e-Mathematics Engineering for Effective Learning

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Abstract Education in digital age requires a strong focus on "engineering of learning", as an emergent field of study and design of effective innovative learning experiences and environments. In this respect, the chapter addresses mathematics education in e-learning settings according to a tetrahedron model, as extension of the classical didactical triangle, to which adds a fourth vertex: the 'author' (A). The introduction of the vertex Author is due to our view that full exploitation of e-environment and its integration with results from research in mathematics education requires properly designed didactical interventions, based on a scientific approach, such as Didactic Engineering. Moreover, such exploitation should consider the centrality of the student, which means that the vertex-positions of the tetrahedron can be assumed also by the student along the learning process. We discuss the didactic engineering work from the perspectives of the tetrahedron faces and taking into account the dynamicity of the vertex-positions.

Keywords e-learning • Didactic engineering • Didactical tetrahedron • Mathematics education

Introduction

This paper concerns the learning/teaching process in the context of e-learning. Many definitions have been given to the term 'e-learning' and sometimes it has been used as synonymous to 'online learning' or 'distance learning' or 'web-based learning'. Lately 'e-learning' has also been referred to 'communication' and 'connectivity' issues. Thus, first of all, we want to make the boundaries of our context precise. Then, we will draw our attention to the teaching/learning process which occurs in a technology-enhanced environment consisting in an online teaching platform added with facilities of the Web2.0 and (eventually) with online mathematical software. From now on, we will refer to it as 'e-environment', and in general, we will put the prefix 'e-' to a word meaning that such word refers to

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G. Aldon et al. (eds.), *Mathematics and Technology*, Advances in Mathematics Education, DOI 10.1007/978-3-319-51380-5_16

e-environment (for instance, ‘e-tool’ is a tool included in the e-environment). So, in this paper, e-learning means teaching/learning process in an e-environment.

Looking at the experiences in e-learning applied to various domain, including mathematics (Borba and Llinares 2012; Garcia Peñalvo 2008; Juan et al. 2012a, b), we can individuate two main strands. One is guided by an exploration of the teaching and learning opportunities offered by the e-learning platform, which has characterized the early studies, where great attention has been devoted to resources (namely learning objects) that exploited the interactive features of the e-tools. The other one has drawn attention to learning activities that integrated the delivery of some resources within a more complex structure of engagement of the students by means of cooperative and collaborative features of the platforms such as forum and wiki.

What we observe in any case is that the researchers have devoted great effort to the design of the resources or learning experiences in a “Design Research” (Kelly et al. 2008) view. This means that they move along a path consisting in planning a set of resources or activities, experiment their potential in order to support learning and analyse the data that can give rise to theories.

We suppose that such approach has been greatly affected by the focus on the technologies rather than on the learning process. For some part, it is due to historical and social constraints since e-tools came before than their didactic consideration and conveyed certain – perhaps unconscious – traditionalistic view of teaching of their high-tech designers (Chevallard and Ladage 2008):

> Much to the contrary, a sound view of didactic engineering calls for the reverse: didactic functions, not structures, must be considered first. (p. 168)

Thus we are firmly convinced that now we need a research methodology allowing us to integrate results from the mathematics education research into learning process in the new global environment empowered by technologies (in their continual and fast change) and to validate the design. The use of a methodology should yield robust teaching/learning proposals (also in terms of strategies) that can be shared and reproduced in the teaching/learning community. In this respect, we claim that *didactic engineering* could be a good research and development tool.

The paper has been structured as follows. After a brief presentation of the Didactic Engineering approach and its relation with the Design Research, we discuss a tetrahedron model of the teaching/learning process in e-environment, that add to the classical vertices a new one, the Author. Then we discuss how each face of the tetrahedron is related to phases of the didactic engineering approach and which the role of the Author is.

Didactic Engineering and Design Research

In the recent years a substantial debate on how the two approaches of "Didactic Engineering" (DE) and "Design Research" (DR) relates each another is going on (Artigue 2015; Godino et al. 2013; Margolinas and Drijvers 2015).

Didactic Engineering (Artigue 1992, 1994, 2009) emerged in the France context of mathematics education in the early eighties to indicate an approach to didactical work in mathematics similar to the engineer's work. This implies that mathematics didacticians should act like engineers, who in order to realise a fixed (learning) project rely on the (disciplinary and didactical) scientific knowledge of the domain, accepting to be submitted to a scientific testing. At the same time they are forced to manage more complex objects than those of science and to face problems not yet tackled by science.

It is worthwhile to recall that Didactic Engineering has become a polysemic notion, that can be development-oriented or research-oriented as pointed out by Artigue (1994):

> designating both productions for teaching derived or based on research and a specific research methodology based on classroom experimentations. (p. 30)

Thus it is conceived as a practice of controlled theory-based intervention, consisting in the design of teaching sequences, the setting up of didactical tools and resources organised and structured in a period of time coherently to the aims of reaching fixed learning objectives, their monitoring and evaluation. The results allow to test theory-based hypotheses and to produce teaching resources scientifically validated.

It is also referred to as a research methodology of qualitative type, in which an essential role is played by the "didactic realisations in classroom" as investigation practises of the elaborate theoretical hypotheses.

At the beginning the base-theory for Didactic Engineering was the Theory of Didactic Situations (TDS) (Brousseau 1997), which contributed to shape the structure of the design (methodology), consisting in the following phases: (1) preliminary phase consisting in epistemological, cognitive and didactical analysis of the mathematical knowledge to be taught; (2) design and *a priori* analysis of the teaching/learning situations; (3) implementation and experimentation; (4) *a posteriori* analysis and validation.

Comparing Didactic Engineering with Design Research, Godino et al. (2013) argued that what mainly distinguishes them is that Didactic Engineering is grounded on a theory (such as Theory of Didactic Situations) whilst Design Research does not. In Didactic Engineering the underlying theory guides the *a priori* analysis and the consequent design and the expected results, whilst in Design Research the design is supported by various interpretive frameworks and theories emerge from the data. This latter thing justifies the foreseen internal validation step in Didactic Engineering, whilst Design Research does not. Anyway, Didactic Engineering and Design Research face similar paradigmatic questions, concerning the improvement of learning in context by instructional design intertwined with educational research. Godino and colleagues conclude that Didactic Engineering can be seen an instantiation of Design Research. Changing the base-theory, they envisage various Didactic Engineering differing in didactic design. Thus they introduce a new notation, as reported in the following:

> We introduce the notation DE (TSD) to indicate the dependence of the “Didactic engineering” from the Theory of Didactic Situations. This will help to express possible generalizations of didactic engineering changing the base-theory used to support the instructional design. (p. 6)

So we can have DE (ATD), DE (SM), DE (IA) according respectively to the Anthropological Theory of Didactic (Chevallard 1992, 2006), the Theory of Semiotic Mediation (Bartolini and Mariotti 2008), the Instrumental Approach (Drijvers et al. 2010a; Drijvers and Trouche 2008; Trouche 2004), and so many others. We also can have a DE based on a network of theories (Bikner-Ahsbahs 2010), that can provide complementary insights and, hence, deepens analysis and understanding of some didactical phenomena.

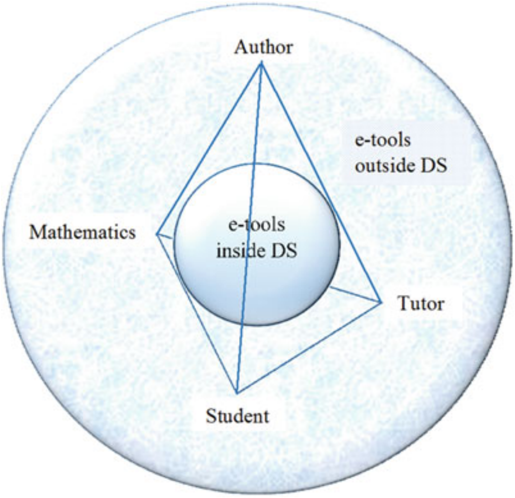
Considering a didactic engineering approach according to various based-theories seems us a very interesting idea because it allows researchers to define a unified methodology to construct a wide range of proposals for learning experiences with two rewards in our view. On one hand, they are designed in the frame of a reference theory, and tested and validated according to scientific criteria. On the other hand, they do not assume a one-sided insight of learning, without referring to one learning theory only, so keeping away from the original drawback of the use of e-learning environments.

The Tetrahedron Model

Following Albano et al. (2013), we assume that the *didactic system* (DS) concerning the teaching/learning process in e-environments can be modelled in a systemic way by a tetrahedron (see Fig. 1). It includes the main classical entities: some mathematical knowledge, that is Mathematics (M); someone who is expected to learn M, that is the Student (S); someone who is supposed to officially help S to learn M, that is the Tutor (T). A further character has been introduced, the Author (A), consisting in a team who is in charge of planning, developing and managing the *didactic organization*.

The addition of the new vertex A has created three new triangular faces, besides the classical didactic triangle. The function of the tetrahedron is not explicative or descriptive of educational experience, but mainly methodological: each vertex of the system can be used as the observer’s position that looks at the inter-relationships within the face generated by the other three vertices, though none of the elements involved can be completely separated from the others.

Fig. 1 The tetrahedron model



We want to underline that technology has brought the Student to have a central role in the learning process and to look for a larger topos, not only restricted to someone who learn, but enlarged as someone who can be engaged in other didactic functions (Chevallard and Ladage 2008). This means, in our systemic view, that the vertices of the tetrahedron are not static figures, but we consider them just as *position*, that is as a system's element (for deepening discussion on teacher's position, see chapter "The Professional Development of Mathematics Teachers: Generality and Specificity" this volume), that can be played also by Student in some situations along the learning process. Transforming the didactic triangle into a didactic tetrahedron through the addition of the Author lets more evident the crucial role of the students, since looking at the tetrahedron faces allows to image, design, observe learning situations where students can play a position different from S. For instance, a student can be in the position Author when she creates resources that modify the *milieu* used to reach the knowledge M. Analogously, a student is in the position Tutor when she is expected or required to officially support some other students, that is she is no more in a symmetric relationship among peers but she assumes an asymmetric role.

The introduction of the vertex Author is due to our view that full exploitation of e-environment and its integration with results from research in mathematics education require properly designed didactical interventions, based on a scientific approach both in terms of *didactical transposition* (Chevallard 1989) and in terms of *didactic engineering*. The vertex Author has a key role in the design and in the validation of the teaching/learning resources and situations, with both contents and activities empowered by technology features. Differing from the classical teaching, the didactic engineering should take into account the chance for students to move in other vertex-positions of the tetrahedron. This adds a dimension of dynamicity where learning does not come out from the fruition of a ready-made product but it is the outcome of a construction where students have chance to carry out didactic functions, suitably supported, guided and supervised, impacting on the didactic organization itself.

Whilst Author, although is a team including in herself various competencies, from pedagogical, to disciplinary, to technological ones, can be thought as a collective subject, that is a single entity, we can think at Student and at Tutor both as a single entity and as a community. The first one is a community of "peers" who are engaged in learning. The second one refers to all the persons dedicated to facilitate student learning – in face-to-face or online settings, from the traditional teacher and any other educational figure (coach, advisor, and so on). We refer to them as singles or as community depending on the teaching strategy adopted, such as individual learning versus group learning, or on the size of the learning group to be supported which can require more than one tutor. Moreover, if Author and Tutor are not the same person, they should anyway intensely interact to succeed in their roles.

In our approach, technology is both outside and inside the teaching/learning system, thus we represent it as a couple of spheres: one outside and one inside.

The external sphere refers to the immersion of humans in a technology-enriched world. It contains all kind of e-tools (for a flavour, see the list presented in Chevallard and Ladage 2008), which permeate our life in all its aspects and our identity too (some years ago, I was astonished when a friend said to me: "*you do not exist if you do not exist on Facebook*"). There is no doubt that nowadays we cannot conceive to being human without being connected to and interacting with the global virtual word created by the web, as noted in Borba (2012).

Although technology is everywhere and affects our life with respect to what we are and what we do, we want to distinguish the intention of use of some technologies, related to the *didactic system*, represented by the tetrahedron.

Thus we introduce an internal sphere. It refers to the use of particular e-tools with an explicit *didactic intent*, that means that they are exploited with the intention to teach in a specific educational context (Chevallard 1989). It is worthwhile to note that the e-tools have not been invented for educational purposes but, more probably, for business, and then their meaningful didactic use is not granted, but it has to be attained (Borba et al. 2013; Chevallard and Ladage 2008).

According to Arzarello et al. (2012), now we assist to a shift from visible to invisible technologies, putting their add-value in their particular use with respect to learning purposes.

This is why we do not consider technology as a new vertex in the model, as others in literature do (Rezat and Sträßer 2012; Ruthven 2012; Tchoshanov 2013), but as a sphere which is into the background.

Due to nature itself of Author, we can assume a certain use of means from the external sphere in order to support group work. It is quite natural that their use comes out spontaneously (for instance, think of the use of Dropbox or GoogleDocs for constructing shared documents). Some kind of natural interactions within the external sphere can occur inside the communities of Student and Tutor (for instance, the use of Facebook group as a notice board in order to share and keep up-to-date on organizational information about a course). According to the tetrahedron model, when interactions inside Student or between Student and Tutor are guided by didactic strategies they will be frame in the internal sphere.

The tangent point of internal sphere with the tetrahedron addresses the add-value that technologies can give to the relations among the actors of the face in the learning process, which, according to Arzarello et al. (2012), is:

> based on the methodology, or in their use as a tool to support well-defined teaching strategies, focused on the learning process of students or teachers engaged in professional training. (p. 3)

In this respect, we are firmly convinced that it is more and more appropriate the vision of mathematics education as a ‘design science’ (Wittmann 1995), which should combine and relate to one another the design of “artificial objects” – moving from teaching units (ibid, p. 362) to e-activities when they make use of e-environment – and empirical research. This science should produce proposals for the classroom, re-usable in other situations, not only test them (Lesh and Sriraman 2010) and this should hold for online classroom in e-environments (Borba 2012).

An Overview of the Tetrahedron Vertices

In this section, we are going to define the new vertex and to specify some characteristics of the other vertices that differ from the actors in the traditional didactic system.

The Author is a collective subject with different professional skills, including educational expert in relation to the knowledge domain, instructional design and management, ICT and pedagogical/sociological expertise. This subject acts as “scriptwriter” of didactical experiences mediated by technology.

The importance of teamwork inside the collective author is noted by Borba (2013), which identify as weakness of Internet the low design and pedagogical quality of online interactive mathematics contents. This can be avoided by the simultaneous work of a variety of experts, such as mathematics educators and human-computer designers, who can take into account and integrate both didactic objectives and interface design principles. Also Schoenfeld (2009) hopes for a synergy between educational researchers and educational designers. The richness of figure Author allows to create a variegated scenario of pedagogical expectations concerning knowledge, of professional or ideological beliefs, of implicit philosophies that supply an enrichment of the e-environment and carry out robust and well-engineered products to made available to the targeted learners. We consider that the comparison, the discussion, the thoughts that can occur among the different experts above, with assorted expertise and experiences, affect the decisions about resources and methodologies to be implemented with respect to a fixed didactic goal, and achieve in such way the construction of a rich and deep product.

Besides Author, the didactic system should include someone who teaches. We prefer to use the word "tutor" instead of "teacher" for making evident the change of the role of the teacher in an e-environment. She is the person who supports student learning and facilitates interaction between learners, and sometimes completely disappears (Arzarello et al. 2012) in the so-called self-directed learning or learning without teaching. In the framework of instrumental orchestration, we can say that the tutor is also the one in charge of the *didactical performance* (Drijvers et al. 2010a):

> A *didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals. (p. 215)

Conversely, the didactic system should include someone who learns: the Student, who also refers to those involved in continuing education program. What seems as important is the 'active' role she has in an e-environment. On the one hand, we have the shift from the centrality of the teacher to the centrality of the learner, which has in her hands the control of the learning process (Arzarello et al. 2012). On the other hand, learners change their role, becoming active in constructing knowledge too (Jahnke et al. 2014).

Finally, all the previous figures move around the element that gives reason of the didactic system, that is Mathematics Knowledge. Although Mathematics mainly refers to the academic mathematical knowledge, resulting from the scientific research, with its specific structural, methodological, historical and cultural characteristics, we cannot ignore that its didactic transposition is no more under the umbrella of a certain certification through official academic or school channels. With the advent of Web2.0 and of the communication infrastructures growth, it has become prerogative of anybody due to the availability of authoring and sharing tools (Borba et al. 2013).

An Overview of the Relations Inside the Tetrahedron

As seen in Albano et al. (2013), despite the fact that the didactical tetrahedron represents a whole, looking at each face allows us to consider the interactions of three characteristics at once and to think of a range of configurations of the teaching/learning process in e-environment.

The Author-Mathematics-Student face refers to a configuration where learning comes out from the interactions of the learner with the mathematics knowledge, which has been transposed in some digital resources by the Author or that she herself can author. No support of any Tutor is planned. Thus this face may refer to a configuration of self-directed learning.

This requires the learner to have an attitude of personal responsibility of her learning and a good level of the self-regulation competency. She is supposed to be able to control her own process of acquisition, elaboration and exploitation of her knowledge in relation to the learning goals, self-posed, to the awareness of her knowings, to the attitude towards failure in the past and expectations for the future (Schunk 1990; Zimmerman 1990). It is worthwhile to note that the lack of such competency, in terms of bad management of personal knowings and negative attitudes, seems the key of repeated unsuccessful behaviours (Zan 2000).

Looking at Author-Mathematics-Student does not mean that the whole learning process we are dealing with is in the previous configuration. More generally, it can also refer to the possibility for the learner of autonomously and freely moving, of choosing, designing and managing her learning in the e-environment. Self-regulation capability requires that Student should be free to construct dynamically her own learning path. In this view, the role of the Author is not just who edits some digital resources, but she becomes an arranger of technology-enhanced contexts where the resources can foster the achievement of a precise purpose.

Besides, Student should be also free to construct her own resources, realizing her own transposition and being herself part of the Author. Students do not restrict themselves to receive and elaborate objects (such as in the case of the book), but they can produce new ones from scratch or starting from those available. For instance, students can rewrite learning objects, traditionally viewed as read-only, by adding personal annotations (not only text-based) and share the so constructed new objects with peers (Borba et al. 2013). Further examples can be found in the frame of the narrative approach to mathematics learning, based on the educational use of story problems (Zan 2011) and of digital storytelling (Robin 2008): students are involved in the creation of digital story problems about real-life situations whose solution needs to apply some mathematics, giving rise to complex interactive resources (Gould and Schmidt 2010). The resources created by the students can be validated by the tutor or not, before being shared, according to the settings of the context in which they arise.

The Mathematics-Student-Tutor face refers to a learning configuration that requires the Tutor's interventions so that learning comes out by the interactions among students, tutor and mathematics. Interactions can occur between tutor and student or group of students, or among students, or among students and resources and e-tools. The tutor's mediation depends on the base-theory that has guided the design of the learning activity, and in an individual or cooperative/collaborative setting. Due to the availability of technological tools allowing online monitoring and recording of the learner's work, the tutor can support students' interaction by means of the *Spot-and-show* orchestration. She can access student work, identify interesting pieces and deliberately use them to set up students' interactions – for instance, ask for explaining the reasoning or providing reactions or feedback on the work (Drijvers et al. 2010a).

In any case, we can say that this face is characterized by the communication and we can frame learning in the so called *discoursive* approach (Sfard 2001).

The features of the new communication means and the various contexts in which mathematical communication occur highlight the importance of some characteristics of the mathematical discourse: not only multisemioticity, but also multimodality and multivariety, which are crucial for developing competence in mathematics (Albano and Ferrari 2013; Arzarello 2006; Duval 2006, Way 2014).

The Author-Mathematics-Tutor face refers to the relation of the author and the tutor with mathematics, consisting in the didactic transposition, that is in the design of the learning situations in the e-environment. Here the Author focuses her efforts on the configuration and exploitation of suitable e-tools and on the analysis, the design and the implementation of educational resources and activities (with or without tutor). It is well known that face-to-face teaching methods cannot be simply 'transferred' in e-environment, but they need to be revised and modified in order to integrate technology and pedagogy and take really advantage of the e-tools (Kahiigi et al. 2008). Moreover, we should observe that even problems trivial in paper and pencil environment can assume new light in e-environment, because of e-tools constraints (Borba 2012).

Didactic transposition should also take into account the shift from text-based communication to multimodal communication. Mathematical resources should combine various semiotic systems (text, graphs and figures, symbols) but also interactivity, animation, videos because it is what young people experience of in their ordinary life and thus it is what they expect in learning context (Borba et al. 2013).

The interactions between the author and the tutor is very important since the tutor can report her own feedback of usage of the didactical proposals. Such feedbacks contribute to a continuous re-design in order to make the proposals able to produce the desired learning. This means that the educational resources and activities are not static objects but they evolve according to the practices' feedback.

Finally, it is important to point out that the didactical proposals, related to certain goals, should be as many and various as possible, according to various approaches and methods, in order to support various students' needs, profiles and preferences, which in the constructivist perspective is viewed being the driver of mathematical learning (Balacheff and Sutherland 1999).

The Author-Student-Tutor face mainly refers to the relations of Tutor with the digital resources authored by Author on one hand and with Student on the other hand. Here we can frame the *documentational genesis* (Gueudet and Trouche 2009): Tutor appropriates and reshapes resources initially made available by Author and builds schemes of their utilization, for a given class of situations, across a variety of contexts, giving rise to the so called *document*. Such product becomes a new resource and it can be involved in a further process of documentational genesis, producing new document. The tutor arranges the documents she generated.

Finally, we note that the relation of Author with Tutor and Student can allow to collect students' and tutors' feedbacks about the resources initially authored by Author and to adjust/refine them in profitable way.

Didactic Engineering Work According to the Tetrahedron Model

In this section, we analyse, according to the systemic view given by the tetrahedron model, the phases of DE (TDS), previously listed, that are: (1) epistemological, cognitive, didactical analysis of the mathematical knowledge to be taught, (2) design and *a priori* analysis of the teaching/learning situations, (3) implementation and experimentation, (4) *a posteriori* analysis and validation.

In particular, we try to exploit the methodological function of the systemic view given by the tetrahedron, looking at the didactic engineering work as we are observing it from each vertex and thus in terms of the opposite face and of the relations among the related elements. This means that in what follows, said X, Y and Z any triplet of vertices of the tetrahedron, we are guided by the question: *assuming the perspective of the face X-Y-Z, what contribution can be given to the didactic engineering work?*

It is worthwhile to remark that the relations and processes inside the face can be affected by the fact that the vertices are not static figures but are considered as positions that can be assumed also by the students. Therefore the didactic engineering work, which can be seen as linear (although back and forth paths can be foreseen among the design and the validation steps), benefits from the systemic tetrahedron model as, conversely to its origin, it is no more teacher's prerogative but it conveys the relations among various positions. Thinking at each of the didactic engineering phases from the various perspectives given by the different faces of the tetrahedron allows to actually put the student in the centre of the whole teaching/learning process, as she can participate in such engineering work.

Author-Mathematics-Tutor: Preparing Mathematics for Students

We consider this face as the most important for the development of teaching products, as the relations among the vertices evoke a process of preparation of mathematics for the students, meaning not only contents by also learning activities that allow the students to reach their own mathematical knowing (referring to an individual's knowledge).

Looking at DE, the first two phases consists in a conceptual work that guides hypotheses concerning student's learning and consequently choices on the design. This conceptual work is constantly referred along the whole engineering effort, in a back and forth path as long as the learning situation is implemented and experimented, in order to revise the choices made.

Generalizing to learning activities what Artigue (1994) says about the Didactic Engineering applied to the teaching contents, the DE work starts with a preliminary phase, which is charge of the Author. It consists in outlining the epistemology (ideas on knowledge and learning) of the “new” element of teaching/learning practice to be designed, starting from examining what already exists and its drawbacks.

Then *a priori* analysis along three dimensions should be carried out: the *epistemological*, related to the characteristics and the way of functioning of the mathematical knowledge; *cognitive*, related to the Student targeted; *didactical*, related to the specificity of the educational system where the teaching/learning process occurs.

The face Author-Mathematics-Tutor seems to contribute mainly in the first dimension, as it concerns mathematics per se. The Author, by means of the internal competence as domain expert, can analyse the feature of the knowledge at stake, going in more details on the specific subjects considered.

The interaction with the Tutor allows to give a certain contribute to the other dimensions too. In fact, as the cognitive dimension takes into account the features of the learner, the analysis can benefits from the teaching experience of the Tutor. She can devise to consider the characteristics of a *generic* learner to whom the knowledge at stake is foreseen to be taught. Analogously, the Tutor can contribute to didactical analysis a certain configuration of the educational system, according to her experience.

After the analysis is completed, the phase of design starts, guided by the outcomes of the previous conceptual work. Here, the DE work involves a certain number of choices, at two levels: macrodidactic or global one, affecting the whole design; microdidactic or local one, guiding the organization of a specific session of a learning activity.

According to Theory of Didactic Situations, the design should include the definition of a *milieu* and of a learning situation, as the ideal model of the system relations among Student, Tutor and the *milieu*, that we call *Learning Situation Model* (LSM). The milieu was originally defined as the set of anything is acting on the student or the student is acting on (Brousseau 1997). Its function appears extremely important as student’s knowing comes out as personal answers to the constraints of the milieu rather than to the teacher’s expectations. Therefore, it consists in a set of resources to be used in order to grasp the knowledge to be learnt.

From the technological point of view, the definition of the milieu requires on one hand the use of digital resource and on the other hand the configuration of the e-tools to be used, at least in terms of the features of such tools needed to the situation performs, producing such a way a particular instantiation of e-environment.

It is worthwhile to note that the characteristics of the digital resources, shaping mathematical contents, initially designed by Author, consists in their openness for re-design can be designed by Tutor (actually assuming the Author position), both individually and collectively (Pepin et al. 2014).

Generally speaking, we note two aspects of the e-tools. The first concerns the fact that the choice of the e-tools that students are required to use is not neutral with respect to envisaging the learning situations, as seen in Borba (2012). The second one affects the functioning of the e-tools inside the milieu: they can act both in antagonistic and in cooperative way (Drijvers et al. 2010b).

Author-Student-Tutor: Refining Mathematics for Students

Looking at DE work from the Author-Student-Tutor, generally we can say that it contributes wherever we refer to concrete cases and characteristics for each of the positions, not just hypothetical categories or models.

Therefore, at level of the *a priori analysis*, this face contributes to the cognitive and didactical dimensions. In fact, the Tutor and the Author can make more precise the analysis along these dimensions considering specific characteristics of the learners engaged and of the educational system in which the learners are framed (that can depend on social or demographic issues).

The reference to real cases allows the Author to make actual choices of specific e-tools (for instance, in case of a teaching platform, Moodle versus IWT). This leads the Learning Situation Model to be more concrete and thus to go towards an implementation of Learning Situation Model, that we call *Learning Situation Instantiation* (LMI), which will be actually experimented.

Actually, the Tutor and the Author realize an instantiation of the design made in Author-Mathematics-Tutor face, taking into account the target Student and the context where the Learning Situation Instantiation takes place. Hence, some choices previously outlined in the Learning Situation Model are made more precise and the e-environment is definitely set up, choosing the actual e-tools to be used and defining the modality of their use in order to obtain the features foreseen in the design. Analogously for other elements of the milieu, generated for instance by the didactical transposition: in order to set up the milieu, the Tutor can choose resources, previously designed on Author-Mathematics-Tutor face, and she can re-design them because of the customization to the needs of the Student. This way she continues the design phase.

From the technological viewpoint, the e-environment has made ready for the Learning Situation Instantiation, thus e-tools satisfying the Learning Situation Model requirements are set and arranged as well as the way of using concretely according to what defined in the Model.

Finally, we also note that, in DE view, this face realizes the back and forth process of continuous comparison of what is expected in the design and what actually occurs, allowing suitable reciprocal adjustments of Learning Situation Model and Instantiation. In fact, the Author, getting in touch with the Tutor and the Student, can collect feedbacks from both of them (also exploiting the technological empowerment) and refine the design as well as the implementation according to what emerges from the data analysis.

Author-Mathematics-Student: Mathematics Construction Without Official Help

Retracing the DE work from the perspective of this face put emphasis on the possible absence of the Tutor position in the learning process.

From the design point of view, it implies to devise, if the case, piece of the learning situation, for which no support from the tutor is expected. Knowings come from various kind of engagement of the Student with Mathematics: (a) Student can interact with the milieu, suitably set up by the Author, and Mathematics knowing comes out by the interaction; (b) Student can assume the Author position, authoring herself mathematical resources, both starting from scratch and modifying and personalizing resources already created by the Author; (c) Student can work together with peers in cooperative and collaborative activities occurring in the learning situation.

In all the previous cases, the absence of Tutor does not mean that Student does not receive any help, but that it can come implicitly from the engagement in the situation (i.e. the responses of the milieu, the feedback of peers, etc.).

Moving along the DE work, here we can observe the experimentation of the above cases and thus collect the data for the a posteriori analysis: in fact, the Author can benefits from direct or indirect observations (for instance, students' products/ protocols or log files available in the e-environment) of what happens between Student and Mathematics and can use these data for adjusting the design of the learning situation.

Mathematics-Student-Tutor: Mathematics Construction Officially Supported

Differing from the previous perspective, looking at DE work from this face emphasizes the a-symmetric relation Student-Tutor with respect to knowledge construction.

The design should take care of what concerns in particular the didactic part of the learning situation, where Mathematics knowing comes out from the interaction between the Students and the milieu, made viable by the Tutor. Thus, a troublesome issue of this face is the didactic contract, consisting in specific behaviour of the teacher expected from the learner and the behaviour of the learner expected from the teacher (Brousseau 1997). In e-environment, it can be affected by more than one variable. A key variable depends on the role of the Tutor, who can or cannot be the same person in charge of the Student's learning assessment (for instance, this is certainly the case of distance online courses). In the latter case, the traditional a-symmetric situation between teacher and learner is not so stressed and the *didactic contract* between Tutor and Student is remarkable modified because it is no more communicated by the assessment phase. Moreover, we highlight that the contemporary interaction of Student with the Tutor and with the digital resources of the milieu, by which the Author shaped Mathematics, can bring to appear two different didactic contracts, one between Student and the digital resources and one between Student and Tutor, which can conflict each other (Cazes et al. 2006).

A further key issue in this face concerns the dealings between the Tutor and the Student, at peer level too. In fact, we want to warn on a misconception that can arise from the familiarity of the digital natives with the social networks. One can be led to confuse their habit of continuous online social interactions with a natural disposal to forms of collaborative learning. Although there can be found some cases of positive correlation between collaborative affordances of e-environment and learning, some other studies show the importance of deepening the issue of collaboration, even in the phases of analysis and design (Borba et al. 2013).

Moving along the DE work, also in this case we can observe the experimentation and gather data for the a posteriori analysis, also exploiting the direct involvement of Tutor.

A Case Study

In this section we want to discuss a case study as an instantiation of the proposed theory. We have used the DE approach according to the tetrahedron model in a scholar experience that was piloted in Italy in blended courses at the Universities of Salerno and of Piemonte Orientale (Albano and Ferrari 2008).

As already noted, the tetrahedron model gives a systemic view; consequently the starting point of a teaching/learning process does not lie on a fixed face. In our case study, the insight for the experience we are going to describe come in Mathematics-Student-Tutor face of the tetrahedron. The position of Student was taken by the first-year engineering students attending a trimester intensive module in mathematics which concerns topics from linear algebra and calculus. Being in the position of Tutor, we should face the evidence of the Student belief that, especially in those contexts where mathematics has seen as a “service domain”, instrumental approach is enough and a more in-depth understanding was unnecessary, despite the failures at the exams. As the assessment focused on conceptual understanding, the same belief was exactly the reason why they failed. We decided to investigate more but, asking the students why they failed, someone said “because of strange and unexpected questions”, referring to traditional examination questions which aim to explore if the student has understood some theorem’s statement or proof. Examples can be questions like the following: “what theorem guarantees the validity of this passage?” or “what means this expression?” or “where did you use this hypothesis in the context of the proof?” It was just the previous someone’s answer to launch the DE work, whose outcome is the teaching/learning experience of our case study.

The initial underlying idea was to elaborate learning situations aiming to foster the students to face topics in a more critical way and to change their attitude from rote learning to critical learning.

Therefore we start with the preliminary phase of DE. Remaining in the perspective of the Mathematics-Student-Tutor face and looking at their interaction, we contribute, in the position of Tutor, to the didactical and cognitive dimensions of the *a priori* analysis.

Concerning the first one, the analysis stresses the specificity of the Italian University system, mainly transmission-based traditional lectures, attended by more than one hundred of people, especially in Faculty such as Engineering. This is true also for the exercises sessions, which remain transmission-based (the teacher shows some solving techniques for typical exercises). In our case study, the course consists in 90 h face-to-face lectures (60 devoted to theory and 30 to solving procedures) and it deals with a large amount of mathematical concepts from basic linear algebra and calculus II. No institutional space is devoted to practice some problematic approach, such as posing questions, although the latter is needed for being successful in assessment. The students are supposed to enter University being endowed with such competency, but the analysis shows that it was true no more. Then it is clear the impact on the exam results and on the students' feeling of "strange and unexpected questions".

Concerning the second dimension, the analysis highlights the Student difficulties during oral discussion of theorems' statement and proof within the exam session, whose roots can be found in what Paul (1990) well expresses:

> What students often learn well – that school is a place to repeat back what the teacher or textbook said and to follow the correct steps in the correct order to get the correct answer – blocks them from thinking seriously about what they learn. (p. 808)

This is a transversal difficulty, not strictly specific of topics at stake, but related to the way learning is approached by students whose many drawbacks conflict with the undergraduate learning approach requiring to grasp complex knowledge going-in-depth and critically thinking.

To complete the *a priori* analysis, we move on the Author-Mathematics-Tutor in order to focus on the epistemological dimension. To this aim, we need to co-opt in the Author position, besides ourselves, other colleagues experts in mathematics and in its education, with especially regards to the subjects we are interested in, and also to take into account the existing literature too. The analysis puts emphasis on some features of the linear algebra and calculus at stake in the given course. For instance, a higher abstraction and complexity of the mathematical objects in linear algebra (the n-dimensional space), a huge usage of symbolic representations (letters instead of numbers), the management of and the dealing with generic objects (reasoning on the basis of the properties of the object and not of its instantiation), the need of coordinating various semiotic representations (algebraic and geometric ones), and so on.

Them we start the design of a new learning situation and the previous analysis guides the didactic organization. We adopt to integrate the face-to-face lectures with on-line time restricted activities to be performed throughout the course.

We aim to design a Learning Situation Model (LSM) that can simulate the oral dissertation of a theorem and its proof during examination. As the analysis highlighted that the students lacked the practice of posing questions in order to understand a topic, we decide, as pedagogical choice at global level (that is concerning the whole design), to frame the situation in cooperative learning (Dillenbourg et al. 1996), in particular in a role-play setting. The envisaged LSM requires the student to play subsequently three roles: the teacher who makes questions, teacher aiming to evaluate student's learning concerning a given topic; the student who gives answers to the teacher aiming to prove her understanding of the topic; the teacher who assesses the student's answers with respect to learning outcomes. The pedagogical choice done suggest to look at this part of design in Author-Mathematics-Student perspective, as no official help is foreseen. Thus, we fine-tune the design fixing some further global choices. The LSM consists in three consecutive tasks corresponding to the roles described above (from now on, we call 'round' an occurrence of all the three tasks). For each task the students are required to produce resources to be used by peers at the next task: in the first, the student make some questions as if she has to assess someone other's learning outcomes; in the second step, the student gives answers to the questions posed by a peer; in the third step, the student checks the correctness of the outcomes (both question and answer) of two peers. All the students play contemporarily the same role, i.e. all of them performed individually the same task at the same time, preferably addressing different topics. At the end, they go to the next task. Once a round finished, a new one can start. The model devises that for each task the devolution is activated by explicitly asking the student to act in the role at stake.

We also define the related global milieu, which consists in a teaching platform, equipped with the following e-tools needed to implement the design: (i) a means that allows implementing cooperative and time-restricted activities; (ii) a repository for sharing resources; (iii) communication tools for interactions among participants. We want to note that the local milieu, i.e. the resources needed during the activities, is not made ready from the Author, but it is supposed to be constructed gradually and for each task it consists in the products of the students themselves playing the previous roles. In this respect, the student assumes the Author position.

Moving to the Mathematics-Student-Tutor face, we think if some official help for the students could be devised. We decide that the Tutor did not intervene during the tasks, leaving them in the context of peer-to-peer interactions, but she gives comments, suggestions or corrections only at the end of the round. This way the Student can benefit from the implicit help of peers playing the third role and form the explicit Tutor's help improving the products of next round, both in terms of the Mathematics and of the methodology used.

At this point, the Learning Situation Model has been completed and we require to move towards the Learning Situation Instantiation, in order to actual implement the situation.

To this aim, in the perspective of the Author-Mathematics-Tutor face, we need to make concrete the devised global milieu. Co-opting and exploiting technological experts within the Author position, we choose the teaching platform and the e-tools to be used. In our case study, we select the platform IWT, available at the University of Salerno. The role-play has been implemented using at each step the

IWT module ‘homework’ that allows the students to submit their work. Then the Tutor, acting as technical support, is in charge of receiving the submissions in each step and then randomly distributing them to the students as resources for the next step. Further, a shared repository, with access restricted only to students involved in the activities, has been set up in order to allow the Tutor, at the end of each round, to make available the products of the students suitably annotated.

We note that, as the global milieu is affected by the technology at disposal, we can have various implementation of the learning situation. In fact, the same situation has been implemented at the University of Piemonte Orientale using the platform Moodle and the ‘workshop’ module, which overcome the Tutor technical support.

Let us consider now the local milieu that is the resources needed for the Learning Situation Instantiation. According to model we defined, the setting of the local milieu can be seen as an ongoing process in the Author-Mathematics-Student perspective. In fact, as the first task do not foresee to deliver any resource specifically designed, the student can freely surf among mathematical resources, concerning the mathematics theorem she has to address, available inside or outside the e-environment (learning objects in the platform, various resources on the web, to her notes during face-to-face lectures etc.). On the contrary, the second and the third tasks devises the delivery of a specific resource, consisting in a product randomly chosen among the ones submitted by the students in the previous sessions. Thus the Student contributed to set up the local milieu, assuming the Author position, as she is required to author resources, completely from scratch in the first session or starting from a peer-authored one of the other two sessions.

Once completed the implementation phase, we experiment the Learning Situation Instantiation. In order to do this, we refined some global and local choices, listed below, taking into account the specificity of Student and of our didactic context:

(a) the duration of each task and of a round of the learning situation: we fixed in 2 day the time needed to the student in order to perform each task; then one day was foreseen to technically distribute the submitting products in order to begin the next task; thus, each round lasted 9 days;
(b) the number of rounds to experiment along the blended course: according to the duration of the face-to-face course and of each round, and taking into account that the experimentation did not start at beginning of the course, we chose to make three rounds;
(c) the mathematical contents to work on: we selected a list of 19 theorems, 11 from the linear algebra and 8 from calculus, consisting in the main theorems whose understanding, in terms of statement and proof, was required to the student in assessment phase;
(d) the partition of the previous list into three sub-lists, each of them to be used in one round; we did not made this partition a priori, but the sub-lists were determined according to the flow of the face-to-face lectures.

Therefore the experimentation starts assigning to each student one of the theorem from the first sub-list (for instance, Steinitz lemma). The task of the first activity requires "*Prepare a file containing four questions which you consider useful to verify that a student has understood the claim and the proof of the Steinitz lemma, as you are a teacher who wants to assess a student's learning about such topic*.". For the next task, each student receives one the previous output file and she is required to "*Answer to the questions contained in the attached file as they are the questions posed by a teacher during an examination and you want to prove that you grasp knowledge about the theorem at stake showing to know how to answer*". The last task expects: "*The attached file contains some questions and answers concerning a given theorem. Correct as you are a teacher during an examination who wants to assess both the questions and the answers with respect the given theorem*.". For the *a posteriori* analysis and validation, we look at the students' engagement from the Author-Matematics-Student face, and at the Tutor's revision after the round finished, from the perspective of Mathematics-Student-Tutor. We conducted it using mainly qualitative methodology, examining at cognitive level the protocols produced by the students and interviewing at affective level some of them regarding the roles played (for more extensive reading, see Albano et al. 2007; Albano and Pierri 2014). The analysis highlighted the add-value of the first task both at cognitive and affective levels, with rich products, addressing thinking and reasoning mathematically, and communication and representation, and leading students to go-in-depth facing a topic.

The analysis also highlighted a drawback concerning the didactic contract caused by technological constraints of the specific e-tool, homework, that need a mandatory score to the learner's product. This brought to refine the implementation in order to focus on the formative assessment, avoiding the drawback.

Conclusions

In the last years, the emergence of coming back to the centrality of the methodology with respect to the technology asks for the mathematics educators to be much more 'design scientists' in their use of e-environment for supporting learning.

The need of combine technology features with well-known theory in mathematics education strongly requires the design of learning situations based on a scientific approach. We assume that a didactic engineering approach should be proper. In this paper, we have looked at the didactic engineering work from the systemic perspective given by a didactic tetrahedron. This latter allows having a systemic view of the learning situations to be designed and validated, taking into account new characteristics of the learning process in a pervasive technology-enriched world.

The didactic tetrahedron makes evident a new figure, the Author, which wants to draw attention to the necessary synergy among technological and educational experts in order to balance and full exploit the benefits of tools and methodologies.

Finally, we underline that the vertices of the tetrahedron refer no more only to static figures but to "dynamic position", as learners can play the role of Author (creating resources) and of Tutor (coaching their peers) during their learning process.

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