Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers

Fernando Hitt, Mireille Saboya, and Carlos Cortés

Abstract This paper discusses mathematical task design in a collaborative environment (the ACODESA teaching method), where activities with both paper and pencil and technology play a central role in learning mathematics. The use of problem situations under a sociocultural framework in the mathematics classroom requires careful mathematical task design to develop mathematical abilities in the classroom, promote diversified thinking, and achieve balance between pencil and paper and technological activities within an activity theory framework. While the task design approach examined in this paper is general, it is exemplified through mathematics teaching tasks appropriate for secondary school entry level.

Keywords Task design • Paper and pencil • Technology • ACODESA • Socio-cognitive conflict

Introduction

The literature on mathematics education regarding problem solving is evolving. As mentioned in chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA", Brownell (1942) makes distinctions among the concepts of exercise, problem and puzzle, thus focusing on issues related to primary school level. This led to a new trend linked to the solving of arithmetic word problems and gave birth to, among others, the current problem solving approach. Thus, a new paradigm linked to problem solving emerged, where the distinction between exercise and problem was, and is, preponderant. However,

F. Hitt (✉) • M. Saboya
Département de mathématiques (GROUTEAM), Université du Québec à Montréal, Montréal, QC, Canada
e-mail: hitt.fernando@uqam.ca; saboya.mireille@uqam.ca

C. Cortés
Facultad de Ciencias, Universidad Michoacana de San Nicolás de Hidalgo, Morelia, México
e-mail: jcortes@zeus.umich.mx

G. Aldon et al. (eds.), *Mathematics and Technology*, Advances in Mathematics Education, DOI 10.1007/978-3-319-51380-5_4

this distinction is not so simple, in that, depending on the problem, either convergent thinking (using closed type problems) or divergent thinking (problems with multiple solutions or open problems) could be generated. The latter approach can be related to the Theory of Didactic Situations (TDS) (Brousseau 1997) and even the emergence of the notion of the epistemological obstacle (Brousseau 1983). The design of mathematical tasks under this paradigm took a unique approach. How to detect an epistemological obstacle in pupils' activity? How to encourage pupils to overcome a certain kind of epistemological obstacle? What kind of activity is needed to promote the overcoming of such an obstacle?

Gradually, design problems became more and more important in research on mathematics education. For example, in his notion of conceptual field, Vergnaud (1990) notes that a concept is developed through a set of problems, a set of operators, and a system of signs. Thus, the type of problems that are proposed in the classroom will determine to some extent the mathematical concept pupils are constructing. In the mid 1980s (as seen in Mason et al. 1982; Schoenfeld 1985) the trend for problem solving took on great force, with, for example, research on problem solving (see Kilpatrick 1985) generating such curriculums as *Standards in the USA* (NCTM 2000). According to Kilpatrik (ibid.), "A problem is generally defined as a situation in which a goal is to be attained and a direct route to the goal is blocked" (p. 2).

A different approach to the foregoing is promoted by the Freudenthal School, which promoted the resolution of problems in context, where, under this approach, the study of mathematical modelling process is essential in a strand known as "Realistic mathematics." Gravemeijer and Doorman (1999) describe the characteristics of the current Freudenthal School. Realistic mathematics is likely to have strongly influenced the notion of problem situation, in which the solution is not necessarily unique. Indeed, realistic mathematics promoted other kinds of curricula linked more closely to the notion of problem situation. The ensuing discussion leads to the question as to whether an exercise, a problem and a problem situation are.

Exercise, Problem or Problem Situation

Advances in mathematics education brought about the need to carefully identify the definition of an exercise, a problem or problem situation. A definition depends on the theoretical framework that has been selected. Given the interest here in definitions linked to mathematics learning environments when using both paper and pencil and technology, this paper seeks to associate these definitions with the notions of non-institutional and institutional representation in order to then link this to Leontiev (1978) and Engeström's (1999) activity theory.

Exercise If reading a mathematical statement immediately suggests a procedure to follow, it can be said that the task is an exercise.

Problem If reading a mathematical statement does not induce the reader to immediately think of a procedure to follow, and requires them to transform the statement and/or use institutional representations and/or produce non-institutional representations to understand and make progress in the proposed task, it can be said that it is a problem.

Problem Situation If the reading of a mathematical statement as in the case of a problem, neither provides a procedure to follow, but in this case, a model must be built (possibly not unique), needed to interpret the phenomenon linked to the statement, then it can be said that this is a problem situation.

This distinction enables the identification of the differences among mathematical tasks that should be considered when designing an activity for the mathematics classroom. The followers of problem solving were more interested in the resolution of problems, as defined above. A different perspective was provided by Lesh and Doerr (2003), Blum et al. (2007), and Lesh and Zawojewski (2007), among others, which dealt with problem solving and modelling, and presented an approached to realistic mathematics and what is meant by the term problem situation.

Indeed, from the perspective of this study, the three types of tasks mentioned above are required for the organisation of mathematical activities in the classroom. The difficulty arises in the organisation of those types of tasks that is needed in order to follow a fixed syllabus. A possible way to overcome this problem may be for the teacher to use the proposition outlined in Simon (1995) and Simon and Tzur (2004) as related to a Hypothetical Trajectory of Learning, which is discussed in the subsequent sections.

One of the first problems to overcome is the fact that the expert (in this case the mathematics teacher) has already constructed different types of thinking (arithmetic, algebraic, geometric) that allow her/him to transform their representations effectively. The beginner (the pupil) has not necessarily built these official representations, and, even if they have, the difficulty arises when they are required to handle them efficiently (as a competence). Generally, learning theories based on the concept of representation focus on the efficient use of institutional representations in the construction of knowledge (as is the case, for example, in Duval's 1995 work which focuses on the notion of register). In the context of our approach, non-institutional and institutional representations are of great importance to the construction of knowledge; also a collaborative learning process is of great significance in a socio-cultural environment, to the refinement of the evolution of the non-institutional representations in which they are promoted to the level of formal representation.

Institutional Representation Representation found in textbooks, websites, software use, or those used by mathematics teachers.

Non-institutional Representation Representation that emerges spontaneously during the resolution of a non-routine mathematical task as a result of a functional representation that has been generated by the action of understanding or solving a task.

Functional and Spontaneous Representation A functional representation is a mental representation linked to an activity. From reading the statement of the task, a need and purpose emerge, which, in terms of Leontiev (1978), mediate the activity undertaken by the individual as a whole. A mental representation is constructed and linked to other concepts, providing the spontaneous representation as a product. The manipulation of objects or artefacts mediates the generation of mental processes which become increasingly complex, as do their external productions.

Leontiev's work (1978) on activity theory is immersed in a sociocultural perspective on learning. Leontiev was interested in the subject and object relationship, while it is in the work of Engeström (1999) where the variable community was explained in the model (see the next section related to ACODESA[1]).

Socio-cognitive Conflict In the past, many researchers, such as Piaget, Inhelder, Brunner and others, were interested in the notion of cognitive conflict. In Bruner's theoretical framework (1966), cognitive conflict occurred when the individual was aware of a mismatch between the enactive, iconic or symbolic representation related to the activity. This study takes Varela et al. (1991) definition of enactive:

> Cognition is not the representation of a pregiven world by a pregiven mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs (p. 9)

In the context of this study, the term iconic could refer to a drawing related to the situation, or a symbol as an institutional representation, with the teacher (expert) easily noticing mismatches between different modes of representation. However, this study is interested in the processes of communication pupils use to point out a mismatch between the spontaneous representations they produce, thus creating a socio-cognitive conflict.

Method of Teaching ACODESA (Collaborative Learning, Scientific-Debate, Self-Reflection)

Looking within a sociocultural framework, in order to organise mathematical work in the classroom and create a form of socio-mathematical norms, it is important to follow a specific educational model. This study is interested in individual work immersed in a collaborative learning structure for the consolidation of knowledge. Our experience has shown us that these aims are not easy to achieve in the mathematics classroom. Thus, the authors designed a teaching model known as ACODESA which is related to an approach involving collaborative learning, scientific debate and self-reflection (see Hitt 2007, 2013; Hitt and Gonzalez-Martin

[1]Acronym which comes from the French abbreviation of *Apprentissage collaboratif, Débat scientifique, Autoréflexion*.

2015) and which includes several steps to be implemented in the mathematics classroom when solving a mathematical task. It is described below in more depth:

1. *Individual work*. Production of spontaneous non-necessary institutional representations related to the task, with prediction processes encouraged.

The design of mathematics classroom situations should follow a structured plan for the use of both paper and pencil and technology. The activity starts when reading the statement of the situation. This mental activity, as mediated by paper and pencil, produces the spontaneous representations linked to the activity of understanding and searching for a goal, even if this is not a well-defined or easy process. Reference to the use of paper and pencil is made in a broad sense[2]. Thus, the use of paper and pencil is intended to be a mediator between pupils' mental representations (i.e. functional representations) as linked to the situation and the activity of understanding, and thus promotes the production of spontaneous representations linked to actions that are not necessarily institutional (Hitt 2013; Hitt and Gonzalez-Martin 2015). This first stage provides the pupil with preliminary ideas that she/he discusses with other members of her/his team. Following an approach where activity and communication go hand in hand (activity theory) creates a link between activities, motives, actions, objectives and operations in the context of Leontiev's work in this area. This stage and that described below are crucial to the production of spontaneous representations and to the commencement of the process of their evolution.

2. *Teamwork* on the same task. Process of prediction, argumentation and validation. Pupils refine their representations in response to their results.

Teamwork helps to refine both the initial ideas and the ability to follow a path towards the resolution of the problem situation. The functional representations that gave rise to spontaneous representations in the individual phase initiate a new process of refinement, which takes into account both the manipulation of physical objects and communication with others. This process is linked to argumentation (persuasion in many cases), prediction and validation, and both testing and taking a position. It is at this stage where cultural norms come into play directly, with teamwork and organisation crucial for the distribution of partial tasks. The question then arises as to how many people are to be allocated to each team. For example, Sela and Zaslavsky (2007) show the difference between teams of two and four people, stressing the fact that, in a two-person team, participation is more balanced, while, with four people, there is an immediate tendency that one of them may take a leadership role with the others becoming followers. As such, teams of two or three people are suggested. It is necessary for the team members to determine who

[2]Touchscreens are used more and more in schools (see the chapter on this matter in Bairral et al., this volume). The paper and pencil component can be converted to the use of an electronic notebook in the production of (not exclusively) institutional representations. Currently, there are some electronic devices, such as notebooks, that can be connected to an iPad for simultaneous use with other applications.

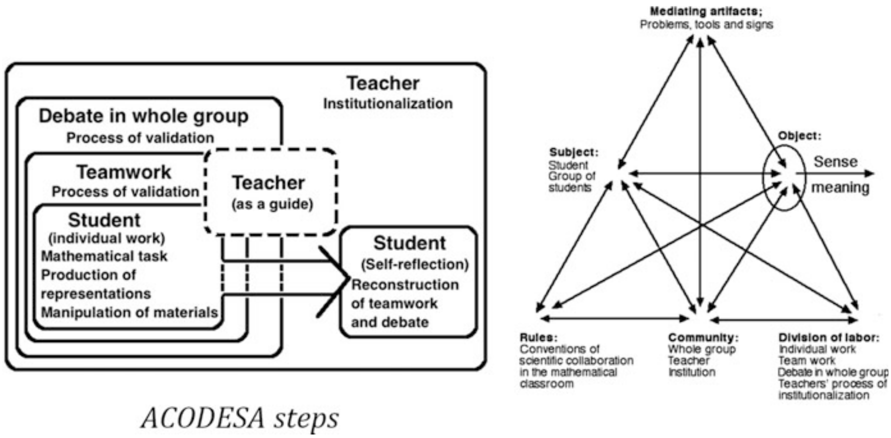


Fig. 1 ACODESA and the Engeström model as adapted to the aims of this study

manipulates physical objects (and how they are manipulated), who uses the computer (e.g., see Hoyles 1988), who notes the progress of the team, and who comes forward to present the achievements of the team for plenary discussion. In fact, it is here that both activity theory and Engeström's (1999) model are very important (see Fig. 1). At this stage, the teacher's role is to guide rather than provide their opinion on how the teams performed.

3. *Debate* (could become scientific debate). This is related to a process of argumentation and validation and the refinement of representations. According to Legrand (2001), the teacher's role is crucial at this stage for the promotion of scientific debate. In general, if the design of the task is related to a problem situation or an open problem, different results from the teams will be presented for discussion. In general, teams will have a natural tendency to protect their results, with the teacher required to regulate the discussion (socio-cultural norms) and decrease the persuasion and argumentation that can lead to prediction and validation. Again, spontaneous representations that have surely undergone a process of refinement first through working in small teams can be refined in large group discussion.
4. *Self-reflection* (individual work – the reconstruction of what has been carried out in the classroom).

Given that the literature has shown, in the classroom, consensus to be ephemeral (Thompson 2002; Hitt and Gonzalez-Martin 2015; Hitt et al. 2015), this study included a stage involving a reconstruction process activity. The teacher must collect everything produced during the previous stage and provide a new copy of the task. Karsenty (2003) demonstrates that after a certain period of time, adults forget the mathematics they have learned. The question as to how to build stable knowledge is one that led to this stage being implemented here and also to the importance attached here to individual reconstruction. It is at this stage that the notion of historicity has a strength action; where the pupil has been influenced by a socio-cultural process of learning and is prone to a sociocultural construction of knowledge. This stage also requires reconstruction related to achievements in terms of individual work, teamwork and plenary discussion designed to strengthen knowledge.

5. *Process of institutionalisation*. The teaching undertaken by the teacher takes the pupils' results into account and uses the official representations.

In a sociocultural knowledge construction process, where the pupil is an active actor in that environment (activity theory), a mathematical concept is not produced through a dogmatic presentation by the teacher. Institutionalisation occurs at the end of those preliminary stages, where the teacher takes pupils' productions into account while refining the concept and, if necessary, providing both the institutional position and its official representations.

ACODESA takes Engeström's model into consideration in the organization of pupils' classroom activities by placing special attention on the artefacts they use.

Task Design

As seen in previous sections, task design is not a new feature in mathematics education. For example, when conducting a teaching experiment, it is important to build a hypothetical model to guide the researcher in the teaching process. More precisely, as described above, both Simon (1995) and Simon and Tzur (2004) proposed the Hypothetical Learning Trajectory (THA) method, which allows the teacher to organise and design mathematical activities for use in the mathematics classroom.

Interested in the learning of mathematics in a sociocultural environment and given the technology involved, researchers in this study considered, for example, the following elements, as described by Arcavi and Hadas (2000, pp. 25–27), as being of fundamental importance to a design based on a Dynamic Geometrical System (DGS):

1. Visualization. "Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information".
2. Experimentation. Besides visualization, playing in dynamic environments enables students to learn to experiment.
3. Surprise. It is unlikely that students will fruitfully direct their own experimentation from the outset. Curriculum activities, such as problem situations, should be designed in such a way that the kinds of questions students are asked can make a significant contribution to the depth and intensity of a learning experience.

4. Feedback. Surprises of the kind described above arise from a disparity between an explicit expectation of a certain action and the outcome of that action. The feedback is provided by the environment itself, in that it reacts as requested.
5. Need for proof and proving. Dreyfus and Hadas (1996) discuss and exemplify how one can capitalize on such student surprises in order to instil and nurture the need for justification and proof.

An analysis of the above characteristics reveals that the DGS is an important element. Under this view, Duval's 2002 approach to Arcavi and Hadas' mathematical visualisation process is very pertinent, since it relates to the discrimination of visual variables on a register as possibly associated with corresponding elements on another register.

The problem with these approaches is that spontaneous representations in the resolution of problem situations are not fully considered in these contexts. These spontaneous representations generally do not belong to a register. This study is interested in the unofficial representations that pupils produce in a paper and pencil environment (Hitt 2013; Hitt and Gonzalez-Martin 2015) and the evolution toward institutional representations (e.g., those on a computer screen) through a process of communication with others and the use of technology.

As the notion of learning with which this study is concerned is linked to collaborative work, other perspectives must also be considered, such as those of Prusak et al. (2013), who, with respect to the creation of tasks to promote productive argument, suggest the following:

1. The creation of *collaborative situations,*
2. The design of activities that trigger *socio-cognitiveconflicts,*
3. The provision of tools for checking hypotheses.

Indeed, for the perspective of this study, Arcavi and Hadas, as well as Duval and Prusak et al., can be taken into account in both paper and pencil and technological approaches (Hitt and Kieran 2009; Hitt et al. chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") formulated using ACODESA.

In this context, ***visualisation*** refers to the ability to represent, transform, and find significant visual variables that may be associated with other elements from another separate representation through a process of communication with others. This thus promotes an evolution where the mathematical activity in question is "seen" and creates an improved approach to the resolution process.

Healy and Sutherland (1990), on one side, and both Hitt (1994) and Hitt et al. (in this volume), on the other, illustrate how pupils or pre-service teachers "see" the task of constructing a process for the generalisation of polygonal numbers differently. For example, both Hitt (1994) and Hitt et al. (chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") found different approaches, such as that related to changing the number of elements on the diagonal in order to obtain the next triangular number (Fig. 2), or that focusing on the number of elements on the base or on one side of the

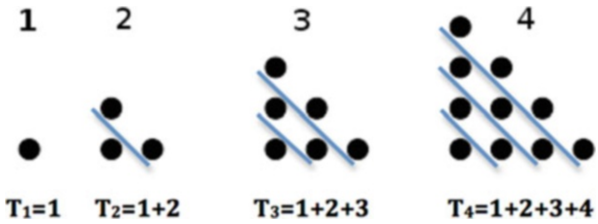


Fig. 2 Process of visualising and articulating visual information in a numerical approach

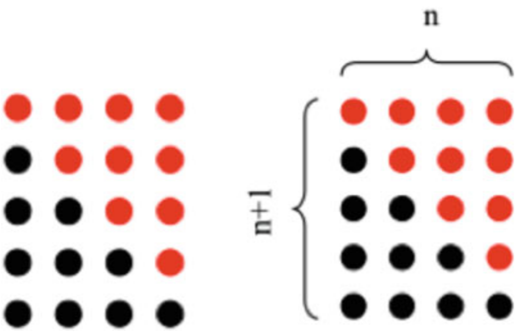


Fig. 3 Transformation of the triangular number to find a general rule

triangular arrangement. Both Healy and Sutherland (Idem) and Hitt (1994) used a triangular arrangement, specifically with an equilateral triangle, while this study used an isosceles triangle rectangle. This arrangement generated the conjecture (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") relating to calculating any triangular number using the formula for calculating the area of a triangle (base * height/2). Pupil conjecture thus created a socio-cognitive conflict, as pupils pointed out that calculating T6 and T8 (triangular numbers 6 and 8) visually did not obtain the same result.

The expert (the mathematics teacher) "sees" triangular numbers institutionally in order to complete a rectangular array, as seen in Fig. 3. The visual triangular number is duplicated and a transformation performed, thus obtaining a similar arrangement that is able to show a rectangular arrangement (Fig. 3), thus revealing the conclusion that:

$$T_n = \frac{n(n+1)}{2}.$$

Pupils' visual processes do not necessarily agree with the ways in which teachers visualise. The teacher uses official representations that enable her/him to be efficient in handling the institutional representations. They, as experts, are able to articulate representations that have developed ways to "see" into the passage, distinguish from one representation to another. Thus, the expert is able to immediately identify the important visual variables (as described by Duval) to be transformed and/or converted into another representation.

The question thus arises in terms of how to develop this expertise by our pupils. The purpose of this study was to create socio-cultural norms in the mathematics classroom through the design of activities that promote a learning process based on the manipulation of physical objects, the production of representations, and the processing of devices for the efficient use of such representations in order that pupils are able to solve problems and problem situations. Furthermore, the aim was to ensure that:

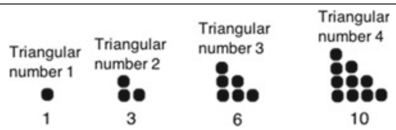
1. individually, the pupil begins, as a result of the preparatory work undertaken in relation to the mathematical activity, to attack the same activity from a socio-cultural perspective using teamwork.
2. by comparing their results with other pupils (in teams of two or three), the pupil possibly creates socio-cognitive conflicts involving productive arguments, with action and communication linked through objectives that they have to follow.
3. the plenary discussion furthers productive arguments, as well as anticipatory processes, the promotion of reconciliation among representations, validation processes, the production of counter-examples and the ability to check hypotheses. Once again, action and communication go together.
4. self-reflection promotes the strengthening of knowledge in order to stabilize it, with historicity (that which was undertaken collaboratively as an essential element of the process of reconstruction) a main component of reconstruction.
5. the process of institutionalisation enables the review of that which has been undertaken by pupils in order to promote the official representations and communication that will further advance their mathematical knowledge.

Considering these characteristics, the design of the activities used in this study begins with a presentation page (the front page). General pupil information is obtained in order to identify their work on an individual basis, as well as their results from the teamwork activity. It can be useful to include, on this page, instructions for the use of different colour inks when working either individually or with others in order to identify any development or evolution.

During the first stage, the mathematical task begins with the promotion of diversified thinking and, therefore, requires an open problem or problem situation. The statement outlining the activity will promote the production of functional representations that will trigger the production of spontaneous representations. This study proposes a block of five questions which allows pupils to individually create their own strategies (for a full outline of the experiment, see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA"). The task design depends on the use of artifacts in the construction of knowledge. For example, in Hitt and Gonzalez-Martin (ibid.), pupils used a rope, flexible wire and a rule, as well as paper and pencil, when attacking the mathematical task. Another example, as seen in Hitt and Kieran (Idem), sees the first stage designed to generate a strategy for a paper and pencil environment. This was then confronted with a second stage that featured the pupils' own algebraic productions as well as those provided by a CAS calculator, thus requiring them to reconcile their own productions with those produced technologically, as well as requiring team discussion. A third stage is related to the promotion of a specific conjecture and the need to convince others, with proof not taught at this educational level.

This study aimed to explore this approach with pupils who are beginning secondary school and, as such, are yet to be introduced to algebra, with the design intended to promote the construction of the concept of a variable through a process of collaborative learning under a sociocultural approach (see chapters "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" and "Problems Promoting the Devolution of the Process of Mathematisation: An Example in Number Theory and a Realistic Fiction"). In fact, researchers in this study considered it necessary to construct an algebraic-geometric-arithmetic thinking before developing an "exclusively" algebraic thinking detached from arithmetic itself. As such, the design of this experiment took into account Healy and Sutherland's (1990) work, who followed an Excel-based approach to polygonal numbers as well as Hitt's (1994) paper and pencil model which also used an applet that exclusively generated the value of any polygonal number. This experimentation also was implemented with a Mexican population in order to generate a comparison with the type of strategies used by those pupils who have already taken an algebra course (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" for details).

A first block was thus designed to promote visualisation, abstraction and generalisation processes from a perspective that seeks to create diversified thinking (see Fig. 4).



1) Look carefully at these numbers. What is the fifth triangular number? Make a representation. Explain how you proceeded.
2) In your opinion, how are the triangular numbers constructed? What do you observe?
3) What is the 11th triangular number? Explain how you find it.
4) You have to write a SHORT email to a friend describing how to calculate the triangular number 83. Describe what you would write. YOU DO NOT HAVE TO DO THE CALCULATIONS!
5) How do you calculate any triangular number (we still want a SHORT message here).

Fig. 4 First task design block for the generation of diversified thinking and spontaneous representations

Develops the same ideas as in the previous section but using Excel (or CAS).
Here we ask you to find:

	A	B	C	D	E	F	G
1	Nombres polygonaux						
2	Position	1	2	3	4	5	
3	Triangulaire	1					
4							
5							

What would you do to discover the 6^{th}, 7^{th}, and 8^{th} triangular number?
Is it possible to calculate the triangular number 30, triangular 83, and triangular 120?
How do you do this?
What are the limitations and possibilities of this approach?
Provide the operations to be performed in order to undertake this calculation with any polygonal number.

Fig. 5 Second part of the task

It is expected that spontaneous representations and personal strategies make their appearance during this first stage. Based on the same questions, it is expected that pupils will work in teams before engaging in plenary discussion.

In the example considered here, teamwork is required in the second block of questions. The aim is to promote in pupils the ability to generate the iteration processes related to a spreadsheet environment (Excel or CAS), similar to that obtained in Healy and Sutherland (1990).

As we can see in the two blocks of questions (see Fig. 5), the pupils generate different types of strategies. It is intended that pupils acquire a broad vision of how to address a problem situation and the various products linked to different strategies in order to promote different kinds of representations.

A comparison was sought between the strategies used in Healy and Sutherland (Excel and secondary school pupils) and Hitt (1994), which involved a group of secondary and primary teachers using Excel, and another group of teachers using paper and pencil and an applet. Generally, there are several kinds of generalisations used to calculate a triangular number:

- trig. $\Delta n = $ na before + position (Healy and Sutherland 1990),
- $Tri(n) = 1 + 2 + 3 + \ldots + n$ (Hitt 1994),

It is noteworthy that the task generates the production of different types of representations, with the type depending on the technological environment. This is the case with pupil production in this new approach to the construction of polygonal numbers (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA").

In the third block of the task, interest focused on the use of an applet that gives pupils the opportunity to immediately verify their generalisation strategies, or to request a polygonal number, etc. Thus pupils are able to receive immediate

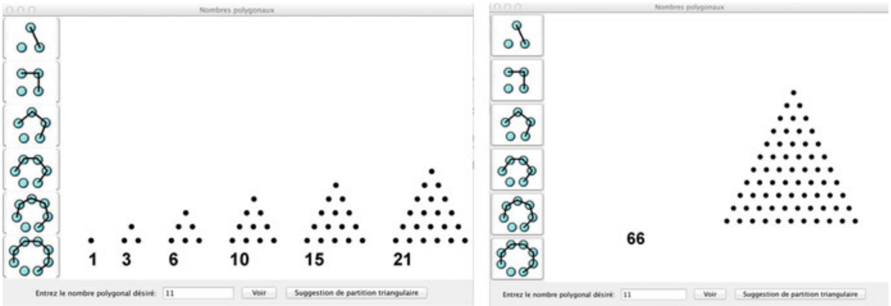


Fig. 6 Examples of the use of the POLY applet with polygonal numbers

a) Here are the five first triangular numbers.

Find a formula to calculate the numerical value of any triangular number. You can use the POLY applet to find the formula.

APPROACH (OPERATIONS, DRAWINGS…)

Write the rule or formula you found:

Using your rule or formula, calculate the following triangular numbers.

Position	Corresponding value
Triangular 10	
Triangular 20	

With the formula, can you calculate the triangular number 120?

Triangular 120 = ______________

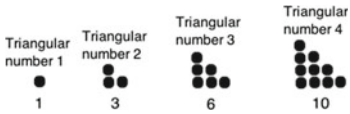
Fig. 7 Third block of questions and use of the POLY applet

feedback on the veracity of their conjecture using the applet. The applet (see Cortés and Hitt 2012) is to be used precisely in this 3rd block.

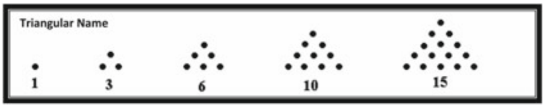
The applet is able to request the first four polygonal numbers selected (triangular, square, pentagonal, etc.) and is also able to request a "large polygonal number" (see Fig. 6). Paper and pencil work with the use of the applet allows pupils to check their guesses. If the pupil's conjecture does not agree with the result given by the applet, the pupil must return to their team and review the process that led to the construction of their conjecture, which, thus, fosters productive communication among pupils.

Pupils are asked to use the Poly applet for the proceeding set of questions in which the arrangement of the triangular numbers was changed, using an equilateral triangle (which corresponds to the institutional representations that pupils usually encounter in textbooks) (Fig. 7).

Here there are the first four triangular numbers



1) What is the 11th triangular number? Explain how you found it.
2) Write a SHORT email to a friend describing how to calculate the triangular numbers 30, 83 and 120. Describe what you would write. YOU DO NOT HAVE TO DO THE CALCULATIONS!
3) How do you calculate any triangular number (we still want a SHORT message here).
4) The following configuration of a triangular number can be found in some textbooks:



Does your strategy always enable you to calculate any triangular number?

Fig. 8 Task designed for the self-reflection stage (reconstruction activity)

From a psychological point of view, the framing of the triangular numbers, which does not leave enough space after the first 5 examples, promotes a tendency to abandon the drawings (see Hitt 1994), while the presentation of the activity promotes the generalisation process.

Building on strategies produced by Hitt (idem) has lead to the following output (it is important to stress that this study is carried out with primary and secondary school teachers and focuses on pupil performance, with chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" discussing secondary school pupils):

- $f(x) = \frac{x^2+x}{2}$ (Hitt 1994)

A summary questionnaire, which does not include the use of technology, is used for the self-reflection stage. Pupils are expected to be able to rebuild their representations, as well as any algebraic expressions that they have produced, thus enabling them to calculate any triangular number (Fig. 8).

As stated above, the reconstruction stage is very important. Research results (Karsenty 2003; Thompson 2002; Hitt and González-Martín 2015) show the fragility of knowledge and the importance of implementing, in the mathematics classroom, activities that can strengthen the construction of such knowledge.

In the case of pupils who are beginning to study algebra, validation can be restricted, while, in the case of the use of the task with pre-university students and/or future teachers, one can request demonstrations using mathematical induction processes. For example, the applet does not work when using large numbers. Furthermore, working with both the official representations of the polygonal

Table 1 Generalisation for calculating any n polygonal number of p sides

Calculation of polygonal n	Expression for generalisation
$T_n = \frac{n(n+1)}{2}$	$T(3 - sides)_n = \frac{n(n-1)}{2}$
$C_n = n^2$	$C(4 - sides)_n = n^2 = \frac{2n(n+0)}{2} = \frac{n(2n+0)}{2}$
$P_n = \frac{n(3n-1)}{2}$	$P(5 - sides)_n = \frac{n(3n-1)}{2}$
$H_n = n^2 + n(n-1)$	$H(6 - sides)_n = \frac{2n(2n-1)}{2} = \frac{n(4n-2)}{2}$
$E_n = \frac{n(5n-3)}{2}$	$E(7 - sides)_n = \frac{n(5n-3)}{2}$
...	...
	$Polygonal(p - sides)_n = \frac{n((p-2)n-(p-4))}{2}$

numbers and the construction of algebraic expressions associated with those numbers, another block of questions could be added. These would request a further and higher generalisation process (see Table 1), which would be built as a single algebraic expression that enables any polygonal number to be calculated.

Conclusion

This paper proposes task design elements to be developed in the mathematics classroom under a sociocultural approach. While some authors point out the importance of creating sociocognitive conflicts in the mathematics classroom, they suggest an organisational schema for performing an activity, with, for example, Prusak et al. (2013) proposing the following for a 75-min class:

> For the first 15–20 minutes, the instructor facilitated a whole class discussion to create a shared understanding of the activity; then, for approximately 5 minutes, each student engaged in the task individually; during the following 45 minutes, students worked in dyads or triads, solving tasks collaboratively and writing a common justification on a worksheet; for the final 5–10 minutes, there was a plenary, where the instructor led a whole class discussion to summarise. (p. 270)

In contrast to the methodological approach outlined above, the methodological approach advocated here takes into account the fundamental point that *consensus is ephemeral* and, as such, it is therefore necessary to consider a knowledge reconstruction stage (referred to as self-reflection in this methodology) in order to strengthen and stabilize knowledge (Karsenty 2003; Thompson 2002; Hitt and González-Martín 2015).

This task design is more related to problem situations that generate diversified thinking and, as a possible consequence, socio-cognitive conflicts in a process of action and communication. To overcome a socio-cognitive conflict, the authors of this study suggest the promotion of signification processes, as described by Radford (2003), in the mathematical classroom (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" on this issue). Consequently, some of our problem situations may take more than one session of a course. In fact, the task design in Hitt and Gonzalez-Martin (ibid.) aimed to create a chain of activities that encompassed the concept of covariation between variables, function in context, and mathematical modelling, over the course of 13 class sessions.

Practice has shown that, as a method such as ACODESA is not easy to implement in the mathematics classroom, it is very important that, working together, researchers and teachers can create learning situations such as those suggested in this chapter for the mathematics teacher. Generally, it is not possible to fully present in research articles the complete activity implemented in an experiment, due to a lack of space. The problem situations dealt with here usually occupy several pages permitting regulate, in some extent, pupils' productions and promoting their evolution.

References

Arcavi, A., & Hadas, N. (2000). Computer mediated learning: An example of an approach. *International Journal of Computers for Mathematical Learning, 5*(1), 25–45.

Bairral, M., Arzarello, F., & Assis, A. (in press). Domains of manipulation in touchscreen devices and some didactic, cognitive and epistemological implications for improving geometric thinking. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), Mathematics and technology: A C.I. E.A.E.M. Sourcebook. Cham: Springer.

Blum, W., Galbraith, P., Henn, H., & Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education*. The 14th ICMI study. New York: Springer.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en Didactique des Mathématiques, 4*(2), 164–198.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. 1970–1990. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds. & Trans.). Dordrecht: Kluwer.

Brownell, W.-A. (1942). Problem solving. In N. B. Henry (Ed.), *The psychology of learning* (41st Yearbook of the National Society for the Study of Education. Part 2, pp. 415–443). Chicago: University of Chicago Press.

Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Belkapp Press.

Cortés, C., & Hitt, F. (2012). *POLY: Applet pour la construction des nombres polygonaux*. Morelia: UMSNH.

Dreyfus, T., & Hadas, N. (1996). Proof as answer to the question why. *International Review on Mathematical Education, 96*(1), 1–5.

Duval, R. (1995). *Sémiosis et pensée humaine: Registres sémiotiques et apprentissage intellectuels*. Neuchâtel: Peter Lang.

Duval, R. (2002). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 311–336). México: PME-NA and Cinvestav-IPN.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen, & R. L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19–38). Cambridge: Cambridge University Press.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111–129.

Healy, L., & Sutherland, R. (1990). The use of spreadsheets within the mathematics classroom. *International Journal of Mathematics Education in Science and Technology, 21*(6), 847–862.
Hitt, F. (1994). Visualization, anchorage, availability and natural image: Polygonal numbers in computer environments. *International Journal of Mathematics Education in Science and Technology, 25*(3), 447–455.
Hitt, F. (2007). Utilisation de calculatrices symboliques dans le cadre d'une méthode d'apprentissage collaboratif, de débat scientifique et d'auto-réflexion. In M. Baron, D. Guin, & L. Trouche (Éds.), Environnements informatisés et ressources numériques pour l'apprentissage. conception et usages, regards croisés (pp. 65–88). Paris: Hermès.
Hitt, F. (2013). Théorie de l'activité, interactionnisme et socioconstructivisme. Quel cadre théorique autour des représentations dans la construction des connaissances mathématiques ? *Annales de Didactique et de Sciences Cognitives, 18*, 9–27.
Hitt, F., & González-Martín, A. S. (2015). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflexion) method. *Educational Studies in Mathematics, 88*(2), 201–219.
Hitt, F., & Kieran, C. (2009). Constructing knowledge via a peer interaction in a CAS environment with tasks designed from a Task-Technique-Theory perspective. *International Journal of Computers for Mathematical Learning, 14*(2), 121–152.
Hitt, F., Cortés, C., & Saboya, M. (2015). Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and technology: A CIEAEM sourcebook*. Cham: Springer.
Hoyles, C. (1988). *Girls and computers*. London: University of London.
Karsenty, R. (2003). What adults remember from their high school mathematics? The case of linear functions. *Educational Studies in Mathematics, 51*(1), 117–144.
Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching matematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–16). Hillsdale: Lawrence Erlbaum.
Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI Study* (pp. 127–135). Dordrecht: Kluwer.
Leontiev, A. (1978). *Activity, counciousness, and personality*. Englewood Cliffs: Prentice Hall.
Lesh, R., & Doerr, H. (2003). *Beyond constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Hillsdale: Lawrence Erlbaum.
Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Greenwich: Information Age.
Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: Author.
Prusak, N., Hershkowits, R., & Schwarz, B. (2013). Conceptual learning in a principled design problem solving environment. *Research in Mathematics Education, 15*(3), 266–285.
Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37–70.
Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic.
Sela, H., & Zaslavsky, O. (2007). Resolving cognitive conflict with peers. Is there a difference between two and four ? In *Proceedings of PME 31, 4, 169–176, 8th–13th July, 2007*. Seoul: PME.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.
Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104.

Thompson, P. (2002). Some remarks on conventions and representations. In F. Hitt (Ed.), *Mathematics visualisation and representations* (pp. 199–206). Mexico: PME-NA and Cinvestav-IPN.

Varela, F.-J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge: MIT Press.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathé matiques, 10*(23), 133–170.