Improving the Teaching of Mathematics with the Use of Technology: A Commentary

Sixto Romero

Abstract Current trends in mathematics education have emphasized the importance of using technology as a means by which students can work in other "pencil and paper" environments and can draw conclusions that will benefit them in the learning process. The non-use of new technologies may prevent the achieving more ambitious goals. The aim of the four chapters presented by Sabena, Lobo da Costa and co-authors, Hitt and co-authors and, Kotarinou and Stathopoulou is to show how the use of technology can help in the teaching and learning of mathematics, provided that process is well directed by the teacher.

Keywords Algorithm • Learning and teaching • Mathematical model • Mathematical task • Spatial competence • Sociocultural context • Technology

Introduction

In chapter "Early Child Spatial Development: A Teaching Experiment with Pro grammable Robots", Sabena presents the development of spatial skills in young children inspecting the educational capabilities provided by programmable robots. In chapter "Mediation of Technological Resources in Lessons on Polyhedra: Anal ysis of Two Teaching Actions", Lobo da Costa, Pimentel and Mendonça, through the mediation of technology resources prepared in geometry class, allow a greater understanding of the shares in the T/L a process through reflective practice teacher as a fundamental agent management framework that needs the reported activity. Hitt, Saboya and Cortés, in chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers" analyzing the design of mathematical tasks in a collaborative environment (the teaching method ACODESA) propose a methodology in which individual and social approaches are envisaged in the construction of mathematical knowledge. Finally Kotarinou and Stathopoulou present the axiomatic definition of Hyperbolic geometry through the Poincare model as an introduction to

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Non-Euclidean geometry developed abstractly from the set of knowledge that emerged in the study of Euclid's fifth postulate.

Comments on Chapter "Early Child Spatial Development: A Teaching Experiment with Programmable Robots"

As a first comment on the contribution of Sabena, it is necessary to reflect on the concept of space. It appears as a fundamental skill that accompanies the development of cognitive skills throughout the growth of children. In every stage of development, it is essential to know who we are and what our role in life is. It is important to note that when we lose consciousness the first thing we ask is: "Where am I?" because knowing who we are, where we are, at what stage of our existence we are, are the three basic issues allowing the contextualization of our own existence notions.

Even if it seems logical and natural for adults to evolve in space, the question of the development of the concept of space is an important issue for the learning process in the first stage of the life (Romero 2000).

For Piaget, acquiring the spatial notion is intrinsically linked to the acquisition of knowledge, and it is through this knowledge that the child's development begins at an early age. "The existence of multiple perspectives relating to various individuals is therefore already involved in the child's effort to represent space to himself. Moreover, to represent to himself space or objects in space is necessarily to reconcile in a single act the different possible perspectives on reality and no longer to be satisfied to adopt them successively" (Piaget 1954).

The notion of space (Parzysz 1991) can only be understood in terms of the construction of objects, and would need to begin by describing this to understand the first: only the degree of objectification that the child attributes to things informs us about the degree of externality according to the space. This cognitive beginning is enriched as the child grows and learns about space. For Craig (1995): "... knowledge of spatial relationships is achieved during the preschool period. This is logical because it is the age at which learning concepts like: inside, outside, near, far, up, down, above and below . . ." (p. 394).

Piaget dedicated two volumes to study the development of spatial knowledge, based on performing a large quality of different experiments. In 1947, in collaboration with Inhelder he writes "The representation of space in the child", and deals with how ontogenetic development arises in topological relationships, projective and Euclidian. In his second work, in 1948, with Inhelder and Szeminska ("Spontaneous geometry in the child"), he studies the genesis of Euclidean geometry, that is, the conservation of length measurement, as well as surface and volume.

Based on the psychological work of Piaget, Inhelder, Lucart and Vygotsky, as well as on the didactical approach of Arzarello, among others, Sabena supports the hypothesis that the reality in early childhood is full of different spatial cognitive aspects and requires different specific skills that must necessarily be related. She focuses on the development of spatial competences of children, and explores the educational potential offered by programmable robots. Cognitive aspects are in the first plane and in particular the delicate relationship between space (Hershkowitz et al. 1996) and everyday experience versus space as a mathematical notion.

Analyzing Sabena's experiment, it occurs that mathematics teaching with technology has to deal with a set of scientific and technical knowledge. Throughout the last century it gained increasing importance in everyday life as well as in the development of modern society. Teacher training in mathematics education requires relatively specific attention to the acquisition of knowledge. In general, educational programs with different materials (providing structured information to students by simulating phenomena) offer an environment more or less sensitive to the circumstances of the students' work, and especially, more or less rich in possibilities for interaction among young children; but all of these share essential characteristics:

- They use the material as a support in which students perform the activities.
- They are interactive, immediately responding to the students' actions and permitting dialogue and exchange of information between the material used and the child.
- They can identify the children's work and adapt to their rhythm and activities.

They are easy to use because a minimum of knowledge is required to perform the tasks (De La Fuente 2010). Thus, the author of this chapter emphasizes that high-tech gadgets surround today's young people and hardly attracted by simple mechanisms. Robots represent a technological element of great attraction to be very close to the type of devices that they use daily. Robotics is a branch of the scientific and technological knowledge that studies the design and construction of machines capable of performing repetitive tasks, where high precision is needed, dangerous work for human beings or unrealizable tasks without intervention of a machine. In the work of Sabena ("Early Child Spatial Development: a Teaching Experiment with Programmable Robots") the spatial development of skills shown by exploring the educational potential through programmable robots, places value on how the experience with a robot has influenced the children conceptualization of the concept of space.

Comments on Chapter "Mediation of Technological Resources in Lessons on Polyhedra: Analysis of Two Teaching Actions"

The theoretical framework of Lobo da Costa, Pimentel and Mendonça's work is based on Zeichner and Serrazina's ideas. It is a very attractive example in which the mediation of technological resources used in geometry classes is studied;

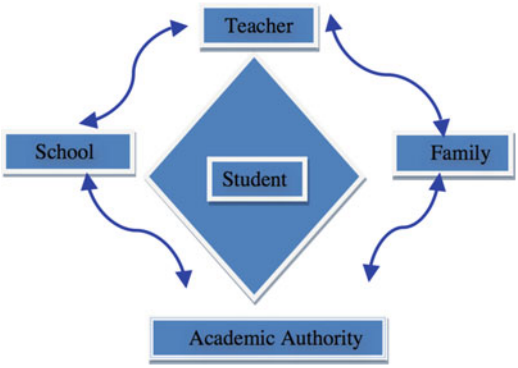


Fig. 1 Relation between the different actors in a process of teaching

particularly for dealing with three dimensional solids like polyhedral, prisms and pyramids, in elementary school.

As a reflection and following the scheme of the previous chapter, it is necessary to indicate that the presence of technology in education is no longer a novelty but a reality (De Lange et al. 1993). The contexts of the teaching-learning have changed their single appearance in the classroom, at least materially. The main issues are the new mathematics education processes and the way to involve all the agents (Fig. 1).

Having high expectations of the technological means, giving it potential for the treatment of information, should not prevent assessment and reflection on the ability to transform information. The objects are not simply the media or technology (NCTM 2000). The objects of evaluation and reflection are the active agents involved, and the contexts of teaching and learning we designed and put into practice and, ultimately the use of technological resources for the generation of knowledge. The ending aim is always education.

Research presented by Lobo da Costa, Pimentel and Mendonça analyses the role of technological resources in the geometry classroom, specifically that which is based on the concept of polyhedral. The mathematical content, practice and technology used during the experience are presented in detail. The categories analyzed were the class routines, interactions with students in order to see how the mathematical content was developed and the technology used.

They emphasize that, according to Serrazina and Oliveira (2005), teachers, in order to manage better their time should be responsible for the activities, contents and class organization proposed to students. Activities imposed by the teaching staff or by the central bodies of education are not always well received by teachers. A literal reproduction of what is stated in the recommendation to students in order to meet the curriculum planned and imposed by academic authorities is mainly observed in both groups of this experimental study, with few time spent in manipulations and collective discussions.

It is important that the authors of the study do not compare mediations, since they are linked to confirm the personal characteristics in the way in which technological resources were used. However, from the analysis of the incidence and mediation of technology resources that teachers use, they conclude that the main features are: the reality of the classroom, the student interest, the number of students per class, the breach in prior knowledge of the students, the need for compliance with the prescribed plan of studies and the time available; and these will be considered as factors that interfere in the mediation.

The presented experience, as Volkert (2008) points out, shows the intrinsic difficulties of solid geometry impeding the introduction of systematic teaching. Solid geometry is much more complicated than its homologue on a plane. Also, the problem of intuition and evidence is far more complex and problematic in Solid geometry. So the history of Euler's theorem is a very good illustration of this theme. These difficulties can be taken together with others like for instance, in secondary, spatial geometry is relegated and in some cases completely absent.

We can emphasize that in the chapter developed by Lobo da Costa, Pimentel and Mendonça the use of physical objects, models and figures is the main tool for teachers to help students understand the geometric concepts, hence the ability to display (or spatial imagination) is imperative to learn geometry. The display is very useful in any area of mathematics and especially in the field of geometry. The teaching of elementary geometry has always been based on intensive use of objects, figures, diagrams, charts, etc. to help understand the concepts, properties, relationships or formulas studied. Thus, as indicated by Hershkowitz et al. (1996), geometry appears to students as the science that studies the physical space and the convenience of using graphical representations to help the understanding of geometric concepts extends beyond elementary Euclidean geometry as developed by Kotarinou and Stathopoulou in chapter "ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching".

As a personal opinion based on the experience I have accumulated since 1975, by collaborative work with teachers from different levels of education, the almost complete unanimity among mathematics teachers that adequate display capability is an essential tool for geometrical learning that is rarely accompanied by a reflection on the learning processes of visualization. This is not an innate ability that can be let develop spontaneously, but a model is necessary, as the display is a complex activity in which several elements are necessary to be understood and learnt in order to be used.

Comments on Chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers"

The third chapter started by making a first reflection on problem solving as a way to mathematical modeling. The research in Mathematics Education has focused its attention for some time on designing activities based on mathematical modeling of real situations, with the conviction of obtaining greater assurance in profit by our students of mathematics learning, and therefore teaching. One of the most complex problems that education faces in different educational levels where the teaching of mathematics is concerned relates the way of articulating the contents with other areas of knowledge and even with mathematics in itself.

For our students, most content organized into topics are disconnected from the real world and science applications, as a consequence this means that they do not conceive the utility of mathematics in their training. In recent years, research in Mathematics Education realizes that one of the striking issues is the design of activities based on the modeling of real situations. In many countries and in different conditions, its inclusion in the curriculum has allowed the development of diverse types of cognitive capabilities, metacognitive and crosscutting to help understanding the role of mathematics in today's society (Aravena and Caamaño 2007; Blomhoj 2004; Keitel 1993). Therefore, today's society must provide the role to deal with problem solving, make estimates, and take decisions, and face a mathematization of culture and the surrounding environment. That is, modelling mathematics is tending to promote understanding (Niss 1989) of the concepts and methods, thus allowing a more comprehensive overview of mathematics.

Over the course of history, mathematics has occupied a prominent place in school curricula. It has achieved this prominence, not because of the importance in itself but for cultural and social reasons.

We collect the idea of Jean Pierre Kahane, French mathematician and professor emeritus at the University Paris Sud Orsay, a former student of the Ecole Normale Superieure, and member of the Academie des Sciences (mathematics section) since 1998 when he asserts:

> the reflection on the teaching of mathematics is done from all angles, from all status: it can be from the daily work in the classroom, difficulties of teachers and students of all educational levels. It can be done through a detailed examination, test study; or extra-curricular activities, the gymkhanas, rallies, competitions, olympics, ultimately all manifestations of animation and diffusion of mathematics; or the role and evolution of the mathematical sciences in the whole of science and society. (Gras et al. 2003, p. 5)

As in France, in many countries, teachers grouped or not in Societies of Teachers, Editors of publications in Mathematics Education have taken initiatives in order to make proposals and initiatives in the field of Problem Solving and Mathematical Modelling (Romero and Romero 2015) to improve the binomial teaching/learning of mathematics. Problem solving has a long tradition in mathematics. George Polya considered Euclid's Elements as a collection of problems (a sequence of statements and solutions). Together with Gabor Szegö, he produced under the title of *Exercise Analysis*, a collection graduate of problems.

The authors, Hitt, Saboya and Cortés, utilize problematic situations in the sociocultural context of mathematics class that requires careful design to develop skills in the classroom, promote diversified thinking and achieve a balance between the pencil, paper and technological activities referred to in the theoretical framework of the activity (Balacheff 2000). The ACODESA methodology presented in the chapter differs in five main phases (Individual work, Teamwork on the same task, Debate, Auto-reflection and Process of institutionalization), and the design of the activities under this perspective and with the use of technology is not a trivial

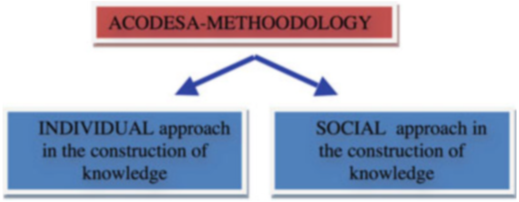


Fig. 2 ACODESA method of teaching, seeing the individual in a social context of learning

task in the mathematics classroom. A comprehensive work to develop the activity and the details that need to be provided to present a complete vision of the activity need a significant space that is not always available in a research context. Deficient communications in all aspects involved in the development of problem solving activities makes it more difficult for teachers to follow those activities.

In the design of tasks, they are taking into account Arcavi and Hadas (2000) suggestions; based on a Dynamic Geometric System that stands out for the elements of visualization, experimentation, surprise, evaluation, need testing and demonstration, as key elements of the analysis detailed. Also, the prospect of collaborative work (Prusak et al. 2013) allows for the design and creation of tasks (Kieran et al. 2015), suggesting problematic situations that enrich the visualization of the problem (Fig. 2).

The authors present very appropriate examples. The use of the concept of triangular number as one that may be in the form of an equilateral triangle with other figurative numbers were studied by Pythagoras and the Pythagoreans, who considered sacred 10 written in a triangular shape, and they called Tetraktys.

The dynamism presents examples, related to:

- Visualizing information through a numerical approach.
- Find a generic pattern.
- Affirm that generally the tasks of connecting the different representations of a concept, is not considered by many teachers as fundamental in the construction of mathematical knowledge and, in particular teachers minimize the task of the conversion among representations.

Hitt et al., proposed that the task of the conversion, among representations, would enable the development of mathematical visualization processes. This visualization has to do with mental processes and transformation productions on paper, on the blackboard or on the computer, generated from a reading of mathematical statements or graphics, promoting the interaction between representations for a better understanding of mathematical concepts involved.

In conclusion, the tasks and the methodology proposed by the authors of this chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers" inculcate in students the learning of mathematisation, defined as problem solving that triggers a process of:

> Identification of relevant mathematical concepts and then progressively simplify reality in order to transform the problem into one susceptible to locate an a mathematical solution ... by finding regularities and patterns, [. . .] It need to use various competencies for mathematisation task. (OECD 2004, pp. 27, 28 and 29)

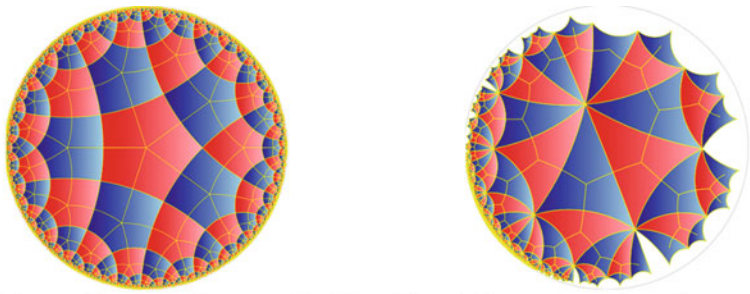
Comments on Chapter "ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching"

A teaching experiment about axiomatic foundation of Hyperbolic geometry and its basic notions, using 'Drama in Education' conventions to motivate and actively engage all of the students, is presented in this chapter "ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching". The fundamental purpose of the work presented by Kotarinou and Stathopoulou, using, as a case study, the introduction of Hyperbolic geometry through the Poincaré model, is to show that the creation of new problematic situations with the use of technology allows more dynamic teaching of geometry in the classroom, improving understanding.

It is interesting to know the theoretical framework in which the activity is presented by the authors. There are many comparisons between Euclidean geometry and Hyperbolic. For example, it could well be that Hyperbolic geometry was actually true in our world cosmological scale. However, the proportionality constant between the deficit angle and a triangle area should be extraordinarily small in this case, and Euclidean geometry would be an excellent approximation to this geometry for any ordinary scale. In the Poincaré model H^2, all the hyperbolic space is represented in a disc of the radius, $r = 1$. The edge of the disc represents the infinite. Within the disk all the postulates of Euclid are satisfied except the 5th (the parallel postulate):

1. It can draw a straight line through two points.
2. It can prolong a straight line indefinitely from a finite straight line.
3. You can draw a circle with given center and known radius.
4. All right angles are equal.
5. *If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.*

In H^2 the sum of the internal angles of a triangle is lower than 180. More surprisingly, two lines with different directions may be parallel. Poincaré model to visualize these aspects of Hyperbolic geometry, but being all the space within a disk, the lines are righteous actually are perceived as curves (hence they are called "Geodesic"). And the metric that allows us to measure distances within the Poincaré disk is not Euclidean. These ideas can be shown and manipulated in a relatively easy way with the use of appropriate software. The time spent by students working with computers is really very important for the visualization, recognition



Hyperbolic tessellation {5,4} generated in C # and WPF *Hyperbolic rotation and translation with Möbius transformations*

Fig. 3 Transformations in Poincaré's disk (https://rastergraphics.wordpress.com/2012/06/27/geometria-hiperbolica-disco-de-poncare/)

and exploration of a non-Euclidean geometry, a geometry that is not in our daily life (Fig. 3).

Kotarinou and Stathopoulou point out that students who carried out the experience came to understand the principles of Hyperbolic geometry through the Poincaré model with the analysis of worksheets. The experiment shows that most participants adequately responded to most of the issues of the circle (the center and its rays), the apparent decline in segments of specific length and the sum of the angles of a triangle. Some students gave further explanation of certain phenomena especially those who had read the book Flatterland, discussed in a previous class period, but not accessible to many. Therefore, it is important to note that the implementation of Hyperbolic geometry in the Poincaré model are useful for the following concepts:

- The hyperbolic space H^2 is a disk of radius, $r = 1$, centered at the origin in the Euclidean plane R^2, called Poincaré disk.
- The points in the hyperbolic space H^2 are points in the Euclidean plane that are within the Poincaré disk.
- The lines passing through two points in H^2 are Euclidean circles passing through two points on disk and are orthogonal to the Poincaré disc.
- The lines passing through the origin (i.e., the center of the Poincaré disk) are circles of radius $r = \infty$, they are Euclidean lines.
- The angles are Euclidean, the measure of angle formed between two geodesics (hyperbolic lines) is the angle between the tangents of the circles at the point where they are intercepting.
- The inversion of a point on the circle is an isometry (preserves angles and distances) and is interpreted as the reflection of a point in a hyperbolic line.

It should be noted as very positive the use of ICT (interactive Java) by students to display the model of Poincaré (axioms and basic concepts of non-Euclidean geometry), thus creating an interactive environment, ultimately providing a new tool in teaching the axiomatic basis of hyperbolic geometry. The working group of students with worksheets, exploring the Poincaré's model has enabled them to draw points, lines, segments, angles and lines perpendicular to a given line. Thus writing the comments on the construction of cycles, line segments of equal length and measuring sides and angles of a triangle has allowed students to understand the axiomatic basis of Hyperbolic geometry in an enjoyable manner, creating a relaxed environment and satisfaction in students. This chapter presented by Kotarinou and Stathopoulou is interesting because the experience presented deals with a new practice leading to new paradigms and new tools with new technologies that have helped the process of students' visualization and therefore the understanding of the geometric concepts presented (Gutiérrez 2006).

Conclusion

First, it can be concluded from the above Chapters of Sabena, Lobo da Costa et al., Hitt et al., and Kotarinou and Stathopoulou, that if the conception of the role of the teacher is close to traditional transmitter and organizer of knowledge and practical activities, where visualization is rarely used in the classroom, the assessment will be related to working methods explained in class, impeding autonomy of the students.

Enquiries from Presmeg (1997) identifying various types of mental imagery is used by students to solve mathematical problems. The most commonly used in geometry are:

– *Concrete images* (pictures in the mind): figurative mental images of real objects.
– *Kinetic images*: mental images associated with muscle activity as a movement of a hand, head, etc. For example, when a student, describing a figure with parallel segments, places the hands stretched parallel and moves them up and down.
– *Dynamic images*: mental images in which the displayed image (or any of its elements) is a moving object. Unlike the kinetic images, these images do not provoke physical movement, but are only displayed in the mind.

For his part, Bishop (1989) describes two processes taking place when using images:

– *Interpretation of figurative information*: the process that takes place when trying to read, understand and interpret an image to extract information from it.
– *Visual information processing*: the process that takes place when converting non-visual information in images, or transforming an image already formed into another image.

The experience at different levels of education (Blomhoj 2004) shows that the treatment of theoretical aspects can be a tool for the practice of teaching problem solving as a path to mathematical modeling. The role played by teachers and researchers in mathematics education should perform interesting work in many mathematical domains such as, for instance, problem solving, almost unexplored in the Primary and/or Secondary school (Romero and Castro 2008), which can produce in an original and creative way, activities enriching the process of teaching. Many of these domains can be planned so that they can become powerful generators of important skills, not only mathematics but crosscutting, *number theory, graph theory and optimization theory chaos, topology, data processing, coding theory and cryptography, fractals mathematical models*, or competences presented by Sabena, Hitt et al., Lobo da Costa et al., and Kotarinou and Stathopoulou.

As a final comment related to learning objectives, it is necessary:

– To analyze, to delve into the research methods in mathematics: particularization, finding general laws, building models, generalization, using analogies, conjectures and demonstrations, among others.
– To use mathematical models for the mathematization of reality and problem solving (Romero et al. 2015); experiencing their validity and usefulness, criticizing limitations, improving them and communicating findings and conclusions.

Moreover, when asked to bring the issue to the classroom, we must be explicit regarding the educational goals we are demanding:

(a) To practice problem solving as the most genuine activity in any specific field of mathematics, where the technology can be an impressive and a fantastic aid.
(b) To bring the students to approach mathematical knowledge, prioritize and solve challenges, search for explanatory models, inquiry and discovery.
(c) To tackle the aspects of the creation process and/or detection in mathematics we must focus on bringing into the classroom in order to achieve the educational goals we have set ourselves (Watanabe and Mcgaw 2004).

> What the teacher says in class is not unimportant, but what students think is a thousand times more important. The ideas must be born in the minds of the students and the teacher should act only as a midwife. This principle is based on let the students discover by themselves as much as feasible under the given circumstance. (Unknown, http://lovelypokharacity.blogspot.com.es)

References

Aravena, D. M., & Caamaño, E. C. (2007). Mathematical modeling with students of secondary of the Talca commune. *Estudios Pedagógicos, 32*(2), 7–25.

Arcavi, A., & Hadas, N. (2000). Computer mediated learning: An example of an approach. *International Journal of Computers for Mathematical Learning, 5*(1), 25–45.

Balacheff, N. (2000). Entornos informáticos para la enseñanza de las matemáticas: complejidad didáctica y expectativas. In N. Gorgorió & J. Deulofeu (Eds.), *Matemáticas y educación. Retos y cambios desde una perspectiva internacional* (pp. 91–108). Ed. Grao: Barcelona.

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7–16.

Blomhoj, M. (2004). Mathematical modelling. A theory for practice. In B. Clarke, D. Clarke, G. Emanuelsson, B. Johnansson, D. Lambdin, F. Lester, A. Walby, & K. Walby (Eds.), *International perspectives on learning and teaching mathematics* (pp. 145–159). Göteborg: National Center for Mathematics Education.

Craig, G.J. (1995). *Early childhood, children today*. Ed. Prentice Hall. University of Michigan.
De Lafuente, C. (2010). Modelos matemáticos, resolución de problemas y proceso de creación y descubrimiento en matemáticas. In *Conexiones y aprovechamiento didáctico en secundaria. Construcción de modelos matemáticos y resolución de problemas* (pp. 123–154). Madrid: Ministerio de Educación.
De Lange, J., Keitel, C., Huntley, I., & Niss, M. (1993). *Innovations in maths education by modelling and applications*. London: Ellis Horwood.
Gras, R., Bardy, P., Parzysz, B., & Pecal, M. (2003). *Pour un enseignement problématisé des Mathématiques au Lycée (Vol. 150 and 154)*. Paris: *APMEP*.
Gutiérrez, A. (2006). La investigación sobre enseñanza y aprendizaje de la geometría. In P. Flores, F. Ruiz, & M. De la Fuente (Eds.), *Geometría para el siglo XXI* (pp. 13–58). Badajoz: FESPM-SAEM.
Hershkowitz, R., Parzysz, B., & Van Dormolen, J. (1996). Space and shape. In A. J. Bishop, K. Clements, C. Jeitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Vol. 4, pp. 161–204). Dordrecht: Kluwer.
Keitel, C. (1993). Implicit mathematical models in social practice and explicit mathematics teaching by applicatins. In J. Lange, C. Keitel, I. Huntley, & M. Niss (Eds.), *Innovations in maths education by modelling and applications* (pp. 19–30). Chichester: Horwood Publishing.
Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI Study-22* (pp. 19–81). Dordrecht: Springer.
NCTM. (2000). *Principles and standards for school mathematics*. Reston: NCTM.
Niss, M. (1989). Aims and scope of applications and modelling in mathematics curricula. In W. Blum, J. Berry, R. Biehler, I. Huntley, G. Kaiser-Messmer, & L. Profke (Eds.), *Applications and modelling in learning and teaching mathematics* (pp. 22–31). Chichester: Horwood Publishing.
OECD. (2004). *Problem solving for tomorrow's world* (pp. 27–29). First measures of cross-circular competencies from PISA 2003, Paris.
Parzysz, B. (1991). *Representations of space and student's conceptions at high school level. Educational Studies in Mathematics, 22*(6), 575–593.
Piaget, J. (1954). *The construction of reality in the child*. New York: Basic Books.
Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 299–312). Mahwah: Erlbaum.
Prusak, N., Hershkowits, R., & Schwarz, B. (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education, 15*(3), 266–285.
Romero, S. (2000). Matematización de la cultura. Límites y asedios a la racionalidad. *Revista Epsilón. SAEM Thales, 48*, 409–420.
Romero, S. & Castro, F. (2008). Modelización matemática en secundaria desde un punto de vista superior. El problema de Dobogókó. *Modelling in Science Education and Learning*, *1*, 11–23.
Romero, S., & Romero, J. (2015). ¿Por una enseñanza problematizada y modelizada de las matemáticas? *Revista UNO, 69*, 33–43.
Romero, S., Rodríguez, I. M., Salas, I. M., Benítez, R., & Romero, J. (2015). La Resolución de Problemas (RdP's) como herramienta para la modelización matemática: ejemplos de la vida real. *Modelling in Science Education and Learning, 8*(2), 51–66.
Serrazina, M. L., & Oliveira, I. (2005). O curriculo de Matemética do ensino básico sob o olhar da competéncia matemática. In A. C. Costa, I. Pesquita, M. Procópio, & M. Acúrcio (Eds.), *O professor e o desenvolvimento curricular* (pp. 40–51). Lisbon: APM.
Volkert, K. (2008). The problem of solid geometry. http://www.unige.ch/math/EnsMath/Rome2008/WG1/Papers/VOLK.pdf. Accessed 5 May 2016.
Watanabe, R. & Mcgaw, B. (2004). Student learning: Attitudes, engagement and strategies. In R. Watanabe & B. Mcgaw (Eds.), *Learning for tomorrow's world. First results from PISA 2003* (pp. 109–158). OCDE. https://www.oecd.org/edu/school/programmeforinternationalstudentassessmentpisa/34002216.pdf. Accessed 5 May 2016.