The Street Lamp Problem: Technologies and Meaningful Situations in Class

Elisa Gentile and Monica Mattei

Abstract The chapter describes a problem solving activity posed with the use of a Dynamic Geometry Software to middle school students. The problem leads students to face a meaningful situation to be explored, and forces them to make conjectures, to discuss and to formulate an argument. The activity starts with the manipulation of materials (paper and pencil, pictures and flashlights) and continues with the transposition of this exploration through technology. We discuss the use of problem solving activities to improve the argumentation skills and the added value of technology in exploration activities.

Keywords Problem solving • Geometry • Discussion • Meta-Didactical Transposition

Introduction

The activity in this chapter belongs to an international research project entitled "Problem Solving with GeoGebra", which involved two different countries, Australia and Italy, with the aim of engaging in-service secondary school teachers in professional development based on best practices in mathematics. This research project is connected to a national project, named PLS (Piano nazionale Lauree Scientifiche – National Programme for Scientific Degrees), born in 2004 from the collaboration among the Italian Ministry of Education, the National Conference of Headmasters of Science and Technology University Faculties and Confindustria[1] with two aims: to increase the number of students enrolled in Scientific Departments and to improve the professional development of teachers, promoting collaborations between school teachers and university teachers.

[1]Confindustria is the main association representing manufacturing and service industries in Italy.

E. Gentile (✉)
S.S. I grado "Holden", Chieri, Italy
e-mail: elisa.gentile@icloud.com

M. Mattei
S.S. I grado "Don Bosco", S. Benigno Canavese, Italy
e-mail: mattei_monica@libero.it

G. Aldon et al. (eds.), *Mathematics and Technology*, Advances in Mathematics Education, DOI 10.1007/978-3-319-51380-5_10

The project was mainly focused on teachers and their professional development, during and after a short course led by researchers and teacher-researchers. The course was addressed to in-service teachers that voluntary choose to attend an 18-h professional development workshop for teachers that took place in several afternoons during the school year. The project involved two communities: the community of researchers, who designed the tasks and the educational programme, and the community of teachers who attended the course. The teachers were also asked to experiment with the activity in their classes, and to reflect on what transpired throughout the activity with the other teachers and the researchers. The teachers were observed during both the course meetings and during the didactical experimentation in the teachers' classrooms; the resulting data were analysed using techniques of "Meta – Didactical Transposition" (Aldon et al. 2013; Arzarello et al. 2012, 2014).

The teaching experiment performed is an adaptation to a middle school context of an open-ended problem, "The street lamp problem". The street lamp problem has been studied previously by the team of researchers in Turin, originally addressed to higher secondary school students (14–19 years old) in order to involve them in a problem-solving activity, activating their argumentation skills. Since this research focused on lower secondary school (11–13 year old students), we needed to adapt the problem to this context. In particular, we paid attention to maintaining the "openness" of the problem and the idea of problem solving, but we inserted additional questions to slightly guide the students (and the teachers) to better understand the problem.

In this chapter we analyse both the students' side, reporting what happened in class, and the teachers' side, focusing on the development of their professionalism.

Overview of Research in Mathematics Education with Technologies

The CIEAEM Manifesto (2000) reflected about the changing role and the importance of technology related to mathematical education. One of the key questions was:

> How can the development and spread of new information technologies really give better access to mathematical knowledge for all? (CIEAEM 2000, p. 7).

The importance of technology in mathematical education was then underlined by the National Council of Teachers of Mathematics in its two positions, proposed in 2008 and 2011 (NCTM 2011). In the most recent one we can read:

> It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication. Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students. (NCTM 2011, p. 1).

A very important study about technology and mathematics education was the first ICMI Study in 1985 (Churchhouse et al. 1986). After that many frameworks followed, emphasizing different aspects of the integration between technology and didactical practice. For example, the CIEAEM Manifesto (2000) considers modern technology as a tool to support, facilitate, organise and rationalise learning and teaching.

In the position about technology, NCTM (2011) highlights that numerous studies, even more recently, have shown that a mindful use of technologies in class can support both advanced mathematical thinking (problem solving, reasoning, arguing, justifying and even proving) and the acquisition of mathematical procedures. Furthermore, technological tools used with didactical intent complement mathematical teaching-learning, and prepare students for their future lives in which technology will play a crucial role.

The simple availability of technology is not sufficient for effective teaching-learning process (NCTM 2011); both the teacher and the curriculum can change the nature of the pedagogical action, mediating the use of technological tools. According to these points, the focus of the research is now about the role of the teacher in constructing effective teaching-learning environments using technology (Artigue et al. 2009; Clark-Wilson et al. 2014; Drijvers et al. 2010). Therefore, it is important to involve teachers in a professional development programme based not only on the technology itself but also on didactical methodologies, best practices, task design and so on (Drijvers et al. 2010). NCTM (2011) pointed out this key concept in this except:

> Programs in teacher education and professional development must continually update practitioners' knowledge of technology and its application to support learning. This work with practitioners should include the development of mathematics lessons that take advantage of technology-rich environments and the integration of digital tools in daily instruction, instilling an appreciation for the power of technology and its potential impact on students' understanding and use of mathematics. (pp. 1–2)

Teaching and Learning with Tools: DGS as an Example

In the last years a great number of studies concerning learning with tools (not only technological ones) have been carried out, especially in the Italian reality. A very important document by UMI (Union of Italian Mathematicians) was produced during years 2000 through 2003 (see UMI 2001, 2003), collecting key ideas for curriculum improvement. Some of these ideas were included in the official document (Guidelines) of the Italian Ministry of Education during its last review of the National Curriculum (in 2012 for the first cycle of education and in 2010 for the second one). The UMI documents pointed out that "basic"[2] materials could be used

[2]We are using the word "basic" without a negative meaning but, on the contrary, with the meaning of simple and easy to find in every house or classroom. Nevertheless, the Italian word used for defining these materials (UMI 2003) can be translated with the word "poor".

as a meaningful starting point not only in primary schools but also at other levels of education. The integration of these materials with technological tools can enhance the teaching-learning process.

We can locate Dynamic Geometry Software (DGS), micro-worlds designed for specific educational tasks, in the theoretical and political context described above. DGS allows students to explore, investigate and observe; to look for invariants, regularities or patterns; and to formulate conjectures and test them within the software. Knowledge is embodied in this software in ways that facilitate students facing it directly, constructing mathematical meanings and objects in the process of using the software (Bartolini Bussi et al. 2004). Marrades and Gutierrez (2000) underlined this as a non-traditional learning environment:

> The contribution of DGS is two-fold. First, it provides an environment in which students can experiment freely. They can easily check their intuition and conjectures in the process of looking for patterns, general properties, etc. Second, DGS provides non-traditional ways for students to learn and understand mathematical concept and methods. (p. 8)

Many research studies have been carried out regarding the role of DGS in proving mathematical theorems (Arzarello et al. 1999; Marrades and Gutierrez 2000; Paola and Robutti 2001; Sinclair and Robutti 2013). The contribution of DGS in constructing knowledge and in promoting justifying competencies is widely recognised among the community of researchers. About this topic, Marrades and Gutierrez (2000) stated:

> DGS environment may help students use different types of justification, setting the basis for them to move from the use of basic to more complex types of empirical justifications, or even to deductive ones. (p. 96)

Sinclair and Robutti (2013) pointed out that the role of the teacher is crucial: the teacher needs to help students develop "schemes of use" (Rabardel 1995). That is, students have to learn not only how to do a specific action (e.g. dragging, measuring) but also the reasons behind their actions, why some actions are not available on every object (e.g. non-draggable points), how and when measuring is useful, and furthermore to learn the limits of using measures with DGS for proofs and justifications. It is important to introduce the scheme of use in a cognitive and metacognitive way, rather than to teach the students a sequence of instructions and rules and then expecting them to reflect on the exploration made.

In this chapter we focus mainly on the role of the integration between "basic" materials and DGS and the emerging of justifying approaches in middle school students.

Realistic Mathematics Education

Although the problem was not created under the framework of Realistic Mathematics Education (RME), a Dutch approach to Mathematics Education (see Van den Heuvel-Panhuizen and Drijvers 2014) that is rarely employed in Italy, this theoretical framework came up during the discussion of the teaching experiment at CIEAEM 66 Conference held in 2014, and we decided to analyse our data in light of this approach, since we can recognize some common ideas with our framework. In fact, the question posed at the CIEAEM meeting was about the reality behind the problem, and we emphasized that the problem was designed to involve students as actors in the learning process, representing a meaningful situation through a "realistic" problem.

Van den Heuvel-Panhuizen and Drijvers (2014) explain in this way the meaning of "realistic problems", which we believe matches with the intent of our activity:

> Although "realistic" situations in the meaning of "real-world" situations are important in RME, "realistic" has a broader connotation here. It means students are offered problem situations which they can imagine. [. . .] It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problem are experientially real in the student's mind. (p. 521)

The International Research Project and Its Theoretical Framework

The open problem analysed belongs to an international research project that considered the interactions between the community of researchers, who designed the educational programme, and the community of teachers who attended the professional development workshop. The "Meta-Didactical Transposition" (Aldon et al. 2013; Arzarello et al. 2012, 2014) is the framework used to analyse the data collected through the observation of the teachers.

The Meta-didactical Transposition Model

Meta-Didactical Transposition (MDT) is a new model for framing teacher education projects. Its focus is the interaction between the *praxeologies* of the researchers and the *praxeologies* of the teachers (in-service or pre-service training), and the dynamics between internal and external components (Aldon et al. 2013; Arzarello et al. 2012, 2014). It is an adaptation of the Anthropological Theory of the Didactic (ATD) by Chevallard (1999) to teacher education. Its main theoretical tool is the notion of *praxeology*, which can be described using two levels:

1. the "know how" (*praxis*): a family of similar *problems* to be studied and the *techniques* available to solve them;
2. the "knowledge" (*logos*): the "discourses" that describe, explain and justify the techniques that are used for solving that task. The "knowledge level" can be further decomposed in two components: *Technologies* and *Theories*.

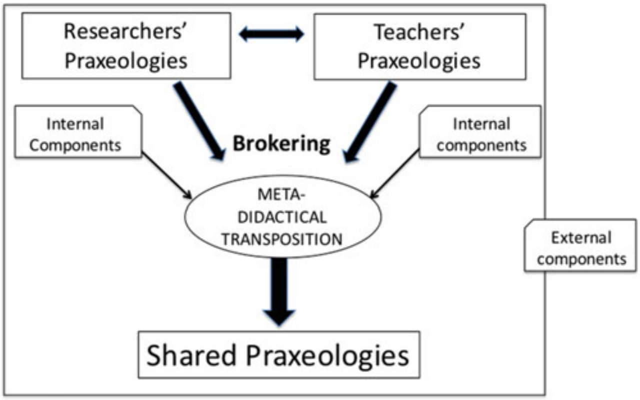


Fig. 1 Internal and external components in MDT

In other words, a *praxeology* consists in a Task, a Technique and a more or less structured argument that justifies or frames the Technique for that Task.

The MDT model considers the *meta-didactical praxeologies*, which consist of the tasks, techniques and justifying discourses that develop during the process of teacher education, and focus on the mechanisms in which the *praxeologies* of the researchers' community are transposed to the community of teachers, and how this implementation transforms the professionalism of teachers. In this way, we can observe a shift from the "savoir savant" to the mathematical and pedagogical knowledge necessary for teaching.

There are two communities involved in this project: the community of teachers (who are in training) and the community of teacher-researchers (who designed the task, act as trainers and observe the teachers). Each of these communities has its own *praxeologies*; the challenge at the end of the project is to create *shared praxeologies*, thanks to the *brokers*.

A *broker* is a person who belongs to more than one community (e.g. a teacher-researcher belongs to the community of mathematics experts and to the community of school-teachers). *Brokers* are able to make new connections across communities and facilitate the sharing of knowledge and practices from one community to the other. The creation of such connections by the *brokers* is called *brokering*.

Some of the components of the two communities' *praxeologies* can change during the educational programme and move from external to become internal (Fig. 1), in terms of the community to which they refer.

Institutional Context

One of the current Italian paradigms for the research in Mathematics Education is "Research for innovation" (Arzarello and Bartolini Bussi 1998), based on teaching experiments in classroom that involve school teachers in every phase of the research, with different roles: teacher-researchers (working with the group of researchers), teacher-trainers (doing education programmes for teachers) and teachers (involved in teacher programmes as learners and working in class as teachers). Sometimes the same teachers may have different roles in different phases of the project: for example, a teacher-researcher can be also teacher-trainer during the process of professional development in the education programme for teachers.

National Curriculum – Grade 1–8

Since in this chapter we discuss a problem proposed to lower secondary school pupils, it is important to understand this problem in the context of the lower secondary school curriculum. We present here a brief analysis of the Italian national curriculum for mathematics education.

In September 2012 the Italian Ministry of education released a new version of the National Curriculum for the first cycle of education (from 3 to 14 years old). The National Curriculum is organized into "Goals for the development of competences" and "Learning Objectives", and explains the expected knowledge and competence at the end of lower secondary school. The National Curriculum is also accompanied by a description of the main ideas of the teaching-learning process and of the different school subjects.

Here you can find some quotations from the National Curriculum for the lower secondary school excerpted because of their relevance for the framework of the activity we proposed (bold by the authors).

> The resolution of problems is a characteristic of mathematical practice. Problems need to be understood as **real and significant issues**, related to everyday life, and not just as repetitive exercises or questions that are answered simply by recalling a definition or a rule. Gradually, stimulated by the teacher's guidance and the discussion with peers, the student will learn to deal with difficult situations with confidence and determination, representing them in several ways, **conducting** appropriate **explorations**, dedicating the time necessary for precise identification of what is known and what to find, **conjecturing solutions** and results, identifying possible strategies.
>
> Particular attention will be devoted to the development of the ability to **present and discuss** with their peers the solutions and the procedures followed.
>
> The conscious and motivated use of calculators and **computers** must be encouraged appropriately [. . .] to check the accuracy of mental and written calculations and to **explore the world of numbers and shapes**.
>
> The development of an adequate vision of mathematics is of a great importance. This vision does not reduce mathematics to a set of rules to be memorized and applied, but recognizes mathematics as a framework to address **significant problems** and to **explore and perceive relationships and structures** that are found and occur in nature and in the creations of men.

Furthermore, we framed our activity with the following *Goal for the development of competencies:*

> To explain the procedure followed, also in written form, maintaining control on both the problem-solving process, both on the results.

and the following *Learning Objective*:

> To know the definitions and properties (angles, axes of symmetry, diagonals,...) of the main plane figures (triangles, quadrilaterals, regular polygons, circles).

The National Curriculum provides clear instructions: the teaching of Mathematics must start from meaningful situations to stimulate and involve students, and to give significance to the topics. In particular, in our experimentation we encouraged the use of technological devices, since the use of technology can effectively support the reaching of some of the National Curriculum goals. As a matter of fact, using a dynamic Geometry software like GeoGebra, students are main actors in their learning process: they can easily explore situations, generalize problems, make and check conjectures.

Class Context

We proposed this activity to 12 year-old pupils belonging to two different schools. One class, whose teacher was Monica, came from "Istituto Don Bosco" in San Benigno Canavese (Turin). It was a 25-student class, including 4 boys with learning disabilities. During the school year they showed interest and curiosity in front of Maths problems, especially involving real situations. In the first part of the year, students started to use GeoGebra as a tool for exploring the geometrical content of the curriculum in an active way. They showed, first of all, astonishment and then a strong desire to learn how the software works.

The other class, whose teacher was Elisa, came from "Scuola Media Holden" in Chieri (Turin). The class was composed of 2 students: a male and a female. They were interested in and curious about the activities proposed during maths lessons. They were used to working with a laboratory methodology and to discussing results and ideas with the teacher. They started to use GeoGebra to explore Geometrical properties (such as angles, perpendicular and parallel lines, etc.) as a support for manipulation of materials (paper folding, paper and pencil, etc.). Both the classes experienced the activity in the second part of the school year, in the same week of April.

The Street Lamp Problem

The street lamp problem, as we said before, is an open problem. The starting situation is a meaningful situation for the students: the municipal technician has to put a unique street lamp in a triangular pedestrian area, designed by the previous administration. The technician has to find the best point for the street lamp in order

Fig. 2 The pedestrian area, covered with grass

to light up the entire triangular area. This is the text of the problem given to the students of lower secondary school:

> The City Council has decided to build a small triangular pedestrian area planned by the previous administration. The registered project foresees only one street lamp as illumination for the whole area. Here there is the picture of the pedestrian area (Fig. 2).
>
> Can you help the technician, who will have to deal with the installation, to find the exact point where the street lamp should be placed?
>
> Part 1: You can use the picture of the pedestrian area and an electric flashlight to simulate the street lamp. Explain how you will proceed to find the best place to locate the street lamp.
>
> Part 2: Now open the file GeoGebra *Streetlamp.ggb*. You will find the pedestrian area to be lit. Together with your group try to find, using GeoGebra, the best point.
>
> What are the operational guidelines that you could give to the municipal technician to identify the point to put the lamp in? What are the relationships of that point with the triangle that defines the pedestrian area?
>
> Part 3: In your opinion, does the position of the point depend on the shape of the pedestrian area? What happens if the triangular shape changes? Be careful! It always remains a triangle but with a different shape! Try to explore the situation with GeoGebra: draw in a new sheet a generic triangle and save the file as *Park.ggb*. Explain what you have discovered and give reason for your answers.

In order to guide our young students, we divided the problem into three parts, beginning with the exploration with "basic" materials and arriving at the use of GeoGebra. In this activity the use of GeoGebra was thought not only to establish confirmation of previous conjectures but also to enable exploration of a more general situation. We also added the sentence related to the operational guidelines to be given to the technician as a way to foster students' argumentation skills: forcing them to explain to a third person how to find the exact point can help them to more deeply understand the geometrical properties of that point (e.g. it is the intersection of the perpendicular bisectors, it is equidistant from the vertices, etc.).

Design of the Open Problem

The design of the problem involved the community of teacher-researchers together with university researchers; they worked to construct the project and the activities

(Bardelle et al. 2014). The streetlamp problem is a transformation of an OECD Pisa item, expected to have only one answer (the circumcentre, see OECD 2003) within an open-ended problem, focusing on multiple solution methods and argumentation skills.

The task in the OECD (2003) Pisa Test was:

> The City council has decided to construct a streetlamp in a small triangular park so that it illuminates the whole park. Where should it be placed? (p. 26)

The problem has been transformed into a more open one, working mainly on three aspects: exploration (with "basic materials" and with GeoGebra), different solutions and discussion.

The idea of giving more space to exploration with both "basic" materials and GeoGebra has been made explicit by adding the sentences:

> You can use the picture of the pedestrian area and an electric flashlight to simulate the street lamp. Explain how you will proceed to find the best place to locate the streetlamp. [. . .] Now open the file GeoGebra *Streetlamp.ggb*. You will find the pedestrian area to be lit.

While the idea of giving more space to different solutions depending on the constraints has been made explicit by adding:

> Together with your group try to find, using GeoGebra, the best point. [. . .] In your opinion, does the position of the point depend on the shape of the pedestrian area?

The idea of giving more space for discussion has been suggested by the following request:

> What happens if the triangular shape changes? [. . .] Explain what you have discovered and give reason for your answers.

Giving more space to exploration meant to let students face the problem for a first time with the use of paper, pencil and an electric flashlight to simulate the lamp, for a second time using a DGS such as GeoGebra to analyse the problem from a static point of view and for a third time using GeoGebra that enables and even cries out for a dynamic perspective where constraints can change.

This exploration with "basic" materials and technological tools helps the students to grasp the dynamicity of the problem and to consider different solutions depending on the shape of the pedestrian area and on the constraints they fixed.

The OECD Pisa item was focused on the transformation of the problem into a mathematical problem: "*locating the centre of a circle that circumscribes the triangle*" (see OECD 2003 pp. 26–27). The reformulation, instead, is focused on the argumentation skills of the students. In fact the problem does not have a clear set of information to start with (e.g. Is the park inside a residential area? Is it possible to put the lamp outside the pedestrian area? ...). The different solutions depend on the choices made by students, on the ideas they consider relevant for the problem, and on the constraints they fix. Having different possible solutions forces the students' argumentation skills, and requires them to develop a strategy for defending their solutions, explaining their reasons, justifying their choices and even proving.

The methodology we used in designing the activity is based on the idea of a "mathematics laboratory" (UMI 2003) not as a physical place, external to the class, but as an approach to mathematics itself:

> A mathematics laboratory is not intended as opposed to a classroom, but rather a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people (students and teachers), structures (classrooms, tools, organisation and management), ideas (projects, didactical planning and experiments). [. . .] In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students). (UMI 2003, p. 28).

The tools of the laboratory can be "basic" materials (transparent sheets, paper folding, grid paper, use of pins and twines), mathematical machines[3] or technological tools, such as DGS or CAS. During and after the laboratory, the "mathematical discussion" (Bartolini Bussi 1996) is the key point, in fact through the discussion it is possible to construct meanings and common ideas.

Aim of the Activity

The problem as posed is related to the exploration of a contextualized situation that, regarding mathematical content, leads to the centres of a triangle, focusing on their geometrical properties.

In Monica's classroom, pupils had already studied triangles and triangles' centres, whereas in Elisa's classroom only triangles and the concepts of perpendicular bisector of a segment, angle bisector, median and altitude had been introduced. Then, in the first situation the problem-solving aim was meant to consolidate acquired knowledge with the testing of the students' competences in using known mathematical concepts within unknown contexts. In the second situation, the aim was more aptly describe as construction of mathematical objects along with a co-construction and discovery of related geometrical properties.

The aim of this activity was not to create students skilled in the use of GeoGebra, but rather to support the development of skills requisite to exploring, conjecturing, justifying and arguing; the aim was also to construct a curriculum around a meaningful problem, powerful in engaging students in a specific context and stimulating their problem-solving competencies. Within this use of technology, the focus was not on the tool *per se*, but on the learning process mediated by the tool, on the new possibilities opened by the tool and on the mathematical objects constructed with the tool. Exploration, argumentation, justification and explanation are the key concepts in this problem-centred activity: it allows students to do maths and to build a piece of knowledge through finding solutions by themselves,

[3]For further information, see UMI (2003, p. 28).

exploring, arguing and justifying their choices. The power of open-ended problems is that the solution depends not only on the problem itself, but also on the interpretation of the problem that students make, on the constraints they fix, on the assumptions they make. Furthermore, the use of a dynamic software environment generates a learning environment that is naturally open-ended because of the way that the dynamic software demands the changing of constraints. It allows the creation of a family of related problems that share characteristics but, at the same time, provoke new directions of exploration caused by the changing constraints.

This use of exploration problems, matched with the use of technology, since the lower secondary school, can help students to face with proofs and can improve their proving competencies, that will become central in the further studies.

Although in the text of the problem there is a reference to the real world, the focus was not to create a realistic problem, plausible from the point of view of the real life. The main aim was to create a problem able to involve students as actors in the learning process and to shift to them the responsibility of learning. In this sense we can say that the problem is not "real", meaning belonging to real-life, but is "realistic" because it is meaningful for the students, according to RME approach (Van den Heuvel-Panhuizen and Drijvers 2014).

Description of the Activity

The activity was organized into 4 phases: three of them were developed by group work while the last one was collective.

1. Analysis of the situation using "basic" materials. Students explored the open problem with "basic" materials: paper and pencil, a flashlight and the picture of the park.
2. Exploration of the problem with static use of GeoGebra.
3. Exploration of the generic situation with dynamic use of GeoGebra.
4. Collective discussion in order to construct together the meanings of the objects involved in the activity.

Research Questions and Observation's Methodology

The research questions we asked ourselves at the beginning of the teaching experiment can be divided into two categories:

Students related

- What is the value added by this activity to the competence of our students?
- Is the use of technology an added value to the activity?

Teachers related

- Had the brokering been performed fostering the creation of shared *praxeologies*?

During the activity, in order to observe and analyse both students and teachers' works, we used a logbook to record, day by day, the things done, the materials used and to write observations about our behaviour as well as the behaviour of the students. Since we gave them forms, with some questions that guided the exploration of the problem, to work with and to fill in, we also collected them to reflect on our experimentation in teaching methodology. Elisa observed Monica's class during the activity, while Elisa's lessons were videotaped.

Critical Analysis

In order to answer out research questions, we critically analyse the activity, focusing on both the work of the students and the teachers.

Critical Analysis of the Activity in Monica's Classroom

Students were divided into working groups of 4–5 people and they were asked to fill in a report giving a shared answer to the questions. We are going to analyze these protocols focusing on the most interested passages.

First Phase

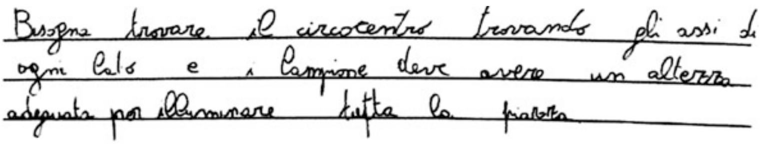
As soon as the students received the flashlight, they started using it to simulate the lamp. First, they noticed that the lamp can be put perpendicular to the ground or oblique: this aspect disoriented them since they were used to exercises with only one solution. Discussing within the group and then all together, guided by the teacher, they agreed that the perpendicular position lights up better than the oblique one. The teacher, in order to encourage them, explained that in this kind of activity there is not a right answer or a wrong one but "every" answer, if justified, is right.

Then they drew some of the fundamental elements of the triangle and two different conjectures emerged concerning the best point: four groups over six chose the barycentre and the other two the circumcentre.

During the previous lesson, the teacher showed, using a cardboard triangle and a pencil as a support, the physical property of the barycentre of being a point of equilibrium. This demonstration suggested students and could, reasonably, had influenced their choice.

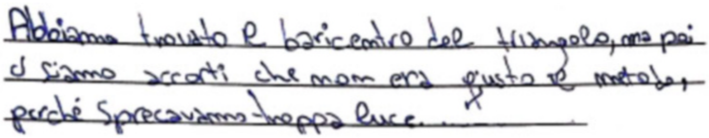
From their protocols we can notice that while students were working in a mathematical context they were making considerations concerning the real context. For instance, in Marta's group protocol (that chose the barycentre) we read:

> The lamp however must be high to light up more the fixed area.


Bisogna trovare il circocentro trovando gli assi di ogni lato e il lampione deve avere un altezza adeguata per illuminare tutta la piazza

[The lamp needs to have an adequate altitude to light up the whole park.]

Fig. 3 Alessandro's group solution


Abbiamo trovato il baricentro del triangolo, ma poi ci siamo accorti che non era giusto il metodo, perché sprecavamo troppa luce.

[We have found the barycentre but, then, we noticed that this was not the right method, since we were wasting too light.]

Fig. 4 Umberto's group consideration

And in Umberto's one:

> Putting the lamp in the centre [barycentre], we notice that lifting it up we are able to light up all the park.

Also Alessandro's group, that chose the circumcentre instead, noticed what is shown in Fig. 3.

Second Phase

We gave the students a GeoGebra file with the picture of the pedestrian area and asked them to work on it. They reproduced with GeoGebra the same construction made with paper and pencil. The static use of GeoGebra helped students to clearly visualize their conjectures and to reflect on the suitability of the choice made. Sometimes, after a discussion with peers, they changed their minds as we are going to analyse.

For example, Umberto's group wrote (see Fig. 4).

They discussed together looking for a better solution. With the help of GeoGebra, they built several triangle's centres and drew some circumferences. They agreed that the best one, with their constraint of wasting as least as possible light, was the circumcentre. Finally they wrote:

> The circumference we have drawn fits perfectly with the triangle.

They noticed that the circumference passed through all the three vertices (Fig. 5), linking together their geometrical knowledge with the exploration of a realistic problem.

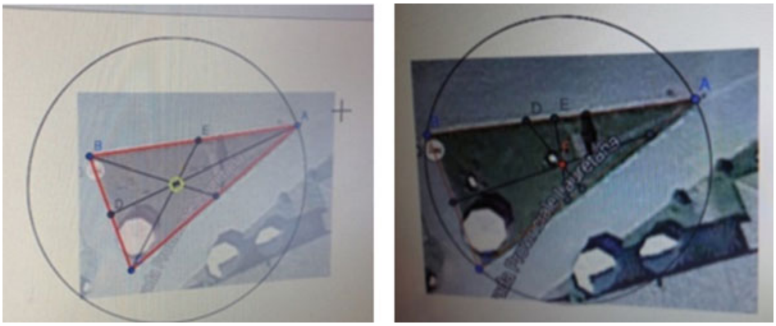


Fig. 5 Solutions in Umberto's group: first barycentre and finally circumcentre

The technological tool helped students to approach to the problem in different ways and to connect mathematical objects to their meanings.

The first solution of Giulia's group was the barycentre, but after a discussion they decided to search for a circumference passing through all the vertices, they drew it with GeoGebra and then they used their mathematical knowledge about the circumference circumscribed to a triangle to find the point:

> We found the barycentre, then we noticed that it wasn't the best solution. We drew the circumference through three points [the vertices] and we used the tool perpendicular bisector on every side of the triangle to find the intersection.

Third Phase

Dragging the triangle drawn with GeoGebra, students were able to explore different situations, making observations that were not possible with the only use of paper and pencil. We are going to report two meaningful quotations in order to support our assertion.

Marta's group (that moved to circumcentre) wrote:

> In the case of the triangle representing the park, the lamp was inside the area, but changing the shape of the triangle we saw that the circumcentre is outside. But if the lamp is higher, even if it were outside the park, it will light up everything [all the park]. We also noticed that, the lamp [put] outside the park lights besides it also the surroundings.

While Alessandro's group wrote:

> [The lamp] can light the park even if it is outside but, in the reality, such high lamps do not exist.

Paying attention to students' work and listening carefully to their discussion, as teacher we noticed that the more they explored, the more they became curious and interested. Some groups, as we have reported, wondered which was the connection between the abstract situation (the triangle, the centre of the circumference and the circumference) and the real situation where we have to use a real lamp. Comparing the geometrical solution found with the real situation, students noticed that there were some problems in putting the street lamp outside of the pedestrian area and, furthermore, they wondered about the maximum possible height of a lamp.

Fourth Phase

We guided the students to explain their solutions in order to convince the other classmates about their ideas. This part of the activity deeply involved students' ability of justify and argue. Almost the students took actively part in the discussion, explaining their ideas or making observations. We report the most interesting considerations:

> The barycentre is not always the best solution! In some cases you have to waste a lot of light in order to light up all the area.

Another student said:

> If the circumcentre is outside the park, you need a very high lamp that could not exist in the real world.

This two sentences point out that students are simultaneously reasoning on two levels: the mathematical one and the realistic one, considering geometrical properties and real problems. Other considerations raised, concerning the situation where the circumcentre is outside the park: "We cannot put the lamp in another property", "Or in a river" beat one classmate "Or in the middle of a motorway" said another.

The ending of the activity was not the choice of ONE solution, but of a SET of solutions and a SET of justifications for those constructions:

- The barycentre seemed to be a suitable point since it was always inside the triangle. Students noticed that in some cases the lamp lights up a big area around the park but they agreed that this was an added value;
- A group of students agreed that the circumcentre is always the best solution, even if it is outside the park, because the circumference passes through all the vertices;
- Other students agreed that the circumcentre is the best point in the case of an acute angled triangle while, in the case of an obtuse angled triangle, the best choice is the barycentre.

Finally we briefly asked them (because the lesson was ending) a personal opinion about the activity. Most of the students were rather surprised from the activity proposed: since schoolbooks usually have closed problems, at the beginning they felt disoriented. Then, they told to have appreciated the use of technology because it allowed them to explore in order to find the point.

Furthermore, students with learning disabilities, that were often bored and distracted during traditional lessons, were actively involved in group working, and in one case a student acted as leader working with GeoGebra.

Critical Analysis of the Activity in Elisa's Classroom

Elisa's students have already studied the fundamental elements of triangles (angle bisectors, perpendicular bisectors, medians and altitudes) but they never faced the triangle's centres, nor in a theoretical way, neither in an exploration activity. The street lamp problem was used as a starting point for the discovery of such centres. Since the class was very small, composed of only two students, they worked in pair. On one hand this represented an advantage: in fact, it allowed the teacher to follow students' reasoning very closely, on the other hand it represented a disadvantage: the collective discussion was less rich because no other point of view was present.

The teacher introduced the activity leveraging on the "realistic" connotation of the problem (in the RME meaning), trying to involve the students as actors:

T: This is a realistic problem, we have to try to understand how to solve this problem, knowing that there is not only one correct answer. This is not a "standard" problem, like an exercise… you finish it and you get the result… that is the same to the one written in the book. Here, we have to let our brain work…
V: Right!
T: The same happens in our everyday life… in our real life we do not have the result at the end of the book, right?

The exploration phase is very important and it is important to do this activity at first time manipulating some materials. As soon as Edoardo picked up the flashlight, he moved it up and down, looking at the light on the picture of the park (Fig. 6).

E: Up or down?

Although the initial idea was to put the streetlamp vertical (as the flashlight in Edoardo's picture) the students engaged a discussion to decide what kind of streetlamp use.

T: Try to discuss… I will do in this way… I will do in that way…
E: I will do this [puts the flashlight on one vertex]
V: Yes, but… here [points the farthest vertex] there is no light…

Fig. 6 Edoardo with the flashlight

Fig. 7 The model of the lamp

E: However… if we put the lamp here [points where Valentina pointed before] it does not light up there [points the vertex in which he put the lamp before]
[…]
V: Up… [puts the flashlight perpendicular to the sheet, but it lights up also outside the park]
E: It is too high!
V: But it lights up everything!
E: And, what about here? [he comes back to his original idea – a vertex]
V: No!

During the exploration with "basic" materials they used the flashlight and the fingers or a pen to simulate the lamp (Fig. 7)

T: Think about… what does the lamp look like?
E: It is high, like this [points the picture of the lamp on the paper]
V: A straight line and then like this… [puts the flashlight down] […] Maybe we can use Edoardo's finger…

The discussion continued, with the teacher posing some level-raising questions and helping the students to make a decision about the kind of streetlamp. The shape of the lamp represented the first constraint chosen by the students, as underlined by the words of the teacher.

T: And... how can you choose the point?
[…]
E: I got it! We will put the lamp here [points the centre of the triangle, with the flashlight perpendicular to the sheet]
[…]
E: Let's try to have a different lamp…
V: As I told before!
T: … Have we decided that we like more this kind of lamp? Ok, so we have done a CHOICE:
how does our lamp look like? Our streetlamp is one of those with the light bulb hanging down. And now… where do we put it?

Fig. 8 The construction of angle bisectors

The students decided to draw the angle bisectors in order to find the point (Fig. 8). Probably they chose in a first time the angle bisectors because the teacher worked a lot on this topic, constructing them in several ways: using paper folding, tracing paper, compass, GeoGebra, also exploring the property of their points of being equidistant from the sides of the angle.

After drawing the angle bisectors and discovering that they all meet in the same point, they pointed out that the height of the flashlight/lamp was an important variable for the problem in order to light up the entire park.

During the second phase the students worked on the GeoGebra file prepared by the teacher, with the same picture of the park used with the flashlight. Students used GeoGebra as a static instrument reproducing the same construction made with the flashlight and the compass. In this phase they never tried to drag the triangle, because the picture of the park (underlying the triangle) forced them to focus on that specific triangle. The previous activity with "basic" material helped students in this technological phase, the mediation of these instruments enabled them to find a first solution to the problem connecting the image of the circular light of the flashlight with the concept of circumference. In particular, Valentina used the flashlight also with the screen of the computer and Edoardo found the mathematical object connected and represented it in GeoGebra.

T: And now, that is the point you have chosen, how can we manage...
V: In GeoGebra there is not a lamp-tool...[puts the flashlight near the screen of the pc, representing the same situation explored before with paper and pencil]
E: I got it... [draws a circumference]
V: Edo, what have you done?
E: I drew a circumference
T: What circumference?
E: Passing through the farthest point
V: From the lamp
E: The circumference has to pass through the point A, because it is the farthest and then we are sure that the circumference contains the other two points... In fact, if I draw a circumference passing through C, something remains out... (Fig. 9).

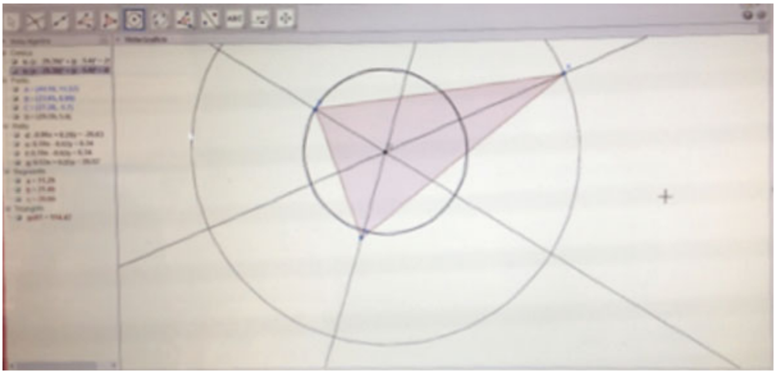


Fig. 9 The right radius of the circumference

Once this solution was found, the students were happy and thought to have solved the problem. The teacher suggested some further reflections on the question.

T: And, in your opinion, is this the BEST point? What is the meaning of BEST point?

The students, then, checked other possible situations and drew the perpendicular bisectors (apparently without a particular reason). They discovered that the circumference in that case passed through all the three vertices and decided that this one was the most beautiful solution.

V: We have found a new point! [. . .] The circumference now "takes" everything! And it is also smaller than the other one! (Fig. 10)
T: What has happened?
V: With the perpendicular bisector. . . the circumference now "takes" all the points [points at the vertices] instead before it takes only the point A. Now there is more light, while before, with the bigger circumference, the light was less intense. So this one is PERFECT.
T: Why do you like this point more than the other?
V: Because it is more centred, the circumference is smaller and it lights up more the park!
T: And what other characteristics does this point have?
V: If we do a smaller circumference, then it does not pass any more through all the vertices. This point is BEAUTIFUL.

Then the students investigated the properties of this centre (circumcentre) while they were trying to explain to the technician how to reach the point, and discovered that it has the same distance from the vertices. The teacher continued asking questions in order to connect the geometrical situation with the realistic one.

T: How would you explain to the technician how to find the point?
E: He has to construct the perpendicular bisectors.
T: Yes... and the technician will say to you "I do not know how to construct a perpendicular bisector".

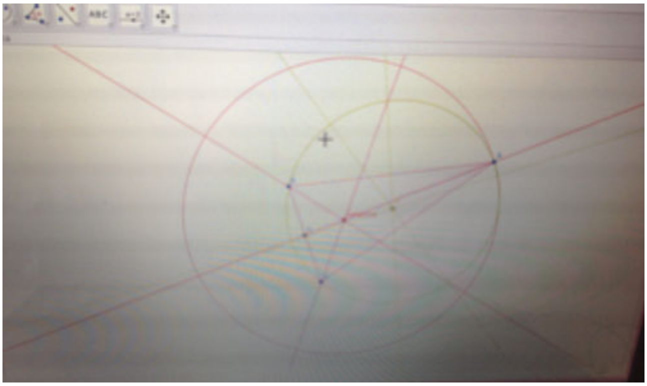


Fig. 10 Circumcentre versus incentre

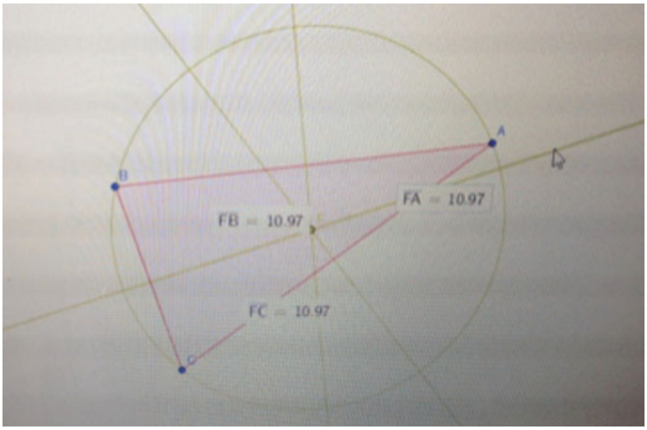


Fig. 11 The distances from the centre

E: I will say to him "Take half of each side of the park. . ."
T: And then? [. . .] What kind of point is the centre?
E: It has the same distance from A, B and C [the vertices], because if the circumference centred there passes through A, B and C. . . then the distance is the same.
V: It is perfect.
E: Let's ask GeoGebra. . . [uses the distance tool to verify if the point is equidistant from the vertices. Figure 11]
T: Can we say to the technician how to construct the point?
E: With the perpendicular bisector tool.
T: But does he have this tool?
V: No.[. . .]
E: Walk away from the sides of the park, perpendicularly, starting in the midpoint.

The students explored the problem from a dynamic point of view. Valentina suggested to draw a lot of different triangles, but Edoardo immediately replied that

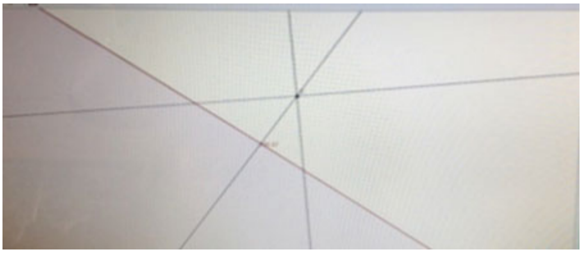


Fig. 12 Obtuse angled triangle: the circumcentre is outside (zoom tool)

they could use the dragging tool, he drew a generic triangle and moved the vertices around. They reproduced the same construction for the circumcentre and they noticed that, dragging the triangle, there were some situations in which this point seemed not so "beautiful" as before: the streetlamp was outside the park and the circumference was big. They explored with GeoGebra in order to find what kind of triangle was it and they argued it was the obtuse angled triangle. They dragged the triangle unless it was "less obtuse angled" (measuring the angle with GeoGebra) and they verified with the zoom tool that the circumcentre was still outside (Fig.12).

At first they recognised that, when the circumcentre is outside the park, the streetlamp needs to be taller in order to light up the whole area. The teacher asked if there were other problems in putting the streetlamp outside and then they decided that it would be not suitable to have it outside the park, then they moved back to the incentre for the obtuse angled triangle.

Through this activity the teacher became aware of some aspects of their students she never observed before: Edoardo, who has some difficulties with calculations, procedures and sequential activities, showed wide intuition and a great accuracy in the geometrical construction, while Valentina became more self-confident, in particular facing problems, instead of being "afraid" of problems such in previous experiences, and solved the task with determination.

Elisa's students, used to laboratory and discussion, were able to discover by themselves that, for instance, the three angle bisectors of a triangle meet in a unique point, that, in a generic triangle, this point is not the same as the intersection of perpendicular bisectors or medians or altitudes and that the circumcentre is equidistant from the vertices. Only at the end of the activity, during the institutionalization discussion, the teacher gave the "names" to these points and formalized definitions and properties.

Comment About the Activity Experienced

The technology represents a key element of this teaching experiment. Technology is involved in the activity with the use of a DGS – GeoGebra – to explore the problem. GeoGebra has the power, as others DGS, of being dynamic, so the students can manipulate dynamically the shapes they constructed by dragging them, they can also modify the shape (enlarge, restrict, etc.) keeping unchanged the construction protocol.

The manipulation in this activity occurred twice, the first time was a concrete action with materials, while the second was a construction and dragging activity carried out with the software. Within the first part, students focus the problem and try to find a solution that will be confirmed, rejected or modified by the observation of the dynamic situation represented with technology.

The integration between "basic" materials and GeoGebra helped students to construct knowledge, and the dynamic use of GeoGebra gave students space to explore, conjecture and argue. One of the added values of this kind of activity is the mediation of instruments and technology (think about Valentina with the flashlight on the screen). The first phase pointed out that the tools we named "poor/basic" (in the meaning of simple) are instead very "rich" elements for the comprehension of the problem. But the use of technology offers more possibilities to investigate the problem with constraints changing over time. Without technological tools the activity's solution could be very different, the dynamicity of the software helped students to emphasize the critical aspects, such as the obtuse angled triangle case and to grasp the variability of the situation over time. For instance, when they used the picture of the triangle it was not A generic triangle, but it was THE particular triangle drawn. When they draw instead a triangle with GeoGebra, it was really a generic one: using the dragging it can change, but maintaining its own properties as a triangle. Looking at the experience, we noticed that students were able to use their knowledge in a real situation, different from the one in which they have learnt it, improving their competences. Finally, they have been able to manage a collective discussion, sharing their ideas and constructing together the meanings. As teachers we noticed that open-ended problems give the possibility of discussing about various aspects, even different from those designed.

Critical Analysis of the Teachers

We tried to find some answers to the research questions analysing the data collected during the teacher-training course: written materials (the beginning questionnaire and the logbook) and also video materials (the beginning interview).

We applied the MDT model to Monica, who belonged to the teachers' community while Elisa belonged to the teacher-researchers' community and acted as a *broker* during the educational programme.

Initially, the use of GeoGebra in lower secondary school and the use of open-ended problems are external components for the teachers, as we can recognize in the following excerpt from Monica's interview:

I: Do you use technology in your class? What kind of software?
M: Although I've been teaching for many years, this is the first year I use technology in class. This year we have the Interactive White Board (IWB) in class and I also

attended some courses to learn how to use GeoGebra in class. We don't have a computer lab big enough to contain all the students, so I worked in class with the IWB, showing the files and the constructions. Students downloaded the software in their personal devices and used it to solve some homework.

Also the focus on the National Curriculum was an external component:

I: Does your Annual Programme of Education follow the National Guidelines?

M: When we wrote the Annual Programme, we followed the previous year's programme. When I started to work in this school the other teachers working here before had already written the programme and I didn't change anything. We never compared the National Guidelines with our Programme. Actually the reference with the Guidelines is missing, but I know the National Guidelines and I think the Programme follows their main ideas.

The laboratorial methodology (group work and discussion) was also an external component for Monica:

I: Are you used to collective discussion? What kind of activity do you manage with collective discussion?

M: I like that students compare their ideas and reasons, but I think that in a middle school (maybe due to the age of the students) it is difficult to manage effective discussions. Students are interested, but they are not able to organize properly a discussion, they have to learn to talk one at a time and to listen to their mates. You waste a lot of time trying to manage the mess and this persuades me not to use the discussion. [...] Sometimes I use it during science lessons.

I: Are you used to group work? Do you think it is useful?

M: I never used group work with this class. They are 25 students and for reasons of time and organization I avoided it. Maybe group work is useful. I have always the problem of managing time: group work needs a lot of time.

At the beginning Monica was sceptical and worried about proposing the activity to her students due to its openness and, furthermore, because the students were very young (12 years old). But she accepted the challenge. At the end of the educational programme the National Curriculum, the use of GeoGebra in middle school classes and the laboratorial methodology became internal components in her *praxeologies* as we can notice in these excerpts from Monica's logbook.

During the activity the students seemed very interested and involved, working seriously on the task given, arguing and justifying their solutions in an accurate way. I felt very involved in this activity; they worked with interest and curiosity and this gave me a great satisfaction and an incentive to repeat in the future this kind of experience. I'm going to design other activities like this one and I will use group work for other tasks.

Elisa acted as a *broker*, being a teacher as Monica but also a member of the researchers' community (as a teacher-researcher). She discussed with Monica and the other teachers, sharing ideas and doubts, reflecting on their didactical practice. The action of *brokering* was performed by the teacher-researchers during the face-to-face sessions of the course and also through the Moodle platform with forums and discussions.

Among the *preaxeologies* of the researcher community, we choose to analyse the *praxeology* of *designing a task for the teachers*. We can recognize the four elements identified in ATD (Chevallard 1999):

Task: designing the activity for teachers and students;
Technique: finding a problem considered linked to the topics of Curriculum; opening a close-ended problem, adapting it to the aims of the project, the methodology to induce, the use of GeoGebra and the institutional constrictions;
Technology: institutional (the new curriculum), from research about exploring, conjecturing, arguing, proving, the use of mathematics laboratory and the use of GeoGebra;
Theory: research elements such as: open problem, conjecturing and arguing, mathematics laboratory, meta-didactical transposition with the related literature as background.

This *praxeology* became a *shared praxeology* when Monica, during the educational course, designed tasks for her own students, in particular Monica took part in the following year to another PLS educational programme, focused on Task design for students.

Conclusion

During the activity, students worked in two different environments: the paper and pencil environment and the technological environment. Technological tools allowed students to explore a variety of different situations simply by dragging the construction made in the specific case. With DGS they can easily represent a generic situation and then study how it changes, test the different ideas and solutions found and validate those most appropriate to their model while justifying choices. Both paper and pencil and technology are important tools for problem solving, but the real potential stands in their integration. Using only paper and pencil or only technology, students do not achieve the same results as they do when using them together. The key point is the mediation and integration of the two environments.

Furthermore, the experience was useful for teachers and students alike. Monica experienced a new approach and new *praxeologies*, improving her professionalism as a teacher, while her pupils were involved with a leading role in the activity: they have made decisions, discussed, argued and mobilized their competencies. Elisa had the opportunity of observing again her didactical practice and to reflect further upon it.

Taking part in an international project is a great opportunity for sharing ideas, methodologies, doubts and for the construction of shared *praxeologies*, that will be, from now on, a critical component of the *praxeologies* of the teachers involved in the training.

References

Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F., Robutti, O., Sabena, C., & Soury-Lavergne, S. (2013). The meta-didactical transposition: A model for analysing teacher education programs. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of PME 37* (Vol. 1, pp. 97–124). Kiel: PME.

Artigue, M., Drijvers, P., Lagrange, J.-B., Mariotti, M. A., & Ruthven, K. (2009). Technologies numériques dans l'enseignement des mathématiques, où en est-on dans les recherches et dans leur intégration? In C. Ouvrier-Buffet & M.-J. Perrin-Glorian (Eds.), *Approches plurielles en didactique des mathématiques: Apprendre à faire des mathématiques du primaire au supé rieur: Quoi de neuf?* (pp. 185–207). Paris: Université Paris Diderot Paris 7.

Arzarello, F., & Bartolini Bussi, M. G. (1998). Italian trends in research in mathematics education: A national case study in the international perspective. In J. Kilpatrick & A. Sierpinska (Eds.), *Mathematics education as a research domain: A search of identity* (pp. 243–262). Dordrecht: Kluwer.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (1999). Dalle congetture alle dimostrazioni. Una possibile continuità cognitiva. *L'Insegnamento della Matematica e delle Scienze integrate, 22*(3), 209–233.

Arzarello, F., Cusi, A., Garuti, R., Malara, N., Martignone, F., Robutti, O., & Sabena, C. (2012). *Vent'anni dopo: Pisa 1991 – Rimini 2012. Dalla ricerca in didattica della matematica alla ricerca sulla formazione degli insegnanti*. Paper presented at the 21st Seminario Nazionale di Ricerca in Didattica della Matematica, Rimini, 26–28 Jan 2012.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programs. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 347–372). Berlin: Springer.

Bardelle, C., Beltramino, S., Berra, A., Dalè, M., Ferrando, E., Gentile, E., Idrofano, C., Mattei, M., Panero, M., Poli, L., Robutti, O., & Trinchero, G. (2014). How a street lamp, paper folding and GeoGebra can contribute to teachers' professional development. *Quaderni di Ricerca in Didattica, 24*(Supplemento 1), 354–358.

Bartolini Bussi, M. G. (1996). Mathematical discussion and perspective drawing in primary school. *Educational Studies in Mathematics, 31*(1), 11–41.

Bartolini Bussi, M. G., Chiappini, G., Paola, D., Reggiani, M., & Robutti, O. (2004). Teaching and learning mathematics with tools. In L. Cannizzaro, A. Pesci, & O. Robutti (Eds.), *Research and teacher training in mathematics education in Italy: 2000–2003* (pp. 138–169). Bologna: UMI.

CIEAEM. (2000). *50 years of CIEAEM: where we are and where we go: Manifesto 2000 for the Year of Mathematics*. Berlin: Freie Universitðt Berlin. http://www.cieaem.org/?q=system/files/cieaem-manifest2000-e.pdf. Accessed on July 2015.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Churchhouse, R. F., Cornu, B., Howson, A. G., Kahane, J.-P., van Lint, J. H., Pluvinage, F., Ralston, A., & Yamaguti, M. (Eds.). (1986). *The influence of computers and informatics on mathematics and its teaching*. Cambridge: Cambridge University Press.

Clark-Wilson, A., Aldon, G., Cusi, A., Goos, M., Haspekian, M., Robutti, O., & Thomas, M. (2014). The challenges of teaching mathematics with digital technologies. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of PME 38 and PME-NA 36* (Vol. 1, pp. 87–116). Vancouver: PME.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.

Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*(1), 87–125.

NCTM. (2011). *Technology in teaching and learning mathematics: A position of the NCTM*. http://www.nctm.org/about/content.aspx?id=31734. Accessed 30 July 2015.

OECD. (2003). *The PISA (2003) assessment framework: Mathematics, reading, science and problem solving knowledge and skills*. Paris: OECD.

Paola, D., & Robutti, O. (2001). La dimostrazione alla prova: Itinerari per un insegnamento integrato di algebra, logica, informatica, geometria. *Quaderni del MPI, 45*, 97–202.

Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains*. Paris: Colin.

Sinclair, N., & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In M. A. K. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 571–596). New York: Springer.

UMI. (2001). *La Matematica per il cittadino: Attività didattiche e prove di verifica per un nuovo curricolo di matematica. Scuola Primaria e Scuola Secondaria di primo grado*. Lucca: Liceo Vallisneri.

UMI. (2003). *La Matematica per il cittadino: Attività didattiche e prove di verifica per un nuovo curricolo di matematica. Ciclo Secondario*. Lucca: Liceo Vallisneri.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 521–525). Dordrecht: Springer.