Pocket Calculator as an Experimental *Milieu*: Emblematic Tasks and Activities

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Abstract In this chapter, we present and analyze calculator-based tasks and activities conceived as means for the learning of mathematics in several grades of primary and secondary school. The tasks or activities have been experimented with students and pre-service teachers. The intent is to show how a set of calculator-based tasks can be organized in a way that they promote the development of theoretical aspects. The results show that a high variety of numerical activities can be proposed in such a way, but that a further institutional promotion is necessary. The analyses are based on the concept of 'milieu' by Brousseau (Theory of didactical situations in mathematics. Kluwer, Dordrecht, 1997) with an anthropological approach (Chevallard Y, Recherches en Didactique des Mathématiques, 19(2):221–266, 1999; Lagrange JB, Educational Studies in Mathematics 43(1):1–30, 2000).

Keywords Calculator • Arithmetics • Fractions • Early algebra • Theory of situations • Learning milieu • Adidacticity • Anthropologic approach • Praxeologies • Teaching

Introduction

In a considerable amount of countries, a relatively large number of primary and secondary mathematics teachers do not consider it important to teach how to use a calculator; they presumably assume that this is something pupils learn from their classmates or outside school. At least, this is the case in the French speaking part of Switzerland. The consequence, observed in higher secondary school, is that the calculator skills of the students are not as far developed as they should be at that point; for example, a few of the observed students showed difficulties to successfully enter expressions such as $\sqrt{2}-1$, but entered $\sqrt{2-1}$ instead. This lack of competences could possibly lead to the situation that when they study formal calculations with square roots and try to check that $\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)=1$, the

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calculator is of no help for them. We are of the opinion that this situation is a cause of social inequalities. A small group of students are able to use their calculators as a powerful checking tool while others limit their use to basic operations; this phenomenon has been observed on the use of symbolic calculators by Guin and Trouche (1999). This is why we are of the opinion that working with calculators on a regular basis is necessary in primary and secondary school. Furthermore, we believe that calculators are highly beneficial tools in regard to mathematics learning on the premise that their use is well introduced to the students and that the calculators are incorporated within appropriate tasks or activities (that is, well-designed sequence of tasks). In recent years, we regularly observed lessons where the teacher aimed to integrate such kind of tasks and activities at different school levels. Further, we integrated them in our pre-service teacher workshops: the secondary student teachers were asked to adapt one calculator-based activity into their teaching[1] which was then evaluated on in a feedback discussion. In this chapter, we synthesize and analyze the results of different calculator-based teaching experiments with primary and secondary school students (Del Notaro and Floris 2011; Weiss and Floris 2008) as well as within teachers' training (Floris 2015). Our analysis aims to answer the main research question whether the use of calculators enhances the learning of mathematics and how it does so. We interpret the research question by integrating the theoretical perspective, selected didactical situations, and praxeological anthropology. These aspects of our interpretation will be thoroughly described in the next section.

Theoretical Background

Our main theoretical reference is Brousseau's theory of didactical situations (1997). This theory emphasizes the role of the *adidactical milieu* in the teaching-learning process of mathematics; the gist of this theory is that, in the end, taught knowledge has to be transferable and applicable to the real, non-didactical world. But, this can only be realized when the non-didactical world is – at least partly – integrated into classroom activities. This is due to the fact that mathematics is mainly a procedural science, i.e. it is impossible to merely memorize questions and their answers, but it requires to learn how to suggest solutions to an infinite variety of possible questions – even with respect to simple additions.

As one example for adidactical feedback, we refer to the task where we asked the students to enlarge a tangram puzzle (Ibid); the students were assigned to groups, and each student of each group had to enlarge one piece of the puzzle. The feedback was provided by the final assembly of the puzzle: in case that it was impossible to put the pieces of the puzzle together because their sizes did not match, the former

[1]In Geneva, secondary teachers follow a two-year training and in the second year they teach half-time in school.

mathematical enlargement procedure, e.g. the addition of constant values to the measurements, was thereby invalidated. In this situation – the adidactical milieu – the nullification of the applied mathematical procedure is not provided by a teacher, but by the situation itself. Here, it is important to acknowledge that the milieu is not limited to the external situation, but it also refers to the goals defined by the teacher and to students' prior knowledge.

In everyday teaching, the adidactical part of the didactical milieu often is of understated significance. In consequence, the learner is held accountable to create individual "milieus", but in case that this option is out of reach for the individual learner the learning outcome is weak. The conjecture of Brousseau's theory is that an adidactical milieu provides rich feedback and thereby successfully supports learning processes. On the basis of these hypotheses, the research question evolves to how the integration of calculators into the classroom can be a successful learning milieu. We will aim to provide an answer by giving examples of different grades where the calculator was used to assist this adidactical learning milieu.

Our methodology is mainly qualitative; our data basis was students' and teachers' gestures, calculator manipulations, and utterances. Finally, we compare our findings to Brousseau's theory to assess the learning potential of the proposed tasks alongside with their feedback.

In our opinion, an analysis of effective teaching methods needs to integrate the anthropological approach by Chevallard (1999). According to this approach, praxeology (from Greek 'praxis' and 'logos') is a four-part mathematical concept which includes a type of tasks, technique, technology, and theory. The first two components are practically oriented whereas, here, technology is the discourse that justifies or explains the technique. It becomes theory when the discourse is more structured. In the educational context, and in the domain of applying technological tools, Lagrange (2000) reduces praxeology to three components: tasks, techniques and theories, i.e. Chevallard's last two components are being combined. Lagrange further considers a study of Rabardel (1995) which analyses the process of tools' transformation to effective working instruments. Thus, within the anthropological approach, discourse is said to link technique and theory; this assumption entails the conjecture that this linkage enhances the mathematical quality of learning processes in the long term. Furthermore, the concept of praxeology is especially helpful for the analysis of the introduction of new techniques.

In summary, the conditions for a profitable learning milieu according to the above mentioned theories are:

LM1 A task, or a set of tasks, that involves some sort of adidactical feedback which is independent from the teacher.

LM2 A more or less explicit presence of a tight mathematical link between theory and technique.

In the following paragraphs, we discuss a set of selected examples. First, we present two summarized examples and then present as well as thoroughly analyze two further examples.

A First Example: A Milieu for Place-Value Notation

The following tasks are intended for students at the end of primary school or early secondary school (i.e. 11–12 years-old students). We suggest that the reader takes out a calculator and solves them while thinking about the mathematical properties involved.

1. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 89454.
2. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 80404.
3. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 892054.
4. Type in the number 4.56 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 4.056.

As a whole, these tasks refer to the domain of place-value notation, which is one of the main domains in primary school. This set of tasks proposes a learning milieu for this fundamental arithmetical property because the use of this property is necessary to solve the tasks. In the course of a workshop, these tasks have been proposed to students and pre-service teachers in primary and secondary education with the underlying intention to prepare them to integrate the calculator into their teaching. We were surprised of the outcome that some of them had difficulties with solving the tasks, even despite their mathematics knowledge. One of our possible interpretations of this is linked to Brousseau's didactic contract (1997):

> It is the set of the reciprocal obligations and sanctions that each partner in the didactic situation imposes, or believes to impose, explicitly or implicitly, on others, and those that are imposed on him or her, or he or she believes that they are imposed on him or her. (Translation by Indiogine 2010, n.p.)

At primary school, without specific instructions on a different handling of the calculator, it is only used to obtain the result of a direct calculation. What is proposed in the example above is an inversion of a common task: the result is given and the operation is asked for. It is what Brousseau refers to as breaking the didactic contract and further elaborates on the students' and teachers' perplexity about this. Activities which propose such disruptions are interesting because they introduce adidacticity into the milieu; or, rephrased slightly different, ignorance triggers learning processes. Here, the teacher has the choice either to instruct the students to find the solution by themselves, or to indicate possible solving strategies (e.g. let a fellow student propose an answer). An alternative to level out the state of not knowing is to propose a slightly easier task beforehand:

1(a) Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 89264.

Evidently, the kind of feedback that is given by the calculator is in agreement with the definition of feedback of LM1, a calculator coming clearly from the non-didactical world (see previous section). With respect to LM2, on the basis of the anthropological approach, we focus on the theoretical aspect. This refers to the fact that the numbering position within the tasks links the way to solve them and explains the solving strategy at the same time. For example, in case that the task requires to change the digit two into a four, you have to add 200 because the digit '2' is at the hundreds place of the positional notation system.

A Second Example: Milieu for Arithmetics Operation Properties

At the beginning of this section, we provide a second example for a possible integration of calculators into the classroom setting. Again, we suggest that the reader takes out a calculator and solves it while thinking about the mathematical properties involved:

1. Determine all digits of the numbers 7^{12}, 7^{13}, 7^{14}, etc. in their standard (base 10) expression.

One possible answer could be that a student thinks that the requested answer for 7^{12} is 13841287200 (see Fig. 1).

In such a case, the feedback provided by the calculator is inadequate. This is why the teacher has to supplement the students with sufficient validation techniques. For example, one technique could be to determine the digits of 7^{10} and 7^{11} first, and then determine the result for 7^{12} with the help of the calculator. Or the teacher could pose the question if it is possible that the last digit of the displayed result is zero. This is how the milieu is enriched with paper-and-pencil calculations, which are necessary to give the right answer to this task. In this situation, the students' calculations are expedient with a mixture of calculator and paper-and-pencil calculations. With the help of the TI-30XSMultiView[2], it is possible to get the results to 7^{11}, i.e. $7^{11} = 1977326743$. This result can then be used in order to work out all digits of 7^{12} by computing

$$\begin{aligned} 7^{12} &= 7(7^{11}) = 7 \times (1977326743) = 7 \times (1977326740 + 3) \\ &= 7 \times 197732674 \times 10 + 21 = 1384128718 \times 10 + 21 = 13841287201. \end{aligned}$$

[2]It is the official calculator in the schools of Geneva, provided to all 10 years-old students.

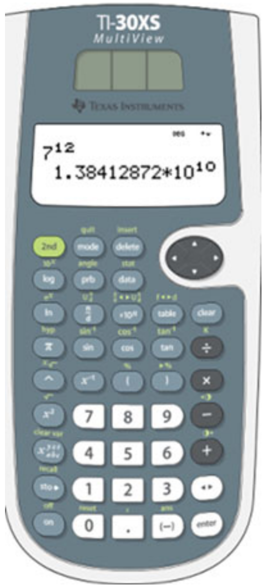


Fig. 1 Display for the result of 7^{12}

This kind of calculation requires the students' ability to apply certain operation properties correctly, in this case distributivity. As a result, the integration of the calculator into mathematics teaching is not only meant to be a calculation tool, but its application needs to be carefully instructed. Hence, the following task could be proposed afterwards to further strengthen the new didactical milieu which is dialectic of calculator and paper-and-pencil work:

2. Without multiplication and with a minimum of operations, please calculate the following products on your calculator: 387×204 and 87×199.

A Milieu at Primary School: Division Without Multiplication

Description of an Experiment

1. Is it possible to equal 24 by repeating the sequence "+ 6 = ?"[3] (See Fig. 2)
2. Is it possible to equal 24 by repeating the sequence "+ 7 = ?" (See Fig. 3)

These two tasks are examples from a long-term experiment with six to seven years-old students. The experiment lasted over the time span of about four months and was held once a week. The task was to identify all possible integers n that equal

[3]Starting from zero, that is after a reset of memory.

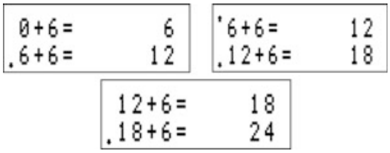


Fig. 2 Repeating the sequence " +6 =? "

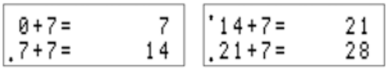


Fig. 3 Repeating the sequence " +7 = ? "

Target	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
18	1	2	3			6			9									18							
19	1																		19						
20	1	2		4	5					10										20					
21	1		3				7														21				
22	1	2									11											22			
23	1																						23		
24	1	2	3	4		6		8				12												24	
25	1				5																				25

Fig. 4 Correct results for the targets between 18 and 25

24 in the sequence " +n = ? ". The calculation result was then changed to another value. It was possible to choose between individual work or group work. This phase was followed by a public collection and discussion of results on the blackboard. A final table is presented in Fig. 4. During the discussion phase, wrong propositions were given by students which were then peer-reviewed by the other students and under teacher's management.

After the final agreement on the correctness of the table, the students were asked to express their findings, for example all the '1' in the first column, the alternate occurrences of '2', lines with only two numbers called 'poor' targets.

In the next session, another set of targets was proposed.

This kind of work, described in Del Notaro and Floris (2011) enriched the classroom study with various arithmetic properties such as parity, multiples and divisors, and primes. The use of the calculator played an important role in order to check properties and to discard wrong ideas. The following feedback instructions were summarized on the blackboard:

1. Make sure that for odd targets there are only odd divisors, and that there are even and odd divisors for even targets.
2. The 'poor' numbers are primes (a prime number is odd, except for "2" – but not all odd numbers are prime).

The 90-minutes working sessions were held weekly. After three months, the teacher introduced the idea of writing also the number of repetitions to reach the target together with the chosen number, for example adding 6 (time ‘2’) to get 12 (Fig. 5). This provided a new possibility to control one’s results: if there is a pair (a, b), there must be a pair (b, a).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
11	1 11										11 1														
12	1 12	2 6	3 4	4 3		6 2						12 1													
13	1												13												

Fig. 5 Writing the number of key iterations to reach the target

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
11	1 11										11 1														
12	1 12	2 6	3 4	4 3		6 2						12 1													
13	1 13												13 1												
14	1 14	2 7					7 2							14 1											
15	1 15		3 5		5 3										15 1										
16	1 16	2 8		4 4				8 2								16 1									
17	1 17																17 1								
18	1 18	2 9	3 6			6 3			9 2									18 1							
19	1 19																		19 1						
20	1 20	2 10		4 5	5 4					10 2										20 1					
21	1 20		3 7				7 3														21 1				
22	1	2									11											22			

Fig. 6 A table for multiplication control

By studying the above table (Fig. 6), the following new rules were identified:

1. Check with your calculator: multiplying the two numbers (in a column) results in the target.
2. Check that there is the reverse correspondent of each multiplication (commutative rule).
3. When there is only one pair and its reverse, the number is prime.
4. A target with a pair type (a, a) is a square and a is its square root.

The usual phenomena of a frantic search (called ‘fishing’ by Artigue 1997; Meissner 2005) was observed, mostly instead of the use of already institutionalized knowledge (such as provided in the list of control rules). But at mid-term, cognitive changes could be observed in the students’ actions, especially on their first approach to multiplication.

This activity provides powerful adidactical feedback. The target can be reached and mathematical properties can be verified with the calculator and the judicious use of different tables. As a result, the properties of LM1 and LM2 are completely fulfilled. The learning milieu is in the sense of Brousseau (1997) because the teacher’s input triggered the students to pose various questions which were answered by the milieu itself.

Detailed Analysis of an Example of Simplification of Fractions

The Type of Tasks

We focus on simplifying procedures for great fractions in case that a calculator is available. For numerators and denominators beyond 100, students of lower secondary school are generally not able to make use of a stored repertoire of mathematical results to find a common divisor of numerator and denominator; therefore, they are urged to apply other procedures to perform this type of task. In a former research (Weiss and Floris 2008), a series of simplifying fractions have been proposed for different types of students – at the age of about 15 years – with permission to use the calculator (Fig. 7).

It often occurred that students considered a fraction like 187/340 irreducible because they did not consider common divisors beyond ten. These students are

Fig. 7 Which fractions are irreducible?

a) $\frac{2500}{7500} =$	b) $\frac{72}{108} =$
c) $\frac{241}{150} =$	d) $\frac{176}{165} =$
e) $\frac{256}{243} =$	f) $\frac{749}{7000} =$
g) $\frac{187}{340} =$	h) $\frac{110}{264} =$

Fig. 8 A theoretical transparent simplification

$$\frac{637}{1183}=\frac{7\cdot 7\cdot 13}{7\cdot 13\cdot 13}=\frac{7}{7}\cdot\frac{7}{13}\cdot\frac{13}{13}=1\cdot\frac{7}{13}\cdot 1=\frac{7}{13}$$

subject to the basic didactic contract in which the teacher proposes fractions simplified by 2, 3, 5, 7, or 10. The tasks which were proposed in this teaching experiment – a bit beyond this contract – aimed to extend the mathematical knowledge of students. In this specific case, it aimed for the awareness of mathematical procedures to make any fraction irreducible; hence, the decomposition of the numerators and denominators into prime factors, which is deeper founded into a theoretical context[4] than the use of GCD (Fig. 8).

The mathematics curriculum of the French part of Switzerland includes the study of divisibility of integers and of their decomposition in a product of primes. But, these mathematical procedures remain rather isolated and algorithmic and are not linked to other parts of the curriculum (Floris 2013).

Analysis: Milieu, Praxeology, Instrumentation

We claim that the set of tasks in Fig. 7 promotes an experimental learning milieu for the simplification of any numerical fraction. Following Brousseau (1997), there is here a fundamental aspect, an essential basic knowledge, corresponding to the prime factorization of integers. By proposing these tasks, the teacher introduces the students to these techniques as well as to the advantages of their use. The feedback (LM1), however, is not entirely adidactical. The teacher is required to assist students, for example by suggesting to look for other common divisors in case that they stop with ten. On the basis of the students' first attempts to solve those tasks, the teacher can then present the calculation of Fig. 8 and ask the students to revise the tasks in the same way.

To set the task according to the LM2 condition, we first need to analyze this calculation on the basis of Lagrange's (2000) three components of praxeology. First of all, the task aims to make any fraction irreducible. As a first step in Fig. 8, the subtask is to decompose numerator and denominator, the technique being the algorithm of successive divisions by all prime factors taken in increasing order. The underlying theory is the theorem that the decomposition exists and that it is unique. At this school level, this theory is generally not made explicit, and in this specific case it is replaced by the use of the algorithm (because it always works). As a second step, the subtask is to obtain a product of fractions by using the definition of this product (technique) which is justified by a definition of a fraction (theory). As a third step in Fig. 8, the subtask is to replace a/a fractions with "1", using the corresponding property (technique), which is also justified by a definition of

[4]See detailed praxeological analysis below.

fraction. Finally, the neutrality of the number one is used. A more refined analysis – which integrates properties of the integer ring as associativity and commutativity of multiplication – could be made.

This analysis shows the importance of the teacher's involvement into the students' working process, e.g. with respect to the choice of hints and degree of institutionalization (i.e. theoretical statements). These have to be observed to evaluate the theoretical level of the mathematical procedures. Furthermore, the entire curriculum on fractions is called into question: how and when are fractions and their operations defined? In some school programs in Switzerland, the multiplication of fractions is taught after simplification, whereas fractions themselves are defined in a rather intuitive way (e.g. as parts of pizzas, etc.) never followed by a rigorous definition.[5]

Inspired by this work, a pre service teacher proposed a similar set of fractions to his students; and, after showing them how to factorize by decompositions, he asked them to directly get the decomposition with the application of a symbolic calculator.[6] Further, the students were asked to check their results on this task as well as on other 'complicated' fractions. He also introduced them to Geogebra and Aplusix[7] as a means of generating decompositions. Finally, he presented a mathematical technique to obtain simplification using the symbolic features of the calculator (MATHPRINT mode, see Fig. 11 below). Here again, a contract disruption helps to promote an experimental milieu for learning as well as enriches the theoretical part of the praxeology corresponding to simplification of fractions – that is, in particular, the linkage with the decompositions of integers and the awareness towards the existence and unicity of any complete simplification of a fraction.

Activities on Non Decimal Numbers and the Limits of a Calculator

Scientific calculators intended for scholarly mathematics treat fractions and square roots in a problematic way which is why quite a lot of primary teachers – or even graduates of mathematics – think that they have certain knowledge whether a decimal development is infinite or not.[8] What we present in the following section are activities aimed to analyze these peculiarities. Further, the analysis aims to give

[5]Our favourite one being 'a / b is a real solution of equation b x = a with a, b integers and b different from zero, positive real numbers being defined as lengths'.

[6]A TI-92 in this case.

[7]There is a CAS part in Geogebra (Geogebra.org); Aplusix is a useful program allowing direct control of numerical equalities and algebraic equivalences (Aplusix.com). Other tools can be easily found on the web, e.g. www.calculatorsoup.com/calculators/math/prime-factors.php.

[8]They also are of the opinion that transcendent functions are programmed according to their Taylor series. Most of them ignore the CORDIC algorithms (https://en.wikipedia.org/wiki/CORDIC).

a detailed insight on the workings of scientific calculators in order to understand them and to use them successfully. During the analysis, it was rather difficult to separate this manipulative learning from the mathematics one, so the reader will have to understand it while working out the activities. The first activity is intended for twelve to thirteen years-old students. The second activity is designed for older students when studying square roots.

Tasks on the Decimals of 3/7

1. Transform 3/7 into a decimal notation with the use of your calculator. Enter the result into the third row of your calculator (here TI-30XSMultiView) and multiply this result by 7 (Fig. 9). What can you conclude?
2. Transform 3/7 into a decimal notation with the use of your calculator. Then, immediately multiply the result by 7 (press * and then 7, see Fig. 10). Explain the difference to what had happened before.
3. Is the decimal notation of 3/7 periodic? If yes, determine the period. Can it be done using the calculator?

Comments

The activity described above highlights how the calculator manages approximations. As for the long multiplications proposed above in this chapter (see second example), it suggests a negotiation between the answers of the calculator and those that can be obtained by using the usual paper-and-pencil algorithms. It leads to an increased knowledge on the functioning of the calculator in case of hidden decimals:

> TI30XS Multiview™ uses internally 13 digits for calculations and it displays 10 in the results. If the first hidden digit is 5, 6, 7, 8 or 9, the digit displayed as the right of the screen will be increased by 1, it is the rounding rule. (Calculator guide book)

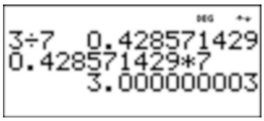


Fig. 9 Checking the decimal result of 3:7

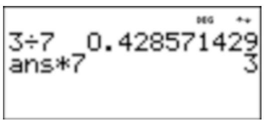


Fig. 10 Multiplying directly by 7 the decimal calculator result of 3:7

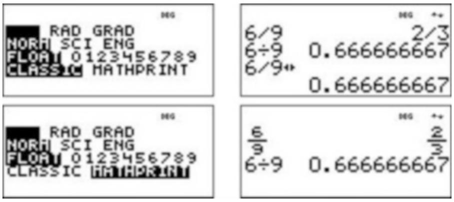


Fig. 11 Different treatments of fractions and divisions in CLASSIC (above) and MATHPRINT mode

Another feature of the TI30XS Multiview™ is the possibility to partially work in a non decimal world, called the MATHPRINT mode (Fig. 11). These features are generally ignored by teachers and students at this level (lower secondary schools) but could be presented after the completion of the previous activity on 3/7. It was often the case that pre service teachers proposed these tasks in their classroom. They further introduced the MODE menu and its different features. The discussion about these tasks among the teachers was quite interesting (Floris 2015). Some of them (mainly the teachers of lower secondary level) said that they were reluctant to propose activities like the previous one, or the comparison between the calculator results of $10^{20}+1-10^{20}$ and $10^{20}-10^{20}+1$, to their students. Their reluctancy relates to a loss of confidence in using the calculator. Others emphasized that the limits or errors of the calculators did not happen erratically but in precise cases. Moreover, they claimed that working on these examples has to go alongside with an understanding and teaching of the concerned features. In the following section, we will present a similar activity for fifteen to sixteen years-old students.

Analysis

For this activity, the focus of the analysis is on learning how the calculator processes numbers with more than ten decimals and study periodical decimal expressions. Additionally, the framing of decimal numbers with similar questions as in the activity below will be considered. At this level, students simply consider two decimals in their calculations when solving problems, and these decimals are not always correctly rounded. It is considered to be a part of the didactic contract and this is why teachers generally accept it. From the point of view of the milieu, the activity seems to implement an uncertainty, but many students do not note this and accept the situation without stepping back. At this stage of their scholarly learning process, students already developed the habit to use the calculator only for calculations, but what is required here is a thoughtful, reflexive approach. This phenomenon is described as instrumentalisation by Rabardel (1995). Furthermore, the mathematical treatment requires a long division that corresponds to a didactic contract disruption. Thus, the relation between calculator tasks and paper-and-pencil tasks is quite straightforward: They are either combined for a (complex) calculation, or for checking one. This could explain the unwillingness of some teachers to address this with their students. From the perspective of the praxeological approach, it is interesting to note that the technique corresponding to the treatment of the third question is long division, and that the theory is a property of this operation, i.e. the recurrence of rests.

This analysis shows the high efforts of teachers to construct enriching learning milieus with the help of these tasks. They have to consider it as a basis for a sequence of lessons on the properties of decimal numbers, framing, periodicity and long division. In this way, the adidactical feedback (LM1) can be constructed by the combination of mathematical properties already familiar to the students and the use of the calculator. Such a medium term study could further entail a search for all decimal figures of 1:31, or 1:29 by again combining the application of calculators and paper-and-pencil tasks. This would improve the students' handling skills of such mathematical tools in agreement to the instrumentation process of Rabardel.

Activity Around the Square Root of 8

1. Is $\sqrt{8}$ equal to 2? Justify.
2. Is $\sqrt{8}$ equal to 3? Justify.
3. Is $\sqrt{8}$ equal to 2,5? Justify.
4. Find two numbers with three decimal digits which frame $\sqrt{8}$.
5. Is $\sqrt{8}$ equal to 2,828427125?
6. Do the following task with the calculator in CLASSIC mode:
   Calculate the square root of 8. Then, put the result directly to the square.
   Compare your result with the result of task 5. What can you conclude?
7. Find the best possible framing of $\sqrt{8}$ using the calculator.
8. What is the decimal value of $\sqrt{8}$?
9. With the calculator in CLASSIC mode, calculate $\sqrt{8}+\sqrt{7},$ and then $\frac{1}{\sqrt{8}-\sqrt{7}}$. Are the results reliable?
   What conjecture can we draw from these results?
   Can we prove this conjecture?
   In MATHPRINT mode, does the calculator confirm the conjecture?
   Explain the results given in MATHPRINT mode (Fig. 12).
10. Identify a generalization of the conjecture established in point 9.

This activity can be analyzed identically to the previous one. It was proposed to high school students, and while they were working on the task we could observe difficulties linked to the didactical contract. Probably, these were due to the sparse experiences with calculators of the concerned students. Nevertheless, the activity was chosen by many teachers who were highly interested to improve their students' instrumentation of calculators because it is a part of the calculus chapters of the

Fig. 12 Calculator outputs for the activity (question 9)

high school curriculum. With the aim of efficiency, they mainly proposed the above-mentioned version of the set of tasks.

Experimental Milieu with Calculator for Early Algebra[9]

In this section, we aim to present the advantages of a compulsory school calculator like TI30XII concerning the study of algebra. In some countries, there has been a recent shift in the curricula towards new ideas for the introduction of algebra. In the 1960s, the New Math reform proposed a structured approach to algebra which was based on the properties of the sets of numbers (i.e. integers and rational numbers). The letter calculation rules were then worked on in isolation. However, the current proposal, called ‘early algebra’, endorses a dialectic between the numerical and algebraic conceptual domains. The literal calculations are considered both a production tool of number sequences and a description tool of numerical properties (e.g. for any integer n, the expressions 2n and 2n+1 equal sequences of even numbers, respectively odd, and thus describe the parity). Furthermore, it is possible to express algebraic properties in the numerical world; for example, $25=5^2$ expresses the fact that 25 is a square, or $333 = 3 \times 111$ expresses that 333 is a multiple of three (i.e. divisible by three):

> One of the major goals of early algebra is generalizing number and set ideas. It moves from particular numbers to patterns in numbers. This includes generalizing arithmetic operations as functions, as well as engaging children in noticing and beginning to formalize properties of numbers and operations such as the commutative property, identities, and inverses. (Wikipedia ‘early algebra’ in 2017)

But even in case that the theoretical aspect is attached less weight in many classrooms, the formal approach towards the algebraic domain has changed insignificantly – apart from the integration of a few motivational activities at the beginning of schoolbook chapters (these are mostly based on the formulas for areas and perimeters). However, these are not connected to the main domain that is being addressed in such a chapter. The study of computational techniques still predominates; this provokes the impression of some students to see algebra as a series of rules or laws which are devoid of meaning and poorly articulated within the numerical frame (Pilet 2012).

[9]See http://ase.tufts.edu/education/earlyalgebra/.

Calculation Programs

Within this new perspective on algebra, the notion of calculation programs is a key element. The name 'calculation program' – rather than formula – was chosen to emphasize the dialectic between the numerical and algebraic conceptual domains. This idea is at the heart of the so-called 'square-edged' activity, where the aim is to establish a method to identify the number of small coloured tiles on the edge regardless of the size of the square (Fig. 13). A detailed analysis of this activity can be found in Eduscol (2008).

This activity can be integrated into the study of literal calculations at various points, whenever it is most suitable with respect to the learning process. For example, in the Swiss textbooks of the year 2000, it was proposed at the beginning of the section on Algebra. In the year 2010, however, the activity was integrated at the end of the section as an 'application' (which is further specified in the teacher's comments). After the analysis of the results of a profound diagnostic test, Pilet (2012) proposes such a task for the reworking of the meaning of algebraic manipulations. In addition to that, we propose to integrate this 'square-edged' activity in an early algebraic setting to create awareness for literal computing at the beginning of secondary school because it further provides precise numeric challenges. This is because the activity asks students to predict the number of small coloured tiles for a square of sides of 6, 11, 37, 88, or 2012 tiles. It showed that higher values led students to abandon calculation procedures based on counting. At this level, the goal is not necessarily to introduce letters. For a square of 37 tiles per side, such a procedure may limit entries to obtain solely calculations like: $4 \times 37–4$ or 37+37+35+35 or 37+36+36+35 or 36+36+36+36. For numeric values exceeding ten, the use of the calculator can be accepted. A possible approach for teachers could be to ask students to identify different calculations and then explain why the results are equal. We would expect explanations such as 37+37+35+35=37+36+36+35 or 37+37+37+37–4=36+36+36+36 which is equal to (37–1)+(37–1)+(37–1)+(37–1). The calculator is used to validate the calculation programs and their equivalence.

With the variation of tasks, these records may achieve a calculation program status, that is a 'model' or 'pattern' of calculation which associates each calculation with a diagram like the following (Fig. 14).

We observed the working on this activity in a class of twelve to thirteen years-old students who are said to have difficulties in mathematics. They worked in

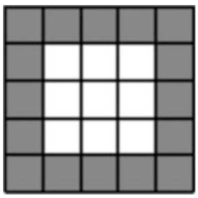


Fig. 13 'Square-edged' activity

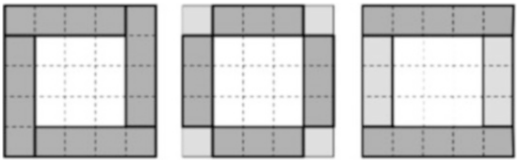


Fig. 14 Different patterns of calculations for the number of tiles in the edge

groups of three to four students. It showed that for some of them, the calculator was a helpful tool in regard to discussions because it assisted them to work out similarities to the calculation programs. The writings of the groups exemplify the successful dialectic of numerical and algebraic conceptual domains (see Fig. 15).

At this point, Eduscol (2008) proposes that teachers introduce a literal symbol:

> The production of a formula appears as an answer to the question of the general description of a situation involving specific numerical values and the use of letters solves the problem of the appointment of the variables involved in the situation.

However, this is not a mandatory recommendation at this point of the learning process, i.e. after the introduction of the topic. Our personal observations showed that an immediate introduction can overextend students; thus, the idea to integrate letters into calculations or equations is solely induced by the teacher without the students having an actual need for it.[10]

An alternative could be that the teacher claims that the numerical expressions are equal and proposes to study this type of scriptures further; for example, this could be achieved by tasks on the properties of sums of consecutive numbers – with the focus on numbers. For example, the sum of three consecutive numbers is equal to the triplication of the middle number because $88 + 89 + 90 = (89–1) + 89 + (89 + 1) = 89 + 89 + 89–1 + 1 = 3 \text{ x } 89$. The aim of these activities is to establish an early algebraic perspective on numerical expressions. This is linked to the idea that algebra is a kind of modelling of the numbers world within a numeric-algebraic dialectic whose lack or weakness is related to the difficulties of many students (Pilet 2012). The calculators which are used in secondary schools nowadays, which are types of calculators with two or more displayed lines, is of great assistance for this kind of calculating (see Fig. 16). It follows that these feedbacks and linkages help to fulfill properties LM1 and LM2.

Additionally, the calculator can be useful when working on symbolic calculations; for example, by pressing the key TABLE one gains access to the feature to create a sort of spreadsheet that introduces formula[11] (Fig. 17).

[10]There is a way to enhance this 'symbol gap'. Following Brousseau's formulation phase (1997), the teacher may propose a contest between groups: he will choose one student from each group, and then give a value for the number of tiles of a side. The student that gives the quicker answer gets a point for her group. The groups have the right and the time to prepare a method. In the subsequent validation phase the contest is about the best methods.

[11]The TI30X Multiview, given to all students in the schools of Geneva.

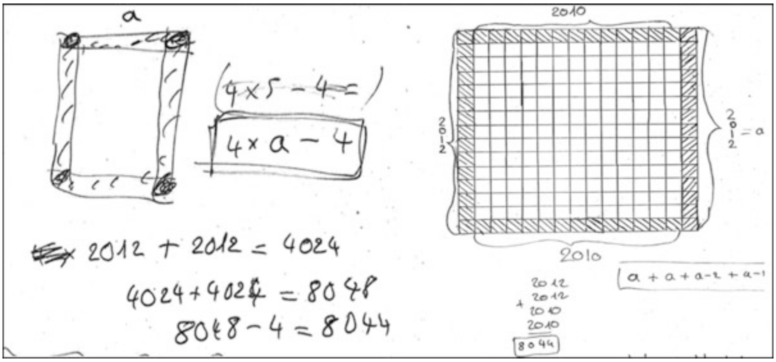
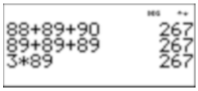


Fig. 15 Early algebra: dialectic between numerical and algebraic conceptual domains

Fig. 16 Multiple lines display in nowadays school calculators



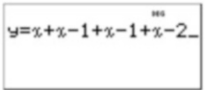
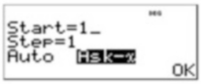
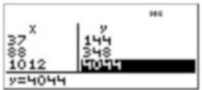
First, TABLE key displays "y=" on the screen, and it is possible to enter a formula using the variable key "x"	
Table initialisation: Auto mode displays a sequence (depending on the values of "Start" and "Step"). "Ask" requires from the user to enter a value.	
Here, "Ask" mode is chosen, and the entered values are 37, 88 and 1012. Y values are generated in correspondence to the programmed formula.	

Fig. 17 How to use TABLE key (with the example of the study of the sum of three consecutive integers)

From Calculation Programs to the Modelling of Arithmetic Properties

The dialectic of numerical and algebraic conceptual domains can be realized in connection to the TABLE key of the calculator, for example by asking students to produce lists of even, odd, or multiples of a given number (Fig. 18). Modelling with formulas can lead to a more logic-based type of thinking on properties such as the thesis that the sum of even numbers is an even number, or that the sum of an even number and an odd number is an odd number, etc.

What can further be studied are the properties of sums of consecutive numbers whether using letters or not. It is interesting to compare this aforementioned work

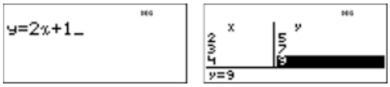


Fig. 18 Programming a sequence of odd numbers using TABLE key

FA107 Après, avant

Soit un nombre entier *n*.

a) Comment écrire le nombre entier qui le suit immédiatement ?
b) Comment écrire le nombre entier qui le précède immédiatement ?
c) Comment écrire le cinquième de *n* ?
d) Comment écrire le carré de *n* ?

Fig. 19 Classical work on algebraic scriptures (Let n be an integer (**a**) How to write (express) the immediately following number? (**b**) How to write (express) the immediately preceding number? (**c**) How to write (express) the fifth of n? (**c**) How to write (express) the square of n?)

with the following example, which is proposed in a current textbook of secondary school (Fig. 19 from CIIP 2012, p. 99).

In such exercises, the dialectic with numbers is not explicitly integrated. The decision to combine the results of this activity with the use of the calculator, constructing paper-and-pencil numerical tables, or integrating the use of the TABLE key function, is the teacher's responsibility.

Another operation that is provided by TABLE is the comparison of calculation programs. For example, in regard to the 'square edged' activity, we can introduce the different formulas that are obtained within the activity and then observe whether the values are the same. This motivates the study of literal transformations like their justification based on properties such as distributivity or commutativity. An analysis of this has shown that this new point of view offers an enhancing relationship between algebraic techniques and the properties founding them.

As TABLE allows the integration of only one formula at a time, this necessarily involves paper-and-pencil transcriptions as well as working in groups of two, three or four students who are assigned to program one formula each. One could argue that it would be better to use a spreadsheet, but the formulas of such a software are not written as polynomials in x which is why they require a technical introduction. Furthermore, it would entail to work in the school's computer room, unless your classroom provides tablets or laptops.

From Calculation Programs to Equations

Problems like "Which number did I think of?", allow to proceed to the notion of an equation:

I think of a number, add its double, divide the result by 3, and add 75. I come up with 80! Which number did I think of? Why?

One possibility to resolve this problem is to approach it arithmetically by starting from the end and reverse the operations. This allows students to grasp the type of problem and then move on to a different one which suggests the use of two different calculation programs at the same time. In such a case, the arithmetical approach would be awkward:

I think of a number. I multiply it by 3. I add 10. I get the original number sevenfold, plus 30. Which number did I think of? Why?

This type of problem allows a wide variation of different statements and promotes working on the numerical properties of the 'facts'. The task set will also comprise 'math-magic' tricks:

Think of a number, add 2000, divide the result by 20, subtract 100, and multiply all by 20. You end up with the number you thought of at the beginning! How do you explain it?

Or

Think of a number between 1 and 9. Double it. Add 2 to the result. Multiply the new result again by 5, add 12, multiply the new result by 10, subtract 220.

Compare the number you get started with the number you had thought! Can you explain?

These statements proposing sequence operations allow easy translation into calculation programs.

An interesting presentation for a same kind of problem is the following activity, "The Lost Number":

I type the following sequence into my calculator:

6	×	?	-	3	-	2	×	?	+	7	Enter

Provided that the two grey boxes mask the same number and the calculator gives 24 as result, can you identify this number?

Can you identify the number in case that the result is 592, 1.2, 69.2, -163.6, or 88?

In case that students aim to solve the task by random trials with the calculator – with or without the TABLE key – these random trials become rather time consuming as soon as the given result is something else than an integer. In fact, for many teachers, this activity aims to motivate the use of equations and is supposed to disqualify the use of the calculator. This intent, however, stems from a teaching position that does not consider the numerical-algebraic dialectic.

Equations, Equalities, Calculation Program DATA Key and Spreadsheet

With problems such as “Which number did I think of?”, or “The Lost Number”, the notion of equation can be introduced by further maintaining a numerical-algebraic dialectic. The question remains how exactly the calculator can be of use here. We already observed that the TABLE key only allows the display of a single column. On the TI34X Multiview, the DATA key provides a small spreadsheet (see Fig. 20). However, the use of this key is not self-explanatory and therefore requires instructions. But this effort is advisable to take in case that the DATA key functions will be further integrated into the classroom teaching in other contexts. These are, for example, numerical equations resolutions, proportionality, and programming a formula (functions):

Comments About the Milieu for Algebra

In agreement with our research question and with the conditions LM1 and LM2, it clarified in the course of our study that a milieu has to entail a variety of activities that link numbers and letters on the basis of arithmetical properties; this is a long-term project. We already presented selected options of how the calculator could be of help, but a large scope research to assess how a learning milieu could be set up for the student (LM1) is still pending. What needs to be constantly considered is whether the numerical expressions are truly providing the correct feedback.

The proposed sets of tasks link naturally with theory, hence the condition LM2 is satisfied.

Didactic Building of a Milieu with Calculator: An Example

Is the Calculator a Milieu?

All tasks or activities that are presented here share that they require the use of the calculator as a mathematical learning tool. From this given, the question arises whether the calculator itself is a learning milieu. According to Brousseau (1997), a learning milieu consists of various elements that will help teaching, in particular, the results of actions of the student such as calculations, drawings, or manipulations.

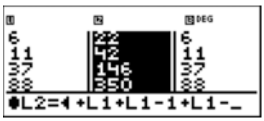


Fig. 20 Programming formulas using DATA key

Fig. 21 A didactic inquiry for different scriptures of numbers

999+1= 1000	999999+1= 1000000
99999999+1 =1000000000	9999999999 +1= 1×10^10
1000+1=1001	10000000+1= 10000001
100000000+1 = 100000001	'10000000000 +1= 1×10^10

This learning milieu further entails the elaboration on the connection between the tasks proposed by the teacher and what is actually achieved by the students. The results that are generated with the calculator can be considered a part of that milieu. The calculator itself, however, is not a milieu just as paper-and-pencil calculations are not a milieu either. They require a linkage with to the other properties of a milieu, and the use of the calculator further requires an official classroom status. The TI-30 calculator, for example, provides the answer 1×10^{10} to the operation $9999999999+1$. But without a prior introduction by the teacher, this result is not a part of the learning milieu. Nevertheless, the teacher should be able to provide a suitable answer to the meaning of this result in case that a student, for example of upper primary school, is interested in that; it is a basic part of the didactic contract. A didactic work that would take such an interest into account cannot be straightforward. It would go back to the number of digits of an integer, leading to small-range working theorems, such as "performing an addition, the number of digits does not increase, or it increases by 1". It would also include investigations of the number of digits displayed by the calculator. Figure 21 provides a sketch of one possible way to create a learning milieu.

Multiplication proves the most interesting operation for such an investigation. One could pursue answers to the question of how the number of digits of the product relate to the number of the digits of the factors.

A study this type, however, would only be of anecdotic interest. It could only be meaningful in the context of a medium length teaching process which includes technical work with mathematical properties. This is how the study on arithmetic properties with primary school pupils presented above was structured. In the example in Fig. 21, the theory corresponds to the positional writing of numbers in base ten and all mathematical properties on which it is based, particularly those of the ring structure. The reformation of the curricula in the 1970s has clearly shown that it is a long process to change the workings of a (mathematical) institution. This interjection does not include that we advocate the return of calculations in different bases, or the introduction of the study of the rings.

Considering the notion of the learning milieu, we observed that it cannot operate sustainably without the presence of praxeologies (Chevallard 1999) formed by tasks, techniques, a vocabulary describing actions, and properties that relativize the results of these actions (e.g. "by adding the same even number several times, you always get an even number as a result"). This is the logos or theoretical part of praxeologies. We highly believe in the value of material results such as physical objects, traces on a blackboard, on paper, or on the screen of a calculator. The physical part of the milieu is essential to recall the actions performed and to help the development of conjectures (i.e. a table, a list on the blackboard or calculator displays). The study on the process of the "course à vingt" (Brousseau 1997) highlights this role of the milieu. In this respect, the use of calculators which display the mathematical transactions is very important, as well as the opportunity to present the results to the whole class using an emulator and a data projector. Kieran and Guzman (2007) highlighted this in their calculator-based experiment for lower secondary school.

Result

The main research question was how the working with calculators in the classroom can become a learning milieu.

Therefore, we presented a survey of selected qualitative studies as well as examples on the use of calculators at different learning levels. In the theoretical part at the beginning, we specified the conditions LM1 (feedback of the milieu) and LM2 (links with theory) as a basis for a learning milieu. A high variety of examples illustrated how these conditions could be totally or partially fulfilled. In the 'target' examples, the necessary requirements for a learning milieu are fulfilled in a complete manner and the calculator is an essential tool. The theoretical output is impressing at this school level: the students handled properties of divisors, prime numbers, and square roots. In other examples, the feedback of the milieu showed to be more problematic which was mostly due to interferences of the didactic contract and flaws in the teacher's management. In these cases, what needs to be prepared is an accurate didactic engineering in order to propose challenging and theoreticaly rich tasks. Such tasks are provided in the fractions and decimals examples.

From a methodological point of view, it was experienced how the properties LM1 and LM2 could be effective as means of analysing the learning potentialities of calculator activities.

Conclusion

Students' experiences set out the basis to create learning milieus for mathematics. These are the reality in which they anticipate their actions and act. Technology, even a pocket calculator, complicates the learning situation by adding specific feedback which can be valid and therefore useful, but sometimes also surprising.

The tool "calculator" cannot be successfully integrated in an instant. It requires long-term planning which needs to be integrated into the curricula. In Switzerland, this is accomplished in the new "Plan d'études romand" (CIIP 2010), but in a rather minimalist manner and without links to specific mathematic subjects. However, the present contribution showed that calculator activities could improve the study of arithmetic properties in a significant way such as fraction operations, square roots, approximation, and algebra. Due to the status quo, the integration of the calculator is in the sole responsibility of the teacher. In our pre-service institution in Geneva, they are prepared for this, but, as we have demonstrated, a long-term institutional strategy is still necessary to transform individual efforts to an effective instrumentation of calculators for the students.

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