_1 \leqslant t_2 \implies \phi(t_1) \leqslant \phi(t_2)$,
(ii) T and E have order isomorphic sets of levels and ϕ preserves this correspondence,
(iii) for each vertex t of E, ϕ maps $\{u \in E : u \leqslant t\}$ onto $\{u \in T : u \leqslant \phi(t)\}$, and for any leaf l of E, ϕ maps $[l, r]$ onto $[\phi(l), \phi(r)]$,
(iv) $\varsigma(\phi(t)) = \varsigma(t)$ for non-leaves, and $F(\phi(t)) = F(t)$ for leaves.

Definition 3.16 The coding tree $(T, \leqslant, \lhd, F, \varsigma, \ll)$ **encodes** the (coloured) linear order $(X, \leqslant)$ if there is an expanded coding tree $(E, \leqslant, \lhd, F, \varsigma, \ll)$ associated with T such that X is (colour and order-) isomorphic to the set of leaves of E under the (obvious) branch order.

It is clear from Theorem 3.14 that the set of leaves of $E = I^R$ is (order- and colour-)isomorphic to X, so according to this definition, if E is associated with the coding tree T, then T encodes X. What remains is to show that any coding tree encodes some (countable, lower 1-transitive) coloured linear order by showing how to find an associated expanded coding tree; that any two such encoded orders are isomorphic; and that an expanded coding tree arises from some coding tree with which it is associated. This is accomplished in what follows.

Given a coding tree, we build an expanded coding tree that it is associated with by using a rich enough family of functions on branches, where here by *branch* we understand a maximal chain containing a leaf (so in fact a branch is just the set of all points above some given leaf). For each of the labels $\mathbb{Z}$, $\mathbb{Q}$, ω^*, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$ we suppose that 'default values' in it have been chosen. For $\mathbb{Z}$ and $\mathbb{Q}$ just one default value is chosen, for ω^* and $\dot{\mathbb{Q}}$ two, the greatest element and one other, for $\mathbb{Q}_n$ there will be n default values, one of each colour, and for $\dot{\mathbb{Q}}_n$ there are $n+1$ default values, the greatest element, and one for each colour. The idea is that by insisting that, except at finitely many places, the default values are taken, the overall cardinality remains countable. Note that there is exactly one default value for each child of a vertex with that label in the coding tree.

Definition 3.17 A **decoding function** is a function, f, defined on a branch B of T, which contains a leaf, such that for each $x \in B$:

- if $\varsigma(x) = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{Q}_n$, then $f(x) \in \varsigma(x)$,
- if $\varsigma(x) = \omega^*, \dot{\mathbb{Q}}$, or $\dot{\mathbb{Q}}_n$, then $f(x) \in \varsigma(x)$ and it is the greatest member of $\varsigma(x)$ if and only if B passes through the right child of x,
- if $\varsigma(x) = \mathbb{Q}_n$ or $\dot{\mathbb{Q}}_n$ and $f(x)$ is not its greatest member, then its colour equals that of the child of x in B,
- if $\varsigma(x) = \vee, \wedge$, or $-$, then $f(x)$ is the member of $\varsigma(x)$ (viewed as a linear order) corresponding to which child of x lies in B,
- the set of non-default values taken by f is finite.

We note that if x is a leaf, or is labelled by *lim*, the value of f is unimportant and we consider it to be undefined.

Theorem 3.18 *Every colour coding tree T encodes a countable (coloured) linear order.*

Proof The linear order encoded by T is taken to be the set Σ_T of decoding functions on T ordered by first difference (from top down). Let us spell out precisely what this means. If f_1, f_2 are decoding functions with domains B_1 and B_2 respectively, then we let $f_1 < f_2$ if for some parent vertex $x \in B_1 \cap B_2, f_1(y) = f_2(y)$ for all $y > x$, and $f_1(x) < f_2(x)$. We observe that from this definition it follows that if for decoding functions f_1 and $f_2, f_1(x) = f_2(x)$, then the same child of x lies in the domains of both f_1 and f_2. If $f \in \Sigma_T$, then f is coloured by taking for $F(f)$ the F-colour of the leaf in its domain. If $B = \operatorname{dom} f$, we note that $(\forall x \in B)(F(f) \in F(x))$.

Let us remark that the definition of the ordering makes sense. Suppose that $f_1 \neq f_2$ have domains B_1 and B_2. By completeness of T, $B_1 \cap B_2$ has a greatest lower bound x say in T. In fact x must be the least member of $B_1 \cap B_2$. For if $x \notin B_1$, and x' is the member of B_1 on the same level as x, then the least upper bound y of x and x' satisfies $y > x$, so there is z in $B_1 \cap B_2$ such that $y > z > x$, contrary to $x', z \in B_1$. Hence $x \in B_1$ and similarly $x \in B_2$.

Suppose that f_1 and f_2 agree on $B_1 \cap B_2$. Then as $f_1 \neq f_2$, x is not a leaf so there are incomparable $x_1 \in B_1$ and $x_2 \in B_2$, whose least upper bound must be x. Hence x ramifies downwards, so is labelled by ω^*, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$, $\vee$ or $\wedge$. Since $f_1(x) = f_2(x)$, it follows as remarked above that the children of x in B_1 and B_2 are equal, contrary to minimality of x in $B_1 \cap B_2$. We deduce that f_1 and f_2 do not agree on $B_1 \cap B_2$. Since decoding functions differ from the default value only finitely often, there is therefore a greatest point at which they differ, which must be labelled $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$, and the definition decides the ordering of f_1, f_2 (the precise details depending on the value of $\varsigma(x)$).

In order to satisfy the definition of 'encodes' we must produce an expanded coding tree, which is given by

$$E = \{(t, f \restriction (t, r]) : f \in \Sigma_T, t \in B, B = \text{dom} f\}$$

with labelling of vertices given by the label in T of the first component. For leaves this means that the labelling is as just defined for Σ_T, and so provided we can show that E is an expanded coding tree associated with T, it will follow that Σ_T is the coloured linear order encoded by T. The lth level of E is the set of all $(t, f \restriction (t, r])$ such that t lies in the lth level of T.

Clauses 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 14 follow from the definition and the corresponding clauses for the coding tree, and the verification that E is a complete tree (clause 2) is as in [5]. Clause 6 is immediate from the definition.

The truth of clause 10 follows from the fact that any isomorphism between the trees below vertices t_1 and t_2 of T induces an isomorphism between the trees of E below $(t_1, f_1 \restriction (t, r])$ and $(t_2, f_2 \restriction (t, r])$ for any decoding functions f_1 and f_2.

The mapping ϕ is given by $\phi((t, f \restriction (t, r])) = t$. This preserves root, leaves, and labels. Also $t_1 \leqslant t_2 \implies \phi(t_1) \leqslant \phi(t_2)$ and clearly for each vertex t of E, ϕ maps $\{u \in E : u \leqslant t\}$ onto $\{u \in T : u \leqslant \phi(t)\}$ and for any leaf l of E, ϕ maps $[l, r]$ onto $[\phi(l), \phi(r)]$. Therefore E is associated with T and Σ_T is order isomorphic to the set of leaves of E. Hence T encodes Σ_T. □

Theorem 3.19 *Any two countable coloured linear orderings encoded by the same colour coding tree T are isomorphic.*

Proof Suppose that X_1 and X_2 are countable coloured orders both encoded by T. Then they may be viewed as the sets of leaves of expanded coding trees E_1 and E_2 where there are association functions ϕ_1, ϕ_2 from E_1, E_2, respectively, to T. We argue by back-and-forth.

Let P be the family of all (level and label-preserving) isomorphisms from a finite subset of E_1 to E_2 such that

(i) the root of E_1 lies in the domain of p, and p takes it to the root of E_2,
(ii) the domain and range of p contain all their ramification points,
(iii) if $t \in \mathrm{dom}\, p$, then $\phi_1(t) = \phi_2(p(t))$,
(iv) all points of the domain or range of p are the root, or leaves, or parent vertices,
(v) if $t \in \mathrm{dom}\, p$ is a ramification point, then there is an isomorphism of the set of children of t in E_1 to the set of children of $p(t)$ in E_2 such that if $u < t$ is in dom p, then the isomorphism takes the child of t above u to the child of $p(t)$ above $p(u)$.

We shall show that if $p \in P$ and $t \in E_1$ is not labelled *lim*, then there is an extension q of p in P such that $t \in \mathrm{dom}\, q$. This is the 'forth' step, and the 'back' step similarly adds a point to the range. Since the sets of parent vertices and leaves are countable, it follows by back-and-forth that there is an isomorphism θ from the set of such points in E_1 to those in E_2. This extends to the desired full isomorphism by continuity at *lim* points.

Given our p and t, if $t \in \mathrm{dom}\, p$, then we let $q = p$. Otherwise since the root r lies in dom p, there is at least one vertex v of dom p above t, so there is a least such.

Case 1: There is also a vertex of dom p below t. Let u be the greatest such. Then, since $p(u)$, $p(v)$ exist, and the association maps are level-preserving, there is a unique point t' such that $p(u) < t' < p(v)$ and $\phi_1(t) = \phi_2(t')$. Let $q = p \cup \{(t, t')\}$.
Case 2: v is minimal in dom p. We know that $\phi_1(v) = \phi_2(p(v))$ and $\phi_1(t) < \phi_1(v)$. Also ϕ_2 maps $\{u \in E_2 : u \leq p(v)\}$ onto $\{u \in T : u \leq \phi_2(p(v))\}$. Hence there is $u \leq p(v)$ such that $\phi_1(t) = \phi_2(u)$ and we let $q = p \cup \{(t, u)\}$.
Case 3: There is no vertex of dom p below t, v is not minimal in dom p, $u \in \mathrm{dom}\, p$, $u < v$ say, and the least upper bound of u and t is v. We may suppose that t is a child of v, since we may repeat this argument to obtain an extension to the descendants of v. Clause (v) tells us which child of $p(v)$ t should be mapped to under q.
Case 4: There is no vertex of dom p below t, v is not minimal in dom p, $u \in \mathrm{dom}\, p$, $u < v$, but the least upper bound w of u and t is not equal to v.

We can extend p to p' in P so that $w \in \mathrm{dom}\, p'$ using Case 1, and this now reduces to Case 3. □

Theorem 3.20 *A coloured ordering* $(X, \leqslant, F)$ *encoded by the colour coding tree* T *is countable and lower* 1*-transitive.*

Proof Countability is immediate. For lower 1-transitivity, let $a, b \in X$ be such that $F(a) = F(b)$ and consider the initial segments $X_a = (-\infty, a]$ and $X_b = (-\infty, b]$ which we have to show are isomorphic.

By Theorem 3.19, we may suppose that X is the specific coloured chain arising from the construction using decoding functions, so that it is defined to be the set of all functions on the branches of T which take a default value at all but finitely many points, so a and b are now viewed as functions on branches B_1, B_2 of T, having leaves x_a, x_b, respectively, and $F(a) = F(x_a)$, $F(b) = F(x_b)$. If x_a and x_b are in the same level, then the levels occurring in B_1 and B_2 are the same. Otherwise, by clauses 8 and 9, $F(x_a) = F(x_b) \neq \bar{c}$, x_a and x_b are in levels that are at most one apart, and the one on the lower level, x_b say, is the left child of a vertex labelled $\vee$. So the levels of B_2 are the same as for B_1 except for the parent of x_b.

If $x \in B_1$, let $\Gamma_x^a = \{f \in (-\infty, a] : f(x) < a(x) \wedge (\forall z > x) f(z) = a(z)\}$ and similarly for Γ_y^b, where $y \in B_2$. If i is a level, and x, y are the elements of B_1, B_2 respectively in that level, we may also write these as Γ_i^a, Γ_i^b. By the definition of the ordering, it is clear that $(-\infty, a]$ is the disjoint union of $\{a\}$ and all the Γ_i^a, and furthermore that $i < j \Rightarrow \Gamma_i^a > \Gamma_j^a$ (where this means that every element of Γ_i^a is greater than every element of Γ_j^a). Since the same is true of the Γ_i^b, to show that $(-\infty, a] \cong (-\infty, b]$ it therefore suffices to show that $\Gamma_i^a \cong \Gamma_i^b$ for each i, and the desired isomorphism from $(-\infty, a]$ to $(-\infty, b]$ is obtained by patching together all the individual isomorphisms. This still works in the exceptional case in which x_a and x_b are in levels that are one apart, since the 'extra' Γ_i^bs are actually empty since x_b is the *left* child of its parent.

Let $x \in B_1$ lie in level i, and let y be the element of B_2 in the same level, where neither of these are leaves. It follows by property 7 that if $\varsigma(x)$ or $\varsigma(y)$ is not a coloured linear order, then $\varsigma(x) = \wedge$ and $\varsigma(y) = -$ (or the other way round), or $\varsigma(x) = \varsigma(y)$.

Case 1: $\varsigma(x) = \lim$.
As just remarked, $\varsigma(x) = \varsigma(y) = \lim$, which gives $\Gamma_i^a = \Gamma_i^b = \varnothing$.

Case 2: $\varsigma(x) = \vee, -$ or $\wedge$. By the above remark again, $\varsigma(x) = \varsigma(y)$, or else one is $\wedge$ and the other $-$. First consider $\vee$. It clearly suffices to observe that B_1 and B_2 either both contain the left child of x, y, respectively, or both contain the right child. For if B_1 contains the left child z of x, which by property 8 is a leaf, labelled 'red' say, then properties 9 and 11 imply that no descendant of the right children of x or y is coloured red, so as $F(a) = F(b)$, it follows that B_2 contains the left child of y.

If $\varsigma(x) = \varsigma(y) = -$, then $\Gamma_i^a = \Gamma_i^b = \varnothing$.

Next suppose that $\varsigma(x) = \varsigma(y) = \wedge$. A similar argument applies as for $\vee$, except that this time in the non-empty case we invoke the existence of an isomorphism between the middle forests of x and y (see clause 10).

It remains to consider the case where $\varsigma(x) = \wedge$ and $\varsigma(y) = -$. Let u and v be the left and right children of x, respectively. Then v is a singleton labelled 'blue' say. Properties 9 and 11 ensure that no descendant of y is labelled blue, and it follows that u lies in B_1, and consequently, $\Gamma_i^a = \Gamma_i^b = \varnothing$ once more.

Case 3: $\varsigma(x)$ and $\varsigma(y)$ are both coloured linear orders, and hence they are lower-isomorphic. By examining each of the possible cases, we see that in fact the *open* intervals $(-\infty, a(x))$ and $(-\infty, b(y))$ are order- and colour-isomorphic (of order-type ω^*, $\mathbb{Q}$, or $\mathbb{Q}_n$), and we choose an order-isomorphism $\varphi : (-\infty, a(x)) \to (-\infty, b(y))$. We now invoke the existence of an isomorphism between the left forests at x and y. Since this preserves labels, subtrees at children of x and y which correspond under φ are isomorphic, so φ may be extended to an isomorphism ψ

between these left forests. Writing x_j^a, x_j^b for the elements of B_1, B_2 in level j, we can now define an isomorphism Φ from Γ_i^a to Γ_i^b by letting

$$\Phi(f)(\psi(x_j)) = \begin{cases} b(x_j^b) & \text{if } j > i \\ \varphi(f(x_j)) & \text{if } j = i \\ f(x_j) & \text{if } j < i \end{cases}$$

where $f \in \Gamma_i^a$.

Finally we apply the methods from [5] to show that Γ_i^a is mapped 1–1 onto Γ_i^b by Φ and this gives the result. □

Theorem 3.21 *If X is the linear order encoded by the colour coding tree $(T, \leqslant)$, then the points coloured $\bar{c}$ are dense in X, and if $a < b$ are consecutive, then $F(b) = \bar{c}$.*

Proof As in the previous theorem we may suppose that $X = \Sigma_T$.

Let $a < b$ with the object of finding a point in between coloured $\bar{c}$. Let B_1 and B_2 be the branches on which a and b are defined. As above there is a point $x \in B_1 \cap B_2$ such that $a \restriction (x, r] = b \restriction (x, r]$ and $a(x) < b(x)$. We deduce that x is labelled by a non-trivial linear order, which must therefore be $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$, $\vee$, or $\wedge$. It follows that $\bar{c} \in F(x)$. To begin with, suppose that $a(x)$ and $b(x)$ are not consecutive members of $\varsigma(x)$. Then there is a leaf l below x labelled $\bar{c}$ and some $f \in \Sigma_T$ such that $a \restriction (x, r] = f \restriction (x, r] = b \restriction (x, r]$ and $a(x) < f(x) < b(x)$. Thus $a < f < b$ and $F(f) = \bar{c}$. For since $a(x)$ and $b(x)$ are not consecutive, $\varsigma(x) = \mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$. If x has just one child y, then $F(x) = F(y)$, so $\bar{c} \in F(y)$. Choose a branch B containing y whose leaf is coloured $\bar{c}$, and let f have domain B, and $a(x) < f(x) < b(x)$. Thus $F(f) = \bar{c}$. If $\varsigma(x) = \mathbb{Q}_n$, we may choose a child y of x with $\bar{c}$ in its label, and carry on as before. If $\varsigma(x) = \omega^*$, $\dot{\mathbb{Q}}$, or $\dot{\mathbb{Q}}_n$, we may similarly use clause 13 of Definition 3.12. So from now on we suppose that $a(x)$ and $b(x)$ are consecutive. This implies that $\varsigma(x) = \mathbb{Z}$, ω^*, $\vee$, or $\wedge$.

Next suppose that some $y \in B_2$ strictly below x is labelled $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$. Then $\bar{c} \in F(y)$ so there is a leaf l below y coloured $\bar{c}$, and we choose $f \in \Sigma_T$ such that $f \restriction (y, r] = b \restriction (y, r]$ and $f(y) < b(y)$. Then $a < f < b$ and $F(f) = \bar{c}$.

Otherwise, all labels of non-leaf vertices in B_2 strictly below x lie in $\{\vee, \wedge, -, \lim\}$. It follows from clause 11 that there are only finitely many such points, and since any vertex labelled lim clearly has infinitely many below it in any branch, only $\vee$, $\wedge$, $-$ can arise. By clause 7, the same applies to all vertices strictly below x.

(i) If $\varsigma(x) = \mathbb{Z}$, let y be the child of x in T. If y is a leaf, then it is coloured $\bar{c}$, so $F(a) = F(b) = \bar{c}$. Otherwise $\varsigma(y)$ exists and equals $\vee$, $\wedge$, or $-$. By clause 11, $\varsigma(y) \neq \vee$. If $\varsigma(y) = \wedge$, then y has two children u, v, which are left and right, and by clause 11, u and v are both leaves, but this contradicts clause 8.

If $\varsigma(y) = -$, then y has just one child which is a leaf labelled $\bar{c}$, so $F(a) = F(b) = \bar{c}$.

(ii) If $\varsigma(x) = \omega^*$, let y, y' be the left and right children of x in T. If $y \in B_1, B_2$ or $y' \in B_1, B_2$, then we argue as in the previous case. Otherwise $y \in B_1$ and

$y' \in B_2$. By clause 13, $\bar{c} \in F(y), F(y')$, so if y is a leaf, then $F(a) = \bar{c}$ and if y' is a leaf, then $F(b) = \bar{c}$. By clause 11, $\varsigma(y), \varsigma(y') \neq \vee$. If $\varsigma(y) = -$ then the child of y is labelled $\bar{c}$, so $F(a) = \bar{c}$, and similarly, if $\varsigma(y') = -$ then $F(b) = \bar{c}$. The only remaining case is where $\varsigma(y) = \varsigma(y') = \wedge$. Let u and v be the left and right children of y'. Then u and v are both leaves, contrary to clause 8.

(iii) If $\varsigma(x) = \vee$, let y, y' be the left and right children of x in T. Then $y \in B_1$ and $y' \in B_2$. By clause 8, y is a leaf, $F(y) \neq \bar{c}$, and y' is not a leaf, so it is labelled $-$ and has a unique leaf below it coloured $\bar{c}$. Thus $F(b) = \bar{c}$. If y' is labelled $\wedge$, then its children u and v are both leaves, contrary to clause 8.

(iv) If $\varsigma(x) = \wedge$, let y, y' be the left and right children of x in T, so $y \in B_1$ and $y' \in B_2$. By clause 11, y and y' are leaves, contrary to clause 8.

For the final statement, we know that a and b are consecutive. Hence $a(x)$ and $b(x)$ are consecutive and the above argument shows that $\varsigma(x) = \mathbb{Z}, \omega^*, \vee$, or $\wedge$, and all vertices strictly below x are labelled $\vee$, $\wedge$, or $-$. We do not need to consider the cases where we found a point strictly in between a and b, or where we already know that $F(b) = \bar{c}$, which cuts things down considerably. So, no cases under (i) or (iii) now arise. For (ii), where $\varsigma(x) = \omega^*$, and y, y' are the children, the only case to be considered has $y \in B_1$ and $y' \in B_2$, and $\varsigma(y') = \wedge$ with children u, v. As a and b are consecutive, $u \in B_2$, and hence as above, $F(b) = \bar{c}$. Finally (iv) cannot arise, since $\varsigma(x) = \wedge$ and y, y' are the children of x, they must both be leaves, which again contradicts clause 8. □

Now that we have shown how any colour coding tree encodes a coloured linear order of the right kind, we revert to our earlier discussion, where we showed how to find an expanded coding tree I^R corresponding to a given branch-coloured chain. It remained to show that we could find a coloured coding tree associated with I^R. In the same way that an expanded coding tree is formed by 'fattening' the given coding tree, the reverse process is done by 'collapsing' the given expanded coding tree. The idea is that we identify all the children of vertices labelled $\mathbb{Z}$ or $\mathbb{Q}$, all the left children of vertices labelled ω^* or $\dot{\mathbb{Q}}$, and all the children (left children) having the same colour in $\mathbb{Q}_n$ or $\dot{\mathbb{Q}}_n$. For this we begin by choosing for each vertex with one of these labels one of its left children of each isomorphism type and for each of its children x a fixed isomorphism of the tree below x to the tree below the chosen child for that isomorphism type (thus for $\mathbb{Z}$, ω^*, $\mathbb{Q}$ or $\dot{\mathbb{Q}}$ just one left child is chosen, and for $\mathbb{Q}_n$, $\dot{\mathbb{Q}}_n$ there are n). For each level l an equivalence relation $\simeq_l$ is given by identifying two vertices below that level if their images under the fixed isomorphism are equal, and vertices on higher levels are only equivalent to themselves. Then $x \simeq y$ if there is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that for each $i < n$, $x_i \simeq_{l_i} x_{i+1}$ for some level l_i. Intuitively, the trees below vertices with these labels are 'collapsed', thereby reversing the way in which a coding tree gives rise to an expanded coding tree.

Theorem 3.22 *The set of $\simeq$-classes on an expanded coding tree whose leaves are (isomorphic to) $(X, \leqslant)$ is a coloured coding tree for $(X, \leqslant)$.*

Proof The new labels and relations introduced in the colour classification, $\vee, \wedge, -$, do not have more than one left child. The children of vertices with these labels do not, therefore, need to be 'collapsed'. In addition we do not need to be concerned with middle children as each of the labels only has one of these. Hence the proof that the set of $\simeq$-classes on an expanded coding tree of $(X, \leqslant)$ is a colour coding tree for $(X, \leqslant)$ is essentially the same as that in [5]. Most properties of the colour coding tree follow immediately from the corresponding property of the expanded coding tree. ☐

3.4 Colour Lower Isomorphism Classes

We have shown that every colour coding tree represents a countable lower 1-transitive coloured linear order and that every countable lower 1-transitive branch-coloured chain can be represented by a colour coding tree. We next see how lower isomorphic coloured linear orders interact, and how this is exhibited in the coding tree.

Theorem 3.23 *Let $T(X), T(Y)$ be two colour coding trees, and $L : T(X) \to \Lambda$, $L' : T(Y) \to \Lambda'$ be functions from a vertex to its level in $T(X), T(Y)$. Suppose that $\phi : \Lambda \to \Lambda'$ is an order isomorphism, such that the following hold:*

- *if $x \in T(X)$, $y \in T(Y)$, $\phi(L(x)) = L'(y)$ then $\varsigma(x) = \varsigma(y)$ or one is $\wedge$ and the other is $-$, or they are lower isomorphic coloured linear orders,*
- *if $x \in T(X)$ and $y \in T(Y)$ are leaves having the same colour, $L'(y) = \phi(L(x))$ or $\phi(L(x))$ and $L(y)$ are at most one level apart, and if so, then the vertex on the lower level is the left child of a vertex labelled $\vee$,*
- *if $x \in T(X)$, $y \in T(Y)$ and $\phi(L(x)) = L'(y)$ then the left forests of x and y are isomorphic. In addition if $\phi(L(x)) = L'(y)$ and $\varsigma(x) = \varsigma(y) = \wedge$ and $F'(x) = F'(y)$ then the middle forests are also isomorphic.*

Then the coloured linear orders $(X, \leqslant)$, $(Y, \leqslant)$ encoded by $T(X), T(Y)$, respectively, are lower isomorphic.

This is verified using arguments from Theorem 3.20.

If a relation, $\sim$ corresponding to an invariant partition π is defined on one member, X, of a lower isomorphism class of branch-coloured chains, we can easily extend it to be defined on any other member, Y, of the class. If $v < w \in Y$, then there is $y \in X$ with $F(w) = F(y)$. There is therefore an isomorphism $\varphi : (-\infty, w] \to (-\infty, y]$, and we let $v \sim w$ if $\varphi(v) \sim \varphi(w)$. This is well defined since π is preserved under lower isomorphisms.

Hence if π is an invariant partition of a branch of a 1-transitive tree it may be viewed as an invariant partition of all the branches of the tree. It is therefore defined on the whole tree and is preserved by automorphisms of the tree.

Theorem 3.24 *Let $(X, \leqslant)$, $(Y, \leqslant)$ be lower isomorphic branch-coloured chains. Let $T(X)$, $T(Y)$ be the labelled refining invariant trees of X and Y, and Λ, Λ' the families of their levels. Then there is an order isomorphism $\phi : \Lambda \to \Lambda'$ such that the three conditions given in Theorem 3.23 hold.*

Proof The isomorphism between the levels is given by means of the correspondence between their invariant partitions just mentioned. The three conditions are verified by the same methods used in the proof of Theorem 3.14. □

Corollary 3.25 *Any lower isomorphism class of branch-coloured chains is countable.*

Proof This follows from the correspondence between the branch-coloured chains and their colour coding trees. By the theorem, in a lower isomorphism class, the set of levels of their colour coding trees is fixed. We consider the initial segment X determined by a $\bar{c}$ point, and its colour coding tree T. This has a rightmost branch B, and any other coding tree T' for a member X' of the class differs from T at some greatest point x, and here the labels must be ω^*, $\mathbb{Z}$, or $\dot{\mathbb{Q}}$, $\mathbb{Q}$, or $\dot{\mathbb{Q}}_n$, $\mathbb{Q}_n$, or $\wedge$, $-$, respectively. The last case is impossible, because by clause 8, the right child of a $\wedge$-labelled vertex is a leaf not coloured $\bar{c}$. In the first three cases, T' is uniquely determined by T and x, since by the third clause of Theorem 3.23, the left forests at x in T and T' are isomorphic, which implies that any other colour coding tree for a branch-coloured chain which is lower isomorphic to X and whose colour coding tree first differs from T at x must be isomorphic to T'.

Since there are only countably many possibilities for x, it follows that the lower isomorphism class is countable. □

4 Cones Types in 1-Transitive Trees

A key ingredient in the study of countable 1-transitive trees $(A, \leqslant)$ is an analysis of their possible ramification behaviour, and in this section we describe this in terms of the types of cones there can be at any vertex of A^+. We let $\mathfrak{R}$ be the set of orbits of points of A^+ under the action of Aut(A), view this as a set of colours, and let $F : A^+ \to \mathfrak{R}$ be the 'colouring' function which takes each point to the orbit containing it. Thus $F(x) = F(y)$ if and only if x and y are in the same orbit under Aut(A).

In Sect. 3 we considered partitions which are invariant under lower isomorphisms, and when building coding trees for colour lower 1-transitive linear orders we chose a rich enough family of such partitions, to fully describe the linear order in question. Each one of these partitions defines a level in the coding tree of the linear order, and we write $\sim_i$ for the partition corresponding to level i. We let the set of levels of the coding tree be $(I, \leqslant)$. As remarked earlier, this linear order need not be well-founded or conversely well-founded. We showed in Theorem 3.24 that all the branches of a 1-transitive tree have coding trees with order-isomorphic sets of levels, which we may therefore identify. Recall that $i < j$ means that $\sim_i$ refines $\sim_j$.

In [10] there were two types of cone at points of A^+ which needed to be distinguished, corresponding to 'special ramification points' (members of A^+ having a cone with a least element), and the rest. In the current context however we need in addition to take account of the possible levels at which these occur, and it turns out that this is also sufficient to distinguish the two types of cone just mentioned.

Definition 4.1 If $a \in A^+$, then we let $C_i(a)$ be set of all cones C at a such that for every branch B through C,

(i) $\{x \in B : a < x\}$ has a least $\sim_i$-class but no least $\sim_j$-class for any $j < i$, or
(ii) $\{x \in B : a < x\}$ has no least $\sim_j$-class for any j, and i is the least member of I such that some $\sim_i$-class of B contains a and also intersects $C \cap B$, or
(iii) $\{x \in B : a < x\}$ has no least $\sim_j$-class for any j, i has no successor in I, $C \cap B$ has no least $\sim_j$-class for any $j \leq i$, but for all $j > i$, every $\sim_j$-class containing a also intersects $C \cap B$.

We remark that by a 'branch through C' we mean a branch (maximal chain) of A^+ whose intersection with C is a maximal chain of C. It is easy to see that the conditions about the existence of least $\sim_i$-classes are independent of which branch through C we take, since any two of them meet strictly above a (this is the definition of 'cone'). The distinction between 'special' and 'normal' cones which was made in [9], generalizing the weakly 2-transitive situation, was that $C \in C_i(a)$ is special if (i) applies, and is otherwise normal.

To illustrate further the meaning of the three clauses in the definition, we remark that the first two apply provided that i has a successor, which has a discrete label (which can be $\mathbb{Z}$, ω^*, or $\vee$, though not $\wedge$) for clause (i), or a dense label ($\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$) for clause (ii), and clause (iii) applies if i has no successor. This is further explained in Lemma 4.3. If x and y lie in a colour coding tree T, then $x <_{left} y$ means that x is a descendant of a left child of y.

Lemma 4.2 *Any cone C of a 1-transitive tree A at $a \in A^+$ lies in $C_i(a)$ for some $i \in I$. Furthermore, i, and which of the three clauses applies, are uniquely determined from C.*

Proof Let B be a branch of C. If the set $P = \{j \in I : B$ has a least $\sim_j$-class which is strictly above $a\}$ is non-empty, then by the colour lower 1-transitivity of the branches and by the density of $\bar{c}$, P has at most two members. Call the least such i and note that $C \in C_i(a)$ according to clause (i), and in this case, i is clearly unique.

If $P = \varnothing$, let $P' = \{j \in I : B$ has a least $\sim_j$-class which also intersects $C \cap B\}$. Then $P' \neq \varnothing$ as one sees by considering the trivial invariant partition into just one set (B). By completeness of the coding tree of B, P' has an infimum, i say. The $\sim_i$-class containing a is then equal to the intersection of the family of $\sim_j$-classes containing a for $j \in P'$. If this set is not equal to $\{a\}$, then $i \in P'$, and $C \in C_i(a)$ by clause (ii).

If this set is equal to $\{a\}$, then i does not have a successor in I, but still $C \in C_i(a)$ by clause (iii). ◻

Lemma 4.3 *If $(A, \leqslant)$ is a countable 1-transitive tree and $a \in A^+$ is such that $F(a) = c_j$ then:*

- *$C_i(a) \neq \emptyset$ fulfils clause (i) in Definition 4.1 if and only if there is a branch B of A^+ through a, and there are $y \leq x' <' x$ in $T(B)$, y a leaf coloured c_j, $x' <_{left} x$, $L(x') = i$, $\neg(\exists z)(y <_{left} z < x)$, $\varsigma(x) = \mathbb{Z}$, ω^* or $\vee$, and either $y = x'$, or x' is labelled $\wedge$ or $-$, and $y <' x'$,*
- *$C_i(a) \neq \emptyset$ fulfils clause (ii) in Definition 4.1 $\iff$ i has a successor in I and there is a branch B of A^+ through a, and there are $y \leq x' <' x$ in $T(B)$, y a leaf coloured c_j, $x' <_{left} x$, $L(x') = i$, $\neg(\exists z)(y <_{left} z < x)$, and $\varsigma(x) = \mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$,*
- *$C_i(a) \neq \emptyset$ fulfils clause (iii) in Definition 4.1 $\iff$ i has no successor in I.*

Proof If $C \in C_i(a)$ is a cone fulfilling Definition 4.1(i), B a branch through C, then there is a least $\sim_i$-class y' in $\{t \in B : a < t\}$. Let x' be the representative in $T(B)$ of the $\sim_i$-class containing a, y the representative of $\{a\}$, and x be the least upper bound in $T(B)$ of x' and the representative of y'. Then since there are no points of B lying between the $\sim_i$-classes of a and y', x is labelled by $\mathbb{Z}$, ω^* or $\vee$. The minimality of i ensures that there is no z satisfying $y <_{left} z < x$, and from this last property we deduce that x' is not labelled $\mathbb{Z}$, ω^*, $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$. This leaves us with the remaining options stated.

If $C \in C_i(a)$ is a cone fulfilling Definition 4.1(ii), B a branch through C, then there is a least $\sim_i$-class containing a and points greater than it, and we let x be the representative of this in $T(B)$. By completeness, there is a greatest invariant partition in which a is not equivalent to points to its right, and so i is a successor level. If its label was $\mathbb{Z}$, ω^*, or $\vee$, then this would reduce to the previous clause, so we deduce that it must be $\mathbb{Q}$, $\dot{\mathbb{Q}}$, $\mathbb{Q}_n$, or $\dot{\mathbb{Q}}_n$. □

Before giving examples to illustrate the definition in our general setting, let us see how it works out for weakly 2-transitive trees. In the case where A^+ has pairs of consecutive elements, all the branches are (even colour-) isomorphic, of order-type $\mathbb{Q}_n(2)$, meaning that one of the colours consists of pairs coloured (red, $\bar{c}$) (and $1 \leq n \leq \aleph_0$). The coding tree for this coloured linear order is shown in Fig. 4a.

In this case, all coloured branches are isomorphic, and have the coding tree shown. There are just three invariant partitions, all restricted, of which the partition into $\{$red, $\bar{c}\}$-pairs and all other coloured singletons is the only non-trivial one. There are two types of cone at the red points, those with a least member, and those with no least member (this latter may or may not actually arise in the tree). The level for the former corresponds to the partition into singletons, and for the latter to the partition into just one set, so the distinction between 'special' and 'normal' is recast

Fig. 4 Colour coding trees for (**a**) $\mathbb{Q}_n(2)$ and (**b**) $\mathbb{Z}.(\text{red} + \mathbb{Z})$

here via the levels. If the $\bar{c}$ points do not ramify, we can say that they have just one cone, but that too corresponds to the partition into just one set. There are no cones corresponding to the non-trivial invariant partition. The full specification of the weakly 2-transitive tree is then given by the ramification orders at red points for the two types of cone and at points of all other colours. In our general description, this will be done by means of a sequence of cardinals indexed by levels, corresponding to each colour.

Now consider further examples in the more general setting. First suppose that the branches of A have order-type $\mathbb{Z}^2$. Then in the completion there are (red) points in between the copies of $\mathbb{Z}$, and we suppose that the tree ramifies at these points, one of which is a. The coding tree is shown in Fig. 4b. There are three non-trivial invariant partitions of any branch B. The first is obtained by coalescing all the copies of $\mathbb{Z}$ to single points, and the other two are given by in addition relating each copy of $\mathbb{Z}$ to the red point immediately above it, or to the red point immediately below it, respectively. The last is however not restricted, so does not count. The finer of the two that remain is clearly the first. In this case, any 1-transitive tree whose branches are of this coloured order-type will only have one type of cone at each red vertex, and one type at each $\bar{c}$ vertex. The cones at the red vertices a will lie in $C_k(a)$ where $\sim_k$ is the partition into copies of $\mathbb{Z}$ and singleton red points (which is finer than the partition which adjoins the red points to the $\mathbb{Z}$-block below it), and the cones at the $\bar{c}$ vertices have least elements also coloured $\bar{c}$, and correspond to the trivial partition into singletons. There is an easy generalization of this example to $\mathbb{Z}^n$, where there are $n-1$ options for different orbits of ramification point.

More interesting examples come about if we allow the branches not all to be isomorphic. For instance, consider a 1-transitive tree having branches in order-types $\omega^*.(\omega^*.(\mathbb{Z}+\text{red}))+\mathbb{Z}.(\mathbb{Z}+\text{red})$ and $\mathbb{Z}.(\omega^*.(\mathbb{Z}+\text{red}))$. The initial segments of both of these are isomorphic to $\omega^*.(\omega^*.(\mathbb{Z}+\text{red}))+\omega^*.(\mathbb{Z}+\text{red})+\omega^*$, so they are colour lower isomorphic. The final segment of such a coloured linear order determined by a red point may be (at least) $\omega.(\mathbb{Z}+\text{red})$ or $\mathbb{Z}.(\mathbb{Z}+\text{red})$, and so there will be cones given by at least two distinct levels in a 1-transitive tree having branches of these two coloured order types.

We are now in a position to define the 'cone type' of a point a, which we denote by $C.T.(a)$. This is the sequence indexed by I, and whose ith entry is $|C_i(a)|$. Recall that $(I, \leqslant)$, the set of the levels of a coding tree, always has a greatest member, r, its root, corresponding to the partition into just one piece.

Definition 4.4 If $a \in A^+$, then the **cone type** of a is the sequence:

$$(\bar{\alpha}) = (\alpha_i)_{i\in I} = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r)$$

where $\alpha_i = |C_i(a)|$.

Definition 4.4 is illustrated in Figs. 5 and 6. In Fig. 5 we show a 1-transitive tree whose branches are $\mathbb{Q}.(\text{red}+\mathbb{Z})$. The cone type of the red points is $(0, 2, 0, 0)$, and the cone type of the $\bar{c}$ points is $(1, 0, 0, 0)$. The dots indicate densely many copies of red $+\ \mathbb{Z}$, and the coding tree of the branches is given on the right. Note that this has

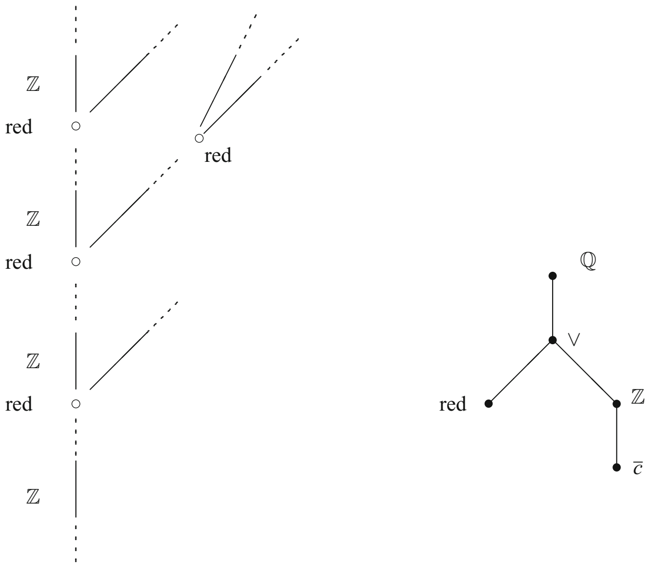

Fig. 5 A 1-transitive tree with $\mathbb{Q}.(\text{red} + \mathbb{Z})$ branches and their coding tree

height 4 and so the type sequences have length 4. The red points have ramification order 2, and both cones have a least $\sim_2$-class. The $\overline{c}$ points do not ramify, and are each covered by a point, that is, by a $\sim_1$-class.

In Fig. 6 we show a 1-transitive tree whose branches are $\mathbb{Q}_2(\mathbb{Q}_2(\mathbb{Z}, \text{red}) + \omega^*, \mathbb{Q}_2(\mathbb{Z}, \text{red}))$. The cone type of the red points is $(0, 0, 2, 0)$. The cone type of the $\overline{c}$ points is $(1, 0, 0, 1)$. There is an interdense mixture of red points and $\mathbb{Z}$s (shown by $\sim$), and also an interdense mixture of $\mathbb{Q}_2(\mathbb{Z}, \text{red})$ and $\mathbb{Q}_2(\mathbb{Z}, \text{red}) + \omega^*$ (shown by $\natural$). The coding tree of the branches is given on the right. The red points have a least $\sim_3$-class above them which they are related to. The $\overline{c}$ points are covered by a point, that is, by a $\sim_1$-class and they are also covered by a $\sim_4$-class which they are related to.

5 The Characterization

In this section we introduce the notion of a 'structured tree', and show that this precisely characterizes which countable proper trees are 1-transitive. In order to provide a meaningful 'classification' however, more information about the tree is needed, and the notion of the 'type' of a countable 1-transitive tree is introduced, which is sufficient to describe it uniquely up to isomorphism. In Sect. 5 it is determined precisely which types can arise, which will therefore provide a classification of all countable 1-transitive trees, our principal goal.

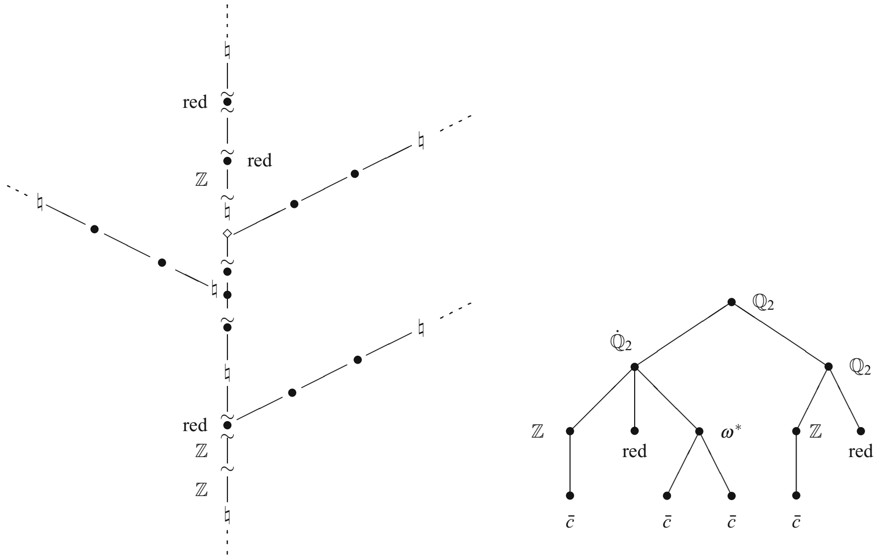

Fig. 6 A 1-transitive tree with $\mathbb{Q}_2(\mathbb{Q}_2(\mathbb{Z}, \textit{red}) + \omega^*, \mathbb{Q}_2(\mathbb{Z}, \text{red}))$ branches and their coding tree

Definition 5.1 A countable tree $(A, \leq)$ is **structured** if it is proper, and there is a colouring function $F : A^+ \to \Re$ such that:

(i) the set Υ of branches of A^+ (up to isomorphism) is a non-empty subset of some colour lower isomorphism class of branch-coloured chains,
(ii) $(\forall x \in A^+)(F(x) = \overline{c} \Leftrightarrow x \in A)$,
(iii) $(\forall x, y \in A^+)(F(x) = F(y) \Rightarrow C.T.(x) = C.T.(y))$,
(iv) if $x, y \in A^+$ and $F(x) = F(y)$ and x lies in a branch B of A^+, then there is a branch B' of A^+ such that $[x, +\infty) \cap B \cong [y, +\infty) \cap B'$ (every final segment of a member of Υ occurs above every point of A).

Lemma 5.2 *All countable 1-transitive trees are structured.*

Proof (i) As A is assumed proper, there are incomparable points $x, y \in A$. As A is a tree, there is $z \in A$, $z < x, y$. Thus x is not least, and z is not greatest, so by 1-transitivity, A has no least or greatest, and the same follows for any branch B.

Suppose that $x < y$ in B. By [14] Lemma 2.4.7, and since no point ramifies downwards, $(x, y] \cap A \neq \emptyset$, from which it follows that there is a point z of A with $x < z \leq y$. Furthermore, if x and y are consecutive, it follows that $y \in A$.

The fact that any two members of Υ are lower isomorphic follows from the definition of colours as orbits, and similarly all members of Υ are lower 1-transitive.

(ii) This is the definition of the colour $\bar{c}$, and the set of these points forms an orbit by 1-transitivity of A.
(iii) Any isomorphism preserves cone types.
(iv) We apply to B any automorphism which takes x to y. □

If $(A, \leq)$ is a countable 1-transitive tree and Υ is a non-empty family of lower isomorphic branch-coloured chains, then $T(A)$ and $T(\Upsilon)$ stand for the sets of coding trees of branches of A^+ and members of Υ, respectively. By Theorem 3.24, all members of $T(\Upsilon)$ have the same sets of levels, so we may talk unambiguously of the levels of $T(\Upsilon)$ to mean those of the coding tree of any $B \in \Upsilon$.

Definition 5.3 A **type** is a triple: $t = (\Upsilon, \Re, (\bar{\alpha}^{c_j})_{j\in J})$ such that:

- Υ is a non-empty subset of a colour lower isomorphism class of branch-coloured chains such that for each parent level of $T(\Upsilon)$ there are $X \in \Upsilon$ and $x \in T(X)$ at that level, such that $\varsigma(x) \neq \omega^*, \dot{\mathbb{Q}}, \dot{\mathbb{Q}}_n$;
- $\Re = \{c_j : j \in J\}$ is the colour set of Υ, indexed by J,
- $\bar{\alpha}^{c_j} = (\alpha_1^{c_j}, \ldots, \alpha_i^{c_j}, \ldots, \alpha_r^{c_j})$ where $i \in I$ and $(I, \leqslant)$ is the set of levels of $T(\Upsilon)$ and $(\forall j \in J)(\forall i \in I)(0 \leqslant \alpha_i^{c_j} \leqslant \aleph_0)$;
- $\sum_{j\in J} \sum_{i\in I} \alpha_i^{c_j} > 1$;

Definition 5.4 We **associate** a type $t(A)$ with a countable 1-transitive tree $(A, \leqslant)$ as follows:

- Υ is the set of isomorphism types of the branches of A^+,
- $\Re$ is the colour set of A^+,
- for each $c_j \in \Re$, $\bar{\alpha}^{c_j}$ is the cone type of each $a \in A^+$ with colour c_j.

Lemma 5.5 *For any countable 1-transitive tree* $(A, \leq)$*,* $t(A)$ *is a type.*

Proof First note that by the definition of the colour of a member of A^+ as an orbit, two points in the same orbit evidently give rise to the same sequence $\bar{\alpha}^{c_j}$. The fact that $\sum_{j\in J} \sum_{i\in I} \alpha_i^{c_j} > 1$ follows from the fact that A is a proper tree.

Next we show that for each parent level $i+1$ of $T(\Upsilon)$ there are $X \in \Upsilon$ and $x \in T(X)$ such that $L(x) = i+1$ and $\varsigma(x) \neq \omega^*, \dot{\mathbb{Q}}, \dot{\mathbb{Q}}_n$. For let B_0 be any branch of A. If B_0 has a $\sim_{i+1}$-class with no top $\sim_i$-class within it, then this B_0 provides the desired member of Υ. Otherwise we build another branch B' as required. Choose any $\sim_{i+1}$-class $(x_1)_{i+1}$ of B_0 so that x_1 lies in its top $\sim_i$-class, and choose $x_0 \in B$ of the same colour as x_1 so that $x_0 \sim_{i+1} x_1$ but $x_0 \not\sim_i x_1$. As $F(x_0) = F(x_1)$, there is an automorphism taking x_0 to x_1, and the image B_1 of B_0 contains x_1, but x_1 is no longer in the top $\sim_i$-class of its $\sim_{i+1}$-class in B_1, but its image x_2 *is*. Now repeat the argument and let x_n be the image of x_0 under the automorphism applied n times. Thus $x_0 < x_1 < x_2 < \ldots$. Let B' be a branch containing all the x_ns. Then in B' the $\sim_{i+1}$-class of x_0 contains all the x_n. Furthermore, there is an invariant partition containing $\bigcup_{n\in\omega}(x_n)_{\sim_i}$, which is a refinement of $\sim_{i+1}$, so as $\sim_i$ is a child of $\sim_{i+1}$, $\bigcup_{n\in\omega}(x_n)_{\sim_i} = (x_0)_{\sim_{i+1}}$. Hence $(x_0)_{\sim_{i+1}}$ has no greatest $\sim_i$-class. □

The characterization theorem will use back-and-forth, using approximations of the following kind.

Definition 5.6 A colour order isomorphism, ϕ, between subsets of A^+, where A is a 1-transitive tree, is **good** if its domain and range are finite unions of branches of A^+.

Lemma 5.7 *If $a, b \in A^+$, where A is a 1-transitive tree, and $C \in C_i(a)$, $C' \in C_i(b)$ then $C \cong C'$.*

Proof The proof is by back-and-forth. In the first case, suppose that a and b do not lie in the least $\sim_i$-classes of C, C', respectively, and choose $x \in C$, $y \in C'$ which *do* lie in these classes of C, C' and such that $F(x) = F(y)$. Then $(-\infty, x] \cong (-\infty, y]$. Furthermore $(-\infty, x] \cap C \cong (-\infty, y] \cap C'$ by the invariance of $\sim_i$. Let $\theta_1 : (-\infty, x] \cap C \to (-\infty, y] \cap C'$ be an isomorphism. Put $\phi(x) = y$ and for all $z \in (-\infty, x] \cap C$ put $\phi(z) = \theta_1(z)$. Then dom(ϕ) contains all its ramification points. Now x lies in a branch B of A^+. By Lemma 5.2, there is a branch, B', of A such that $[x, +\infty) \cap B \cong [y, +\infty) \cap B'$. Letting θ_2 be such an isomorphism we may put $\phi(z) = \theta_2(z)$ for all $z \in [x, +\infty) \cap B$. Then ϕ is a good isomorphism.

Now suppose that ϕ is a good isomorphism and $u \in C \setminus$ dom ϕ and we extend ϕ to include u in the domain (with a similar argument for the range). Then there is $z \in$ dom ϕ such that $z < u$ and for all $v \in$ dom ϕ with $v < u$, $v \leqslant z$. Since ϕ is a good isomorphism, $F(z) = F(\phi(z))$ and hence $C.T.(z) = C.T.(\phi(z))$. If x lies in a branch B of A then, by Lemma 5.2, there is a branch, B', of A such that $[x, +\infty) \cap B \cong [y, +\infty) \cap B'$, and we let θ_3 be such an isomorphism. Let ϕ' be the extension of ϕ given by $\phi(v) = \theta_3(v)$ for all $v \in [z, +\infty) \cap B$. Then $\phi'(u) = \theta_3(u)$ and ϕ' is the desired good isomorphism extending ϕ.

Now we move to the case in which a, b do lie in the least $\sim_i$-classes of C, C'. We first deal with the case where i has a successor in I. Let $x \in C$, $y \in C'$ be such that x, y are points in the least $\sim_i$-classes of C, C', respectively, and $F(x) = F(y)$. For any u, v in the least $\sim_i$-classes of C, C', respectively, and any $j < i$ we know that $\neg(a \sim_j u)$ and $\neg(b \sim_j v)$. If i has an immediate predecessor k, then the $\sim_k$-classes are dense and there is no least $\sim_k$-class. If i has no immediate predecessor, then there are certainly no $\sim_j$-classes, $j < i$ in a discrete relationship to a or b. Hence $(-\infty, x] \cap C \cong (-\infty, y] \cap C'$ by the invariance of $\sim_i$. Let $\theta_1 : (-\infty, x] \cap C \to (-\infty, y] \cap C'$ be an isomorphism. Put $\phi(x) = y$ and for all $z \in (-\infty, x] \cap C$ let $\phi(z) = \theta_1(z)$. Now x lies in a branch B of A. By Lemma 5.2 there is a branch, B', of A such that $[x, +\infty) \cap B \cong [y, +\infty) \cap B'$. Letting θ_2 be such an isomorphism, we let $\phi(z) = \theta_2(z)$ for all $z \in [x, +\infty) \cap B$, and then ϕ is a good isomorphism.

Suppose now that ϕ is a good isomorphism between finite unions of branches of C and C' and $u \in C \setminus$ dom ϕ. We extend ϕ to include u in the domain. There is $z \in$ dom ϕ such that $z < u$ and for all $x \in$ dom ϕ with $x < u$, $x \leqslant z$. Now, since ϕ is a good isomorphism, $F(z) = F(\phi(z))$ and hence $C.T.(z) = C.T.(\phi(z))$. By Lemma 5.2 there is a branch B' of A, such that $[x, +\infty) \cap B \cong [y, +\infty) \cap B'$. Let θ_2 be such an isomorphism, and let ϕ' be the extension of ϕ given by $\phi'(x) = \theta_2(x)$ for all $x \in [z, +\infty) \cap B$.

We now look at the case where i has no successor level in I. The families of invariant partitions which exist on branches of C and C' are equal. For each of these partitions, $\sim_j$, $j > i$, every final segment of a part of $\sim_j$ that occurs above points coloured by $F(a)$ occurs above a and within C, and likewise for b. By assumption, there is no least $\sim_j$ partition. We may therefore choose $j > i$ and a final segment, $\sigma = [a, +\infty) \cap B$, of a $\sim_j$ -class. Hence there is a final segment $\sigma' = [b, +\infty) \cap B'$ in C' of a $\sim_j$-class isomorphic to σ. We may now continue by back-and-forth, as before, to obtain $C \cong C'$. □

Theorem 5.8 *1-transitive trees* $(A_1, \leqslant)$ *and* $(A_2, \leqslant)$ *are isomorphic if and only if they have the same type.*

Proof The fact that isomorphic trees have the same type is immediate (since anything definable, first or second order, is preserved by an isomorphism).

Conversely, suppose they have the same type $(\Upsilon, \Re, (\bar{\alpha}^{c_j})_{j \in J})$. We show by back-and-forth using good isomorphisms as approximations that $A_1 \cong A_2$. To start we may choose any member B of Υ. By assumption there are branches B_1 of A_1 and B_2 of A_2 isomorphic to B, and hence there is an isomorphism from B_1 to B_2, which is necessarily good.

For the 'forth' step (and 'back' is similar) suppose that ϕ is a good isomorphism from a finite union of branches of A_1 to a finite union of branches of A_2, and suppose that $u \in A_1 \setminus \text{dom}(\phi)$. Let $z \in \text{dom}(\phi)$ be the ramification point where $(-\infty, u]$ meets $\text{dom}(\phi)$; that is, such that z is the greatest member of $\text{dom}(\phi)$ less than u. Now $\phi(z)$ has the same colour as z and so $C.T.(\phi(z)) = C.T.(z)$. Let C be the cone at z containing u. By Lemma 5.7 there is a colour-preserving order-isomorphism $\theta : C \to C'$ where C' is a cone at $\phi(z)$ not containing any point of $\text{range}(\phi)$ above $\phi(z)$. Let ϕ' be the extension of ϕ to $\text{dom}(\phi) \cup C$ given by $\phi'(z) = \theta(z)$ for $z \in C$. We obtain an extension of ϕ to a good isomorphism having u in its domain by restricting ϕ' to $\text{dom}(\phi) \cup B$ where B is some branch of A_1 containing u. □

We have shown that the type of a countable 1-transitive tree determines it uniquely. By essentially the same proof, we may derive the following.

Theorem 5.9 *All countable structured trees are 1-transitive.*

6 The Construction

In this section we determine which types actually arise from countable 1-transitive trees. We shall see that the definition given so far is insufficient. We recall that in the case of 2-transitive trees, or even weakly 2-transitive ones, the 'construction' of a tree corresponding to a given type was relatively straightforward. Namely, one starts with a branch (a copy of $\mathbb{Q}$, or a coloured version); then adds the correct number of branches above all ramification points, taking care in the weakly 2-transitive case that the correct cones have least elements; and then one just repeats over countably many steps. The fact that all the branches are (even colour-)isomorphic makes these cases relatively unproblematical.

An initial complication in the 1-transitive case is that Υ need not necessarily form the whole of a lower isomorphism class of branch-coloured chains, as we show by an example below, but whether this necessarily happens or not depends on the ramification behaviour. The following lemma captures a key point in trying to pin down the possible values of Υ.

Lemma 6.1 *Suppose that $(A, \leq)$ is a countable 1-transitive tree in which, for every $a < b$ in A there is $c > a$ in A incomparable with b such that $[a, b]$ and $[a, c]$ are isomorphic (as coloured chains). Then the family Υ of isomorphism types of branches of A is the whole of some lower isomorphism class of branch-coloured chains.*

Proof Let B be a branch-coloured chain which is lower isomorphic to a branch of A. Choose a cofinal sequence $b_0 < b_1 < b_2 < \dots$ of points of B coloured $\bar{c}$. We shall inductively choose corresponding points $x_0 < x_1 < x_2 < \dots$ in A. The main point is to ensure that $\{y \in A : (\exists n) x_n \geq y\}$ is a branch, and for that purpose we shall ensure that it has no upper bound in A. Let A be enumerated as $\{a_n : n \in \omega\}$. Choose $x_0 \not\leq a_0$ in A. This exists since A has no maximal point, so we may actually take $x_0 > a_0$. Since A is 1-transitive and B is lower isomorphic to a branch of A, $(-\infty, x_0] \cong (-\infty, b_0]$.

Now assume inductively that $x_0 < x_1 < x_2 < \dots < x_n$ in A have been chosen in such a way that $[x_i, x_{i+1}] \cong [b_i, b_{i+1}]$ for each $i < n$. Since A is 1-transitive and B is lower isomorphic to a branch of A, $(-\infty, b_{n+1}] \cong (-\infty, x_0]$. Let x' be the image of b_n under this isomorphism. By composing with an isomorphism taking x' to x_n, we find $x'_{n+1} > x_n$ such that $[b_n, b_{n+1}] \cong [x_n, x'_{n+1}]$. By assumption there is $x''_{n+1} > x''$ incomparable with x'_{n+1} such that $[x_n, x'_{n+1}] \cong [x_n, x''_{n+1}]$. Since x'_{n+1} and x''_{n+1} are incomparable, they cannot both be below a_{n+1}, so we let x_{n+1} be x'_{n+1} if $x'_{n+1} \not\leq a_{n+1}$ and x''_{n+1} otherwise. □

The lemma gives us a hint as to how to find a countable 1-transitive tree in which Υ is a proper subset of a colour lower isomorphism class of branch-coloured chains. We can also do this in the monochromatic case. Consider the lower 1-transitive chains whose coding trees are shown in Fig. 7 which form a colour lower isomorphism class of branch-coloured chains (one can see this as in the proof of Corollary 3.25), having order-type $\omega^*(\omega^* \cdot \mathbb{Z} + \omega^*) + \mathbb{Z}^2$, $\mathbb{Z}(\omega^* \cdot \mathbb{Z} + \omega^*)$, and $\omega^*(\omega^* \cdot \mathbb{Z} + \omega^*) + \omega^* \cdot \mathbb{Z}$, respectively. We can build a countable 1-transitive tree $(A, \leq)$, illustrated in Fig. 8, having all branches isomorphic to either B_1 or B_2, but

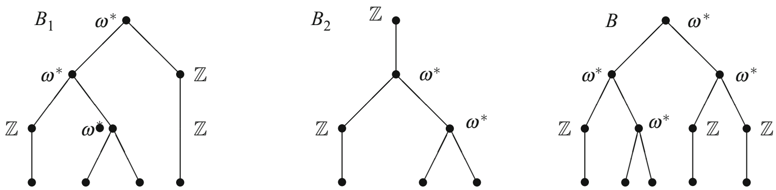

Fig. 7 A colour lower isomorphism class of branch-coloured chains

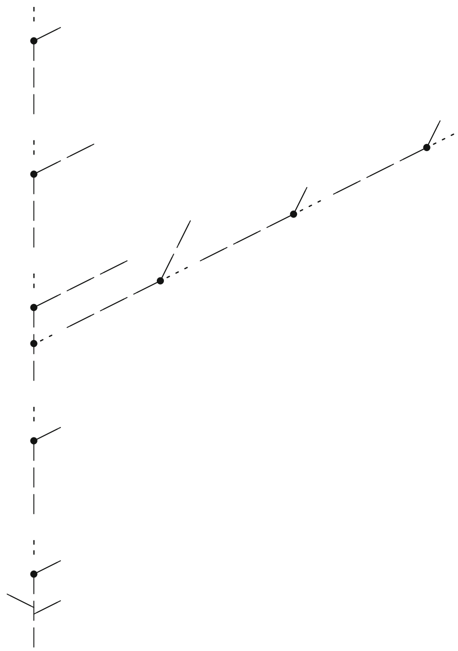

Fig. 8 A tree having branches of type B_1 and B_2 but not B

not B. In terms of the coding trees present, by the first clause of Definition 5.3, Υ must have a member having a label in the second level down with no endpoint, and likewise for the third level down, so B_1 and B_2 are unavoidable.

To construct such A, start with a branch of order-type B_1, and follow through in countably many stages. At each stage, vertices which ramify with ramification order 2 are no longer touched. For those which do not yet ramify we add another branch through them of one of these types in such a way that one of the two cones there has a least member and the other doesn't. We can also ensure while doing this that there are branches of both types, B_1 and B_2, passing through each such vertex (which means that we have to add the correct *final* segments). This clearly results in a tree A which ramifies only at points of A, with all ramification orders equal to 2, and such that there are branches in order-types B_1 and B_2 passing through all vertices. Furthermore this suffices to characterize A up to isomorphism by a back-and-forth argument, and this also establishes 1-transitivity. It remains to show that A has no branches in type B. We inevitably add branches that were not 'explicitly' included during the construction, since we only added countably many and A has $2^{\aleph_0}$ branches. However the point is that these other ones, which arose 'by accident' must actually all be isomorphic either to B_1 or B_2.

Suppose for a contradiction that B does arise as a branch of A. Now B has a final segment in type ω, $x_0 < x_1 < x_2 < \ldots$ say. By construction, there is exactly one other cone at each x_i which does not have a least member. Since $\omega^* \cdot \mathbb{Z}$ is a convex subset of each branch of A, there is a convex subset $\{y_n : n \in \omega\} \cup \{z_n : n \in \mathbb{Z}\}$ of A. By 1-transitivity there is an automorphism of A taking y_0 to x_0, and since x_{n+1}

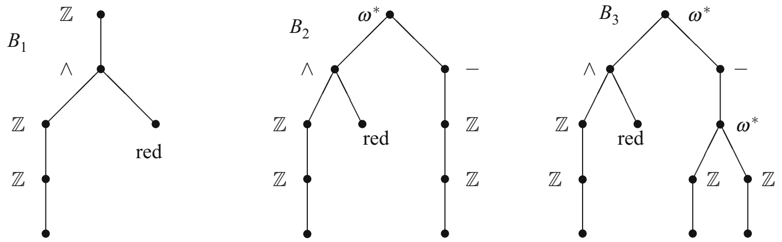

Fig. 9 Another example as in Fig. 7, but involving the label

is the unique point of A immediately above x_n (as the other cone there has no least member), we see inductively that y_n is taken to x_n by this automorphism. But now the image of z_0 is greater than every x_n, contrary to $\{x_n : n \in \omega\}$ a final segment of B.

We give a further example involving $\wedge$ labels to illustrate a related point. Consider the three colour coding trees shown in Fig. 9. The branch-coloured chain encoded by the first coding tree has to occur in any countable 1-transitive proper tree having a branch colour lower isomorphic to an order encoded by them because of the first clause of Definition 5.3, but the second and third can be omitted. Also see further discussion of this example below concerning the ramification of $\bar{c}$ points.

The argument enshrined in the above discussion can now be suitably adapted to lead to the general characterization of which types are types of 1-transitive trees. We first make the following remark, which is related to the techniques needed to finish our characterization.

Lemma 6.2 *Suppose that Υ is the family of branch-coloured chains arising as branches of a countable 1-transitive proper tree $(A, \leq)$, and that B is a branch-coloured chain lower colour isomorphic to the members of Υ. Then B is isomorphic to an initial segment of some member of Υ.*

Proof We adapt the proof of Lemma 6.1. Choose a cofinal sequence $b_0 < b_1 < b_2 < \ldots$ of points of B coloured $\bar{c}$ and choose corresponding points $x_0 < x_1 < x_2 < \ldots$ in A inductively. Choose x_0 in A. Since A is 1-transitive and B is lower isomorphic to a branch of A, $(-\infty, x_0] \cong (-\infty, b_0]$.

Now assume inductively that $x_0 < x_1 < x_2 < \ldots < x_n$ in A have been chosen in such a way that $[x_i, x_{i+1}] \cong [b_i, b_{i+1}]$ for each $i < n$. Since A is 1-transitive and B is lower isomorphic to a branch of A, $(-\infty, b_{n+1}] \cong (-\infty, x_0]$. Let x' be the image of b_n under this isomorphism. By composing with an isomorphism taking x' to x_n, we find $x_{n+1} > x_n$ such that $[b_n, b_{n+1}] \cong [x_n, x_{n+1}]$. Let X be a branch of A containing all x_i. Then $\{x \in A : (\exists n)(x \leq x_n)\}$ is an initial segment of X (proper or not) isomorphic to B.

This is the same proof as before, but making no attempt to ensure that the points of A do not lie above all x_n. □

To formulate our main theorem, we require the notion of 'ambiguity'. If $t = (\Upsilon, \Re, (\bar{\alpha}^{c_j})_{j \in J})$ is a type, and B is a branch-coloured chain which is colour lower isomorphic to the members of Υ, then a point x of B is said to be **ambiguous** (with respect to t) if either it is c_j-coloured and (x, ∞) begins with a $\sim_i$-class where $\alpha_i^{c_j} \geq 2$, or there are $z > y > x$ in B, and $i \neq i'$ in I, such that $[x, y]$ and $[x, z]$ are isomorphic (as coloured chains), and (z, ∞), (y, ∞) begin with $\sim_i$-, $\sim_{i'}$-classes, respectively.

We illustrate the need for consideration of this notion by some examples. If the $\bar{c}$ points ramify, then the branch-coloured chain encoded by the coding trees B_2 and B_3 in Fig. 9 both have cofinal sets of ambiguous points in the first sense, so must arise in any such 1-transitive tree. Thus in the first sense, whether or not a point is ambiguous depends on the value of the ramification order $\bar{\alpha}^{c_j}$. Moving on to the second sense, here this does not depend on the ramification orders. As an easy (monochromatic) example, we just take $\mathbb{Q}.\omega^*$, x any point, and $y < z$ the top two points in a copy of ω^* greater than the copy containing x. Then (x, y) and (x, z) are isomorphic (to $k + \mathbb{Q}.\omega^*$ for some finite k), and (y, ∞) has a least member but (z, ∞) does not, so that all x are ambiguous. This illustrates what 'ambiguous' means; however, it still does not show how this condition is required in Theorem 6.4 below. For the lower isomorphism class of branch-coloured chains comprises precisely $\mathbb{Q}.\omega^*$ and $\mathbb{Q}.\omega^* + \mathbb{Z}$, and both of these are already required as branches of A by virtue of the first clause of Definition 5.3 (since one has only an ω^* label on the middle level, and other has only a $\dot{\mathbb{Q}}$ label on the root). So to illustrate the real point, we require a more complicated example.

The order encoded by the left-hand coding tree shown in Fig. 10 has a cofinal set of ambiguous points in the second sense, is lower isomorphic to the order encoded by the right hand coding tree, and is embeddable in it as a proper initial segment [see Lemmas 6.2 and 6.3(ii)]. To demonstrate the existence of ambiguous points in the order encoded by the left-hand tree, we take x, y, and z in the final part of the encoded order. For definiteness, x and z are both in the encoding of the rightmost branch with $x < z$ (that is, they are top points of distinct copies of $\omega^*.\mathbb{Z} + \omega^*$), and y is taken to be the predecessor of z, which lies in the encoding of the second branch from the right. Then (x, y) and (x, z) are both isomorphic to $\mathbb{Q}.(\omega^*.\mathbb{Z} + \omega^*) + \omega^*.\mathbb{Z} + \omega^*$, and once more, (y, ∞) has a least member but (z, ∞) does not. There are many ambiguous points in the earlier part of this ordering, but since we require a *cofinal* set of ambiguous points, we have to have some of the sequence in the final part, so we have concentrated on it straight away. Note that since the right-hand linear order fulfils all the requirements for the first clause of Definition 5.3, this example shows why the ambiguity condition really is required to pin down which branch-coloured chains can or cannot be omitted. Note further that this example is still monochromatic, but is also easy to find coloured examples.

Two of the key steps in the argument for our main theorem are given in the following lemma.

Lemma 6.3 *Suppose that Υ is the family of branch-coloured chains arising as branches of a countable 1-transitive proper tree $(A, \leq)$, and that B is a branch-coloured chain lower colour isomorphic to the members of Υ. Then if either of the following conditions holds, $B \in \Upsilon$:*

(i) B has a cofinal set of ambiguous points,
(ii) B is not isomorphic to a proper initial segment of any member of Υ.

Proof (i) Let $A = \{a_n : n \in \omega\}$ be an enumeration of A. We use the method of Lemma 6.1, and see that the hypothesis that ambiguity holds cofinally fulfils exactly what is required to ensure that B arises as a branch of A. This time we let $b_0 < b_1 < b_2 < \ldots$ be a cofinal sequence of ambiguous points of B. Then for each n, one of the two clauses in the definition of 'ambiguous' applies. If it is the second, then there are points $z > y > b_n$ as in the definition. By passing to a suitable infinite subsequence of (b_n), we may suppose that if this case holds, then $z < b_{n+1}$. Now we choose corresponding points $x_0 < x_1 < x_2 < \ldots$ of A so that $(-\infty, b_0] \cong (-\infty, x_0]$ and $[b_n, b_{n+1}] \cong [x_n, x_{n+1}]$ for each n. The point x_0 is chosen of the same colour as b_0 (rather than lying in A), and $x_0 \not\leq a_0$. This is easily arranged using 1-transitivity of the tree.

Now suppose that x_n has been chosen, and we show how to find x_{n+1}. First let $x' \leq x_0$ have the same colour as b_{n+1}, so that $(-\infty, b_{n+1}] \cong (-\infty, x']$, and let x'' be the image of b_n under this isomorphism. Since b_n, x'', and x_n all have the same colour, there is an automorphism of A taking x'' to x_n, and if we let x'_{n+1} be the image of x' under this map, $[b_n, b_{n+1}]$ and $[x_n, x'_{n+1}]$ are colour isomorphic.

Knowing that b_n is ambiguous allows us to choose x_{n+1} in place of x'_{n+1} so that it does not lie below a_{n+1}. The key point is that we can find incomparable extensions x'_{n+1}, x''_{n+1} of x_n so that $[b_n, b_{n+1}] \cong [x_n, x'_{n+1}] \cong [x_n, x''_{n+1}]$. In the first case, b_n is c_j-coloured and (b_n, ∞) begins with a $\sim_i$-class where $\alpha_i^{c_j} \geq 2$. So there are (at least) 2 cones at x_n beginning with a $\sim_i$-class, one of them containing x'_{n+1}, and we let x''_{n+1} be the corresponding point in another such cone. In the second case, there are $z > y > b_n$ in B, and $i \neq i'$ in I, such that $[b_n, y]$ and $[b_n, z]$ are isomorphic (as coloured chains), and (z, ∞), (y, ∞) begin with $\sim_i$-, $\sim_{i'}$-classes, respectively. By the assumption made above, we may take $z < b_{n+1}$. Taking the images under the isomorphism, there are $z' > y' > x_n$ such that $z' < x'_{n+1}$ fulfilling the same conditions. Now y' and z' have the same colour (since $[b_n, y']$ and $[b_n, z']$ are colour isomorphic), and (z', ∞) begins with a $\sim_i$-class. Hence there is a cone at y' beginning with a $\sim_i$-class, and this cone is disjoint from (y', x'_{n+1}) since $i \neq i'$ (and (y', ∞) begins with a $\sim_{i'}$-class). Since this cone at y' and the cone at z' containing

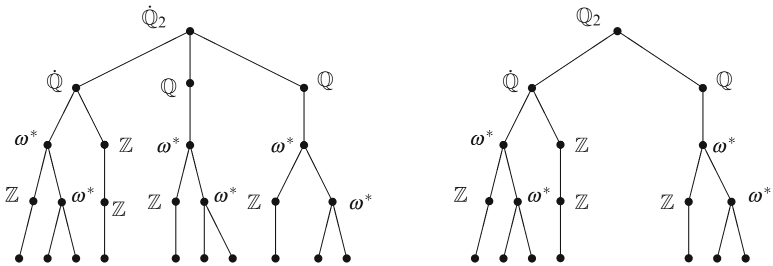

Fig. 10 Illustration of the notion of ambiguity

x'_{n+1} correspond to the same i, by Lemma 5.7 they are isomorphic, and we can find a point x''_{n+1} above y' corresponding to x'_{n+1}. Since $[b_n, y']$ and $[b_n, z']$ are isomorphic, $[b_n, b_{n+1}] \cong [x_n, x'_{n+1}] \cong [x_n, x''_{n+1}]$.

In each case we let x_{n+1} be one of x'_{n+1}, x''_{n+1} which is not below a_{n+1}.

Then $X = \{y \in A : (\exists n) x_n \geq y\}$ is a branch of A isomorphic to B, as required.

(ii) This follows at once from Lemma 6.2.

□

Theorem 6.4 *Let $t = (\Upsilon, \Re, (\bar{\alpha}^{c_j})_{j \in J})$ be a type. Then there is a countable, proper 1-transitive tree having type t if and only if any branch-coloured chain B lower isomorphic to the members of Υ and having a cofinal set of ambiguous points lies in Υ.*

Proof The truth of the given condition for the type of any countable 1-transitive proper tree was proved in the previous lemma.

Conversely, suppose that $t = (\Upsilon, \Re, (\bar{\alpha}^{c_j})_{j \in J})$ is a type fulfilling the two stated properties, and we construct the desired tree $(A, \leq)$ as the union of a countable sequence of trees. Start with a single branch, taken to be any member A_0 of Υ. At a general step, we shall have a tree A_n, and all points of earlier A_ms, coloured or not, will ramify correctly. In passing from A_n to A_{n+1} we ensure that all the newly added points also ramify correctly. Let x be a j-coloured point of A_n which does not yet have the correct ramification. For each i we add branches at x so that there are $\alpha_i^{c_j}$ in all (the branch that is already above it may already be of the desired kind, in which case the number we *add* is actually one less than $\alpha_i^{c_j}$). The section that we add has to be so that the total branch (including $(-\infty, x]$) is of the correct order-type. For this we add a final segment taken at a c_j-coloured point of the branch to be added.

To conclude the construction we have to ensure that all members of Υ occur above all points, and also that members of the colour lower isomorphism class containing Υ which do not lie in Υ do *not* occur.

The first is achieved by enumerating the requirements dynamically. That is, at each point, we list all the members of Υ (which is possible by Corollary 3.25), and we 'promise' to include them all at some stage above the current one. The choice of which branches to extend is now made based on the promises made at the current point and earlier ones; this is a standard technique, which is formally carried out using a 'book-keeping function' to keep track of the dove-tailing.

To achieve the second, we argue as in the discussion preceding the statement of the theorem. The idea is that we only *explicitly* added branches in Υ, and we have to see that we didn't add any others by accident. Consider a branch-coloured chain B in the same isomorphism class as the members of Υ, but which does not lie in Υ. Then by assumption, the set of ambiguous points of B is bounded, and by Lemma 6.2, B is isomorphic to an initial segment of some member of Υ. We have to show that B is not isomorphic to any branch of A. Suppose otherwise, and let B' be such a branch.

Then B' is isomorphic to a proper initial segment of some branch $X' \in \Upsilon$ of A by an isomorphism f say. Choose a cofinal sequence x_n of points of B', such that all points $\geq x_0$ are unambiguous, and by 1-transitivity of A, let g be an automorphism of A which takes x_0 to $f(x_0)$. Since B' is a branch and g is an automorphism, gB' is also a branch. Since $B' \notin \Upsilon$, there is a greatest point $x \in \bar{A}$ lying in $gB' \cap X'$.

Since x is not ambiguous (as a member of gB', and using the first clause in the definition of what this means) there are distinct $i, i' \in I$ such that the cones at x containing gB' and X' lie in $C_i(x)$ and $C_{i'}(x)$, respectively. It follows that $f^{-1}x \neq g^{-1}x$, for as f and g are both isomorphisms, if $f^{-1}x = g^{-1}x = y$ then $f(y, \infty)$ and $g(y, \infty)$ would have to lie in the same cone at x. We deduce that $gf^{-1}x \neq x$. Since $x, gf^{-1}x \in gB'$, they are comparable. Assume that $gf^{-1}x < x$. (If $gf^{-1}x > x$, then $fg^{-1}x < x$, and we use a similar argument with f and g interchanged.) Since $x > gf^{-1}x \geq fx_0$ in gB', and fx_0 is unambiguous, $(gf^{-1}x, \infty)$ must begin with a $\sim_i$ class. However, also $x > gf^{-1}x \geq fx_0$ in fB', so the same argument shows that $(gf^{-1}x, \infty)$ must begin with a $\sim_{i'}$ class. Since $i \neq i'$, this gives a contradiction. □

In conclusion we remark that we can now deduce that the definition of 'branch-coloured chain' precisely captured what was intended, that is, arising as a branch of some countable proper 1-transitive tree.

Corollary 6.5 *A countable coloured chain* $(X, \leq, F)$ *is isomorphic to a branch of a countable 1-transitive tree if and only if it is a branch-coloured chain.*

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On Ore's Theorem and Universal Words for Permutations and Injections of Infinite Sets

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Dedicated to the memory of Rüdiger Göbel

Abstract We give a simple proof that any injective self-mapping of an infinite set M can be written as a product of an injection and a permutation of M both having infinitely many infinite orbits (and no others). This implies Ore's influential theorem that each permutation of M is a commutator, a similar result due to Mesyan for the injections of M, and a result on which injections f of M can be written in the form $f = x^m \cdot y^n$.

Keywords Commutators • Infinite symmetric group • Ore's theorem • Permutations • Monoid of injections • Universal words

1 Introduction

For words $w = w(x_1, \cdots, x_n)$ in free variables $x_1, \cdots, x_n$, it often leads to difficult problems to describe groups G for which each element $g \in G$ is expressible in the form $g = w(g_1, \cdots, g_n)$ for some $g_1, \cdots, g_n \in G$. In the case of commutators $w = x_1^{-1} \cdot x_2^{-1} \cdot x_1 \cdot x_2$, this is known to be true for all finite and infinite alternating groups [12], all semi-simple complex Lie groups [13], all semi-simple connected algebraic groups [14], and many others; recently, it was established for all finite non-abelian simple groups [6], thereby confirming Ore's conjecture.

Ore [12] showed that, in contrast to the finite symmetric groups S_n somewhat surprisingly, each element of the infinite symmetric groups $S(M)$ of all permutations of an infinite set M is a commutator. His proof involved a non-trivial case analysis of cycle types. Here, we wish to provide a simple geometric proof of an extension of this result. We will consider the monoids $\mathrm{Inj}(M)$ of all injections of an infinite set M. An Ore-type result for these monoids $\mathrm{Inj}(M)$ was recently established in Mesyan [8]; see [3, 9] for consequences and descriptions of the normal subsemigroups of $\mathrm{Inj}(M)$. Our main result will be a simple proof showing that each injection $f \in \mathrm{Inj}(M)$ can

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be written as a product $f = g \cdot h$ with an injection $g \in \mathrm{Inj}(M)$ and a permutation $h \in S(M)$ each having infinitely many infinite orbits (and no others). This result itself also follows from a general result given in [8] which, however, involves a more complicated case analysis of possible orbits and previous results for $S(M)$. Our idea is to take as underlying set $M = \mathbb{Z} \times \mathbb{Z}$ (for the crucial case that M is countable) and to represent f in a suitable form. This idea was also used for the symmetric group $S(M)$ in [2] and in [4; Sects. 3, 4 and 5] with applications for extension results on coverings of surfaces. As an immediate consequence of the above result we obtain an Ore-type result for $\mathrm{Inj}(M)$, Ore's result for $S(M)$, and a description of all elements f of $\mathrm{Inj}(M)$ which can be written in the form $f = x^m \cdot y^n$ with $x, y \in \mathrm{Inj}(M)$.

2 Background

Here we summarize the notation and background results, as needed subsequently.

Let M be an infinite set, $\mathrm{Inj}(M)$ the monoid of all injective maps of M, and $S(M)$ the symmetric group of all permutations of M. Let $f \in \mathrm{Inj}(M)$. If $x \in M$, the set $\{y \in M \mid xf^i = y \text{ or } yf^i = x \text{ for some } i \geq 0\}$ is called the *f-orbit* of x, or an *orbit* of f. We call an orbit a *forward orbit*, if it is the f-orbit of some x such that $x \notin Mf$. Note that then this orbit equals $\{xf^i \mid i \geq 0\}$ and is infinite. This gives a bijection between $M \backslash Mf$ and the set of forward orbits of f. We have the following important observation.

Proposition 2.1 *Let* $f, g \in \mathrm{Inj}(M)$. *Then*

$$|M \backslash Mfg| = |M \backslash Mf| + |M \backslash Mg|.$$

Proof We have

$$M \backslash Mfg = (M \backslash Mf)g \mathbin{\dot{\cup}} (M \backslash Mg).$$

□

As usual, for $g \in \mathrm{Inj}(M)$ and $h \in S(M)$, we let $g^h = h^{-1}gh$. We say that two injections $f, g \in \mathrm{Inj}(M)$ are *conjugate* if $f = g^h$ for some $h \in S(M)$. We let $g^{S(M)} = \{g^h \mid h \in S(M)\}$, the set of conjugates of f. Next we wish to describe when two elements of $\mathrm{Inj}(M)$ are conjugate.

We let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Given $f \in \mathrm{Inj}(M)$, we call any orbit U of f with $U \subseteq Mf$, i.e., which is not a forward orbit, a *closed orbit*; then clearly $f\restriction_U \in S(U)$. We define $\bar{f}$ to be the map from $\mathbb{N}_\infty$ to the cardinals by letting $\bar{f}(n)$ be the number of closed orbits of size n of f, for each $n \in \mathbb{N}_\infty$. Recall that $|M \backslash Mf|$ is the number of forward orbits of f.

The following result, which is well known for permutations, describes that two elements of $S(M)$, resp., $\mathrm{Inj}(M)$ are conjugate if and only if they have the same "orbit structure".

Proposition 2.2 *(a) Let* $f, g \in S(M)$. *Then* f *and* g *are conjugate if and only if* $\bar{f} = \bar{g}$.

(b) (Mesyan [8]) *Let* $f, g \in \mathrm{Inj}(M)$. *Then* f *and* g *are conjugate if and only if* $\bar{f} = \bar{g}$ *and* $|M\backslash Mf| = |M\backslash Mg|$.

Proof Note that (a) is a special case of (b). We indicate the proof of (b) for the convenience of the reader. If $f = g^h$ for some $h \in S(M)$, then h maps the orbits of g onto the orbits (of the same length) of f. Hence $\bar{f} = \bar{g}$ and $|M\backslash Mf| = |M\backslash Mg|$. Conversely, given a length-preserving and forwardness-preserving bijection π from the orbits of g onto the orbits of f, for each orbit U of g, choose elements $x_U \in U$, $y_U \in U\pi$ (and such that $x_U \notin Mg$, $y_U \notin Mf$ in case U is a forward orbit), put $x_U h = y_U$ and extend h uniquely to a permutation of M satisfying $hf = gh$. □

3 The Main Result

In this section we will provide a simple proof for the following result.

Theorem 3.1 *Let* M *be an infinite set. Then every injection* $f \in \mathrm{Inj}(M)$ *is a product* $f = g \cdot h$ *of an injection* $g \in \mathrm{Inj}(M)$ *and a permutation* $h \in S(M)$ *both having infinitely many infinite orbits (and no others). We also have* $f = h{\cdot}g$ *with* $g \in \mathrm{Inj}(M)$, $h \in S(M)$ *as described before.*

We note that Theorem 3.1 is a special case of the main result of Mesyan [8] whose proof, however, involves a detailed analysis of the orbit structure of elements of $\mathrm{Inj}(M)$ and uses previous results on $S(M)$.

For our proof of Theorem 3.1, if M is countable, we take $M = \mathbb{Z} \times \mathbb{Z}$, the integer plane. We will show that for any $f \in \mathrm{Inj}(M)$ there is a conjugate f' of f which moves each element of M at most one unit up or down. For this, we construct f' with the same "orbit structure" as f by employing a Cantor-like enumeration of $\mathbb{Z} \times \mathbb{Z}$ or of suitable subsets (like half planes). For the case that $f \in S(M)$, this is also described in [2] and in [4; Sects. 3, 4, and 5].

Lemma 3.2 *Let* $M = \mathbb{Z} \times \mathbb{Z}$. *Then for each* $f \in \mathrm{Inj}(M)$ *there is* $f' \in \mathrm{Inj}(M)$ *such that* $\bar{f} = \bar{f'}$, $|M\backslash Mf| = |M\backslash Mf'|$ *and* $(i,j)f' \in \mathbb{Z} \times \{j-1, j, j+1\}$ *for each* $(i,j) \in M$.

Proof If f has infinitely many orbits, it is easy to construct such an injection f' satisfying even $(i,j)f' \in \mathbb{Z} \times \{j\}$ for each $(i,j) \in M$, i.e., the orbits of f' are all contained in the horizontal lines of $M = \mathbb{Z} \times \mathbb{Z}$. Therefore now let f have only finitely many orbits. Consequently, f has at least one infinite orbit.

First, let f have only one forward orbit (and no others). Then consider the "infinite spiral"

$$(0,0) \to (1,0) \to (1,1) \to (0,1) \to (-1,1) \to (-1,0) \to (-1,-1) \to$$
$$(0,-1) \to (1,-1) \to (2,-1) \to (2,0) \to \cdots$$

which gives f'.

This construction leaves a lot of freedom for changes enabling us to deal with the other cases. For instance, assume that $f \in S(M)$ has precisely one infinite closed orbit (and no others). Then let $f' \in S(M)$ act on the upper half plane $\mathbb{Z} \times \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, similarly as above, like

$$(0,0) \to (1,0) \to (1,1) \to (0,1) \to (-1,1) \to (-1,0) \to (-2,0) \to$$
$$(-2,1) \to (-2,2) \to (-1,2) \to (0,2) \to \cdots .$$

By a similar enumeration of the lower half plane $\mathbb{Z} \times \{-n \mid n > 0\}$, we define the pre-images of $(0,0)$ under f'.

Now if f has $k+1$ infinite orbits ($k > 0$), we can define f' such that it has each half-line $\mathbb{N} \times \{i\}$ ($i = 1, \cdots, k$) as an infinite orbit and has the set $M \setminus \bigcup_{i=1}^{k} \mathbb{N} \times \{i\}$ as the remaining infinite orbit, in each case realizing forwardness or closedness as necessary.

Finally, for the finite orbits of f (note that by our assumption, f has only finitely many orbits), we can take a suitably large interval in $\mathbb{N} \times \{0\}$ to realize the corresponding orbits of f', and use the complement of this interval for the infinite orbits of f'. □

Now we can show Theorem 3.1.

Proof of Theorem 3.1 It suffices to consider the case that M is countable. Indeed, if M is uncountable and $f \in \mathrm{Inj}(M)$, by a standard argument we can split $\dot{\bigcup}_{i\in I} M_i$ into pairwise disjoint f-invariant countable sets M_i, so $f\restriction_{M_i} \in \mathrm{Inj}(M_i)$. Then by the result of the countable case, for each $i \in I$ write $f\restriction_{M_i} = g_i \cdot h_i$ with an injection $g_i \in \mathrm{Inj}(M_i)$ and a permutation $h_i \in S(M_i)$ both having infinitely many infinite orbits (and no others). Then $g = \bigcup_{i\in I} g_i \in \mathrm{Inj}(M)$ and $h = \bigcup_{i\in I} h_i \in S(M)$ satisfy $f = g \cdot h$ as claimed.

So, let M be countable. We may assume that $M = \mathbb{Z} \times \mathbb{Z}$. Let $f \in \mathrm{Inj}(M)$. By Lemma 3.2, there is $f' \in \mathrm{Inj}\, M$ moving each point $x \in M$ at most one unit up or down such that $\bar{f} = \bar{f'}$ and $|M \setminus Mf| = |M \setminus Mf'|$. Then, $f \in f'^{S(M)}$ by Proposition 2.2.

Now define $h : M \to M$ by letting $(i,j)h = (i, j+2)$ for each $(i,j) \in M$. So $h \in S(M)$ has infinitely many infinite orbits (and no others). Now consider $g = f' \cdot h \in \mathrm{Inj}(M)$. Since f' moves each point $x = (i,j) \in M$ at most one unit up or down and h moves each point two units up, we obtain $xg \in \mathbb{Z} \times \{j+1, j+2, j+3\}$, so g moves each point at least one unit up. Hence g has only infinite orbits, and all elements $(i,0), i \in \mathbb{Z}$, lie in different orbits of g, thus g has infinitely many infinite orbits. So $f' = g \cdot h^{-1}$ as claimed, and the first statement of the result follows.

For the second statement, write $f = g \cdot h = h \cdot (h^{-1}gh)$; then $g^h \in \mathrm{Inj}(M)$ as claimed. □

Let C_∞ be the conjugacy class in $S(M)$ comprising all permutations of M with infinitely many infinite orbits (and no others). Note that if in Theorem 3.1 $f \in S(M)$ is a permutation, by the proof of Theorem 3.1 (or by Proposition 2.1) we obtain $f = g \cdot h$ with permutations $g, h \in S(M)$. Hence, as an immediate consequence of Theorem 3.1 we have:

Corollary 3.3 (Gray [5]). *Let M be an infinite set. Then $S(M) = C_\infty^2$.*

By subsequent work of Bertram, Göbel and the author, the author, and Moran, culminating in Moran [10], all conjugacy classes C in $S(M)$ were described satisfying $S(M) = C^2$.

4 Ore's Theorem and Universal Words

Here we will derive Ore's theorem and results on universal words for $S(M)$ and $\text{Inj}(M)$ as immediate consequences of Theorem 3.1. First we have:

Corollary 4.1 (Ore [12]) *Let M be an infinite set. Then each element $f \in S(M)$ is a commutator $f = [g, h]$.*

Proof By Theorem 3.1 (or Corollary 3.3), write $f = g^{-1} \cdot k$ with $g, k \in C_\infty$. Then $k = h^{-1}gh$ for some $h \in S(M)$ and $f = [g, h]$. □

Mesyan [8] gave a general result describing when an arbitrary injection $f \in \text{Inj}(M)$ can be written as a product of two injections $g, h \in \text{Inj}(M)$ both having at least one infinite orbit. As an immediate consequence, he obtained the subsequent Ore-type result for $\text{Inj}(M)$ which we wish here to deduce from Theorem 3.1.

Corollary 4.2 (Mesyan [8]) *Let M be an infinite set and $f \in \text{Inj}(M)$. Then f can be written in the form $f = g^a \cdot g^b$ for some $g \in \text{Inj}(M)$ and $a, b \in S(M)$ if and only if $|M \backslash Mf|$ is either an even integer or infinite.*

Proof Clearly, if $f = g^a \cdot g^b$ is of the form described, by Proposition 2.1 we have $|M \backslash Mf| = 2 \cdot |M \backslash Mg|$ as claimed.

Now let $|M \backslash Mf|$ be even or infinite. If $f \in S(M)$, the result is immediate by Corollary 3.3. Hence assume $f \in \text{Inj}(M) \backslash S(M)$, so f has at least two infinite forward orbits. Split $M = M_1 \dot{\cup} M_2$ in such a way that $|M_1| = |M_2|$, both M_1 and M_2 are f-invariant, and M_1 and M_2 contain the same number of infinite forward orbits of f. By Theorem 3.1, write $f\restriction_{M_1} = g_1 \cdot h_1$ and $f\restriction_{M_2} = h_2 \cdot g_2$ with injections $g_i \in \text{Inj}(M_i)$ and permutations $h_i \in S(M_i)$ such that $|M_i \backslash M_i f| = |M_i \backslash M_i g_i|$, and g_i, h_i have infinitely many infinite orbits (and no others), for $i = 1, 2$. Let $g = g_1 \cup h_2$ and $g' = h_1 \cup g_2$. Then $g, g' \in \text{Inj}(M)$ satisfy

$$|M \backslash Mg| = |M_1 \backslash M_1 g_1| = |M_1 \backslash M_1 f| = |M_2 \backslash M_2 f| = |M_2 \backslash M_2 g_2| = |M \backslash Mg'|$$

and g, g' each has infinitely many infinite closed orbits (and no other closed orbits). Hence $f = g \cdot g' = g \cdot g^b$ for some $b \in S(M)$ as claimed. □

Let G be a group and $w = w(x_1, \cdots, x_n)$ a word in the free group over $x_1, \cdots, x_n$. Then w is said to be *G-universal*, if for each $g \in G$ there are $g_1, \cdots, g_n \in G$ such that $g = w(g_1, \cdots, g_n)$. By Corollary 4.1, the commutator word $w = [x, y]$ is

$S(M)$-universal for infinite sets M. Clearly, no power $w = x^n$ ($n \geq 2$) is $S(M)$-universal. As a further immediate consequence of Corollary 3.3, we have:

Corollary 4.3 (Silberger [15]) *Let M be an infinite set and $w = x^m \cdot y^n$ with $m, n \neq 0$. Then w is $S(M)$-universal.*

Proof Let $f \in S(M)$. Write $f = g \cdot h$ with $g, h \in C_\infty$. Since $g^m, h^n \in C_\infty$, they are conjugate to g and h and the result follows. □

We note that we could also obtain Corollary 4.3 as follows. First, write $f \in S(M)$ as a product $f = g \cdot h$ of two involutions $g, h \in S(M)$ each having infinitely many 2-orbits. Note that the m-th power of a cycle of length $2m$ consists of m disjoint 2-cycles. Hence we can write $g = a^m$ with $a \in S(M)$ having only orbits of length $2m$ and, possibly, fixed points. Similarly, $h = b^n$ with $b \in S(M)$ having only orbits of length $2n$, and, possibly, fixed points. In the above proof of Corollary 4.3, we have obtained that $f = a^m \cdot b^n$ with $a, b \in C_\infty$. Extensions of this result are contained in [2]. Mycielski [11] and Lyndon [7], cf. [1], showed that each word $w = w(x_1, \cdots, x_n)$ which does not reduce to a power is $S(M)$-universal.

Now consider a semigroup S and a word $w = w(x_1, \cdots, x_n)$ in the free semigroup over $x_1, \cdots, x_n$. We say that $g \in S$ is a *w-element*, if there are $g_1, \cdots, g_n \in S$ such that $g = w(g_1, \cdots, g_n)$. Given a free semigroup word $w(x_1, \cdots, x_n)$, let $e(x_i)$ be the sum of the exponents of x_i in w, for $i = 1, \cdots, n$. Clearly, by Proposition 2.1, if $f \in \mathrm{Inj}(M)$ is a w-element, then either $M \backslash Mf$ is infinite or $|M \backslash Mf| \in \langle e(x_1), \cdots, e(x_n) \rangle$, the subsemigroup of $(\mathbb{N}, +)$ generated by $e(x_1), \cdots, e(x_n)$. Now we show that for products of powers, we also have the converse.

Corollary 4.4 *Let M be an infinite set, $m, n \geq 1$, and $f \in \mathrm{Inj}(M)$. Then f is a $x^m \cdot y^n$-element if and only if $M \backslash Mf$ is infinite or $|M \backslash Mf| \in \langle m, n \rangle$.*

Proof As noted before, if $f = g^m \cdot h^n$ with $g, h \in \mathrm{Inj}(M)$, by Proposition 2.1 we have

$$|M \backslash Mf| = m \cdot |M \backslash Mg| + n \cdot |M \backslash Mh|$$

which is infinite or in $\langle m, n \rangle$. Conversely, assume that $|M \backslash Mf| = k \cdot m + \ell \cdot n$ for some $k, \ell \geq 0$. First assume that $k, \ell > 0$. We include the case that $M \backslash Mf$ is infinite here by letting $k = \ell = \infty$. We split $M = M_1 \dot{\cup} M_2$ into two disjoint f-invariant subsets M_1 and M_2 such that M_1 (resp. M_2) contains $k \cdot m$ (resp. $\ell \cdot n$) infinite forward orbits of f. By Theorem 3.1, we can write $f \restriction_{M_1} = g_1' \cdot h_1'$ and $f \restriction_{M_2} = h_2' \cdot g_2'$ with injections $g_i' \in \mathrm{Inj}(M_i)$ and permutations $h_i' \in S(M_i)$ each having infinitely many infinite orbits (and no others), for $i = 1, 2$. In particular,

$$|M_1 \backslash M_1 g_1'| = |M_1 \backslash M_1 f| = k \cdot m$$

and

$$|M_2 \backslash M_2 g_2'| = |M_2 \backslash M_2 f| = \ell \cdot n.$$

Consequently, $g_1' \cup h_2' \in \mathrm{Inj}(M)$ has $k \cdot m$ forward orbits, infinitely many infinite closed orbits and no others. Choose any $g' \in \mathrm{Inj}(M)$ which has k forward orbits if $M \backslash Mf$ is finite, infinitely many forward orbits if $M \backslash Mf$ is infinite, and in any case infinitely many infinite closed orbits and no others. Then $g_1' \cup h_2'$ is conjugate to g'^m. Therefore, $g_1' \cup h_2' = g^m$ for some $g \in \mathrm{Inj}(M)$. Similarly, we have $h_1' \cup g_2' = h^n$ for some $h \in \mathrm{Inj}(M)$. Hence $f = g^m \cdot h^n$.

If $k = 0$ or $\ell = 0$ (but not both), we can apply a similar (but simpler) argument, using Theorem 3.1 directly for M. Finally, if $k = \ell = 0$, i.e., $f \in S(M)$, the result is immediate by Corollary 4.3. □

In view of Corollary 4.4 and the results of Mycielski and Lyndon for $S(M)$ the following question arises.

Let $w = w(x_1, \cdots, x_n)$ be a free semigroup word, $n \geq 2$, and let $f \in \mathrm{Inj}(M)$ satisfy $|M \backslash Mf| \in \langle e(x_1), \cdots, e(x_n) \rangle$. Does it follow that f is a w-element?

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An Extension of M. C. R. Butler's Theorem on Endomorphism Rings

Manfred Dugas, Daniel Herden, and Saharon Shelah

Abstract We will prove the following theorem: Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.

Keywords Endomorphism rings • Butler's theorem

Mathematical Subject Classification (2010): Primary 20K20, 20K30; Secondary 16S60, 16W20

1 Introduction

The main purpose of this paper is to honor the memory of Rüdiger Göbel, a dear friend and colleague, who passed away much too early. He made significant contributions on *realizing rings as endomorphism rings of abelian groups and modules* in many different settings. Most of this work can be found in the excellent monographs [4] and [5]. When Rüdiger came to Essen University, he started a successful research seminar. Among the first batch of papers studied was A. L. S. Corner's celebrated paper [2], where he proved that each countable torsion-free reduced ring R is the endomorphism ring of a countable torsion-free reduced abelian

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group G. If the additive group of such a ring R has finite rank n, then the group G can be constructed such that G has rank $\leq 2n$. Corner also provided examples of rings R such that the corresponding group G must have rank equal to $2n$. On the other hand, Zassenhaus [9] proved that for every ring R with identity and free additive group of finite rank, there is some abelian group M such that $R \subseteq M \subseteq R \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R = \mathrm{End}_{\mathbb{Z}}(M)$, i.e., R and M have the same rank.

Soon after [9] was written, Butler [1] generalized Zassenhaus's result replacing "free abelian of finite rank" by "locally free abelian of finite rank". Reid and Vinsonhaler [7] extended this result by replacing the ring of integers by certain Dedekind domains. More recently, Zassenhaus's result was generalized in [3] to rings with free additive groups of countable rank, whose elements are all algebraic over $\mathbb{Q}$. We will combine the results in [3] and [7] to obtain:

Theorem 1.1 *Let D be the ring of algebraic integers of a finite Galois field extension F of $\mathbb{Q}$ and E a D-algebra such that E is a locally free D-module of countable rank and all elements of E are algebraic over F. Then there exists a left D-submodule $M \supseteq E$ of $FE = E \otimes_D F$ such that the left multiplications by elements of E are the only D-linear endomorphisms of M.*

After reading Corner's paper [2], it became a goal of Rüdiger's to remove the cardinality barrier in this result. Eventually, this was accomplished by utilizing powerful combinatorial tools such as the diamond principle and Shelah's Black Box.

2 The Results

Notation 2.1 *Let D denote a countable Dedekind domain of characteristic zero and with infinitely many prime ideals. Let F be its field of fractions. It follows that for any prime ideal P of D, the localization D_P of D at P is a PID with unique maximal ideal pD_P for some $p \in P$. Let $\widehat{D}_P$ denote the P-adic closure of D_P. Let $f(x) \in F[x]$. Then $f(x) \in D_P[x]$ with the leading coefficient a unit in D_P for all but finitely many prime ideals P of D. Define $N_P(f)$ to be the number of roots of $f(x)$ in $\widehat{D}_P$. We call D an* ***admissible domain*** *if for all $f(x) \in F[x]$ the set of prime ideals P of D with $N_P(f) \geq 1$ is infinite. If E is some D-module, then we call E* ***torsion-free*** *if $se = 0$ for $s \in D$ and $e \in E$ implies $s = 0$ or $e = 0$. Moreover, E is called* ***locally free****, if the localization $E_P = E \otimes_D D_P$ is a free D_P-module for all prime ideals P of D. If R is some ring and $a \in R$, we define the map $a\cdot$ from R to R to be the left multiplication by the element a, i.e., $(a\cdot)(x) = ax$ for all $x \in R$.*

Our main result will be the following:

Theorem 2.1 *Let D be an admissible domain and E a countable, torsion-free and locally free D-algebra such that each $a \in E$ is algebraic over F. Then there exists a locally free left E-submodule M of $FE = E \otimes_D F$ such that $E \subseteq M$ and $\mathrm{End}_D(M) = E\cdot$, the ring of left multiplications by elements of E.*

2.1 The Proof of Theorem 1.1

Before we turn to the proof of Theorem 2.1, note that Theorem 1.1 will be an immediate consequence provided:

Proposition 2.2 *Let D be the ring of algebraic integers of some finite Galois field extension of $\mathbb{Q}$. Then D is an admissible Dedekind domain.*

We need to show for all $f(x) \in F[x]$ the existence of infinitely many prime ideals P of D with $N_P(f) \geq 1$. We will line up some results from algebraic number theory to obtain this proposition. Note that for any $f(x) \in F[x]$ there exists some $d \in D$ with $df(x) \in F[x]$. Thus, we may restrict to polynomials $f(x) \in D[x]$. Furthermore, any polynomial is a product of irreducible ones and we may restrict to irreducible $f(x) \in D[x]$.

We recall the following, well-known version of Hensel's Lemma [6, Proposition 2, p. 43]:

Lemma 2.3 *Let $1 \in S$ be a commutative ring and $\mathfrak{m}$ an ideal of S such that S is complete in the $\mathfrak{m}$-adic topology. Let $f(x) \in S[x]$ and $a \in S$ be such that $f(a) \in f'(a)^2\mathfrak{m}$. Then there exists some $b \in S$ such that $f(b) = 0$ and $b - a \in f'(a)^2\mathfrak{m}$.*

Applying this to our situation:

Remark 2.4 Let P be a prime ideal of D and let $f(x) \in D[x]$ be irreducible of degree n over F. Then $f(x)$ has only simple roots and thus has non-zero discriminant $\Delta(f)$. Let P be a prime ideal of D such that $\Delta(f) \notin P$. Then $f(x) \bmod P$ has no multiple roots. Assume that $a \in \widehat{D}_P$ is such that $f(a) \in p\widehat{D}_P$. Then $f'(a) \notin p\widehat{D}_P$ and we may apply Lemma 2.3 to obtain $b \in \widehat{D}_P$ with $f(b) = 0$ and $b - a \in p\widehat{D}_P$. Thus, for irreducible $f(x) \in D[x]$, $f(a) \in p\widehat{D}_P$ implies $N_P(f) \geq 1$.

By the above it is sufficient to show that for any irreducible $f(x) \in D[x]$, there are infinitely many prime ideals P of D such that $f(x) \bmod P$ has a root in D/P. Hensel's Lemma will then provide a root of $f(x)$ in $\widehat{D}_P$.

First we recall some well-known definitions that are in [6] and many other sources.

Let k be an algebraic number field and K a Galois extension of k with Galois group G. Let $\mathcal{O}_k$ ($\mathcal{O}_K$) denote the ring of algebraic integers in k (K). Let $\mathfrak{p}$ be a prime (ideal) of $\mathcal{O}_k$ and $\mathfrak{P}$ a prime of $\mathcal{O}_K$ lying over $\mathfrak{p}$. Then $\mathcal{O}_K/\mathfrak{P}$ is a finite extension of the finite field $\mathcal{O}_k/\mathfrak{p}$ and thus a finite field of order $n_{\mathfrak{P}}$ with cyclic Galois group $G = \langle\overline{\sigma}\rangle$ over $\mathcal{O}_k/\mathfrak{p}$ where $\overline{\sigma}(x) = x^{n_{\mathfrak{P}}} \bmod \mathfrak{P}$. Let $G_{\mathfrak{P}} = \{g \in G : g\mathfrak{P} = \mathfrak{P}\}$ denote the *decomposition group* of $\mathfrak{P}$ and $T_{\mathfrak{P}} = \{g \in G : \overline{g} = \mathrm{id}_{\mathcal{O}_K/\mathfrak{P}}\}$ the *inertia group* of $\mathfrak{P}$. Then there exists some coset $\sigma T_{\mathfrak{P}} \in G_{\mathfrak{P}}/T_{\mathfrak{P}}$ which induces $\overline{\sigma}$. Any element of that coset is called a *Frobenius automorphism* which we denote by $\sigma(\mathfrak{P}, K/k)$. Now we need a celebrated theorem due to Chebotarev [6, Theorem 10, page 169]:

Theorem 2.5 (Chebotarev) *Let K be a Galois extension of k with Galois group G. Let $\emptyset \neq C \subseteq G$ be some set invariant under conjugations with $|C| = c$ and $[K : k] = n$. Let*

$$M = \{\text{primes } \mathfrak{p} \text{ of } k \mid \mathfrak{p} \text{ is unramified in } K \text{ and there is some prime } \mathfrak{P} \text{ of } K \text{ lying over } \mathfrak{p} \text{ such that } \sigma(\mathfrak{P}, K/k) \in C\}.$$

Then the set M has a density and this density is $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$ for all $C \subsetneq G$.

The definition of density in this context can be found in [6, page 167]. All we need to know is that only infinite sets have a positive density.

Now we need a result from [8]. We maintain our current notations.

Theorem 2.6 ([8, Theorem 1]) *Let $f(x) \in \mathcal{O}_k[x]$ have degree $n \geq 2$ and be irreducible over k. Let $N_\mathfrak{p}(f)$ be the number of roots of $f(x)(\text{mod } \mathcal{O}_k/\mathfrak{p})$ in $\mathcal{O}_k/\mathfrak{p}$. Let*

$$P_0(f) = \{\mathfrak{p} \text{ prime in } \mathcal{O}_k \mid N_\mathfrak{p}(f) = 0\}.$$

Then $P_0(f)$ has density $\frac{c}{n}$. Moreover, $0 < \frac{c}{n} < 1$.

This shows that the set of all primes $\mathfrak{p}$ **not** in $P_0(f)$ has positive density and thus is infinite, completing the proof of Proposition 2.2.

Here is an outline of Serre's argument [8, page 432]: First, disregard all (finitely many) primes $\mathfrak{p}$ of $\mathcal{O}_k$ that are ramified or contain non-zero coefficients of $f(x)$. Let K be the splitting field of $f(x)$ over k with Galois group G and $\sigma = \sigma(\mathfrak{P}, K/k)$. Moreover, let X be the set of the n distinct roots of $f(x)$ in K. It turns out that $N_\mathfrak{p}(f)$ is the number of fixed points of $\sigma \restriction X$. Now put

$$G_0 = \{g \in G \mid g \restriction X \text{ has no fixed point}\}$$

and note that G_0 is invariant under conjugation, with $G_0 \subsetneq G$ since $\text{id}_K \notin G_0$. Now apply Theorem 2.5 with $C = G_0$.

2.2 *The Proof of Theorem 2.1*

We start with an easy observation.

Proposition 2.7 *Let $1 \in S$ be a commutative ring, A some S-algebra, and $\tau \in A$. Let $f(x) = \sum_{i=0}^m f_i x^i \in S[x]$, the polynomial ring over S. Then*

$$f(x) = f(\tau) + (x - \tau)(f_m \tau^{m-1} + g(\tau, x))$$

where $g(\tau, x) \in \text{span}_{\mathbb{Z}[x, f_0, \ldots, f_m]}\{\tau^j : 0 \leq j \leq m - 2\}$.

Proof We evaluate

$$f(x) = f((x-\tau)+\tau) = \sum_{i=0}^{m} f_i[(x-\tau)+\tau]^i = \sum_{i=0}^{m} f_i\left[\sum_{j=0}^{i}\binom{i}{j}(x-\tau)^j\tau^{i-j}\right]$$

$$= \sum_{i=0}^{m} f_i\left[\tau^i + \sum_{j=1}^{i}\binom{i}{j}(x-\tau)^j\tau^{i-j}\right]$$

$$= f(\tau) + \sum_{i=0}^{m} f_i(x-\tau)\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}$$

$$= f(\tau) + (x-\tau)\left[\sum_{i=0}^{m} f_i\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}\right].$$

The highest power of τ that might occur in $\sum_{j=1}^{i}\binom{i}{j}(x-\tau)^{j-1}\tau^{i-j}$ is τ^{i-1}. Note that

$$\sum_{j=1}^{m}\binom{m}{j}(x-\tau)^{j-1}\tau^{m-j} = \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k(-\tau)^{j-1-k}\right]\tau^{m-j}$$

$$= \sum_{j=1}^{m}\binom{m}{j}\left[\sum_{k=0}^{j-1}\binom{j-1}{k}x^k\tau^{m-1-k}(-1)^{j-1-k}\right].$$

Thus τ^{m-1} only occurs for $k=0$ and with coefficient $\sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$. Recall that $\sum_{j=0}^{m}\binom{m}{j}(-1)^j = 0$ and thus $1 = \binom{m}{0} = -\sum_{j=1}^{m}\binom{m}{j}(-1)^j = \sum_{j=1}^{m}\binom{m}{j}(-1)^{j-1}$.

This shows that $f(x) = f(\tau) + (x-\tau)\left[f_m\tau^{m-1} + g(\tau,x)\right]$ where $g(\tau,x) \in \mathrm{span}_{\mathbb{Z}[x,f_0,\dots,f_m]}\{\tau^j : 0 \le j \le m-2\}$. □

Corollary 2.8 *Same notation as in the proposition. Let S be an integral domain with Q its field of fractions and $c \in S$ such that $f(c) \neq 0 = f(\tau)$. Then*

$$(c-\tau)^{-1} = \frac{1}{f(c)}(f_m\tau^{m-1} + g(\tau,x)) \in QA.$$

We also want to list:

Proposition 2.9 *Let F be a field and V some vector space over F. If $\tau \in \mathrm{End}_F(V)$ is algebraic over F, then τ has only finitely many eigenvalues.*

Proof There exists some monic polynomial $f(x) \in F[x]$ such that $f(\tau) = 0$. Let $0 \neq v \in V$ be an eigenvector of τ with eigenvalue λ. Then $I = \{g(x) \in F[x] : g(\tau \restriction_{vF}) = 0\} = (x-\lambda)F[x]$ is an ideal of $F[x]$ and $f(x) \in I$. This shows that λ is a root of $f(x)$, of which there are only finitely many. □

Lemma 2.10 *Let $\tau \in \mathrm{End}_D(E^+)$ such that τ is algebraic over F. Let $0 \neq e \in E$ and Π a finite number of prime ideals of D. Then there exists a prime ideal $P \notin \Pi$ of D and $c \in D$ such that $c - \tau$ is an automorphism of FE^+ and $e \notin E_P(c - \tau)$. Moreover, $E_P(c - \tau)^{-1} \subseteq p^{-k}E_P$ for some natural number k where $PD_P = pD_P$.*

Proof Let $g(x) = \sum_{i=0}^n g_i x^i \in F[x]$ be the minimal polynomial of τ over F with $g_n = 1$. Let $V = eF[\tau]$, a finite dimensional τ-invariant F-subspace of FE. Put $\theta = \tau \restriction_V$, the restriction of τ to V, and $f(x) = \sum_{i=0}^m f_i x^i \in F[x]$ the monic minimal polynomial of θ. Then $f(x)$ is a divisor of $g(x)$ and the set of all prime ideals Q of D for which $h(x) \notin D_Q[x]$ for any monic divisor $h(x)$ of $g(x)$ is finite. We may enlarge Π to contain the finitely many exceptions. By Proposition 2.7, we have, for any $s \in D$, that $g(s) = (s - \tau)(\tau^{n-1} + \sum_{i=0}^{n-2} s_i \tau^i)$ where $s_i \in D_Q$ for all prime ideals $Q \notin \Pi$. We infer that $s - \tau$ is an automorphism of FE^+ whenever $g(s) \neq 0$. In this case, we have that $E_Q(s - \tau)^{-1} \subseteq \frac{1}{g(s)} E_Q$. A similar statement holds for $s - \theta$.

Since D is admissible, there is an infinite set of prime ideals Q of D such that $f(x)$ has a root γ in the Q-adic completion of the discrete valuation domain D_Q. We choose such a prime ideal $P \notin \Pi$. Let $P = D \cap pD_P$ for some $p \in P$.

Let $V = eF[\tau] = e \cdot \mathrm{span}_F\{1, \tau, \tau^2, \ldots, \tau^{m-1}\}$ be the τ-invariant subspace of FE generated by e. Note that $\{e, e\theta, e\theta^2, \ldots, e\theta^{m-1}\}$ is a basis of V over F.

Let $V_P = V \cap E_P$, which is a free D_P-module of rank m. Let $W_P = e \cdot \mathrm{span}_{D_P}\{1, \theta, \theta^2, \ldots, \theta^{m-1}\}$, a free D_P-module of rank m. Since D_P is a PID, the Stacked Basis Theorem for finite rank free modules holds and we infer that $p^h V_P \subseteq W_P$ for some natural number h.

Let $\gamma_0 \in D$ be such that $\gamma \equiv \gamma_0 \bmod p^{h+1}D_P$. Then $f(\gamma_0 + p^{h+j}) \equiv 0 \bmod p^{h+1}D_P$ for all natural numbers $j \geq 1$. We infer the existence of some $c \in D$ such that

(1) $g(c) \neq 0$
(2) $f(c) \equiv 0 \bmod p^{h+1}D_P$.

Note that this implies $f(c) \neq 0$ and $g(c) \equiv 0 \bmod p^{h+1}D_P$ as well.

It follows from the above that $c - \tau \in \mathrm{End}_F(FE^+)$ is bijective with $E_P(c - \tau)^{-1} \subseteq \frac{1}{g(c)} E_P$. Moreover, $c - \theta \in \mathrm{End}_F(V)$ is bijective as well.

Assume that $e(c - \theta)^{-1} \in E_P$.

Since $e(c - \theta)^{-1} \in eF[\theta] = V$ as well, we infer that $e(c - \theta)^{-1} \in V_P$ and thus $p^h e(c - \theta)^{-1} = \frac{p^h}{f(c)} \left[e\theta^{m-1} + e\psi \right] \in W_P$ for some $\psi \in \mathrm{span}_{D_P}\{1, \theta, \theta^2, \ldots, \theta^{m-2}\}$. This is a contradiction since $\frac{1}{p} e\theta^{m-1} \notin W_P$. □

Corollary 2.11 *Let Π be a finite set of prime ideals of D and $0 \neq \psi \in \mathrm{End}_F(FE^+)$ such that $1\psi = 0$. Let $t \in E$ be such that $0 \neq t\psi$. Then there is a prime ideal $P \notin \Pi$ of D and a free D_P-submodule M_P of FE^+ such that*

(1) $E_P \subseteq M_P$,
(2) $M_P\psi \nsubseteq M_P$ and
(3) For each $x \in FE$ we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.

Note that (2) holds for any $\varphi \in \mathrm{End}_F(FE^+)$ in place of ψ such that $1\varphi = 0$ and $t\psi = t\varphi$.

Proof Let $0 \neq e = t\psi$. We may assume that $e \in E$. Define $\tau \in \mathrm{End}_F(FE^+)$ by $\tau(x) = xt$ for all $x \in FE$. Then $\tau E \subseteq E$ since E is a ring. Since t is algebraic over F, so is τ and we can apply Lemma 2.10 and find a prime ideal $P \notin \Pi$ and $c \in D$ such that $\sigma = c - \tau \in \mathrm{End}_F(FE^+)$ is bijective, $e \notin E_P\sigma$ and $E_P\sigma \subseteq E_P$. Moreover, $E_P\sigma^{-1} \subseteq p^{-k}E_P$ for some natural number k. We infer $p^kE_P \subseteq E_P\sigma \subseteq E_P$.

Let $M_P = p^{-k}E_P\sigma$. Since σ is injective, M_P is a free D_P-module.

Then $E \subseteq E_P \subseteq p^{-k}E_P\sigma = M_P$ and (1) holds. Moreover, $E_P \cdot M_P \subseteq M_P$ since the multiplication in FE is associative.

Since $1\psi = 0$, we have $-1\sigma\psi = -(c1 - \tau)\psi = t\psi = e$ and $p^{-k}e \in M_P\psi$ but $p^{-k}e \notin p^{-k}E_P\sigma = M_P$. This shows that $M_P\psi \nsubseteq M_P$ and we have (2).

Let $x \in FE$. Then $x(p^{-k}E_P\sigma) = p^{-k}(xE_P)\sigma$ is contained in $p^{-k}E_P\sigma$ if and only if $xE_P \subseteq E_P$ by the injectivity of σ. Since $1 \in E_P$, this holds if and only if $x \in E_P$, and (3) follows. □

Let $\mathrm{End}^0(FE^+) = \{\varphi \in \mathrm{End}_F(FE^+) : 1\varphi = 0\}$ be the set of all linear transformations of FE^+ that map the identity element of E to zero. Then $\mathrm{End}_F(FE^+) = \mathrm{End}^0(FE^+) \oplus ((FE^+)\cdot)$. There exists a countable subset $1 \notin B$ of E such that $FE = \mathrm{span}_F(B \cup \{1\})$. Note that if $0 \neq \varphi \in \mathrm{End}^0(FE^+)$, then there exists some $b \in B$ such that $b\varphi \neq 0$. Moreover, $b\varphi$ is an element of the *countable* (E is countable, cf. Notation 2.1 and Theorem 2.1) set FE. This shows that there exists a countable list $\{\varphi_n : n \in \mathbb{N}\}$ of elements of $\mathrm{End}^0(FE^+)$ such that for all $\tau \in \mathrm{End}^0(FE^+)$ there exists some $n \in \mathbb{N}$ and $b \in B$ such that $\tau(b) = \varphi_n(b) \neq 0$. We apply Corollary 2.11 repeatedly to find a sequence of distinct prime ideals P_n of D and free D_{P_n}-modules M_{P_n} with properties

(1_n) $E_{P_n} \subseteq M_{P_n}$
(2_n) $M_{P_n}\varphi_n \nsubseteq M_{P_n}$ and
(3_n) If $x \in FE$, then $xM_{P_n} \subseteq M_{P_n}$ if and only if $x \in E_{P_n}$.

If Q is a prime ideal not in the list $\{P_n : n \in \mathbb{N}\}$, we put $M_Q = E_Q$. Then we have

(1) $E_P \subseteq M_P$ for all prime ideals P of D and also
(3) For each $x \in FE$, we have $xM_P \subseteq M_P$ if and only if $x \in E_P$.

Now let $M = \bigcap_P M_P$, where the intersection runs over all prime ideals P of D. Then $E \subseteq M$ by (1), and M is locally free since all M_P are free D_P-modules. Recall that $\mathrm{End}_D(M) = \bigcap_P \mathrm{End}_{D_P}(M_P)$. By (3) we get that

$$((FE)\cdot) \cap \mathrm{End}_D(M) = (E\cdot).$$

Let $0 \neq \psi \in \mathrm{End}^0(FE^+)$. Then there exists some $n \in \mathbb{N}$ such that, for some $b \in B$, we have $b\psi = b\varphi_n \neq 0$. By ($2_n$), we have that $M_{P_n}\psi \nsubseteq M_{P_n}$ which shows that $\mathrm{End}^0(FE^+) \cap \mathrm{End}_D(M) = \{0\}$. Let $\varphi \in \mathrm{End}_D(M)$. Then $\varphi = \psi + (x\cdot)$ for some $x \in FE$ and $\psi \in \mathrm{End}^0(FE^+)$. Pick $0 \neq s \in D$ with $sx \in E$. Then $s\varphi = s\psi + s(x\cdot) = s\psi + (sx\cdot)$, where $sx \in E$, and we infer

$$s\varphi - (sx\cdot) = s\psi \in \mathrm{End}^0(FE^+) \cap \mathrm{End}_D(M) = \{0\}.$$

Thus $\psi = 0$ and $\varphi = x\cdot$ for some $x \in E$ by condition (3). We conclude that $\mathrm{End}_D(M) = E\cdot$, as promised. □

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The Jacobson Radical's Role in Isomorphism Theorems for p-Adic Modules Extends to Topological Isomorphism

Mary Flagg

Abstract For a complete discrete valuation domain R, a class of R-modules is said to satisfy an isomorphism theorem if an isomorphism between the endomorphism algebras of two modules in that class implies that the modules are isomorphic. A class satisfies a Jacobson radical isomorphism theorem if an isomorphism between only the Jacobson radicals of the endomorphism rings of two modules in that class implies that the modules are isomorphic. Jacobson radical isomorphism theorems exist for subclasses of the classes of torsion, torsion-free and mixed modules which satisfy an isomorphism theorem. Warren May investigated the use of the finite topology in isomorphism theorems, and showed that the topological setting allows an isomorphism theorem for a broader class of reduced mixed modules than the algebraic isomorphism alone. The purpose of this paper is to prove that the parallels that exist between isomorphism theorems and Jacobson radical isomorphism theorems extend to the topological setting. The main result is that the class of reduced modules over a complete discrete valuation domain which contain an unbounded torsion submodule and are divisible modulo torsion satisfy a topological Jacobson radical isomorphism theorem.

Keywords Mixed modules • Endomorphism rings • Isomorphism theorem • Jacobson radical • Finite topology

1 Introduction

The celebrated Baer-Kaplansky Theorem [1, 9] states that any isomorphism between the endomorphism rings of two torsion modules over a discrete valuation domain is induced by an isomorphism between the modules. Generalizing Baer-Kaplansky, a class of groups or modules is said to satisfy an isomorphism theorem if an isomorphism between the endomorphism rings of two objects in that class implies

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that the groups or modules are isomorphic. An isomorphism theorem is said to be strong if every isomorphism between the endomorphism rings is induced by an isomorphism between the groups or modules. Wolfson [17] showed that the class of torsion-free modules over a complete discrete valuation domain satisfies a strong isomorphism theorem. Files [2], May and Toubassi [15] and May [12, 13] defined special classes of mixed modules over a discrete valuation domain which satisfy isomorphism theorems.

In a different direction, Hausen et al. [8] asked whether the whole endomorphism ring was required for an isomorphism of the ring to imply the underlying modules were isomorphic. They showed that given a torsion group G with an unbounded basic subgroup and a second torsion group H, if there exists an isomorphism between only the Jacobson radicals of the endomorphism ring of G and H, $\Phi : J(End(G)) \rightarrow J(End(H))$, then Φ is induced by an isomorphism $\phi : G \rightarrow H$. Information from only the Jacobson radical is significant for two reasons. First, the primitive idempotents that are central to the proof of the Baer-Kaplansky Theorem are not in the Jacobson radical. Second, a complete characterization of the Jacobson radical of an endomorphism ring is only known in special cases.

Let R be a complete discrete valuation domain. A class of R-modules will be said to satisfy a Jacobson radical isomorphism theorem if an isomorphism between the Jacobson radicals of the endomorphism rings of two modules in the class implies the modules are isomorphic. Hausen et al. [8] translate directly to imply a strong Jacobson radical isomorphism theorem for torsion R-modules with an unbounded basic submodule. In [3], Flagg proved that the class of torsion-free modules which are not divisible satisfies a strong Jacobson radical isomorphism theorem. Flagg [5] shows that there are two classes of mixed R-modules which satisfy a Jacobson radical isomorphism theorem.

In Section 108 of [6] on isomorphism theorems for torsion groups, Fuchs comments that "some generalization is expected to hold if the endomorphism rings are furnished with the finite topology." In [14], W. May investigates the role of the finite topology in isomorphism theorems for reduced modules over a discrete valuation domain. The purpose of this paper is to investigate the Jacobson radical's role in this topological isomorphism theorem.

To help the reader appreciate the role of the Jacobson radical in isomorphism theorems, this paper begins by summarizing the main parallels between isomorphism theorems using the whole endomorphism ring and Jacobson radical isomorphism theorems. Section 2 gives the reader a sample of the classes of modules which satisfy an isomorphism theorem and a list of classes which satisfy a Jacobson radical isomorphism theorem. Section 3 explains the basic ideas and results of May's topological isomorphism theorem in [14]. Section 4 defines the needed terminology and fundamental results needed to investigate Jacobson radical isomorphism theorems for reduced mixed modules. Finally, Sect. 5 proves a topological Jacobson radical isomorphism theorem for reduced mixed modules over a complete discrete valuation domain which have unbounded torsion submodules and are divisible modulo torsion.

2 Parallels Between Isomorphism Theorems and Jacobson Radical Isomorphism Theorems

Let R be a discrete valuation domain with unique prime p and quotient field Q. All modules will be left R-modules and endomorphisms will be written as acting from the right. The ring R is viewed as a topological ring in the p-adic topology, and R is said to be complete if it is a complete topological ring, which will be denoted $\hat{R}$ in this section.

Modules over discrete valuation domains are studied as generalized abelian groups ($\mathbb{Z}$-modules). Their structure is simpler due to the presence of a single prime in R contrasted with infinity many primes in $\mathbb{Z}$. Terminology standard for abelian groups is used, however it is defined with respect to the ring R instead of with respect to $\mathbb{Z}$. Properly, one should use terms like R-torsion or R-divisible, yet this paper will follow the standard conventions in the literature. Since an R-module is also an abelian group, and its properties as a group may be different from its properties as an R-module, a few definitions are in order for clarity. An element x of the R-module M is said to be torsion if there exists a nonzero $r \in R$ such that $rx = 0$. The set of all torsion elements of M is a submodule of M called the torsion submodule. The module M is said to be divisible if $M = rM$ for all $r \in R$. A module is said to be reduced if it has no divisible submodules. A submodule A of M is said to be pure (analogous to p-purity in abelian groups) if $rx = a$ with $r \in R$ and $a \in A$ is solvable in A whenever it is solvable in M. A basic submodule of M is defined as a natural analog of p-basic subgroup of an abelian group using these definitions of torsion, divisibility and purity.

Given a module M, its endomorphism ring $End_R(M)$ will be considered an R-algebra in the natural way. Isomorphisms are assumed to be R-algebra isomorphisms, which require maps corresponding to multiplication by elements in the ring R to be invariant under isomorphism. The Jacobson radical of the endomorphism ring of M, denoted $J(End_R(M))$ is a two-sided ideal of $End_R(M)$, and hence a ring without identity. The Jacobson radical also has an R-algebra structure inherited from the whole endomorphism ring. Isomorphisms between the Jacobson radicals of the endomorphism rings of two modules will also be considered R-algebra isomorphisms.

When defining classes of R-modules which satisfy an isomorphism theorem, it is necessary to separate the torsion and torsion-free cases since the endomorphism ring of the divisible torsion module Q/R is isomorphic to the endomorphism ring of the torsion-free module $\hat{R}$. Separate, but similar, techniques are used to prove the isomorphism theorems for torsion or torsion-free modules. However, these techniques rely on the particular properties of endomorphism rings of torsion or torsion-free modules, and neither translates directly to the mixed module case. Theorems for mixed modules must be considered separately since they require a third, completely different proof technique.

2.1 Torsion or Torsion-Free Modules

Kaplansky begun the subject of isomorphism theorems for modules over a discrete valuation ring with the following theorem.

Theorem 2.1 (Kaplansky [9])
Let R be a complete discrete valuation ring, M and N faithful primary R-modules. Then any R-isomorphism between $E(M)$ and $E(N)$ is induced by an isomorphism of M and N.

The proof of Theorem 2.1 relies on recognizing certain maps in the endomorphism ring of the module. In the case that the torsion module M has an unbounded basic submodule, there exists a decomposition

$$M = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \langle b_3 \rangle \oplus \cdots \oplus \langle b_j \rangle \oplus M_j$$

with $M_j = \langle b_{j+1} \rangle \oplus M_{j+1}$ and such that $o(b_i) = p^{m_i}$ with $1 \leq m_1 < m_2 < m_3 < \ldots$. Then there exists idempotents π_i that project onto the summand $\langle b_i \rangle$ and diagonal maps ζ_{ij} which map b_i onto b_j or $p^{m_j - m_i} b_j$ depending on whether $j < i$ or $j > i$. Recognizing the primitive idempotents and the diagonal maps from their ring theoretic properties is the key to Kaplansky's theorem.

Hausen et al. [8] were able to recognize the diagonal maps in the Jacobson radical of the endomorphism ring of a p-group which contains an unbounded basic subgroup. This is the foundation of their proof of their Jacobson radical isomorphism theorem.

Theorem 2.2 (Hausen et al. [8])
Given a p-group G with an unbounded basic subgroup and a second p-group H. If there exists an isomorphism $\Phi : J(End(G)) \to J(End(H))$, then there exists an isomorphism $\phi : G \to H$ which induces Φ.

Theorem 2.2 translates directly to imply that a Jacobson radical isomorphism theorem exists for the class of torsion R-modules which contain an unbounded basic submodule. If the torsion module M is the direct sum of a bounded plus a divisible module, the Jacobson radical $J(End(M))$ is not as strongly tied to the structure of M. Schultz [16] showed that in many cases the Jacobson radical determines the structure of the module. However, his proof involves identifying a torsion module by its Ulm invariants, the isomorphism is not constructive. Furthermore, Hausen and Johnson [7] showed that isomorphisms between the Jacobson radicals of the endomorphism rings of two bounded modules are not usually induced by an isomorphism between the modules.

In the case of torsion-free R-modules, the pathologies present in torsion-free finite rank groups carry over to modules over a discrete valuation ring if the ring is not complete. If the ring is complete, indecomposable $\hat{R}$-modules are isomorphic to the ring $\hat{R}$. Wolfson used this fact and Kaplansky's method to prove his isomorphism theorem for torsion-free modules.

Theorem 2.3 (Wolfson [17])
Let R be a complete discrete valuation domain and let M and N be torsion-free R-modules. If there exists an isomorphism $\Phi : End(M) \to End(N)$, then there exists an isomorphism $\phi : M \to N$ which induces Φ.

The primitive idempotents in the endomorphism ring of a torsion-free R-module are not in the Jacobson radical of its endomorphism ring. However, if $\pi \in End(M)$ is a primitive idempotent, then $p\pi \in J(End(M))$, and may be identified by its ring theoretic properties when the endomorphism ring is torsion-free as an R-module. The author used these maps to prove a Jacobson radical isomorphism theorem for torsion-free modules which are not divisible.

Theorem 2.4 (Flagg [3])
Let R be a complete discrete valuation domain and let M be a torsion-free R-module which is not divisible. If N is a torsion-free module such that there exists an isomorphism $\Phi : J(End(M)) \to J(End(N))$, then there exists an isomorphism $\phi : M \to N$ which induces Φ.

Theorem 2.4 is the strongest theorem possible for torsion-free R-modules since the Jacobson radical of the endomorphism ring of a divisible torsion-free R-module is 0.

2.2 Isomorphism Theorems for Mixed Modules

Isomorphism theorems for classes of mixed modules are truly the exceptions rather than the norm. Yet, there are classes of mixed modules over a discrete valuation domain for which an isomorphism theorem exists. For an isomorphism theorem to exist, some extra assumptions are required on the structure of the module. In particular, isomorphism theorems exist for classes of modules which contain a finitely generated nice submodule with quotient totally projective and torsion. In this case, a homomorphism on the nice submodule which does not decrease heights may be extended to a homomorphism of the module. The ample supply of homomorphisms with specific properties gives the connections needed to show the isomorphic endomorphism rings imply the underlying modules are isomorphic.

The first step to proving an isomorphism theorem for mixed modules, assuming the torsion submodule is nontrivial, is to use the Baer-Kaplansky theorem to show that if the endomorphism rings of two mixed modules are isomorphic, then there exists an isomorphism between the torsion submodules of the two modules. The Jacobson radical also contains enough information to determine the torsion submodule of a module. Note that this theorem does not require the ring R to be complete.

Theorem 2.5 (Flagg [5])
Let R be a discrete valuation domain. Let M and N be R-modules with torsion submodules T_M and T_N, respectively. If T_M has an unbounded basic submodule and there exists an R-algebra isomorphism $\Phi : J(E(M)) \to J(E(N))$, then there exists an isomorphism $\phi : T_M \to T_N$ which induces Φ on the torsion.

The reader is referred to May [11, 13] for a more thorough discussion of categories of mixed modules over a discrete valuation domain which satisfy an isomorphism theorem. Only a portion of this discussion is needed to explain the parallels between theorems using the whole endomorphism ring and those using only the Jacobson radical.

In [15], May and Toubassi showed that there exists a class of mixed modules of torsion-free rank one with an isomorphism theorem.

Theorem 2.6 (May and Toubassi [15])
Let R be a discrete valuation ring and let M be an R-module of torsion-free rank one with simply presented torsion submodule. If N is an R-module of torsion-free rank one, then every algebra isomorphism $\Phi : E(M) \to E(N)$ is induced by an isomorphism $\theta : M \to N$ such that $\Phi(\alpha) = \theta\alpha\theta^{-1}$ for every $\alpha \in E(M)$.

The isomorphism constructed by Theorem 2.6 is constructive. If the R-module is allowed to be of larger torsion-free rank, the techniques of [15] do not translate. Especially in the case of a nonsplit mixed module, new tools were needed. The tool that has proved to be the most useful is the technique of embedding reduced mixed modules in their cotorsion hulls, which will be described in the next section. This technique allowed May to prove that there are more classes of mixed modules which satisfy an isomorphism theorem, the following being one example.

Theorem 2.7 (May [12] Corollary B Part (3))
Let R be a complete discrete valuation domain and let M be a reduced Warfield module which is neither torsion nor torsion-free. Then every isomorphism of $E(M)$ with $E(N)$ is induced by an isomorphism of M with N.

The technique of embedding the modules into their cotorsion hulls is also available using only the Jacobson radicals of the endomorphism rings, see [5] for details. Since the Jacobson radical contains no more information than the whole endomorphism ring, classes of modules with a Jacobson radical isomorphism theorem are subclasses of those which satisfy an isomorphism theorem using the endomorphism ring. Jacobson radical isomorphism theorems have been proved for two classes of mixed modules which are subclasses of those in Theorems 2.6 and 2.7.

Theorem 2.8 (Flagg [5])
Let R be a complete discrete valuation domain and let M and N be mixed modules of torsion-free rank one. Assume M has a totally projective torsion submodule which is unbounded. Let J_M and J_N be the Jacobson radicals of their respective endomorphism rings. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$, then there exists an isomorphism $\phi : M \to N$ which induces Φ.

Theorem 2.9 (Flagg [5])
Let R be a complete discrete valuation domain. Let M and N be reduced Warfield modules of finite torsion-free rank. Assume that M has an unbounded torsion submodule. If there exists an R-algebra isomorphism $\Phi : J_M \to J_N$*, then* $M \cong N$.

2.3 Setting Isomorphism Theorems for Mixed Modules in the Cotorsion Hull of the Torsion Submodule

The technique of embedding the reduced modules in the cotorsion hull of their common torsion submodule and studying the modules as submodules of the same cotorsion module is the proper setting for isomorphism theorems for reduced mixed modules. This section reviews the key features of this embedding in the full endomorphism ring case, since it is the setting for the topological isomorphism theorem. Let M be a reduced R-module with nontrivial torsion submodule T. The cotorsion hull of M is the module $M^* = Ext_R^1(Q/R, M)$. Recall that there exists an embedding of M into M^* such that M^*/M is torsion-free and divisible. If M/T is divisible, then $M^* \cong Ext^1(Q/R, T) = T^*$. The image under the embedding has the following properties.

Lemma 2.10 *Let M be a reduced R-module with torsion submodule T such that* M/T *is divisible. Let* $\upsilon : M \to T^*$ *be the embedding of M into* T^**. For notational simplicity, identify M with its image* $M\upsilon$*. Then M is a pure submodule of* T^* *containing T and* T^*/M *is torsion-free and divisible.*

The embedding of M into T^* induces an embedding of endomorphism rings $E(M) \to E(T^*)$. Every endomorphism $\alpha \in E(M)$ extends uniquely to an endomorphism $\alpha^* \in E(T^*)$. Thus, as a subring of $E(T^*)$, the endomorphism ring of M has the following form.

Lemma 2.11 *Let M be a reduced mixed R-module with torsion submodule T such that* M/T *is divisible. Then, as a submodule of* T^*, $E_M = \{\alpha^* \in E^* : M\alpha^* \subseteq M\}$.

An isomorphism theorem starts with a reduced mixed module M and another module N with the property that $E(M) \cong E(N)$. The module M is assumed to have some special structure in order to make an isomorphism theorem possible. The module N may be arbitrary, but may also need to be of the same general type as M. For example, in [15] M is a module of torsion-free rank one with a simply presented torsion submodule, but N is only required to be of torsion-free rank one. For the following discussion, M will be a reduced mixed module which is divisible modulo torsion, and N will be arbitrary. These are the minimum properties needed to embed both modules into the cotorsion hull of their common torsion submodule. Furthermore, this is the form that will be needed for the topological isomorphism theorem in Sect. 3. May [12] shows that the embedding has the following form.

Lemma 2.12 *Let M be a reduced R-module with torsion submodule T_M such that M/T_M is divisible. Let N be an R-module with torsion submodule T_N. If there exists an isomorphism $\Phi : E(M) \to E(N)$, then N has the following properties.*

1. *N is reduced, N/T_N is divisible.*
2. *There exists an isomorphism $\theta : T_M \to T_N$ such that $t_n(\alpha\Phi) = t_n\theta^{-1}\phi\theta$ for all $\alpha \in E(M)$ and for all $t_n \in T_N$.*
3. *Let T be the common torsion submodule of M and N. Then, M and N may be embedded into T^* as pure submodules containing T.*
4. *The embedding into T^* induces an embedding of the endomorphism rings $E(M)$ and $E(N)$ such that $E(M) = E(N) = \{\alpha^* \in E(T^*) : M\alpha^* \subseteq M\} = \{\alpha^* \in E(T^*) : N\alpha^* \subseteq N\}$*

From the embedding in Lemma 2.12, one way to show an isomorphism theorem exists is to show that if M and N are not the same submodule of T^*, then there is an endomorphism in $E(M)$ that is not in $E(N)$, creating a contradiction.

3 Topological Isomorphism Using the Finite Topology

Given a module M, the finite topology on E_M is a linear topology with the collection of sets $U_x = \{\alpha \in E_M : x\alpha = 0\}$ for all $x \in M$ as the subbase at 0. The topology on the Jacobson radical and other ideals will be the subspace topology inherited from the finite topology on the endomorphism ring. Topological isomorphisms are assumed to be R-algebra isomorphisms which are continuous in the finite topology.

To investigate Fuchs' question about isomorphism theorems in the topological setting, May [14] first notes that the Baer-Kaplansky Theorem and Wolfson's Theorem may be combined. This is possible since the endomorphism algebras of the modules Q/R and $\hat{R}$ are isomorphic as R-algebras, but not homeomorphic in the finite topology.

Theorem 3.1 (May [14]) *Assume that R is complete. If M is either a torsion module or a torsion-free module, then every topological isomorphism of $End_R(M)$ with $End_R(N)$ is induced by an isomorphism of M with N.*

The interesting question then becomes whether an isomorphism theorem exists for a class of mixed modules in the topological setting which does not exist without the topology. The focus of May's investigation was on reduced mixed modules, without necessarily assuming any extra restrictions on their structure. Let M be a reduced R-module with a nonzero torsion submodule T. If R is not complete and M/T is not divisible, then the pathologies of torsion-free modules still carry over in the topological setting. If R is complete and M/T is not divisible, then M/T, and hence M has a summand isomorphic to R, and the primitive idempotent in $E(M)$ that projects onto that direct summand may be used to construct an isomorphism between two modules with isomorphic endomorphism rings (topology not required as long as the module is assumed to be properly mixed- neither torsion nor torsion-free). Therefore, the real question of how the topological setting extends the scope of isomorphism theorems is when M/T is divisible. In that case, the machinery of the embedding into the cotorsion hull is available.

Assume that our reduced module M is divisible modulo torsion. If there exists a topological isomorphism $\Phi : E(M) \to E(N)$ for some R-module N, then Lemma 2.12 shows that M and N may be regarded as submodules of T^* with equal endomorphism rings. The question is whether the finite topology helps identify submodules of T^* by their endomorphism rings. The ideal $Hom(M, T)$ is the key to the topological isomorphism theorem. Although this ideal is not easy to characterize ring theoretically in the algebraic setting, it has a particularly nice characterization in the topological setting.

Lemma 3.2 *Let M be a reduced R-module with torsion submodule T such that M/T is divisible. Then $Hom(M, T)$ is the ideal of $E(M)$ consisting of all $\alpha \in E(M)$ such that $p^n\alpha \to 0$ as $n \to \infty$. The ideal $Hom(M, T)$ is invariant under topological isomorphism.*

Thus, if $\Phi : E(M) \to E(N)$ is a topological isomorphism in the finite topology, then the restriction map $\Phi : Hom(M, T) \to Hom(N, T)$ is also a topological isomorphism. The problem is that this isomorphism is still not necessarily enough to show that M and N are isomorphic. May [14] addresses this issue by defining a natural "hull" for a reduced module M that is a unique submodule of T^* containing M.

Lemma 3.3 (May [14] Section 2)
Let M be a reduced mixed module with nontrivial torsion submodule T.

1. *There exists a maximum reduced module $\widehat{M}$ containing M with torsion submodule T such that the induced map $Hom(\widehat{M}, T) \to Hom(M, T)$ is a topological isomorphism. Any two such maximum modules are isomorphic by a unique isomorphism which is the identity on M.*
2. *$\widehat{M}$ is an $\hat{R}$-module, $\widehat{M}/M$ is torsion-free divisible and we may regard $\widehat{M}$ as a unique submodule of M^*.*
3. *Every $\alpha \in E(M)$ extends uniquely to $\hat{\alpha} \in E(\widehat{M})$, in fact $\hat{\alpha}$ is the restriction to $\widehat{M}$ of the unique extension of α to $\alpha^* \in E(M^*)$.*

May then proves that $\widehat{M} \cong \widehat{N}$ under topological isomorphism.

Theorem 3.4 (May [14] Theorem 1)
Let M be a reduced module over a discrete valuation domain R with nonzero torsion submodule. Assume that M is divisible modulo torsion. Then every topological isomorphism of $End_R(M)$ with $End_R(N)$ is induced by an isomorphism of $\widehat{M}$ with $\widehat{N}$. If R is complete, then the assumption that M is divisible modulo torsion is unnecessary.

If R is complete, then a suitable hypothesis on M allows the hulls to be replaced by the modules.

Theorem 3.5 (May [14] Theorem 2)
Assume that R is complete and that M is a reduced module. Let $\Phi : End_R(M) \to End_R(N)$ be a topological isomorphism.

1. *If the first Ulm submodule M^1 is a cotorsion module, then Φ is induced by an injection of N into M.*
2. *If M^1 is a cotorsion module, then every topological isomorphism of $End_R(M)$ is inner.*
3. *If M^1 has bounded torsion and finite torsion-free rank, then Φ is induced by an isomorphism of N with M. In particular, this is true if $M^1 = 0$.*

4 The Background for a Topological Jacobson Radical Isomorphism Theorem for Mixed Modules

In order to examine the Jacobson radical's role in Theorems 3.4 and 3.5, it is necessary to give the background on how an isomorphism between the Jacobson radicals of two reduced mixed modules enables the use of the embedding of the modules into the cotorsion hull of their common torsion submodule. From this point forward, assume that R is a complete discrete valuation domain. For an R-module M, let T_M be its torsion submodule, E_M be its endomorphism ring and J_M be the Jacobson radical of E_M.

The Jacobson radical version of the topological isomorphism theorem is for the following class of R-modules.

Definition 4.1 Let R be a complete discrete valuation domain. Let $\mathfrak{D}(R)$ be the class of all reduced R-modules M such that T_M is unbounded and M/T_M is divisible.

Note that the class $\mathfrak{D}(R)$ includes unbounded, reduced torsion modules.

4.1 *Ideals of the Endomorphism Ring*

In this section the ideals of the endomorphism ring and the ideals of the Jacobson radical that play a key role in the isomorphism theorem will be defined. Note that proper ideals of the endomorphism ring are rings without identity, and also have an R-algebra structure inherited from the whole endomorphism algebra. For a module M, the Jacobson radical J_M is a two-sided ideal of the endomorphism ring E_M. Therefore, any ideal of the endomorphism ring which is contained in J_M is also an ideal of the ring J_M.

Let I be an ideal of E_M, and let tI be the set of all maps $\alpha \in I$ with the property that there exists a positive integer n such that $p^n\alpha = 0$. Considering I as an R-module, tI is simply the torsion submodule of I. More importantly, it is straightforward to check that tI is also an ideal of E_M. In particular tJ_M is an ideal of the ring E_M and hence an ideal of J_M.

The first ideal is the nilradical N_M, which is the sum of all nilpotent ideals. It is well known that $N_M \subseteq J_M$. In the topological setting, there is an ideal of E_M that lies between N_M and J_M that is invariant under topological isomorphism. In Section 22 of [10], the authors define this weaker version of nilpotence as follows:

Definition 4.2 Given a module M, a map $\alpha \in E_M$ is called *locally nilpotent* if for every $x \in M$ there exists a positive integer k depending on x such that $x\alpha^k = 0$. An ideal I is locally nilpotent if all elements of I are locally nilpotent. Let the *local nilradical* L_M be the sum of all locally nilpotent ideals.

Lemma 4.3 *For a module M, $N_M \subseteq L_M \subseteq J_M$ and $tN_M = tL_M = tJ_M$.*

Proof Nilpotence implies local nilpotence, so $N_M \subseteq L_M$. The containment $L_M \subseteq J_M$ is proved in [10]. The relationship between the torsion ideals follows directly from the fact that $tN_M = tJ_M$, proved in [4]. □

The next ideal is the key to recognizing whether the quotient module M/T_M is divisible. Recall that if the torsion free R-module M/T_M is divisible, then it is a vector space over Q, the quotient field of R.

$$H_M = \{\alpha \in tJ_M : MtJ\alpha = 0\}$$

H_M is a two-sided ideal of J_M. The following result is proved in [4]. The "divisible as an R-module" phrase is included here to remind the reader of our definition of divisibility with respect to the ring R, not as an abelian group.

Proposition 4.4 *Let M be a module over a discrete valuation domain with torsion submodule T which contains an unbounded basic submodule. Then $H_M = 0$ if and only if M/T is divisible as an R-module.*

The ideal of the endomorphism ring central to May's argument in [14] is the set of all maps from M to its torsion submodule T_M, $Hom(M, T_M)$. The ideal $Hom(M, T_M)$ is not contained in the Jacobson radical. However, the next ideal is defined to capture the important maps into the torsion which are in the Jacobson radical. Define

$$\Gamma_M = L_M \cap Hom(M, T_M)$$

which is an ideal of J_M. The ideal Γ_M is central to the Jacobson radical version of the topological isomorphism theorem.

Lemma 4.5 *Given a module M with torsion submodule T and endomorphism ring E_M. Let $\alpha \in E_M$ and let n be a positive integer.*

1. *The map $\alpha \in Hom(M, T)$ if and only if $p^n\alpha \to 0$ as $n \to \infty$.*
2. *The map $\alpha \in L_M$ if and only if $\alpha^n \to 0$ as $n \to \infty$.*
3. *The map $\alpha \in \Gamma_M$ if and only if $p^n\alpha \to 0$ and $\alpha^n \to 0$ as $n \to \infty$*

The relationships between the other ideals and Γ will be important, so they are listed here.

Lemma 4.6 *For a reduced module M with an unbounded torsion submodule, the following relationships exist between the ideals of the Jacobson radical of the endomorphism ring of M:*

- $pHom(M, T_M) \subseteq \Gamma_M$.
- $H_M \subseteq tJ_M \subseteq \Gamma_M$

4.2 Embedding into the Cotorsion Hull using the Jacobson Radical

Let $M \in \mathfrak{D}(R)$ and let N be a reduced module. If $E_M \cong E_N$, then N is also in $\mathfrak{D}(R)$ and the standard procedure is to use the Baer-Kaplansky theorem to construct an isomorphism $\phi : T_M \rightarrow T_N$ and use ϕ to embed N into T^* as detailed in Lemma 2.12. An isomorphism between the Jacobson radicals of two modules affords the same technique. The following results are from Flagg [5]

Proposition 4.7 *Let $M \in \mathfrak{D}(R)$ and let N be a module such that there exists an R-algebra isomorphism $\Phi : J_M \rightarrow J_N$. Then $N \in \mathfrak{D}(R)$ and there exists an isomorphism $\phi : T_M \rightarrow T_N$ which induces Φ on the torsion.*

Corollary 4.8 *Given modules M and N in $\mathfrak{D}(R)$ with an R-algebra isomorphism $\Phi : J_M \rightarrow J_N$. Then the isomorphism ϕ given by Proposition 4.7 implies there is an embedding of N into T^* such that*

- *N is a pure submodule of T^* containing T and N/T is torsion-free and divisible.*
- *Under the embedding $J_M = J_N$ as subrings of E^*.*

Note that the image of J_M in $E(T^*)$ is not assumed to be the Jacobson radical of $E(T^*)$, simply a subring.

5 The Topological Jacobson Radical Isomorphism Theorem

Given modules M and N in the class $\mathfrak{D}(R)$, an algebraic isomorphism between the Jacobson radicals of their endomorphism rings implies $T = T_M \cong T_N$ by Proposition 4.7 and that M and N are pure submodules of T^* containing T with M/T and N/T torsion-free divisible and $J_M = J_N$ by Theorem 4.8. The similarity of this situation with the setting of the full endomorphism ring topological isomorphism theorem in the cotorsion hull leads one to question if the equality of only the Jacobson radicals is sufficient to show that the modules are isomorphic.

5.1 The Hull $\widehat{M}$ from a Jacobson Radical Perspective

Let M be a module in the class $\mathfrak{D}(R)$. This section shows that the same associated module $\widehat{M}$ defined in Lemma 3.3 may also be defined from the result in Lemma 4.5 that the ideal Γ_M is invariant under topological isomorphism.

Lemma 5.1 *Let M be a module in the class $\mathfrak{D}(R)$ with torsion submodule T, and let M' be a reduced module such that $M \subseteq M'$ and $tM' = T$. The following properties for the module M' are equivalent.*

1. *The induced map $\Gamma_{M'} \to \Gamma_M$ is a topological isomorphism.*
2. *M' has the TF Divisible and Open Sets Properties listed below:*

   TF Divisible Property: *M'/M is torsion-free and divisible.*
   Open Sets Property: *For every $x \in M'$ there exists finitely many elements $y_1, \ldots, y_n \in M$ such that $x\beta' = 0$ for every $\beta' \in Hom(M', T^*)$ such that $M\beta' \subseteq T$ and $y_i\beta' = 0$ for $1 \leq i \leq n$.*

3. *The induced map $Hom(M', T) \to Hom(M, T)$ is a topological isomorphism.*

Proof The equivalence of Properties 2 and 3 is Lemma 1 of [14]. To show that Property 1 is equivalent to the others is where the justification is needed. First, assume that the induced map $Hom(M', T) \to Hom(M, T)$ is a topological isomorphism. Then the restricted map $\Gamma_{M'} \to \Gamma_M$ is a topological isomorphism by Lemma 4.5.

To complete the proof, assume $\iota : \Gamma_{M'} \to \Gamma_M$ given by $\beta'\iota = \beta'|_M$ for all $\beta' \in \Gamma_{M'}$ is a topological isomorphism. The proof will be complete if we show that the TF Divisible and Open Sets Properties hold for M'. By Lemma 4.6, $tJ_{M'}$ is in $\Gamma_{M'}$ and is mapped onto tJ_M under isomorphism. By Proposition 4.4, $M \in \mathfrak{D}(R)$ implies $H_M = 0$. By isomorphism, $H_{M'} = 0$ and another application of Proposition 4.4 shows M'/T is torsion-free and divisible. Since M' is reduced and the torsion of M' is unbounded, $M' \in \mathfrak{D}(R)$. The fact that $M'/M \cong (M'/T)/(M/T)$ implies M'/M is also torsion-free and divisible.

To show that the Open Sets Property holds, first note that since M' is reduced and M'/M is torsion-free and divisible, M' may be embedded into M^* as a pure submodule containing M. Then, since M/T is divisible, $M^* \cong T^*$, so both M and M' may be identified as submodules of T^* with $M \subseteq M'$. The set of maps $\beta' \in Hom(M', T^*)$ such that $M'\beta' \in T$ is simply $Hom(M', T)$ identified as a subring of $End(T^*)$. The Open Sets Property would follow directly from a topological isomorphism between $Hom(M', T)$ and $Hom(M, T)$, but only a topological isomorphism between $\Gamma_{M'}$ and Γ_M is assumed. Therefore, extra steps are needed. Let $x \in M'$. Since M'/T is divisible, there exists a $c \in M'$ such that $x + T = pc + T$ and thus $x = pc + t$ for some $t \in T$. Let $U_c = \{\alpha \in \Gamma_{M'} : c\alpha = 0\}$. As an open set in $\Gamma_{M'}$, its image under the isomorphism ι is open in Γ_M. There exists a set $U_Y = \{\phi \in \Gamma_M : y\phi = 0, \forall y \in Y\}$ for a finite set $Y = \{y_1, y_2, \ldots y_n \in M\}$ which is contained in $U_c\iota$. By the topological isomorphism and for all $\phi \in \Gamma_{M'}$, if

$\phi|_M = \phi\iota \in U_Y$, then $\phi \in U_c$. In other words, if $y_i\phi = 0$ for all $y_i \in Y$, then $c\phi = 0$. For the given $x = pc + t \in M'$, let $Z = Y \cup \{t\}$. Then, for any $\beta' \in Hom(M', T)$, $p\beta' \in \Gamma_{M'}$ by Lemma 4.6. The fact that $x\beta' = c(p\beta') + t\beta'$ means that if $Z\beta' = 0$, then $t\beta' = 0$. By the topological isomorphism, $U_Y\phi = 0$ implies $c\phi = c(p\beta') = 0$. Hence, $x\beta' = 0$ as desired. □

Lemma 5.2 *Let M be a module in the class* $\mathfrak{D}(R)$.

1. *There exists a maximal module* $\widehat{M}$ *with* $M \subseteq \widehat{M}$ *and* $t\widehat{M} = T$ *such that the equivalent properties of Lemma 5.1 hold. Any two such modules are isomorphic by a unique isomorphism which is the identity on M.*
2. $\widehat{M}/M$ *is torsion-free and divisible and* $\widehat{M}$ *may be regarded as a unique submodule of* T^*.
3. *Every* $\alpha \in E_M$ *extends uniquely to an* $\overline{\alpha} \in End(\widehat{M})$ *and it is the restriction of the unique extension* $\alpha^* \in E^*$ *to the submodule* $\widehat{M}$.

Proof The module $\widehat{M}$ is defined to be the sum of all submodules of $M' \subseteq T^*$ which satisfy the properties of Lemma 5.1 and the proof is the same as that in [14]. □

5.2 The Proof of the Theorem

The machinery is now in place to prove a Jacobson radical version of the topological isomorphism theorem.

Theorem 5.3 *Let M be a module in the class* $\mathfrak{D}(R)$ *and let N be a reduced R-module. For a module* $A \in \mathfrak{D}(R)$, *let the module* $\widehat{A}$ *corresponding to A be the hull defined in Lemma 5.2. If there exists a topological isomorphism* $\Phi : J_M \to J_N$, *then* Φ *is induced by an isomorphism* $\phi : \widehat{M} \to \widehat{N}$.

Proof Let M and N be R-modules with $M \in \mathfrak{D}(R)$ and let $\Phi : J_M \to J_N$ be a topological isomorphism. By Proposition 4.7, $N \in \mathfrak{D}(R)$ and there exists an isomorphism $\phi : T_M \to T_N$ that induces Φ on the torsion. Let $T = T_M$ be identified as the common torsion submodule of M and N. Then, embed M in $T^* \cong M^*$ and embed N into T^* using the isomorphism ϕ and Corollary 4.8. Under this embedding, $J_M = J_N$ as subrings of $End(T^*)$. Since the isomorphism Φ is topological, Lemma 4.5 implies $\Gamma_M\Phi = \Gamma_N$ and thus $\Gamma_M = \Gamma_N$ as subrings of $End(T^*)$.

Define $\widehat{M}$ as the maximal submodule of T^* with the property that $M \subseteq \widehat{M}$, $t\widehat{M} = T_M = T$, $\widehat{M}/M$ is torsion-free and divisible and the induced map $\Psi_M : \Gamma_{\widehat{M}} \to \Gamma_M$ is a topological isomorphism using Lemma 5.2. Since $N \in \mathfrak{D}(R)$, also define $\widehat{N}$ with the corresponding properties. As submodules of T^* under the embedding, $M \subseteq \widehat{M}$ and $N \subseteq \widehat{N}$. Embed $\Gamma_{\widehat{M}}$ and $\Gamma_{\widehat{N}}$ into $End(T^*) = E^*$ using the isomorphisms Ψ_M and Ψ_N, which implies $\Gamma_{\widehat{M}} = \Gamma_M = \Gamma_N = \Gamma_{\widehat{N}}$ as subrings of E^*. The theorem will be proved if $\widehat{M} = \widehat{N}$ as submodules of T^*.

To show $\widehat{M} = \widehat{N}$, the symmetry of the argument shows that is sufficient to prove $\widehat{M} \subseteq \widehat{N}$. Let $N' = \widehat{M} + N$. If we can show N' has the TF Divisible and Open Sets Properties of Lemma 5.1, then by definition of $\widehat{N}$, $N' \subseteq \widehat{N}$ and thus $\widehat{M} \subseteq \widehat{N}$.

First, by definition of N', N'/T is torsion-free and divisible. Then $N'/N \cong (N'/T)/(N/T)$ is torsion-free and divisible. Second, it is necessary to show that N' has the Open Sets Property of Lemma 5.1. Note that $x \in N'$ if and only if $x = a + n$ for some $a \in \widehat{M}$ and some $n \in N$. Elements of N satisfy the Open Sets Property trivially, so without loss of generality, assume $x \in \widehat{M}$. Let $\nu' \in Hom(N', T)$ and recall that N' is viewed as a submodule of T^* which means that $Hom(N', T)$ is equal to the set of all maps in $Hom(N', T^*)$ which map into the torsion. To show that N' has the Open Sets Property, it is necessary to show that for the given x and any map $\nu' \in Hom(N', T)$ there exists a finite set of elements $\overline{Y} = \{y_1, y_2, \ldots, y_k\} \subseteq N$ such that $x\nu' = 0$ whenever $y_i\nu' = 0$ for $1 \leq i \leq k$. The complication is that $x \in \widehat{M}$ and the isomorphism Ψ_M does not directly connect open sets in N with open sets in $\widehat{M}$. Also, the map $\nu' \in Hom(N', T)$ not necessarily in $\Gamma_{N'}$, which also must be addressed in the following argument.

The module $\widehat{M}$ is divisible modulo T, thus we may choose elements $c \in \widehat{M}$ and $t \in T$ such that $x = pc + t$. Then, $x\nu' = c(p\nu') + t\nu'$. The map $\nu' \in Hom(N', T)$ has a unique extension to $\nu^* \in E^*$. The restriction of $p\nu'$ to N is an element of Γ_N, and $p\nu'|_N = p\nu^*|_N$. The equality of the ideals $\Gamma_{\widehat{M}} = \Gamma_M = \Gamma_N = \Gamma_{\widehat{N}}$ shows that $p\nu^* \in \Gamma_{\widehat{M}}$ and $p\nu^* \in \Gamma_M$. The module $\widehat{M}$ satisfies Lemma 5.2 so for the element $c \in \widehat{M}$ and the map $p\nu^* \in \Gamma_{\widehat{M}}$, there exists a finite set Z of elements of M such that if $z_i p\nu^* = 0$ for all $z_i \in Z$, then $cp\nu^* = 0$. The isomorphism $\Phi : \Gamma_M \to \Gamma_N$ is topological, so since $U_Z = \{\alpha \in \Gamma_M : Z\alpha = 0\}$ is open in Γ_M in the finite topology, then there exists an open set $U_Y \in \Gamma_N$ for a finite set of elements $y_1, y_2, \ldots, y_n$, with $y_i \in N$ for all $1, 2, \ldots, n$, such that $U_Y(\Phi^{-1}) \subseteq U_Z$. In other words, if $y_i(p\nu^*) = 0$ for all $y_i \in Y$, then $zp\nu^* = 0$ for all $z \in Z$, which means $cp\nu^* = 0$. Let $\overline{Y} = Y \cup \{t\}$. Then, if $a\nu^* = 0$ for all $a \in \overline{Y}$, then $x\nu' = x\nu^* = c(p\nu^*) + t\nu^* = 0$. This shows N' has the Open Sets Property, proving the theorem. ◻

The hulls $\widehat{M}$ and $\widehat{N}$ are the same submodules of T^* as given in May [14]. The following is May's Lemma 2.12.

Lemma 5.4 *Assume that R is complete and let M be a reduced module with unbounded torsion. If M^1 is a cotorsion module, then $\widehat{M} = M$.*

Note that Lemma 5.4 is valid for modules in the class $\mathfrak{D}(R)$. Thus the proof is unchanged. A direct consequence of Lemma 5.4 and Theorem 5.3 is the following theorem.

Theorem 5.5 *Assume that R is complete and $M \in \mathfrak{D}(R)$. Let $\Phi : J_M \to J_N$ be a topological isomorphism.*

1. *If the first Ulm submodule M^1 is a cotorsion module, then Φ is induced by an injection of N into M.*
2. *If M^1 is a cotorsion module, then every topological isomorphism of E_M is inner.*
3. *If M^1 has bounded torsion and finite torsion-free rank, then Φ is induced by an isomorphism of N with M. In particular, this is true if $M^1 = 0$.*

Proof Given $M \in \mathfrak{D}(R)$ with a topological isomorphism $\Phi : J_M \to J_N$, Theorem 5.3 shows that Φ is induced by an isomorphism $\phi : \widehat{M} \to \widehat{N}$. By Lemma 5.4, $\widehat{M} = M$. Thus $\phi^{-1} : \widehat{N} \to M$ is an isomorphism and since $N \subseteq \widehat{N}$, $\phi^{-1} : N \to M$ induces Φ. The fact that $\widehat{M} = M$ in the case of M^1 cotorsion proves (2). To prove (3), note that if M^1 has bounded torsion and finite torsion-free rank, it is cotorsion. Since there exists an injection $N \to M$, $N^1 \subseteq M^1$, so it is of the same form and cotorsion. Hence, Lemma 5.4 shows $\widehat{N} = N$. Then $\phi : M \to N$ is an isomorphism inducing Φ. □

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A Note on Hieronymi's Theorem: Every Definably Complete Structure Is Definably Baire

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Abstract We give an exposition and strengthening of P. Hieronymi's Theorem: if C is a nonempty closed set definable in a definably complete expansion of an ordered field, then C satisfies an analogue of Baire's Category Theorem.

Keywords Baire • Definably complete

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1 Introduction

The real line is Dedekind Complete: every subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R} \cup \{\pm\infty\}$. The notion of being Dedekind Complete is clearly not first-order. People have been studying a weaker, but *first-order*, version of Dedekind Completeness: a structure $\mathbb{K}$ expanding an ordered field is **Definably Complete** (**DC**) if every *definable* subset of $\mathbb{K}$ has a least upper bound in $\mathbb{K} \cup \{\pm\infty\}$.

Examples of DC structures are: all expansions of the real field, o-minimal structures, and ultra-products of DC structures.

DC structure were introduced in [14], where it was further observed that definable completeness is equivalent to the intermediate value property for definable functions; it is also shown in [2, 3, 8, 11, 14, 17] that most results of elementary real analysis can be generalized to DC structures (see Sect. 2 for some examples). Several people have also proved definable versions of more difficult results: for instance, in [1] they transferred a theorem on Lipschitz functions by Kirszbraun and Helly, in [8, 9] we considered Wilkie's and Speissegger's theorems on o-minimality of Pfaffian functions (see also [10] for a more expository version), while in [7] we

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considered Hieronymi's dichotomy theorem and Lebesgue's differentiation theorem for monotone functions (and some other results from measure theory).

On the other hand, not every first-order property of structures expanding the real field can be generalized to DC structures: for instance, [13] shows that there exists a first-order sentence which true in any expansion of the real field but false in some o-minimal structures (see also [16] for a related result).

In this note we will focus on a first-order version of Baire Category Theorem. Tamara Servi and I conjectured in [8] that every DC structure is definably Baire (see Definition 1.1). In [12], Philipp Hieronymi proved our conjecture. The aim of this note is to give an alternative proof of Hieronymi's Theorem, together with a generalization of Hieronymi's and Kuratowski–Ulam's theorems to definable closed subsets of $\mathbb{K}^n$.

We recall the relevant definitions.

Definition 1.1 ([8]) Let $A \subseteq B \subseteq \mathbb{K}^n$ be definable sets.

A is nowhere dense in B if the closure of A has interior (inside B); otherwise, A is somewhere dense in B.

A is **definably meager** in B if there exists a definable increasing family $\left(Y_t : t \in \mathbb{K}\right)$ of nowhere dense subsets of B, such that $A \subseteq \bigcup_t Y_t$).

- A is **definably residual** in B if $B \setminus A$ is definably meager in B.
- B is **definably Baire** if every nonempty open definable subset of B is not definably meager in B (or, equivalently, in itself).

A is an $\mathscr{F}_\sigma$ subset of B if there exists a definable increasing family $\left(Y_t : t \in \mathbb{K}\right)$ of closed subsets of B, such that $A = \bigcup_t Y_t$; $A \subseteq B$ is a $\mathscr{G}_\delta$ subset of B if $B \setminus A$ is an $\mathscr{F}_\sigma$ subset of B; if we don't specify the ambient space B, we mean that $B = \mathbb{K}^n$.

Fact 1.1 ([8] §3) *A finite Boolean combination of closed definable subsets of $\mathbb{K}^n$ is $\mathscr{F}_\sigma$ in $\mathbb{K}^n$. Moreover, for every $n \in \mathbb{N}$, the family of $\mathscr{F}_\sigma$ subsets of $\mathbb{K}^n$ is closed under finite union and intersections. Besides, if $X \subseteq \mathbb{K}^n$ is $\mathscr{F}_\sigma$ and $g : \mathbb{K}^n \to \mathbb{K}^m$ is a definable continuous function, then $g(X)$ is also $\mathscr{F}_\sigma$.*

The main result of this note is the following.

Theorem 1.2 (Baire Category) *Let $C \subseteq \mathbb{K}^n$ be a nonempty $\mathscr{G}_\delta$ subset of $\mathbb{K}^n$. Then, C is definably Baire.*

Notice that the case when $C = \mathbb{K}^n$ in Theorem 1.2 is exactly Hieronymi's Theorem (see [12]). We will prove Theorem 1.2 in Sect. 5.

We denote by $\Pi^{m+n}_m : \mathbb{K}^{n+m} \to \mathbb{K}^m$ the projection onto the first m coordinates and, given $C \subseteq \mathbb{K}^{n+m}$ and $\bar{x} \in \mathbb{K}^m$, by $C_{\bar{x}} := \{\, \bar{y} \in \mathbb{K}^n : \langle \bar{x}, y \rangle \in C \,\}$ the fiber of C at $\bar{x}$.

Theorem 1.3 (Kuratowski–Ulam) *Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E := \Pi^{n+m}_m(C)$. Let $F \subseteq C$ be a definable set. Let*

$$T := T^m_C(F) := \{\, \bar{x} \in E : F_{\bar{x}} \text{ is definably meager in } C_{\bar{x}} \,\}.$$

Assume that F is definable meager in C. If either F or C is $\mathscr{F}_\sigma$ in $\mathbb{K}^{m+n}$, then T contains some $T' \subseteq C$ such that T' is definable and definably residual in E.

Notice that the case when $C = \mathbb{K}^{m+n}$ in Theorem 1.3 is [8, Theorem 4.1]. We will prove Theorem 1.3 in Sect. 4.

As an application, we prove the following results.

Corollary 1.4 *Let C be a nonempty, closed, bounded, and definable subset of $\mathbb{K}^{m+n}$, and $A := \Pi^{n+m}_m(C)$. Define $f : A \to \mathbb{K}^n$, $f(\bar{x}) := \operatorname{lex\,min}(C_{\bar{x}})$. Let E be the set of $\bar{x} \in A$ such that either $\bar{x}$ is an isolated point of A, or f is continuous at $\bar{x}$. Then, E is definably residual in A, and therefore it is dense in A.*

Proof By [2, 1.9] (see Fact 2.3), E is definably residual in A. By Theorem 1.2, A is definably Baire, and every definably residual subset E of a definably Baire set A is dense in A. □

Corollary 1.5 *Let $F \subseteq C \subseteq \mathbb{K}^{m+n}$ be nonempty definable, closed subsets of $\mathbb{K}^{m+n}$. Let $E := \Pi^{n+m}_m(C)$. Assume that E is closed inside $\mathbb{K}^m$, and that the set*

$$T' := T'^m_C(F) := \{\, \bar{x} \in E : F_{\bar{x}} \text{ has no interior inside } C_{\bar{x}} \,\}$$

is not dense in E. Then, F has interior inside C.

Proof By Theorem 1.2 (applied to each fiber $C_{\bar{x}}$), $T' = T^m_C(F)$. By Theorem 1.2 again (applied to the set E), T' is not definably residual inside E. Thus, by Theorem 1.3, F is not meager inside C; therefore, F is somewhere dense inside C, and thus it has interior inside C. □

Question 1.6 What is the most general form of Theorem 1.3? For instance, can we drop the assumption that either F or C are $\mathscr{F}_\sigma$ subsets of $\mathbb{K}^{m+n}$? Can we prove that the set $T^m_C(F)$ is definable?

Definition 1.7 ([3] §4) A pseudo-$\mathbb{N}$ set is a set $\mathscr{N} \subset \mathbb{K}_{\geq 0}$, such that $\mathscr{N}$ is definable, closed, discrete, and unbounded.

A quasi-order $\langle D, \trianglelefteq \rangle$ is a forest if, for every $a \in D$, the set $\{\, c \in D : c \trianglelefteq a \,\}$ is totally ordered by $\triangleleft$.

The following lemma is at the core of the proof: we hope it may be of independent interest.

Lemma 1.8 (Leftmost Branch) *Let $\mathscr{N}$ be a pseudo-$\mathbb{N}$ set. Let $\trianglelefteq$ be a definable quasi-order of $\mathscr{N}$ (i.e., $\trianglelefteq$ is a reflexive and transitive binary relation on $\mathscr{N}$, whose graph is definable). Assume that $\trianglelefteq$ is a forest. Then, there exists a definable set $E_0 \subseteq \mathscr{N}$, such that:*

1. *the minimum of $\mathscr{N}$ is in E_0;*
2. *for every $d \in E_0$, the successor of d in E_0 (if it exists) is*

$$n(d) := \min\{\, e \in \mathscr{N} : d < e \;\&\; d \triangleleft e \,\};$$

conversely, if $n(d)$ exists, then it is the successor of d in E_0.

Furthermore, E_0 is unique, satisfying the above conditions. Besides, $\leq$ and $\trianglelefteq$ coincide on E_0 (and, in particular, E_0 is linearly ordered by $\trianglelefteq$).

If moreover we have

$$(*) \quad \textit{For every } d \in \mathscr{N} \textit{ there exists } e \in \mathscr{N} \textit{ such that } d \lhd e,$$

then E_0 is unbounded (and hence cofinal in $\mathscr{N}$).

We call the set E_0 defined in the above lemma the leftmost branch of $\trianglelefteq$ (inside $\mathscr{N}$); notice $\mathscr{N}$ has a minimum, and every element of $\mathscr{N}$ has a successor in $\mathscr{N}$ (see [3, §4]).

The proof of a particular case of the above lemma is given in [12, Definition 12, Lemma 14, Definition 15, Lemma 16, Lemma 17], where $d \lhd e$ if "$f(e)$ extends $f(d)$" (in [12] terminology). We will give a sketch of the proof in Sect. 3. P. Hieronymi pointed out a mistake in a previous version of these notes, when we did not require the condition that $\trianglelefteq$ is a forest in Lemma 1.8 (see Sect. 3.1).

Let $\mathscr{N}$ be a pseudo-$\mathbb{N}$ set. While it is quite clear how to prove statements about elements of $\mathscr{N}$ by (a kind of) induction (see [3, Remark 4.15]), a priori it is not clear how to construct (definable) sets by recursion: Lemma 1.8 gives a way to produce a definable set E_0 whose definition is recursive; this will allow us to prove that $\mathbb{K}$ is definably Baire (see Sect. 5.1). However, to prove that a $\mathscr{G}_\delta$ set $C \subseteq \mathbb{K}^n$ is definably Baire we need to use a different method (since, for technical reason, the proof in Sect. 5.1 requires the assumption that C contains a dense pseudo-enumerable set, and we do not know if the assumption holds for C), that relies on Theorem 1.3, used inductively (see Sect. 5.2).

In [7] we gave a completely different proof of Hieronymi's Theorem, based on our Dichotomy Theorem: either $\mathbb{K}$ is "unrestrained" (i.e., $\mathbb{K}$ is, in a canonical way, a model of the first-order formulation of second-order arithmetic, and therefore any of the classical proofs of Baire's Category Theorem generalize to $\mathbb{K}$), or $\mathbb{K}$ is "restrained" (and many "tameness" results from o-minimality hold in $\mathbb{K}$, allowing a relatively straightforward proof of Hieronymi's Theorem). When we are in the unrestrained situation, the same reasoning gives a proof of Theorem 1.2. However, when $\mathbb{K}$ is restrained, it was not clear how to prove Theorem 1.2 for $C \neq \mathbb{K}^n$.

2 Preliminaries

Fact 2.1 ([8] Proposition 2.11) *Let $U \subseteq \mathbb{K}^n$ be open and definable. U is definably meager in $\mathbb{K}^n$ iff U is definably meager in itself.*

Definition 2.1 A d-compact set is a definable, closed, and bounded subset of $\mathbb{K}^n$ (for some n).

The following fact will be used many times without mentioning it explicitly.

Fact 2.2 (**[14]**)

1. *Let $X \subseteq \mathbb{K}^n$ be a d-compact set, and $f : X \to \mathbb{K}^m$ be a definable and continuous function. Then, $f(X)$ is d-compact.*
2. *Let $\big(X(t) : t \in \mathbb{K}\big)$ be a definable decreasing family of nonempty d-compact subsets of $\mathbb{K}^n$. Then, $\bigcap_t X(t)$ is nonempty.*

Definition 2.2 Let $a \in \mathbb{K}^n$, $X \subseteq \mathbb{K}^n$, and $r > 0$. Define

$$B(a; r) := \{x \in \mathbb{K}^n : |x - a| < r\};$$

$$\overline{B}(a; r) := \{x \in \mathbb{K}^n : |x - a| \leq r\};$$

$$B_X(a; r) := X \cap B(a; r);$$

$$\overline{B}_X(a; r) := X \cap \overline{B}(a; r).$$

Given $Y \subseteq X$, denote by $\mathrm{cl}_X(Y)$ (resp., $\mathrm{int}_X(Y)$) the topological closure (resp., the interior) of Y inside X, and denote $\mathrm{cl}(Y) := \mathrm{cl}_{\mathbb{K}^n}(Y)$.

Remark 2.3 Let X be a topological space and $A \subseteq X$. A is somewhere dense in X iff there exists $V \neq \emptyset$ an open subset of X, such that, for every $W \neq \emptyset$ open subset of W, $W \cap A \neq \emptyset$.

Lemma 2.4 *1. Let X be a topological space, U be a dense subset of X, and $A \subseteq X$ be any subset. T.f.a.e.:*

a. A is nowhere dense in X;
b. $A \cap U$ is nowhere dense in X;
c. $A \cap U$ is nowhere dense in U.

2. Let $X \subseteq \mathbb{K}^n$ be a definable, U be a dense open definable subset of X, and $A \subseteq X$ be any definable subset. T.f.a.e.:

a. A is definably meager in X;
b. $A \cap U$ is definably meager in X;
c. $A \cap U$ is definably meager in U.

3. Let $X \subseteq \mathbb{K}^n$ be a definable set and $U \subseteq X$ be a definable dense open subset of X. Then, X is definably Baire iff U is definably Baire.

Proof (1) follows from Remark 2.3.
(2) follows from (1).
(3) follows from (2). □

Corollary 2.5 *Let $A \subseteq X \subseteq \mathbb{K}^n$ be definable nonempty sets. Assume that A is a dense subset of X.*

1. *If A is definably Baire, then X is also definably Baire.*
2. *If X is definably Baire and A is a $\mathcal{G}_\delta$ subset of X, then A is also definably Baire.*

Proof

(1) is clear from Lemma 2.4(1).
(2) Assume, for a contradiction, that A is not definably Baire. We can easily reduce to the case when A is definably meager in itself. Let $F := X \setminus A$. By our assumption on A, F is an $\mathscr{F}_\sigma$ subset of X with empty interior; thus, F is definably meager in X. Since X is definably Baire and $X = A \cup F$, A is not definably meager in X. Since A is definably meager in itself, $A = \bigcup_t C_t$, for some definable increasing family $\big(C_t : t \in \mathbb{K}\big)$ of nowhere dense subsets of A. By Lemma 2.4(1), each C_t is also nowhere dense in X, contradicting the fact that A is not definably meager in X. □

Definition 2.6 Let $E \subseteq \mathbb{K}^m$ and $f : E \to \mathbb{K}^n$ be definable. Given $\varepsilon > 0$, define

$$\mathscr{D}(f;\varepsilon) := \{\,\bar{a} \in E : \forall \delta > 0\, f(B_E(\bar{a};\delta)) \not\subseteq B(f(\bar{a});\varepsilon)\,\}.$$

Fact 2.3 ([2] 1.9) *Let $C \subset \mathbb{K}^{m+n}$ be a nonempty d-compact set and $E := \Pi^{m+n}_m(C) \subset \mathbb{K}^m$. Define $f : E \to \mathbb{K}$, $f(x) := \operatorname{lex\,min}(C_{\bar{x}})$. Then, for every $\varepsilon > 0$, $\mathscr{D}(f;\varepsilon)$ is nowhere dense in E.*

Conjecture 2.7 Let X_1, X_2 be definable subsets of $\mathbb{K}^n$, and $X := X_1 \cup X_2$. If both X_1 and X_2 are definably Baire, then X is also definably Baire.

3 Proof of Lemma 1.8

We will proceed by various reductions. Define

$$E := \{\, d \in \mathscr{N} : (\forall e \in \mathscr{N})\; e < d \to d \not\trianglelefteq e\,\}.$$

Let $e_0 := \min(\mathscr{N})$. Define

$$E_1 := \{\, d \in E : e_0 \trianglelefteq d\,\}.$$

Define E_2 as the set of elements $d \in E_1$, such that d is the minimum of the set $\{\, d' \in E_2 : d' \trianglelefteq d \;\&\; d \trianglelefteq d'\,\}$. Notice that E_2 satisfies the following conditions, for all $d, d' \in E_2$:

1. $e_0 \in E_2$;
2. $e_0 \le d$;
3. $n(d) \in E_2$;
4. $d \trianglelefteq d' \to d \le d'$;
5. $\trianglelefteq$ is a partial order on E_2;
6. $\langle E_2, \trianglelefteq\rangle$ is a forest.

For every $a, b \in E_2$, define $a \perp b$ if $a \not\trianglelefteq b$ and $b \not\trianglelefteq a$. Given $a, b \in E_2$ such that $a \perp b$, define $c(a,b) := \min\{ a' \in E_2 : a' \trianglelefteq a \ \&\ a' \perp b \}$ (where the minimum is taken w.r.t. $\leq$). Notice that if $a \perp b$, $a' \trianglelefteq a$, and $a' \perp b$, then $c(a,b) \leq a'$ and, by (iv), either $c(a,b) \trianglelefteq a'$, or $c(a,b) \perp a'$; moreover, $c(a,b) \leq a$, $c(a,b) \trianglelefteq a$, and $c(a,b) \perp b$.

Finally, define E_0 as the set of $a \in E_2$, such that, for every $b \in E_2$, if $b \perp a$, then $c(a,b) < b$.

We have to show that E_2 is the leftmost branch of $\trianglelefteq$ inside $\mathscr{N}$. W.l.o.g., we can assume that $\mathscr{N} = E_2$.

Claim 3.1 *For every $a \in E_0$, $n(a) \in E_0$.*

Assume not. Let $a \in E_0$ be such that $b := n(a) \notin E_0$. Thus, by definition, there exists $d \in \mathscr{N}$ such that $c := c(b,d) \geq d$. If $d \trianglelefteq a$, then $d \trianglelefteq b$, absurd. If $a \trianglelefteq d$, then $d \geq b$ because $b = n(a)$, also absurd. If $d \perp a$, then, since $a \in \mathscr{N}$, we have that $c' := c(a,d) < d$. Moreover, $c' \trianglelefteq a \triangleleft b$ and $c' \perp d$; thus, by definition, $c \leq c'$, and therefore $c' > d$, absurd.

The next claim is the only place where we use the fact that $\mathscr{N}$ is a forest.

Claim 3.2 *Let $b \in E_0$ and $a \in \mathscr{N}$ with $a \trianglelefteq b$. Then, $a \in E_0$.*

Assume not. Let $d \in \mathscr{N}$ such that $d \perp a$ and $c := c(a,d) \geq d$. Since $c \leq a$ and $a \neq d$, we have $d < a$, and therefore $d < b$. If $d \trianglelefteq b$, then, since $\langle \mathscr{N}, \trianglelefteq \rangle$ is a forest, we have $d \trianglelefteq a$, absurd. Thus, we have $d \perp b$. Since $b \in E_0$, we have $c' := c(b,d) < d$. Moreover, since $c \perp d$ and $c \trianglelefteq b$, the definition of c' implies $c' \leq c$, and therefore, since $\langle \mathscr{N}, \trianglelefteq \rangle$ is a forest, $c' \trianglelefteq c$. Conversely, since $c' \trianglelefteq a$ and $c' \perp d$, the definition of c implies $c \leq c'$, and therefore $c = c' < d$, absurd.

Claim 3.3 *$\leq$ and $\trianglelefteq$ coincide on E_0.*

Assume not. By (iv), there exist $a, b \in E_0$, such that $a \perp b$; let $a \in E_0$ be minimal such there exists $b \in E_0$ with $a \perp b$. Let $c := c(b,a)$. By Claim 3.2, since $c \trianglelefteq b$, we have $c \in E_0$. Since $b \in E_0$, we have $c < a$, contradicting the minimality of a. □

3.1 A Counterexample: The Forest Hypothesis is Necessary

The following counterexample is due to P. Hieronymi.[1] We show that the conclusion of Lemma 1.8 may fail if we remove the assumption that $\langle \mathscr{N}, \trianglelefteq \rangle$ is a forest, even under the assumption that $\mathbb{K}$ expands the reals and satisfies strong "tameness" condition (i.e., d-minimality).

Let $P := \{ 2^{2^n} : n \in \mathbb{N} \}$ and let $\mathbb{K} := \langle \mathbb{R}, P \rangle$ be the expansion of the real field by a predicate for P; by [15], $\mathbb{K}$ is d-minimal. It is quite clear that P is a pseudo-$\mathbb{N}$ set. We now define a partial ordering on P. For every $a \in P$, denote by $s(a)$ the successor

[1]Private communication.

of a in P, i.e. $s(a) = a^2$. Let $a, b \in P$; define $a \trianglelefteq b$ iff either $a = b$ or $b > s(a)$ (that is, if $b \geq a^4$). It is clear that $\langle P, \trianglelefteq \rangle$ is a partially ordered set, satisfying (*), and that the leftmost branch Q of $\langle P, \trianglelefteq \rangle$ is the set of elements with even index, i.e. $Q := \{ 2^{4^n} : n \in \mathbb{N} \}$. However, the set Q is not definable in $\mathbb{K}$: see [4, Lemma 2.2] and [15] for the details.

4 Proof of Theorem 1.3

Lemma 4.1 *Let $m, n \in \mathbb{N}_{\geq 1}$. Let $\pi := \Pi^{m+n}_m$. Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E := \pi(C)$. Let $F \subseteq C$ be a d-compact definable set. Define*

$$T' := T'^{m}_{C}(F) := \{ \bar{x} \in E : \operatorname{int}_{C_{\bar{x}}}(F_{\bar{x}}) = \emptyset \}.$$

If $\operatorname{int}_C(F) = \emptyset$, then T' is definably residual in E.

Proof The proof of the Lemma is similar to [8, §4, Case 1]. Fix $\varepsilon > 0$; define

$$F(\varepsilon) := \{ \langle \bar{x}, \bar{y} \rangle \in F : B_C(\bar{y}; \varepsilon) \subseteq F_{\bar{x}} \};$$
$$X(\varepsilon) := \operatorname{cl}_C(F(\varepsilon)) = \operatorname{cl}_F(F(\varepsilon));$$
$$Y(\varepsilon) := \pi(X(\varepsilon)) \subseteq E.$$

Since $E \setminus T' \subseteq \bigcup_{\varepsilon > 0} Y(\varepsilon)$, we only have to prove the following:

Claim 4.2 *$Y(\varepsilon)$ is nowhere dense in E.*

Since F is d-compact and $X(\varepsilon)$ is closed in F, $X(\varepsilon)$ is d-compact. Since $Y(\varepsilon) = \pi(X(\varepsilon))$, $Y(\varepsilon)$ is also d-compact, and therefore it is closed in E. Assume, for a contradiction, that $Y(\varepsilon)$ is somewhere dense in A: thus, $U := \operatorname{int}_E(Y(\varepsilon)) \neq \emptyset$. Define $f : U \to \mathbb{K}^n$, $\bar{x} \mapsto \operatorname{lex\,min}(X(\varepsilon)_{\bar{x}})$; notice that $\Gamma(f) \subseteq X(\varepsilon)$. By Fact 2.3, $\mathscr{D}(f; \varepsilon/4)$ is nowhere dense in U. Thus, there exist $\bar{a} \in U$ and $\delta > 0$ such that $B_E(\bar{a}, \delta) \subseteq U \setminus \operatorname{cl}_E(\mathscr{D}(f; \varepsilon/4))$, and $\delta < \varepsilon/4$. Let $\bar{b} := f(\bar{a})$; thus, $\langle \bar{a}, \bar{b} \rangle \in \Gamma(f) \subseteq X(\varepsilon) \subseteq F$. The following Claim 4.3 contradicts the fact that F is nowhere dense in C, and therefore Claim 4.2 will follow.

Claim 4.3 *$B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1) \subseteq F$, for some $\delta_1 > 0$.*

Choose $\delta_1 > 0$ such that $\delta_1 < \delta$ and $f(B_E(\bar{a}; \delta_1)) \subseteq B(\bar{b}; \delta)$ (δ_1 exists because $\bar{a} \notin \mathscr{D}(f; \varepsilon/4)$). Let $\langle \bar{x}, \bar{y} \rangle \in B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1)$. Thus, $\bar{x} \in E$, $|\bar{x} - \bar{a}| < \delta_1$, $y \in C_{\bar{x}}$, and $|\bar{y} - \bar{b}| < \delta_1$. Therefore, $\bar{x} \in B_E(\bar{a}; \delta) \subseteq U \setminus \operatorname{cl}_A(\mathscr{D}(f; \varepsilon/4))$. Thus,

$$|\bar{y} - f(\bar{x})| \leq |\bar{y} - \bar{b}| + |\bar{b} - f(\bar{x})| \leq \delta_1 + \delta < 2\delta < \varepsilon,$$

and therefore $\bar{y} \in B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon)$. Since $\langle \bar{x}, f(\bar{x}) \rangle \in X(\varepsilon)$, we have $B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon) \subseteq F_{\bar{x}}$; thus, Claim 4.3 is proven.

Claim 4.4 $E \setminus T' \subseteq \bigcup_{\varepsilon 0} Y(\varepsilon)$.

Let $\bar{x} \in E \setminus T'$. Since $F_{\bar{x}}$ is d-compact, we have, $\mathrm{int}_{C_{\bar{x}}}(F_{\bar{x}}) \neq \emptyset$. Let $\bar{y} \in \mathrm{int}_{C_{\bar{x}}}(F_{\bar{x}})$ and $\varepsilon > 0$ such that $B_{C_{\bar{x}}}(\bar{y}; \varepsilon) \subseteq F_{\bar{x}}$. Thus, $\langle \bar{x}, \bar{y} \rangle \in F(\varepsilon) \subseteq X(\varepsilon)$, and $\bar{x} \in Y(\varepsilon)$.

Thus, by Claims 4.2 and 4.4, T' is definably residual in E. □

Proof (Proof of Theorem 1.3) The proof of the Lemma is as in [8, §4, Case 2].

Case 1: F is $\mathscr{F}_\sigma$ in $\mathbb{K}^{m+n}$. Then, $F = \bigcup_{s>0} F(s)$, for some definable increasing family $\big(F(s) : s \in \mathbb{K}\big)$ of d-compact sets. By the proof of Lemma 4.1, for each $s \in \mathbb{K}$, $E \setminus T^m(F(s)) \subseteq \bigcup_{\varepsilon>0} Y(s, \varepsilon)$, where $Y(s, \varepsilon)$ is a definable family of nowhere dense subsets of E, which is increasing in t and decreasing in ε. Thus, $T^m(F)$ is definably residual in E.

Case 2: C is $\mathscr{F}_\sigma$ in $\mathbb{K}^{m+n}$. By definition, there exists $F' \subseteq C$, such that F' is a definably meager $\mathscr{F}_\sigma$ subset of C, and $F \subseteq F'$; thus, by replacing F with F', w.l.o.g. we can assume that F is $\mathscr{F}_\sigma$ in C. Then, since C is an $\mathscr{F}_\sigma$ set, F is $\mathscr{F}_\sigma$ also in $\mathbb{K}^{m+n}$, and we can apply Case 1.

□

5 Proof of Theorem 1.2

Lemma 5.1 *Let $m \in \mathbb{N}$ and $C \subseteq \mathbb{K}^n$ be a definable nonempty set. Assume that, for every $a \in C$, there exists $U \subseteq C$, such that U is a definable neighborhood (in C) of a which is definably Baire. Then, C is definably Baire.*

Proof Let $V \subseteq C$ be a definable open nonempty subset of C. Assume, for a contradiction, that V is definably meager in itself. Let $a \in V$ and let U be a definable neighborhood (in C) of a which is definably Baire. Let $W := \mathrm{int}_C(U \cap V)$. Notice that W is a nonempty open subset of C. Since V is definably meager in itself and W is an open subset of V, W is also definably meager in itself. Since W is open in U, W is meager in U. Since W is a nonempty open subset of U and U is definably Baire, W is not definably meager in U, absurd. □

Lemma 5.2 *Let $C \subseteq \mathbb{K}^n$ be definable, closed (in $\mathbb{K}^n$), and nonempty. If C is not definably Baire, then there exists $E \subseteq C$, such that E is definable, nonempty, d-compact, and definably meager in itself.*

Proof By Lemma 5.1, there exists a d-compact B such that $C' := B \cap C$ is not definably Baire; thus, by replacing C with C', w.l.o.g. we can assume that C is d-compact. Let $U \subseteq C$ be definable, nonempty, and open in C, such that U is definably meager in itself. Let $E := \mathrm{cl}(U) = \mathrm{cl}_C(U)$. By assumption, U is an open and dense subset of E; thus, by Lemma 2.4(2) (applied to $A = X = E$), E is definably meager in itself. □

5.1 The Case $n = 1$

The first step in the proof of Theorem 1.2 is the case when $m = 1$ and C is closed. Thus, we have to prove the following lemma.

Lemma 5.3 *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Then, C is definably Baire.*

The remainder of this subsection is the proof of the above lemma.

Definition 5.4 ([3] §4) Let $C \subseteq \mathbb{K}^n$ be a definable set. C is at most pseudo-enumerable if there exists a pseudo-$\mathbb{N}$ set $\mathcal{N}$ and a definable surjective function $f : \mathcal{N} \to C$. C is pseudo-finite if it is closed, discrete, and bounded. C is pseudo-enumerable if it is at most pseudo-enumerable but not pseudo-finite. A family of sets $\big(C(t) : t \in T\big)$ is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable) if it is a definable family and its index set T is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable).

We need a few results on pseudo-enumerable sets and families.

Fact 5.1 ([3, 5])

1. *The union of two pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable) sets is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable).*
2. *Every definable discrete subset of $\mathbb{K}^n$ is at most pseudo-enumerable.*
3. *The image of a pseudo-finite set under a definable function is also pseudo-finite.*
4. *If $\big(C(t) : t \in T\big)$ is an pseudo-finite family of nowhere dense sets, then $\bigcup_{t \in T} C(t)$ is also nowhere dense.*

Lemma 5.5 *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. If C is definably meager in itself, then there exists a pseudo-enumerable set $P \subseteq C$, such that P is dense in C.*

Proof Assume that C is definably meager in itself. Let $U := \mathrm{int}_{\mathbb{K}}(C)$ and $E := C \setminus U$. Notice that E is definable, closed, and nowhere dense in $\mathbb{K}$. If U is nonempty, then, since U is open in C and definably meager in itself, then, by [3, Proposition 6.4], there is a pseudo-enumerable set $P_0 \subseteq U$, such that P_0 is dense in U. Let P_1 be the set of endpoints of $\mathbb{K} \setminus E$; notice that P_1 is at most pseudo-enumerable. Since E is nowhere dense in $\mathbb{K}$, P_1 is dense in E. Define $P := P_0 \cup P_1$. Since P is the union of two at most pseudo-enumerable sets, P is also pseudo-enumerable pseudo-enumerable, and it is dense in C. □

Lemma 5.6 *Let $C \subseteq \mathbb{K}^n$ be definable, nonempty, and definably meager in itself. Then, C has no isolated points.*

Let $U \subseteq C$ be a nonempty definable open subset of C. Then, U is not pseudo-finite: that is, there is no discrete and d-compact subset D of $\mathbb{K}$, such that there is a definable surjective function $f : D \to U$.

Proof Clear. □

Definition 5.7 Let $C \subseteq \mathbb{K}^n$ be a nonempty definable set. Let $\mathscr{A} := \big(A_i : i \in I\big)$ be a definable family of subsets of C. We say that $\mathscr{A}$ is a **weak basis** for C if:

1. for every $i \in I$, $\operatorname{int}_C(A_i) \neq \emptyset$;
2. if $U \subseteq C$ is a nonempty open subset of C, then there exists $i \in I$ such that $A_i \subseteq U$.

Lemma 5.8 *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Assume that C is definably meager in itself. Then, C has a pseudo-enumerable weak basis of d-compact sets.*

Proof By Lemma 5.5, there exists $P \subseteq C$ which is pseudo-enumerable and dense in C; thus, we can write $P := \{p_i : i \in \mathscr{N}\}$, for some pseudo-$\mathbb{N}$ set $\mathscr{N} \subset \mathbb{K}_{\geq 1}$ and some definable function $i \mapsto p_i$. For every $i \in \mathscr{N}$, define $A_i := \overline{B}_C(p_i; 1/i)$. Let $\mathscr{A} := \big(A_i : i \in \mathscr{N}\big)$. The lemma follows from the following claim.

Claim 5.9 *Let $U \subseteq C$ be a definable open nonempty subset of C. Then, there exists $i \in \mathscr{N}$ such that $C_i \subseteq U$.*

Choose $i_0 \in \mathscr{N}$ such that $q := p_{i_0} \in U$. Choose $r > 0$ such that $B_C(q; 3r) \subseteq U$.

Claim 5.10 *There exists $i \in \mathscr{N}$ such that $p_i \in B_C(q; r)$ and $i > 1/r$*

We know that C has no isolated points. Let $F := \{q\} \cup \{j \in \mathscr{N} : j \leq r\}$; notice that F is d-compact and discrete; thus, $V := B_C(q; r) \setminus F$ is open in C, and, since C is not pseudo-finite, V is nonempty. Thus, by density, there exists $i \in \mathscr{N}$ such that $p_i \in V$, proving the claim.

Then,

$$\overline{B}_C(p_i; 1/i) \subseteq B_C(p_i; 2r) \subseteq B_C(q; 3r) \subseteq U.$$

□

We also need a choice function for open sets.

Lemma 5.11

1. *Let $\bar{b}$ be a set of parameters and let $X \subseteq \mathbb{K}^n$ be a $\bar{b}$-definable set. Assume that X is nonempty and open. Then, there exists $a \in X$ which is $\bar{b}$-definable.*
2. *Let $\big(X(t) : t \in T\big)$ be a definable family of subsets of $\mathbb{K}^n$. Assume that each $X(t)$ is nonempty and open. Then, there is a definable function $f : T \to \mathbb{K}$ such that, for every $t \in T$, $f(t) \in C(t)$.*

The above lemma remains true if we weaken the hypothesis from "X open" (or each $X(t)$ open) to "X constructible" (or each $X(t)$ constructible, i.e., a finite Boolean combination of open sets), see [6]; however, the proof is more involved and we won't use the more general version.

Proof (2) follows from (1) and standard compactness arguments. Thus, we only have to show (1). It is trivial to see that it suffices to do the case when $n = 1$, and therefore we will assume that $n = 1$.

W.l.o.g., we may assume that X is bounded. For every $r > 0$, let $U(r) := \{x \in X : \overline{B}(x; r) \subseteq X\}$; since X is open, $X = \bigcup_{r>0} U(r)$. Define $r_0 := \inf\{r > 0 : U(r) \neq \emptyset\}$. Notice that $U(\frac{1}{2}r_0)$ is $\bar{b}$-definable, d-compact, nonempty, and contained in X; thus, it has a minimum element a, which is therefore $\bar{b}$-definable and in X. □

We now turn to the proof of Lemma 5.3 proper. Assume, for a contradiction, that $C \subseteq \mathbb{K}$ is nonempty, definable, closed, but it is not definably Baire. Let $E \subseteq C$ be as in Lemma 5.2. By replacing C with E, w.l.o.g. we can also assume that C is also d-compact and definably meager in itself.

By Lemma 5.8, there exists a pseudo-$\mathbb{N}$ set $\mathscr{N} \subset \mathbb{K}_{\geq 1}$ and a definable family $\mathscr{A} := \big(A_i : i \in \mathscr{N}\big)$, such that $\mathscr{A}$ is a weak basis for C of d-compact sets. Moreover, since C is definably meager in itself, there exists a definable decreasing family $\big(U_j : j \in \mathscr{N}\big)$, such that each U_j is a dense open subset of C, and $\bigcap_j U_j = \emptyset$.

Since each U_d is open and dense, and A_d has nonempty interiors, $A_d \cap U_d$ has nonempty interior. Since $\mathscr{A}$ is a weak basis, there exists $e \in \mathscr{N}$ such that $A_e \subseteq U_d \cap A_d$; since moreover C has no isolated points, we can find e as above such that $e \geq d$. For every $d \in \mathscr{N}$, let $g(d)$ be the minimum element of $\mathscr{N}$, such that $g(d) \geq d$ and $A_{g(d)} \subseteq U_d \cap A_d$. Notice that $\mathscr{A}' := \big(A_{g(d)} : d \in \mathscr{N}\big)$ is also a weak basis; thus, by replacing $\mathscr{A}$ with $\mathscr{A}'$ (and each A_d with $A_{g(d)}$), we can assume that $A_d \subseteq U_d$ for every $d \in \mathscr{N}$.

For every $d \in \mathscr{N}$, notice that $E_d := \bigcup_{e \leq d, e \in \mathscr{N}} \operatorname{bd}(A_d)$ is a pseudo-finite union of closed nowhere dense subsets of C; thus, $A_d \setminus E_d$ is non-empty and open, and therefore, by Lemma 5.11, there is a definable function $f : \mathscr{N} \to \mathbb{K}$ such that $f(d) \in A_d \setminus E_d$ for every $d \in \mathscr{N}$.

For every $a \in \mathscr{N}$, define

$$T(a) := \{d \in \mathscr{N} : d \leq a \;\&\; f(a) \in A_d\}.$$

Notice that each $T(a)$ is a pseudo-finite set, and that $a = \max(T(a))$.

We now define the following partial order on $\mathscr{N}$: $a \trianglelefteq b$ if $T(a)$ is an initial segment of $T(b)$, that is:

$$\forall c \leq a \quad c \in T(a) \leftrightarrow c \in T(b).$$

Lemma 5.12 $\big(\mathscr{N}, \trianglelefteq\big)$ *is a partially ordered set, which is a forest and satisfies condition (*) in Lemma 1.8.*

Proof $a \trianglelefteq a$ by definition.

Notice that $a \trianglelefteq b$ implies $a \leq b$, by definition. Moreover, if $a \trianglelefteq b$, then $a \in T(b)$, since $a \in T(a)$.

Claim 5.13 *If $a \trianglelefteq b$ and $b \trianglelefteq c$, then $a \trianglelefteq c$.*

In fact, let $d \leq a$. Then, $d \in T(a)$ iff $d \in T(b)$ iff $d \in T(c)$.

Thus, $\trianglelefteq$ is a partial order.

The fact that $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest is clear.

We now prove that $\trianglelefteq$ satisfies (*). Let $a \in \mathcal{N}$ and $b \leq a$. For every $a \in \mathcal{N}$, let

$$J(a) := \bigcap_{b \in T(a)} \operatorname{int} A_b \setminus \bigcup_{b \leq a \ \& \ b \notin T(a)} A_b.$$

Since each A_b is closed, and the set $\{ b \in \mathcal{N} : b \leq a \ \& \ b \notin T(a) \}$ is pseudo-finite, we have that $J(a)$ is an open set. We claim that $J(a)$ is nonempty. It suffices to prove the following claim.

Claim 5.14 $f(a) \in J(a)$.

In fact, by our choice of f, we have that, for every $b \leq a$, $b \in T(a)$ iff $f(a) \in A_b$ iff $f(a) \in \operatorname{int} A_b$; the claim is then obvious from the definition of $J(a)$.

Since C has no isolated points, the set $\{f(b) : b \leq a\}$ is pseudo-finite, and $J(a)$ is open and nonempty, the set $J'(a) := J(a) \setminus \{f(b) : b \leq a\}$ is also open and nonempty; thus, there exists $b \in \mathcal{N}$ such that $A_b \subseteq J'(a)$. The lemma then follows from the following claim.

Claim 5.15 $a \triangleleft b$.

The fact that $b > a$ is clear from the fact that $f(c) \notin A_b$ for every $c \leq a$.

Let $c \leq a$. We have to show that $c \in T(b)$ iff $c \in T(a)$. If $c \in T(a)$, then $J(a) \subseteq \operatorname{int} A_c$, therefore $f(b) \in \operatorname{int}(A_c)$, and thus $c \in T(b)$. Conversely, if $c \in T(b)$, then $f(b) \in \operatorname{int} A_c \cap J(a)$; thus, $J(a) \cap \operatorname{int} A_c \neq \emptyset$, and therefore, by definition of $J(a)$, we have $c \in T(a)$. ☐

We now continue the proof of Lemma 5.3. By Lemma 5.12, we can apply Lemma 1.8 to the partial order $\trianglelefteq$: denote by E_0 the leftmost branch of $\trianglelefteq$ inside $\mathcal{N}$.

For every $a \in E_0$, let $F_a := \bigcap_{d \in E_0 \ \& \ d \leq a} A_d$. Then, $f(a) \in F_s$, the family $\big(F_a : a \in E_0\big)$ is a definable decreasing family of d-compact nonempty sets. Therefore, by Fact 2.2

$$\emptyset \neq \bigcap_{a \in E_0} F_a = \bigcap_{d \in E_0} A_d \subseteq \bigcap_{d \in E_0} U_d = \bigcap_{d \in \mathcal{N}} U_d = \emptyset,$$

absurd. ☐

5.2 *The Inductive Step*

Lemma 5.16 *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be nonempty and d-compact. Then, C is definably Baire.*

Proof We will prove the lemma by induction on m.

Let $1 \leq m \in \mathbb{N}$. We denote by $(5.16)_m$ the instantiation of Lemma 5.16 at m. Notice that $(5.16)_1$ follows from Lemma 5.3. Thus, we assume that we have already proven $(5.16)_m$ and $(5.16)_1$; we need to prove $(5.16)_{m+1}$.

Let $C \subseteq \mathbb{K}^{m+1}$ be d-compact and nonempty. We have to show that C is definably Baire. Assume not. Let F be a definable nonempty open subset of C, such that F is definably meager in C. Define $\pi := \Pi^{m+1}_m$, and $E := \pi(C)$. By Theorem 1.3, the set $S := \{\bar{x} \in E : F_{\bar{x}}$ is not definably meager in $C_{\bar{x}}\}$ is definably meager in E. Since F is open in C, $F_{\bar{x}}$ is open in $C_{\bar{x}}$ for every $\bar{x} \in E$; thus, by $(5.16)_1$, $S = \pi(F)$. Notice that E is also d-compact and nonempty; thus, by $(5.17)_m$, E is definably Baire. Since moreover S is open in E, the fact that S is definably meager in E implies that S is empty, contradicting the fact that F is nonempty. □

Lemma 5.17 *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be closed, nonempty, and definable. Then, C is definably Baire.*

Proof We want to apply Lemma 5.1; thus, given $a \in C$; it suffices to find $A \subseteq C$, such that A is a definable neighborhood of a inside C, and A is definably Baire. Fix $r > 0$ (e.g., $r = 1$); Let $A := \overline{B}_C(a; r)$. It is clear that A is a definable neighborhood of a inside C. Moreover, A is d-compact; thus, by Lemma 5.16, A is definably Baire. □

Proof (Proof of Theorem 1.2) Let $Y := \mathrm{cl}(C)$. By Lemma 5.17, Y is definably Baire. Since C is dense in Y, the conclusion follows from Corollary 2.5. □

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Cotorsion and Tor Pairs and Finitistic Dimensions over Commutative Rings

László Fuchs

In memoriam Rüdiger Göbel

Abstract Some of the most familiar cotorsion and Tor pairs on integral domains do not extend to rings R with divisors of zero. A closer look shows that for some of them the validity hinges on the strength of torsion-freeness in R which in turn is closely related to one of the finitistic projective and weak dimensions of the classical ring Q of quotients of R. This interesting fact was observed by Bazzoni–Herbera; in their paper (Bazzoni and Herbera, Isr J Math 174:119–160, 2009) a number of important results of this kind can be found, explicitly or implicitly. We not only complement some of them with new results, but we also give different proofs for most of them, restricted to commutative rings. We consider three distinct versions of torsion-freeness, three finitistic dimensions of Q, and investigate their influence on some cotorsion and Tor pairs of major interest.

Keywords Cotorsion and Tor pairs • Flat, torsion-free, torsion, and divisible modules • Projective and weak dimension, Finitistic dimension • Injective, pure-injective and weak-injective modules • Matlis-, Enochs- and Warfield-cotorsion modules

Mathematical Subject Classification (2010): Primary 13C13; Secondary 13C11

1 Introduction

Since the groundbreaking paper [13] by Luigi Salce, the theory of cotorsion and Tor pairs has grown into a fast developing and challenging area of mathematics. It started with abelian groups and modules over integral domains, but soon it was recognized that a number of results retain their validity if zero-divisors are admitted, or even if commutativity is abandoned—as witnessed by the monograph

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Göbel–Trlifaj [9]. However, additional conditions seem to be required for some more sophisticated theorems to carry over to non-domain cases, as is demonstrated by several sections in [9] where the statements and the proofs are restricted to integral domains. Bazzoni–Herbera [1] deserve the credit for pointing out that the problem of extending numerous theorems to rings R with zero-divisors is intimately related to the category of modules over the classical ring Q of fractions of R, in particular, to the vanishing of its finitistic dimensions (in the domain case, Q-modules are vector spaces, so all these dimensions are 0). A number of most relevant results related to this question are contained, explicitly or implicitly, in their paper [1].

We wish to take a more systematic approach to the problem as to what extent results on familiar cotorsion and Tor pairs over integral domains retain their validity and main properties if we move to arbitrary commutative rings R with zero-divisors. In particular, what extra conditions on R needed for a particular result to reach the same conclusions as for domains. Some of the required conditions turn out to be closely connected to the way torsion-freeness is treated in R. Therefore, the point of view of torsion-freeness seems to be a natural approach to the problem. As it is clear from [1], torsion-freeness is intimately related to the finitistic dimensions of Q-modules. Perhaps we get a clearer picture on how certain cotorsion and Tor pairs on integral domains depend on the category of Q-modules if we organize the results according to the finitistic dimensions. We do exactly this in our main Theorems 4.1, 5.2, and 6.4 that include most relevant results by Bazzoni–Herbera [1] on the dependence on the finitistic dimensions. Several results in this paper cover the first time a non-domain case, and a number of proofs are new.

When dealing with modules over integral domains, one of the most pleasant properties is undoubtedly the injectivity of Q-modules that is very frequently used in the arguments. However, the results in [1] and in this note are convincing evidence that not this injectivity, but the finitistic dimensions of Q are the crucial key to carry over important features from integral domains to rings with zero-divisors.

We consider the three most frequently used versions of torsion-freeness (that are also the relevant ones): M is torsion-free if

version 1: $\operatorname{Tor}_1^R(R/Rr, M) = 0$ for all non-zero-divisors $r \in R$;
version 2: $\operatorname{Tor}_1^R(P, M) = 0$ for all modules P of projective dimension ≤ 1;
version 3: $\operatorname{Tor}_1^R(F, M) = 0$ for all modules F of weak dimension ≤ 1.

Note that flat modules are torsion-free in either version, and so are the Q-modules. Also, all three torsion-free classes are closed under submodules, extensions, and direct limits.

We provide new proofs for most results we need from Bazzoni–Herbera [1]—this paper is our main source of ideas. In the proofs, we shall use a natural isomorphism from Fuchs–Lee [6] that seems to fit to our topic perfectly: it is our main tool of handling Q-modules when treated both as R- and Q-modules. The other natural isomorphism playing substantial role in our arguments is well known by Cartan–Eilenberg, see (1). Both can be derived from the same long exact sequence (*).

Let R, S be any (associative) rings with identity, and ${}_RA_S, B_S, C_R$ modules as indicated by the subscripts. In Fuchs–Lee [6, Lemma 2.1] it was shown that if $\mathrm{Ext}^1_S(A,B) = 0$, then the sequence

$$0 \to \mathrm{Ext}^1_S(C \otimes_R A, B) \to \mathrm{Ext}^1_R(C, \mathrm{Hom}_S(A,B)) \to \qquad (*)$$

$$\to \mathrm{Hom}_S(\mathrm{Tor}^R_1(C,A), B) \to \mathrm{Ext}^1_S(HA, B) \to \dots$$

of abelian groups is exact (the meaning of H is now irrelevant). In the case of injective B_S, the hypothesis on Ext is trivially satisfied, while the first and the last Ext vanish. This leads to the familiar natural isomorphism

$$\mathrm{Ext}^1_R(C, \mathrm{Hom}_S(A,B)) \cong \mathrm{Hom}_S(\mathrm{Tor}^R_1(C,A), B) \qquad (1)$$

valid for all modules ${}_RA_S, C_R$ and injective B_S (Cartan–Eilenberg [4, p. 120]). In the special case when $S = \mathbb{Z}$ and $B = \mathbb{Q}/\mathbb{Z}$, we have

$$\mathrm{Ext}^1_R(C, A^\flat) \cong (\mathrm{Tor}^R_1(C,A))^\flat \qquad (1')$$

where $M^\flat = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denotes the *character module* of M.

On the other hand, if the Tor in (*) vanishes, then we obtain the isomorphism

$$\mathrm{Ext}^1_S(C \otimes_R A, B) \cong \mathrm{Ext}^1_R(C, \mathrm{Hom}_S(A,B)); \qquad (2)$$

thus (2) holds for the modules ${}_RA_S, B_S, C_R$ provided that

$$\mathrm{Ext}^1_S(A,B) = 0 \quad \text{and} \quad \mathrm{Tor}^R_1(C,A) = 0. \qquad (2')$$

(By the way, the left Ext in (2) is 0 if the right one is 0, even if the condition on Tor is not satisfied.) (2) will be used for commutative rings $R = S$, in which case all the occurring groups are R-modules.

In order to keep this paper reasonable in size, we have not explored relations of our topic to other relevant questions, like being of finite type, tilting modules, etc. Several results can be extended straightforwardly to non-commutative rings that admit (left or right) Ore-rings of quotients.

For unexplained terminology and results we refer to Enochs–Jenda [5], and primarily to Göbel–Trlifaj [9].

2 Definitions and Notations

Throughout, R will denote an arbitrary commutative ring with identity; if we wish to exclude divisors of zero, then we say R is a domain. The notation $R^\times$ is used for the set of non-zero-divisors of R. R-Mod stands for the category of R-modules. Q

will denote the classical ring of quotients of R, and $K = Q/R$. Evidently, Q is a flat R-module and w.d.$K = 1$. We will use the notations p.d. (projective dimension), i.d. (injective dimension), w.d. (weak dimension); gl. will indicate 'global' dimension.

For a non-negative integer n, $\mathscr{P}_n(R)$ (or briefly $\mathscr{P}_n$) denotes the class of R-modules of p.d.$\leq n$, and $\mathscr{F}_n$ the class of modules of w.d.$\leq n$. In particular, $\mathscr{P}_0$ denotes the class of projective R-modules. $\mathscr{F} = \mathscr{F}_0$ is the notation for the flat, and $\mathscr{T}\mathscr{F}_i$ $(i = 1, 2, 3)$ for the various torsion-free classes listed in the introduction. We say M is *torsion-free* if $M \in \mathscr{T}\mathscr{F}_1$ and *strongly torsion-free* if $M \in \mathscr{T}\mathscr{F}_3$. Every $M \in \mathscr{T}\mathscr{F}_i$ for either i satisfies $\mathrm{Tor}_1^R(K, M) = 0$. $\mathscr{D}$ is the class of divisible, $\mathscr{H}\mathscr{D}$ the class of h-divisible, and $\mathscr{W}\mathscr{I}$ the class of weak-injective modules (definitions below).

A module T is said to be *torsion* if for every $x \in T$ there is an $r \in R^{\times}$ such that $rx = 0$. In any module M, the set of elements annihilated by some $r \in R^{\times}$ form a submodule (called the *torsion submodule* $t(M)$ of M), and there is an exact sequence $0 \to t(M) \to M \to M/t(M) \to 0$ where $M/t(M)$ has no torsion. Clearly, M is a torsion module exactly if $t(M) = M$.

An R-module D is called *divisible* if $rD = D$ for each $r \in R^{\times}$. It is *h-divisible* if every homomorphism $R \to D$ extends to a homomorphism $Q \to D$; or, equivalently, D is an epimorphic image of a direct sum of copies of Q. Thus Q is a generator of the category $\mathscr{H}\mathscr{D}$ of h-divisible R-modules. It also follows that $D \in \mathscr{D}$ if and only if $\mathrm{Ext}_R^1(R/Rr, D) = 0$ for all $r \in R^{\times}$, and $D \in \mathscr{H}\mathscr{D}$ whenever $\mathrm{Ext}_R^1(K, D) = 0$. We call M *strongly divisible* if $\mathrm{Ext}_R^1(P, M) = 0$ for all $P \in \mathscr{P}_1$, and *h-reduced* if it contains no h-divisible submodule $\neq 0$.

Observe that the h-divisible torsion-free R-modules M are exactly the Q-modules, thus they satisfy both $\mathrm{Hom}_R(Q, M) \cong M$ and $Q \otimes_R M \cong M$. For Q-modules, flatness over R and over Q are equivalent.

An R-module M is said to be *weak-injective* if $\mathrm{Ext}_R^1(A, M) = 0$ for all $A \in R$-Mod with w.d.$A \leq 1$ (Lee [11]). Weak-injective modules are h-divisible: $\mathscr{W}\mathscr{I} \subseteq \mathscr{H}\mathscr{D}$.

F.dim(Q) (resp. f.dim(Q)) will denote the *big* (resp. *little*) *finitistic dimension* of Q, i.e. the supremum of the projective dimensions of the Q-modules of finite projective dimensions (resp. those having projective resolutions with finitely generated modules). We will use the notation Fw.dim(Q) for the supremum of the weak dimensions in Q-Mod of finite weak dimensions. Observe that both Fw.dim(Q) = 0 and F.dim(Q) = 0 imply f.dim(Q) = 0. (Cf. Proposition 5.3 and Corollary 6.6.)

For a class $\mathscr{C}$ of R-modules, define

$$\mathscr{C}^{\perp} = \{M \in R\text{–Mod} \mid \mathrm{Ext}_R^1(C, M) = 0\ \forall C \in \mathscr{C}\},$$

$$^{\perp}\mathscr{C} = \{M \in R\text{–Mod} \mid \mathrm{Ext}_R^1(M, C) = 0\ \forall C \in \mathscr{C}\},$$

and

$$\mathscr{C}^{\mathsf{T}} = \{M \in R\text{–Mod} \mid \mathrm{Tor}_1^R(C, M) = \mathrm{Tor}_1^R(M, C) = 0\ \forall C \in \mathscr{C}\}.$$

A pair $(\mathscr{A}, \mathscr{B})$ is said to be a *cotorsion pair* if both $\mathscr{A} = ^{\perp}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\perp}$ hold, and a *Tor pair* if both $\mathscr{A} = ^{\mathsf{T}}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\mathsf{T}}$. To simplify notation, we will write, e.g. $\mathrm{Ext}_R^1(\mathscr{A}, \mathscr{B}) = 0$ meaning that $\mathrm{Ext}_R^1(A, B) = 0$ for all $A \in \mathscr{A}, B \in \mathscr{B}$.

When we deal with character modules, it will be convenient to have the following list available (keep in mind that character modules are always pure-injective).

Lemma 2.1 (i) *M is a torsion module if and only if $M^\flat$ is h-reduced.*
(ii) *M is torsion-free if and only if $M^\flat$ is h-divisible.*
(iii) *M is strongly torsion-free if and only if $M^\flat$ is weak-injective.*
(iv) *M is flat if and only if $M^\flat$ is injective.*
(v) *$M \in \mathscr{P}_1^{\top}$ if and only if $M^\flat$ is strongly divisible.*
(vi) *M is divisible exactly if $M^\flat$ is torsion-free.*

Proof Everything follows from the isomorphisms $\mathrm{Hom}_R(A, M^\flat) \cong (A \otimes_R M)^\flat$ and $\mathrm{Ext}^1_R(A, M^\flat) \cong (\mathrm{Tor}^R_1(A, M))^\flat$. See [7] for details. □

As far as weak-injectivity is concerned, one would like to know when it is equivalent to injectivity. If R is a domain, then each of (a)–(c) in the next lemma characterizes Prüfer domains.

Lemma 2.2 *For a commutative ring R, these conditions are equivalent:*

(a) *weak-injective R-modules are injective;*
(b) *strongly torsion-free R-modules are flat;*
(c) gl.w.d.$R \leq 1$.

Proof (a) $\Rightarrow$ (b) Let F be strongly torsion-free, and $A \in \mathscr{F}_1$. Then $\mathrm{Tor}^R_1(A, F) = 0$, thus $\mathrm{Ext}^1_R(A, F^\flat) = 0$ for such an A (see (1′)). This means that $F^\flat$ is weak-injective, so injective by (a). This is equivalent to the flatness of F.

(b) $\Rightarrow$ (c) Let $0 \to H \to F \to M \to 0$ be a free presentation of an arbitrary R-module M. Here H is strongly torsion-free (as submodule of a free module), so flat by (b). But then $M \in \mathscr{F}_1$.

(c) $\Rightarrow$ (a) This is clear in view of the definition of weak-injectivity. □

3 Cotorsion Modules

An R-module M is called *Matlis-cotorsion* if it satisfies $\mathrm{Ext}^1_R(Q, M) = 0$. It is *Enochs-cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat R-modules F, and *Warfield-cotorsion* if $\mathrm{Ext}^1_R(A, M) = 0$ for all $A \in \mathscr{T}\mathscr{F}_1$. The respective classes are denoted as $\mathscr{M}\mathscr{C}, \mathscr{E}\mathscr{C}$, and $\mathscr{W}\mathscr{C}$; they are all different, in general, even for domains. The inclusions $\mathscr{W}\mathscr{C} \subseteq \mathscr{E}\mathscr{C} \subseteq \mathscr{M}\mathscr{C}$ are obvious.

The cotorsion pairs $(\mathscr{F}_0, \mathscr{E}\mathscr{C}), (\mathscr{F}_1, \mathscr{W}\mathscr{I}), (\mathscr{T}\mathscr{F}_1, \mathscr{W}\mathscr{C})$ are perfect, thus the modules over any commutative ring admit both covers and envelopes for these pairs (cf. [9, Chaps. 3–4]).

Warfield [14] characterized the Warfield-cotorsion modules over integral domains as Matlis-cotorsion of i.d.$\leq$ 1. This is no longer true for arbitrary commutative rings under a generalized torsion-freeness. The precise result is Theorem 3.2.

Lemma 3.1 *A torsion-free divisible R-module is Warfield-cotorsion if and only if it is injective.*

Proof To prove necessity, let M be any torsion-free divisible R-module (i.e. a Q-module). With an arbitrary module C, we use (2) to argue that

$$\mathrm{Ext}^1_R(C \otimes_R Q, M) \cong \mathrm{Ext}^1_R(C, \mathrm{Hom}_R(Q, M)) \tag{3}$$

which holds as $\mathrm{Tor}^R_1(Q, C) = 0$ and $\mathrm{Ext}^1_R(Q, M) = 0$, the latter because Q is Q-projective. If M is also Warfield-cotorsion, then the first Ext is 0 for every C ($C \otimes_R Q$ is always torsion-free), and since M is torsion-free divisible, the right-hand Ext is simply $\cong \mathrm{Ext}^1_R(C, M)$. Thus $\mathrm{Ext}^1_R(C, M) = 0$ for all modules C, i.e. M is injective if it is as stated. □

The Warfield-cotorsion modules can now be characterized in the following way.

Theorem 3.2 *Over a commutative ring R, a module M is Warfield-cotorsion if and only if*

(a) *M is Matlis-cotorsion, i.e.* $\mathrm{Ext}^1_R(Q, M) = 0$;
(b) i.d.$M \leq 1$; *and*
(c) $\mathrm{Hom}_R(Q, M)$ *is injective.*

If M is h-reduced, then (c) *is automatically satisfied.*

Proof If M is Warfield-cotorsion, then (a) is obvious as Q is flat. For (b), consider an exact sequence $0 \to H \to F \to N \to 0$, where N is arbitrary and F is free. We derive the exact sequence

$$0 = \mathrm{Ext}^1_R(H, M) \to \mathrm{Ext}^2_R(N, M) \to \mathrm{Ext}^2_R(F, M) = 0,$$

where the first Ext vanishes because H is torsion-free. Hence (b) follows. If we apply (3) with a torsion-free C, then we can conclude that $\mathrm{Hom}_R(Q, M)$ is Warfield-cotorsion if so is M (even if M is not torsion-free). As $\mathrm{Hom}_R(Q, \mathrm{Hom}_R(Q, M)) \cong \mathrm{Hom}_R(Q, M)$, (c) follows from the preceding lemma.

Conversely, suppose M satisfies (a)–(c). For every torsion-free R-module A, the standard exact sequence $0 \to R \to Q \to K \to 0$ induces the exact sequence $0 \to A \to Q \otimes_R A \to K \otimes_R A \to 0$. Hence

$$\mathrm{Ext}^1_R(Q \otimes_R A, M) \to \mathrm{Ext}^1_R(A, M) \to \mathrm{Ext}^2_R(K \otimes_R A, M) = 0,$$

where (b) implies that the last Ext is 0. The first Ext is likewise 0, because Q is flat and so (a) implies the isomorphism

$$\mathrm{Ext}^1_R(Q \otimes_R A, M) \cong \mathrm{Ext}^1_R(A, \mathrm{Hom}_R(Q, M)) \tag{4}$$

(see (2)). By virtue of (c), the right-hand side Ext vanishes. Consequently, so does the left-hand side, and thus $\mathrm{Ext}^1_R(A, M) = 0$, completing the proof of the first claim. The second claim is an obvious corollary. □

The next lemma shows that the three mentioned definitions of torsion-freeness would coincide if in their definitions via Tor only torsion modules were used.

Lemma 3.3 *For an R-module M, the following are equivalent:*

(a) M is torsion-free;
(b) $\mathrm{Tor}_1^R(A, M) = 0$ *for all torsion R-modules* $A \in \mathscr{F}_1$;
(c) $\mathrm{Tor}_1^R(A, M) = 0$ *for all torsion R-modules* $A \in \mathscr{P}_1$.

Proof $(a) \Rightarrow (b)$ Let $A \in \mathscr{F}_1$ be an R-module. As is well known, then $\mathrm{i.d.}A^\flat \leq 1$, and in addition, $A^\flat$ is Matlis-cotorsion as a character module. If A is also torsion, then by Lemma 2.1, $A^\flat$ is h-reduced, so Warfield-cotorsion (Theorem 3.2). Thus, for every torsion-free M, $0 = \mathrm{Ext}_R^1(M, A^\flat) \cong (\mathrm{Tor}_1^R(M, A))^\flat$. Hence we derive the desired $\mathrm{Tor}_1^R(A, M) = 0$.

The implications $(b) \Rightarrow (c) \Rightarrow (a)$ are trivial. □

Corollary 3.4 *If* $A^\flat$ *is Warfield-cotorsion whenever* $A \in \mathscr{F}_1$, *then in the preceding lemma the hypothesis that A is torsion can be dropped.*

Proof This is pretty obvious from the preceding proof. □

We will need the following less known fact whose proof relies on Warfield-cotorsion modules. It makes it possible to descend from h-divisible modules to all divisible modules. (It is perhaps worth while emphasizing that we do *not* claim that a divisible submodule of an h-divisible module is pure.)

Lemma 3.5 *A module is divisible if and only if it can be embedded in an h-divisible module as a pure submodule.*

Proof One way the claim is obvious. For the converse, we first show that any module M can be embedded as a submodule in an h-divisible D such that D/M is a direct sum of copies of K. In fact, if $M \cong F/H$ for a free module F and its submodule H, then $D \cong (F \otimes_R Q)/(H \otimes_R R)$ is a good choice.

Now let M be divisible in the exact sequence $0 \to M \to D \to \oplus_\kappa K \to 0$ with an h-divisible D and some cardinal κ. In the induced exact sequence

$$0 \to (\oplus_\kappa K)^\flat \to D^\flat \to M^\flat \to 0$$

of character modules, $M^\flat$ is torsion-free (as M is divisible). If C is any torsion-free module, then in the isomorphism $\mathrm{Ext}_R^1(C, (\oplus_\kappa K)^\flat) \cong (\mathrm{Tor}_1^R(C, \oplus_\kappa K))^\flat$ the right-hand side is 0. Consequently, the Ext vanishes, so that $(\oplus_\kappa K)^\flat$ is Warfield-cotorsion. Hence the displayed sequence splits; this fact is well known to be equivalent to the pure-exactness of the original sequence. □

4 The Case Fw.dim(Q) = 0

We first discuss the strongest version of torsion-freeness that we are going to consider: the case when $\mathcal{TF}_1 = \mathcal{TF}_3$. This turns out to be equivalent to Fw.dim(Q) = 0, i.e. the finitistic weak dimension of Q is 0 (every Q-module of finite weak dimension is flat). Here is the main theorem:

Theorem 4.1 *For any commutative ring R, the following conditions are equivalent:*

(i) $(\mathcal{F}_1, \mathcal{TF}_1)$ *is a Tor pair; i.e.* $\mathcal{TF}_1 = \mathcal{TF}_3$: *torsion-free modules are strongly torsion-free;*
(ii) $M \in \mathcal{F}_1$ *implies* $M^\flat$ *is Warfield-cotorsion;*
(iii) $M \in \mathcal{F}_1$ *implies* $\mathrm{Hom}_R(Q, M^\flat)$ *is injective;*
(iv) $M \in \mathcal{F}_1$ *implies* $Q \otimes_R M$ *is flat as an R- and as a Q-module;*
(v) *an exact sequence* $0 \to D \to M \to N \to 0$ *is pure-exact whenever* $D \in \mathcal{D}$ *and* $N \in \mathcal{F}_1$;
(vi) *divisible* (*h-divisible*) *pure-injective modules are weak-injective;*
(vii) $M^\flat$ *is weak-injective* (*if and*) *only if M is torsion-free;*
(viii) Fw.dim(Q) = 0.

Proof (i) ⇔ (ii) If $M \in \mathcal{F}_1$, then (i) implies $\mathrm{Tor}_1^R(C, M) = 0$ for all torsion-free C. Hence from $\mathrm{Ext}_R^1(C, M^\flat) \cong (\mathrm{Tor}_1^R(C, M))^\flat$ we conclude that $M^\flat$ is Warfield-cotorsion. The converse follows from the last isomorphism.

(ii) ⇔ (iii) $M^\flat$ is always Matlis-cotorsion, and satisfies i.d.$M \leq 1$ whenever $M \in \mathcal{F}_1$. Therefore, by Theorem 3.2, $M^\flat$ is Warfield-cotorsion if and only if $\mathrm{Hom}_R(Q, M^\flat)$ is injective.

(iii) ⇔ (iv) Clearly,

$$\mathrm{Hom}_R(Q, M^\flat) = \mathrm{Hom}_R(Q, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(Q \otimes_R M, \mathbb{Q}/\mathbb{Z}) = (Q \otimes_R M)^\flat$$

is injective if and only if $Q \otimes_R M$ is R-flat. It is then also Q-flat.

(ii) ⇒ (v) The second part of the proof of Lemma 3.5 works with N replacing $\oplus K$ provided we make sure that $N^\flat$ is Warfield-cotorsion. Clearly, $N^\flat$ is Matlis-cotorsion of i.d.≤ 1 (as $N \in \mathcal{F}_1$), and the rest follows from (ii).

(v) ⇒ (vi) If M is divisible, then by (v) it is pure in its weak-injective envelope W (since $W/M \in \mathcal{F}_1$). If it is also pure-injective, then it is a summand of W, so necessarily $M = W$.

(i) ⇔ (vi) and (i) ⇔ (vii) follow at once from (ii) and (iii) in Lemma 2.1.

(iv) ⇔ (viii) First suppose the Q-module N has w.d. 1 as a Q-, and hence also as an R-module. Then (iv) implies $N = Q \otimes_R N$ is flat. Next suppose w.d.$_Q N = n > 1$; then in the exact sequence $0 \to H \to F \to N \to 0$ of Q-modules with free Q-module F, we have w.d.$_Q H = n - 1$, so w.d.$_Q H = 0$ by induction hypothesis. Hence also w.d.$_R N \leq 1$, so from (iv) we infer that N is flat as a Q-module. Conversely, for an R-module $M \in \mathcal{F}_1$, (viii) implies $Q \otimes_R M$ is a flat Q-module, and hence it is also flat as an R-module. □

The ring of quotients of semiprime Goldie rings (see [10]) is semisimple artinian, so these Goldie rings are easy examples of rings satisfying the conditions in Theorem 4.1. In this case Q is moreover injective (which need not be true for rings covered by Theorem 4.1).

Lee [12, Theorem 3.4] proved that over a domain, a module is weak-injective if and only if its flat cover is injective (for flat cover, see [3]). The corresponding result here is as follows.

Proposition 4.2 *Assume R is a commutative ring for which* $\text{Fw.dim}(Q) = 0$*. An R-module is weak-injective if and only if its flat cover is weak-injective as a Q-module.*

Proof Let $0 \to H \to F \xrightarrow{\alpha} M \to 0$ be a flat cover sequence of the R-module M; thus F is flat and H is Enochs-cotorsion. If M is weak-injective, then it is Enochs-cotorsion as well, and so is F as an extension of an Enochs-cotorsion R-module by another one. Since M is h-divisible, it follows that the cover F must also be h-divisible, since every map $Q \to M$ factors through α. Thus F is a Q-module, and evidently also Enochs-cotorsion as a Q-module. We argue with (4), after choosing an R-module $A \in \mathscr{F}_1$. By Theorem 4.1, $Q \otimes_R A$ is a flat Q-, and hence a flat R-module, so the left-hand side of (4) vanishes. The claim follows at once.

Conversely, if F is a weak-injective Q-module, then consider $\text{Ext}^1_R(A, F) \cong \text{Ext}^1_R(A \otimes Q, F)$ for any R-module $A \in \mathscr{F}_1$ (isomorphism by (2)). By hypothesis, $A \otimes Q$ is a flat Q-module, so the second Ext vanishes, and it follows that F is weak-injective as an R-module. From the induced exact sequence $0 = \text{Ext}^1_R(A, F) \to \text{Ext}^1_R(A, M) \to \text{Ext}^2_R(A, H) = 0$ we obtain that M is likewise weak-injective. □

Over a domain, torsion-free weak-injective modules are injective, so Lee's mentioned theorem is a simple corollary to Proposition 4.2.

5 The Case f.dim(Q) = 0

In this section we consider the case when $\text{f.dim}(Q) = 0$ which turns out to be equivalent to $\mathscr{T}\mathscr{F}_1 = \mathscr{T}\mathscr{F}_2$. We start with the following lemma.

Lemma 5.1 *If* $\text{f.dim}(Q) = 0$*, then* $\text{Ext}^1_R(P, D) = 0$ *for all $P \in \mathscr{P}_1$ with finitely generated projective resolutions and for all $D \in \mathscr{H}\mathscr{D}$.*

Proof Let $P \in \mathscr{P}_1$ and D torsion-free divisible. By (2), we have

$$\text{Ext}^1_R(P \otimes_R Q, D) \cong \text{Ext}^1_R(P, \text{Hom}_R(Q, D)) \cong \text{Ext}^1_R(P, D), \qquad (5)$$

since the conditions $\text{Ext}^1_R(Q, D) = 0$ and $\text{Tor}^R_1(P, Q) = 0$ are satisfied, and in our case, $\text{Hom}_R(Q, D) \cong D$. In view of the hypothesis, $P \otimes_R Q$ is Q-projective whenever P has projective resolution with finitely generated modules, so the first Ext vanishes, and therefore so does the last Ext.

Finally, as $P \in \mathscr{P}_1$, $\mathrm{Ext}^1_R(P, D)$ vanishes also for epic images of torsion-free divisible D, thus for all modules in $\mathscr{H}\mathscr{D}$. □

Let us point out that $(\mathscr{P}_1, \mathscr{T}\mathscr{F}_2)$ is usually not a Tor pair, but $(\varinjlim \mathscr{P}_1, \mathscr{T}\mathscr{F}_2)$ is always such a pair where $\varinjlim \mathscr{P}_1$ denotes the collection of direct limits of modules in $\mathscr{P}_1$. In fact, by the definition of $\mathscr{T}\mathscr{F}_2$, we have $\mathscr{T}\mathscr{F}_2 = \mathscr{P}_1^{\mathsf{T}} = (\varinjlim \mathscr{P}_1)^{\mathsf{T}}$, and by Göbel and Trlifaj [9, Theorem 4.5.6], $\varinjlim \mathscr{P}_1 = ^{\mathsf{T}}((\varinjlim \mathscr{P}_1)^{\mathsf{T}})$; hence the claim follows. Here we have made use of the familiar fact that Tor commutes with direct limits.

Most parts of the following theorem were proved by Bazzoni–Herbera, see [1, Theorem 6.7]. Our proof (with the exception of (ii) ⇒ (v)) is different.

Theorem 5.2 *The following conditions are equivalent:*

(i) $(\varinjlim \mathscr{P}_1, \mathscr{T}\mathscr{F}_1)$ *is a Tor pair, i.e.* $\mathscr{T}\mathscr{F}_1 = \mathscr{T}\mathscr{F}_2$*;*
(ii) $P \in \mathscr{P}_1$ *implies that* $Q \otimes_R P$ *is Q-flat;*
(iii) *for a P having a projective resolution with finitely generated modules,* $P \in \mathscr{P}_1$ *implies* $Q \otimes_R P$ *is Q-projective;*
(iv) $P \in \mathscr{P}_1$ *implies* $\mathrm{Ext}^1_R(P, D) = 0$ *for torsion-free divisible pure-injective D;*
(v) $\mathrm{f.dim}(Q) = 0$.

Proof (i) ⇒ (ii) By hypothesis, $\mathrm{Tor}^R_1(P, D) = 0$ for all $P \in \mathscr{P}_1$ and torsion-free $D \in \mathscr{D}$. In view of the flatness of Q over R, $\mathrm{Tor}^R_1(P, D) = 0$ implies that also $\mathrm{Tor}^Q_1(Q \otimes_R P, D) = 0$ for all Q-modules D; this is equivalent to the flatness of $Q \otimes_R P$ as a Q-module.

(ii) ⇔ (iii) One way, the claim follows from the well-known fact that finitely presented flat modules are projective. Conversely, (iii) implies (ii) for those $P \in \mathscr{P}_1$ that have finitely generated projective resolutions. Since tensor product commutes with direct limits, (ii) holds as well for the direct limits of such P's. It is well known that every $P \in \mathscr{P}_1$ can be obtained as a direct limit of this kind. Noting that direct limits of flat modules are flat, (ii) is immediate.

(ii) ⇔ (iv) For $P \in \mathscr{P}_1$, (ii) implies that $\mathrm{Ext}^1_R(Q \otimes_R P, D) = 0$ for torsion-free divisible pure-injective D, as $Q \otimes_R P$ is also R-flat. From (5) we conclude that (iv) holds. Conversely, assume (iv). D in (iv) is a summand of $E^\flat$ for some torsion-free divisible E, i.e. for some Q-module E. Therefore, (iv) implies $\mathrm{Tor}^R_1(P, E) = 0$ for all $P \in \mathscr{P}_1$ and all $E \in Q$-Mod. Since Q is R-flat, we also have $\mathrm{Tor}^R_1(Q \otimes_R P, E) = 0$, whence (ii) is obvious.

(ii) ⇒ (v) Assume that there is a Q-module D of Q-projective dimension $n \geq 1$ with finitely generated projective resolution, and n is the minimal such number. Select an exact sequence $0 \to H \to F \to D \to 0$ with free Q-module F; then the Q-module H has p.d. $\leq n - 1$. Thus by the choice of n, H is Q-projective, and therefore $\mathrm{p.d.}_Q D = 1$. From [1, Lemma 6.2] we can conclude that there exists an R-module $P \in \mathscr{P}_1$ such that $D \cong Q \otimes_R P$. By (ii), D is Q-flat, moreover Q-projective (as finitely presented), a contradiction.

(v) ⇒ (i) If (v) holds, then from Lemma 5.1 we obtain $\mathrm{Ext}^1_R(P, D) = 0$ for all $P \in \mathscr{P}_1$ with finitely generated projective resolutions and for all torsion-free $D \in \mathscr{D}$.

In particular, this holds if $D = E^\flat$ for any torsion-free divisible E (cf. Lemma 2.1). Hence isomorphism (1) assures that $\mathrm{Tor}_1^R(P, E) = 0$ for all such P and all torsion-free divisible E. If M is an arbitrary torsion-free R-module, then from the injection $M \to Q \otimes_R M$ we derive that $\mathrm{Tor}_1^R(P, M) \to \mathrm{Tor}_1^R(P, Q \otimes_R M)$ is also an injection. Hence $\mathrm{Tor}_1^R(P, M) = 0$ for all $P \in \mathscr{P}_1$ with finitely generated projective resolutions as well as for the direct limits of such P's. Consequently, $\mathrm{Tor}_1^R(P, M) = 0$ for all $P \in \varinjlim \mathscr{P}_1$ and for all torsion-free M. □

A comparison of two finitistic dimensions leads to

Proposition 5.3 (Bazzoni-Herbera [1, Theorems 6.7–6.8]) *The conditions in Theorem 4.1 are equivalent to:*

(ix) $\mathrm{f.dim}(Q) = 0$ *and* $\varinjlim \mathscr{P}_1 = \mathscr{F}_1$.

Proof $\mathrm{f.dim}(Q) = 0$ is equivalent to $\mathrm{Tor}_1^R(\varinjlim \mathscr{P}_1, \mathscr{T}\mathscr{F}_1) = 0$ (Theorem 5.2), which is the same as $\mathrm{Tor}_1^R(\mathscr{F}_1, \mathscr{T}\mathscr{F}_1) = 0$ provided (ix) holds. This is equivalent to (i) in Theorem 4.1.

Conversely, if conditions (i)–(viii) of Theorem 4.1 hold, then (i) being stronger than condition (i) of Theorem 5.2, $\mathrm{f.dim}(Q) = 0$ is obvious. In view of Theorem 5.2, $(\varinjlim \mathscr{P}_1, \mathscr{T}\mathscr{F})$ is a Tor pair, whence $\varinjlim \mathscr{P}_1 = \mathscr{F}_1$ follows from Theorem 4.1. □

Bazzoni–Herbera [1, Lemma 8.3] prove that $\mathrm{f.dim}(Q) = 0$ whenever R is noetherian. They also point out that orders in von Neumann regular rings are examples of rings for which $\varinjlim \mathscr{P}_1 = \mathscr{F}_1$.

6 The Case F.dim(Q) = 0

The remaining case in our project is when $\mathrm{F.dim}(Q) = 0$. This seems to be related to the cotorsion pair $(\mathscr{P}_1, \mathscr{D})$ rather than to the torsion-free questions. The equality $\mathscr{T}\mathscr{F}_2 = \mathscr{T}\mathscr{F}_3$ is equivalent to the coincidence of the classes $\varinjlim \mathscr{P}_1$ and $\mathscr{F}_1$, and it is a well-known open problem to characterize rings for which these classes coincide.

To explore consequences of the condition $\mathrm{F.dim}(Q) = 0$, we start with two preliminary lemmas. The first lemma is well known.

Lemma 6.1 *If an R-module P satisfies* $\mathrm{Ext}_R^1(P, D) = 0$ *for all* $D \in \mathscr{H}\mathscr{D}$*, then* $P \in \mathscr{P}_1$.

Proof For an arbitrary N, let $0 \to N \to E \to D \to 0$ be an injective resolution. In the induced exact sequence $\mathrm{Ext}_R^1(P, D) \to \mathrm{Ext}_R^2(P, N) \to \mathrm{Ext}_R^2(P, E) = 0$, Ext^1 vanishes by hypothesis, hence $\mathrm{Ext}_R^2(P, N) = 0$ for all N, i.e. $P \in \mathscr{P}_1$. □

The next lemma deals with h-divisible modules.

Lemma 6.2 *If* $\mathrm{F.dim}(Q) = 0$*, then* $\mathrm{Ext}_R^1(P, D) = 0$ *holds for all* $P \in \mathscr{P}_1$ *and* $D \in \mathscr{H}\mathscr{D}$.

Proof See the proof of Lemma 5.1. □

By a *filtration* of a module M is meant a well-ordered ascending chain $\{M_\alpha \mid \alpha < \tau\}$ of submodules (for some ordinal τ) with $M_0 = 0, \cup_{\alpha<\tau} M_\alpha = M$ such that the chain is continuous in the sense that $M_\alpha = \cup_{\beta<\alpha} M_\beta$ whenever $\alpha < \tau$ is a limit ordinal. From [2, Proposition 3.3] we derive the following lemma:

Lemma 6.3 *If $(\mathscr{P}_1, \mathscr{D})$ is a cotorsion pair, then every $P \in \mathscr{P}_1$ has a filtration $\{P_\alpha \mid \alpha < \tau\}$ for some ordinal τ such that $P_{\alpha+1}/P_\alpha \in \mathscr{P}_1$ is countably presented for each $\alpha + 1 < \tau$.* □

We do not need the fact (but we may point out) that under the hypothesis of the preceding lemma, it also follows that every $P \in \mathscr{P}_1$ admits a tight system in the sense [8, Chap. VI.5]. This is a consequence of Hill's lemma [9, Theorem 4.2.6] if combined with the preceding lemma.

We can now extend a main result by Bazzoni–Herbera [1, Proposition 6.3] where the equivalence of conditions (ii) and (iii) in the next theorem was established.

Theorem 6.4 *For a commutative ring R, the following are equivalent:*

(i) $\mathrm{Ext}^1_R(\mathscr{P}_1, D) = 0$ *for all torsion-free divisible R-modules D*;
(ii) *if $P \in \mathscr{P}_1$, then $Q \otimes_R P$ is Q-projective;*
(iii) $\mathrm{F.dim}(Q) = 0$.

Proof (i) ⇒ (ii) Suppose $P \in \mathscr{P}_1$ and D is torsion-free divisible. By hypothesis, the last Ext in (5) vanishes for all $P \in \mathscr{P}_1$, and hence the first Ext is also 0. As both $P \otimes_R Q$ and D are Q-modules, we may view the first Ext in Q-Mod. If we do this, then we can conclude that $P \otimes_R Q$ must be Q-projective (as D can be any Q-module).

(ii) ⇒ (iii) Imitate the proof of (ii) ⇒ (v) in Theorem 5.2 above.

(iii) ⇒ (i) As $P \otimes_R Q$ is torsion-free divisible for every $P \in R$-Mod, by hypothesis it is a projective Q-module whenever $P \in \mathscr{P}_1$. Therefore, the first Ext in (5) is 0 for all $P \in \mathscr{P}_1$. Consequently, $\mathrm{Ext}^1_R(P, D) = 0$ holds for every torsion-free divisible D and for every $P \in \mathscr{P}_1$. □

The next result characterizes the rings for which $(\mathscr{P}_1, \mathscr{D})$ is a cotorsion pair.

Theorem 6.5 *The following are equivalent for every ring R:*

(a) *$(\mathscr{P}_1, \mathscr{D})$ is a cotorsion pair;*
(b) $\mathrm{F.dim}(Q) = 0$, *and every $P \in \mathscr{P}_1$ has a filtration $\{P_\alpha \mid \alpha < \tau\}$ with countably presented factors in $\mathscr{P}_1$.*

Proof That (a) implies (b) is evident in view of Theorem 6.4 and Lemma 6.3.

Assume (b). By Theorem 6.4, we have $\mathrm{Ext}^1_R(\mathscr{P}_1, D) = 0$ for all torsion-free h-divisible R-modules D. We continue by recalling the following powerful result (see Göbel–Trlifaj [9, Theorem 5.2.16]). *Assume P is countably presented, and $\mathscr{C}$ is a class of R-modules closed under countable direct sums. If*

$$\mathrm{Ext}^1_R(P, C) = 0 \quad \textit{for all } C \in \mathscr{C},$$

then $\mathrm{Ext}^1_R(P, D) = 0$ *holds also for every pure submodule D of any $C \in \mathscr{C}$.*

In view of Lemma 6.2, this result can be applied to the case where $P \in \mathscr{P}_1$ is countably presented, and $\mathscr{C} = \mathscr{H}\mathscr{D}$. With the aid of Lemma 3.5, we argue that then $\mathrm{Ext}^1_R(P, D) = 0$ holds for all countably presented $P \in \mathscr{P}_1$ and for all $D \in \mathscr{D}$. As by (b) every $P \in \mathscr{P}_1$ has a filtration with countably presented factors in $\mathscr{P}_1$, it remains to invoke a well-known lemma by Eklof on the vanishing of Ext when the first argument is the union of a chain (see, e.g., [9, Lemma 3.1.2]) to conclude that then every $P \in \mathscr{P}_1$ satisfies $\mathrm{Ext}^1_R(P, D) = 0$, i.e. $(\mathscr{P}_1, \mathscr{D})$ is a cotorsion pair. □

By virtue of the remark following Lemma 6.3, (b) in Theorem 6.5 can be replaced by the following condition:

(b′) F.dim$(Q) = 0$ *and every* $P \in \mathscr{P}_1$ *admits a tight system.*

Bazzoni–Herbera [1, Theorem 8.6] point out that in case the ring R is noetherian, F.dim$(Q) = 0$ holds exactly if Q is artinian.

The comparison of Theorems 5.2 and 6.4 leads to necessary and sufficient criteria for the simultaneous vanishing of two finitistic dimensions of Q:

Corollary 6.6 *For a commutative ring R, the following are equivalent:*

(α) F.dim$(Q) = 0$*;*
(β) f.dim$(Q) = 0$ *and Q-modules are R-pure-injective;*
(γ) f.dim$(Q) = 0$ *and flat Q-modules of the form* $Q \otimes_R P$ $(P \in \mathscr{P}_1)$ *are Q-projective.*

□

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Permutation Groups Without Irreducible Elements

A.M.W. Glass and H. Dugald Macpherson

Abstract We call a non-identity element of a permutation group irreducible if it cannot be written as a product of non-identity elements of disjoint support. We show that it is indeed possible for a sublattice subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$ to have no irreducible elements and still be transitive on the set of pairs $\alpha < \beta$ in $\mathbb{R}$. This answers a question raised in "The first-order theory of ℓ-permutation groups", a Conference talk by the first author.

Keywords Order-preserving permutation • ℓ-permutation group

Mathematical Subject Classification (2010): 20B22, 06F15

1 Permutation Groups Without Irreducible Elements

Let $(\Omega, \leqslant)$ be a totally ordered set and G be a subgroup of $\mathrm{Aut}(\Omega, \leqslant)$. Let 1 be the identity element of $\mathrm{Aut}(\Omega, \leq)$ and $g \in G \setminus \{1\}$. Then g is said to be *irreducible* if $g = g_1 g_2$ with $g_1, g_2 \in G$ and $\mathrm{supp}(g_1) \cap \mathrm{supp}(g_2) = \emptyset$ implies $g_1 = 1$ or $g_2 = 1$. Note that if $G = \mathrm{Aut}(\Omega, \leq)$, then $g \in G$ is irreducible if and only if g has a single supporting interval; i.e., there is $\sigma \in \mathrm{supp}(g)$ such that the convexification in Ω of $\{\sigma g^n \mid n \in \mathbb{Z}\}$ is $\mathrm{supp}(g)$. We prove:

Theorem 1.1 *There is an ℓ-subgroup of* $\mathrm{Aut}(\mathbb{R}, \leq)$ *that is transitive on ordered pairs $\alpha < \beta$ and has no irreducible elements.*

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Here, an *ℓ-subgroup* of $\mathrm{Aut}(\mathbb{R}, \leq)$ is a subgroup G of $\mathrm{Aut}(\mathbb{R}, \leq)$ such that $g_+ \in G$ whenever $g \in G$, where $\alpha g_+ := \alpha g$ if $\alpha g \geq \alpha$ and $\alpha g_+ = \alpha$ if $\alpha g \leq \alpha$ ($\alpha \in \mathbb{R}$). In particular, G is a lattice-ordered group where $f \vee g = (fg^{-1} \vee 1)g$ and $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$. For background on ordered permutation groups and ℓ-groups see [1].

Proof Let $g \in \mathrm{Aut}(\mathbb{R}, \leq)$. We say that g has period $n \in \mathbb{Z}_+$ if $(\alpha + n)g = \alpha g + n$ for all $\alpha \in \mathbb{R}$. Let

$$G := \{g \in \mathrm{Aut}(\mathbb{R}, \leq) \mid (\exists m \in \mathbb{Z}_+)(g \text{ has period } m)\}.$$

Then G is transitive on ordered pairs $\alpha < \beta$ in $\mathbb{R}$ and it is easily checked that $(G, \mathbb{R})$ is an ℓ-permutation group. Obviously, if $f \in G$ fixes no point in $\mathbb{R}$, then it must be irreducible. So G has irreducible elements. On the other hand, if $g \in G$ has period m and is not the identity but fixes $\alpha_0 \in \mathbb{R}$ (and so fixes $\alpha_0 + km$ for all $k \in \mathbb{Z}$), define $g_1, g_2 \in G$, each with periods $2m$, as follows:

$$g_1(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \end{cases}$$

$$g_2(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \end{cases}.$$

Then g_1 and g_2 have disjoint supports and $g = g_1 g_2$, so g is reducible. Thus if $H := \{g \in G \mid 0g = 0\}$, then H has no irreducible elements. Now H acts faithfully on $\mathbb{R}_+$ and $(H \upharpoonright \mathbb{R}_+, \mathbb{R}_+)$ (the permutation group induced by H on $\mathbb{R}_+$) is an ℓ-permutation group that is transitive on ordered pairs $\alpha < \beta$ in $\mathbb{R}_+$. Consequently we obtain an ℓ-permutation group $(H^*, \mathbb{R})$ that is transitive on pairs $\alpha < \beta$ in $\mathbb{R}$ and has no irreducible elements. For let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be an order-preserving bijection between $\mathbb{R}$ and $\mathbb{R}_+$ and $h^* \in \mathrm{Aut}(\mathbb{R}, \leq)$ be given by $\alpha h^* = (\alpha\varphi)h\varphi^{-1}$ ($\alpha \in \mathbb{R}$, $h \in H$). Then the desired properties transfer from H (acting on $\mathbb{R}_+$) to $H^* = \{h^* : h \in H\}$ (acting on $\mathbb{R}$).

The above proof can similarly be adapted to ℓ-permutation groups $(L, \mathbb{Q})$ that are transitive on pairs $\alpha < \beta$ in $\mathbb{Q}$ and have no irreducible elements.

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R-Hopfian and L-co-Hopfian Abelian Groups (with an Appendix by A.L.S. Corner on Near Automorphisms of an Abelian Group)

Brendan Goldsmith and Ketao Gong

In memoriam Rüdiger Göbel

Abstract The notions of Hopfian and co-Hopfian groups are well known in both non-commutative and Abelian group theory. In this work we begin a systematic investigation of natural generalizations of these concepts and, in the case of Abelian p-groups, give a complete characterization of the generalizations in terms of the original concepts. The final section of the paper contains an unpublished result of A.L.S. Corner on near automorphisms which has been useful in a number of contexts.

Keywords Hopfian and co-Hopfian groups • Ker-Direct and Im-Direct groups • Rickart and Dual Rickart modules

Mathematical Subject Classification (2010): Primary 20K30; Secondary 20K10

1 Introduction

A standard, and often useful, strategy in mathematics is to seek to investigate notions that are in some sense a generalization of finiteness. Thus, in topology one looks at compactness, in group theory local finiteness is investigated and similarly in many other areas. The starting point is usually to seek some relevant property that finite objects possess and then to look to see if there are non-finite objects possessing the same property. In this paper we seek to employ the same strategy; here our setting is

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the category of all groups $\mathscr{G}$, although we shall focus primarily on the subcategory of Abelian groups $\mathscr{A}b$. Indeed all our comments relating to $\mathscr{A}b$ can be interpreted in the category of modules over a fixed ring R but we shall not carry this out in detail. The property that we wish to explore is the familiar one in the category of sets $\mathscr{S}$: a set S is finite if, and only if, every one-one function $S \to S$ is invertible, if, and only if, every onto function $S \to S$ is invertible. The comparable statements in the category $\mathscr{G}$ would be that G being finite is equivalent to:

(i) every monic endomorphism of a group G is an automorphism;
(ii) every epic endomorphism of a group G is an automorphism.

These equivalences are, of course, not true: multiplication by the prime p in the additive group of integers is a monomorphism which is not an automorphism and the same multiplication in the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is an epimorphism which is not monic. Nevertheless, these conditions can be used to 'select' certain classes of groups which are not necessarily finite. Groups satisfying (i) are usually now referred to as *co-Hopfian groups* and those satisfying (ii) are called *Hopfian groups*. It is well known and easy to establish that the properties (i) and (ii) can be translated into the following equivalent conditions:

(i)' G cannot have a proper isomorphic subgroup and (ii)' G cannot have a proper isomorphic factor group.

There is, of course, a third condition which subsumes both of (i)' and (ii)': (iii)' G cannot have a proper isomorphic subdirect factor (or summand as is the more usual terminology in the Abelian situation). It is straightforward to show that this latter condition is equivalent to

(iii) if ϕ and ψ are endomorphisms of G and $\phi\psi = 1_G$, the identity endomorphism of G, then $\psi\phi = 1_G$.

Groups satisfying condition (iii) are usually referred to as *directly finite groups*.

Hopfian, co-Hopfian and directly finite groups have been the subject of intensive investigation for many years—see, for example, the discussions in [1–3, 7–10, 13, 14]. The reader should note that the complements of these notions have also been studied under various names: Abelian groups which are not directly finite have been studied previously by Beaumont and Pierce [2] under the terminology *ID-group*—the context suggests this was intended to mean 'isomorphic direct summand', while in the context of non-Abelian group theory an alternative terminology, due to Peter Neumann [13], is *badly non-Hopfian*.

The conditions (i) and (ii) can be re-formulated to say that a group G is co-Hopfian [Hopfian] if every monomorphism [epimorphism] ϕ of G has a two sided inverse and this leads naturally to the following definition, where in an obvious notation the letters "R, L" stand for "right" and "left", respectively. Note that in this paper, maps are always written on the left.

Definition 1.1 A group G is said to be R-Hopfian [L-Hopfian] if for every surjection $\phi \in \mathrm{End}(G)$, there is an endomorphism ψ of G such that $\phi\psi = 1_G$ [$\psi\phi = 1_G$].

Observe firstly that if G is Hopfian, then certainly G is both R-Hopfian and L-Hopfian. Moreover, if G is L-Hopfian and ϕ is a surjection, then the equation $\psi\phi = 1_G$ implies that ϕ is also an injection, so that ϕ is an automorphism of G. Consequently the class of L-Hopfian groups coincides with the class of Hopfian groups.

We have a dual situation here where Hopficity is replaced by co-Hopficity:

Definition 1.2 A group G is said to be R-co-Hopfian [L-co-Hopfian] if for every injection $\phi \in \mathrm{End}(G)$, there is an endomorphism ψ of G such that $\phi\psi = 1_G$ [$\psi\phi = 1_G$].

Again it is easy to see that a group G is R-co-Hopfian if, and only if, it is co-Hopfian. Thus we concentrate on the concepts of R-Hopficity and L-co-Hopficity. In particular, we look closely at the situation when the groups being considered are also Abelian p-groups for an arbitrary prime p. Our principal result, Theorem 3.11, shall be a classification of R-Hopfian and L-co-Hopfian p-groups in terms of Hopfian and co-Hopfian p-groups.

An important tool in our investigation will be a weakening of the classical notions of (Ker)-direct and (Im)-direct Abelian groups. Recall that an Abelian group G is said to be (Ker)-direct [(Im)-direct] if the kernel [image] of each endomorphism of G is a direct summand of G. Rangaswamy observed in [15], or see [6, Lemma 112.1], a connection between these notions and the (von Neumann) regularity of the endomorphism ring of G: The endomorphism ring of a group G is regular if, and only if, G is both (Ker)-direct and (Im)-direct. In fact, the same observation had been made in the context of module theory by Azumaya in the late 1940s. Recently, in that same context, the notions of (Ker)-direct and (Im)-direct modules have been called *Rickart modules* and *dual Rickart modules*, respectively—see [11, 12]. We shall come back to this in Sect. 2.

We finish off this introduction by noting that notation in the paper is standard as in the two volumes of Fuchs [5, 6]; in particular mapping is consistently written on the left and for an Abelian group G, the ring of endomorphisms of G shall be denoted by $\mathrm{End}(G)$. *With the exception of the first two results in Sect. 2 below, all groups will be additively written Abelian groups.*

Acknowledgment: the authors would like to thank Peter V. Danchev who suggested that a concept similar to what is now called R-Hopficity might be of interest. They also would like to thank Peter Vámos for drawing their attention to references [11, 12].

2 Elementary Results

The notion of direct finiteness provides the connection between Hopficity and R-Hopficity (and dually between co-Hopficity and L-co-Hopficity).

Proposition 2.1 *(i) An arbitrary group G is Hopfian if, and only if, it is R-Hopfian and directly finite;*

(ii) An arbitrary group G is co-Hopfian if, and only if, it is L-co-Hopfian and directly finite.

In particular, if the endomorphism monoid of G is commutative, then G is R-Hopfian [L-co-Hopfian] if, and only if, it is Hopfian [co-Hopfian].

Proof (i) If G is Hopfian, then every surjection has an inverse, so G is certainly R-Hopfian. However, if $\alpha\beta = 1_G$ for some endomorphisms α, β, then α is surjective and so, by the Hopficity of G, it has an inverse α^{-1}. It follows immediately that $\beta = \alpha^{-1}$ and so $\beta\alpha = 1_G$, whence G is directly finite.

Conversely, given any surjection ϕ of G, R-Hopficity ensures the existence of an endomorphism ψ such that $\phi\psi = 1_G$. By direct finiteness, we have that $\psi\phi$ is also equal to 1_G and so ϕ is invertible with inverse ψ. Since ϕ was arbitrary, we have that G is Hopfian.

The proof of (ii) runs dually and is left to the reader, while the particular case when the endomorphism monoid of G is commutative is then immediate. □

Corollary 2.2 *A group which is not a non-trivial semidirect product is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian]. In particular, the group* $\mathbb{Z}(p^\infty)$ *is not R-Hopfian for any prime p and* $\mathbb{Z}$ *is not L-co-Hopfian.*

Proof The necessity is immediate in both cases and doesn't require the semidirect product condition. Conversely suppose that G is R-Hopfian [L-co-Hopfian]. It suffices by Proposition 2.1 to show that G is directly finite. Suppose then that $\phi\psi = 1_G$ for endomorphisms ϕ, ψ of G. Then $\psi\phi$ is an idempotent endomorphism which cannot be the trivial map and so the fact that G is not a non-trivial semidirect product, forces $\psi\phi = 1_G$, as required. □

From now on all groups will be additively written Abelian groups.

Corollary 2.3 *A reduced group G such that G/pG is finite for all primes p is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian].*

Proof We show that the hypotheses on G ensure that G cannot have a proper isomorphic direct summand and so G is directly finite and then the result follows from Proposition 2.1. Suppose then that $G \cong H \oplus G$ for some $H \leq G$. Then $G/pG \cong H/pH \oplus G/pG$ and so by the finiteness of the latter term, we conclude that $H/pH = \{0\}$ for all primes p. Thus H is divisible and hence, as G is reduced, $H = \{0\}$. Thus G is directly finite, as required. □

In response to a question of Baumslag [1, Problem 3], Corner [3, Example 1] exhibited a non-Hopfian torsion-free group having automorphism group of order 2. Using Corner's example and Proposition 2.1 we can establish:

Example 2.4 There is a torsion-free group G with automorphism group of order 2, but G is not R-Hopfian.

Proof Corner's example of a non-Hopfian torsion-free group with automorphism group of order 2 has the property that its full endomorphism ring is isomorphic to the polynomial ring $\mathbb{Z}[X]$; in particular the endomorphism ring is commutative and so the group is directly finite. Since it is non-Hopfian, it cannot be R-Hopfian by Proposition 2.1. □

As noted in [9], the endomorphism ring of a group does not determine its Hopficity since there are also Hopfian groups, hence R-Hopfian groups, with endomorphism ring $\mathbb{Z}[X]$ which can be obtained using Corner's realization theorem. This also applies to R-Hopficity: the group in Example 2.4 is not R-Hopfian but has endomorphism ring $\mathbb{Z}[X]$.

In light of Proposition 2.1 we would expect R-Hopfian and L-co-Hopfian groups to share some properties known for Hopfian and co-Hopfian groups. Our first result is an analogue of such a property of Hopfian groups.

Proposition 2.5 *A direct summand of an R-Hopfian [L-co-Hopfian] group G is again R-Hopfian [L-co-Hopfian].*

Proof We handle the L-co-Hopfian case leaving the analogous proof for R-Hopfian groups to the reader. Suppose then that $G = H \oplus S$ and let α be an arbitrary injection in $\mathrm{End}(H)$. Then $\psi = \alpha \oplus 1_S$ is an injection in $\mathrm{End}(G)$ and so there is a $\phi \in \mathrm{End}(G)$ such that $\phi\psi = 1_G$. Using the standard matrix representation of endomorphisms of a direct sum, this means that

$$\begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 1_S \end{pmatrix} = \begin{pmatrix} 1_H & 0 \\ 0 & 1_S \end{pmatrix}, \quad \text{where } \phi = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}.$$

Thus $\mu\alpha = 1_H$, and so, since α was an arbitrary injection in $\mathrm{End}(H)$, H is L-co-Hopfian. □

Corollary 2.6 *A torsion R-Hopfian group is reduced and an L-co-Hopfian group has trivial dual.*

Proof The results are immediate from the fact that no quasi-cyclic group $\mathbb{Z}(p^\infty)$ is R-Hopfian while $\mathbb{Z}$ is not L-co-Hopfian. □

It is also possible to relate R-Hopficity [L-co-Hopficity] of a group G to the corresponding properties of subgroups of the form nG:

Proposition 2.7 *If G is R-Hopfian [L-co-Hopfian], then, for each natural number n, the subgroup nG is R-Hopfian [L-co-Hopfian].*

Proof We only handle with R-Hopfian case, the L-co-Hopfian case is analogous. If $\phi : nG \to nG$ is epic, then it follows from the proof of Proposition 113.3 in [6], that there exists an epic $\psi : G \to G$ such that $\psi \restriction nG = \phi$. Since G is R-Hopfian, ψ must have a right inverse, θ say. But then the restriction $\theta \restriction nG$ is the required right inverse of ϕ. □

Theorem 2.8 *If G is a group which has no n-bounded pure subgroups for a given integer n and nG is R-Hopfian [L-co-Hopfian], then G is R-Hopfian [L-co-Hopfian]. The requirement of no n-bounded pure subgroups cannot be omitted.*

Proof We only prove the R-Hopfian case, the L-co-Hopfian case is analogous. Suppose that nG is R-Hopfian and $\phi : G \to G$ is a surjection. Then $\alpha = \phi \restriction nG : nG \to nG$ is a surjection and since nG is R-Hopfian , there is an endomorphism of nG, β say, with $\alpha\beta = 1_{nG}$. Now it follows from the proof of [6, Proposition 113.3] (or see [4, Lemma 2.11]) that there is an endomorphism ψ of G with $\psi \restriction nG = \beta$. Now for all $x \in G$, $\phi\psi(nx) = \phi\beta(nx) = \alpha\beta(nx) = nx$ and so $n(\phi\psi - 1_G) = 0$. Thus $\phi\psi$ is an n-map in the sense of Corner—see the Appendix to this paper—and it follows from Theorem A15 of that appendix that if G has no nonzero n-bounded pure subgroups, then $\phi\psi$ is an automorphism, θ say, and so $\phi\psi\theta^{-1} = 1_G$. Hence ϕ has a right inverse $\psi\theta^{-1}$ and so G is R-Hopfian.

For the second part of the result take $n = p$, a prime and set $G = \mathbb{Z}(p) \oplus \mathbb{Z}(p^2)^{(\aleph_0)}$. It follows from the discussions in Sect. 3 below that $pG \cong \mathbb{Z}(p)^{(\aleph_0)}$ is both R-Hopfian and L-co-Hopfian but G itself is neither R-Hopfian nor L-co-Hopfian. □

We also have the easy but useful:

Proposition 2.9 *If $G = \bigoplus_{i \in I} H_i$ and each H_i is fully invariant in G, then G is R-Hopfian [L-co-Hopfian] if, and only if, each H_i is R-Hopfian [L-co-Hopfian].*

Proof The necessity is immediate from Proposition 2.5 while the sufficiency follows from the fact that every endomorphism of G can be expressed in the form $\bigoplus_{i \in I} \phi_i$, where ϕ_i is an endomorphism of H_i. □

The following notions, which are weaker than the corresponding notions mentioned in the introduction, will play a key role in our investigations.

Definition 2.10 A group G is said to be *(sKer)-direct* if the kernel of each surjective endomorphism of G is a direct summand of G; it is said to be *(mIm)-direct* if the image of each monic endomorphism of G is a direct summand of G.

The following theorem gives a complete characterization of R-Hopficity [L-co-Hopficity] in terms of these groups, the proof is well known and hence omitted.

Theorem 2.11 *A group G is R-Hopfian [L-co-Hopfian] if, and only if, it is (sKer)-direct [(mIm)-direct].*

It follows from the characterization in Theorem 2.11 that the classes of R-Hopfian [L-co-Hopfian] groups are large since they necessarily contain the classes of (Ker)-direct [(Im)-direct] groups. Groups which are (Im)-direct have been classified by Rangaswamy [15] and include groups G where the torsion subgroup $t(G)$ is a direct sum of elementary p-groups for various primes p, $G/t(G)$ is divisible and every endomorphic image of G is maximally disjoint from a pure subgroup of G; the class of (Ker)-direct groups does not seem to have been classified but it is easy to see that in addition to the torsion-free divisible groups and the elementary groups, groups which are free (of arbitrary rank) and torsion-free algebraic compact groups are (Ker)-direct and hence R-Hopfian.

Note that the class of (Im)-direct [(Ker)-direct] groups is strictly contained in the class of L-co-Hopfian [R-Hopfian] groups, indeed containment within the class of co-Hopfian [Hopfian] groups is strict. There are even finite examples: if G is a cyclic group of order p^2, then multiplication by p has both an image and a kernel which are not summands but the group G is both co-Hopfian and Hopfian.

The classification of torsion-free co-Hopfian groups is an easy exercise: they are precisely the class of finite-dimensional $\mathbb{Q}$-vector spaces. Similarly it is easy to classify the torsion-free L-co-Hopfian groups:

Theorem 2.12 *A torsion-free group is L-co-Hopfian if, and only if, it is divisible.*

Proof The sufficiency is straightforward since all divisible groups are (Im)-direct and hence L-co-Hopfian.

Conversely suppose that G is a torsion-free L-co-Hopfian group. For each natural number n, let ϕ_n denote the endomorphism of G corresponding to multiplication by n. Then ϕ_n is monic and hence there is an endomorphism ψ of G with $\psi\phi_n = 1_G$. However, ϕ_n is central in $\text{End}(G)$ and so $\phi_n\psi = 1_G$. Hence ϕ_n is a unit in $\text{End}(G)$ and so G is n-divisible. Since n was arbitrary, G is then divisible. □

Corollary 2.13 *A torsion-free group is both R-Hopfian and L-co-Hopfian if, and only if, it is divisible.*

Proof This follows immediately from Theorem 2.12 and the fact that torsion-free divisible groups are R-Hopfian since, as observed above, they are (Ker)-direct. □

Our next example shows us that no simple characterization of groups which are both R-Hopfian and L-co-Hopfian is likely to be achieved. We refer to [6, §112] for the notion of π-regularity.

Example 2.14 A group having a left π-regular endomorphism ring is both R-Hopfian and L-co-Hopfian; in fact it is both Hopfian and co-Hopfian.

Proof If $\text{End}(G)$ is π-regular, then it follows from Proposition 112.9 [6], that for any endomorphism ϕ of G, we have a positive integer m and a decomposition $G = \text{Ker}\,\phi^m \oplus \text{Im}\phi^m$. If ϕ is onto this forces $\text{Ker}\,\phi^m = 0$, whence $\text{Ker}\,\phi = 0$ and ϕ is an automorphism. A similar argument using ϕ monic establishes the result. □

3 Torsion Groups

We begin with an example that shows that arbitrary direct sums of R-Hopfian [L-co-Hopfian] groups need not be R-Hopfian [L-co-Hopfian].

Example 3.1 The group $B = \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^{i_n})$, with $i_1 < i_2 < \cdots < i_t < \ldots$, is neither R-Hopfian nor L-co-Hopfian.

Proof Let e_n denote a generator of the group $\mathbb{Z}(p^{i_n})$ and consider the endomorphism ϕ of B which acts as the left Bernoulli shift: $e_1 \mapsto 0, e_2 \mapsto e_1, \ldots, e_{n+1} \mapsto e_n \ldots$; then ϕ is surjective but, as the kernel is not a summand of B, by Theorem 2.11, B is not R-Hopfian.

The proof that B is not L-co-Hopfian is similar using the right algebraic Bernoulli shift: $e_n \mapsto p^{(i_{n+1}-i_n)} e_{n+1}$, and the fact that its image is not a summand. □

Note that it follows immediately from Example 3.1 that an unbounded direct sum of cyclic p-groups can never be R-Hopfian nor L-co-Hopfian: any such group must contain a summand of the form of B above and summands inherit R-Hopficity and L-co-Hopficity by Proposition 2.5.

Our first result establishes the unsurprising fact that homocyclic p-groups are both R-Hopfian and L-co-Hopfian. There are two possible different approaches to proving this: a direct, and possibly more insightful, approach and an approach utilizing Theorem 2.8. Considering the merits of both approaches we use the direct proof for establishing L-co-Hopficity and use Theorem 2.8 for showing R-Hopficity.

Proposition 3.2 *A homocyclic p-group A is both R-Hopfian and L-co-Hopfian.*

Proof First note that in a homocyclic p-group A of exponent n, an element $a \in A$ is divisible by p^k if and only if $p^{n-k}a = 0$ (for $0 \leq k \leq n$). We deal with the L-co-Hopfian case first. It follows from Theorem 2.11 that it will suffice to show that A is (mIm)-direct. Let $A = \bigoplus_{i \in I} \langle e_i \rangle$, where the order of each e_i is p^n. Let ϕ be an injective endomorphism of A; since A is bounded it will suffice to show that the image $\phi(A)$ is pure in A.

Pick an element $a \in \phi(A) \cap p^k A$, then $a = \phi(x)$ for some $x \in A$, and $a = p^k a'$ for some $a' \in A$. Multiplying by p^{n-k}, we have $p^{n-k}a = p^{n-k}\phi(x) = p^n a' = 0$. Hence, $\phi(p^{n-k}x) = 0$ and since ϕ is injective, we have $p^{n-k}x = 0$, and thus x is divisible by p^k. Therefore, $a = \phi(x) = \phi(p^k y) = p^k \phi(y) \in p^k \phi(A)$ and $\phi(A)$ is pure in A, as required.

For R-Hopficity observe that if the exponent of A equals 1, then A is elementary and thus is certainly R-Hopfian. If the exponent of A is $n > 1$, then $p^{n-1}A$ is R-Hopfian and, as A clearly has no p^{n-1}-bounded pure subgroups, it follows from Theorem 2.8 that A is R-Hopfian. □

Our next result shows that there are considerable restrictions on the p-groups which can be R-Hopfian or L-co-Hopfian.

Proposition 3.3 *A direct sum of an infinite rank homocyclic p-group and a cyclic p-group of smaller exponent is neither R-Hopfian nor L-co-Hopfian. Consequently a direct sum of two homocyclic p-groups of infinite rank and of different exponents is neither R-Hopfian nor L-co-Hopfian.*

Proof The arguments for R-Hopficity and L-co-Hopficity are broadly similar so we give details of just the L-co-Hopficity case.

Since a direct summand of an L-co-Hopfian group is again L-co-Hopfian, it suffices to show that a direct sum of a countable rank homocyclic p-group and a cyclic p-group of smaller exponent is not L-co-Hopfian. Suppose then that $G = \langle e\rangle \oplus \bigoplus_{i=1}^{\infty}\langle f_i\rangle$, where $o(e) = p^n, o(f_i) = p^{n+k}$ for each i and $k > 0$. Consider the map $\phi : G \to G$ as follows (similar to the forward shift):

$$e \mapsto p^k f_1, f_i \mapsto f_{i+1} (i \geq 1).$$

It is easy to see that ϕ is a monomorphism. Now suppose on the contrary that there is an endomorphism ψ with $\psi\phi = 1_G$. So on the one hand, $\psi\phi(e) = e$, on the other hand,$\psi\phi(e) = \psi(p^k f_1) = p^k\phi(f_1)$, hence $e = p^k\phi(f_1)$, this is not possible since the height of e in G is 0, but the height of $p^k\phi(f_1)$ in G is $\geq k > 0$.

The final statement follows immediately from the fact that a direct sum of two homocyclic p-groups of infinite rank and of different exponents has a summand which is a direct sum of an infinite rank homocyclic p-group and a cyclic p-group of smaller exponent. □

The following technical lemma will simplify arguments we require later.

Lemma 3.4 *Let A be an R-Hopfian [L-co-Hopfian group] and B an arbitrary group. If a surjective endomorphism [monic endomorphism]* Φ *of* $A \oplus B$ *has a matrix representation of the form* $\Phi = \left(\begin{smallmatrix}\alpha & \gamma\\ \delta & \beta\end{smallmatrix}\right)$, *where* β *is an automorphism of B, then* Φ *has a right [left] inverse* Ψ, *i.e.,* $\Phi\Psi = 1$ $[\Psi\Phi = 1]$.

Proof We give only the argument for L-co-Hopficity, the argument for R-Hopficity is dual. So assume that Φ represents a monic endomorphism of $A \oplus B$.

Pre-multiplying Φ by the invertible matrix $\Delta = \left(\begin{smallmatrix}1 & -\gamma\beta^{-1}\\ 0 & \beta^{-1}\end{smallmatrix}\right)$ and post-multiplying it by the invertible matrix $\Sigma = \left(\begin{smallmatrix}1 & 0\\ -\beta^{-1}\delta & 1\end{smallmatrix}\right)$ reduce Φ to a diagonal matrix $\Delta\Phi\Sigma = \left(\begin{smallmatrix}\alpha-\gamma\beta^{-1}\delta & 0\\ 0 & 1\end{smallmatrix}\right)$ which is again injective.

Claim that $\alpha-\gamma\beta^{-1}\delta$ is injective. Suppose, on the contrary, that there is a nonzero element $a \in A$ with $(\alpha - \gamma\beta^{-1}\delta)(a) = 0$, then the injection $\Delta\Phi\Sigma$ maps the nonzero element $(a, 0)$ to $(0, 0)$—contradiction.

Now since A is L-co-Hopfian, there is an endomorphism μ of A such that $\mu(\alpha - \gamma\beta^{-1}\delta) = 1$. If $\Gamma = \left(\begin{smallmatrix}\mu & 0\\ 0 & 1\end{smallmatrix}\right)$, then $\Gamma\Delta\Phi\Sigma = 1$. Hence $\Sigma\Gamma\Delta$ is the required left inverse of Φ. □

Proposition 3.5 *(i) If A is an R-Hopfian [L-co-Hopfian] group, B a Hopfian [co-Hopfian] group and* $\mathrm{Hom}(A, B) = 0$ $[\mathrm{Hom}(B, A) = 0]$, *then* $A \oplus B$ *is R-Hopfian [L-co-Hopfian];*

(ii) *If A is an R-Hopfian [L-co-Hopfian] p-group of exponent n and B is a Hopfian [co-Hopfian] p-group that has no p^n-bounded pure subgroups, then $A \oplus B$ is R-Hopfian [L-co-Hopfian].*

Proof We deal first with the R-Hopficity.

(i) An arbitrary surjection of $A \oplus B$ has the form $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ 0 & \sigma \end{smallmatrix}\right)$ and this forces σ to be a surjection of B. Since B is Hopfian this implies that σ is an automorphism of B. It follows from Lemma 3.4 that Ψ has a right inverse and so $A \oplus B$ is R-Hopfian, as required.

(ii) Let $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ \rho & \sigma \end{smallmatrix}\right)$ be an arbitrary surjective endomorphism of $A \oplus B$. Then $\rho(A) + \sigma(B) = B$, and so $p^n\rho(A) + p^n\sigma(B) = p^nB$, implying that $\sigma(p^nB) = p^nB \neq 0$. Thus $\sigma \restriction p^nB$ is a surjection of the non-trivial Hopfian group p^nB and so $\sigma \restriction p^nB$ is an automorphism of p^nB. By Fuchs [6, Proposition 113.3], there is an automorphism ϕ of B with $\phi \restriction p^nB = \sigma \restriction p^nB$. Hence $p^n(\sigma - \phi) = 0$, and as ϕ is an automorphism of B, σ is a p^n-map of B in the sense of Corner—see the Appendix. Since B has no p^n-bounded pure subgroups, it follows from Theorem A.14 of the Appendix that σ is an automorphism of B. It follows immediately from Lemma 3.4 that Δ has a right inverse and so $A \oplus B$ is R-Hopfian.

The argument for L-co-Hopficity in part (i) follows a similar argument to that used for R-Hopficity noting that in this case endomorphisms of $A \oplus B$ have matrix representations of the form $\Delta = \left(\begin{smallmatrix} \mu & \nu \\ \rho & \sigma \end{smallmatrix}\right)$ with $\nu = 0$.

For part (ii) the argument is entirely dual to that used for R-Hopficity. □

We can now classify those direct sums of cyclic p-groups which are R-Hopfian [L-co-Hopfian]; the situation parallels that in Hopfian [co-Hopfian] groups where the only direct sums of cyclic groups which are Hopfian [co-Hopfian] are the finite groups and so Hopficity and co-Hopficity coincide for such groups.

Theorem 3.6 *A direct sum of cyclic p-groups G is R-Hopfian [L-co-Hopfian] if, and only if, it has the form $G = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, where $B_1 = \bigoplus_{\kappa_1} \mathbb{Z}(p^{n_1})$ for some cardinal κ_1 which may be infinite and each B_i $(2 \leq i \leq k)$ is of the form $B_i = \bigoplus_{\kappa_i} \mathbb{Z}(p^{n_i})$ with κ_i finite and $n_1 < n_2 < \cdots < n_k$. In particular, a direct sum of cyclic groups is R-Hopfian if, and only if, it is L-co-Hopfian.*

Proof For the sufficiency note that B_1 is R-Hopfian [L-co-Hopfian] and $B_2 \oplus \cdots \oplus B_k$ is a Hopfian [co-Hopfian] p-group which has no p^{n_1}-bounded pure subgroups and thus the result follows from Proposition 3.5(ii) above.

Conversely, suppose that $G = \bigoplus_{i=1}^{\infty} B_i$ is R-Hopfian [L-co-Hopfian] and each B_n is homocyclic of exponent n. It follows, as noted after Example 3.1, that almost all the B_n are zero. Let B_r be the first homocyclic component of infinite rank; if no such exists, then G is a finite group and clearly has the desired form. It follows from Proposition 3.3 that each B_i $(1 \leq i < r)$ must be zero since summands of R-Hopfian [L-co-Hopfian] groups are again R-Hopfian [L-co-Hopfian]. Furthermore, it follows from the same proposition that no B_j with $j > r$ can be of infinite rank. Thus G is of the claimed form.

The final statement follows from the fact that the classifications of R-Hopficity and L-co-Hopficity coincide for the class of direct sums of cyclic groups. □

Recall that a reduced p-group G is said to be *semi-standard* if for each $n < \omega$, the Ulm invariant $f_n(G)$ is finite; it is well known that both Hopfian and co-Hopfian p-groups are necessarily semi-standard.

Our next result shows that Hopficity and R-Hopficity [co-Hopficity and L-co-Hopficity] coincide for semi-standard p-groups.

Proposition 3.7 *A semi-standard p-group is Hopfian [co-Hopfian] if, and only if, it is R-Hopfian [L-co-Hopfian].*

Proof The necessity is clear in both cases. For the sufficiency, by Proposition 2.1, it is enough to prove that every semi-standard p-group G is directly finite. Suppose $G \cong G \oplus K$. Then $f_\sigma(G) = f_\sigma(G) + f_\sigma(H)$ for all ordinals σ. If $\sigma < \omega$, we must have $f_\sigma(K) = 0$ as the cardinals in question are finite. Hence a basic subgroup of K is the zero subgroup; since K is reduced, we are forced to have $K = 0$. □

We can now classify R-Hopficity [L-co-Hopficity] for reduced p-groups in terms of Hopficity [co-Hopficity].

Proposition 3.8 *A reduced p-group G is R-Hopfian [L-co-Hopfian] if, and only if, it is of the form $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$, where κ is a cardinal which may be infinite and H is Hopfian [co-Hopfian] and all Ulm invariants $f_i(H)(i < m)$ are zero.*

Proof The condition on the Ulm invariants of H ensure that H has no p^m-bounded pure subgroups and so the sufficiency follows from Proposition 3.5(ii).

Conversely suppose that G is a reduced R-Hopfian [L-co-Hopfian] p-group. Let B_{i_k} be the first nonzero infinite homogeneous component of a basic subgroup of G; if no such component exists, then G is semi-standard and hence Hopfian [co-Hopfian] by Proposition 3.7 above, so we are finished in that case. It follows from Proposition 3.3 that B_{i_n} $(n > k)$ cannot be infinite since $B_{i_k} \oplus B_{i_n}$ is a summand of G. Furthermore, B_{i_j} $(j < k)$ cannot be nonzero by the same proposition. Simplifying notation by writing $i_n = m$, we conclude that $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$ and that H is semi-standard and all Ulm invariants $f_i(G)$ $(i < m)$ are zero. □

Proposition 3.8 can be re-phrased to say that a reduced R-Hopfian [L-co-Hopfian] p-group G differs from a reduced Hopfian [co-Hopfian] p-group in that it may have at most one infinite homogeneous component and this corresponds to the summand of G of least exponent. Notice also that although a reduced R-Hopfian [L-co-Hopfian] p-group G can be of arbitrarily large cardinality, there is an integer n such that the cardinality of p^nG is at most $2^{\aleph_0}$, the cardinality of the continuum.

Note that it is not necessary to specify that the group be reduced in the case of R-Hopficity: the group $\mathbb{Z}(p^\infty)$ is not R-Hopfian for any prime p. For L-co-Hopfian groups we need some further work to handle the situation where the group may have a divisible summand.

Lemma 3.9 *The group $G = \mathbb{Z}(p^n) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^\infty)$ is not L-co-Hopfian.*

Proof Write G as $G = \langle e\rangle \oplus \mathbb{Z}(p^\infty)f_1 \oplus \mathbb{Z}(p^\infty)f_2 \oplus \mathbb{Z}(p^\infty)f_3 \oplus \cdots$, where the order of e is p^n. Consider the forward shift mapping $\phi : G \to G$, $e \mapsto 1/p^n f_1, f_1 \mapsto f_2, f_2 \mapsto f_3, \cdots$. Then ϕ is an injective endomorphism of G. Suppose on the contrary that there is an endomorphism ψ with $\psi\phi = 1_G$. Then $\psi\phi(e) = e$, that is, $\psi(1/p^n f_1) = e$, but $1/p^n f_1 = p(1/p^{n+1} f_1)$, so $\psi(1/p^n f_1) = p\psi(1/p^{n+1} f_1) = px$ for some $x \in G$, this is impossible since e is not divisible by p. □

Theorem 3.10 *If A is a non-trivial, reduced L-co-Hopfian p-group and D is a divisible p-group, then the direct sum $A \oplus D$ is L-co-Hopfian if, and only if, D is of finite rank, $D \cong \bigoplus_n \mathbb{Z}(p^\infty)$, for some finite n.*

Proof The sufficiency follows from the fact that a finite rank divisible p-group D is actually co-Hopfian: any injective endomorphism of D has image whose rank is equal to that of D and, since the image is a summand, it must be the whole of D, so that the injection is an automorphism. Now apply Proposition 3.5(i) and it follows immediately that $A \oplus D$ is L-co-Hopfian.

Conversely, suppose for a contradiction, that $A \oplus D$ is L-co-Hopfian but that D has infinite rank. Then there is a summand of $A \oplus D$ of the form $\mathbb{Z}(p^n) \oplus \bigoplus_{\aleph_0} \mathbb{Z}(p^\infty)$ and this summand is also L-co-Hopfian. This, however, contradicts Lemma 3.9. Thus D has finite rank, as required. □

We summarize the preceding results as:

Theorem 3.11 *A p-group G is R-Hopfian if, and only if, it is of the form $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$, where κ is a cardinal which may be infinite, H is Hopfian and all Ulm invariants $f_i(H)(i < m)$ are zero and D is a finite direct sum of copies of $\mathbb{Z}(p^\infty)$.*

A p-group G is L-co-Hopfian if, and only if, it is of the form (i) $G = \bigoplus_\kappa \mathbb{Z}(p^m) \oplus H \oplus D$, $\bigoplus_\kappa \mathbb{Z}(p^m) \oplus H$ non-trivial, where κ is a cardinal which may be infinite, H is reduced co-Hopfian and all Ulm invariants $f_i(H)(i < m)$ are zero and D is a finite direct sum of copies of $\mathbb{Z}(p^\infty)$; or of the form (ii) $G = \bigoplus_\kappa \mathbb{Z}(p^\infty)$, where κ is an arbitrary cardinal.

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Appendix: Near Automorphisms of an Abelian Group

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Introductory Remarks

The notion of a near automorphism was discussed by Corner in this paper dating from sometime around the early 1960s. It should be noted that it is a different concept from that used nowadays in the theory of torsion-free Abelian groups *of finite rank. The note was discovered among Corner's papers after his death. The hand-written material has been transcribed by Brendan Goldsmith and some comments (in italics) have been added. Please note that in this Appendix, mappings are always written on the right.*

The Main Result

Let G be an Abelian group and ι the identity map on G.

Definition An endomorphism ε of G is called a **near automorphism** if $q(\varepsilon-\iota) = 0$ for some integer $q \geq 1$.

We characterize those groups whose near automorphisms are all automorphisms and those whose monomorphic or epimorphic near automorphisms are all automorphisms.

Theorem A.12 *Every near automorphism of G is an automorphism if, and only if, G has no nonzero bounded pure subgroups.*

Theorem A.13 *The following are equivalent:*

(M) Every monic near automorphism of G is an automorphism
(E) Every epic near automorphism of G is an automorphism
(B) G has no bounded pure subgroup of infinite rank.

If q is a positive integer, we call an endomorphism ε of G a **q-map** if $q(\varepsilon-\alpha) = 0$ for some automorphism α of G. We say that a group G is **q-bounded** if $qG = 0$. It is clear that Theorems A.12 and A.13 are contained in the more precise:

Theorem A.14 *Every q-map of G is an automorphism if, and only if, G has no nonzero q-bounded pure subgroup.*

Theorem A.15 *The following are equivalent:*

(M_q) *Every monic q-map of G is an automorphism*
(E_q) *Every epic q-map of G is an automorphism*
(B_q) *G has no q-bounded pure subgroup of infinite rank.*

The proofs of Theorems A.14, A.15 are based on two lemmas which are largely computational in nature. In the proof of Lemma A.16 *below, Corner used the unexplained term* $E(x)$ *in relation to the element x of a p-group G; clearly this was intended to mean the* exponent, *i.e., the least integer n such that* $p^n x = 0$. *In modern notation this is often denoted by either* $e(x)$ *or* $O(x)$.

Lemma A.16 *If G has no nonzero q-bounded pure subgroup and if* ϕ *is an endomorphism of G such that* $q\phi = 0$, *then* $\phi^n = 0$ *for some integer* $n \geq 1$.

Proof (i) If G is torsion-free, then $q\phi = 0$ implies that $\phi = 0$; so we may take $n = 1$.

(ii) Suppose that G is a p-group and let p^k be the highest power of p dividing q. Since multiplication by qp^{-k} affects an automorphism of G, therefore $p^k\phi = 0$; and it is clear that G has no p^k-bounded pure subgroup. We prove that $\phi^{k+1} = 0$. If $k = 0$ there is nothing to prove; so we suppose that $k \geq 1$.

Note first that if $x \in G$ and $E(x) = 1$, then $h_G(x) \geq k$. For if $h_G(x) = l < k$, then $x = p^l y$ for some $y \in G$, and it is clear that $\langle y \rangle$ is a pure subgroup of order p^{l+1}, a factor of p^k, contrary to our hypothesis.

Let $\mathscr{P}(n)$ denote the proposition: $x \in G, E(x) = n \leq k \Rightarrow x\phi^n = 0$. We prove $\mathscr{P}(n)$ by induction on n. Since $\mathscr{P}(0)$ is trivial, we may suppose that $1 \leq n \leq k$ and that $\mathscr{P}(r)$ is true for $r < n$. If $x \in G$ and $E(x) = n$, then $E(p^{n-1}x) = 1$, so $h_G(p^{n-1}x) \geq k$ and therefore $p^{n-1}x = p^k z$ for some $z \in G$. So $p^{n-1}(x\phi) = z(p^k\phi) = 0$, whence $E(x\phi) \leq n - 1$ and so $(x\phi)\phi^{n-1} = 0$, i.e., $x\phi^n = 0$.

Since for each $x \in G$ we have $p^k(x\phi) = 0$, so that $E(x\phi) \leq k$, therefore it follows that $x\phi^{k+1} = (x\phi)\phi^k = 0$. Thus $\phi^{k+1} = 0$.

(iii) If G is mixed, write $q = \prod_p p^{k(p)}$ and set $n = 1 + \max_p k(p)$. For each prime p, the p-component T_p of the torsion subgroup T of G is mapped into itself by ϕ, so that ϕ induces an endomorphism of T_p. Since this endomorphism of T_p is annihilated by $p^{k(p)}$, ϕ^n vanishes on T_p by (ii). Consequently ϕ^n vanishes on T. But the endomorphism of the torsion-free group G/T induced by ϕ is annihilated by q and so vanishes by (i). Thus ϕ^n vanishes on T and induces the zero endomorphism of G/T; so $\phi^n = 0$ by the Five Lemma.[1]

□

Lemma A.17 *Let ε be an endomorphism of G with $q(\varepsilon - 1) = 0$, let A be a maximal q-bounded pure subgroup of G, and B a direct complement of the direct summand A: $G = A \oplus B$. Then*

$$G = A + \operatorname{Im}\varepsilon \ \text{ and } \ B \cap \operatorname{Ker}\varepsilon = 0.$$

Moreover, there exist endomorphisms $\lambda, \mu, \lambda', \mu'$ of G such that

$$\lambda = 1_A \lambda 1_A = (1_A + \mu 1_B)\varepsilon,$$
$$\lambda' = 1_A \lambda' 1_A = \varepsilon(1_A + 1_B \mu'),$$

where $1_A, 1_B$ are the projections of G onto A, B corresponding to the direct decomposition $G = A \oplus B$.

[1]In the original hand-written note there was a section giving the standard Five Lemma with the additional claim that mappings which are zero on a subgroup and its factor group must be zero on the whole group. This (erroneous) claim had been crossed out. However, it is trivial to show that if the map ϕ is zero on a subgroup H of G and induces the zero map on G/H, then $\phi^2 = 0$ and this clearly suffices here. BG.

Proof Write $\varepsilon = 1 + \phi$ so that $q\phi = 0$. Then $1_B\phi 1_B$ may be regarded as an endomorphism of B. Since $q(1_B\phi 1_B) = 0$ and B has no nonzero q-bounded pure subgroup, it follows from Lemma A.16 that

$$(1_B\phi 1_B)^{n-1} = 0 \text{ for some integer } n \geq 2. \tag{1}$$

Write $\theta = 1_B - (1_B\phi 1_B) + (1_B\phi 1_B)^2 - \cdots + (-1_B\phi 1_B)^{n-2}$; since $1_B\varepsilon 1_B = 1_B + 1_B\phi 1_B$, we have, in view of (1)

$$\theta 1_B\varepsilon 1_B = 1_B\varepsilon 1_B\theta = 1_B. \tag{2}$$

The first claims now follow easily: since $1_B = 1 - 1_A$, for any $x \in G$ we have $x = x1_A + x1_B = (x - x\theta 1_B\varepsilon)1_A + (x\theta 1_B)\varepsilon \in A + G\varepsilon$. And if $y \in B \cap \operatorname{Ker}\varepsilon$, then $y = y1_B = y1_B\varepsilon 1_B\theta = y\varepsilon 1_B\theta = 0$.

Since $1_B 1_B = 1_B$, we may write (1) in either of the forms $1_B(\phi 1_B)^{n-1} = 0$ or $(1_B\phi)^{n-1}1_B = 0$. Pre- and post-multiplying by ϕ, these become

$$(\phi 1_B)^n\phi = 0 \qquad \text{and} \qquad \phi(1_B\phi)^n = 0. \tag{3}$$

Substituting $\phi 1_B = \varepsilon 1_B + 1_A - 1$ in the first of these, we find that

$$0 = (-\phi 1_B)^n\phi 1_A = [1 - (\varepsilon 1_B + 1_A)]^n\phi 1_A = (\sum_{r=0}^{n}(-)^r\binom{n}{r}(\varepsilon 1_B + 1_A)^r)\phi 1_A$$
$$= (\varepsilon - 1)1_A + (\varepsilon 1_B + 1_A)(\sum_{r=1}^{n}(-)^r\binom{n}{r}(\varepsilon 1_B + 1_A)^{r-1})\phi 1_A,$$

whence

$$[1_A - 1_A\sum_{r=1}^{n}(-)^r\tbinom{n}{r}(\varepsilon 1_b + 1_A)^{r-1}\phi 1_A] = \varepsilon[1_a + 1_B\sum_{r=1}^{n}(-)^r\tbinom{n}{r}(\varepsilon 1_b + 1_A)^{r-1}\phi 1_A].$$

Taking λ' to be the left-hand side, and μ' to be the summation on the right, we see that $\lambda' = \varepsilon(1_A + 1_B\mu')$; and it is clear that $\lambda' = 1_A\lambda' 1_A$. The proof of the corresponding statement for $(1_A + 1_B\mu)\varepsilon$ is similar. □

With Lemmas A.16, A.17 established, it is now easy to give the desired proof of Theorem A.14.

Proof of Theorem A.14 ($\Leftarrow$) Let G be a group with no nonzero q-bounded pure subgroup, and let ε be a q-map of G, so that $q(\varepsilon - \alpha) = 0$ for some automorphism α. Since $q(\varepsilon\alpha^{-1} - 1) = 0$, and ε is an automorphism, monomorphism or epimorphism if, and only if, $\varepsilon\alpha^{-1}$ is one, it is enough to consider the case $\alpha = 1$. Then $q(\varepsilon - 1) = 0$. In Lemma A.17 we may take $A = 0, B = G$. Then we have $G = \operatorname{Im}\varepsilon$, $\operatorname{Ker}\varepsilon = 0$; which proves that ε is an automorphism of G.

($\Rightarrow$) Let G be a group with a q-bounded pure subgroup $A > 0$. Then A is a direct summand of G, so it has a direct complement B (say). If $\imath_B$ is the corresponding projection of G onto B, then $\imath_B$ is not an automorphism of G; but it is a q-map because $q(\imath_B - \imath) = q\imath_A = 0$. □

The proof of Theorem A.15 proceeds by showing firstly that (B_q) *implies both* (M_q) *and* (E_q)*; the reverse implications are established using a counter-positive argument.*

Proof of Theorem A.15 (1) Suppose first that G satisfies (B_q), i.e., that G has no q-bounded pure subgroup of infinite rank, and let ε be a q-map of G which is either (i) a monomorphism or (ii) an epimorphism. We prove that in either case ε is an automorphism of G. As in the case of Theorem A.14 we may suppose that $q(\varepsilon - \imath) = 0$. Take A, B as in Lemma A.17, and let $\lambda, \lambda', \mu, \mu'$ be endomorphisms as given in that lemma. Note that A being a q-bounded pure subgroup of G is of finite rank, and so is in fact a finite group.

(i) If $x \in A \cap \operatorname{Ker}\lambda$, then $0 = x\lambda = (x + x\mu\imath_B)\varepsilon$, whence $x + x\mu\imath_B = 0$ because ε is a monomorphism, so $x = -x\mu\imath_B \in A \cap B$, i.e., $x = 0$. Now it follows from the properties of λ given in Lemma A.17 that λ may be regarded as an endomorphism of A; what we have just proved shows that, so regarded, λ is a monomorphism. Since A is finite, it follows that λ maps A onto itself. Consequently, given any $x \in A$, there exists $y \in A$ such that $x = y\lambda$, i.e., $x = y(\imath_A + \mu\imath_B)\varepsilon \in G\varepsilon$. So $A \leq G\varepsilon$; whence $G = G\varepsilon$. Thus the monomorphism ε is also an epimorphism; so it is an automorphism, as required.

(ii) If $x \in A$, then, because ε is given to be an epimorphism, there exists $y \in G$ such that $x = y\varepsilon$; but $x = x\imath_A$, so from the properties of λ' we find that $x = y\varepsilon\imath_A = y\lambda' - y\varepsilon\imath_B\mu' = y\lambda' - x\imath_B\mu'$; and since $x \in A = \operatorname{Ker}\imath_B$, it follows that $x = y\lambda'$. Thus λ', regarded as an endomorphism of the finite group A, is epic, and therefore monic; so $A \cap \operatorname{Ker}\lambda' = 0$. Now, if $x \in \operatorname{Ker}\varepsilon$, we have from the properties of λ' that $x\imath_A\lambda' = x\varepsilon(\imath_A + \imath_B\mu') = 0$, so $x\imath_A = 0$ and therefore $x = x\imath_B \in B \cap \operatorname{Ker}\varepsilon$; whence $x = 0$. We conclude that the epimorphism ε is also a monomorphism, and so an automorphism.

(2) Suppose that G does not satisfy (B_q), so that G admits a q-bounded pure subgroup A (say) of infinite rank. Now A is a direct sum of cyclic groups of orders dividing q; passing to a direct summand of A, if necessary, we may suppose that A is a direct sum of countably many isomorphic cyclic subgroups, say $A = \bigoplus_{i=1}^{\infty}\langle e_i\rangle$. Let B be a direct complement of A in G. Now it is clear that A admits monomorphic q-maps which are not epic, and epimorphic q-maps which are not monic; e.g., the endomorphisms defined by $e_i \mapsto e_{i+1}$ $(i \geq 1)$ and by $e_1 \mapsto 0$, $e_i \mapsto e_{i-1}$ $(i \geq 2)$. If we extend such an endomorphism of A to the whole of G by requiring it to coincide with the identity on B, then the resulting endomorphism of G is clearly a monomorphic or epimorphic q-map, but not an automorphism. □

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On the Abelianization of Certain Topologist's Products

Wolfgang Herfort and Wolfram Hojka

Abstract For the topologist's product $\circledast_i G_i$ where each G_i is the group of p elements, a description of its abelianization is provided. It turns out that the latter is isomorphic to $\left(\bigoplus_i \mathbb{Z}(p)\right) \oplus P/S$, where $P = \prod_i \mathbb{Z}$ is the *Specker* group and $S = \bigoplus_i \mathbb{Z}$.

Keywords Wild homology • Shrinking wedge • Topologist's product • Higman completeness • Cotorsion • Algebraically compact • Specker group

Mathematical Subject Classification (2010): 20K25, 20E06, 20F10, 57M30, 08A45

1 Introduction

The *topologist's product* $G = \circledast_{i \geq 1} G_i$, for a given sequence of groups G_i, has its origin in work of Griffiths and Higman from the 1950s, see [17, 19]. Given spaces X_i, good at their base point, with fundamental groups G_i, the topologist's product describes the fundamental group of their *shrinking wedge*. For a detailed explanation see [17, Sect. 6] and [5, Sect. 2].

To algebraically define $\circledast_{i \geq 1} G_i$, one first considers the canonical inverse system (Γ_n, p_n) of free products $\Gamma_n := *_{i=1}^n G_i$ and bonding maps $p_n : \Gamma_n \to \Gamma_{n-1}$ with kernel the normal closure of G_n in Γ_n. In the inverse limit $\varprojlim_n \Gamma_n$ the topologist's product consists exactly of those coherent sequences, for which the number of factors from any one group G_i is bounded (see [22, p. 532]).

There has been interest in computing the structure of the abelianization $\mathrm{Ab}(G)$ particularly when all G_i are isomorphic to $\mathbb{Z}$ (thus computing the singular homology of the Hawaiian earring) and Eda and Kawamura in [14] describe it in the form

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$\mathrm{Ab}(G) \cong P \oplus P/S$. Let us recall that P stands for the *Specker* group, the cartesian product $\mathbb{Z}^{\mathbb{N}} = \prod_{i\geq 1} \mathbb{Z}$, while S is the direct sum $\mathbb{Z}^{(\mathbb{N})} = \bigoplus_{i\geq 1} \mathbb{Z}$. S, in a canonical fashion, appears as a subgroup of P.

The main objective of the present article is to derive a description of the abelianization of $G = \circledast_{i\geq 1} G_i$ when every G_i, instead of being cyclic of *infinite* order, is a copy of the cyclic group $\mathbb{Z}(p)$ of order p, where p is a fixed prime. When e.g. $p = 2$, one would thus consider the singular homology of a shrinking wedge of projective planes. Here is what we want to prove:

Theorem 1.1 *Let $G = \circledast_{i\geq 1} \mathbb{Z}(p)$ where p is a fixed prime. Then*

$$\mathrm{Ab}(G) \cong \left(\bigoplus_{i\geq 1} \mathbb{Z}(p)\right) \oplus P/S.$$

Notice the striking difference to [14]: when the G_i were all $\mathbb{Z}$, the first summand of the formula was a *product* of the factors, here, however, it is a *sum*!

2 Historical Perspective

The topologist's product made its first implicit appearance in the literature in an article [19] by Higman. There the case where all factor groups are the integers is used as a counterexample to a question on freely irreducible groups. A few years later, Griffiths in [17] established the mentioned link to topology and thus the name of these products: he showed them to be the fundamental groups of shrinking wedges of spaces each required to be good at the base point. The "shrinking" property is usually thought of in metric terms, namely that the diameters of the wedged spaces converge to 0; but it is also topologically induced by the Tychonoff-product of spaces, as opposed to using the weaker box topology for regular weak wedges (see [5, Sect. 2]). Griffiths' argument contained a gap that was closed by Morgan and Morrison in [22]. In [9] and a series of subsequent papers, Eda introduced and developed an *infinite word calculus* that allowed many novel arguments. For example, he used it to prove a non-commutative version of Chase's lemma in [10] and to define a non-commutative analogue to slender or cotorsion-free groups in [13]. It further led to more information about maps to free products in [11].

Independently, Cannon and Conner in [2–4] devised a more topologically motivated approach to these infinite word groups and their properties, also considering generalizations to order types of a larger cardinality.

The inverse limit of free products used in the definition of the topologist's product $\circledast_{i\geq 1} \mathbb{Z}$ has also appeared in the description of arbitrary one-dimensional spaces given by Curtis and Fort in [6], who proved that their fundamental groups are locally free. Infinite words have also been useful for further studies of these fundamental groups as well as of those of planar spaces [15, 24].

The special case $\circledast_{i\geq 1}\mathbb{Z}$ corresponds to the fundamental group of the Hawaiian earring and has received the greatest attention. A complete description of its homology group H_1 as the abelianization of $\circledast_{i\geq 1}\mathbb{Z}$ has been accomplished in [14]. More recently, research has been extended to higher-dimensional variations and there are some results on problems related to their homology groups in [1, 18, 21].

The present article avoids making use of infinite words, instead relying on a technique already developed by Higman and adapted in our Lemma 3.5. Another difference to the approach outlined in [8, 9, 14] is in how algebraic compactness is confirmed. Instead of going through the more cumbersome calculation that the group is $\hat{\mathbb{Z}}$-adically complete modulo its Ulm subgroup, Theorem 3.3 directly connects the solvability of infinite systems of equations to cotorsion. Then, as is known, torsion-freeness implies the algebraic compactness.

We have restricted our attention to the case of a countable index set, topologically corresponding to the first countable setting. Going beyond that presents multiple obstacles. First, competing definitions can claim naturality, either the free complete product $⋇_{i\in I} G_i$ defined in [9] or the topologically defined alternative "$\circledast_{i\in I} G_i$" (in Eda's notation: $⋇^{\sigma}_{i\in I} G_i$). Secondly, certain properties are not necessarily preserved: Shelah and Strüngmann in [23] proved that for an uncountable index set I and nontrivial groups G_i the group $G := ⋇_{i\in I} G_i$ always admits an epimorphism onto $\mathbb{Z}$. In particular, if all G_i are torsion groups, then the normal closure N of the subgroup generated by their union must be contained in the kernel. This means that G/N also maps onto $\mathbb{Z}$. Then the abelianization of G/N cannot be cotorsion, for no cotorsion group admits an epimorphism to $\mathbb{Z}$. On the other hand, when I is countable, this abelianization *is* cotorsion—a crucial fact to be used during the proof of Lemma 4.1(d) below.

3 Preliminaries

Before giving a proof let us recall a few facts from the very recent article [18].

Definition 3.1 Let us call a group G *Higman-complete* if for any sequence $f_1, f_2, \ldots \in G$, and for a given sequence of words $w_1, w_2, \ldots$, there exists a sequence $h_1, h_2, \ldots \in G$ such that all equations

$$h_i = w_i(f_i, h_{i+1})$$

hold simultaneously.

Lemma 3.2 *If G is Higman-complete, then so is every epimorphic image. In particular, its abelianization* $\mathrm{Ab}(G)$ *is Higman-complete.*

Proof Let N be a normal subgroup of G and $h_i = w_i(f_i, h_{i+1})$ be a system of equations for elements in G/N as in the definition. Every constant $f_i \in G/N$ can be lifted to a $\tilde{f}_i \in G$, and by assumption the system $\tilde{h}_i = w_i(\tilde{f}_i, \tilde{h}_{i+1})$ admits a sequence of $\tilde{h}_i$ as a solution, whose images $h_i := \tilde{h}_i N/N$ form a solution sequence of our given system of equations in G/N. Hence G/N is Higman-complete.

The second statement follows by letting N be the commutator subgroup G' of G. □

The next result, Theorem 3 in [18], is somewhat surprising as it exposes an unexpected link between classical abelian group theory and wild topology. We allow ourselves to reproduce the algebraic part of the argument from that article.

Theorem 3.3 *An abelian group A is Higman-complete if and only if it is cotorsion.*

Proof Suppose first that A is Higman-complete. It suffices to show that any exact sequence $0 \to A \to G \to \mathbb{Q} \to 0$ of abelian groups splits. Consider A embedded as a subgroup of G. $\mathbb{Q}$ possesses a presentation generated by the countably many $x_i := \frac{1}{i!}$ and with the relations $x_i - (i+1)x_{i+1}$, for $i \geq 1$. Lift the x_i to elements ξ_i in G. The relations in $\mathbb{Q}$ translate into

$$\xi_i = (i+1)\xi_{i+1} + a_i$$

for suitable elements a_i in A. Since A is by assumption Higman-complete the infinite system of equations

$$h_i = (i+1)h_{i+1} + a_i$$

admits a solution sequence h_i in A. The elements $z_i := \xi_i - h_i \in G$ satisfy the relations

$$z_i = (i+1)z_{i+1}.$$

The $\mathbb{Z}$-module, say Q_0, generated by $Z := \{z_1, z_2, \ldots\}$ projects modulo A onto $\mathbb{Q}$. We still need to show that $A \cap Q_0 = 0$. Any $q_0 \in A \cap Q_0$ can be presented in the form $q_0 = \lambda z_j$ for some $\lambda \in \mathbb{Z}$ and $j \in \mathbb{N}$, due to the relations among the elements in Z. Modulo A this tells us that $\lambda x_j = 0$ and, since $\mathbb{Q}$ is torsion-free, we must have that $\lambda = 0$, i.e., $q_0 = 0$. Hence $Q_0 \simeq \mathbb{Q}$ and thus the extension splits, as claimed.

Conversely, assume now that A is cotorsion. In the abelian group A, any system of equations as in Definition 3.1 is of the type of a system of equations $h_i = d_i h_{i+1} + f_i$ with $d_i \in \mathbb{Z}$ and $f_i \in A$.

There is an algebraically compact group G such that $A \simeq G/N$ for a suitable subgroup N of G. Lift the elements f_i to elements $\tilde{f}_i \in G$. Since every finite subsystem of the system of equations $\tilde{h}_i = d_i\tilde{h}_{i+1} + \tilde{f}_i$ admits a solution, the algebraic compactness of G implies the existence of a sequence of $\tilde{h}_i$ in G solving all equations; the sequence of their images $h_i \in A$ constitutes a solution sequence of the system $h_i = d_i h_{i+1} + f_i$ in A. Hence A is Higman-complete. □

One of the two indispensable tools for the proof of Theorem 1.1 will be Theorem 8 in [18]:

Theorem 3.4 *Let $G = \circledast_{i \geq 1} G_i$ be the topologist's product of groups G_i of nontrivial groups of cardinality at most the continuum. Then the abelianization of the quotient group G/N where N is generated as a normal subgroup of G by the subgroups G_i is isomorphic to P/S.*

Our last preparation consists of a *purely algebraic* proof that G/N is Higman-complete. The latter fact also follows from Theorem 4 in [18] – but one would have to first represent G/N as the fundamental group of a certain topological space. Here we avoid such a detour. The following lemma is essentially Lemma 1 in [19], a variation for more general equations is used in [20, Lemma 6].

Lemma 3.5 *Given a Higman system $h_i = w_i(f_i, h_{i+1})$ in the topologist's product $G := \circledast_{i\geq 1} G_i$ the following statements hold:*

(a) If, for each $i \in \mathbb{N}$, the element f_i belongs to $\circledast_{j\geq i} G_j$, then the system has a unique solution sequence in G such that also h_i is contained in $\circledast_{j\geq i} G_j$.
(b) Projecting G onto G/N, the equation system has a solution sequence in G/N, in other words, G/N is Higman-complete.

Proof (a) Put $X_n := *_{j=1}^n G_j$, let $\phi_n : X_n \to X_{n-1}$ be the canonical projection when factoring the normal closure of G_n in X_n. Recall that G is a subgroup of $\hat{G} := \varprojlim_n X_n$. For given n, a solution sequence $(h_i^{(n)})_i$ can be defined in X_n setting $h_j^{(n)} = 1$ for $j \geq n+1$ and computing $h_j^{(n)}$ for $j \leq n$ from the equations. One observes $h_j^{(n-1)} = \phi_n(h_j^{(n)})$ for j and n arbitrary. Thus the compatibility relations of the inverse system are fulfilled giving rise to a solution sequence (h_j) in $\hat{G}$. Since the number of elements from G_j appearing in X_n is bounded, when n runs through $\mathbb{N}$, the members h_j all belong to G. Uniqueness follows by projecting a given solution sequence (h_j) to X_n. It then turns out that such a projection must agree with the solution $(h_j^{(n)})$.

(b) As G/N results from factoring the normal subgroup of G generated by $\bigcup_j G_j$, one can represent every element in G/N by elements in the subgroup $\circledast_{j\geq i} G_j$ of G. Now the result follows from part (a). □

4 Proof of Theorem 1.1

When $G = \circledast_{i\geq 1} G_i$, then the kernel of the canonical epimorphism from G onto $\prod_{i\geq 1} G_i$ will be denoted by K in the sequel.

Lemma 4.1 *Let $G = \circledast_{i\geq 1} G_i$ be the topologist's product of groups $G_i \cong \mathbb{Z}(p)$ for p a fixed prime. The following statements hold true:*

(a) $\operatorname{tor}(G/G') = \operatorname{tor}(G)G'/G'$.
(b) $\operatorname{tor}(G/G') = \bigoplus_{i\geq 1} \langle a_i G'/G' \rangle$ *where a_i is a generator of G_i.*
(c) The factor group K/G' is torsion-free.
(d) There is a torsion-free algebraically compact subgroup T of G/G' such that

$$G/G' \cong \operatorname{tor}(G/G') \oplus T.$$

Proof (a) Consider first tor(G), the subgroup generated by the elements of finite order. By Conner et al. [5, Lemma 22] such elements are conjugate into a factor G_i and thus the kernel N as used in Theorem 3.4 coincides with tor(G). Then certainly tor$(G)G'/G'$ is included in tor(G/G'). Suppose next that $y \in$ tor(G/G'). Then there are $m \in \mathbb{N}$ and $x \in G$ such that $x^m \in G'$ and $xG' = y$. Since $x^m \in G'$ it also belongs to $G'N$. As a consequence of Theorem 3.4, Ab$(G/N) \cong G/G'N$ is torsion-free. Therefore x itself belongs to $G'N$, as needed.

(b) is implied from (a) by observing that $N =$ tor(G) is generated by the elements a_i.

(c) Pick any $y \in$ tor(K/G'). Then, by what we have just proven, there is $x \in G$ with $y = xG'$ and $x = g'a_{i_1} \dots a_{i_k}$ for pairwise different generators a_{i_j} of G_{i_j} and some g' in G'. Since $x \in K$ and g' belongs to K it turns out that no a_{i_j} may appear and hence $x = g'$, i.e., $x \in G'$ showing that K/G' is indeed torsion-free.

(d) Observe first that $G/K = \prod_{i\geq 1} G_i$ is a vector space over the field with p elements. Therefore, as tor(G/G') maps injectively into G/K under the canonical epimorphism from G/G' onto G/K, there is vector space complement, say U, for tor$(G/G')/(K/G')$ in G/K. Let T be the preimage of U in G/G'. By construction $T \cap$ tor$(G/G') \subseteq K/G'$, and, as K/G' is torsion-free by what we have shown earlier, the desired decomposition of G/G' is established.

Finally, for proving that T is algebraically compact, one observes from the just established decomposition and from item (a) the isomorphisms

$$T \cong (G/G')/(\text{tor}(G)G'/G') \cong G/\text{tor}(G)G'.$$

Since, by part (b) of Lemma 3.5, $G/N = G/$tor(G) is Higman-complete, Lemma 3.2 in conjunction with Theorem 3.3, implies that $G/$tor$(G)G'$, and hence T, are both cotorsion. Therefore, as T is torsion-free, it is algebraically compact, by Fuchs [16, 54.5]. □

Since the factors $G_i \cong \mathbb{Z}(p)$ are all abelian and the index set is countable, our subgroup K of G agrees with $G^{\sigma\,\prime}$ in [9]. Lemma 4.7 ibidem shows that if the factors are, in contrast to here, n-slender, the quotient K/G' is complete modulo its first Ulm subgroup. Now it is known that a group is algebraically compact if and only if, modulo its maximal divisible subgroup D, it is complete and Hausdorff, and in that case, this subgroup D also coincides with the first Ulm subgroup, see [7, Theorem 2.2]. Thus, the following statement can be seen as a parallel to [9, Lemma 4.7].

Corollary 4.2 *The factor group K/G' is algebraically compact.*

Proof Since G/K is abelian of exponent p, by the very definition of K, so is the subquotient $T/(K/G')$. Hence $T/(K/G')$ is algebraically compact and reduced. Next observe that the maximal divisible subgroup D of G/G' is a subgroup of K/G'. Since T is cotorsion by Lemma 4.1(d), so is the quotient T/D, and as $(K/G')/D$ is a subgroup of the reduced group T/D with quotient isomorphic to the reduced group $T/(K/G')$, we infer from [16, 54.(B)] that $(K/G')/D$ is cotorsion. Therefore, by Fuchs [16, 21.2], $K/G' = L \oplus D$, with both, $L \cong (K/G')/D$ and the divisible D cotorsion, so K/G' is cotorsion. Recall from Lemma 4.1(c) that K/G' is also torsion-free and hence it is algebraically compact. □

Turning to the proof of the main result.

Proof (of Theorem 1.1) Recall that, in the case of all G_i cyclic of finite order, the normal subgroup N of G generated by the union $\bigcup_i G_i$ coincides with $\mathrm{tor}(G)$. As noted in Lemma 4.1(d) and (a), one has $G/G' \cong \mathrm{tor}(G)G'/G' \oplus T = NG'/G' \oplus T$. Therefore $G/NG' \cong T$. On the other hand, as a consequence of Theorem 3.4, $G/NG' \cong P/S$. Hence

$$G/G' \cong \left(\bigoplus\nolimits_{i\geq 1} \mathbb{Z}(p)\right) \oplus P/S,$$

as has been claimed. □

Let us mention that for $G_i \cong \mathbb{Z}$ the proof in [14] depends on Lemma 4.11 in [9]. There has been discussion about its correctness in [13] and [12]. The results in the present paper and their derivation do not depend upon that lemma.

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Some Remarks on dp-Minimal Groups

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In memoriam Rüdiger Göbel

Abstract We prove that ω-categorical dp-minimal groups are nilpotent-by-finite, a small step in the general direction of proving this for NIP ω-categorical groups. We also show that in dp-minimal definably amenable groups, f-generic global types are strongly f-generic.

Keywords Omega-categorical groups • Dp-minimal groups • Definably amenable groups

Mathematical Subject Classification (2010): 03C45, 20A15, 03C60, 03C35

1 Introduction

Many results in the model theory of algebraic structures have the form: if A is an algebraic structure with some model theoretic property, then A satisfies some nice algebraic properties. There are many examples of such results, e.g., every ω-stable infinite field is algebraically closed [18]. Here our algebraic structure is a group G and the model theoretic property is dp-minimality, which we define now.

A (complete, first order) theory T is *dp-minimal* if the following cannot happen. There are two formulas $\varphi(x, y)$, $\psi(x, z)$ with x a singleton (y and z perhaps not), and sequences $\langle a_i \mid i < \omega \rangle$ and $\langle b_j \mid j < \omega \rangle$ such that $|a_i| = |y|$, $|b_j| = |z|$ for all $i, j < \omega$

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and for every $i, j < \omega$ there is some element $c_{i,j}$ (all in the monster model $\mathfrak{C} \models T$) such that for all $i', j', i, j < \omega$, $\varphi\left(c_{i,j}, a_{i'}\right)$ holds iff $i = i'$ and $\psi\left(c_{i,j}, b_{j'}\right)$ holds iff $j = j'$.

At first this definition might seem arbitrary, so we will give some motivation. Recall that T is *NIP* (without the independence property) or *dependent*, if the following cannot happen. There is a formula $\varphi(x, y)$ and sequences $\langle a_i \mid i < \omega \rangle$, $\langle b_s \mid s \subseteq \omega \rangle$ (all in the monster model $\mathfrak{C}$ of T) such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

NIP plays an important role in current research in model theory. For more on general NIP, see [27].

Strong dependence is a strengthening of NIP, where one assumes not only that there is no formula $\varphi(x, y)$ as in the definition, but moreover, that there are no $\langle \varphi_i(x, y_i) \mid i < \omega \rangle$ and $\left\langle a_{i,j} \,\middle|\, i, j < \omega \right\rangle$ in $\mathfrak{C}$ (where $\left|a_{i,j}\right| = |y_i|$) such that for every $\eta : \omega \to \omega$, there is some $b_\eta \in \mathfrak{C}^{|x|}$ such that $\varphi_i\left(b_\eta, a_{i,j}\right)$ holds iff $\eta(i) = j$. (Checking that if T is strongly dependent then it is dependent is a nice exercise in the definitions.)

Dp-minimality is then a natural subclass of strong dependence, which was first properly defined and studied in [20].

Dp-minimal theories are in some sense the simplest case of NIP theories, but still they include all o-minimal and c-minimal theories and the theory of the p-adics (see [27, Example 4.28]). This restrictive yet still interesting assumption about T yields many conclusions, evident by the amount of research done in the area, sometimes with the additional assumption of a group or field structure. See, e.g., [9, 12–15, 25, 26] to name a few examples.

This note contains some results (mostly) on dp-minimal groups, contributing to the general research in the area.

In Sect. 2 we prove that all dp-minimal ω-categorical groups are nilpotent-by-finite. In Sect. 2.7 we prove a general result on NIP groups: there is a finite A with $C(A)$ abelian.

In Sect. 3 we prove that in definably amenable dp-minimal groups, being f-generic is the same as being strongly f-generic.

All definitions are given in the appropriate sections.

2 ω-Categorical dp-Minimal Groups

2.1 Introduction

It is well known that stable ω-categorical groups are nilpotent-by-finite by Baur et al. [4], Felgner [10] where in [4] it is proved that ω-categorical ω-stable groups are abelian-by-finite (and it is conjectured that this is true for stable ω-categorical groups as well). In [19] Macpherson proves that ω-categorical NSOP groups (and in particular simple in the model theoretic sense) are also nilpotent-by-finite.

Krupinski generalized the stable case in [17, Theorem 3.4] by proving that every ω-categorical NIP group that has fsg (finitely satisfiable generics) is nilpotent-by-finite. In [8] Krupinski and Dobrowolski extended this result and removed the NIP hypothesis.[1]

In this section we will go in the other direction and remove the assumption of fsg. However, our proof requires the stronger assumption of dp-minimality and not just NIP.

2.2 What we Get From ω-Categoricity

We will need the following facts about ω-categorical theories.

Suppose that T is ω-categorical.

1. (Ryll–Nardzewski, see, e.g., [28, Theorem 4.3.1]) For all $n < \omega$, there are at most finitely many $\emptyset$-definable sets in n variables.
2. If $M \models T$ is saturated (in particular, countable) and $X \subseteq M^n$ is invariant under $\mathrm{Aut}(M)$, then X is $\emptyset$-definable.
3. By (2), an ω-categorical theory T eliminates $\exists^\infty$, which means that for all $\varphi(x, y)$ there is some $n < \omega$ such that for all $a \in M \models T$, $\varphi(M, a)$ is infinite iff $|\varphi(M, a)| \geq n$.

A structure M is *ω-categorical* if its theory is.

Suppose that $(G, \cdot)$ is an ω-categorical group. Then, it follows easily from (1) that $(G, \cdot)$ is locally finite (every finitely generated subgroup is finite).

We will use the following fact about locally finite groups.

Fact 2.1 *[16, Corollary 2.5] If G is an infinite locally finite group (every finitely generated subgroup is finite), then G contains an infinite abelian subgroup.*

2.3 Equivalent Conditions for Being Nilpotent-by-Finite

Remark 2.1 If G is nilpotent-by-finite, and $H \equiv G$ then H is nilpotent-by-finite. Why? Suppose that $\lambda^+ = 2^\lambda > |G|, |H|$ and let G^* be a saturated extension of G of size λ^+, which, we may assume, also contains H. Then it is enough to show that G^* is nilpotent-by-finite (being nilpotent-by-finite transfers to subgroups). Suppose that $G_0 \leq G$ is nilpotent of finite index. Let $\left(S^*, S_0^*\right)$ be a saturated extension of (G, G_0) of size λ. Then $S_0^* \leq S^*$ is nilpotent of finite index in S^* and $G^* \cong S^*$.

If there is no such λ, we can either force its existence or use special models instead (see [11, Theorems 10.4.4, 10.4.2]).

[1] On the face of it, they asked that the group is generically stable. However, by Krupiński [17, Remark 1.8], under NIP and ω-categoricity, a definable group has fsg iff it is generically stable.

Suppose that G is any group. Let $G^{00}_{\emptyset}$ be the intersection of all $\emptyset$-type-definable subgroups of G (in $\mathfrak{C}$). When G is ω-categorical, it must be $\emptyset$-definable of finite index (so we can talk about it in G without going to a saturated extension). However, if G is NIP then, by Shelah [23], $G^{00}_{\emptyset} = G^{00}_{A}$ for any small set A.

Fact 2.2 *[21, Theorem 5.2.8] Let M and N be normal nilpotent subgroups of a group G, then L = MN is a nilpotent group.*

Corollary 2.2 *Assume G is a nilpotent-by-finite ω-categorical group with* $G = G^{00}_{\emptyset}$*, then G is a nilpotent group.*

Proof By Fact 2.2 the product of all normal nilpotent subgroups of finite index of G is itself a nilpotent group which is also $\emptyset$-definable (by ω-categoricity) and of finite index, thus G is nilpotent. □

Proposition 2.3 *Suppose that* $\mathscr{C}$ *is a class of countable ω-categorical NIP groups (in the pure group language) satisfying: if* $G \in \mathscr{C}$, $H \trianglelefteq G$ *definable (over ∅), then* $G/H \in \mathscr{C}$ *and* $H \in \mathscr{C}$*. Then the following statements are equivalent:*

1. *Every* $G \in \mathscr{C}$ *is nilpotent-by-finite.*
2. *Every infinite characteristically simple* $G \in \mathscr{C}$ *is abelian.*
3. *Every infinite* $G \in \mathscr{C}$ *contains an infinite ∅-definable abelian subgroup.*

Proof (2) implies (1) is essentially Krupinski's argument from [17]. Suppose that $G \in \mathscr{C}$ and we wish to show that it is nilpotent-by-finite. We may of course assume that G is infinite. We may assume that $G^{00} = G$ so that G has no definable subgroups of finite index.

By ω-categoricity, we can write $\{e\} = G_0 \lneq G_1 \lneq \cdots \lneq G_n = G$ where the groups G_i are 0-definable, and this is a maximal (length-wise) such chain. The groups G_i are invariant under $\mathrm{Aut}(G)$ so normal, and by assumption $G_i \in \mathscr{C}$. The proof is now by induction on $n \geq 1$. For $n = 1$, this follows immediately from (2) (i.e., G_1 will be abelian).

Now note that by the induction hypothesis if $H \trianglelefteq G$ is $\emptyset$-definable and non-trivial then G/H is nilpotent: G/H is in $\mathscr{C}$ (as $G = G^{00}$, G/H is infinite). Also $(G/H)^{00} = (G/H)$. But the maximal length of a chain as above which suits G/H must be shorter than n. Hence G/H is nilpotent-by-finite and by Corollary 2.2 G/H is itself nilpotent.

Hence we may assume that $Z(G)$ is trivial (otherwise $G/Z(G)$ is nilpotent and so G is too). This in turn implies that G_1 is infinite (if not, then $C_G(G_1)$ is of finite index in G, and hence equals G, but then $G_1 \subseteq Z(G)$). Now we use (2) on G_1 to finish: G_1 is abelian and G/G_1 is nilpotent, so both are solvable, and hence G is solvable. However, by Archer and Macpherson [2, Theorem 1.2], if G is not nilpotent-by-finite (equivalently, nilpotent, since we already assumed $G = G^{00}$), it interprets the infinite atomless boolean algebra, and has IP.

(3) implies (2) is obvious.

(1) implies (3). Without loss of generality, $G = G^{00}$.

Claim Either $Z(G)$ is infinite or $G/Z(G)$ is centerless.

Proof If $x \in G$ with $y^{-1}xy \in xZ(G)$ for all $y \in G$, then $C(x)$ has a finite index in G. Hence $C(x) = G$ so $x \in Z(G)$. □

If $Z(G)$ is infinite, we are done. Otherwise, by the claim $G/Z(G)$ is centerless, but by (1) and Corollary 2.2 , G is nilpotent so we have a contradiction. □

Remark 2.4 Note that the class of all (countable) NIP ω-categorical groups satisfy the conditions in Proposition 2.3. So does the class of ω-categorical dp-minimal groups (taking quotients of the universe M, as opposed to e.g., M^2, preserves dp-minimality).

Remark 2.5 In [17], Krupinski proved that (2) in Proposition 2.3 holds for the class of NIP ω-categorical groups with fsg. His proof uses a classification theorem on ω-stable characteristically simple groups due to Wilson and Apps [1, 29] (see remarks after Problem 2.16). By that theorem and [17, Proposition 3.2], it follows that ω-categorical characteristically simple groups with NIP are p-groups for some p. By the argument in the proof of [17, Proposition 3.1], it follows that for such groups G, if $a_1, \ldots, a_n \in G$ then $C(a_1) \cap \cdots \cap C(a_n)$ is infinite.

However, we will avoid using the classification theorem, and prove (3) directly for dp-minimal ω-categorical groups.

Remark 2.6 Note also that Fact 2.1 alone is not enough, even though it may seem so in light of the fact that if G is both NIP and contains an infinite abelian subgroup, then G contains an infinite definable abelian subgroup by Shelah [24, Claim 4.3]. However this subgroup is not necessarily $\emptyset$-definable.

2.4 *What we Get From dp-Minimality and NIP*

The only use of dp-minimality in the proof is the following basic observation.

Fact 2.3 ([27, Claim in proof of Proposition 4.31]) *If $(G, \cdot)$ is a dp-minimal group, then for every definable subgroups $H_1, H_2 \leq G$ either $[H_1 : H_1 \cap H_2] < \infty$ or $[H_2 : H_1 \cap H_2] < \infty$.*

We will also use the Baldwin-Saxl lemma, which is true for all NIP groups.

Fact 2.4 ([3]) *Let $(G, \cdot)$ be NIP. Suppose that $\varphi(x, y)$ is a formula and that $\{\varphi(x, c) \mid c \in C\}$ defines a family of subgroups of G. Then there is a number $n < \omega$ (depending only on φ) such that any finite intersection of groups from this family is already an intersection of n of them.*

2.5 Proof of the Main Result

Theorem 2.7 *If $(G, \cdot)$ is an infinite dp-minimal ω-categorical group, then G contains an infinite $\emptyset$-definable abelian subgroup.*

By Proposition 2.3 we get the following.

Corollary 2.8 *If $(G, \cdot)$ is a dp-minimal ω-categorical group, then G is nilpotent-by-finite.*

For the proof we work in a countable (so ω-saturated) model. So fix such a group G. By ω-categoricity, there is a minimal infinite $\emptyset$-definable subgroup $G_0 \leq G$ (i.e., G_0 contains no $\emptyset$-definable infinite subgroups), so we may assume that $G = G_0$.

Lemma 2.9 *For every $a, b \in G$ either $[C(a) : C(b) \cap C(a)] < \infty$ or $[C(b) : C(b) \cap C(a)] < \infty$.*

Proof *This follows directly from Fact 2.3.* □

Let $X = \{a \in G \mid |C(a)| = \infty\}$. By elimination of $\exists^\infty$, X is definable. By Fact 2.1, X is infinite. For $a, b \in X$, by Lemma 2.9, it follows that either $C(a) \cap C(b)$ has finite index in $C(a)$ or in $C(b)$. In either case, $C(a) \cap C(b)$ is infinite. Since $C(a) \cap C(b) \subseteq C(ab)$, it follows that X is a group. By our assumption on G (it contains no infinite $\emptyset$-definable subgroups), $G = X$.

Compare the following corollary with Remark 2.5.

Corollary 2.10 *For every $a_0, \ldots, a_{n-1} \in G$, $\bigcap \{C(a_i) \mid i < n\}$ is infinite.*

Proof By induction on n. For $n = 1$ and $n = 2$ we just gave the argument. For larger n it is exactly the same: $\bigcap \{C(a_i) \mid i < n\}$ has infinite index in one of $\bigcap \{C(a_i) \mid i < n-1\}$ or $\bigcap \{C(a_i) \mid 1 \leq i < n\}$, both infinite. □

For every $a \in G$ let $H_a = \{b \in G \mid [C(a) : C(b) \cap C(a)] < \infty\}$, and define

$$C^0(a) = \bigcap \{C(b) \mid b \in H_a\}.$$

Observe that $C^0(a)$ is definable over a since H_a is definable.

By ω-categoricity there exists some n_* such that for all $a, b \in G$, $[C(a) : C(b) \cap C(a)] < n_*$ iff $[C(a) : C(b) \cap C(a)] < \infty$.

Main Lemma 2.1 *For every $a, b \in G$ either $C^0(a) \subseteq C^0(b)$ or $C^0(b) \subseteq C^0(a)$.*

Proof By Fact 2.3 either $[C(a) : C(b) \cap C(a)] < n_*$ or $[C(b) : C(b) \cap C(a)] < n_*$. Suppose that the former happens. Then $C^0(a) \subseteq C^0(b)$: if $d \in H_b$, $[C(b) : C(b) \cap C(d)] < n_*$, so

$$\begin{aligned}[C(a) : C(a) \cap C(d)] &\leq [C(a) : C(a) \cap C(d) \cap C(b)] \\ &\leq [C(a) : C(a) \cap C(b)] \cdot [C(b) : C(b) \cap C(d)] < n_*^2.\end{aligned}$$

Hence $d \in H_a$. □

Lemma 2.11 *For every $a \in G$ the group $C^0(a)$ is infinite. Moreover,* $\left[C(a) : C^0(a)\right] < \infty$.

Proof By Fact 2.4, there is some N such that for every $k < \omega$ and every $a_i \in G$ for $i < k$, $\bigcap \{C(a_i) \mid i < k\} = \bigcap \{C(a_i) \mid i \in I_0\}$ where $I_0 \subseteq k$ is of size $\leq N$. Find $a_1, \ldots, a_N \in H_a$ with $\bigcap \{C(a_i) \mid i < N\} \cap C(a)$ of maximal index in $C(a)$ (this index is bounded by n_*^N). Let $D = \bigcap \{C(a_i) \mid i < N\} \cap C(a)$. Then, for every $b \in H_a$, $C(b) \cap D$ equals to some sub-intersection D' of size N, but then $[C(a) : D'] = [C(a) : D]$ so $D' = D$ and hence $\bigcap \{C(b) \mid b \in H_a\} = D$ and in particular it is infinite and of finite index in $C(a)$. □

Proof (Proof of Theorem 2.7.) Split into two cases.

Case 1: The set $Y = \left\{a \in G \,\middle|\, a \in C^0(a)\right\}$ is infinite.
In this case, note that if $a, b \in Y$ then by Main Lemma 2.1, we may assume that $C^0(a) \subseteq C^0(b) \subseteq C(b)$ (because $b \in H_b$). But then $a \in C(b)$, so Y is an infinite commutative $\emptyset$-definable set. Hence the group generated by Y must be abelian, and it must be G by our choice of G, so we are done.

Case 2: The set Y is finite.
Pick some $a_0 \notin Y$. By induction on $n < \omega$, choose $a_n \in C^0(a_{n-1}) \setminus Y$. We can find such elements by Lemma 2.11. For $n < \omega$, if $C^0(a_n) \subseteq C^0(a_{n+1})$ then $a_{n+1} \in C^0(a_{n+1})$ which cannot be, so by Main Lemma 2.1, $C^0(a_{n+1}) \subseteq C^0(a_n)$.
Let $K > \left[C(a_n) : C^0(a_n)\right]$ for all $n < \omega$. As $a_K \in C^0(a_i)$ for all $i < K$, $a_i \in C(a_K)$. Hence for some $i < j < K$, $a_i^{-1} a_j \in C^0(a_K) \subseteq C^0(a_i)$. But $a_j \in C^0(a_i)$ as well, so $a_i \in C^0(a_i)$—contradiction.

□

2.6 Concluding Remarks

Problem 2.12 Can we generalize this result to work under weaker assumptions than ω-categoricity, such as elimination of $\exists^\infty$? (i.e., assume that G is a dp-minimal group eliminating $\exists^\infty$, also for imaginaries, with an infinite abelian subgroup, then does it contain an infinite $\emptyset$-definable subgroup?)

Problem 2.13 Can one improve this to showing that every dp-minimal ω-categorical group is abelian-by-finite?

Remark 2.14 Any abelian-by-finite group is stable, so if we can solve Problem 2.13 positively, then it would mean that any dp-minimal omega-categorical group is stable. Why? suppose that $(G, \cdot)$ is a group with $H \leq G$ abelian of finite index. Then there is a normal subgroup of H with finite index in G (this is a standard exercise in group theory), and so we may assume that H is normal. Let $R = \mathbb{Z}[G]$, the group ring of G over $\mathbb{Z}$, whose elements we write as sums $\sum_{i<n} a_i g_i$ where

$a_i \in \mathbb{Z}$ and $g_i \in G$. Put a structure of a $\mathbb{Z}[G]$-module on H by letting $\left(\sum_{i<n} a_i g_i\right) \cdot h = \sum_{i<n} a_i \cdot h^{g_i}$ (where $h^g = g^{-1}hg$). As a module, H is stable (see [28, Example 8.6.6]). Now, G can be interpreted in this structure (with parameters). How? Suppose $[G : H] = n$. Then G is the union of $g_i H$ where $\{g_i \mid i < n\}$ are representatives for the different cosets of H in G. For each $i, j < n$ there is a unique $k(i,j) < n$ and $h(i,j) \in H$ such that $g_i \cdot g_j = g_{k(i,j)} h(i,j)$. So now interpret G as $\{c_i \mid i < n\} \times H$, where the c_i's are distinct elements from H and the product is given by $(c_i, h) \cdot (c_j, h') = \left(c_{k(i,j)}, h(i,j)\, h^{g_j} h'\right)$. Since h^{g_j} is just $g_j \cdot h$ in the module, this group is definable in H. The map $(c_i, h) \mapsto g_i h$ is then an isomorphism from this group to G.

Problem 2.15 Is there a (pure) group $(G, \cdot)$ which is ω-categorical NIP and unstable?

Problem 2.16 What about inp-minimal groups? Inp-minimality is the analogous notion to dp-minimality for NTP_2 [5], so it makes sense that this result still holds there, as it does in both the simple (by Macpherson [19]) and NIP case.

Remark 2.17 Problem 2.16 was solved by Frank Wagner, who shared his proof with us in a private communication. We decided to still include the following discussion, as it might be useful for any future generalizations to NTP_2.

Using the same notation as in [17], we let $B(F)$ be the group of all continuous functions from the cantor space 2^ω into a finite simple non-abelian group F. We also let $B^-(F)$ be the group of all such functions sending a fixed point $x_0 \in 2^\omega$ to $e \in F$. By Wilson [29] and Apps [1, Theorem 2.3] they are characteristically simple and ω-categorical, and in fact by a theorem of Wilson [29], a countably infinite ω-categorical characteristically simple group is either isomorphic to one of them, is an abelian p-group or is a perfect p-group. Neither groups is nilpotent-by-finite. If they were nilpotent-by-finite, then there would be a normal nilpotent subgroup of finite index, so they would be nilpotent (by Corollary 2.2). But then they must be abelian, which they are not.

It is worthwhile to note the following.

Proposition 2.18 *For a finite simple non-abelian group F, both $B(F)$ and $B^-(F)$ have TP_2 and in particular are not inp-minimal.*

For the proof we will need the following simple criterion for having TP_2.

Lemma 2.19 *Suppose that A is some infinite set in $\mathfrak{C}$ and $\varphi(x, y)$ is a formula such that for some $k < \omega$, for every sequence $\langle A_i \mid i < \omega\rangle$ of pairwise disjoint subsets of A, there are $\langle b_i \mid i < \omega\rangle$ such that $A_i \subseteq \varphi(\mathfrak{C}, b_i)$ and $\{\varphi(x, b_i) \mid i < \omega\}$ is k-inconsistent. Then T has TP_2.*

Proof We may enumerate A as $\langle a_s \mid s \in \omega^\omega \wedge |\mathrm{supp}(s)| < \omega\rangle$, where $\mathrm{supp}(s) = \{i \in \omega \mid s(i) \neq 0\}$. Let $A_{i,j} = \{a_s \mid s(i) = j\}$. Then for each $i < \omega$, $\left\{A_{i,j} \,\middle|\, j < \omega\right\}$ are mutually disjoint. By assumption we can find $b_{i,j}$ for $i, j < \omega$ such that $\left\{\varphi\left(x,b_{i,j}\right)\,\middle|\,j<\omega\right\}$ are k-inconsistent and $A_{i,j} \subseteq \varphi_{i,j}\left(\mathfrak{C},b_{i,j}\right)$. Then $\left\langle\varphi\left(x,b_{i,j}\right)\,\middle|\,i,j<\omega\right\rangle$ witness the tree property of the second kind. □

Proof We do the proof for $B(F)$. The proof for $B^{-}(F)$ is similar.

Let $\varphi(x,y)$ be the formula $x \neq e$ and $x \in C(C(y))$.

Fix some $g \in F$, $g \neq e_F$. Suppose that $s \subseteq 2^{\omega}$ is a clopen subset, and let $f_s \in B(F)$ be such that $f_s \restriction s$ is constantly g and $f \restriction 2^{\omega}\backslash s$ is constantly e_F. Then $C(f_s)$ contains (in fact equals) all functions f' such that $f'(s) \subseteq C_F(g)$ (so outside of s there are no restrictions on f'). Hence if $f' \in C(C(f_s))$ then $f' \restriction 2^{\omega}\backslash s$ is constantly e_F.

It follows that if $s_1 \cap s_2 = \emptyset$ are two clopen subsets, then $C(C(f_{s_1})) \cap C(C(f_{s_2})) = \{e\}$.

On the other hand, if $s_1 \subseteq s_2$, then $f_{s_1} \in C(C(f_{s_2}))$.

Fix a sequence of pairwise disjoint clopen sets $\langle s_i \mid i<\omega\rangle$. Then we see that for any choice of finite pairwise disjoint subsets A_n, $n<\omega$ such that A_n are finite, $f_n = f_{\bigcup\{s_i \mid i \in A_n\}}$ satisfies $\{\varphi(x,f_n) \mid n<\omega\}$ is 2-inconsistent but $f_{s_i} \models \varphi(x,f_n)$ if $i \in A_n$. By compactness, we get such f_n's for every choice of pairwise disjoint subsets A_n for $n<\omega$ (not necessarily finite). By Lemma 2.19 we are done. □

2.7 A Theorem on NIP Groups

We end this section with a general remark on NIP groups (without any other assumptions).

Theorem 2.20 *Suppose that $(G,\cdot)$ is an NIP group (or more generally, a type-definable group in an NIP theory). Then there is some finite set A such that $C(A)$ is abelian.*

Proof Assume that G is a type-definable group, defined by the type $\pi(x)$ (the multiplication $\cdot_G$ and the unit e_G are definable).

Let M be any $|\pi|^{+}$-saturated model, so that $G(M) \prec G(\mathfrak{C})$ by the Tarski-Vaught test.

Let p_0 be a partial type containing the formulas $x \in C(A)$ for all finite sets A of $G(M)$. The partial type p_0 is finitely satisfiable in $G(M)$ (witnessed by e_G). Let S be the set of all global types in $S_G(\mathfrak{C})$ containing p_0 and f.s. in $G(M)$. All of these types are in particular invariant over M, so their product is well defined. (For the precise definition of a product of global invariant types, see [27, 2.2.1], but one can understand it from the proof.)

Claim For $p,q \in S$, $p(x) \otimes q(y) \models x \cdot y = y \cdot x$.

Proof We need to show that if $N \supseteq M$, $a \models q|_N$, $b \models p|_{Na}$ then $a \cdot b = b \cdot a$. If not, then $b \notin C(a)$, so for some $b_0 \in G(M)$, $b_0 \notin C(a)$, so $a \notin C(b_0)$—contradiction. □

By Simon [27, Lemma 2.26], it follows that for any $a \models p, b \models q$, $a \cdot b = b \cdot a$ (the proof there works just fine for type-definable groups, because it only uses that the formula for multiplication is NIP, but multiplication is definable).

By compactness for every $p(x), q(y) \in S$ there are formulas $\psi_{p,q}(x) \in p, \varphi_{p,q}(y) \in q$ such that for every $a \models \psi_{p,q}$, $b \models \varphi_{p,q}$ in G, $a \cdot b = b \cdot a$. Fix p. By compactness (as S is closed), there is a finite set of types $q_i \in S$ for $i < n$ such that $\varphi_p = \bigvee_{i<n} \varphi_{p,q_i}$ contains S. Let $\psi_p = \bigwedge_{i<n} \psi_{p,q_i}$. Again by compactness there are p_i for $i < m$ such that $\bigvee_{i<m} \psi_{p_i}$ contains S. Let $\chi = \left(\bigwedge_{i<m} \varphi_{p_i}\right) \wedge \left(\bigvee_{i<m} \psi_{p_i}\right)$, then χ contains S and for every $a, b \models \chi$ in $G(\mathfrak{C})$, $a \cdot b = b \cdot a$. (This is the same as in the proof of [27, Proposition 2.27].)

It cannot be that for all finite $A \subseteq G(M)$, $\neg\chi(M) \cap C(A) \neq \emptyset$ (otherwise we can define a type, f.s. in M, containing p_0, so in S, but not satisfying χ). Hence there is some finite $A \subseteq G(M)$ such that $C(A)(M) \models \chi$. Hence $C(A)(M)$ is abelian, but as $G(M) \prec G(\mathfrak{C})$, so is $C(A)$. $\square$

Remark 2.21 When the group G is an ω-categorical characteristically simple NIP group, then by Proposition 2.18, and the remark before it (or just [17, Fact 0.1 and Proposition 3.2]), Krupinski's proof of [17, Proposition 3.1] gives us that for any finite set A, $C(A)$ is infinite. Together with Theorem 2.20, we know that we can find some A such that $C(A)$ is abelian and infinite.

3 f-Generic is the Same as Strongly f-Generic

Assume that G is definably amenable and dp-minimal. The main theorem here says that any definable $X \subseteq G$ which divides over a small model also G-divides. This means that are at most boundedly many global f-generic types and a global type is f-generic iff it is strongly f-generic (see Corollaries 3.6 and 3.8).

Let us first recall the definitions. Throughout we assume T is NIP, and we work in a monster model $\mathfrak{C}$.

Definition 3.1 A definable group G is definably amenable if it admits a G-invariant Keisler measure on its definable subsets.

A *Keisler measure* is a finitely additive probability measure on definable subsets of G. We will not use this definition, so there is no need for us to get too deeply into Keisler measures. Instead we will use the following characterization from [7] given in terms of G-dividing.

Definition 3.2 For $X \subseteq G$ definable, we say that it G-divides if there is an indiscernible sequence $\langle g_i \mid i < \omega \rangle$ of elements from G over the parameters defining X such that $\{g_i X \mid i < \omega\}$ is inconsistent (equivalently, remove the indiscernibility assumption and replace it with k-inconsistency). Similarly, we say that X *right-G-divides* if there is a sequence as above such that $\langle X g_i \mid i < \omega \rangle$ is inconsistent.

Fact 3.1 *[7, Corollary 3.5]Let G be a group definable in an NIP theory. Then if G is definably amenable then the family of G-dividing subsets of G forms an ideal. Hence in this case any non-G-dividing partial type can be extended to a global one.*

As an example which relates to the previous section, we note that any countable ω-categorical group is locally finite and hence it is amenable by Runde [22, Example 1.2.13] and so any group elementarily equivalent to it is definably amenable [27, Example 8.13].

Definition 3.3 A global type is called *f-generic* if it contains no G-dividing formula.

Remark 3.4 ([7, Proposition 3.4]) If G is definably amenable, then p is f-generic iff all its formula are f-generic, which means that for every $\varphi \in p$, no translate of φ forks over M where M is some small model containing the parameters of φ. It is also proved there that a formula is f-generic iff it does not G-divide, so we will use these terms interchangeably. Similarly, we will write right-f-generic for non-right-G-dividing.

Fact 3.2 ([7, Proposition 3.9]) *When G is definably amenable then a global type is f-generic type iff it is G^{00}-invariant.*

Theorem 3.5 *Suppose that G is dp-minimal and definably amenable. Then if $\varphi(x,c)$ forks over a small model M, then it G-divides.*

Proof Suppose not.

By assumption (and as forking equals dividing over models, see [6]), there is an M-indiscernible sequence $\langle c_j \mid j<\omega \rangle$ such that $\langle \varphi(x,c_j) \mid j<\omega \rangle$ is inconsistent.

However, $\varphi(x,c_j)$ is still f-generic.

We now divide into two cases: either there is a formula $\psi_0(x,b)$ which is right-f-generic but not f-generic (call this case 0), or not (case 1). In case 0, let $\zeta_0(x,y,b) = \psi_0(y^{-1}x,b)$.

Note that if case 0 does not occur, then every G-dividing formula also right-G-divides. As X is f-generic iff X^{-1} is right-f-generic, this means that every right-G-dividing formula also G-divides.

In case 1, choose a formula $\psi_1(x,b)$ for which, for every formula $\chi(x) \supseteq G^{00}$ (with no parameters) both $\psi_1(x,b) \wedge \chi(x)$ and $\neg\psi_1(x,b) \wedge \chi(x)$ are f-generic (such a formula exists, as otherwise there is a unique non-G-dividing type concentrating on G^{00}, so the number of f-generic types is bounded, but by assumption there are unboundedly many). Let $\zeta_1(x,y,z,b) = \psi_1(y^{-1}x,b) \wedge \neg\psi_1(z^{-1}x,b)$. We may assume that $\langle c_j \mid i<\omega \rangle$ is indiscernible over Mb.

Depending on the case, let $\zeta(x,y,z,b)$ be either ζ_0 or ζ_1 (so z might be redundant). Construct a sequence $\langle I_i, g_i, h_i \mid i<\omega \rangle$ such that:

- In case 0, $g_i \in G$. In case 1, $g_i, h_i \in G^{00}$.
- I_i is indiscernible, $I_i = \langle e_{i,j} \mid j<\omega \rangle$, $e_{i,j} \models \varphi(x,c_j)$ for all $i,j<\omega$.
- $e_{i,j} \models \zeta(x,g_i,h_i,b)$ for all $i,j<\omega$.
- $e_{i',j} \not\models \zeta(x,g_i,h_i,b)$ for all $i',i,j<\omega$ whenever $i'>i$.

(In case 0, we only need g_i.) How?

Note that for any $g, h \in G^{00}$, $\zeta(x, g, h, b)$ does G-divide by Fact 3.2 (this is trivially true in case 0).

By compactness it is enough to construct such a sequence for $i < n$. Suppose we have $\langle I_i, g_i, h_i \,|\, i < n \rangle$. Let $\xi(x) = \bigvee_{i<n} \zeta(x, g_i, h_i, b)$. Then $\xi(x)$ does G-divide by Fact 3.1. Hence $\varphi(x, c_j) \setminus \xi(x)$ is not empty for all j, and hence we may find a sequence $e_{n,j} \models \varphi(x, c_j) \setminus \xi(x)$. Consider the sequence $I = \langle (e_{0,j}, \ldots, e_{n,j}, c_j) \,|\, j < \omega \rangle$. By Ramsey and compactness there is an $Mh_{<n}g_{<n}b$-indiscernible sequence I' with the same EM-type as I over $Mh_{<n}g_{<n}b$. There is an automorphism taking $\langle c'_j \,|\, j < \omega \rangle$ to $\langle c_j \,|\, j < \omega \rangle$ over Mb, and applying it we are in the same situation as before (changing $h_{<n}g_{<n}$ and $e_{i,j}$) but now $I_n = \langle e_{n,j} \,|\, j < \omega \rangle$ is indiscernible. This takes cares of all the bullets except the third one, for which needs to find g_n, h_n.

In case 0, the set $\left\{ \psi_0(x, b) \cdot e_{n,j}^{-1} \,\middle|\, j < \omega \right\}$ is consistent (as $\psi_0(x, b)$ does not right-G-divide), so contains some $g \in G$, hence $ge_{n,j} \models \psi_0(x, b)$, i.e., $e_{n,j} \models g^{-1} \cdot \psi_0(x, b) = \zeta_0(x, g^{-1}, b)$ so let $g_n = g^{-1}$.

In case 1, the set $\left\{ (\chi(x) \wedge \psi_1(x, b)) \cdot e_{n,j}^{-1} \,\middle|\, G^{00} \subseteq \chi(x), j < \omega \right\}$ is consistent (as G-dividing = right-G-dividing in this case) so again we can find $g \in G^{00}$ realizing it, so in particular $e_{n,j} \models g^{-1} \cdot \psi_1(x, b)$. Similarly, there is some $h \in G^{00}$ such that $e_{n,j} \models h^{-1} \cdot (\neg\psi_1(x, b))$. Finally, choose $g_n = g^{-1}$ and $h_n = h^{-1}$.

This finishes the construction.

Now by Ramsey and compactness we may assume that $\langle I_i g_i h_i \,|\, i < \omega \rangle$ is indiscernible over $Mb\langle c_j \,|\, j < \omega \rangle$ and that $\langle \langle e_{i,j} \,|\, i < \omega \rangle c_j \,|\, j < \omega \rangle$ is indiscernible over $Mb \langle g_i h_i \,|\, i < \omega \rangle$.

Let $\xi(x, y, y', z, z', b) = \zeta(x, y, z, b) \setminus \zeta(x, y', z', b)$. Then for every $i > 0$ and $j < \omega$, $e_{i_0,j_0} \models \xi(x, g_i, h_i, g_{i-1}, h_{i-1})$ iff $i_0 = i$ and $e_{i_0,j_0} \models \varphi(x, c_j)$ iff $j = j_0$ contradicting dp-minimality. □

For the next corollary, we recall that in the context of NIP, definably amenable groups, a global type is called *strongly f-generic* if it is f-generic and does not fork over some small model (this is not the original definition, but see [7, Proposition 3.10].

Corollary 3.6 *If G is a dp-minimal definably amenable group, then any global f-generic 1-type p is strongly f-generic.*

Proof Take any small model M. Then p cannot divide over M. □

Remark 3.7 Theorem 3.5 does not hold for a group definable in a dp-minimal theory. Consider $T = RCF$, and let $G = R^2$ where R is a saturated model of T. Example 3.11 in [7] gives a G-invariant type $r(x, y)$ (so does not G-divide) which is not invariant over any small model M.

Corollary 3.8 *Suppose that G is a dp-minimal definably amenable group. Then there are boundedly many global f-generic types.*

Proof Fix some small model M. This follows by NIP and (the proof of) Corollary 3.6, as there are boundedly many global types non-forking over M (by NIP they must be invariant over M). □

Corollary 3.9 *Suppose G is a dp-minimal definably amenable group. Then there are boundedly many G-invariant Keisler measures.*

Proof Suppose that there are unboundedly many such measures μ_i. Fix some small model M. By Erdös-Rado, we may find a formula $\varphi(x, y)$, a type $p \in S_y(M)$ some numbers $\alpha \neq \beta \in [0, 1]$ and a sequence of G-invariant Keisler measures $\langle \mu_i \mid i < \omega \rangle$ such that for all $i < j < \omega$, $\mu_i \left(\varphi\left(x, a_{i,j}\right)\right) = \alpha$ and $\mu_j \left(\varphi\left(x, a_{i,j}\right)\right) = \beta$ for some $a_{i,j} \models p$ in $\mathfrak{C}$.

Then there are $a, b \models p$ such that $\alpha = \mu_0(\varphi(x, a)) \neq \mu_1(\varphi(x, a)) = \beta$ and $\alpha = \mu_1(\varphi(x, b)) \neq \mu_2(\varphi(x, b)) = \beta$. In particular $\mu_1(\varphi(x, a) \triangle \varphi(x, b)) \neq 0$. As μ_1 is G-invariant, $\varphi(x, a) \triangle \varphi(x, b)$ does not G-divide (see [7, Theorem 3.38], but this follows easily from the definitions), but it forks by NIP. □

Corollary 3.10 *Suppose that $(F, +, \cdot)$ is a dp-minimal field. Then every additive f-generic set (i.e., with respect to $(F, +, 0)$) is also multiplicatively f-generic.*

Proof Note that any abelian group is definably amenable (see [27, Example 8.13]).

Suppose that X is additively f-generic. For any $a \in F^\times$, $a \cdot X$ is also additively f-generic. Hence, if X is not multiplicatively f-generic, then there is an indiscernible sequence $\langle a_i \mid i < \omega \rangle$ over the parameters defining X such that $\{a_i X \mid i < \omega\}$ is inconsistent (so k-inconsistent for some $k < \omega$). Increasing the sequence to any length, by Fact 3.1, we get unboundedly many additively f-generic global types. Contradicting Corollary 3.8. □

Problem 3.11 Is there a dp-minimal group which is not definably amenable?

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Square Subgroups of Decomposable Rank Three Groups

Fatemeh Karimi

Abstract Let $A = A_1 \oplus A_2$ be a torsion-free abelian group of rank three, where $r(A_2) = 2$ and let $\Box A$ be its square subgroup as defined in the introduction below. The main objective of this paper is to calculate the square subgroup of A and show that for a fixed rank two group A_2, this subgroup can vary when $t(A_1)$ changes. Moreover, we generalize these results to some decomposable torsion-free groups of arbitrary rank and use the rank one groups belonging to a maximal independent set to determine all multiplications on a group.

Keywords Square subgroup • Nil modulo a subgroup • Rank

Mathematical Subject Classification (2010): 20K15

1 Introduction

For the purposes of this paper all groups are abelian and written additively. A ring R is said to be a ring on A if the group A is isomorphic to the additive group of R. In this situation we write $R = (A, *)$ where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative. In general, we call a group A a nil group if there is no ring on A other than the zero ring. A generalization of the notion of nil group was considered by Feigelstock [4]. In fact, let B be a subgroup of A, then A is nil modulo B if $A * A \subseteq B$ for every ring $(A, *)$ on A. Clearly A is a nil group if and only if A is nil modulo $\{0\}$. Feigelstock [4] shows that if B is a divisible subgroup of A and A is nil modulo B, then A/B is a nil group. Also he goes on to ask if this is true in general. In other words, does A nil modulo B imply that A/B

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is a nil group? Stratton and Webb [7] showed that the answer to this question is no. However, the question has a positive answer if either A is a torsion group or B a direct summand of A.

It is clear that if A is nil modulo B_1 and B_2, then A is nil modulo $B_1 \cap B_2$. This suggests the following definition of the square subgroup $\Box A$, as

$$\Box A = \cap\{B \subseteq A \mid A \text{ is nil modulo } B\}.$$

Clearly $\Box A$ is the smallest subgroup with the property that A is nil modulo $\Box A$ and $A * A \subseteq \Box A$, for any ring $(A, *)$ on A. The square subgroup was initially studied by Stratton and Webb [7] and subsequently Aghdam and Najafizadeh [1] showed that the square subgroup of any non-homogenous and indecomposable torsion-free group A of rank two is a pure subgroup of A and that $A/\Box A$ is a nil group. They studied $\Box A$ by classifying A according to the cardinality of the typeset.

In this paper we investigate the square subgroup of a decomposable rank three torsion-free group $A = A_1 \oplus A_2$ and show that for a fixed rank two group A_2, the square subgroup can be different when the type, $t(A_1)$, of A_1 changes. Moreover, we find cases in which $\Box A$ is a pure subgroup of A and observe that $A/\Box A$ is not a nil group in most of these cases.

2 Notation and Preliminaries

Let A be a torsion-free abelian group, then the typeset of A is the partially ordered set of types, i.e.,

$$T(A) = \{t(a) \mid 0 \neq a \in A\},$$

where $t(a)$ denotes the type of any non-zero element a in A. We also write $h_p^A(a)$ for the p-height of the element a in A and $\langle a\rangle^*$ for the pure subgroup of A generated by a. A type $t \in T(A)$ is said to be maximal if for all $\mu \in T(A)$, $\mu \geq t$ implies that $\mu = t$. A good reference for basic facts about type and other undefined concepts is [5, pp. 109].

Proposition 2.1 *Let A be a torsion-free group of finite rank. Then the length of every chain in $T(A)$ is at most equal to the rank of A.*

Proof See [3, Proposition 1]. □

Theorem 2.2 *A torsion-free ring of rank one is either a zero ring or isomorphic to a subring of the rational number field. A torsion-free group of rank one is not a nil group if and only if its type is idempotent.*

Proof See [5, Theorem 121.1]. □

Lemma 2.3 *Let G be a subgroup of $\mathbb{Q}$. If $1/b, 1/d \in G$ and $(b, d) = 1$, then $1/bd \in G$.*

Proof Obvious. □

We recall some definitions and results previously used in [2]. Let $A = A_1 \oplus A_2$ be a rank three torsion-free group and $\{x, y, z\}$ be independent elements, where $x \in A_1$ and $y, z \in A_2$. Each element g of A has the unique representation $g = ux + vy + wz$, where u, v, w are rational numbers. Let,

$$U_0 = \{u_0 \in \mathbb{Q} : u_0 x \in A\}, \quad U = \{u \in \mathbb{Q} : ux + vy + wz \in A \quad \text{for some} \quad v, w \in \mathbb{Q}\},$$
$$V_0 = \{v_0 \in \mathbb{Q} : v_0 y \in A\}, \quad V = \{v \in \mathbb{Q} : ux + vy + wz \in A \quad \text{for some} \quad u, w \in \mathbb{Q}\}.$$
$$W_0 = \{w_0 \in \mathbb{Q} : w_0 z \in A\}, \; W = \{w \in \mathbb{Q} : ux + vy + wz \in A \quad \text{for some} \quad u, v \in \mathbb{Q}\}.$$

Then U_0, V_0 and W_0 are subgroups of U, V and W, respectively, and U, U_0, V, V_0, W, W_0 are called the groups of rank one belonging to the independent set $\{x, y, z\}$.

Observe that in general $U_0 \leq U$; i.e., U_0 is a subgroup of U. But in our case since $A = A_1 \oplus A_2$, A_1 is of rank one, $x \in A_1$ and U, U_0 are rank one subgroups belonging to it, we have $A_1 = \langle x \rangle_*$ which implies $U = U_0$.

Proposition 2.4 *Let A, B be subgroups of $\mathbb{Q}$ such that $1 \in A \cap B$. Suppose there exists a non-zero integer n such that $nA \leq B$. If m is the least positive integer such that $mA \leq B$, then the following statements hold:*

(a) Let p be a prime number such that $\alpha = h_p^A(1) < \beta = h_p^B(1)$, then for all $k \leq \beta$, $(1/p^{k-\alpha})(mA) \leq B$. Furthermore, p does not divide m.
(b) If $B \leq A$, then $mA = B$ and $1/m \in A$.
(c) Let d be a positive integer such that d divides m and $1/d \in B$. If $B^2 = B$ then $d = 1$.

Proof See [1, Proposition 2.6]. □

Theorem 2.5 *Let A be a torsion-free group of rank two. If A is non-nil, then $T(A)$ contains a unique minimal member and at most three elements. If the typeset consists of:*

(a) one type, then the type must be idempotent.
(b) two types, then one is minimal and the other is maximal.
(c) three types, then one of the types is minimal and the other two types are maximal. In this case at least one of the maximal types is idempotent.

Proof See [6, Theorem 3.3]. □

Lemma 2.6 *Let A be a torsion-free group, $0 \neq x \in A$ and U, U_0 be rank one groups belonging to x. Then $t(x) = t(U_0)$.*

Proof Let $t(x) = (k_1, k_2, k_3, \cdots)$. Then for any $i = 1, 2, \cdots$, there exist some $y_i \in A$ such that $x = (p_i)^{k_i} y_i$. This yields $1/(p_i)^{k_i} \in U_0$ and hence $t(U_0) \geq t(x)$. Similarly it is easy to see that $t(U_0) \leq t(x)$, because if $1/(p_i)^{l_i} \in U_0$, then $(1/(p_i)^{l_i})x \in A$ which means that $l_i \leq k_i$. □

3 Main Results

Lemma 3.1 *Let A be a torsion-free group of rank three and* $\{x, y, z\}$ *a maximal independent set of A with rank one groups* U, U_0, V, V_0, W, W_0 *belonging to it. If* $U_0^2 = U_0$ *and there exists an integer m such that* $mU = U_0$*, then the multiplication:*

$$x^2 = m^2 x, xy = yx = y^2 = xz = zx = z^2 = zy = yz = 0,$$

yields a ring on A such that $A^2 = U_0 x$.

Proof Let $a_1 = u_1 x + v_1 y + w_1 z$ and $a_2 = u_2 x + v_2 y + w_2 z$ be two non-zero elements of A. Now $a_1 a_2 = u_1 u_2 m^2 x$ and by $u_1, u_2 \in U$ we obtain $m^2 u_1 u_2 = (mu_1)(mu_2) \in (mU)^2 = U_0^2 = U_0$ which yields $m^2 u_1 u_2 x \in U_0 x \subseteq A$. Thus the product gives a ring on A such that $A^2 \subseteq U_0 x$.

Now by $U_0^2 = U_0$ and $mU = U_0$ we obtain $(mU)^2 = U_0^2 = U_0$ and therefore any $u_0 \in U_0$ can be written in the form $u_0 = (mu_1)(mu_2)$ for some $u_1, u_2 \in U$. By the definition of U, there exist elements $u_1 x + v_1 y + w_1 z$ and $u_2 x + v_2 y + w_2 z$ in A such that $(u_1 x + v_1 y + w_1 z)(u_2 x + v_2 y + w_2 z) \in U_0 x$ which yields $U_0 x \subseteq A^2$. Consequently, $A^2 = U_0 x$ as required. □

Remark 3.2 Note that in Lemma 3.1 the condition $U_0^2 = U_0$ can be replaced by $t^2(x) = t(x)$.

From Theorem 2.5, we know that if A_2 is a non-nil rank two group, then its typeset contains at most three elements. In the succeeding we will consider the cases where $T(A_2)$ contains either two or three elements and we will obtain the square subgroup of the group $A = A_1 \oplus A_2$, when the type of the rank one group A_1 changes.

Theorem 3.3 *Let* $A = A_1 \oplus A_2$ *be a rank three group such that*

$$T(A_2) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\},$$

$t_0 < t_i$ *for* $i = 1, 2$*, and* t_1 *and* t_2 *are maximal in* $T(A_2)$*. If* $t(A_1)$ *is non-idempotent and incomparable with* t_0, t_1, t_2*, then* $\Box A = \langle y \rangle^*$*, where* $0 \neq y \in A_2$ *with* $t(y) = t_1$.

Proof Let $t(A_1) = t, x \in A_1, y, z \in A_2$ with $t(y) = t_1, t(z) = t_2$ and let $U(= U_0), V, V_0, W, W_0$ be the rank one groups belonging to $\{x, y, z\}$. Then for any ring $R = (A, .)$ we will have:

$$xy = yx = xz = zx = yz = zy = z^2 = x^2 = 0, y^2 = ry,$$

for some $r \in \mathbb{Q}$; because in this case

$$T(A) = \{t_0, t_1, t_2, t, t \cap t_1, t \cap t_2, t \cap t_0 \mid t_1^2 = t_1, t_2^2 \neq t_2, t^2 \neq t\},$$

where $t_0 < t_i$ for $i = 1, 2$, and t_1, t_2 and t are maximal in $T(A)$. But $t(x^2) \geq t(x).t(x) = t^2 \geq t$ and $t^2 \neq t$. So if $x^2 \neq 0$, then

$$t(x^2) \geq t^2 \geq t, \quad t^2 \neq t.$$

Thus $t(x^2)$ would be a maximal element of $T(A)$ which it is not equal to t. (Because if $t = t(x^2) \geq t^2$, then using the fact $t^2 \geq t$, we obtain $t^2 = t$, a contradiction.) Now if $t(x^2) = t_1$ or t_2, then either t_1 or $t_2 = t(x^2) \geq t$, which is impossible since t is incomparable to both t_1 and t_2. This means $x^2 = 0$; similarly $z^2 = 0$. Moreover, if $xz \neq 0$, then a similar argument would give $t(xz) \geq t_2$ to be a maximal element of $T(A)$ which is not equal to t_2. (Because if $t(xz) = t_2$, then $t_2 = t(xz) \geq t$, which is impossible, as t and t_2 are incomparable.) So $t(xz) = t$ or t_1. But in this case we obtain t or $t_1 = t(xz) \geq t_2$, which is impossible, because t and t_2 or t_1 and t_2 are incomparable. Therefore $xz = 0$. In the same way zx, yz, zy, xy and yx are all zero.

Furthermore, $t(y^2) \geq t_1.t_1 = t_1^2 = t_1$ and the maximality of t_1 in $T(A)$ yields if $y^2 \neq 0$, then $t(y^2) = t(y)$. Thus y^2 and y are dependent and so $y^2 = ry$, for some $r \in \mathbb{Q}$.

Now if $a_1 = u_1x + v_1y + w_1z, a_2 = u_2x + v_2y + w_2z$ are two arbitrary elements of A, then $a_1a_2 = rv_1v_2y$, which implies $A^2 \subseteq \langle y\rangle^*$. From the definition of $\Box A$, this yields $\Box A \subseteq \langle y\rangle^*$.

Now suppose that $v \in V$. Then there exists $u, w \in \mathbb{Q}$ such that $ux + vy + wz \in A$. So $(ux + vy + wz)y = rvy$, hence $rv \in V_0$ for all $v \in V_0$, which gives $rV \subseteq V_0 \subseteq V$ and therefore there exists a positive integer k such that $kV \subseteq V_0$. If m is the least such integer, Proposition 2.4(b) yields $mV = V_0$ and so by Lemma 3.1 we can construct a ring multiplication $\cdot$ on A satisfying $A \cdot A = V_0y = \langle y\rangle^*$. Since the subgroup $\Box A$ contains the product A_*A for *any* multiplication $*$ on A, it follows immediately that for this particular product $\cdot$, $\langle y\rangle^* = A \cdot A \subseteq \Box A$. Thus we have established that $\Box A$ is equal to $\langle y\rangle^*$, as required. □

Remark 3.4 Note that for the group $A = A_1 \oplus A_2$ in Theorem 3.3, any ring on A is a ring on A_2 and any ring on A_2 is a ring on A, because $xy = yx = xz = zx = yz = zy = z^2 = x^2 = 0$.

Remark 3.5 A generalization of the above theorem is:

Let $A = A_1 \oplus (\oplus_{i \in I} A_i)$ such that $r(A_1) = 2$ and A_is are rank one nil groups with incomparable non-idempotent types and $t(A_i)$s are incomparable with the members of $\mathrm{T}(A_1) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\}, t_0 < t_i$ for $i = 1, 2$ and t_1 and t_2 are maximal in $T(A_1)$. If $0 \neq y \in A_1$ with $t(y) = t_1$, then $\Box A = \langle y\rangle^*$.

Theorem 3.6 *Let $A = A_1 \oplus A_2$ be a rank three torsion-free group such that*

$$T(A_2) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\},$$

$t_0 < t_i$ for $i = 1, 2$, and t_1 and t_2 are maximal in $T(A_2)$. If $t(A_1)$ is idempotent and incomparable with t_0, t_1, t_2, then $\Box A = A_1 \oplus \langle y\rangle^*$, in which $0 \neq y \in A_2$ with $t(y) = t_1$.

Proof Let $t(A_1) = t$ and let $x \in A_1, y, z \in A_2$ be elements as in the previous theorem. Now by standard type-based arguments similar to those used in the proof of Theorem 3.3, we can show that for any ring $R = (A, .)$:

$$xy = yx = xz = zx = yz = zy = z^2 = 0, x^2 = sx, y^2 = ry,$$

for some $r, s \in \mathbb{Q}$. By considering the multiplication of two arbitrary non-zero elements of A, we will obtain $A^2 \subseteq A_1 \oplus \langle y\rangle^*$, which means A is nil modulo $A_1 \oplus \langle y\rangle^*$ and so $\Box A \leq A_1 \oplus \langle y\rangle^*$. Now from $U = U_0$, taking $m = 1$ in Lemma 3.1, we get a ring on A with $A^2 = A_1$. So, as observed in the proof of Theorem 3.3, $A_1 \subseteq \Box A$.

Moreover, if $v \in V$, then there exist $u, w \in \mathbb{Q}$ such that $ux + vy + wz \in A$. But $(ux + vy + wz)y = rvy$, and so $rv \in V_0$ for all $v \in V$. Hence $rV \subseteq V_0 \subseteq V$. Then there exists a positive integer k such that $kV \subseteq V_0$ and if m is the least such integer, Proposition 2.4(b) yields $mV = V_0$. Therefore by Lemma 3.1 we can construct a ring on A satisfying $A^2 = V_0 y = \langle y\rangle^*$. As noted in the proof of Theorem 3.3, this yields $\langle y\rangle^* \subseteq \Box A$, and so $A_1 \oplus \langle y\rangle^* = \Box A$, as claimed. □

Theorem 3.7 *If $A = A_1 \oplus (\oplus_{i\in I} A_i)$, where $r(A_1) = 2$, and the A_is are rank one groups such that the types $t(A_i)$ are incomparable with each other and with the members of $T(A_1) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\}$; $t_0 < t_i$ for $i = 1, 2$ and t_1 and t_2 are maximal in $T(A_1)$, then $\Box A = \langle y\rangle^* \oplus (\oplus_{j\in J\subseteq I} A_j)$, where the types $t(A_j)$ are idempotent and $0 \neq y \in A_1$ with $t(y) = t_1$.*

Proof Let $x_i \in A_i$, $t(A_i) = t_i$ and $t_j^2 = t_j$ for any $j \in J \subseteq I$. Then as $t^2(x_j) = t(x_j)$, we can define a ring $(A, .)$ such that $A.A = \langle x_j\rangle^* = A_j \subseteq \Box A$. Moreover, since $t^2(y) = t(y)$, we have $\langle y\rangle^* \subseteq \Box A$. This means $\langle y\rangle^* \oplus (\oplus_{j\in J\subseteq I} A_j) \subseteq \Box A$.

We establish the reverse inclusion. For two indices $l \neq k$, we have $t(x_l x_k) \geq t_l t_k \geq t_l, t_k$ and $t_l t_k \neq t_l, t_k$; the inequality holds because t_l and t_k are incomparable. But the types t_i and the members of $T(A_1)$ are incomparable which gives that the maximal elements of $T(A)$ are just t_1, t_2 and t_i, for all $i \in I$. So if $x_l x_k \neq 0$, then $t(x_l x_k)$ should be a maximal element of $T(A)$ which is not equal to t_l and t_k. So if $x_l x_k \neq 0$, then $t(x_l x_k) = t_1$ or t_2 or t_m for some $(l, k \neq) m \in I$. This means one of t_1, t_2, $t_m = t(x_l x_k) \geq t_l, t_k$, which yields a contradiction because of incomparability of the maximal elements of $T(A)$. Therefore for any ring on A we would have $x_l x_k = 0$.

Moreover, arguing in a similar way, we cannot have $0 \neq x_i^2 \in \oplus_{(i\neq)k\in K} A_k \oplus A_1$, for some finite $K \subseteq I$. Because in this case we could write $x_i^2 = \sum_{k\in K} r_k x_k + a$ for some $r_k \in \mathbb{Q}$, $a \in A_1$. So $\bigcap_{k\in K} t_k \bigcap t(a) = t(x_i^2) \geq t_i^2 \geq t_i$ and therefore $t_i \leq t_k$, for all $k \in k$ and $t_i \leq t(a)$, which is impossible since $t(a) = t_0$ or t_1 or t_2. This means that if $x_i^2 \neq 0$, then it must be in A_i and this could only happen when $i \in J$.

Also if we choose $y, z \in A_1$ such that $t(y) = t_1$ and $t(z) = t_2$ then, similar to the proof of the previous theorem, we can see that for all rings on A, $x_i z = z x_i = x_i y = y x_i = z^2 = 0$, which means that if $\{y, z, x_i \mid y, z \in A_1, i \in I\}$ is a maximal independent set of A and $a_1 = \alpha_1 y + \beta_1 z + \sum_{i \in K} \gamma_i x_i$, $a_2 = \alpha_2 y + \beta_2 z + \sum_{l \in K'} \gamma_l x_l$, $(K, K'$ finite subset of $I)$, are two arbitrary elements of A, with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_i, \gamma_l \in \mathbb{Q}$, for all $i \in K, l \in K'$, then for any ring $(A, .)$ we will have $a_1.a_2 \in \langle y \rangle^* \oplus (\oplus_{j \in J \subseteq I} A_j)$ and so $\Box A \subseteq \langle y \rangle^* \oplus (\oplus_{j \in J \subseteq I} A_j)$. □

Theorem 3.8 *Let* $A = A_1 \oplus A_2$ *be a rank three torsion-free group such that* $T(A_2) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\}, t_0 < t_i$ *for* $i = 1, 2$ *and* t_1 *and* t_2 *are maximal in* $T(A_2)$. *If* $t(A_1) = t_1$ *then* $\Box A = A_1 \oplus \langle y \rangle^*$, *where* $0 \neq y \in A_2$ *and* $t(y) = t_1$.

Proof In this case by $t(xy) \geq t(x)t(y) = t_1^2 = t_1$, if xy is non-zero, then $t(xy) = t_1$ and $xy \in A_1 \oplus \langle y \rangle^*$. This yields $xy = l'x + ly$, for some $l', l \in \mathbb{Q}$. Similarly, $y^2 = l'_1 x + l_1 y, x^2 = l'_2 x + l_2 y, yx = l'_3 x + l_3 y$, for some $l_1, l_2, l_3, l'_1, l'_2, l'_3 \in \mathbb{Q}$, and by the proof of Theorem 3.3, $zx = xz = zy = yz = z^2 = 0$. Therefore for any ring $R = (A, .)$ the usual type-based calculations show that:

$$xy = l'x + ly, y^2 = l'_1 x + l_1 y, x^2 = l'_2 x + l_2 y, yx = l'_3 x + l_3 y, zx = xz = zy = yz = z^2 = 0,$$

for some $l, l_1, l_2, l_3, l', l'_1, l'_2, l'_3 \in \mathbb{Q}$. Thus A is nil modulo $A_1 \oplus \langle y \rangle^*$, and so $\Box A \subseteq A_1 \oplus \langle y \rangle^*$.

Moreover, A_1 is a rank one group with idempotent type, so by Theorem 2.2, there is a ring $S = (A_1, *)$ such that $0 \neq x * x = rx \in A_1$, for some $r \in \mathbb{Q}$. Therefore the multiplication

$$x^2 = rx, y^2 = z^2 = yz = zy = xy = yx = xz = zx = 0,$$

gives a ring structure $(A, .)$ on A such that by Lemma 3.1, $A^2 = U_0 x = \langle x \rangle^* = A_1$. Similarly, the multiplication

$$y^2 = l_1 y, z^2 = x^2 = yz = zy = xy = yx = zx = xz = 0$$

yields a ring on A such that $A^2 = \langle y \rangle^*$. Hence $\langle y \rangle^*, A_1 \subseteq \Box A$ and this completes the proof. □

Theorem 3.9 *Let* $A = A_1 \oplus A_2$ *be a rank three torsion-free group such that* $T(A_2) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\}, t_0 < t_i$ *for* $i = 1, 2$ *and* t_1 *and* t_2 *are maximal in* $T(A_2)$. *If* $t(A_1) = t_2$, *then* $\Box A = \langle y \rangle^*$, *where* $0 \neq y \in A_2$ *and* $t(y) = t_1$.

Proof The proof is essentially identical to that of Theorem 3.3. In fact, using the types, for any ring $R = (A, .)$ we again have $y^2 = ry, x^2 = z^2 = yz = zy = xy = yx = xz = zx = 0$, for some $r \in \mathbb{Q}$. □

Our next result shows, *inter alia*, that the square subgroup is not completely determined by abstract properties of types. The outcome may depend on arithmetical properties of the individual types but we have avoided this kind of calculation by referring to ring structures which may arise from such properties.

Theorem 3.10 *Let* $A = A_1 \oplus A_2$ *be a rank three torsion-free group such that* $T(A_2) = \{t_0, t_1, t_2 \mid t_1^2 = t_1, t_2^2 \neq t_2\}, t_0 < t_i$ *for* $i = 1, 2$, *and* t_1 *and* t_2 *are maximal in* $T(A_2)$. *If* $t(A_1) = t_0$, *non-idempotent, then* $\square A = \langle y\rangle^*$ *or* $\square A = \langle y\rangle^* \oplus \langle z\rangle^*$ *where* $0 \neq y, z \in A_2$ *with* $t(y) = t_1$ *and* $t(z) = t_2$.

Proof In this case because of $t(A_1) = t_0$, we have $T(A) = \{t_0, t_1, t_2\} = T(A_2)$ and for any ring $R = (A, .)$ we must have $yz = zy = z^2 = 0, y^2 = ry$, for some $r \in \mathbb{Q}$. Moreover, since $t(x^2) \geq t^2(x) = t_0^2 \geq t_0$ and $t_0^2 \neq t_0$, we have that if $x^2 \neq 0$, then $t(x^2) = t_1$ or t_2 which means $x^2 = ly$ or $x^2 = l_1 z$, for some $l, l_1 \in \mathbb{Q}$. Similarly, $t(xy) \geq t(x)t(y) = t_0 t_1 \geq t_1$ and $t(xz) \geq t_2$; so that $xy = sy$ and $xz = s_1 z$ for some $s, s_1 \in \mathbb{Q}$. Therefore, $\square A \subseteq \langle y\rangle^* \oplus \langle z\rangle^*$.

On the other hand as $t^2(y) = t(y)$, one can define a ring on A by setting the other products to zero. Hence we have $\langle y\rangle^* \subseteq \square A$. Now if $xz = s_1 z \neq 0$, then considering the ring R on A which is obtained from the multiplications

$$yz = zy = z^2 = y^2 = xy = yx = zx = 0, xz = s_1 z;$$

for any two non-zero elements $a = u_1 x + v_1 y + w_1 z, a' = u_2 x + v_2 y + w_2 z$ of $A, aa' = s_1 u_1 w_2 z \in A$. So $s_1 UW \leq W_0$ and $s_1 W \leq W_0 \leq W$. Therefore, there exists a ring on A with $A^2 = W_0 z$ and hence $\langle z\rangle^* \subseteq \square A$. Consequently, $\langle y\rangle^* \oplus \langle z\rangle^* \subseteq \square A$ and in this case we get the equality, completing a part of the proof.

But if for any ring on $A, xz = zx = 0$ and $x^2 = l_1 z \neq 0$, for some $l_1 \in \mathbb{Q}$, then considering the ring which is obtained from the multiplications

$$xy = yx = xz = zx = y^2 = z^2 = yz = zy = 0, x^2 = l_1 z,$$

we see that for any two non-zero elements $a = u_1 x + v_1 y + w_1 z, a' = u_2 x + v_2 y + w_2 z$ of $A, aa' = l_1 u_1 u_2 z \in A$. This implies $l_1 U^2 \subseteq W_0, t(U^2) \leq t(W_0)$ and so there exists a least positive integer m such that

$$mU^2 \leq W_0, \quad mU^2 \leq U^2 \cap W_0 \leq U^2. \tag{1}$$

Now Proposition 2.4(b) implies that

$$mU^2 = U^2 \cap W_0, \qquad \frac{1}{m} \in U^2. \tag{2}$$

Let $\chi_{W_0}(1) = (k_1, k_2, \cdots, k_i, \cdots)$ and $\chi_U(1) = (m_1, m_2, \cdots, m_i, \cdots)$ be the height sequences of 1 in W_0 and U, respectively, then

$$\chi_{U^2}(1) = (2m_1, 2m_2, \cdots, 2m_i, \cdots).$$

We prove $1/(p_i)^{\gamma_i} z \in \square A$ for all γ_i such that, $0 \le \gamma_i \le k_i \quad (i = 1, 2, 3, \cdots)$. To do this, we consider two cases for each fixed i, $k_i \le 2m_i$ or $2m_i < k_i$. Firstly suppose that $k_i \le 2m_i$, then we define a multiplication over A as follows:

$$x^2 = mz,\ y^2 = yz = zy = z^2 = xz = zx = xy = yx = 0.$$

Let $g = ux + vy + wz$ and $g' = u'x + v'y + w'z$ be arbitrary elements of A, so $gg' = muu'z$. By (1) $muu' \in W_0$, so the product actually lies in A, which yields a ring structure on A. Since $k_i \le 2m_i$, so $1/(p_i)^{\gamma_i} \in U^2 \cap W_0$ and in view of (2), $1/(p_i)^{\gamma_i} \in mU^2$. Consequently $1/(p_i)^{\gamma_i} = mu_1u_2$ for some $u_1, u_2 \in U$. On the other hand, there exist $w_1, w_2 \in W$ such that $a = u_1x + v_1y + w_1z$ and $a' = u_2x + v_2y + w_2z$ belong to A, so $aa' = u_1u_2x^2 = mu_1u_2z = (1/(p_i)^{\gamma_i})z$.

This means that in this case we obtained a ring on A such that for any $\gamma_i \le k_i$, $(1/(p_i)^{\gamma_i})z = aa'$, for some $a, a' \in A$. But by $\chi_{W_0}(1) = (k_1, k_2, \cdots, k_i, \cdots)$, we have $1/(p_i)^{\gamma_i} \in W_0$, for every $\gamma_i \le k_i$. Therefore from

$$\langle \frac{1}{p_i^{\gamma_i}} z \mid 0 \le \gamma_i \le k_i \rangle = W_0 z = \langle z \rangle^*,$$

we deduce that $\langle z \rangle^* \subseteq A^2$. But every element of A^2 belongs to $\square A$, and so $\langle z \rangle^* \subseteq \square A$. Since we have already established that $< y >^* \subseteq \square A$, we again get that

$$\square A = < y >^* \oplus < z >^* .$$

In the other case, i.e., $2m_i < k_i$, by (a) in Proposition 2.4, p_i does not divide m. By (2), $1/m \in U^2$ so that $\frac{1}{m} = \frac{1}{m'm''}$ where $1/m'$, $1/m'' \in U$. If $m_i = \infty$, then $k_i = \infty$ and so $2m_i = k_i$, which is in contradiction with $2m_i < k_i$, thus $m_i < \infty$. Now since $1/(p_i)^{m_i} \in U$ and p_i does not divide m, so $(p_i, m') = (p_i, m'') = 1$, hence by Lemma 2.3

$$1/(p_i^{m_i} m'),\ 1/(p_i^{m_i} m'') \in U. \tag{3}$$

Define another multiplication over A by:

$$x^2 = m/(p_i)^{\gamma_i - 2m_i} z,\quad yx = xy = xz = zx = y^2 = z^2 = yz = zy = 0.$$

Since $2m_i < k_i$, it follows from (1) that $mU^2 \le W_0$ and thus we deduce from Proposition 2.4(*a*) that $(mU^2)(1/(p_i)^{\gamma_i - 2m_i}) \le W_0$. Hence the product x^2 lies in A and the multiplication above yields a ring structure on A. By (3), there exist $w_1, w_2 \in W$ such that

$$a_1 = \frac{1}{(p_i)^{m_i} m'} x + v_1 y + w_1 z \in A, \quad a_2 = \frac{1}{(p_i)^{m_i} m''} x + v_2 y + w_2 z \in A$$

and since $m'm'' = m$, so $a_1a_2 = m/(p_i^{\gamma_i}m'm'')z = (1/p_i^{\gamma_i})z$. Consequently in this case, similar to the previous case, $\langle z\rangle^* \subseteq \Box A$ and again we have the equality $\Box A =< y >^* \oplus < z >^*$.

But if $x^2 = ly$ and $xz = zx = 0$, then for any ring on A, we have

$$x^2 = ly, xz = zx = zy = yz = z^2 = 0, xy = sy, yx = s_1y, y^2 = ry,$$

for some non-zero $l, s, s_1, r \in \mathbb{Q}$. Now if $a_1 = u_1x + v_1y + w_1z$ and $a_2 = u_2x + v_2y + w_2z$ are two arbitrary elements of A,

$$a_1a_2 = u_1u_2x^2 + u_1v_2xy + v_1u_2yx + v_1v_2y^2 = u_1u_2ly + u_1v_2sy + v_1u_2s_1y + v_1v_2ry,$$

which is an element of $\langle y\rangle^*$. This means $A^2 \subseteq \langle y\rangle^*$ and so $\Box A \subseteq \langle y\rangle^*$ in this case.

As already observed the reverse inequality holds, so in this situation we get $\langle y\rangle^* = \Box A$ which completes the proof. □

Remark 3.11 From the proof of Theorem 3.10 we see that if there are two elements of our group A such that their product is equal to a rational multiple of g (one or both of these two elements could be g itself), then $\langle g\rangle^* \subseteq \Box A$. This is an important point for proving the next theorem.

Using some arguments similar to those used in the proofs of the previous theorems and noting Remark 3.11, we have the following result:

Theorem 3.12 *Let $A = A_1 \oplus A_2$ be a rank three torsion-free group with A_2 indecomposable of rank two, $T(A_2) = \{t_1, t_2 \mid t_1^2 \neq t_1, t_2^2 \neq t_2\}$ and $t_1 < t_2$. If $t(A_1)$ is either incomparable to the elements of $T(A_2)$, or $t(A_1) \in T(A_2)$, then $\Box A = A_1 \oplus \langle z\rangle^*$ or $\Box A = \langle z\rangle^*$, where $0 \neq z \in A_2$ and $t(z) = t_2$.*

Proof We consider following cases for $t(A_1) = t$:

(I) The type t is non-idempotent and incomparable with t_1, t_2
In this case using an argument similar to that used in the proof of Theorem 3.3, for any ring $R = (A, .)$ we have:

$$y^2 = sz, x^2 = z^2 = yz = zy = xz = zx = yx = xy = 0, \quad (s \in \mathbb{Q})$$

which yields $\Box A \subseteq \langle z\rangle^*$. Moreover, as in the proof of Theorem 3.10, from $y^2 = sz$, for some non-zero $s \in \mathbb{Q}$, we can show that $\langle z\rangle^* \subseteq \Box A$. Thus we have $\Box A = \langle z\rangle^*$.

(II) The type t is idempotent and incomparable with t_1 and t_2.
In this case $T(A) = \{t_1, t_2, t, t_1 \cap t, t_2 \cap t\}$ which means that the only maximal elements of $T(A)$ are t_2 and t.

Now, for example, $t(xy) \geq t(x)t(y) = tt_1 \geq t, t_1$, and $tt_1 \neq t, t_1$, the last inequality holding because t and t_1 are incomparable. So if $xy \neq 0$, then $t(xy)$ is a maximal element of $T(A)$ which is not equal to t. This means $t(xy)$ would be equal to t_2, but this is impossible by the incomparability. Therefore for any ring on A, $xy = 0$. Similarly $zy = yz = 0$ since, for example, $t(yz) \geq t_1 t_2 \geq t_1, t_2$, and $t_1 t_2 \neq t_1, t_2$ as t_1 and t_2 are both non-idempotent, infinitely many of their components are finite.

But $t(y^2) \geq t_1^2 \geq t_1$ and $t_1^2 \neq t_1$. So if $y^2 \neq 0$, then we would have $t(y^2) = t_2$. Hence $y^2 = sz$, for some non-zero $s \in \mathbb{Q}$. Note that if $y^2 \neq 0$, then $t(y^2) \neq t$, because t and t_1 are incomparable.

In the same way, $t(x^2) \geq t^2 = t$ and if $x^2 \neq 0$, then $t(x^2) = t$. This gives $x^2 \in \langle x \rangle^*$ and hence $x^2 = rx$, for some non-zero $r \in \mathbb{Q}$. This means that:

$$x^2 = rx, y^2 = sz, yz = zy = z^2 = yx = xy = xz = zx = 0, \quad (r, s \in \mathbb{Q})$$

for any ring on A. Therefore $A^2 \subseteq A_1 \oplus \langle z \rangle^*$, i.e., $\square A \subseteq A_1 \oplus \langle z \rangle^*$.

Now as mentioned in Remark 3.11, from $y^2 = sz$, for some non-zero $s \in \mathbb{Q}$, we can show that $\langle z \rangle^* \subseteq \square A$. Moreover, from $t^2 = t$, we can find a ring $(A, .)$ such that $A.A = \langle x \rangle^* = A_1$ and so $A_1 \subseteq \square A$. Gathering all this together we obtain $\square A = A_1 \oplus \langle z \rangle^*$.

(III) The type $t = t_1$.

Then for any ring $R = (A, .)$:

$$y^2 = s_1 z, x^2 = s_2 z, xy = s_3 z, yx = s_4 z, yz = zy = z^2 = xz = zx = 0,$$

where $(s_1, s_2, s_3, s_4 \in \mathbb{Q})$ and, exactly as before, this implies $\square A = \langle z \rangle^*$.

(IV) The type $t = t_2$.

In this case $T(A) = \{t_1, t_2\} = T(A_2)$ and for any ring on A, $t(y^2) \geq t_1.t_1 = t_1^2 \geq t_1$ and $t_1^2 \neq t_1$, which means that if $y^2 \neq 0$, then $t(y^2) = t_2$ and so $y^2 \in \langle x, z \rangle^*$. Hence $y^2 = rx + sz$, for some $r, s \in \mathbb{Q}$.

But $t(xy) \geq t_2 t_1 \geq t_1, t_2$ and $t_1 t_2 \neq t_1, t_2$, because t_2 and t_1 are non-idempotent and infinitely many of their components are finite numbers. This yields $xy = 0$, because otherwise $t(xy)$ would then be a maximal element of $T(A)$ which is not equal to t_2. In the same way $yx = xz = zx = yz = zy = z^2 = 0$, for any ring on A. So, as we have observed previously, if R is a ring on A, then

$$y^2 = rx + sz, x^2 = xy = yx = xz = zx = yz = zy = z^2 = 0,$$

where $(r, s \in \mathbb{Q})$. As before we deduce that $\square A \subseteq A_1 \oplus \langle z \rangle^*$.

Now note that since $rx \in A_1, sz \in \langle z \rangle^*$ we could define two rings $R_1 = (A, *_1)$ and $R_2 = (A, *_2)$ via the relations

$$y *_1 y = rx, x *_1 x = x *_1 y = y *_1 x = x *_1 z = z *_1 x = y *_1 z = z *_1 y = z *_1 z = 0,$$

and

$$y *_2 y = sz, x *_2 x = x *_2 y = y *_2 x = x *_2 z = z *_2 x = y *_2 z = z *_2 y = z *_2 z = 0,$$

in which $A *_1 A = \langle x \rangle^*$ and $A *_2 A = \langle z \rangle^*$, so that $A_1 \subseteq \square A$ and $\langle z \rangle^* \subseteq \square A$. This finally proves $\square A = A_1 \oplus \langle z \rangle^*$.

$\square$

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An Invariant on Primary Abelian Groups with Applications to Their Projective Dimensions

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Abstract Nunke's Problem asks when the torsion product of two abelian p-groups is a dsc group of length $\lambda \leq \omega_1$. For countable values of λ this was completely solved by the author in previous work, where a new invariant was defined using transfinite induction on filtrations of subgroups. Though the natural extension of these ideas to the case of groups of length ω_1 does not tell us when the torsion product is a dsc group, it is shown to be sufficient to describe when the product is a C_{ω_1}-group of p^{ω_1}-projective dimension at most one.

Keywords Abelian p-group • Projective dimension • Torsion product • Invariants • Filtrations • Nunke's problem • dsc-group • C_{ω_1}-group

1 Introduction

In this work all group considered will be abelian p-groups for some fixed prime p. The terminology and notation will generally follow [3], and when α is an ordinal we will assume some familiarity with the theory of p^α-purity which can be found, for example, in [5]. Finally, we will be using a modicum of pretty basic set-theoretic techniques which can be found, for example, in [2].

The group G is a *dsc* if it is a direct sum of countable groups. If $\lambda \leq \omega_1$, then a group G is a C_λ*-group* iff for every $\alpha < \lambda$, any p^α*-high* subgroup H of G (that is, one maximal with respect to $H \cap p^\alpha G = 0$) is a dsc group. If λ is a limit ordinal, this is equivalent to requiring that $G/p^\alpha G$ is p^α-projective for all $\alpha < \lambda$.

We will denote the torsion product of the groups A and B by the convenient, albeit non-standard, notation $A \triangledown B$. A classical problem of Nunke asks when the torsion product $A \triangledown B$ is a direct sum of cyclics, or more generally, a dsc group of length $\lambda \leq \omega_1$. In particular, it was shown that for this to occur, A and B must be C_λ-groups. For countable values of λ, the problem was, in a sense, solved in [10]. There an invariant was defined for every group G, written L^λ_G, consisting of a class of

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finite subsets of the regular cardinals. This reduced the question from one regarding the torsion product to one about computing the values of this invariant (see [10], Theorem 3.10). Unfortunately, even for some very familiar groups, computing this invariant almost immediately leads to undecidable statements from set theory.

Even earlier, in [8] it was shown that the case $\lambda = \omega_1$ is intimately connected to a set-theoretic statement known as *Kurepa's Hypothesis* (KH). There are several equivalent ways to express this statement. For example, it asserts that there is a tree of height ω_1 having at least ω_2 branches, but whose levels are all countable. Equivalently, KH asserts the existence of a family $\mathscr{F}$ of subsets of ω_1 such that $|\mathscr{F}| \geq \omega_2$, but for every countable $\lambda < \omega_1$, $\{X \cap \lambda : X \in \mathscr{F}\}$ is countable. KH is known to be true in the constructible universe, but to be undecidable over ZFC (in fact, KH is a consequence of $\diamondsuit^+$, which is true in V=L). In ([8], Theorem 13) it was established that the denial of KH is equivalent to the statement that $A \triangledown B$ is a dsc group for all p^{ω_1}-bounded C_{ω_1}-groups A and B. This is only one of a longer list of algebraic statements that were shown to be equivalent to $\neg$KH.

The purpose of this note is to extend the methods of [10], which depended strongly upon the countability of the lengths of the groups in question, to the case of C_{ω_1}-groups, and to use this extension to add to the list of algebraic statements that are equivalent to $\neg$KH.

We begin by reviewing some well-known properties of p^λ-purity. In the next two results $E : 0 \to A \to B \to G \to 0$ will denote a short exact sequence.

Lemma 1.1 ([9], Lemma 1) *If λ is an ordinal and H is a totally projective group of length λ, then the short exact sequence E is p^λ-pure iff the corresponding sequence $E \triangledown H : 0 \to A \triangledown H \to B \triangledown H \to G \triangledown H \to 0$ is splitting exact.*

The proof of this in [9] assumed that the sequence E is pure-exact. However, this assumption is unnecessary. Essentially, this is due to the fact that for $n < \omega$, E is p^n-pure iff $E[p^n] : 0 \to A[p^n] \to B[p^n] \to G[p^n] \to 0$ is splitting exact. (In fact, the finite case will not be needed in this work).

Lemma 1.2 ([5], Theorem 93) *If $\lambda \leq \omega_1$ is a limit ordinal and G is a C_λ-group, then E is p^λ-pure iff it is λ-balanced (that is, for all $\alpha < \lambda$, $p^\alpha E : 0 \to p^\alpha A \to p^\alpha B \to p^\alpha G \to 0$ is exact).*

In several other papers we have used the language of balanced-projective dimension (or b.p.d. for short; see, for example, [8]), but here we will use the language of p^λ-projective dimension (or p^λ-p.d. for short). By Lemma 1.2, however, these ideas are equivalent in the context of C_{ω_1}-groups, which is the focus of this work.

The following was the first important result on Nunke's Problem, which has been rephrased using our terminology:

Lemma 1.3 ([12], Theorem 6) *Suppose A and B are reduced groups, B has length $\lambda \leq \omega_1$ and $p^\lambda A \neq 0$. Then $A \triangledown B$ is a dsc group iff A is a C_λ-group and B is a dsc group.*

We quote another simple result that is pertinent to our discussions.

Lemma 1.4 ([6], Corollary 9) *Suppose A and B are C_{ω_1}-groups of cardinality at most ω_1. If $p^{\omega_1}A = p^{\omega_1}B = 0$, then $A \bigtriangledown B$ is a dsc group.*

Whenever λ is an infinite ordinal there is a p^λ-pure short exact sequence $0 \to M_\lambda \to H_\lambda \to Z_{p^\infty} \to 0$, where H_λ is the "generalized Prüfer group" of length λ. The group M_λ is called a λ-*elementary S-group*. It is well known that M_λ is totally projective exactly when $\lambda = \alpha + n$, where α is a limit ordinal of countable cofinality and $n < \omega$.

Lemma 1.5 *Suppose G is a group and H is a dsc group of length $\lambda \leq \omega_1$.*

(a) [9], Theorem 2) *$G \bigtriangledown H$ is a dsc group iff G is a C_λ-group.*
(b) [6], Theorem 19) *$G \bigtriangledown M_{\omega_1}$ is a dsc group iff G is a C_{ω_1}-group with p^{ω_1}-p.d. at most one.*

In [6] the class of C_{ω_1}-groups with p^{ω_1}-p.d. at most one was denoted by $\mathscr{C}_1$ and in [9] it was denoted by $\mathscr{F}$; we will use the latter notation. This class will be central in our application of the invariant L_G^λ to the case where $\lambda = \omega_1$ is uncountable.

Observe that if G is a C_{ω_1}-group with $p^{\omega_1}G \neq 0$, then since M_{ω_1} is not a dsc group, it follows from Lemma 1.3 that $G \bigtriangledown M_{\omega_1}$ fails to be a dsc group. Therefore, by Lemma 1.5(b), any group in $\mathscr{F}$ must be p^{ω_1}-bounded.

We mention a few additional well-known properties of this class. Recall that a cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$; otherwise, it is *singular*.

Lemma 1.6 *Suppose A, B, and G are C_{ω_1}-groups.*

(a) *If $|G| \leq \omega_1$ and $p^{\omega_1}G = 0$, then $G \in \mathscr{F}$.*
(b) [6], Theorem 21) *If $G \in \mathscr{F}$ and A is an isotype subgroup of G, then $A \in \mathscr{F}$.*
(c) [6], Theorem 23) *If $A, B \in \mathscr{F}$, then $A \bigtriangledown B$ is a dsc group.*
(d) *If $|G| = \kappa$ is regular and $G \in \mathscr{F}$, then G has a filtration $\{A_i\}_{i \leq \kappa}$ consisting of p^{ω_1}-pure subgroups such that for all $i < \kappa$, $G/A_i \in \mathscr{F}$.*
(e) [9], Theorem 15) *If $|G|$ is singular and $A \in \mathscr{F}$ for every isotype subgroup A of G such that $|A| < |G|$, then $G \in \mathscr{F}$.*

First, (a) follows immediately from Lemmas 1.4 and 1.5(b). Next, in (d), use Lemma 1.5 and fix decompositions of $G \bigtriangledown M_{\omega_1}$ and $G \bigtriangledown H_{\omega_1}$ into countable summands, and then choose the filtration so that for each i, $A_i \bigtriangledown M_{\omega_1}$ and $A_i \bigtriangledown H_{\omega_1}$ are direct sums of subcollections of the terms in these decompositions. The result then follows from Lemmas 1.1 and 1.5. Finally, (d) is a variation on Shelah's Singular Compactness Theorem.

We now provide some additional motivation for our interest in the class $\mathscr{F}$. If G is a group, then a subgroup $B \subseteq G$ will be said to be ω_1-*basic* if

(a) $B \in \mathscr{F}$;
(b) B is p^{ω_1}-pure in G; and
(c) G/B is divisible.

By way of comparison, if $\lambda < \omega_1$ is a countably infinite ordinal, then any C_λ-group G has a λ-*basic subgroup* B; that is, B is a p^λ-bounded dsc group that is

p^{λ}-pure in G and G/B is divisible (see, for example, [13]). Comparing this with the above, it might be supposed that condition (a) should require that B actually be a dsc group. However, it is well known that a dsc group is complete in the ω_1-topology (which uses $\{p^{\alpha}G\}_{\alpha<\omega_1}$ as a neighborhood base of $0 \in G$); so if G is a reduced C_{ω_1}-group that has an ω_1-basic subgroup B that is actually a dsc, then $G = B$ will be a dsc group. In other words, only dsc groups would have ω_1-basic subgroups.

The reason for our interest in this terminology is the following observation.

Proposition 1.7 *A group is a C_{ω_1}-group iff it has an ω_1-basic subgroup.*

Proof Suppose G is a C_{ω_1}-group. By an obvious induction on $\alpha < \omega_1$ we can construct a smoothly ascending chain of subgroups $\{H_{\alpha}\}_{\alpha<\omega_1}$ of G such that for all α, $H_{\alpha+1}$ is p^{α}-high in G. In particular, each $H_{\alpha+1}$ is a dsc group; and since H_{α} is isotype in $H_{\alpha+1}$ and of countable length, it is also a dsc group. If $B = \cup_{\alpha<\omega_1} H_{\alpha}$, then B is isotype in G, so it is a C_{ω_1}-group, as well. Clearly, each H_{α} has b.p.d. equal to 0. So by [4], Theorem 5.2), B has b.p.d. at most 1; that is, $B \in \mathscr{F}$.

Whenever $\alpha < \omega_1$ is infinite, G/H_{α} will be divisible; so $(G/H_{\alpha})/(B/H_{\alpha}) \cong G/B$ will also be divisible. In addition, for all $\alpha < \omega_1$, $G[p] = (p^{\alpha}G)[p] + H_{\alpha+1}[p] \subseteq (p^{\alpha}G)[p] + B[p]$, so by [5], Theorem 91), B is p^{ω_1}-pure in G.

The converse is even easier and is left to the reader. □

For countably infinite values of λ, a λ-basic subgroup of a C_{λ}-group G can be constructed precisely as in Proposition 1.7 as the union of a chain of p^{α}-high subgroups for all $\alpha < \lambda$. Since λ is countable, the p^{λ}-p.d. of G will be at most 1 (this follows since M_{λ} will be a dsc group). And since B will be a dsc group, its p^{λ}-p.d. will be 0, that is, one less than the p^{λ}-p.d. of G. In a parallel fashion, if G is a C_{ω_1}-group and B is ω_1-basic in G, then the p^{ω_1}-p.d. of G will be at most 2 (this follows since $M_{\omega_1} \bigtriangledown M_{\omega_1}$ will be a dsc group by Lemma 1.4). And the p^{ω_1}-p.d. of B will be at most 1, that is, one less than the p^{ω_1}-p.d. of G.

Clearly, there are many ways in which ω_1-basic subgroups are much less well-behaved than λ-basic subgroups when λ is countable. For example, since a λ-basic subgroup is a dsc group, it can be fully described by cardinal invariants. On the other hand, it follows from Lemma 1.6(b) that every isotype subgroup of a reduced dsc group is in $\mathscr{F}$. And unfortunately, this class of isotype subgroups is known to be fully as complicated as the entire class of all abelian p-groups.

So for the ordinal ω_1, in some respects $\mathscr{F}$ is the appropriate analogue of the class of p^{λ}-bounded dsc groups for countable ordinals $\lambda < \omega_1$. And again, if $\lambda < \omega_1$ and G is a C_{λ}-group, then L_G^{λ} can be used to decide if G is a dsc. It is therefore logical to consider the following question, which is the main purpose of this paper:

If G is a C_{ω_1}-group, can we use the techniques of [10] to determine when $G \in \mathscr{F}$?

To this end, we will define a slightly amended version of the invariant $L_G^{\omega_1}$, which we call J_G. Let $\mathscr{Q}$ be the class of all regular cardinals $\kappa > \aleph_1$ (the definition of L_G^{λ} in [10] used $\mathscr{R} = \mathscr{Q} \cup \{\aleph_1\}$). Any successor cardinal $\kappa > \aleph_1$ is in $\mathscr{Q}$, and it is actually consistent with ZFC that all elements of $\mathscr{Q}$ are successor cardinals (that is, there are no regular limit cardinals). If $\kappa \in \mathscr{Q}$, then $C \subseteq \kappa$ is a *CUB* if it is closed and unbounded in the order topology; and $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all CUB subsets $C \subseteq \kappa$. A regular cardinal $\kappa \in \mathcal{Q}$ is said to be *weakly Mahlo* if $\kappa \cap \mathcal{Q}$ is stationary in κ; such cardinals played a very significant role in the definition of the invariant L_G^λ in [10].

Let $\mathcal{Q}_f$ be the class of finite subsets of $\mathcal{Q}$. Suppose $T \in \mathcal{Q}_f$; if $T = \emptyset$, let $\mu(T) = \aleph_1$, and if $T \neq \emptyset$, let $\mu(T)$ be its greatest element. Let $T' = T - \{\mu(T)\}$, and if $i < \mu(T)$, let $T_i = (T' \cup \{i\}) \cap \mathcal{Q}$ (that is, $T_i = T'$ if $i \notin \mathcal{Q}$ and $T_i = T' \cup \{i\}$ when $i \in \mathcal{Q}$). In particular, if $T \in \mathcal{Q}_f$ is non-empty and $i < \mu(T)$, then $\mu(T_i) < \mu(T)$. These are almost exactly the same definitions as in [10], the main difference being that in that earlier work, $\mu(\emptyset)$ was defined to be $\aleph_0$ instead of $\aleph_1$.

There are two particularly important (and extreme) classes contained in $\mathcal{Q}_f$; $0_{\mathcal{Q}} = \emptyset$ and $1_{\mathcal{Q}} = \mathcal{Q}_f$. For a group G, we continue the parallel with the definition of $L_G^{\omega_1}$ from [10], by defining a subclass $J_G \subseteq \mathcal{Q}_f$ by transfinite induction on $\kappa := \mu(T)$. To begin with the base case:

(J-0) If $\kappa = \aleph_1$ (that is, $T = \emptyset$), then $T \in J_G$ iff $p^{\omega_1} G \neq 0$.

Next, suppose $T \neq \emptyset$ and for all groups H we have defined all the elements $S \in J_H$ such that $\mu(S) < \kappa$; in particular, when $i < \kappa$ this holds for $S = T_i$. We then define T to be in J_G iff one of two things occurs:

(J-1) $\Upsilon_T^{\omega_1}(G) := \{i < \kappa : T_i \in J_G\}$ is stationary in κ; or
(J-2) G has a subgroup A of cardinality κ, with a filtration $\{A_i\}_{i<\kappa}$ such that

$$\Lambda_T^{\omega_1}(A) := \{\, i < \kappa : T_i \in J_{A/A_i} \,\}$$

is stationary in κ.

These conditions are nearly identical to the conditions which define $L_G^{\omega_1}$; the primary difference is that we start applying (J-1) and (J-2) at $\kappa = \aleph_2$ instead of $\kappa = \aleph_1$. We are only concerned with whether $\Upsilon_T^{\omega_1}(G)$ and $\Lambda_T^{\omega_1}(A)$ are or are not stationary. Since any two filtrations agree on a CUB subset $C \subseteq \kappa$, if this is true for one filtration, then it is true for them all. As a consequence, if κ is not weakly Mahlo, then (J-1) holds exactly when $T' \in J_G$, and (J-2) is equivalent to requiring that $\{\, i < \kappa : T' \in J_{A/A_i} \,\}$ be stationary in κ.

The following makes clear the relation between these two ways of defining our invariants.

Proposition 1.8 *If G is a group, then*

$$J_G = L_G^{\omega_1} \cap \mathcal{Q}_f = \{T \in L_G^{\omega_1} : \aleph_1 \notin T\}.$$

Proof Let T be in $\mathcal{Q}_f$. We prove via induction on $\kappa := \mu(T)$ that $T \in J_G$ iff $T \in L_G^{\omega_1}$.

First, if $\kappa = \aleph_1$, then $T = \emptyset$, and either condition is equivalent to $p^{\omega_1} G \neq 0$.

Next, if $\kappa > \aleph_1$, then T is in either collection iff one of our other two conditions is satisfied. Observe that $\kappa^- := (\aleph_1, \kappa)$ is a CUB in κ. In addition, if $i \in \kappa^-$, then

$$T_i = (T' \cup \{i\}) \cap \mathscr{R} = (T' \cup \{i\}) \cap \mathscr{Q}.$$

So by induction, T satisfies one of the last two defining conditions for $L_G^{\omega_1}$ iff it satisfies the corresponding condition for J_G. This completes the induction. □

In fact, an alternate approach to our discussions would be to simply define $J_G = L_G^{\omega_1} \cap \mathscr{Q}$.

We are aiming to generalize the following result:

Theorem 1.9 ([10], Theorem 1.6) *If $\lambda < \omega_1$ is a countable ordinal, then a C_λ-group G is a dsc group iff $L_G^\lambda = 0_{\mathscr{R}}$.*

The original statement of this result assumed $\lambda = \omega_0$, but as was noted at the end of the paper, the result and its proof immediately generalize to any countable ordinal. In order to translate that argument to our current framework, we begin with a summary of some properties of J_G that parallel properties for L_G^λ when λ is countable:

Lemma 1.10 (cf. [11], Lemma 2.1) *Suppose G and H are groups.*

(a) If $T \in J_G$, $S \in \mathscr{Q}_f$ and $T \subseteq S$, then $S \in J_G$.
(b) $J_G = 1_{\mathscr{Q}}$ precisely when $p^{\omega_1}G \neq 0$.
(c) If G is a subgroup of H, then $J_G \subseteq J_H$.
(d) $J_{G\oplus H} = J_G \cup J_H$.
(e) If $T \in J_G$ iff there is a subgroup $A \subseteq G$ such that $|A| \leq \mu(T)$ and $T \in J_A$.
(f) If G is reduced and T is an element of J_G that is minimal under inclusion, then there is a subgroup $A \subseteq G$ such that $|A| = \mu(T)$ and $T \in J_A$.

To justify these statements, note that (a)–(d) follow from Proposition 1.8 and [11], Lemma 2.1(a,b,c,e)) since they all hold for $L_G^{\omega_1}$.

For (e) and (f), suppose first that $p^{\omega_1}G \neq 0$; then there is a subgroup $A \subseteq G$ of cardinality at most $\aleph_1$ such that $p^{\omega_1}A \neq 0$. So $J_A = 1_{\mathscr{Q}}$ and (e) follows immediately. Turning to (f), $p^{\omega_1}G \neq 0$, together with the minimality of T, implies that $T = \emptyset$. And since G is reduced, $|A| = \aleph_1 = \mu(T)$ and (f) follows.

Finally, if G is p^{ω_1}-bounded, then (e) and (f) follow from Proposition 1.8 and [11], Lemma 2.1(f,g)) since they both hold for $L_G^{\omega_1}$.

It should be emphasized that the obvious generalization of Theorem 1.9 to $\lambda = \omega_1$ does not hold; in other words, it is not true that a C_{ω_1}-group G is a dsc group iff $L_G^{\omega_1} = 0_{\mathscr{R}}$. For example, if X is a reduced countable group of length $\lambda < \omega_1$, then [11], Proposition 2.2) implies that $L_X^{\omega_1} \subseteq L_X^\lambda = 0_{\mathscr{R}}$, so that $L_X^{\omega_1} = 0_{\mathscr{R}}$. So if $H = \bigoplus_{i\in I} X_i$ is any reduced dsc group, then by [11], Lemma 2.6), we will again have $L_H^{\omega_1} = 0_{\mathscr{R}}$—so one of these implications does, in fact, hold. On the other hand, if G is any subgroup of such a reduced dsc group H, then [11], Lemma 2.1(c)) together with the above implies that $L_G^{\omega_1} = 0_{\mathscr{R}}$. However there are many such subgroups which are C_{ω_1}-groups but not dsc groups (for example, the elementary S-group M_{ω_1}). On the other hand, as we will see, if we replace the condition of being a dsc group by the condition of being in $\mathscr{F}$, we do obtain a valid statement.

Before we state and prove this result we mention a key step in the argument, which is a variation on Fodor's Lemma. Suppose $\kappa \in \mathscr{Q}$ and $W \subseteq \kappa$ is a stationary subset. A function $f : W \to \mathscr{Q}_f$ such that $f(i) \subseteq i$ for all $i \in W$ will be called *regressive.*

Lemma 1.11 ([10], Lemma 1.5) *Suppose $\kappa \in \mathscr{Q}$ and $W \subseteq \kappa$ is a stationary subset. If $f : W \to \mathscr{Q}_f$ is a regressive function, then there is a stationary subset $W' \subseteq W$ such that $f(i) = f(j)$ for all $i, j \in W'$.*

Theorem 1.12 *If G is a C_{ω_1}-group, then the following are equivalent:*

(a) $G \in \mathscr{F}$;
(b) $J_G = 0_{\mathscr{Q}}$;
(c) $\aleph_1 \in T$ for all $T \in L_G^{\omega_1}$.

Proof The equivalence of (b) and (c) follows immediately from Proposition 1.8.

We now verify (a) implies (b). For $T \in \mathscr{Q}_f$, we show $T \notin J_G$ for all $G \in \mathscr{F}$ by induction on $\kappa := \mu(T)$.

Suppose first that $\kappa = \aleph_1$; that is, $T = \emptyset$. If $G \in \mathscr{F}$, then we know that $p^{\omega_1}G = 0$, and this immediately implies that $\emptyset \notin J_G$. So we may assume $\kappa > \aleph_1$.

If $G \in \mathscr{F}$, then for all $i < \kappa$, $\mu(T_i) < \kappa = \mu(T)$, so by induction, T_i will not be in J_G. Therefore, $\Upsilon_T^{\omega_1}(G)$ will be empty, so that (J-1) does not hold. Next, consider a subgroup $A \subseteq G$ of cardinality κ, as in (J-2). Expanding A a bit (without changing its cardinality), we may assume that A is isotype in G. By Lemma 1.6(b), $A \in \mathscr{F}$. By Lemma 1.6(d), we can construct a p^{ω_1}-pure filtration $\{A_i\}_{i<\kappa}$ of A such that each $A/A_i \in \mathscr{F}$. So again by induction, for all $i < \kappa$, $T_i \notin J_{A/A_i}$, that is, $\Lambda_T^{\omega_1}(A)$ will be empty. Hence (J-2) also fails, and $T \notin J_G$, as required.

Conversely, we show that if G is a C_{ω_1}-group with $J_G = 0_{\mathscr{R}}$, then $G \in \mathscr{F}$ by induction on $\gamma = |G|$. Note that since $\emptyset \notin J_G$, G will be p^{ω_1}-bounded.

First, if $\gamma = \aleph_1$, then $G \in \mathscr{F}$ follows directly from Lemma 1.6(a).

Suppose, then, that $\gamma > \aleph_1$. If A is any isotype subgroup of G with $|A| < \gamma$, then by Lemma 1.10(c), $J_A \subseteq J_G = 0_{\mathscr{Q}}$; so by induction, $A \in \mathscr{F}$. And if γ is singular, then $G \in \mathscr{F}$ follows from Lemma 1.6(e).

Assume, then, that γ is regular. Since G is a C_{ω_1}-group, we can conclude that $G \bigtriangledown H_{\omega_1}$ is a dsc group. It follows that we can find a filtration of G, $\{A_i\}_{i<\gamma}$, with the property that for each $i < \gamma$, $A_i \bigtriangledown H_{\omega_1}$ is a summand of $G \bigtriangledown H_{\omega_1}$. In other words, by Lemma 1.1 this is a filtration of G consisting of p^{ω_1}-pure subgroups. By adding 0 terms at the beginning and repeating terms as necessary, we may assume that $|A_i| \leq |i|$ for all $i < \gamma$; in particular, $A_0 = 0$. Again, by induction and Lemma 1.10(c), each $A_i \in \mathscr{F}$.

We claim that

$$\{i < \gamma : \forall (i < j < \gamma)\, A_j/A_i \in \mathscr{F}\}$$

contains a CUB. If we have verified this, then after possibly relabeling, we may assume this set is all of γ. By Lemma 1.5(b), we can conclude that for all i,

$(A_{i+1}/A_i) \triangledown M_{\omega_1}$ will be a dsc group. Therefore, each p^{ω_1}-pure sequence

$$0 \to A_i \triangledown M_{\omega_1} \to A_{i+1} \triangledown M_{\omega_1} \to (A_{i+1}/A_i) \triangledown M_{\omega_1} \to 0$$

must necessarily split. This implies that

$$G \triangledown M_{\omega_1} \cong \bigoplus_{i<\gamma} \big((A_{i+1}/A_i) \triangledown M_{\omega_1}\big)$$

will also be a dsc group, so that $G \in \mathscr{F}$ by Lemma 1.5(b).

We now show that the denial of this claim leads to a contradiction. So assume

$$\mathscr{S} := \{i < \gamma : \exists (i < j < \gamma)\, A_j/A_i \notin \mathscr{F}\}$$

is stationary in γ.

If $i \in \mathscr{S}$, then it determines an $i < j < \gamma$. Since $A_j \in \mathscr{F}$, $A_j \triangledown M_{\omega_1}$ will be a dsc group. This implies that there is a p^{ω_1}-pure subgroup $X \subseteq A_j$ containing A_i such that $|X| = |A_i| \leq |i|$ and $X \triangledown M_{\omega_1}$ is a summand of $A_j \triangledown M_{\omega_1}$. Therefore, $(A_j/X) \triangledown M_{\omega_1} \cong (A_j \triangledown M_{\omega_1})/(X \triangledown M_{\omega_1})$ is also a dsc group. Since A_j/A_i is not in $\mathscr{F}$, $(A_j/A_i) \triangledown M_{\omega_1}$ will not be a dsc group. And since

$$(A_j/A_i) \triangledown M_{\omega_1} \cong ((X/A_i) \triangledown M_{\omega_1}) \oplus ((A_j/X) \triangledown M_{\omega_1}),$$

we can conclude that $(X/A_i) \triangledown M_{\omega_1}$ also fails to be a dsc group.

By induction $J_{X/A_i} \neq 0_{\mathscr{R}}$, so let $S^i \in J_{X/A_i}$; and since $|X/A_i| \leq |i|$, by Lemma 1.10(f) we may assume that $\mu(S^i) \leq |i|$, as well. Therefore, the function $i \mapsto S^i - \{i\}$ will be regressive, so by Lemma 1.11, there is a stationary subset $\hat{\mathscr{S}} \subseteq \mathscr{S}$ such that $S^i - \{i\}$ is constant for all $i \in \hat{\mathscr{S}}$; call this common value $\hat{S}$. Let $T = \hat{S} \cup \{\gamma\}$. So if $i \in \hat{\mathscr{S}}$, then $S^i \subseteq (\hat{S} \cup \{i\}) \cap \mathscr{Q} = T_i$, which implies that $T_i \in J_{X/A_i} \subseteq J_{G/A_i}$. Therefore, $\hat{\mathscr{S}}$ will be contained in $\Lambda_T^{\omega_1}(G)$. Since $\hat{\mathscr{S}}$ is stationary, we can conclude that $T \in J_G$, so that $J_G \neq 0_{\mathscr{R}}$. This contradicts our hypotheses, completing the argument. □

In the following, Theorem 1.12 asserts the equivalence of (d) with (b), and hence with the other conditions from [8], Theorem 13).

Corollary 1.13 *The following are equivalent:*

(a) Kurepa's Hypothesis fails;
(b) Every p^{ω_1}-bounded C_{ω_1}-group has p^{ω_1}-p.d. at most 1;
(c) For all p^{ω_1}-bounded C_{ω_1}-groups G and H, $G \triangledown H$ is a dsc group;
(d) For any C_{ω_1}-group G, J_G is either $1_{\mathscr{Q}}$ or $0_{\mathscr{Q}}$.

Theorem 1.12 and its proof clearly parallel the proof of [10], Theorem 1.9). In that earlier paper, (p^{ω}-)pure filtrations are used to describe when G is Σ-cyclic, and in this case, p^{ω_1}-pure filtrations are used to describe when $G \triangledown M_{\omega_1}$ is a dsc group.

Again, the length λ version of Nunke's Problem asks when $G \bigtriangledown H$ is a dsc group of length λ. As mentioned above, this has been completely solved when $\lambda < \omega_1$ is countable; so only the case of $\lambda = \omega_1$ remains unresolved. As in [10] (and other places), if $\mathscr{C}$ and $\mathscr{D}$ are classes contained in $\mathscr{Q}_f$ or $\mathscr{R}_f$, we let

$$\mathscr{C} \cdot \mathscr{D} = \{X \cup Y : X \in \mathscr{C}, Y \in \mathscr{D}, X \cap Y = \emptyset\}.$$

We want to generalize the following:

Theorem 1.14 ([10], Theorem 3.10) *If G and H are groups and $\lambda < \omega_1$ is a countable ordinal, then $G \bigtriangledown H$ is a dsc group of length λ iff G and H are C_λ-groups of length at least λ and $L_G^\lambda \cdot L_H^\lambda = 0_{\mathscr{R}}$.*

However, as we have seen, in some respects the proper analogue of the statement "X is a dsc group of countable length λ" is actually "X is in $\mathscr{F}$" (that is, X is a C_{ω_1}-group of p^{ω_1}-p.d. at most 1). So, we ask:

For C_{ω_1}-groups G and H, can we describe when $G \bigtriangledown H$ is in $\mathscr{F}$?

We answer this affirmatively in the next result.

Theorem 1.15 *If G and H are C_{ω_1}-groups, then the following are equivalent:*

(a) $G \bigtriangledown H \in \mathscr{F}$;
(b) $J_{G \bigtriangledown H} = 0_{\mathscr{Q}}$;
(c) $J_G \cdot J_H = 0_{\mathscr{Q}}$;
(d) $\aleph_1 \in T$ *for all* $T \in L_G^{\omega_1} \cdot L_H^{\omega_1}$.

Proof The equivalence of (a) and (b) is Theorem 1.12. The equivalence of (c) and (d) is Proposition 1.8. Finally, the equivalence of (b) and (c) is established by observing that the same translation of the proof of the above Theorem 1.9 found in [10] which gave our proof of Theorem 1.12 can be applied to the proof of the above Theorem 1.14 found in [10] to establish this equivalence—for example, replacing pure filtrations of the group G when $\lambda = \omega_0$ with p^{ω_1}-pure filtrations of the C_{ω_1}-group G when $\lambda = \omega_1$ constructed using subgroups $A \subseteq G$ such that $A \bigtriangledown H_{\omega_1}$ is a direct sum of a subcollection of the terms in a fixed decomposition of $G \bigtriangledown H_{\omega_1}$ into countable groups. □

As was the case for the case where $\lambda < \omega_1$ is countable, the above result can be extended to the case of several groups.

Corollary 1.16 *If $G_1, \ldots, G_n$ are C_{ω_1}-groups, then $G_1 \bigtriangledown \cdots \bigtriangledown G_n$ is in $\mathscr{F}$ iff $J_{G_1} \cdots J_{G_n} = 0_{\mathscr{Q}}$.*

Again, as in [10], Corollary 3.7), this follows from observing that the arguments used there to establish Theorem 1.15 can actually be applied with any given number of terms.

It is worth emphasizing that given C_{ω_1}-groups G and H, $L_G^{\omega_1}$ and $L_H^{\omega_1}$ do *not* determine when their torsion product is a dsc group, except in the trivial case when this always happens. To see this, note that if KH fails, then by Corollary 1.13, $G \bigtriangledown H$ will always be a dsc group whenever G and H are p^{ω_1}-bounded C_{ω_1}-groups. On the other hand, if KH holds, then let G be some p^{ω_1}-bounded C_{ω_1}-group of p^{ω_1}-p.d. 2 (that is, $G \notin \mathscr{F}$). Note that $L^{\omega_1}_{H_{\omega_1}} = L^{\omega_1}_{M_{\omega_1}} = 0_{\mathscr{R}}$. However by Lemma 1.5, $G \triangledown H_{\omega_1}$ is a dsc group, but $G \triangledown M_{\omega_1}$ is not.

We now discuss some properties of the elements of J_G for a group G. We will see that there are very real restrictions on what sets can appear in these classes. This is in marked contract to the situation when $\lambda < \omega_1$ is countable. For example, in the constructible universe we have the following result, which was stated for $\lambda = \omega$, but in fact holds for any countable λ:

Theorem 1.17 ([1], Theorem 10) *Assuming the axiom of constructability (V=L), suppose $M \subseteq \mathscr{R}_f$ is an antichain (i.e., no two elements of M are comparable under inclusion). If $\cup M$ does not contain any weakly Mahlo cardinals, then there is a group G such that M is precisely the collection of elements of L^{ω}_G that are minimal under inclusion.*

What the last result says is that if we are in the constructible universe and we stay away from weakly Mahlo cardinals, then any conceivable such invariant can be realized for some group G.

To see how different the situation is for groups of length $\lambda = \omega_1$, we begin by reviewing some terminology from [8]. If B is a subgroup of the C_{ω_1}-group G, κ is a cardinal and $\aleph_1 = |B| \leq \kappa \leq |\overline{B}|$ (where the closure is taken in the ω_1-topology), then we say B is a *κ-Kurepa subgroup* of G. Let $\nu_G > \aleph_1$ be the smallest cardinal such that G does not have a ν_G-Kurepa subgroup. In other words, $\nu_G = \sup\{|\overline{B}|^+ :$ B is a subgroup of G of cardinality $\aleph_1\}$, where $|\overline{B}|^+$ is the successor cardinal to $|\overline{B}|$.

Our discussion is based on the next result, which has been recast in our current terminology.

Theorem 1.18 ([7], Theorem 8) *If G is a p^{ω_1}-bounded C_{ω_1}-group of cardinality $\kappa \in \mathscr{Q}$ with $\nu_G \leq |G|$, then G has a filtration consisting of closed subgroups.*

We use this to put some serious constraints on the possible elements of J_G. We begin by looking at the singletons. Let $\rho_G \geq \nu_G$ be the smallest *non-weakly Mahlo* cardinal such that G does not have a ρ_G-Kurepa subgroup. So, if ν_G is not weakly Mahlo, then $\rho_G = \nu_G$; and if it is weakly Mahlo, then $\rho_G = \nu_G^+$. Clearly, $\nu_G, \rho_G \leq (2^{\aleph_1})^+$. Let ν, $\rho \leq (2^{\aleph_1})^+$ be the supremum of the values of ν_G, ρ_G, respectively, over all p^{ω_1}-bounded C_{ω_1}-groups G. It is easy to check that by taking direct sums there is, in fact, a single such group G for which $\nu = \nu_G$ and $\rho = \rho_G$.

Lemma 1.19 *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group and $\kappa \in \mathscr{Q}$.*

(a) If $\kappa < \rho_G$, then $\{\kappa\} \in J_G$.
(b) If $\kappa \geq \nu_G$ and $\{\kappa\} \in J_G$, then κ is weakly Mahlo.

Proof Considering (a), suppose $\kappa < \rho_G$. First, if $\kappa < \nu_G$, then there is a κ-Kurepa subgroup B of G. There is clearly a subgroup A' of $\overline{B}$ containing B of cardinality κ. If we let A be any isotype subgroup of G containing A', also of cardinality κ, then it easily follows that $A^{\omega_1}_{\{\kappa\}}(A)$ contains all but an initial segment of κ. So by (J-2), we have $\{\kappa\} \in J_G$.

Next, suppose $\kappa = \nu_G$ is weakly Mahlo; so $\mathscr{S} = \kappa \cap \mathscr{Q}$ is stationary in κ. It follows from the last paragraph that if $i \in \mathscr{S}$, then $\{\kappa\}_i = \{i\} \in J_G$. In other words, $\mathscr{S} \subseteq \Upsilon^{\omega_1}_{\{\kappa\}}(G)$; so that by (J-1), we have $\{\kappa\} \in J_G$. This completes the proof of (a).

Now, suppose $\kappa \geq \nu_G$ is not weakly Mahlo; this means that there is a CUB subset $C \subseteq \kappa$ such that $C \cap \mathscr{Q} = \emptyset$. To verify (b) we need to show that $\{\kappa\}$ is not in J_G.

Observe that if $i \in C$, then since $i \notin \mathscr{Q}$, we have $\{\kappa\}_i = \emptyset$. Since G is p^{ω_1}-bounded, it follows that $\{\kappa\}_i = \emptyset \notin J_G$. Therefore, $\Upsilon^{\omega_1}_{\{\kappa\}}(G)$ is not stationary, showing that $\{\kappa\}$ does not satisfy (J-1).

Suppose now that (J-2) holds for some isotype subgroup $A \subseteq G$ of cardinality κ. By Theorem 1.18, A has a closed p^{ω_1}-pure filtration $\{A_i\}_{i<\kappa}$; so A/A_i is a p^{ω_1}-bounded C_{ω_1}-group for all $i < \kappa$. Now if $i \in C$, then $\{\kappa\}_i = \emptyset$ will not be a member of J_{A/A_i}. This implies that $\Lambda^{\omega_1}_{\{\kappa\}}(A)$ will not be stationary in κ, showing that $\{\kappa\}$ does not satisfy (J-2), either. □

Corollary 1.20 *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group and $\kappa \in \mathscr{Q}$ is not weakly Mahlo. Then $\{\kappa\} \in J_G$ iff $\kappa < \rho_G$.*

We now characterize completely the singletons that can appear in one of these invariants.

Theorem 1.21 *If $\kappa \in \mathscr{Q}$, then $\{\kappa\} \in J_G$ for some p^{ω_1}-bounded C_{ω_1}-group G iff $\kappa < \rho$.*

Proof First, assume $\kappa < \rho$. It was observed above that there is a p^{ω_1}-bounded C_{ω_1}-group G such that $\rho = \rho_G$. So this half follows immediately from Lemma 1.19(a).

We actually prove the contrapositive of the converse by induction on κ; in other words: if $\kappa \geq \rho$, then $\{\kappa\} \notin J_G$ for any p^{ω_1}-bounded C_{ω_1}-group G.

First, if κ is not weakly Mahlo, then since $\kappa \geq \rho \geq \rho_G$, the result follows from Corollary 1.20. So we may assume that κ is weakly Mahlo. And since ρ is not weakly Mahlo, we can actually conclude that $\kappa > \rho$.

If $\rho \leq i < \kappa$, then $\{\kappa\}_i$ will be either $\emptyset$ or $\{i\}$. But since G is p^{ω_1}-bounded, $\emptyset \notin J_G$; and by our induction hypothesis, $\{i\} \notin J_G$ when $i \in \mathscr{Q}$. Therefore, $\Upsilon^{\omega_1}_{\{\kappa\}}(G)$ is not stationary, showing that $\{\kappa\}$ does not satisfy (J-1).

Suppose now that (J-2) holds for some isotype subgroup $A \subseteq G$ of cardinality κ. By Theorem 1.18, A has a closed p^{ω_1}-pure filtration $\{A_i\}_{i<\kappa}$; so A/A_i is a p^{ω_1}-bounded C_{ω_1}-group for all $i < \kappa$. Again, if $\rho \leq i < \kappa$, then $\{\kappa\}_i$ will be either $\emptyset$ or $\{i\}$. But since A/A_i is p^{ω_1}-bounded, $\emptyset \notin J_{A/A_i}$; and again by our induction hypothesis, $\{i\} \notin J_{A/A_i}$ when $i \in \mathscr{Q}$. This implies that $\Lambda^{\omega_1}_{\{\kappa\}}(A)$ will not be stationary in κ, showing that $\{\kappa\}$ does not satisfy (J-2), either. □

We next extend the above discussion to arbitrary elements of J_G.

Theorem 1.22 *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group, $T \in J_G$ and τ is the least element of T. Then we have,*

(a) $\tau < \rho$.
(b) *If τ is not weakly Mahlo, $\sigma \in \mathscr{Q}$, $\sigma < \tau$ and $S = (T - \{\tau\}) \cup \{\sigma\}$, then $S \in J_G$.*

Proof Note that since G is p^{ω_1}-bounded, $\emptyset \notin J_G$, that is $T \neq \emptyset$. As usual, we induct on $\kappa := \mu(T)$. If $T = \{\tau\}$, then (a) follows immediately from Theorem 1.21. Next, in (b), by Corollary 1.20 we have $\tau < \rho_G$. Therefore, if $\sigma < \tau$, then $\sigma < \rho_G$, so that $\{\sigma\} = S \in J_G$ holds by Lemma 1.19(a).

So suppose $\kappa > \tau$, and the result holds for all $\hat{T} \in J_G$ with $\mu(\hat{T}) < \kappa$.

As usual, we now examine the two possible reasons that T might be in J_G. Suppose first that (J-1) holds. If i is any element of $\Upsilon_T^{\omega_1}(G)$ with $\tau < i < \kappa$, then $T_i \in J_G$. Note that τ will also be the smallest element of T_i and $\mu(T_i) < \kappa$. So, by induction on κ, (a) holds as desired. As to (b), note that for all $i \in \Upsilon_T^{\omega_1}(G)$ with $i > \tau$, the same substitution that converts T to S also converts T_i to S_i. So by induction, $\Upsilon_T^{\omega_1}(G) \subseteq \Upsilon_S^{\omega_1}(G)$. So $\Upsilon_S^{\omega_1}(G)$ is a stationary subset of κ, and $S \in J_G$, as stated.

Suppose now that (J-2) holds. Let A be an isotype subgroup of G of cardinality κ as in that statement. Let $\{A_i\}_{i<\kappa}$ be a filtration of A consisting of p^{ω_1}-pure subgroups. Observe that if $p^{\omega_1}(A/A_i) \neq 0$ for all the is in some stationary subset of κ, then we could conclude that $\{\kappa\} \in J_G$. And since $\{\kappa\} \subseteq S$, by Lemma 1.10(a), this would imply that $S \in J_G$, as desired. Therefore, by restricting to some CUB subset of κ, we may assume that A/A_i is a p^{ω_1}-bounded C_{ω_1}-group for all $i < \kappa$. Now choose $i \in \Lambda_T^{\omega_1}(A)$ with $i > \tau$. Therefore, $T_i \in J_{A/A_i}$, τ is the smallest element of T_i and $\mu(T_i) < \kappa$. So again, since the substitution that takes T to S also takes T_i to S_i, induction implies that $\Lambda_T^{\omega_1}(A) \subseteq \Lambda_S^{\omega_1}(A)$. Therefore, the latter is a stationary subset of κ, so that $S \in J_G$, completing the proof. □

This means that the least element of any set in J_G must be less than ρ. In particular, we have the following immediate consequence:

Corollary 1.23 *Suppose $2^{\omega_1} = \omega_2$ and G is p^{ω_1}-bounded C_{ω_1}-group that is not in $\mathscr{F}$ (i.e., $J_G \neq 0_{\mathscr{Q}}, 1_{\mathscr{Q}}$). If $T \in J_G$, then $\omega_2 \in T$.*

We can use Theorem 1.22 to clarify and simplify some of the ideas found in previous papers. The following, for example, appeared as part of [7],Theorem 19).

Corollary 1.24 *We have $\nu \leq \aleph_{n+1}$ iff $G_1 \triangledown \cdots \triangledown G_n \in \mathscr{F}$ whenever $G_1, \ldots, G_n$ are p^{ω_1}-bounded C_{ω_1}-groups.*

Proof Suppose first that $\nu \leq \aleph_{n+1}$; in particular, $\rho = \nu$. If $T^j \in J_{G_j}$ for $j = 1, 2, \ldots, n$, then by Theorem 1.22(a), the smallest element of T^j will be an element of $Z = \{\aleph_2, \aleph_3, \ldots, \aleph_n\}$. However, since Z only has $n - 1$ elements, $T^1, \ldots, T^n$ cannot be pairwise disjoint. Therefore, $J_{G_1} \cdots J_{G_n} = 0_{\mathscr{Q}}$, which by Corollary 1.16 implies that $G_1 \triangledown \cdots \triangledown G_n$ is in $\mathscr{F}$.

Conversely, if $\nu > \aleph_{n+1}$, then by Theorem 1.21, for $j = 1, \ldots, n$, we can find a p^{ω_1}-bounded C_{ω_1}-group G_j such that $\{\aleph_{j+1}\} \in J_{G_j}$. It follows that $\{\aleph_2, \aleph_3, \ldots, \aleph_{n+1}\} \in J_{G_1} \cdots J_{G_n}$, which again by Corollary 1.16 implies $G_1 \triangledown \cdots \triangledown G_n$ is not in $\mathscr{F}$. □

Observe that the $n = 1$ case of Corollary 1.24 is just the equivalence of conditions (a) and (b) in Corollary 1.13; i.e., the statement that ¬KH iff every p^{ω_1}-bounded C_{ω_1}-group is in $\mathscr{F}$.

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The Valuation Difference Rank of a Quasi-Ordered Difference Field

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Abstract There are several equivalent characterizations of the valuation rank of an ordered or valued field. In this paper, we extend the theory to the case of an ordered or valued *difference* field (that is, ordered or valued field endowed with a compatible field automorphism). We introduce the notion of *difference rank*. To treat simultaneously the cases of ordered and valued fields, we consider quasi-ordered fields. We characterize the difference rank as the quotient modulo the equivalence relation naturally induced by the automorphism (which encodes its growth rate). In analogy to the theory of convex valuations, we prove that any linearly ordered set can be realized as the difference rank of a maximally valued quasi-ordered difference field. As an application, we show that for every regular uncountable cardinal κ such that $\kappa = \kappa^{<\kappa}$, there are 2^κ pairwise non-isomorphic quasi-ordered difference fields of cardinality κ, but all isomorphic as quasi-ordered fields.

Keywords Quasi-ordered field • Valued field • Ordered field • Rank of an ordered • Abelian group • Hahn group • Hahn field • Automorphism • Isometry

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1 Introduction

The theory of convex valuations and coarsenings of valuations is a special chapter in classical valuation theory. It is a basic tool in algebraic and real algebraic geometry. Surveys can be found in [20, 21, 23]. This special chapter is in turn closely related to ordered algebraic structures, see [8]. In particular, an important isomorphism invariant of an ordered or valued field is its rank as a valued field, which has several equivalent characterizations: via the ideals of the valuation ring, the value group, or the residue field, see [27].

This can be extended to ordered and valued fields with extra structure, giving a characterization completely analogous to the above, but taking into account the corresponding induced structure on the ideals, value group, or residue field. For example, in [14, Chapter 3] the notion of the exponential rank of an ordered exponential field is introduced and analysed in light of the above classical tools. The exponential rank measures the growth rate of the given exponential function, and is thus closely related to asymptotic analysis in the sense of Hardy [10].

In this paper, we push this analogy to the case of an ordered or valued difference field. We work with quasi-ordered fields, see [7]. In Sect. 2 we review classical notions and results on ordered or valued fields. We thereby present a uniform approach via quasi-orders, treating simultaneously the cases of ordered and valued fields. Theorem 2.2 gives a characterization of the rank of a quasi-ordered field in terms of coarsenings of its natural valuation. Descending down to the value group of the quasi-ordered field, and yet further down to the value set Γ of the value group, the rank and principal rank are finally characterized by the chain Γ, see Theorems 2.7 and 2.12. In Sect. 3 we start by a key remark regarding equivalence relations defined by monotone maps on chains. We describe in Theorem 3.4 the rank of a quasi-ordered field via the equivalence relations induced by addition and multiplication on the field. This approach allows us to develop in Sect. 4 the notion of difference compatible valuations and introduce the difference rank. We characterize in Theorem 4.2 the difference rank, in analogy to Theorem 2.2. [26, Lemma 1] is a special case of our Corollary 4.8 on weak isometries. Corollary 4.9 describes the set of fixed points of an automorphism σ in terms of its difference rank, whereas Corollary 4.11 examines the special case of ω-increasing or ω-contracting automorphisms. In the last Sect. 5 we describe the principal difference rank, see Theorem 5.3 and its Corollaries 5.5, 5.4 and 5.6. In Theorem 5.8 we construct large families of quasi-ordered difference fields with distinct difference ranks.

Some closing comments are in place. The theory of well-quasi orders [16] is currently a highly developed part of combinatorics with surprising applications in logic, mathematics and computer science. Quasi-ordered algebraic structures are interesting in their own right, and we will continue our investigations of these fascinating objects. Quasi-orders [2] appear in the literature also as *preorders*, see, e.g., [8, p.1]. However, we will not use this terminology, in order to avoid confusion with the notion of preorders appearing in real algebraic geometry (e.g. in [15]). The theory of quasi-ordered abelian groups is closely related to that of C-groups [11] and has already found interesting applications in [22] to the study of the asymptotic couple associated with a valued differential field. Throughout the paper, Hahn groups and Hahn fields play a fundamental role. The group of automorphisms of Hahn structures have been extensively studied, see [3, 5, 12, 26]. In future work, we will analyse the behaviour of the difference rank as function defined on these automorphism groups.

2 The Rank of a Quasi-Ordered Field

A **quasi-order (q.o.)** on a set S is a binary relation $\preceq$ which is reflexive and transitive. Throughout this paper, we will deal only with **total quasi-order**, i.e. either $a \preceq b$ or $b \preceq a$, for any $a,\ b \in S$. We will omit henceforth 'total'. Note that an order is a q.o which is in addition anti-symmetric. In the latter case, we say that S is an ordered set or a **chain**. The **induced equivalence relation** is defined by $a \asymp b$ if and only if ($a \preceq b$ and $b \preceq a$). We shall write $a \prec b$ if $a \preceq b$ but $b \asymp a$ fails. Note that $\preceq$ induces canonically a total order on $S/\asymp$. Conversely if $\asymp$ is an equivalence relation on a set S such that $S/\asymp$ is a total order, then $\asymp$ induces canonically a q.o. on S. A subset E of S is **$\preceq$-convex** if for all a, b, c in S, if $a \preceq c \preceq b$ and $a,\ b \in E$, then $c \in E$. We shall write convex instead of $\preceq$-convex if the context is clear.

A **quasi-ordered field** $(K, \preceq)$ is a field K endowed with a quasi-order $\preceq$ which satisfies the following compatibility conditions, for any $a, b, c \in K$.

qo1 If $a \asymp 0$, then $a = 0$.
qo2 If $0 \preceq c$ and $a \preceq b$, then $ac \preceq bc$.
qo3 If $a \preceq b$ and $b \not\asymp c$, then $a + c \preceq b + c$.

From **qo2** one deduces that if $a \preceq b$ and $0 \preceq c \preceq d$, then $ac \preceq bc \preceq bd$, so $ac \preceq bd$.

Given a valuation w on K we denote the **valuation ring** by K_w , its **group of units** $K_w^\times$ by $\mathcal{U}_w$, its **valuation ideal** (i.e. its unique maximal ideal) by I_w , its **value group** by $w(K^\times)$ and **residue field** K_w/I_w by Kw .

An ordered field $(K, \leq)$ is a q.o. field. The valuation on a valued field (K, w) induces a quasi-order: $a \preceq_w b$ if and only if $w(b) \leq w(a)$, i.e. if and only if $ab^{-1} \in K_w$. Fakhruddin [7] showed that if $\preceq$ is a q.o. on a field K, then $\preceq$ is either an order or there is a (unique up to equivalence of valuations) valuation v on K such that $\preceq = \preceq_v$. The dichotomy is achieved by considering the equivalence class E_1 of 1 with respect to $\asymp$. In the order case, $E_1 = \{1\}$ and $\asymp$ is just equality. The quasi-order is said to be a **proper quasi-order (p.q.o.)** if $E_1 \neq \{1\}$. In this case, $E_1 \neq \{1\}$ is a non-trivial subgroup of $K^\times$ and $K^\times/E_1$ is an ordered abelian group. Then $\mathcal{U}_v$ is just E_1 and $v(K^\times)$ is $K^\times/E_1$. In the p.q.o case $a \succeq 0$ for all $a \in K$.

Given two valuations v and w on K, recall that w is said to be a **coarsening** of v (w is coarser than v) or that v a **refinement** of w (v is finer than w) if $K_v \subseteq K_w$. In case the inclusion of the valuation rings is strict, we add the predicate strict in the terminology coarser and finer. Note that w is coarser than v if and only if $a \preceq_v b$ implies $a \preceq_w b$. If $\sim_1$ and $\sim_2$ are two equivalence relations defined on the same set, then $\sim_1$ is said to be **coarser** than $\sim_2$ (or $\sim_2$ **finer** than $\sim_1$) if $\sim_2$-equivalence implies $\sim_1$-equivalence.

Let us now fix a q.o. $\preceq$ on K. A valuation w on K is called **convex** with respect to $\preceq$ if its valuation ring K_w is convex. It is called **compatible** with $\preceq$ (or $\preceq$ is compatible with w or w and $\preceq$ are compatible) if for all $a, b \in K$:

$$0 \preceq b \preceq a \quad \Longrightarrow \quad w(a) \leq w(b)\ .$$

Equivalently, w is compatible with $\preceq$ if and only if for all $a, b \in K$:

$$0 \preceq b \preceq a \quad \Longrightarrow \quad b \preceq_w a\ .$$

Remark 2.1

(i) If $\preceq$ is an order, then this is the usual notion of compatibility for orders and valuations, see, e.g., [19, 20, 23], or [24].

(ii) If $\preceq = \preceq_v$ is a p.q.o., then w compatible with $\preceq_v$ just means that for all $a, b \in K$, $v(a) \leq v(b) \quad \Longrightarrow \quad w(a) \leq w(b)$. This in turn just means that $K_v \subseteq K_w$ or w is a coarsening of v, equivalently $\asymp_w$ is coarser than $\asymp_v$.

The following gives the characterization of valuations compatible with a quasi-order. Theorem 2.2 is in complete analogy to the characterization of valuations compatible with an order. So for $\preceq$ an order, we omit the proof and refer the reader to [19, Proposition 5.1], or [20, Theorem 2.3 and Proposition 2.9], or [23, Lemma 3.2.1], or [24, Lemma 7.2] or [6, Proposition 2.2.4].

Theorem 2.2 *Let $(K, \preceq)$ be a q.o. field and w a valuation on K. The following assertions are equivalent:*

(1) w is compatible with $\preceq$,

(2) w is convex,

(3) I_w is convex,

(4) $I_w \prec 1$,

(5) the quasi-order $\preceq$ induces canonically via the residue map $a \mapsto aw$ a quasi-order on the residue field Kw.

Proof Assume $\preceq = \preceq_v$ is a p.q.o. Compatible valuations are clearly convex, this follows from the definitions. Conversely if w is convex and $0 = v(1) \leq v(a)$, i.e. $a \preceq 1$, then $a \in K_w$ by convexity. So w is a coarsening of v. This establishes the equivalence of (1) and (2).

If w is convex, $a \preceq b$ with $b \in I_w$, then $0 < w(b) \leq w(a)$ by compatibility, so $a \in I_w$. Conversely assume I_w convex, and let $a \preceq b$ with $b \in K_w \setminus I_w$. If $a \notin K_w$ then $a^{-1} \in I_w$. Now $b^{-1} \preceq a^{-1}$, so $b^{-1} \in I_w$, a contradiction. This establishes the equivalence of (2) and (3).

If I_w is convex, then w is a coarsening of v, so $I_w \subseteq I_v \prec 1$. Conversely, assume $I_w \prec 1$ and let $a \preceq b$ with $b \in K_w$. If $a \notin K_w$, then $a^{-1} \in I_w$. So $a^{-1}b \in I_w$ whence $a^{-1}b \prec 1$. Multiplying by a gives $b \prec a$, a contradiction. This establishes the equivalence of (3) and (4).

Now let w be a coarsening of v. Then v induces canonically a valuation v/w on the residue field Kw, defined by $v/w(aw) := \infty$ if $aw = 0$ and $v/w(aw) := v(a)$

otherwise ([6] p. 44) . The p.q.o. $\preceq_{v/w}$ is precisely the induced well-defined quasi-order in (5), i.e. $aw \preceq_{v/w} bw$ if and only if $a \preceq_v b$ holds. Conversely, let $\preceq_{v/w}$ be a p.q.o. on Kw induced by the residue map. This means that $aw \preceq_{v/w} bw$ if and only if $a \preceq_v b$ holds. Then w is a coarsening of v (see [6, p. 45]). This establishes the equivalence of (1) and (5). □

Remark 2.3 If $\preceq$ is an order, then the induced quasi-order in (5) is also an order, if $\preceq$ is a p.q.o., then the induced quasi-order in (5) is also a p.q.o.

Let $(K, \preceq)$ be a q.o. field. We define its **natural valuation**, denoted by v, to be the finest $\preceq$-convex valuation of K. If $(K, \leq)$ is ordered, then the natural valuation is the valuation v whose valuation ring K_v is the convex hull of $\mathbb{Q}$ in K. In this case, the natural valuation on K satisfies $v(x+y) = \min\{v(x), v(y)\}$ if sign(x) = sign(y) and for all $a, b \in K : a \geq b > 0 \implies v(a) \leq v(b)$. It is characterized by the fact that the induced order on its residue field Kv is archimedean, i.e. the only equivalence classes for the archimedean equivalence relation (see definition below following Lemma 2.5) are those of 0 and 1. If w is a coarsening of a convex valuation, then w also is convex. Conversely, a convex subring containing 1 is a valuation ring, see [6, Section 2.2.2]. The set $\mathscr{R}$ of all valuation rings K_w of convex valuations $w \neq v$ (i.e. all strict coarsenings of v) is totally ordered by inclusion. Its order type is called the **rank of the ordered field** K. For convenience, we will identify it with $\mathscr{R}$. For example, the rank of an archimedean ordered field is empty since its natural valuation is trivial (i.e. its valuation ring is the field itself). The rank of the rational function field $K = \mathbb{R}(t)$ with any order is a singleton: $\mathscr{R} = \{K\}$. Theorem 2.2 is a characterization of the elements of the rank of the ordered field $(K, \leq)$. Note that the rank of $(K, \leq)$ is invariant under isomorphisms of ordered fields.

If $(K, \preceq)$ is p.q.o., then the unique valuation v such that $\preceq = \preceq_v$ is the natural valuation. A compatible valuation w is a coarsening of v. We define the **rank of the valued field** (K, v) to be the (order type of the) totally ordered set $\mathscr{R}$ of all strict coarsenings of v. Thus, Theorem 2.2 is a characterization of the elements of the rank of (K, v). Note that the rank of (K, v) is invariant under isomorphisms of valued fields. As we recalled in the proof of Theorem 2.2, the natural valuation v induces canonically a valuation v/w on the residue field Kw and v is the **compositum** of w and v/w (see [6, pp. 44–45]) . The p.q.o. $\preceq_{v/w}$ is precisely the induced quasi-order in Theorem 2.2(5). If $w = v$, then v/w is trivial. Thus v is characterized by the fact that the induced p.q.o on its residue field Kv is **trivial**, i.e. the only equivalence classes of $\asymp$ are those of 0 and 1.

Remark 2.4 The maximal ideals I_w appearing in Theorem 2.2(4) are prime ideals of the valuation ring K_v. The strict coarsenings K_w of K_v are the localizations of K_v at the prime ideals $\{0\} \subseteq I \subset I_v$, [6, Lemma 2.3.1 p. 43], [27, Theorem 15, p. 40]. Thus the rank is also isomorphic to the totally ordered (by reverse inclusion) set of prime ideals of K_v which are strictly contained in the maximal ideal I_v.

We now want to characterize the rank by going down to the value group. Let v be the natural valuation on the q.o. field $(K, \preceq)$. We set $G = v(K^\times)$. Recall that the set of all convex subgroups $G_w \neq \{0\}$ of the value group G is totally ordered by inclusion. Its order type is called the **rank** of G, it is an isomorphism invariant, see [8] or [23]. To every convex valuation ring K_w, we associate a convex subgroup $G_w := \{v(a) \mid a \in K \wedge w(a) = 0\} = v(\mathscr{U}_w)$. We call G_w the **convex subgroup associated with** w. Note that $G_v = \{0\}$. Conversely, given a convex subgroup G_w of $v(K^\times)$, we define $w : K \to v(K^\times)/G_w$ by $w(a) = v(a) + G_w$. Then w is a convex valuation with $v(\mathscr{U}_w) = G_w$ (and v is strictly finer than w if and only if $G_w \neq \{0\}$). We call w the **convex valuation associated with** G_w. We summarize the above discussion in the following lemma, for more details, see [6], or [8] or [23].

Lemma 2.5 *The correspondence $K_w \mapsto G_w$ is an order preserving bijection, thus $\mathscr{R}$ is (isomorphic to) the rank of G.*

We now want to characterize the rank by going further down to the value set of the value group. Recall that on the negative cone $G^{<0}$ of an ordered abelian group G, the **archimedean equivalence** relation $\sim$ is defined by: $a \sim b$ if and only if there is $n \in \mathbb{N}$ such that $a \geq nb$ and $b \geq na$. Let v_G be the map $a \mapsto [a]_\sim$, where $[a]_\sim$ denotes the equivalence class of a. The order on $\Gamma := G^{<0}/\sim$ is the one induced by the order of $G^{<0}$. We call $v_G(G^{<0}) := \Gamma$ the **value set of** G. By convention we also write $v_G(G) := \Gamma \cup \{\infty\}$ extending the archimedean equivalence relation to the positive cone of G by setting $v_G(g) := v_G(-g)$ and $v_G(0) = \infty > \Gamma$. The map v_G on G satisfies the ultrametric triangle inequality, and in particular we have: $v_G(x + y) = \min\{v_G(x), v_G(y)\}$ if $\mathrm{sign}(x) = \mathrm{sign}(y)$. We call v_G the **natural valuation on** G.

We now recall the relation between the rank of G and the value set Γ of G. To $G_w \neq \{0\}$ a convex subgroup, we associate $\Gamma_w := v_G(G_w^{<0})$ a non-empty final segment of Γ. Conversely, if Γ_w is a non-empty final segment of Γ, then $G_w = \{g \mid g \in G, v_G(g) \in \Gamma_w\} \cup \{0\}$ is a convex subgroup, with $\Gamma_w = v_G(G_w)$. Let us denote by Γ^{fs} the set of non-empty final segments of Γ, totally ordered by inclusion. We summarize the above discussion in the following lemma, for more details, see [6], or [8] or [23].

Lemma 2.6 *The correspondence $G_w \mapsto \Gamma_w$ is an order preserving bijection, thus the rank of G is (isomorphic to) Γ^{fs}.*

Combining Lemmas 2.5 and 2.6 we obtain the following result. Note that Theorem 2.7 will also follow, by a different argument, from Theorem 3.4 in the next section.

Theorem 2.7 *The correspondence $K_w \mapsto \Gamma_w$ is an order preserving bijection, thus $\mathscr{R}$ is (isomorphic to) Γ^{fs}.*

A final segment which has a least element is a **principal final segment**. It is of the form $\{\gamma' \mid \gamma' \in \Gamma, \gamma' \geq \gamma\}$, for some $\gamma \in \Gamma$. Let Γ^* denote the set Γ with its reversed ordering. The proof of the following Lemma is now routine.

Lemma 2.8 *The map from Γ to Γ^{fs} defined by $\gamma \mapsto \{\gamma' \mid \gamma' \in \Gamma, \gamma' \geq \gamma\}$ is an order reversing embedding. Its image is the set of principal final segments. Thus Γ^* is (isomorphic to) the totally ordered set of principal final segments.*

For the notions and results in this last paragraph of the section, we refer the reader to [8] or [23] for more details. Recall that a convex subgroup G_w of G is called **principal generated by** g, $g \in G$, if G_w is the minimal convex subgroup containing g. The **principal rank** of G is the subset of the rank of G consisting of all principal $G_w \neq \{0\}$.

Lemma 2.9 *Let $G_w \neq \{0\}$ be a convex subgroup. Then G_w is principal if and only if $v_G(G_w) = \Gamma_w$ is a principal final segment.*

Lemma 2.10 *The map $G_w \mapsto \min \Gamma_w$ is an order reversing bijection from the principal rank of G onto Γ. Thus the principal rank of G is (isomorphic to) Γ^*.*

We set: $\mathbf{P}_K := K^{\succeq 0} \setminus K_v$, where $K^{\succeq 0} := \{a \in K \,; a \succeq 0\}$. A $K_w \in \mathscr{R}$ is **principal generated by** a for $a \in \mathbf{P}_K$ if K_w is the smallest (convex) subring containing a. We observe:

Lemma 2.11 *Let $K_w \in \mathscr{R}$. Then, K_w is principal generated by a if and only if $K_w = \{b \in K : \exists n \in \mathbb{N}_0 \text{ s.t. } b \preceq_v a^n\}$.*

Proof It is enough to verify that $\{b \in K : \exists n \in \mathbb{N}_0 \;\; b \preceq_v a^n\}$ is a subring of K. Let $b_1 \preceq_v a^{n_1}$ and $b_2 \preceq_v a^{n_2}$. Then $b_1 b_2 \preceq_v a^{n_1+n_2}$ and $b_1 + b_2 \preceq_v a^{max\{n_1,n_2\}}$. Clearly, this ring contains K_v and a. □

The **principal rank** of K is the subset $\mathscr{R}^{\mathrm{pr}}$ of $\mathscr{R}$ consisting of all principal $K_w \in \mathscr{R}$. Combining the last three lemmas we obtain:

Theorem 2.12 *The correspondence $K_w \mapsto \Gamma_w$ is an order preserving bijection between $\mathscr{R}^{\mathrm{pr}}$ and the principal rank of G, thus $\mathscr{R}^{\mathrm{pr}}$ is (isomorphic to) Γ^*.*

Note that Theorem 2.12 will also follow, by a different argument, from Theorem 3.4 in the next section.

Remark 2.13 It is straightforward to verify that an order preserving isomorphism $\psi : \Gamma_1 \to \Gamma_2$ induces an order preserving isomorphism $\psi^{\mathrm{fs}} : \Gamma_1^{\mathrm{fs}} \to \Gamma_2^{\mathrm{fs}}$ [25, p.19]. Thus Γ determines Γ^{fs} up to isomorphism. It follows from Theorems 2.7 and 2.12 that if two q.o. fields have isomorphic principal ranks, then they have isomorphic ranks. In the next section we shall hence focus our attention on the principal rank.

3 The Principal Rank via Equivalence Relations

We begin by the following key observation:

Remark 3.1 Let φ be a map from a q.o. ordered set $(S, \preceq)$ into itself, and assume that φ is q.o. preserving, i.e. $a \preceq a'$ implies $\varphi(a) \preceq \varphi(a')$, for all $a, a' \in S$. Assume that φ has an orientation or is **oriented**, i.e. $\varphi(a) \succeq a$ for all $a \in S$ (φ is a **right shift**) or $\varphi(a) \preceq a$ for all $a \in S$ (φ is a **left shift**). We set $\varphi^0(a) := a$ and $\varphi^{n+1}(a) := \varphi(\varphi^n(a))$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is then straightforward that the following defines an equivalence relation on S:

(i) If φ is a right shift, set $a \sim_\varphi a'$ if and only if there is some $n \in \mathbb{N}_0$ such that $\varphi^n(a) \succeq a'$ and $\varphi^n(a') \succeq a$ (equivalently for some $n, m \in \mathbb{N}_0$, $\varphi^n(a) \succeq a'$ and $\varphi^m(a') \succeq a$),

(ii) If φ is a left shift, set $a \sim_\varphi a'$ if and only if there is some $n \in \mathbb{N}_0$ such that $\varphi^n(a) \preceq a'$ and $\varphi^n(a') \preceq a$ (equivalently for some $n, m \in \mathbb{N}_0$, $\varphi^n(a) \preceq a'$ and $\varphi^m(a') \preceq a$).

(iii) The equivalence classes $[a]_\varphi$ of $\sim_\varphi$ are $\preceq$-convex and closed under application of φ. By the $\preceq$-convexity, the quasi-order of S induces an order on $S/\sim_\varphi$ such that $[a]_\varphi \prec [b]_\varphi$ if and only if $a' \prec b'$ for all $a' \in [a]_\varphi$ and $b' \in [b]_\varphi$.

Note that if φ is the identity map $\mathbb{I}$, then the equivalence relation $\sim_{\mathbb{I}}$ is just $\asymp$ associated with the q.o., and is the finest one such that $S/\sim_{\mathbb{I}}$ is an ordered set.

We exploit Remark 3.1 to give an interpretation of the rank and principal rank as quotients via an appropriate equivalence relation, thereby providing—as promised in the previous section—alternative proofs for Theorems 2.7 and 2.12. It is precisely this approach that we will generalize to the difference rank in Sect. 5. Let v be the natural valuation on the q.o. field $(K, \preceq)$. Recall that $\mathbf{P}_K$ denotes $K^{\succeq 0} \setminus K_v$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{P}_K & \xrightarrow{\ \varphi\ } & \mathbf{P}_K \\
\downarrow v & /\!/\!/ & \downarrow v \\
G^{<0} & \xrightarrow{\ \varphi_G\ } & G^{<0} \\
\downarrow v_G & /\!/\!/ & \downarrow v_G \\
v_G(G) & \xrightarrow{\ \varphi_\Gamma\ } & v_G(G)
\end{array}$$

with $\varphi(a) := a^2$ for all $a \in \mathbf{P}_K$, φ is a right shift,

$\varphi_G(v(a)) := v(\varphi(a))$ for all $a \in \mathbf{P}_K$,

that is $\varphi_G(g) = 2g$ for all $g \in G^{<0}$, φ_G is a left shift and

$\varphi_\Gamma(v_G(g)) := v_G(\varphi_G(g))$ for all $g \in G^{<0}$,

that is $\varphi_\Gamma(\gamma) = \gamma$ for all $\gamma \in \Gamma$, so that φ_Γ is just the identity map.

By Remark 3.1, we can work with the equivalence relations associated with the following oriented maps: the q.o. preserving map φ and the order preserving maps φ_G and φ_Γ (as defined on the right-hand side of the above diagram). Note that $\sim_{\varphi_G}$ is just archimedean equivalence on G and $\sim_{\varphi_\Gamma}$ is just equality on Γ. The following straightforward observation will be useful for the proof of Theorem 3.4 below:

Lemma 3.2 *The equivalence classes of $\sim_\varphi$ are closed under multiplication.*

Proof The proof is similar to that of Lemma 2.11. Let $a, b \in \mathbf{P}_K$, and without loss of generality assume that $a \preceq b$ and $a \sim_\varphi b$. We show that $ab \sim_\varphi a$. Let $n \in \mathbb{N}_0$, such that $b \preceq a^{2^n}$. By axiom qo2, $ab \preceq b^2$. Thus $b^2 \preceq a^{2^n} b$ and $ab \preceq a^{2^n} b$. So, $ab \preceq a^{2^{n+1}}$. Since $1 \preceq b$, by axiom qo2, we get that $a \preceq ab$. Therefore, $ab \sim_\varphi a$.

□

Remark 3.3 We note that

$$\varphi_G^n(v(a)) \;=\; v(\varphi^n(a)) \text{ and } \varphi_\Gamma^n(v_G(g)) \;=\; v_G(\varphi_G^n(g)) \tag{1}$$

thus

$$a \sim_\varphi a' \text{ if and only if } v(a) \sim_{\varphi_G} v(a') \text{ if and only if } v_G(v(a)) \sim_{\varphi_\Gamma} v_G(v(a')) \tag{2}$$

Thus we have an order reversing bijection from $\mathbf{P}_K/\sim_\varphi$ onto $\Gamma/\sim_{\varphi_\Gamma}= \Gamma$. Thus the chain $[\mathbf{P}_K/\sim_\varphi]^{\text{is}}$ of non-empty initial segments of $\mathbf{P}_K/\sim_\varphi$ ordered by inclusion is isomorphic to Γ^{fs}. In particular, initial segments which have a last element are in bijective correspondence to principal final segments. Thus the subchain of $[\mathbf{P}_K/\sim_\varphi]^{\text{is}}$ of initial segments which have a last element is isomorphic to Γ^*.[1]. Therefore, as promised in the previous section, Theorems 2.7 and 2.12 will now follow from the following result:

Theorem 3.4 *The rank $\mathscr{R}$ is isomorphic to the chain $[\mathbf{P}_K/\sim_\varphi]^{\text{is}}$ and the principal rank $\mathscr{R}^{\text{pr}}$ is isomorphic to the subchain of $[\mathbf{P}_K/\sim_\varphi]^{\text{is}}$ of initial segments which have a last element.*

Proof First we note that if K_w is a convex valuation ring, then clearly $K_w^{\succ 0} \setminus K_v^{\succ 0}$ is an initial segment of $\mathbf{P}_K$. Moreover by Lemma 2.11 if K_w is principal generated by a, then $[a]_{\sim_\varphi}$ is the last class. Furthermore, if K_w intersects an equivalence class $[a]_{\sim_\varphi}$ then it must contain it, since the sequence $a^n; n \in \mathbb{N}_0$ is cofinal in $[a]_{\sim_\varphi}$ and K_w is a convex subring. We conclude that $(K_w^{\succ 0} \setminus K_v^{\succ 0})/\sim_\varphi$ is an initial segment of $\mathbf{P}_K/\sim_\varphi$. Conversely set $\mathscr{I}_w = \{[a]_\varphi \mid a \in K_w^{\succ 0} \setminus K_v^{\succ 0}\}$. Given $\mathscr{I} \in [\mathbf{P}_K/\sim_\varphi]^{\text{is}}$, we show that there is a convex valuation ring K_w such that $\mathscr{I}_w = \mathscr{I}$. Given $\mathscr{I}$, let $(\bigcup \mathscr{I})$ denote the set theoretic union of the elements of $\mathscr{I}$ and $-(\bigcup \mathscr{I})$ the set of additive inverses. Set $K_w = -(\bigcup \mathscr{I}) \cup K_v \cup (\bigcup \mathscr{I})$. We claim that K_w is the required ring. Clearly, $\mathscr{I}_w = \mathscr{I}$. Further K_w is convex (by its construction), and strictly contains K_v. We leave it to the reader, using Lemmas 3.2 and 2.11, to verify that K_w is a ring, and that K_w is principal generated by a if $[a]_{\sim_\varphi}$ is the last element of $\mathscr{I}$. □

4 The Difference Analogue of the Rank

In this section, we develop a difference analogue of what has been reviewed above. That is, we develop a theory of difference compatible valuations, in analogy to the theory of convex valuations. The automorphism will play the role that multiplication plays in the previous case.

[1]Note that the subchain of $[\mathbf{P}_K/\sim_\varphi]^{\text{is}}$ of initial segments which have a last element is isomorphic to $[\mathbf{P}_K/\sim_\varphi]$ itself.

Let $(K, \preceq)$ be a q.o. field and σ be a **q.o. preserving** field automorphism of K, that is, $a \preceq a'$ if and only if $\sigma(a) \preceq \sigma(a')$, for all $a,\ a' \in K$. We say that $(K, \preceq, \sigma)$ is a **q.o. difference field.**

Remark 4.1 Let $(K, \leq, \sigma)$ be an ordered difference field. Recall that the natural valuation v on K is defined by archimedean equivalence. Since archimedean equivalence is preserved under order preserving automorphisms, we see that σ is also $\preceq_v$ preserving (so that $(K, \preceq_v, \sigma)$ is a q.o. difference field). The converse fails: Consider the field of real Laurent series $K := \mathbb{R}((t))$ endowed with the lexicographic order and the corresponding natural valuation $v_{\min}$ (see definitions following Corollary 4.11 below). The map $t \mapsto (-t)$ defines a field automorphism σ on K which clearly preserves $v_{\min}$ but not the lexicographic order on K.

Now let $(K, \preceq, \sigma)$ be a non-trivial (i.e. $\sigma \neq$ identity) q.o. difference field and v its natural valuation. By definition, σ satisfies for all $a, b \in K\ : v(a) \leq v(b)$ if and only if $v(\sigma(a)) \leq v(\sigma(b))$ and thus induces an order preserving automorphism σ_G and σ_Γ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbf{P}_K & \xrightarrow{\ \sigma\ } & \mathbf{P}_K \\
\downarrow v & \cong & \downarrow v \\
G^{<0} & \xrightarrow{\ \sigma_G\ } & G^{<0} \\
\downarrow v_G & \cong & \downarrow v_G \\
v_G(G) & \xrightarrow{\ \sigma_\Gamma\ } & v_G(G)
\end{array}
$$

with $\sigma_G(v(a)) := v(\sigma(a))$ for all $a \in \mathbf{P}_K$,

and

$\sigma_\Gamma(v_G(g)) := v_G(\sigma_G(g))$ for all $g \in G^{<0}$.

Now let w be a convex valuation on K. Say w is **σ-compatible** if for all $a, b \in K\ : w(a) \leq w(b)$ if and only if $w(\sigma(a)) \leq w(\sigma(b))$. Thus w is σ-compatible if and only if σ preserves the q.o. $\preceq_w$.

The subset $\mathscr{R}_\sigma := \{\, K_w \in \mathscr{R}\ ;\ w$ is σ- compatible $\}$ is the **σ-rank** of $(K, \preceq, \sigma)$. Similarly, the subset of all convex subgroups $G_w \neq \{0\}$ such that $\sigma_G(G_w) = G_w$, i.e. G_w is **σ_G-invariant**, is the **σ-rank** of G. Finally, we denote by σ_Γ-Γ^{fs} the subset of non-empty final segments Γ_w such that $\sigma_\Gamma(\Gamma_w) = \Gamma_w$, i.e. Γ_w is **σ_Γ-invariant**.

The following Theorem 4.2, Lemmas 4.5 and 4.6 are analogues of Theorem 2.2, Lemmas 2.5 and 2.6, respectively. They are verified by straightforward computations, using basic properties of valuations rings on the one hand and of automorphisms on the other (e.g. $\sigma(A \setminus B) = \sigma(A) \setminus \sigma(B)$, $\sigma(A) \subseteq B$ if and only if $A \subseteq \sigma^{-1}(B)$ and $\sigma(A) \subseteq B$ if and only if $\sigma(-A) \subseteq -B$). The equivalence of (1) and (7) in Theorem 4.2 follows from the compatibility of σ with w on the one hand, and from the definition of the induced q.o. on Kw on the other. We call K_w σ-compatible if any of the equivalent conditions below holds.

Theorem 4.2 *The following assertions are equivalent for a convex valuation w :*

(1) w *is* σ*–compatible*
(2) w *is* σ^{-1}*–compatible*
(3) $\sigma(K_w) = K_w$
(4) $\sigma(I_w) = I_w$
(5) $\sigma(\mathscr{U}_w) = \mathscr{U}_w$
(6) $\sigma(K_w^{\succ 0} \setminus K_v^{\succ 0}) = K_w^{\succ 0} \setminus K_v^{\succ 0}$
(7) *the map* $\sigma w : Kw \to Kw$ *defined by* $aw \mapsto \sigma(a)w$ *is well-defined and is a q.o. (with respect to the induced q.o. on* Kw *) preserving field automorphism of* Kw *.*

Remark 4.3 Let $(K, \leq, \sigma)$ be an ordered field with natural valuation v. In this case, condition (7) on σw in Theorem 4.2 is referring to the induced order on the residue field Kw. Consider instead the following condition:

(8) the map $\sigma w : Kw \to Kw$ defined by $aw \mapsto \sigma(a)w$ is well-defined and is a q.o. (with respect to the q.o. $\preceq_{v/w}$ on Kw) preserving field automorphism of Kw .

We observe that (7) implies (8). Indeed, σw is assumed to be order preserving on Kw by (7). Now $(Kw)(v/w) = Kv$ (see [17, Lemma 2.1]). Therefore v/w has archimedean residue field and is thus the natural valuation on the ordered field Kw. By Remark 4.1 we obtain the assertion.

Remark 4.4 The maximal ideals I_w appearing in Theorem 4.2(4) are σ-invariant prime ideals (also called transformally prime ideals in [4]) of the valuation ring K_v and the coarsenings K_w are just the localizations of K_v at those σ-invariant prime ideals, see [6, Lemma 2.3.1 p. 43]. Thus the σ- rank is also characterized by the chain of σ-invariant prime ideals of K_v.

Lemma 4.5 *The correspondence* $K_w \mapsto G_w$ *is an order preserving bijection from* $\mathscr{R}_\sigma$ *onto the* σ_G*-rank of* G*.*

Lemma 4.6 *The correspondence* $G_w \mapsto \Gamma_w$ *is an order preserving bijection from the* σ_G*-rank of* G *onto* σ_Γ*-*Γ^{fs}*.*

We deduce from Lemmas 4.5 and 4.6 that the σ-rank is the order type of σ_Γ-Γ^{fs}:

Theorem 4.7 *The correspondence* $K_w \mapsto \Gamma_w$ *is an order preserving bijection from* $\mathscr{R}_\sigma$ *onto* σ_Γ*-*Γ^{fs}*.*

We now exploit this observation. An automorphism σ is an **isometry** if $v(\sigma(a)) = v(a)$ for all $a \in K$, equivalently σ_G is the identity automorphism, and a **weak isometry** if σ_Γ is the identity automorphism. Every isometry is a weak isometry. Note that if Γ is a rigid chain (i.e. the only order preserving automorphism is the identity map), then σ is necessarily a weak isometry. If σ is a weak isometry, then $\sigma_\Gamma(v_G(g)) = v_G(\sigma_G(g)) = v_G(g)$, thus g is archimedean equivalent to $\sigma_G(g)$ for all g, and so every convex subgroup is σ_G-invariant.

Corollary 4.8 *If* σ *is a weak isometry, then* $\mathscr{R}_\sigma = \mathscr{R}$*.*

Corollary 4.9 *The correspondence* $K_w \mapsto \min \Gamma_w$ *is an order (reversing) isomorphism from* $\mathscr{R}_\sigma \cap \mathscr{R}^{\mathrm{pr}}$ *onto the chain* $\{\gamma \; ; \; \sigma_\Gamma(\gamma) = \gamma\}$ *of fixed points of* σ_Γ.

Proof By Lemma 2.9, set $\min \Gamma_w := \gamma_0$. By Lemmas 4.5 and 4.6, Γ_w in invariant under σ_Γ. Since σ_Γ is order preserving, we must have $\sigma_\Gamma(\gamma_0) = \gamma_0$ □

At the other extreme σ is said to be ω**-increasing** if $a^n \prec \sigma(a)$ for all $n \in \mathbb{N}_0$ and all $a \in \mathbf{P}_K$, and ω**-contracting** if σ^{-1} is ω-increasing.

Remark 4.10 Note that σ is ω-increasing (respectively, ω-contracting) if and only if σ_Γ is a **strict left shift**, that is, $\sigma_\Gamma(\gamma) < \gamma$ for all $\gamma \in \Gamma$ (respectively, a **strict right shift**, i.e. $\sigma_\Gamma(\gamma) > \gamma$ for all $\gamma \in \Gamma$). Thus if σ ω-increasing or ω-contracting, then σ_Γ has no fixed points.

Corollary 4.11 *If* σ *is* ω*-increasing or* ω*-contracting, then* $\mathscr{R}_\sigma \cap \mathscr{R}^{\mathrm{pr}}$ *is empty.*

Recall that the **Hahn group** [9] over the chain Γ and components $\mathbb{R}$, denoted $\mathbf{H}_\Gamma\mathbb{R}$, is the totally ordered abelian group whose elements are formal sums $g := \sum g_\gamma 1_\gamma$, with well-ordered support $g := \{\gamma \; ; \; g_\gamma \neq 0\}$. Here $g_\gamma \in \mathbb{R}$ and 1_γ denotes the characteristic function on the singleton $\{\gamma\}$. Addition is pointwise and the order lexicographic. Similarly, given a field F, the field of **generalized power series** over the ordered abelian group G (or **Hahn field** over G) with coefficients in F, denoted $\mathbb{F} := F((G))$, is the field whose elements are formal series $s := \sum s_g t^g$, with well-ordered support $s := \{g \; ; \; s_g \neq 0\}$. Addition is pointwise, multiplication is given by the usual convolution formula. The field $\mathbb{F}$ has the same characteristic as that of F. The canonical valuation $v_{\min}$ on $\mathbb{F}$ is defined by $v_{\min}(s) := \min$ support s for $s \neq 0$. Its value group is G and its residue field is F. Thus $(\mathbb{F}, \preceq_{v_{\min}})$ is a q.o. field. If F is an ordered field, its order extends to the lexicographic order on $\mathbb{F}$: a series s is positive if and only if the coefficient of $t^{v_{\min}(s)}$ is positive in F. Thus, in that case $(\mathbb{F}, \leq)$ is an ordered field. Hahn fields are maximally valued: they admit no proper immediate extension, that is, no proper valued field extension preserving the value group and the residue field. They were extensively studied, e.g. by Hahn [9] and in the seminal paper of Kaplansky [13].

Lemma 4.12 *Any order preserving automorphism* σ_Γ *of the chain* Γ *lifts to an order preserving automorphism* σ_G *of the Hahn group* G *over* Γ*, and* σ_G *lifts in turn to a q.o. preserving automorphism* σ *of the Hahn field over* G.

Proof Set $\sigma_G(\sum g_\gamma 1_\gamma) := \sum g_\gamma 1_{\sigma_\Gamma(\gamma)}$. It is straightforward to verify that the thus defined σ_G induces the given automorphism σ_Γ on Γ. Thus σ_G is a lifting of σ_Γ. Now set $\sigma(\sum s_g t^g) := \sum s_g t^{\sigma_G(g)}$. Again, it is clear that σ induces σ_G on G. Thus σ is a lifting of σ_G as asserted. □

Corollary 4.13 *Given any order type* τ *there exists an ordered difference field* $(K, \leq, \sigma)$*, and also a p.q.o. difference field* $(K, \preceq, \sigma)$ *such that the order type of* $\mathscr{R}_\sigma \cap \mathscr{R}^{\mathrm{pr}}$ *is* τ.

Proof Set $\mu := \tau^*$, and consider, e.g. the linear ordering $\Gamma := \sum_\mu \mathbb{Q}^{\geq 0}$, that is, the concatenation of μ copies of the non-negative rationals. Fix a non-trivial order automorphism η of $\mathbb{Q}^{>0}$. Define σ_Γ to be the uniquely defined order automorphism of Γ fixing every $0 \in \mathbb{Q}^{\geq 0}$ in every copy and extending η on every copy. It is clear that the order type of the chain of fixed points (the zeros in every copy) of σ_Γ is μ. Set e.g. $G := \mathbf{H}_\Gamma \mathbb{R}$. By Lemma 4.12, σ_Γ lifts canonically to σ_G on G. Now consider, e.g. the ordered field $\mathbb{F} := \mathbb{R}((G))$. Again by Lemma 4.12, σ_G lifts canonically to an order automorphism σ of $\mathbb{F}$. This is our required σ, by Corollary 4.9. To obtain a p.q.o difference field, take F any field and the corresponding $(\mathbb{F}, \preceq_{v_{\min}}, \sigma)$. □

In the next section, we will exploit appropriate equivalence relations to define the principal difference rank and construct difference fields of arbitrary principal difference rank.

5 The σ-Rank and Principal σ-Rank via Equivalence Relations

Let $(K, \preceq, \sigma)$ be a q.o. difference field. As promised in Sect. 3, we now exploit Remark 3.1 to give an interpretation of the σ- rank and define the principal σ-rank as quotients via appropriate equivalence relations. Our aim is to state and prove the analogues to Theorems 3.4, 2.7 and 2.12. We recall that the q.o. preserving maps considered in Remark 3.1 are assumed to be oriented. Moreover, scrutinizing the proof of Theorem 3.4 we quickly realize that we need Lemma 5.2 below, an analogue of Lemma 3.2. Thus we need further assumptions on σ, to ensure that σ satisfies Lemma 5.2. For simplicity from now on we will assume that σ or σ^{-1} satisfy $\sigma(a) \succeq a^2$ for all $a \in \mathbf{P}_K$. Note that this implies that $\sigma(a) \succ a$, so σ is an oriented strict right-shift. Note that our condition on σ is fulfilled for ω-increasing or ω-contracting automorphisms.

A convex subring $K_w \neq K_v$ is σ**-principal generated by** a for $a \in \mathbf{P}_K$ if K_w is the smallest convex σ-compatible subring containing a. The σ**-principal rank** of K is the subset $\mathscr{R}_\sigma^{\text{pr}}$ of $\mathscr{R}_\sigma$ consisting of all σ-principal $K_w \in \mathscr{R}$. We will use the analogue of Remark 3.3:

Remark 5.1 The maps σ, σ_G and σ_Γ are q.o. preserving and we can define the corresponding equivalence relations $\sim_\sigma$, $\sim_{\sigma_G}$ and $\sim_{\sigma_\Gamma}$. As before we have

$$a \sim_\sigma a' \text{ if and only if } v(a) \sim_{\sigma_G} v(a') \text{ if and only if } v_G(v(a)) \sim_{\sigma_\Gamma} v_G(v(a')) \tag{3}$$

Thus we have an order reversing bijection from $\mathbf{P}_K/\sim_\sigma$ onto $\Gamma/\sim_{\sigma_\Gamma}$. Thus the chain $[\mathbf{P}_K/\sim_\sigma]^{\text{is}}$ of initial segments of $\mathbf{P}_K/\sim_\sigma$ ordered by inclusion is isomorphic to $(\Gamma/\sim_{\sigma_\Gamma})^{\text{fs}}$. As before, the subchain of initial segments which have a last element is isomorphic to $(\Gamma/\sim_{\sigma_\Gamma})^*$.

Lemma 5.2 *The equivalence classes of $\sim_\sigma$ are closed under σ and under multiplication.*

Proof The condition on σ implies by induction that $\sigma^n(a) \succeq a^{2^n}$. Thus given $n \in \mathbb{N}_0$, there exists $l \in \mathbb{N}_0$ such that $\sigma^l(a) \succeq a^n$. Thus $a \sim_\sigma \sigma(a)$. So the equivalence classes of σ are closed under σ . Recall that the natural valuation v_G on G satisfies $v_G(x+y) = \min\{v_G(x), v_G(y)\}$ if sign(x) = sign(y). Again one easily deduces from this fact and the equivalences (3) above that the equivalence classes of σ are closed under multiplication. Indeed assume that $a \sim_\sigma b$ and $a \sim_\sigma c$. We want to show that $a \sim_\sigma bc$. Set $x := v(b)$, $y := v(c)$ and $z := v(a) \in G^{<0}$. By the first equivalence in (3), it is enough to show that $v(a) \sim_{\sigma_G} v(bc)$, i.e. that $x + y \sim_{\sigma_G} z$. By the second equivalence in (3), it is enough to show that $v_G(x + y) \sim_{\sigma_\Gamma} v_G(z)$. Without loss of generality $v_G(x + y) = v_G(x)$. But since $a \sim_\sigma b$ it follows by (3) that $v_G(x) \sim_{\sigma_\Gamma} v_G(z)$ as required. □

We can now prove the analogue of Theorem 3.4:

Theorem 5.3 *The σ-rank $\mathscr{R}_\sigma$ is isomorphic to $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ and the principal σ-rank $\mathscr{R}_\sigma^{\mathrm{pr}}$ is isomorphic to the subset of $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ of initial segments which have a last element.*[2]

Proof First we note that if K_w is a convex σ-compatible valuation ring, then clearly $K_w^{\succ 0} \setminus K_v^{\succ 0}$ is an initial segment of $\mathbf{P}_K$. Furthermore, if K_w intersects a σ-equivalence class $[a]_{\sim_\sigma}$ then it must contain it, since the sequence $\sigma(a)^n; n \in \mathbb{N}_0$ is cofinal in $[a]_{\sim_\sigma}$ and K_w is a convex subring. We conclude that $(K_w^{\succ 0} \setminus K_v^{\succ 0})/{\sim_\sigma}$ is an initial segment of $\mathbf{P}_K/{\sim_\sigma}$ and moreover $[a]_{\sim_\sigma}$ is the last class in case K_w is σ- principal generated by a. Conversely set $\mathscr{I}_w = \{[a]_\sigma \mid a \in K_w^{\succ 0} \setminus K_v^{\succ 0}\}$. Given $\mathscr{I} \in [\mathbf{P}_K/{\sim_\sigma}]^{\mathrm{is}}$, we show that there is a σ-compatible convex valuation ring K_w such that $\mathscr{I}_w = \mathscr{I}$. Given $\mathscr{I}$, let $(\bigcup \mathscr{I})$ denote the set theoretic union of the elements of $\mathscr{I}$ and $-(\bigcup \mathscr{I})$ the set of additive inverses. Set $K_w = -(\bigcup \mathscr{I}) \cup K_v \cup (\bigcup \mathscr{I})$. We claim that K_w is the required ring. Clearly, $\mathscr{I}_w = \mathscr{I}$. Further K_w is convex (by its construction), and strictly contains K_v. We leave it to the reader, using Lemma 5.2, to verify that K_w is a σ-compatible subring, and that K_w is σ-principal generated by a if $[a]_{\sim\sigma}$ is the last element of $\mathscr{I}$. □

We now deduce from this theorem combined with Remark 5.1 the promised analogues of Theorems 2.7 and 2.12 respectively:

Corollary 5.4 *$\mathscr{R}_\sigma$ is (isomorphic to) $(\Gamma/\sim_{\sigma_\Gamma})^{\mathrm{fs}}$.*

Corollary 5.5 *$\mathscr{R}_\sigma^{\mathrm{pr}}$ is (isomorphic to) $(\Gamma/\sim_{\sigma_\Gamma})^*$.*

We call the order type of $(\Gamma/\sim_{\sigma_\Gamma})$ the **rank** of the automorphism σ_Γ . We now can construct ω-increasing automorphisms of arbitrary principal difference rank. Corollary 5.6 below, compared to Corollary 4.11 demonstrates the discrepancy between the chains $\mathscr{R}_\sigma^{\mathrm{pr}}$ and $\mathscr{R}_\sigma \cap \mathscr{R}^{\mathrm{pr}}$.

[2]Note that the subchain of $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ of initial segments which have a last element is isomorphic to $[\mathbf{P}_K/\sim_\sigma]$ itself.

Corollary 5.6 *Given any order type τ there exists a maximally valued ordered field endowed with an ω-increasing automorphism of principal difference rank τ.*

Proof Set $\mu := \tau^*$, and consider, e.g. the linear ordering $\Gamma := \sum_\mu \mathbb{Q}$, that is, the concatenation of μ copies of the non-negative rationals. Let ℓ be, e.g., translation by -1 on $\mathbb{Q}$. Define σ_Γ to be the uniquely defined order automorphism of Γ extending ℓ on every copy. It is clearly a strict left shift of rank μ. Set, e.g. $G := \mathbf{H}_\Gamma\mathbb{R}$. Then by Lemma 4.12 σ_Γ lifts canonically to σ_G on G. Now set, e.g. $K := \mathbb{R}((G))$. By Lemma 4.12, Remark 4.10 and Corollary 5.5, σ_G lifts canonically to an ω-increasing automorphism of K of principal difference rank $\mu^* = \tau$. □

Example 5.7 Consider the chain $\Gamma = \mathbb{Z} \times \mathbb{Z}$ (the lexicographic product of two copies of $\mathbb{Z}$). We endow Γ with the automorphisms $\tau((x, y)) := (x - 1, y)$ and $\sigma((x, y)) := (x, y - 1)$. The rank of τ is one and that of σ is $\mathbb{Z}$. Both are strict left shifts. Lifting those automorphisms to $G := \mathbf{H}_\Gamma\mathbb{R}$ and then to $K := \mathbb{R}((G))$ as in the proof of Corollary 5.6, we obtain ω-increasing automorphisms of K of distinct principal difference ranks.

For a regular uncountable cardinal κ, let us denote by G_κ the κ-bounded Hahn group, that is, the subgroup of $G = \mathbf{H}_\Gamma\mathbb{R}$ consisting of elements with support of cardinality $< \kappa$. Similarly, we denote by $\mathbb{R}((G))_\kappa$ the κ-bounded Hahn field, i.e. the subfield of $K = \mathbb{R}((G))$ consisting of series with support of cardinality $< \kappa$. If $\kappa = \kappa^{<\kappa}$ then $\mathbb{R}((G_\kappa))_\kappa$ has cardinality κ, see [1].

We now generalize Example 5.7. In [18, Corollary 14], we construct for every infinite cardinal κ a chain Γ of cardinality κ which admits of family of 2^κ strict left shift automorphisms, of pairwise distinct ranks. Lifting those automorphisms to $\mathbb{R}((G_\kappa))_\kappa$, we conclude as in [18, Theorem 9]:

Theorem 5.8 *Let $\kappa = \kappa^{<\kappa}$ be a regular uncountable cardinal and Γ be any chain of cardinality κ which admits a family of 2^κ strict left shift automorphisms of pairwise distinct ranks. Then the corresponding κ-bounded Hahn field $\mathbb{R}((G_\kappa))_\kappa$ of cardinality κ admits a family of 2^κ ω-increasing automorphisms of distinct principal difference ranks.*

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The Lattice of U-Sequences of an Abelian p-Group

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Abstract Let G be a reduced abelian p-group. In a rare blemish in Kaplansky's monograph *Infinite abelian groups* it is stated that the supremum of a finite number of U-sequences of G is taken pointwise. We provide an algorithm to show how the supremum of an arbitrary set of U-sequences should be calculated and use it to show that the lattice of U-sequences is distributive. This enables us to correct the proof of Kaplansky's result that, when G is fully transitive, its lattice of fully invariant subgroups is distributive. We also prove, even when G is not fully transitive, that its lattice of large subgroups is distributive and we extend many of these results to non-reduced groups.

Keywords Abelian p-group • U-sequence • Lattice of U-sequences • Lattice of fully invariant subgroups

1 Introduction

The U-sequences of an abelian p-group, G, were introduced by Kaplansky in [4]. When G is reduced, the case that will mostly concern us here, a strictly increasing sequence of ordinals (each less than the length of G) possibly followed by ∞ symbols is called a *U-sequence of G* if it satisfies the gap condition that

$$\text{whenever } \alpha_i + 1 < \alpha_{i+1}, \text{ we have } f_G(\alpha_i) \neq 0. \tag{1}$$

Here, for each ordinal α,

$$f_G(\alpha) = \dim_{\mathbb{F}_p}\{(p^\alpha G)[p]/(p^{\alpha+1}G)[p]\}$$

is the classical Ulm invariant of G at α, with $\mathbb{F}_p$ being the field of p elements. Kaplansky showed that, under the natural pointwise ordering, the U-sequences of G form a complete lattice in which the infimum of any number of U-sequences

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is taken pointwise [4, Lemma 26]. Finding their supremum is trickier, however, and Kaplansky gave an example [4, Exercise 74, p. 65] of an infinite set of U-sequences whose supremum is not taken pointwise. In a most unusual mistake in this beautiful monograph it is stated at the foot of p.60 that finite suprema are taken pointwise. (The mistake also invades Exercise 76 of [4]. A counterexample was given in Remark 17.18, p. 2979 of [3] and others are given below.) Remarkably, there seems to be no clear published algorithm for calculating the supremum, $\bigvee \boldsymbol{u_i}$, of an arbitrary set of U-sequences, $\boldsymbol{u_i}$, and one of the aims of this paper is to provide such an algorithm. We examine the special case of finding the supremum of a pair of U-sequences $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ and use this to prove that the lattice of U-sequences of an abelian p-group is distributive. One corollary establishes the truth of Kaplansky's result in Theorem 25 of [4] that the lattice of fully invariant subgroups of a fully transitive group is indeed distributive. (His own proof relied on suprema being taken pointwise.) A second corollary is that the lattice of large subgroups of an arbitrary abelian p-group is distributive. Most of the present paper is concerned with reduced groups, but a final section extends many results to non-reduced groups. A subsequent paper will show how a method of finding $\boldsymbol{u_1} \vee \boldsymbol{u_2}$ can be modified to construct covers of elements in the U-sequence lattice and to describe all the characteristic subgroups of a transitive abelian 2-group.

Our terminology and notation is consistent with that of [2] and [4]. Recall that the *length* of an abelian p-group, G, is the least ordinal λ such that $p^\lambda G = p^{\lambda+1} G$. Here $p^\lambda G$ is the maximal divisible subgroup of G and when this subgroup is zero, G is said to be *reduced*. It is useful to have a name for sequences that are akin to U-sequences but do not necessarily satisfy the gap condition (1). Here such sequences are referred to as V-sequences. Thus, when G is reduced, a *V-sequence* is any strictly increasing sequence of ordinals, each less than λ, possibly followed by ∞ symbols. An ordinal, α, will be called a *jump number of* G whenever the Ulm invariant $f_G(\alpha) \neq 0$. Let $\mathbf{v}$ be any sequence. Unless otherwise stated, we shall always denote its terms by $v_0, v_1, v_2, \ldots$. If $v_i + 1 < v_{i+1}$, we say that $\boldsymbol{v}$ *jumps* from v_i to v_{i+1} and that v_i is a *jump number of* $\boldsymbol{v}$. Thus a V-sequence, $\boldsymbol{v}$, is a U-sequence if and only if each jump number of $\boldsymbol{v}$ is a jump number of G. The non-negative integers will be denoted by $\mathbb{N}$.

2 Constructing Suprema of U-Sequences

Given a U-sequence, $\boldsymbol{u}$, of an abelian p-group G we can construct the set $G(\boldsymbol{u}) \equiv \{x \in G : U(x) \geq \boldsymbol{u}\}$. It is easy to verify that $G(\boldsymbol{u})$ is a fully invariant subgroup of G. Kaplansky's famous Theorem 25 of [4] shows that, when G is fully transitive, each of its fully invariant subgroups has the form $G(\boldsymbol{u})$, so that the lattice of fully invariant subgroups is anti-isomorphic to the lattice of U-sequences. The following simple example illustrates this anti-isomorphism and also provides a counterexample to Kaplansky's claim that the supremum of a finite number of U-sequences is taken pointwise.

Let $G = \mathbb{Z}(p) \oplus \mathbb{Z}(p^3)$ be the direct sum of cyclic groups of orders p and p^3. Then G is fully transitive with length 3 and jump numbers 0 and 2. The lattice of U-sequences and the inverted lattice of fully invariant subgroups of G are as shown:

U-sequences		Fully invariant subgroups	
$\{\infty, \infty, \ldots\}$		$\{0\}$	
$\{2, \infty, \infty\}$		p^2G	
$\{0, \infty, \ldots\}$	$\{1, 2, \infty, \ldots\}$	$G[p]$	pG
$\{0, 2, \infty, \ldots\}$		$G[p^2]$	
$\{0, 1, 2, \infty, \ldots\}$		G	

Note that if $\boldsymbol{u} = \{0, \infty, \ldots\}$ and $\boldsymbol{v} = \{1, 2, \infty, \ldots\}$, then $\boldsymbol{u} \wedge \boldsymbol{v} = \{0, 2, \infty, \ldots\}$ is taken pointwise, but the supremum of $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u} \vee \boldsymbol{v} = \{2, \infty, \ldots\}$, not the pointwise maximum $\{1, \infty, \ldots\}$, which is not a U-sequence as 1 is not a jump number of G.

To state our algorithm for calculating suprema, we need the following lemma and its important corollary. Both are true for reduced and non-reduced groups.

Lemma 2.1 *Let G be an abelian p-group of length λ and let σ be an ordinal such that $\sigma + n < \lambda$ for all positive integers n. Then infinitely many of the Ulm invariants*

$$f_G(\sigma), f_G(\sigma + 1), \ldots, f_G(\sigma + n), \ldots \tag{2}$$

are non-zero.

Proof As $p^{\sigma+n}G \neq 0$ for all integers n, $p^{\sigma}G$ is unbounded. Let B be a basic subgroup of $p^{\sigma}G$. Then B contains cyclic summands of arbitrarily large orders. Each of these summands is bounded and pure in B, so is pure in $p^{\sigma}G$. Hence each is a summand of $p^{\sigma}G$, so infinitely many of the invariants

$$f_{p^{\sigma}G}(0), f_{p^{\sigma}G}(1), f_{p^{\sigma}G}(2), \ldots, f_{p^{\sigma}G}(n), \ldots$$

are non-zero. But these latter invariants are precisely those in (2) above. □

Corollary 2.2 *Let G be an abelian p-group and let σ be any ordinal less than the length of G. Then G has a jump number that is no less than σ. Moreover, if α is the least such jump number, then there is an integer $n \geq 0$ such that $\alpha = \sigma + n$.*

Proof If $p^{\sigma}G$ is unbounded, the result follows from the proof of Lemma 2.1. If $p^{\sigma}G$ is bounded, then there is an integer $m \geq 0$ such that $p^{\sigma+m}G \neq 0$ and $p^{\sigma+m+1}G = 0$, so $\sigma + m$ is a jump number of G, giving the result. □

For the remainder of this section G will always be a reduced p-group.

There are several different algorithms for calculating the supremum, $\bigvee \boldsymbol{u}_i$, of an arbitrary set of U-sequences, $\boldsymbol{u}_i$. They all begin with the same first three simple steps. Their fourth steps differ, with more efficient algorithms being more complicated to describe. Here we choose simplicity and state a procedure, **Algorithm A**, that makes least demands on the reader.

Step 1: Form the pointwise supremum, $\boldsymbol{m} = \max \boldsymbol{u}_i$.
Step 2: Construct the least strictly increasing sequence of ordinals and ∞ symbols that is pointwise no less than $\boldsymbol{m}$.
Step 3: Replace any ordinals greater than or equal to the length, λ , of G by ∞ symbols. Call the resulting sequence $\boldsymbol{v}$.

For example, let $G = H_{\omega+1} \oplus H_{\omega+2}$, where H_σ is the generalized Prüfer group of length σ. Then G has length $\omega + 2$ and its jump numbers consist of all the non-negative integers together with ω and $\omega + 1$. For each $i \in \mathbb{N}$, let $\boldsymbol{u}_i = \{i, i+3, i+7, \infty, \infty, \ldots\}$.
Step 1 gives $\boldsymbol{m} = \{\omega, \omega, \omega, \infty, \infty, \ldots\}$. Step 2 yields the sequence $\{\omega, \omega+1, \omega+2, \infty, \infty, \ldots\}$.
Step 3 gives the sequence $\boldsymbol{v} = \{\omega, \omega+1, \infty, \ldots\}$. In this particular example, $\boldsymbol{v} = \bigvee \boldsymbol{u_i}$, but **usually a fourth step is needed.**

We pause here to observe that in all cases $\boldsymbol{v}$ is a V-sequence as defined near the end of the introduction. It is easy to see that $\bigvee \boldsymbol{u}_i$ is the least U-sequence that is (pointwise) no less than $\boldsymbol{v}$. Thus the problem of calculating suprema reduces to that of finding the least U-sequence $\boldsymbol{u}$ that is no less than a given V-sequence $\boldsymbol{v}$ of G. Note too that if $G(\boldsymbol{v}) = \{x \in G : U(x) \geq \boldsymbol{v}\}$, then $G(\boldsymbol{v})$ is the fully invariant subgroup $G(\boldsymbol{u})$. See also Exercise 5, p. 13 of [2] Volume II, where the printed inequality should read $\boldsymbol{u} \geq \boldsymbol{v}$.

We are now ready to carry out the fourth step of Algorithm A.

Step 4: Let $\boldsymbol{v} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be a V-sequence of G. We wish to construct the least U-sequence $\boldsymbol{u} = \{u_0, u_1, u_2, \ldots\}$ such that $\boldsymbol{v} \leq \boldsymbol{u}$. To avoid triviality we may suppose that α_0 is an ordinal less than the length, λ, of G. (Otherwise $\boldsymbol{v}$ is already the U-sequence of ∞ symbols.) Let γ_1 be the least jump number of G such that $\alpha_0 \leq \gamma_1$. Corollary 2.2 ensures that γ_1 exists and that there is an integer $k_1 \geq 0$ such that $\gamma_1 = \alpha_0 + k_1$. Hence there is a unique integer $n(1) \geq 0$ such that $\alpha_{n(1)} \leq \gamma_1 < \alpha_{n(1)+1}$. Now $\alpha_0 + n(1) \leq \alpha_{n(1)} \leq \alpha_0 + k_1$, so $n(1) \leq k_1$. Take $\alpha_0 + \{k_1 - n(1)\}, \alpha_0 + \{k_1 - n(1)\} + 1, \ldots, \gamma_1$ to be the leading terms $u_0, u_1, \ldots, u_{n(1)}$ of $\boldsymbol{u}$ and define

$$\begin{aligned}\boldsymbol{v}_1 &= \{\alpha_0 + \{k_1 - n(1)\}, \alpha_0 + \{k_1 - n(1)\} + 1, \ldots, \gamma_1, \alpha_{n(1)+1}, \alpha_{n(1)+2}, \ldots\} \\ &= \{u_0, u_1, \ldots, u_{n(1)}, \alpha_{n(1)+1}, \alpha_{n(1)+2}, \ldots\}.\end{aligned}$$

We can think of this process as placing γ_1 in position $n(1)$ and backtracking by subtracting 1 at a time until we reach position zero at the start of the V-sequence $\boldsymbol{v_1}$.

We now treat $\alpha_{n(1)+1}$ in the same way as α_0 above to produce the next run of consecutive ordinals of $\boldsymbol{u}$. Continuing in this way, we produce the terms of $\boldsymbol{u}$ inductively. At the ith stage we let γ_i be the least jump number of G such that $\alpha_{n(i-1)+1} \leq \gamma_i$. Corollary 2.2 shows that $\gamma_i = \alpha_{n(i-1)+1} + k_i$ for some $k_i \in \mathbb{N}$, so there is a unique integer $n(i)$ such that $n(i-1)+1 \leq n(i)$ and $\alpha_{n(i)} \leq \gamma_i < \alpha_{n(i)+1}$. Arguing as above, we get $n(i) - n(i-1) - 1 \leq k_i$, so we may take the consecutive ordinals $\alpha_{n(i-1)+1} + \{k_i - n(i) + n(i-1) + 1\}, \ldots, \gamma_i$ as $u_{n(i-1)+1}, \ldots, u_{n(i)}$. This can be thought of as placing γ_i in position $n(i)$ and backtracking by subtracting 1 at a time until we come to the place we had reached in the previous iteration. An example below illustrates how the desired U-sequence $\boldsymbol{u} = \bigvee \boldsymbol{u_i}$ is built up. Meanwhile we define $\boldsymbol{v_i}$ to be

$$\begin{aligned}\boldsymbol{v_i} &= \{\alpha_0 + \{k_1 - n(1)\}, \dots, \gamma_1; \alpha_{n(1)+1} + \{k_2 - n(2) + n(1) + 1\}, \dots, \gamma_2; \dots \\ &\quad \dots; \alpha_{n(i-1)+1} + \{k_i - n(i) + n(i-1) + 1\}, \dots, \gamma_i; \alpha_{n(i)+1}, \alpha_{n(i)+2}, \dots\} \\ &= \{u_0, \dots, u_{n(1)}; u_{n(1)+1}, \dots, u_{n(2)}; \dots \\ &\quad \dots; u_{n(i-1)+1}, \dots, u_{n(i)}; \alpha_{n(i)+1}, \alpha_{n(i)+2}, \dots\}\end{aligned}$$

where the ellipses immediately preceding the terms $\gamma_1, \gamma_2, \dots, \gamma_i$ denote runs of consecutive ordinals.

For example, let $G = \mathbb{Z}(p^2) \oplus \mathbb{Z}(p^3) \oplus \mathbb{Z}(p^5) \oplus \mathbb{Z}(p^9) \oplus \mathbb{Z}(p^{17}) \oplus \mathbb{Z}(p^{33}) \oplus \dots \oplus \mathbb{Z}(p^{2^k+1}) \oplus \dots$ The jump numbers of G are $1, 2, 4, 8, 16, 32, \dots, 2^k, \dots$

Define U-sequences $\boldsymbol{u_n}$ for $n \in \mathbb{N}$ as follows. Let $\boldsymbol{u_0} = \{1, 2, 3, 4, \dots\}$.

If $n = 2^i$ for some $i \in \mathbb{N}$, let $\boldsymbol{u_n} = \{1, 2, \dots, n, 2n + 1, 2n + 2, 2n + 3, \dots\}$.
If $n = 2^i + j$ for some $i, j \in \mathbb{N}$ such that $1 \le j < 2^i$, then let

$$\boldsymbol{u_n} = \{1, 2, 3, \dots, 2^i, 2^{i+1} - j + 1, 2^{i+1} - j + 2, \dots, 2^{i+1}, 2n+1, 2n+2, 2n+3, \dots\}.$$

Thus

$$\begin{aligned}
\boldsymbol{u_0} &= \{1, 2, 3, 4, 5, \;\; 6, \;\; 7, \;\; 8, \;\; 9, 10, 11, \dots\} \\
\boldsymbol{u_1} &= \{1, 3, 4, 5, 6, \;\; 7, \;\; 8, \;\; 9, 10, 11, 12, \dots\} \\
\boldsymbol{u_2} &= \{1, 2, 5, 6, 7, \;\; 8, \;\; 9, 10, 11, 12, 13, \dots\} \\
\boldsymbol{u_3} &= \{1, 2, 4, 7, 8, \;\; 9, 10, 11, 12, 13, 14, \dots\} \\
\boldsymbol{u_4} &= \{1, 2, 3, 4, 9, 10, 11, 12, 13, 14, 15, \dots\} \\
\boldsymbol{u_5} &= \{1, 2, 3, 4, 8, 11, 12, 13, 14, 15, 16, \dots\} \\
\boldsymbol{u_6} &= \{1, 2, 3, 4, 7, \;\; 8, 13, 14, 15, 16, 17, \dots\} \\
\boldsymbol{u_7} &= \{1, 2, 3, 4, 6, \;\; 7, \;\; 8, 15, 16, 17, 18, \dots\} \\
\boldsymbol{u_8} &= \{1, 2, 3, 4, 5, \;\; 6, \;\; 7, \;\; 8, 17, 18, 19, \dots\} \\
\boldsymbol{u_9} &= \{1, 2, 3, 4, 5, \;\; 6, \;\; 7, \;\; 8, 16, 19, 20, \dots\} \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}$$

It is easy to check that Step 1 gives $\boldsymbol{m} = \max \boldsymbol{u_n} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, \dots\}$. This pointwise maximum jumps after each odd integer, so is far from being a U-sequence, as the jump numbers of G are powers of 2. Steps 2 and 3 leave $\boldsymbol{m}$ unchanged, so

$$\boldsymbol{v} = \boldsymbol{m} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, \dots\}.$$

Applying Step 4, we have

$\alpha_0 = 1$. As $\gamma_1 = 1$ is in position $n(1) = 0$ of $\boldsymbol{m}$, put it in position 0 of $\vee \boldsymbol{u}_n$.

$\vee \boldsymbol{u_n}$ starts $\{1;$

$\alpha_1 = 3$. As $\gamma_2 = 4$ lies between positions $n(2) = 1$ and $n(2) + 1 = 2$, put it in position 1.

$\vee \boldsymbol{u_n}$ starts $\{1; 4;$

$\alpha_2 = 5$. As $\gamma_3 = 8$ lies between positions $n(3) = 3$ and $n(3) + 1 = 4$, put it in position 3 and work back through consecutive ordinals.

$\vee \boldsymbol{u_n}$ starts $\{1; 4; 7, 8;$

$\alpha_4 = 9$. As $\gamma_4 = 16$ lies between positions $n(4) = 7$ and $n(4) + 1 = 8$, put it in position 7 and work back through consecutive ordinals.

$\vee \boldsymbol{u_n}$ starts $\{1; 4; 7, 8; 13, 14, 15, 16;$

$\alpha_8 = 17$. As $\gamma_5 = 32$ lies between positions $n(5) = 15$ and $n(5) + 1 = 16$, put it in position 15 and work back through consecutive ordinals.

$\vee \boldsymbol{u_n}$ starts $\{1; 4; 7, 8; 13, 14, 15, 16; 25, 26, 27, 28, 29, 30, 31, 32;$

Continuing in this way, we get more and more terms of the supremum $\boldsymbol{u} = \vee \boldsymbol{u_n}$.

Theorem 2.3 *With the notation of Step 4 above, $\boldsymbol{u}$ is the least U-sequence of G such that $\boldsymbol{v} \le \boldsymbol{u}$.*

Proof The construction of $\boldsymbol{u}$ ensures that its only jump numbers are jump numbers of G, so $\boldsymbol{u}$ is a U-sequence. Suppose that there is a U-sequence $\boldsymbol{w}$ of G such that $\boldsymbol{v} \le \boldsymbol{w} \le \boldsymbol{u}$. It is sufficient to prove that $\boldsymbol{v_i} \le \boldsymbol{w}$ for all i, for this would imply that $\boldsymbol{u} \le \boldsymbol{w}$, so that $\boldsymbol{w} = \boldsymbol{u}$.

We begin by showing that $\boldsymbol{v_1} \le \boldsymbol{w}$. If $\gamma_1 = \alpha_{n(1)}$ and α_0 is a jump number of G, then $n(1) = 0$ and $u_0 = \gamma_1 = \alpha_0 \le w_0$, so $\boldsymbol{v_1} \le \boldsymbol{w}$.

If $\gamma_1 = \alpha_{n(1)}$ and α_0 is not a jump number of G, then no ordinal, δ, in the range $\alpha_0 \le \delta < \gamma_1$ is a jump number of G. As $\alpha_0 \le w_0$ and

$$\alpha_0 \le \alpha_{n(1)-1} \le w_{n(1)-1} \le u_{n(1)-1} < u_{n(1)} = \gamma_1,$$

none of the ordinals $w_0, \ldots, w_{n(1)-1}$ is a jump number of G. This and the fact that $\boldsymbol{w}$ is a U-sequence shows that $w_{n(1)} = w_j + \{n(1) - j\}$ for all integers j in the range $0 \le j \le n(1)$. But $\gamma_1 = \alpha_{n(1)} \le w_{n(1)} \le u_{n(1)} = \gamma_1$, so $w_{n(1)} = \gamma_1$ and $w_j = \alpha_0 + \{k_1 - n(1)\} + j = u_j$, which gives $\boldsymbol{v_1} \le \boldsymbol{w}$.

If $\alpha_{n(1)} < \gamma_1$, then no ordinal, δ, in the range $\alpha_0 \le \delta < \gamma_1$ is a jump number of G and, as above, $w_{n(1)} = w_j + \{n(1) - j\}$ for all j such that $0 \le j \le n(1)$. Since $\gamma_1 < \alpha_{n(1)+1}$, we have

$$\alpha_0 \le w_0 \le w_{n(1)} \le u_{n(1)} = \gamma_1 < \alpha_{n(1)+1} \le w_{n(1)+1}.$$

If $w_{n(1)} < \gamma_1$, then $w_{n(1)}$ would not be a jump number of G, yet $\boldsymbol{w}$ would jump from $w_{n(1)}$ to $w_{n(1)+1}$, contradicting the fact that $\boldsymbol{w}$ is a U-sequence. Hence $w_{n(1)} = \gamma_1$, so

$$w_j + \{n(1) - j\} = w_{n(1)} = \gamma_1 = u_{n(1)} = u_j + \{n(1) - j\}$$

for $0 \le j \le n(1)$, whence $w_j = u_j$ and $\boldsymbol{v_1} \le \boldsymbol{w}$ as desired.

Similar arguments show in succession that $\boldsymbol{v_2} \le \boldsymbol{w}, \ldots, \boldsymbol{v_i} \le \boldsymbol{w}$ for all i. This gives $\boldsymbol{u} \le \boldsymbol{w}$, so $\boldsymbol{w} = \boldsymbol{u}$ and the proof is complete. □

Whilst Algorithm A described in Steps 1–4 above enables us to find the supremum of an arbitrarily large family of U-sequences, a different procedure, **Algorithm B**, is often useful for finding the supremum, $\boldsymbol{a} \vee \boldsymbol{b}$, of a pair of U-sequences $\boldsymbol{a}$ and $\boldsymbol{b}$. In this situation, $\boldsymbol{m} = \max(\boldsymbol{a}, \boldsymbol{b})$ is necessarily a V-sequence, for it is strictly increasing and each of its ordinals is less than the length of G. Suppose that $\boldsymbol{m} \equiv \{m_0, m_1, m_2, \ldots\}$ is not already a U-sequence and that its gaps occur immediately after the terms $m_{n(1)}, m_{n(2)}, \ldots$. Defining $n(0) = -1$ so that $m_{n(0)+1} = m_0$, we see that, apart from perhaps ending in ∞ symbols, the terms of $\boldsymbol{m}$ fall into blocks of consecutive ordinals, with a typical block running from $m_{n(i-1)+1}$ to $m_{n(i)}$. If possible,

let β_i be the greatest jump number of G such that $m_{n(i-1)+1} \le \beta_i < m_{n(i)}$
and γ_i be the least jump number of G such that $m_{n(i)} < \gamma_i < m_{n(i)+1}$.
If β_i exists, then there is an integer s such that $\beta_i = m_s$.

To construct $\boldsymbol{a} \vee \boldsymbol{b}$ from $\boldsymbol{m}$, we leave unchanged each block of terms $m_{n(i-1)+1}, \ldots, m_{n(i)}$ that ends in a jump number of G and replace each block for which $m_{n(i)}$ is not a jump number of G by

$$\begin{cases} m_{n(i-1)+1}, \ldots, \beta_i, \gamma_i - n(i) + s + 1, \ldots, \gamma_i & \text{if } \exists \beta_i \text{ and } \exists \gamma_i \\ m_{n(i-1)+1}, \ldots, \beta_i, m_{n(i)+1} - n(i) + s, \ldots, m_{n(i)+1} - 1 & \text{if } \exists \beta_i \text{ and } \nexists \gamma_i \\ \gamma_i - n(i) + n(i-1) + 1, \ldots\ldots\ldots\ldots, \gamma_i, & \text{if } \nexists \beta_i \text{ and } \exists \gamma_i \\ m_{n(i)+1} - n(i) + n(i-1), \ldots\ldots\ldots\ldots, m_{n(i)+1} - 1 & \text{if } \nexists \beta_i \text{ and } \nexists \gamma_i \end{cases}$$

In the special case when the block consists of the single term $m_{n(i)}$, we replace it by γ_i if this exists and by $m_{n(i)+1} - 1$ otherwise. Lemma 2.1 and Corollary 2.2 imply that when $m_{n(i)}$ is not a jump number of G, either γ_i exists together with all lower ordinals down to $\gamma_i - n(i) + n(i-1) + 1$, or $m_{n(i)+1}$ is an ordinal and each of the ordinals $m_{n(i)+1} - n(i) + n(i-1), \ldots, m_{n(i)+1} - 1$ exists.

Before proving that Algorithm B gives $\boldsymbol{a} \vee \boldsymbol{b}$ as desired, we give an example of its use, showing that each of the four possibilities for the existence or non-existence of β_i and γ_i can actually arise.

Let $G = \mathbb{Z}(p^3) \oplus \mathbb{Z}(p^4) \oplus \mathbb{Z}(p^8) \oplus \mathbb{Z}(p^{10}) \oplus (\mathbb{Z}(p^{13}) \oplus \mathbb{Z}(p^{14}) \oplus \mathbb{Z}(p^{21}) \oplus \mathbb{Z}(p^{28}) \oplus \mathbb{Z}(p^{29}) \oplus \mathbb{Z}(p^{34})$, which has length 34 and jump numbers 2, 3, 7, 9, 12, 13, 20, 27, 28 and 33. Let $\boldsymbol{a}, \boldsymbol{b}$ be the U-sequences

$$\boldsymbol{a} = \{0, 1, 2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 29, 30, 31, 32, 33, \infty, \ldots\}$$

$$\boldsymbol{b} = \{1, 2, 3, 4, \ \ 5, \ \ 6, \ \ 7, \ \ 8, \ \ 9, 19, 20, 23, 24, 25, 26, 27, 33, \infty, \ldots\}$$

Their pointwise maximum, $\boldsymbol{m}$, is given by

$$\boldsymbol{m} = \{1, 2, 3, 4; 11, 12, 13, 14, 15; 19, 20; 23, 24, 25; 29, 30; 33; \infty, \ldots\}$$

Here we have six blocks of consecutive ordinals (marked by semicolons in the display). The jump numbers of $\boldsymbol{m}$ are $m_{n(1)} = 4, m_{n(2)} = 15, m_{n(3)} = 20, m_{n(4)} = 25, m_{n(5)} = 30$ and $m_{n(6)} = 33$. Of these six values of $m_{n(i)}$, only those given by $i = 3$ and $i = 6$ are jump numbers of G. Hence the blocks corresponding to $i = 1, 2, 4$ and 5 need to be replaced.

When $i = 1$, we have $\beta_i = 3$, since 3 is the greatest jump number of G amongst the terms 1, 2, 3. Also $\gamma_1 = 7$, as this is the least jump number of G between 4 and 11. Thus we replace the block of terms 1, 2, 3, 4 by 1, 2, 3, 7.

When $i = 2$, we examine the block of terms 11, 12, 13, 14, 15. As 13 is the greatest jump number of G amongst the terms 11, 12, 13, 14, we have $\beta_2 = 13$. There is no jump number of G between 15 and 19, so γ_2 does not exist. Hence we replace the block 11, 12, 13, 14, 15 by 11, 12, 13, 17, 18.

When $i = 4$, β_i does not exist, since neither 23 nor 24 is a jump number of G, but $\gamma_4 = 27$, for this is the least jump number of G between 25 and 29. Hence we replace the block 23, 24, 25 by 25, 26, 27.

When $i = 5$, β_i does not exist, since 29 is not a jump number of G, and γ_5 does not exist, as there is no jump number of G between 30 and 33. Thus we replace the block 29, 30 by 31, 32.

Leaving unchanged the two blocks 19, 20 and 33 that end in jump numbers of G, we get

$$\boldsymbol{a} \vee \boldsymbol{b} = \{1, 2, 3, 7, 11, 12, 13, 17, 18, 19, 20, 25, 26, 27, 31, 32, 33, \infty, \ldots\}.$$

If $\boldsymbol{u}$ is the result of applying Algorithm B above to $\boldsymbol{m}$, it is not immediately clear that $\boldsymbol{u}$ is a U-sequence, let alone that it is equal to $\boldsymbol{a} \vee \boldsymbol{b}$. Certainly $\boldsymbol{u}$ will be a U-sequence unless all three of the following conditions are satisfied.

(a) $\boldsymbol{m}$ has successive jump numbers $m_{n(i)}$ and $m_{n(i+1)}$, neither of which is a jump number of G,
(b) neither γ_i nor β_{i+1} exists, and
(c) $m_{n(i)+1} - 1$ is not a jump number of G.

In these circumstances (a) shows that each block of terms ending in $m_{n(i)}$ and in $m_{n(i)+1}$ must be replaced. The non-existence of γ_i shows that the last term in the first block's replacement is $m_{n(i)+1} - 1$, whilst the non-existence of β_{i+1} shows that the first term in the next block's replacement exceeds $m_{n(i)+1}$. These facts coupled with (c) would prevent $\boldsymbol{u}$ from being a U-sequence. But our next result shows that (a), (b) and (c) cannot all be satisfied when $\boldsymbol{m}$ is the pointwise maximum of just two U-sequences.

Lemma 2.4 *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be U-sequences of an abelian p-group G. Suppose that $\boldsymbol{m} = \max(\boldsymbol{a}, \boldsymbol{b})$ has successive jumps immediately after the ordinals $m_{n(i)}$ and $m_{n(i+1)}$ and that neither of these ordinals is a jump number of G. Then γ_i exists or β_{i+1} exists.*

Proof Suppose (as we may) that $a_{n(i)} \leq b_{n(i)}$, so that $m_{n(i)} = b_{n(i)}$. As $\boldsymbol{m}$ jumps immediately after this term without $m_{n(i)}$ being a jump number of G, $\boldsymbol{a}$ overtakes $\boldsymbol{b}$ at this point. so $a_{n(i)} < b_{n(i)}$ and $b_{n(i)+1} = b_{n(i)} + 1$, whilst

$$b_{n(i)+1} < a_{n(i)+1} = m_{n(i)+1}.$$

At $\boldsymbol{m}$'s next jump, $\boldsymbol{b}$ overtakes $\boldsymbol{a}$. We have $b_{n(i+1)} < a_{n(i+1)} = m_{n(i+1)}$ and $a_{n(i+1)+1} = a_{n(i+1)} + 1$ whilst

$$a_{n(i+1)+1} < b_{n(i+1)+1} = m_{n(i+1)+1}.$$

Now $b_{n(i+1)}$ is a jump number of the U-sequence $\boldsymbol{b}$, so must be a jump number of G. It lies in the range $m_{n(i)} = b_{n(i)} < b_{n(i+1)} < a_{n(i+1)} = m_{n(i+1)}$. Also $m_{n(i)} < m_{n(i)+1} \leq m_{n(i+1)}$. So either our jump number $b_{n(i+1)}$ lies in the range $m_{n(i)} < b_{n(i+1)} < m_{n(i)+1}$, in which case γ_i exists, or else it lies in the range $m_{n(i)+1} \leq b_{n(i+1)} < m_{n(i+1)}$, in which case β_{i+1} exists. □

Theorem 2.5 *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be U-sequences of an abelian p-group G. Let $\boldsymbol{m} = \max(\boldsymbol{a}, \boldsymbol{b})$ be their pointwise maximum and $\boldsymbol{u}$ be the result of applying Algorithm B to $\boldsymbol{m}$. Then $\boldsymbol{u} = \boldsymbol{a} \vee \boldsymbol{b}$.*

Proof Lemma 2.4 and the immediately preceding remarks show that $\boldsymbol{u}$ is a U-sequence. It is sufficient to prove that $\boldsymbol{u}$ is the least U-sequence that is no less than $\max(\boldsymbol{a}, \boldsymbol{b})$.

With the notation of Algorithm B, suppose that the jump number $m_{n(i)}$ of $\boldsymbol{m}$ is not a jump number of G. Let $\boldsymbol{u_i}$ be the result of replacing the single block of terms $m_{n(i-1)+1}, \ldots, m_{n(i)}$ as described in Algorithm B and leaving all other blocks unchanged. Clearly $\boldsymbol{u} = \max \boldsymbol{u_i}$ where this pointwise maximum is taken over all i for which $m_{n(i)}$ is not a jump number of G. Let $\boldsymbol{w}$ be a U-sequence such that $\boldsymbol{m} \leq \boldsymbol{w}$. It is sufficient to prove that $\boldsymbol{u_i} \leq \boldsymbol{w}$, for this would yield $\boldsymbol{u} \leq \boldsymbol{w}$ as desired.

Suppose first that $\beta_i (= m_s)$ exists. Then $(\boldsymbol{u_i})_j = m_j \leq w_j$ for all j such that $n(i-1) + 1 \leq j \leq s$. If, in addition, γ_i exists, then, as $m_{n(i)}$ is not a jump number of G, there is no such jump number between β_i and γ_i. If the set of terms

$$w_{s+1}, \ldots, w_{n(i)} \tag{3}$$

contains a least one, say w_t, that is a jump number of G, then

$$\beta_i = m_s < m_{s+1} \leq w_{s+1} \leq w_t,$$

so $\gamma_i \leq w_t$. Thus, when $s + 1 \leq j \leq t$, we have $\gamma_i \leq w_t = w_j + t - j$, so

$$(\boldsymbol{u_i})_j = \gamma_i - n(i) + j \leq w_j - n(i) + t \leq w_j,$$

whilst when $t \leq j \leq n(i)$ we have $(\boldsymbol{u_i})_j \leq (\boldsymbol{u_i})_{n_i} = \gamma_i \leq w_t \leq w_j$ as desired. On the other hand, , if none of the terms (3) is a jump number of G, then for $s+1 \leq j \leq n(i)$ we have $\gamma_i < m_{n(i)+1} \leq w_{n(i)+1} = w_j + n(i) - j + 1$, so $(\boldsymbol{u_i})_j = \gamma_i - n(i) + j \leq w_j$.

Slightly modified arguments show that $\boldsymbol{u_i} \leq \boldsymbol{w}$ when at least one of the ordinals β_i, γ_i does not exist. □

3 Distributivity

Kaplansky's Theorem 25 in [4] showed that when a reduced abelian p-group, G, is fully transitive, there is an anti-isomorphism between the lattice of its fully invariant subgroups and the lattice of its U-sequences. His proof that each lattice satisfies an appropriate infinite distributive law relied on the supremum of a finite number of U-sequences being taken pointwise. Even the supremum of two U-sequences is not taken in this way, so questions of distributivity are reopened. In Theorem 17.17 of [3], Grinshpon and Krylov established distributivity in the case when G is separable. Effectively, their neat proof uses a lattice (where both suprema and infima are taken pointwise) of certain maps from the jump numbers of a separable p-group, G, into $\mathbb{N}$. A modified version of their proof could be used to show that the lattice of large subgroups of an arbitrary p-group is distributive, a result proved in Corollary 4.4 below. We begin with the following general theorem that appears to be new.

Theorem 3.1 *Let G be a reduced abelian p-group, $\boldsymbol{a}$ be a U-sequence of G and let $\boldsymbol{b_r}(r \in I)$ be any collection of U-sequences where I is an indexing set. Then $\boldsymbol{a} \vee (\bigwedge \boldsymbol{b_r}) = \bigwedge(\boldsymbol{a} \vee \boldsymbol{b_r})$.*

Proof Let G have length τ. Denote by H_α the generalized Prüfer group of length α, let $L = \bigoplus_{\alpha \leq \tau} H_\alpha$ and $H = G \oplus L$. Then L and H have length τ and α is a jump number of both L and H for all ordinals α less than τ. Now suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are U-sequences of G. They are necessarily U-sequences of H, so we can talk about $\boldsymbol{u} \vee_G \boldsymbol{v}$ and $\boldsymbol{u} \vee_H \boldsymbol{v}$. Theorem 2.5 shows how to compute the first one, whilst the second is simply the pointwise maximum of $\boldsymbol{u}$ and $\boldsymbol{v}$.

If $\{\boldsymbol{u}_i\}_{i \in I}$ is any collection of U-sequences of G, then we can talk about $\bigwedge_G \boldsymbol{u}_i$ and $\bigwedge_H \boldsymbol{u}_i$. Each is equal to the pointwise minimum of the $\boldsymbol{u}_i$, so we do not have to specify the group using a subscript.

With $\boldsymbol{a}$ and $\boldsymbol{b_r}(r \in I)$ as in the statement of the theorem,

$$\boldsymbol{a} \vee_H (\bigwedge \boldsymbol{b_r}) = \bigwedge(\boldsymbol{a} \vee_H \boldsymbol{b_r}),$$

since each is computed using pointwise maxima and minima. Thus

$$H(\boldsymbol{a} \vee_H (\bigwedge \boldsymbol{b_r})) = H(\bigwedge(\boldsymbol{a} \vee_H \boldsymbol{b_r})),$$

whence

$$H(\boldsymbol{a}) \cap (\sum H(\boldsymbol{b_r})) = \sum (H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})).$$

So

$$G(\boldsymbol{a}) \cap (\sum G(\boldsymbol{b_r})) \subseteq G \cap H(\boldsymbol{a}) \cap (\sum H(\boldsymbol{b_r})) = G \cap (\sum (H(\boldsymbol{a}) \cap H(\boldsymbol{b_r}))). \quad (4)$$

As $H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})$ is a fully invariant subgroup of $G \oplus L$, we have

$$H(\boldsymbol{a}) \cap H(\boldsymbol{b_r}) = \{G \cap H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})\} \oplus \{L \cap H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})\}.$$

But $G \cap H(\boldsymbol{u}) = G(\boldsymbol{u})$ for all U-sequences $\boldsymbol{u}$ of G, so

$$H(\boldsymbol{a}) \cap H(\boldsymbol{b_r}) = \{G(\boldsymbol{a}) \cap G(\boldsymbol{b_r})\} \oplus \{L \cap H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})\}.$$

Hence $\quad G \cap \sum (H(\boldsymbol{a}) \cap H(\boldsymbol{b_r})) = \sum (G(\boldsymbol{a}) \cap G(\boldsymbol{b_r})).$

From (4), $\quad G(\boldsymbol{a}) \cap (\sum G(\boldsymbol{b_r})) \subseteq \sum (G(\boldsymbol{a}) \cap G(\boldsymbol{b_r})).$

The reverse inclusion is trivial, so

$$G(\boldsymbol{a} \vee_G (\bigwedge \boldsymbol{b_r})) = G(\boldsymbol{a}) \cap (\sum G(\boldsymbol{b_r})) = \sum (G(\boldsymbol{a}) \cap G(\boldsymbol{b_r})) = G(\bigwedge (\boldsymbol{a} \vee_G \boldsymbol{b_r})),$$

which gives the desired result. □

Corollary 3.2 *Let G be a reduced fully transitive abelian p-group. Then the lattice of fully invariant subgroups of G is distributive and satisfies the infinite distributive law $A \cap (\sum B_r) = \sum (A \cap B_r)$.*

4 Non-reduced Groups

To deal with a non-reduced p-group G of length λ, we follow Kaplansky [4, p. 57] and assign all non-zero elements in the maximal divisible subgroup $p^\lambda G$ the height λ. We also define the Ulm invariant $f_G(\lambda)$ to be $\dim_{\mathbb{F}_p}(p^\lambda G)[p]$ and modify our original definition of a U-sequence of G by allowing λ itself to be one of the terms and also letting the sequence level off at this height before possibly ending in a strings of ∞s. With these conventions we can now extend some earlier results to non-reduced groups.

It is useful to have information about the fully invariant subgroups of G.

Lemma 4.1 *Let $G = D \oplus R$, where $D = p^\lambda G \neq 0$ is divisible and R is reduced. If F is a fully invariant subgroup of G, then F is of the form (i) $F = D \oplus K$, where K*

is fully invariant in R or (ii) $F = D[p^r] \oplus M$ *where r is some natural number and M is some* p^r*-bounded fully invariant subgroup of R. Moreover, every such subgroup F of the form (i) or (ii) is fully invariant in G.*

Proof The only aspect of the proof that is not straightforward is to show that if $F \cap D = D[p^r]$, then $p^r M = p^r(F \cap R) = 0$. If this were not so, the reduced group $F \cap R$ would have a cyclic summand $\langle x \rangle$ of order p^n for some integer $n > r$. Choose a quasicyclic summand D_1 of D and define a map $\phi : (F \cap R) \to D_1$ by mapping $x \mapsto y$, where y generates $D_1[p^n]$, and ϕ annihilates the complement of $\langle x \rangle$. By the injectivity of D_1, ϕ extends to a map (which we continue to call ϕ) from R into $D_1 \subseteq D$. As $n > r$, we have $y \notin D[p^r]$. But this is impossible since $y \in \phi(F \cap R)$ but $y \notin F$. □

Let D and R be as in Lemma 4.1. Corresponding to each U-sequence $\boldsymbol{u}$ of G there is a fully invariant subgroup F of G given by

$$F = G(\boldsymbol{u}) = \{x \in G : U(x) \geq \boldsymbol{u}\}.$$

A key observation is that, provided we include certain special cases stated below, any U-sequence of G has the form

$$\boldsymbol{u} = \{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \underbrace{\lambda, \lambda, \ldots, \lambda}_{n}, \infty, \infty, \ldots\}$$

Here there are m ordinals α_i less than λ and n copies of λ. As special cases we allow $m = 0$, $n = 0$, $m = \infty$ (in which case $n = 0$) and $n = \infty$. Note that the jump numbers of G are precisely those of R augmented by the ordinal λ. Note too that α_{m-1} is necessarily a jump number of G, for either $\alpha_{m-1} + 1 = \lambda$, in which case α_{m-1} is the greatest jump number of R or $\alpha_{m-1} + 1 < \lambda$ and the result follows from the gap condition for $\boldsymbol{u}$. This implies that

$$\boldsymbol{v} = \{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}, \infty, \infty, \ldots\}$$

is a U-sequence of R,

$$\boldsymbol{w} = \{\underbrace{\lambda, \lambda, \ldots, \lambda}_{m+n}, \infty, \infty, \ldots\}$$

is a U-sequence of D, $G(\boldsymbol{u}) \cap R = R(\boldsymbol{v})$ and $G(\boldsymbol{u}) \cap D = D(\boldsymbol{w})$. Hence the lattice of all subgroups of the form $G(\boldsymbol{u})$ embeds in the product of the lattice of subgroups of R of the form $R(\boldsymbol{v})$ with that of subgroups of D of the form $D(\boldsymbol{w})$. Now the lattice of $R(\boldsymbol{v})$'s is anti-isomorphic to the lattice of U-sequences of R, so by Theorem 3.1, it satisfies the appropriate infinite distributive law. The lattice of $D(\boldsymbol{w})$'s is a chain, so it and the lattice of subgroups $G(\boldsymbol{u})$ also satisfy this law. This enables us to extend Theorem 3.1 as follows.

Theorem 4.2 *Let G be an arbitrary abelian p-group, $\boldsymbol{a}$ be a U-sequence of G and let $\boldsymbol{b_r}(r \in I)$ be any collection of U-sequences where I is an indexing set. Then $\boldsymbol{a} \vee (\bigwedge \boldsymbol{b_r}) = \bigwedge(\boldsymbol{a} \vee \boldsymbol{b_r})$.*

Recall Kaplansky's example (Exercise 74, p. 65 of [4]) that the dual infinite distributive law is false. Let $G = H_{\omega+1} \oplus H_{\omega+2}$, where H_σ is the generalized Prüfer group of length σ. The jump numbers of G consist of all ordinals α such that $\alpha \leq \omega + 1$. Now take U-sequences $\boldsymbol{a} = \{1, \omega + 1, \infty, \infty, \ldots\}$ and $\boldsymbol{b}_r = \{r, r + 1, \infty, \infty, \ldots\}$ for each positive integer r. We have $\bigvee \boldsymbol{b}_r = \{\omega, \omega + 1, \infty, \infty, \ldots\}$, so $\boldsymbol{a} \wedge (\bigvee \boldsymbol{b_r}) = \{1, \omega + 1, \infty, \infty, \ldots\}$. But $\boldsymbol{a} \wedge \boldsymbol{b_r} = \{1, r + 1, \infty, \infty, \ldots\}$, so $\bigvee(\boldsymbol{a} \wedge \boldsymbol{b_r}) = \{1, \omega, \infty, \infty, \ldots\}$.

The following corollary shows that both finite distributive laws hold.

Corollary 4.3 *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be U-sequences of an abelian p-group G. Then*

(i) $\boldsymbol{a} \vee (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\boldsymbol{a} \vee \boldsymbol{c})$ *and*
(ii) $\boldsymbol{a} \wedge (\boldsymbol{b} \vee \boldsymbol{c}) = (\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{c})$.

Proof (i) is a special case of Theorem 4.2. (ii) follows from (i) by duality. (See Lemma 4.3 of [1]). □

Corollary 4.4 *Let G be an arbitrary abelian p-group. Then the lattice of large subgroups of G is distributive and satisfies the infinite distributive law*

$$A \cap (\sum B_r) = \sum(A \cap B_r).$$

Proof If the length of G is finite, then $G = B \oplus D$ where B is bounded and D is divisible. Theorem 1.9 of [5] shows that the lattice of large subgroups of G is isomorphic to the lattice of fully invariant subgroups of B. Since B is fully transitive, Corollary 3.2 gives the result.

If G has infinite length, Lemma 2.5 of [5] shows that the lattice of large subgroups of G is anti-isomorphic to the lattice of U-sequences of G that consist entirely of non-negative integers. Theorem 4.2 completes the proof. □

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Strongly Non-Singular Rings and Morita Equivalence

Bradley McQuaig

Abstract The focus of this paper is to characterize the rings R such that, for every ring S Morita-equivalent to R, the classes of torsion-free and non-singular right S-modules coincide.

Keywords Torsion-free • Non-singular • Morita-equivalent • Baer-ring

1 Torsion-Freeness and Non-Singularity Under Morita Equivalence

There are various ways to extend the notion of torsion-freeness from integral domains to non-commutative rings. Following Hattori [7], we will say that a right R-module M over a ring R is *torsion-free* if $Tor_1^R(M, R/Rr) = 0$ for every $r \in R$. Goodearl takes a different approach in [6] by considering the *singular submodule*

$$Z(M) = \{x \in M \mid xI = 0 \textit{ for some essential right ideal } I \textit{ of } R\}$$

of M. The module M is *singular* if $Z(M) = M$ and *non-singular* if $Z(M) = 0$. Finally, the *right singular ideal of* R is $Z_r(R) = Z(R_R)$, and R is *right non-singular* if it is non-singular as a right R-module. Every right non-singular ring has a maximal right ring of quotients Q^r. Furthermore, a right non-singular ring R is a *right Utumi-ring* if every $\mathscr{S}$-closed right ideal of R is a right annihilator where a submodule U of M is *$\mathscr{S}$-closed* if M/U is non-singular. If R is a right and left non-singular ring, then $Q^r = Q^l$ if and only if R is a right and left Utumi-ring [6, Theorem 2.38].

Albrecht, Dauns, and Fuchs investigated in [1] the rings for which the classes of torsion-free and non-singular right S-modules coincide. However, torsion-freeness is not preserved under a Morita-equivalence, whereas non-singularity is a Morita-invariant property [5, Example 5.4]. Here, two rings are *Morita-equivalent* if their

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module categories are equivalent. It is the goal of this paper to characterize the rings R for which the classes of torsion-free and non-singular right S-modules coincide for every ring S Morita-equivalent to R.

By Albrecht et al. [1, Theorem 3.7], the classes of torsion-free and non-singular right R-modules coincide if and only if R is a right Utumi right p.p.-ring without an infinite set of orthogonal idempotents. A ring R is a *right p.p.-ring* if every principal right ideal is projective as a right R-module, while it is a *Baer-ring* if every right (or left) annihilator ideal is generated by an idempotent. In the classification of the rings R for which this property holds for every Morita-equivalent ring S, strongly non-singular and semi-hereditary rings will play an important role. A right non-singular ring R is *right strongly non-singular* if its maximal right ring of quotients is a perfect left localization, where Q^r is a *perfect left localization of* R if it is flat as a right R-module and the multiplication map $Q^r \otimes_R Q^r \to Q^r$ is an isomorphism. A ring R is *right semi-hereditary* if every finitely generated right ideal is projective as a right R-module.

Theorem 1.1 *The following are equivalent for a ring R:*

a) R is a right strongly non-singular, right semi-hereditary, right Utumi-ring not containing an infinite set of orthogonal idempotents.
b) The classes of torsion-free right S-modules and non-singular right S-modules coincide for every ring S Morita-equivalent to R.
c) For every $0 < n < \omega$, $Mat_n(R)$ *is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents.*

Moreover, if R is such a ring, then the corresponding left conditions are also satisfied.

Proof $a) \Rightarrow b)$ and $c)$: We first show that every ring S Morita-equivalent to R also is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents. Since R has finite right Goldie-dimension by Albrecht et al. [1, Theorem 3.7], its maximal right ring of quotients $Q^r(R)$ is semi-simple Artinian [10]. Because $Q^r(R)$ and $Q^r(S)$ are Morita-equivalent, $Q^r(S)$ is semi-simple Artinian as well since these properties are Morita-invariant [2]. Thus, $Q^r(S)$ is regular, and S is right non-singular [10].

Let $\mathscr{F} : Mod_R \to Mod_S$ and $\mathscr{G} : Mod_S \to Mod_R$ be an equivalence. If M is a finitely generated non-singular right S-module, then $\mathscr{G}(M)$ is a finitely generated non-singular right R-module [2]. Since R is right strongly non-singular, $\mathscr{G}(M)$ is isomorphic to a finitely generated submodule of a free right R-module [6, Theorem 5.17]. Thus, $\mathscr{G}(M)$ is a projective right R-module since R is right semi-hereditary [8]. Hence, $M \cong \mathscr{F}\mathscr{G}(M)$ is projective. Therefore, S is a right strongly non-singular right semi-hereditary ring [6].

Since S is a right non-singular ring with a semi-simple Artinian maximal right ring of quotients, S has finite right Goldie dimension [10], and thus cannot contain an infinite set of orthogonal idempotents. Therefore, S is a right strongly non-singular right p.p.-ring not containing an infinite set of orthogonal idempotents. As shown in [1], this yields that a right S-module is torsion-free if and only if it is non-singular, and b) holds by Albrecht et al. [1, Theorem 5.2].

In particular, $Mat_n(R)$ is a right strongly non-singular right p.p.-ring without an infinite set of orthogonal idempotents by Albrecht et al. [1, Theorem 4.2]. Consequently, $Mat_n(R)$ is a Baer-ring by Albrecht et al. [1, Theorem 3.7]. Moreover, [1, Theorem 4.2] shows that every $\mathscr{S}$-closed one-sided ideal of $Mat_n(R)$ is generated by an idempotent. Hence, every right ideal of $Mat_n(R)$ is a right annihilator and every left ideal of $Mat_n(R)$ is a left annihilator. Therefore, $Mat_n(R)$ is a right and left Utumi-ring.

b) ⇒ *a*): Assume the classes of torsion-free right S-modules and non-singular right S-modules coincide for every ring S Morita-equivalent to R. Since $Mat_n(R)$ is Morita-equivalent to R for $0 < n < \omega$, the classes of torsion-free right $Mat_n(R)$-modules and non-singular right $Mat_n(R)$-modules coincide for $0 < n < \omega$. Hence, $Mat_n(R)$ is a right Utumi p.p.-ring not containing an infinite set of orthogonal idempotents [1]. In particular, R is a right Utumi-ring without an infinite set of orthogonal idempotents. Furthermore, we know R is right semi-hereditary since $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$ [4].

It remains to be seen that R is right strongly non-singular. For this, consider a finitely generated non-singular right R-module M. In view of [6, Theorem 5.18], it suffices to show that M is projective. Let $0 \to U \to F \to M \to 0$ with $F = R^n$ be an exact sequence of right R-modules. It induces the exact sequence

$$0 \to Hom_R(F, U) \to Hom_R(F, F) \to Hom_R(F, M) \to 0.$$

Observe that F is a progenerator of Mod_R. Therefore, if $S = End_R(F) \cong Mat_n(R)$, then $\mathscr{F} : Mod_R \to Mod_S$ defined by $\mathscr{F}(M) = Hom_R(F, M)$ with inverse equivalence $\mathscr{G} : Mod_S \to Mod_R$ given by $\mathscr{G}(N) = N \otimes_S F$ is a Morita-equivalence between R and S [2]. Thus, $Hom_R(F, M)$ is a non-singular right S-module, and hence torsion-free by assumption. Observe that $Hom_R(F, M) \cong S/Hom_R(F, U)$ is cyclic as an S-module since $Hom_R(F, U)$ is a right ideal of the right S-module S. Now, as a right p.p.-ring not containing an infinite set of orthogonal idempotents, S is also a left p.p.-ring [4]. Thus, the cyclic torsion-free right S-module $Hom_R(F, M)$ is projective by Albrecht et al. [1, Corollary 3.4]. Therefore, $M \cong \mathscr{G}\mathscr{F}(M) = \mathscr{G}(Hom_R(F, M))$ is a projective right R-module and we conclude that R is right strongly non-singular.

c) ⇒ *a*): Suppose $Mat_n(R)$ is a right and left Utumi Baer-ring for $0 < n < \omega$ and does not contain an infinite set of orthogonal idempotents. Clearly, $Mat_n(R)$ is a right p.p.-ring, and thus R is right semi-hereditary [4]. Furthermore, since $Mat_n(R)$ satisfies these conditions for every $0 < n < \omega$, $R \cong Mat_1(R)$ is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents. Thus, every $\mathscr{S}$-closed one-sided ideal of R is an annihilator and hence generated by an idempotent. Therefore, since R is a right and left p.p.-ring, R is right strongly non-singular [1]. □

Corollary 1.2 *The following are equivalent for a ring R which does not contain an infinite set of orthogonal idempotents:*

a) R is a right and left Utumi, right semi-hereditary ring.
b) For every $0 < n < \omega$, $Mat_n(R)$ is a Baer-ring, and $Q^r(R)$ is torsion-free as a right R-module.

Proof *a*) $\Rightarrow$ *b*): Suppose R is right and left Utumi and right semi-hereditary. Then, R is a right p.p.-ring and hence right non-singular. Moreover, since R satisfies the idempotent condition, it is also a left p.p.-ring and hence left non-singular. Therefore, $Q^r(R) = Q^l(R)$ since R is both right and left Utumi [6], and $Q^r(R) = Q^l(R)$ is semi-simple Artinian and torsion-free since R is a right Utumi right p.p.-ring satisfying the idempotent condition [1]. Consequently, R is right strongly non-singular by Albrecht et al. [1, Theorem 4.2]. Now, R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents. By Theorem 1.1, $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$.

b) $\Rightarrow$ *a*): Assume $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$, and $Q^r(R)$ is torsion-free as a right R-module. Since $Mat_n(R)$ is a Baer-ring, it is both a right and left p.p.-ring. Hence, R is both right and left semi-hereditary [4]. It then readily follows that R is right and left non-singular, and by assumption we have that $R \cong Mat_1(R)$ is a Baer-ring. To see that R is right Utumi, let I be a proper $\mathscr{S}$-closed right ideal of R. Then, R/I is non-singular as a right R-module. Furthermore, R/I is cyclic and thus finitely generated. Hence, R/I is isomorphic to a submodule of a free Q^r-module by Stenström [10, Chap. XII, Proposition 7.2]. Since Q^r is assumed to be torsion-free as a right R-module, it follows from [1, Proposition 3.3] that I is generated by an idempotent $e \in R$. Hence, $I = ann_r(1 - e)$ since R is a right p.p.-ring. Therefore, R is right Utumi. Observe that the argument works for $\mathscr{S}$-closed left ideals as well, and so R is also left Utumi. □

The next example illustrates why it is necessary to consider right semi-hereditary rings in Theorem 1.1.

Example 1.3 Let $R = \mathbb{Z}[x]$. As an integral domain, R is a strongly non-singular p.p.-ring not containing an infinite set of orthogonal idempotents [1, Corollary 3.10]. By Albrecht et al. [1, Theorem 4.2], the classes of torsion-free and non-singular right R-modules coincide, and by Albrecht et al. [1, Theorem 3.7] R is right Utumi. However, R is not semi-hereditary since the ideal $x\mathbb{Z}[x] + 2\mathbb{Z}[x]$ of $\mathbb{Z}[x]$ is not projective. This implies that $S = Mat_2(R)$ is not a right or left p.p.-ring [4], and hence not a Baer ring. Therefore, Theorem 1.1 does not hold if R is not assumed to be right semi-hereditary.

Moreover, this example shows that the classes of torsion-free and non-singular S-modules do not necessarily coincide, even if R has this property and S is Morita-equivalent to R.

In [3, Theorem 4.3.5], Birkenmeier, Park, and Rizvi show that $Mat_n(R)$ is a Baer-ring precisely when every finitely generated torsionless right R-module is projective. A right R-module is *torsionless* if it is isomorphic to a submodule of R^I for some set I. In case that R has finite right Goldie-dimension, this condition is equivalent to R being right semi-hereditary:

Corollary 1.4 *The following are equivalent for a ring R with finite right Goldie dimension:*

a) R is right semi-hereditary.
b) Every finitely generated torsionless right R-module is projective.

Proof In view of [3, Theorem 4.3.5], it needs to be shown that a ring R with finite right Goldie dimension is right semi-hereditary if and only if $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$. Now, R is right semi-hereditary if and only if $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$ [9]. Hence, R is right semi-hereditary whenever $Mat_n(R)$ is a Baer-ring. On the other hand, note that $Mat_n(R)$ has finite right Goldie dimension since every ring Morita-equivalent to R also has finite dimension. Thus, $Mat_n(R)$ does not contain an infinite set of orthogonal idempotents. Therefore, if R is right semi-hereditary, $Mat_n(R)$ is a right p.p-ring not containing an infinite set of orthogonal idempotents, and it follows from [9, Theorem 1] that $Mat_n(R)$ is a Baer-ring. □

Clearly, the conditions in part *a*) of Theorem 1.1 imply that every finitely generated torsionless module is projective since these conditions imply that $Mat_n(R)$ is a Baer-ring. However, the condition on the torsionless modules in [3] is not enough to ensure that the coincidence of torsion-freeness and non-singularity is preserved by Morita-equivalence. The following examples provide rings for which the conditions of Theorem 1.1 fail, even though every finitely generated torsionless module is projective.

Example 1.5 Let $R = F^I$ for some field F and an infinite index-set I. Then R is a commutative semi-hereditary ring which is its own maximal ring of quotients. Thus, R is strongly non-singular, and all finitely generated torsionless R-modules are projective. Therefore, $Mat_n(R)$ is a Baer-ring for all $n < \omega$, but R does not satisfy Theorem 1.1 since it has infinite Goldie dimension.

Example 1.6 [4] Let $K = F(y)$ for some field F and consider the endomorphism f of K determined by $y \mapsto y^2$. The ring we consider is $R = K[x]$ with coefficients written on the right and multiplication defined according to $kx = xf(k)$ for any $k \in K$. Observe that $yx = xy^2$. It can be shown that $Rx \cap Rxy = 0$, and hence $Rxy \oplus Rxyx \oplus Rxyx^2 \oplus \ldots \oplus Rxyx^k \oplus \ldots$ is an infinite direct sum of left ideals of R. Thus, R has infinite left Goldie-dimension. On the other hand, every right ideal of R is a principal ideal [4], and thus R is right Noetherian. It then follows from [1, Theorem 3.7] that R is a right Utumi Baer ring and Q^r is semi-simple Artinian. However, R having infinite left Goldie-dimension but finite right Goldie-dimension implies that $Q^r \neq Q^l$ [1, Proposition 4.1]. Therefore, R cannot be left Utumi [6, Theorem 2.38].

Thus, we have a right Utumi Baer-ring which is not left Utumi, and so this ring fails to satisfy the conditions of Theorem 1.1. However, since R is a Baer-ring and every right ideal is principal, R is right semi-hereditary. Therefore, every finitely generated torsionless right R-module is projective by Corollary 1.4. Observe that Example 1.6 also illustrates why it is necessary in Theorem 1.1 to include the requirement $Mat_n(R)$ is both right and left Utumi.

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The Class of $(2, 3)$-Groups with Homocyclic Regulator Quotient of Exponent p^2

Ebru Solak

Dedicated to the memory of Rüdiger Göbel

Abstract The class of almost completely decomposable groups with a critical typeset of type $(2, 3)$ and a homocyclic regulator quotient of exponent p^2 is shown to be of bounded representation type. There are only 5 near-isomorphism classes of indecomposables and they are of rank 5 or 6.

Keywords Almost completely decomposable group • Indecomposable • Bounded representation type

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1 Introduction

A torsion-free abelian group G is an additive subgroup of a rational vector space. The dimension of the spanned subspace is called rank [7].

Completely decomposable groups are direct sums of groups of rank 1 and *almost completely decomposable groups* are torsion- free abelian groups that contain a finite rank completely decomposable subgroup of finite index. Every almost completely decomposable group G contains a canonical completely decomposable fully invariant subgroup $\mathrm{R}(G)$, the *regulator* of G. In this paper we deal exclusively with almost completely decomposable groups with p-primary *regulator quotient* $G/\mathrm{R}(G)$, the so-called *p-local* case.

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The set of all types of elements of a torsion-free abelian group G is called *typeset* of G. For almost completely decomposable groups the finite set of types of the direct summands of rank 1 of the regulator is called *critical typeset*. The typeset of an almost completely decomposable group is the closure of the critical typeset relative to intersection of types. A type τ is *p-locally free* if $pG \neq G$ for any group G of rank 1 and type τ.

Two p-local almost completely decomposable groups G and H are *nearly isomorphic* if there is an integer n relatively prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n$ and $gf = n$. The group G is indecomposable if and only if G is nearly isomorphic to an indecomposable group [1, 12.9].

By well-known theorems of Arnold, [1, 12.9], [8, 10.2.5] and Faticoni-Schultz [6] if G is a p-local almost completely decomposable group, then the direct decompositions of G with indecomposable summands are unique up to near isomorphism and two groups that are nearly isomorphic have identical decompositions up to near isomorphism of summands.

As was shown in [2] most of these classes contain indecomposable groups of arbitrarily large rank, the unbounded case in which it is hopeless to try to describe all near-isomorphism classes of indecomposable groups. This leaves some special subclasses that may have a finite number of near-isomorphism classes of indecomposable groups. Bounded classes are those that contain only finitely many near-isomorphism types of indecomposables.

Almost completely decomposable groups with an inverted forest as critical typeset are investigated in [5] to be bounded or unbounded. There are some few classes that are not known to be unbounded or not.

Let p be a prime, $(2,3) = (\tau_1 < \tau_2, \tau_3 < \tau_4 < \tau_5)$ a set of p-locally free types, partially ordered as indicated. Let $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5$ where R_i is homogeneous completely decomposable of finite rank ≥ 1 and type τ_i. A p-local almost completely decomposable group G is called a *$(2,3)$-group* if $\mathrm{R}(G) \cong R$. Such a group has a regulating regulator cf., [9] and, up to near isomorphism, unique indecomposable decompositions.

A $(2,3)$-group with a homocyclic regulator quotient of exponent p^2 is called a $(2,3)$-p^2-hc-group. *Homocyclic* means that the regulator quotient is the direct sum of cyclic groups all of the same order. This paper is devoted to a classification of indecomposable $(2,3)$-p^2-hc-groups, thereby confirming bounded representation type in this case. More precisely, we present a complete collection of near-isomorphism types of indecomposable homocyclic $(2,3)$-p^2-groups. There are precisely five near-isomorphism classes of indecomposables and they are of rank 5 and 6.

Our method consists in turning the decomposition question into an equivalence problem for matrices.

2 Coordinate Matrices

We describe $(2,3)$-groups by a so-called *coordinate matrix*. This is done by a relation matrix connecting a p-basis of the regulator with a basis of the regulator quotient, cf. [3] for details. Each column of this coordinate matrix belongs to a basis element of the regulator, so it corresponds to a type.

The critical typeset is $(\tau_1 < \tau_2, \tau_3 < \tau_4 < \tau_5)$. We obtain the coordinate matrix written as columns in the form $[\alpha_1|\alpha_2\|\beta_1|\beta_2|\beta_3]$ where the columns of α_1 and α_2 correspond to τ_i for $i = 1, 2$, respectively and the columns of β_1, β_2 and β_3 correspond to τ_i for $i = 3, 4, 5$.

Two $(2,3)$-p^2-hc groups are nearly isomorphic if and only if their coordinate matrices are equivalent via an equivalence relation defined by certain row and column operations listed below, see [3, Theorem 12].

Moreover, any entry a_{ij} in the coordinate matrix δ may be replaced by an integer congruent to a_{ij} modulo p^2, in particular $p^2 = 0$.

Remark 2.1 We call transformations of rows and of columns of a coordinate matrix of a $(2,3)$-hc-group G *allowed* if the transformed coordinate matrix is the coordinate matrix of a group H where G and H are nearly isomorphic. Then the following row and column operations on the coordinate matrix of a homocyclic $(2,3)$-group are allowed:

1. Any multiple of a row may be added to any other row.
2. Any row or column may be multiplied by an integer relatively prime to p.
3. Any multiple of a column of α_1 may be added to another column of $\alpha = [\alpha_1 | \alpha_2]$ and any multiple of a column of α_2 may be added to another column of α_2.
4. Any multiple of a column of β_1 may be added to another column of $\beta = [\beta_1|\beta_2|\beta_3]$, any multiple of a column of β_2 may be added to another column of $[\beta_2|\beta_3]$ and any multiple of a column of β_3 may be added to another column of β_3.

Now we state the Regulator Criterion in [3, Lemma 13], in the special case of $(2,3)$-groups.

Lemma 2.2 *Let G be a $(2,3)$-group. Then G has a regulating regulator. Let $r =$ $\mathrm{rank}(G/R)$. The completely decomposable subgroup $R = R_{\tau_1} \oplus R_{\tau_2} \oplus R_{\tau_3} \oplus R_{\tau_4} \oplus R_{\tau_5}$ of finite index in G is the regulator of G if and only if $R_{\tau_1} \oplus R_{\tau_2}$ and $R_{\tau_3} \oplus R_{\tau_4} \oplus R_{\tau_5}$ are pure in G, and this holds if and only if α and β of a coordinate matrix $[\alpha\|\beta]$, relative to any p-basis of R both have p-rank r.*

An integral matrix $A = [a_{i,j}]$ is called *p-reduced* if

1. there is at most one 1 in a row and column and all other entries are in $p\,\mathbb{Z}$,
2. if an entry 1 of A is at the position (i_s, j_s), then $a_{i_s,j} = 0$ for all $j > j_s$ and $a_{i,j_s} = 0$ for all $i < i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\,\mathbb{Z}$ for all $j < j_s$ and all $i > i_s$.

Thus in a p-reduced matrix, the entries to the left and below of a 1 are in $p\mathbb{Z}$. By elementary row transformations upward and elementary column transformations to the right A can be transformed into a p-reduced matrix, cf. [3, Lemma 14].

3 Standard Coordinate Matrices

Line means a row or a column. The matrix $A = [a_{i,j}]$ has a *cross* at (i_0, j_0) if $a_{i_0,j_0} \neq 0$ and $a_{i_0,j} = 0$, $a_{i,j_0} = 0$ for all $i \neq i_0$ and $j \neq j_0$. The entry a_{i_0,j_0} is called *cross entry*. By *"$x \in A$ leads to a cross"* we mean that this entry x can be used to produce a cross by allowed line transformations, i.e., x is afterward the cross entry.

We apply transformations to annihilate entries. While doing this, some other entries that were originally zero may become non-zero; those entries are called *fill-ins*.

A matrix is *decomposed* if it is of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Here either one of the matrices A, B is allowed to have no rows or no columns, i.e., the decomposed matrices include the special cases $[0\ B]$, $\left[\begin{smallmatrix} 0 \\ B \end{smallmatrix}\right]$, $[A\ 0]$, $\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$. A matrix A is called *decomposable* if there are row and column permutations that transform it to a decomposed form, i.e., there are permutation matrices P, Q such that PAQ is decomposed.

Lemma 3.1 *[4, Lemma 3.1] A p-local almost completely decomposable group with an inverted forest as critical typeset is decomposable if and only if there exists a decomposable coordinate matrix.*

A torsion-free abelian group is called *clipped* if it has no summand of rank 1.

Proposition 3.2 *A* $(2,3)$-p^2-hc-*group* G *without summands of rank* ≤ 4 *has a coordinate matrix*

$$\left[\alpha_1 | \alpha_2 \| \beta_1 | \beta_2 | \beta_3\right] = \left[\begin{array}{cc|cc||c|c|c} I_A & 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & pI & I_B & 0 & pB_1 & B_2 & B_3 \\ 0 & 0 & 0 & I_C & pC_1 & C_2 & C_3 \end{array}\right] \tag{1}$$

such that

1. *the sizes of the identity matrices I_A, I_B, I_C all are near isomorphism invariants of G and the sum of the sizes of I_A, I_B, I_C is the rank of the regulator quotient;*
2. *the submatrix $[\beta_1 \mid \beta_2]$ is p-reduced and the submatrix of β_3, obtained by omitting the* 0*-rows is the identity matrix, in particular, the blocks A_3, B_3, C_3 are completely determined by $[\beta_1 \mid \beta_2]$;*
3. $[\beta_1 \mid \beta_2]$ *has no* 0*-line and there is no cross in* $[\beta_1 \mid \beta_2]$.

Proof As G is clipped neither α nor β can contain a 0-column. All elementary row and column transformations are allowed in α_1, hence α_1 may be assumed to be in Smith Normal Form. Moreover, we may assume α to be p-reduced, hence there are 0-rows in α_2 to the right of I_A. The Regulator Criterion requires that the submatrix of α_2, obtained by omitting its 0-rows, can be transformed to the identity matrix by column transformations in α_2, hence without changing α_1. The claimed form of α is now established.

(1) It can be shown that the sizes of the identity matrices I_A, I_B, I_C all are near-isomorphism invariants of G, cf. [10, Proposition 4.3].
(2) Row transformations upward in α create fill-ins in α that can be removed by suitable allowed column transformations in α. Hence β may be transformed by row transformations upward and the usual allowed column transformations. So we may assume that β is in p-reduced form. Using allowed column transformations, we produce zeros to the right of any 1 in $[\beta_1 \mid \beta_2]$ and the submatrix remaining after omitting all zero rows from β_3 may be changed to the identity matrix. This can be done without changing $[\beta_1 \mid \beta_2]$ or α, cf. [3, Lemma 25].
(3) Clearly, $[\beta_1 \mid \beta_2]$ has no 0-line, because there are no summands of rank ≤ 4. A cross in $[\beta_1 \mid \beta_2]$ displays summands of rank ≤ 3 if the cross entry is 1, and summands of rank 4 if the cross entry is not a unit modulo p. Hence $[\beta_1 \mid \beta_2]$ has no 0-line.

It remains to show that the entries of β_1 in B_1 and C_1 all are in $p\mathbb{Z}$. A $1 \in C$ leads to a cross in β. Hence the entries of C all are in $p\mathbb{Z}$. In turn, a $1 \in B$ leads to a cross in β_1. In both cases there are summands of rank ≤ 3.

A coordinate matrix of a $(2, 3)$-p^2-hc-group as in Proposition 3.2 is called *standard*. Note that in a standard coordinate matrix the form of α_2, β_3 is completely determined by $\alpha_1, [\beta_1 \mid \beta_2]$, respectively.

Line transformations of $[\beta_1 \mid \beta_2]$ are called *α-allowed* if after executing such, α can be returned to its previous form by column transformations of α. All column transformations of β_1 and all column transformations of β_2 are automatically α-allowed.

The following row transformations are α-allowed.

1. Any line may be multiplied by a unit.
2. Any row transformation may be applied to A, B, C, respectively.
3. Any multiple of a row in C may be added to any other row.
4. Any multiple of a row in B may be added to any row in $B \cup A$.
5. Any p-multiple of a row in A may be added to a row in B.
6. Any p-multiple of a row in B may be added to a row in C.

We may state [3, Proposition 27] for $(2, 3)$-groups as follows:

Proposition 3.3 *A $(2, 3)$-group is decomposable if and only if there exists a standard coordinate matrix $[\alpha_1 | \alpha_2 \| \beta_1 | \beta_2 | \beta_3]$ with decomposable $[\beta_1 \mid \beta_2]$.*

We next list the near-isomorphism classes of indecomposable $(2, 3)$-p^2-hc-groups. We define the type of a group G using the invariants of their standard coordinate matrix and $[\beta_1 \mid \beta_2]$. G is of type $\left(\operatorname{rk} G/R, \operatorname{rk} G, \left[X\right], [\beta_1, \beta_2]\right)$, where $\left[X\right]$ is a part of $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ indicating which block rows are present:

1. $\begin{bmatrix} 1 & | & 0 & || & 1 & | & 0 & | & 0 \\ 0 & | & 1 & || & p & | & p & | & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ p & p \end{bmatrix}\right)$
2. $\begin{bmatrix} 1 & 0 & | & 0 & || & p & | & 0 & | & 1 \\ 0 & p & | & 1 & || & p & | & 1 & | & 0 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} A \\ B \end{bmatrix}, \begin{bmatrix} p & 0 \\ p & 1 \end{bmatrix}\right)$
3. $\begin{bmatrix} 1 & | & 0 & || & p & | & 1 & | & 0 \\ 0 & | & 1 & || & 0 & | & p & | & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix}\right)$
4. $\begin{bmatrix} 1 & | & 0 & || & p & | & 0 & | & 1 \\ 0 & | & 1 & || & p & | & 1 & | & 0 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} p & 0 \\ p & 1 \end{bmatrix}\right)$
5. $\begin{bmatrix} p & | & 1 & 0 & || & p & | & 0 & | & 1 \\ 0 & | & 0 & 1 & || & p & | & 1 & | & 0 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} B \\ C \end{bmatrix}, \begin{bmatrix} p & 0 \\ p & 1 \end{bmatrix}\right)$

Theorem 3.4 *There are precisely the* 5 *near-isomorphism types in the list above of indecomposable* $(2, 3)$*-groups with homocyclic regulator quotient of exponent* p^2.

Proof Let G be an indecomposable $(2, 3)$-group with homocyclic regulator quotient of exponent p^2. The group G cannot have summands of rank ≤ 4 because the critical typeset has cardinality 5. By Proposition 3.2 we assume a standard coordinate matrix for G is $[\alpha_1|\alpha_2\|\beta_1|\beta_2|\beta_3]$. There is no cross in $[\beta_1|\beta_2]$. At least two of the blocks A, B, C are present because the presence of just one block allows the transformation of $[\beta_1|\beta_2]$ to Smith Normal Form , i.e., a cross or a 0-line.

Note that mostly we want to change certain submatrices either to a 0-matrix or to a matrix of the form pI. The phrase *"we form the Smith Normal Form of A"* means that by a sequence of allowed elementary row and column transformations we obtain a diagonal matrix with diagonal entries 0 or p. It is tacitly included that all the line transformations are allowed and that originally "normed" blocks can be reestablished. If blocks split into subblocks, then if possible we keep the original name to avoid overwhelming indexing. Forming the Smith Normal Form, in general, splits blocks. Block names, like pC, are place holders only and are re-used again and again with changing values. By Proposition 3.3 we only have to consider $[\beta_1|\beta_2]$ and hence we apply α-allowed line transformations to $[\beta_1|\beta_2]$. Our technique is to form successively Smith Normal Forms of subblocks and this is done by α-allowed transformations only. While forming successively Smith Normal Forms we get fill-ins in the blocks that are already obtained in Smith Normal Form and we always proceed in such a way that those fill-ins can be removed by successive transformations. The phrase *"we can annihilate"* or *"it allows to annihilate"* tacitly includes that the occurring fill-ins can be removed by subsequent transformations and the previously "normed" blocks are reestablished. Note that sometimes fill-ins occur that have a prefactor p^2, hence can be and are replaced by 0, because we deal with groups with regulator quotient of exponent p^2.

Starting with Equation (1) we first form the Smith Normal Form for C_1. The Smith Normal Form of C_1 is $\begin{bmatrix} pI & 0 \\ 0 & 0 \end{bmatrix}$. We use the pI in the Smith Normal Form of C_1 to annihilate in pB_1 and then we form the Smith Normal Form of the rest of pB_1 to get $\begin{bmatrix} pI & 0 \\ 0 & 0 \end{bmatrix}$. Thus we obtain

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & A_2 \\ 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & C_2 \\ 0 & 0 & 0 & C_3 \end{array}\right] \begin{array}{c} A \\ B^1 \\ B^2 \\ C^1 \\ C^2 \end{array}$$

There is no 0-column in A_{13}. A unit in A_{13} leads to a cross in $[\beta_1|\beta_2]$. Hence the entries of A_{13} are in $p\,\mathbb{Z}$ and we write pA_{13}. The Smith Normal Form of pA_{13} is $\left[\begin{smallmatrix} pI \\ 0 \end{smallmatrix}\right]$. Thus we get

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{ccc|c} A_{11} & A_{12} & pI & A_2 \\ A_{14} & A_{15} & 0 & A_3 \\ 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & C_2 \\ 0 & 0 & 0 & C_3 \end{array}\right] \begin{array}{c} A^1 \\ A^2 \\ B^1 \\ B^2 \\ C^1 \\ C^2 \end{array}$$

All the non-zero entries of A_{11}, A_{14}, A_{12} and A_{15} are units due to the presence of pI in the B^1- and C^1-row. A unit in A_{15} leads to a cross in $[\beta_1|\beta_2]$. Hence $A_{15} = 0$. If there is a unit in A_{12} we annihilate with this unit in the A^1-row and below in the B^1-row and this leads to a cross in $[\beta_1|\beta_2]$. Hence $A_{12} = 0$ and we obtain

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{ccc|c} A_{11} & 0 & pI & A_2 \\ A_{14} & 0 & 0 & A_3 \\ 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & C_2 \\ 0 & 0 & 0 & C_3 \end{array}\right] \begin{array}{c} A^1 \\ A^2 \\ B^1 \\ B^2 \\ C^1 \\ C^2 \end{array}$$

The Smith Normal Form of A_{14} is $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We now create zeros in the A^1-row above the I in the Smith Normal Form of A_{14}. Hence we get

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccc|c} 0 & A_{11} & 0 & pI & A_2 \\ I & 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & A_4 \\ 0 & 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & 0 & C_2 \\ 0 & pI & 0 & 0 & C_2' \\ 0 & 0 & 0 & 0 & C_3 \end{array}\right] \begin{array}{c} A^1 \\ A^{21} \\ A^{22} \\ B^1 \\ B^2 \\ C^{11} \\ C^{12} \\ C^2 \end{array}$$

Due to the presence of pI in the C^{12}-row we may assume that the entries of A_{11} are either units or zero. Hence the Smith Normal Form of A_{11} is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and

$$
\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccccc|c} 0 & I & 0 & 0 & pI & 0 & A_{21} \\ 0 & 0 & 0 & 0 & 0 & pI & A_{22} \\ I & 0 & 0 & 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & pI & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & 0 & 0 & 0 & C_2 \\ 0 & pI & 0 & 0 & 0 & 0 & C'_{21} \\ 0 & 0 & pI & 0 & 0 & 0 & C'_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & C_3 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{21} \\ A^{22} \\ B^1 \\ B^2 \\ C^{11} \\ C^{12} \\ C^{12}_* \\ C^2 \end{array}
$$

The blocks A_{21} and A_3 can be annihilated by the respective identity matrices in the A^{11}- and A^{21}-rows. We can annihilate pI in the A^{11}-row by using I in the same row. Then the fifth column of β_1 is zero, hence not present. Note that a unit in C_3 leads to a cross in $[\beta_1|\beta_2]$, hence we write pC_3. We may combine column 1 and column 2 of β_1 to one column, and we get

$$
\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & A_{22} \\ 0 & 0 & 0 & 0 & A_4 \\ 0 & 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & 0 & C'_{21} \\ 0 & pI & 0 & 0 & C'_{22} \\ 0 & 0 & 0 & 0 & pC_3 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^1 \\ B^2 \\ C^{12} \\ C^{12}_* \\ C^2 \end{array}
$$

A unit in C'_{21} causes a direct summand of rank ≤ 4. Hence the entries of C'_{21} are in $p\,\mathbb{Z}$ and we write pC'_{21}. Thus we get

$$
\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccc|c} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & A_{22} \\ 0 & 0 & 0 & 0 & A_4 \\ 0 & 0 & pI & 0 & B_2 \\ 0 & 0 & 0 & 0 & B_3 \\ pI & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & C'_{22} \\ 0 & 0 & 0 & 0 & pC_3 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^1 \\ B^2 \\ C^{12} \\ C^{12}_* \\ C^2 \end{array}
$$

Due to the presence of pI in the C_*^{12}-row the entries of C'_{22} are units or zero. There is no zero row in C'_{22} to avoid a cross. Hence the Smith Normal Form of C'_{22} is $\begin{bmatrix} I & 0 \end{bmatrix}$. We annihilate with $I \subset C'_{22}$ in A_{22}, B_2, pC'_{21} and pC_3. The Smith Normal Form of the rest of pC_3 is $\begin{bmatrix} pI & 0 \end{bmatrix}$. We annihilate with $pI \subset pC_3$ in pC'_{21}. Thus we get

$$\left[\beta_1 | \beta_2\right] = \left[\begin{array}{cccc|ccc} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & A_{22} & A_{23} \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A_{43} \\ 0 & 0 & pI & 0 & 0 & B_{22} & B_{23} \\ 0 & 0 & 0 & 0 & B_{31} & B_{32} & B_{33} \\ pI & 0 & 0 & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^1 \\ B^2 \\ C^{12} \\ C_*^{12} \\ C^2 \end{array}$$

A unit in B_{33} leads to a cross in $[\beta_1 \mid \beta_2]$, so we write pB_{33}. The entries of B_{32} are either units or zero due to the pI in the C^2-row. But a unit in B_{32} leads to a cross in $[\beta_1 \mid \beta_2]$, so we write pB_{32}. Then we annihilate with pI in the C^2-row the block matrix pB_{32}. Hence $B_{32} = 0$. Thus

$$\left[\beta_1 | \beta_2\right] = \left[\begin{array}{cccc|ccc} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & A_{22} & A_{23} \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A_{43} \\ 0 & 0 & pI & 0 & 0 & B_{22} & B_{23} \\ 0 & 0 & 0 & 0 & B_{31} & 0 & pB_{33} \\ pI & 0 & 0 & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^1 \\ B^2 \\ C^{12} \\ C_*^{12} \\ C^2 \end{array}$$

A unit in B_{31} allows to annihilate in pB_{33} and in A_{41} and this leads to as summand $[\beta_1 \mid \beta_2] = \left[\begin{array}{c|c|c} 0 & 1 & B \\ p & 1 & C \end{array}\right]$ and the p-reduced form might be given as $[\beta_1 \mid \beta_2] = \left[\begin{array}{c|c|c} p & 0 & B \\ p & 1 & C \end{array}\right]$ and a summand of type cf. (5) in the list. Omitting those summands we obtain $B_{31} = 0$. Thus we get

$$\left[\beta_1 | \beta_2\right] = \left[\begin{array}{cccc|ccc} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & A_{22} & A_{23} \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A_{43} \\ 0 & 0 & pI & 0 & 0 & B_{22} & B_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & pB_{33} \\ pI & 0 & 0 & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^1 \\ B^2 \\ C^{12} \\ C_*^{12} \\ C^2 \end{array}$$

The Smith Normal Form of B_{23} is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. We annihilate with $I \subset B_{23}$ in B_{22}, pB_{33}, pC'_{21} and in A_{23} and then we form the Smith Normal Form of the rest of B_{22} that is $\begin{bmatrix} I & 0 \end{bmatrix}$. Thus we get

$$\begin{bmatrix} \beta_1 | \beta_2 \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 & 0 & A_{22} & 0 & A_{23} \\ 0 & 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & A_{43} & A'_{43} \\ 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pB_{33} \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^{11} \\ B^{12} \\ B^{2} \\ C^{12} \\ C^{12}_{*} \\ C^{21} \\ C^{22} \end{array}$$

We may annihilate with I in the B^{12}-row the matrix pI below in the C^{21}-row. Then the C^{21}-row is not present. The columns (3) and (4) are combined. Thus we get

$$\begin{bmatrix} \beta_1 | \beta_2 \end{bmatrix} = \left[\begin{array}{cccc|cccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & A_{22} & A_{23} \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & A'_{43} \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pB_{33} \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & pC'_{21} \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^{1} \\ B^{2} \\ C^{12} \\ C^{12}_{*} \\ C^{22} \end{array}$$

There is no 0-row in pB_{33} and in pC'_{21} to avoid a direct summand of rank ≤ 4. Hence the Smith Normal Form of pC'_{21} is $\begin{bmatrix} pI & 0 \end{bmatrix}$. We annihilate with $pI \subset pC'_{21}$ in pB_{33} and produce the Smith Normal Form of pB_{33} that is $\begin{bmatrix} pI & 0 \end{bmatrix}$. Thus $\left[\begin{array}{c} pB_{33} \\ \hline pC'_{21} \end{array}\right] = \left[\begin{array}{ccc} 0 & pI & 0 \\ \hline pI & 0 & 0 \end{array}\right]$ and we get

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccc|cccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & A_{22} & A_{23} & A'_{23} & A''_{23} \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & A'_{43} & A''_{43} & A'''_{43} \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^{1} \\ B^{2} \\ C^{12} \\ C^{12}_{*} \\ C^{22} \end{array}$$

A unit in A'''_{43} causes a cross, so we write pA'''_{43}. In turn a unit in A''_{43} leads to a cross, hence $A''_{43} = 0$. Then a unit in A''_{23} allows to annihilate first in pA'''_{43} and then in its row. This leads to a summand of rank 3, so $A''_{23} = 0$. Thus we get

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{cccc|cccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & A_{22} & A_{23} & A'_{23} & 0 \\ 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & A'_{43} & 0 & pA'''_{43} \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{12} \\ A^{22} \\ B^{1} \\ B^{2} \\ C^{12} \\ C^{12}_{*} \\ C^{22} \end{array}$$

A unit in A'_{23} leads to a direct summand of rank 3, so $A'_{23} = 0$. But then the B^2-row together with the fifth column of β_2 is not present to avoid a cross. The Smith Normal Form of A'_{43} is $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We annihilate with $I \subset A'_{43}$ first in its row and then in its column except of pI below. Thus we get

$$\left[\beta_1|\beta_2\right] = \left[\begin{array}{ccccc|cccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 & 0 & A_{22} & 0 & A_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & 0 & 0 & pA'''_{43} \\ 0 & 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 \end{array}\right] \begin{array}{c} A^{11} \\ A^{11}_{*} \\ A^{12} \\ A^{22} \\ A^{22}_{*} \\ B^{1} \\ C^{12} \\ C^{12}_{**} \\ C^{12}_{*} \\ C^{22} \end{array}$$

The submatrix pI in the C^{12}-row can be annihilated by pI on the left. Then the A^{22}-row together with the fourth column of β_2 is not present to avoid a cross. Then in turn, the A^{11}-row and the C^{12}-row together with the first column of β_1 are not present to avoid a summand of rank ≤ 4.

A zero column in A_{23} leads to $[\beta_1 \mid \beta_2] = \left[\begin{smallmatrix} 1 & | & 0 & | & A \\ p & | & p & | & C \end{smallmatrix}\right]$ and a summand of type cf. (1) in the list. Omitting this summand we may assume that the Smith Normal Form of A_{23} is $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. We annihilate with $I \subset A_{23}$ in A_{22}. Thus we get

$$\left[\beta_1 \middle| \beta_2\right] = \left[\begin{array}{ccccc|ccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & pI & 0 & 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{41} & A_{42} & A'_{42} & 0 & pA''_{43} \\ 0 & 0 & pI & 0 & 0 & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ 0 & pI & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 \end{array}\right] \begin{array}{c} A_*^{11} \\ A^{12} \\ A_*^{12} \\ A_*^{22} \\ B^1 \\ C^{12} \\ C_*^{12} \\ C^{22} \end{array}$$

We annihilate with pI in β_1 in the C^{12}-row the pI on the right in the same row. This leads to a summand of rank ≤ 4. Hence the A^{12}-row together with the fourth column of β_1 and the fourth column of β_2 are not present. Moreover, the A_*^{11}-row, the C^{12}-row and the first column of β_1 are not present to avoid a direct summand of rank ≤ 4. A unit in A'_{42} leads to a summand of rank ≤ 4. Hence $A'_{42} = 0$. Then a unit in A_{22} leads to a summand of type $\left(2, 5, \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} p & | & 1 & | & A \\ 0 & | & p & | & C \end{smallmatrix}\right]\right)$, cf. (3) in the list. Omitting this summand we get $A_{22} = 0$. Then the A_*^{12}-row, the C^{22}-row and the corresponding columns are not present to avoid a cross. Thus we get

$$\left[\beta_1 \middle| \beta_2\right] = \left[\begin{array}{cc|ccc} 0 & 0 & A_{41} & A_{42} & pA''_{43} \\ 0 & pI & 0 & I & 0 \\ pI & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} A_*^{22} \\ B^1 \\ C_*^{12} \end{array}$$

A unit in A_{42} leads to $[\beta_1 \mid \beta_2] = \left[\begin{smallmatrix} 0 & | & 1 & | & A \\ p & | & 1 & | & B \end{smallmatrix}\right]$ and a summand of type cf. (2) in the list. Omitting this summand we may assume that $A_{42} = 0$. Then in turn a unit in A_{41} leads to $[\beta_1 \mid \beta_2] = \left[\begin{smallmatrix} 0 & | & 1 & | & A \\ p & | & 1 & | & C \end{smallmatrix}\right]$ or in p-reduced form $[\beta_1 \mid \beta_2] = \left[\begin{smallmatrix} p & | & 0 & | & A \\ p & | & 1 & | & C \end{smallmatrix}\right]$ and a summand of type cf. (4) in the list. Omitting this summand we may assume that $A_{41} = 0$. But then a $p \in pA''_{43}$ leads to a cross in $[\beta_1 \mid \beta_2]$. All the other row constellations cause direct summands of rank ≤ 4. This finishes the proof.

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Unbounded Monotone Subgroups of the Baer–Specker Group

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Abstract We consider special subgroups of the Baer–Specker group $\mathbb{Z}^\omega$ of all integer valued functions on ω, which L. Fuchs called monotone groups (Fuchs, Infinite abelian groups. Academic, Boston, MA, 1973, Specker, Portugaliae Math 9:131–140, 1950). Together with R. Göbel the author defined an equivalence relation between monotone groups which corresponds to a behavior of homomorphisms from a monotone group into an abelian group (Göbel and Wald, Symp Math 23:201–239, 1979). The group $\mathbb{Z}^\omega$ and the subgroup B of all bounded functions form two equivalence classes with just a single member. A third class is build by all bounded monotone groups, which are monotone groups where the growth of all elements is bounded by the growth of some given function b. An unbounded monotone group different from $\mathbb{Z}^\omega$ can be constructed by an ultrafilter of ω. So the number of equivalence classes of monotone groups is at least 4. In Göbel and Wald (Math Z 172:107–121, 1980) it is proved that the number is $2^{2^{\aleph_0}}$ if the Continuum Hypothesis, CH, or alternatively Martin's Axiom is assumed. Later A. Blass and C. Laflamme showed that it is relatively consistent with ZFC, that the number of equivalence classes is 4. In this case all unbounded monotone groups different from $\mathbb{Z}^\omega$ are equivalent (Blass and Laflamme, J Symb Log 54:54–56, 1989). Further investigations on monotone groups by O. Kolman and the author led to a special technical assumption on monotone groups (Kolman and Wald, Isr J Math 217, to appear). In the present paper we call these monotone groups *comfortable* and show that the existence of a monotone group that is not comfortable, is independent of ZFC.

Keywords Abelian groups • Baer-Specker group • Continuum hypothesis

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1 Preliminaries

The Baer–Specker group $\mathbb{Z}^\omega$ is the direct product of countably many copies of the additive group of integers, $\mathbb{Z}$. The elements of $\mathbb{Z}^\omega$ can be considered as functions from ω to $\mathbb{Z}$. We define a relation $\preceq$ as follows: for two function $a, b \in \mathbb{Z}^\omega$ let $a \preceq b$ if and only if there is a factor $k > 0$ such that $|a(n)| \leq k\bar{b}(n)$ for almost all $n \in \omega$, where $\bar{b}$ is defined by $\bar{b}(n) = \max\{1, |b(0)|, \ldots, |b(n)|\}$ for $n \in \omega$. We call $\bar{b} \in \mathbb{Z}^\omega$ the *monotonization* of b. Here *almost all* means that the assertion holds for all but a finite number of exceptions. A subgroup $M \subseteq \mathbb{Z}^\omega$ is called *monotone* if $a \preceq b \wedge b \in M$ implies $a \in M$. Of course $\mathbb{Z}^\omega$ is monotone and so also is B the subgroup of all bounded elements of $\mathbb{Z}^\omega$. A function $a \in \mathbb{Z}^\omega$ is *bounded* if the set of values $a(n)$ is finite. Further examples can be defined by $M_b = \{a \in \mathbb{Z}^\omega : a \preceq b\}$ for a given $b \in \mathbb{Z}^\omega \setminus B$—see [2, 3, 12]

Specker's paper [12] gave the inspiration for the definition of slenderness for abelian groups G or, more generally, for M-slenderness where M is a monotone group. Recall that an abelian group G is called M-slender if every homomorphism from M to G maps almost all of the special functions e_n to 0; the function $e_n \in \mathbb{Z}^\omega$ is defined by $e_n(m) = 1$ if $n = m$ and $e_n(m) = 0$ for $m \in \omega \setminus \{n\}$. A slender group is a $\mathbb{Z}^\omega$-slender group. It is known that M-slender abelian groups cannot contain a cotorsion group and hence they are torsionfree and reduced. If $M \neq B$, all torsionfree reduced abelian groups with cardinality $< 2^{\aleph_0}$ are M-slender. The monotone group B is a special case, because B is free and no non-trivial abelian group is B-slender—see [2, 3, 8, 10, 12].

We say that two monotone groups M and N are equivalent if M-slenderness and N-slenderness are the same. Another way to define this equivalence relation is to say that a subgroup of M is isomorphic to N and a subgroup of N is isomorphic to M. There are two equivalence classes with one element, $\{B\}$ and $\{\mathbb{Z}^\omega\}$. The latter follows from Specker's result that monotone groups $\neq \mathbb{Z}^\omega$ are slender but of course $\mathbb{Z}^\omega$ itself is not. A third equivalence class is given by the set of all so called bounded monotone groups $\neq B$. A monotone group is bounded if it is contained in some M_b, see [3]. The author gave a construction of an unbounded monotone group $\neq \mathbb{Z}^\omega$ under the assumption of the continuum hypothesis CH in [14]. Later the technique used there was extended to construct $2^{2^{\aleph_0}}$ examples which are pairwise not equivalent. In addition the assumption on CH could be dropped for the construction of a single example [4]. In contrast to the CH-case, Blass and Laflamme showed that it is consistent with ZFC that all unbounded monotone groups $\neq \mathbb{Z}^\omega$ are equivalent [1]. Hence the number of equivalence classes is undecidable in ZFC. This number can be 4 or it can be $2^{2^{\aleph_0}}$; see also *History of the Continuum in the Twentieth Century* by J. Steprāns [13].

We finish this section of preliminaries by noting that our terminology and notation are standard and follow that used by Fuchs in [2]; in particular, all references to groups are to additively written abelian groups.

2 The Result

A known characterisation of slender groups is that an abelian group is slender if, and only if, it is torsionfree and reduced and does not contain a subgroup which is isomorphic to the group of p-adic integers, J_p, for some prime p or to the Baer–Specker group $\mathbb{Z}^\omega$—see [9]. A similar result for M-slenderness ($M \neq \mathbb{Z}^\omega$) would be that an abelian group is M-slender if, and only if, it is torsionfree and reduced and does not contain a subgroup isomorphic to M. Here the p-adic integers need not be mentioned because no J_p can be embedded in any $M \neq \mathbb{Z}^\omega$—see [14]. Kolman and the author showed in a recent paper that the suggested statement holds for a bounded monotone $M \neq B$, but also in other cases—see [6]. The technical requirement for the proof given in [6] about the monotone group leads to the following definition:

Definition 2.1 A monotone subgroup M of $\mathbb{Z}^\omega$ is called *comfortable*, if there is a monotone injection $s\colon \omega \to \omega$ and an unbounded monotone function $c \in \mathbb{Z}^\omega$ such that $\Phi_c^s(M) \subseteq M$, where the homomorphism $\Phi_c^s\colon \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ is defined by

$$\Phi_c^s(x) = \textstyle\sum_{n\in\omega} c(n)x(n)e_{s(n)} \text{ for } x \in \mathbb{Z}^\omega .$$

Notice that we can build infinite sums of the functions e_n quite naturally. For example the sum $\sum_{n\in\omega} x(n)e_n$ is just the function x. The result mentioned above is the following theorem:

Theorem 2.2 *Let $M \neq \mathbb{Z}^\omega$ be a comfortable monotone group. Then an abelian G is M-slender if and only if G is torsionfree and reduced and G does not contain a subgroup isomorphic to M.*

The aim of this work is to investigate whether there exists a monotone group which is not comfortable; we prove the following theorem:

Theorem 2.3 *The assertion that there is a monotone group $\neq B$ which is not comfortable, is not decidable in ZFC.*

At first let us state some basic facts.

Proposition 2.4 *The following hold*

1. *to be comfortable is a property of equivalence classes of monotone groups;*
2. *every bounded monotone group $M \neq B$ group is comfortable;*
3. *every monotone group $M \neq \mathbb{Z}^\omega$ can be enlarged to a comfortable subgroup $N \neq \mathbb{Z}^\omega$.*

Proof We start with two equivalent monotone groups M and N and suppose that M is comfortable. By the equivalence of M and N it follows that we can embed M into N and N into M. It is known that the embeddings can be chosen such that they are of a special kind. There are two monotone injections $s_1, s_2\colon \omega \to \omega$ such that $\sum_{n\in\omega} x(n)e_{s_1(n)} \in N$ for $x \in M$ and $\sum_{n\in\omega} x(n)e_{s_2(n)} \in M$ for $x \in N$. Because M is comfortable there is a third monotone injection s and an unbounded monotone $c \in \mathbb{Z}^\omega$ such that $\Phi_c^s(M) \subseteq M$. Composing these three morphisms, we get $\Phi_{c'}^{s'}(N) \subseteq N$ for $s' = s_1 \circ s \circ s_2$ and $c' = c \circ s_2$. Hence N is also comfortable.

Now let $M \neq B$ be a bounded monotone group. We choose two unbounded monotone function $b, c \in \mathbb{Z}^\omega$ such that $M \subseteq M_b$ and $c^2 \in M$; here the square c^2 is defined pointwise. We can construct a monotone injection $s : \omega \to \omega$ such that $b(n) \leq c(s(n))$ for all $n \in \omega$. Then $\Phi^s_{c'}(b) \preceq c^2$ if we define $c' = c \circ s$. It follows that $\Phi^s_{c'}(M) \subseteq M$.

To prove the third property we start with a monotone group $M \neq \mathbb{Z}^\omega$. We define c and s by $c(n) = n$ and $s(n) = n^2$ for $n \in \omega$ and a further monotone group N by

$$N = \{x \in \mathbb{Z}^\omega : (\exists y \in M)(x \preceq cy)\} .$$

Clearly $M \subseteq N$ and $N \neq \mathbb{Z}^\omega$.

We show that $\Phi^s_c(cy) \preceq cy$ if $y \in M$ is monotone, which immediately implies $\Phi^s_c(N) \subseteq N$. Consider $\Phi^s_c(cy)(n)$ for $n \in \omega$ and suppose $\Phi^s_c(cy)(n) \neq 0$. Then we must have that $n = s(m)$ for some m and therefore

$$\Phi^s_c(cy)(n) = c(m)c(m)y(m) = m^2 y(m) \leq m^2 y(s(m)) =$$
$$s(m)y(s(m)) = c(s(m))y(s(m)) = c(n)y(n) = (cy)(n) .$$

Hence $\Phi^s_c(cy)(n) \leq cy(n)$ for all $n \in \omega$. This completes the proof. □

One conclusion of this is that in the set-theoretical universe used by Blass and Laflamme in which all unbounded monotone groups $\neq \mathbb{Z}^\omega$ are equivalent, every monotone group $\neq B$ is comfortable. Hence this assertion is relative consistent with ZFC. In the next two paragraphs we show that the opposite is also consistent. We give a construction of an uncomfortable monotone group $\neq B$ working under the assumption of CH. This completes the proof, that the existence of such a group is independent of ZFC.

3 Technical Preparation

Definition 3.1 A *diagonal stepper*, or in short a *stepper*, is an unbounded function $\chi : \omega \to \omega$ with the property that for all $n \in \omega$, either $\chi(n) = \chi(n+1)$ or $\chi(n) < n < \chi(n+1)$. We call the numbers $n \in \omega$ with $\chi(n) < n < \chi(n+1)$ the *jump points* of χ.

For a diagonal stepper χ the sets $\chi^{-1}(c) = \{n \in \omega : \chi(n) = c\}$ are intervals. Let $[h, g]$ be such a nontrivial interval, then g is a jump point and has the property $\chi(g) < g < \chi(g+1)$. For h we have $h = 0$ or $h = g' + 1$ where g' is a further jump point. In both cases it follows that $c = \chi(h) = \chi(g)$ lies itself in the interval $[h, g]$ and hence c is a fixed point of χ. Hence between two jump points there lies a fixed point and of course between any two fixed points there lies a jump point. Let $\mathrm{jump}(\chi)$ denote the set of jump points and $\mathrm{fix}(\chi)$ the set of fixed points. Now let us assume that we have two disjoint infinite subsets J and F of ω with the property that between any two elements of J lies an element of F and between two elements of

F lies an element of J. Assume, in addition, that F contains the smallest element of $J \cup F$. Then there is a uniquely defined diagonal stepper χ such that $\mathrm{jump}(\chi) = J$ and $\mathrm{fix}(\chi) = F$.

Definition 3.2 We say that a diagonal stepper χ_2 *follows* a diagonal stepper χ_1, if the sets $\mathrm{jump}(\chi_2) \setminus \mathrm{jump}(\chi_1)$ and $\mathrm{fix}(\chi_2) \setminus \mathrm{fix}(\chi_1)$ are finite or in other word, almost all jump points of χ_2 are jump points of χ_1 and almost all fixed points of χ_2 are fixed points of χ_1.

In the case that χ_2 follows χ_1 we have the inequality

$$\chi_2(g) \leq \chi_1(g) < g < \chi_1(g+1) \leq \chi_2(g+1) \tag{1}$$

for almost all jump points g of χ_2.

Lemma 3.3 *Let $(\chi_n)_{n\in\omega}$ be a family of diagonal steppers such that χ_{n+1} follows χ_n for all $n \in \omega$. Then a further diagonal stepper χ exists which follows all χ_n.*

Proof We construct two sequences $(g_n)_{n\in\omega}$ and $(r_n)_{n\in\omega}$ with the following properties:

1. $g_m < g_n$ and $r_m < r_n$ if $m < n$;
2. $r_n < g_n < r_{n+1}$ for all $n \in \omega$;
3. $r_n \in \mathrm{fix}(\chi_m)$ for all $m \leq n$;
4. $g_n \in \mathrm{jump}(\chi_m)$ for all $m \leq n$.

As we mentioned above there is a diagonal stepper χ such that $\{g_n : n \in \omega\} = \mathrm{jump}(\chi)$ and $\{r_n : n \in \omega\} = \mathrm{fix}(\chi)$. This diagonal stepper follows all χ_n as desired. □

Next we will consider the role of diagonal steppers within $\mathbb{Z}^\omega$.

Definition 3.4 We say a diagonal stepper χ *overcrosses* an unbounded monotone function $a \in \mathbb{Z}^\omega$, if for all $k > 0$ and almost all $g \in \mathrm{jump}(\chi)$

$$k\chi(g) \leq a(g) \tag{2}$$

and

$$ka(g+1) \leq \chi(g+1)\,. \tag{3}$$

In particular $a \not\leq^* \chi$ and $\chi \not\leq^* a$.

We can formulate the following simple proposition.

Proposition 3.5 *Let χ_1 be a diagonal stepper and $a \in \mathbb{Z}^\omega$ be an unbounded monotone function.*

1. *There is a stepper χ_2 which follows χ_1 and overcrosses a.*
2. *If χ_1 overcrosses a and χ_2 is a second stepper which follows χ_1, then χ_2 overcrosses a, too.*

Proof For the first part we define χ_2 by selecting suitable $r \in \text{fix}(\chi_1)$ as the fix points and $g \in \text{jump}(\chi_1)$ as the jump points of χ_2. For a recursive definition we start with $r_0 = \min(\text{fix}(\chi_1))$ and assume that for some $n \in \omega$, all $r_m \in \text{fix}(\chi_1)$ have already been chosen if $m \leq n$ and $g_m \in \text{jump}(\chi_1)$ if $m < n$. Let g_n be the least $g \in \text{jump}(\chi_1)$ such that $g > r_n$ and $(n+1)r_n \leq a(g)$, and let r_{n+1} be the least $r \in \text{fix}(\chi_1)$ such that $r > g_n$ and $(n+1)a(g_n+1) \leq r$. Notice that by this construction $\chi_2(g_n) = r_n$ and $\chi_2(g_n+1) = r_{n+1}$. It follows that χ_2 overcrosses a because, for a given $k > 0$, equations (ii) and (iii) hold for g_n if $n \geq k-1$.

The second part follows from a combination of (i), (ii) and (iii). □

The conclusion of this section is a further proposition, which will be essential for the construction in the next section.

Proposition 3.6 *Let χ_1 be a diagonal stepper and $A \subset \mathbb{Z}^\omega$ be a countable set of unbounded monotone functions. Then there is a stepper χ_2 which follows χ_1 and overcrosses all $a \in A$.*

Proof We assume that $A = \{a_n : n \in \omega\}$ and construct a sequence $(\chi'_n)_{n\in\omega}$ as in Lemma 3.3, by applying 3.5 1 such than χ'_n overcrosses a_n. We start this construction with χ'_0 as a follower of χ_1 and end, utilizing Lemma 3.3, with a stepper χ_2 which follows all χ'_n. By 3.5 2 χ_2 overcrosses all a_n. □

4 Construction of a Monotone Group Which is Not Comfortable

For our construction we assume the continuum hypothesis CH. This means that we have only ω_1 many of the various types of objects that are the focus of our investigation. Hence there is an ω_1-enumeration $(\Phi_\nu)_{\nu\in\omega_1}$ of all the possible homomorphisms Φ_c^s with corresponding c_ν and s_ν excluding the case that s is the identity on ω. By transfinite induction we will construct a sequence $(\chi_\nu)_{\nu\in\omega_1}$ of diagonal steppers such that

1. χ_ν follows χ_μ if $\mu < \nu$;
2. χ_ν overcrosses $\chi_\mu \circ s_\nu$ if $\mu < \nu$;
3. χ_ν overcrosses the monotonization of $\Phi_\nu(\chi_\nu)$.

Let M be the monotone group which is generated by the χ_ν. We will show that for all $\nu \in \omega_1$, $\Phi_\nu(\chi_\nu) \notin M$.

To achieve the first two conditions of the construction, we only have to apply the last propositions of the previous section, but for the third condition, we need the following lemma.

Lemma 4.1 *Let s be a monotone non-trivial injection of ω into ω, c an unbounded monotone function and χ_1 a diagonal stepper. Then there is a diagonal stepper χ_2 which follows χ_1 and overcrosses the monotonization of $\Phi_c^s(\chi_2)$.*

Here non-trivial mean $s \neq id_\omega$; notice that for $s = id_\omega$ the lemma is not true.

Proof We define χ_2 by an increasing sequence in ω which alternately gives the fixed points and the jump points of χ_2. We choose the first fixed point of χ_2 as a fixed point of χ_1 which is not a fixed point of s. Because s is monotone, injective and not the identity, s has a nonfixed point and after this, no further fixed point can occur. Assume all fixed points and jump points of χ_2 are defined up to a fixed point r. We choose some jump point g of χ_1 such that $g > s(r)$ as the jump point of χ_2 which follows the fixed point r. After this we choose the next fixed point of χ_2 as a fixed point r' of χ_1 such that $r' \geq gc(g)r$. Obviously, χ_2 follows χ_1.

Now let a be the monotonization of $\Phi_c^s(\chi_2)$. To see that χ_2 overcrosses a, we compare the two functions at the positions g and $g+1$ if g is a jump point of χ_2. Starting with such a g, let r be the last fixed point of χ_2 before g. By the construction of g as described above, we have $g > s(r)$. Thus $s(r)$ lies in the interval $[r, g]$. We choose the largest $n \in \omega$ such that $s(n) \leq g$. Then $r \leq n \leq s(n) \leq g$. Because the function χ_2 is constant on this interval $[r, g]$, we have $\chi_2(n) = r = \chi_2(g)$. Looking at a we have

$$a(g) = a(s(n)) = c(n)\chi_2(n) = c(n)\chi_2(g).$$

It follows that for a given $k > 0$, for almost all $g \in \mathrm{jump}(\chi_2)$

$$k\chi_2(g) \leq a(g) \ .$$

Now we consider a and χ_2 at some $g+1$. If $m \in \omega$ is the greatest integer for which $s(m) \leq g+1$, then $a(g+1) = c(m)\chi_2(m)$. Of course $n \leq m$. We have the cases $\chi_2(m) \neq r$ and $\chi_2(m) = r$. In the first case it follows that $g < m$ and hence $g \leq s(g) < s(m) \leq g+1$. Therefore $s(g) = g$. This is a contradiction to this construction, which started beyond the last fixed point of s. So the second case $\chi_2(m) = r$ holds and hence $m \leq g$. Let r' be the fixed point of χ_2 which follows g. Then $gc(g)r \leq r'$ by the special construction of r'. If we put all this together we get

$$ga(g+1) = gc(m)\chi_2(m) = gc(m)r \leq gc(g)r \leq r' = \chi_2(g+1) \ .$$

Now for a given $k > 0$ and almost all $g \in \mathrm{jump}(\chi_2)$ one gets

$$ka(g+1) \leq ga(g+1) \leq \chi_2(g+1) \ .$$

Hence χ_2 overcrosses a, which completes the proof. □

To prove that the monotone group M is not comfortable we assume, for a contradiction, that $\Phi_c^s(M) \subseteq M$. First we consider the case that $s \neq id_\omega$. We can choose $\nu \in \omega_1$ such that $\Phi_c^s = \Phi_\nu$ and get $\Phi_\nu(\chi_\nu) \in M$. Let a be the monotonisation of $\Phi_\nu(\chi_\nu)$. Then $a \in M$ and hence $a \preceq \max\{\chi_\mu : \mu \in E\}$ for a finite set $E \subset \omega_1$. We can select some $k > 0$ and some $N \in \omega$ such that

$$a(n) \leq k \max\{\chi_\mu(n) : \mu \in E\} \tag{4}$$

for all $n \geq N$.

Let κ be the maximum of E. Since χ_κ follows all χ_μ for $\mu \in E$, we can select N large enough such that all jump points g of χ_κ with $g \geq N$ are also jump points of the χ_μ for $\mu \in E$ and $k\chi_\nu(g) < a(g)$. The latter follows from the fact that χ_ν overcrosses a by applying equation (ii) to $k+1$ instead of k. Now consider such a g. As χ_μ follows χ_ν if $\mu \geq \nu$, we have $\chi_\mu(g) \leq \chi_\nu(g)$ for such μ. By the definition of the function a, the first $n \in \omega$ for which $a(n) = a(g)$ is of the form $s(g'+1)$, where g' is a jump point of χ_ν. For this $n = s(g'+1)$, it follows for $\mu \geq \nu$ that

$$k\chi_\mu(n) \leq k\chi_\mu(g) \leq k\chi_\mu(g) < a(g) = a(n)$$

Now, we consider the case $\mu < \nu$. Because the stepper χ_ν overcrosses the composition $\chi_\mu \circ s$, for almost all relevant g' we have

$$k\chi_\mu(s(g+1)) = k(\chi_\mu \circ s)(g'+1) \leq \chi_\nu(g'+1)\ .$$

We combine this with the fact, that

$$a(n) = a(s(g'+1)) = c(g'+1)\chi_\nu(g'+1)\ .$$

Because c is unbounded, for almost all of the considered n

$$k\chi_\mu(n) = k\chi_\mu(s(g+1)) \leq k\chi_\nu(g'+1) < c(g'+1)\chi_\nu(g'+1) = a(n)\ .$$

We get a contradiction to (iv) for infinitely many specially selected $n \in \omega$ and see that the assumption $\Phi_c^s(M) \subseteq M$ fails in the case of $s \neq id_\omega$.

It remains to prove that $\Phi_c^{id}(M) \nsubseteq M$ for unbounded monotone $c \in \mathbb{Z}^\omega$; we show that $\Phi_c^{id}(\chi_0) \notin M$. Otherwise, there is a finite $E \subset \omega_1$ such that $\Phi_c^{id}(\chi_0) \preceq \max\{\chi_\mu : \mu \in E\}$. As above, there is some $k > 0$ such that

$$c(n)\chi_0(n) = \Phi_c^{id}(\chi_0)(n) \leq k \max\{\chi_\mu(n) : \mu \in E\} \tag{5}$$

for almost all $n \in \omega$. Let κ be the maximum of E. We look at this equation in the cases where n is a jump point g of χ_κ. Since χ_κ follows all the other involved χ_μ, almost all g are jump points of the χ_μ. As all involved χ_μ follow χ_0, we have $\chi_\mu(g) \leq \chi_0(g)$ for infinitely many suitable g. For these g, relation (v) becomes

$$c(g)\chi_0(g) \leq k\chi_0(g)$$

and we have a contradiction to the assumption that c is unbounded and monotone.

5 Final Remarks

The first remark is that our construction works also if we assume Martin's Axiom (MA) instead of CH. Martin's Axiom, which was introduced and investigated by Martin, Solovay and Tennenbaum, is true if CH holds and is consistent with ZFC+¬CH—see [5, 7, 11]. The only thing we need is a generalization of Lemma 3.3 to sequences $(\chi_\nu)_{\nu\in\lambda}$ where λ is an ordinal less than $2^{\aleph_0}$. We consider the sequences $(\text{fix}(\chi_\nu))_{\nu\in\lambda}$ and $(\text{jump}(\chi_\nu))_{\nu\in\lambda}$. Because χ_μ follows χ_ν, if $\nu < \mu$, these sequences have the property that $\text{fix}(\chi_\mu) \setminus \text{fix}(\chi_\nu)$ and $\text{jump}(\chi_\mu) \setminus \text{jump}(\chi_\nu)$ are finite, for $\nu < \mu$. It is a known fact that in MA such sequences of sets have lower bounds, in the sense that there are infinite sets F and J, such that $F\setminus\text{fix}(\chi_\nu)$ and $J\setminus\text{jump}(\chi_\nu)$ are finite for all $\nu \in \lambda$ [5, Exercise 24.17, p. 261]. Now we can define a diagonal stepper χ if we choose fixed points from F and jump points from J alternately. Of course, χ follows χ_ν for all $\nu \in \lambda$. As a consequence of this generalization, Proposition 3.6 holds for a set A if $|A| < 2^{\aleph_0}$.

A second remark is that it is still an open question whether Theorem 2.2 fails for uncomfortable monotone groups.

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Clusterization of Correlation Functions

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Abstract Using the Zhu recursion formulas for correlation functions for vertex operator algebras, we introduce a cluster algebra structure over a non-commutative set of variables.

Keywords Vertex algebras • Correlation functions

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1 Introduction

The deep theory of cluster algebras [5] is connected to many different areas of mathematics. In particular, it has intersections with the theory of Riemann surfaces, the moduli spaces of local systems, higher Teichmüller theory, stability structures, Donaldson–Thomas invariants, dilogarithm identities, and many others, [1–4, 7–9]. Several applications of cluster algebras in conformal field theory are known [3, 9]. Non-trivial but natural definition of seeds and mutations this notion allows to apply this kind of relations in various algebraic configurations. In some sense cluster algebras unify alternative ways of description of previously known structures.

The rich theory of vertex operator algebras which constitute an algebraic language of the conformal field theory are also known. Being a natural generalization for Lie algebras, vertex algebras represent a version of Fourier analysis with non-commutative modes. The expansion of vertex operators in terms of modes allows us to operate in an algebraic manner with analytic structures associated with powers of formal parameters attached to modes. This serves as a tool relating complicated algebraic relations vertex operator algebra modes with descriptions of algebraic-geometry objects.

Since both cluster algebras and vertex algebras represent two classes of quite universal algebraic instrumentation, one would be naturally interested in possible

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connections between these two machineries. In this note we would like to sketch a way to relate cluster algebras [5] with vertex operator algebras [6]. We formulate definition of a vertex operator cluster algebra which possesses a structure similar to an ordinary cluster algebra. The seeds are defined over non-commutative variables, coordinates around marked points, and matrix elements of a number of vertex operators. In [6] it was proven that one can describe a vertex operator algebra by the set of all its correlation functions.

1.1 Cluster and Vertex Operator Algebras

Let $\mathbb{P}$ be an abelian group with binary operation $\oplus$. Let $\mathbb{ZP}$ be the group ring of $\mathbb{P}$ and let $\mathbb{QP}(x_1, \ldots, x_n)$ be the field of rational functions in n variables with coefficients in $\mathbb{QP}$. A seed is a triple $(\mathbf{x}, \mathbf{y}, B)$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a basis of $\mathbb{QP}\,(x_1, \ldots, x_n)$, $\mathbf{y} = \{y_1, \ldots, y_n\}$, is an n-tuple of elements $y_i \in \mathbb{P}$, and B is a skew-symmetrizable matrix. Given a seed $(\mathbf{x}, \mathbf{y}, B)$ its mutation $\mu_k(\mathbf{x}, \mathbf{y}, B)$ in direction k is a new seed $(\mathbf{x}', \mathbf{y}', B')$ defined as follows. Let $[x]_+ = max(x, 0)$. Then we have $B' = (b'_{ij})$ with $b'_{ij} = b_{ij}$ for $i = k$ or $j = k$, and $b'_{ij} = b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+$, otherwise. For new coefficients $\mathbf{y}' = \left(y'_1, \ldots, y'_n\right)$, with $y'_j = y_k^{-1}$ if $j = k$, $y'_j = y_j y_k^{[b_{kj}]+} (y_k \oplus 1)^{-b_{kj}}$ if $j \neq k$, and $\mathbf{x} = \{x_1, \ldots, x_n\}$, where $x'_k = \left(y_k \prod_{i=1}^{n} x_i^{[b_{ik}]+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]+} \right) ((y_k \oplus 1)x_k)^{-1}$. Mutations are involutions, i.e., $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}, \mathbf{y}, B)$.

A vertex operator algebra (VOA) [6] is determined by a quadruple $(V, Y, \mathbf{1}, \omega)$, where is a linear space endowed with a $\mathbb{Z}$-grading with $V = \bigoplus_{r \in \mathbb{Z}} V_r$ with $\dim V_r < Y\infty$. The state $\mathbf{1} \in V_0$, $\mathbf{1} \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ for formal variable z so that for any vector $u \in V$ we have a vertex operator $Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}$. The linear operators (modes) $u(n) : V \to V$ satisfy creativity $Y(u, z)\mathbf{1} = u + O(z)$, and lower truncation $u(n)v = 0$, conditions for each u, $v \in V$ and $n \gg 0$. For the conformal vector ω one has $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, where $L(n)$ satisfies the Virasoro algebra for some central charge C: $[\, L(m), L(n)\,] = (m - n)L(m + n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}\mathrm{Id}_V$, where Id_V is identity operator on V. Each vertex operator satisfies the translation property $Y(L(-1)u, z) = \partial_z Y(u, z)$. The Virasoro operator $L(0)$ provides the $\mathbb{Z}$-grading with $L(0)u = ru$ for $u \in V_r$, $r \in \mathbb{Z}$. Finally, the vertex operators satisfy the Jacobi identity which we omit here. These axioms imply locality, $(z_1 - z_2)^N Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^N Y(v, z_2)Y(u, z_1)$, skew-symmetry, $Y(u, z)v = e^{zL(-1)} Y(v, -z)u$, associativity $(z_0 + z_2)^N Y(u, z_0 + z_2)Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w$, and commutativity $u(k)Y(v, z) - Y(v, z)u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j)v, z) z^{k-j}$, conditions for u, v, $w \in V$ and integers $N \gg 0$. For $v = \mathbf{1}$ one has $Y(\mathbf{1}, z) = \mathrm{Id}_V$. Note also that modes of homogeneous states are graded operators on V, i.e., for $v \in V_k$, $v(n) : V_m \to$

$V_{m+k-n-1}$. In particular, let us define the zero mode $o(v)$ of a state of weight $wt(v) = k$, i.e., $v \in V_k$, as $o(v) = v(wt(v) - 1)$, extending to V additively.

1.2 Correlation Functions of Genus Zero and One Riemann Surfaces

We define the restricted dual space of V by Frenkel [6]. Let V be a vertex operator algebra. $V' = \bigoplus_{n \geq 0} V_n^*$, where V_n^* is the dual space of linear functionals on the finite dimensional space V_n. Let $\langle ., . \rangle$ denote the canonical pairing between V' and V. Define matrix elements for $v' \in V'$, $v \in V$ and n vertex operators $Y(v_1, z_1)$, …, $Y(v_n, z_n)$ by $\langle v', Y(v_1, z_1) \ldots Y(v_n, z_n) v \rangle$. Choosing $v = \mathbf{1}$ and $v' = \mathbf{1}'$ we obtain the n-point correlation function on the sphere: $F_V^{(0)}(v_1, z_1; \ldots; v_n, z_n) = \langle \mathbf{1}', Y(v_1, z_1) \ldots Y(v_n, z_n) \mathbf{1} \rangle$. Here the upper index of $F^{(0)}$ stands for the genus. For $u \in V_n$,

$$u(k) : V_m \to V_{m+n-k-1}. \tag{1}$$

Hence it follows that for $v' \in V'_{m'}$, $v \in V_m$, and $u \in V_n$ we obtain a monomial $\langle v', Y(u, z) v \rangle = C^u_{v'v} z^{m'-m-n}$, where $C^u_{v'v} = \langle v', u(m + n - m' - 1) v \rangle$. Recall now the following formal expansion: for variable x, y we adopt the convention that $(z_1 + z_2)^m = \sum_{n \geq 0} \binom{m}{n} z_1^{m-n} z_2^n$, i.e., for $m < 0$ we formally expand in the second parameter z_2. Using the vertex commutator property, i.e., $[u(m), Y(v, z)] = \sum_{i \geq 0} \binom{m}{i} Y(u(i) v, z) z^{m-i}$, one can also derive [10] a recursive relationship. In [10] we find a recurrent formula expressing an $n + 1$-point matrix element on the sphere as a finite sum of n-point matrix elements [10, Lemma 2.2.1]. For $v_1, \ldots, v_n \in V$, and a homogeneous $v \in V$, we find

$$\begin{aligned} &\langle v', Y(v_1, z_1) \ldots Y(v_n, z_n) v \rangle \\ &= \sum_{r=2}^{n} \sum_{m \geq 0} f_{wt(v_1),m}(z_1, z_r) \cdot \langle v', Y(v_2, z_2) \ldots Y(v_1(m)\, v_r, z_r) \ldots Y(v_n, z_n) v \rangle \\ &\quad + \langle v', o(v_1)\, Y(v_2, z_2) \ldots Y(v_n, z_n) v \rangle, \end{aligned} \tag{2}$$

where $f_{wt(v_1),m}(z_1, z_r)$ is a rational function defined by $f_{n,m}(z, w) = \frac{z^{-n}}{m!} \left(\frac{d}{dw} \right)^m \frac{w^n}{z-w}$. $\iota_{z,w} f_{n,m}(z, w) = \sum_{j \in \mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1}$. In order to consider modular-invariance of n-point functions at genus one, Zhu introduced [10] a second "square-bracket" VOA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ associated with a given VOA $(V, Y(,), \mathbf{1}, \omega)$. The new square bracket vertex operators are $Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1)$, with $q_z = e^z$, while the new conformal vector is $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and

$m \geq 0$, $v[m] = m! \sum\limits_{i \geq m} c(wt(v), i, m) v(i)$, where $\sum\limits_{m=0}^{i} c(wt(v), i, m) x^m = \binom{wt(v)-1+x}{i}$. For $v_1, \ldots, v_n \in V$ the genus one n-point function [10] has the form

$$F_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau) = Tr_V \left(Y(q_1^{L(0)} v_1, q_1) \ldots Y(q_n^{L(0)} v_n, q_n) \, q^{L(0)-C/24} \right),$$

for $q = e^{2\pi i \tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parameter. Then the genus one Zhu recursion formula is given by the following [10]. For any $v, v_1, \ldots, v_n \in V$ we find for an $n+1$-point function

$$\begin{aligned} & F_V^{(1)}(v, z; v_1, z_1; \ldots; v_n, z_n; \tau) \\ & = \sum_{r=1}^{n} \sum_{m \geq 0} P_{m+1}(z - z_r, \tau) \cdot F_V^{(1)}(v_1, z_1; \ldots; v[m] v_r, z_r; \ldots; v_n, z_n; \tau) \\ & \quad + F_V^{(1)}\left(o(v); v_1, z_1; \ldots; v_n, z_n; \tau\right), \end{aligned} \tag{3}$$

$F_V^{(1)}\left(o(v); v_1, z_1; \ldots; v_n, z_n; \tau\right) = Tr_V \Big(o(v) \; Y(q_1^{L(0)} v_1, q_1) \ldots Y(q_n^{L(0)} v_n, q_n)$ $q^{L(0)-C/24} \Big)$. In this theorem $P_m(z, \tau)$ denote higher Weierstrass functions defined by $P_m(z, \tau) = \frac{(-1)^m}{(m-1)!} \sum\limits_{n \in \mathbb{Z}_{\neq 0}} \frac{n^{m-1} q_z^n}{1 - q^n}$.

2 Cluster Structure for a Vertex Operator Algebra Correlation Functions

Fix a vertex operator algebra V. Choose n-marked points p_i, $i = 1, \ldots, n$ on a compact Riemann surface. In the vicinity of each marked point p_i define a local coordinate z_i with zero at p_i. Consider n-tuples $\mathbf{v} \equiv \{v_1, \ldots, v_n\}$, of arbitrary states $v_i \in V$, and local corresponding vertex operators $\mathbf{Y}(\mathbf{v}, \mathbf{z}) \equiv \{Y(v_1, z_1), \ldots, Y(v_n, z_n)\}$, with coordinates $\mathbf{z} \equiv \{z_1, \ldots, z_n\}$ around p_i, $i = 1, \ldots, n$. We define a *vertex operator cluster algebra seed*

$$\left(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})\right), \tag{4}$$

where $F_n(\mathbf{v}, \mathbf{z}) \equiv F_n(v_1, z_1; \ldots; v_n, z_n)$ is an n-point correlation function (matrix element for the sphere case) for n states v_i. Now, define the mutation $\mu_k(v, m, z)$:

$$\left(\mathbf{v}', \mathbf{Y}(\mathbf{v}', \mathbf{z}), F_n'(\mathbf{v}', \mathbf{z})\right) = \mu_k(v, m, z) \left(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})\right), \tag{5}$$

of the seed (4) in direction $k \in 1, \ldots, n$ for $v \in V$, according to the Zhu reduction formula for corresponding Riemann surface genus, e.g., for the sphere as in (2), for the torus as in (3), etc. Namely, for $\mathbf{v}$, we define $\mathbf{v}'$ as the mutation of $\mathbf{v}$ in direction $k \in 1, \ldots, n$ as

$$\mathbf{v}' = \mu_k(v, m, z)\mathbf{v} = (v_1, \ldots, v(m)v_k \ldots, v_n), \tag{6}$$

for some $m \geq 0$. Note that due to the lower truncation property we get a finite number of terms as a result of the action of $v(m)$ on v_r. For the n-tuple of vertex operators we define

$$\mathbf{Y}(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z)\mathbf{Y}(\mathbf{v}, \mathbf{z}) = (Y(v_1, z_1), \ldots, Y(v(m)v_k, z_k), \ldots, Y(v_n, z_n)) . \tag{7}$$

The mutation

$$F'_n(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z)F_n(\mathbf{v}, \mathbf{z}), \tag{8}$$

is defined by summing over mutations in all possible directions with auxiliary functions $f(\mathrm{wt}\ \ \mathrm{v}, m, k, z)$, $k \in 1, \ldots, n$ and all $m \geq 0$:

$$\begin{aligned} F'_n(\mathbf{v}', \mathbf{z}) &= \mu_k(v, m, z)F_n(v_1, z_1; \ldots; v_n, z_n) \\ &= \sum_{k=1}^{n} \sum_{m \geq 0} f(\mathrm{wt}\ \ \mathrm{v}, m, k, z)F_n(v_1, z_1; \ldots; v(m)v_k, z_k; \ldots; v_n, z_n) + \widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}), \end{aligned} \tag{9}$$

where $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z})$ denote higher terms in the Zhu reduction formula for a specific genus of a Riemann surfaces used in the consideration. In particular, for the genus zero case we have $f(\mathrm{wt}\ \ \mathrm{v}, m, k, z) = f_{\mathrm{v},m}(z, z_k)$ for some $m \geq 0$, $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(0)}(o(v); \mathbf{v}, \mathbf{z}) = \langle \mathbf{1}', o(v) \quad Y(v_1, z_1) \ldots Y(v_n, z_n)\mathbf{1} \rangle$, while for the genus one Riemann surface we take and $f(\mathrm{wt}\ v, m, k, z) = P_{m+1}(z - z_k; \tau)$ given by $P_m(z, \tau)$, $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(1)}(o(v); \mathbf{v}, \mathbf{z}) = Tr_V\,(o(v)Y(v_1, z_1) \ldots Y(v_n, z_n))$. The mutation $\mu_k(v, m, z)$ defined by (6)–(9) is an involution, i.e.,

$$\mu_k(v, m, z)\mu_k(v, m, z)\,(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})) = (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z}))\,,$$

subject a few conditions. As the first condition, one can take $v(m)v(m)v_k = v_k$, $k = 1, \ldots, n$ for the actions (6)–(7). The simplest case, in particular, for $v \in V_k$, for some specific $k = 1, \ldots, n$, when $k - m - 1 = 0$, then $v(m) = o(v) \equiv v(\mathrm{wt}\ v - 1)$. Then due to the property (1), $v(m)v(m) : V_p \longrightarrow V_p$. Note that when we sum in (9) over mutations in all possible directions $k \in 1, \ldots, n$ and all $m \geq 0$, we obtain a correlation function (matrix element for the sphere) of rank $n + 1$ (see (2) and (3)) with extra $v \in V$ inserted at a point p with corresponding local coordinate z:

$$F^{(g)}_{n+1}(v, z; v_1, z_1; \ldots; v_n, z_n; \tau)$$

$$= \sum_{k=1}^{n} \sum_{m \geq 0} f(\text{wt } v, m, k, z) \cdot F_n^{(g)}(v_1, z_1; \ldots; v(m)v_k \ldots; v_n, z_n; \tau) + \widetilde{F}_n^{(g)}(v, z; \mathbf{v}, \mathbf{z}).$$

When we reduce $F_n^{(g)}(v_1, z_1; \ldots; v(m)v_k \ldots; v_n, z_n)$ in (9) to the partition function $F_0^{(g)}$ (i.e., the zero point function) according to the Zhu reduction formulas ((2) or (3)), we obtain multiple action of modes $\prod_{m \geq 0} v_r(m)$ on various v_k as well as products of $f(\text{wt } v_r, m_r, r, z_r)$ functions as a result of action on z_k.

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