Chapter 6
Synthesis of Model Detection Filters

This chapter presents general synthesis procedures of residual generators which solve the model detection problems formulated in Chap. 4. Similarly to Chap. 3, the synthesis procedures are described in terms of input–output models. The numerical aspects of equivalent state-space representation based synthesis algorithms are essentially the same as for the synthesis algorithms of fault detection and isolation filters, and the discussion of related computational techniques is also covered in Chap. 7.

6.1 Nullspace-Based Synthesis

We assume the overall residual generator filter $Q(\lambda)$ has the TFM of the form (4.3), which corresponds to a bank of N individual filters as in (4.2). Furthermore, for $i = 1, \ldots, N$, the i-th filter driven by the j-th model has the internal form in (4.4). Let $R_u^{(i,j)}(\lambda)$ and $R_d^{(i,j)}(\lambda)$ be the TFMs defined in (4.4) and (4.5). A useful parametrization of all individual filters can be obtained on the basis of the conditions $R_u^{(i,i)}(\lambda) = 0$ and $R_d^{(i,i)}(\lambda) = 0$ for $i = 1, \ldots, N$ in (4.11). For each filter with the TFM $Q^{(i)}(\lambda)$, these conditions are equivalent to

$$Q^{(i)}(\lambda) \begin{bmatrix} G_u^{(i)}(\lambda) & G_d^{(i)}(\lambda) \\ I_{m_u} & 0 \end{bmatrix} = 0\,. \tag{6.1}$$

Therefore, $Q^{(i)}(\lambda)$ must be a left annihilator of the TFM

$$G^{(i)}(\lambda) := \begin{bmatrix} G_u^{(i)}(\lambda) & G_d^{(i)}(\lambda) \\ I_{m_u} & 0 \end{bmatrix}. \tag{6.2}$$

Let $r_d^{(i)}$ be the normal rank of $G_d^{(i)}(\lambda)$. It follows that there exists a maximal full row rank left annihilator $N_l^{(i)}(\lambda)$ of size $\left(p-r_d^{(i)}\right)\times(p+m_u)$ such that $N_l^{(i)}(\lambda)G^{(i)}(\lambda) = 0$. Any such $N_l^{(i)}(\lambda)$ is a *rational basis* of $\mathcal{N}_L(G^{(i)}(\lambda))$, the left (rational) nullspace of

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$G^{(i)}(\lambda)$. Using this fact and Theorem 5.1, we have the following straightforward parametrization of all component filters:

Theorem 6.1 *For $i = 1, \ldots, N$, let $N_l^{(i)}(\lambda)$ be a basis of $\mathcal{N}_L(G^{(i)}(\lambda))$, with $G^{(i)}(\lambda)$ defined in (6.2). Then, each filter $Q^{(i)}(\lambda)$ satisfying condition (i) of (4.11) can be expressed in the form*

$$Q^{(i)}(\lambda) = V^{(i)}(\lambda)N_l^{(i)}(\lambda), \quad i = 1, \ldots, N, \tag{6.3}$$

where $V^{(i)}(\lambda)$ is a suitable TFM.

The parametrization result of Theorem 6.1 underlies the nullspace method based synthesis procedures of model detection filters. All synthesis procedures of the model detection filters, presented in this book, rely on the initial factored forms

$$Q^{(i)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)Q_1^{(i)}(\lambda), \quad i = 1, \ldots, N, \tag{6.4}$$

where each $Q_1^{(i)}(\lambda) = N_l^{(i)}(\lambda)$ is a basis of $\mathcal{N}_L(G^{(i)}(\lambda))$, while each factor $\overline{Q}_1^{(i)}(\lambda)$ has to be subsequently determined. The nullspace-based first step allows to reduce the synthesis problems of model detection filters formulated for the multiple models (4.1) to simpler problems, which allow to easily check the solvability conditions.

Using the factored form (6.4), the model detection filters in (4.2) can be rewritten in the alternative forms

$$\mathbf{r}^{(i)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)Q_1^{(i)}(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix} = \overline{Q}_1^{(i)}(\lambda)\overline{\mathbf{y}}^{(i)}(\lambda), \quad i = 1, \ldots, N, \tag{6.5}$$

where

$$\overline{\mathbf{y}}^{(i)}(\lambda) := Q_1^{(i)}(\lambda)\begin{bmatrix} \mathbf{y}(\lambda) \\ \mathbf{u}(\lambda) \end{bmatrix}. \tag{6.6}$$

For $y(t) = y^{(j)}(t)$, both the residual signal $r^{(i)}(t)$ in (6.5) and $\overline{y}^{(i)}(t)$ in (6.6) depend on all system inputs $u^{(j)}(t)$, $d^{(j)}(t)$ and $w^{(j)}(t)$ via the system output $y^{(j)}(t)$. The internal form (4.4) of the i-th filter for the j-th model can be expressed as

$$\widetilde{\mathbf{r}}^{(i,j)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)\widetilde{\mathbf{y}}^{(i,j)}(\lambda),$$

where

$$\widetilde{\mathbf{y}}^{(i,j)}(\lambda) := Q_1^{(i)}(\lambda)\begin{bmatrix} \mathbf{y}^{(j)}(\lambda) \\ \mathbf{u}^{(j)}(\lambda) \end{bmatrix}.$$

Using the expression of $\mathbf{y}^{(j)}(\lambda)$ from (4.1), we obtain

$$\widetilde{\mathbf{y}}^{(i,j)}(\lambda) = \overline{G}_u^{(i,j)}(\lambda)\mathbf{u}^{(j)}(\lambda) + \overline{G}_d^{(i,j)}(\lambda)\mathbf{d}^{(j)}(\lambda) + \overline{G}_w^{(i,j)}(\lambda)\mathbf{w}^{(j)}(\lambda), \tag{6.7}$$

with

$$\left[\overline{G}_u^{(i,j)}(\lambda) \middle| \overline{G}_d^{(i,j)}(\lambda) \middle| \overline{G}_w^{(i,j)}(\lambda) \right] := Q_1^{(i)}(\lambda) \left[\begin{array}{c|c|c} G_u^{(j)}(\lambda) & G_d^{(j)}(\lambda) & G_w^{(j)}(\lambda) \\ I_{m_u} & 0 & 0 \end{array} \right]. \tag{6.8}$$

The system (6.7) can be interpreted as the internal form of the i-th filter driven by the j-th model, corresponding to the partial synthesis $Q_1^{(i)}(\lambda)$. For $j = i$, the particular choice of $Q_1^{(i)}(\lambda)$ as a left nullspace basis of $G^{(i)}(\lambda)$ in (6.2) ensures that $\overline{G}_u^{(i,i)}(\lambda) = 0$ and $\overline{G}_d^{(i,i)}(\lambda) = 0$.

At this stage we can assume that both $Q_1^{(i)}(\lambda)$ and the TFMs (6.8) are proper and stable. This can be always achieved using $Q_1^{(i)}(\lambda) = M^{(i)}(\lambda)N_l^{(i)}(\lambda)$ (instead $Q_1^{(i)}(\lambda) = N_l^{(i)}(\lambda)$), where $M^{(i)}(\lambda)$ is a stable and proper TFM such that

$$M^{(i)}(\lambda)\big[\, N_l^{(i)}(\lambda) \mid \overline{G}_u^{(1)}(\lambda) \;\; \overline{G}_d^{(1)}(\lambda) \;\; \overline{G}_w^{(1)}(\lambda) \mid \cdots \mid \overline{G}_u^{(N)}(\lambda) \;\; \overline{G}_d^{(N)}(\lambda) \;\; \overline{G}_w^{(N)}(\lambda) \,\big]$$

is stable and proper. Such an $M^{(i)}(\lambda)$ can be determined as the denominator matrix of a stable and proper LCF (see Sect. 9.1.6).

Relying on the parametrization result of Theorem 6.1, we have the following straightforward characterization of the model detectability of the multiple model (4.1) in terms of the N multiple models (6.7):

Proposition 6.1 *For the multiple model (4.1), let $Q_1^{(i)}(\lambda) = N_l^{(i)}(\lambda)$, $i = 1, \ldots, N$, be rational bases of $\mathcal{N}_L(G^{(i)}(\lambda))$, with $G^{(i)}(\lambda)$ defined in (6.2), and let (6.7) be the multiple model associated to the i-th residual. Then, the multiple model (4.1) with $w^{(j)} \equiv 0$ for $j = 1, \ldots, N$, is model detectable if and only if, for $i = 1, \ldots, N$*

$$\left[\overline{G}_u^{(i,j)}(\lambda) \; \overline{G}_d^{(i,j)}(\lambda) \right] \neq 0 \quad \forall j \neq i\,. \tag{6.9}$$

6.2 Solving the Exact Model Detection Problem

Using Proposition 6.1, the solvability conditions of the *exact model detection problem* (EMDP) formulated in Sect. 4.4.1 for the multiple model (4.1), can be also expressed in terms of the multiple models (6.7), according to the following corollary to Theorem 4.2:

Corollary 6.1 *For the multiple model (4.1) with $w^{(j)} \equiv 0$ for $j = 1, \ldots, N$, the EMDP is solvable if and only if for the multiple model (6.7), with $w^{(j)} \equiv 0$ for $j = 1, \ldots, N$, the following conditions hold for $i = 1, \ldots, N$*

$$\left[\overline{G}_u^{(i,j)}(\lambda) \; \overline{G}_d^{(i,j)}(\lambda) \right] \neq 0 \quad \forall j \neq i\,. \tag{6.10}$$

The synthesis procedure of the N component filters $Q^{(i)}(\lambda)$, $i = 1, \ldots, N$, employs a common computational approach. Accordingly, the i-th filter $Q^{(i)}(\lambda)$ is determined in the factored form

$$Q^{(i)}(\lambda) = Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda) \,,$$

where: $Q_1^{(i)}(\lambda)$ is a rational basis of $\mathcal{N}_L\big(G^{(i)}(\lambda)\big)$, the left nullspace of $G^{(i)}(\lambda)$ defined in (6.2); $Q_2^{(i)}(\lambda)$ ensures that $Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ has least McMillan degree; and, $Q_3^{(i)}(\lambda)$ is chosen such that $Q^{(i)}(\lambda)$ is stable and the corresponding $R^{(i,j)}(\lambda)$ defined in (4.6), for $j = 1, \ldots, N$, $j \neq i$, are stable and nonzero. Using Proposition 6.1, the existence condition of the i-th filter is satisfied if $Q_1^{(i)}(\lambda)G^{(j)}(\lambda) \neq 0, \forall j \neq i$.

There exists some freedom in determining model detection filters which solve the EMDP. For example, the number of outputs of the i-th filter $Q^{(i)}(\lambda)$ can be chosen arbitrarily between 1 and $p - r_d^{(i)}$, where $r_d^{(i)} := \operatorname{rank} G_d^{(i)}(\lambda)$, provided the model detectability conditions are fulfilled. Also, least-order scalar output model detection filters can be employed to ensure that the overall bank of filters has the least achievable global order. However, filters with more outputs can occasionally provide a better sensitivity condition (see later) for model detection.

The **Procedure EMD**, given below, determines the N filters $Q^{(i)}(\lambda)$, $i = 1, \ldots, N$, and the corresponding internal forms $R^{(i,j)}(\lambda) := \big[\, R_u^{(i,j)}(\lambda) \;\; R_d^{(i,j)}(\lambda) \,\big]$, for $i, j = 1, \ldots, N$, with the i-th filter having a maximal row dimension q_{max}.

Procedure EMD: Exact synthesis of model detection filters

Inputs : $\{G_u^{(j)}(\lambda), G_d^{(j)}(\lambda)\}$, for $j = 1, \ldots, N$; q_{max}
Outputs: $Q^{(i)}(\lambda)$, for $i = 1, \ldots, N$; $R^{(i,j)}(\lambda)$ for $i, j = 1, \ldots, N$

For $i = 1, \ldots, N$

1) Compute a $(p - r_d^{(i)}) \times (p + m_u)$ minimal basis matrix $Q_1^{(i)}(\lambda)$ for the left nullspace of $G^{(i)}(\lambda)$ defined in (6.2), where $r_d^{(i)} := \operatorname{rank} G_d^{(i)}(\lambda)$; set $Q^{(i)}(\lambda) = Q_1^{(i)}(\lambda)$ and compute $R^{(i,j)}(\lambda) = Q^{(i)}(\lambda)G^{(j)}(\lambda)$ for $j = 1, \ldots, N$.
Exit if $R^{(i,j)}(\lambda) = 0$ for any $j \in \{1, \ldots, N\}$, $j \neq i$ (no solution exists).
2) Choose a $\min\big(q_{max}, p - r_d^{(i)}\big) \times (p + m_u)$ rational matrix $Q_2^{(i)}(\lambda)$, such that $Q_2^{(i)}(\lambda)Q^{(i)}(\lambda)$ has least McMillan degree and $Q_2^{(i)}(\lambda)R^{(i,j)}(\lambda) \neq 0$ for $j = 1, \ldots, N$, $j \neq i$; compute $Q^{(i)}(\lambda) \leftarrow Q_2^{(i)}(\lambda)Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_2^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$.
3) Choose a proper and stable invertible rational matrix $Q_3^{(i)}(\lambda)$ such that $Q_3^{(i)}(\lambda)Q^{(i)}(\lambda)$ has a desired stable dynamics and $Q_3^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$ are stable; compute $Q^{(i)}(\lambda) \leftarrow Q_3^{(i)}(\lambda)Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_3^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$.

The computational algorithms underlying **Procedure EMD** are essentially the same as those used for the synthesis of fault detection filters (see **Procedure EFD**) and rely on state-space representations as in (2.19) of the component models. These algorithms are amply described in Sects. 7.4–7.6, and therefore, we restrict our discussion on specific aspects of Steps 2) and 3). To determine filters with least dynamical orders at Step 2), a straightforward systematic approach is to build successive candidate filters $Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ with increasing McMillan degrees and check the specific *admissibility condition* $Q_2(i)(\lambda)Q_1^{(i)}(\lambda)G^{(j)}(\lambda) \neq 0$ (or equivalently $Q_2^{(i)}(\lambda)\big[\, \overline{G}_u^{(i,j)}(\lambda) \;\; \overline{G}_d^{(i,j)}(\lambda) \,\big] \neq 0$) for all $j \neq i$. The least possible order of the fault detection filter $Q^{(i)}(\lambda)$ is uniquely determined by the fulfilment of the above admissibility condition. Since $Q_3^{(i)}(\lambda)$ is invertible, its choice plays no role in ensuring admissibility. However, the final orders of the individual filters can occasionally further increase at Step 3), if the cancellation of unstable poles in the component models is necessary, in accordance with the formulated requirements for the EMDP. As in the case of solving the EFDP, a least-order filter synthesis can be always achieved by a scalar output filter. Since the choice of $Q_2^{(i)}(\lambda)$ is not unique, an appropriate parametrization of $Q_2^{(i)}(\lambda)$ allows to make an optimal choice of free parameters (e.g., to achieve other desirable features; see Remark 6.1). Further aspects of selecting suitable $Q_2^{(i)}(\lambda)$, in accordance with the employed type of nullspace basis, are discussed in Sect. 5.2, in the context of solving the EFDP.

Remark 6.1 Assume that all component models in (4.1) are stable and all vectors $d^{(i)}(t)$, $i = 1, \ldots, N$, have dimension m_d. In this case, the norm of $R^{(i,j)}(\lambda)$ has a simple interpretation as a weighted distance between the i-th and j-th models. In accordance with Theorem 6.1, $Q^{(i)}(\lambda)$ can be expressed as $Q^{(i)}(\lambda) = V^{(i)}(\lambda)N_l^{(i)}(\lambda)$, with the nullspace basis $N_l^{(i)}(\lambda)$ chosen in a form similar to (5.5), as

$$N_l^{(i)}(\lambda) = N_{l,d}^{(i)}(\lambda)\big[\, I_p \;\; -G_u^{(i)}(\lambda) \,\big] \,,$$

where $N_{l,d}^{(i)}(\lambda)$ is a $(p - r_d^{(i)}) \times p$ TFM representing a basis of $\mathcal{N}_L\big(G_d^{(i)}(\lambda)\big)$. This choice leads to

$$\begin{aligned} R^{(i,j)}(\lambda) &= Q^{(i)}(\lambda)G^{(j)}(\lambda) \\ &= V^{(i)}(\lambda)N_{l,d}^{(i)}(\lambda)\big[\, G_u^{(j)}(\lambda) - G_u^{(i)}(\lambda) \;\; G_d^{(j)}(\lambda) - G_d^{(i)}(\lambda) \,\big] . \end{aligned} \tag{6.11}$$

If we define the distance between the i-th and j-th models as

$$\operatorname{dist}\,(G^{(i)}(\lambda), G^{(j)}(\lambda)) := \big\|\big[\, G_u^{(j)}(\lambda) - G_u^{(i)}(\lambda) \;\; G_d^{(j)}(\lambda) - G_d^{(i)}(\lambda) \,\big]\big\| \,,$$

then, the norm of $R^{(i,j)}(\lambda)$ can be interpreted as a weighted distance between the TFMs of the i-th and j-th models. An ideal model detection filter $Q(\lambda)$ of the form (4.3) would monotonically map the distances between two models to the corresponding norms of $R^{(i,j)}(\lambda)$, that is, if the distances of the j-th and k-th models to the i-th model satisfy

$$\operatorname{dist}\,(G^{(i)}(\lambda), G^{(j)}(\lambda)) < \operatorname{dist}\,(G^{(i)}(\lambda), G^{(k)}(\lambda)) \,,$$

then the weighted distances satisfy

$$\|R^{(i,j)}(\lambda)\| < \|R^{(i,k)}(\lambda)\| \,.$$

Moreover, the fulfilment of the symmetry conditions

$$\|R^{(i,j)}(\lambda)\| = \|R^{(j,i)}(\lambda)\|, \;\forall i \neq j\,,$$

is also highly desirable. A model detection filter having these properties, can be employed to reliably identify the nearest model from a given set of models to the actual plant model.

Ensuring the monotonic distance mapping and symmetry properties can be seen as a global synthesis goal of model detection filters, and can be targeted in various ways, as—for example, by an optimal choice of the free parameters of the weighting functions $V^{(i)}(\lambda)N_{l,d}^{(i)}(\lambda)$, or by choosing each filter $Q^{(i)}(\lambda)$ to enforce a certain isometry (i.e., distance preserving) property (e.g., by choosing $V^{(i)}(\lambda)N_{l,d}^{(i)}(\lambda)$ a co-inner matrix). □

Remark 6.2 A properly designed model detection system as in Fig. 4.1 (e.g., with the model detection filter determined using **Procedure EMD**), is always able to identify the exact matching of the current model with one (and only one) of the N component models. However, in practice, we often encounter the situation that the actual (or true) model will never match exactly any of the N component models, and therefore, the best we can aim is to correctly figure out the nearest model to the actual one. Assume that the actual model has $\widetilde{G}_u(\lambda)$ and $\widetilde{G}_d(\lambda)$, the TFMs from the control-input-to-output and disturbance-input-to-output, respectively. Therefore, $\widetilde{G}_u(\lambda)$ and $\widetilde{G}_d(\lambda)$ can be expressed in terms of their deviations to the N component models for $j = 1, \ldots, N$ as

$$\widetilde{G}_u(\lambda) = G_u^{(j)}(\lambda) + \Delta G_u^{(j)}(\lambda), \quad \widetilde{G}_d(\lambda) = G_d^{(j)}(\lambda) + \Delta G_d^{(j)}(\lambda)\,.$$

Assuming the N component models are mutually distinct, there exists for each $i = 1, \ldots, N$, a largest $\delta^{(i)} > 0$ such that the following conditions simultaneously hold

$$\big\|\big[\, \Delta G_u^{(i)}(\lambda) \;\; \Delta G_d^{(i)}(\lambda) \,\big]\big\|_\infty \leq \delta^{(i)}, \quad \big\|\big[\, \Delta G_u^{(j)}(\lambda) \;\; \Delta G_d^{(j)}(\lambda) \,\big]\big\|_\infty > \delta^{(i)}, \forall j \neq i\,.$$

The size of $\delta^{(i)}$ defines the family of all sufficiently nearby models to the i-th model which are distinguishable (using the $\mathcal{H}_\infty$-norm based distance) from the rest of models. In the case when the nearest model to the actual model is the i-th model (i.e., the above inequalities are fulfilled), it is highly desirable that the model detection filter ensures that the i-th evaluation signal, $\theta_i \approx \|r^{(i)}\|_2$, has the least value among the N components of θ, and thus, allow to identify the i-th model as the nearest one to the current model. The attainability of this goal usually depends on the concrete problem to be solved. With the interpretation of the norm of $R^{(i,j)}(\lambda)$ in Remark 6.1 as a weighted distance between the i-th and j-th models, a prerequisite to fulfill the above goal is the use of a model detection filter able to monotonically map the distances between the models to the corresponding norms of the internal representations (i.e., to $R^{(i,j)}(\lambda)$). □

Example 6.1 To illustrate the effectiveness of the proposed nullspace-based synthesis approach of model detection filters, we consider the detection and identification of loss of efficiency of flight actuators using a model detection based approach. The fault-free state-space model describes the continuous-time lateral dynamics of an F-16 aircraft with the matrices

$$A^{(1)} = \begin{bmatrix} -0.4492 & 0.046 & 0.0053 & -0.9926 \\ 0 & 0 & 1.0000 & 0.0067 \\ -50.8436 & 0 & -5.2184 & 0.7220 \\ 16.4148 & 0 & 0.0026 & -0.6627 \end{bmatrix}, \quad B_u^{(1)} = \begin{bmatrix} 0.0004 & 0.0011 \\ 0 & 0 \\ -1.4161 & 0.2621 \\ -0.0633 & -0.1205 \end{bmatrix},$$

$$C^{(1)} = I_4, \qquad D_u^{(1)} = 0_{4\times 2}\,.$$

The four state variables are the sideslip angle, roll angle, roll rate and yaw rate, and the two input variables are the aileron deflection and rudder deflection. The individual failure models correspond to different levels of surface efficiency degradation. For simplicity, we build a multiple model with $N = 9$ component models on a coarse two-dimensional parameter grid for N values of the parameter vector $\rho := [\rho_1, \rho_2]^T$. For each component of ρ, the chosen three grid points are $\{0, 0.5, 1\}$. The component system matrices in (2.19) are defined for $i = 1, 2, \ldots, N$ as: $E^{(i)} = I_4$, $A^{(i)} = A^{(1)}$, $C^{(i)} = C^{(1)}$, and $B_u^{(i)} = B_u^{(1)}\Gamma^{(i)}$, where $\Gamma^{(i)} = \text{diag}\big(1-\rho_1^{(i)}, 1-\rho_2^{(i)}\big)$ and $\big(\rho_1^{(i)}, \rho_2^{(i)}\big)$ are the values of parameters (ρ_1, ρ_2) on the chosen grid

ρ_1 :	0	0	0	0.5	0.5	0.5	1	1	1
ρ_2 :	0	0.5	1	0	0.5	1	0	0.5	1

For example, $\big(\rho_1^{(1)}, \rho_2^{(1)}\big) = (0, 0)$ corresponds to the fault-free situation, while $\big(\rho_1^{(9)}, \rho_2^{(9)}\big) = (1, 1)$ corresponds to complete failure of both control surfaces. It follows, that the TFM $G_u^{(i)}(s)$ of the i-th system can be expressed as

$$G_u^{(i)}(s) = G_u^{(1)}(s)\Gamma^{(i)}, \tag{6.12}$$

where

$$G_u^{(1)}(s) = C^{(1)}\big(sI - A^{(1)}\big)^{-1}B_u^{(1)}$$

is the TFM of the fault-free system. Note that $G_u^{(N)}(s) = 0$ describes the case of complete failure.

We applied the **Procedure EMD** to design $N = 9$ model detection filters of least dynamical order with scalar outputs. At Step 1), nullspace bases of the form

$$Q_1^{(i)}(s) = \big[\, I_4 \;\; -G_u^{(i)}(s) \,\big] = \big[\, I_4 \;\; -G_u^{(1)}(s)\Gamma^{(i)} \,\big]$$

have been chosen as initial designs. The internal forms corresponding to these designs are

$$R_1^{(i,j)}(s) := Q_1^{(i)}(s)\begin{bmatrix} G_u^{(j)}(s) \\ I_2 \end{bmatrix} = G_u^{(j)}(s) - G_u^{(i)}(s) = G_u^{(1)}(s)\big(\Gamma^{(j)} - \Gamma^{(i)}\big)\,.$$

At this stage, the norms $\big\|R_1^{(i,j)}(s)\big\|_\infty$ monotonically map the distances between the i-th and j-th component models, as can be also seen in Fig. 6.1.

At Step 2) we target to preserve the monotonic mapping of norms (as in Fig. 6.1) after updating $Q_1^{(i)}(s)$, by choosing the updating filter $Q_2^{(i)}(s)$ such that $Q_2^{(i)}(s)Q_1^{(i)}(s)$ has least-order. For this purpose, with a suitably chosen row vector h, a linear combination of the basis vectors has been formed as $X^{(i)}(s) = hQ_1^{(i)}(s)$, and then a proper rational row vector $Y^{(i)}(s)$ has been determined such that $Q_2^{(i)}(s)Q_1^{(i)}(s) := X^{(i)}(s) + Y^{(i)}(s)Q_1^{(i)}(s)$ has least McMillan degree and $Q_2^{(i)}(s)R_1^{(i,j)}(s) \neq 0$ for all $j \neq i$. The resulting $Q_2^{(i)}(s)$ is simply $Q_2^{(i)}(s) = h + Y^{(i)}(s)$. For this computation, minimal dynamic cover techniques described in Sect. 7.5 have been used. After some trials with randomly generated h, the value

$$h = [\, 0.7645 \;\; 0.8848 \;\; 0.5778 \;\; 0.9026 \,]$$

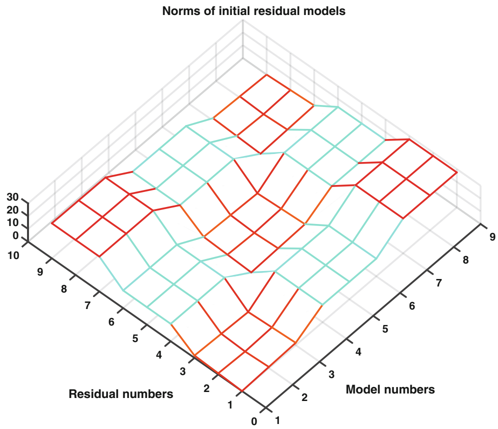

Fig. 6.1 Norms of residual models for the initial full order synthesis

led to a satisfactory dynamics of a first-order updated filter, without the need of further stabilization. Due to the particular forms of $G_u^{(i)}(s)$ in (6.12), the same $Q_2^{(i)}(s) := Q_2^{(1)}(s)$, $i = 1, \ldots, N$, can be used for all models. The resulting final filters are given by

$$Q^{(i)}(s) = Q_2^{(i)}(s)Q_1^{(i)}(s) = \left[\, Q_2^{(1)}(s) \quad -Q_2^{(1)}(s)G_u^{(1)}(s)\Gamma^{(i)} \,\right], \tag{6.13}$$

where, for convenience, we set $Q^{(N)}(s)$ as

$$Q^{(N)}(s) = \left[\, Q_2^{(1)}(s) \; 0 \,\right],$$

with a first-order state-space realization, although $Q^{(N)}(s) = \left[\, h \;\; 0 \,\right]$ was also possible.

The final internal filters $R^{(i,j)}(s)$ result as

$$R^{(i,j)}(s) = Q_2^{(1)}(s)G_u^{(1)}(s)\big(\Gamma^{(j)} - \Gamma^{(i)}\big), \quad i, j = 1, \ldots, N$$

and preserve the monotonic mapping of distances, as in Fig. 6.1.

For practical use, the N filters $Q^{(i)}(s)$ have been scaled such that the corresponding row blocks $R^{(i,j)}(s)$ fulfill the condition $\min_{j=1:N, i\neq j} \left\| R^{(i,j)}(s) \right\|_\infty = 1$. This amounts to replace $Q^{(i)}(s)$ by $Q^{(i)}(s)/\gamma_i$ and $R^{(i,j)}(s)$ by $R^{(i,j)}(s)/\gamma_i$, for $j = 1, \ldots, N$, where $\gamma_i = \min_{j=1:N, i\neq j} \left\| R^{(i,j)}(s) \right\|_\infty$. This scaling also enforces the symmetry conditions $\left\| R^{(i,j)}(s) \right\|_\infty = \left\| R^{(j,i)}(s) \right\|_\infty$ for all $i \neq j$.

In Fig. 6.2 the step responses from u_1 (aileron) and u_2 (rudder) are presented for the 9×9 block array, whose entries are the rescaled TFMs $R^{(i,j)}(s)$. Each column corresponds to a specific model for which the step responses of the N residuals are computed. The achieved typical structure matrix for model detection (with zeros down the diagonal) can easily be read out from this signal-based assessment.

The script **Ex6_1** in Listing 6.1 solves the EMDP considered in this example. The script **Ex6_1figs** (not listed) generates the plots in Figs. 6.1 and 6.2. ◊

Listing 6.1 Script **Ex6_1** to solve the EMDP of Example 6.1 using **Procedure EMD**

```
% Uses the Control Toolbox and the Descriptor System Tools

% define lateral aircraft model without faults G_u
A = [-.4492 0.046 .0053 -.9926;
      0 0 1 0.0067;
      -50.8436 0 -5.2184 .722;
      16.4148 0 .0026 -.6627];
Bu = [0.0004 0.0011; 0 0; -1.4161 .2621; -0.0633 -0.1205];
C = eye(4); p = size(C,1); mu = size(Bu,2);
% define the LOE faults Γ^(i)
Gamma = 1 - [ 0  0 0 .5 .5 .5 1  1 1;
              0 .5 1  0 .5  1 0 .5 1 ]';
N = size(Gamma,1);
% define multiple physical fault model G_u^(i) = G_u Γ^(i)
sysu = ss(zeros(p,mu,N,1));
for i=1:N
    sysu(:,:,i,1) = ss(A,Bu*diag(Gamma(i,:)),C,0);
end

% setup initial full order model detector Q_1^(i) = [ I  -G_u^(i) ]
Q1 = [eye(p) -sysu];

% form a linear combination of hQ_1^(i) with the rows of Q_1^(i)
% to obtain a minimum order synthesis, by solving a minimum
% dynamic cover problem; the result is a least-order Q^(i) = Q_2^(i) Q_1^(i)
h = [ 0.7645   0.8848   0.5778   0.9026];
tol = 1.e-7;                        % set tolerance
Q = ss(zeros(1,p+mu,N,1));
for i = 1:N-1
  Q(:,:,i,1) = glmcover1([h;eye(p)]*Q1(:,:,i,1),1,tol);
end
Q(1,1:p,N,1) = Q(1,1:p,1,1);   % set Q^(N) = [ Q_2^(1) 0 ]

% compute internal forms R^(i,j) and their norms
R = ss(zeros(1,mu,N,N));
for i = 1:N
  for j = 1:N
    temp = Q(:,:,i,1)*[sysu(:,:,j,1); eye(mu)];
    R(:,:,i,j) = gir(temp,tol);
  end
end

% scale Q^(i) and R^(i,j)
distinf = norm(R,inf);
for i=1:N
  gammai = 1/min(distinf(i,[1:i-1 i+1:N]));
  Q(:,:,i,1) = gammai*Q(:,:,i,1);
  for j = 1:N
     R(:,:,i,j) = gammai*R(:,:,i,j);
  end
end
```

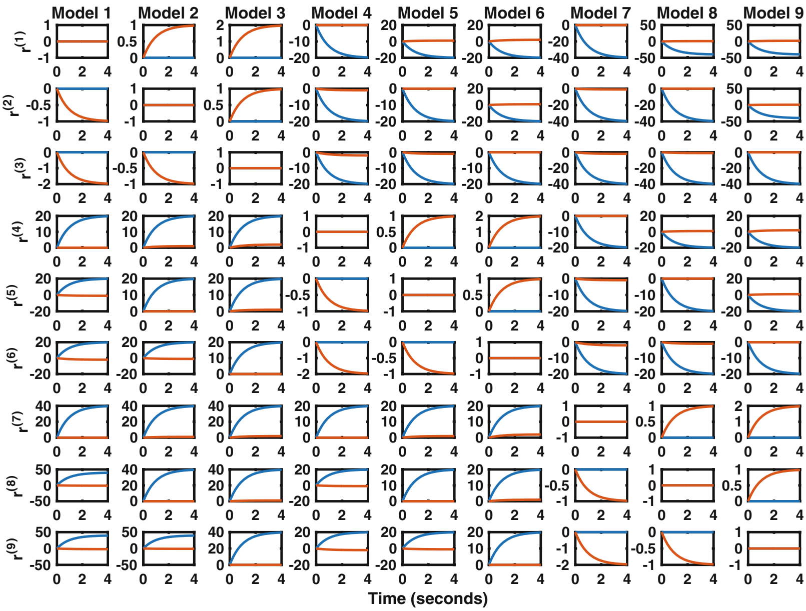

Fig. 6.2 Step responses of $R^{(i,j)}(s)$ from u_1 (*blue*) and u_2 (*red*) for least-order syntheses

6.3 Solving the Approximate Model Detection Problem

Using Proposition 6.1, the solvability conditions of the *approximate fault detection problem* (AMDP) formulated in Sect. 4.4.2 for the multiple model (4.1), can be also expressed in terms of the multiple models (6.7), according to the following corollary to Theorem 4.3:

Corollary 6.2 *For the multiple model (4.1) the AMDP is solvable if and only if for the multiple models (6.7) the following conditions hold for* $i = 1, \ldots, N$

$$\left[\, \overline{G}_u^{(i,j)}(\lambda)\ \overline{G}_d^{(i,j)}(\lambda) \,\right] \neq 0 \quad \forall j \neq i\,. \tag{6.14}$$

We have seen in the proof of Theorem 4.3, that a solution of the AMDP can be determined by solving the related EMDP with $w^{(j)} \equiv 0$ for $j = 1, \ldots, N$, using, for example, the **Procedure EMD**. However, potentially better solutions can be obtained by trying to maximize the gap between the requirements for high sensitivity to non-current models and strong attenuation of noise signals for the current model. An optimization-based approach, similar to that used for the solution of the AFDP, can be used to achieve this goal.

Consider the parametrization (6.4) of the i-th filter as $Q^{(i)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)Q_1^{(i)}(\lambda)$. With the notation used in (6.8), we obtain from (4.5)

$$R^{(i,j)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)\overline{R}^{(i,j)}(\lambda) , \tag{6.15}$$

where

$$\overline{R}^{(i,j)}(\lambda) := \left[\, \overline{G}_u^{(i,j)}(\lambda)\ \overline{G}_d^{(i,j)}(\lambda)\ \overline{G}_w^{(i,j)}(\lambda) \,\right] . \tag{6.16}$$

The above choice of $Q^{(i)}(\lambda)$ ensures that

$$\overline{R}^{(i,i)}(\lambda) = \left[\, 0\ 0\ \overline{G}_w^{(i,i)}(\lambda) \,\right]. \tag{6.17}$$

Let $\gamma_i > 0$ be an admissible level for the effect of the noise signal $w^{(i)}(t)$ on the residual $r^{(i)}(t)$ in the case when the i-th model is the current model. In the light of (6.17), such a limitation can be imposed, for example, as a constraint of the form

$$\| R_w^{(i,i)}(\lambda)\|_{2/\infty} \leq \gamma_i \ , \tag{6.18}$$

where $R_w^{(i,j)}(\lambda)$ is defined in (4.5). Using (6.15)–(6.17), $R_w^{(i,i)}(\lambda)$ can expressed as $R_w^{(i,i)}(\lambda) = \overline{Q}_1^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$, and therefore, (6.18) becomes

$$\left\| \overline{Q}_1^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda) \right\|_{2/\infty} \leq \gamma_i \, . \tag{6.19}$$

For $\gamma_i > 0$ it is always possible, via a suitable scaling of the i-th filter, to use the normalized value $\gamma_i = 1$.

In the absence of noise, the influence of the j-th model on the i-th residual can be characterized by the associated gain $\left\| \left[\, R_u^{(i,j)}(\lambda) \quad R_d^{(i,j)}(\lambda) \,\right] \right\|_{2/\infty}$. Therefore, as a measure of the global sensitivity of the i-th residual to the rest of $N-1$ models different from the i-th model, the minimum values of these gains can be employed. Using the parametrization (6.4) of the i-th filter, the following sensitivity measure can be defined

$$\zeta_1^{(i)}\big(\overline{Q}_1^{(i)}(\lambda)\big) := \min_{1\leq j\leq N,\, j\neq i} \left\| \overline{Q}_1^{(i)}(\lambda)\left[\, \overline{G}_u^{(i,j)}(\lambda)\ \ \overline{G}_d^{(i,j)}(\lambda) \,\right] \right\|_{2/\infty}, \tag{6.20}$$

where the dependence of $\zeta_1^{(i)}$ of the choice of the filter $\overline{Q}_1^{(i)}(\lambda)$ is explicitly emphasized. The requirement $\zeta_1^{(i)} > 0$ for $i = 1, \ldots, N$ can be interpreted as an alternative characterization of the *model detectability* of the N component models.

We can formulate several optimization problems (for different combinations of employed norms) to address the computation of a satisfactory (or even optimal) solution of the AMDP, having the goal of maximizing the model sensitivities (6.20) under the noise attenuation constraints (6.19). In what follows, we only discuss the $\mathcal{H}_\infty$-norm based synthesis, for which we give a detailed computational procedure.

The synthesis of the i-th filter, can be individually addressed, by solving for each $i = 1, \ldots, N$ the following constrained optimization problem: Given $\gamma_i \geq 0$, determine $\beta_i > 0$ and a stable and proper filter $\overline{Q}_1^{(i)}(\lambda)$ such that

$$\beta_i = \max_{\overline{Q}_1^{(i)}(\lambda)} \left\{ \zeta_1^{(i)}\big(\overline{Q}_1^{(i)}(\lambda)\big) \,\Big|\, \|\overline{Q}_1^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)\|_\infty \leq \gamma_i \right\}. \tag{6.21}$$

The gap $\eta_i := \beta_i/\gamma_i$ can be interpreted as a measure of the quality of i-th filter in differentiating between the i-th model and the rest of models in the presence of noise. For $\gamma_i = 0$, the above formulation includes the exact solution (i.e., of the EMDP) and the corresponding gap is infinite.

To solve the formulated N optimization problems (6.21), we devise a synthesis procedure based on successive simplifications of the original problem by reducing it to simpler problems with the help of the factorized representations of the filters (6.4). The existence conditions of Corollary 6.2 can be immediately checked. In this context, we introduce a useful concept to simplify the presentation. A filter $Q^{(i)}(\lambda)$ is called *admissible* if the corresponding $\big[\, R_u^{(i,j)}(\lambda) \;\; R_d^{(i,j)}(\lambda) \,\big]$ in (4.5) are all nonzero for $j \neq i$. Tests as those of Corollary 6.2 can be used to check admissibility. Assume that the test indicates the solvability of the AMDP.

Let q_i be the desired number of residual components for the i-th filter with output $r^{(i)}(t)$. As in the case of an EMDP, if a solution of the AMDP exists, then, in general, the use of a scalar output fault detection filter (thus, $q_i = 1$) is always possible. However, larger values of q_i can be advantageous, because may provide more free parameters which can be appropriately tuned. In general, the choice of q_i must satisfy $q_i \leq p - r_d^{(i)}$, where $r_d^{(i)} := \text{rank}\, G_d^{(i)}(\lambda)$. In the **Procedure AMD** to solve the AMDP, given in what follows, the choice $q_i \leq r_w^{(i)}$ is enforced, in the case when $r_w^{(i)} := \text{rank}\, \overline{G}_w^{(i,i)}(\lambda) > 0$. This choice is only for convenience and leads to a simpler synthesis procedure.

As next step, the factor $\overline{Q}_1^{(i)}(\lambda)$ is determined in the product form $\overline{Q}_1^{(i)}(\lambda) = \overline{Q}_2^{(i)}(\lambda)Q_2^{(i)}(\lambda)$, where the $r_w^{(i)} \times (p - r_d^{(i)})$ factor $Q_2^{(i)}(\lambda)$ is determined such that $Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$ has full row rank $r_w^{(i)}$, the product $Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ is admissible, and, has the least possible McMillan degree. If this latter requirement is not imposed, then a simple choice is $Q_2^{(i)}(\lambda) = H^{(i)}$, where $H^{(i)}$ is an $r_w^{(i)} \times \big(p - r_d^{(i)}\big)$ full row rank constant matrix (e.g., chosen as a randomly generated matrix with orthonormal rows). This corresponds to building $Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ as $r_w^{(i)}$ linear combinations of the left nullspace basis vectors contained in the rows of $Q_1^{(i)}(\lambda)$.

At this stage, the optimization problem to be solved falls in one of two categories. The *standard case* is when $Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$ has no unstable zeros on the boundary of the stability domain $\partial\mathbb{C}_s$ (i.e., the extended imaginary axis in the continuous-time case, or the unit circle centered in the origin in the discrete-time case). The *nonstandard case* corresponds to the presence of such zeros. This categorization is revealed at the next step, which also involves the computation of the respective zeros.

The quasi-co-outer–co-inner factorization of the full row rank $Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$ is

$$Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda) = G_{wo}^{(i)}(\lambda)G_{wi}^{(i)}(\lambda), \tag{6.22}$$

where the quasi-co-outer factor $G_{wo}^{(i)}(\lambda)$ is invertible, having only zeros in $\overline{\mathbb{C}}_s$, and $G_{wi}^{(i)}(\lambda)$ is co-inner. The factor $\overline{Q}_2^{(i)}(\lambda)$ is chosen in the product form $\overline{Q}_2^{(i)}(\lambda) = \overline{Q}_3^{(i)}(\lambda)Q_3^{(i)}(\lambda)$, with $Q_3^{(i)}(\lambda) = \big(G_{wo}^{(i)}(\lambda)\big)^{-1}$ and $\overline{Q}_3^{(i)}(\lambda)$ to be determined. Using (6.16), we define

$$\widetilde{R}^{(i,j)}(\lambda) := Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)\overline{R}^{(i,j)}(\lambda)\,, \tag{6.23}$$

with the component blocks defined as

$$\big[\, \widetilde{R}_u^{(i,j)}(\lambda) \mid \widetilde{R}_d^{(i,j)}(\lambda) \mid \widetilde{R}_w^{(i,j)}(\lambda) \,\big] := Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)\big[\, \overline{R}_u^{(i,j)}(\lambda) \mid \overline{R}_d^{(i,j)}(\lambda) \mid \overline{R}_w^{(i,j)}(\lambda) \,\big].$$

This allows to express $\zeta_1^{(i)}$ in (6.20) as $\zeta_1^{(i)}\big(\overline{Q}_1^{(i)}(\lambda)\big) = \zeta_3^{(i)}\big(\overline{Q}_3^{(i)}(\lambda)\big)$, where

$$\zeta_3^{(i)}\big(\overline{Q}_3^{(i)}(\lambda)\big) := \min_{1\leq j\leq N,\, j\neq i} \big\| \overline{Q}_3^{(i)}(\lambda)\big[\, \widetilde{R}_u^{(i,j)}(\lambda) \mid \widetilde{R}_d^{(i,j)}(\lambda) \,\big] \big\|_\infty . \tag{6.24}$$

It follows, that $\overline{Q}_3^{(i)}(\lambda)$ can be determined as the solution of

$$\beta_i = \max_{\overline{Q}_3^{(i)}(\lambda)} \left\{ \zeta_3^{(i)}\big(\overline{Q}_3^{(i)}(\lambda)\big) \,\middle|\, \big\|\overline{Q}_3^{(i)}(\lambda)\big\|_\infty \leq \gamma_i \right\},$$

where we used that

$$\big\|\overline{Q}_3^{(i)}(\lambda)Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)\big\|_\infty = \big\|\overline{Q}_3^{(i)}(\lambda)G_{wi}^{(i,i)}(\lambda)\big\|_\infty = \big\|\overline{Q}_3^{(i)}(\lambda)\big\|_\infty .$$

In the standard case, we can always ensure that the partial filter defined by the product of stable factors $Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ is stable. However, $\widetilde{R}^{(i,j)}(\lambda)$ is generally not stable, unless all component systems of the multiple model (4.1) are stable. In such a case, $\overline{Q}_3^{(i)}(\lambda)$ can be simply determined as $\overline{Q}_3^{(i)}(\lambda) = Q_4^{(i)}$, where $Q_4^{(i)}$ is a constant matrix representing the optimal solution of the simpler problem

$$\beta_i = \max_{Q_4^{(i)}} \left\{ \zeta_3^{(i)}\big(Q_4^{(i)}\big) \,\middle|\, \big\|Q_4^{(i)}\big\|_\infty \leq \gamma_i \right\},$$

such that the resulting filter $Q^{(i)}(\lambda) = Q_4^{(i)}Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ is admissible. For square $Q_4^{(i)}$, the choice $Q_4^{(i)} = \gamma_i I$ is the simplest optimal solution. If $\widetilde{R}^{(i,j)}(\lambda)$ is unstable or improper, the solution approach for the nonstandard case, discussed below, can be used.

The following result, given without proof, is similar to Theorem 5.2. The proof is similar to the proofs in the case of solving AFDPs in continuous- and discrete-time, see [77] and [78], respectively.

Theorem 6.2 *Using the parametrization (6.4) of the i-th filter and the notation in (6.16), let $Q_2^{(i)}(\lambda)$ be such that $\big\| Q_2^{(i)}(\lambda)\big[\, \overline{G}_u^{(i,j)}(\lambda) \;\; \overline{G}_d^{(i,j)}(\lambda) \,\big]\big\|_\infty > 0$ for all $j \neq i$, and, additionally, $Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$ has full row rank and has no zeros on the boundary of the stability domain. Then, for $\gamma_i > 0$, the optimal solution of the optimization problem (6.21) is*

$$\overline{Q}_{1,opt}^{(i)}(\lambda) := \gamma_i \, \big(G_{wo}^{(i)}(\lambda)\big)^{-1} Q_2^{(i)}(\lambda) \, ,$$

where $G_{wo}^{(i)}(\lambda)$ is the co-outer factor of the co-outer–co-inner factorization (6.22).

In the nonstandard case, both the partial filter $\widetilde{Q}^{(i)}(\lambda) := Q_3^{(i)}(\lambda)Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ and the corresponding $\widetilde{R}^{(i,j)}(\lambda)$ in (6.23) for $j = 1, \ldots, N$, can result unstable or improper due the presence of poles of $Q_3^{(i)}(\lambda) = \big(G_{wo}^{(i)}(\lambda)\big)^{-1}$ in $\partial\mathbb{C}_s$ (i.e., $G_{wo}^{(i)}(\lambda)$ has zeros in $\partial\mathbb{C}_s$). In this case, $\overline{Q}_3^{(i)}(\lambda)$ is chosen in the form $\overline{Q}_3^{(i)}(\lambda) = Q_5^{(i)} Q_4^{(i)}(\lambda)$, where $Q_4^{(i)}(\lambda)$ results form a LCF with stable and proper factors

$$\big[\, \widetilde{Q}^{(i)}(\lambda) \;\; \widetilde{R}^{(i,1)}(\lambda) \ldots \widetilde{R}^{(i,N)}(\lambda) \,\big] = \big(Q_4^{(i)}(\lambda)\big)^{-1} \big[\, \widehat{Q}^{(i)}(\lambda) \;\; \widehat{R}^{(i,1)}(\lambda) \ldots \widehat{R}^{(i,N)}(\lambda) \,\big] ,$$

while $Q_5^{(i)}$ is a constant matrix which solves

$$\beta_i = \max_{Q_5^{(i)}} \Big\{ \zeta_5^{(i)}\big(Q_5^{(i)}\big) \;\Big|\; \big\| Q_5^{(i)} Q_4^{(i)}(\lambda) \big\|_\infty \leq \gamma_i \Big\} \, ,$$

where

$$\zeta_5^{(i)}\big(Q_5^{(i)}\big) := \min_{1 \leq j \leq N, j \neq i} \big\| Q_5^{(i)} Q_4^{(i)}(\lambda) \big[\, \widetilde{R}_u^{(i,j)}(\lambda) \mid \widetilde{R}_d^{(i,j)}(\lambda) \,\big] \big\|_\infty .$$

The choice of a diagonal $Q_4^{(i)}(\lambda)$, with all its diagonal elements having $\mathcal{H}_\infty$-norms equal to 1, significantly simplifies the solution of the above problem. In this case, the choice $Q_5^{(i)} = \gamma_i I$ is always possible.

In the standard case, the dynamical order of the resulting filter $Q^{(i)}(\lambda)$ is the McMillan degree of $Q_3^{(i)}(\lambda)$, provided $Q_4^{(i)}(\lambda)$ is chosen a constant matrix. This order results from the conditions that $Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda)$ has full row rank and $Q_2^{(i)}(\lambda)Q_1^{(i)}(\lambda)$ has least-order and is admissible. For each candidate $Q_2^{(i)}(\lambda)$, the corresponding optimal $Q_3^{(i)}(\lambda)$ results automatically, but the different "optimal" filters for the same level γ_i of noise attenuation performance can have significantly differing optimal performance levels β_i. Finding the best compromise between the achieved order and the achieved performance (measured via the gap β_i/γ_i), should take into account that larger orders and larger number of detector outputs q_i may potentially lead to better performance.

The **Procedure AMD**, given in what follows, allows the synthesis of least-order model detection filters, by solving the AMDP employing an $\mathcal{H}_\infty$ optimization-based approach. This procedure includes also the **Procedure EMD**, in the case when, an exact solution exists. Similar synthesis procedures, relying on alternative optimization-based formulations, can be devised by only adapting appropriately the last computational step of **Procedure AMD**.

Procedure AMD: Approximate synthesis of model detection filters

Inputs : $\{G_u^{(j)}(\lambda), G_d^{(j)}(\lambda), G_w^{(j)}(\lambda)\}$, for $j = 1, \ldots, N$; q_{max}
Outputs: $Q^{(i)}(\lambda)$, for $i = 1, \ldots, N$; $R^{(i,j)}(\lambda)$ for $i, j = 1, \ldots, N$

For $i = 1, ..., N$

1) Compute a $(p - r_d^{(i)}) \times (p + m_u)$ minimal proper stable basis $Q_1^{(i)}(\lambda)$ for the left nullspace of $G^{(i)}(\lambda)$ defined in (6.2), where $r_d^{(i)} := \operatorname{rank} G_d^{(i)}(\lambda)$; set $Q^{(i)}(\lambda) = Q_1^{(i)}(\lambda)$, compute $\overline{G}_w^{(i,i)}(\lambda) = Q_1^{(i)}(\lambda)\begin{bmatrix} G_w^{(i)}(\lambda) \\ 0 \end{bmatrix}$, and
$$R^{(i,j)}(\lambda) = [\, R_u^{(i,j)}(\lambda) \mid R_d^{(i,j)}(\lambda) \mid R_w^{(i,j)}(\lambda)\,]$$
$$= Q_1^{(i)}(\lambda)\left[\begin{array}{c|c|c} G_u^{(j)}(\lambda) & G_d^{(j)}(\lambda) & G_w^{(j)}(\lambda) \\ I_{m_u} & 0 & 0 \end{array}\right], \quad j = 1, \ldots, N$$
Exit if $[\, R_u^{(i,j)}(\lambda)\;\; R_d^{(i,j)}(\lambda)\,] = 0$ for any $j \in \{1, \ldots, N\}$, $j \neq i$ (no solution)
2) Compute $r_w^{(i)} = \operatorname{rank} \overline{G}_w^{(i,i)}(\lambda)$; if $r_w^{(i)} = 0$, set $q_1^{(i)} = \min(p - r_d^{(i)}, q_{max})$; else, set $q_1^{(i)} = r_w^{(i)}$; choose a $q_1^{(i)} \times (p - r_d^{(i)})$ rational matrix $Q_2^{(i)}(\lambda)$ such that $Q_2^{(i)}(\lambda)[\, R_u^{(i,j)}(\lambda)\;\; R_d^{(i,j)}(\lambda)\,] \neq 0$ for $j = 1, \ldots, N$, $j \neq i$, $Q_2^{(i)}(\lambda)Q^{(i)}(\lambda)$ has least McMillan degree, and, if $r_w^{(i)} > 0$, then $\operatorname{rank}\, Q_2^{(i)}(\lambda)\overline{G}_w^{(i,i)}(\lambda) = r_w^{(i)}$; compute $Q^{(i)}(\lambda) \leftarrow Q_2^{(i)}(\lambda)Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_2^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$.
3) If $r_w^{(i)} > 0$, compute the quasi-co-outer–co-inner factorization (6.22) with $G_{wo}^{(i)}(\lambda)$ invertible and having only zeros in $\overline{\mathbb{C}}_s$, and $G_{wi}^{(i)}(\lambda)$ co-inner; with $Q_3^{(i)}(\lambda) = \left(G_{wo}^{(i)}(\lambda)\right)^{-1}$ compute $Q^{(i)}(\lambda) \leftarrow Q_3^{(i)}(\lambda)Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_3^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$.
4) Choose a square rational matrix $Q_4^{(i)}(\lambda)$ such that $Q_4^{(i)}(\lambda)Q^{(i)}(\lambda)$ has a desired stable dynamics and $Q_4^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$ are stable; compute $Q^{(i)}(\lambda) \leftarrow Q_4^{(i)}(\lambda)Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_4^{(i)}(\lambda)R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$.
5) If $r_w^{(i)} > 0$, choose $Q_5^{(i)} \in \mathbb{R}^{\min(q_{max}, r_w^{(i)}) \times q_1^{(i)}}$ such that $\|Q_5^{(i)}Q_4^{(i)}(\lambda)\|_\infty = \gamma_i$ and $\beta_i = \min_{1\leq j\leq N, j\neq i} \left\|Q_5^{(i)}[\, R_u^{(i,j)}(\lambda)\;\; R_d^{(i,j)}(\lambda)\,]\right\|_\infty > 0$; compute $Q^{(i)}(\lambda) \leftarrow Q_5^{(i)}Q^{(i)}(\lambda)$ and $R^{(i,j)}(\lambda) \leftarrow Q_5^{(i)}R^{(i,j)}(\lambda)$ for $j = 1, \ldots, N$, $j \neq i$; else, set $\beta_i = \infty$.

Remark 6.3 For the selection of the threshold τ_i for the component $r^{(i)}(t)$ of the residual vector, a similar approach to that described in Remark 5.11 can be used. The i-th residual, which results when the j-th model is the current one, is

$$\mathbf{r}^{(i)}(\lambda) = R_u^{(i,j)}(\lambda)\mathbf{u}(\lambda) + R_d^{(i,j)}(\lambda)\mathbf{d}^{(j)}(\lambda) + R_w^{(i,j)}(\lambda)\mathbf{w}^{(j)}(\lambda), \tag{6.25}$$

where $R_u^{(i,j)}(\lambda)$, $R_d^{(i,j)}(\lambda)$, and $R_w^{(i,j)}(\lambda)$ are formed from the columns of $R^{(i,j)}(\lambda)$ corresponding to the inputs u, $d^{(j)}$ and $w^{(j)}$, respectively. To determine the false alarm bound for the i-th residual, we can use the residual which results for the i-th filter if the i-th model is the current one. Taking into account that $R_u^{(i,i)}(\lambda) = 0$ and $R_d^{(i,i)}(\lambda) = 0$, we obtain

$$\mathbf{r}^{(i)}(\lambda) = R_w^{(i,i)}(\lambda)\mathbf{w}^{(i)}(\lambda)\,. \tag{6.26}$$

If we assume, for example, a bounded energy noise input $w^{(i)}(t)$ such that $\|w^{(i)}\|_2 \leq \delta_w^{(i)}$, then the false alarm bound $\tau_f^{(i)}$ for the i-th residual vector component $r^{(i)}(t)$ can be computed as

$$\tau_f^{(i)} = \sup_{\|w^{(i)}\|_2 \leq \delta_w^{(i)}} \|R_w^{(i,i)}(\lambda)\mathbf{w}^{(i)}(\lambda)\|_2 = \|R_w^{(i,i)}(\lambda)\|_\infty \delta_w^{(i)}\,. \tag{6.27}$$

The setting of the thresholds to $\tau_i = \tau_f^{(i)}$ for $i = 1, \ldots, N$ ensures no false alarms in detecting the i-th model, provided sufficient control, disturbance or noise activity is present such that

$$\|r^{(j)}\|_2 > \tau_f^{(j)}, \quad \forall j \neq i\,.$$

Therefore, to enhance the decision-making process it must be additionally checked that the control input u has a certain minimum energy, i.e., $\|u\|_2 > \underline{\delta}_u$, where $\underline{\delta}_u$ is the least size of the acceptable control inputs. A conservative (worst-case) estimate of $\underline{\delta}_u$ can be determined by enforcing

$$\left\| R_u^{(i,j)}(\lambda)\mathbf{u}(\lambda)\right\|_2 \geq \left\| R_d^{(i,j)}(\lambda)\mathbf{d}^{(j)}(\lambda)\right\|_2 + \left\| R_w^{(i,j)}(\lambda)\mathbf{w}^{(j)}(\lambda)\right\|_2$$

for $\left\|d^{(j)}\right\|_2 \leq \delta_d^{(j)}$ and $\left\|w^{(j)}\right\|_2 \leq \delta_w^{(j)}$, $\forall\, i, j$ with $j \neq i$. A possible choice is

$$\underline{\delta}_u = \max_{i,j;i\neq j} \frac{\left\| R_d^{(i,j)}(\lambda)\right\|_\infty \delta_d^{(j)} + \left\| R_w^{(i,j)}(\lambda)\right\|_\infty \delta_w^{(j)}}{\left\| R_u^{(i,j)}(\lambda)\right\|_\infty}\,.$$

□

Example 6.2 This is basically the same multiple model as that used in Example 6.1, however with only two measured outputs, namely, the sideslip angle and roll angle, and additional input noise and output noise. The fault-free state-space model describes the continuous-time lateral dynamics of a F-16 aircraft with the matrices

$$A^{(1)} = \begin{bmatrix} -0.4492 & 0.046 & 0.0053 & -0.9926 \\ 0 & 0 & 1.0000 & 0.0067 \\ -50.8436 & 0 & -5.2184 & 0.7220 \\ 16.4148 & 0 & 0.0026 & -0.6627 \end{bmatrix}, \quad B_u^{(1)} = \begin{bmatrix} 0.0004 & 0.0011 \\ 0 & 0 \\ -1.4161 & 0.2621 \\ -0.0633 & -0.1205 \end{bmatrix},$$

$$C^{(1)} = \begin{bmatrix} 57.2958 & 0 & 0 & 0 \\ 0 & 57.2958 & 0 & 0 \end{bmatrix}, \quad D_u^{(1)} = 0_{2\times 2}, \quad D_w^{(1)} = [\, 0_{2\times 4} \;\; I_2 \,].$$

The component system matrices in (2.19) are defined for $i = 1, 2, \ldots, N$ as: $E^{(i)} = I_4$, $A^{(i)} = A^{(1)}$, $C^{(i)} = C^{(1)}$, $B_w^{(i)} = B_w^{(1)}$, $D_w^{(i)} = D_w^{(1)}$, and $B_u^{(i)} = B_u^{(1)}\Gamma^{(i)}$, where $\Gamma^{(i)} = \mathrm{diag}\,\big(1-\rho_1^{(i)}, 1-\rho_2^{(i)}\big)$ and $\big(\rho_1^{(i)}, \rho_2^{(i)}\big)$ are the values of parameters (ρ_1, ρ_2) on the chosen grid points $\{0, 0.5, 1\}$ for each component of $\rho := [\rho_1, \rho_2]^T$. The values $\big(\rho_1^{(1)}, \rho_2^{(1)}\big) = (0, 0)$ correspond to the fault-free situation. The TFMs $G_u^{(i)}(s)$ and $G_w^{(i)}(s)$ of the i-th system can be expressed as

$$G_u^{(i)}(s) = G_u^{(1)}(s)\Gamma^{(i)}, \quad G_w^{(i)}(s) = G_w^{(1)}(s)\,, \tag{6.28}$$

where

$$G_u^{(1)}(s) = C^{(1)}\big(sI - A^{(1)}\big)^{-1}B_u^{(1)}, \quad G_w^{(1)}(s) = C^{(1)}\big(sI - A^{(1)}\big)^{-1}B_w^{(1)} + D_w^{(1)}\,.$$

We applied the **Procedure AMD** to design $N = 9$ model detection filters of least dynamical order with scalar outputs. At Step 1), nullspace bases of the form

$$Q_1^{(i)}(s) = \big[\, I_2 \;\; -G_u^{(i)}(s) \,\big] = \big[\, I_2 \;\; -G_u^{(1)}(s)\Gamma^{(i)} \,\big]$$

have been chosen as initial designs. The internal forms corresponding to these designs are

$$R_{u,1}^{(i,j)}(s) := Q_1^{(i)}(s)\begin{bmatrix} G_u^{(j)}(s) \\ I_2 \end{bmatrix} = G_u^{(1)}(s)\big(\Gamma^{(j)} - \Gamma^{(i)}\big), \quad R_{w,1}^{(i,j)}(s) := Q_1^{(i)}(s)\begin{bmatrix} G_w^{(j)}(s) \\ 0 \end{bmatrix} = G_w^{(1)}(s)\,.$$

At Step 2), the choice $Q_2^{(i)}(s) = I$ ensures that $Q_2^{(i)}(s)G_w^{(1)}(s)$ has full row rank and no zeros. Therefore, the co-outer–co-inner factorization (6.22) of $Q_2^{(i)}(s)G_w^{(1)}(s)$ computed at Step 3) allows to obtain the optimal solution for $\gamma_i = 1$ (see Theorem 6.2) as

$$Q^{(i)}(s) = \big(G_{wo}^{(i)}(s)\big)^{-1}Q_1^{(i)}(s)\,.$$

The final internal forms of the filters, $R^{(i,j)}(s) = \big[\, R_u^{(i,j)}(s) \;\; R_w^{(i,j)}(s) \,\big]$, result for $i, j = 1, \ldots, N$ with

$$R_u^{(i,j)}(s) = \big(G_{wo}^{(i)}(s)\big)^{-1}G_u^{(1)}(s)\big(\Gamma^{(j)} - \Gamma^{(i)}\big), \qquad R_w^{(i,j)}(s) = G_{wi}^{(i)}(s)\,,$$

and, therefore, $R_u^{(i,j)}(s)$ preserves the monotonic mapping of distances between the i-th and j-th models. The performance of each filter $Q^{(i)}(s)$ is given by the resulting gap $\eta_i = \beta_i/\gamma_i (= \beta_i)$, where $\beta_i = \min_{j=1:N, i\neq j} \big\| R_u^{(i,j)}(s) \big\|_\infty$. For the resulting design, we have $\eta_i = 0.0525$, for $i = 1, \ldots, N$.

Each of the filters $Q^{(i)}(s)$ has McMillan degree 4, and therefore, the overall filter $Q(s)$ has the same complexity as a filter based on a bank of Kalman filters. A Kalman-filter-based approach is well suited in the case when the input and measurement noise are Gaussian white noise processes. Assuming the input noise has a covariance of $\Sigma_x = 0.01^2 I_4$ and the measurement noise has a covariance of $\Sigma_y = 0.2^2 I_2$, then N Kalman-filters-based residual generators $\widetilde{Q}^{(i)}(s)$, with state-space realizations of the form

$$\begin{aligned} \dot{x}_e^{(i)}(t) &= (A^{(i)} - K^{(i)}C^{(i)})x_e^{(i)}(t) + K^{(i)}y(t) + B^{(i)}u(t)\,, \\ r^{(i)}(t) &= C^{(i)}x_e^{(i)}(t) - y(t) \end{aligned}$$

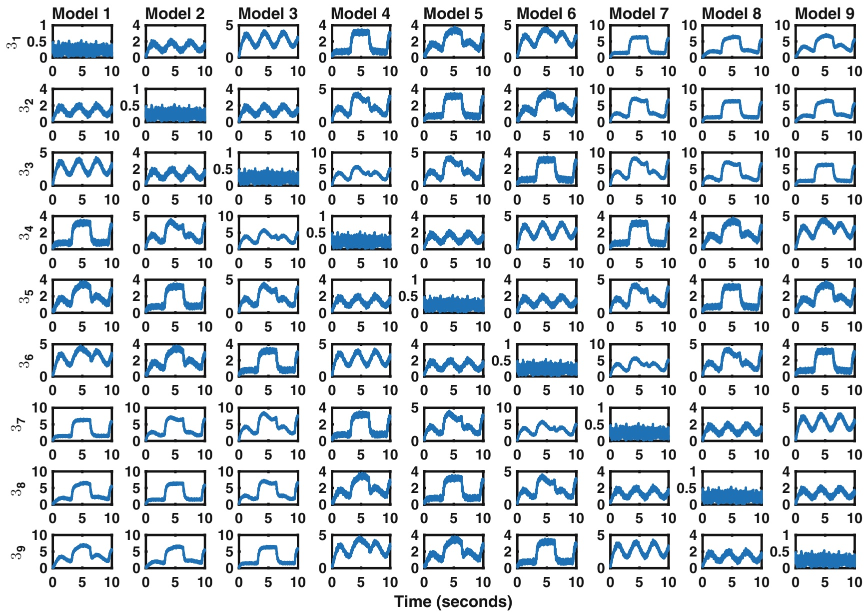

Fig. 6.3 Time responses of evaluation signals for optimal syntheses

can be determined, where the optimal gains $K^{(i)}$ result by solving suitable algebraic Riccati equations. The achieved gaps for these filters are $\widetilde{\eta}_i = 0.0152$, and therefore, below the values achieved in the optimal $\mathcal{H}_\infty$ synthesis. An optimal synthesis with second-order scalar output residual generators achieves a gap of 0.0323.

In Fig. 6.3 the time responses of the residual evaluation signals $\theta_i(t)$ are presented, where $\theta_i(t)$ are computed using a Narendra-type evaluation filter (3.40) with input $\|r^{(i)}(t)\|_2^2$ and parameters $\alpha = 0.9$, $\beta = 0.1$, $\gamma = 10$ (see Sect. 3.6). The control inputs have been chosen as follows: $u_1(t)$ is a step of amplitude 0.3 added to a square wave of period 2π, and $u_2(t)$ is a step of amplitude 1.5 added to a sinus function of unity amplitude and period π. The noise inputs are zero mean white noise of amplitude 0.01 for the input noise and 0.03 for the measurement noise. Each column corresponds to a specific model for which the time responses of the N residual evaluation signals are computed. The achieved typical structure matrix for model detection (with zeros down the diagonal) can easily be read out from this signal based assessment, even in the presence of noise.

The script **Ex6_2** in Listing 6.2 solves the AMDP considered in this example. The script **Ex6_2KF** (not listed) generates the analysis results for the Kalman filter-based synthesis and the least-order optimal synthesis. ◊

Listing 6.2 Script **Ex6_2** to solve the AMDP of Example 6.2 using **Procedure AMD**

```
% Uses the Control Toolbox and the Descriptor System Tools

% define lateral aircraft model without faults G_u
A = [-.4492 0.046 .0053 -.9926;
     0 0 1 0.0067;
     -50.8436 0 -5.2184 .722;
     16.4148 0 .0026 -.6627];
Bu = [0.0004 0.0011; 0 0; -1.4161 .2621; -0.0633 -0.1205];
```

```
[n,mu] = size(Bu); p = 2; mw = n+p; m = mu+mw;
Bw = eye(n,mw);
C = 180/pi*eye(p,n);  Du = zeros(p,mu); Dw = [zeros(p,n) eye(p)];
% define the LOE faults Γ^(i)
Gamma = 1 - [ 0  0 0 .5 .5 .5 1  1 1;
              0 .5 1  0 .5  1 0 .5 1 ]';
N = size(Gamma,1);
% define multiple physical fault model G_u^(i) = G_u Γ^(i) and G_w^(i) = G_w
sysuw = ss(zeros(p,m,N,1));
for i=1:N
    sysuw(:,:,i,1) = ss(A,[Bu*diag(Gamma(i,:)) Bw],C,[Du Dw]);
end

% optimal H-inf design
% setup initial full order model detector Q_1^(i) = [ I  -G_u^(i) ]
Q1 = [eye(p) -sysuw(:,1:mu)];

% perform optimal synthesis (standard case)
R = ss(zeros(p,mu+mw,N,N)); Q = ss(zeros(p,p+mu,N,1));
tol = 1.e-7;
for i = 1:N
  rwi = gir(Q1(:,1:p,i,1)*sysuw(:,mu+1:m,i,1),tol);
  [gi,go] = goifac(rwi,1.e-7);
  Q(:,:,i,1) = gminreal(go\Q1(:,:,i,1),tol);
  for j = 1:N
    R(:,:,i,j) = gir(Q(:,:,i,1)*[sysuw(:,:,j,1); eye(mu,m)],tol);
  end
end

% scale Q^(i) and R^(i,j); determine gap
distinf = norm(R(:,1:mu),inf);
beta = zeros(N,1);
for i=1:N
  scale = min(distinf(i,[1:i-1 i+1:N]));
  distinf(i,:) = distinf(i,:)/scale;
  Q(:,:,i,1) = Q(:,:,i,1)/scale;
  for j = 1:N
     R(:,:,i,j) = R(:,:,i,j)/scale;
  end
  beta(i) = scale;
end
gap = beta
```

6.4 Notes and References

Section 6.1. The nullspace-based computational paradigm, which underlies the synthesis procedures presented in this chapter, has been discussed for the first time in the author's papers [144, 151] in the context of solving fault detection and isolation problems. The resulting factorized form of the component filters is similar to that for fault detection filters (see (5.1)) and is the basis of numerically reliable integrated computational algorithms. Specific numerical aspects of these algorithms are presented in Chap. 7. The parametrization of component filters given in Theorem 6.1

is similar to that used for solving FDI synthesis problems stated in Theorem 5.1. The nullspace-based characterization of model detectability in Proposition 6.1 can be interpreted as an extension of a special version of Theorem 3.5 for a particular structure matrix S.

Section 6.2. The nullspace-based synthesis method to solve the EMDP using least-order component filters has been proposed in [142]. The multiple model used in Example 6.1 has been used in [70] to address a fault tolerant control problem using interacting multiple-model Kalman filters. A solution with $N = 25$ models, allowing a more accurate identification of the degree of loss of efficiency, has been presented in [142].

Section 6.3. The solution method of the AMDP using an optimization-based method, summarized in **Procedure AMD**, represents a straightforward adaptation of the synthesis method for solving the AFDIP given in **Procedure** AFDI. The Kalman filter-based multiple-model approaches have been investigated by Wilsky in [161], where the Baram's proximity measure, introduced in [4], has been used to define the distance between two stochastic models. This measure is also the basis for discriminating among stochastic models in recently proposed methods for robust multiple-model adaptive control [41].