L-System Fractals as Geometric Patterns: A Case Study

Anna Alfieri

Abstract Digital technologies are impacting all aspects of personal, social and professional life by now, spreading out at an incredible speed. We should take into account all these changes in the teaching and learning processes of mathematics, implying new challenges and responsibilities. In this paper, the role of technology in a mathematics education activity is analysed into two aspects: in terms of its operational features for presenting mathematical content, to research for information on the web, to work in an e-learning environment with students, and then for enhancing cognitive learning processes in the student, through the digital manipulation of geometric objects. The context is the L-system fractal theory.

Keywords L-system fractals • Technology • Education

Introduction

New technologies strongly influence our social and professional lifestyle, becoming necessary in our communication, as considerably as in our relationships with authorities and Institution. They also affect the learning process and teacher's work inside and outside school, contributing to an evolution of the relationship between teachers and students, and creating new challenges and responsibilities.

Many questions arise regarding the impact of technology on mathematics education. Artigue (2013) highlights some crucial questions that have been shared and discussed within our community:

> How to avoid an increasing divorce between social practices and school practices, with the resulting negative effects on the image of the discipline and the students' motivation for engaging in its study? How to benefit in mathematics education from the incredible amount of accessible information and resources? How to benefit from the changing modes of social communication and Internet facilities for creating and supporting communities of mathematics teachers, and definitively break with the vision of the teacher as a solitary worker? (p. 5)

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The issues introduced by Artigue (Ibid.) led me to renew the mathematics education methodology through ICT. In fact, the most relevant aims in my educational activities are:

1. Developing a mathematics education open to the external world, remaining anchored to the epistemology of the discipline;
2. Giving students more autonomy in geometric knowledge;
3. Using new technologies to better address the specific needs students may have.

From Solitary Teaching to a Community Mathematics Project

In this evolving technological context, Artigue (Ibid.) highlights:

> Learning mathematics is not achieved just by accessing mathematical information or even direct answers on the Internet. Accepting to live in the digital era is accepting to have this flux of information entering the school world. (p. 5)

This "flux of information" also leads to discover countless technological tools that teachers can test with their students in the classroom, even if often the teacher is left alone in this researching process.

For this reason, many teachers are looking for training course to introduce new technologies in their learning and teaching processes; they need to find community with other teachers to share experiences, to exchange results, and to know the issues that have been addressed by other teachers in similar educational contexts.

In Italy, for example, the national M@t.abel project is a very important educational program, involving several mathematics teachers. The M@t.abel project, performed by INDIRE (Istituto Nazionale Documentazione, Innovazione, Ricerca Educativa), introduces teachers to mathematical training through examples of real class activities. The teachers work together in a technological platform and are included in virtual classes, managed by a pool of tutors (including the author of this chapter), in order to discuss and share their experiences cooperatively (Arzarello et al. 2006). M@t.abel is an important national activity that turns solitary teachers into members of "communities of practice" (Wenger 2000).

Another interesting national project is Matematica & Realtà (M&R), managed by the department of mathematics at the University of Perugia. This project deals with making proposals to develop innovative and educational connections between mathematics and the real world (Brandi and Salvadori 2004). The Liceo Scientifico "Luigi Siciliani" in Catanzaro (Italy) has been taking part in the M&R project for 8 years with a pool of teachers (me included).

The central focus of M&R is mathematical models, defined as a result of a rationalization and abstraction process, which allows teachers and students together to analyze the problem, to describe it, and to create a representation with the universal symbolic language of mathematics (Ibidem).

The M&R project employs a variety of activities, including: mathematical modelling and best maths presentation competitions. A central feature of the project is the creations of mathematical laboratories, organized in each participating school; in these laboratories, the teacher proposes different themes, related to M&R, depending on the age of the pupils and in connection to compulsory mathematical content. For instance:

1. Linear mathematical models of reality (for 15–16 years old students);
2. Iterative functions and fractal geometry (for 16–17 years old students);
3. Nonlinear mathematical (exponential and trigonometric) models of reality (for 17–18 years old students).

In Liceo, many different mathematical laboratories have taken place since 2006, attended by approximately 1000 students. The M&R project is one of the most important extra-time activities in the school to entail additional credit to the students.

The Role of Technology: From Classroom to Cloud-Learning

My experience in the project concerns geometry, in particular fractal geometry, geometric properties of fractal figures, and geometric transformations. In this chapter, I am going to describe the educational activity concerning Lindenmayer Fractals (or L-system fractals, as they are commonly called), in order to examine the role of new technologies in this specific teaching and learning process. Technological tools can have several educational functionalities, three of which are suggested by Drijvers (2012):

1. The tool function for doing mathematics, which refers to outsourcing work that could also be done by hand,
2. the function of learning environment for practicing skills,
3. the function of learning environment for fostering the development of conceptual understanding. (p. 3)

In the M&R activity, the teaching and learning of fractal geometry have dealt with through several stages; in each of these, technology plays a relevant role characterized by a significant use of ICT. The different “functionalities” allow me to split the teaching activities into following steps.

Working Steps

The educational activity in the L-Fractal M&R project has three different steps:

Table 1 Connection among steps of activity, role of technology and tools

Step activity	Role of technology	Tools
Fractals class	Presenting fractal theory	Interactive with e-board, laptop
Fractals webquest	Researching for data and information	Internet
Fractals cloud learning	Working and sharing in virtual environment	Google doc, Google drive, PowerPoint, Paint

1. *Fractals class*, an extra-time course about fractal geometry;
2. *Fractals webquest*, the design of a final project about L-system fractal theory;
3. *Fractals cloud learning*, a methodology used to work online.

In my experience, each of these steps is related to one of the three different facets of using technology and its tools. *In Fractals class* technology is used to present geometrical objects and their properties; *in fractals webquest* it is employed for researching information on the web; and *in fractals cloud learning* it is used to share and to work with students in an e-learning environment (see Table 1).

These connections are explained in the following sections. Actually, these different types of the use of technology are not distinct and are deeply intertwined. However, in the design of the activities, the use of a specific facet of technology has characterized the corresponding phase more than any other.

Fractals Class: Technology for Learning by Presenting

Fractals class is a basic course about fractal geometry, which I have been developing in my school since 2006. It requires 16 h (2 hour per week in a single lesson). Many 16–17 years old students choose to attend the course voluntarily (see Fig.1).

The principal aims of the course are:

1. Modelling the world around us using affine transformations;
2. Building the most famous fractals (Sierpinski's Triangle, Koch's Snowflake, etc.);
3. Plotting fractals with software tools;
4. Making conjectures and simulations using free software;
5. Mixing traditional teaching and new technologies.

The geometrical content, dealt with in the fractals class, includes:

1. Geometric transformations and matrices (composition of geometric transformations), the inverse of a geometric transformation, affine transformations (rotation, contraction, translation);
2. Iterated function system, codes of fractals, evolution of an iterative process of figures patterns and attractors;
3. Fractal properties.

Fig. 1 Fractals class: during the lesson

During this step, the teacher works using the following educational strategies:

1. *Interactive lesson and practice*

The teacher explains the basic concepts of fractal theory: affine transformations and the most famous fractal codes. Then she enhances the learning process through some exercises. At this stage, the student is encouraged to work alone.

2. *Cooperative learning*

The teacher arranges students in small groups to work together, in order to improving communication and cooperation. Cooperative learning fosters their participation and their involvement without inhibition, promoting skills development of the students. Peer-to-peer communication multiplies the value of the educational message and increases its effectiveness: sending and receiving communications spread throughout the network of students, and are not limited to the first sender.

In this step, the presentation of fractal theory is simplified by using interactive whiteboards (IWB) during the lessons. Some aspects of direct teaching, such as explaining, modelling, directing and instructing, are facilitated by the IWB, or more specifically, the software accessed via a large screen presentation device. (Wood and Ashfield 2007). The quality and clarity of multimedia resources may offer enhanced visual material for presenting to a large audience, and the teacher is able to move between varieties of electronic resources, with greater speed in comparison to non-electronic tools. In this step, technologies, such as the IWB, may change the pedagogic practice to make easier the learning process of several mathematical contents.

Fractals WebQuests: Technology for Learning by Searching For

At the conclusion of the fractals class, the teacher offers an advanced course about L-system fractals. This is a new step of the activity, called *fractals webquest*. The teacher assigns a final project work concerning L-system fractals which would enable students who complete the assignment to participate in the annual M&R competition, *"Best Maths Presentation*", a special feature of the yearly M&R National Congress at the University of Perugia. Each year, there are many groups of students, from the fractals class, who wish to experience this unique educational opportunity.

During this step, the role of the teacher is to support students in cultivating the critical skills, necessary for appropriately utilizing media tools inside and outside of school. Living digitally means that, when a question springs up or a problem arises, we already have instant access to relevant information or even the correct answer. For this reason, the student must learn to consciously explore and use this flux of information (Fig. 2).

The webquest educational approach has the primary goal of discovering additional information on a specific topic, and to create a presentation, using the collected data. Bernie Dodge at the University of San Diego (USA) brought international recognition to webquest pedagogy in the mid-1990s, defining webquests as:

> An inquiry-oriented activity in which most or all of the information used by learners is drawn from the Web. WebQuests are designed to use learners' time well, to focus on using information rather than looking for it, and to support learners' thinking at the levels of analysis, synthesis and evaluation. (Dodge 2001, p. 6)

The Internet is a chaotic space, in which anything or nothing can be found. The network carries a plethora of information, presents endless realities, news, and experiences; the Internet lives on exchanges, enriching overall products with multiple subscribers.

This chaotic knowledge must be deciphered, selected, structured; otherwise, without the skills of artfully curating what you find and search could seem sterile and superficial. The webquest helps students to avoid getting lost in the network and to use their time efficiently. In order to achieve such efficiency and clarity of purpose, a good webquests contains at least these accompanying parts (Dogde 2001):

1. An **introduction** that sets the stage and provides some background information.
2. A **task** that is doable and interesting.
3. A **description** of the process, the learners should get through in carrying out the task.
4. A **conclusion** that brings closure to the quest.

The strategy employed during this step of my activity is based on two kinds of webquests:

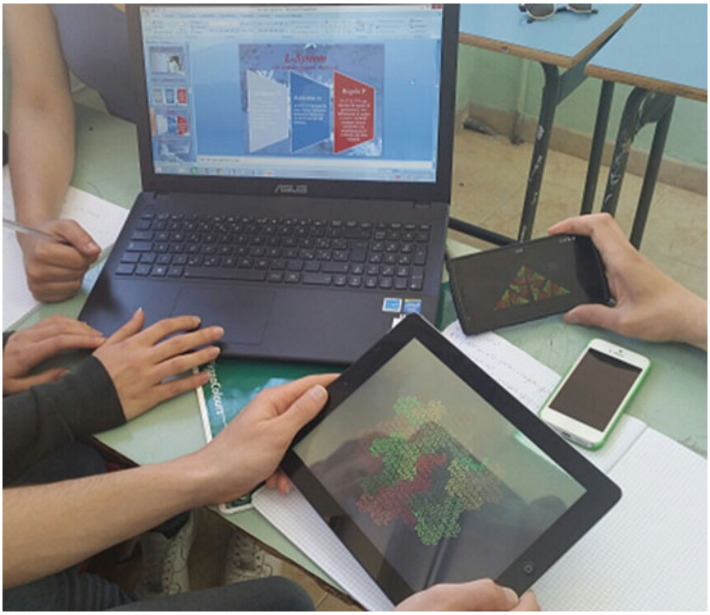

Fig. 2 Fractals Webquest: During the Lesson

1. *Short Term WebQuests.*

It consists of two lessons to gather data and to elaborate their structure. The project work about L-system fractals is divided into four different parts:

1. Fractals in general (what a fractal is and the most important properties of fractals);
2. Mathematical content (affine transformations and matrices);
3. L-system fractals (theoretical definitions about L-system fractals and their codes);
4. Logo (or Turtle) language and free software for plotting images of fractals.

2. *Longer Term WebQuests.*

At this phase, the students collect materials about L-sytem fractals from the Web. Each student has to analyse, study and summarize the documents respecting the sequence of previus points in *Short WebQuest*, without the teacher's help. After completing this part, the learner is able to create an original presentation about L-system fractals, using the material downloaded from Internet.

The role of the teacher, in this step, is to guide students' choices, and to introduce a timeline.

In this activity, students acquire the ability to look for information on the web or in data sources, including web documents, databases and freeware software, to select the most relevant parts, and to apply the most suitable results of their search in developing their topic.

Fractals Cloud-Learning: Technology for Learning by Sharing

After fractals class and the fractals webquests, the teacher works with the groups of students to prepare multimedia presentations for the national M&R congress. *Fractals cloud-learning* is the name of this step.

New technologies enable people to customize the working/learning environment using a range of instruments to match personal interests and demands. This is the reason why it has been explored the educational potential of *'cloud computing'*, in our activities. Web 2.0 tools offer the opportunity to interact and to cooperate with one another in a social media dialogue as creators of user-generated content in a virtual community, in contrast with websites that limit users to a passive view of the content (Despotović-Zrakić at all. 2013; Katz 2009). Examples of Web 2.0 technologies include social networking sites, blogs, wikis, video-sharing sites. Web 2.0 or cloud-based technologies support that trend begins with the emergence of the Internet: a shift away from large organizational control of the instructional function toward the individual user.

These emerging technologies, not necessarily created for higher education, support, require individual creativity and autonomy, and foster the growing trend towards user-generated content and knowledge, in a way that many institutionally developed products do not.

They also have the potential to promote sharing, openness, transparency and collective knowledge construction. A part of their proliferation can be ascribed to the low-cost instructional innovation; the emerging technologies enable, along with ease of utilization in a higher education environment of shrinking budgets and increased competition for information technology budget.

In my educational activity, Google Docs® and Google Drive® (see Fig. 3) are applied and used as a part of an e-learning environment, where students and teacher toghether can exchange and share ideas and information and they can work in a synchronous way – despite not being in the same classroom – in order to achieve the final version of the multimedial presentation about L-system fractals.

Google Drive® is used as storage space for students' files, while Google Docs® is used as a learning and teaching tool for working between teacher and students simultaneously. About the role of technological tools, in general, Railean (2012) claims that:

> The role of these tools for teachers is to provide a learning environment for team work as a need for each child in order to develop self-regulated skills. Imitation, cooperation, confrontation, discussions and sharing are all part of the development of the individual and his or her socialization. These tools play an important role in their cognitive, affective activities. (p. 22)

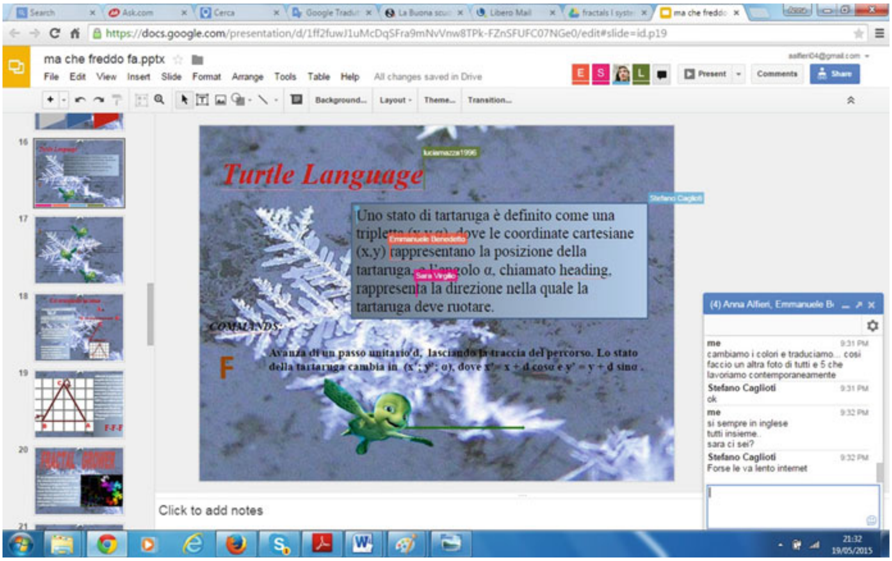

Fig. 3 Fractals cloud-Learning: a discussion board in Google Doc®

The advantages, in addition, of using Google Docs® are:

1. Many people can work at the same time on the same document and everyone can see people's changes as they make them, and every change is saved automatically.
2. Everyone can also propose changes directly by suggesting an edit without editing the text. These suggestions won't change the original text until the document owner approves them.
3. Everyone can collaborate in real time over chat, too. If more than one person has the document open, he just has to click to open a group chat. Instant feedback is possible without leaving the document.

In Google Docs®, the teacher acts more systematically as advisor, guide and supervisor, as well as a provider of the frameworks in the learning process of her students.

The Role of Technology in the Cognitive Learning Process During the Project

During this educational activity, I recognize another important role of technology in the knowledge of Geometry: the use of ICT enhances cognitive learning processes of the students and allows themselves to discover in autonomy some geometrical properties. In fact, during the development of the L-fractal theory, I observe a dialectic between technology and the construction of the geometric content: there are some moments in which technology supportes the comprehension of theory and there are others in which the use of technology transforms theoretical properties in a repetitive training of procedures, these must be reduced. In order to discuss this aspect, a summary of the L-system fractals is necessary.

What an L-System Fractal Is

L-system fractal geometry regards a mathematical theory of plant development. Fractals are geometrical figures, characterized by unlimited repetition of the same shape on lower sequence. Fractal's proprieties are: self-similarity; scaling laws and non integer dimension (Mandelbrot 1977; Gowers 2004).

Aristid Lindenmayer (1925–1989) was a Hungarian biologist who created a formal language called Lindenmayer System or L-system to generate fractals.

The central concept of L-system is the rewriting process, that is a technique for defining complex objects by successively reproducing parts of a simple initial object using a set of rewriting rules or productions (Prusinkiewicz and Lindenmayer 1990).

In L-system, a string can be defined as an ordered triplet $\mathbf{G} = (\mathbf{V}, \boldsymbol{\omega}, \mathbf{P})$ in which:

1. $\mathbf{V}$ is a finite set of symbols called "alphabet";
2. $\boldsymbol{\omega} \in \mathbf{V}^{+}$ is a non-empty word called axiom ($\mathbf{V}^{+}$ is the set of all non-empty words over V);
3. $\mathbf{P}$ is a finite set of production**:** $\mathbf{P} \subset \mathbf{V}\ \mathbf{x}\ \mathbf{V}^{*}$, $\mathbf{V}^{*}$ is the set of all words over V. $\mathbf{P}$ defines how the variables can be replaced with combinations of constants and other variables. A production $(\mathbf{a}, \boldsymbol{\omega}) \in \mathbf{P}$ is written as $\mathbf{a} \rightarrow \boldsymbol{\omega}$**.** The letter **a** and the word $\boldsymbol{\omega}$ are called the predecessor and the successor of this production, respectively. It is assumed that for any letter $\mathbf{a} \in \mathbf{V}$, there is at least one word $\boldsymbol{\omega} \in \mathbf{V}^{*}$ such that $\mathbf{a} \rightarrow \boldsymbol{\omega}$.

The Connection Between L-System Fractals and Technology Through Productions of Students

Many software of computer graphics, based on L-system theory, are available on the Web, they allow us realistic visualization of plant structures and their development processes. During the activities, the generated fractal images fascinate the students for their colors and shapes, similar to real plants, but the use of technology, for creating these patterns, supportes them, especially, in reasoning and solving some problems, in developing of their curiosity. According to the position of NCTM (2011) (National Council of Teachers of Mathematics):

> It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication. Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students. (p. 1)

In the Italian educational program, regarding high school mathematics curriculum, there are the same goals: problem solving ability, creative thinking and logical thinking to enhance mathematical ability of modelling reality. For achieving these purposes, during my educational activity, the following theoretical issues from L-system theory are proposed and discussed with students:

(a) *Converting logical and formal rules of L-system language in geometrical patterns: making fractals;*
(b) *Proving that if the starting figure changes, the fractal does not: fractals as attractors;*
(c) *Modelling reality: fractals as geometric patterns.*

Every issue is the expression of a specific goal and for everyone the technology plays a relevant role, that I summarize in the table below (see Table 2). Goals, issues and contents are connected and solved by a digital manipulating of geometric objects. Actually, it is difficult to separate the single goal from the single content and single issue; they are interconnected in fractal theory. This is a necessary attempt to analyse the "dialectic" relation between the use of technology and the comprehension of L-system fractal theory, at this school level.

The issues (a), (b), (c) will be discussed separately in order to document what happens during the activity and to describe the role of technology in the comprehension of fractals. For the description, I am going to use many images from two final multimedia presentations, made by students during the M&R activities:

1. "*To make a tree… it takes an L-system fractal*"
2. "*Fractal snowfall in Catanzaro*"

Both works were presented at M&R National Congress in Perugia in 2012 and 2014, the first topic was also presented at European Mathematical Congress for Students in Gothenburg.

Both works are an example of mixing affine transformations and L-system fractals for modelling real world: in the first case the world of trees and in the second one the world of snowflakes.

(a) *Converting logical and formal rules of L-system language in geometrical patterns: making fractals.*

In this issue, the formal language and the geometric patterns are compared: the passage from the numerical codes and formal rules of the L-system fractal to the visualization of the geometric patterns is allowed by technology.

In fact, one of the geometric system that computer graphics used for the L-system's generation is called Turtle Geometry.

Table 2 Connection among goals, content, issues and solution by technology

Goals	L-system fractals contents	Issues	Solution by technology
Logical thinking	String, **ω** axiom, **p** production, turtle-language	Converting logical and formal rules in graphical patterns	Computing codes in graphical patterns
Problem solving ability	Affine transformation, contraction, rotation, translation	Proving that the fractal does not change, if the starting figure changes	Applying new codes in geometric transformations
Creative thinking	Logo-language	Modelling reality by fractals	Plotting graphical patterns

A state of the turtle is defined as a triplet (x, y, α) where the Cartesian coordinates (x, y) represent its position, and the angle α, called the heading, is the direction the turtle is facing in.

Given step size *d* and the angle increment *δ*, the turtle can respond to commands represented by the following symbols:

1. **F** (it moves forward a step of length *d* the state changes to ($x' = x + d \cdot \cos\alpha$, $y' = y + d \cdot \sin\alpha, \alpha$) a line segment between points (x, y) and (x', y') is drawn);
2. **f** (it moves forward a step of length *d* without drawing a line);
3. **+** (it turns left by angle δ, the state changes to $(x, y, \alpha + \delta)$);
4. **–** (It turns right by angle δ, the state changes to $(x, y, \alpha - \delta)$).

Given a string *ν*, the initial state of the turtle (x_0, y_0, α_0) and fixed parameters *d* and *δ*, the *turtle interpretation* of *ν* is the figure drawn by the turtle in response to the string *ν*. Specifically, this method can be applied to interpret strings which are generated by the L-system (Prusinkiewicz, 1999). This language has been used to generate many fractal figures, during the project work, with students, like into following examples.

Example 1: Quadratic Koch

1. Axiom **ω: F − F − F – F**, start angle 0°, turn angle 90° (it corresponds to the initiator or starter figure of the fractal) (Fig.4a);
2. Production **p: F → F − F + F + FF − F − F + F** (it corresponds to the generator of the fractal) (Fig.4b);
3. Quadratic Koch island at the fifth generation (Fig.4c).

Example 2: A snowflake

1. Axiom **ω: F--F--F--F--F**, start angle 18°, turn angle 36° (initiator or starter figure of the fractal) (Fig.5a);
2. Production **p: F=F--F--F-----F+F--F** (generator of the fractal) (Fig.5b);
3. Snowflake at the fifth generation (Fig.5c).

In order to manipulate geometric objects and to plot the fractal figures, the students use a free software, called Fractal Grower. It is a Java program (http://www.cs.unm.edu/) created by the University of New Mexico.

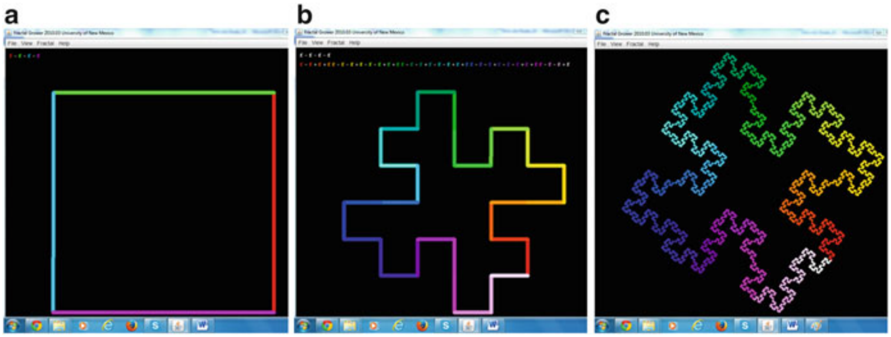

Fig. 4 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A quadratic Koch island

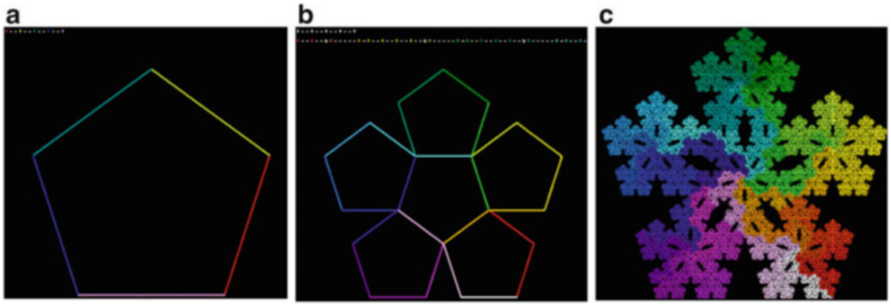

Fig. 5 (**a**) Axiom or Initiator. (**b**) Production, Generator. (**c**) A snowflake

The software allows students to visualize immediately the codes of fractals in figures and realize if they are corrected or not. Visualization is very significant aspect in learning geometric process. According to Duval (2000), three kinds of cognitive processes involved in it can be distinguished: visualization processes, construction processes with tools, and reasoning. Duval (2000) also analyses the role of visualization in the solution processes of a geometry problem and distinguishes several approaches to a diagram in geometry:

> An immediate perceptual approach that may be an obstacle for the geometric interpretation of the diagram, an operative approach that is used for identifying sub-configurations useful for solving the problem and a discursive approach that is related to the statement describing the givens of the problem. (p. 64)

The construction of fractals, in its visualizazion aspect, is a geometric problem solved by a "discursive approach". The students, in fact, understand and convert the affine transformations in the codes of fractals and transform them in images successively. Only the use of technological tools can help students to solve and to interpret these steps correctly. Besides they make some conjectures about geometrical properties of fractal figures. For example, changing the start angle or the turn angle or the turtle code, they are able to verify new hypotheses and visualize their solutions quickly. In this case, the visualization improves the theoretical knowledge of formal languages in order to turn them into geometric information and vice versa. In addition, I would highlight that technology has a role as more “reorganizer” than an “amplifier” (Dörfler 1993; Pea 1987). Fuglestad (2007) describes these functionalities by:

> The amplifier metaphor means doing the same as before, more efficiently, but not fundamentally changing the objects and tasks we work on, whereas seeing ICT tools as organizers implies fundamental changes in objects to work on, and the way we work. For example in using a graph plotting program as an amplifier the software produces quickly the graph as the end product, whereas seen as a reorganizer the function graph itself is seen as a new object which can be manipulated either directly or by setting parameters. (p. 250)

The features of the software used in the activity allowed us to manipulate geometric objects directly, setting codes and formal language and experiencing new in order to reduce the moment of repetitive application of the same geometric rules and to understand better the fractal theory.

(b) *Proving that if the starting figure changes, the fractal does not: fractal as an attractor*

In the recursive process of fractals, the rules of production are more important than an axiom. The starting figure is not decisive for the fractal; it could be a triangle or a square, in any case, after a few iterations, the figure converges to the same fractal as an attractor. This important feature of fractal is not evident to the student, unless he uses technology like in Examples 3 and 4.

Example 3: A triangle as starting figure

1. Axiom **ω: F++F++F**, start angle: 90°, turn angle 60° (a triangle corresponds as initiator or starting figure of the fractal) (Fig. 6a).
2. Production **p: F=F-F++F-F** (it corresponds to the generator of the fractal) (Fig. 6b).
3. Koch fractal at sixth generation (Fig. 6c).

Example 4:A square as starting figure

If the students change the starting figure, using a square:

axiom **ω: f++f++f++f**, start angle: 0, turn angle 45°, with the same production and same number of iterations, they obtain the same attractor (see Fig. 7a–c).

Through the manipulation of geometric objects graphically (translating, turning or reducing them), the students transform numerical codes in dynamic figures and find out some properties of geometric objects in fractal theory. In fact, the technology offers them the opportunity to learn and explore the fractal geometry in autonomy.

(c) *Modelling reality: fractals as geometric patterns.*

Modelling real world means to study shapes and patterns, to discover similarities and differences among objects, to analyse the components of a form and to recognize their different representations (Barnsley 1993; Edgar 2008; Prusinkiewicz and Lindenmayer 1990; Steen 1990). According to Steen (ibid.):

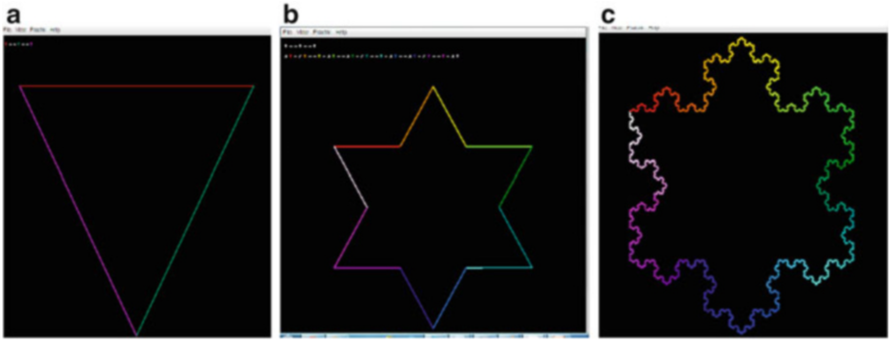

Fig. 6 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A snowflake

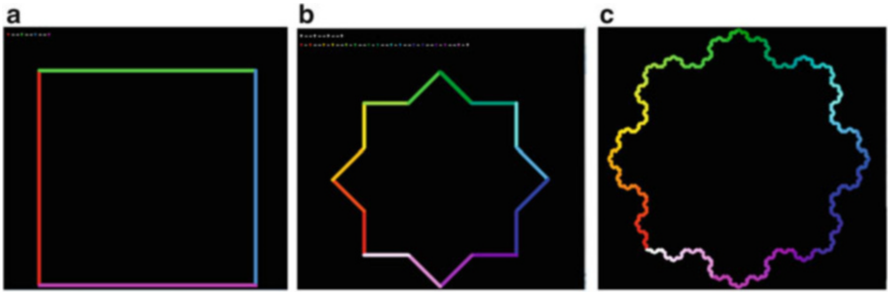

Fig. 7 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A snowflake

> Patterns are evident in the simple repetition of a sound, a motion, or a geometric figure, as in the intricate assemblies of molecules into crystals, of cells into higher forms of life, or in countless other examples of organizational hierarchies. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels. (p. 139)

For describing the complexity of the nature, we need to enforce the turtle language by adding new symbols (Prusinkiewicz 1999):

[(It pushes the current state of the turtle onto a pushdown stack. The information saved on the stack contains the turtle's position and orientation);

] (It pops a state from the stack and make it the current state of the turtle. No line is drawn, although in general the position of the turtle changes);

! (It branches out smaller pattern in the same).

Trees and snowflakes representations generated by the students are very similar to real shapes, as reported in the Examples 5, 6, and 7:

Example 5
Axiom: **f** (line as initiator), start angle: 0° turn angle: 20°
Production: **f** =**f [+f] f[-f][f]** (as generator) (see Fig. 8)

Fig. 8 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A tree. (**d**) A fractal tree branch at fifth generation

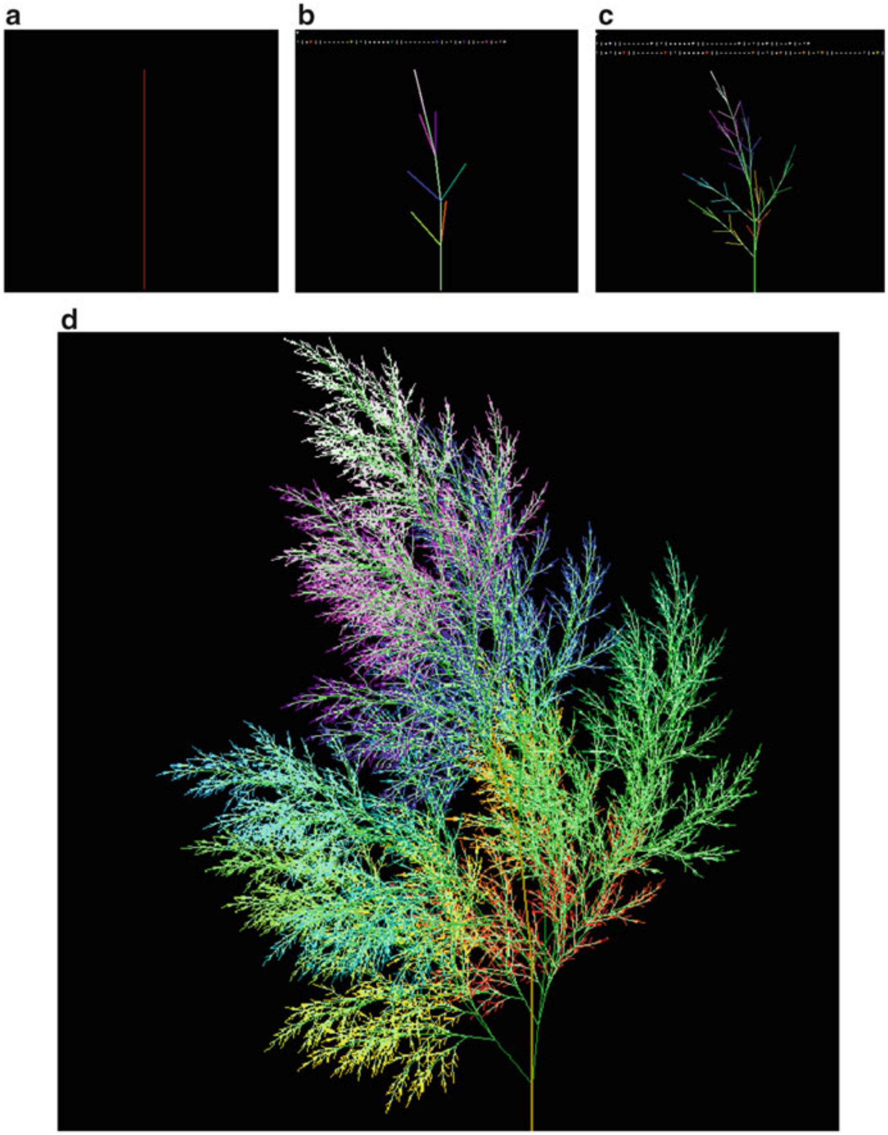

Fig. 9 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A tree. (**d**) A fractal tree at fifth generation

Example 6

Axiom: **f** (line as initiator), start angle: 0° turn angle: 7°

Production: **f** = **![+f][------f]![+++++f][-------f]-![+f][--f]-!f** (as generator)(see Fig. 9)

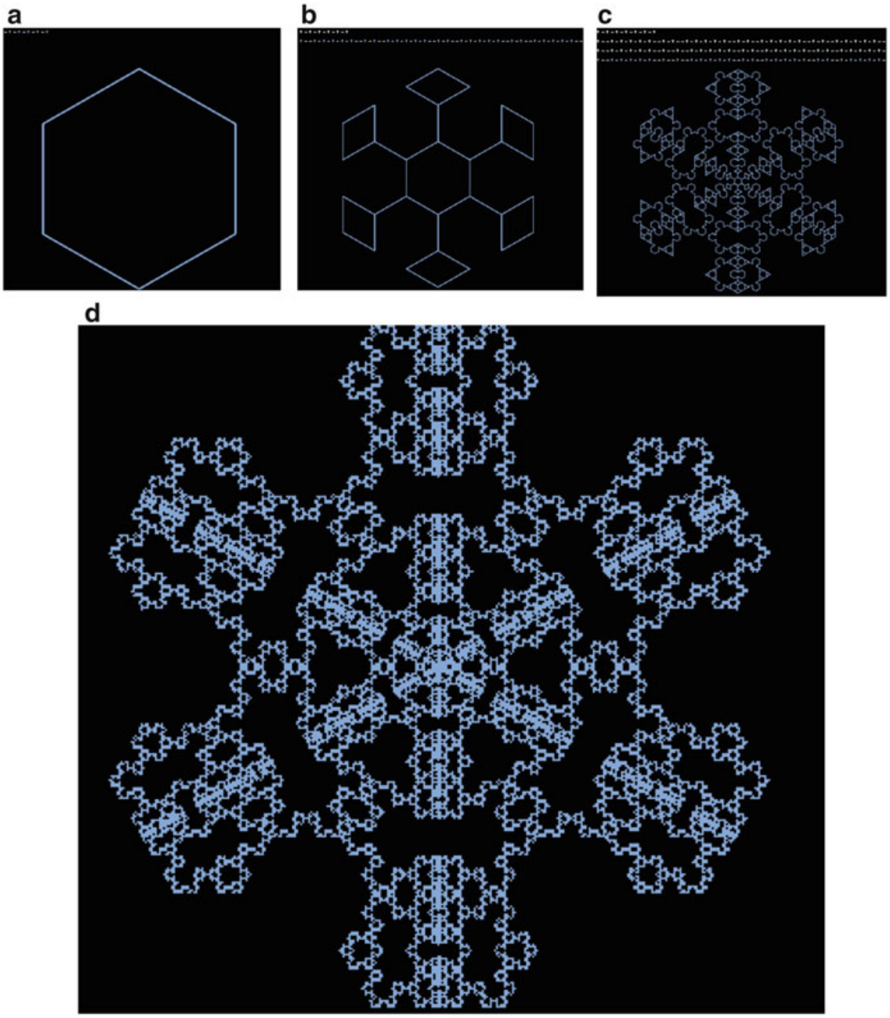

Fig. 10 (**a**) Axiom or Initiator. (**b**) Production, generator. (**c**) A snowflake. (**d**) A fractal snowflake at fifth generation

Example 7

Axiom: **ω:f+f+f+f+f+f** (initiator), start angle: 0°, turn angle: 60°

Production: **f=f++f-f-f-f-f++f** (as generator) (see Fig. 10)

Some links of the videos about L-system fractals, made by students and posted on youtube.com, are reported below:

https://youtu.be/1tNfXrp2JXI
https://youtu.be/EYCHgLr3YhY
https://youtu.be/4E7ECRwEbu0
https://youtu.be/UXWDpmlc2-k
https://youtu.be/DQTf-QOxmzA

In my activity, ICT are essential in order to plot fractals with many iterative applications, to manipulate fractal codes for creating different figure, to make conjectures for applying affine transformations to the figures. These applications increase the sense of independence and autonomy in the students and stimulate their creativity and imagination.

Conclusion

The educational experience described in this chapter, based on L-system fractals and ICT is extremely rewarding. In fact, discovering mathematics in real life through the study of geometric shapes, increasing students' geometric knowledge, and also showing them that there are new educational approaches to geometry mediated by new technologies are the principal aims achieved at the end of this educational activity.

Web 2.0 tools are significant in my activities with students, for:

1. Introducing theoretical content about L-system fractals, an uncommon topic at this school level;
2. Turning formal language into geometric figures and analysing fractal properties;
3. Working in e-learning and teaching spaces (using of Google Docs®).

The students are able to make conjectures, to create fractals and to model the real world through geometric shapes. At the beginning, they study the theoretical content and then check their results by multimedia tools.

The outcomes achieved during these activities are: strengthening mathematics skills (geometric transformations, iterative processes and functions) in the sense of the definition given by Programme for International Student Assessment (PISA):

> Mathematical skill is the ability of an individual to identify and understand the role that mathematics plays in the real world, to operate based assessments and to use mathematics and confront it in ways that meet the needs of the life of that individual as citizen exercising a constructive role, committed and based on the reflection.

The educational activity, applied in this project, features an inclusive approach between theacher and student, among students, it stimulates students' curiosity, it fosters the knowledge of geometrical properties in autonomy and it enhances their sense of self-confidence. In addition, the teacher also gains from this experience:

– Discovering new educational approaches with students in order to make mathematical content interesting and more accessible;
– Experimenting with new methodology to renew his/her own teaching of mathematical topics.

New challenges are still to be experienced regarding the educational approach described in this chapter, for example to extend it to more mathematical topics and above all to involve the whole class, not only those students who voluntarily participate in any optional activity.

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