Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive, and Epistemological Implications for Improving Geometric Thinking

Marcelo Bairral, Ferdinando Arzarello, and Alexandre Assis

Abstract In this chapter, we discuss the results of a research project which investigates aspects of students' cognitions during the process of solving tasks dealing with a Dynamic Geometric Environment with touchscreen (DGEwT). In this chapter, we discuss data from two teaching experiments carried out with Brazilian and Italian high school students dealing with GeoGebraTouch (GT) and a Geometric Constructer (GC) software. With the focus on strategies used by students to solve the proposed tasks, we suggest two domains: Constructive and relational. Furthermore, we suggest the drag-approach as an important form of manipulation to improve geometrical thinking. Finally, we present a selected variety of representative examples of didactic, cognitive, and epistemological implications for learning and researching with the use of DGEwT.

Keywords Mobile devices • Manipulation on screen • Sketchometry • GeoGebra App • Geometric Constructer

Introduction

The significance of the gesture in supporting mathematical reasoning in a technological context is an emerging field of research in mathematics education, particularly in the interaction with touchscreen learning devices. As a past improvement,

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we have had a first shift from paper-and-pencil to dynamic geometry environments which include drag-and-drop activities (e.g. Cabri Géomètre, Sketchpad, etc.). Today, we experience a further shift and continuous improvements with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, or Sketchometry) that allow a variety of simultaneous finger actions.

The emergence of multi-touch devices provides new insights as well as challenges in mathematics learning and instruction. For instance, simulating rotation and other kinds of rotating movements on screen are made possible by means of touchscreen devices (Bairral et al. 2015a). Due to the fact that students and teachers become increasingly familiar with multi-touch technology and manipulation, we believe that looking for types of manipulation can provide new epistemological insights in regard to the geometrical conceptualizing through the application of touchscreen devices.

We recognize the touchscreen manipulation as a human action: embodied and multimodal. It can also reveal the mathematical thinking of learners while working on tasks with multi-touch devices. In this chapter, we illustrate some strategies used by students who applied rotation actions in order to solve tasks on GeoGebraTouch, or by students who dealt with the Geometric Constructor software to solve a Varignon Theorem task.

Interaction, Motion and Geometric Learning with DGEwT

With the focus on the user, there are differences between handling a usual PC – where dragging is produced with the help of a mouse – and making use of the touch screen of a tablet – where they can use their fingers in order to move figures. Additionally, it makes a difference whether users can use more than one finger – as in multi-touch environments – or only one finger. In this section, we reflect on how we dealt with these singularities theoretically.

Interaction on Touchscreen Devices

To click the mouse or to touch a screen are increasingly common routines of our daily lives. Each form of such handling implies different sensory perceptions; (the) sensitivity differs whether one uses a wired or wireless mouse, touches the screen of an ATM or that of a cell phone.

With the focus on their usage, environment mobile touchscreen user interfaces employ a specialized interaction model. The interaction of current mobile touchscreens, for example, is based on the computer's recognition and tracking of the location of the user's input on the display.

Adopting an embodied cognition perspective in our research, we highlight reciprocal connections between touchscreen manipulation and cognition. Contrary

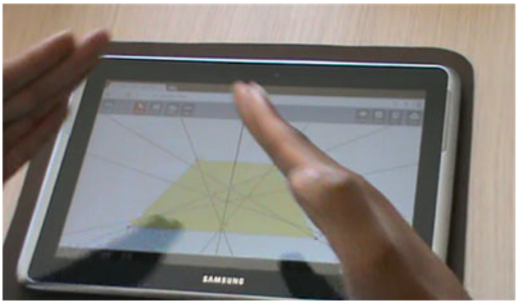


Fig. 1 Student's construction and embodiment reflection on GC

to what happens by in clicking, manipulating touchscreen interfaces implies a continuity of action, such as the spatiality of the screen, the simultaneity as well as the combination of movement, and – depending on the resource device – the response speed of the device. The following picture depicts one student's gestures who tried to explain one of the properties of the isosceles trapezoid (Bairral and Arzarello 2015) (Fig. 1).

The student uses his hands to represent the two sides that are not parallel. Although the picture does not show manipulation on screen, it describes specific configurations and actions of the fingers performing an action (Sinclair and Pimm 2014) with the construction made.

When we manipulate the screens of our devices by means of touchscreen technology, we perform a set of movements. These movements are not necessarily gestures such as signs or expressions of joy, silence, or doubt. Some of the manipulations that we perform induce specific mathematical cognition, for example when we want to enlarge or reduce the size of a picture with the help of an image editor (e.g. Paintbrush), or by means of touchscreen manipulation.

On such occasions, we either pull the image diagonally, upwards, or downwards; or we click on one of its vertices, so both dimensions – width and height – are reduced or enlarged proportionately. In case that we do not perform this type of movement, i.e. if we manipulate only one dimension, the result will be a deformed image.

Nevertheless, although all these manipulations are based on the same mathematical concept (the method of the diagonal as a way to generate similar figures), they are not necessarily of the same value with respect to cognition (the action of enlarging without deforming), epistemology (the simultaneous changing of different elements of the shape, e.g. points, sides, angles, areas, etc.), or space (work and manipulating area on the screen).

In order to guide our analysis of this process of embodiment expression, we can find support in Damásio (2010) for whom "also the most stable aspects of bodily function are represented in the brain, in the form of maps, thus contributing with images for the mind" (p. 39). Damásio further states that "Complex brains like ours naturally create explicit maps of the structures that make up the body, with a greater or lesser degree of detail. Inevitably, the brain also maps the functional states that are naturally taken up by these corporal components" (Idem, p. 119).

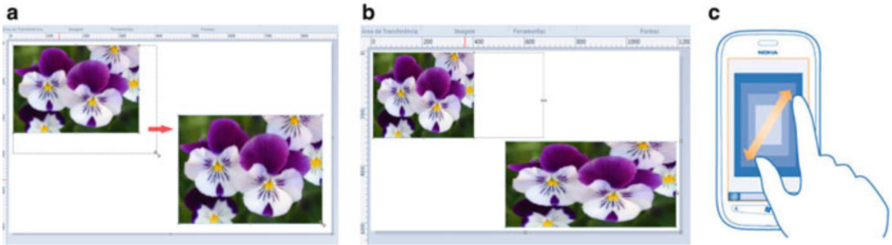


Fig. 2 (**a**) Illustration of an enlargement in a drawing program; (**b**) Distortion in a drawing program; (**c**) Enlargement through sliding on the screen (Bairral and Arzarello 2015)

We could argue that the brain mapped the fact that the touchscreen device is going to enlarge the figure or that a soft and quick lateral touch will make the screen slide to one side. The size of the screen, or the user's familiarity with it, can have an impact on the ways of manipulation. This is the spatial dimension, i.e. the screen handling and interaction area (Tang et al. 2010).

In case of the widening of an image by means of clicks, the shape illustrated in Fig. 2 (*a* and *b*) involves the actions of selecting, clicking and dragging on a point. When we touch the screen with only one hand (Figs. 2c and 3a), or both hands (Fig. 3b), on the screen, we map a specific area on the screen. Even in case that manipulation is done in order to see specific, punctual details of an object on the screen, the movement of this second action involves a simultaneous manipulation of dots.

Still, in regard to the enlarging of an image, although the simultaneous manipulation with two fingers (Fig. 3a) is the most usual, the second enlarging strategy (Fig. 3b) also follows the cognitive orientation structure of moving in a diagonal direction.

In the same way that simultaneous touchscreen manipulation of points on the screen brings about implications of an epistemological order, it also makes our cognitive structures more complex, for example through the simultaneous movement of various elements (e.g. angles, sides, area, etc.) in a figure. These movements will depend on the performance – the response speed – of each device (Bairral et al. 2015a, b, c).

Ways of Manipulation on Screen

Most current tabletop interaction techniques rely on a three state model: contact-down, contact-move, and contact-up – more akin to mouse dragging (Tang et al. 2010). In other words, interaction occurs in response to two dimensions of the input action (Yook 2009; Park 2011). This enables some basic or active finger actions for input such as tap, double tap, long tap (hold), drag, flick, and multi-touch (rotate). They are summarized in Table 1.

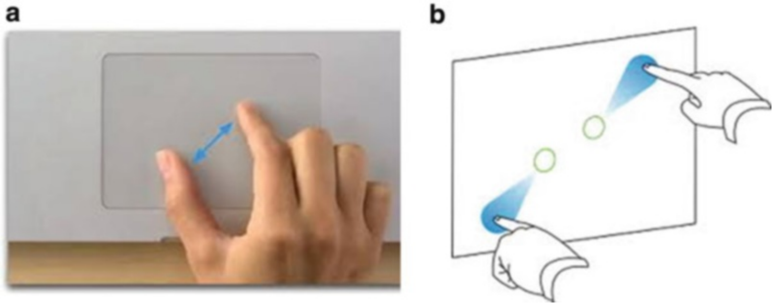


Fig. 3 (**a**, **b**) Sliding on the screen to enlarge with two fingers (Source: *Google picture*)

Table 1 Yook framework quoted by Park (2011, p. 23)

Action	Type	Motion	
Basic	Refers to tap and hold which are the basic ways of interacting with a touch interface	Tap (single)	Closed
		Tap (double)	
		Hold (single)	
		Hold (multi)	
Active[a]	It is a combination of the basic action and the performed finger action, which includes drag, flick, free, or rotate	Drag	Open
		Flick	
		Free	
		Rotate	

[a]According to Yook's (2009) framework the four active actions can be associated to multi hold manipulation

Manipulation – as a basic action – refers to a designated closed motion that occurs in response to the user's input, e.g. scale, flip, move, or push. Open motion occurs in relation to the user's input by reflecting the spatial and temporal quality of the finger action. In relation to geometrical thinking that we observed on students who were dealing with DGEwT, we named the basic action as constructive domain and the active action as relational.

After interpreting and using Yook's (2009) framework, which identifies each type of touchscreen in relation to geometrical thinking throughout the proposed tasks, we provide a scheme that includes another alternative to the drag approach and three further options for the rotating action (Fig. 4).

Due to the nature of the geometrical proposal, we identified that touches of the relational domain were predominant, while touches such as drag free, flick, or rotate occurred only a few times.

Regarding the usage of single or multi touch fingers, we observed (Arzarello et al. 2014; Assis 2016) that students manipulated the figures using mainly one or two fingers only (Tang et al. 2010). Due to the fact that they occasionally worked in

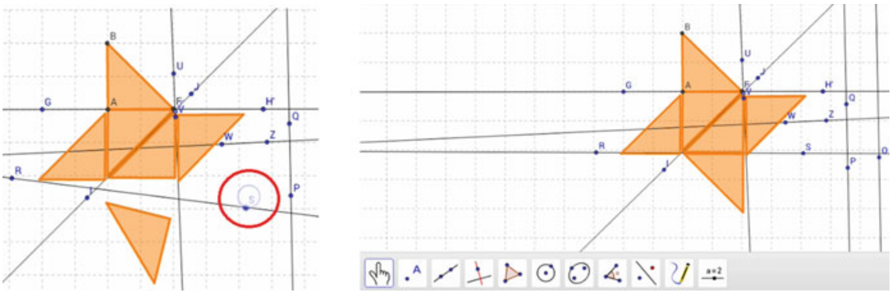


Fig. 4 Ways of manipulation on DGEwT (Arzarello et al. 2013)

pairs, it occurred that students also shared fingers (e.g. they used one finger each) or hands to manipulate a figure. This especially occurred when the shape had multiple geometric objects or constructions.[1]

The dominant approach was dragging. The usage of rotating appeared a few times; those appearances differed in a way that allowed us to distinguish three different ways of rotating which are illustrated in the scheme below. For example, we observed students doing rotation into some shape. We are of the opinion that looking for different types of manipulation provides new epistemological insights on the geometrical conceptualization within the use of touchscreen devices.

Even though we are not only looking for alternative kinds of touch that represent mathematical concepts (e.g. rotation), we agree with Boncoddo et al. (2013) that a particular way of manipulation may serve as an important function of grounding mathematical ideas in bodily form which may communicate spatial and relational concepts. Specifically for geometrical thinking – inspired by Hostetter and Alibali (2008) – we consider it important to stress that in touchscreen devices manipulations are based on visuospatial images: linguistic factors influence gestures, and ways of touchscreen manipulation can be regarded as intentional communications.

Performing Rotation on Touchscreen Devices

Although rotating appeared only a few times, these appearances allow us to distinguish three different kinds of rotation while working with a Geometric Constructer (GC) multi-touch device (Arzarello et al. 2013, 2014): rotation using one finger; rotation using two fingers, but one of the two fingers is fixed: and rotation with two fingers, with both in movement, as it is illustrated in the schemes below (Fig. 5).

[1]To see this kind of motion, please download the video: https://youtu.be/qC-G96NssJk

Fig. 5 Ways of rotation on GC or on GeoGebraTouch

Table 2 Examples of students' rotating on GC

Rotation types using GC	Example	Geometric process
Rotation using one finger		Student constructs and moves the selected point with the index finger
Rotation using two fingers, but one of the two fingers is fixed		Student keeps index finger fixed and moves the middle finger to observe what happens
Rotation with two fingers (both in movement)		Student selects two points and rotates the shape

Arzarello et al. (2014, p. 46)

In the above table (Table 2), we illustrate each kind of rotation by relating them to the geometric process that was applied in order to solve the task with the help of the Geometric Construct software.

Although the first two types seem equal on mathematical terms, we are of the opinion that – cognitively – they can provide different insights in regard to the use of the fingers. In order to grasp the fingers actions conceptually, we need to determine the centre of rotation individually in each point that we are about to rotate. With the use of two fingers, this cannot not be done beforehand.

Following this idea, this kind of finger movement results in new concepts in regard to the way we deal with rotation. Following the same idea, we agree with Sinclair and Pimm (2014) that this type of manipulation – two fingers in movement – involves contact with a screen and perform an action. Due to the fact that mobile touchscreen devices provide a wider range of freedom with respect to manipulation, we conclude that this particular kind of rotation may serve as an important function in order to ground mathematical ideas in bodily form. It may also reveal spatial as well as relational concepts (Boncoddo et al. 2013) in the field of plane transformations.

To investigate on manipulation on screen, especially if it is the case that students use more than two fingers, may be an interesting and challenging issue in future mathematics education research with touchscreen mobile devices. As we argued before, due to the nature of the software GC (multi-touch) as well as due to the geometrical task on the Varignon Theorem that was proposed beforehand, we observed that rotation manipulation occurred a few times (Arzarello et al. 2014). To solve the task, students did not apply the rotation action or other related plane transformation concepts.

Finally, on a theoretical basis, it seems important to highlight that the process of performing an action (Sinclair and Pimm 2014) by applying a concept such as constructing or other kinds of geometric strategies within DGEwT led us to assume that:

- Our brain has the ability to adjust to its environment; the touches on screen broaden the formerly established concepts of our brain (Damásio 2010).
- Human actions, as well as geometric concepts, are multimodal; what distinguishes them is that geometric concepts are also multimodal in their realization – the transition from virtual to actual. Indeed, by this transition due to which they become objects of thought and consciousness, geometric concepts are provided with certain features. These have to be put into practice by means of sensuous, multimodal, and material activities (Radford 2014, p. 354).
- In geometrical reasoning, there is a profound symbiosis of symbolic, analytical constraints, and figural properties. It is important to consider three categories of mental entities when referring to geometrical figures: the definition, the image – based on the perceptive-sensorial experience, e.g. the image of a drawing – and the figural concept (Fischbein 1993). Figural concepts do not evolve naturally in a way that they represent their ideal model. Consequently, one of the main tasks of mathematics education in the domain of geometry is to create types of didactical environments which systematically provoke a close cooperation between these two aspects up to the point where they fuse into unitary mental objects (Fischbein 1993, pp. 160–161).
- The interaction with a figure on screen can be differentiated according to the altering options by which subjects perceive them. Arzarello et al. (2012) point out two main cognitive and epistemic modalities according to which the figures on the screen were perceived and treated accordingly. A modality is ascending – from the environment to the subject – when the user explores the situation – such

as a figure on the screen – open-mindedly in order to see whether the situation itself has the potential to reveal something that is of interest to the user. The situation is descending – from the subject to the environment – when the user explores the situation with a conjecture in mind. In the first case, the applied actions have an *explorative* nature, i.e. to see if something happens; in the second case, they have a *checking* nature, i.e. to see if the conjecture is corroborated or refuted.

- In the transitional process from an inductive to a deductive approach, the drag-approach screen manipulation should be seen as a cognitive tool to empower learners to make assumptions and verify argumentations during the process of solving the task (Arzarello et al. 2014).

Teaching Experiments with DGEwT

In this section, we summarise the results from two teaching experiments (TE) that we conducted in Brazil (Rio de Janeiro)[2] and in Italy (Torino) with High School students working with two touchscreen devices: GeoGebraTouch (GwT) and Geometric Constructer (GC) (Table 3).

The analytical process that is presented as TE 1 and TE 2 was mainly based on the videodata.[3] We are of the opinion that the analysis should consider the interaction with touchscreen devices as paths of interaction rather than points of interactions. In most cases, it would be mathematically inappropriate to reduce the data of an entire process to a single point. In each session, optionally on their own or in pairs, the students worked on proposed activities.

TE 1: High School Students Dealing with GeoGebraTouch

This TE was conducted with High School students of 15–17 years of age at the *Instituto de Educação Rangel Pestana* (Nova Iguaçu, Rio de Janeiro, Brazil). None of them have had previous experiences with a dynamic geometry environment (DGE) or scholarly induced knowledge on plane transformations. In each session, the students worked on assigned activities with GeoGebraTouch which is described in Table 4.

Each session lasted two hours; in each lesson the students were asked to complete three activities similar to the one illustrated above. We observed all of

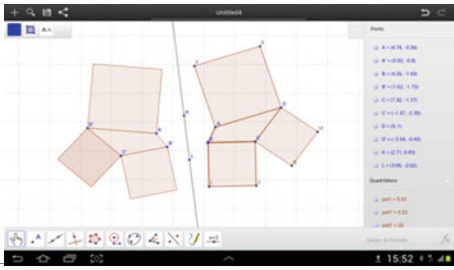
[2]In Brazil we are working with prospective mathematics teachers as well as with Sketchometry devices. We decided not to discuss data from their TE in this chapter.

[3]In recent analyses we used SCR PRO (Assis 2016) as a strategy to review some details that emerged from the video analysis.

Table 3 Teaching experiments information

	TE 1	TE 2
Age	15–17 years old	16–17 years old
Amount of hours of research session	8 h, 4 sessions	6 h, 3 sessions
Device	GwT	GC
Touch feature	Single touch	Multi-touch
Name of the Institution	Instituto de Educação Rangel Pestana	Liceo Volta
Previous experience with software	All of them had no previous experience with DGE	All of them had previous experience with Cabri
Sources for data collection	Written answers for each task Videotape Sheet of icon Software Recorder Pro (SCR PRO) for tracking touches on screen	Written answers for each task Videotape
Geometric content	Rotation and plan transformation	Quadrilaterals
Proposed and analyzed task in this chapter	Star	Varignon Theorem

Table 4 GeoGebra touchscreen features

Software	Interface	Device features
GeoGebra touch		Runs and allows save constructions off-line Version used on the analyzed task in this paper: 4.3 Single touch only

the students' manipulations on the screen and identified their kinds of actions (e.g. tap, hold, drag, flick, free, and rotate). Our analysis of this teaching experiment focuses on the student's strategies to solve the tasks, e.g. the application of rotation or other plane transformation concepts.

Besides alternative kinds of rotation applied by students to solve the geometric tasks that differ to the ones discussed in the previous section, further – curricular and cognitive – justifications to analyze students performing rotation or other plane transformations are the following:

- Rotation and other gyrating movements on screen are often applied due to the various alternatives of handling touchscreen devices (Kruger et al. 2005; Tang et al. 2010).
- Rotation and other plane transformations remained unaddressed in Brazilian geometry classrooms so far.

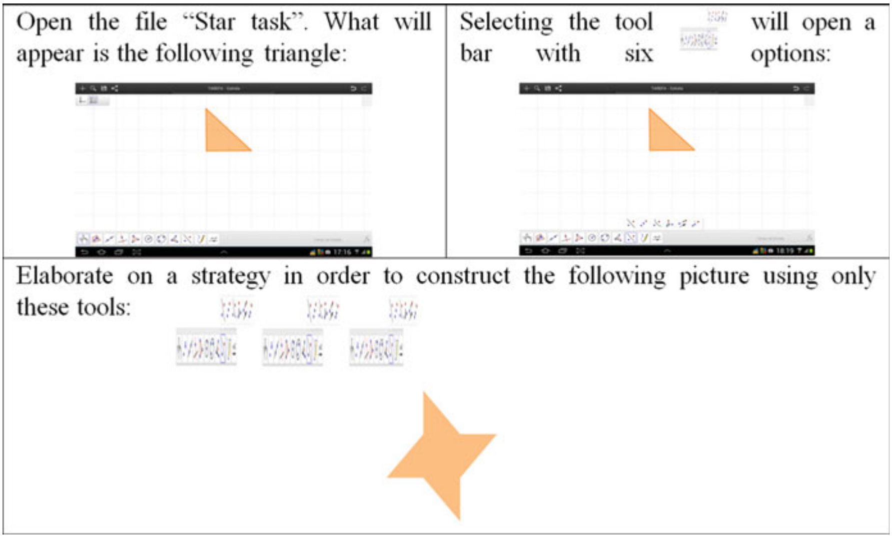


Fig. 6 Star task

- Touchscreen devices provide possibilities of gyrating movements on screen, or with the device itself, which might result in new insights on embodied cognition.
- Rotation and other plane transformations are concepts that involve intrinsically embodied motions.

The proposed task with GeoGebra is the following (Figs. 6 and 7):

The analysis in this TE process was mainly based on (1) the videodata of students working with the GeoGebraTouch software, (2) written answers on each task, and (3) the use of the students' lists of icons.

In the following pictures, as well as with the timing intervals, we illustrate and describe how the student Adriano deals with the task on GeoGebra by using single touch. He starts (12:14) to construct lines and reflects triangles related to them. By moving the line (27:34) he tries to locate the triangle coincident to the other; but since these actions remain unsuccessful, he decides to restart the construction.[4]

Using reflection tool and moving the line trying to adjust the reflected triangle | Restarting the construction | observing and adjusting

[4]The whole video is available on https://www.youtube.com/watch?v=qC-G96NssJk

Writing an answer on the tasks | Student's list of icons[a] | Videodata[b]

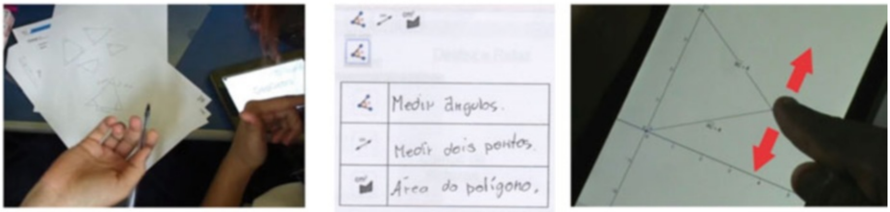


Fig. 7 Three main sources of data collection
[a]List containing all GeoGebra icons (Appendix 3). Each student had his/her own list and during each TE they filled it in and reviewed it to their own accord
[b]The *red arrows* indicate motions on screen performed by students

Interestingly, he keeps his left finger below a point on the line while using his right finger to rotate the line. Throughout this entire process, he carefully observes and adjusts his actions. The next figures illustrate how Adriano constructs lines and uses reflection to move the triangles.

28:28-28:33 | 35:51 | 38:16

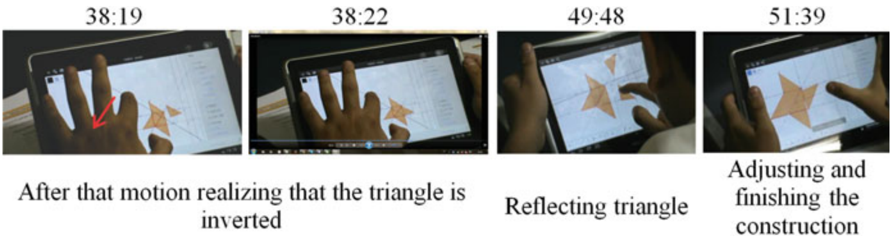
Constructing lines and using the reflection tool | Using the reflection tool and line by two points, reflecting the triangle afterwards | Applying rotation motion

The student constructs lines (28:28) and uses the reflection tool (28:33) in order to move the triangle. Afterwards, he constructed additional lines and repeated the process of reflecting those triangles (35:51). In the next three figures, we illustrate how Adriano applies a rotation motion by keeping his thumb on the line. At 38:17, we observe him making a rotation motion with his index finger to move the triangle and complete the shape (38:18).

Rotate touch

38:16 | 38:17 | 38:18

The next set of pictures show how Adriano uses his constructions (38:18) in order to finalize the task.



Again using his index finger, Adriano selects the line (38:19) and translates it so that the triangles connect. He creates another line and reflects the triangle (49:48). Afterwards, he adjusts and finishes the construction in accordance with the task statement.

TE 2: High School Students Working with the Geometric Constructer

The GC is a free dynamic geometry software developed in Japan by Yasuyuki Iijima at the Aichi University of Education (Iijima 2012). We chose the GC software because it is software which incorporates all the potentialities of usual DGE in a touch-screen device. With the term 'potentialities' we refer to the two main features (Arzarello et al. 2014): (i) the possibility of using more than one digit (multi-touch) on screen to interact with the software, and (ii) the possibility to design constructions as opposed to mere explorations.

With the GC, we are able to construct basic geometrical objects (e.g. points, segments, lines, or circles), measure such, drag, make traces of geometrical objects, etc. Below, we summarize students' working processes as well as results while dealing with the Varignon Theorem[5] task. This entails the illustration of selected, representative aspects of their geometrical thinking captured by means of their manipulations. These are described in the following chart (Table 5).

The analytical process was done in two main steps: (1) identification of each type of manipulation (Arzarello et al. 2014; Park et al. 2011; Yook 2009) and (2) construction of the timeline (Appendices 1 and 2) to describe the global cognitive movement throughout interaction on GC software.

Based on the videodata, we created a timeline which illustrates the ways of touchscreen and shows geometric aspects from students' interaction with the GC software (Arzarello et al. 2014, p. 47). In the following two charts, we illustrate

[5]In quadrilateral ABCD, the middle points (E, F, G and H) on each side have been drawn, forming quadrilateral EFGH. What characteristics does EFGH have? What happens if ABCD is a rectangle? What if it is a square? What if it is any quadrilateral? Demonstrate.

Table 5 Example from students working on the Varignon Theorem[a]

	Screen example	
Task	High School student	Undergraduate student
Varignon Theorem		
Geometric strategy	Student constructs the diagonals AC and BD by tapping (with one finger) on point A and C, and then on point B and D	Student using different colors to edit the construction and measuring internal angles from the quadrilateral EGHF

[a]The whole video is available on http://www.gepeticem.ufrrj.br/portal/materiais-curriculares/varignon-touchscreen-no-construtor-geometrico-2/

parts of a timeline which shows students' actions in order to solve task 3[6] by means of the GC software. The analysis has shown that they perform four types of basic actions: tap single, scale, hold single, and hold multi (Fig. 8).

Although, in order to construct a geometrical figure (e.g. a point, a line, an angle, or a circle, etc.) with the GC software the user has to use the software icons, we observed all the manipulation directly on the screen. For instance, we didn't consider touch on the icon as a case of the tap or hold action. Instead, we observed more than a single kind of touch at a time, but in order to categorize them clearly we selected the type of touch that was predominant in that specific situation.

Due to the nature of the task, which was situated in the domain of open construction and exploration, the types of touches that we predominantly identified were on the relational domain; for example, drag free, drag approach, and flick. Rotation did not occur in the process of solving this task. As we can see in Fig. 9, the drag approach was dominant (e.g. in interval 8:31–15:02).

[6]Build a quadrilateral ABCD. On each of its sides build a square external to the quadrilateral with one side coincident to the side of the quadrilateral. Consider the centers of the squares that have been built: R, S, T, U. Consider the quadrilateral RSTU: what can you observe? What commands do you use in order to verify your conjecture? This activity was thought as a task to introduce curiosity among students for the Napoleon Theorem, which was explored on the next assigned task.

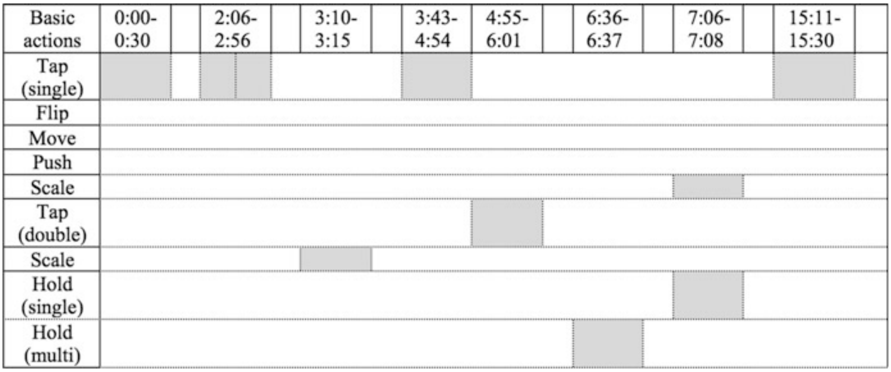


Fig. 8 Part of the timeline illustrating basic actions

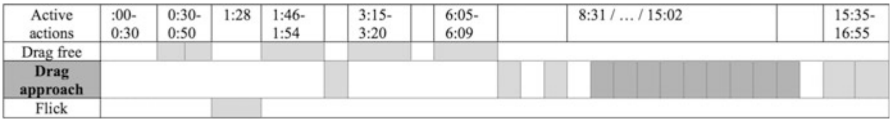


Fig. 9 Part of the timeline illustrating active actions

Domains of Manipulation and Geometric Thinking Within DGEwT

In the following screenshots, we show a summarized approach of students dealing with the software features. They further illustrate types of manipulation in order to identify conceptual reasons which prove that the EFGH shape is a parallelogram. These screenshots illustrate four different approaches towards the picture given to them (see footnote 4) (Fig. 10).

The analysis of the timelines (see Appendix) shows the progress of the altering approaches of touches. The students' constructions, strategies, and reasoning either moved from basic to active, or from active to basic actions.

We built on the two types (basic or active) of finger actions (Table 6) to say that the cognitive process with GC could be seen in two interrelated domains of manipulation: firstly, in the constructive domain, where students basically refer to tap and hold which are the basic or isolated ways of constructing geometric objects (point, line, circle, shape etc.) with a touch interface. Secondly, the relational domain is a combination of the constructional and the performed touches which thereby include drag, flick, free, or rotational approaches. The Table 6 below illustrates how we moved from a global observation – by means of a timeline – to a descriptive one – with the focus on some cognitive processes concerning the two domains of touches (Bairral and Arzarello 2015).

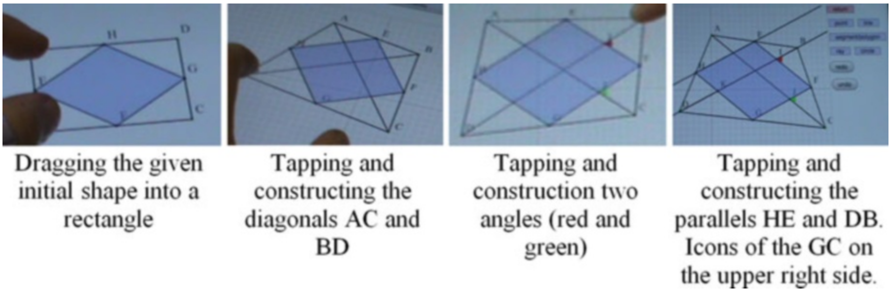
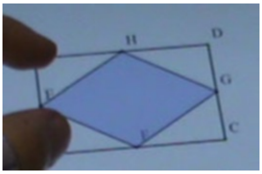


Fig. 10 Summarized student's drag approach and reasoning on Varignon task

Table 6 Relating domains of touches, cognitive processes, and motions

Domain of manipulation	Geometric process	Motion	Example of touches and students' strategy descriptions	
Constructive	Discrete construction and "isolated" observation (perception)	Closed, predetermined (specific goal, basic construction)		Student constructing angle to observe relation among diagonals and the side of quadrilateral ABCD
Relational	Related construction and global observation	Open, but focused on emergent conceptual demand of the task		Student using two fingers and dragging point AB to the left to transform the initial shape – a square – into a rectangle and observing what happens with shape EFGH

Even though we did not expect this, we observed that students also constructed geometric objects in the relational domain (Arzarello et al. 2014); they also showed more interacting and reflecting about the construction in this particular domain. Due to the nature of the geometry tasks we identified a predominance of touchscreen types on the relational domain; touches such as drag free, flick, or rotate occurred few times.

In the construction domain, students act as discrete observants; they focus on some specific construction, a constructed object, or touch something on the screen. In contrast to the relational domain, their manipulations seem more focused on their questioning, on conceptual understanding as well as on other emerging demands concerning their manipulation as a whole. The manipulations in regard to the construction domain seem focused on only predetermined motions, although motion through relational manipulations facilitates motion that is ‘open’ in the sense of that it can generate more unpredictable processes.

The Drag Approach Way of Touch and Semiotic Bundles in Geometric Tasks in DGEwT

The drag approach[7] seems to be a useful kind of touch in regard to the relational domain. It is a kind of manipulation that students apply when they are confronted with a specific geometric property, shape, or construction. During this process, we identified that the usage of the drag approach was dominantly applied when students aimed to clarify their reasoning.

Table 7 illustrates a student’s strategy to adjust his constructions of the star task on GeoGebra (see Fig. 6) by applying the drag approach. The drag-approach is a type of screen manipulation on the relational realm. Even when a student uses only one finger, the drag-approach works as a refreshing, quite stabilizing and reflecting way to a deep understanding of the geometric properties that emerge from the manipulation on drag free or other ways of touchscreen use. It seems to be an appropriate tool to facilitate mathematical justification, prove, and further geometric discoveries.

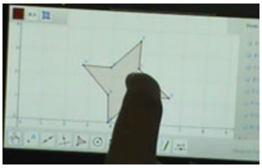
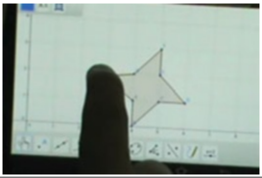
According to Arzarello et al. (2009), a semiotic bundle is a system of signs – with Peirce’s comprehensive notion of the sign – that is produced by one or more interacting subjects and evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher (Arzarello et al. 2009, p. 100).

The way of touch could not be identified as the only cognitive resource in students’ learning processes. Rather, pictorial representations, cultural artifacts, speaking, writing and gestures are examples of tools of a bundle of semiotic resources (Arzarello et al. 2009) that contribute to an understanding of the process of knowledge construction as well as for the development of tasks that foster the improvement of the geometric thinking within DGEwT as we show in Table 8.

In other assigned tasks on rotation or other kinds of plane transformation, we observed students applying composed forms of transformations. The picture above illustrates how manipulation on a touchscreen, the device, its features, and other artefact mediators are intertwined in the process of construction and performing plane transformation strategies with the software. While observing students applying rotation and reflection we came to the conclusion that looking for specific types

[7] Inspired by Arzarello et al. (2002).

Table 7 Illustrating student's drag-approach and reasoning on star task

Screen shots	
 	Student constructs and moves the vertices freely (without the use of the grid squared)
 	Student adds the square grid on his construction and adjusts the star's vertices in some points (intersections of the grid)

of manipulation – as well as including the concept of the semiotic bundle – can provide new epistemological insights on geometrical conceptualizing in DGEwT.

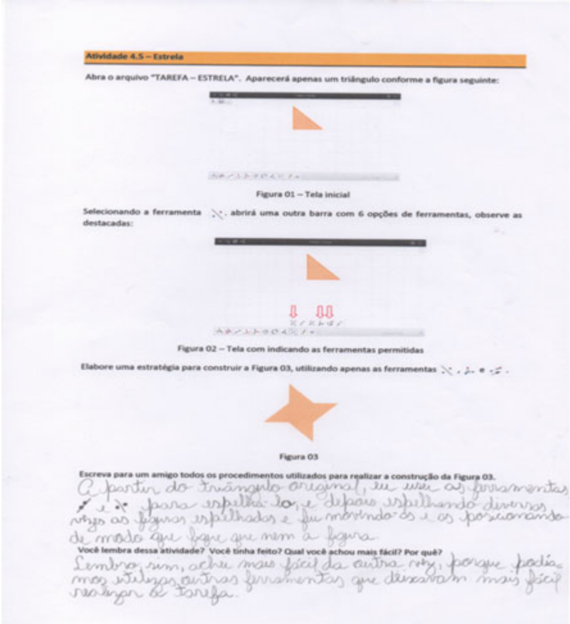
Conclusion and Implications of DGEwT

In the next sections, we present a variety of implications that summarize the main results which emerged from the two teaching experiments illustrated in the previous sections of this chapter.

Didactical Implications

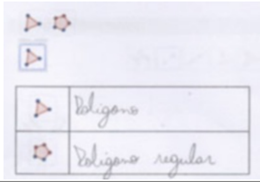
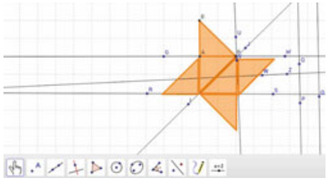
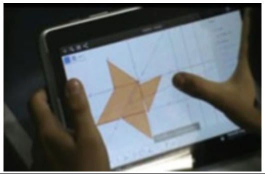
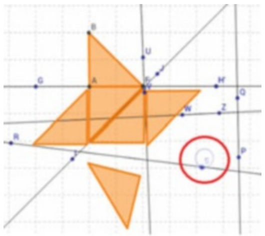
In mathematics education, a considerable amount of research stresses the key role of the task in each environment – with or without ICTs. The pedagogical importance of carrying out research on touchscreen use is not that it is trendy. Rather than that, it is

Table 8 Exemplifying semiotic bundle used on TE 1

Semiotic bundle resources using DGEwT	
Student's writing	 Atividade 4.5 – Estrela Abra o arquivo "TAREFA – ESTRELA". Aparecerá apenas um triângulo conforme a figura seguinte: Figura 01 – Tela inicial Selecionando a ferramenta, abrirá uma outra barra com 6 opções de ferramentas, observe as destacadas: Figura 02 – Tela com indicando as ferramentas permitidas Elabore uma estratégia para construir a Figura 03, utilizando apenas as ferramentas , e . Figura 03 Escreva para um amigo todos os procedimentos utilizados para realizar a construção da Figura 03. A partir do triângulo original, eu usei as ferramentas e para espelhá-lo, e depois espelhando diversas vezes as figuras espelhadas e fui movendo-as e as posicionando de modo que fique que nem a figura. Você lembra dessa atividade? Você tinha feito? Qual você achou mais fácil? Por quê? Lembro sim, achei mais fácil da outra vez, porque podíamos utilizar outras ferramentas que deixavam mais fácil realizar a tarefa.
	"From the original triangle, I used the tools  *and*  *to reflect it, and then after mirroring the pictures several times, I was moving and fixing the figure to the correct position".*

(continued)

Table 8 (continued)

Student's notes on list of icons	
Picture	
Screenshot	
Line of mirror/mirror[a]	
Tracking on Recorder Pro	

Ways of touch preforming rotation	Using one finger in motion		Free rotation
			Rotation to the left
	Two fingers in motion		Thumb is fixed and the index finger is gyrating to the right
			Both thumb and index fnger are rotating to the left

[a]Translation of how the student named the GeoGebra icon used to construct symmetry

important to design new ways of instruction with that type of technology in order to empower learners with high abilities to acquire mathematical knowledge (Leung 2011).

Since our research is embedded in the dynamic geometry environment (Ibid), manipulation in this kind of environment should be regarded as a cognitive tool in order to empower learners with amplified abilities to explore. Also, again in agreement with Leung, we are of the opinion that a mathematical task within GC becomes meaningful when it involves conjecturing, activities which require an explanation, and that provokes learners to engage in situated discourses in order to communicate their mathematical reasoning or argumentation. We aimed to fulfil these requirements with the provision of tasks such as the Varignon Theorem.

To solve a task, which involves the concept of rotation, using GeoGebra with single touch (as discussed on TE 1), we observed that the students used their fingers – no more than two (Tang et al. 2010) – similar to the students who dealt with the software GC in an open task (see TE 2).

In the TE 1 – due to the fact that the students were unacquainted with DGE, the list of icons (see Appendix 3) was didactically helpful for them. During each teaching experiment, they had the opportunity to remember the functionality of the tool, review it, and add new items to the list. Throughout the sessions, we observed that they resorted to the list to identify the most appropriate tool to apply in order to fulfil the task. Besides defining the functions of a specific icon, they further took notes on the geometric concept or strategy that underlied such icon. Revisiting and rewriting their notes on the list of icons can also be considered a process of learning.

Besides cognitive challenges and constraints with respect to the used software, we identified that the use of DGEwT can also provide new pedagogical issues in regard to the wording of mathematical instructions. In addition, our identification of the different types of manipulation can lead to improvements of the software, basically related to the drag action and touch (Iijima 2012).

Cognitive and Epistemological Implications

The cognitive process of solving geometric tasks within DGEwT could be seen in two intertwined domains of manipulation (Arzarello et al. 2014; Bairral et al. 2015a, b, c): the construction domain which refers to tap and hold as the basic or isolated ways of constructing a geometric object, and the relational domain which is a combination of the constructional domain and the performance on the touchscreen. Although the students dealt with the device naturally, their manipulation was apparently restricted by software constrains (or facilitated by the possibilities offered) or by the proposed geometry task.

In respect to the two TE illustrated in this chapter, we are of the opinion that any kind of manipulation that promotes open motion, e.g. relational ways of touching, are appropriate in order to provide new epistemological challenges concerning geometric knowledge as well as altering kinds of proving. Since the drag approach is a relational action, it seems to be an appropriate tool to improve justification and proving competences within the mathematics classrooms setting that uses touchscreen devices. As one restriction to that – depending on the aim of the teacher –, the task has to be selected carefully, and the teacher should promote that the students work independently on the task by experimenting with altering kinds of touches. Identifying in which geometric constructions the manipulation with more than two fingers occurs may be another interesting issue for future studies. What was not being discussed by this study is the issue of the two domains of manipulation analyzing kinds of touches on different touchscreen devices.

As simultaneous touchscreen manipulation of spots on the screen brings about implications of an epistemological order, it also adds complexity to our cognitive structures. This particular feature was observed by one of the students in our research. According to him, "in a very complex figure, moving several elements at the same time can become a bit difficult". Besides this cognitive implication, the use of touchscreen devices in the teaching of mathematics brings about transformations in didactic and epistemological realms, but the necessary educational research is still needed.

Another relevant issue that needs to be considered is the way how using a multi-touch-screen allows alterations on the task design in a substantial way. More precisely, multi-touch screen devices allow a design of geometrical problems in a way that differs from familiar ones in such ways that the combination with non-multi-touch screen environments would be very difficult. For instance, from TE 1 we are intrigued how students – without previous instruction concerning rotation and reflection – apply these two concepts, mostly in form of a composition of the two.

Research Implications

Our prior assumption was that the single touch provided by GeoGebra would restrict our possible observations of altering kinds of rotational manipulation on the screen. However, as we illustrated in TE 1, even students without previous experiences with rotation, or reflection, used those concepts intuitively, isolated, or even a mixed variation of the two tools (Assis 2016).

Usually in Brazil, plane transformations (e.g. isometries) are conceptually mapped in the following sequence: reflection, axial symmetry, rotation, and translation. The composition of plane transformations is underexplored in geometry lessons when the instruction uses traditional resources. In that sense, DGEwT seems to be a powerful resource for changing tasks as well as the nature of the geometric understanding concerning plane transformations. In our current analysis, we provided tasks where students had to apply the concept of rotation. In this paper, we present results from students dealing with GeoGebra touch to solve the proposed task.

In a more recent analysis (Bairral et al. 2015a, b, c), we further observed that the drag approach manipulation – as discussed within the TE 2 – could be applied using only one finger. The application apparently depends on the device features and the task proposal. This sort of touch should be seen as a cognitive tool that empowers learners to conjecture and explore their line of argumentation during the process of solving the task. This allows us to ascertain that the drag approach provided by the preconditions of a multi-touch environment can suitably support and improve the students' justifying (i.e. exploring) and proving (i.e. conjecturing) performances (Bairral and Arzarello 2015).

According to Arzarello et al. (2012), using DGE there is an alternation between an ascending and a descending modality: when there is a shift to a descending one, this is, possibly marked by the production of an *abduction*, which can also determine the transition from an inductive to a deductive approach. Within DGEwT the only difference seems to be in the time according to which such exchange takes place: in touch-screen modalities, the changes seem to happen more frequently than in mouse modalities. Possibly, this can have cognitive consequences similar to those ascertained by Arzarello (2009) in TI-inspired environments in comparison to Cabri-géomètre ones; but this statement is in need of further investigation before being an assured scientific result. At the moment, it is only a plausible conjecture.

To achieve our aim – which was to observe the development of geometric thinking –, the next step after the identification of each kind of manipulation was to construct timelines (see Appendices 1 and 2) and to gain information of the global cognitive movement of the interaction with the device. For each analyzed activity, we constructed one separate timeline. Depending on the type of task, some kinds of touches were not classifiable, but in all the timelines that we constructed we noticed a clear accumulation of active actions. In summary, the timeline has been methodologically and didactically important in order to:

- Illustrate the global cognitive movement related to the various kinds of touches (e.g. from constructive to relational and vice-versa) throughout the students' working on the tasks.
- Show selected local cognitive movements of the kinds of touches throughout a variety of geometric aspects in certain intervals.
- Allow researchers to determine and record certain intervals where students' geometrical thinking focused on the relationship of touches on the screen with other semiotic resources.

Another resource used for data collection was the Screen Recorder Pro device (SCR PRO), which allows to capture, in addition to the audio, the touches on the surface of the tablet (see Table 8). In the PRO version, the application does not limit the recording time and should take into account the ability of the device itself. However, the application installation requires a procedure that changes the tablet configuration. This feature was utilized in implementations carried out with the GC, since the acquisition and installation have been carried out only after the period in which implementations are made with the touch GeoGebra.

Final Remarks

Mathematics applied by students to solve a task in a paper-and-pencil environment differs from the mathematics applied on a touchscreen device. In this chapter, we highlighted two intertwined domains of manipulation – the constructive and the relational domain – for geometrical thinking development with DGEwT. The constructive domain refers to tap and hold; these are the basic, or isolated, ways of constructing geometric objects. The relational domain is a combination of the geometric construction and the performed touch. In the relational realm, the drag approach appears as a useful way of touch to improve geometric thinking. With this type of manipulation, students can make use of one or more than one finger.

We are of the belief that it is not important that the teacher monitors the students' application of certain types of touches on the screen. By taking device features and performances into account, we conclude that teachers need to be aware of the singularity of each kind of touch while proposing tasks that aim to trigger the students' intrinsic motivation to work into mathematics activities that enhance findings, reflections, and the development of mathematical thinking in its various aspects (Bairral et al. 2015a, b, c).

Inspired by Fischbein (1993), we argue that logic, image, and manipulation – on screen or gesturing on it – should be inseparable from geometrical reasoning with touch devices. In this process, it is important to interpret geometrical figures as mental entities which possess conceptual and figural properties (Fischbein 1993, p. 160).

Our brain adjusts to its surrounding environment (Damásio 2010); this implies that the touches on the screen or other touch performances add new mappings to the brain. These should be taken into account regarding teaching and learning processes. As a proofing example, the following picture illustrates how students interact with a touch device and its features of manipulation – as well as performing action (Sinclair and Pimm 2014) with hands on the screen – without previously established knowledge on plane transformation which we also illustrated in the TE 1 (Fig. 11).

In this geometrical process, the students apply figural concepts for executing constructions and transformations. They use images based on their perceptive-sensorial experience (Fischbein 1993). In this process – a sensorial process – motion and manipulation on screen make up an important cognitive function and, by becoming objects of thought and consciousness, geometric concepts are endowed with particular determinations; they have to be actualized in sensuous multimodal and material activity (Radford 2014, p. 354).

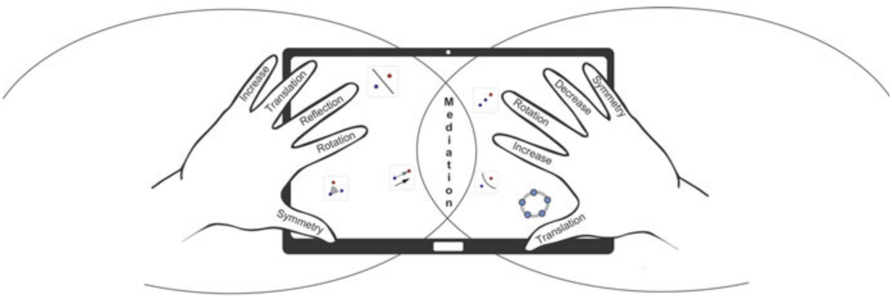
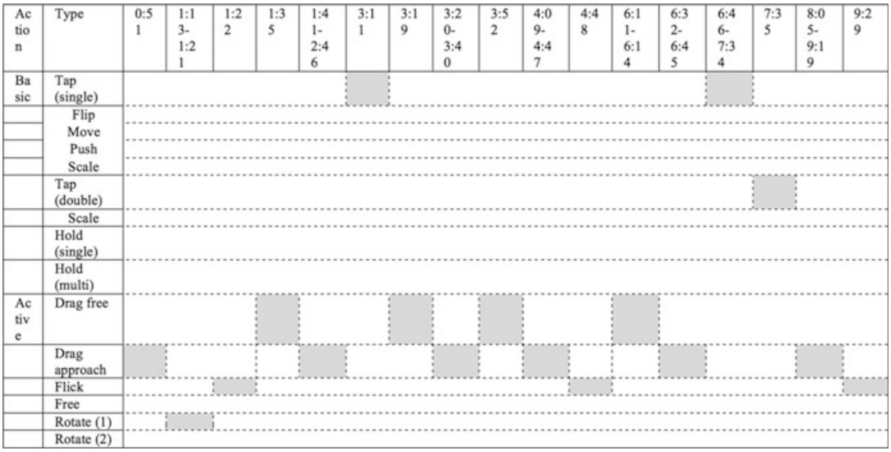


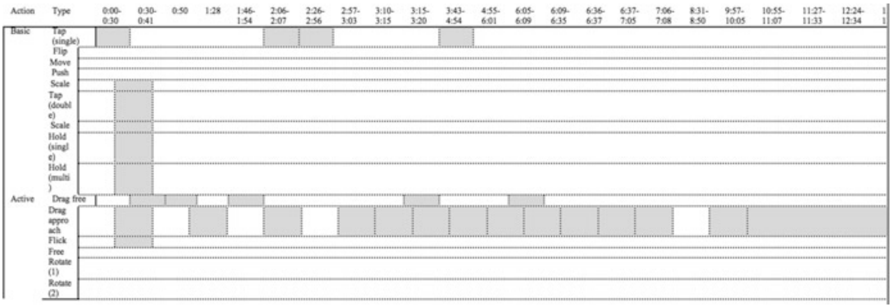
Fig. 11 Manipulation on touch devices interplaying symbolic, analytical and figural properties

Appendices

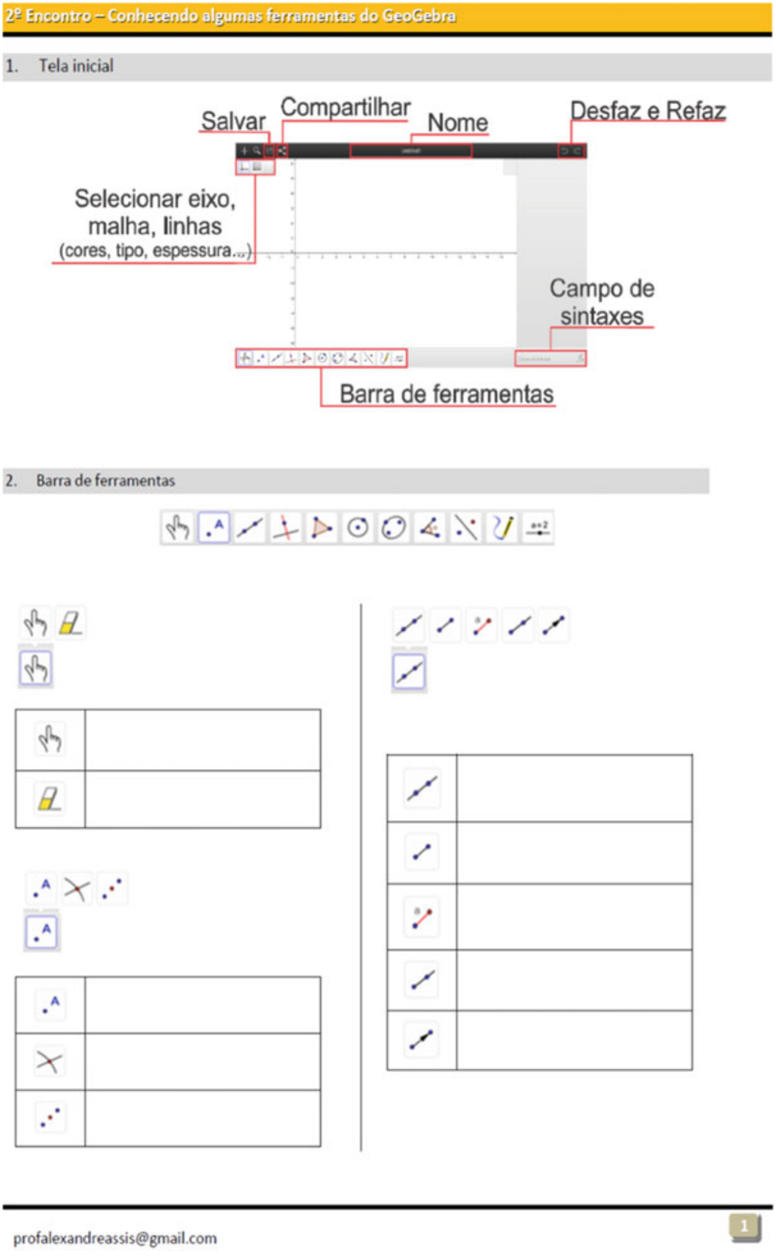
Appendix 1: Timeline of the Varignon Theorem Task (Discussed on TE 2)

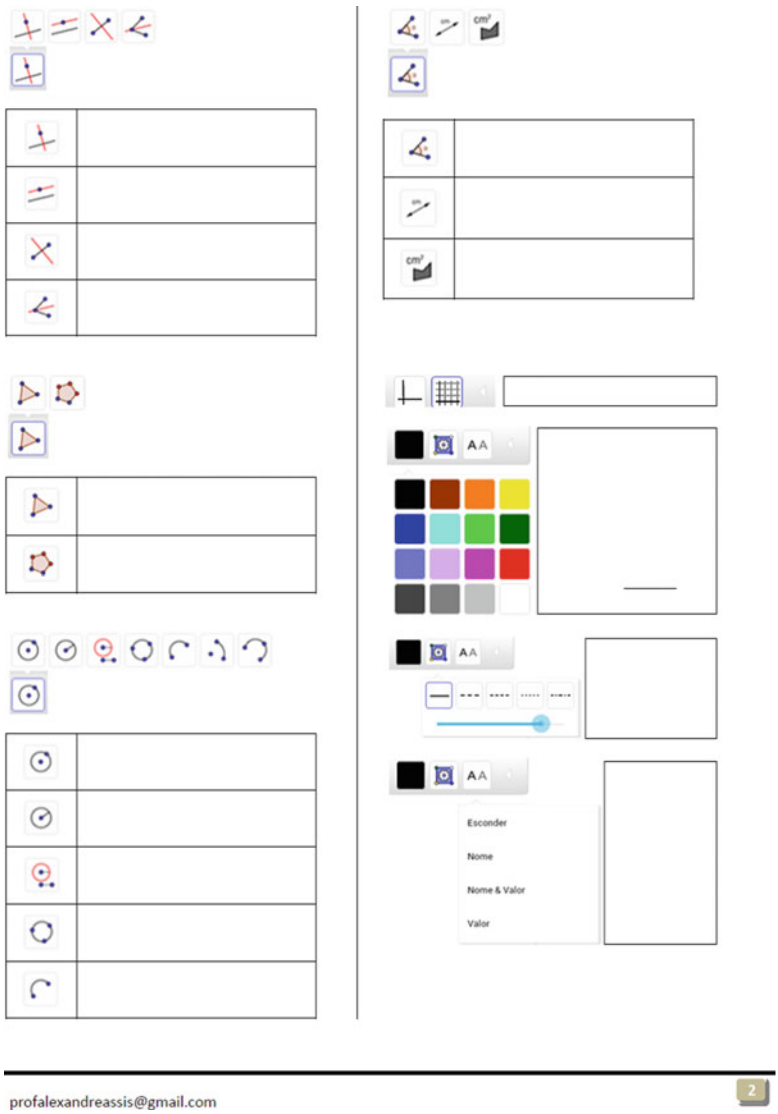


Appendix 2: Timeline of the Task Shown in note (a) of Table 5



Appendix 3: List of Icons Elaborated for TE with GeoGebraTouch





References

Arzarello, F. (2009). New technologies in the classroom: Towards a semiotics analysis. In B. Sriraman & S. Goodchild (Eds.), *Relatively and philosophically earnest: Festschrift in honor of Paul Ernest's 65th birthday* (pp. 235–255). Charlotte: IAP.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM, 34*(3), 66–72.
Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.
Arzarello, F., Bartolini Bussi, M. G., Leung, A. Y. L., Mariotti, M. A., & Stevenson, I. (2012). Experimental approaches to theoretical thinking: Artefacts and proof. In G. Hanna & M. de Villliers (Eds.), *Proof and proving in mathematics education* (pp. 97–137). Dordrecht: Springer.
Arzarello, F., Bairral, M., Danè, C., & Yasuyuki, I. (2013). Ways of manipulation touchscreen in one geometrical dynamic software. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th international conference on technology in mathematics teaching* (pp. 59–64). Bari: University of Bari.
Arzarello, F., Bairral, M., & Dané, C. (2014). Moving from dragging to touchscreen: Geometrical learning with geometric dynamic software. *Teaching Mathematics and its Applications, 33*(1), 39–51.
Assis, A. R. (2016). *Alunos do ensino médio trabalhando no geogebra e no geometric constructer: Mãos e rotações em touchscreen*. Unpublished M.Ed. thesis, Universidade Federal Rural do Rio de Janeiro, Seropédica.
Bairral, M., & Arzarello, F. (2015). The use of hands and manipulation touchscreen in high school geometry classes. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME 9* (pp. 2460–2466). Prague: Charles University.
Bairral, M. A., Arzarello, F., & Assis, A. (2015a). High school students rotating shapes in Geogebra with touchscreen. *Quaderni di Ricerca in Didattica*, 25(Supplemento 2), 103–108.
Bairral, M., Assis, A. R., & da Silva, B. C. (2015b). *Mãos em ação em dispositivos touchscreen na educação matemática*. Seropédica: Edur.
Bairral, M. A., Assis, A., & da Silva, B. C. (2015c). *Toques para ampliar interações e manipulações touchscreen na aprendizagem em geometria*. Paper presented at VI SIPEM, Pirenópolis, Brazil, 15–19 November 2015. http://www.sbembrasil.org.br/visipem/anais/story_ content/external_files/TOQUES%20PARA%20AMPLIAR%20INTERA%C3%87%C3%95ES%20E%20MANIPULA%C3%87%C3%95ES%20TOUCHSCREEN%20NA%20APRENDIZAGEM%20EM%20GEOMETRIA.pdf. Accessed 8 Apr 2016.
Boncoddo, R., Williams, C., Pier, E., Walkington, C., Alibali, M., Nathan, M., Dogan, M. F., & Waala, J. (2013). Gesture as a window to justification and proof. In M. C. S. Martinez & A. C. Superfine (Eds.), *Proceedings of PME-NA 35* (pp. 229–236). Chicago: University of Illinois.
Damásio, A. R. (2010). *O livro da consciência: A construção do cérebro consciente*. Porto: Temas e Debates.
Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.
Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review, 15*(3), 495–514.
Iijima, Y. (2012). *GC/HTML5: Dynamic geometry software which can be used with Ipad and PC: Feature of software and some lessons with it*. Paper presented at ICME-12, Seoul, 8–15 July 2012.
Kruger, R., Carpendale, S., Scott, S. D., & Tang, A. (2005). Fluid integration of rotation and translation. In *Proceedings of the SIGCHI conference on human factors in computing systems* (pp. 601–610). New York: ACM.
Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM–The International Journal on Mathematics Eduation, 43*(3), 325–336.
Park, D. (2011). *A study on the affective quality of interactivity by motion feedback in touchscreen user interfaces*. PhD dissertation. Graduate School of Culture Technology. KAIST (Korea). Retrieved from. http://descarteslab.kaist.ac.kr/contents/thesis/PhDdissertation_DoyunPark.pdf

Park, D., Lee, J., & Kim, S. (2011). Investigating the affective quality of interactivity by motion feedback in mobile touchscreen user interfaces. *International Journal of Human-Computer Studies, 69*(12), 839–853.

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM – The International Journal on Mathematics Education, 46*(3), 349–361.

Sinclair, N., & Pimm, D. (2014). Number's subtle touch: Expanding finger gnosis in the era of multi-touch technologies. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of PME 38 and PME-NA 36* (Vol. 5, pp. 209–216). Vancouver: PME.

Tang, A., Pahud, M., Carpendale, S., & Buxton, B. (2010). *VisTACO: Visualizing tabletop collaboration.* Paper presented at 10th international conference on interactive tabletops and surfaces, Saarbrücken, Germany, 7–10 Nov 2010.

Yook, H. J. (2009). *A study on the types of interactive motions in mobile touch interface*. Unpublished PhD thesis, Hongik University.