

Boundary Terms of the Einstein–Hilbert Action

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Abstract The Einstein–Hilbert action for general relativity is not well posed in terms of the metric g_{ab} as a dynamical variable. There have been many proposals to obtain an well posed action principle for general relativity, e.g., addition of the Gibbons–Hawking–York boundary term to the Einstein–Hilbert action. These boundary terms are dependent on what one fixes on the boundary and in particular on spacetime dimensions as well. Following recent works of Padmanabhan we will introduce two new variables to describe general relativity and the action principle with these new dynamical variables will turn out to be well posed. Then we will connect these dynamical variables and boundary term obtained thereof to existing literature and shall comment on a few properties of Einstein–Hilbert action which might have been unnoticed earlier in the literature. Before concluding with future prospects and discussions, we will perform a general analysis of the boundary term of Einstein–Hilbert action for null surfaces as well.

1 Introduction

Action principle is the starting point of any field theory. Along with the action functional one need to fix the spacetime volume, its boundary and what variable should be fixed on the boundary. When the boundary conditions imposed on an action are compatible with the derived field equation(s), we refer that action principle as well posed. It turns out that the widely used action principle for general relativity, the Einstein–Hilbert action is *not* well posed. To be more precise, with Ricci scalar as the gravitational Lagrangian, derivation of Einstein’s equations requires fixing both metric and its first derivative on the boundary — inconsistent with Einstein’s equations.

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This feature arises, since the action principle for general relativity is peculiar. It contains second derivatives of the dynamical variables, the metric g_{ab} , unlike any other existing Lagrangians. At first glance it seemed quite exotic, since the field equations derived from an action which has second derivatives of the dynamical variable are supposed to have third order derivatives, leading to existence of ghost fields. However it is again the structure of the action principle for general relativity that comes to rescue. The Ricci scalar can be separated into a bulk term and a surface term. The bulk term has the structure Γ^2 , where Γ_{bc}^a are the connection coefficients and along with being quadratic it contains only first derivatives of the metric. In any action principle the surface terms do not contribute to the derivation of field equations, so Einstein's equations also have second derivatives of the metric. However all the second derivatives of the metric hides in the surface term and it is the surface term that leads to boundary contribution. Hence quite naturally, in the case of Einstein–Hilbert action one ends up fixing both the metric and its derivative on the boundary.

The above arguments pose the problem but also solves it — it suffices to remove the surface term and consider a new action functional for general relativity, namely, $L = R - L_{\text{sur}}$, as proposed by Einstein in 1916 [1]. Then one obtains Einstein's equations without worrying about the boundary terms. But the problem with the above approach is that, the action is not invariant under diffeomorphism, while we want every action to have the symmetries that the underlying system has. Fortunately, the boundary term that one need to add to the Einstein–Hilbert action is by no means unique. Any boundary term that kills all the normal derivatives of the metric on the boundary surface is good enough for our purpose and there could be infinitely many of them as demonstrated by Charap and Nelson in [2]. The most popular boundary term that keeps the action invariant under diffeomorphism and also makes it well posed is the Gibbons–Hawking–York term [3–5]. The Gibbons–Hawking–York term depends on the extrinsic curvature K of the boundary surface and is given by $2K\sqrt{|h|}$, where h stands for the determinant of the induced metric on the boundary surface. Note that even though the Gibbons–Hawking–York boundary term is invariant under diffeomorphism, is not covariant in a strict sense, because of its dependence on the foliation. Further, the Gibbons–Hawking–York term was guessed and then shown to yield a well posed variational principle without a first principle derivation. This gap was filled by providing a direct derivation of the Gibbons–Hawking–York boundary term from the action itself in [6] while another important issue, the boundary term for null boundaries has been tackled recently in [7]. Even then the structure of the boundary term can change depending on what one needs to fix on the boundary, the induced metric or the conjugate momentum and it also changes depending on the spacetime dimensions. In this work we will try to provide a broad overview on the possible boundary term structures of the Einstein–Hilbert action along with what one needs to fix on the boundary surfaces. This will be performed for both null and non-null cases, besides discussing some other important features of the Einstein–Hilbert action.

The paper is organized as follows, in Sect. 2 we will present various boundary terms used in various dimensions for an well-posed action of general relativity and their possible connections. Then in Sect. 3 we will explicitly demonstrate some

common notions in the context of general relativity starting from the well known $(1 + 3)$ decomposition. Finally we comment on the nature of the boundary terms in the context of null surfaces in Sect. 4 before concluding with a brief discussion.

Notation: We will work in D spacetime dimensions in Sect. 2, while the rest of the analysis will be performed in four spacetime dimensions following the mostly positive signature $(-, +, +, +, \dots)$. The fundamental constants c , G and \hbar have been set to unity.

2 Reconciling Boundary Terms for the Einstein–Hilbert Action

The origin of boundary value problem for general relativity is due to the fact that Einstein–Hilbert action contains second derivatives of the metric — as a consequence one needs to fix both the metric and its derivatives on the boundary rendering the action ill posed. The above problem arises for using the metric as a fundamental variable and hence to obtain a well posed variational principle we have to add boundary terms to the Einstein–Hilbert action. However, it is possible to rewrite the Einstein–Hilbert action in the momentum space and the resulting variational principle becomes well posed. The momentum space representation of the Einstein–Hilbert action can be obtained by introducing two new variables [8] (see [9] for a generalization to Lanczos–Lovelock gravity),

$$f^{ab} = \sqrt{-g}g^{ab}; \quad N_{bc}^a = Q_{be}^{ad}\Gamma_{cd}^e + Q_{ce}^{ad}\Gamma_{bd}^e = -\Gamma_{bc}^a + \frac{1}{2}(\Gamma_{bd}^d\delta_c^a + \Gamma_{cd}^d\delta_b^a), \quad (1)$$

where f^{ab} is a tensor density and N_{bc}^a stands for a linear combination of the connections. Note that the above relation holds for any number of spacetime dimensions as $Q_{cd}^{ab} = (1/2)(\delta_c^a\delta_d^b - \delta_d^a\delta_c^b)$ is independent of spacetime dimensions. However the inverse relation connecting Γ_{bc}^a in terms of N_{bc}^a depends on the spacetime dimensions and reads in general,

$$\Gamma_{ab}^c = -N_{ab}^c + \frac{1}{D-1}(N_{ad}^d\delta_b^c + N_{bd}^d\delta_a^c), \quad (2)$$

which reduces to the expression in [8] for $D = 4$. Then the expressions for various curvature components are also modified. For example, the Ricci tensor can be expressed in terms of N_{ab}^c such that,

$$R_{ab} = -\left(\partial_c N_{ab}^c + N_{ad}^c N_{bc}^d - \frac{1}{D-1}N_{ac}^c N_{bd}^d\right), \quad (3)$$

reducing to the one given in [8] for four spacetime dimensions. These variables can be used in the action principle as well, in which case the Einstein–Hilbert Lagrangian density becomes $\sqrt{-g}R = f^{ab}R_{ab}$, where R_{ab} can be written in terms of N_{ab}^c following Eq. (3). This leads to momentum space representation of the Einstein–Hilbert action, which follows from the result that $N_{ab}^c = \partial(\sqrt{-g}R)/\partial(\partial_c f^{ab})$ and hence the set (f^{ab}, N_{ab}^c) acts as a set of canonically conjugate variables. Further Einstein–Hilbert action when varied reads in terms of variations of these canonically conjugate variables as,

$$\begin{aligned} \delta \left(\int_{\mathcal{V}} d^D x \sqrt{-g} R \right) &= \int_{\mathcal{V}} d^D x R_{ab} \delta f^{ab} - \int_{\mathcal{V}} d^D x f^{ab} \nabla_c \delta N_{ab}^c \quad (4) \\ &= \int_{\mathcal{V}} d^D x R_{ab} \delta f^{ab} - \int_{\partial\mathcal{V}} d^{D-1} x \bar{n}_c f^{ab} \delta N_{ab}^c, \quad (5) \end{aligned}$$

where \mathcal{V} stands for the spacetime volume under interest with boundary being denoted by $\partial\mathcal{V}$. The last term has been obtained through the use of the following relation $f^{ab} \nabla_c \delta N_{ab}^c = \partial_c (\sqrt{-g} g^{ab} \delta N_{ab}^c)$. Also \bar{n}_c in the final expression is the unnormalized normal. If the surface $\partial\mathcal{V}$ is some $\phi = \text{constant}$ surface, then $\bar{n}_c = \delta_c^\phi$. With suitable normalization one obtains, $\bar{n}_c = \varepsilon(1/N)n_c$, where n_c is the normalized normal, $\varepsilon = \pm 1$ depending on the normal being spacelike or timelike and N is $\sqrt{|g^{\phi\phi}|}$. Thus note that one can obtain the Einstein’s equations provided N_{ab}^c is fixed at the boundary, leading to an well posed action principle for general relativity, since N_{ab}^c and f^{ab} are treated as independent variables.

On the other hand, it is also well known that the variation of the Einstein–Hilbert action leads to $\delta(2K\sqrt{h})$, where K is the extrinsic curvature of the boundary surface and h is the determinant of the induced metric on that surface, along with variations of the induced metric with proper coefficients as the boundary term [6]. Thus for being consistent one must have the $f^{ab} \delta N_{ab}^c$ to yield $\delta(2K\sqrt{h})$ along with variations of the induced metric. It is not at all clear a priori, how this can be achieved. In order to fill this gap we would like to connect the boundary term obtained above in Eq. (5) with the standard literature. As a first step towards the connection, we will present a simplified analysis and shall subsequently provide a general derivation.

2.1 A Warm-Up Example: Analysis in Synchronous Frame

Before jumping into the formal derivation let us consider an explicit example as a warm-up. Let us use all the gauge degrees of freedom due to diffeomorphism to eliminate four degrees of freedom from the metric and reduce it to synchronous form, in which the line element reads,

$$ds^2 = -d\tau^2 + h_{\alpha\beta}(\tau, x^\mu) dx^\alpha dx^\beta. \quad (6)$$

As explicitly demonstrated in [10], any metric can be written in the synchronous coordinate system. The boundary $\partial\mathcal{V}$ of the full spacetime volume can be taken to be $\tau = \text{constant}$ hypersurface in this coordinate system, such that the unnormalized normal becomes $\bar{n}_c = \delta_c^\tau$ and hence the surface term reads,

$$\bar{n}_c f^{ab} \delta N_{ab}^c = f^{ab} \delta N_{ab}^0 = -\sqrt{h} \delta N_{00}^0 + \sqrt{h} h^{\alpha\beta} \delta N_{\alpha\beta}^0, \quad (7)$$

where in obtaining the last line we have used the synchronous frame metric as in Eq. (6). From the definition of N_{bc}^a in terms of connections as in Eq. (1) and the metric in Eq. (6) it follows that,

$$N_{00}^0 = \Gamma_{0\alpha}^\alpha = -K; \quad N_{\alpha\beta}^0 = -\Gamma_{\alpha\beta}^0 = K_{\alpha\beta}. \quad (8)$$

Thus one can substitute both N_{00}^0 and $N_{\alpha\beta}^0$ in the boundary term which finally leads to,

$$\begin{aligned} \bar{n}_c f^{ab} \delta N_{ab}^c &= \sqrt{h} \delta K + \sqrt{h} h^{\alpha\beta} \delta K_{\alpha\beta} \\ &= \delta \left(2K \sqrt{h} \right) + \sqrt{h} \left(K^{\alpha\beta} - K h^{\alpha\beta} \right) \delta h_{\alpha\beta}. \end{aligned} \quad (9)$$

This shows the equivalence of the boundary term with (f^{ab}, N_{ab}^c) as the dynamical variables with the standard boundary term. The above expression explicitly shows that one needs to add $2K\sqrt{h}$ as the boundary term to the Einstein–Hilbert action and as a consequence one needs to fix only the spatial part of the metric $h_{\alpha\beta}$ on the boundary $\partial\mathcal{V}$, i.e., on $\tau = \text{constant}$ surfaces.

However the above derivation is a special case and more importantly the boundary term even though is independent of coordinate choices depends heavily on foliation, thus it is not clear from the above result whether the same conclusion should hold for arbitrary foliation as well. This is precisely what we will prove next.

2.2 Boundary Terms: A General Analysis

As explained above the demonstration in synchronous frame is a specific one among many possible foliations and one needs to provide a general analysis for an arbitrary foliation to grasp the complete structure. To proceed with the general analysis, we will start with the boundary term and shall write N_{ab}^c in terms of the connections. Using the fact that variations of the connections are tensors one can ultimately write down the boundary term in terms of the normal and variations in the metric tensor,

$$\int_{\partial\mathcal{V}} d^{D-1} x \bar{n}_c f^{ab} \delta N_{ab}^c = - \int_{\partial\mathcal{V}} d^{D-1} x \bar{n}_c \nabla_d \left(-\delta g^{cd} + g^{cd} g_{ik} \delta g^{ik} \right), \quad (10)$$

where the following algebraic identity, $-g^{ab}\delta N_{ab}^c = \nabla_d(-\delta g^{cd} + g^{cd}g_{ik}\delta g^{ik})$ have been used in order to arrive at the final result. Given the above Eq.(10) we can immediately incorporate the normal inside the covariant derivative and the above expression reads,

$$-\int_{\partial\mathcal{V}} d^{D-1}x\bar{n}_c f^{ab}\delta N_{ab}^c = \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\sqrt{h}\left\{\nabla_d(-n_c\delta g^{cd} + n^d g_{ik}\delta g^{ik}) - \nabla_d n_c(-\delta g^{cd} + g^{cd}g_{ik}\delta g^{ik})\right\}, \quad (11)$$

where $\varepsilon = -1$ for spacelike hypersurfaces and is $+1$ for timelike hypersurfaces respectively. The variations of the metric can be divided into two pieces, variations in the induced metric h_{ij} and variations in the normal n^i . Using the contractions properly and the fact that $\delta(n_i n^i) = 0$, we immediately obtain the following expression for the boundary term of the Einstein–Hilbert action,

$$-\int_{\partial\mathcal{V}} d^{D-1}x\bar{n}_c f^{ab}\delta N_{ab}^c = \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta(2K\sqrt{h}) - \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\sqrt{h}(K_{ab} - Kh_{ab})\delta h^{ab} + \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\sqrt{h}D_i(-n_c h_b^i \delta g^{bc} + 2n_k h_i^j \delta g^{kl}). \quad (12)$$

The last term is again a surface term and would contribute only on the two surface and hence is neglected. It is useful and instructive to define the momentum conjugate to the induced metric h_{ab} on the hypersurface $\partial\mathcal{V}$ as,

$$\Pi_{ab} = \sqrt{h}(K_{ab} - Kh_{ab}). \quad (13)$$

Note that $n_a \Pi^{ab} = 0$. Thus finally using the expression for Π_{ab} and neglecting the surface term, we obtain the simplified version of the boundary term from Eq.(12) in the most general case as,

$$-\int_{\partial\mathcal{V}} d^{D-1}x\bar{n}_c f^{ab}\delta N_{ab}^c = \int_{\partial\mathcal{V}} d^3x\varepsilon\delta(2K\sqrt{h}) - \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\Pi_{ab}\delta h^{ab}. \quad (14)$$

The result in the synchronous frame can be derived immediately from the above relation by substituting $\varepsilon = -1$, since $\tau = \text{constant}$ surfaces are spacelike. However note that the two-dimensional surface terms identically vanishes in the synchronous frame. The above result suggests that if we add the boundary term $-2\varepsilon K\sqrt{h}$ to the Einstein–Hilbert action the normal derivatives of the metric will be removed from the boundary and one needs to fix only the induced metric h^{ab} . It is important to emphasis at this stage that fixing h^{ab} is different from fixing h_{ab} . Since by construction we have $n_a \propto \nabla_a \phi$, and $n_a h^{ab} = 0$, this suggests $h^{ab} = h^{\alpha\beta}$, where α, β are spacetime indices

excluding ϕ , while h_{ab} has all the metric components. Due to the momentum and Hamiltonian constraints of general relativity one cannot fix all the metric components on the hypersurfaces and hence the correct variational principle would be the one which fixes only h^{ab} , i.e., $h^{\alpha\beta}$ on the boundary $\partial\mathcal{V}$.

Let us now illustrate the fact that $2\varepsilon K\sqrt{h}$ is not the only boundary term that can lead to a well-posed action principle for general relativity, there are infinitely many. However for our illustration we will pick two of them. Since we are working in a D dimensional spacetime we have the following identity, $\Pi_{ab}h^{ab} = -(D-2)K\sqrt{h}$. We can use the above identity to convert the original result in Eq. (14) to two different results,

$$\begin{aligned}
 -\int_{\partial\mathcal{V}} d^{D-1}x\bar{n}_c f^{ab}\delta N_{ab}^c &= \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left(2K\sqrt{h}\right) - \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left(\Pi_{ab}h^{ab}\right) \\
 &+ \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon h^{ab}\delta\Pi_{ab} \\
 &= \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left(DK\sqrt{h}\right) \\
 &+ \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon h^{ab}\delta\Pi_{ab} .
 \end{aligned} \tag{15}$$

The above result depicts that one can also add $-D\varepsilon K\sqrt{h}$ as the boundary term to the Einstein–Hilbert action and hence obtain an well-posed variational principle if Π_{ab} is fixed at the boundary. Note that as we have argued earlier, the only non-zero components of h^{ab} are $h^{\alpha\beta}$ and hence one need to fix only $\Pi_{\alpha\beta}$ at the boundary $\partial\mathcal{V}$. This result can also be casted in a different form, for that we need to use the identity, $\Pi_{ab}\delta h^{ab} = -\Pi^{ab}\delta h_{ab}$. Use of which enables one to write Eq. (14) in the following form

$$\begin{aligned}
 -\int_{\partial\mathcal{V}} d^{D-1}x\bar{n}_c f^{ab}\delta N_{ab}^c &= \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left(2K\sqrt{h}\right) + \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left(\Pi^{ab}h_{ab}\right) \\
 &- \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon h_{ab}\delta\Pi^{ab} \\
 &= \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon\delta \left[(4-D)K\sqrt{h}\right] \\
 &- \int_{\partial\mathcal{V}} d^{D-1}x\varepsilon h_{ab}\delta\Pi^{ab} .
 \end{aligned} \tag{16}$$

This is another form of the boundary contribution recently discussed in [11] which essentially follows from the original boundary term in terms of the canonically conjugate variables (f^{ab}, N_{ab}^c) . In this case the boundary term one has to add to the Einstein–Hilbert action corresponds to, $(4-D)\varepsilon K\sqrt{h}$, with the peculiarity that at $D=4$ this term identically vanishes. While in this case one need to fix Π^{ab} at the boundary $\partial\mathcal{V}$. Hence the original boundary term from which all possible versions of

the boundary terms including the well-known $2\varepsilon K\sqrt{h}$ can be derived is the $f^{ab}\delta N_{ab}^c$ combination. Further we have shown two explicit examples in which one can add different boundary term at the expense of fixing either Π^{ab} or Π_{ab} at the boundary (Table 1). Even though it is tempting to assume $h_{ab}\delta\Pi^{ab} = -h^{ab}\delta\Pi_{ab}$, this relation is actually not correct. This can be seen from the following algebraic manipulation straightforwardly,

$$\begin{aligned} h^{ab}\delta\Pi_{ab} &= h^{ab}\delta(h_{ac}h_{bd}\Pi^{cd}) = h_{cd}\delta\Pi^{cd} + 2\Pi^{ac}\delta h_{ac} \\ &= h_{cd}\delta\Pi^{cd} - 2\Pi_{ac}\delta h^{ac} = -h_{cd}\delta\Pi^{cd} + \delta\left[(4-2D)K\sqrt{h}\right], \end{aligned} \quad (17)$$

reconciling the two results presented in Eqs. (15) and (16) respectively. Through this exercise we have achieved two important goals, which are,

- By introducing the canonically conjugate variables (f^{ab}, N_{ab}^c) , one obtains the Einstein's equations from variations of f^{ab} , while variations of N_{ab}^c leads to the boundary term. Hence the Einstein–Hilbert action becomes action in the momentum space such that one need to fix the momentum N_{ab}^c at the boundary. However there were no clear consensus how this boundary term is related to the existing ones, e.g., the Gibbons–Hawking–York boundary term. In this section we have explicitly demonstrated the connection, by deriving the Gibbons–Hawking–York counter term starting from the boundary term consisting of $f^{ab}\delta N_{ab}^c$.
- Secondly, in most of the literatures people always take the Gibbons–Hawking–York boundary term to be the only boundary term possible. In the last part of this section we have explicitly demonstrated two more boundary terms. Our result clearly shows that the structure of the boundary term depends crucially on what one fixes at the boundary. If one fixes the induced metric h^{ab} , then Gibbons–Hawking–York term is the only option. But if one fixes the conjugate momentum, then depending on whether one fixes Π^{ab} or Π_{ab} , one arrives to different boundary terms. In particular when Π^{ab} is fixed one need not add any boundary term in four dimensions, which is a peculiar feature of general relativity.

Thus we have reconciled the possible boundary terms that one can add to the Einstein–Hilbert action. Their non-uniqueness and derivation from a first principle starting from Einstein–Hilbert action in momentum space has also been presented. We will now turn to the $(1+3)$ decomposition of the Einstein–Hilbert action and related comments.

Table 1 A comparison of various boundary terms of Einstein–Hilbert action

Bulk term	Surface term	Boundary term ^a	What to fix on boundary	Well-posed action
$R_{ab}\delta f^{ab}$	$-\bar{n}_c f^{ab}\delta N_{ab}^c$	None	N_{ab}^c	$\sqrt{-g}R$
$G_{ab}\delta g^{ab}$	$\varepsilon\delta(2K\sqrt{h})$ $-\varepsilon\Pi_{ab}\delta h^{ab}$	$\varepsilon\delta(2K\sqrt{h})$	h^{ab}	$\sqrt{-g}R - \varepsilon\delta(2K\sqrt{h})$
$G_{ab}\delta g^{ab}$	$\varepsilon\delta(DK\sqrt{h})$ $\varepsilon h^{ab}\delta\Pi_{ab}$	$\varepsilon\delta(DK\sqrt{h})$	Π_{ab}	$\sqrt{-g}R - \varepsilon\delta(DK\sqrt{h})$
$G_{ab}\delta g^{ab}$	$\varepsilon\delta[(4-D)K\sqrt{h}]$ $-\varepsilon h_{ab}\delta\Pi^{ab}$	$\varepsilon\delta[(4-D)K\sqrt{h}]$	Π^{ab}	$\sqrt{-g}R - \varepsilon\delta[(4-D)K\sqrt{h}]$

^aNote that in the last case for $D = 4$ no boundary term is needed and Einstein–Hilbert action is well posed, with Π^{ab} fixed on the boundary (see also [9])

3 (1+3) Decomposition, Time Derivatives and Canonical Momenta

In general relativity space and time are treated on an equal footing. However for many application, e.g., canonical quantization schemes, one need the notion of time and hence the splitting of four dimensional spacetime into one time and three spatial coordinates becomes immediate. This has been performed successfully by Arnowitt, Deser and Misner (henceforth referred to as ADM) in a seminal work [12], in which the ten independent metric components are split into three pieces — $h_{\alpha\beta}$, N^α and N , such that, the line element becomes

$$ds^2 = -N^2 dt^2 + h_{\alpha\beta} (dx^\alpha + N^\alpha dt) (dx^\beta + N^\beta dt) . \quad (18)$$

Thus note that the spatial metric $g_{\alpha\beta}$ is just $h_{\alpha\beta}$, the off-diagonal entries are $N_\alpha \equiv h_{\alpha\beta} N^\beta$, while the temporal component of the metric becomes, $g_{00} = -N^2 + h_{\alpha\beta} N^\alpha N^\beta$. For the inverse metric the temporal component is simple but not the spatial components such that,

$$g^{tt} = -\frac{1}{N^2}, \quad g^{t\alpha} = \frac{N^\alpha}{N^2}, \quad g^{\alpha\beta} = \left(h^{\alpha\beta} - \frac{N^\alpha N^\beta}{N^2} \right) . \quad (19)$$

The next point one can address from the ADM splitting corresponds to the (1 + 3) decomposition of the Einstein–Hilbert action. This would require projection of the Riemann tensor components on the spacelike hypersurface, leading to ${}^{(3)}R$, the Ricci scalar of the spacelike hypersurface and invariants like $K_{ab}K^{ab}$, K^2 constructed out of the extrinsic curvature components [12, 13]

$$\begin{aligned} \sqrt{-g}R &= \sqrt{-g} \left[{}^{(3)}R + K_{ab}K^{ab} - K^2 - 2\nabla_i (Kn^i + a^i) \right] \\ &= \sqrt{-g}L_{\text{ADM}} - 2\sqrt{-g}\nabla_i (Kn^i + a^i) , \end{aligned} \quad (20)$$

where n_i is the normal to the spacelike hypersurface and a^i is the corresponding acceleration. Thus the Einstein–Hilbert Lagrangian can be written in terms of the ADM Lagrangian and an additional boundary term which coincides with the Gibbons–Hawking–York counter term since $n_i a^i = 0$. It is well known that the ADM Lagrangian does not contain time derivatives of N and N^α and hence their conjugate momenta vanish. Thus these variables are non-dynamical. However we have just witnessed that boundary terms are not unique, one can in principle add any boundary term that cancels the normal derivative. Then a natural question arises — are the time derivatives of N and N^α zero for any possible boundary term? If not can they be dynamical? These questions get firm ground as the following example is considered.

Dynamical or Non-dynamical?

Let us consider a cosmological spacetime. Being homogeneous and isotropic it is described by a single function, the scale factor $a(t)$. The line element for cosmological spacetime by imposition of these symmetry conditions become,

$$ds^2 = -dt^2 + a^2(t) [dr^2 + r^2 d\Omega^2], \quad (21)$$

where the spatial section has been assumed to be flat for simplicity. The above metric is manifestly in ADM form, with $N = 1$, $N^\alpha = 0$ and $h_{\alpha\beta} = a^2(t)\delta_{\alpha\beta}$ respectively. Thus it is evident that N and N^α are not dynamical, all the dynamics comes from the scale factor $a(t)$ as expected. One can now introduce a new coordinate r , such that $R = a(t)r$ and write the metric in the (t, R, θ, ϕ) coordinate system such that,

$$ds^2 = - (1 - H^2 R^2) dt^2 - 2HR dt dR + dR^2 + R^2 d\Omega^2. \quad (22)$$

Surprisingly, now the metric is again in ADM form but with a completely different structure. This time the spatial metric is flat, i.e., $h_{\alpha\beta} = \delta_{\alpha\beta}$ and hence cannot have any dynamics. On the other hand, one obtains $N = 1$ and $N^\alpha = HR\delta_R^\alpha$ and would conclude that cosmological spacetime is non-dynamical! This explicitly shows that the standard argument for ADM variables N and N^α to be non-dynamical based on their time derivatives is misleading.

To resolve the dilemma we will explicitly illustrate, depending on the boundary term, Einstein–Hilbert action do contains time derivatives of N and N^α but they are *not* dynamical. For this purpose we make use of the following decomposition of the Einstein–Hilbert action,

$$\sqrt{-g}R = \sqrt{-g}g^{ab} \left(\Gamma_{ja}^i \Gamma_{ib}^j - \Gamma_{ab}^i \Gamma_{ij}^j \right) + \partial_c \left\{ \sqrt{-g} \left(g^{ik} \Gamma_{ik}^c - g^{ck} \Gamma_{km}^m \right) \right\}. \quad (23)$$

Here the first term is quadratic in the connection and is known as the Γ^2 Lagrangian, while the second term is the boundary term and contains normal derivatives of the metric as elaborated in [13]. Thus an alternative to Gibbons–Hawking–York boundary term is the total divergence term introduced above and hence a possible well-posed Lagrangian corresponds to the Γ^2 Lagrangian. We will show that this Lagrangian depends on time derivatives of N and N^α . To achieve this we shall expand out the Γ^2 Lagrangian in terms of the ADM variables and separate out the time derivatives of N and N^α . Any term X which contains time derivatives of N and N^α will be denoted by $[X]_{t,d}$. By Expressing all the connections in terms of the ADM variables we find that only Γ_{tt}^t and Γ_{tt}^α depends on time derivatives of N and N^α . Hence the time derivative part for the full Γ^2 Lagrangian reads,

$$[\sqrt{-g}L_{\text{quad}}]_{t,d} = \frac{\sqrt{h}}{N^2} \partial_t N \partial_\alpha N^\alpha - \sqrt{h} \frac{\partial_t N^\alpha \partial_\alpha N}{N^2} + \frac{\partial_t N^\alpha}{N} \partial_\alpha \sqrt{h}. \quad (24)$$

Hence we have explicitly demonstrated, that the Γ^2 Lagrangian contains time derivatives of N and N^α . Then one question naturally arises, how is that the ADM Lagrangian does not contain these time derivative terms, as evident from the expression for L_{ADM} ? The answer to this question is hiding in the boundary terms, since they are not identical. Thus in order to understand this, we will have to compare the two boundary terms, the surface term in Eq. (23) and the Gibbons–Hawking–York boundary term, that separate Γ^2 Lagrangian and ADM Lagrangian, respectively, from the Einstein–Hilbert Lagrangian $\sqrt{-g}R$.

Let us now evaluate the Einstein and the Gibbons–Hawking–York boundary terms using the ADM variables. We shall not evaluate the integrands of the surface integrals, but the corresponding divergence terms present in the bulk Lagrangians given by Eqs. (20) and (23) respectively. One can again use the Christoffel symbols to calculate $Kn^i + a^i$ required for evaluating the Gibbons–Hawking–York term in divergence form. Performing the same, terms in the Gibbons–Hawking–York boundary contribution containing time derivatives of N and N^α has the expression

$$\left[-2\partial_i \{ \sqrt{-g} (Kn^i + a^i) \} \right]_{t,d} = \sqrt{h} \frac{\partial_t N \partial_\alpha N^\alpha}{N^2} - 2\sqrt{h} \frac{\partial_t \partial_\alpha N^\alpha}{N} - 2 \frac{\partial_t \sqrt{h} \partial_t N}{N^2} + 2 \frac{\partial_t N}{N^2} N^\alpha \partial_\alpha \sqrt{h} - 2 \frac{\partial_t N^\alpha \partial_\alpha \sqrt{h}}{N}. \quad (25)$$

Having derived the relevant expressions related to Gibbons–Hawking–York boundary term, let us next concentrate on the boundary term in the Einstein–Hilbert action given in Eq. (23), which has the expression $\partial_i (\sqrt{-g}V^i)$, where $V^i = g^{ab}\Gamma_{ab}^i - g^{im}\Gamma_{mk}^k$. Computation of each individual components of the boundary term which contains time derivatives of N and N^α are thus given by

$$\begin{aligned} \left[\partial_i \left(\sqrt{-g} V^i \right) \right]_{t,d} &= -\frac{2}{N^2} \partial_t N \partial_t \sqrt{h} + \frac{\sqrt{h}}{N^2} \partial_t N \partial_\alpha N^\alpha - \frac{\sqrt{h}}{N} \partial_t \partial_\alpha N^\alpha + \frac{2}{N^2} \partial_t N N^\alpha \partial_\alpha \sqrt{h} \\ &\quad - \frac{2}{N} \partial_t N^\alpha \partial_\alpha \sqrt{h} + \frac{\sqrt{h}}{N^2} \partial_\alpha N \partial_t N^\alpha - \frac{\sqrt{h}}{N} \partial_\alpha \partial_t N^\alpha - \frac{\partial_t N^\alpha \partial_\alpha \sqrt{h}}{N}. \end{aligned} \quad (26)$$

Hence, from Eqs. (25) and (26), we finally arrive at the total contribution from the boundary terms

$$\begin{aligned} \left[\partial_c \left(\sqrt{-g} V^c \right) + 2\partial_i \left\{ \sqrt{-g} \left(K n^i + a^i \right) \right\} \right]_{t,d} \\ = -\frac{\sqrt{h}}{N^2} \partial_t N \partial_\alpha N^\alpha + \sqrt{h} \frac{\partial_t N^\alpha \partial_\alpha N}{N^2} - \frac{\partial_t N^\alpha \partial_\alpha \sqrt{h}}{N}. \end{aligned} \quad (27)$$

Thus, we observe that the surface terms in Einstein–Hilbert action in Einstein’s original decomposition and ADM decomposition are different. The difference contains time derivatives of N^α and N . These time derivatives should exactly match the time derivatives in Γ^2 Lagrangian as we know that the ADM Lagrangian does not have time derivatives of N and N^α . Evaluating time derivatives in ADM Lagrangian using Eqs. (24) and (27), we obtain

$$\begin{aligned} \left[\sqrt{-g} L_{\text{ADM}} \right]_{t,d} &= \left[\sqrt{-g} R + 2\partial_i \left\{ \sqrt{-g} \left(K n^i + a^i \right) \right\} \right]_{t,d} \\ &= \left[\sqrt{-g} L_{\text{quad}} + \partial_c \left(\sqrt{-g} V^c \right) + 2\partial_i \left\{ \sqrt{-g} \left(K n^i + a^i \right) \right\} \right]_{t,d} \\ &= 0, \end{aligned} \quad (28)$$

which confirms the ADM Lagrangian does not contain any time derivatives of N and N^α and demonstrates that the time derivatives of N and N^α in the Γ^2 action arise because of the difference in surface terms.

Since the Γ^2 Lagrangian contains time derivatives of N and N^α , it is pertinent to ask what are the conjugate momenta corresponding to N and N^α . From Eq. (24), the conjugate momenta for N and N^α turn out to be

$$p_{(N)} = \frac{\partial \left(\sqrt{-g} \Gamma^2 \right)}{\partial \left(\partial_t N \right)} = \frac{\sqrt{h}}{N^2} \partial_\alpha N^\alpha \quad (29)$$

$$p_{\alpha(N^\alpha)} = \frac{\partial \left(\sqrt{-g} \Gamma^2 \right)}{\partial \left(\partial_t N^\alpha \right)} = -\sqrt{h} \frac{\partial_\alpha N}{N^2} + \frac{1}{N} \partial_\alpha \sqrt{h}. \quad (30)$$

Note that the conjugate momenta to N and N^α do not depend on time derivatives of N and N^α respectively. Hence, these relations cannot be inverted to obtain $\partial_t N$ and $\partial_t N^\alpha$ in terms of $p_{(N)}$ and $p_{\alpha(N^\alpha)}$. Returning back to our example of cosmological spacetime, this means that H is indeed non-dynamical and that is clear since in terms of Hubble parameter, the Einstein’s equations contain only single time derivative of H . Thus we conclude:

Even though the ADM Lagrangian does not contain time derivatives of N and N^α , the quadratic Lagrangian L_{quad} differing from the ADM Lagrangian by total derivative do contains time derivatives of N and N^α . However, the corresponding canonical momentums are non-invertible, i.e., one cannot obtain time derivatives of N and N^α in terms of their canonical momentum. Hence follows their non-dynamical nature.

This explicitly demonstrates standard statements, showing truth in non-dynamical behavior of N and N^α but also demonstrating existence of time derivatives of non-dynamical variables.

4 Null Surfaces: Completing the Circle

The boundary terms and ADM decomposition discussed earlier depends crucially on the timelike (or spacelike) nature of the boundary surface. However, the most ubiquitous surfaces in general relativity are the null surfaces, e.g., in a black hole spacetime the standard boundary would consist of the surface at infinity and the event horizon, which is a null surface. The limit of non-null surfaces to null surfaces is not at all straightforward, since many quantities including the extrinsic curvature, induced metric can either blow up or vanish on the null surface if proper care is not taken. Thus it is important to consider the boundary term from a first principle in connection to null hypersurfaces. The first step towards this direction was taken in [6] by constructing a general formalism and its explicit implementation was carried out in [7]. There it was argued that for a null vector ℓ_a (i.e., $\ell^a \ell_a = 0$) the boundary term one should add corresponds to $2\sqrt{q}(\Theta + \kappa)$, where q stands for the determinant of the induced metric on the null surface, Θ stands for the expansion of the null geodesics and κ is the non-affinity parameter. Since null surfaces are intrinsically two-dimensional, use of a single vector field ℓ_a is not sufficient. One need to introduce another auxiliary vector field k_a , satisfying $k_a k^a = 0$ and $\ell_a k^a = -1$. In the above derivations it has been assumed that the null surface is preserved under variations, i.e., the following three conditions hold: $\delta(\ell_a \ell^a) = 0$, $\delta(\ell_a k^a) = 0$ and finally $\delta(k_a k^a) = 0$. In this work we will relax all these assumptions and shall investigate the effect of these constraints on the boundary term and degrees of freedom on the boundary. We will start with the general expression for boundary term of Einstein–Hilbert action having the form [14]

$$\begin{aligned} \sqrt{-g} Q[\ell_c] &= \sqrt{-g} \nabla_c (\delta u^c) - 2\delta (\sqrt{-g} \nabla_a \ell^a) + \sqrt{-g} [\nabla_a \ell_b - g_{ab} (\nabla_c \ell^c)] \delta g^{ab} \\ &= Q_1 + Q_2 + Q_3, \end{aligned} \quad (31)$$

where, $\delta u^a = \delta \ell^a + g^{ab} \delta \ell_b$. We have separated the boundary term in three natural combinations, one is a divergence term, Q_1 , second one corresponds to total variation Q_2 and finally the degrees of freedom term Q_3 respectively. We will explore each of these terms and subsequently shall evaluate the boundary term on the null surface following the convention, if some relation holds *only* on the null surface it will be denoted by $A := 0$. As explained above we will assume the following conditions on the null surface only, $\ell_a \ell^a := 0$, $\ell_a k^a := 0$ and $k_a k^a := 0$ respectively, but we would not assume anything about off the null surface relations, i.e., variations can be arbitrary. Then one can introduce the *partial* projector P_b^a through the vectors ℓ^a and k^a as, $P_b^a = \delta_b^a + k^a \ell_b$ and can write the first divergence term Q_1 in Eq. (31) as:

$$Q_1 := \partial_\alpha (\sqrt{-g} P_d^a \delta u^d) - \delta (\sqrt{-g} k^c \partial_c \ell^2) + (k^c \partial_c \ell^2) \delta \sqrt{-g} + \sqrt{-g} \delta k^c \partial_c \ell^2 - \partial_c (\sqrt{-g} k^c) \delta \ell^2, \quad (32)$$

while the second term can also be expressed using the partial projector P_b^a and then the complete boundary term on using the variation of $\sqrt{-g}$, takes the following form

$$\sqrt{-g} Q [\ell_c] := \partial_\alpha (\sqrt{-g} P_d^a \delta u^d) - 2\delta (\sqrt{-g} P_b^a \nabla_a \ell^b) + \sqrt{-g} \delta k^c \partial_c \ell^2 - \partial_c (\sqrt{-g} k^c) \delta \ell^2 + \sqrt{-g} (\nabla_a \ell_b - g_{ab} \{P_d^c \nabla_c \ell^d\}) \delta g^{ab}. \quad (33)$$

Note that the first term is a pure surface term — it has no component along the normal ℓ_a . Then we can decompose the metric in terms of the induced metric q_{ab} and the null vectors ℓ^a and k^a as: $g_{ab} = q_{ab} - \ell_a k_b - \ell_b k_a$. Thus variations of the metric now gets transformed to variations of the induced metric and the null vectors. One important point to keep in mind is the fact that $\delta \ell^a = g^{ab} \delta \ell_b + \ell_b \delta g^{ab}$ but *not* $g^{ab} \delta \ell_b$. Using the properties of the null vectors outside variation and decomposition of $\nabla_a \ell_b$ in terms of the extrinsic curvature ultimately lands us into the following expression for the boundary term

$$\begin{aligned} \sqrt{-g} Q := & \partial_\alpha (\sqrt{-g} P_d^a \delta u^d) - 2\delta (\sqrt{-g} P_b^a \nabla_a \ell^b) \\ & + \sqrt{-g} [\Theta_{ab} - (P_d^c \nabla_c \ell^d) q_{ab}] \delta q^{ab} \\ & - \sqrt{-g} \{k^m \nabla_m \ell_a + k^n \nabla_a \ell_n + (k^m k^n \nabla_m \ell_n) \ell_a - 2(P_d^c \nabla_c \ell^d) k_a\} \delta \ell^a \\ & + \sqrt{-g} \{k^m \nabla_m \ell^a + k^n \nabla^a \ell_n + (k^m k^n \nabla_m \ell_n) \ell^a - 2(P_n^m \nabla_m \ell^n) k^a\} \delta \ell_a \\ & + \sqrt{-g} \{\partial_c \ell^2\} \delta k^c - \partial_c (\sqrt{-g} k^c) \delta \ell^2. \end{aligned} \quad (34)$$

Before commenting on the structure of the boundary term let us quickly check one possible limit we have derived in our earlier works [7]. This corresponds to the situation in which $\ell_a = \nabla_a \phi$, implying $\delta \ell_a = 0$ and also $\delta \ell^2 = 0 = \delta (\ell^a k_a)$, such that we have $P_b^a \nabla_a \ell^b = \Theta + \kappa$. Under imposition of these conditions, the boundary term reduces to:

$$\begin{aligned}
\sqrt{-g}Q[\nabla_c\phi] &:= \partial_\alpha(\sqrt{-g}P^\alpha_a\delta u^a) - 2\delta[\sqrt{-g}(\Theta + \kappa)] \\
&+ \sqrt{-g}[\Theta_{ab} - (\Theta + \kappa)q_{ab}]\delta q^{ab} \\
&- 2\sqrt{-g}\{k^m\nabla_m\ell_a - (\Theta + \kappa)k_a\}\delta\ell^a. \tag{35}
\end{aligned}$$

This is exactly what we had derived by various other routes in [7]. Having checked the consistency with earlier derived results we now concentrate on the physical implications of Eq. (34). The first term as emphasized earlier corresponds to another boundary term¹ and contributes only on the two surface without much significance. The second term is the boundary term that one should add (negative of that term, to be precise) to the Einstein–Hilbert action as evaluated with volume encompassing null boundaries. The rest of the terms are related to degrees of freedom and what one should fix on the null surface. Among them fixing induced metric is expected, with its conjugate momentum being $\pi_{ab} = \sqrt{-g}[\Theta_{ab} - (P_n^m\nabla_m\ell^n)q_{ab}]$. In this case as well one can write the last term as a total divergence leading to a different boundary term and conjugate momentum to fix on the boundary. Unlike the cases of timelike or spacelike surfaces the situation is not so simple for null surfaces, since even after fixing the induced metric one needs to fix the components of the null vectors as well. But one can improve on that. Since the normalization of the null vector is arbitrary one can always choose ℓ_a to be a pure gradient such that $\delta\ell_a = 0$. Further since the choice of k^a is arbitrary one might choose it such that its expansion vanishes and further with $\delta(k^a\ell_a) = 0$. As these seemingly natural conditions are being satisfied the boundary term simplifies a lot, ultimately leading to,

$$\begin{aligned}
\sqrt{-g}Q &:= \partial_\alpha(\sqrt{-g}P^\alpha_a\delta u^a) - 2\delta(\sqrt{-g}P^a_b\nabla_a\ell^b) \\
&+ \sqrt{-g}[\Theta_{ab} - (P^c_d\nabla_c\ell^d)q_{ab}]\delta q^{ab} \\
&- \sqrt{-g}\{k^m\nabla_m\ell_a + k^n\nabla_n\ell_n + (k^mk^n\nabla_mk_n)\ell_a - 2(P^c_d\nabla_c\ell^d)k_a\}\delta\ell^a. \tag{36}
\end{aligned}$$

Hence along with q_{ab} one need to fix the components of the null vector ℓ^a . One more point should be noted, since $\delta\ell_a = 0$, one obtains $\delta(\ell_a\ell^a) = \ell_a\delta\ell^a$ and hence any contribution from $\delta\ell^2$ can be dumped into the contribution from $\delta\ell^a$. Hence this suggests that on the null surface one need to fix the induced metric q^{ab} as well as ℓ^a . This has interesting consequences for degrees of freedom on the null surfaces à la degrees of freedom on spacelike or timelike surfaces. One interesting consequence could be, as the diffeomorphisms are gauged away one can eliminate the four degrees of freedom in $\delta\ell^c$, keeping the true (physical) degrees of freedom in the two metric

¹This kind of terms are also present in the the calculation for spacelike (or timelike) surfaces, see for example the last term of Eq. (12).

q_{ab} of the null surface. This can have interesting implications for black hole entropy, which we will pursue elsewhere.

5 Concluding Remarks

The peculiarity of the Einstein–Hilbert action can be traced back to its boundary terms. In the standard treatments it is often overlooked that Einstein–Hilbert action is not well posed, one has to add boundary terms to get an well posed action for gravity. There have been parallel results on this issue, one is the well-known Gibbons–Hawking–York boundary term, while the other is recent and more promising from a thermodynamic hindsight which invokes two new variables f^{ab} and N_{ab}^c to describe gravity, with $f^{ab}\delta N_{ab}^c$ as the boundary term. In this work we have explicitly derived the equivalence between these two formalisms in any spacetime dimensions. Further we have also demonstrated the argument that “boundary terms are not unique” by constructing two more boundary terms starting from the Gibbons–Hawking–York term. To our surprise these boundary terms depends strongly on the spacetime dimensions and even can vanish in $D = 4$. Then we have elaborated the meaning of another statement made often in the literature, “the ADM variables N and N^α are not dynamical”. The standard argument goes by saying that the ADM Lagrangian does not depend on time derivatives of N and N^α . We have shown that one can add boundary terms to the ADM Lagrangian leading to a new Lagrangian which contain time derivatives of N and N^α , (so it might appear they can be made dynamical by adding boundary terms) but still they are non-dynamical as conjugate momentums cannot be inverted. This finishes our discussion on spacelike or timelike surfaces and we turn to the case of null surfaces. In earlier works regarding boundary term on null surfaces, various assumption about variations of the null vectors were imposed, here we have derived the structure of the boundary term for most general variation. Imposing some minimal restrictions we could show that besides the induced metric, the null vector ℓ^a contains additional degrees of freedom. If they can be removed by diffeomorphism (as [7] suggests) then the induced metric might contain all the physical degrees of freedom associated with null surfaces, having greater implications for emergent paradigms of gravity [15–18].

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Dedication

One of the great quality of Prof. Padmanabhan is his ability to ask the correct question. This work stems from such questions asked by him during our discussions: “What is the connection between various action principles for general relativity? Why N and N^α are non-dynamical? What one should fix on a null surface?”. I have tried to answer them in this work and I respectfully dedicate it to Prof. Padmanabhan on the occasion of his 60th birthday.

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