Advances in Mathematics Education

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Uwe Gellert *Editors*

Mathematics and Technology

A C.I.E.A.E.M. Sourcebook

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Introduction

Fernando Hitt

Abstract Technology and its use in mathematics education has been the subject of study for many decades. In the past some researchers wondered how to transform a constructivist theory that could include technological resources, e.g., "*Constructivism in the computer age*" (Forman and Putfall 1988); the development of this theoretical framework and new theoretical frameworks attempt to better explain the phenomena of learning and teaching in a technological environment (Artigue 2000, 2002a, b; Baron et al. 2007). The effort of teachers and researchers is enormous, and the influence on the educational environment with respect to technological resources is not as pleased as they expect it. And this despite the fact that speeches of education administrators, educational reforms and programs of study and teacher associations (see, e.g. NCTM 2008, 2011), make a special emphasis on the importance of using technology in the learning and teaching of mathematical concepts.

Keywords Mathematics and technology • E-learning • Task-design • Representations • Paper-and-pencil

Technology and its use in mathematics education has been the subject of study for many decades. In the past some researchers wondered how to transform a constructivist theory that could include technological resources, e.g., "*Constructivism in the computer age*" (Forman and Putfall 1988); the development of this theoretical framework and new theoretical frameworks attempt to better explain the phenomena of learning and teaching in a technological environment (Artigue 2000, 2002a, b; Baron et al. 2007). The effort of teachers and researchers is enormous, and the influence on the educational environment with respect to technological resources is not as pleased as they expect it. And this despite the fact that speeches of education administrators, educational reforms and programs of study and teacher associations (see, e.g. NCTM 2008, 2011), make a special emphasis on the importance of using technology in the learning and teaching of mathematical concepts.

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Under this perspective, the mathematics teacher, who wants to use technology in the mathematics classroom in a reasoned way, must take into account a lot of variables that allow him/her to reach a broad view of the problems of teaching and learning mathematics in technological environments. If we take the famous phrase of Euclid (II century BC) formulated by King Ptolemy: “There is no real way to learn geometry,” we could apply here: “There is no real way to know how to use technology in the mathematics classroom”. The choice of which technology to use in the mathematics classroom and why, should take into account different variables for a reasoned choice. The variables involved can be of different types, cognitive (to answer the why), economic (f.e., use of free computer packages or commercial), social (f.e., promote individual learning, including e-learning and/or collaborative learning or both) or institutional (f.e., linked to the curriculum). The technology is present in our daily lives, therefore it is important to reflect on what we could do in the mathematics classroom to support teaching and learning of mathematics in technological environments.

The CIEAEM (International Commission for the Study and Improvement of Teaching Mathematics) aware of those problems above mentioned promoted reflexion about the use of technology in the mathematics classroom in their last three congress CIEAEM 65, 66, and 67. As a product of selecting important articles presented at those meetings, the editorial committee present this volume, covering different properties related to the use of technology in mathematics education:

- as tools allowing a new kind of dynamic representation and giving opportunities to teachers to emphasize particular knowledge construction, as elements of the learning environment of students, offering an opportunity to comprehend mathematical concepts in a dynamic way,
- but also as tools allowing a new way of communication between the different actors, and facilitating dealing with information and information processing.

The first part of the book deals with the role of technology in the teaching and learning of mathematics while the second part treats of information and communication properties of technology.

Technology, a Tool for Teaching and Learning Mathematics

With the development of theories such as “*Instrumental genesis*” (Rabardel 1995), acquisition of resources for teaching (Gueudet and Trouche 2010) etc., are given new theoretical elements that allow us to better tackle this phenomenon related to the lack of influence in the educational environment. One aspect that has been noted by researchers is the lack of mathematical tasks on technological environment that mathematics teachers could use in the mathematical classroom (Aldon 2009, 2010, 2011; Artigue 2000, 2002a, b; Hitt 2007; Hitt 2011; Hitt and Kieran 2009). Precisely related to this point, both in the academic community, and in CIEAEM (65, 66 and 67) congresses, we have seen the growing concern of teachers and researchers in relation to the "task design" that includes the technological variable as an important element in the teaching and learning mathematics, not neglecting the use of paper and pencil. In that context, the issue of the use of paper and pencil and it's replacement (or it's complement) by touchscreen technologies is addressed and research has to answer questions about new gestures and their relationships with the construction of knowledge through personal representations; those representations are crucial in the mathematical modelling process. Even if in the past socio-constructivist and sociocultural theoretical approaches were constructed, in this era, research related to those approaches in a technological milieu is presented in the CIEAEM community; as you can find in this volume.

In relation about what has been said, a key question would be, how the proposed mathematical task in a technological environment influences the acquisition of knowledge? What elements are important to retain in the design of mathematical tasks in computing environments? How to construct a task depending of the milieu? The editors of this volume, aware of the importance of this problem have been proposed as the first part of the book the theme Technology, a tool for teaching and learning mathematics. We initiate the book with the following aspect.

Teaching Mathematics

In relation with the theme "teaching with mathematics" we present four chapters. The first chapter is related to the concept of space and with the development of the child, the acquisition of this concept. In this chapter special attention is made by the author (Sabena) to relate her research with tasks used in textbooks and the Italian National Curriculum. The bee robot used by Sabena reminded us the turtle used by Papert but the playground is no more the screen of a computer but the local space of children. Sabena, taking a Vigotskian perspective, analyses authors that in the past were aware about the development of space like Piaget and Inhelder. And, under this Vigotskian perspective, Sabena highlights several important notions, as anticipation and control that children developed in her teaching experiment. In chapter "Mediation of technological resources in lessons on polyhedra: analysis of two teaching actions", the authors Lobo da Costa, Pimentel and Mendonça, pay attention to teachers' actions mediated by technological resources. Their subject is related to three-dimensional geometry, specifically with polyhedral-prisms. They show the problems where two different teachers have to control the activity of the children in class and want to reach the objective they already had fixed. They conclude about the mediators of technological resources that should be paid attention like the reality of the classroom, students' interest, the number of students per class, the knowledge of the students, the need to cope with the prescribed curriculum and available time. Authors of chapter "Task design in a paper and pencil and technological environment to promote inclusive learning: An example with polygonal numbers", Hitt, Saboya, and Cortés, address an issue that seems very important in this computer age; the authors stress the importance of task-design related to the use of technology and related to a sociocultural approach to promote learning (method ACODESA of teaching). Under this perspective, a problem situation seems more adequate to promote learning under this perspective. Then, under this approach, the authors propose a careful task-design, which is different to that of an exercise or a problem. Then, in this chapter, authors present their theoretical and methodological approach and in chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA", they show the results of their experimentation in secondary education, following the method ACODESA of teaching. In chapter "ICT and liminal performative space for Hyperbolic Geometry's teaching", Kotarinou and Stathopoulou are showing Instructional Computer Technology together with 'Drama in Education' (DiE), drama taken as a milieu to develop a mathematical understanding of non-Euclidean geometries. The authors conclude that an approach to non-Euclidean geometry is difficult for the students, because they are confronted with contradictory statements emerging from different axiomatic systems. The role of the combination of ICT and DiE played a major role to cop with this difficulty. In conclusion of this theme the Sixto Romero's chapter: Improving the teaching of mathematics with the use of technology: A commentary gives a global vision of this first and important part of the book.

Learning Mathematics

Related to the second part of the same theme about Technology, a tool for teaching and learning mathematics, the eight following chapters are related to the section: learning mathematics. The first chapter of this section is chapter "Domains of manip ulation in touchscreen devices and some didactic, cognitive and epistemological implications for improving geometric thinking" and the authors, Bairral, Arzarello, and Assis, are addressing an important issue related to technology in this century, that is to say, touchscreen devices. The mathematical content related to this chapter is geometry. The gestures we made when solving a mathematical task using a touchscreen device is analysed by these authors, from a didactic, cognitive and epistemological perspective. The authors of this chapter show that the process leading to the solution of a mathematical task differs in a pencil and paper approach from that about a touchscreen device. They stress two intertwined domains of manipulation as the results of their experimentation, the constructive domain and the relational domain directly related to geometrical thinking. In chapter "Graphs in primary school: Playing with technology", Ferrarelo shows how introduce elements of Graph theory in primary school, presenting the mathematical activities in an enjoyable milieu. Technology permits this enjoyable approach in teaching and learning. This approach needs a careful task-design to be effective with the aim of the teacher. Under this perspective, Ferrarelo is addressing important aspects of mathematics activity in primary school, that is creativity and independence. In chapter "Pocket calculators as an experimental *milieu*: Emblematic tasks and activities", we have an important approach to the use of calculators. The author (Floris) is showing interesting activities to use in primary and secondary level and also in teachers training. Taking into account Brousseau's notion of 'milieu', Floris shows aspects of the process of instrumentation and instrumentalisation when using a calculator. He also shows how mathematical expectations are modified when using technology. Chapter "The street lamp problem: Technologies and meaningful situations in class" is related to experimentation with teachers and students, using an open-ended problem "The street lamp problem". The authors of this chapter, Gentile and Mattei, analyse the performance of students of a lower secondary school when manipulating materials as paper and pencil, pictures and flashlights before using a DGS software. They conclude about the importance of using DGS to represent a generic situation and they study the situation changes. The activity in this technological milieu permits the conjecture and process of validation. They postulate that both, paper and pencil and technology are important in the process of learning mathematics. The Meta-Didactical Transposition (MDT) offers a framework allowing to express the relationships of researchers and teachers working at different levels of a same educational project; this viewpoint brings to the chapter a meta analysis of the geometrical activity in term of teacher training. In chapter "A framework for failed proving processes in a Dynamic Geom etry Environment", also in a Dynamic Geometric Environment, Chartouny, Osta and Raad, analysed students' performances related to an open geometry problem. They are interested about students' arguments when proving: *Deductive justifications by structural thought experiment*, and *failed proving process*. This last led them to divide their analysis in several stages, the analysis of *Failed construction, Failed conjecture*, and *Failed proof*. The authors claim that focusing on those stages can help teachers and researchers to anticipate and to undertake analysis of students' errors to better teach and better understand the students' knowledge construction. In chapter "Disclosing the "ræmotionality" of a mathematics teacher using technology in her classroom activity" we have a completely different approach to analyse how technology influences teachers' cognition and affect, De Simone analyses a teacher activity from this perspective. The teacher uses a DGS (GeoGebra) and an applet (Virtual scale) in a 9th grade class about linear equations. De Simone shows how technology affects the epistemic emotionality, the teleological emotionality and the communicative emotionality of the teacher. Authors of chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA", Hitt, Cortés and Saboya, address an important topic related to the articulation of arithmetic and algebra in a technological milieu. In their experimentation, the authors utilise the ACODESA method of teaching, showing with detail what means to learn in a sociocultural approach, and how to promote a mathematical activity related to prediction, argumentation, conjecture and validation in a technological environment, to construct a cognitive structure were arithmetical thinking is a support to control the algebraic activity. This eleven chapter is related to the third chapter where the authors presented their theoretical approach about Activity theory in its 5th generation and their methodological approach about task-design and that of teaching. In chapter "L-system fractals as geometric patterns: A case study", Alfieri takes into consideration Artigue's (2013) crucial questions about learning mathematics in a technological environment. The author shows us how to promote geometric knowledge using technology and relating this knowledge with real life taking into account L-system fractals. Under this approach, students conjectured and modelled shapes of real world. To finalize this section we have the commentary chapter written by Appelbaum, who discusses the content of the section, posing general questions and answering trough the results given by different authors. The questions Appelbaum analyse are: can we tame technology? Can learning with technology help us to better comprehend technologies, and how to classify, categorize, exploit, and control them? Learning with/out/for technology?

Communication and Information

Communication Inside and Outside the Classroom

New paradigms of this century have highlighted the importance of communication both inside and outside the classroom, whether researchers or teachers follow a constructivist, socio-constructivist or sociocultural paradigm, communication is an essential part of their research program (Aldon et al. 2009). Technology plays an important role in this communication, and simultaneously imposes its way of being used. In the past, this communication outside the classroom was not fully efficient because we did not have mobile resources as we currently have. Then, the use of digital whiteboards, platforms as Moodle, tablets and smartphones has transformed the way we communicate both inside and outside the mathematics classroom. How to make an efficient use of these resources in the classroom and beyond? Under these ideas the CIEAEM committee has proposed this theme in their congress.

In relation with this theme, the three next chapters look deeper into the communication potentiality of technology. The first of them is chapter "e-mathematics engineering for effective learning" written by Albano. She addresses the problem of learning mathematics in a technological environment, taking into account the methodology of Didactic Engineering, she is extending the classical didactical triangle (Student, Mathematics and Tutor for each vertex) to a tetrahedron where the added vertex represents the author. She shows the need of a fourth vertex if we are immersed in an e-learning environment, and as a consequence, she proposes a Didactical tetrahedron as model, expanding in a way this methodological approach when immersed in a technological environment. Chapter "Learning paths and teaching bridges: The emergent mathematics classroom within the open system of a globalised virtual social network" is written by Moutsios-Rentzos, Kalavasis, and Sofos, they investigate the views of primary teachers, principals and school advisors with respect to social networking sites (SNS). Using a questionnaire in their study, they analyse what kind of preferences and teachers, principals and advisors related to the teaching in primary school, privilege uses of networking sites. They answer the question: in which ways (if any) the aforementioned SNS realities and epistemic views about mathematics seem to be linked with SNS and mathematics as a school course? The authors answer this question analysing data from three points of view: symbolic/normative, pragmatic representations, desired/intentioned actions and from a mathematics didactics and general didactics perspective. In chapter "e-collaborative forums as mediators when solving algebraic problems", Royo, Coll, and Giménez, investigate an e-collaborative learning related to algebraic problem solving. They follow students' interventions when solving algebraic problems in a forum, permitting the students to express their points of view and letting them to reflect before answering to a question. The authors think this approach permit the students to have more confidence and deep reflection because the students have more responsibility when proposing or answering to the whole group.

Information's Tools, to Inform Oneself and to Inform Others

Another important aspect detected by the editors of this volume, is on how teachers appropriate themselves the information to teach courses and to communicate with their colleagues. The issue of the documentation of teachers in the digital era has to be addressed as well as the documentation of students. Technology modifies the way information is transmitted and mathematical education has to take into account the new ways of learning through connected networks as well as new ways of teaching with an extensive documentation (Aldon 2010; Gueudet and Trouche 2009; Trouche and Drijvers 2010). The tools to learn and inform others are important in the process of teaching. The problems facing the educational environment for information to flow in both directions are huge. How the researchers can appropriate themselves about the teachers' experiences and in turn, how the teachers can appropriate themselves about research results. Bilateral information represents one of the biggest problems to solve in the educational environment.

In relation with this theme, we have two chapters. Aldon, Durand-Guerrier, and Ray wrote the chapter "Problems promoting the devolution of the process of mathematisation: An example in number theory and a realistic fiction". The authors address the big problem of learning about modelling. They studied this topic related to modelling phenomena from both mathematical and real-life situations, and modelling a phenomenon in a fictional context in a technological environment. Using Brousseau's theoretical approach in a collaborative milieu, the authors gave, in the mathematical classroom, a particular importance to the devolution of the problem. The experimentation of these authors is immersed in collaborative research among researchers, teachers and students. Addressing the problem of learning about modelling, authors explain also how the project is developed in this collaborative research approach. In chapter "A classroom activity to work with real data and diverse strategies in order to build models with the help of the computer", Ginovart, uses mathematical models that depending on the parameters these models can be applied to different contexts. The task-design was implemented to elaborate the tasks related to real data that should by analysed using technological devices. This permitted the students to recalculate as necessary to better model the phenomenon in question, this search of a better model promoted visualization processes in the students. Finally, related to this theme, Hahn analysed five chapters. Highlighting the importance of questioning the notion of problem, the notion of knowledge, the notion of activity and device, in her chapter, she made a global analysis of the five previous chapters, emphasising the differences of approaches and theirs particularities.

Technology and Teachers' Professional Development

The last session, but not the least, takes an interest in teachers training in the digital era. Teacher education is a political issue that policy makers have to take into consideration in the cultural and social context of the society. The injunctions, often insistent, of introducing technology in the classroom have to face the reality of teachers' technological knowledge and the necessity of teachers' training (Clark-Wilson et al. 2014). It is well known that we can divide the teachers into three populations, those that completely reject the use of technology, those who believe that problems of teaching and learning mathematics are solved immediately with technology, and those who believe that they must make a rational use of technological resources in order to have a real impact on teaching and learning mathematics. How to convince the mathematics teacher about the importance of the use of technological resources? How to convince the enthusiastic teacher who must make a rational use of technology? What problems of teaching and learning will have to confront future mathematics teachers in the use of technological resources?

Related to this theme we have five chapters, chapter "A study on statistical technological and pedagogical content knowledge on an innovative course on quan titative research methods" written by Serradó, Meletiou-Mavrotheris, and Paparistodemou. Their study is double, first, it is related to the affordance a Quantitative Research Methods course to develop students' Statistical Technological and Pedagogical Content Knowledge (STPACK), and second, to investigate the effects of the STPACK model in graduate education studies. The packages used in their experimentation are Fathom, TinkerPlots, and Probability Explorer. That software allowed them to design activities to use them as amplifiers and reorganisers from a statistical perspective. In chapter "The professional development of mathematics teachers: Generality and specificity", Polo stresses the professional development of mathematics teachers from a general and a specific point of view. Two groups of teachers were analysed from a pedagogical, psychological and sociological role as a teacher. Polo's concerns is that there is a lack of integration of the different theoretical models used to describe the role of the teacher and a poor relation between prospective teachers and teachers in service. In chapter "Integration of digital technologies in mathematics teacher education: The reconstruction processes of previous trigonomet rical knowledge", Lobo da Costa, Esteves, and Brisola studied the integration of digital technologies in the classroom with two case studies, one with prospective teachers and the other with a teacher in service. Their theoretical approach is related also with TPACK and they added the Rabardel's perspective about the instrumentation theory. The results show that non-traditional design training can help to develop the professional knowledge in prospective teachers and in-service teachers. In chapter "Formative assessment and technology: Reflections developed through the collaboration between teachers and researchers", Aldon, Cusi, Morselli, Panero, and Sabena, addresses the problem of formative assessment in relation to a European project FaSMEd. This project is immersed in a collaborative research between teachers and researchers. Their study uses a three-dimensional framework where one of the axis is related to « functionalities of the technology ». Their results show how different functionalities of technology enable the development of formative assessment strategies and more important a characterisation of the dynamics that intervene in a collaborative research study, where the researchers learn from the teachers' practices and the teacher learn from the theoretical approach of the researchers. In chapter "Teaching intriguing geometric loci with DGS", Ferrarelo, Mammana, Pennisi, and Taranto, stress a teaching experiment developed in several high schools in the South of Italy. Their theoretical approach is based on the TPACK framework, using DGS software. Some of the results show the importance of high school teachers to work with university professors in a collaborative task-design perspective. Teachers think that the software GeoGebra is useful to better understand mathematical topics. The technological approach attracted attention and interest of the students, and that motivated them to participate actively in the experiment. Finally to close this theme, FitzSimons wrote a chaper from an integrative perspective taking into account the five previous chapters related to the theme, and taking into consideration the whole structure of the book, arriving to the conclusion that: the five chapters that are the subject of this commentary have much to offer teachers and researchers alike, and she commends each team for its innovative work on behalf of the students concerned and, hopefully, students of the future.

Finally, Aldon presents a global conclusion related to all the chapters, giving special attention to what the commentators had stressed in their analysis.

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Part I
Technology, a Tool for Teaching and Learning Mathematics: A. Teaching

Early Child Spatial Development: A Teaching Experiment with Programmable Robots

Cristina Sabena

Abstract This contribution addresses young children development of spatial competences, and investigates the didactic potentialities offered by a programmable robot. The theoretical framework addresses the delicate relationship between space as lived in everyday experience versus space as a mathematical notion, and takes a multimodal perspective on mathematics teaching and learning. An experimental study has been conducted in kindergarten school. The qualitative data analysis of video-recordings constitutes the background against which children spatial development is discussed.

Keywords Multimodality • Kindergarthen mathematics • Spatial thinking • Programmable robots

Introduction

Attention on early years mathematics is emerging in recent times in research, as witnessed by the new Thematic Working Group in CERME, (http://www.cerme8.metu.edu.tr/wgpapers/wg13_papers.html), and the ICMI Study 23 Conference (http://www.umac.mo/fed/ICMI23/). As in the latter case, the focus of attention is placed in particular on the development of whole numbers competences, which are fundamental steps for children mathematics education. Less attention is given to other competences, such as the spatial ones.

Spatial competences develop through a complex process, requiring long-time experiences in meaningful contexts. Kindergarten and the first year of primary school are the proper places for these experiences, constituting the base on which the learning of geometry can be grounded, first as modeling of spatial properties, and then as theoretical elaboration specific on the mathematics field. However, especially when starting primary school, spatial competences are often overlooked (at least in Italy, but this may not be an isolated case), being the major efforts put on numbers.

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This chapter focuses on children development of spatial competences, and explores the didactic potentialities offered by programmable robots. Cognitive aspects will be on the foreground, and in particular the delicate relationship between space as lived in everyday experience versus space as a mathematical notion will be addressed. On the background of psychological results on spatial conceptualisation in children, and taking a multimodal perspective on mathematics teaching and learning, an experimental study has been conducted in kindergarten school. The study was based on the teaching experiment methodology and explored the didactic potentialities offered by a programmable robot with a bee-shape, with respect to children development of spatial competences. In the following, after a theoretical discussion on young children spatial thinking development, the methodology of the teaching experiment will be described, and a case study data analysis will be provided, from video-recordings and collected written materials of the classroom activities in a kindergarten school.

The Development of Spatial Thinking in Early Years

The complexity of children spatial conceptualization processes has been pointed out by research in psychology and education for several years. Great differences in different theorizing in the field prevent researchers from reducing these processes to simple and linear models of learning, based on rigid pre-determined steps. Concerning spatial relationships, we can consider three different fields of experiences, which correspond to three different kinds of space, requiring each specific perceptive and exploration modalities (Bartolini Bussi 2008):

- The *body space*, that is the internal reference frame relative to the awareness of body movements, its parts, and to the construction of the body schema;
- *Specific external spaces*, including different kinds of living spaces (the house, the town, the school,. . .) and different representative spaces (the sheet of paper, squared papers, the computer screen, . . .);
- *Abstract spaces*, that are the geometrical models developed within mathematics science in its history.

The first two kinds of spaces refer to actual spaces in real world, the latter one belongs to the world of mathematics. Such a categorization must not be thought as a sort of hierarchical scale, or as a developmental sequence. On the contrary, according to Lurçat (1980): "it appears difficult to imagine a development in which the body schema is constructed before, to allow then the knowledge of external world" (p. 30, translation by the author). As a matter of fact, several studies agree in recognizing a fundamental role to the experiences that the child makes both in his/her family and in specific educational settings, and suggest to go beyond linear models, which position abstract space at the end of a developmental process (in the stage of formal operations, in the Piagetian case). A discussion in this direction may be found in Lurçat (1980), and in Donaldson (2010).

Recent strands in cognitive sciences place perception and everyday experiences with the body as grounding pillars for more abstract knowledge conceptualization, included the mathematical knowledge. In particular, the embodied cognition perspective (Lakoff and Núñez 2000) proposes a model for the "embodied mind", as a radical criticism of the dualism between the mind and the body of classical cognitivist approaches.

If mathematics is no longer a purely "matter of head", it becomes of paramount importance to carry out mathematical activities in suitable contexts in which children can interact with different kinds of space and spatial thinking. Concerning the external space, we can distinguish further between *macro-spaces* and *micro-spaces* (Bartolini Bussi 2008):

- *macro-spaces* are those in which the subject is embedded (the subject being part of the macro-space); their exploration is carried out through movement, and their perception is only local and partial, requiring usually to coordinate different points of views;
- *micro-spaces* are external to the subject; their exploration is carried out through manipulation, and their perception is global.

A park is an example of macro-space, whereas a sheet of paper and a book page are examples of micro-space. As an intermediate category, called *meso-space*, we can consider the big posters often used in classroom for group-work: children can enter into them, but also look at them at distance. The essential aspects in this distinction are the different modalities of perception and exploration: the school garden, for instance, can be an example of macro-space—when the child is playing within it—or of micro-space, when the child is observing it from a window above.

The body space and the external spaces share fundamental differences with respect to abstract spaces: as a matter of fact, they can be perceived and explored, and are featured by *fundamental directions* (vertical and horizontal) and by *typical objects* (e.g. a door in a room, a fridge in a kitchen). On the contrary, abstract spaces (like the geometrical ones) are isotropous and homogeneous, i.e. do not have any privileged directions, nor special points. These features may be sources for difficulties for students, when facing tasks with figures in non-prototypical positions, as in the assessing item reported in Fig. 1 from Italian National test INVALSI 2012–2013, grade 5): *Four isosceles triangles are cut from a paper sheet, with the same base and different heights. In each case, the height of the triangle is the double of the previous one. In triangle A the height measures 2 cm. Which is the total length of the paper sheet?*

Among the advantages of introducing reference systems like the Cartesian one in the geometrical space, we find the introduction of privileged points (in particular, the origin point) and special directions (those parallel to the axis).

In mathematics, reference systems are *objective* or *absolute*, in the sense that they do not depend of the position of the subject using them. Objective references are the product of the historical-cultural development of society and have to be introduced by the teacher starting from the subjective references (which depend on our position in the external space).

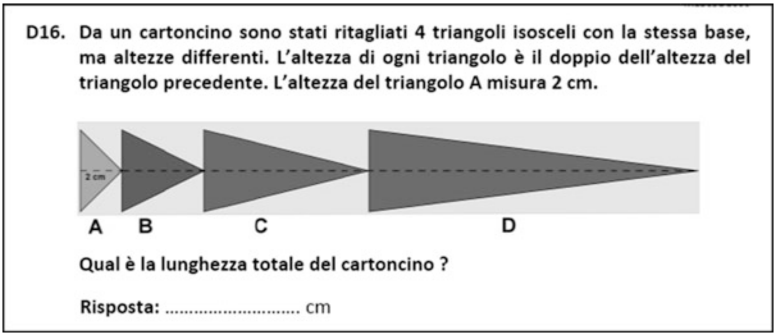

Fig. 1 Non prototypical positions in an Italian National Assessment item (INVALSI 2012–2013), grade 5

According to Lurçat (1980), our subjective references depend heavily not only to external objects (e.g. a door in a room), but also on our ways to project our body schema into objects. Subjective reference systems can be *egocentric*, if the description is provided according to the subject position (e.g. "to my left") or *allocentric*, when the reference is made with respect to another object or person (e.g. "to the left of the house"). Egocentric systems are the first to develop in children, but not the only ones. While Piaget and Inhelder (1956) claimed that children until 8–9 years of age are incapable of decentralize with imagination and so of correctly using allocentric references, following studies have refuted this conclusion, and proved that also children aged 3 are able to decentralize, if faced with problems comprehensible to them (for a discussion, see Donaldson 2010). Being able to coordinate egocentric and allocentric perspectives constitutes an important competence for spatial and geometrical development, and in Italian curriculum is placed as a goal for Primary school (MIUR 2012). An example of task requiring this competence is reported in Fig. 2, again from the Italian National Assessment test. Two children are looking at an object from different positions and the students are asked what the girl is seeing, thus activating an allocentric perspective:

On the base of this discussion of results from psychology, we can ground the hypothesis that the reality faced by young children (and indeed, by all of us) is full of *cognitively-different* spatial contexts, which require different related specific competences. In order to reach this goal, Lurçat (1980) underlines the importance of choosing carefully the requests to the child in the spatial activity:

> . . .not all spatial behaviours necessarily imply a knowledge on space. In order to have knowledge, a suitable activity is necessary: for instance, going in a place, locating objects, positioning in the space of places and objects [. . .]. As in other psychical fields, it does not exist an age for the development, which can be considered independent from the concrete conditions of existence (p. 16, translation by the author).

An educational implication of this perspective is that in order to develop the necessary different spatial competences, children need to be involved since their early childhood in dedicated activities with dedicated task design. For instance, in

Fig. 2 Allocentric perspective required in an Italian National Assessment item (INVALSI 2012–2013), grade 2

order to foster the passage from ego-based to allocentric references and lay the foundations of objective reference systems, activities on the change of points of view, such as the realization of maps of familiar places, can be proposed already at the kindergarten.

Along with meaningful experiences in different spaces, languages constitute a second fundamental set of sources of knowledge, including verbal and non-verbal means of communications.

The key role of verbalization not only as a communicative means but also for thinking processes has widely been discussed in Vygotskian studies (e.g. Vygotsky 1934), and stressed by Lurçat (1980) concerning spatial development:

> It seems hard to separate, in the appropriation of the environment realized by the young child, these two sources of knowledge, the one practice, the other verbal, since both converge early in the first months of life (pp. 15–16, translation by the author).

For mathematics, we know the importance of symbols and graphical representations of various kinds—in particular for geometry, of geometrical figures and Cartesian plane systems. Each of these representations situates in a specific way in the external space of the child: usually, school lessons heavily exploit bi-dimensional micro-spaces, such as the blackboard, the book sheet, or more recently the computer/tablet screen. The passage from experience and perception in the tri-dimensional (macro-) space to these representation spaces is a very complex process, so far little studied in literature. Also at primary school, this passage is often taken for granted and in many cases written representations are used but not problematized.

In such a passage, on the one hand the use of artefacts can be exploited as didactic resources in the development of children spatial competences, and on the other hand gestures and embodied means of expression may play an important role in synergy with verbal language, according to a multimodal perspective (Arzarello et al. 2009; Bazzini et al. 2010; Sabena et al. 2012). The role of artefacts will be discussed in the next section.

The role of embodied resources such as gestures, gazes, and body postures in thinking processes (and of course in communicative ones) has been pointed out in psychological literature with cognitive and linguistic focus (McNeill 1992, 2005). The study of gestures and embodied resources in synergy with verbal language has gained a certain attention also in mathematics education, in an increasing variety of contexts, such as: students solving problems (Radford 2010), students and teachers interacting (Arzarello et al. 2009; Bazzini et al. 2010; Bazzini and Sabena 2015), the teacher's lectures (Pozzer-Ardenghi and Roth 2010), considering not only the semantic but also the logical aspects of mathematical thinking (Arzarello and Sabena 2014). For what concerns spatial tasks, *iconic and pointing gestures* come to the fore: iconic gestures are those ones which resembling the semantic content they refer to, and pointing gestures are usually performed with the index forefinger and have the function of indicating something in the actual context.

The Teaching Experiment: Methodology

On the base of the outlined theoretical frame, an experimental study has been planned and carried out in a kindergarten school in Northern Italy, with the goal of studying the didactical possibilities for children spatial conceptualization offered by programmable robot toys.

The study is based on the teaching-experiment methodology. The activities have been organized around a programmable robot[1] with a bee-shape (Fig. 3a), a technological artefact new to the children. The robot is a kind of tri-dimensional and touchable version of the well-known Logo turtle by Papert (1984), and its movement can be programmed through buttons placed on the upper part (Fig. 3b): they are four arrows for onward and backward steps, right and left turns, and a pause of one second. The robot bee can move on a plane with 15 cm-long steps (the same measure of its length). Steps are marked by a quick stop, which creates a silent pause with respect to the noise of the movement, and by the lightening of its eyes (see Fig. 4b). Pushing the green button "GO", the robot executes the previously programmed sequence. A specific button ("clear") allows the user to clear the memory from past commands.

[1]Bartolini Bussi and Baccaglini-Frank (2015) carried out a study with the same artefact in primary school, about the introduction of the definition of rectangle.

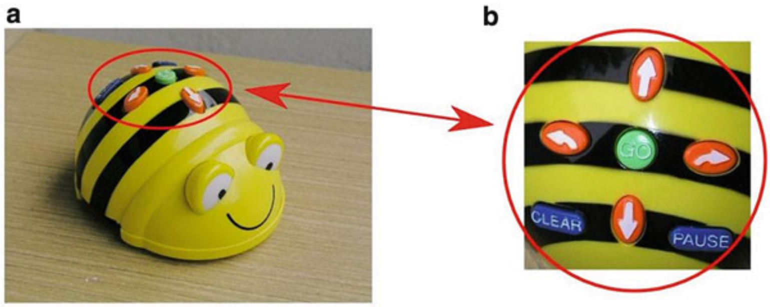

Fig. 3 (**a**, **b**) The programmable robot used in the teaching-experiment

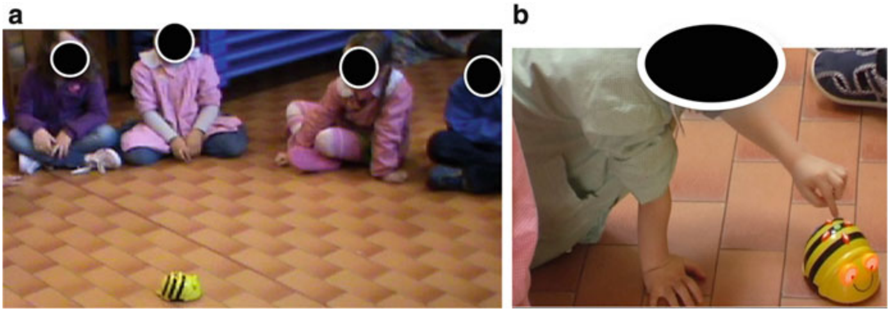

Fig. 4 (**a**, **b**) Initial exploration of the artefact in the meso-space

The teaching-experiments involved four classrooms of 5 years old children, and was carried out with the collaboration of the four teachers, four Master students in Primary school education, and the author. Being inserted in the usual school activities, the experiments had didactic as well as research goals.

From a didactic point of view, the activities had the general goals of promoting competences related to spatial thinking, but also problem-solving. These competences were linked to the use of a new artefact, in the context of exploring it through a playful environment. Concerning spatial thinking, the passage from egocentric to allocentric reference systems is particularly involved, in particular when the robot is not oriented parallel to and with the same orientation as the children. On the other hand, the activity of programming in advance the movements of the robot, and checking afterwards the consequences of the choices, by means of observing the obtained movement, offers a suitable context for stimulating and developing anticipation and control processes, which are at the base of successful problem-solving (Martignone and Sabena 2014).

The didactic dimension intertwines with the research one. The study had mainly an explorative character of the potentialities and the limits of the artefact-based activities with respect to the identified didactical goals. Such an analysis needs to consider the specific activities proposed to children, and the role of the teacher in their management.

The key-role of the teacher in using with success artefacts in the mathematics teaching and learning has been pointed out and stressed by Bartolini Bussi and Mariotti (2008):

> The role of the teacher is crucial, in fact the evolution of signs, principally related to the activity with artefacts, towards mathematics signs, is not expected to be neither spontaneous nor simple, and for this reason seems to require the guidance of the teacher (ibid., p. 755).

Adopting a Vygotskian perspective, Bartolini Bussi and Mariotti elaborate the Theory of Semiotic Mediation, according to which the fundamental elements of didactical activities involving artefacts are the signs that emerge when using the artefact, and above all the role of cultural mediator accomplished by the teacher when using the artefact as a *tool of semiotic mediation*: this expression refers to the fact that when using an artefact (for accomplishing a certain task) new meanings emerge. These meanings are linked to the use of the artefacts but can be general and can evolve under the guidance of the teacher:

> Any artefact will be referred to as a tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a design didactical intervention (ibid., p. 754).

An important didactic feature of this theory is the “mathematical discussion” (Bartolini Bussi 1998), in which the whole classroom is collectively engaged in discussing the personal meanings emerged from an activity, relating them—with the essential guidance of the teacher—to the mathematical signs.

The teaching experiment with the robot has been planned sharing the same Vygotskian view, assigning great relevance to the peer as well as teacher-students interaction, and focusing on the evolution of signs developed during the technology-based activities. Due to the young age, the specific mathematical contents have been limited, and the discussions have regarded more general competences, at the base of spatial and logical thinking.

The activities have been video-recorded and the obtained videos have been analysed in detail. Furthermore, children written drawings related to the activities have been collected and analysed.

The Teaching Experiment: Analysis

Children were organized in groups of about 10–12, with one or two bee-robots at disposal. For each group, the activities developed along 5–6 one-hour meetings,[2] for a period of about 1 month. Most of activities involved the whole group, with the

[2]In Italy, usually we use the term “lesson” starting from Primary school, were formal education begins (also with textbooks, notebooks, and so on). In kindergarten, activities unfold in a less formal way.

Fig. 5 Egocentric perspective kept during the comparison of steps lengths

coordination of the teacher, and only in some cases individual work was required (e.g. to produce a drawing).

The first meeting was always dedicated to the introduction of the new artefact. The initial exploration of the robot has been carried out letting the children play with the robot. In some groups, the activity was organized around a table, while in others children were sitting in a circle on the floor (see Fig. 4a): the resulting delimitation of space produced a sort of meso-space, since the children could globally perceive it with their sight, but also enter into it and explore it with their body.

One of the games played in this context was "sending the bee to my friend (name)". In this game, each child had to name a friend, and to program the bee so to be able to send it where stated. We observed that when programming, every child always started positioning herself/himself behind the robot (as in Fig. 4b). It is the most natural choice, since it keeps the cognitive burden low: in this way, in fact, the reference system introduced by the robot (allocentric system) is coincident with the child one (egocentric system). We kept therefore this choice in those activities focusing on more specific aspects of the artefact, such as estimating the length of the steps, compared with those of the teacher or of the children (see Fig. 5).

Other games required the imitation with one own body of some movements made by the robot, with or without verbal description. The imitation is simple if the child is oriented in the same way of the bee-robot (for instance, if the child is following the robot), because grounded on the ego-based reference system. When the robot is oriented differently with respect to the child, the task increases in difficulty, because it requires reproducing, during one's own movement, an external point of view. In other terms, it requires coordinating the egocentric system not only with an allocentric one, but with a mobile allocentric one: it is a coordination constantly in need of control and adjustments. In our experiences, verbalization has constituted an important supporting tool: when accompanying the bee-robot movement with a verbal description (such as 'onwards, onwards, onwards, turn right'), the task was more easily faced by children. However, verbal indications were of little help for children with difficulties in knowing right from left (a problem for which the bee-robot could not offer any support).

Fig. 6 'The bee-game': Ego-centric perspective to program the movement

With each group of children, at least a couple of meetings were dedicated to an activity on a poster showing a path to be travelled by the robot. The paths were all structured with lengths multiple of 15 cm (the exact dimension of the robot, and of its steps) and with right angle turns, so to be viable by the robot in an exact number of steps and rotations. These choices were meant to ask the children to program rotations, which were never activated in the explorative phase, avoiding problems provoked by non-perpendicular turnings—impossible to program with the bee-robot.

An example is the 'Bee game'[3] (Fig. 6), a sort of Snake and ladders game. The game setting facilitated the introduction of the rule of 'moving the bee only through its buttons' (and not pushing or rotating it with the hands, as the children were tempted to do. . .). In our intentions, the race setting would have also fostered the need of programming as many segments of the path as possible, in order to reach a farther place. For instance, if the first roll of the dice gives '3', the children have to program the sequence 'two onwards, turn left, one onward'. However, in our experiments the children did not fulfil this expectation. Indeed, in all groups children preferred to program one segment at a time: in the given example, programming two steps onwards, observing the robot movement, then programming one turn leftwards, observing the turn, and then programming the final two steps. Figure 6b shows a child while programming this last segment: again, the ego-based perspective is taken by the child in order to carry out the task.

Probably we missed the occasion of challenging the children, by introducing an additional rule, such as 'programming the robot sitting always in the black arrow place'. This request would have forced the children to coordinate their egocentric perspective with the moving perspective of the robot (allocentric for the children).

The activities were alternated with collective discussions, which constituted occasions for reflection on what happened. Discussion organized *before* to carry out new activities are of particular interest. In a group, a guided discussion

[3]In Italian the popular game Snake and ladders is called 'Gioco dell'oca' ('The goose game').

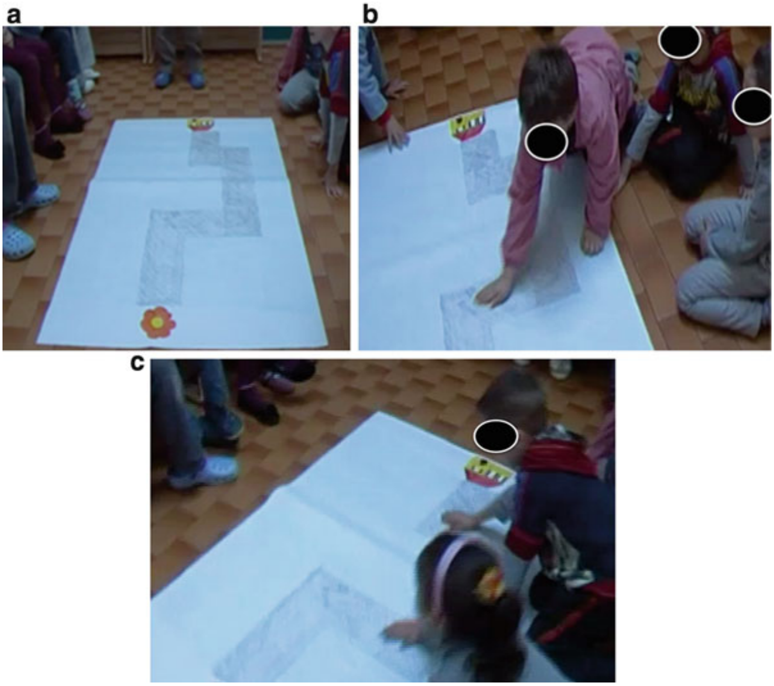

Fig. 7 (**a**–**c**) The setting of the activity "Let's help the bee to reach the flower"

introduced the activity on the path. The bee-robot was not on the scene (Fig. 7a): the discussion constituted a moment of reflection for the children, during which the development of the spatial competences is realized by observing and describing the present scene, but also recalling past experiences with the artefact, and anticipating potential actions through imagination. We are going to analyse in greater details what happened.

The teacher guides the discussion with the goal of making the children to observe that the path is not linear. As a matter of fact, in the previous activity children moved the robot using only the "onward arrow", without turns. The general goal of the activity is to make the students program more complex sequences involving turns, such as 'forward-forward-turn left-forward'.

1. *Teacher*: Today we explore this (*looking at the poster*). What comes to your mind looking at this?
2. *Stefano*: It is a road
3. *Viviana*: A flower and a house
4. *Teacher*: And whose is the house?
5. *All the children*: The bees!
6. *Teacher*: And how is it this road? Is it straight?
7. *All children*: Noooo!
8. *Stefano*: It has some curves (*with his hand he is traveling the road,* Fig. 7b)

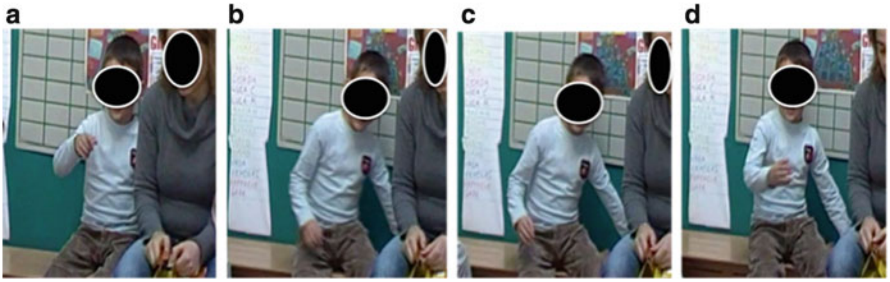

Fig. 8 (**a**–**d**) Fabio's gestures accompany the introduction of the terms « straight » and « turn ». In pictures **b**–**d** also the body rotation can be observed

9. *Cristina:* It makes like this and like this (*she travels the road with her hands, as Stefano is doing*)
10. *Other children do not make any verbal comment, but touch the entire path with their hands* (Fig. 7c).

The teacher's questions have the goal to help the children becoming aware of the characteristic of the road along which they will make the bee travel. Though not explicit, they play an important role with respect to the anticipatory thinking needed to program the robot. The children immediately answer to the question with very poor descriptions, made of the list of the elements of the poster, without relating them each other. They describe the road with deictic terms ("like this") that contain little information without the co-timed gesture. In order to push them to provide more suitable verbal descriptions, the teacher closes her eyes and asks them to better explain:

1. *Teacher*: And then? Let's do like this: I close my eyes and you tell me how is the road, because I do not know it...Is there a starting point? And an arrival? Explain to me.
2. *Fabio*: The start is in the house and maybe over there (*pointing gestures*) where there is the flower, it is the arrival.
3. *Teacher*: But in this way I would not be able to arrive: you must explain well.
4. *Fabio*: You must go straight (*pointing gesture,* Fig. 8a), then turn (*moving and turning his body,* Fig. 8b, c*, and making a turning gesture with right hand,* Fig. 8d), go still a bit straight, then turn again, go straight and you are arrived at the flower.

1. *Teacher*: But I don't know where to turn, how can I understand which part to turn...
2. *The children continue to explain mainly with deictic terms such as "here", "there", accompanied by gestures.*
3. *Teacher*: No, no, if you had to explain it only with words?
4. *Chiara*: Left and right

5. *Teacher*: Left and right, or towards. . .So, explain me better, you can do it: not like "I make some curves", but how many, I go straight and how far, or rightwards, or towards the benches, towards the door. . .

The first description provided in describing the poster referred to *static* elements: the house, the flower, and the road (lines 2–6). Soon after (from line 9), when pushed to better describe the path, a *dynamic* perspective is brought to the fore: children use dynamic pointing gestures (also materially touching the poster) and then words referring to the motion along the path (e.g. Fabio in line 13).

The teacher suggests some reference points, such as the starting and the arrival points (line 10), and insists on asking the children to provide a clear explanation ("explain well"). In line 13 Fabio introduces two verbs that characterize the movement of the robot: going straight, and turning. The introduction of these two terms is accompanied by two specific gestures: a deictic gesture made with the extended index (Fig. 8a), and a dynamic gesture, combined with the full-body rotation (Fig. 8b–d). The body movement and the hand gesture are the only semiotic resources that express the information about the direction of the rotation (leftwards). The teacher insists constantly about more accurate verbal descriptions, making this goal explicit to the children (line 15), and giving some indication on what aspects to mention: quantifying (line 17: "I go straight and how far"), subjective ("rightwards"), and objective references ("towards the benches, towards the door"). Analysing the following part of the video, we can see that children will seize only the subjective references, whereas for the quantification they will go by trial and error with the bee-robot.

The first path is run with the bee-robot programmed only with straight short traits, so the teacher asks to make more elaborate programs. But before asking to program the entire path, she sets an intermediate goal, consisting in programming until the third square, indicated with e deictic gesture on the poster (Fig. 9).

Fig. 9 The teacher indicates an intermediate goal to reach

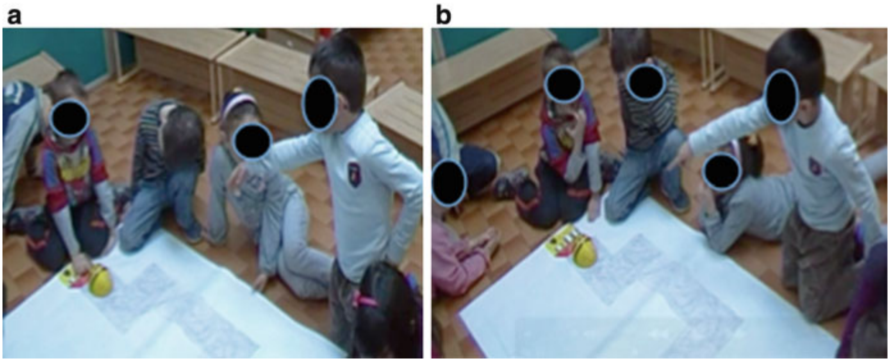

Fig. 10 (**a**, **b**) Fabio's gestures during his verbal instructions

The first attempts fails. Fabio then wants to give instructions to his mates, and the teacher pushes him to state precisely what he is saying:

1. *Teacher*: Think at which arrows you have to push.
2. *Fabio*: So, you have to push once the arrow, right
3. *Teacher*: To go forward, straight, backwards. . .how?
4. *Fabio*: Forward. Then you must do the. . .left. . .here (*he indicates the direction to the left of the robot,* Fig. 10a), then you must do again
5. *Anna*: Straight
6. *Fabio*: Straight and. . .and then we arrive here (*indicates on the path the third square, that is the arrival point stated by the teacher*).

The teacher asks to Fabio to repeat his proposal, so that all children can listen to it, before to check with the robot. Fabio would like to act directly on the robot, but the teacher insists that he gives the instructions from his place: the child accompanies then the verbal instructions with deictic gestures (Fig. 10b), and his mates follow them. We observe that Fabio is placed on the side with respect to the path direction: his egocentric reference system is therefore not aligned with that of the robot. Looking at the video we can clearly see that the child meets difficulties in accomplishing this task: to overcome them, he speaks slowly, and tries to incline his body so to position himself in the same direction as the robot (this can only be guessed by Fig. 10b, but is clearly visible in the video). The problem of *programming many steps consecutively*, when rotations are included, seems therefore strictly linked to the problem of *coordinating different reference systems*. The specific requests of giving instruction to others, while remaining far from the robot and in a different position, allowed Fabio to face the difficulties of the task, and to overcome them successfully, activating and developing his spatial competences, intertwined with the anticipatory thinking. As we can see from Chiara's intervention (line 22), also other children participated to Fabio's endeavour, either listening carefully, or pushing the robot buttons, or suggesting words, or checking in their mind his instructions: through the social interactive context, the request made by the teacher to an individual child, becomes a resource for making all children facing the complex task, each according to their actual capacities and specific attitudes.

Conclusion

Through teaching experiments in kindergarten, some potentialities and limits of robot-based activities for the development of spatial conceptualization were investigated. They were intertwined with anticipation and control competences, crucial to problem-solving in different fields.

Programming tasks, in fact, require children to imagine the consequences of their own actions, and allow later them to verify their correctness (in our case, through the observation of the robot actual motion). *Anticipatory processes*, that are cognitive processes carried out while imagining the consequences of our actions in a hypothetical future, are of paramount importance in problem-solving activities (Martignone and Sabena 2014). Their counterpart is *control processes*, which can be activated when checking if the actual robot motion does correspond to the programmed sequence of steps. In the light of our experimentation, we can affirm that robotic artefacts can offer great potentialities for the activation of these kinds of processes, but such activation requires an acute attention that in 5-years-old children is still in its initial development. Many children, in fact, showed great difficulty in keeping in mind even a small sequence of commands, and this difficulty made impossible to them to activate suitably control strategies.

For what concerns spatial conceptualization, robotic activities carried out in the material world can foster in children the intertwining and coordination between different reference systems. As discussed in the first part of the chapter, the coordination between different reference systems and points of view is necessary in order to face geometry problems.

A first remark regards the activations of different reference systems. In our observations, in order to face the proposed tasks, children always spontaneously took the egocentric perspective. Of course, to make sense of what their mates or the teacher were doing with the artefact, children were often in the need of coordinating their ego-based perspective with the allocentric one assumed by the robot. However, our findings suggest that specific constraints have to be set up on the task in order to ‘force’ children to actively work with allocentric perspective: for instance, have the children to imitate the movement of the robot when is not parallel to them, or to program it from a certain fixed position.

Both ego- and allo-centric perspectives are subjective reference systems, used in the space of reality. As discussed above, geometrical space requires the use of objective references. In the proposed activities, we did not focus on the passage from subjective to objective references. Some hints have been made by the teachers (as the one documented in the analysed episode), but with no success. Our

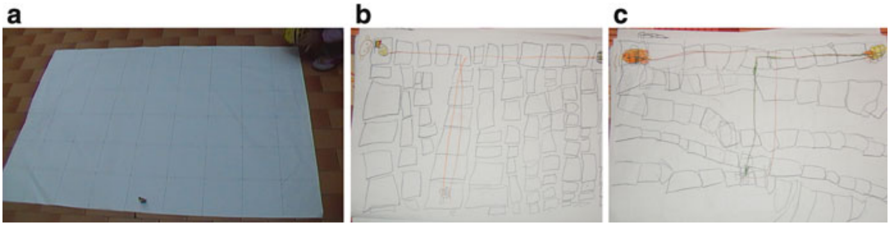

Fig. 11 (**a–c**) The grid used during the activities with the bee-robot and two drawings made afterwards by children

impression is that specific activities need to be designed in order to reach this goal, possibly in later age.

A second remark concerns two *different spatial conceptualizations* that emerged during the artefact-based activities: a *static and global* one, and a *dynamic and paths-based* one. The two perspectives do not constitute a dichotomy. For instance, in line 8 in the excerpt above Stefano is blending both of them: his words are referring to a global feature, and the gestures expressing dynamic ones (Fig. 7b). In the overall experimentations, gestures have often offered a window into the children's conceptualization of space, and new spatial terms have often been used the first time accompanied by corresponding gestures (as Fabio in line 13).

Evidence of how the experience with the robot paths has influenced the children's conceptualization of space can be seen also in several children drawings. Figure 11(b, c) reports the drawings made by two children, who had had the robot moving on a grid made by straight lines (Fig. 11a). In the children's drawings, the grid looses its global features and becomes a sequence of steps.

The paths-based perspective has been certainly fostered by the use of the bee-robot, and future research is needed to investigate its role in early spatial thinking. Studies in cognitive science within the embodied mind approach have shown that motion constitutes the source domain of many concepts, and that also static objects are often conceptualized in terms of motion[4] (Lakoff and Núñez 2000). Starting with motion activities seems thus promising for children spatial development.

The last remark concerns the crucial role of the specific requests made to the children. For instance, we encountered a great "resistance" from children to program sequences of steps that could include one or more turning: they preferred to divide the path in straight parts, and program each of them separately. Rotations in particular were never spontaneously linked to following onward steps. Probably programming an entire long sequence requires cognitive capacities still under construction by the children, but maybe the main difficulty lies in the fact that the goal of reaching a certain place through a single program sequence had not any understandable 'sense' for the children (Donaldson 2010). We could observe that even when this goal was proposed within a competitive setting (like a team

[4]Talmy (2000) has called 'fictive motion' the cognitive mechanism underlying the description of a static object (e.g. a path, in our example) in motion terms (e.g. 'it starts. . .it goes. . .').

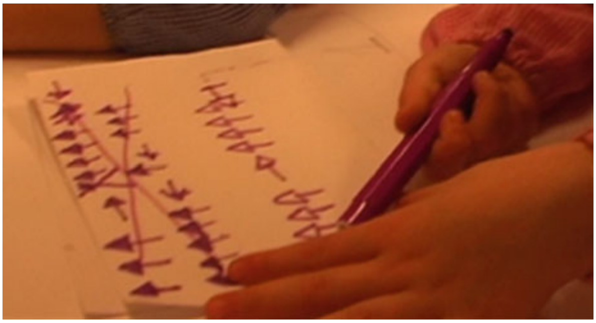

Fig. 12 Written signs representing the programmed sequence of steps

competition), the children did not undertake it. As a matter of fact, programming a certain artefact using less time as possible can be a goal for adults, which are often under time pressure. In the case of children, pleasure was given in using the robot as long as possible, because they liked it. In our task design, we initially underestimated this essential dimension, and not a few times the goals that we had chosen for the activities were completely neglected by the children.

The mediation of the teacher has therefore been necessary to introduce the possibility itself of articulated programs, and to make their benefits explicit to the children. The teacher mediation in the activities was accomplished through natural language, as well as embodied resources such as gestures, as in the analysed episode, but also through the introduction of written signs to register the commands given or to be given to the robot (see an example in Fig. 12).

The different resources (words, gestures, written signs) intertwined in complex interpretative processes of the programming code used by the robot, represented by the arrows buttons (Fig. 4b), and its actual movement. The introduction of written signs has not been here discussed, and requires further examinations. It has a limited scope for kindergarten level, but it constitutes an interesting didactic path for primary school, since it regards the delicate passage from experiences in macro-space of reality to the use of micro-space of representation, the fundamental background of much geometric activity.

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Mediation of Technological Resources in Lessons on Polyhedra: Analysis of Two Teaching Actions

Nielce Meneguelo Lobo da Costa, Maria Celia Pimentel de Carvalho, and Tânia Maria Mendonça Campos

Abstract This chapter aims at discussing the mediation of technological resources done in geometry classes, in particular, for the teaching of polyhedral-prisms and pyramids – in primary school. The focus is to understand teaching actions and the use of technological resources by observing two teachers on geometry classes and analyzing results from this "in loco" survey. Zeichner's ideas about teacher's reflective practice and the view of Serrazina about the teacher as a manager of curriculum defined the theoretical framework. The methodology was qualitative, grounded in the complexity theory. We observed two different teachers' actions and the mediation of available resources in order to help students' participation and the development of the classes.

Keywords Teaching practice • Technological resoucers • Geometry classes • Primary school • Mediation

Introduction

Mathematics provides the students with the possibility of dealing with everyday situations, criticizing, developing creativity and skills such as communication, reasoning, phenomena interpretation and solving routine problems among others, leading those students to reason according to their existing and/or intuitive knowledge.

The study of Mathematics in elementary education intends to develop some students' skills, such as communicating, thinking in a variety of ways, interpretation phenomena, solving daily problems, among other activities that make them reason through their existing knowledge and intuition or make them try to construct new knowledge. This way the students get mobilized in different situations of learning.

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In order to assist students to develop these skills, the teacher needs to propose interesting and challenging teaching situations. The technological resources selected by the teacher to do this, may help the students to establish the connection between the classroom, the society and daily life. This will leading them to perceive mathematics as a tool for thought and for facing challenging situations. The students will become agent of social change. The use of technological resources is central in the composition of teacher's pedagogical practices and another critical issue is how to do the mediation of the chosen resources.

According to Ma (1999), in order to improve the mathematical knowledge of the students, one should begin by improving the teachers' own knowledge of the subject, that is, elementary school mathematics. Such knowledge regards the teachers' initial education and continuing education programs. The act of thinking would help update their practice in the classroom, both in planning and in relation to their peers.

Therefore, we started thinking about the matter: *What is the current state of integration of technological resources regarding teaching and learning geometry in elementary schools?* With this question in mind we start to research what really happens in two class related to the use of technological resources by primary teachers.

We regard as technological resources, every material that a teacher may choose to use in class in order to help students mobilize their existing knowledge as well as to assimilate and integrate new knowledge. We consider as technological resources, materials like chalk, chalkboard, ruler etc. We follow Moran (2007) which defines technological resources as the means, the support and the tools that teachers use in the classroom. We agree that writing with chalk on the blackboard is a communication technology and good organization makes writing and learning easier. They also mention that the teacher´s way of looking, making gestures and speaking, may generate good or bad results in the action, knowledge acquisition. So, they may help develop the students' learning process.

In this chapter, mediation of technological resources in geometry classes will be analyzed from the action of two primary teachers in the same school, notably the 5th grade. This text is an enlarged version of Carvalho et al. (2013).

The Research

The investigation that supports this chapter aimed at investigating the pedagogical practice of teaching in elementary schools, particularly at the use of technological resources and the mediation of these resources. We have observed two different teacher's actions and the mediation of available resources in order to help students' participation and the development of the classes. The focus of this chapter is to understand teaching actions and the use of technological resources by observing these two teachers in geometry classes, and to analyze results from this "in loco" survey.

The research theoretical basis stems from the following: Zeichner's (1993) reflective teacher and Serrazina's (1998) view on teachers and their relationship with curricular development. Serrazina considers reflection as an important component of the teachers' professional development, since it will enable them to enhance their theoretical and methodological knowledge, thus constantly deepening their reflections.

Zeichner (1993) also shares the idea of reflection for professional development. He believes that teachers act according to their personal theories, so in order to understand their teaching practices, it is necessary to analyze the situation in which teachers do their work. The author goes on to say that when the teacher does not reflect on his or her teaching practices, they are doomed to follow the same procedure class after class, while applying a curriculum they have not chosen.

The research methodology was qualitative grounded in the principles of the Morin's (2006) complexity theory, assumed by Moraes and Valente (2008) as a way to do qualitative research.

For them search, from the perspective of complexity, is to assume the interpretative character and epistemological dimension that asserts that knowledge is not copying of reality, but rather a result of the action that considers the individual cognitive structures. Also within this classification, consider the methodological dimension of research the predominance of qualitative methods, whereas the dialogue with the quantitative methods, if there are theoretical and methodological compatibility for both. The strategy is the action method open adaptive and evolutionary knowledge, contemplating not only the process but the product as well. The search procedures adopted in this respect are flexible and revisable in every stage of the investigation. Uncertainty is always present in the pursuit of scientific truth. This means that Moraes and Valente (2008), claim that the entire "objectivity is always an objectivity in parentheses, since the observer, whether consciously or not, is always included in the system that distinguishes" (p. 8), participating in the reality to be investigated. These authors still summarize it by saying that the researcher faces:

> a relational, indeterminate, non-linear, diffuse and unpredictable dynamic reality. This multidimensional reality is possessed of a complex nature, consisting of different levels: a macro physical, a microphysical and a virtual one. Thus, the complexity pervades the different levels of reality. It is also a constitutive factor of life that allows this common tessitura and the existence of different life nourishing streams of life and are nurturers of their relational, interdependent and self-organizing processes. (p. 21)

The validity is directly linked to the accuracy of the results requiring the rationale of the essential concepts, the definition of data to be collected, the processes and tools used for the collection of such data, organization, analysis and interpretation of data, the tools used and how the data are analyzed. In this sense, it is worth emphasizing the non-neutrality of the observer researcher, since he is the filter and the reading of reality that is personal and not neutral.

The investigation was done in two phases: documentation and field research in an elementary public school in São Paulo city by observing two 5th grade teachers in action in 16 lessons with children aged between 9 and 10 and some of the school

1. Lesson procedures (classroom learning environment, what is said, how it is said).
Order and work habits revealed.
2. Interactions with the students
The role of an elementary school teacher
✓ Proposing questions, showing models to be followed, negotiating/imposing criteria, supervising activities, providing information/resources, clearing doubts.
The role of elementary school students
✓ Proposing questions, being mere executers/spectators, participating in decisions, proposing initiatives, managing their own activities and others.
✓ *Student-student interaction in the classroom regarding the job at hand.*
3. Development form of the mathematical content
The teacher explains significance criteria
✓ For the proposed learning tasks
✓ For the proposed situation
Evaluation carried out at the observed situation
4. References regarding innovation – technology resources

Fig. 1 Class observation protocol (Source: Private collection, Adapted from Hernandez et al. 2000)

meetings. We collected the data, in the phase of field research, by a questionnaire, semi-structured interviews with the two teachers, classrooms audio records, video, images, and a researcher field book.

The analysis was carried out from three main points: mathematical content, the practice and the technology used during the classes. The categories analyzed were the class routines, the interactions with the students, how the math contents were developed and the technology used (see Fig. 1). The observation protocol based on Hernández et al. (2000) guided us to compose those categories. We selected the following analysis aspects: lesson procedures (class routine, surface organization and work habits), interactions with the students (the role of the teacher, the role of the students, student-student interaction), how the students behave concerning the task, clear significance and evaluation criteria.

The Research Subject

The characterization of the research subjects Teacher Piera and Teacher Ana (fictitious names) was done from the semi-structured interview and from the initial questionnaire:

- Teacher Piera is a pedagogue and a specialist in educational psychology. She has been teaching primary school for 34 years and prefers to teach fifth graders. She considers the use of technological resources essential to the education process, she is interested in learning using technological resources, but she has not used educational software aimed at teaching math in her classes. For her the technological resources that can foster learning are ludic activities, games, dynamic, calculators, concrete material, music, drama, etc.

- Teacher Ana is a pedagogue and has been teaching primary school for 17 years. She also prefers to work with fifth graders. She does not use mathematical software in her classes, but uses other digital features such as calculators and DVDs besides the non-digital resources as books, golden bead material, solid geometric shapes, etc.

We observe that both teachers are experienced in teaching and they have good academic background. Both prefers to teach fifth grade students because they affirm it is easier to control the classes, and that the students are more obedient and complying. Both teachers declared that they used different technological resources to teach mathematics, but they did not use software to teach geometry at the time of the survey.

Mediation of Technological Resources: The Episodes

In this chapter, we chose to discuss episodes of classroom situations, in which both teachers approached the same geometry content. In them we were able to observe the actions of both observed teachers and here we focused the discussion on different mediations and classroom management styles.

The following reported episodes are related to Teacher Piera's fourth observed math lesson and to teacher's third observed math lesson.

The observed classes established the same objectives as the competences and abilities to be developed by the students and indicated the use of the same technological resources. (See Fig. 2)

We started discussing the episode of Teacher Piera's mediation, related to the situation she developed with her students, described below.

Piera's Mediation to Explore Prisms

The lesson focusing on prisms started with the teacher asking the students to open the Student's book to page p. 28 (see Fig. 3).

Content: Space and form: prisms
Skills: Recognizing similarities and differences between polyhedral (such as prisms, pyramids and others). Identify relationships between the number of elements (such as faces, vertices and edges of a polyhedron).
Resources: Student's book (p. 28 and 29), chalk, blackboards, notebooks, pencils, erasers, scissors, cardboard, a set of prisms in wood or cardboard, scrap.

Fig. 2 Content, skills and resources (Source: Carvalho, p. 93)

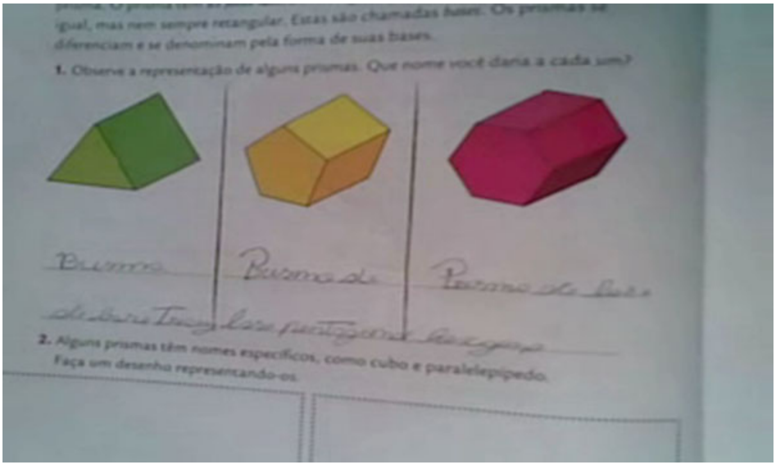


Fig. 3 Student's book (São Paulo, 2010) (Source: Carvalho, p. 97)

It took a few minutes for students to calm down in their seats. The teacher asked the students to listen while she read the student's book.

Teacher Piera read:

– Page 28, prisms and their denominations, you noticed . . .

She interrupted her reading to remember what happened in the previous lesson:

– Do you remember that in the last class we saw the boxes that Patricia, the little girl built? So the first box she built, we built one too, didn't we?

A student corrected her:

– Two boxes.

And the teacher objected vehemently:

– ONE!!! This lilac one, which she built. From the rectangular box she made another, but we just built the lilac one. We put together a little box, okay? Now, today we are going to study prisms.

Teacher Piera read the first sentence of the book and prompted the students, stopping in the middle of a sentence:

– Prisms are geometric figures that have the faces _____???

And asked the students, in a more emphatic way:

– What are faces again, people?

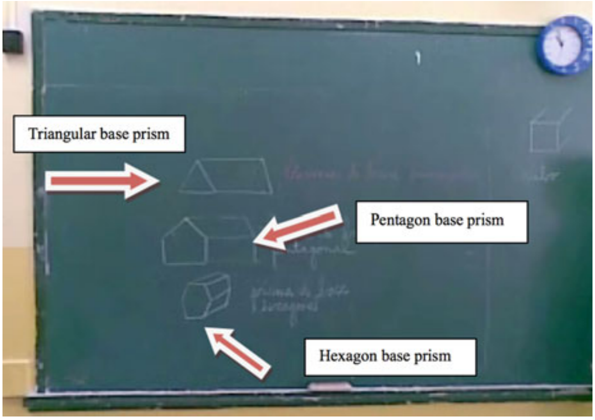


Fig. 4 Blackboard with prisms (Source: Carvalho 2012, p. 95)

Some students answered:

– The sides.
– That's it! Look, remember that we saw them? A prism is any geometric figure that has rectangular faces, okay? It's a rectangle, some are bigger, others are smaller.

It is worth mentioning that the teacher meant the lateral faces although she did not say that. She drew a triangular based prism, a pentagonal base and another of hexagonal base on the blackboard (see Fig. 4).

She explained the concept then:

– We saw, the other day, what Patricia did; she built a little box that had a base. . .

So, she interrupted the explanation and asked:

– What is it, when you have three sides?

Some students said:

– A triangle.
– A triangle, right? A triangular base and sides of the box are ...
– Rectangular. Some students responded.
– We've seen a box like that, but before that, we saw that other one. When you have five sides, what is that?
– A PENTAGON, the students said.
– Look at the base, the two bases are pentagons, because the faces are rectangular, so they are thin.

And she continued:

– Well ... and then?

Now she interrupted to comment with a student:

– You will know how to do all of this, and you will make beautiful boxes.

The teacher continued:

– Prisms are different; they are called according to the shape of their bases. For example, this one is a triangular based prism, why?

Some students shouted the answer at the same time and the teacher replied:

– The bases, eh, there are three bases, so they are triangular. This one is a prism of what type of base?

A student replied:

– A pentagon.

The teacher continued:

– Pentagonal because the base has 5 ...?
– SIIIIIDES! students groaned with boredom.

The teacher said:

– And when you have 5 sides, it is a...?
– Pentagooooon, again they groaned.

Teacher Piera started to talk with the students about the concept of the base of a prism with three, four, five and six edges, which had been seen in the previous class. She asked the students about these concepts, and the students responded in unison.

Then she wrote, under each picture drawn on the blackboard, the answers to be copied by students in their book.

A student criticized the design of a prism the teacher had made on the blackboard, and she said to him:

– Igor, you know better than I do?

She continued:

– I ask you to come here to the blackboard and draw it for us.

And the student went to the board and nailed it. After he has finished drawing, the teacher said that he was too bold to defy her.

We believe that, at this point, one possibility would be for the teacher to have validated the student's boldness pointing out that he had the courage to apply her skills.

The teacher returned to the chalkboard to conceptualize the figures she had drawn. Students responded to questions she asked. She used the same procedure to explain the concept of triangular and pentagonal prisms as she did to explain the hexagonal prism.

So, she asked the students what was the concept of prism by saying:

– Why, again, do you call it Prism?

A student replied:

– Because the bases are rectangular.

And the teacher explained:

– The sides, the faces are rectangular, only the bases are the ones that form different figures, the two bases, now the sides, the faces are rectangles.

It is worth mentioning that by saying "faces" the teacher is referring to the sides and when she says "sides" she uses language that can cause confusion, but those terms are familiar to the students once they have been used when studying two-dimensional figures.

The teacher asked if everyone had already filled the answers in their books and went on to explain the exercise number 2. She explained the exercise in Fig. 5, by writing it on the blackboard.

Piera guided the students and drew on the blackboard showing how faces should be placed on the paper or on the blackboard, in order to represent the 3-dimentional figure (see Fig. 6).

In reality, the teacher used the cavalier perspective for the cube[1]. Geometric concepts to develop the cavalier perspective are supposed to assist in visualizing and in solving problems. However, it has not been possible to identify if she knows rudiments of perspective and if she uses it consciously.

Teacher Piera said to the class:

– What we are doing here [on the board] are 3-dimensional drawings, isn't it "chic"?

Continuing the dialogue, the teacher asked the students to give examples of rectangular boxes they knew. And they said:

– Box and loaf of bread.

[1]You can set the cavalier perspective as an oblique cylindrical projection on a plane parallel to one of the main faces of the object. Most representations of geometric figures in the books are in cavalier perspective. In cavalier perspective, there are the following properties: (1) figures and segments parallel to the plane of projection (paper plane) are represented in true greatness; congruent figures, situated in different planes, but parallel to the plane of the paper, have congruent representations-this is contrary to the vision, but according to the reality of the objects; (2) perpendicular threads to the plane of the paper are represented by oblique segments (if adopted, making angles of 30° to the bottom edge of the paper), and has reduced its length (if adopted, the reduction of 50); (3) parallel to each other and straight segments are represented by straight parallel segments and each other (it is a cylindrical projection); (4) keep the midpoints of the segments and the centroid of figures; (5) as Convention, trace the lines that are visible to the observer and trace the invisible lines. Fonte: http://www.apm.pt/apm/geometria/inoveg/egtext1.html

2. Alguns prismas têm nomes específicos, como cubo e paralelepípedo.
Faça um desenho representando-os.

Proposal of item 2: some prisms have specific names as cube and rectangular box. Make a drawing representing them.

Fig. 5 Student's book – Personal answer (Source: São Paulo 2010, p. 28)

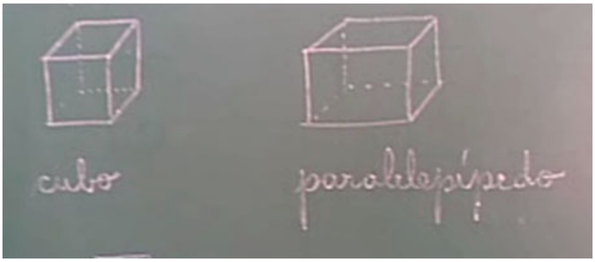


Fig. 6 Cube and rectangular box on the blackboard (Source Carvalho 2012, p. 100)

After this conversation, the students drew a cube on their student's book (see Fig. 7).

She continued showing a worksheet with the printed figure saying:

– Now we are going to try to make a cube. Don't ruin your paper because we do not have any more to spare. What we are going to do first is: paint the figure, then after painting you're going to cut on those outer black lines, after cutting it...,
– Hey, guys, if this is not done, it is impossible to form a solid.

After the guidelines given by the teacher, some students tried to reverse the instructions, for example, first cutting before painting, which the teacher did not allow. It was possible to notice that, at this time of the lesson, the students were involved in the task. Although the students were sitting as if they were working in pairs, each one had his or her own material (Fig. 8).

In Fig. 9 we can see worksheets being painted, cut and pasted by the students.

The teacher was talking informally around the classroom watching the students, while they built the solids. She asked what prism was being built and all in unison they responded that it was a cube.

While the students worked on the solid's plan, Piera copied the constant table on p. 29 of the student's book in order to explain it later.

Fig. 7 On the left, the drawing of a cube made by a student (Source: Carvalho 2012, p. 99)

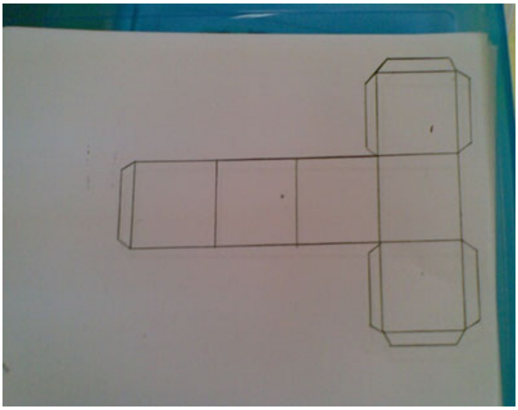


Fig. 8 2D planning of a cube (Source: Carvalho 2012, p. 100)

Teacher Piera picked up a cube she had made and, taking into account, one by one of the items, which the table requested, she filled the number of sides, number of bases, and total number of faces. The students participated responding orally while they calculated their scores and added numbers in their own assembled cubes. The students counted the items of the other figures in the book.

Then she read exercises 2 and 3 on p. 29 of the book and explained the concept of vertex, “where any two lines meet”.

We emphasize in the Student’s book, that item 2 shown in the figure above, proposes a discussion among the students about the information collected on item 1 table (see Fig. 10) and students should write the conclusion they have reached.

However, the teacher chose not to discuss that item. That is, it was not discussed among the students nor did the teacher make any reference to it at the time. She told the students to go straight to item 3.

So, the students answered item 3 based on the figure of a prism (see Fig. 11).

Item 3 proposes the following questions, concerning the figure on the left “How many vertices are there in this prism?” and about the figure on the right “How many edges are there?”

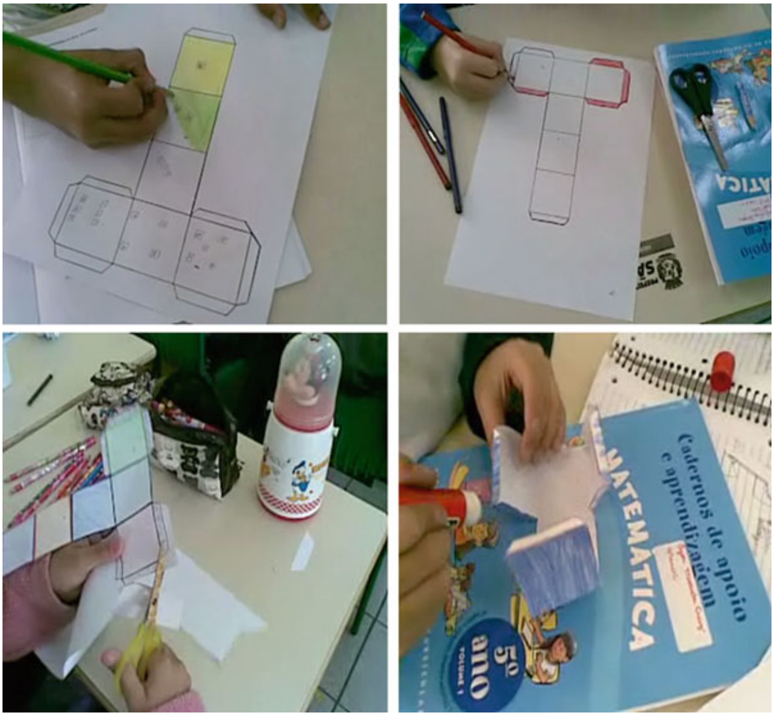


Fig. 9 Students working on painting, cutting and assembling the cube (Source: Carvalho 2012, pp. 101–102)

Contando o número de faces de um prisma

1. Observando as representações dos prismas da página anterior, preencha a tabela abaixo:

prisma	número de faces laterais	número de bases	número total de faces
prisma de base triangular			
prisma de base pentagonal			
prisma de base hexagonal			
cubo			
paralelepípedo			

Prism	Number of faces	Number of bases	Total number of faces
Triangular base prism			
Pentagon base prism			
Hexagon base prism			
Cube			
Parallelepiped			

Counting the number of faces of a prism

1. Observing the representations of the prisms on the previous page fill in the table below:

Fig. 10 Table from the Student's book copied on the blackboard (Source: São Paulo 2010, p. 29)

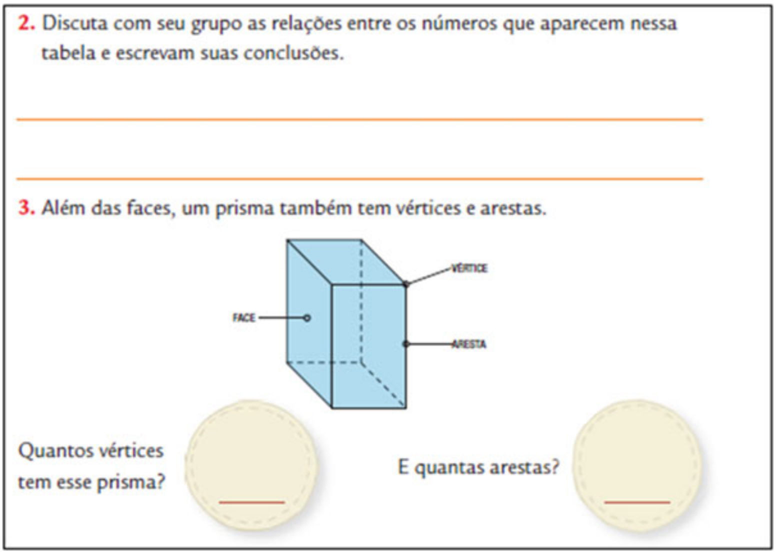


Fig. 11 Exercises from the Student's book, page 29 (Source: Carvalho 2012, p. 104)

Teacher Piera went on drawing the prism on the blackboard and showing where the vertices and the edges were, and suggested that the students touch them in order to feel their forms with their hands. She asked them to show in their production, where the side of the prism was. She asked where the base was, and answered it herself: "they are the two edges."

The teacher asked:

– Where's the edge?

A student replied:

– It is this line here, showing it in the prism.

The teacher continued by explaining:

– And the vertex, which is the meeting point of the two lines. Every time two lines meet, there is a line here and here is another one, this is a vertex. See how easy it is to learn geometry?

After the explanations, she explored in exercise 3, the blue figure, for students to fill out the answer (see Fig. 11).

We have analyzed that when conducting this activity, Teacher Piera's mediation strategy was to avoid the discussion and lead the students to finish the activity since class time was running out. The teacher assigned the homework for the next day and considered the class was over.

Figure 12 shows that the end of the class could have been considered a time of evaluation of the dossier, which was not observed by the teacher.

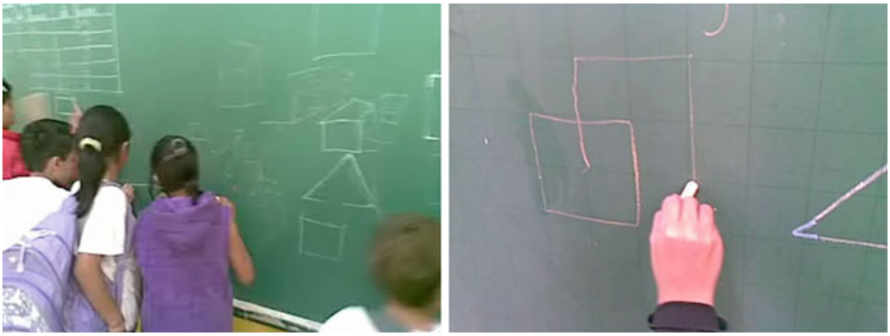


Fig. 12 Students drawing prisms on the blackboard after class (Source: Carvalho 2012, p. 105)

As soon as the bell rang to announce the end of the class, some students literally ran to the blackboard and began to draw geometric figures, showing evidence of their interest, which was not explored by the teacher; and making evident the need for more discussions on the activities (see Fig. 12).

After the lesson, we talked to Teacher Piera and conducted the following interview, about this lesson and the students' development:

Researcher: *Teacher Piera, do you think your students get more involved in this kind of activity?*

Teacher Piera: *Yes, we get a little crazy, but they participate and easily understand, you see? Now I am thinking that the school should also provide some geometric solids, to show to the students, some bigger ones, so that they could handle and touch.*

Researcher: *Do you teachers bring the solids?*

Teacher Piera: *Yes, the school does not have them, but I think it's a question of asking, because when we ask, they buy them, that is when they have the money, but they buy them. For the next budget, if I remember correctly we have asked. The school used to have several solids, they were kept in a bag, in class students formed groups and each student could have and handle at least one geometric solid of each type.*

From Teacher Piera's answers, we can interpret that she considers the type of activity developed in this class as appropriate to promote more active participation on part of the students and, also, to foster learning. However she analyzes that such an activity is more difficult to mediate ("we get a little crazy"), since students are working in groups with more autonomy. That hinders the focus of discussions. In addition, we observed that the teacher declared the lack of school's necessary technological resources, which hinders the development of the activity in class. However, she assumes that this may be due to a failure of the school's own teachers, who do not request such resources at a time when the school has funds to purchase school supplies. This highlights the need for more coordination between teachers and pedagogical coordinator, in order to provide the technological resources that will be used in class.

Also from her answers, we can observe that, in spite of years of practice, the teacher doesn't feel safe and cannot deal very well with situations in which students bring their previous knowledge. As she said to herself, that kind of lesson she gave gets students involved, but drives her a little crazy and she blames the entity for not providing materials that will help the class to be more dynamic. She made it clear that she prefers to conduct the classes according to the school's established curriculum.

We have noticed that the teacher's role was to drive students' development of concepts and rationale for which answers were given. Apparently the students managed their own activity, because the teacher was not always available to check all their tasks in all their notebooks. We have highlighted the action of some students of running to the blackboard when the lesson ended. They felt at ease to show each other their ability to make drawings of prisms.

We can validate what Nacarato (2011) calls the questioning process, which is a tool that aids the formation of autonomous individuals, who will act critically and reflectively, with competence to propose changes when necessary, i.e. have the ability to change the environment in which they live.

The established skills in this class have been developed, with emphasis on materials such as chalks, blackboards, notebooks, pencils and erasers. At the end of the class, the students used scissors, bond paper with 2D geometric shapes.

Ana's Mediation to Explore Prisms

Ana's lesson was the third observed lesson in the research. She began by asking students to open the Students' book top. 25, and worked the content of operations involving natural numbers, additive field, multiplicative field and situations involving composition.

In the second part of the lesson the content of space and form was developed – polyhedral – prisms.

The teacher read the textbook then went on to work the content of p. 26 and 27 as shown in Fig. 13.

In the Student's book there is space for students to create their reply. Teacher went on to discuss p. 26 and, after reading the statement, she asked the students:

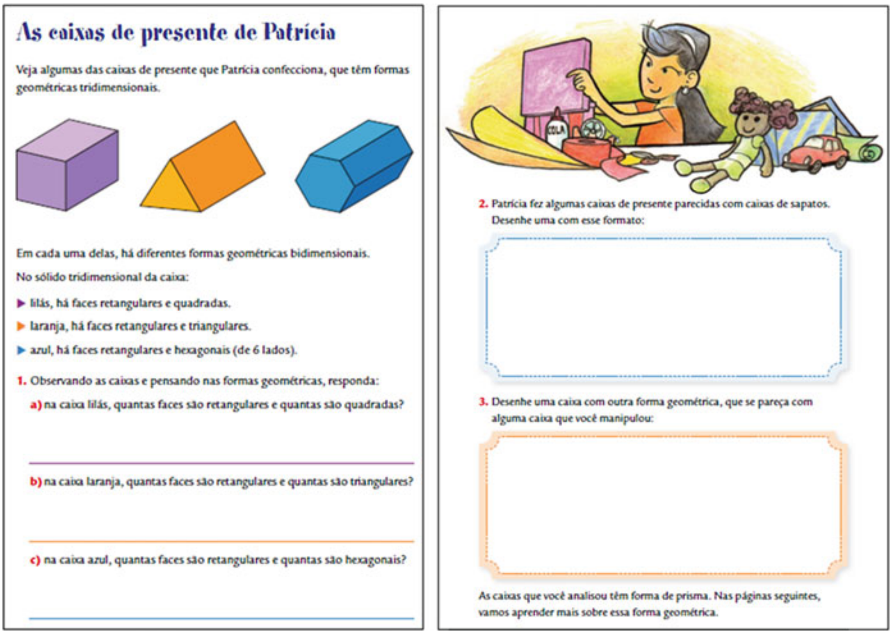
– What is your idea of three-dimensional [things]?

A student answered:

– It's the third dimension.

The teacher repeated her question:

– And what does it mean? Have you ever heard about a 3D movie? Have you seen it? So, what is 3D?



As caixas de presente de Patrícia

Veja algumas das caixas de presente que Patrícia confecciona, que têm formas geométricas tridimensionais.

Em cada uma delas, há diferentes formas geométricas bidimensionais.

No sólido tridimensional da caixa:

- lilás, há faces retangulares e quadradas.
- laranja, há faces retangulares e triangulares.
- azul, há faces retangulares e hexagonais (de 6 lados).

1. Observando as caixas e pensando nas formas geométricas, responda:

a) na caixa lilás, quantas faces são retangulares e quantas são quadradas?

b) na caixa laranja, quantas faces são retangulares e quantas são triangulares?

c) na caixa azul, quantas faces são retangulares e quantas são hexagonais?

2. Patrícia fez algumas caixas de presente parecidas com caixas de sapatos. Desenhe uma com esse formato:

3. Desenhe uma caixa com outra forma geométrica, que se pareça com alguma caixa que você manipulou:

As caixas que você analisou têm forma de prisma. Nas páginas seguintes, vamos aprender mais sobre essa forma geométrica.

Fig. 13 Student's book (Source: São Paulo 2010, pp. 26–27)

Another student tried to answer with a question:

– Is it the three-dimensional?

The teacher asked:

– And what is 'three-dimensional'?

A third student answered:

– Does it have to do with 3D glasses?

Teacher said:

– It has 3 what? – Three dimensions, three measurements.

The intention of the lesson was to help develop the skill of recognizing similarities and differences among polyhedrons, such as prisms, pyramids and others. To build the concept of 3D – three dimensions, teacher brought back the concept of 2D – two dimensions, by drawing and explaining how to represent the third dimension like Fig. 14.

Next, the teacher invited the students to look up the definition of those concepts in the dictionary and then, she drew a cube on the blackboard saying that it had three measurements: height, width and depth. She went on to ask the students to count the sides while she showed them on the blackboard. She explained to the students that they were seeing the representation of the cube on the blackboard in two dimensions.

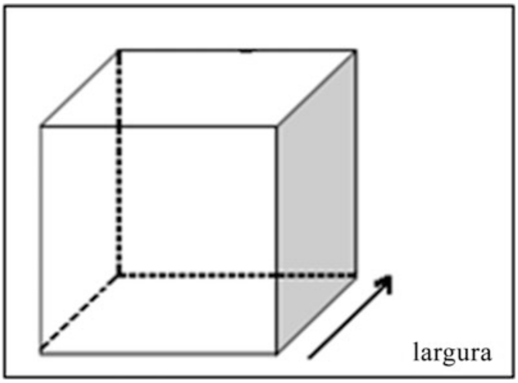


Fig. 14 Tridimensional figure represented in 2D (Source: Private collection)

She reinforced the idea of the difference between a two-dimensional and a three-dimensional figure, pointing out that the first had two measurements and the latter, three, in her own words. Ana related ‘two’ to “bi” (Greek radical for ‘twice’), as in ‘twice champions’ and “tri” (Greek radical for ‘three times’) to ‘three times’ as in ‘three times champion’.

Teacher explained that any side of an object that is drawn in three-dimensional shape can be seen. With the drawing of the cube she had made she said, “this figure has three dimensions: length, width and height.” She asked the students to tell her which sides they were, and at the same time she showed them on the blackboard. She explained that it is possible to see only the height and the length and then drew a square on the board, while saying:

– I drew a die. How many dimensions are there?

Some students answered in unison:

– Threeeeeeeee!!!!

Teacher Ana asked where the three dimensions were and explained that what they saw on the blackboard was just a representation of two dimensions. She demonstrated that if represented by a drawing, which she called width, now completed a three-dimensional figure.

Then she related this explanation to activities done in previous lessons, in which two-dimensional figures were studied, in this case, polygons such as Pentagon, Hexagon, among others.

The teacher finished off with a drawing on the board saying:

– You can see that all figures have three measurements; all of them have height, width and depth. In each of them, there are different three-dimensional geometric shapes. The different two-dimensional geometric shape measurements are height and width. If you look at them in front view, they have two dimensions and when looking sideways, they have three dimensions (sic).

Fig. 15 An example: shoebox (Source: Carvalho 2012, p. 145)

Teacher Ana read the written explanation on p. 26 and asked the students to do the exercises on that page and the following one after reading the statement aloud for them. She demanded that they "work in silence" and waited for a few minutes.

At this point, a student interrupted the teacher, announcing that he had already done the exercise and showed it to her. She said vehemently:

– Wait ... you are not supposed to do it now. If you did, you keep your mouth shut and let the others hear the explanation!

A student asked for an explanation about the third exercise on p. 27, in which a design of a geometric figure similar to what was seen during class was proposed. When replying, teacher picked up the shoebox (Fig. 15) and asked students to imagine another way of drawing a rectangular box. While students worked, the teacher walked around their desks, watching the results.

After the end of the period given for students to solve the exercises, she corrected the exercises on page 26 on the blackboard, with oral participation of the students.

Teacher orally discussed what kinds of polyhedral there are and asked students if any of them would know now how to define what is a three-dimensional figure.

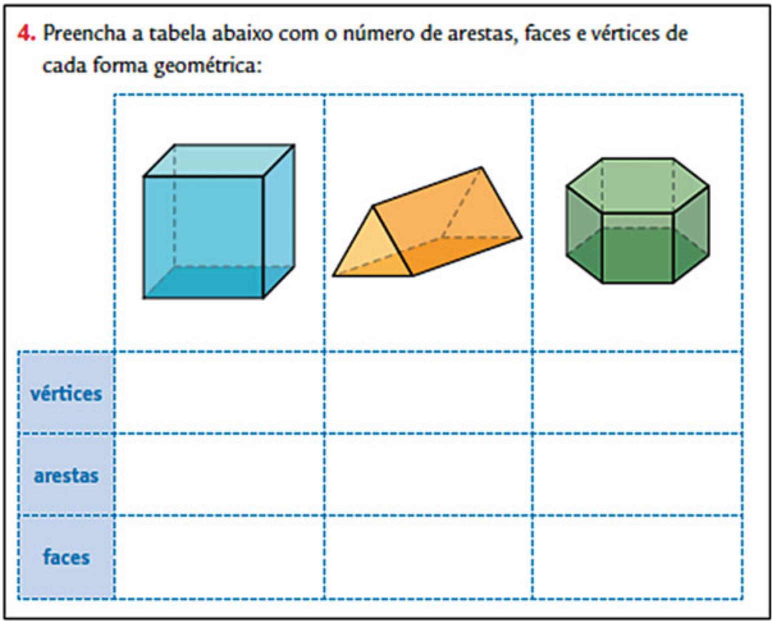
The students responded in a beat:

– An object that has height, width and length.

Then teacher drew a checkered cube on the blackboard and asked the students to identify its dimensions. She explored, always orally, the figures from the same page of the book, then she directed the students to work individually and do exercise 1 on that page and also two other exercises on the next page, concerning polyhedral.

On the blackboard, she provided the answers to (a), (b), (c), of exercise 1 p. 26, with oral participation of students and their imagination, once they used the figures that are in the book. Because of that, some of the faces of the figures represented could not be seen and, sometimes, the children identified incorrect dimensions. At the end, she announced that on the following week, the students would crop and make their own solids, which actually happened, as you can see in Figs. 18 and 19.

She resumed the exercise correction on p. 27 and read the instructions on item 2. She proposed that the students do the exercise, whose response should be



4. Preencha a tabela abaixo com o número de arestas, faces e vértices de cada forma geométrica:

vértices			
arestas			
faces			

Fig. 16 Student's book (Source: São Paulo 2010, p. 45)

personal, but teacher herself immediately showed on the blackboard how to make the drawing and drew the suggested box on the board.

She declared that she was not a designer, so they should have no expectation of a good drawing and went on from desk to desk to check if students had copied the rectangular box or the cube.

Then she handed flyers with multiplication calculation exercises for homework. She showed how these calculations were to be made, giving as an example, one of items of the flyer on the blackboard (Fig. 16).

In order to analyze the categories that showed up in this class were the ecology of the room that remained the same from the previous class, meaning, students' desks were arranged in rows. In her performing role as a teacher, she kept the students' attention focused on her that is, she led their reasoning, suggesting that they imagine other forms of solid, and asked them not to say out loud what they've imagined or thought. Actually, class management has been routinely like this: the teacher tries to work with the whole class with a few highlights of interventions from individual students.

As for the technological resources, the teacher made the dictionary available for students to look up the definition of three-dimensional shapes. It was a positive action, because she was able to show the importance of the use of such a resource.

In the fourth period observed, she worked the contents of transactions involving natural numbers – multiplicity field. Space and shape-polyhedral-geometric figures, were developed in the second part of the class with the following report. She

Fig. 17 Video related to geometric figures of the buildings of São Paulo city (Source: Carvalho 2012, p. 149)

addressed the concept of prism again and its different faces, the concept of edge and vertex, studied in the previous lesson, to resolve the item 4 of p. 45.

The proposal of item 4: fill the table with the number of edges, faces and vertex of each geometric shape: teacher drew the prisms of that exercise on the blackboard and went on, along with the students, filling in the table, they in their book and she on the blackboard.

She moved on to correcting item 5 on p. 46-multiplication by two-digit numbers. She put the math problem on the board and did the calculation with the students. She proceeded in the same way to check items 6 and 7 of the same page. In order to resolve item 6 she justified for students that they were not very familiar with division calculations yet, but those were going to start showing up more frequently, so she would teach them little by little how to proceed.

Following that, some examples of geometric figures were seen in a video that is part of the school's support material. The students were able to identify them by associating them to some buildings in the city of São Paulo (Fig. 17).

After showing the video, teacher explored the content of p. 48 and 49, showing the figures with the marked buildings forming prisms. She identified FIESP (SP) building is not a pyramid as it is commonly called, because the side faces are quadrilaterals and not triangles.

After that, the students did the exercises in their Student's book (see Fig. 18).

After showing the video the teacher suggested that the students study for the June test and for that they should be identifying all geometric shapes that are part of everyday life.

After the class we talked to the teacher about the use of the computer lab. This dialogue can be read below:

Researcher: *Teacher, what is the sequence of these planned activities?*

Teacher Ana: *With these activities, we're working the concept of three dimensions, next lesson I will give the measurement of the edges, then I can ask the students how to calculate the perimeter of a face, and also the area. In each class I will teach them something new.*

Researcher: *May I ask something about the computer lab? I know that there are specific classes for the students in the computer lab with an IT (Information Technology) teacher. And what about the other*

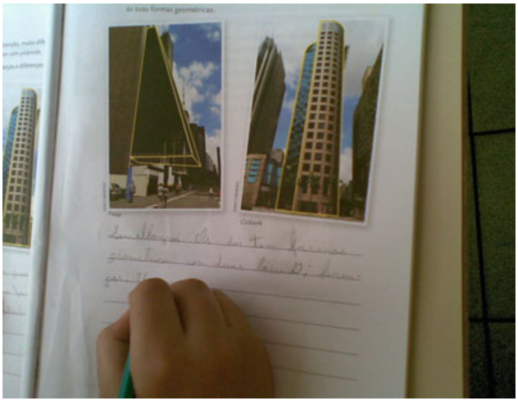


Fig. 18 Exercises corresponding to the video assisted by student (Source: Carvalho 2012, p. 150)

teachers? Is there some kind of incentive from the principal or the pedagogical coordinator so that teachers can, use digital technology to teach mathematics or science for example?

Teacher Ana: *Yes, there is. I'm currently developing a project with my students in Portuguese Language about water and I want to let them search the internet after they finish the activity proposal for the IT teacher in the lab class.*

Researcher: *Then is it possible to develop classes in a computer lab?*

Teacher Ana: *Yes, but last year the IT teacher complained that she couldn't let everyone go online to a specific website at the same time, because that was overloading the system, and it caused it to get too slow. There are assistants who help the IT teacher and I think they could give a hand to the students who finish the lab activity, so that they can do some kind of inquiry for me.*

Researcher: *Oh, Yes!*

Teacher Ana: *Tomorrow, when I get home, I'm going to prepare this lesson. According to the principal, we, can use the computer lab, however when we use it, we are responsible for the equipment. So we prefer to be in the computer lab with the IT teacher, because in reality the work should be done together, right? So, I know there are barriers that we will face in the course. It's hard because there is a part of the book that asks for technological resources and we have to have this feature available at school. I'm going to talk to the pedagogical coordinator and ask for help. Maybe if the assistants could give me some support in the laboratory, we could effectively with the students.*

Researcher: *Oh, I see!*

Teacher Ana: *If the students do something wrong in the computer lab, I will justify to the Director that the equipment is there to be used by the students. My fear is that a student may make some stupid mistake and then we are held responsible for it, right?*

We emphasize that neither Teacher Piera, nor Teacher Ana do any work in the computer lab. The time that students are given for this practice is guided by the IT teacher advisor. Either way, teachers could request the lab in other schedules, but they don't for the reasons explained above.

The analysis of the data in teacher's class departing from the defined categories reveals the usual organization routine and habits, namely desks arranged in rows as in all other classes observed. The students' action development was focused on teacher's action, that is, students performed the task proposal when she so guided.

TV and video were used as technological resources in class. After watching them, the students did the corresponding exercises in Students book, managing their own tasks.

In her role as a teacher, Teacher offered information and resource models to suggest that the students investigate other buildings around town; she related it to the content of a similar situation in the previous lesson, clarified and explained a content sense concepts criteria studied in this class. The evaluation happened through the questions and answers during class.

On the fifth class each student received a sheet on which a planned figure was printed. The teacher distributed among the students several models of Polyhedra (cubes, two types of pyramids, prisms of triangular and hexagonal base), each planned on a page. Students were told to paint the sides of the figure, and cut them, then paste the tabs to assemble and complete the polyhedron.

After assembling solids teacher kept all the productions in the classroom cabinet, as evidenced by the photo in Fig. 19.

At the end of this class, teacher did not make any comments on operation of solids. She did not wrap up the subject, either. Probably because there was no time for it. However, considering the categories established for analysis, it was found that the routine in this class was different from previous ones. Class proceeded dynamically, the interaction teacher/students and students/students was one of

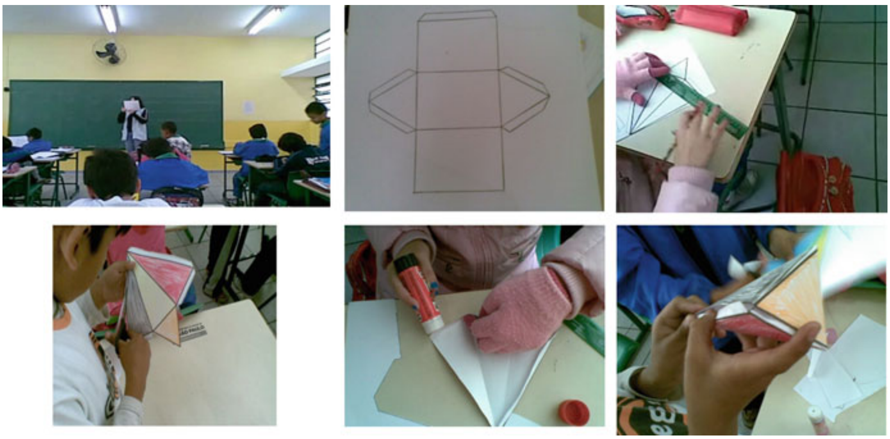


Fig. 19 Teacher and students during explanation of exercise (Source: Private collection)

Fig. 20 Solid built and stored in the classroom cabinet and a student showing your prism (Source: Private collection)

sharing during task execution proposal. The students were able to participate in making decisions; they initiated and ran the activity itself. Various technological resources were used, and none of them was innovative or digital. There was no formal assessment of the established competences (Fig. 20).

Conclusion

Analyzing the episode observed, i.e. the lesson in Piera's class and looking into the categories for analysis, we can say that class management of the school class was made to centralize the students' attention on the figure of the teacher. Teacher ruled the class, seeking dialogue with the students and getting them to understand the content, making use of questions and answers. The order and work habits were the routine, which led the students to have a passive role since they merely copied the blackboard and accompanied explanations.

Analyzing the mediation of technological resources in the classroom done by Teacher Piera we identify her difficulty in bringing students to manifest or expose their knowledge and show their doubts, unless it was done at her command. The students should only respond if their teacher asked a question, that is, during all her time in class. We infer that this may be a reflection of her fear of losing control of the class and of dispersing the students' attention. Another question that may have influenced this attitude was the need to develop the curriculum planned for that school year due to time, which she considered scarce.

Considering the way in which technological resources were used in Teacher Piera's class, we commented to her, that she should lead the class so that the students could freely explore the solids. However, that would require rearranging class time, and dropping the established routine, which could cause her to lose control of the class. We observed in the classroom, where the technological resources were offered to teach polyhedra, the exploitation of solids by the students was controlled by the teacher, taking place under her command.

However, analysis of mediation showed that in Teacher's Ana class activities happened with participation of students, who were encouraged to express their ideas, explaining what they knew about the subject matter and could devote joyfully to the painting activity, cutting, gluing and assembling solids. Though, Teacher Ana also sought to centralize and direct the discussions and keep the students' attention under her control at times. Teacher Ana deals better than Teacher Piera with the time for class issue and considers it enough to fulfill the curriculum. However, manipulation and collective discussion on the geometric characteristics of the polyhedra could not be exploited by the Teacher Ana because the class time ran out.

We realize that using the school laboratory seemed difficult for both teachers, because there weren't monitors or somebody to support them. This situation discouraged them from going to the lab with only the students. Teacher Ana could use videos in the classroom as a technological resource and Piera couldn't.

We observed that Teacher Piera did not feel confident using these features, since she would have to rearrange her lesson plan, which would require more time and she was afraid of losing control of the class.

We emphasize that, for teachers to better manage their time, according Serrazina and Oliveira (2005), they must be responsible for the activities they will propose to their students, i.e. they must take ownership of the curriculum and believe in the activities they will propose. Activities imposed by the pedagogical teams or by the central organs of Education are not always well received by teachers. In both classes, we could observe a verbatim reproduction of what is stated in the Students' book, in order to meet the expected curriculum and official statements to the State of São Paulo.

We highlight that logically our intention was not to compare the mediations, especially since they are also linked to personal characteristics of the teachers. However, from the analysis of the mediation of technological resources that teachers used, we can conclude that features, such as the reality of the classroom, students' interest, the number of students per class, the previous knowledge gap among students, the need for compliance with the prescribed curriculum and the available time will interfere in mediation.

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Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers

Fernando Hitt, Mireille Saboya, and Carlos Cortés

Abstract This paper discusses mathematical task design in a collaborative environment (the ACODESA teaching method), where activities with both paper and pencil and technology play a central role in learning mathematics. The use of problem situations under a sociocultural framework in the mathematics classroom requires careful mathematical task design to develop mathematical abilities in the classroom, promote diversified thinking, and achieve balance between pencil and paper and technological activities within an activity theory framework. While the task design approach examined in this paper is general, it is exemplified through mathematics teaching tasks appropriate for secondary school entry level.

Keywords Task design • Paper and pencil • Technology • ACODESA • Socio-cognitive conflict

Introduction

The literature on mathematics education regarding problem solving is evolving. As mentioned in chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA", Brownell (1942) makes distinctions among the concepts of exercise, problem and puzzle, thus focusing on issues related to primary school level. This led to a new trend linked to the solving of arithmetic word problems and gave birth to, among others, the current problem solving approach. Thus, a new paradigm linked to problem solving emerged, where the distinction between exercise and problem was, and is, preponderant. However,

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this distinction is not so simple, in that, depending on the problem, either convergent thinking (using closed type problems) or divergent thinking (problems with multiple solutions or open problems) could be generated. The latter approach can be related to the Theory of Didactic Situations (TDS) (Brousseau 1997) and even the emergence of the notion of the epistemological obstacle (Brousseau 1983). The design of mathematical tasks under this paradigm took a unique approach. How to detect an epistemological obstacle in pupils' activity? How to encourage pupils to overcome a certain kind of epistemological obstacle? What kind of activity is needed to promote the overcoming of such an obstacle?

Gradually, design problems became more and more important in research on mathematics education. For example, in his notion of conceptual field, Vergnaud (1990) notes that a concept is developed through a set of problems, a set of operators, and a system of signs. Thus, the type of problems that are proposed in the classroom will determine to some extent the mathematical concept pupils are constructing. In the mid 1980s (as seen in Mason et al. 1982; Schoenfeld 1985) the trend for problem solving took on great force, with, for example, research on problem solving (see Kilpatrick 1985) generating such curriculums as *Standards in the USA* (NCTM 2000). According to Kilpatrik (ibid.), “A problem is generally defined as a situation in which a goal is to be attained and a direct route to the goal is blocked” (p. 2).

A different approach to the foregoing is promoted by the Freudenthal School, which promoted the resolution of problems in context, where, under this approach, the study of mathematical modelling process is essential in a strand known as “Realistic mathematics.” Gravemeijer and Doorman (1999) describe the characteristics of the current Freudenthal School. Realistic mathematics is likely to have strongly influenced the notion of problem situation, in which the solution is not necessarily unique. Indeed, realistic mathematics promoted other kinds of curricula linked more closely to the notion of problem situation. The ensuing discussion leads to the question as to whether an exercise, a problem and a problem situation are.

Exercise, Problem or Problem Situation

Advances in mathematics education brought about the need to carefully identify the definition of an exercise, a problem or problem situation. A definition depends on the theoretical framework that has been selected. Given the interest here in definitions linked to mathematics learning environments when using both paper and pencil and technology, this paper seeks to associate these definitions with the notions of non-institutional and institutional representation in order to then link this to Leontiev (1978) and Engeström's (1999) activity theory.

Exercise If reading a mathematical statement immediately suggests a procedure to follow, it can be said that the task is an exercise.

Problem If reading a mathematical statement does not induce the reader to immediately think of a procedure to follow, and requires them to transform the statement and/or use institutional representations and/or produce non-institutional representations to understand and make progress in the proposed task, it can be said that it is a problem.

Problem Situation If the reading of a mathematical statement as in the case of a problem, neither provides a procedure to follow, but in this case, a model must be built (possibly not unique), needed to interpret the phenomenon linked to the statement, then it can be said that this is a problem situation.

This distinction enables the identification of the differences among mathematical tasks that should be considered when designing an activity for the mathematics classroom. The followers of problem solving were more interested in the resolution of problems, as defined above. A different perspective was provided by Lesh and Doerr (2003), Blum et al. (2007), and Lesh and Zawojewski (2007), among others, which dealt with problem solving and modelling, and presented an approached to realistic mathematics and what is meant by the term problem situation.

Indeed, from the perspective of this study, the three types of tasks mentioned above are required for the organisation of mathematical activities in the classroom. The difficulty arises in the organisation of those types of tasks that is needed in order to follow a fixed syllabus. A possible way to overcome this problem may be for the teacher to use the proposition outlined in Simon (1995) and Simon and Tzur (2004) as related to a Hypothetical Trajectory of Learning, which is discussed in the subsequent sections.

One of the first problems to overcome is the fact that the expert (in this case the mathematics teacher) has already constructed different types of thinking (arithmetic, algebraic, geometric) that allow her/him to transform their representations effectively. The beginner (the pupil) has not necessarily built these official representations, and, even if they have, the difficulty arises when they are required to handle them efficiently (as a competence). Generally, learning theories based on the concept of representation focus on the efficient use of institutional representations in the construction of knowledge (as is the case, for example, in Duval's 1995 work which focuses on the notion of register). In the context of our approach, non-institutional and institutional representations are of great importance to the construction of knowledge; also a collaborative learning process is of great significance in a socio-cultural environment, to the refinement of the evolution of the non-institutional representations in which they are promoted to the level of formal representation.

Institutional Representation Representation found in textbooks, websites, software use, or those used by mathematics teachers.

Non-institutional Representation Representation that emerges spontaneously during the resolution of a non-routine mathematical task as a result of a functional representation that has been generated by the action of understanding or solving a task.

Functional and Spontaneous Representation A functional representation is a mental representation linked to an activity. From reading the statement of the task, a need and purpose emerge, which, in terms of Leontiev (1978), mediate the activity undertaken by the individual as a whole. A mental representation is constructed and linked to other concepts, providing the spontaneous representation as a product. The manipulation of objects or artefacts mediates the generation of mental processes which become increasingly complex, as do their external productions.

Leontiev's work (1978) on activity theory is immersed in a sociocultural perspective on learning. Leontiev was interested in the subject and object relationship, while it is in the work of Engeström (1999) where the variable community was explained in the model (see the next section related to ACODESA[1]).

Socio-cognitive Conflict In the past, many researchers, such as Piaget, Inhelder, Brunner and others, were interested in the notion of cognitive conflict. In Bruner's theoretical framework (1966), cognitive conflict occurred when the individual was aware of a mismatch between the enactive, iconic or symbolic representation related to the activity. This study takes Varela et al. (1991) definition of enactive:

> Cognition is not the representation of a pregiven world by a pregiven mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs (p. 9)

In the context of this study, the term iconic could refer to a drawing related to the situation, or a symbol as an institutional representation, with the teacher (expert) easily noticing mismatches between different modes of representation. However, this study is interested in the processes of communication pupils use to point out a mismatch between the spontaneous representations they produce, thus creating a socio-cognitive conflict.

Method of Teaching ACODESA (Collaborative Learning, Scientific-Debate, Self-Reflection)

Looking within a sociocultural framework, in order to organise mathematical work in the classroom and create a form of socio-mathematical norms, it is important to follow a specific educational model. This study is interested in individual work immersed in a collaborative learning structure for the consolidation of knowledge. Our experience has shown us that these aims are not easy to achieve in the mathematics classroom. Thus, the authors designed a teaching model known as ACODESA which is related to an approach involving collaborative learning, scientific debate and self-reflection (see Hitt 2007, 2013; Hitt and Gonzalez-Martin

[1]Acronym which comes from the French abbreviation of *Apprentissage collaboratif, Débat scientifique, Autoréflexion*.

2015) and which includes several steps to be implemented in the mathematics classroom when solving a mathematical task. It is described below in more depth:

1. *Individual work*. Production of spontaneous non-necessary institutional representations related to the task, with prediction processes encouraged.

The design of mathematics classroom situations should follow a structured plan for the use of both paper and pencil and technology. The activity starts when reading the statement of the situation. This mental activity, as mediated by paper and pencil, produces the spontaneous representations linked to the activity of understanding and searching for a goal, even if this is not a well-defined or easy process. Reference to the use of paper and pencil is made in a broad sense[2]. Thus, the use of paper and pencil is intended to be a mediator between pupils' mental representations (i.e. functional representations) as linked to the situation and the activity of understanding, and thus promotes the production of spontaneous representations linked to actions that are not necessarily institutional (Hitt 2013; Hitt and Gonzalez-Martin 2015). This first stage provides the pupil with preliminary ideas that she/he discusses with other members of her/his team. Following an approach where activity and communication go hand in hand (activity theory) creates a link between activities, motives, actions, objectives and operations in the context of Leontiev's work in this area. This stage and that described below are crucial to the production of spontaneous representations and to the commencement of the process of their evolution.

2. *Teamwork* on the same task. Process of prediction, argumentation and validation. Pupils refine their representations in response to their results.

Teamwork helps to refine both the initial ideas and the ability to follow a path towards the resolution of the problem situation. The functional representations that gave rise to spontaneous representations in the individual phase initiate a new process of refinement, which takes into account both the manipulation of physical objects and communication with others. This process is linked to argumentation (persuasion in many cases), prediction and validation, and both testing and taking a position. It is at this stage where cultural norms come into play directly, with teamwork and organisation crucial for the distribution of partial tasks. The question then arises as to how many people are to be allocated to each team. For example, Sela and Zaslavsky (2007) show the difference between teams of two and four people, stressing the fact that, in a two-person team, participation is more balanced, while, with four people, there is an immediate tendency that one of them may take a leadership role with the others becoming followers. As such, teams of two or three people are suggested. It is necessary for the team members to determine who

[2]Touchscreens are used more and more in schools (see the chapter on this matter in Bairral et al., this volume). The paper and pencil component can be converted to the use of an electronic notebook in the production of (not exclusively) institutional representations. Currently, there are some electronic devices, such as notebooks, that can be connected to an iPad for simultaneous use with other applications.

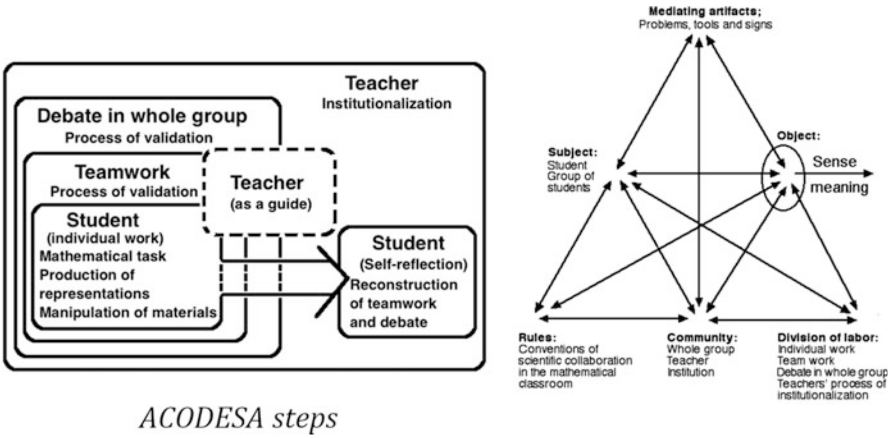


Fig. 1 ACODESA and the Engeström model as adapted to the aims of this study

manipulates physical objects (and how they are manipulated), who uses the computer (e.g., see Hoyles 1988), who notes the progress of the team, and who comes forward to present the achievements of the team for plenary discussion. In fact, it is here that both activity theory and Engeström's (1999) model are very important (see Fig. 1). At this stage, the teacher's role is to guide rather than provide their opinion on how the teams performed.

3. *Debate* (could become scientific debate). This is related to a process of argumentation and validation and the refinement of representations. According to Legrand (2001), the teacher's role is crucial at this stage for the promotion of scientific debate. In general, if the design of the task is related to a problem situation or an open problem, different results from the teams will be presented for discussion. In general, teams will have a natural tendency to protect their results, with the teacher required to regulate the discussion (socio-cultural norms) and decrease the persuasion and argumentation that can lead to prediction and validation. Again, spontaneous representations that have surely undergone a process of refinement first through working in small teams can be refined in large group discussion.
4. *Self-reflection* (individual work – the reconstruction of what has been carried out in the classroom).

Given that the literature has shown, in the classroom, consensus to be ephemeral (Thompson 2002; Hitt and Gonzalez-Martin 2015; Hitt et al. 2015), this study included a stage involving a reconstruction process activity. The teacher must collect everything produced during the previous stage and provide a new copy of the task. Karsenty (2003) demonstrates that after a certain period of time, adults forget the mathematics they have learned. The question as to how to build stable knowledge is one that led to this stage being implemented here and also to the importance attached here to individual reconstruction. It is at this stage that the notion of historicity has a strength action; where the pupil has been influenced by a socio-cultural process of learning and is prone to a sociocultural construction of knowledge. This stage also requires reconstruction related to achievements in terms of individual work, teamwork and plenary discussion designed to strengthen knowledge.

5. *Process of institutionalisation*. The teaching undertaken by the teacher takes the pupils' results into account and uses the official representations.

In a sociocultural knowledge construction process, where the pupil is an active actor in that environment (activity theory), a mathematical concept is not produced through a dogmatic presentation by the teacher. Institutionalisation occurs at the end of those preliminary stages, where the teacher takes pupils' productions into account while refining the concept and, if necessary, providing both the institutional position and its official representations.

ACODESA takes Engeström's model into consideration in the organization of pupils' classroom activities by placing special attention on the artefacts they use.

Task Design

As seen in previous sections, task design is not a new feature in mathematics education. For example, when conducting a teaching experiment, it is important to build a hypothetical model to guide the researcher in the teaching process. More precisely, as described above, both Simon (1995) and Simon and Tzur (2004) proposed the Hypothetical Learning Trajectory (THA) method, which allows the teacher to organise and design mathematical activities for use in the mathematics classroom.

Interested in the learning of mathematics in a sociocultural environment and given the technology involved, researchers in this study considered, for example, the following elements, as described by Arcavi and Hadas (2000, pp. 25–27), as being of fundamental importance to a design based on a Dynamic Geometrical System (DGS):

1. Visualization. "Visualization generally refers to the ability to represent, transform, generate, communicate, document, and reflect on visual information".
2. Experimentation. Besides visualization, playing in dynamic environments enables students to learn to experiment.
3. Surprise. It is unlikely that students will fruitfully direct their own experimentation from the outset. Curriculum activities, such as problem situations, should be designed in such a way that the kinds of questions students are asked can make a significant contribution to the depth and intensity of a learning experience.

4. Feedback. Surprises of the kind described above arise from a disparity between an explicit expectation of a certain action and the outcome of that action. The feedback is provided by the environment itself, in that it reacts as requested.
5. Need for proof and proving. Dreyfus and Hadas (1996) discuss and exemplify how one can capitalize on such student surprises in order to instil and nurture the need for justification and proof.

An analysis of the above characteristics reveals that the DGS is an important element. Under this view, Duval's 2002 approach to Arcavi and Hadas' mathematical visualisation process is very pertinent, since it relates to the discrimination of visual variables on a register as possibly associated with corresponding elements on another register.

The problem with these approaches is that spontaneous representations in the resolution of problem situations are not fully considered in these contexts. These spontaneous representations generally do not belong to a register. This study is interested in the unofficial representations that pupils produce in a paper and pencil environment (Hitt 2013; Hitt and Gonzalez-Martin 2015) and the evolution toward institutional representations (e.g., those on a computer screen) through a process of communication with others and the use of technology.

As the notion of learning with which this study is concerned is linked to collaborative work, other perspectives must also be considered, such as those of Prusak et al. (2013), who, with respect to the creation of tasks to promote productive argument, suggest the following:

1. The creation of *collaborative situations,*
2. The design of activities that trigger *socio-cognitiveconflicts,*
3. The provision of tools for checking hypotheses.

Indeed, for the perspective of this study, Arcavi and Hadas, as well as Duval and Prusak et al., can be taken into account in both paper and pencil and technological approaches (Hitt and Kieran 2009; Hitt et al. chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") formulated using ACODESA.

In this context, ***visualisation*** refers to the ability to represent, transform, and find significant visual variables that may be associated with other elements from another separate representation through a process of communication with others. This thus promotes an evolution where the mathematical activity in question is "seen" and creates an improved approach to the resolution process.

Healy and Sutherland (1990), on one side, and both Hitt (1994) and Hitt et al. (in this volume), on the other, illustrate how pupils or pre-service teachers "see" the task of constructing a process for the generalisation of polygonal numbers differently. For example, both Hitt (1994) and Hitt et al. (chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") found different approaches, such as that related to changing the number of elements on the diagonal in order to obtain the next triangular number (Fig. 2), or that focusing on the number of elements on the base or on one side of the

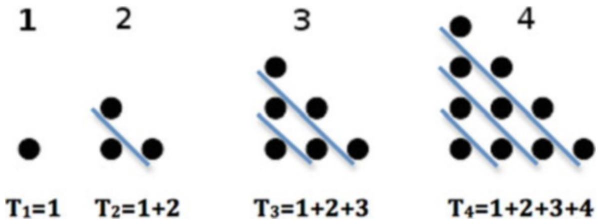


Fig. 2 Process of visualising and articulating visual information in a numerical approach

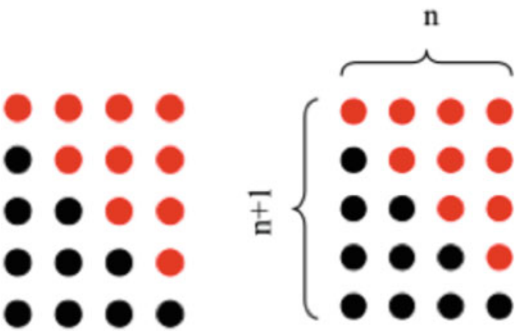


Fig. 3 Transformation of the triangular number to find a general rule

triangular arrangement. Both Healy and Sutherland (Idem) and Hitt (1994) used a triangular arrangement, specifically with an equilateral triangle, while this study used an isosceles triangle rectangle. This arrangement generated the conjecture (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA") relating to calculating any triangular number using the formula for calculating the area of a triangle (base * height/2). Pupil conjecture thus created a socio-cognitive conflict, as pupils pointed out that calculating T6 and T8 (triangular numbers 6 and 8) visually did not obtain the same result.

The expert (the mathematics teacher) "sees" triangular numbers institutionally in order to complete a rectangular array, as seen in Fig. 3. The visual triangular number is duplicated and a transformation performed, thus obtaining a similar arrangement that is able to show a rectangular arrangement (Fig. 3), thus revealing the conclusion that:

$$T_n = \frac{n(n+1)}{2}.$$

Pupils' visual processes do not necessarily agree with the ways in which teachers visualise. The teacher uses official representations that enable her/him to be efficient in handling the institutional representations. They, as experts, are able to articulate representations that have developed ways to "see" into the passage, distinguish from one representation to another. Thus, the expert is able to immediately identify the important visual variables (as described by Duval) to be transformed and/or converted into another representation.

The question thus arises in terms of how to develop this expertise by our pupils. The purpose of this study was to create socio-cultural norms in the mathematics classroom through the design of activities that promote a learning process based on the manipulation of physical objects, the production of representations, and the processing of devices for the efficient use of such representations in order that pupils are able to solve problems and problem situations. Furthermore, the aim was to ensure that:

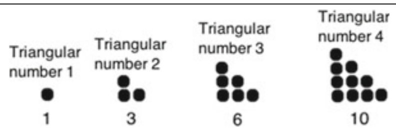
1. individually, the pupil begins, as a result of the preparatory work undertaken in relation to the mathematical activity, to attack the same activity from a socio-cultural perspective using teamwork.
2. by comparing their results with other pupils (in teams of two or three), the pupil possibly creates socio-cognitive conflicts involving productive arguments, with action and communication linked through objectives that they have to follow.
3. the plenary discussion furthers productive arguments, as well as anticipatory processes, the promotion of reconciliation among representations, validation processes, the production of counter-examples and the ability to check hypotheses. Once again, action and communication go together.
4. self-reflection promotes the strengthening of knowledge in order to stabilize it, with historicity (that which was undertaken collaboratively as an essential element of the process of reconstruction) a main component of reconstruction.
5. the process of institutionalisation enables the review of that which has been undertaken by pupils in order to promote the official representations and communication that will further advance their mathematical knowledge.

Considering these characteristics, the design of the activities used in this study begins with a presentation page (the front page). General pupil information is obtained in order to identify their work on an individual basis, as well as their results from the teamwork activity. It can be useful to include, on this page, instructions for the use of different colour inks when working either individually or with others in order to identify any development or evolution.

During the first stage, the mathematical task begins with the promotion of diversified thinking and, therefore, requires an open problem or problem situation. The statement outlining the activity will promote the production of functional representations that will trigger the production of spontaneous representations. This study proposes a block of five questions which allows pupils to individually create their own strategies (for a full outline of the experiment, see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA"). The task design depends on the use of artifacts in the construction of knowledge. For example, in Hitt and Gonzalez-Martin (ibid.), pupils used a rope, flexible wire and a rule, as well as paper and pencil, when attacking the mathematical task. Another example, as seen in Hitt and Kieran (Idem), sees the first stage designed to generate a strategy for a paper and pencil environment. This was then confronted with a second stage that featured the pupils' own algebraic productions as well as those provided by a CAS calculator, thus requiring them to reconcile their own productions with those produced technologically, as well as requiring team discussion. A third stage is related to the promotion of a specific conjecture and the need to convince others, with proof not taught at this educational level.

This study aimed to explore this approach with pupils who are beginning secondary school and, as such, are yet to be introduced to algebra, with the design intended to promote the construction of the concept of a variable through a process of collaborative learning under a sociocultural approach (see chapters "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" and "Problems Promoting the Devolution of the Process of Mathematisation: An Example in Number Theory and a Realistic Fiction"). In fact, researchers in this study considered it necessary to construct an algebraic-geometric-arithmetic thinking before developing an "exclusively" algebraic thinking detached from arithmetic itself. As such, the design of this experiment took into account Healy and Sutherland's (1990) work, who followed an Excel-based approach to polygonal numbers as well as Hitt's (1994) paper and pencil model which also used an applet that exclusively generated the value of any polygonal number. This experimentation also was implemented with a Mexican population in order to generate a comparison with the type of strategies used by those pupils who have already taken an algebra course (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" for details).

A first block was thus designed to promote visualisation, abstraction and generalisation processes from a perspective that seeks to create diversified thinking (see Fig. 4).



1) Look carefully at these numbers. What is the fifth triangular number? Make a representation. Explain how you proceeded.
2) In your opinion, how are the triangular numbers constructed? What do you observe?
3) What is the 11th triangular number? Explain how you find it.
4) You have to write a SHORT email to a friend describing how to calculate the triangular number 83. Describe what you would write. YOU DO NOT HAVE TO DO THE CALCULATIONS!
5) How do you calculate any triangular number (we still want a SHORT message here).

Fig. 4 First task design block for the generation of diversified thinking and spontaneous representations

Develops the same ideas as in the previous section but using Excel (or CAS).
Here we ask you to find:

	A	B	C	D	E	F	G
1	Nombres polygonaux						
2	Position	1	2	3	4	5	
3	Triangulaire	1					
4							
5							

What would you do to discover the 6^{th}, 7^{th}, and 8^{th} triangular number?
Is it possible to calculate the triangular number 30, triangular 83, and triangular 120?
How do you do this?
What are the limitations and possibilities of this approach?
Provide the operations to be performed in order to undertake this calculation with any polygonal number.

Fig. 5 Second part of the task

It is expected that spontaneous representations and personal strategies make their appearance during this first stage. Based on the same questions, it is expected that pupils will work in teams before engaging in plenary discussion.

In the example considered here, teamwork is required in the second block of questions. The aim is to promote in pupils the ability to generate the iteration processes related to a spreadsheet environment (Excel or CAS), similar to that obtained in Healy and Sutherland (1990).

As we can see in the two blocks of questions (see Fig. 5), the pupils generate different types of strategies. It is intended that pupils acquire a broad vision of how to address a problem situation and the various products linked to different strategies in order to promote different kinds of representations.

A comparison was sought between the strategies used in Healy and Sutherland (Excel and secondary school pupils) and Hitt (1994), which involved a group of secondary and primary teachers using Excel, and another group of teachers using paper and pencil and an applet. Generally, there are several kinds of generalisations used to calculate a triangular number:

- trig. Δn = na before + position (Healy and Sutherland 1990),
- $Tri(n) = 1 + 2 + 3 + \ldots + n$ (Hitt 1994),

It is noteworthy that the task generates the production of different types of representations, with the type depending on the technological environment. This is the case with pupil production in this new approach to the construction of polygonal numbers (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA").

In the third block of the task, interest focused on the use of an applet that gives pupils the opportunity to immediately verify their generalisation strategies, or to request a polygonal number, etc. Thus pupils are able to receive immediate

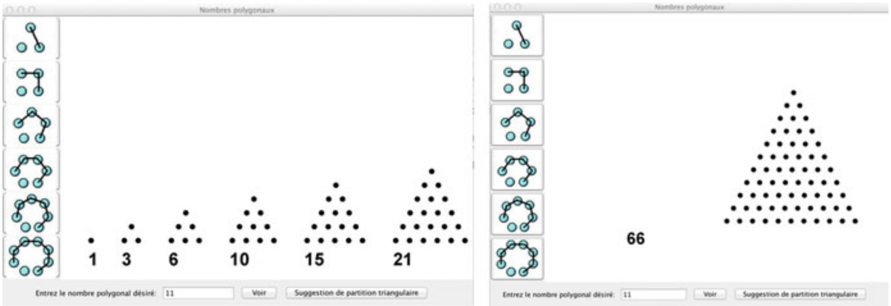


Fig. 6 Examples of the use of the POLY applet with polygonal numbers

a) Here are the five first triangular numbers.

Find a formula to calculate the numerical value of any triangular number. You can use the POLY applet to find the formula.

APPROACH (OPERATIONS, DRAWINGS…)

Write the rule or formula you found:

Using your rule or formula, calculate the following triangular numbers.

Position	Corresponding value
Triangular 10	
Triangular 20	

With the formula, can you calculate the triangular number 120?

Triangular 120 = ____________

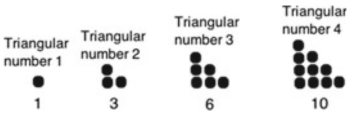
Fig. 7 Third block of questions and use of the POLY applet

feedback on the veracity of their conjecture using the applet. The applet (see Cortés and Hitt 2012) is to be used precisely in this 3rd block.

The applet is able to request the first four polygonal numbers selected (triangular, square, pentagonal, etc.) and is also able to request a "large polygonal number" (see Fig. 6). Paper and pencil work with the use of the applet allows pupils to check their guesses. If the pupil's conjecture does not agree with the result given by the applet, the pupil must return to their team and review the process that led to the construction of their conjecture, which, thus, fosters productive communication among pupils.

Pupils are asked to use the Poly applet for the proceeding set of questions in which the arrangement of the triangular numbers was changed, using an equilateral triangle (which corresponds to the institutional representations that pupils usually encounter in textbooks) (Fig. 7).

Here there are the first four triangular numbers

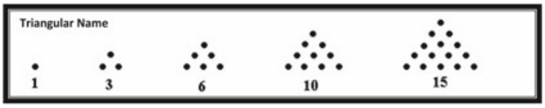


1) What is the 11th triangular number? Explain how you found it.

2) Write a SHORT email to a friend describing how to calculate the triangular numbers 30, 83 and 120. Describe what you would write. YOU DO NOT HAVE TO DO THE CALCULATIONS!

3) How do you calculate any triangular number (we still want a SHORT message here).

4) The following configuration of a triangular number can be found in some textbooks:



Does your strategy always enable you to calculate any triangular number?

Fig. 8 Task designed for the self-reflection stage (reconstruction activity)

From a psychological point of view, the framing of the triangular numbers, which does not leave enough space after the first 5 examples, promotes a tendency to abandon the drawings (see Hitt 1994), while the presentation of the activity promotes the generalisation process.

Building on strategies produced by Hitt (idem) has lead to the following output (it is important to stress that this study is carried out with primary and secondary school teachers and focuses on pupil performance, with chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" discussing secondary school pupils):

- $f(x) = \frac{x^2+x}{2}$ (Hitt 1994)

A summary questionnaire, which does not include the use of technology, is used for the self-reflection stage. Pupils are expected to be able to rebuild their representations, as well as any algebraic expressions that they have produced, thus enabling them to calculate any triangular number (Fig. 8).

As stated above, the reconstruction stage is very important. Research results (Karsenty 2003; Thompson 2002; Hitt and González-Martín 2015) show the fragility of knowledge and the importance of implementing, in the mathematics classroom, activities that can strengthen the construction of such knowledge.

In the case of pupils who are beginning to study algebra, validation can be restricted, while, in the case of the use of the task with pre-university students and/or future teachers, one can request demonstrations using mathematical induction processes. For example, the applet does not work when using large numbers. Furthermore, working with both the official representations of the polygonal

Table 1 Generalisation for calculating any n polygonal number of p sides

Calculation of polygonal n	Expression for generalisation
$T_n = \frac{n(n+1)}{2}$	$T(3 - sides)_n = \frac{n(n-1)}{2}$
$C_n = n^2$	$C(4 - sides)_n = n^2 = \frac{2n(n+0)}{2} = \frac{n(2n+0)}{2}$
$P_n = \frac{n(3n-1)}{2}$	$P(5 - sides)_n = \frac{n(3n-1)}{2}$
$H_n = n^2 + n(n-1)$	$H(6 - sides)_n = \frac{2n(2n-1)}{2} = \frac{n(4n-2)}{2}$
$E_n = \frac{n(5n-3)}{2}$	$E(7 - sides)_n = \frac{n(5n-3)}{2}$
...	...
	$Polygonal(p - sides)_n = \frac{n((p-2)n-(p-4))}{2}$

numbers and the construction of algebraic expressions associated with those numbers, another block of questions could be added. These would request a further and higher generalisation process (see Table 1), which would be built as a single algebraic expression that enables any polygonal number to be calculated.

Conclusion

This paper proposes task design elements to be developed in the mathematics classroom under a sociocultural approach. While some authors point out the importance of creating sociocognitive conflicts in the mathematics classroom, they suggest an organisational schema for performing an activity, with, for example, Prusak et al. (2013) proposing the following for a 75-min class:

> For the first 15–20 minutes, the instructor facilitated a whole class discussion to create a shared understanding of the activity; then, for approximately 5 minutes, each student engaged in the task individually; during the following 45 minutes, students worked in dyads or triads, solving tasks collaboratively and writing a common justification on a worksheet; for the final 5–10 minutes, there was a plenary, where the instructor led a whole class discussion to summarise. (p. 270)

In contrast to the methodological approach outlined above, the methodological approach advocated here takes into account the fundamental point that *consensus is ephemeral* and, as such, it is therefore necessary to consider a knowledge reconstruction stage (referred to as self-reflection in this methodology) in order to strengthen and stabilize knowledge (Karsenty 2003; Thompson 2002; Hitt and González-Martín 2015).

This task design is more related to problem situations that generate diversified thinking and, as a possible consequence, socio-cognitive conflicts in a process of action and communication. To overcome a socio-cognitive conflict, the authors of this study suggest the promotion of signification processes, as described by Radford (2003), in the mathematical classroom (see chapter "Integrating arithmetic and algebra in a collaborative learning and computational environment using ACODESA" on this issue). Consequently, some of our problem situations may take more than one session of a course. In fact, the task design in Hitt and Gonzalez-Martin (ibid.) aimed to create a chain of activities that encompassed the concept of covariation between variables, function in context, and mathematical modelling, over the course of 13 class sessions.

Practice has shown that, as a method such as ACODESA is not easy to implement in the mathematics classroom, it is very important that, working together, researchers and teachers can create learning situations such as those suggested in this chapter for the mathematics teacher. Generally, it is not possible to fully present in research articles the complete activity implemented in an experiment, due to a lack of space. The problem situations dealt with here usually occupy several pages permitting regulate, in some extent, pupils' productions and promoting their evolution.

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ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching

Panagiota Kotarinou and Charoula Stathopoulou

Abstract The use of technology tools creates new situations and new dynamics in Geometry's teaching in the classroom, enhancing the ways of its understanding. In this chapter, the experience of using ICT together with 'Drama in Education' (DiE) in a teaching experiment, regarding the axiomatic definition of Hyperbolic Geometry through Poincaré's Disk, in a class of 11th grade students is described. The use of 'Drama in Education' techniques created a space appropriate to transform traditional classroom practices. In this space, a liminal space, students became more active and involved in (re)negotiating different discourses, their own learning processes and conceptions of Hyperbolic Geometry while interactive Java software allowed them to explore a non Euclidean Space. The use of ethnographic research techniques (i.e. participant observation and interviewing) helped us to gather empirical evidence concerning students' experiences. Moreover, our research revealed considerable evidence that it was Drama techniques which motivated students and offered them fuller participation in the teaching process, while ICT helped them visualize the Poincaré's Disk and through it understand key elements of Hyperbolic Geometry.

Keywords Interactive java • Liminal performative space • Hyperbolic geometry • Poincaré model

Introduction

The awareness that mathematics appear as a difficult school subject dictates, inter alia, the revision of teaching in order to enhance students' interest and their active participation in class. As pointed out by Mayer (2005), one of the biggest problems in learning in the school context is to motivate students to fully commit themselves

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to the learning process, without this becoming a boomerang in the long term, causing them resentment from the school experience (Kohn 1993; Appelbaum and Clark 2001).

We consider the integration, into the same scene where the teaching of mathematics occurs, of all available resources and techniques, as an enrichment of teaching mathematics and simultaneously challenge the perception of students about the nature of mathematics. In such a context, the students face the challenge of seeing mathematics as a continuous spectrum that penetrates various aspects of life both now and in the future, touching both individual and social needs. Talking specifically about geometry, and in particular about the teaching of Geometry, "structure of traditional geometry has never been a convincing didactical success... to my opinion it failed because its deductivity could not be reinvented by the learner but only imposed" (Freudenthal 1971, pp. 417–418).

In recent years, alternative approaches in geometry teaching have been studied. The use of new technologies (Jones 2011; Laborde et al. 2006; Oldknow 2008), the study of the applications of geometry in various sectors (Fletcher 1971), the use of the History of Geometry applications with appropriate material from historical sources, as well as the use of the arts, have created new educational situations, involving students actively in the process of teaching / learning.

This chapter presents a project focused on axiomatic foundation of hyperbolic geometry, designed by the two authors and implemented by the first one in a classroom of 11th grade students. This interdisciplinary project was implemented with the use of new technologies and "Drama in Education" techniques, which we claim that created an in-between/liminal space where new practices, new discourses and new tools emerged, while new technologies helped students' visualization and hence understanding of geometrical concepts we dealt. The challenge of the uniqueness of Euclidean space for the interpretation of the world was an additional provocation which students were asked to handle in this new context.

Subsequently, we will briefly present the importance of the teaching of Hyperbolic geometry in school and the role of new technologies for its teaching and we will describe the liminal space created by Drama techniques. We will close with the presentation of the design and implementation of the research, concluding with the discussion of results.

Teaching Hyperbolic Geometry Through ICT

The discovery of non-Euclidean Geometries is a rupture in the history and evolution of mathematics, through the separation of reality from mathematical space and through the conscious realization that mathematical structures, in their role as models, are the new mediating artifacts to explore space (Hegedus and Moreno-Armella 2011). As characteristically Hegedus and Moreno state "With Euclidean ontology a mirror was placed between the world and mathematics. Non Euclidean Geometry broke the mirror" (ibid, p. 379).

The teaching of Geometry at school imposes, as an absolute and undeniable truth, that Euclidean geometry is the model, which interprets and represents our space (Thomaidis 1992). The teaching of non-Euclidean geometries would help students recognize that there are several other geometries and spaces, other than the Euclidean ones, and to realize that mathematics is not an absolute truth (Kazim 1988, cited in Thomaidis et al. 1989). Directing students to investigate properties of other geometries, in order to see how the basic axioms and definitions lead to quite different –and often contrary– results, helps students to gain an appreciation of the Euclidean geometry as one of the many axiomatic systems (NCTM 1989, cited in Gray and Sarhangi n.d.). Comparisons between similar geometrical concepts in the various axiomatic systems contribute to a better understanding of these concepts (Lénárt 2004, 2007).

The Hyperbolic Geometry has been chosen for introducing students to non-Euclidean geometries, because it is the "closest" in the Euclidean geometry paradigm, including changes in only one of its postulate; the famous fifth postulate (Dwyer and Pfeifer 1999). Offering a "world" in which all shapes are altered, Hyperbolic Geometry can help students reflect on the definitions of geometric objects and thus to understand the typical definitions of shapes (Austin et al. 1993).

Towards the end of the nineteenth century Poincaré attempted to remedy the visualisation problem by creating three models for hyperbolic geometry, while he was investigating different aspects of analysis. One of his most famous and commonly used is the Disc model, which employs the interior of a unit circle for hyperbolic space. The Disc model is particularly interesting since it forms the basis for a series of pictures by Escher, entitled 'Circle Limits' (Stevenson 2000). With a Euclidean model of Hyperbolic Geometry, Poincaré replaced the infinite plane with a finite circular disk where the circumference of the disc represents hyperbolic infinity. In the Disc model hyperbolic lines are shown as either Euclidean 'straight' lines (the diameters of the circle) or the arcs of circles orthogonal to the circumference of the disc. Inside our hyperbolic 'world' the other shapes are defined in the usual manner. The angles are measured in the Euclidean manner, from the angle between the tangent lines of the curves at the vertex of the angle, while the definition of the distance is not the Euclidean one (Davis 1993).

- Poincaré's Disk, like every model, helps us prove the consistency of Hyperbolic geometry as an axiomatic system. In Poincaré's Disk the five postulates of Hyperbolic geometry (with the first four the same of Euclidean ones) are verified and for this reason the Disc constitutes a model of this geometry. It shows therefore that Hyperbolic geometry is consistent, to the extent that Euclidean one is (Davis 1993).
- This model, from teaching perspective, enables us through visualization, to teach students Hyperbolic geometry as a consistent axiomatic system and help them distinguish some of the unusual theorems and properties of this geometry.
- Another model for Hyperbolic geometry is 'Cold Plate Universe' (Gray 1989, apud Stevenson and Noss 1998) which can be obtained by projecting the positive branch of a two-sheet hyperboloid from a point beneath the viewing plane. In the

flat model of ‘Cold Plate universe’ the whole of hyperbolic space is represented, as in Poincaré’s Disk, as the interior of a unit Euclidean circle, with the circumference of the circle representing ‘infinity’. In ‘Cold Plate universe’ the temperature decreases as one moves radially towards the circumference of the circle. Consequently a metal ruler used to mark out distances along the arcs of circles would contract as it is moved out from the centre and the ‘unit length’ would decrease. At the same time the magnitudes to be measured would also contract and the measurement results would remain the same, while the magnitudes could not coincide even if they are equal. However, living in this surface one would be unaware of the variation in length and the change in distance measure would only become apparent, if one could compare the rule with that in a ‘constant temperature’ Euclidean world (Stevenson and Noss 1998, p. 234). In this model distance measure, which varies with position, is a key perceptual feature and is also contrary to our usual perception of distance measures in Euclidean Geometry and our experience of everyday life (Stevenson 1999). In Hyperbolic Geometry then the concept of equality as congruence is changed and the model of ‘temperature’ helps us in teaching, to address the fact that in Poincaré’s Disk while the angle measure is preserved, equal shapes do not necessarily coincide.

- As Hilbert proved in 1901, it is impossible to embed an infinite simply connected surface of constant negative curvature isometrically into Euclidean 3-space. As a sequence unlike the situation in Spherical geometry, we cannot embed the whole hyperbolic plane into Euclidean 3-space and thus to visualize Hyperbolic Geometry, we have to resort to a model (Series 2010).
- Gutiérrez (1996, apud Christou et al. 2007) considers ‘visualization’ in mathematics as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties. Visualization helps us extract useful information from complex or often voluminous data sets, through the use of interactive graphics and imaging (Kaufman 1994). Computer-based learning environments commonly comprise symbolic as well as static and dynamic pictorial representations, frequently combined with the possibility of modifying them interactively (Christou et al. 2007). The real power of computer graphics lies in its ability to accurately represent objects for which physical models are difficult or impossible to build, combined with its ability to allow the user to interact with simulated worlds. Interactive computer graphics can provide new insights into the objects of pure geometry, providing intuitively useful images, and, in some cases, unexpected results (Hanson et al. 1994, p. 74).
- The teaching of Hyperbolic Geometry is usually implemented through models such as the hemisphere model (Lénárt 2004) or the Poincaré’s Disk (Dwyer and Pfeifer 1999; Krauss and Okolika 1977). As a teaching means for understanding the Poincaré’s Disk, the use of new technologies is recommended through relevant software, as Geometer’s Sketchpad (Dwyer and Pfeifer 1999; Gray and Sarhangi n.d.) or Interactive Java software “NonEuclid” (Austin et al. 1993), as well as haptic tools, as the trasparent hemisphere (Lénárt 2004) and

the paintings of the famous Dutch painter Maurits Cornelis Escher (Menguini 1989, apud Furinghetti and Somaglia 1998).

- In this paper, a teaching experiment about axiomatic foundation of Hyperbolic geometry and its basic notions is presented, which was held through its model of Poincaré's Disk and Gray's (1989) model "Cold Plate Universe". In our teaching experiment we used 'Drama in Education' conventions to motivate and actively engage all of the students, with the students having to create 'Radio broadcasts' concerning "Platterland"; a land with Poincaré's Disk shape. In order for the students to conceive Poincaré's Disk and thus key elements of Hyperbolic geometry and to be able to present them through "Radio broadcasts", we exploited the Java applet by Joel Castellanos, Joe Dan Austin and Ervan Darnell and the book Flatterland by Ian Stewart (2002), as well as the Escher paintings Circle Limit I, II, III.

"Drama in Education" and Liminal Space

Drama in Education, according to O′ Neil and Lambert (1990, p. 11), is a mode of learning, in which, through the pupils' active identification with imagined roles and situations, they can learn to explore issues, events and relationships. It is a performative art form with pedagogical character that has as a basic aim the pupils' understanding about human behaviour, themselves and the world they live in (Idem, p. 13). It is also a dynamic and creative methodological tool for the various curricular subject areas through collective actions and lived experiences, putting children in the position of the actor (experience), spectator (judgement), author (meaning) and director (form). DiE combines: Form and Content, Action and Reflection, Logic and Imagination, Thinking and Feeling, Body and Spirit. In Drama, participants create a story, an imaginary world, they perform roles, explore an issue or face problems and decide, act and reflect upon their actions (Avdi and Chadzigeorgiou 2007). Drama in Education enables participants, either during the drama itself or after the drama in a discussion, to look at reality through fantasy, to see below the surface of actions to their meaning (Wagner 1999, p. 1).

The key elements and techniques of DiE derived from the world of traditional theatre, and like any art it is highly disciplined (Wagner 1999). Way (1967), first attempted the separation of Drama from the theatre, considering that:

> Theatre is largely concerned with communication between actors and a audience; Drama is largely concerned with experience by the participants, irrespective of any function of communication to the audience…Theatre is undoubtedly achievable with a few - a very small minority, but there is not a child born anywhere in the world, in any physical or intellectual circumstances or conditions, who cannot do Drama. (p. 3)

DiE is an activity focusing rather on the process than on the outcome-the theatrical performance. In DiE there is no distinction between actor and audience; the learner is both participant and observer, playing a role while interacting with others in role (Andersen 2004, p. 282). The success of a DiE activity is assessed by the ideas, expressions, skills, abilities, imagination and creativity that it causes, rather than pupils' theatrical skills or the aesthetic effect, as in a professional theatrical production.

Drama in Education is not limited to experience, but goes to the awareness of this experience and in this way learning is achieved. Heathcote (1984) notes that:

> I have struggled to perfect techniques which allow my classes…to be able both to experience and reflect upon their experience at the same time: simultaneously to understand their journey while being both the cause and the medium of the work. (p. 127)

In Educational Drama, a series of conventions that freeze or delay the action are used, in order for the students to analyse, interpret and understand it. In Drama, students think about their ideas when they are 'in role' and can better understand the process of thinking, once they leave the role (Andersen 2002). The time for observation and discussion following each activity helps children to distance themselves, to elaborate and understand what preceded it, to judge situations, to evaluate the behaviours and attitudes of others, but also to be self-assessed.

In DiE, the teacher and the students participate in-role to create stories and scenes, through which students can experience the curriculum in an emotionally rich context. The aim for the students is to pay attention and care in an "as if" world, a world that they feel it as real, even if they know that it is not (see also chapter "Problems Promoting the Devolution of the Process of Mathematisation: An Example in Number Theory and a Realistic Fiction" of this book). When students experience a role in this imaginary world, they build faith and feelings for the characters, the situations and conflicts of this imaginary world. Through this belief and through this participation and involvement in Drama, students gain interest in the knowledge of the curriculum.

Liminality and Drama in Education The concept of liminal space in performance studies comes from the field of social anthropology (Arnold Van Gennep) and was introduced by Victor Turner (1982), who came to performance theory from anthropology. Richard Schechner (2002, cited in Gerofsky 2006, p. 7), a key performance theorist with a background in both anthropology and theatre, develops the idea of liminal cultural space as a quality of the actual physical spaces where performances are enacted. Limen is a threshold, the boundary line between two places or (metaphorically) two states of being. As limen in the bottom part of a doorway, which is not a separate place but connects two other places the 'inside' and 'outside', the empty space in ritual and aesthetic performances, becomes actually and conceptually a passageway, a threshold in which action remains, to use Turner's phrase, "betwixt and between". The space expands and becomes a living space which does not necessarily follow the rules and conventions of everyday life. An empty theatre space is liminal, open to all kinds of possibilities – that space by means of performing could become anywhere. Performers can explore personas that are not their quotidian selves and actors can convincingly play in role, because the story space of the stage lies in the expanded limen between truth and falsehood (Gerofsky 2012, p. 244). In performance, this liminal space is expanded and opened up and becomes a space which allows for exploration of the contradictions, paradoxes, transitions and transformations that take place as we pass boundaries (Gerofsky 2015). As Gerofsky declares (2006),

> A classroom can be a liminal space – a space of possibility, a passageway, an expanded marginal space with room for play. Classrooms are designed to allow for flexible spatial arrangements; if we are willing to work in the space of the culturally liminal, a classroom can be as mutable as a theatre space. (p. 7)

In a performative place these liminal spaces are open to an emotional, physical and intellectual involvement of students, which may lead them to a deeper understanding and appreciation of mathematics (Gerofsky 2015).

In this paper we claim that Drama in Education techniques in the Mathematics class can create a new liminal space, as described by Gerofsky (2006), a passage way between worlds, where boundaries are blurry. A passage way between learning and play, between different disciplines, as mathematics and art, between "teacher as knower" and "student as listener", between performers and audience between body and mind, between imaginary and real.

Within this space, where the verb "learn" –linked to school practice– and the verb "play" –linked with non classroom-context– coexist, the use of new technologies is faced as a play by students. In this place, through a constructive dialogue, conditions and prerequisites are created for greater and more effective participation of students in the learning process, as it seems from the results of the project which follows.

The Research: Participants, Setting and Methods

- Empirical data for the research presented in this paper, arose from our endeavours towards exploring the dynamics of Drama in Education Techniques in teaching Geometry in high upper school.
- The research was carried out in a group of 26 eleventh grade students in an urban elementary school in the greater area of Athens, and took place in one academic year.
- The use of ethnographic research techniques (i.e. participant observation and interviewing) helped us to gather empirical evidence concerning students' experiences and mathematics achievement and retention of knowledge. All students' presentations were videotaped and analysed regarding the proper use of mathematical notions in their dialogues.
- In terms of the research methods used, we designed and implemented an interdisciplinary didactical intervention, based on a teaching experiment methodology (Chronaki 2008). The teaching experiment (25 teaching periods) –titled "Is our world Euclidean?"– is focused on a detailed design of the teaching of the axiomatizing of Euclidean and Non-Euclidean geometries as well as the history

of Euclid's fifth postulate. The present paper describes the part which refers to the axiomatic foundation of Hyperbolic Geometry.

The Teaching Experiment

The teaching experiment was carried out, by the researcher in teaching role (first author in this paper), in 6 teaching periods, in Geometry, Literature and Greek Language classes. The teaching aims were to enable students: (a) perceive the axiomatic foundation of Hyperbolic geometry, (b) perceive the role of the postulates in an axiomatic system, (c) redefine Euclidean geometry by comparing the similarities and differences of Hyperbolic geometry with the Euclidean one, (d) perceive the role of a model in mathematics and (e) challenge students' stereotypical images about Geometry.

Specifically, the following stages were encountered as entries to the teaching intervention;

- as introduction to the topic, a lecture enhanced with digital projection was provided by the teacher/researcher,
- subsequently, the students were asked to work in teams, using appropriate bibliographical resources such as digital material and literary books, for them to acquire suitable knowledge regarding their presentations,
- a summing up activity by the teacher/researcher ensued where there was ample chance to discuss ideas at the public space,
- the teams prepared their presentations with drama conventions,
- after rehearsing, students performed their presentations,
- a concluding and reflective session followed.

Briefly the activities of this project included:

- Three digital presentations, concerning: (a) historical data, key concepts of Hyperbolic Geometry and elements of the Poincaré's Disk, (b) works of the painter Escher and (c) basic theorems of Hyperbolic geometry.
- Activities in IT lab using the Interactive Java software "NonEuclid" by Castellanos et al.
- Study of the chapter "Platterland" from Ian Stewart book "Flatterland".
- Radio broadcasts concerning "Hyperbolic" geometry and its Poincaré's Disk.

A more detailed description of these activities follows.

During the Literature class (3 h), a digital presentation with historical data, key concepts and theorems of Hyperbolic geometry, elements of the Poincaré Disc, model of this geometry was conducted by the researcher. In Poincaré's Disk, the notions of a point, straight line, segment, angle, triangle and parallel lines were defined. For the students to perceive that distances are not maintained unchanged with position, we used the model "Cold Plate universe" (Gray 1989, apud

Stevenson and Noss 1998). Finally through the Poincaré's Disk we presented the axiomatic foundation of Hyperbolic geometry and we explained the postulate that replaced the 5th Parallel Postulate. A discussion with students followed about the notion of an axiomatic system, about its consistency, the independence of the axioms, and the meaning of a model of an axiomatic system.

Subsequently, students in groups studied excerpts from the chapter "Platterland" from Ian Stewart book "Flatterland" concerning Poincaré's Disk of Hyperbolic geometry, in order to prepare a radio broadcast with the same name issue. The chapter of Ian Stewart's book, in which the protagonists visit "Platterland" (a land with the shape of Poincare disk) addresses the following mathematical concepts: (a) parallel lines in Poincaré's Disk, (b) the infinite distance of the centre of the circular disc to its circumference, (c) the distance between two points (with the apparent shrinkage of shapes as they approach the circumference), (d) the straight lines as the arcs of circles orthogonal to the circumference of the disc, (e) the independence of Euclide's 5th postulate. Suitable shapes illustrate basic properties that characterize the disk, such as that the shortest path between two points on the disc is not the segment of the Euclidean straight line, but the Poincaré straight line, that parallel lines are not equidistant, with the distance between parallel lines approaching to zero (the equidistant line of a Poincaré straight line, it is not a straight line) and that from a point not on a straight line an infinite number of straight lines passes through it and do not intersect it.

In Geometry class (1 h), the students used ICT (Interactive Java software "NonEuclid" by J. Castellanos et al.) for visualizing the Poincaré's Disk, the axioms and basic concepts of this non-Euclidean geometry. "NonEuclid" creates an interactive environment for ruler and compass constructions in the Poincaré's Disk and thus enables the user to explore non-Euclidean geometry. For these reasons 'NonEuclid" is a tool in the teaching of the axiomatic foundation of Hyperbolic geometry.

The 23 students that were present worked on computers with worksheets, in groups of two (seven groups) or of three (three groups) and explored the Poincaré's Disk by drawing points, lines, segments, angles and perpendicular to a given straight line (see Fig. 1). They also measured segments and angles and wrote their comments about the construction of cycles, line segments of equal length and of the measurement of the sides and angles of a triangle. Finally the axiomatic foundation of Hyperbolic geometry was held through the model.

Before students prepare their texts for "Radio broadcasts", a short summing up activity (1 h) was held, during which, Escher work from the unit Circle Limit Exploration was presented. In these paintings, Poincaré straight lines were identified and the equality of shapes was discussed through the repeated patterns of Escher work (see Fig. 2). Then in Modern Greek Language class (2 h), students' teams prepared, wrote and presented their own texts for the radio broadcasts concerning Hyperbolic geometry and its Poincaré's Disk.

Fig. 1 Working on computers

Fig. 2 Hyperbolic geometry through Escher work

Fig. 3 Radio broadcasts

The six radio broadcasts that were presented, were in the form of a radio show in which, a radio producer discusses with invited scientists or with some residents of Platterland, or makes quiz and receive phone calls from the audience. The radio broadcasts were presented from behind a screen so that the students not be seen by the audience (see Fig. 3).

After students' presentations time was dedicated for reflection (see Fig. 4).

Fig. 4 Time for reflection. . .

Results

The Learning of Mathematical Notions

Students' performance (and comprehension through performance) of the mathematical concepts was evaluated through analysis of the worksheets, via relevant questions in specifically organized interviews and through analysis of the enacted dialogues in students' radio broadcasts.

Analysis of Worksheets All the students' teams had to execute, according to the worksheet, various commands of the software, in order to draw basic shapes in Poincaré's Disk and visualize through them the axioms and some basic theorems of Hyperbolic geometry. In the beginning, they had to draw from the command "construction", straight lines, line segments, angles, the midpoint of a segment the bisector of an angle, the perpendicular to a straight line. Students executed easily all these commands, expressing surprise with the differences they observed between the geometric figures in Poincaré's Disk and the corresponding figures in Euclidean geometry.

Students after learning, through corresponding commands, to measure distance and angles, they had to do five different activities: (a) Draw several circles (b) Draw segment at specific length (c) Measure different rays of a circle and (d) Draw triangles and measure their sides and angles and then write what they observe. This task could only be done through the use of this software, through which students were able to design these schemes, work extremely difficult, if not impossible with conventional means as pencil and paper. In the above questions, five of the ten teams gave answers in four ones, three groups in three and two in only two questions.

Regarding students' understanding the Poincaré's Disk of Hyperbolic geometry, the analysis of the worksheets of the teams showed that the students understood the basic notions of the model, with the majority of them responding properly to most

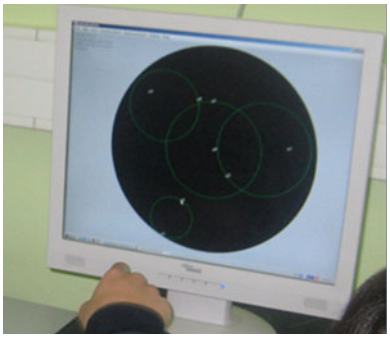

Fig. 5 Drawing circles in Poincaré's Disk

questions relating the circle (the centre and its rays), the apparent shrinking of segments of specific length and the sum of the angles of a triangle. Some students indeed provided additional interpretations of certain phenomena, which they had read in the book Flatterland and we had discussed in a previous class period.

Thereafter we present in detail the students' answers to the various questions. During the construction of different circles (first question) all of the ten groups observed that something different with circles in Poincaré's Disk occurred. They observed that the centre was not where one would expect to be and that it didn't coincide with the centre of the Euclidean circle (see Fig. 5). They write about it as following: "the centre of the circle is not the same with the circle of Euclidean geometry", "the centre is not as in a usual circle". Two of the teams tried additionally to justify it "the centres are more distant (from the Euclidean centre) as we move to the circumference of the Disk. This happens because along the circumference, it is cooled and it shrinks", "Each circle has not the same centre as the circle in Euclidean geometry. This is because as we move away from the centre, we are seemingly changing, but not the distance from the centre, of the points of the circle". A single group responds in a different way "when the centre of the circle is located approximately in the centre of the circular disk, the rays appear symmetrical, but while it is removed from the centre of the circular disk, the radius look different."

In the second question "Draw segments at specific length, and write your comments", all groups answered correctly. One of the groups observed the curvilinear shape of the segment, while the rest of the teams noted that the segments, although of the same length, seemed unequal. Some of their answers, are written in more formal mathematical language and some not: "We drew line segments of the same length and we noticed that the line segment seems smaller near the circumference, in contrast to the segment that is near the centre of the circle. They have the same length but the unit length decreases", "the closer to the centre, the segments appear bigger", "as you go out, leaving the centre, you shrink."

In the third question, students were asked to draw a circle and measure various rays. All of the eight teams who responded have the same answer, writing that the rays are equal even if they seem unequal (see Fig. 6). "Although the rays have the same length, they look different, and not equal to each other", "they are equal, even

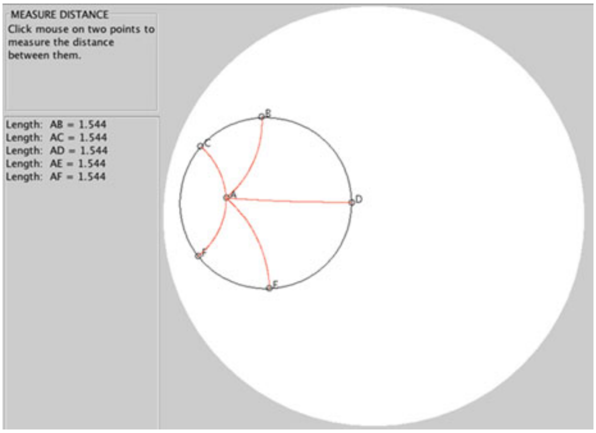

Fig. 6 The rays are equal even if they seem unequal

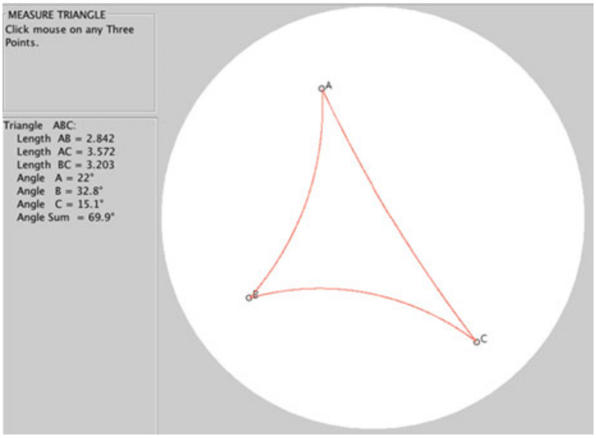

Fig. 7 The sum of the angles of the triangle is less than 180 degrees

though the centre is not the centre of the circle in Euclidean Geometry", "all the rays are of equal length, although they seem unequal."

Finally, the students were asked to draw a triangle, to measure its sides and angles and write their comments (fourth question). Five of the groups made the measurement of the sum of the angles of the triangle and found it to be less than 180 degrees (see Fig. 7), while the other teams did not have time to answer this question.

We asked the teams that they had managed to answer all the questions, to try to draw a rectangle and a square according to the Euclidean definitions of these shapes. The students tried to construct a rectangle but they constantly failed. At first, the students plotted a segment of Poincaré's Disk, as the base of the rectangle. Then they designed, on the same side of the base, two equal segments, perpendicular to it and they joined their edges, or they designed two straight lines e_1 and e_2 perpendicular to the base and a third straight line e_3 perpendicular to e_1 and

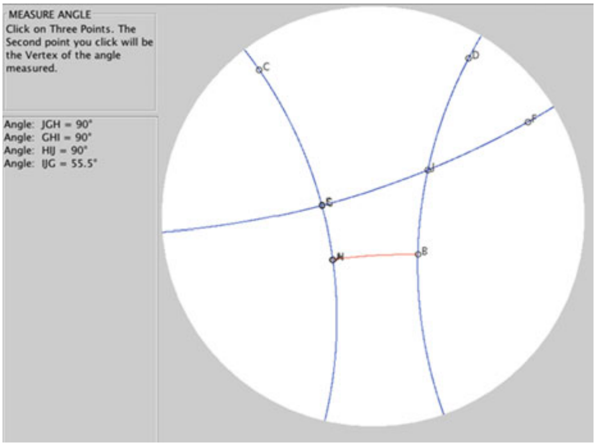

Fig. 8 Trying to draw a rectangle

intersecting e_2 (see Fig. 8). Both drawings, in Euclidean geometry lead to a quadrilateral, though in Hyperbolic the former leads to a shape having two right angles and each of the other two angles less than a right angle and the latter to a quadrilateral with three right angles and the fourth less than a right angle. The finding that there is no rectangular in the hyperbolic world stunned them. From our observation during students' work, we realized that their effort to construct in the model, a straight line, a circle and a rectangle helped them to renegotiate and understand the relevant concepts of Euclidean geometry. Also their rectangle construction effort and the finding that the sum of the angles of a triangle is less than 180 degrees made them understand the role of the Euclidean geometry 5th axiom.

Analysis of Dialogues in Radio Broadcasts Following the analysis of the six radio broadcasts texts that students prepared about Hyperbolic Geometry, we observe that two teams did a full description of the Poincaré's Disk, while the remaining teams referred to only a few concepts, those that they considered important or had been impressed by. In three radio broadcasts students referred to Platterland's shape which had the shape of Poincaré's Disk: "I was impressed that in "Platterland" there are only two dimensions and that while from far away the country it seems to have a certain extent, when actually reaching there, you realize that it is infinite" and also defined straight lines (four teams): 'Defining in this model as straight lines the arcs of circles orthogonal to the circumference of the disc, imagine a circle and curves inside it'. In one broadcast the particular shape of the circle is emphasized: "In Hyperbolic geometry the centre of the circle tends to its circumference, yet its radii again are equal" and also that parallel lines are not equidistant (two teams): "The distance between parallel lines is not constant". Finally in all broadcasts students talk about the apparent decrease in the length of a segment and to justify this phenomenon they use the model "Cold plate", "because the objects shrink, depending on the temperature. In the periphery the temperature is zero degrees Celsius and as they go towards there they shrink" while in two, they talk about the alternative 5th postulate: "From a point not on a straight line an infinite number of parallels lines passes through it". Closing the analysis, we want to add, that two groups in their radio shows, although they referred to Hyperbolic geometry, they gave information about Poincaré's Disk, identifying Hyperbolic geometry with its model.

We will quote as an example, an excerpt from the radio show "Weird and True", in which the students chose as a framework a fictional interview of a reporter with a Mathematician Guest and a resident of Platterland, a country with the shape and properties of the Poincaré's Disk.

The Radio host	Good morning our dear listeners. Today we will analyse in our show, a really special topic, that of the Hyperbolic geometry. I am pleased to announce that we will host a resident of the long distance Platterland and a Mathematician, with a PhD in Non-Euclidean geometries. With the help also of our good colleague we will introduce you to this unknown to us Geometry. Let's listen to what they have to say.
The reporter	I was impressed by the fact that in Platterland there are only two dimensions and that although from away the land appears to a certain extent, actually arriving there, you realize that it is infinite.
Mathematician Guest	In Euclidean geometry what we call a straight line, in their own world we perceive it as an arc that intersects the circumference at right angles. It is noteworthy that Platterland is a circular disc, without its circumference and the points are defined only within it...
The Radio host	Indeed, all this is very interesting. Let us listen now, the experiences of our guest, who is a resident of Platterland.
The resident	The objects of Platterland as they remove to infinity, they shrink, although this can not be perceived by a person moving in Platterland, because as the object shrinks, at the same time the meter shrinks too. So in every measurement we have the same result.
The Radio host	Thank you. Now, we say goodbye to all of you and we renew our appointment, next week.

Analysis of the Interviews Two months after the activities sixteen students were asked questions about Hyperbolic geometry (in 16 semi-structured interviews) for the retention of knowledge from the use of the specific tools to be examined. From their responses it seemed that students were so impressed by the Poincaré's Disk that they identified it with hyperbolic space. The majority of students responded that they were impressed by the shape of the lines in Poincaré's Disk (11 replies) "First of all, curves were considered as straight lines. For those who were in hyperbolic world it seemed like a normal straight line, it was just a matter of how you see things. Do you see them from outside or inside? It's completely different what we call in Euclidean geometry a straight line and what we call in Hyperbolic geometry straight', and the apparent change of segments of the same length (six replies): “As the objects of Platterland remove to infinity, they shrink, although this can not be perceived by a person moving in Platterland, because as the object shrinks, at the same time the meter shrinks too. So in every measurement we have the same result”. Three students mentioned with surprise the sum of the angles of a triangle in Poincaré disc: “In Euclidean it is 180°, in Hyperbolic it is more than 180°”, and the non existence of the square: “Especially this finding about the square; we will never forget that there is no square”, while some students highlight the importance of the fifth postulate: “Changing the 5th postulate, essentially changes the whole theory of geometry”.

Summing up, from the analysis of all the aforementioned data, we can conclude that the students understood the basic concepts of Hyperbolic geometry through Poincaré’s Disk, and it was ICT which played an important role in this, by contributing to the visualization of the model.

Mathematics as a Creation Under Constant Negotiation

Students were actively involved with the new and strange to them non Euclidean geometry,

Sofia The fact that I saw other geometries, basically because I like those weird things, intrigued me; i.e. to see a geometry which I have not seen before, in which the line segment was a curve. I had never seen something like this before. Or to tell me that this circle is infinite, it has infinite points, it is immense. This piqued my interest more and to tell the truth, I started then searching about it at home and on the internet and I liked it more. I put the book of geometry aside, I do not want to see it again in my life and I sat and read something what I had not understood.

Thus the teaching of Hyperbolic geometry provoked students’ perception about mathematics as a science of the absolute truth. Their involvement in this procedure helped students perceive Mathematics as corrigible and as a creation under constant negotiation, modifying thus their epistemological beliefs about mathematics and provoked the dominant belief that Euclidean geometry is the only model which interprets and represents our real world, shaking thereby other certainties.

Stefanos Certainly the plasticity of mathematics emerged and the way mathematics are created and changed depends on the needs of the mathematician, of the scientist and of the human being generally. It is clear that mathematics is a complex notion, which is not restricted to only one way of understanding reality…

Angela Finally there are and alternative views and we cannot say which is absolutely right and which not.

The Role of ICT and Drama in Education in the Project

Our research aim of the teaching experiment was exploring the dynamics of Drama in Education Techniques in teaching Geometry in high upper school. The entire teaching experiment included a number of activities that would give students the appropriate knowledge for their presentations. We wanted the students through our lectures, the work in teams, the study of the relevant bibliography, the study of extracts from a book of "mathematical literature" and the use of new technologies to understand the concepts and presenting them with various techniques of Drama in Education.

Students' Experiences Both from the responses we got in the semi-structured interviews with the pupils, and from the observation during the whole process, it seemed that the pupils were motivated within these expanded contexts, and that they became cooperative through their engagement in new teaching practices. Students referred positively to the whole project and highlighted its multimodality. The following quotes are indicative:

Peter	It was interesting, it was nice that we used in teaching many different methods and tools, which prevented anybody to be bored. It was a row of different things, it was theatre, it was a normal presentation by the teacher, it was computers, even a radio broadcast. I think it is very interesting; it just needs its time.
Giolena	You know something, we had no problem, I personally had no problem that you would tell us, "today we will do a radio show or we will go to computers". I just liked that you would come to do a different lesson, instead of sitting on chairs and get bored.

Angeliki emphasizes the variety of teaching means in the project, which enabled her to express herself by the one which suited more to her identity.

Angeliki	As I have told you before, in Geometry class I was feeling boredom. But in the project, even if it was a sketch, ok the sketch was not my best, because you saw that I cannot talk, I have a problem, I knew that we would read something new and then in computers it was very nice, because although it was geometry we would learn something interesting.

The enthusiasm and the interest of the students during the teaching experiment were observed by their teachers -observers also in the teaching experiment- and the researcher herself.

Maria M. (Greek Language teacher)	I saw them (the students) enchanted. Because they were creating.
Kostas K. (Greek Literature teacher)	I was enchanted by the fact that I saw the children to be so much interested.

The researcher, from the observation and the video recording of students during their work in groups and during preparing their presentations, she saw that the students seemed full of vitality, talking to each other, explaining to each other, teasing each other, laughing, having fun and in general developing important communication.

Drama in Education Techniques From all these activities of the teaching experiment, we wanted to identify those which most attracted students' interest and motivated them to work. So, in the interviews, we addressed the following question to the students: "What was the most significant and interesting activity for you, in the project?"

All the 15 students that were asked in the interviews, answered that the sketches were for them the most important activity, what they liked most and were the motivation for their active participation in the teaching experiment.

Chris I believe that the presentations were the most interesting part.
Tzina This was pretty nice, because when you just write it will not be very different, because who will see it. But when you do a sketch is more interesting, more beautiful. . .
Antonis the presentation for me, sparkled my interest. I think, it missed me something like this, in Geometry class.

When we asked the students to present their thoughts, during the final discussion reflecting the whole project, seven students also stressed that it was the happenings with DiE techniques that impressed them the most, without justifying always their opinion.

Mina For me, from all this, it was the sketches that I liked more.

Some of them stated that they liked the DiE techniques while they were not interested in the subject of Mathematics.

Effie ... for me, mathematics is not the best subject, but I believe that in this way the lesson became more interesting.

In the researcher's question in the interviews: "In the project you could have stopped after having participated in all other activities, till the writing the dialogues of the sketches, but without performing them. Would it be the same for you?" students' answers highlight that what differentiated the "project" were the activities with DiE techniques.

Sofia If we just studied or just wrote, we would have been bored to death and even more.
Nicky [without the presentations] I do not think that anyone would be interested.
Vicky It was completely different than just writing stuff, from comprehending something yourself and trying to pass it to others. It was very nice.

Even Angeliki, the student who found difficulties in expression through theatre techniques, replied that she was negative in non-performing.

Angeliki they were interesting, although I couldn't.

We believe that this students' choice is due to the liminal space created by the DiE techniques, a space with blurry boundaries between learning and play, between mathematics and art, between performers and audience, between imaginary and real, between body and mind. This liminal space, which was a combination of lesson and fun, in which students learned and enjoyed through playing, it supported, according to students, the learning process.

Virgianna It was a combination of lesson and entertainment... In the project, except that we cooperate, you can learn things without realizing it. I was getting knowledge in such a way, it was as entertainment. It was a perfect way for one to have a good time and get knowledge.

Petros recognizes the teaching in this context as a game or fun.

Petros All this dramatic, theatrical thing is like a game. Maybe this entertaining form is missing.

Students, in the final reflection of the whole experiment, emphasized the feeling of well-being which they felt during the project. The phrase "we had a good time," was something that was repeated continuously.

Zoe It was certainly something different. We combined Mathematics with other subjects of the curriculum, as the Greek Language, we cooperated, we had a good time.

Many students in the interviews repeat: "we had a good time" too.

Christos In the project, we are in groups, we had a good time, everyone participated.

John It was a different experience. It was something new. We had a good time.

In our question "What would a visitor have seen in our class during your activities?" the students gave us an image of a class full of joy and energy, highlighting that the whole process was beneficial not only at a cognitive but at an affective level as well.

Giolena I think that an observer would see (in the class), that we had understood what we had learned from you, and that he would see more joy, teamwork, and not so much boredom.

The teaching context with DiE techniques gave opportunity to students for fun, humour, jokes and laughter contributing to students" well-being.

Vicky I liked all this, the way we did all this, with the theatre and the cooperation and that we laughed and that the time passed too quickly, i.e. I was looking at the clock and I was saying, "well how did the time pass?" and I wanted more.

Nikoleta Initially it was more interesting than a common lesson, we laughed.

The students' well-being in the teaching experiment, helped them get involved in the learning process, in contrast with the typical class which usually caused them boredom.

Gina It was very nice, it was something special and I agree with those who previously said that the hour of mathematics that we made these sketches, it was fine, because the time passes better and you learn more things, because when it is a boring lesson you do not learn so many things or you do other things, but when things are like this you get more.

The Role of ICT By observing students as they worked with computers and as they completed their worksheets, we conclude the active participation of students in the activity with computers. Students constantly played with the shapes, trying to understand the new and strange to them geometry, with its different fifth postulate and the different basic theorems. The students had heard about all these topics in the teacher's presentation and they had read in the book Flatterland, about the protagonists' experiences from their visit in Platterland. All these, took a form through the NonEuclid software. The interactivity of the software allowed them to design their own basic shapes and, through the design of many of such schemes, to discover that equality is not identical with congruence and that there are no squares and rectangles as defined in Euclidean geometry. Their exclamations and conversations were indicative for that finding.

In our teaching experiment we wanted to explore further, through interviews, the views of students themselves about their experience with the activity on the computers. From their replies, we see a positive approach to the use of applet NonEuclid in the teaching of Hyperbolic geometry.

Giolena Perfect. I m not good at all with computers, I see a PC and I am afraid that the keyboard will fly away, but this thing regarding Geometry, the shapes that we were drawing lines, was top fun. I'm even considering downloading it and sit alone and play with it.

Mina Fine, it was related with the technology, we are a technology savvy generation.

Sofia Through the sketches and the use of computers we learned more things than in a classical Geometry lesson.

Only one student, Angela, showed her preference for the activity with computers, having informed us that her wish was to study computer science and that she didn't like being exposed through theatrical techniques.

Aggeliki I liked this very much, it was my favourite among all we did. Especially regarding Hyperbolic geometry and all in the plane of Platterland, it was very interesting. With computers Mathematics can become more creative.

We asked them then, if computers helped them understand the Poincaré's Disk and in this way they facilitated the creation of radio show.

Aggeliki Yes, because I got it. When the time came to say the text, I got what it was all about, therefore it was easier to do so, even though I didn't remember what the exact script was. If I didn't understand the text, I would have to memorize the text, whereas now that I understood what the text was talking about I could change things around.

Aggeliki who had stated that it had difficulties with geometry due to the shapes "It's the shapes, I can not manage with so many lines", in this activity with the computers, she dealt and understood many different shapes, "Perhaps", as she added, "because it was something different and new. In geometry we do the same things, so many years now, while with computers it was different, I had not seen this things before".

Mina believes that new technologies helped her understand the text of Platterland, expressing at the same time the complaint that this does not happen in their daily lesson of geometry.

Mina But that was only about Platterland, in our daily lesson about Euclidean geometry, we didn't do nothing.

Finally we had two responses from students, which stated that the computers did not help them in understanding the concepts.

Gina ... most of the things, I understood them by the story, not by the computers.

Tatiana The computers, I had forgotten them. They left me nothing. It was interesting that we did, I had never done something like this before, but I'm more with the arts.

Concluding Remarks

In this chapter, we presented an example of creating a specific space in geometry class –a space formed by a mediating tool different than those used in traditional teaching of geometry. It seemed that this space, a liminal space, inspired students to actively engage in the learning process and acquire complex mathematical concepts, such as that of the foundation of an axiomatic system. Students experienced how hyperbolic geometry is axiomatically founded and how basic axioms and definitions can lead to different and sometimes contradictory results regarding Euclidean geometry. The students' contact with a non-Euclidean geometry was an opportunity for them to renegotiate the basic concepts of Euclidean geometry – as a geometry and not as "the" Geometry—and gain a deeper and more holistic understanding of geometry. Their contact with this geometry enabled them to appreciate the liberation of geometry as a science that tries to describe the spatial properties of the world we live in and contributed to the creation of an image of Geometry as an interesting curriculum subject.

We believe that the most important factor of what it was achieved, was the multimodality of teaching, the utilization at the same time in mathematics classroom, both of new technologies and of DiE techniques. The analysis of our research data suggests that ICT and DiE helped for the mathematical knowledge to be developed in a class of high school students who were involved actively and effectively. More specifically, ICTs have helped students visualize the Poincaré's Disk and through it understand key elements of Hyperbolic geometry, while Drama offered students the motivation for a dynamic learning through the rich experience that involves body, feelings and senses. In this liminal space, created by DiE techniques, with the blurry borders –classroom/non-classroom–, our students in a collaborative framework addressed mathematical concepts –geometric concepts–, wrote texts and they performed them, pretending roles, experiencing mathematics in terms of aesthetics, humour and emotions.

This space, as it was open to an emotional, physical and intellectual engagement of students, inspired students' greater participation in mathematical thinking and expression and led them to a deeper understanding and appreciation of mathematics. In this space the use of interactive tools, as java applet, for the exploration of non Euclidean spaces showed us clearly how ICTs can allow students to experience worlds that otherwise would not be accessible.

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Improving the Teaching of Mathematics with the Use of Technology: A Commentary

Sixto Romero

Abstract Current trends in mathematics education have emphasized the importance of using technology as a means by which students can work in other "pencil and paper" environments and can draw conclusions that will benefit them in the learning process. The non-use of new technologies may prevent the achieving more ambitious goals. The aim of the four chapters presented by Sabena, Lobo da Costa and co-authors, Hitt and co-authors and, Kotarinou and Stathopoulou is to show how the use of technology can help in the teaching and learning of mathematics, provided that process is well directed by the teacher.

Keywords Algorithm • Learning and teaching • Mathematical model • Mathematical task • Spatial competence • Sociocultural context • Technology

Introduction

In chapter "Early Child Spatial Development: A Teaching Experiment with Programmable Robots", Sabena presents the development of spatial skills in young children inspecting the educational capabilities provided by programmable robots. In chapter "Mediation of Technological Resources in Lessons on Polyhedra: Analysis of Two Teaching Actions", Lobo da Costa, Pimentel and Mendonça, through the mediation of technology resources prepared in geometry class, allow a greater understanding of the shares in the T/L a process through reflective practice teacher as a fundamental agent management framework that needs the reported activity. Hitt, Saboya and Cortés, in chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers" analyzing the design of mathematical tasks in a collaborative environment (the teaching method ACODESA) propose a methodology in which individual and social approaches are envisaged in the construction of mathematical knowledge. Finally Kotarinou and Stathopoulou present the axiomatic definition of Hyperbolic geometry through the Poincare model as an introduction to

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Non-Euclidean geometry developed abstractly from the set of knowledge that emerged in the study of Euclid's fifth postulate.

Comments on Chapter "Early Child Spatial Development: A Teaching Experiment with Programmable Robots"

As a first comment on the contribution of Sabena, it is necessary to reflect on the concept of space. It appears as a fundamental skill that accompanies the development of cognitive skills throughout the growth of children. In every stage of development, it is essential to know who we are and what our role in life is. It is important to note that when we lose consciousness the first thing we ask is: "Where am I?" because knowing who we are, where we are, at what stage of our existence we are, are the three basic issues allowing the contextualization of our own existence notions.

Even if it seems logical and natural for adults to evolve in space, the question of the development of the concept of space is an important issue for the learning process in the first stage of the life (Romero 2000).

For Piaget, acquiring the spatial notion is intrinsically linked to the acquisition of knowledge, and it is through this knowledge that the child's development begins at an early age. "The existence of multiple perspectives relating to various individuals is therefore already involved in the child's effort to represent space to himself. Moreover, to represent to himself space or objects in space is necessarily to reconcile in a single act the different possible perspectives on reality and no longer to be satisfied to adopt them successively" (Piaget 1954).

The notion of space (Parzysz 1991) can only be understood in terms of the construction of objects, and would need to begin by describing this to understand the first: only the degree of objectification that the child attributes to things informs us about the degree of externality according to the space. This cognitive beginning is enriched as the child grows and learns about space. For Craig (1995): "... knowledge of spatial relationships is achieved during the preschool period. This is logical because it is the age at which learning concepts like: inside, outside, near, far, up, down, above and below . . ." (p. 394).

Piaget dedicated two volumes to study the development of spatial knowledge, based on performing a large quality of different experiments. In 1947, in collaboration with Inhelder he writes "The representation of space in the child", and deals with how ontogenetic development arises in topological relationships, projective and Euclidian. In his second work, in 1948, with Inhelder and Szeminska ("Spontaneous geometry in the child"), he studies the genesis of Euclidean geometry, that is, the conservation of length measurement, as well as surface and volume.

Based on the psychological work of Piaget, Inhelder, Lucart and Vygotsky, as well as on the didactical approach of Arzarello, among others, Sabena supports the hypothesis that the reality in early childhood is full of different spatial cognitive aspects and requires different specific skills that must necessarily be related. She focuses on the development of spatial competences of children, and explores the educational potential offered by programmable robots. Cognitive aspects are in the first plane and in particular the delicate relationship between space (Hershkowitz et al. 1996) and everyday experience versus space as a mathematical notion.

Analyzing Sabena's experiment, it occurs that mathematics teaching with technology has to deal with a set of scientific and technical knowledge. Throughout the last century it gained increasing importance in everyday life as well as in the development of modern society. Teacher training in mathematics education requires relatively specific attention to the acquisition of knowledge. In general, educational programs with different materials (providing structured information to students by simulating phenomena) offer an environment more or less sensitive to the circumstances of the students' work, and especially, more or less rich in possibilities for interaction among young children; but all of these share essential characteristics:

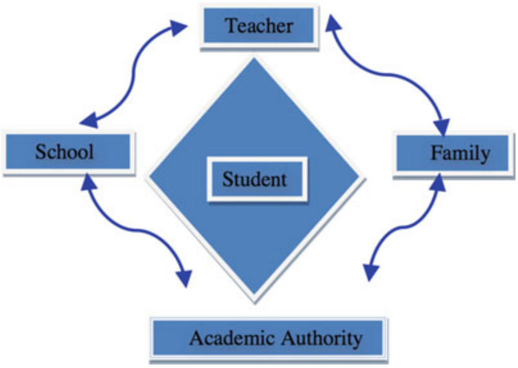
- They use the material as a support in which students perform the activities.
- They are interactive, immediately responding to the students' actions and permitting dialogue and exchange of information between the material used and the child.
- They can identify the children's work and adapt to their rhythm and activities.

They are easy to use because a minimum of knowledge is required to perform the tasks (De La Fuente 2010). Thus, the author of this chapter emphasizes that high-tech gadgets surround today's young people and hardly attracted by simple mechanisms. Robots represent a technological element of great attraction to be very close to the type of devices that they use daily. Robotics is a branch of the scientific and technological knowledge that studies the design and construction of machines capable of performing repetitive tasks, where high precision is needed, dangerous work for human beings or unrealizable tasks without intervention of a machine. In the work of Sabena ("Early Child Spatial Development: a Teaching Experiment with Programmable Robots") the spatial development of skills shown by exploring the educational potential through programmable robots, places value on how the experience with a robot has influenced the children conceptualization of the concept of space.

Comments on Chapter "Mediation of Technological Resources in Lessons on Polyhedra: Analysis of Two Teaching Actions"

The theoretical framework of Lobo da Costa, Pimentel and Mendonça's work is based on Zeichner and Serrazina's ideas. It is a very attractive example in which the mediation of technological resources used in geometry classes is studied;

Fig. 1 Relation between the different actors in a process of teaching



particularly for dealing with three dimensional solids like polyhedral, prisms and pyramids, in elementary school.

As a reflection and following the scheme of the previous chapter, it is necessary to indicate that the presence of technology in education is no longer a novelty but a reality (De Lange et al. 1993). The contexts of the teaching-learning have changed their single appearance in the classroom, at least materially. The main issues are the new mathematics education processes and the way to involve all the agents (Fig. 1).

Having high expectations of the technological means, giving it potential for the treatment of information, should not prevent assessment and reflection on the ability to transform information. The objects are not simply the media or technology (NCTM 2000). The objects of evaluation and reflection are the active agents involved, and the contexts of teaching and learning we designed and put into practice and, ultimately the use of technological resources for the generation of knowledge. The ending aim is always education.

Research presented by Lobo da Costa, Pimentel and Mendonça analyses the role of technological resources in the geometry classroom, specifically that which is based on the concept of polyhedral. The mathematical content, practice and technology used during the experience are presented in detail. The categories analyzed were the class routines, interactions with students in order to see how the mathematical content was developed and the technology used.

They emphasize that, according to Serrazina and Oliveira (2005), teachers, in order to manage better their time should be responsible for the activities, contents and class organization proposed to students. Activities imposed by the teaching staff or by the central bodies of education are not always well received by teachers. A literal reproduction of what is stated in the recommendation to students in order to meet the curriculum planned and imposed by academic authorities is mainly observed in both groups of this experimental study, with few time spent in manipulations and collective discussions.

It is important that the authors of the study do not compare mediations, since they are linked to confirm the personal characteristics in the way in which technological resources were used. However, from the analysis of the incidence and mediation of technology resources that teachers use, they conclude that the main features are: the reality of the classroom, the student interest, the number of students per class, the breach in prior knowledge of the students, the need for compliance with the prescribed plan of studies and the time available; and these will be considered as factors that interfere in the mediation.

The presented experience, as Volkert (2008) points out, shows the intrinsic difficulties of solid geometry impeding the introduction of systematic teaching. Solid geometry is much more complicated than its homologue on a plane. Also, the problem of intuition and evidence is far more complex and problematic in Solid geometry. So the history of Euler's theorem is a very good illustration of this theme. These difficulties can be taken together with others like for instance, in secondary, spatial geometry is relegated and in some cases completely absent.

We can emphasize that in the chapter developed by Lobo da Costa, Pimentel and Mendonça the use of physical objects, models and figures is the main tool for teachers to help students understand the geometric concepts, hence the ability to display (or spatial imagination) is imperative to learn geometry. The display is very useful in any area of mathematics and especially in the field of geometry. The teaching of elementary geometry has always been based on intensive use of objects, figures, diagrams, charts, etc. to help understand the concepts, properties, relationships or formulas studied. Thus, as indicated by Hershkowitz et al. (1996), geometry appears to students as the science that studies the physical space and the convenience of using graphical representations to help the understanding of geometric concepts extends beyond elementary Euclidean geometry as developed by Kotarinou and Stathopoulou in chapter "ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching".

As a personal opinion based on the experience I have accumulated since 1975, by collaborative work with teachers from different levels of education, the almost complete unanimity among mathematics teachers that adequate display capability is an essential tool for geometrical learning that is rarely accompanied by a reflection on the learning processes of visualization. This is not an innate ability that can be let develop spontaneously, but a model is necessary, as the display is a complex activity in which several elements are necessary to be understood and learnt in order to be used.

Comments on Chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers"

The third chapter started by making a first reflection on problem solving as a way to mathematical modeling. The research in Mathematics Education has focused its attention for some time on designing activities based on mathematical modeling of real situations, with the conviction of obtaining greater assurance in profit by our students of mathematics learning, and therefore teaching. One of the most complex problems that education faces in different educational levels where the teaching of mathematics is concerned relates the way of articulating the contents with other areas of knowledge and even with mathematics in itself.

For our students, most content organized into topics are disconnected from the real world and science applications, as a consequence this means that they do not conceive the utility of mathematics in their training. In recent years, research in Mathematics Education realizes that one of the striking issues is the design of activities based on the modeling of real situations. In many countries and in different conditions, its inclusion in the curriculum has allowed the development of diverse types of cognitive capabilities, metacognitive and crosscutting to help understanding the role of mathematics in today's society (Aravena and Caamaño 2007; Blomhoj 2004; Keitel 1993). Therefore, today's society must provide the role to deal with problem solving, make estimates, and take decisions, and face a mathematization of culture and the surrounding environment. That is, modelling mathematics is tending to promote understanding (Niss 1989) of the concepts and methods, thus allowing a more comprehensive overview of mathematics.

Over the course of history, mathematics has occupied a prominent place in school curricula. It has achieved this prominence, not because of the importance in itself but for cultural and social reasons.

We collect the idea of Jean Pierre Kahane, French mathematician and professor emeritus at the University Paris Sud Orsay, a former student of the Ecole Normale Superieure, and member of the Academie des Sciences (mathematics section) since 1998 when he asserts:

> the reflection on the teaching of mathematics is done from all angles, from all status: it can be from the daily work in the classroom, difficulties of teachers and students of all educational levels. It can be done through a detailed examination, test study; or extra-curricular activities, the gymkhanas, rallies, competitions, olympics, ultimately all manifestations of animation and diffusion of mathematics; or the role and evolution of the mathematical sciences in the whole of science and society. (Gras et al. 2003, p. 5)

As in France, in many countries, teachers grouped or not in Societies of Teachers, Editors of publications in Mathematics Education have taken initiatives in order to make proposals and initiatives in the field of Problem Solving and Mathematical Modelling (Romero and Romero 2015) to improve the binomial teaching/learning of mathematics. Problem solving has a long tradition in mathematics. George Polya considered Euclid's Elements as a collection of problems (a sequence of statements and solutions). Together with Gabor Szegö, he produced under the title of *Exercise Analysis*, a collection graduate of problems.

The authors, Hitt, Saboya and Cortés, utilize problematic situations in the sociocultural context of mathematics class that requires careful design to develop skills in the classroom, promote diversified thinking and achieve a balance between the pencil, paper and technological activities referred to in the theoretical framework of the activity (Balacheff 2000). The ACODESA methodology presented in the chapter differs in five main phases (Individual work, Teamwork on the same task, Debate, Auto-reflection and Process of institutionalization), and the design of the activities under this perspective and with the use of technology is not a trivial

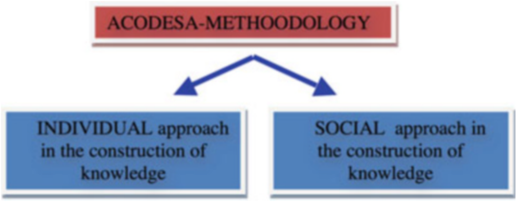


Fig. 2 ACODESA method of teaching, seeing the individual in a social context of learning

task in the mathematics classroom. A comprehensive work to develop the activity and the details that need to be provided to present a complete vision of the activity need a significant space that is not always available in a research context. Deficient communications in all aspects involved in the development of problem solving activities makes it more difficult for teachers to follow those activities.

In the design of tasks, they are taking into account Arcavi and Hadas (2000) suggestions; based on a Dynamic Geometric System that stands out for the elements of visualization, experimentation, surprise, evaluation, need testing and demonstration, as key elements of the analysis detailed. Also, the prospect of collaborative work (Prusak et al. 2013) allows for the design and creation of tasks (Kieran et al. 2015), suggesting problematic situations that enrich the visualization of the problem (Fig. 2).

The authors present very appropriate examples. The use of the concept of triangular number as one that may be in the form of an equilateral triangle with other figurative numbers were studied by Pythagoras and the Pythagoreans, who considered sacred 10 written in a triangular shape, and they called Tetraktys.

The dynamism presents examples, related to:

- Visualizing information through a numerical approach.
- Find a generic pattern.
- Affirm that generally the tasks of connecting the different representations of a concept, is not considered by many teachers as fundamental in the construction of mathematical knowledge and, in particular teachers minimize the task of the conversion among representations.

Hitt et al., proposed that the task of the conversion, among representations, would enable the development of mathematical visualization processes. This visualization has to do with mental processes and transformation productions on paper, on the blackboard or on the computer, generated from a reading of mathematical statements or graphics, promoting the interaction between representations for a better understanding of mathematical concepts involved.

In conclusion, the tasks and the methodology proposed by the authors of this chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers" inculcate in students the learning of mathematisation, defined as problem solving that triggers a process of:

> Identification of relevant mathematical concepts and then progressively simplify reality in order to transform the problem into one susceptible to locate an a mathematical solution ... by finding regularities and patterns, [...] It need to use various competencies for mathematisation task. (OECD 2004, pp. 27, 28 and 29)

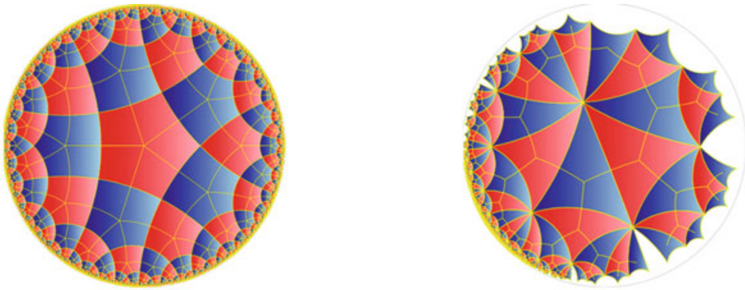
Comments on Chapter "ICT and Liminal Performative Space for Hyperbolic Geometry's Teaching"

A teaching experiment about axiomatic foundation of Hyperbolic geometry and its basic notions, using 'Drama in Education' conventions to motivate and actively engage all of the students, is presented in this chapter "ICT and Liminal Performa tive Space for Hyperbolic Geometry's Teaching". The fundamental purpose of the work presented by Kotarinou and Stathopoulou, using, as a case study, the introduction of Hyperbolic geometry through the Poincaré model, is to show that the creation of new problematic situations with the use of technology allows more dynamic teaching of geometry in the classroom, improving understanding.

It is interesting to know the theoretical framework in which the activity is presented by the authors. There are many comparisons between Euclidean geometry and Hyperbolic. For example, it could well be that Hyperbolic geometry was actually true in our world cosmological scale. However, the proportionality constant between the deficit angle and a triangle area should be extraordinarily small in this case, and Euclidean geometry would be an excellent approximation to this geometry for any ordinary scale. In the Poincaré model H^2, all the hyperbolic space is represented in a disc of the radius, $r = 1$. The edge of the disc represents the infinite. Within the disk all the postulates of Euclid are satisfied except the 5th (the parallel postulate):

1. It can draw a straight line through two points.
2. It can prolong a straight line indefinitely from a finite straight line.
3. You can draw a circle with given center and known radius.
4. All right angles are equal.
5. *If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.*

In H^2 the sum of the internal angles of a triangle is lower than 180. More surprisingly, two lines with different directions may be parallel. Poincaré model to visualize these aspects of Hyperbolic geometry, but being all the space within a disk, the lines are righteous actually are perceived as curves (hence they are called "Geodesic"). And the metric that allows us to measure distances within the Poincaré disk is not Euclidean. These ideas can be shown and manipulated in a relatively easy way with the use of appropriate software. The time spent by students working with computers is really very important for the visualization, recognition



Hyperbolic tessellation {5,4} generated in C # and WPF

Hyperbolic rotation and translation with Möbius transformations

Fig. 3 Transformations in Poincaré's disk (https://rastergraphics.wordpress.com/2012/06/27/geometria-hiperbolica-disco-de-poncare/)

and exploration of a non-Euclidean geometry, a geometry that is not in our daily life (Fig. 3).

Kotarinou and Stathopoulou point out that students who carried out the experience came to understand the principles of Hyperbolic geometry through the Poincaré model with the analysis of worksheets. The experiment shows that most participants adequately responded to most of the issues of the circle (the center and its rays), the apparent decline in segments of specific length and the sum of the angles of a triangle. Some students gave further explanation of certain phenomena especially those who had read the book Flatterland, discussed in a previous class period, but not accessible to many. Therefore, it is important to note that the implementation of Hyperbolic geometry in the Poincaré model are useful for the following concepts:

- The hyperbolic space H^2 is a disk of radius, $r = 1$, centered at the origin in the Euclidean plane R^2, called Poincaré disk.
- The points in the hyperbolic space H^2 are points in the Euclidean plane that are within the Poincaré disk.
- The lines passing through two points in H^2 are Euclidean circles passing through two points on disk and are orthogonal to the Poincaré disc.
- The lines passing through the origin (i.e., the center of the Poincaré disk) are circles of radius $r = \infty$, they are Euclidean lines.
- The angles are Euclidean, the measure of angle formed between two geodesics (hyperbolic lines) is the angle between the tangents of the circles at the point where they are intercepting.
- The inversion of a point on the circle is an isometry (preserves angles and distances) and is interpreted as the reflection of a point in a hyperbolic line.

It should be noted as very positive the use of ICT (interactive Java) by students to display the model of Poincaré (axioms and basic concepts of non-Euclidean geometry), thus creating an interactive environment, ultimately providing a new tool in teaching the axiomatic basis of hyperbolic geometry. The working group of students with worksheets, exploring the Poincaré's model has enabled them to draw points, lines, segments, angles and lines perpendicular to a given line. Thus writing the comments on the construction of cycles, line segments of equal length and measuring sides and angles of a triangle has allowed students to understand the axiomatic basis of Hyperbolic geometry in an enjoyable manner, creating a relaxed environment and satisfaction in students. This chapter presented by Kotarinou and Stathopoulou is interesting because the experience presented deals with a new practice leading to new paradigms and new tools with new technologies that have helped the process of students' visualization and therefore the understanding of the geometric concepts presented (Gutiérrez 2006).

Conclusion

First, it can be concluded from the above Chapters of Sabena, Lobo da Costa et al., Hitt et al., and Kotarinou and Stathopoulou, that if the conception of the role of the teacher is close to traditional transmitter and organizer of knowledge and practical activities, where visualization is rarely used in the classroom, the assessment will be related to working methods explained in class, impeding autonomy of the students.

Enquiries from Presmeg (1997) identifying various types of mental imagery is used by students to solve mathematical problems. The most commonly used in geometry are:

– *Concrete images* (pictures in the mind): figurative mental images of real objects.
– *Kinetic images*: mental images associated with muscle activity as a movement of a hand, head, etc. For example, when a student, describing a figure with parallel segments, places the hands stretched parallel and moves them up and down.
– *Dynamic images*: mental images in which the displayed image (or any of its elements) is a moving object. Unlike the kinetic images, these images do not provoke physical movement, but are only displayed in the mind.

For his part, Bishop (1989) describes two processes taking place when using images:

– *Interpretation of figurative information*: the process that takes place when trying to read, understand and interpret an image to extract information from it.
– *Visual information processing*: the process that takes place when converting non-visual information in images, or transforming an image already formed into another image.

The experience at different levels of education (Blomhoj 2004) shows that the treatment of theoretical aspects can be a tool for the practice of teaching problem solving as a path to mathematical modeling. The role played by teachers and researchers in mathematics education should perform interesting work in many mathematical domains such as, for instance, problem solving, almost unexplored in the Primary and/or Secondary school (Romero and Castro 2008), which can produce in an original and creative way, activities enriching the process of teaching. Many of these domains can be planned so that they can become powerful generators of important skills, not only mathematics but crosscutting, *number theory, graph theory and optimization theory chaos, topology, data processing, coding theory and cryptography, fractals mathematical models*, or competences presented by Sabena, Hitt et al., Lobo da Costa et al., and Kotarinou and Stathopoulou.

As a final comment related to learning objectives, it is necessary:

– To analyze, to delve into the research methods in mathematics: particularization, finding general laws, building models, generalization, using analogies, conjectures and demonstrations, among others.
– To use mathematical models for the mathematization of reality and problem solving (Romero et al. 2015); experiencing their validity and usefulness, criticizing limitations, improving them and communicating findings and conclusions.

Moreover, when asked to bring the issue to the classroom, we must be explicit regarding the educational goals we are demanding:

(a) To practice problem solving as the most genuine activity in any specific field of mathematics, where the technology can be an impressive and a fantastic aid.
(b) To bring the students to approach mathematical knowledge, prioritize and solve challenges, search for explanatory models, inquiry and discovery.
(c) To tackle the aspects of the creation process and/or detection in mathematics we must focus on bringing into the classroom in order to achieve the educational goals we have set ourselves (Watanabe and Mcgaw 2004).

> What the teacher says in class is not unimportant, but what students think is a thousand times more important. The ideas must be born in the minds of the students and the teacher should act only as a midwife. This principle is based on let the students discover by themselves as much as feasible under the given circumstance. (Unknown, http://lovelypokharacity.blogspot.com.es)

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Part II
Technology, a Tool for Teaching and Learning Mathematics: B. Learning

Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive, and Epistemological Implications for Improving Geometric Thinking

Marcelo Bairral, Ferdinando Arzarello, and Alexandre Assis

Abstract In this chapter, we discuss the results of a research project which investigates aspects of students' cognitions during the process of solving tasks dealing with a Dynamic Geometric Environment with touchscreen (DGEwT). In this chapter, we discuss data from two teaching experiments carried out with Brazilian and Italian high school students dealing with GeoGebraTouch (GT) and a Geometric Constructer (GC) software. With the focus on strategies used by students to solve the proposed tasks, we suggest two domains: Constructive and relational. Furthermore, we suggest the drag-approach as an important form of manipulation to improve geometrical thinking. Finally, we present a selected variety of representative examples of didactic, cognitive, and epistemological implications for learning and researching with the use of DGEwT.

Keywords Mobile devices • Manipulation on screen • Sketchometry • GeoGebra App • Geometric Constructer

Introduction

The significance of the gesture in supporting mathematical reasoning in a technological context is an emerging field of research in mathematics education, particularly in the interaction with touchscreen learning devices. As a past improvement,

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we have had a first shift from paper-and-pencil to dynamic geometry environments which include drag-and-drop activities (e.g. Cabri Géomètre, Sketchpad, etc.). Today, we experience a further shift and continuous improvements with the transition to multi-touch environments (e.g. Geometric Constructor, SketchPad Explorer, or Sketchometry) that allow a variety of simultaneous finger actions.

The emergence of multi-touch devices provides new insights as well as challenges in mathematics learning and instruction. For instance, simulating rotation and other kinds of rotating movements on screen are made possible by means of touchscreen devices (Bairral et al. 2015a). Due to the fact that students and teachers become increasingly familiar with multi-touch technology and manipulation, we believe that looking for types of manipulation can provide new epistemological insights in regard to the geometrical conceptualizing through the application of touchscreen devices.

We recognize the touchscreen manipulation as a human action: embodied and multimodal. It can also reveal the mathematical thinking of learners while working on tasks with multi-touch devices. In this chapter, we illustrate some strategies used by students who applied rotation actions in order to solve tasks on GeoGebraTouch, or by students who dealt with the Geometric Constructor software to solve a Varignon Theorem task.

Interaction, Motion and Geometric Learning with DGEwT

With the focus on the user, there are differences between handling a usual PC – where dragging is produced with the help of a mouse – and making use of the touch screen of a tablet – where they can use their fingers in order to move figures. Additionally, it makes a difference whether users can use more than one finger – as in multi-touch environments – or only one finger. In this section, we reflect on how we dealt with these singularities theoretically.

Interaction on Touchscreen Devices

To click the mouse or to touch a screen are increasingly common routines of our daily lives. Each form of such handling implies different sensory perceptions; (the) sensitivity differs whether one uses a wired or wireless mouse, touches the screen of an ATM or that of a cell phone.

With the focus on their usage, environment mobile touchscreen user interfaces employ a specialized interaction model. The interaction of current mobile touchscreens, for example, is based on the computer's recognition and tracking of the location of the user's input on the display.

Adopting an embodied cognition perspective in our research, we highlight reciprocal connections between touchscreen manipulation and cognition. Contrary

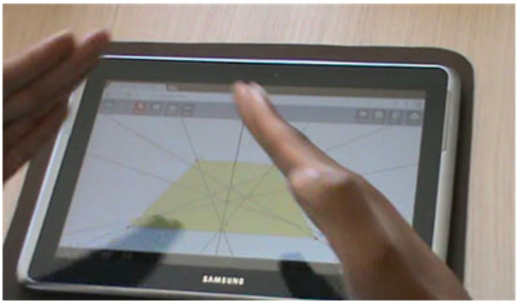


Fig. 1 Student's construction and embodiment reflection on GC

to what happens by in clicking, manipulating touchscreen interfaces implies a continuity of action, such as the spatiality of the screen, the simultaneity as well as the combination of movement, and – depending on the resource device – the response speed of the device. The following picture depicts one student's gestures who tried to explain one of the properties of the isosceles trapezoid (Bairral and Arzarello 2015) (Fig. 1).

The student uses his hands to represent the two sides that are not parallel. Although the picture does not show manipulation on screen, it describes specific configurations and actions of the fingers performing an action (Sinclair and Pimm 2014) with the construction made.

When we manipulate the screens of our devices by means of touchscreen technology, we perform a set of movements. These movements are not necessarily gestures such as signs or expressions of joy, silence, or doubt. Some of the manipulations that we perform induce specific mathematical cognition, for example when we want to enlarge or reduce the size of a picture with the help of an image editor (e.g. Paintbrush), or by means of touchscreen manipulation.

On such occasions, we either pull the image diagonally, upwards, or downwards; or we click on one of its vertices, so both dimensions – width and height – are reduced or enlarged proportionately. In case that we do not perform this type of movement, i.e. if we manipulate only one dimension, the result will be a deformed image.

Nevertheless, although all these manipulations are based on the same mathematical concept (the method of the diagonal as a way to generate similar figures), they are not necessarily of the same value with respect to cognition (the action of enlarging without deforming), epistemology (the simultaneous changing of different elements of the shape, e.g. points, sides, angles, areas, etc.), or space (work and manipulating area on the screen).

In order to guide our analysis of this process of embodiment expression, we can find support in Damásio (2010) for whom "also the most stable aspects of bodily function are represented in the brain, in the form of maps, thus contributing with images for the mind" (p. 39). Damásio further states that "Complex brains like ours naturally create explicit maps of the structures that make up the body, with a greater or lesser degree of detail. Inevitably, the brain also maps the functional states that are naturally taken up by these corporal components" (Idem, p. 119).

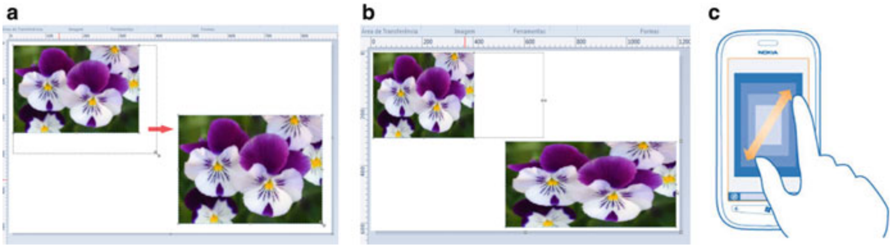


Fig. 2 (**a**) Illustration of an enlargement in a drawing program; (**b**) Distortion in a drawing program; (**c**) Enlargement through sliding on the screen (Bairral and Arzarello 2015)

We could argue that the brain mapped the fact that the touchscreen device is going to enlarge the figure or that a soft and quick lateral touch will make the screen slide to one side. The size of the screen, or the user's familiarity with it, can have an impact on the ways of manipulation. This is the spatial dimension, i.e. the screen handling and interaction area (Tang et al. 2010).

In case of the widening of an image by means of clicks, the shape illustrated in Fig. 2 (*a* and *b*) involves the actions of selecting, clicking and dragging on a point. When we touch the screen with only one hand (Figs. 2c and 3a), or both hands (Fig. 3b), on the screen, we map a specific area on the screen. Even in case that manipulation is done in order to see specific, punctual details of an object on the screen, the movement of this second action involves a simultaneous manipulation of dots.

Still, in regard to the enlarging of an image, although the simultaneous manipulation with two fingers (Fig. 3a) is the most usual, the second enlarging strategy (Fig. 3b) also follows the cognitive orientation structure of moving in a diagonal direction.

In the same way that simultaneous touchscreen manipulation of points on the screen brings about implications of an epistemological order, it also makes our cognitive structures more complex, for example through the simultaneous movement of various elements (e.g. angles, sides, area, etc.) in a figure. These movements will depend on the performance – the response speed – of each device (Bairral et al. 2015a, b, c).

Ways of Manipulation on Screen

Most current tabletop interaction techniques rely on a three state model: contact-down, contact-move, and contact-up – more akin to mouse dragging (Tang et al. 2010). In other words, interaction occurs in response to two dimensions of the input action (Yook 2009; Park 2011). This enables some basic or active finger actions for input such as tap, double tap, long tap (hold), drag, flick, and multi-touch (rotate). They are summarized in Table 1.

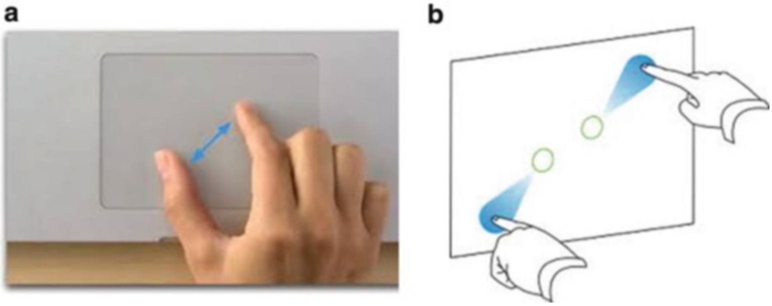


Fig. 3 (**a**, **b**) Sliding on the screen to enlarge with two fingers (Source: *Google picture*)

Table 1 Yook framework quoted by Park (2011, p. 23)

Action	Type	Motion	
Basic	Refers to tap and hold which are the basic ways of interacting with a touch interface	Tap (single)	Closed
		Tap (double)	
		Hold (single)	
		Hold (multi)	
Active[a]	It is a combination of the basic action and the performed finger action, which includes drag, flick, free, or rotate	Drag	Open
		Flick	
		Free	
		Rotate	

[a]According to Yook's (2009) framework the four active actions can be associated to multi hold manipulation

Manipulation – as a basic action – refers to a designated closed motion that occurs in response to the user's input, e.g. scale, flip, move, or push. Open motion occurs in relation to the user's input by reflecting the spatial and temporal quality of the finger action. In relation to geometrical thinking that we observed on students who were dealing with DGEwT, we named the basic action as constructive domain and the active action as relational.

After interpreting and using Yook's (2009) framework, which identifies each type of touchscreen in relation to geometrical thinking throughout the proposed tasks, we provide a scheme that includes another alternative to the drag approach and three further options for the rotating action (Fig. 4).

Due to the nature of the geometrical proposal, we identified that touches of the relational domain were predominant, while touches such as drag free, flick, or rotate occurred only a few times.

Regarding the usage of single or multi touch fingers, we observed (Arzarello et al. 2014; Assis 2016) that students manipulated the figures using mainly one or two fingers only (Tang et al. 2010). Due to the fact that they occasionally worked in

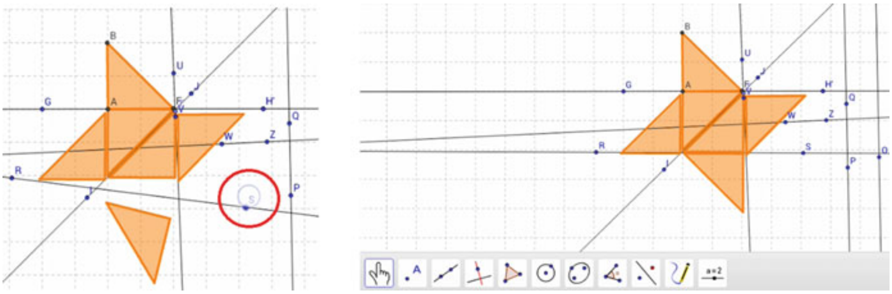


Fig. 4 Ways of manipulation on DGEwT (Arzarello et al. 2013)

pairs, it occurred that students also shared fingers (e.g. they used one finger each) or hands to manipulate a figure. This especially occurred when the shape had multiple geometric objects or constructions.[1]

The dominant approach was dragging. The usage of rotating appeared a few times; those appearances differed in a way that allowed us to distinguish three different ways of rotating which are illustrated in the scheme below. For example, we observed students doing rotation into some shape. We are of the opinion that looking for different types of manipulation provides new epistemological insights on the geometrical conceptualization within the use of touchscreen devices.

Even though we are not only looking for alternative kinds of touch that represent mathematical concepts (e.g. rotation), we agree with Boncoddo et al. (2013) that a particular way of manipulation may serve as an important function of grounding mathematical ideas in bodily form which may communicate spatial and relational concepts. Specifically for geometrical thinking – inspired by Hostetter and Alibali (2008) – we consider it important to stress that in touchscreen devices manipulations are based on visuospatial images: linguistic factors influence gestures, and ways of touchscreen manipulation can be regarded as intentional communications.

Performing Rotation on Touchscreen Devices

Although rotating appeared only a few times, these appearances allow us to distinguish three different kinds of rotation while working with a Geometric Constructer (GC) multi-touch device (Arzarello et al. 2013, 2014): rotation using one finger; rotation using two fingers, but one of the two fingers is fixed: and rotation with two fingers, with both in movement, as it is illustrated in the schemes below (Fig. 5).

[1]To see this kind of motion, please download the video: https://youtu.be/qC-G96NssJk

Fig. 5 Ways of rotation on GC or on GeoGebraTouch

Table 2 Examples of students' rotating on GC

Rotation types using GC	Example	Geometric process
Rotation using one finger		Student constructs and moves the selected point with the index finger
Rotation using two fingers, but one of the two fingers is fixed		Student keeps index finger fixed and moves the middle finger to observe what happens
Rotation with two fingers (both in movement)		Student selects two points and rotates the shape

Arzarello et al. (2014, p. 46)

In the above table (Table 2), we illustrate each kind of rotation by relating them to the geometric process that was applied in order to solve the task with the help of the Geometric Construct software.

Although the first two types seem equal on mathematical terms, we are of the opinion that – cognitively – they can provide different insights in regard to the use of the fingers. In order to grasp the fingers actions conceptually, we need to determine the centre of rotation individually in each point that we are about to rotate. With the use of two fingers, this cannot not be done beforehand.

Following this idea, this kind of finger movement results in new concepts in regard to the way we deal with rotation. Following the same idea, we agree with Sinclair and Pimm (2014) that this type of manipulation – two fingers in movement – involves contact with a screen and perform an action. Due to the fact that mobile touchscreen devices provide a wider range of freedom with respect to manipulation, we conclude that this particular kind of rotation may serve as an important function in order to ground mathematical ideas in bodily form. It may also reveal spatial as well as relational concepts (Boncoddo et al. 2013) in the field of plane transformations.

To investigate on manipulation on screen, especially if it is the case that students use more than two fingers, may be an interesting and challenging issue in future mathematics education research with touchscreen mobile devices. As we argued before, due to the nature of the software GC (multi-touch) as well as due to the geometrical task on the Varignon Theorem that was proposed beforehand, we observed that rotation manipulation occurred a few times (Arzarello et al. 2014). To solve the task, students did not apply the rotation action or other related plane transformation concepts.

Finally, on a theoretical basis, it seems important to highlight that the process of performing an action (Sinclair and Pimm 2014) by applying a concept such as constructing or other kinds of geometric strategies within DGEwT led us to assume that:

- Our brain has the ability to adjust to its environment; the touches on screen broaden the formerly established concepts of our brain (Damásio 2010).
- Human actions, as well as geometric concepts, are multimodal; what distinguishes them is that geometric concepts are also multimodal in their realization – the transition from virtual to actual. Indeed, by this transition due to which they become objects of thought and consciousness, geometric concepts are provided with certain features. These have to be put into practice by means of sensuous, multimodal, and material activities (Radford 2014, p. 354).
- In geometrical reasoning, there is a profound symbiosis of symbolic, analytical constraints, and figural properties. It is important to consider three categories of mental entities when referring to geometrical figures: the definition, the image – based on the perceptive-sensorial experience, e.g. the image of a drawing – and the figural concept (Fischbein 1993). Figural concepts do not evolve naturally in a way that they represent their ideal model. Consequently, one of the main tasks of mathematics education in the domain of geometry is to create types of didactical environments which systematically provoke a close cooperation between these two aspects up to the point where they fuse into unitary mental objects (Fischbein 1993, pp. 160–161).
- The interaction with a figure on screen can be differentiated according to the altering options by which subjects perceive them. Arzarello et al. (2012) point out two main cognitive and epistemic modalities according to which the figures on the screen were perceived and treated accordingly. A modality is ascending – from the environment to the subject – when the user explores the situation – such

as a figure on the screen – open-mindedly in order to see whether the situation itself has the potential to reveal something that is of interest to the user. The situation is descending – from the subject to the environment – when the user explores the situation with a conjecture in mind. In the first case, the applied actions have an *explorative* nature, i.e. to see if something happens; in the second case, they have a *checking* nature, i.e. to see if the conjecture is corroborated or refuted.
- In the transitional process from an inductive to a deductive approach, the drag-approach screen manipulation should be seen as a cognitive tool to empower learners to make assumptions and verify argumentations during the process of solving the task (Arzarello et al. 2014).

Teaching Experiments with DGEwT

In this section, we summarise the results from two teaching experiments (TE) that we conducted in Brazil (Rio de Janeiro)[2] and in Italy (Torino) with High School students working with two touchscreen devices: GeoGebraTouch (GwT) and Geometric Constructer (GC) (Table 3).

The analytical process that is presented as TE 1 and TE 2 was mainly based on the videodata.[3] We are of the opinion that the analysis should consider the interaction with touchscreen devices as paths of interaction rather than points of interactions. In most cases, it would be mathematically inappropriate to reduce the data of an entire process to a single point. In each session, optionally on their own or in pairs, the students worked on proposed activities.

TE 1: High School Students Dealing with GeoGebraTouch

This TE was conducted with High School students of 15–17 years of age at the *Instituto de Educação Rangel Pestana* (Nova Iguaçu, Rio de Janeiro, Brazil). None of them have had previous experiences with a dynamic geometry environment (DGE) or scholarly induced knowledge on plane transformations. In each session, the students worked on assigned activities with GeoGebraTouch which is described in Table 4.

Each session lasted two hours; in each lesson the students were asked to complete three activities similar to the one illustrated above. We observed all of

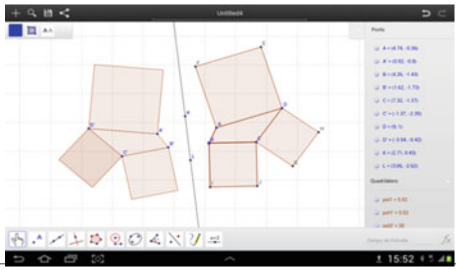
[2]In Brazil we are working with prospective mathematics teachers as well as with Sketchometry devices. We decided not to discuss data from their TE in this chapter.

[3]In recent analyses we used SCR PRO (Assis 2016) as a strategy to review some details that emerged from the video analysis.

Table 3 Teaching experiments information

	TE 1	TE 2
Age	15–17 years old	16–17 years old
Amount of hours of research session	8 h, 4 sessions	6 h, 3 sessions
Device	GwT	GC
Touch feature	Single touch	Multi-touch
Name of the Institution	Instituto de Educação Rangel Pestana	Liceo Volta
Previous experience with software	All of them had no previous experience with DGE	All of them had previous experience with Cabri
Sources for data collection	Written answers for each task	Written answers for each task
	Videotape	
	Sheet of icon	Videotape
	Software Recorder Pro (SCR PRO) for tracking touches on screen	
Geometric content	Rotation and plan transformation	Quadrilaterals
Proposed and analyzed task in this chapter	Star	Varignon Theorem

Table 4 GeoGebra touchscreen features

Software	Interface	Device features
GeoGebra touch		Runs and allows save constructions off-line
		Version used on the analyzed task in this paper: 4.3
		Single touch only

the students' manipulations on the screen and identified their kinds of actions (e.g. tap, hold, drag, flick, free, and rotate). Our analysis of this teaching experiment focuses on the student's strategies to solve the tasks, e.g. the application of rotation or other plane transformation concepts.

Besides alternative kinds of rotation applied by students to solve the geometric tasks that differ to the ones discussed in the previous section, further – curricular and cognitive – justifications to analyze students performing rotation or other plane transformations are the following:

- Rotation and other gyrating movements on screen are often applied due to the various alternatives of handling touchscreen devices (Kruger et al. 2005; Tang et al. 2010).
- Rotation and other plane transformations remained unaddressed in Brazilian geometry classrooms so far.

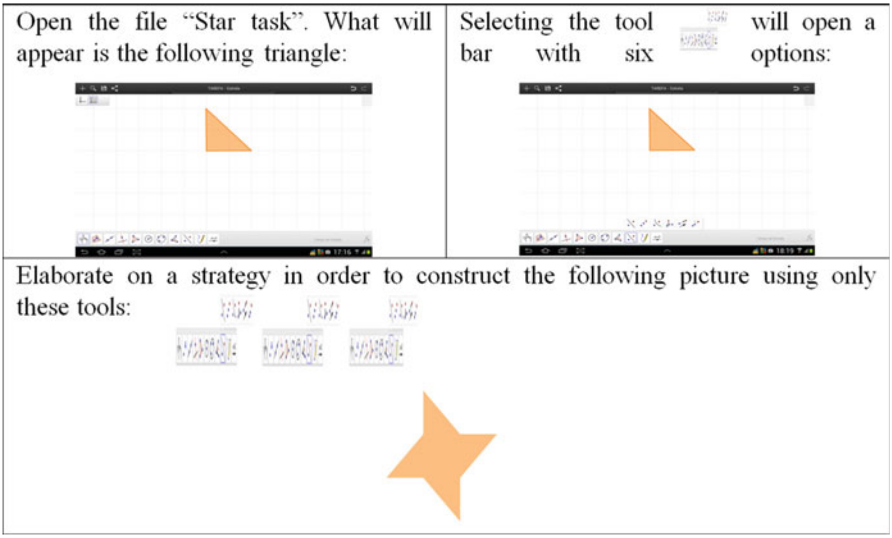


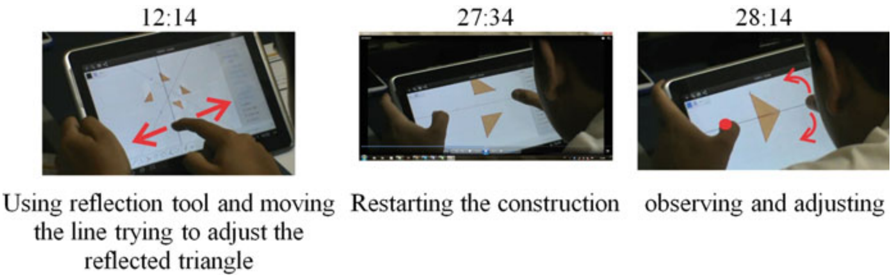
Fig. 6 Star task

- Touchscreen devices provide possibilities of gyrating movements on screen, or with the device itself, which might result in new insights on embodied cognition.
- Rotation and other plane transformations are concepts that involve intrinsically embodied motions.

The proposed task with GeoGebra is the following (Figs. 6 and 7):

The analysis in this TE process was mainly based on (1) the videodata of students working with the GeoGebraTouch software, (2) written answers on each task, and (3) the use of the students' lists of icons.

In the following pictures, as well as with the timing intervals, we illustrate and describe how the student Adriano deals with the task on GeoGebra by using single touch. He starts (12:14) to construct lines and reflects triangles related to them. By moving the line (27:34) he tries to locate the triangle coincident to the other; but since these actions remain unsuccessful, he decides to restart the construction.[4]



[4]The whole video is available on https://www.youtube.com/watch?v=qC-G96NssJk

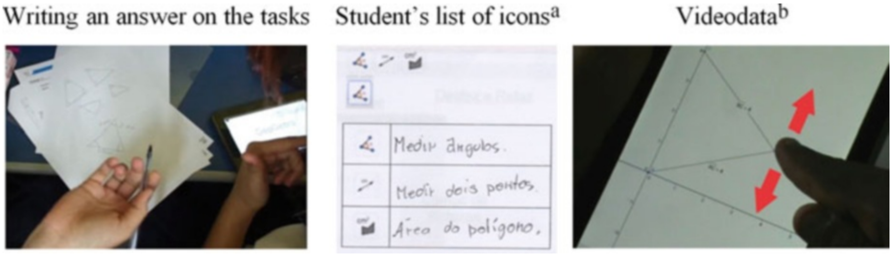
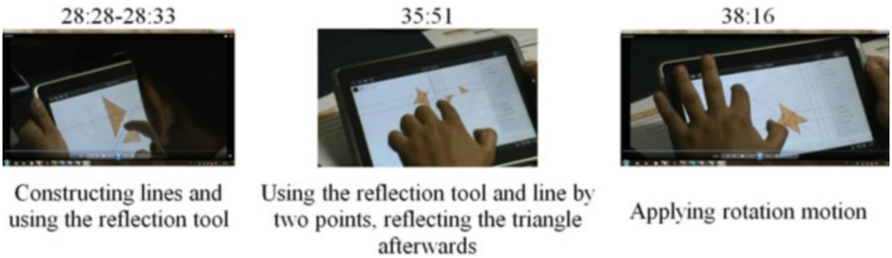


Fig. 7 Three main sources of data collection
[a]List containing all GeoGebra icons (Appendix 3). Each student had his/her own list and during each TE they filled it in and reviewed it to their own accord
[b]The *red arrows* indicate motions on screen performed by students

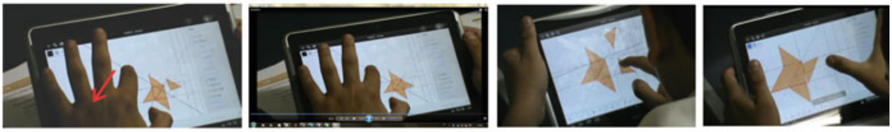
Interestingly, he keeps his left finger below a point on the line while using his right finger to rotate the line. Throughout this entire process, he carefully observes and adjusts his actions. The next figures illustrate how Adriano constructs lines and uses reflection to move the triangles.



The student constructs lines (28:28) and uses the reflection tool (28:33) in order to move the triangle. Afterwards, he constructed additional lines and repeated the process of reflecting those triangles (35:51). In the next three figures, we illustrate how Adriano applies a rotation motion by keeping his thumb on the line. At 38:17, we observe him making a rotation motion with his index finger to move the triangle and complete the shape (38:18).

The next set of pictures show how Adriano uses his constructions (38:18) in order to finalize the task.

38:19 38:22 49:48 51:39



After that motion realizing that the triangle is inverted

Reflecting triangle

Adjusting and finishing the construction

Again using his index finger, Adriano selects the line (38:19) and translates it so that the triangles connect. He creates another line and reflects the triangle (49:48). Afterwards, he adjusts and finishes the construction in accordance with the task statement.

TE 2: High School Students Working with the Geometric Constructer

The GC is a free dynamic geometry software developed in Japan by Yasuyuki Iijima at the Aichi University of Education (Iijima 2012). We chose the GC software because it is software which incorporates all the potentialities of usual DGE in a touch-screen device. With the term 'potentialities' we refer to the two main features (Arzarello et al. 2014): (i) the possibility of using more than one digit (multi-touch) on screen to interact with the software, and (ii) the possibility to design constructions as opposed to mere explorations.

With the GC, we are able to construct basic geometrical objects (e.g. points, segments, lines, or circles), measure such, drag, make traces of geometrical objects, etc. Below, we summarize students' working processes as well as results while dealing with the Varignon Theorem[5] task. This entails the illustration of selected, representative aspects of their geometrical thinking captured by means of their manipulations. These are described in the following chart (Table 5).

The analytical process was done in two main steps: (1) identification of each type of manipulation (Arzarello et al. 2014; Park et al. 2011; Yook 2009) and (2) construction of the timeline (Appendices 1 and 2) to describe the global cognitive movement throughout interaction on GC software.

Based on the videodata, we created a timeline which illustrates the ways of touchscreen and shows geometric aspects from students' interaction with the GC software (Arzarello et al. 2014, p. 47). In the following two charts, we illustrate

[5]In quadrilateral ABCD, the middle points (E, F, G and H) on each side have been drawn, forming quadrilateral EFGH. What characteristics does EFGH have? What happens if ABCD is a rectangle? What if it is a square? What if it is any quadrilateral? Demonstrate.

Table 5 Example from students working on the Varignon Theorem[a]

	Screen example	
Task	High School student	Undergraduate student
Varignon Theorem		
Geometric strategy	Student constructs the diagonals AC and BD by tapping (with one finger) on point A and C, and then on point B and D	Student using different colors to edit the construction and measuring internal angles from the quadrilateral EGHF

[a]The whole video is available on http://www.gepeticem.ufrrj.br/portal/materiais-curriculares/varignon-touchscreen-no-construtor-geometrico-2/

parts of a timeline which shows students' actions in order to solve task 3[6] by means of the GC software. The analysis has shown that they perform four types of basic actions: tap single, scale, hold single, and hold multi (Fig. 8).

Although, in order to construct a geometrical figure (e.g. a point, a line, an angle, or a circle, etc.) with the GC software the user has to use the software icons, we observed all the manipulation directly on the screen. For instance, we didn't consider touch on the icon as a case of the tap or hold action. Instead, we observed more than a single kind of touch at a time, but in order to categorize them clearly we selected the type of touch that was predominant in that specific situation.

Due to the nature of the task, which was situated in the domain of open construction and exploration, the types of touches that we predominantly identified were on the relational domain; for example, drag free, drag approach, and flick. Rotation did not occur in the process of solving this task. As we can see in Fig. 9, the drag approach was dominant (e.g. in interval 8:31–15:02).

[6]Build a quadrilateral ABCD. On each of its sides build a square external to the quadrilateral with one side coincident to the side of the quadrilateral. Consider the centers of the squares that have been built: R, S, T, U. Consider the quadrilateral RSTU: what can you observe? What commands do you use in order to verify your conjecture? This activity was thought as a task to introduce curiosity among students for the Napoleon Theorem, which was explored on the next assigned task.

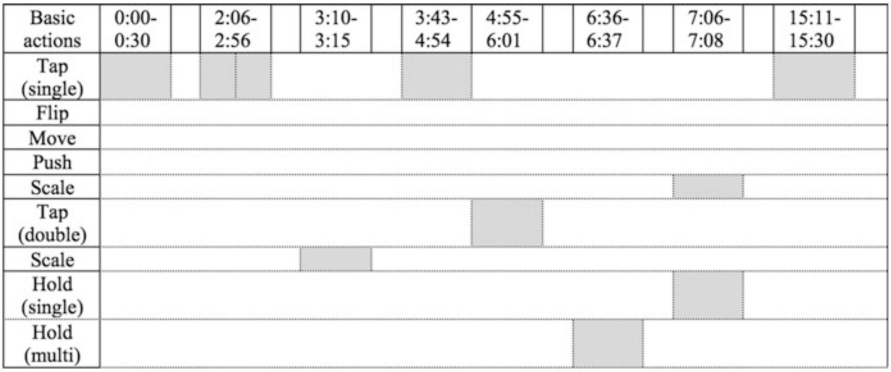


Fig. 8 Part of the timeline illustrating basic actions

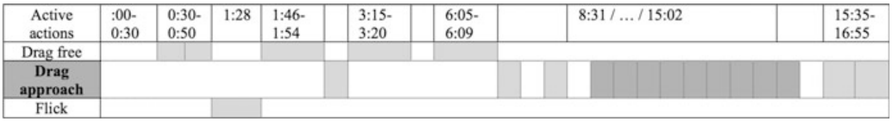


Fig. 9 Part of the timeline illustrating active actions

Domains of Manipulation and Geometric Thinking Within DGEwT

In the following screenshots, we show a summarized approach of students dealing with the software features. They further illustrate types of manipulation in order to identify conceptual reasons which prove that the EFGH shape is a parallelogram. These screenshots illustrate four different approaches towards the picture given to them (see footnote 4) (Fig. 10).

The analysis of the timelines (see Appendix) shows the progress of the altering approaches of touches. The students' constructions, strategies, and reasoning either moved from basic to active, or from active to basic actions.

We built on the two types (basic or active) of finger actions (Table 6) to say that the cognitive process with GC could be seen in two interrelated domains of manipulation: firstly, in the constructive domain, where students basically refer to tap and hold which are the basic or isolated ways of constructing geometric objects (point, line, circle, shape etc.) with a touch interface. Secondly, the relational domain is a combination of the constructional and the performed touches which thereby include drag, flick, free, or rotational approaches. The Table 6 below illustrates how we moved from a global observation – by means of a timeline – to a descriptive one – with the focus on some cognitive processes concerning the two domains of touches (Bairral and Arzarello 2015).

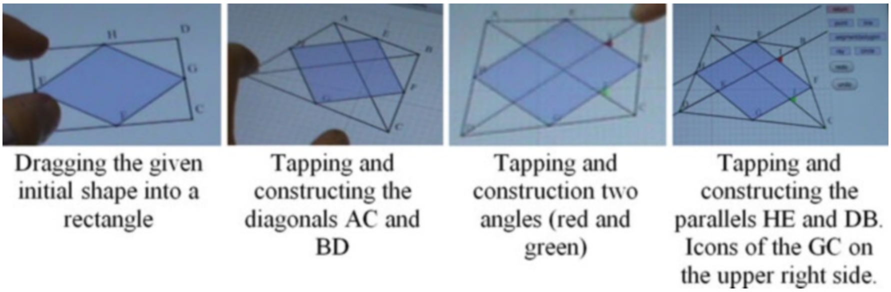
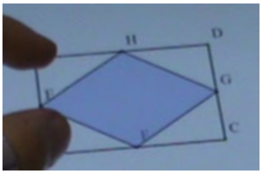


Fig. 10 Summarized student's drag approach and reasoning on Varignon task

Table 6 Relating domains of touches, cognitive processes, and motions

Domain of manipulation	Geometric process	Motion	Example of touches and students' strategy descriptions	
Constructive	Discrete construction and "isolated" observation (perception)	Closed, predetermined (specific goal, basic construction)		Student constructing angle to observe relation among diagonals and the side of quadrilateral ABCD
Relational	Related construction and global observation	Open, but focused on emergent conceptual demand of the task		Student using two fingers and dragging point AB to the left to transform the initial shape – a square – into a rectangle and observing what happens with shape EFGH

Even though we did not expect this, we observed that students also constructed geometric objects in the relational domain (Arzarello et al. 2014); they also showed more interacting and reflecting about the construction in this particular domain. Due to the nature of the geometry tasks we identified a predominance of touchscreen types on the relational domain; touches such as drag free, flick, or rotate occurred few times.

In the construction domain, students act as discrete observants; they focus on some specific construction, a constructed object, or touch something on the screen. In contrast to the relational domain, their manipulations seem more focused on their questioning, on conceptual understanding as well as on other emerging demands concerning their manipulation as a whole. The manipulations in regard to the construction domain seem focused on only predetermined motions, although motion through relational manipulations facilitates motion that is 'open' in the sense of that it can generate more unpredictable processes.

The Drag Approach Way of Touch and Semiotic Bundles in Geometric Tasks in DGEwT

The drag approach[7] seems to be a useful kind of touch in regard to the relational domain. It is a kind of manipulation that students apply when they are confronted with a specific geometric property, shape, or construction. During this process, we identified that the usage of the drag approach was dominantly applied when students aimed to clarify their reasoning.

Table 7 illustrates a student's strategy to adjust his constructions of the star task on GeoGebra (see Fig. 6) by applying the drag approach. The drag-approach is a type of screen manipulation on the relational realm. Even when a student uses only one finger, the drag-approach works as a refreshing, quite stabilizing and reflecting way to a deep understanding of the geometric properties that emerge from the manipulation on drag free or other ways of touchscreen use. It seems to be an appropriate tool to facilitate mathematical justification, prove, and further geometric discoveries.

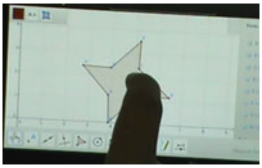
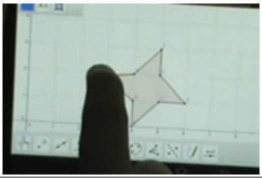
According to Arzarello et al. (2009), a semiotic bundle is a system of signs – with Peirce's comprehensive notion of the sign – that is produced by one or more interacting subjects and evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher (Arzarello et al. 2009, p. 100).

The way of touch could not be identified as the only cognitive resource in students' learning processes. Rather, pictorial representations, cultural artifacts, speaking, writing and gestures are examples of tools of a bundle of semiotic resources (Arzarello et al. 2009) that contribute to an understanding of the process of knowledge construction as well as for the development of tasks that foster the improvement of the geometric thinking within DGEwT as we show in Table 8.

In other assigned tasks on rotation or other kinds of plane transformation, we observed students applying composed forms of transformations. The picture above illustrates how manipulation on a touchscreen, the device, its features, and other artefact mediators are intertwined in the process of construction and performing plane transformation strategies with the software. While observing students applying rotation and reflection we came to the conclusion that looking for specific types

[7]Inspired by Arzarello et al. (2002).

Table 7 Illustrating student's drag-approach and reasoning on star task

Screen shots	
 	Student constructs and moves the vertices freely (without the use of the grid squared)
 	Student adds the square grid on his construction and adjusts the star's vertices in some points (intersections of the grid)

of manipulation – as well as including the concept of the semiotic bundle – can provide new epistemological insights on geometrical conceptualizing in DGEwT.

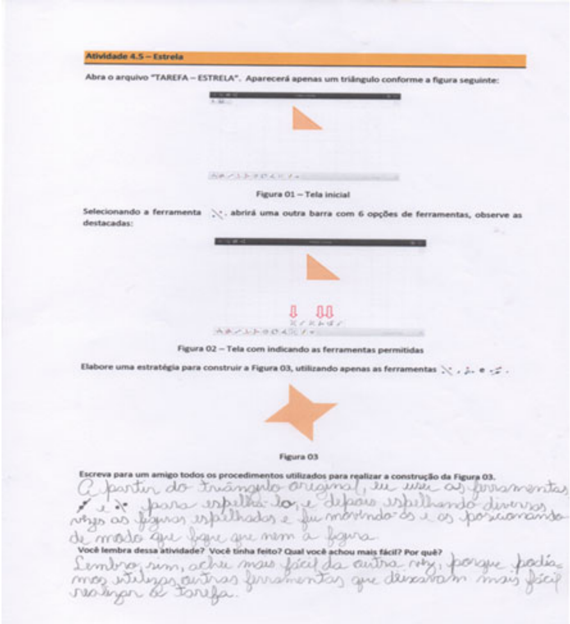
Conclusion and Implications of DGEwT

In the next sections, we present a variety of implications that summarize the main results which emerged from the two teaching experiments illustrated in the previous sections of this chapter.

Didactical Implications

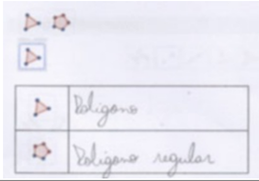
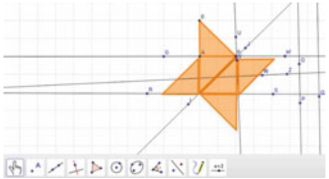
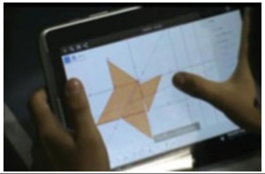
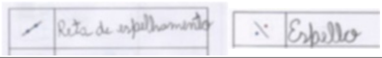
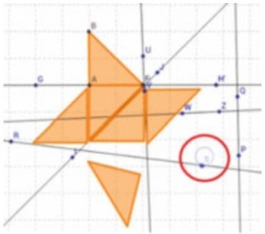
In mathematics education, a considerable amount of research stresses the key role of the task in each environment – with or without ICTs. The pedagogical importance of carrying out research on touchscreen use is not that it is trendy. Rather than that, it is

Table 8 Exemplifying semiotic bundle used on TE 1

Semiotic bundle resources using DGEwT	
Student's writing	 Atividade 4.5 – Estrela Abra o arquivo "TAREFA – ESTRELA". Aparecerá apenas um triângulo conforme a figura seguinte: Figura 01 – Tela inicial Selecionando a ferramenta , abrirá uma outra barra com 6 opções de ferramentas, observe as destacadas: Figura 02 – Tela com indicando as ferramentas permitidas Elabore uma estratégia para construir a Figura 03, utilizando apenas as ferramentas , e . Figura 03 Escreva para um amigo todos os procedimentos utilizados para realizar a construção da Figura 03. A partir do triângulo original, eu usei as ferramentas e para espelhá-lo, e depois espelhando diversas vezes as figuras espelhadas e fui movendo-as e os posicionando de modo que fique igual a figura. Você lembra dessa atividade? Você tinha feito? Qual você achou mais fácil? Por quê? Lembro sim, achei mais fácil da outra vez, porque podíamos utilizar outras ferramentas que deixavam mais fácil realizar a tarefa.
	"From the original triangle, I used the tools  *and*  *to reflect it, and then after mirroring the pictures several times, I was moving and fixing the figure to the correct position".*

(continued)

Table 8 (continued)

Student's notes on list of icons	
Picture	
Screenshot	
Line of mirror/mirror[a]	
Tracking on Recorder Pro	

Ways of touch preforming rotation	Using one finger in motion		Free rotation
			Rotation to the left
	Two fingers in motion		Thumb is fixed and the index finger is gyrating to the right
			Both thumb and index fnger are rotating to the left

[a]Translation of how the student named the GeoGebra icon used to construct symmetry

important to design new ways of instruction with that type of technology in order to empower learners with high abilities to acquire mathematical knowledge (Leung 2011).

Since our research is embedded in the dynamic geometry environment (Ibid), manipulation in this kind of environment should be regarded as a cognitive tool in order to empower learners with amplified abilities to explore. Also, again in agreement with Leung, we are of the opinion that a mathematical task within GC becomes meaningful when it involves conjecturing, activities which require an explanation, and that provokes learners to engage in situated discourses in order to communicate their mathematical reasoning or argumentation. We aimed to fulfil these requirements with the provision of tasks such as the Varignon Theorem.

To solve a task, which involves the concept of rotation, using GeoGebra with single touch (as discussed on TE 1), we observed that the students used their fingers – no more than two (Tang et al. 2010) – similar to the students who dealt with the software GC in an open task (see TE 2).

In the TE 1 – due to the fact that the students were unacquainted with DGE, the list of icons (see Appendix 3) was didactically helpful for them. During each teaching experiment, they had the opportunity to remember the functionality of the tool, review it, and add new items to the list. Throughout the sessions, we observed that they resorted to the list to identify the most appropriate tool to apply in order to fulfil the task. Besides defining the functions of a specific icon, they further took notes on the geometric concept or strategy that underlied such icon. Revisiting and rewriting their notes on the list of icons can also be considered a process of learning.

Besides cognitive challenges and constraints with respect to the used software, we identified that the use of DGEwT can also provide new pedagogical issues in regard to the wording of mathematical instructions. In addition, our identification of the different types of manipulation can lead to improvements of the software, basically related to the drag action and touch (Iijima 2012).

Cognitive and Epistemological Implications

The cognitive process of solving geometric tasks within DGEwT could be seen in two intertwined domains of manipulation (Arzarello et al. 2014; Bairral et al. 2015a, b, c): the construction domain which refers to tap and hold as the basic or isolated ways of constructing a geometric object, and the relational domain which is a combination of the constructional domain and the performance on the touchscreen. Although the students dealt with the device naturally, their manipulation was apparently restricted by software constrains (or facilitated by the possibilities offered) or by the proposed geometry task.

In respect to the two TE illustrated in this chapter, we are of the opinion that any kind of manipulation that promotes open motion, e.g. relational ways of touching, are appropriate in order to provide new epistemological challenges concerning geometric knowledge as well as altering kinds of proving. Since the drag approach is a relational action, it seems to be an appropriate tool to improve justification and proving competences within the mathematics classrooms setting that uses touchscreen devices. As one restriction to that – depending on the aim of the teacher –, the task has to be selected carefully, and the teacher should promote that the students work independently on the task by experimenting with altering kinds of touches. Identifying in which geometric constructions the manipulation with more than two fingers occurs may be another interesting issue for future studies. What was not being discussed by this study is the issue of the two domains of manipulation analyzing kinds of touches on different touchscreen devices.

As simultaneous touchscreen manipulation of spots on the screen brings about implications of an epistemological order, it also adds complexity to our cognitive structures. This particular feature was observed by one of the students in our research. According to him, "in a very complex figure, moving several elements at the same time can become a bit difficult". Besides this cognitive implication, the use of touchscreen devices in the teaching of mathematics brings about transformations in didactic and epistemological realms, but the necessary educational research is still needed.

Another relevant issue that needs to be considered is the way how using a multi-touch-screen allows alterations on the task design in a substantial way. More precisely, multi-touch screen devices allow a design of geometrical problems in a way that differs from familiar ones in such ways that the combination with non-multi-touch screen environments would be very difficult. For instance, from TE 1 we are intrigued how students – without previous instruction concerning rotation and reflection – apply these two concepts, mostly in form of a composition of the two.

Research Implications

Our prior assumption was that the single touch provided by GeoGebra would restrict our possible observations of altering kinds of rotational manipulation on the screen. However, as we illustrated in TE 1, even students without previous experiences with rotation, or reflection, used those concepts intuitively, isolated, or even a mixed variation of the two tools (Assis 2016).

Usually in Brazil, plane transformations (e.g. isometries) are conceptually mapped in the following sequence: reflection, axial symmetry, rotation, and translation. The composition of plane transformations is underexplored in geometry lessons when the instruction uses traditional resources. In that sense, DGEwT seems to be a powerful resource for changing tasks as well as the nature of the geometric understanding concerning plane transformations. In our current analysis, we provided tasks where students had to apply the concept of rotation. In this paper, we present results from students dealing with GeoGebra touch to solve the proposed task.

In a more recent analysis (Bairral et al. 2015a, b, c), we further observed that the drag approach manipulation – as discussed within the TE 2 – could be applied using only one finger. The application apparently depends on the device features and the task proposal. This sort of touch should be seen as a cognitive tool that empowers learners to conjecture and explore their line of argumentation during the process of solving the task. This allows us to ascertain that the drag approach provided by the preconditions of a multi-touch environment can suitably support and improve the students' justifying (i.e. exploring) and proving (i.e. conjecturing) performances (Bairral and Arzarello 2015).

According to Arzarello et al. (2012), using DGE there is an alternation between an ascending and a descending modality: when there is a shift to a descending one, this is, possibly marked by the production of an *abduction*, which can also determine the transition from an inductive to a deductive approach. Within DGEwT the only difference seems to be in the time according to which such exchange takes place: in touch-screen modalities, the changes seem to happen more frequently than in mouse modalities. Possibly, this can have cognitive consequences similar to those ascertained by Arzarello (2009) in TI-inspired environments in comparison to Cabri-géomètre ones; but this statement is in need of further investigation before being an assured scientific result. At the moment, it is only a plausible conjecture.

To achieve our aim – which was to observe the development of geometric thinking –, the next step after the identification of each kind of manipulation was to construct timelines (see Appendices 1 and 2) and to gain information of the global cognitive movement of the interaction with the device. For each analyzed activity, we constructed one separate timeline. Depending on the type of task, some kinds of touches were not classifiable, but in all the timelines that we constructed we noticed a clear accumulation of active actions. In summary, the timeline has been methodologically and didactically important in order to:

- Illustrate the global cognitive movement related to the various kinds of touches (e.g. from constructive to relational and vice-versa) throughout the students' working on the tasks.
- Show selected local cognitive movements of the kinds of touches throughout a variety of geometric aspects in certain intervals.
- Allow researchers to determine and record certain intervals where students' geometrical thinking focused on the relationship of touches on the screen with other semiotic resources.

Another resource used for data collection was the Screen Recorder Pro device (SCR PRO), which allows to capture, in addition to the audio, the touches on the surface of the tablet (see Table 8). In the PRO version, the application does not limit the recording time and should take into account the ability of the device itself. However, the application installation requires a procedure that changes the tablet configuration. This feature was utilized in implementations carried out with the GC, since the acquisition and installation have been carried out only after the period in which implementations are made with the touch GeoGebra.

Final Remarks

Mathematics applied by students to solve a task in a paper-and-pencil environment differs from the mathematics applied on a touchscreen device. In this chapter, we highlighted two intertwined domains of manipulation – the constructive and the relational domain – for geometrical thinking development with DGEwT. The constructive domain refers to tap and hold; these are the basic, or isolated, ways of constructing geometric objects. The relational domain is a combination of the geometric construction and the performed touch. In the relational realm, the drag approach appears as a useful way of touch to improve geometric thinking. With this type of manipulation, students can make use of one or more than one finger.

We are of the belief that it is not important that the teacher monitors the students' application of certain types of touches on the screen. By taking device features and performances into account, we conclude that teachers need to be aware of the singularity of each kind of touch while proposing tasks that aim to trigger the students' intrinsic motivation to work into mathematics activities that enhance findings, reflections, and the development of mathematical thinking in its various aspects (Bairral et al. 2015a, b, c).

Inspired by Fischbein (1993), we argue that logic, image, and manipulation – on screen or gesturing on it – should be inseparable from geometrical reasoning with touch devices. In this process, it is important to interpret geometrical figures as mental entities which possess conceptual and figural properties (Fischbein 1993, p. 160).

Our brain adjusts to its surrounding environment (Damásio 2010); this implies that the touches on the screen or other touch performances add new mappings to the brain. These should be taken into account regarding teaching and learning processes. As a proofing example, the following picture illustrates how students interact with a touch device and its features of manipulation – as well as performing action (Sinclair and Pimm 2014) with hands on the screen – without previously established knowledge on plane transformation which we also illustrated in the TE 1 (Fig. 11).

In this geometrical process, the students apply figural concepts for executing constructions and transformations. They use images based on their perceptive-sensorial experience (Fischbein 1993). In this process – a sensorial process – motion and manipulation on screen make up an important cognitive function and, by becoming objects of thought and consciousness, geometric concepts are endowed with particular determinations; they have to be actualized in sensuous multimodal and material activity (Radford 2014, p. 354).

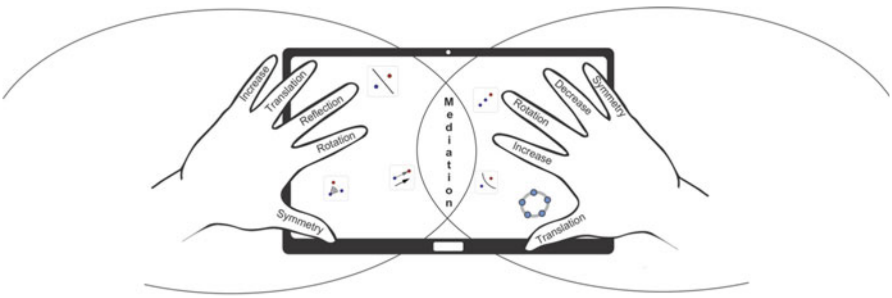
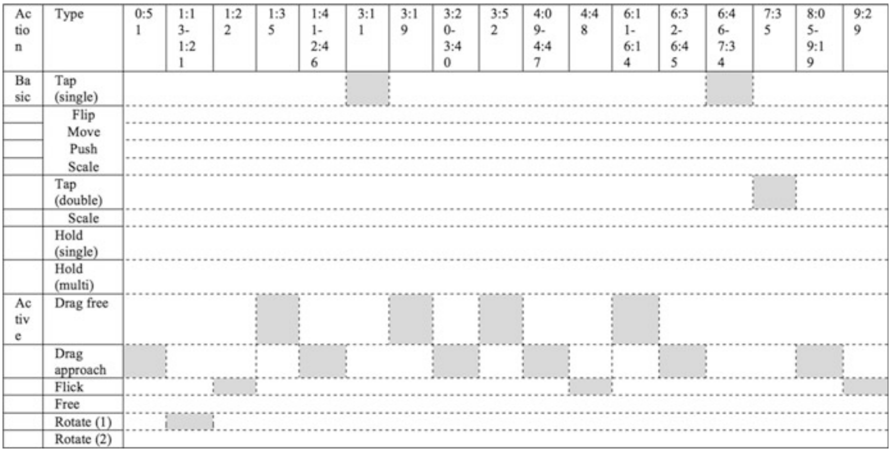


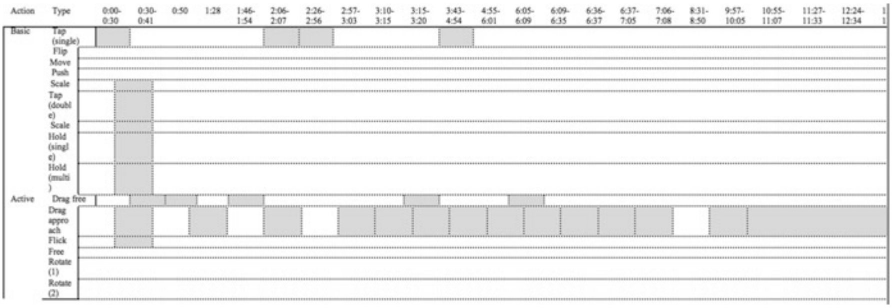
Fig. 11 Manipulation on touch devices interplaying symbolic, analytical and figural properties

Appendices

Appendix 1: Timeline of the Varignon Theorem Task (Discussed on TE 2)



Appendix 2: Timeline of the Task Shown in note (a) of Table 5

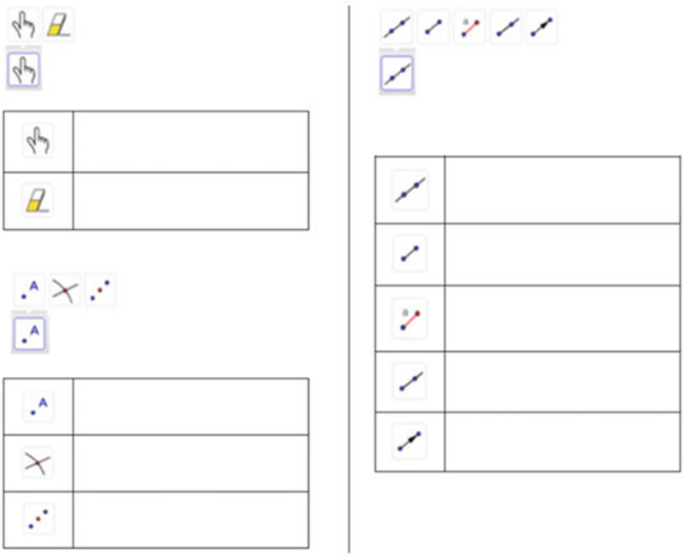


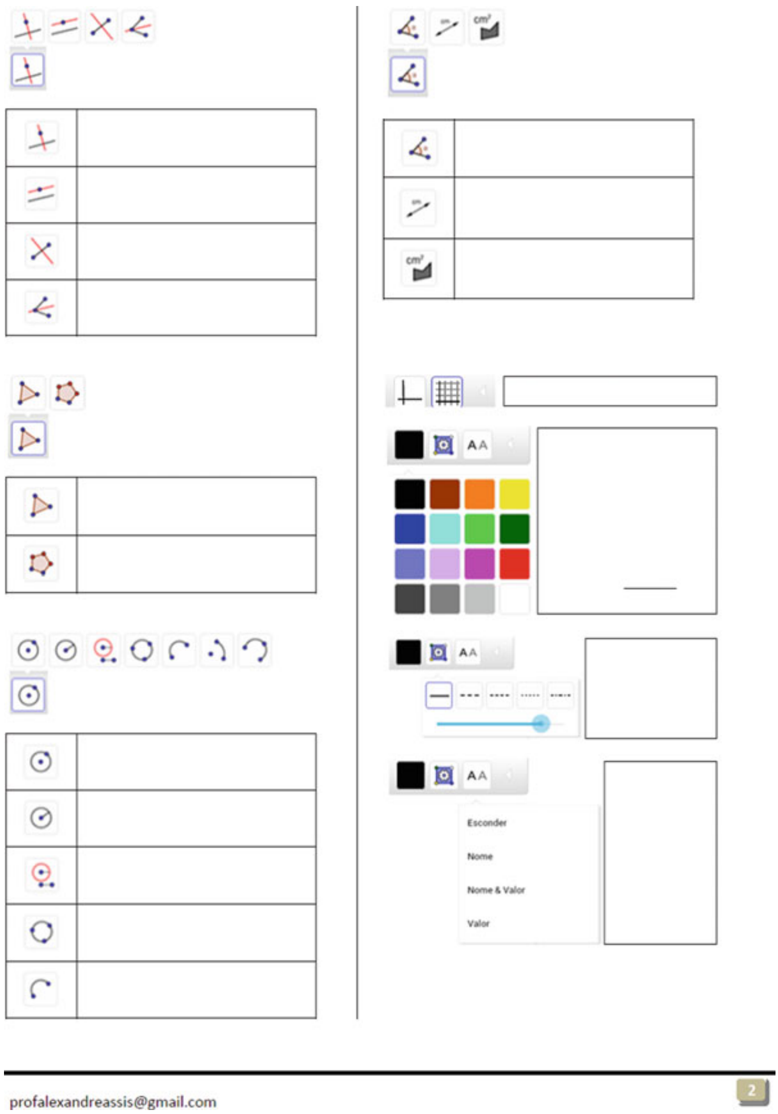
Appendix 3: List of Icons Elaborated for TE with GeoGebraTouch

2º Encontro – Conhecendo algumas ferramentas do GeoGebra

1. Tela inicial

2. Barra de ferramentas





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Graphs in Primary School: Playing with Technology

Daniela Ferrarello

Abstract It is very important to motivate primary school students in a way that they enjoy mathematics, e.g. by encouraging their creative perspective on it. This could be achieved by using everyday tools, i. e. tools children are familiar with in regard to manipulation. With respect to, for example, graph theory teachers can introduce the subject matter by giving the instruction to draw and play with several graphs with paper-and-pencil, or with the help of advanced technology. This technology is a precious tool for conjecturing activities due to the high variety of cases that can be observed and compared simultaneously – which is also a lot less time consuming than with paper-and-pencil activities. In this chapter, we describe selected graph theory activities for third and fourth classes of primary school which were designed for the application of technology; for this purpose, we named these activities mathematics laboratory.

Keywords Graph theory • Primary school • Technology

Graphs in Primary School: Why and How?

Everyone should enjoy learning because if you enjoy what you do – in comparison to doing something because it is your obligation to do so – you get maximum results with a minimum of effort. This is particularly true for children. At the beginning of institutionalized education, i.e. for pre-school children, mathematics is not yet that "monster" as some older children might perceive mathematics. Five-year-old children tend to enjoy mathematics as an internalized experience (Vygotsky 1986) rather then externally suggested to them. Thought and emotion agree; this influences the learning process positively because as Brown (2012, p. 186) states: affect, far from being the "other" of thinking, is a part of it. Affect influences thinking, just as thinking influences affect (see, e.g., Chapter "A Framework for Failed Proving Processes in a Dynamic Geometry Environment", this volume).

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As soon as children enter young adulthood, they commonly develop a negative attitude towards mathematics (Di Martino 2007) which originates from many causes, including the experience of unfit teaching methodologies. These include, for example, learning without manipulating, learning without self-generated hypotheses, or without connecting mathematical concepts to reality (Mammana and Milone 2009a, b). This possibly results in an alienation of pupils from mathematics as well as from the viewpoints of thinking and affection.

Too often, teachers do not integrate all the potentialities of their digital native students, who could make use of their natural inclination towards technology to learn better and faster. On the other hand, the use of technology is already integrated in teaching mathematics: just think of 2.0 classes, or the high variety of software and apps designed for math teaching and learning. Of course, technology is not a panacea for all math related problems, which can be seen by the many ongoing studies that are concerned with the advantages and disadvantages of technology in the classroom practice such as Drijvers' study (2012). Obviously, the use of technology is not self-sufficient in a way to ensure learning to occur in favourable ways.

This strongly technology-based teaching experiment with primary school children is build up on laboratorial modalities as well as learners' real-world experiences that we called "horizontal teaching" (Ferrarello et al. 2014). Horizontal teaching is a kind of teaching in which the teacher aims to envision the student's perspective and set up a learning environment that originates from this shared perspective. Horizontal teaching necessitates that the teacher acquires knowledge on the students' perspective on life, to understand their needs, to analyse their reality. This consequently requires an extra effort of the teacher, but it is effective and promises to enlarge not only the students' knowledge, but also the teacher's knowledge (see Fig. 1).

In this context, we apply the term "reality" not only to physically existent, touchable objects, but also to situations and characters that are familiar to children with the aim to motivate their effort to study. These could be, for example, cartoon characters, personal relationships, etc. And by "envisioning the student's perspective" we refer not only to the application of mathematical concepts familiar to the pupils, but also to code switch to a level of language use to which students can adapt to.

In this sense, the use of technology is intended to be a language that is familiar to pupils and therefore could be applied to the above-mentioned setting up of a learning environment that originates from a shared perspective of teacher's and student's knowledge.

In this respect, we developed a mathematics classroom activity for primary school children on the concept of graphs. In fact, a small variety of graph-theory concepts are sufficiently simple enough to be proposed to primary school children. The basic mathematical concept of a graph is quite straightforward: a graph $G = (N, E)$ consists of a set N, whose elements we call nodes, which we represent as points; and a set E, formed by couples of nodes that we call edges, which are represented as

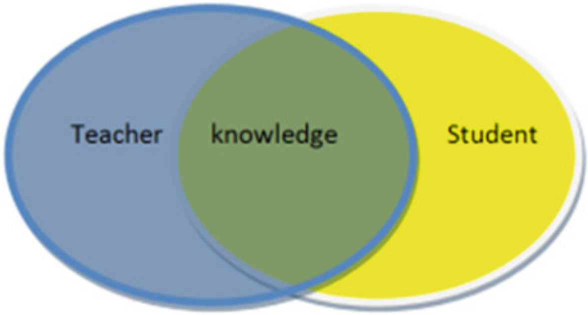


Fig. 1 Horizontal teaching (Ferrarello et al. 2014)

lines; i.e., an edge is a line connecting two nodes A and B in the set N, if the couple {A, B} is in E.

The prior, situational aim of this teaching experiment is to make children enjoy mathematics; the secondary, future-oriented aim is to familiarise children with the concept of modeling real-life situations with the aid of graphs. The mathematical concept of the graph is well suited for this intention because it is easily visualizable, drawable, open for an explorative, creative take on it, and also because it is quite useful to describe real-life situations by schematizing them.

It is not by chance that there is an increasing attention towards graph theory in several international projects (e.g., http://math.illinoisstate.edu/reu/). Not only is there an undeniable effectiveness graphs and their several representations, but also National Standards – in Italy, at least – ask for tools that enable to "represent relations and data and, in significant situations, use the representations to get information, formulate opinions and make decisions" as an aim for students at the end of primary school (Ministero dell'Istruzione, dell'Università e della Ricerca 2012) in order to support a concept of mathematics as "a context to solve and pose significant problems". These tools are provided by graph theory, but despite such precise requests, Italian teachers often do not integrate graph theory into their teaching in primary school; even further, quite a high amount of teachers are not familiar with the basic concepts of graph theory.

The approach of this activity is adapted from the one described in Aleo et al. (2009) and based on mathematics laboratories (Chiappini 2007). For our teaching environment, we altered the activities in such a way that they fit the requirements of eight to nine years-old students. Additionally, they are enriched with the use of technology as described in the following sections. Theoretical references to topics can be found in Higgins (2007) and Wilson (1996).

The next part of the chapter deals with the methodology used in the teaching experiments and further presents the technology provided to the children – namely the software and games used. We give a description of all the aspects and suitable activities of graph theory involved in the courses. Moreover, for a more detailed insight in our study, we describe a complete instructional unit in detail. Finally, we present the main results and draw possible conclusions. A short glossary of the basic definitions we used in our teaching experiment (in alphabetical order), together with a collection of activities, is provided in the appendix at the end of the chapter.

Methodology

The teaching experiment activity has been carried out three times, twice during the school year 2012/2013 and once in school year 2013/2014. It was led by the author – who is not a primary-school teacher – and therefore was both a researcher as well as a teacher during the teaching experiment activities. In regard to practical duties, she was tutored by an assistant. Each teaching experiment activity – which had a total duration of about three month – consisted of 12 weekly meetings; the children participated voluntarily. They came from different morning classes – but from the same grade – and were put together into groups of 14 to 20 children. In total, about 50 students participated.

Some activities, like Eulerian and Semieulerian graphs, were given the time of more than one meeting because the different sub-contents were based upon one another, so the children were given the time to think about the subject matter at home as well as the possibility to enjoy the topic instead of consuming it in a "fast and furious" way. Other activities, like the ones on nodes and edges, were introduced simultaneously because they relate to the same mathematical concept.

The activities were embedded within the laboratorial methodology, which means that they were embodied-mind oriented by making children manipulate the objects and discover their properties. Activities proposed by the teacher (i.e. the author of this chapter) as problems to be solved were introduced to the children on the whiteboard. This led to the situation that some students attempted to solve the posed problem directly on the whiteboard while other students tried to solve it in their exercise books or suggested a strategy to their fellow students. Intermediate and final tests, combined with satisfaction questionnaires, were collected and analyzed. The teaching experiment was not filmed, except for selected moments during the production of a poster.

Technologies

As it was said before, technology is a significant tool for our activity. In agreement with diSessa et al. (1991), "we believed that design, construction, and exploration of dynamic games and simulations would provide a rich context for an initial exploration into what children's science might involve" (p. 3). We used different technologies, e.g. paper-and-pencil, coloured chalks, etc. and dynamic software to handle graphs, online games, etc.

The whole teaching experiment activity was held in classes with Multimedia Interactive Whiteboards in addition to a classical blackboard and chalks. This resulted in children who manipulated graphs with paper-and-pencil, dragging nodes and deforming edges both with a dynamic graph editor and online games namely with the YED Graph Editor (http://www.yworks.com/en/products_yed_ about.html) which is a dynamic software that is designed to draw and explore graphs. Its main advantage is that you can easily draw nodes just with a click; and you can choose nodes of many shapes which represent several objects (e.g. people, geometric shapes, but also any kind of picture you want to import – just by dragging the pictures into the working area). For example, to draw an edge from node A to node B you just have to click on A and – keeping the mouse pressed – move on B – and release.

To drag nodes and to adjust edges with the help of this software is a very explorative and creative take on the mathematical concept of graphs. In our activity, many potentialities of the software there were not intended to be applied; the children were just asked to use the basic functions, i.e. the ones sufficient to our purposes such as drawing graphs, moving nodes, changing shapes to edges, colouring nodes and edges with different colours, creating a random graph, or changing the layout of the graph by putting it in a random shape.

Although the tool bar was in English, children did not encounter any difficulties in using the software. They learned to use the software by mimicking the teacher and trying for themselves afterwards.

Icosien (http://www.freewebarcade.com/game/icosien)

The pictures below illustrate an online game that includes Eulerian, Semieulerian, and Hamiltonian paths in given graphs (see Fig. 2, Fig. 3) by wrapping the string around the nails to create the given shape in each level.

It is not an educational software, it is just a game. Moreover, it is not a game that is intended for children. However, this game was probably the most successful activity within our teaching experiment because children quickly learned how to

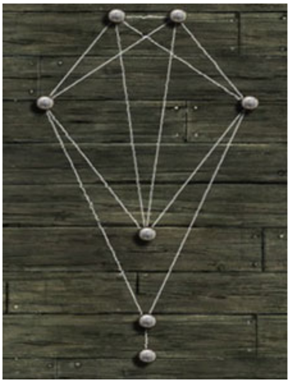


Fig. 2 Semieulerian graph in Icosien

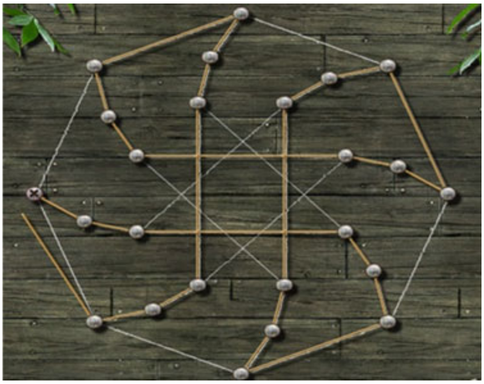


Fig. 3 Hamiltonian graph in Icosien

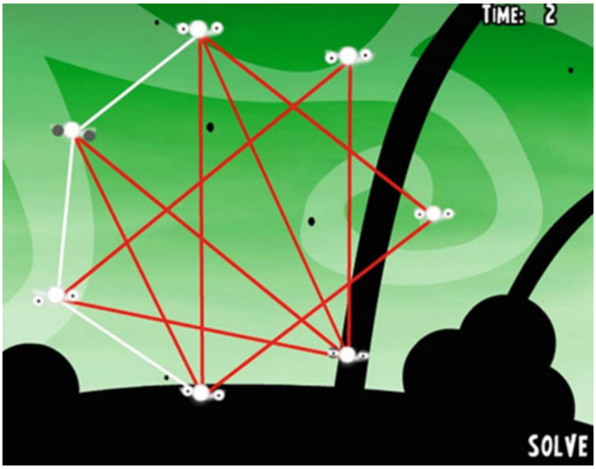


Fig. 4 Planar graph in Fly Tangle

construct and play with Eulerian graphs although they were not able to reach level 5 without knowing "the trick" (see paragraph "An instructional unit: Eulerian and Semieulerian graphs"). After they discovered and explained "the trick" about the degree of the nodes, they succeeded to complete all nine Eulerian levels.

The levels on Hamiltonian graphs were more difficult to complete – even for adults – mostly because of the lack of a clear winning strategy. This is presumably why children were not able to solve more than four levels.

Fly Tangle 3 (http://www.gamesforwork.com/games/play-18303-Fly_Tangle_3-Flash_Game)

Fly Tangle 3 is a game in which one drags nodes of a graph in order to show its planarity (Fig. 4).

Not all children succeeded in the completion of this game. In order to make all the students enjoy the activity and make them practice their mathematical skills

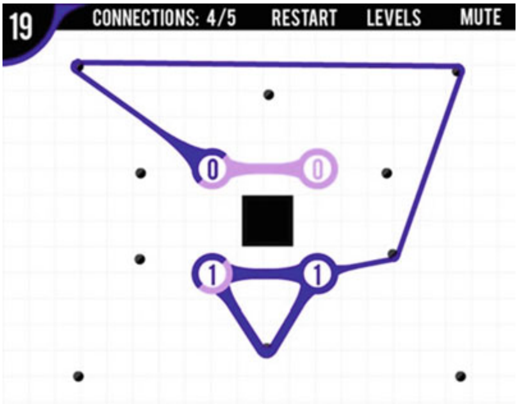


Fig. 5 A planar graph to be constructed in Strand

without risking frustration due to failure, we used YED to draw random planar graphs and changing the layout randomly. Then, the students came to the whiteboard and dragged the nodes to put each graph in a planar layout.

Strand (http://www.kongregate.com/games/ewmstaley/strand)

Strand is another online game whose aim it is to draw planar graphs with nodes of given degrees. This game is quite helpful to understand the basic concept of planar graphs because there are more basic levels to solve compared to Fly Tangle. Moreover, while Fly Tangle graphs are already drawn, Strand graphs are to be constructed. It showed that it was easier for children to build planar graphs step by step instead of adjusting tangled graphs. Furthermore, this game is applicable to reason about degrees. In fact, every node has a variable number that is the number of nodes yet to be connected with it. For instance, in the graph of Fig. 5, there is a missing edge between the two nodes with a "1" degree left, while the "0" degree nodes are complete.

It is worth to note that in many levels there is not a unique solution. This is good with respect to encourage children's creativity.

Activity and Topics

The contents of the project are summarized in Table 1 together with some of their related activities.[1]

In the following, we describe a selected variety of the activities that have shown to be the most intriguing examples arising from "real" problems, by using multiple

[1]Activities written in bold are those we are analyzing within this chapter.

Table 1 Topics and activities

Topics	Activities
Introduction of graphs and basic definitions:	**The Königsberg bridges problem;**
Graphs to solve real problems;	Searching flights between cities by looking at air lines' maps;
Nodes and edges;	
Degree of nodes;	**Matching Disney princesses with their boyfriends;**
Bipartite graphs and matching problems;	
Paths;	The genealogic tree of Dragon Ball cartoon;
Cycles;	football championship.
Trees.	
Eulerian and Semieulerian graphs:	**Pictures of points joined by lines, that one can draw without lifting the pencil from the paper and drawing each line only once;**
Graphs whose edges you all have to visit just once.	**Words or sentences you can discover in a graph whose nodes are letters;**
	Online game Icosien.
Hamiltonian graphs:	The problem to sit around a table with friends both on your right and on your left;
Graphs whose nodes you all have to visit just once.	Violetta's tour.
Planar graphs:	Three cottages problem;
Graphs that can be drawn on the plane in such a way that its edges intersect only at their endpoints.	Online game Fly Tangle;
	Online game Strand.
Graph colouring:	Maps you can colour by using the least numbers of colours such that adjacent regions have different colours.
Colour nodes of a graph in such a way adjacent nodes have different colours.	

software, and online games. As examples, we briefly present the activities 1), and 3) as well as justifying our motivation to use them. The whole path on Eulerian and Semieulerian graphs, including activities 6), 7), 8), is discussed in detail in the following section while other activities are illustrated at the end of the chapter, in the appendix.

Introduction to Graph Theory

Activity (1) The Königsberg Bridges Problem

This puzzle has been introduced by telling the well-known Königsberg bridges story, and making the children try and find a possible path in the map of Königsberg by touching every bridge just once. A schema of the Königsberg city was drawn which included the Pregel river, the islands, and the seven bridges. It was drawn on the blackboard, and in a second step students tried to find a proper path. Then, the children were told how a famous mathematician, named Euler, had the idea to

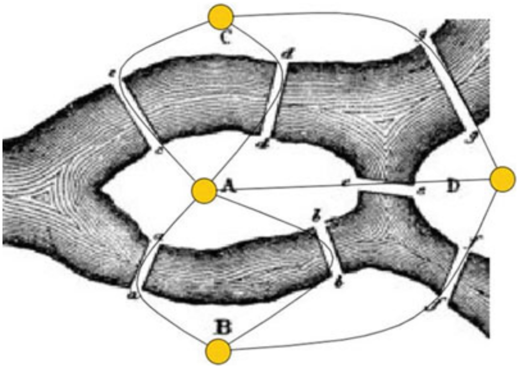


Fig. 6 Model of the Königsberg's problem by graphs (made with YED)

model this problem by assigning a point to every region of the city and a line to every bridge. Children were asked to draw the appropriate graph upon the map of Königsberg by using the appropriate software (in this case YED) to handle graphs.

After the graph (Fig. 6) was completely drawn, the children copied it in their exercise books and continued to work on it by themselves with pencil-and-paper. Whenever some of them claimed to have found the solution, he/she came to the whiteboard to show his/her possible solution. We did not immediately reveal that the problem does not have a single solution, so the children continued to work on the activity at home – also asking their parents about this problem.

Additionally, other examples of graph problems that required every edge to be visited just once were given in order to make pupils aware that similar problems are actually solvable. Later on, the problem of Königsberg was shown to be unsolvable when Eulerian and Semieulerian graphs were introduced.

One of the greater misconceptions of mathematics – since primary school – concerns "the" solution of a problem: every problem – they taught us – has a solution; and it has just one solution, preferably reachable with a particular method and only that method. This idea of mathematics does not really do justice to the complex and creative concepts of mathematics; and, above all, it does not fit the real world. If we want to teach real mathematics in real world situations, we should make students aware that many problems are "open" to no solution, or multiple solutions. The Königsberg's puzzle was one of the "no solution" problems we posed while several "multiple solutions" problems were handled, for instance, by using the game Strand.

Activity (3) The Matching of Disney Princesses with Their Boyfriends

A drawing of Disney princesses and their boyfriends was given to the students (see Fig. 7); each princess had an assigned boyfriend. But, all princesses and their boyfriends were mismatched. So, the students were asked to correct and redraw

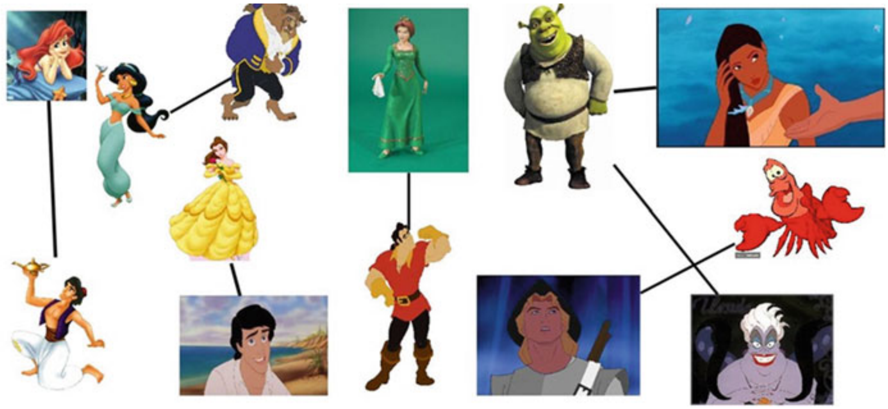


Fig. 7 Wrong connections between princesses and boyfriends (made with YED)

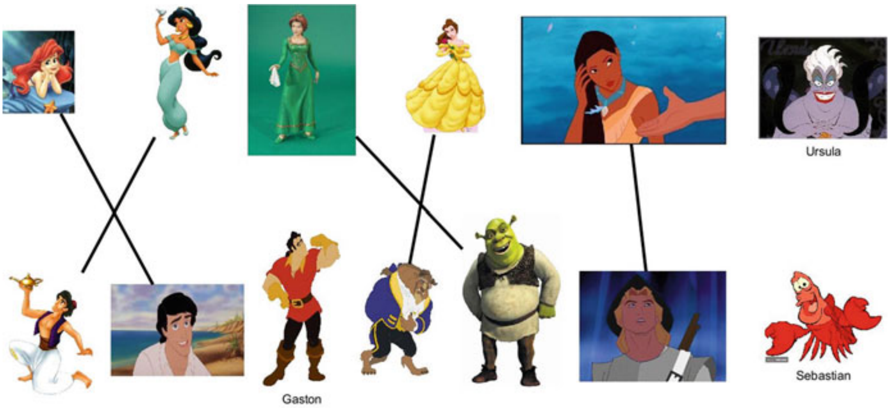


Fig. 8 Right connections between princesses and boyfriends (made with YED)

the connections to end up with the right matches (see Fig. 8). It could be observed that they had a lot of fun with this activity – partly because they had more knowledge about the right matches than the teacher. This graph is useful to introduce the degree of a node.

Teacher: How many boyfriends can a princess have?
Students: Just one!
Moreover, it is useful to introduce isolated nodes.
Teacher: Does Ursula have a boyfriend? Which degree does Ursula have?
Students: Zero!

After the students have finished to match up the princesses with their boyfriends, one can use these graphs for to explain the mathematical notion of a function: if we take Ursula out of the equation and follow the requirement that every princess has to have only one boyfriend, we develop a function from the set of princesses to the set of boyfriends – a non onto function, actually, because of Gaston and Sebastian, who have no girlfriend.

Finally, we adjusted the graph by putting all the princesses upside and all the boyfriends downside (as older students usually do with bipartite graphs), so that the graph is ready to represent a function with a separated domain and image set.

This is one of many examples of graphs that can be used in order to introduce the notions of functions and relations.

In general, the activity was appreciated because of its integration into the students' reality. If it is true that the "concrete" is the "abstract" that becomes familiar – also fictional objects such as cartoon princesses – they thereby become real objects. Presumably, this is why the children manipulated these objects with naturalness and straightforwardness. It was natural for them not only to connect matching characters with a line and correct wrong connections, but also to move nodes in such a way that the set of nodes resulted in two split partitioned sets.

An Instructional Unit: Eulerian and Semieulerian Graphs

In this section, we describe concept of Eulerian and Semieulerian graphs in detail. For matters of clarification, we named Eulerian and Semieulerian graphs "walkable".

At first, we briefly refer to the process of acquiring and using knowledge described by Spijkerboer (2015) with the use of the O.B.I.T. model: Remember, which means to acquire knowledge by the use of appropriate words or images; Understand, which refers to the acquisition of knowledge by practising procedures; relate, i.e. the use of already acquired knowledge by relating it to new situations; and Creative Application, which includes the use of knowledge in order to establish individual approaches of how to deal with a task.

The first two items, Remember and Understand, are linked to the surface approach, and focus on reproduction and memorization in order to relate "what I know" to "how to do it". The other two items, Relate and Creative Application, are part of the deep approach and focus on relationships among different aspects of the content and the competencies to "know what to do with what I know". Quite often, we are tempted to think that the deep approach is better than the surface approach. But, both approaches have a value, especially for kids, who need the surface approach. What we, as teachers, should not do is to stop at the surface approach, but to use also the insight and creativity to make students able to apply their creativity as well. Further, we should give the students time and space to grasp the concept of what they are working on – to deepen their understanding of it, to elaborate on it, and to use it creatively.

As we are going to explain in detail, in this teaching experiment we used both approaches; and we focused on the appropriate use of words and included storytelling to strengthen the chances that the mathematical concepts are being remembered. Further, we included games for practicing the reproduction of solving strategies, we used insight and argumentation to deepen the concept, we used stories as tasks to make it easier to apply knowledge that is already there, and we encouraged students to transfer their newly acquired knowledge to their own daily-life problems or to observe reality in order to draw further connections to the mathematical concept of graphs.

We started with the Remember phase of the O.B.I.T. model: For matters of simplification, we named Eulerian and Semieulerian graphs closed walkable and open walkable, respectively, because the concept of walking is graspable for students while the mathematical technical terms are not. Further, it made them recall the activity where they walked through the edges of the graph and therefore had an idea of what is requested for a graph to be walkable.

The instructional unit proceeds with the following phases:

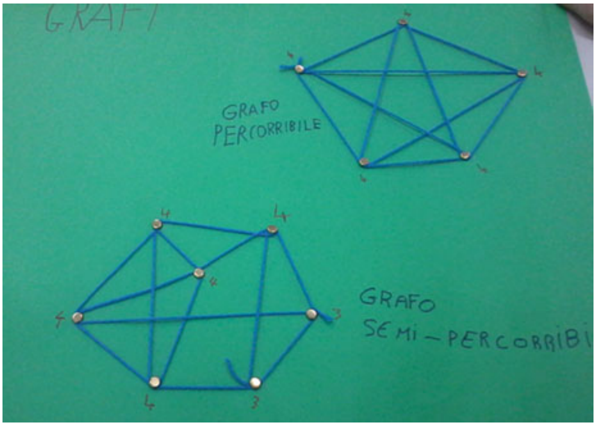
Historic Introduction

The topic was introduced by the seven bridges problem (activity 1). We retold the story and invited the students to think about a possible solution for the inhabitants of Königsberg. As discussed before, the students were guided to solve the problem with an appropriate mathematical model, i.e. a graph. This was when we passed from the real-life situation to the graph of Fig. 6, which is based on the picture of the city of Königsberg with the help of the YED software. This mathematical problem remained unsolved for a couple of lessons. During that time, it sometimes happened that parents asked their children whether they had managed to solve that problem. For us, this meant that children got highly involved into the problem and even thought about it outside of the classroom situation – eager to solve it.

For reasons of motivation, solvable problems – as the classical cabin of Fig. 13 – were presented to the students to make them aware that not all problems are unsolvable or difficult, so that they would have fun and acquire a sense of self-efficacy during the process.

After nodes, edges, and degrees were introduced, the children could practice on those walkable graphs. By analyzing and practicing on these graphs, the students were asked to identify similarities among walkable graphs and to discover that such graphs had only nodes of even degree – which are mathematically defined as Eulerian graphs, but we called them closed walkable –, or just two nodes of odd degree – which are mathematically defined as Semieulerian graphs, but we called them open walkable. Then, students practiced with several graphs, decided whether they were walkable or not by hand, drew walkable graphs without lifting the pencil from the paper (activity 6), and practiced paths in walkable graphs – especially by means of the online game Icosien (activity 8). In addition, we realized such graphs also with strings (see Fig. 9).

Fig. 9 Graphs realised with strings



Paper-and-Pencil Games

A variety of figures was presented to the group of primary school children (see Fig. 10). They were then asked to draw the individual graphs in one go and the premise to pass every edge just once.

In general, we noticed that students were quite excited about this activity. At the beginning, they were convinced that it was always possible, especially after several consecutive successes. When they failed to accomplish to redraw a graph (e.g. graph *e* in Fig. 10) they were convinced that they had made a mistake rather than thinking about the possibility that it might be impossible to do so, so they went on trying and trying. When we claimed that it was impossible to redraw "graph *e*" as a walkable graph, they did not believe it and kept on trying.

Practice with Sentences

After that, we practiced on words and sentences. The graph in Fig. 11 was used to practice paths – especially Semieulerian paths. For this purpose, we posed the question "Can you read the sentence hidden in the graph?"

Starting from a node of an odd degree – in this case G – you end up in the other node of odd degree – in this case E. The sentence that has formed is "Grafo percorribile", which is Italian for «walkable graph». As explained in the following, this graph was further used to introduce loops and multiple edges. When letters occur twice such as "R" in the word "percoRRibile" they have to be repeated, which means that we have a loop in "R", i.e. an edge that connects a node to itself. When there is a sequence of three letters whose first and third letter are the same, as the two I of IBI in the word "percorrIBIle", you have to return to the first letter by passing the second letter. In this case, you need a multiple edge between the first and the second letter.

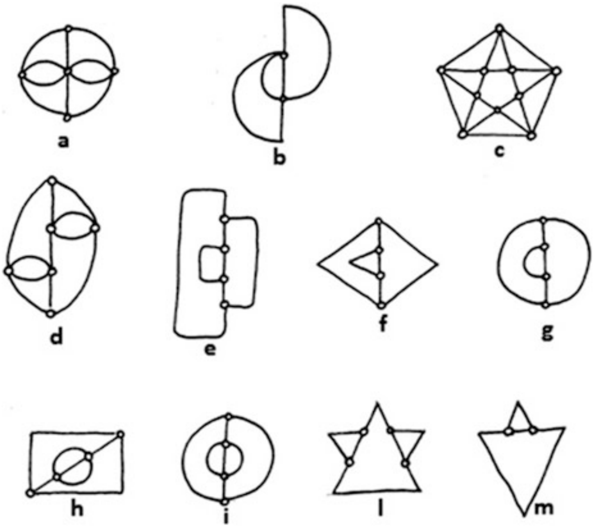


Fig. 10 Graphs to be explored with respect to eulerianity

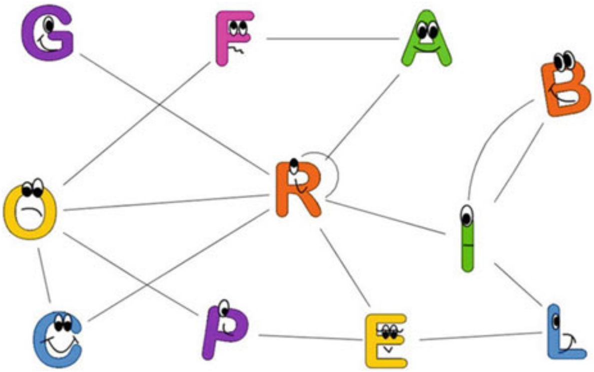


Fig. 11 Semieulerian graph to represent a sentence (made with YED)

Children played not only with sentences as in Fig. 11, but also with anagrams and graphs related to possible configurations of letters, e.g. as in the popular game Ruzzle, where letters can be used consecutively if they are neighbours in vertical, horizontal, or diagonal direction. The graph associated to a Ruzzle level is made of the letters shown in the game board as nodes; two letters are connected by an edge if you can use them consecutively in a word. For example, to form the word "sea" you have to follow the path 's', 'e', 'a'. You can do that because 's' and 'e' are connected and 'e' and 'a' are connected. However, you cannot form the word "tea" because 't' and 'e' are not connected (see Fig. 12).

Figure 12 illustrates a graph that is linked to the highlighted rectangle. After they were shown this example, the students were asked to think about their own set of letters with their individual connections among such letters. The aim was to find a set with the least possible number of letters, but with the most possible number of words. The students did this activity at home; and when they were back in school, they were quite proud to show the word-graph they had produced on their own.

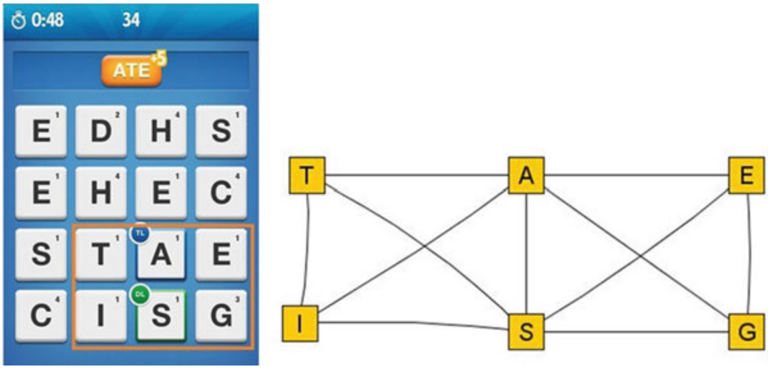


Fig. 12 A Ruzzle frame and a graph related to some letters of the frame

Such examples are considerably useful to practice on paths, but for them it was rather challenging to identify walkable paths. This is why they came up with the idea that not all graphs are walkable even though it was the first concept of graphs that had been introduced to them. Before the introduction of this activity, they had always been induced to consider eulerianity in every graph they were confronted with, even if the task made explicit that they were dealing with, for instance, a Hamiltonian graph, or trees.

Conjecturing by Online Games

The online game "Icosien" proofed to be quite useful to practice on Eulerian and Semieulerian graphs. This is partly because this game clearly indicates actions that are not allowed. For example, if you intend to connect two nodes that are not connected by string, or you want to cross an edge twice the string – that is yellow – becomes red. While working with the online game, the children were independent from the teacher's advises. We intervened only occasionally because it was our aim that every child tries by him/herself – aided by the software or receiving suggestions by classmates if necessary. Rarely, a student quit the game because he/she failed a level. In the majority of cases the atmosphere among the students was collaborative, so if someone needed assistance the other students helped him/her. At the beginning, the children tried to construct graphs by wrapping the string around the nails starting off from an arbitrary node. This solving strategy stopped working out at level 4, and they began to notice that for some graphs – the Eulerian ones – the starting point is irrelevant while for other graphs – the Semieulerian ones – only two nodes were suited to be the starting point. The strategy of odd-degree nodes was evident in the graph of Fig. 13, the cabin they were able to solve also with paper and pencil, and we decided later on to use just this graph for reasoning activities. Little by little, they identified that if there are two nodes with odd degrees (the degree of a node was already introduced in the previous lessons), they needed to start from one of them.

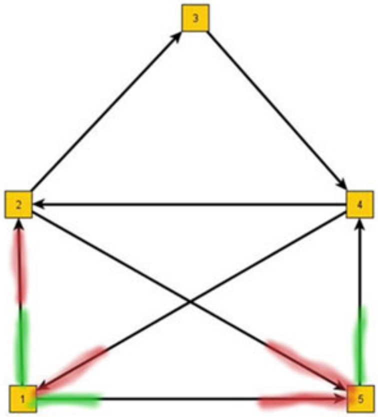


Fig. 13 A graph used to argue on Semieulerian paths

In the end, with the help of this winning strategy, all students were able to solve all levels of the online game. What was striking was that the students were solely satisfied to have established a strategy to win the game; however, they did not show any interest in the motivation why their strategy was successful. Instead, they preferred to think that it was a trick. In other words, they were satisfied with the surface approach, being on the stage of having memorized (Remember in the O.B.I.T. model) the concept and to know how to act to “win the game” (Understand in the O.B.I.T. model).

The following activities were closer linked to the deep approach because it was our aim to establish a full understanding of the concept instead of letting “whatever works” to be sufficient. This turned out to be quite challenging for such young students.

Argumentation by Chalk (In Case That Technology Does Not Work)

Technology was indeed quite useful for experimenting activities; however, it would not have been useful for reasoning. The online game was not suitable for the purpose to argue about odd degrees in Semieulerian graphs because the children would have been too focused on the game itself rather than on reasoning. This is why, instead of the whiteboard and our fingers, we used the blackboard and coloured chalks. Even though we did not explicitly talk about directed graphs, we used oriented edges (see Fig. 13). We argued about Semieularian paths on the basis of students’ examples, i.e. {1,2}, {2,3}, {3,4}, {4,2}, {2,5}, {5,4}, {4,1}, {1,5}. Due to this approach, the students did not encounter any difficulties in using oriented edges; on the contrary, they intuitively made use of arrows to follow a path. Then, we coloured every source with a green chalk and every sink with a red chalk. After that, we focused on the two nodes 1 and 5 – they are the starting source and the ending sink – and counted the green and red edges. In 1, we counted two green edges and a red one because the path first goes through edge {1,2}, then through edge {4,1}, and finally through edge {1,5}. After passing {1,5}, the path abandons node 1 because our path equals a Semieulerian graph. This is because of the odd number of edges. In that case, the first two edges are edges were the path changes its direction – but will pass the edges for a second time – while the last edge indicates the end of the path. We did similar examinations on node 5, which we also pass twice, i.e. in {2,5} and again in {5,4}, and then we end the path in {1,5}. So this second path has one green starting source and two red ending sinks – one of which is used to complete the path.

The other nodes are classified as passing nodes because their amount of being a starting source and ending sink is equal.

This figure proofed helpful to make young students understand the motivation for the two odd-degree nodes of Semieulerian graphs. In regard to Eulerian graphs, we identified that the starting point coincides with the final point, i.e. we have the same number of outgoing and incoming edges. The entire phase of argumentation was teacher-led who stimulated the students with questions, encouraged them to express their thoughts, made them reflect on their own actions and claims, and, finally, thanked them for their insights and reflective reasoning.

Ongoing Test: Eulerian Carnival

Halfway through each activity we integrated a test to monitor each student's learning progress. In the first year, the test was about Eulerian and Semieulerian graphs. The students were given a variety of graphs and were asked whether these are Eulerian or not. In the case of a positive answer, they were asked to give an example. The majority of students completed this test successfully. In the second year, additionally to a selection of classical exercises, we decided to integrate a story which was already used in the first year as a whole class activity – "The Eulerian carnival".

The story taken from Aleo et al. (2009, p. 112) is the following:

> We are in a strange place called Polygonsland, peopled by polygons, namely the Decagon, that is the king of the land, Mr. Equilater Triangle, Mr. Isosceles Triangle, Mr. Square, Mr. Rectangle, Mr. Pentagon and Mr. Hexagon. For the three days of Carnival, the naughty king Decagon, as he usually does every year, announces a contest. The inhabitants of Polygonsland are requested to walk in the path drawn in Fig. 14, by passing from every street, but only once. Each inhabitant starts from an assigned emplacement, as shown in Fig. 14: A for Isosceles Triangle, B for Rectangle, C for Square, D for Pentagon, E for Equilateral Triangle, F for Hexagon.
>
> In the emplacement O there is a treasure, when you pass by O you can take the treasure. It is not requested that O is the last emplacement to visit.
>
> Question 1 Why did we say that the king was naughty? . . . Is the king sure that no one could win? . . . Why? . . .
>
> But during the night Mr. Isosceles Triangle, who is smart and knows graph theory, decides to modify the trace by adding a new street in such a way to win.

Question 2 Which street is built by Isosceles Triangle to win the contest? . . .

So, the first day of the contest Mr. Isosceles Triangle wins. The king is surprised, but the contest goes on. Mr. Equilateral Triangle, who knows graphs too, and understood everything, during the second night also modifies the path, in such a way he wins.

Question 3 Which street is built by Equilateral Triangle to win? . . .

So, the second day of the contest Equilateral Triangle wins. The king is disappointed, but the show must go on. Finally Mr. Rectangle, who is smart as his brothers Triangles, decides that is right to give to all the possibility to win. And during the third and last night modifies the whole path in such a way everyone could win.

Question 4 Which change can Rectangle do in such a way everyone could win? . . .

The solving of the questions required from the children to apply acquired knowledge about Eulerian and Semieulerian graphs. The results were that the students were able to answer to the first question correctly. This was a result that we did not definitely expect because the students were only used to count the degree of the nodes – but while working with Eulerian or Semieulerian graphs – they were not used to more than two nodes of odd degree. Usually, when they identified a graph they could classify, they started to count the degree of nodes until they encountered a node of odd degree which they used as a starting point. The only graphs they had encountered before which were not Eulerian or Semieulerian were the graphs of Fig. 10. And in that case, they had managed those graphs before starting to count the degrees. So, the situation described in question one was unusual.

By the time that they identified that the graph of Fig. 14 neither Eulerian nor Semieulerian, they had already acquired enough knowledge in order to know how to make it Semieulerian. The aim of the task was to build a street. A few children solved the task by deleting an edge so that the two nodes connecting that edge

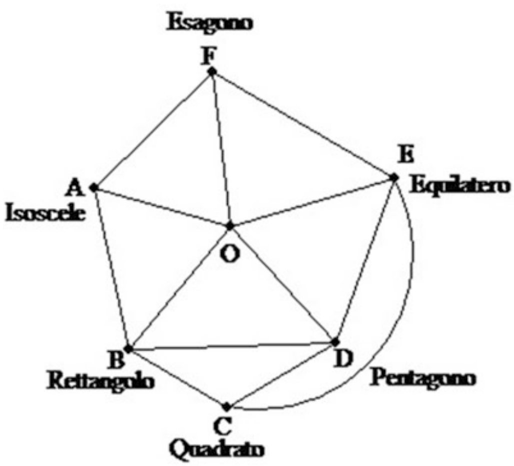


Fig. 14 Path for Polygonsland contest (Aleo et al. 2009, p. 112)

changed their parity. Of course, this was a possible solving strategy and such answers were positively considered.

Some difficulties arose on question four, even though all students were familiar with the fact that an Eulerian path could start anywhere in the graph. We later identified that a few of them had not understood that the request could have been translated into "make the graph Eulerian" – possibly because question two and three both asked for semieulerianity.

Finally, we conclude – on the basis of the test results – that the majority of students were able to relate acquired knowledge with the presented task.

Embodiment with Wool Strings

As a final activity in class, the children created posters which summarized the main ideas that had arisen during the teaching experiment. One of these deliverables was solely on walkable graphs (see Figs. 9 and 15) and was realized by manipulated real wool strings which were wrapped around split pins, similar to the online game Icosien.

In order to make every child an active part of the activity, the class was divided into three homogeneous teams according to the self-assigned preferences of students, i.e. drawers, writers, and thinkers. In each group, students designed their poster together once the teacher assigned the task to each group. The thinkers then defined the graphs to be used. In the whole, they worked as a team, but each student had an individual role. At the beginning of the working progress, each student in each group proposed his/her own graph to the others. After that, they decided which one they want to use. The drawers were asked to apply their ability of drawing which resulted in a mixture of experimenting and purposeful manipulating of the graphs; It was not a mere copy of the thinkers' graphs. The writers were in charge of the title and subtitles, e.g. in Fig. 15 they decided for "inizio" and "fine" – Italian words for "begin" and "end" respectively. As for the wrapping of the strings, all members of the team were asked to participate because it is important to join mind and body as well as putting the concepts they had studied into practical actions. This requirement did not pose any difficulties; all students were quite eager to participate in this activity.

Final "Fighting" with Parents

In the first year in which we carried out this teaching experiment, their children from the very beginning, especially in regard to the online games, involved the parents. In the second year, the teacher asked the students not reveal "the trick" of walkable graphs, so that the parents could be invited to the last meeting of the experiment for a "Children vs Parents Contest" based on the online games. The games that were used for the contest were Icosien and Strand; and the children clearly won the contest. In fact, it showed that the children had not only practised on

Fig. 15 Poster realised by children on “walkable” graphs

Eulaerian and Semieulerian graphs during the teaching experiment, but they also had internalized the underlying definition and reasoning. Further, it showed that the children were very proud to win against their own parents which implemented that they knew something that their parents did not. The parents, on the other side, were proud to see their children so happy and excited about mathematical contents.

Results and Conclusion

This teaching experiment had two major aims. Firstly, mathematical oriented, we wanted the children to be able to represent relations and data and – in significant situations – use representations to get information. Secondly, emotionally oriented, we wanted the children to be happy when doing mathematics. We will briefly discuss both aspects.

The majority of children were able to master the activities as is shown by the ongoing and final tests. Moreover, in the first lessons, when the basic concept had been introduced, the teacher asked the students to name suitable examples from their everyday life. They were able to identify the model of a graph in many settings, which indicates that they mastered the “Creative Application” ability the of O.B.I.T. model. An example is illustrated in Fig. 16. The figure represents a city, but indeed it can be interpreted as a graph. The children successfully connected the studied topic with the real life situation.

Fig. 16 A picture of a city, seen by children as a graph

In the ongoing of the teaching experiment, it has clearly shown that the YED artefact, which was used to draw several graphs, was a helpful tool. At the beginning of each activity, objects were presented by the software as one precise symbol, e.g. pictures of cartoon characters, letters, cities, persons, etc. The software provided a high variety of symbols and included the possibility to drag each object into the working area so that it could be used as a node of a graph. But, whenever a topic had been mastered, the children went on by using standard symbols of nodes – which was usually a dot. They were able to abstract from the real objects and worked on the model rather than on the problem itself; they manipulated not only real objects (e.g. characters, cities, etc.), but also mathematical objects (e.g. graphs). Moreover, they were able to use *conceptual metaphors*: they inferred properties of a certain source domain (i.e. real objects) by manipulating a target domain (i.e. graphs) (Lakoff and Núñez 2001). The software contributed to this transition.

Another feature of the software – the possibility to drag nodes or change the shape of edges – made children aware of isomorphisms among graphs which look different at first sight. As a matter of fact, they were able to transform a graph without deleting or adding an element, but by simply changing the shape. Not every child was able to grasp this concept by him/herself, but at the end of the teaching experiment – after various manipulations of graphs on the YED software – they were all convinced that it is possible to preserve mathematical properties in differently looking graphs.

The major disadvantage that we encountered on the use of the software was the English language; but, as we mentioned earlier, the tools that were required for our purposes were self-explanatory so that the Italian translations were not needed.

As for the online games, we focus on Icosien because it was the most frequently used during the teaching experiment. The wide range of possibilities to freely experiment was highly appreciated by the children, and they eagerly lined up to come to the whiteboard in order to do so. From a teaching and learning point of view, the game is useful because students can experiment by themselves without the constant assistance of a teacher. The game itself restricts actions and signals the end of it. The children are guided without any external intervention. Another disadvantage of such a game is that it makes the students primarily focus on the game itself. Their focus of attention was on winning the game, not on the analysis why a certain property is held. Due to this, we used the game to conjecture, but we did not use it to reason or argue. Even though the use of technology has a high variety of advantages, it showed that time as well as pencil-and-paper activities help students to reason successfully.

From the emotional point of view, students of both schools were enthusiastic to encounter mathematics as we did in our teaching experiment that is without numbers, calculations, or systematic operations, but rich of princesses, relatives, football players, and real-life situations. They encountered “another mathematics” which was different from what they previously experienced. Somehow, they saw mathematics with the eyes of a mathematician. The students understood mathematics as a game. Indeed, the activities required reaching a target while obeying the “rules of the game”. On the basis of previous experiences, teachers often focus too much on the application of rules of mathematics instead of making students analyse the origin of such rules. If mathematics could be understood as a game more frequently, students would be more encouraged to think about how to improve the rules on higher levels as well as about strategies that are important in order to achieve your aim: Aside from playing by the rules it is allowed to use creative strategies to win in the most elegant and fastest way.

Young students appreciated the possibility to play with mathematics through online games – even at home. It happened that children insisted on their parents to play with them, and even a grandfather was invited to play. They further enjoyed the possibility to draw their own graphs once they understood how to do so (see Fig. 17).

Lilia Teacher, I thought of another graph!

With reference to the paper-and-pencil activity in the previous section, after the teacher said that the graph *f* in Fig. 10 was not walkable, Lorenzo, an eight-years-old student, said:

Lorenzo Then graph *g* is not walkable either; they are the same!

This showed that the student was able to identify the model underneath the drawings without any guiding explanation of the teacher.

Technology had a fundamental part in this teaching experiment. As mentioned before, we used “old technology” (e.g. pencils, paper, blackboard, and chalks) to reason and to fully understand, but the application of “new technology” was very useful in the practical part because it made children independent. This was also previously discussed on Icosien. There, the students constructed different graphs without the assistance of the teacher, and they continued to work independently at home – with the online games and with YED. The possibility to drag nodes and/or

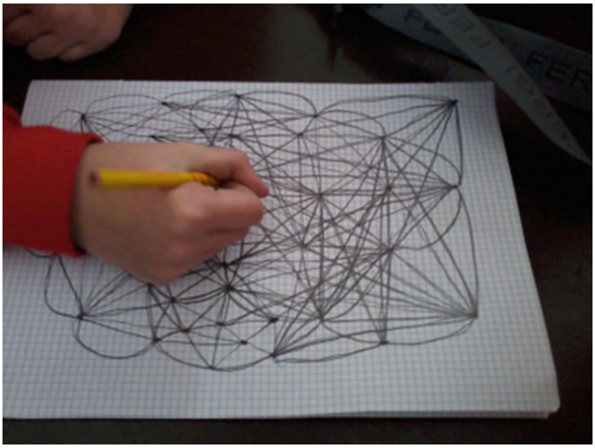


Fig. 17 A student drawing a graph by herself

change shapes to edges made the manipulation of graphs easier; otherwise, one would have been obliged to continuously redraw each representation of the same graph. Moreover, by transforming the graph instead of redrawing it, the students identified that the features of a graph do not change by the mere alteration of its representation.

In addition to observing the students, other teachers of the students have been interviewed. They noticed that the children's logical skills increased as well as their active participation in class. They posed questions more frequently instead of passive listening. So, the students learnt to learn, and they transferred this new skill to their whole learning process.

What was difficult was the argumentation activity. The students were much more interested in experimenting and learning new "tricks" rather than in reasoning on possible explanations. But when they were actively engaged into the discussion and invited to participate actively instead of mere listening, this helped them to concentrate on the topic.

In the second year of my teaching experiment, my tutor – a primary school teacher – took notes at each of my lectures and then organized meetings with other primary school math teachers to share materials and ideas for possible future math sessions. It showed that this teaching experiment was highly appreciated by teachers because primary school teachers – but also high-school teachers – are not familiar with the underlying aspects of graph theory. Further, they appreciated the idea that graphs can be used to model problems from real life situations, to represent relations, to mathematize situations. Additionally, the teachers who took part to these meetings had the opportunity to address questions about innovative teaching/learning processes of mathematics.

Finally, this chapter ends on comments written by children in the final satisfaction questionnaire: "I think that graphs are more funny than games", "I think that this laboratory on graphs helped me to reason more quickly", "Mathematics is beautiful, intriguing and is of help".

Appendix

This appendix provides a short glossary of those concepts in graph theory we dealt with in the chapter and a collection of some activities we carried out with the children.

Short Glossary of Graph Theory

- *Bipartite graphs*: a graph G = (N, E) is said bipartite if it is possible to divide the set N into two subsets N_1 and N_2, in such a way that every edge in E joins a node in N_1 with a node in N_2.
- *Coloring of a graph*: a colouring of a graph is a function from the set of nodes to a set of colours, that assigns a colour to every node in such a way that connected nodes have different colours.
- *Complete graph*: a graph is complete if every couple of nodes is connected by an edge. A complete graph with n nodes is indicated by K_n.
- A bipartite graph with bipartition sets N_1 and N_2 is said *complete bipartite* if every node in N_1 is connected to every node in N_2. A complete bipartite graph with m nodes in N_1 and n nodes in N_2 is indicated by $K_{m,n}$.
- *Cycle*: a cycle is a path that is closed. Moreover, the starting node is the only node repeated in the path (repeated as the ending node). For instance, in the graph G = (N, E) with N = {1, 2, 3, 4, 5} and E = {(1,2), (1,3), (2,3), (2,4), (3,4), (3, 5), (4,5)}, the path C = [(1,2), (2,3), (3,1)] is a cycle, while the path P = [(1,2), (2,3), (3,5), (5,4), (4,3), (3,1)] is not a cycle because not only node 1 is repeated, but also node 3.
- *Degree of a node*: the degree of a node in a graph is the number of edges involving the node.
- *Eulerian and Semieulerian graph*: a graph is called Eulerian if there is a closed path containing every edge of the graph just once. (An Eulerian graph has all the nodes with even degree). A graph is called Semieulerian if there is an open path containing every edge of the graph just once (a Semieulerian graph has exactly two nodes with odd degree).
- *Graph*: A Graph G = (N, E) consists of two sets, N and E. N is called the set of *nodes*, and its elements are represented by points. E is a set of couples of nodes, called *edges*. If two nodes are a couple in E, then the two points representing the two nodes in the edge are joined by a line. Whenever the couples are sorted, the graph is said "directed", otherwise it is said "undirected". When we say just graph, without specifying directed or undirected, we implicitly mean undirected.
- Two nodes A and B can be connected by one edge, in this case we indicate the edge with (A, B), or by two or more edges, in this case we indicate the edges with s_1=(A,B), s_2=(A,B), ... Edges connecting the same couple of nodes are called *multiple edges*, and a graph with multiple edges is said *multigraph*.
- *Hamiltonian graph*: a graph is called Hamiltonian if there is a cycle involving all the nodes (it is not requested that this cycle involves all the edges). This cycle is called *Hamiltonian cycle*.

– *Loop*: a loop is an edge connecting a node with itself.
– *Path*: a path in a graph G = (N, E) is a sequence of consecutive edges, for instance if we have N = {1, 2, 3, 4, 5} and E = {(1,2), (1,3), (2,3), (2,4), (3,4), (3, 5), (4,5)}, a path could be P = [(1,2), (2,4), (4,3)]. A path is said open if the first node of the first edge in the path (starting node) is different by the last node of the last edge in the path (ending node), closed otherwise.
– *Planar graph*: a graph is planar if it can be drawn in a plane without graph edges crossing, i.e. if it can be drawn in such a way an edge can touch another edge just in the common node.
– *Tree*: a tree is a graph without cycles and connected, where a graph is connected if there is always a path between any two nodes.

Some Activities on Graph Theory in Primary School

Activity on Hamiltonian Graphs (We Called Hamiltonian Graphs "Visitable") Violetta's Tour

How can we help Violetta (a pop singer very popular among children) to adjust the trip of her Italian tour in such a way that she stays in every city she planned to visit just once? This problem is connected to reality, not only because the cities are the real stops of Violetta's tour in 2014, but also because we checked for flights between the cities in question (but, for the sake of simplicity, only with just one airline,). The children constructed the graph in Fig. 18 without difficulty by connecting cities joined by a flight.

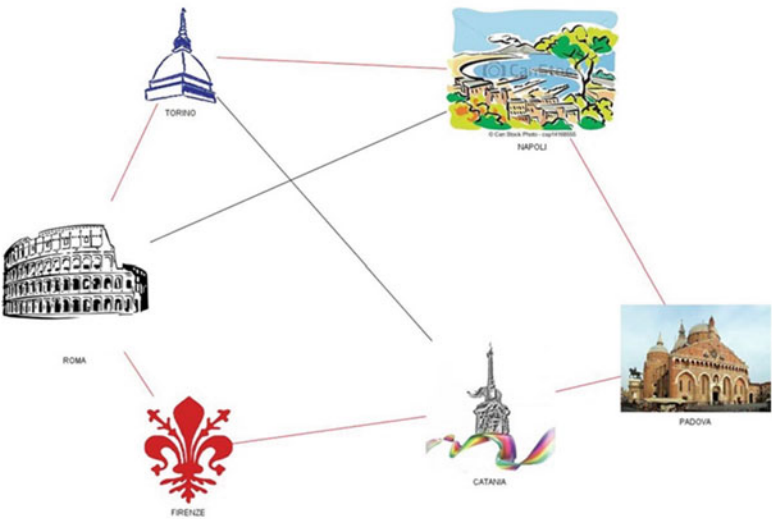


Fig. 18 Hamiltonian graph to represent a tour (made with YED)

The use of the software was irreplaceable. Even when the graphs are simple, with just a low number of nodes – like the previous one – the same activity is much more difficult to be carried out with paper-and-pencil. In fact, the students made first attempts in their exercise books by drawing several cases or by colouring used edges, but this was unsuccessful. As soon as they came to the whiteboard and could drag nodes digitally, the task could be resolved easily. Additionally, several Hamiltonian graphs were explored by means of the online game Icosien. But the students met difficulties in finding Hamiltonian paths because they did not have any strategy to follow.

Activities on Planar Graphs

As for planar graphs, we applied the well-known three cottages problem (another unsolvable problem, after the "seven bridges" impossible path): there are three cottages and three utilities, each cottage has to be connected to each utility, but we want to draw connections in such a way that they intersect just in utilities and cottages.

The children started by constructing the requested bipartite graph in their exercise books: they drew the six nodes in two separated lines, as they were used to due to the Disney's princesses graph. Then, they started to draw the edges from cottages to utilities one by one, avoiding making them touch. All students were successful up to the fifth edge, but they failed at the sixth edge because the problem is not solvable. Then, a few children tried to solve the task at the whiteboard. This time, the edges were already drawn – by the teacher – and the students dragged the nodes, unsuccessfully of course. After that, a modified version of the problem – simplified with three cottages and two utilities – was posed and solved very soon.

This problem was useful to introduce complete graphs with n nodes, named K_n and complete bipartite graphs with n and m nodes in the two sets of nodes, named $K_{n,m}$. The children were asked to analyse complete graphs and were guided to discover the "trick" of triangulations when they have a K_4 inside a graph as shown in Fig. 19.

The "triangulation trick" was often used in the games at school (as fly tangle, see paragraph "Technologies") and in the home-made games which consisted of

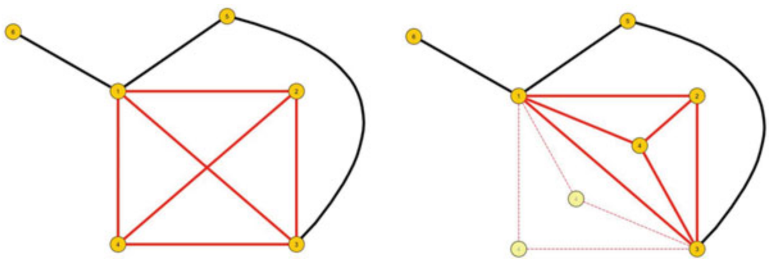


Fig. 19 A planar graph containing a subgraph K_4 (made with YED)

adjusting planar random graphs produced by the YED software. The students were guided to discover that for K_5 – and higher – complete graphs are not planar and so are graphs containing not planar graphs. Similarly, they easily grasped that complete bipartite graphs are planar until $K_{2,3}$ They tried this with the $K_{3,3}$ of the cottages problem and hence found an easy solution for $K_{2,3}$.

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Pocket Calculator as an Experimental *Milieu*: Emblematic Tasks and Activities

Ruhal Floris

Abstract In this chapter, we present and analyze calculator-based tasks and activities conceived as means for the learning of mathematics in several grades of primary and secondary school. The tasks or activities have been experimented with students and pre-service teachers. The intent is to show how a set of calculator-based tasks can be organized in a way that they promote the development of theoretical aspects. The results show that a high variety of numerical activities can be proposed in such a way, but that a further institutional promotion is necessary. The analyses are based on the concept of 'milieu' by Brousseau (Theory of didactical situations in mathematics. Kluwer, Dordrecht, 1997) with an anthropological approach (Chevallard Y, Recherches en Didactique des Mathématiques, 19(2):221–266, 1999; Lagrange JB, Educational Studies in Mathematics 43(1):1–30, 2000).

Keywords Calculator • Arithmetics • Fractions • Early algebra • Theory of situations • Learning milieu • Adidacticity • Anthropologic approach • Praxeologies • Teaching

Introduction

In a considerable amount of countries, a relatively large number of primary and secondary mathematics teachers do not consider it important to teach how to use a calculator; they presumably assume that this is something pupils learn from their classmates or outside school. At least, this is the case in the French speaking part of Switzerland. The consequence, observed in higher secondary school, is that the calculator skills of the students are not as far developed as they should be at that point; for example, a few of the observed students showed difficulties to successfully enter expressions such as $\sqrt{2}-1$, but entered $\sqrt{2-1}$ instead. This lack of competences could possibly lead to the situation that when they study formal calculations with square roots and try to check that $\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)=1$, the

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calculator is of no help for them. We are of the opinion that this situation is a cause of social inequalities. A small group of students are able to use their calculators as a powerful checking tool while others limit their use to basic operations; this phenomenon has been observed on the use of symbolic calculators by Guin and Trouche (1999). This is why we are of the opinion that working with calculators on a regular basis is necessary in primary and secondary school. Furthermore, we believe that calculators are highly beneficial tools in regard to mathematics learning on the premise that their use is well introduced to the students and that the calculators are incorporated within appropriate tasks or activities (that is, well-designed sequence of tasks). In recent years, we regularly observed lessons where the teacher aimed to integrate such kind of tasks and activities at different school levels. Further, we integrated them in our pre-service teacher workshops: the secondary student teachers were asked to adapt one calculator-based activity into their teaching[1] which was then evaluated on in a feedback discussion. In this chapter, we synthesize and analyze the results of different calculator-based teaching experiments with primary and secondary school students (Del Notaro and Floris 2011; Weiss and Floris 2008) as well as within teachers' training (Floris 2015). Our analysis aims to answer the main research question whether the use of calculators enhances the learning of mathematics and how it does so. We interpret the research question by integrating the theoretical perspective, selected didactical situations, and praxeological anthropology. These aspects of our interpretation will be thoroughly described in the next section.

Theoretical Background

Our main theoretical reference is Brousseau's theory of didactical situations (1997). This theory emphasizes the role of the *adidactical milieu* in the teaching-learning process of mathematics; the gist of this theory is that, in the end, taught knowledge has to be transferable and applicable to the real, non-didactical world. But, this can only be realized when the non-didactical world is – at least partly – integrated into classroom activities. This is due to the fact that mathematics is mainly a procedural science, i.e. it is impossible to merely memorize questions and their answers, but it requires to learn how to suggest solutions to an infinite variety of possible questions – even with respect to simple additions.

As one example for adidactical feedback, we refer to the task where we asked the students to enlarge a tangram puzzle (Ibid); the students were assigned to groups, and each student of each group had to enlarge one piece of the puzzle. The feedback was provided by the final assembly of the puzzle: in case that it was impossible to put the pieces of the puzzle together because their sizes did not match, the former

[1]In Geneva, secondary teachers follow a two-year training and in the second year they teach half-time in school.

mathematical enlargement procedure, e.g. the addition of constant values to the measurements, was thereby invalidated. In this situation – the adidactical milieu – the nullification of the applied mathematical procedure is not provided by a teacher, but by the situation itself. Here, it is important to acknowledge that the milieu is not limited to the external situation, but it also refers to the goals defined by the teacher and to students' prior knowledge.

In everyday teaching, the adidactical part of the didactical milieu often is of understated significance. In consequence, the learner is held accountable to create individual "milieus", but in case that this option is out of reach for the individual learner the learning outcome is weak. The conjecture of Brousseau's theory is that an adidactical milieu provides rich feedback and thereby successfully supports learning processes. On the basis of these hypotheses, the research question evolves to how the integration of calculators into the classroom can be a successful learning milieu. We will aim to provide an answer by giving examples of different grades where the calculator was used to assist this adidactical learning milieu.

Our methodology is mainly qualitative; our data basis was students' and teachers' gestures, calculator manipulations, and utterances. Finally, we compare our findings to Brousseau's theory to assess the learning potential of the proposed tasks alongside with their feedback.

In our opinion, an analysis of effective teaching methods needs to integrate the anthropological approach by Chevallard (1999). According to this approach, praxeology (from Greek 'praxis' and 'logos') is a four-part mathematical concept which includes a type of tasks, technique, technology, and theory. The first two components are practically oriented whereas, here, technology is the discourse that justifies or explains the technique. It becomes theory when the discourse is more structured. In the educational context, and in the domain of applying technological tools, Lagrange (2000) reduces praxeology to three components: tasks, techniques and theories, i.e. Chevallard's last two components are being combined. Lagrange further considers a study of Rabardel (1995) which analyses the process of tools' transformation to effective working instruments. Thus, within the anthropological approach, discourse is said to link technique and theory; this assumption entails the conjecture that this linkage enhances the mathematical quality of learning processes in the long term. Furthermore, the concept of praxeology is especially helpful for the analysis of the introduction of new techniques.

In summary, the conditions for a profitable learning milieu according to the above mentioned theories are:

LM1 A task, or a set of tasks, that involves some sort of adidactical feedback which is independent from the teacher.

LM2 A more or less explicit presence of a tight mathematical link between theory and technique.

In the following paragraphs, we discuss a set of selected examples. First, we present two summarized examples and then present as well as thoroughly analyze two further examples.

A First Example: A Milieu for Place-Value Notation

The following tasks are intended for students at the end of primary school or early secondary school (i.e. 11–12 years-old students). We suggest that the reader takes out a calculator and solves them while thinking about the mathematical properties involved.

1. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 89454.
2. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 80404.
3. Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 892054.
4. Type in the number 4.56 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 4.056.

As a whole, these tasks refer to the domain of place-value notation, which is one of the main domains in primary school. This set of tasks proposes a learning milieu for this fundamental arithmetical property because the use of this property is necessary to solve the tasks. In the course of a workshop, these tasks have been proposed to students and pre-service teachers in primary and secondary education with the underlying intention to prepare them to integrate the calculator into their teaching. We were surprised of the outcome that some of them had difficulties with solving the tasks, even despite their mathematics knowledge. One of our possible interpretations of this is linked to Brousseau's didactic contract (1997):

> It is the set of the reciprocal obligations and sanctions that each partner in the didactic situation imposes, or believes to impose, explicitly or implicitly, on others, and those that are imposed on him or her, or he or she believes that they are imposed on him or her. (Translation by Indiogine 2010, n.p.)

At primary school, without specific instructions on a different handling of the calculator, it is only used to obtain the result of a direct calculation. What is proposed in the example above is an inversion of a common task: the result is given and the operation is asked for. It is what Brousseau refers to as breaking the didactic contract and further elaborates on the students' and teachers' perplexity about this. Activities which propose such disruptions are interesting because they introduce adidacticity into the milieu; or, rephrased slightly different, ignorance triggers learning processes. Here, the teacher has the choice either to instruct the students to find the solution by themselves, or to indicate possible solving strategies (e.g. let a fellow student propose an answer). An alternative to level out the state of not knowing is to propose a slightly easier task beforehand:

1(a) Type in the number 89254 on your calculator. Without deleting this number, using mathematical operations, find the shortest way to display the number 89264.

Evidently, the kind of feedback that is given by the calculator is in agreement with the definition of feedback of LM1, a calculator coming clearly from the non-didactical world (see previous section). With respect to LM2, on the basis of the anthropological approach, we focus on the theoretical aspect. This refers to the fact that the numbering position within the tasks links the way to solve them and explains the solving strategy at the same time. For example, in case that the task requires to change the digit two into a four, you have to add 200 because the digit ‘2’ is at the hundreds place of the positional notation system.

A Second Example: Milieu for Arithmetics Operation Properties

At the beginning of this section, we provide a second example for a possible integration of calculators into the classroom setting. Again, we suggest that the reader takes out a calculator and solves it while thinking about the mathematical properties involved:

1. Determine all digits of the numbers 7^{12}, 7^{13}, 7^{14}, etc. in their standard (base 10) expression.

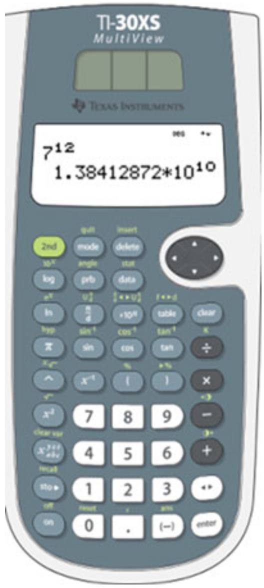
One possible answer could be that a student thinks that the requested answer for 7^{12} is 13841287200 (see Fig. 1).

In such a case, the feedback provided by the calculator is inadequate. This is why the teacher has to supplement the students with sufficient validation techniques. For example, one technique could be to determine the digits of 7^{10} and 7^{11} first, and then determine the result for 7^{12} with the help of the calculator. Or the teacher could pose the question if it is possible that the last digit of the displayed result is zero. This is how the milieu is enriched with paper-and-pencil calculations, which are necessary to give the right answer to this task. In this situation, the students’ calculations are expedient with a mixture of calculator and paper-and-pencil calculations. With the help of the TI-30XSMultiView[2], it is possible to get the results to 7^{11}, i.e. $7^{11} = 1977326743$. This result can then be used in order to work out all digits of 7^{12} by computing

$$\begin{aligned} 7^{12} &= 7(7^{11}) = 7 \times (1977326743) = 7 \times (1977326740 + 3) \\ &= 7 \times 197732674 \times 10 + 21 = 1384128718 \times 10 + 21 = 13841287201. \end{aligned}$$

[2]It is the official calculator in the schools of Geneva, provided to all 10 years-old students.

Fig. 1 Display for the result of 7^{12}



This kind of calculation requires the students' ability to apply certain operation properties correctly, in this case distributivity. As a result, the integration of the calculator into mathematics teaching is not only meant to be a calculation tool, but its application needs to be carefully instructed. Hence, the following task could be proposed afterwards to further strengthen the new didactical milieu which is dialectic of calculator and paper-and-pencil work:

2. Without multiplication and with a minimum of operations, please calculate the following products on your calculator: 387×204 and 87×199.

A Milieu at Primary School: Division Without Multiplication

Description of an Experiment

1. Is it possible to equal 24 by repeating the sequence "+ 6 = ?"[3] (See Fig. 2)
2. Is it possible to equal 24 by repeating the sequence "+ 7 = ?" (See Fig. 3)

These two tasks are examples from a long-term experiment with six to seven years-old students. The experiment lasted over the time span of about four months and was held once a week. The task was to identify all possible integers n that equal

[3]Starting from zero, that is after a reset of memory.

Fig. 2 Repeating the sequence " +6 =? "

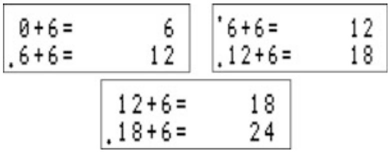
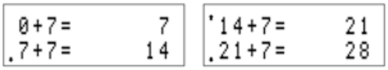


Fig. 3 Repeating the sequence " +7 = ? "



Target	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
18	1	2	3			6			9									18							
19	1																		19						
20	1	2		4	5					10										20					
21	1		3				7														21				
22	1	2									11											22			
23	1																						23		
24	1	2	3	4		6		8				12												24	
25	1				5																				25

Fig. 4 Correct results for the targets between 18 and 25

24 in the sequence " +n = ? ". The calculation result was then changed to another value. It was possible to choose between individual work or group work. This phase was followed by a public collection and discussion of results on the blackboard. A final table is presented in Fig. 4. During the discussion phase, wrong propositions were given by students which were then peer-reviewed by the other students and under teacher's management.

After the final agreement on the correctness of the table, the students were asked to express their findings, for example all the '1' in the first column, the alternate occurrences of '2', lines with only two numbers called 'poor' targets.

In the next session, another set of targets was proposed.

This kind of work, described in Del Notaro and Floris (2011) enriched the classroom study with various arithmetic properties such as parity, multiples and divisors, and primes. The use of the calculator played an important role in order to check properties and to discard wrong ideas. The following feedback instructions were summarized on the blackboard:

1. Make sure that for odd targets there are only odd divisors, and that there are even and odd divisors for even targets.
2. The 'poor' numbers are primes (a prime number is odd, except for "2" – but not all odd numbers are prime).

The 90-minutes working sessions were held weekly. After three months, the teacher introduced the idea of writing also the number of repetitions to reach the target together with the chosen number, for example adding 6 (time ‘2’) to get 12 (Fig. 5). This provided a new possibility to control one’s results: if there is a pair (a, b), there must be a pair (b, a).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
11	1 11										11 1														
12	1 12	2 6	3 4	4 3		6 2						12 1													
13	1												13												

Fig. 5 Writing the number of key iterations to reach the target

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
11	1 11										11 1														
12	1 12	2 6	3 4	4 3		6 2						12 1													
13	1 13												13 1												
14	1 14	2 7					7 2							14 1											
15	1 15		3 5		5 3										15 1										
16	1 16	2 8		4 4				8 2								16 1									
17	1 17																17 1								
18	1 18	2 9	3 6			6 3			9 2									18 1							
19	1 19																		19 1						
20	1 20	2 10		4 5	5 4					10 2										20 1					
21	1 20		3 7				7 3														21 1				
22	1	2									11											22			

Fig. 6 A table for multiplication control

By studying the above table (Fig. 6), the following new rules were identified:

1. Check with your calculator: multiplying the two numbers (in a column) results in the target.
2. Check that there is the reverse correspondent of each multiplication (commutative rule).
3. When there is only one pair and its reverse, the number is prime.
4. A target with a pair type (a, a) is a square and a is its square root.

The usual phenomena of a frantic search (called 'fishing' by Artigue 1997; Meissner 2005) was observed, mostly instead of the use of already institutionalized knowledge (such as provided in the list of control rules). But at mid-term, cognitive changes could be observed in the students' actions, especially on their first approach to multiplication.

This activity provides powerful adidactical feedback. The target can be reached and mathematical properties can be verified with the calculator and the judicious use of different tables. As a result, the properties of LM1 and LM2 are completely fulfilled. The learning milieu is in the sense of Brousseau (1997) because the teacher's input triggered the students to pose various questions which were answered by the milieu itself.

Detailed Analysis of an Example of Simplification of Fractions

The Type of Tasks

We focus on simplifying procedures for great fractions in case that a calculator is available. For numerators and denominators beyond 100, students of lower secondary school are generally not able to make use of a stored repertoire of mathematical results to find a common divisor of numerator and denominator; therefore, they are urged to apply other procedures to perform this type of task. In a former research (Weiss and Floris 2008), a series of simplifying fractions have been proposed for different types of students – at the age of about 15 years – with permission to use the calculator (Fig. 7).

It often occurred that students considered a fraction like 187/340 irreducible because they did not consider common divisors beyond ten. These students are

Fig. 7 Which fractions are irreducible?

a) $\frac{2500}{7500} =$	b) $\frac{72}{108} =$
c) $\frac{241}{150} =$	d) $\frac{176}{165} =$
e) $\frac{256}{243} =$	f) $\frac{749}{7000} =$
g) $\frac{187}{340} =$	h) $\frac{110}{264} =$

Fig. 8 A theoretical transparent simplification

$$\frac{637}{1183}=\frac{7\cdot 7\cdot 13}{7\cdot 13\cdot 13}=\frac{7}{7}\cdot\frac{7}{13}\cdot\frac{13}{13}=1\cdot\frac{7}{13}\cdot 1=\frac{7}{13}$$

subject to the basic didactic contract in which the teacher proposes fractions simplified by 2, 3, 5, 7, or 10. The tasks which were proposed in this teaching experiment – a bit beyond this contract – aimed to extend the mathematical knowledge of students. In this specific case, it aimed for the awareness of mathematical procedures to make any fraction irreducible; hence, the decomposition of the numerators and denominators into prime factors, which is deeper founded into a theoretical context[4] than the use of GCD (Fig. 8).

The mathematics curriculum of the French part of Switzerland includes the study of divisibility of integers and of their decomposition in a product of primes. But, these mathematical procedures remain rather isolated and algorithmic and are not linked to other parts of the curriculum (Floris 2013).

Analysis: Milieu, Praxeology, Instrumentation

We claim that the set of tasks in Fig. 7 promotes an experimental learning milieu for the simplification of any numerical fraction. Following Brousseau (1997), there is here a fundamental aspect, an essential basic knowledge, corresponding to the prime factorization of integers. By proposing these tasks, the teacher introduces the students to these techniques as well as to the advantages of their use. The feedback (LM1), however, is not entirely adidactical. The teacher is required to assist students, for example by suggesting to look for other common divisors in case that they stop with ten. On the basis of the students' first attempts to solve those tasks, the teacher can then present the calculation of Fig. 8 and ask the students to revise the tasks in the same way.

To set the task according to the LM2 condition, we first need to analyze this calculation on the basis of Lagrange's (2000) three components of praxeology. First of all, the task aims to make any fraction irreducible. As a first step in Fig. 8, the subtask is to decompose numerator and denominator, the technique being the algorithm of successive divisions by all prime factors taken in increasing order. The underlying theory is the theorem that the decomposition exists and that it is unique. At this school level, this theory is generally not made explicit, and in this specific case it is replaced by the use of the algorithm (because it always works). As a second step, the subtask is to obtain a product of fractions by using the definition of this product (technique) which is justified by a definition of a fraction (theory). As a third step in Fig. 8, the subtask is to replace a/a fractions with "1", using the corresponding property (technique), which is also justified by a definition of

[4]See detailed praxeological analysis below.

fraction. Finally, the neutrality of the number one is used. A more refined analysis – which integrates properties of the integer ring as associativity and commutativity of multiplication – could be made.

This analysis shows the importance of the teacher's involvement into the students' working process, e.g. with respect to the choice of hints and degree of institutionalization (i.e. theoretical statements). These have to be observed to evaluate the theoretical level of the mathematical procedures. Furthermore, the entire curriculum on fractions is called into question: how and when are fractions and their operations defined? In some school programs in Switzerland, the multiplication of fractions is taught after simplification, whereas fractions themselves are defined in a rather intuitive way (e.g. as parts of pizzas, etc.) never followed by a rigorous definition.[5]

Inspired by this work, a pre service teacher proposed a similar set of fractions to his students; and, after showing them how to factorize by decompositions, he asked them to directly get the decomposition with the application of a symbolic calculator.[6] Further, the students were asked to check their results on this task as well as on other 'complicated' fractions. He also introduced them to Geogebra and Aplusix[7] as a means of generating decompositions. Finally, he presented a mathematical technique to obtain simplification using the symbolic features of the calculator (MATHPRINT mode, see Fig. 11 below). Here again, a contract disruption helps to promote an experimental milieu for learning as well as enriches the theoretical part of the praxeology corresponding to simplification of fractions – that is, in particular, the linkage with the decompositions of integers and the awareness towards the existence and unicity of any complete simplification of a fraction.

Activities on Non Decimal Numbers and the Limits of a Calculator

Scientific calculators intended for scholarly mathematics treat fractions and square roots in a problematic way which is why quite a lot of primary teachers – or even graduates of mathematics – think that they have certain knowledge whether a decimal development is infinite or not.[8] What we present in the following section are activities aimed to analyze these peculiarities. Further, the analysis aims to give

[5]Our favourite one being 'a / b is a real solution of equation b x = a with a, b integers and b different from zero, positive real numbers being defined as lengths'.

[6]A TI-92 in this case.

[7]There is a CAS part in Geogebra (Geogebra.org); Aplusix is a useful program allowing direct control of numerical equalities and algebraic equivalences (Aplusix.com). Other tools can be easily found on the web, e.g. www.calculatorsoup.com/calculators/math/prime-factors.php.

[8]They also are of the opinion that transcendent functions are programmed according to their Taylor series. Most of them ignore the CORDIC algorithms (https://en.wikipedia.org/wiki/CORDIC).

a detailed insight on the workings of scientific calculators in order to understand them and to use them successfully. During the analysis, it was rather difficult to separate this manipulative learning from the mathematics one, so the reader will have to understand it while working out the activities. The first activity is intended for twelve to thirteen years-old students. The second activity is designed for older students when studying square roots.

Tasks on the Decimals of 3/7

1. Transform 3/7 into a decimal notation with the use of your calculator. Enter the result into the third row of your calculator (here TI-30XSMultiView) and multiply this result by 7 (Fig. 9). What can you conclude?
2. Transform 3/7 into a decimal notation with the use of your calculator. Then, immediately multiply the result by 7 (press * and then 7, see Fig. 10). Explain the difference to what had happened before.
3. Is the decimal notation of 3/7 periodic? If yes, determine the period. Can it be done using the calculator?

Comments

The activity described above highlights how the calculator manages approximations. As for the long multiplications proposed above in this chapter (see second example), it suggests a negotiation between the answers of the calculator and those that can be obtained by using the usual paper-and-pencil algorithms. It leads to an increased knowledge on the functioning of the calculator in case of hidden decimals:

> TI30XS Multiview™ uses internally 13 digits for calculations and it displays 10 in the results. If the first hidden digit is 5, 6, 7, 8 or 9, the digit displayed as the right of the screen will be increased by 1, it is the rounding rule. (Calculator guide book)

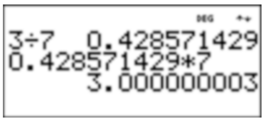


Fig. 9 Checking the decimal result of 3:7

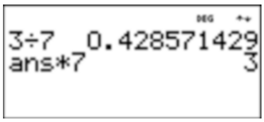


Fig. 10 Multiplying directly by 7 the decimal calculator result of 3:7

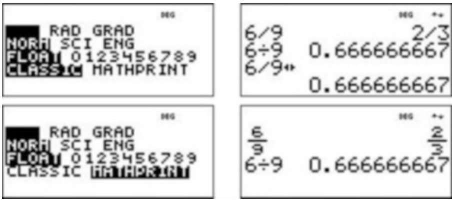


Fig. 11 Different treatments of fractions and divisions in CLASSIC (above) and MATHPRINT mode

Another feature of the TI30XS Multiview™ is the possibility to partially work in a non decimal world, called the MATHPRINT mode (Fig. 11). These features are generally ignored by teachers and students at this level (lower secondary schools) but could be presented after the completion of the previous activity on 3/7. It was often the case that pre service teachers proposed these tasks in their classroom. They further introduced the MODE menu and its different features. The discussion about these tasks among the teachers was quite interesting (Floris 2015). Some of them (mainly the teachers of lower secondary level) said that they were reluctant to propose activities like the previous one, or the comparison between the calculator results of $10^{20}+1-10^{20}$ and $10^{20}-10^{20}+1$, to their students. Their reluctancy relates to a loss of confidence in using the calculator. Others emphasized that the limits or errors of the calculators did not happen erratically but in precise cases. Moreover, they claimed that working on these examples has to go alongside with an understanding and teaching of the concerned features. In the following section, we will present a similar activity for fifteen to sixteen years-old students.

Analysis

For this activity, the focus of the analysis is on learning how the calculator processes numbers with more than ten decimals and study periodical decimal expressions. Additionally, the framing of decimal numbers with similar questions as in the activity below will be considered. At this level, students simply consider two decimals in their calculations when solving problems, and these decimals are not always correctly rounded. It is considered to be a part of the didactic contract and this is why teachers generally accept it. From the point of view of the milieu, the activity seems to implement an uncertainty, but many students do not note this and accept the situation without stepping back. At this stage of their scholarly learning process, students already developed the habit to use the calculator only for calculations, but what is required here is a thoughtful, reflexive approach. This phenomenon is described as instrumentalisation by Rabardel (1995). Furthermore, the mathematical treatment requires a long division that corresponds to a didactic contract disruption. Thus, the relation between calculator tasks and paper-and-pencil tasks is quite straightforward: They are either combined for a (complex) calculation, or for checking one. This could explain the unwillingness of some teachers to address this with their students. From the perspective of the praxeological approach, it is interesting to note that the technique corresponding to the treatment of the third question is long division, and that the theory is a property of this operation, i.e. the recurrence of rests.

This analysis shows the high efforts of teachers to construct enriching learning milieus with the help of these tasks. They have to consider it as a basis for a sequence of lessons on the properties of decimal numbers, framing, periodicity and long division. In this way, the adidactical feedback (LM1) can be constructed by the combination of mathematical properties already familiar to the students and the use of the calculator. Such a medium term study could further entail a search for all decimal figures of 1:31, or 1:29 by again combining the application of calculators and paper-and-pencil tasks. This would improve the students' handling skills of such mathematical tools in agreement to the instrumentation process of Rabardel.

Activity Around the Square Root of 8

1. Is $\sqrt{8}$ equal to 2? Justify.
2. Is $\sqrt{8}$ equal to 3? Justify.
3. Is $\sqrt{8}$ equal to 2,5? Justify.
4. Find two numbers with three decimal digits which frame $\sqrt{8}$.
5. Is $\sqrt{8}$ equal to 2,828427125?
6. Do the following task with the calculator in CLASSIC mode:
   Calculate the square root of 8. Then, put the result directly to the square.
   Compare your result with the result of task 5. What can you conclude?
7. Find the best possible framing of $\sqrt{8}$ using the calculator.
8. What is the decimal value of $\sqrt{8}$?
9. With the calculator in CLASSIC mode, calculate $\sqrt{8}+\sqrt{7},$ and then $\frac{1}{\sqrt{8}-\sqrt{7}}$. Are the results reliable?
   What conjecture can we draw from these results?
   Can we prove this conjecture?
   In MATHPRINT mode, does the calculator confirm the conjecture?
   Explain the results given in MATHPRINT mode (Fig. 12).
10. Identify a generalization of the conjecture established in point 9.

This activity can be analyzed identically to the previous one. It was proposed to high school students, and while they were working on the task we could observe difficulties linked to the didactical contract. Probably, these were due to the sparse experiences with calculators of the concerned students. Nevertheless, the activity was chosen by many teachers who were highly interested to improve their students' instrumentation of calculators because it is a part of the calculus chapters of the

Fig. 12 Calculator outputs for the activity (question 9)

high school curriculum. With the aim of efficiency, they mainly proposed the above-mentioned version of the set of tasks.

Experimental Milieu with Calculator for Early Algebra[9]

In this section, we aim to present the advantages of a compulsory school calculator like TI30XII concerning the study of algebra. In some countries, there has been a recent shift in the curricula towards new ideas for the introduction of algebra. In the 1960s, the New Math reform proposed a structured approach to algebra which was based on the properties of the sets of numbers (i.e. integers and rational numbers). The letter calculation rules were then worked on in isolation. However, the current proposal, called ‘early algebra’, endorses a dialectic between the numerical and algebraic conceptual domains. The literal calculations are considered both a production tool of number sequences and a description tool of numerical properties (e.g. for any integer n, the expressions 2n and 2n+1 equal sequences of even numbers, respectively odd, and thus describe the parity). Furthermore, it is possible to express algebraic properties in the numerical world; for example, $25=5^2$ expresses the fact that 25 is a square, or $333 = 3 \times 111$ expresses that 333 is a multiple of three (i.e. divisible by three):

> One of the major goals of early algebra is generalizing number and set ideas. It moves from particular numbers to patterns in numbers. This includes generalizing arithmetic operations as functions, as well as engaging children in noticing and beginning to formalize properties of numbers and operations such as the commutative property, identities, and inverses. (Wikipedia ‘early algebra’ in 2017)

But even in case that the theoretical aspect is attached less weight in many classrooms, the formal approach towards the algebraic domain has changed insignificantly – apart from the integration of a few motivational activities at the beginning of schoolbook chapters (these are mostly based on the formulas for areas and perimeters). However, these are not connected to the main domain that is being addressed in such a chapter. The study of computational techniques still predominates; this provokes the impression of some students to see algebra as a series of rules or laws which are devoid of meaning and poorly articulated within the numerical frame (Pilet 2012).

[9] See http://ase.tufts.edu/education/earlyalgebra/.

Calculation Programs

Within this new perspective on algebra, the notion of calculation programs is a key element. The name 'calculation program' – rather than formula – was chosen to emphasize the dialectic between the numerical and algebraic conceptual domains. This idea is at the heart of the so-called 'square-edged' activity, where the aim is to establish a method to identify the number of small coloured tiles on the edge regardless of the size of the square (Fig. 13). A detailed analysis of this activity can be found in Eduscol (2008).

This activity can be integrated into the study of literal calculations at various points, whenever it is most suitable with respect to the learning process. For example, in the Swiss textbooks of the year 2000, it was proposed at the beginning of the section on Algebra. In the year 2010, however, the activity was integrated at the end of the section as an 'application' (which is further specified in the teacher's comments). After the analysis of the results of a profound diagnostic test, Pilet (2012) proposes such a task for the reworking of the meaning of algebraic manipulations. In addition to that, we propose to integrate this 'square-edged' activity in an early algebraic setting to create awareness for literal computing at the beginning of secondary school because it further provides precise numeric challenges. This is because the activity asks students to predict the number of small coloured tiles for a square of sides of 6, 11, 37, 88, or 2012 tiles. It showed that higher values led students to abandon calculation procedures based on counting. At this level, the goal is not necessarily to introduce letters. For a square of 37 tiles per side, such a procedure may limit entries to obtain solely calculations like: 4 × 37–4 or 37+37+35+35 or 37+36+36+35 or 36+36+36+36. For numeric values exceeding ten, the use of the calculator can be accepted. A possible approach for teachers could be to ask students to identify different calculations and then explain why the results are equal. We would expect explanations such as 37+37+35+35=37+36+36+35 or 37+37+37+37–4=36+36+36+36 which is equal to (37–1)+(37–1)+(37–1)+(37–1). The calculator is used to validate the calculation programs and their equivalence.

With the variation of tasks, these records may achieve a calculation program status, that is a 'model' or 'pattern' of calculation which associates each calculation with a diagram like the following (Fig. 14).

We observed the working on this activity in a class of twelve to thirteen years-old students who are said to have difficulties in mathematics. They worked in

Fig. 13 'Square-edged' activity

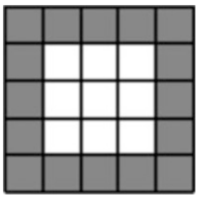
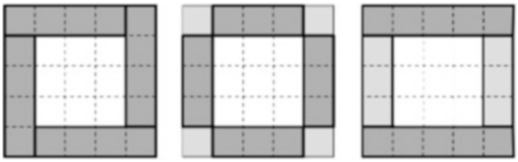


Fig. 14 Different patterns of calculations for the number of tiles in the edge



groups of three to four students. It showed that for some of them, the calculator was a helpful tool in regard to discussions because it assisted them to work out similarities to the calculation programs. The writings of the groups exemplify the successful dialectic of numerical and algebraic conceptual domains (see Fig. 15).

At this point, Eduscol (2008) proposes that teachers introduce a literal symbol:

> The production of a formula appears as an answer to the question of the general description of a situation involving specific numerical values and the use of letters solves the problem of the appointment of the variables involved in the situation.

However, this is not a mandatory recommendation at this point of the learning process, i.e. after the introduction of the topic. Our personal observations showed that an immediate introduction can overextend students; thus, the idea to integrate letters into calculations or equations is solely induced by the teacher without the students having an actual need for it.[10]

An alternative could be that the teacher claims that the numerical expressions are equal and proposes to study this type of scriptures further; for example, this could be achieved by tasks on the properties of sums of consecutive numbers – with the focus on numbers. For example, the sum of three consecutive numbers is equal to the triplication of the middle number because $88 + 89 + 90 = (89–1) + 89 + (89 + 1) = 89 + 89 + 89–1 + 1 = 3 \times 89$. The aim of these activities is to establish an early algebraic perspective on numerical expressions. This is linked to the idea that algebra is a kind of modelling of the numbers world within a numeric-algebraic dialectic whose lack or weakness is related to the difficulties of many students (Pilet 2012). The calculators which are used in secondary schools nowadays, which are types of calculators with two or more displayed lines, is of great assistance for this kind of calculating (see Fig. 16). It follows that these feedbacks and linkages help to fulfill properties LM1 and LM2.

Additionally, the calculator can be useful when working on symbolic calculations; for example, by pressing the key TABLE one gains access to the feature to create a sort of spreadsheet that introduces formula[11] (Fig. 17).

[10]There is a way to enhance this 'symbol gap'. Following Brousseau's formulation phase (1997), the teacher may propose a contest between groups: he will choose one student from each group, and then give a value for the number of tiles of a side. The student that gives the quicker answer gets a point for her group. The groups have the right and the time to prepare a method. In the subsequent validation phase the contest is about the best methods.

[11]The TI30X Multiview, given to all students in the schools of Geneva.

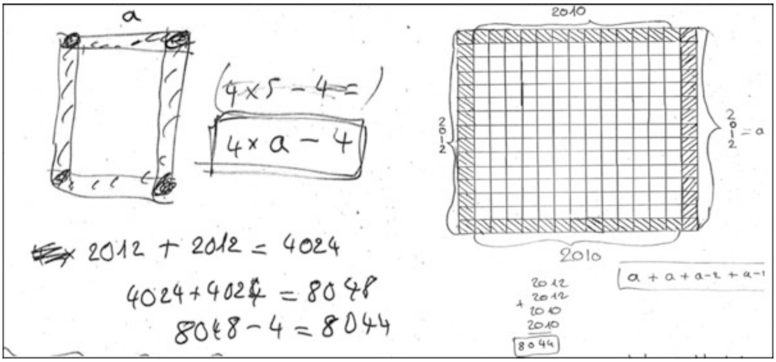
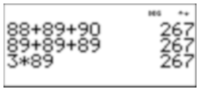


Fig. 15 Early algebra: dialectic between numerical and algebraic conceptual domains

Fig. 16 Multiple lines display in nowadays school calculators



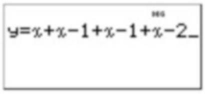
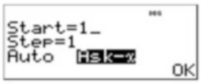
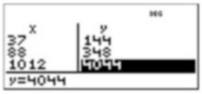
First, TABLE key displays "y=" on the screen, and it is possible to enter a formula using the variable key "x"	
Table initialisation: Auto mode displays a sequence (depending on the values of "Start" and "Step"). "Ask" requires from the user to enter a value.	
Here, "Ask" mode is chosen, and the entered values are 37, 88 and 1012. Y values are generated in correspondence to the programmed formula.	

Fig. 17 How to use TABLE key (with the example of the study of the sum of three consecutive integers)

From Calculation Programs to the Modelling of Arithmetic Properties

The dialectic of numerical and algebraic conceptual domains can be realized in connection to the TABLE key of the calculator, for example by asking students to produce lists of even, odd, or multiples of a given number (Fig. 18). Modelling with formulas can lead to a more logic-based type of thinking on properties such as the thesis that the sum of even numbers is an even number, or that the sum of an even number and an odd number is an odd number, etc.

What can further be studied are the properties of sums of consecutive numbers whether using letters or not. It is interesting to compare this aforementioned work

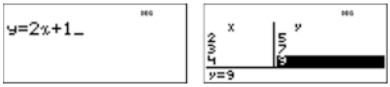


Fig. 18 Programming a sequence of odd numbers using TABLE key

FA107 Après, avant

Soit un nombre entier *n*.

a) Comment écrire le nombre entier qui le suit immédiatement ?
b) Comment écrire le nombre entier qui le précède immédiatement ?
c) Comment écrire le cinquième de *n* ?
d) Comment écrire le carré de *n* ?

Fig. 19 Classical work on algebraic scriptures (Let n be an integer (**a**) How to write (express) the immediately following number? (**b**) How to write (express) the immediately preceding number? (**c**) How to write (express) the fifth of n? (**c**) How to write (express) the square of n?)

with the following example, which is proposed in a current textbook of secondary school (Fig. 19 from CIIP 2012, p. 99).

In such exercises, the dialectic with numbers is not explicitly integrated. The decision to combine the results of this activity with the use of the calculator, constructing paper-and-pencil numerical tables, or integrating the use of the TABLE key function, is the teacher's responsibility.

Another operation that is provided by TABLE is the comparison of calculation programs. For example, in regard to the 'square edged' activity, we can introduce the different formulas that are obtained within the activity and then observe whether the values are the same. This motivates the study of literal transformations like their justification based on properties such as distributivity or commutativity. An analysis of this has shown that this new point of view offers an enhancing relationship between algebraic techniques and the properties founding them.

As TABLE allows the integration of only one formula at a time, this necessarily involves paper-and-pencil transcriptions as well as working in groups of two, three or four students who are assigned to program one formula each. One could argue that it would be better to use a spreadsheet, but the formulas of such a software are not written as polynomials in x which is why they require a technical introduction. Furthermore, it would entail to work in the school's computer room, unless your classroom provides tablets or laptops.

From Calculation Programs to Equations

Problems like "Which number did I think of?", allow to proceed to the notion of an equation:

I think of a number, add its double, divide the result by 3, and add 75. I come up with 80! Which number did I think of? Why?

One possibility to resolve this problem is to approach it arithmetically by starting from the end and reverse the operations. This allows students to grasp the type of problem and then move on to a different one which suggests the use of two different calculation programs at the same time. In such a case, the arithmetical approach would be awkward:

I think of a number. I multiply it by 3. I add 10. I get the original number sevenfold, plus 30. Which number did I think of? Why?

This type of problem allows a wide variation of different statements and promotes working on the numerical properties of the 'facts'. The task set will also comprise 'math-magic' tricks:

Think of a number, add 2000, divide the result by 20, subtract 100, and multiply all by 20. You end up with the number you thought of at the beginning! How do you explain it?

Or

Think of a number between 1 and 9. Double it. Add 2 to the result. Multiply the new result again by 5, add 12, multiply the new result by 10, subtract 220.

Compare the number you get started with the number you had thought! Can you explain?

These statements proposing sequence operations allow easy translation into calculation programs.

An interesting presentation for a same kind of problem is the following activity, "The Lost Number":

I type the following sequence into my calculator:

6	×	?	-	3	-	2	×	?	+	7	Enter

Provided that the two grey boxes mask the same number and the calculator gives 24 as result, can you identify this number?

Can you identify the number in case that the result is 592, 1.2, 69.2, -163.6, or 88?

In case that students aim to solve the task by random trials with the calculator – with or without the TABLE key – these random trials become rather time consuming as soon as the given result is something else than an integer. In fact, for many teachers, this activity aims to motivate the use of equations and is supposed to disqualify the use of the calculator. This intent, however, stems from a teaching position that does not consider the numerical-algebraic dialectic.

Equations, Equalities, Calculation Program DATA Key and Spreadsheet

With problems such as "Which number did I think of?", or "The Lost Number", the notion of equation can be introduced by further maintaining a numerical-algebraic dialectic. The question remains how exactly the calculator can be of use here. We already observed that the TABLE key only allows the display of a single column. On the TI34X Multiview, the DATA key provides a small spreadsheet (see Fig. 20). However, the use of this key is not self-explanatory and therefore requires instructions. But this effort is advisable to take in case that the DATA key functions will be further integrated into the classroom teaching in other contexts. These are, for example, numerical equations resolutions, proportionality, and programming a formula (functions):

Comments About the Milieu for Algebra

In agreement with our research question and with the conditions LM1 and LM2, it clarified in the course of our study that a milieu has to entail a variety of activities that link numbers and letters on the basis of arithmetical properties; this is a long-term project. We already presented selected options of how the calculator could be of help, but a large scope research to assess how a learning milieu could be set up for the student (LM1) is still pending. What needs to be constantly considered is whether the numerical expressions are truly providing the correct feedback.

The proposed sets of tasks link naturally with theory, hence the condition LM2 is satisfied.

Didactic Building of a Milieu with Calculator: An Example

Is the Calculator a Milieu?

All tasks or activities that are presented here share that they require the use of the calculator as a mathematical learning tool. From this given, the question arises whether the calculator itself is a learning milieu. According to Brousseau (1997), a learning milieu consists of various elements that will help teaching, in particular, the results of actions of the student such as calculations, drawings, or manipulations.

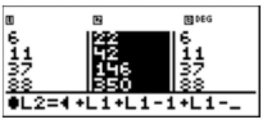


Fig. 20 Programming formulas using DATA key

Fig. 21 A didactic inquiry for different scriptures of numbers

999+1= 1000	999999+1= 1000000
999999999+1 =1000000000	9999999999 +1= 1×10^10
1000+1=1001	1000000+1= 1000001
100000000+1 = 100000001	'1000000000 +1= 1×10^10

This learning milieu further entails the elaboration on the connection between the tasks proposed by the teacher and what is actually achieved by the students. The results that are generated with the calculator can be considered a part of that milieu. The calculator itself, however, is not a milieu just as paper-and-pencil calculations are not a milieu either. They require a linkage with to the other properties of a milieu, and the use of the calculator further requires an official classroom status. The TI-30 calculator, for example, provides the answer 1×10^{10} to the operation $9999999999+1$. But without a prior introduction by the teacher, this result is not a part of the learning milieu. Nevertheless, the teacher should be able to provide a suitable answer to the meaning of this result in case that a student, for example of upper primary school, is interested in that; it is a basic part of the didactic contract. A didactic work that would take such an interest into account cannot be straightforward. It would go back to the number of digits of an integer, leading to small-range working theorems, such as "performing an addition, the number of digits does not increase, or it increases by 1". It would also include investigations of the number of digits displayed by the calculator. Figure 21 provides a sketch of one possible way to create a learning milieu.

Multiplication proves the most interesting operation for such an investigation. One could pursue answers to the question of how the number of digits of the product relate to the number of the digits of the factors.

A study this type, however, would only be of anecdotic interest. It could only be meaningful in the context of a medium length teaching process which includes technical work with mathematical properties. This is how the study on arithmetic properties with primary school pupils presented above was structured. In the example in Fig. 21, the theory corresponds to the positional writing of numbers in base ten and all mathematical properties on which it is based, particularly those of the ring structure. The reformation of the curricula in the 1970s has clearly shown that it is a long process to change the workings of a (mathematical) institution. This interjection does not include that we advocate the return of calculations in different bases, or the introduction of the study of the rings.

Considering the notion of the learning milieu, we observed that it cannot operate sustainably without the presence of praxeologies (Chevallard 1999) formed by tasks, techniques, a vocabulary describing actions, and properties that relativize the results of these actions (e.g. “by adding the same even number several times, you always get an even number as a result”). This is the logos or theoretical part of praxeologies. We highly believe in the value of material results such as physical objects, traces on a blackboard, on paper, or on the screen of a calculator. The physical part of the milieu is essential to recall the actions performed and to help the development of conjectures (i.e. a table, a list on the blackboard or calculator displays). The study on the process of the “course à vingt” (Brousseau 1997) highlights this role of the milieu. In this respect, the use of calculators which display the mathematical transactions is very important, as well as the opportunity to present the results to the whole class using an emulator and a data projector. Kieran and Guzman (2007) highlighted this in their calculator-based experiment for lower secondary school.

Result

The main research question was how the working with calculators in the classroom can become a learning milieu.

Therefore, we presented a survey of selected qualitative studies as well as examples on the use of calculators at different learning levels. In the theoretical part at the beginning, we specified the conditions LM1 (feedback of the milieu) and LM2 (links with theory) as a basis for a learning milieu. A high variety of examples illustrated how these conditions could be totally or partially fulfilled. In the ‘target’ examples, the necessary requirements for a learning milieu are fulfilled in a complete manner and the calculator is an essential tool. The theoretical output is impressing at this school level: the students handled properties of divisors, prime numbers, and square roots. In other examples, the feedback of the milieu showed to be more problematic which was mostly due to interferences of the didactic contract and flaws in the teacher’s management. In these cases, what needs to be prepared is an accurate didactic engineering in order to propose challenging and theoreticaly rich tasks. Such tasks are provided in the fractions and decimals examples.

From a methodological point of view, it was experienced how the properties LM1 and LM2 could be effective as means of analysing the learning potentialities of calculator activities.

Conclusion

Students’ experiences set out the basis to create learning milieus for mathematics. These are the reality in which they anticipate their actions and act. Technology, even a pocket calculator, complicates the learning situation by adding specific feedback which can be valid and therefore useful, but sometimes also surprising.

The tool "calculator" cannot be successfully integrated in an instant. It requires long-term planning which needs to be integrated into the curricula. In Switzerland, this is accomplished in the new "Plan d'études romand" (CIIP 2010), but in a rather minimalist manner and without links to specific mathematic subjects. However, the present contribution showed that calculator activities could improve the study of arithmetic properties in a significant way such as fraction operations, square roots, approximation, and algebra. Due to the status quo, the integration of the calculator is in the sole responsibility of the teacher. In our pre-service institution in Geneva, they are prepared for this, but, as we have demonstrated, a long-term institutional strategy is still necessary to transform individual efforts to an effective instrumentation of calculators for the students.

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The Street Lamp Problem: Technologies and Meaningful Situations in Class

Elisa Gentile and Monica Mattei

Abstract The chapter describes a problem solving activity posed with the use of a Dynamic Geometry Software to middle school students. The problem leads students to face a meaningful situation to be explored, and forces them to make conjectures, to discuss and to formulate an argument. The activity starts with the manipulation of materials (paper and pencil, pictures and flashlights) and continues with the transposition of this exploration through technology. We discuss the use of problem solving activities to improve the argumentation skills and the added value of technology in exploration activities.

Keywords Problem solving • Geometry • Discussion • Meta-Didactical Transposition

Introduction

The activity in this chapter belongs to an international research project entitled "Problem Solving with GeoGebra", which involved two different countries, Australia and Italy, with the aim of engaging in-service secondary school teachers in professional development based on best practices in mathematics. This research project is connected to a national project, named PLS (Piano nazionale Lauree Scientifiche – National Programme for Scientific Degrees), born in 2004 from the collaboration among the Italian Ministry of Education, the National Conference of Headmasters of Science and Technology University Faculties and Confindustria[1] with two aims: to increase the number of students enrolled in Scientific Departments and to improve the professional development of teachers, promoting collaborations between school teachers and university teachers.

[1]Confindustria is the main association representing manufacturing and service industries in Italy.

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The project was mainly focused on teachers and their professional development, during and after a short course led by researchers and teacher-researchers. The course was addressed to in-service teachers that voluntary choose to attend an 18-h professional development workshop for teachers that took place in several afternoons during the school year. The project involved two communities: the community of researchers, who designed the tasks and the educational programme, and the community of teachers who attended the course. The teachers were also asked to experiment with the activity in their classes, and to reflect on what transpired throughout the activity with the other teachers and the researchers. The teachers were observed during both the course meetings and during the didactical experimentation in the teachers' classrooms; the resulting data were analysed using techniques of "Meta – Didactical Transposition" (Aldon et al. 2013; Arzarello et al. 2012, 2014).

The teaching experiment performed is an adaptation to a middle school context of an open-ended problem, "The street lamp problem". The street lamp problem has been studied previously by the team of researchers in Turin, originally addressed to higher secondary school students (14–19 years old) in order to involve them in a problem-solving activity, activating their argumentation skills. Since this research focused on lower secondary school (11–13 year old students), we needed to adapt the problem to this context. In particular, we paid attention to maintaining the "openness" of the problem and the idea of problem solving, but we inserted additional questions to slightly guide the students (and the teachers) to better understand the problem.

In this chapter we analyse both the students' side, reporting what happened in class, and the teachers' side, focusing on the development of their professionalism.

Overview of Research in Mathematics Education with Technologies

The CIEAEM Manifesto (2000) reflected about the changing role and the importance of technology related to mathematical education. One of the key questions was:

> How can the development and spread of new information technologies really give better access to mathematical knowledge for all? (CIEAEM 2000, p. 7).

The importance of technology in mathematical education was then underlined by the National Council of Teachers of Mathematics in its two positions, proposed in 2008 and 2011 (NCTM 2011). In the most recent one we can read:

> It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication. Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students. (NCTM 2011, p. 1).

A very important study about technology and mathematics education was the first ICMI Study in 1985 (Churchhouse et al. 1986). After that many frameworks followed, emphasizing different aspects of the integration between technology and didactical practice. For example, the CIEAEM Manifesto (2000) considers modern technology as a tool to support, facilitate, organise and rationalise learning and teaching.

In the position about technology, NCTM (2011) highlights that numerous studies, even more recently, have shown that a mindful use of technologies in class can support both advanced mathematical thinking (problem solving, reasoning, arguing, justifying and even proving) and the acquisition of mathematical procedures. Furthermore, technological tools used with didactical intent complement mathematical teaching-learning, and prepare students for their future lives in which technology will play a crucial role.

The simple availability of technology is not sufficient for effective teaching-learning process (NCTM 2011); both the teacher and the curriculum can change the nature of the pedagogical action, mediating the use of technological tools. According to these points, the focus of the research is now about the role of the teacher in constructing effective teaching-learning environments using technology (Artigue et al. 2009; Clark-Wilson et al. 2014; Drijvers et al. 2010). Therefore, it is important to involve teachers in a professional development programme based not only on the technology itself but also on didactical methodologies, best practices, task design and so on (Drijvers et al. 2010). NCTM (2011) pointed out this key concept in this except:

> Programs in teacher education and professional development must continually update practitioners' knowledge of technology and its application to support learning. This work with practitioners should include the development of mathematics lessons that take advantage of technology-rich environments and the integration of digital tools in daily instruction, instilling an appreciation for the power of technology and its potential impact on students' understanding and use of mathematics. (pp. 1–2)

Teaching and Learning with Tools: DGS as an Example

In the last years a great number of studies concerning learning with tools (not only technological ones) have been carried out, especially in the Italian reality. A very important document by UMI (Union of Italian Mathematicians) was produced during years 2000 through 2003 (see UMI 2001, 2003), collecting key ideas for curriculum improvement. Some of these ideas were included in the official document (Guidelines) of the Italian Ministry of Education during its last review of the National Curriculum (in 2012 for the first cycle of education and in 2010 for the second one). The UMI documents pointed out that "basic"[2] materials could be used

[2]We are using the word "basic" without a negative meaning but, on the contrary, with the meaning of simple and easy to find in every house or classroom. Nevertheless, the Italian word used for defining these materials (UMI 2003) can be translated with the word "poor".

as a meaningful starting point not only in primary schools but also at other levels of education. The integration of these materials with technological tools can enhance the teaching-learning process.

We can locate Dynamic Geometry Software (DGS), micro-worlds designed for specific educational tasks, in the theoretical and political context described above. DGS allows students to explore, investigate and observe; to look for invariants, regularities or patterns; and to formulate conjectures and test them within the software. Knowledge is embodied in this software in ways that facilitate students facing it directly, constructing mathematical meanings and objects in the process of using the software (Bartolini Bussi et al. 2004). Marrades and Gutierrez (2000) underlined this as a non-traditional learning environment:

> The contribution of DGS is two-fold. First, it provides an environment in which students can experiment freely. They can easily check their intuition and conjectures in the process of looking for patterns, general properties, etc. Second, DGS provides non-traditional ways for students to learn and understand mathematical concept and methods. (p. 8)

Many research studies have been carried out regarding the role of DGS in proving mathematical theorems (Arzarello et al. 1999; Marrades and Gutierrez 2000; Paola and Robutti 2001; Sinclair and Robutti 2013). The contribution of DGS in constructing knowledge and in promoting justifying competencies is widely recognised among the community of researchers. About this topic, Marrades and Gutierrez (2000) stated:

> DGS environment may help students use different types of justification, setting the basis for them to move from the use of basic to more complex types of empirical justifications, or even to deductive ones. (p. 96)

Sinclair and Robutti (2013) pointed out that the role of the teacher is crucial: the teacher needs to help students develop "schemes of use" (Rabardel 1995). That is, students have to learn not only how to do a specific action (e.g. dragging, measuring) but also the reasons behind their actions, why some actions are not available on every object (e.g. non-draggable points), how and when measuring is useful, and furthermore to learn the limits of using measures with DGS for proofs and justifications. It is important to introduce the scheme of use in a cognitive and metacognitive way, rather than to teach the students a sequence of instructions and rules and then expecting them to reflect on the exploration made.

In this chapter we focus mainly on the role of the integration between "basic" materials and DGS and the emerging of justifying approaches in middle school students.

Realistic Mathematics Education

Although the problem was not created under the framework of Realistic Mathematics Education (RME), a Dutch approach to Mathematics Education (see Van den Heuvel-Panhuizen and Drijvers 2014) that is rarely employed in Italy, this theoretical framework came up during the discussion of the teaching experiment at CIEAEM 66 Conference held in 2014, and we decided to analyse our data in light of this approach, since we can recognize some common ideas with our framework. In fact, the question posed at the CIEAEM meeting was about the reality behind the problem, and we emphasized that the problem was designed to involve students as actors in the learning process, representing a meaningful situation through a "realistic" problem.

Van den Heuvel-Panhuizen and Drijvers (2014) explain in this way the meaning of "realistic problems", which we believe matches with the intent of our activity:

> Although "realistic" situations in the meaning of "real-world" situations are important in RME, "realistic" has a broader connotation here. It means students are offered problem situations which they can imagine. [. . .] It is this emphasis on making something real in your mind that gave RME its name. Therefore, in RME, problems presented to students can come from the real world but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problem are experientially real in the student's mind. (p. 521)

The International Research Project and Its Theoretical Framework

The open problem analysed belongs to an international research project that considered the interactions between the community of researchers, who designed the educational programme, and the community of teachers who attended the professional development workshop. The "Meta-Didactical Transposition" (Aldon et al. 2013; Arzarello et al. 2012, 2014) is the framework used to analyse the data collected through the observation of the teachers.

The Meta-didactical Transposition Model

Meta-Didactical Transposition (MDT) is a new model for framing teacher education projects. Its focus is the interaction between the *praxeologies* of the researchers and the *praxeologies* of the teachers (in-service or pre-service training), and the dynamics between internal and external components (Aldon et al. 2013; Arzarello et al. 2012, 2014). It is an adaptation of the Anthropological Theory of the Didactic (ATD) by Chevallard (1999) to teacher education. Its main theoretical tool is the notion of *praxeology*, which can be described using two levels:

1. the "know how" (*praxis*): a family of similar *problems* to be studied and the *techniques* available to solve them;
2. the "knowledge" (*logos*): the "discourses" that describe, explain and justify the techniques that are used for solving that task. The "knowledge level" can be further decomposed in two components: *Technologies* and *Theories*.

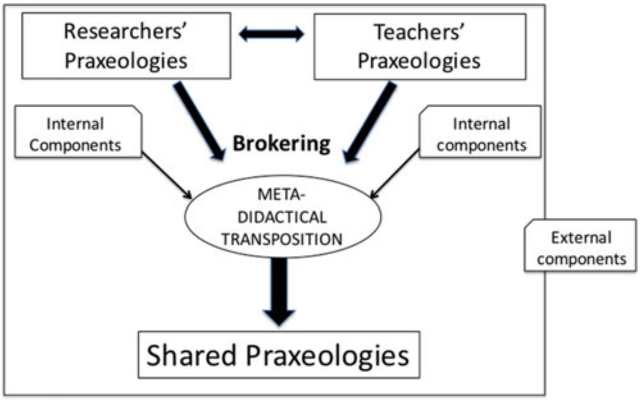


Fig. 1 Internal and external components in MDT

In other words, a *praxeology* consists in a Task, a Technique and a more or less structured argument that justifies or frames the Technique for that Task.

The MDT model considers the *meta-didactical praxeologies*, which consist of the tasks, techniques and justifying discourses that develop during the process of teacher education, and focus on the mechanisms in which the *praxeologies* of the researchers' community are transposed to the community of teachers, and how this implementation transforms the professionalism of teachers. In this way, we can observe a shift from the "savoir savant" to the mathematical and pedagogical knowledge necessary for teaching.

There are two communities involved in this project: the community of teachers (who are in training) and the community of teacher-researchers (who designed the task, act as trainers and observe the teachers). Each of these communities has its own *praxeologies*; the challenge at the end of the project is to create *shared praxeologies*, thanks to the *brokers*.

A *broker* is a person who belongs to more than one community (e.g. a teacher-researcher belongs to the community of mathematics experts and to the community of school-teachers). *Brokers* are able to make new connections across communities and facilitate the sharing of knowledge and practices from one community to the other. The creation of such connections by the *brokers* is called *brokering*.

Some of the components of the two communities' *praxeologies* can change during the educational programme and move from external to become internal (Fig. 1), in terms of the community to which they refer.

Institutional Context

One of the current Italian paradigms for the research in Mathematics Education is "Research for innovation" (Arzarello and Bartolini Bussi 1998), based on teaching experiments in classroom that involve school teachers in every phase of the research, with different roles: teacher-researchers (working with the group of researchers), teacher-trainers (doing education programmes for teachers) and teachers (involved in teacher programmes as learners and working in class as teachers). Sometimes the same teachers may have different roles in different phases of the project: for example, a teacher-researcher can be also teacher-trainer during the process of professional development in the education programme for teachers.

National Curriculum – Grade 1–8

Since in this chapter we discuss a problem proposed to lower secondary school pupils, it is important to understand this problem in the context of the lower secondary school curriculum. We present here a brief analysis of the Italian national curriculum for mathematics education.

In September 2012 the Italian Ministry of education released a new version of the National Curriculum for the first cycle of education (from 3 to 14 years old). The National Curriculum is organized into "Goals for the development of competences" and "Learning Objectives", and explains the expected knowledge and competence at the end of lower secondary school. The National Curriculum is also accompanied by a description of the main ideas of the teaching-learning process and of the different school subjects.

Here you can find some quotations from the National Curriculum for the lower secondary school excerpted because of their relevance for the framework of the activity we proposed (bold by the authors).

> The resolution of problems is a characteristic of mathematical practice. Problems need to be understood as **real and significant issues**, related to everyday life, and not just as repetitive exercises or questions that are answered simply by recalling a definition or a rule. Gradually, stimulated by the teacher's guidance and the discussion with peers, the student will learn to deal with difficult situations with confidence and determination, representing them in several ways, **conducting** appropriate **explorations**, dedicating the time necessary for precise identification of what is known and what to find, **conjecturing solutions** and results, identifying possible strategies.
>
> Particular attention will be devoted to the development of the ability to **present and discuss** with their peers the solutions and the procedures followed.
>
> The conscious and motivated use of calculators and **computers** must be encouraged appropriately [. . .] to check the accuracy of mental and written calculations and to **explore the world of numbers and shapes**.
>
> The development of an adequate vision of mathematics is of a great importance. This vision does not reduce mathematics to a set of rules to be memorized and applied, but recognizes mathematics as a framework to address **significant problems** and to **explore and perceive relationships and structures** that are found and occur in nature and in the creations of men.

Furthermore, we framed our activity with the following *Goal for the development of competencies:*

> To explain the procedure followed, also in written form, maintaining control on both the problem-solving process, both on the results.

and the following *Learning Objective*:

> To know the definitions and properties (angles, axes of symmetry, diagonals,...) of the main plane figures (triangles, quadrilaterals, regular polygons, circles).

The National Curriculum provides clear instructions: the teaching of Mathematics must start from meaningful situations to stimulate and involve students, and to give significance to the topics. In particular, in our experimentation we encouraged the use of technological devices, since the use of technology can effectively support the reaching of some of the National Curriculum goals. As a matter of fact, using a dynamic Geometry software like GeoGebra, students are main actors in their learning process: they can easily explore situations, generalize problems, make and check conjectures.

Class Context

We proposed this activity to 12 year-old pupils belonging to two different schools. One class, whose teacher was Monica, came from “Istituto Don Bosco” in San Benigno Canavese (Turin). It was a 25-student class, including 4 boys with learning disabilities. During the school year they showed interest and curiosity in front of Maths problems, especially involving real situations. In the first part of the year, students started to use GeoGebra as a tool for exploring the geometrical content of the curriculum in an active way. They showed, first of all, astonishment and then a strong desire to learn how the software works.

The other class, whose teacher was Elisa, came from “Scuola Media Holden” in Chieri (Turin). The class was composed of 2 students: a male and a female. They were interested in and curious about the activities proposed during maths lessons. They were used to working with a laboratory methodology and to discussing results and ideas with the teacher. They started to use GeoGebra to explore Geometrical properties (such as angles, perpendicular and parallel lines, etc.) as a support for manipulation of materials (paper folding, paper and pencil, etc.). Both the classes experienced the activity in the second part of the school year, in the same week of April.

The Street Lamp Problem

The street lamp problem, as we said before, is an open problem. The starting situation is a meaningful situation for the students: the municipal technician has to put a unique street lamp in a triangular pedestrian area, designed by the previous administration. The technician has to find the best point for the street lamp in order

Fig. 2 The pedestrian area, covered with grass

to light up the entire triangular area. This is the text of the problem given to the students of lower secondary school:

> The City Council has decided to build a small triangular pedestrian area planned by the previous administration. The registered project foresees only one street lamp as illumination for the whole area. Here there is the picture of the pedestrian area (Fig. 2).
>
> Can you help the technician, who will have to deal with the installation, to find the exact point where the street lamp should be placed?
>
> Part 1: You can use the picture of the pedestrian area and an electric flashlight to simulate the street lamp. Explain how you will proceed to find the best place to locate the street lamp.
>
> Part 2: Now open the file GeoGebra *Streetlamp.ggb*. You will find the pedestrian area to be lit. Together with your group try to find, using GeoGebra, the best point.
>
> What are the operational guidelines that you could give to the municipal technician to identify the point to put the lamp in? What are the relationships of that point with the triangle that defines the pedestrian area?
>
> Part 3: In your opinion, does the position of the point depend on the shape of the pedestrian area? What happens if the triangular shape changes? Be careful! It always remains a triangle but with a different shape! Try to explore the situation with GeoGebra: draw in a new sheet a generic triangle and save the file as *Park.ggb*. Explain what you have discovered and give reason for your answers.

In order to guide our young students, we divided the problem into three parts, beginning with the exploration with "basic" materials and arriving at the use of GeoGebra. In this activity the use of GeoGebra was thought not only to establish confirmation of previous conjectures but also to enable exploration of a more general situation. We also added the sentence related to the operational guidelines to be given to the technician as a way to foster students' argumentation skills: forcing them to explain to a third person how to find the exact point can help them to more deeply understand the geometrical properties of that point (e.g. it is the intersection of the perpendicular bisectors, it is equidistant from the vertices, etc.).

Design of the Open Problem

The design of the problem involved the community of teacher-researchers together with university researchers; they worked to construct the project and the activities

(Bardelle et al. 2014). The streetlamp problem is a transformation of an OECD Pisa item, expected to have only one answer (the circumcentre, see OECD 2003) within an open-ended problem, focusing on multiple solution methods and argumentation skills.

The task in the OECD (2003) Pisa Test was:

> The City council has decided to construct a streetlamp in a small triangular park so that it illuminates the whole park. Where should it be placed? (p. 26)

The problem has been transformed into a more open one, working mainly on three aspects: exploration (with "basic materials" and with GeoGebra), different solutions and discussion.

The idea of giving more space to exploration with both "basic" materials and GeoGebra has been made explicit by adding the sentences:

> You can use the picture of the pedestrian area and an electric flashlight to simulate the street lamp. Explain how you will proceed to find the best place to locate the streetlamp. [. . .] Now open the file GeoGebra *Streetlamp.ggb*. You will find the pedestrian area to be lit.

While the idea of giving more space to different solutions depending on the constraints has been made explicit by adding:

> Together with your group try to find, using GeoGebra, the best point. [. . .] In your opinion, does the position of the point depend on the shape of the pedestrian area?

The idea of giving more space for discussion has been suggested by the following request:

> What happens if the triangular shape changes? [. . .] Explain what you have discovered and give reason for your answers.

Giving more space to exploration meant to let students face the problem for a first time with the use of paper, pencil and an electric flashlight to simulate the lamp, for a second time using a DGS such as GeoGebra to analyse the problem from a static point of view and for a third time using GeoGebra that enables and even cries out for a dynamic perspective where constraints can change.

This exploration with "basic" materials and technological tools helps the students to grasp the dynamicity of the problem and to consider different solutions depending on the shape of the pedestrian area and on the constraints they fixed.

The OECD Pisa item was focused on the transformation of the problem into a mathematical problem: "*locating the centre of a circle that circumscribes the triangle*" (see OECD 2003 pp. 26–27). The reformulation, instead, is focused on the argumentation skills of the students. In fact the problem does not have a clear set of information to start with (e.g. Is the park inside a residential area? Is it possible to put the lamp outside the pedestrian area? ...). The different solutions depend on the choices made by students, on the ideas they consider relevant for the problem, and on the constraints they fix. Having different possible solutions forces the students' argumentation skills, and requires them to develop a strategy for defending their solutions, explaining their reasons, justifying their choices and even proving.

The methodology we used in designing the activity is based on the idea of a "mathematics laboratory" (UMI 2003) not as a physical place, external to the class, but as an approach to mathematics itself:

> A mathematics laboratory is not intended as opposed to a classroom, but rather a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects. A mathematics laboratory activity involves people (students and teachers), structures (classrooms, tools, organisation and management), ideas (projects, didactical planning and experiments). [. . .] In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students). (UMI 2003, p. 28).

The tools of the laboratory can be "basic" materials (transparent sheets, paper folding, grid paper, use of pins and twines), mathematical machines[3] or technological tools, such as DGS or CAS. During and after the laboratory, the "mathematical discussion" (Bartolini Bussi 1996) is the key point, in fact through the discussion it is possible to construct meanings and common ideas.

Aim of the Activity

The problem as posed is related to the exploration of a contextualized situation that, regarding mathematical content, leads to the centres of a triangle, focusing on their geometrical properties.

In Monica's classroom, pupils had already studied triangles and triangles' centres, whereas in Elisa's classroom only triangles and the concepts of perpendicular bisector of a segment, angle bisector, median and altitude had been introduced. Then, in the first situation the problem-solving aim was meant to consolidate acquired knowledge with the testing of the students' competences in using known mathematical concepts within unknown contexts. In the second situation, the aim was more aptly describe as construction of mathematical objects along with a co-construction and discovery of related geometrical properties.

The aim of this activity was not to create students skilled in the use of GeoGebra, but rather to support the development of skills requisite to exploring, conjecturing, justifying and arguing; the aim was also to construct a curriculum around a meaningful problem, powerful in engaging students in a specific context and stimulating their problem-solving competencies. Within this use of technology, the focus was not on the tool *per se*, but on the learning process mediated by the tool, on the new possibilities opened by the tool and on the mathematical objects constructed with the tool. Exploration, argumentation, justification and explanation are the key concepts in this problem-centred activity: it allows students to do maths and to build a piece of knowledge through finding solutions by themselves,

[3]For further information, see UMI (2003, p. 28).

exploring, arguing and justifying their choices. The power of open-ended problems is that the solution depends not only on the problem itself, but also on the interpretation of the problem that students make, on the constraints they fix, on the assumptions they make. Furthermore, the use of a dynamic software environment generates a learning environment that is naturally open-ended because of the way that the dynamic software demands the changing of constraints. It allows the creation of a family of related problems that share characteristics but, at the same time, provoke new directions of exploration caused by the changing constraints.

This use of exploration problems, matched with the use of technology, since the lower secondary school, can help students to face with proofs and can improve their proving competencies, that will become central in the further studies.

Although in the text of the problem there is a reference to the real world, the focus was not to create a realistic problem, plausible from the point of view of the real life. The main aim was to create a problem able to involve students as actors in the learning process and to shift to them the responsibility of learning. In this sense we can say that the problem is not "real", meaning belonging to real-life, but is "realistic" because it is meaningful for the students, according to RME approach (Van den Heuvel-Panhuizen and Drijvers 2014).

Description of the Activity

The activity was organized into 4 phases: three of them were developed by group work while the last one was collective.

1. Analysis of the situation using "basic" materials. Students explored the open problem with "basic" materials: paper and pencil, a flashlight and the picture of the park.
2. Exploration of the problem with static use of GeoGebra.
3. Exploration of the generic situation with dynamic use of GeoGebra.
4. Collective discussion in order to construct together the meanings of the objects involved in the activity.

Research Questions and Observation's Methodology

The research questions we asked ourselves at the beginning of the teaching experiment can be divided into two categories:

Students related

- What is the value added by this activity to the competence of our students?
- Is the use of technology an added value to the activity?

Teachers related

- Had the brokering been performed fostering the creation of shared *praxeologies*?

During the activity, in order to observe and analyse both students and teachers' works, we used a logbook to record, day by day, the things done, the materials used and to write observations about our behaviour as well as the behaviour of the students. Since we gave them forms, with some questions that guided the exploration of the problem, to work with and to fill in, we also collected them to reflect on our experimentation in teaching methodology. Elisa observed Monica's class during the activity, while Elisa's lessons were videotaped.

Critical Analysis

In order to answer out research questions, we critically analyse the activity, focusing on both the work of the students and the teachers.

Critical Analysis of the Activity in Monica's Classroom

Students were divided into working groups of 4–5 people and they were asked to fill in a report giving a shared answer to the questions. We are going to analyze these protocols focusing on the most interested passages.

First Phase

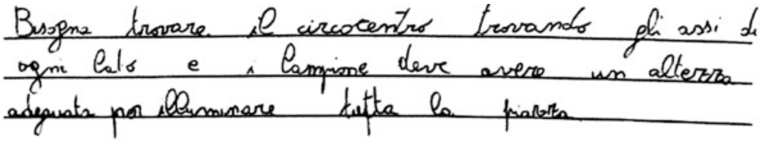
As soon as the students received the flashlight, they started using it to simulate the lamp. First, they noticed that the lamp can be put perpendicular to the ground or oblique: this aspect disoriented them since they were used to exercises with only one solution. Discussing within the group and then all together, guided by the teacher, they agreed that the perpendicular position lights up better than the oblique one. The teacher, in order to encourage them, explained that in this kind of activity there is not a right answer or a wrong one but "every" answer, if justified, is right.

Then they drew some of the fundamental elements of the triangle and two different conjectures emerged concerning the best point: four groups over six chose the barycentre and the other two the circumcentre.

During the previous lesson, the teacher showed, using a cardboard triangle and a pencil as a support, the physical property of the barycentre of being a point of equilibrium. This demonstration suggested students and could, reasonably, had influenced their choice.

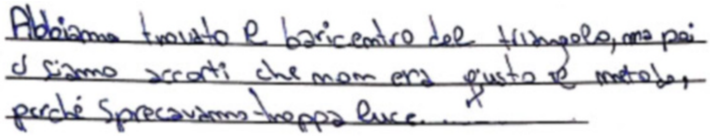
From their protocols we can notice that while students were working in a mathematical context they were making considerations concerning the real context. For instance, in Marta's group protocol (that chose the barycentre) we read:

> The lamp however must be high to light up more the fixed area.



[The lamp needs to have an adequate altitude to light up the whole park.]

Fig. 3 Alessandro's group solution



[We have found the barycentre but, then, we noticed that this was not the right method, since we were wasting too light.]

Fig. 4 Umberto's group consideration

And in Umberto's one:

> Putting the lamp in the centre [barycentre], we notice that lifting it up we are able to light up all the park.

Also Alessandro's group, that chose the circumcentre instead, noticed what is shown in Fig. 3.

Second Phase

We gave the students a GeoGebra file with the picture of the pedestrian area and asked them to work on it. They reproduced with GeoGebra the same construction made with paper and pencil. The static use of GeoGebra helped students to clearly visualize their conjectures and to reflect on the suitability of the choice made. Sometimes, after a discussion with peers, they changed their minds as we are going to analyse.

For example, Umberto's group wrote (see Fig. 4).

They discussed together looking for a better solution. With the help of GeoGebra, they built several triangle's centres and drew some circumferences. They agreed that the best one, with their constraint of wasting as least as possible light, was the circumcentre. Finally they wrote:

> The circumference we have drawn fits perfectly with the triangle.

They noticed that the circumference passed through all the three vertices (Fig. 5), linking together their geometrical knowledge with the exploration of a realistic problem.

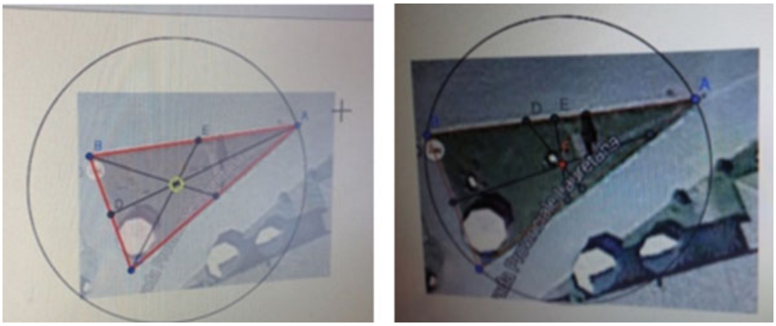


Fig. 5 Solutions in Umberto's group: first barycentre and finally circumcentre

The technological tool helped students to approach to the problem in different ways and to connect mathematical objects to their meanings.

The first solution of Giulia's group was the barycentre, but after a discussion they decided to search for a circumference passing through all the vertices, they drew it with GeoGebra and then they used their mathematical knowledge about the circumference circumscribed to a triangle to find the point:

> We found the barycentre, then we noticed that it wasn't the best solution. We drew the circumference through three points [the vertices] and we used the tool perpendicular bisector on every side of the triangle to find the intersection.

Third Phase

Dragging the triangle drawn with GeoGebra, students were able to explore different situations, making observations that were not possible with the only use of paper and pencil. We are going to report two meaningful quotations in order to support our assertion.

Marta's group (that moved to circumcentre) wrote:

> In the case of the triangle representing the park, the lamp was inside the area, but changing the shape of the triangle we saw that the circumcentre is outside. But if the lamp is higher, even if it were outside the park, it will light up everything [all the park]. We also noticed that, the lamp [put] outside the park lights besides it also the surroundings.

While Alessandro's group wrote:

> [The lamp] can light the park even if it is outside but, in the reality, such high lamps do not exist.

Paying attention to students' work and listening carefully to their discussion, as teacher we noticed that the more they explored, the more they became curious and interested. Some groups, as we have reported, wondered which was the connection between the abstract situation (the triangle, the centre of the circumference and the circumference) and the real situation where we have to use a real lamp. Comparing the geometrical solution found with the real situation, students noticed that there were some problems in putting the street lamp outside of the pedestrian area and, furthermore, they wondered about the maximum possible height of a lamp.

Fourth Phase

We guided the students to explain their solutions in order to convince the other classmates about their ideas. This part of the activity deeply involved students' ability of justify and argue. Almost the students took actively part in the discussion, explaining their ideas or making observations. We report the most interesting considerations:

> The barycentre is not always the best solution! In some cases you have to waste a lot of light in order to light up all the area.

Another student said:

> If the circumcentre is outside the park, you need a very high lamp that could not exist in the real world.

This two sentences point out that students are simultaneously reasoning on two levels: the mathematical one and the realistic one, considering geometrical properties and real problems. Other considerations raised, concerning the situation where the circumcentre is outside the park: "We cannot put the lamp in another property", "Or in a river" beat one classmate "Or in the middle of a motorway" said another.

The ending of the activity was not the choice of ONE solution, but of a SET of solutions and a SET of justifications for those constructions:

- The barycentre seemed to be a suitable point since it was always inside the triangle. Students noticed that in some cases the lamp lights up a big area around the park but they agreed that this was an added value;
- A group of students agreed that the circumcentre is always the best solution, even if it is outside the park, because the circumference passes through all the vertices;
- Other students agreed that the circumcentre is the best point in the case of an acute angled triangle while, in the case of an obtuse angled triangle, the best choice is the barycentre.

Finally we briefly asked them (because the lesson was ending) a personal opinion about the activity. Most of the students were rather surprised from the activity proposed: since schoolbooks usually have closed problems, at the beginning they felt disoriented. Then, they told to have appreciated the use of technology because it allowed them to explore in order to find the point.

Furthermore, students with learning disabilities, that were often bored and distracted during traditional lessons, were actively involved in group working, and in one case a student acted as leader working with GeoGebra.

Critical Analysis of the Activity in Elisa's Classroom

Elisa's students have already studied the fundamental elements of triangles (angle bisectors, perpendicular bisectors, medians and altitudes) but they never faced the triangle's centres, nor in a theoretical way, neither in an exploration activity. The street lamp problem was used as a starting point for the discovery of such centres. Since the class was very small, composed of only two students, they worked in pair. On one hand this represented an advantage: in fact, it allowed the teacher to follow students' reasoning very closely, on the other hand it represented a disadvantage: the collective discussion was less rich because no other point of view was present.

The teacher introduced the activity leveraging on the "realistic" connotation of the problem (in the RME meaning), trying to involve the students as actors:

T: This is a realistic problem, we have to try to understand how to solve this problem, knowing that there is not only one correct answer. This is not a "standard" problem, like an exercise… you finish it and you get the result… that is the same to the one written in the book. Here, we have to let our brain work…
V: Right!
T: The same happens in our everyday life… in our real life we do not have the result at the end of the book, right?

The exploration phase is very important and it is important to do this activity at first time manipulating some materials. As soon as Edoardo picked up the flashlight, he moved it up and down, looking at the light on the picture of the park (Fig. 6).

E: Up or down?

Although the initial idea was to put the streetlamp vertical (as the flashlight in Edoardo's picture) the students engaged a discussion to decide what kind of streetlamp use.

T: Try to discuss… I will do in this way… I will do in that way…
E: I will do this [puts the flashlight on one vertex]
V: Yes, but… here [points the farthest vertex] there is no light…

Fig. 6 Edoardo with the flashlight

Fig. 7 The model of the lamp

E: However… if we put the lamp here [points where Valentina pointed before] it does not light up there [points the vertex in which he put the lamp before]
[…]
V: Up… [puts the flashlight perpendicular to the sheet, but it lights up also outside the park]
E: It is too high!
V: But it lights up everything!
E: And, what about here? [he comes back to his original idea – a vertex]
V: No!

During the exploration with "basic" materials they used the flashlight and the fingers or a pen to simulate the lamp (Fig. 7)

T: Think about… what does the lamp look like?
E: It is high, like this [points the picture of the lamp on the paper]
V: A straight line and then like this… [puts the flashlight down] […] Maybe we can use Edoardo's finger…

The discussion continued, with the teacher posing some level-raising questions and helping the students to make a decision about the kind of streetlamp. The shape of the lamp represented the first constraint chosen by the students, as underlined by the words of the teacher.

T: And... how can you choose the point?
[…]
E: I got it! We will put the lamp here [points the centre of the triangle, with the flashlight perpendicular to the sheet]
[…]
E: Let's try to have a different lamp…
V: As I told before!
T: … Have we decided that we like more this kind of lamp? Ok, so we have done a CHOICE:
how does our lamp look like? Our streetlamp is one of those with the light bulb hanging down. And now… where do we put it?

Fig. 8 The construction of angle bisectors

The students decided to draw the angle bisectors in order to find the point (Fig. 8). Probably they chose in a first time the angle bisectors because the teacher worked a lot on this topic, constructing them in several ways: using paper folding, tracing paper, compass, GeoGebra, also exploring the property of their points of being equidistant from the sides of the angle.

After drawing the angle bisectors and discovering that they all meet in the same point, they pointed out that the height of the flashlight/lamp was an important variable for the problem in order to light up the entire park.

During the second phase the students worked on the GeoGebra file prepared by the teacher, with the same picture of the park used with the flashlight. Students used GeoGebra as a static instrument reproducing the same construction made with the flashlight and the compass. In this phase they never tried to drag the triangle, because the picture of the park (underlying the triangle) forced them to focus on that specific triangle. The previous activity with "basic" material helped students in this technological phase, the mediation of these instruments enabled them to find a first solution to the problem connecting the image of the circular light of the flashlight with the concept of circumference. In particular, Valentina used the flashlight also with the screen of the computer and Edoardo found the mathematical object connected and represented it in GeoGebra.

T: And now, that is the point you have chosen, how can we manage. . .
V: In GeoGebra there is not a lamp-tool. . .[puts the flashlight near the screen of the pc, representing the same situation explored before with paper and pencil]
E: I got it... [draws a circumference]
V: Edo, what have you done?
E: I drew a circumference
T: What circumference?
E: Passing through the farthest point
V: From the lamp
E: The circumference has to pass through the point A, because it is the farthest and then we are sure that the circumference contains the other two points. . . In fact, if I draw a circumference passing through C, something remains out. . . (Fig. 9).

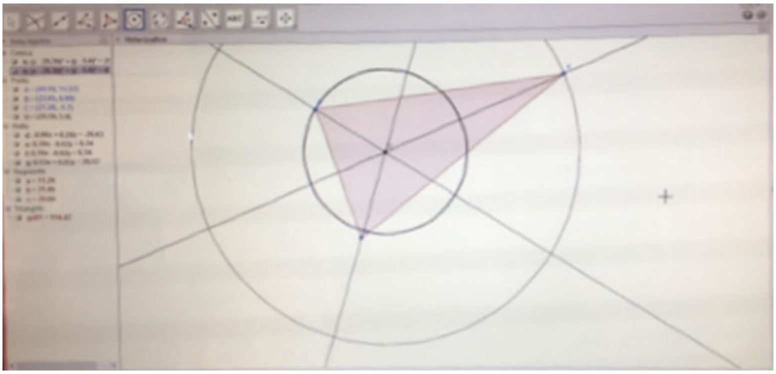


Fig. 9 The right radius of the circumference

Once this solution was found, the students were happy and thought to have solved the problem. The teacher suggested some further reflections on the question.

T: And, in your opinion, is this the BEST point? What is the meaning of BEST point?

The students, then, checked other possible situations and drew the perpendicular bisectors (apparently without a particular reason). They discovered that the circumference in that case passed through all the three vertices and decided that this one was the most beautiful solution.

V: We have found a new point! [. . .] The circumference now "takes" everything! And it is also smaller than the other one! (Fig. 10)
T: What has happened?
V: With the perpendicular bisector. . . the circumference now "takes" all the points [points at the vertices] instead before it takes only the point A. Now there is more light, while before, with the bigger circumference, the light was less intense. So this one is PERFECT.
T: Why do you like this point more than the other?
V: Because it is more centred, the circumference is smaller and it lights up more the park!
T: And what other characteristics does this point have?
V: If we do a smaller circumference, then it does not pass any more through all the vertices. This point is BEAUTIFUL.

Then the students investigated the properties of this centre (circumcentre) while they were trying to explain to the technician how to reach the point, and discovered that it has the same distance from the vertices. The teacher continued asking questions in order to connect the geometrical situation with the realistic one.

T: How would you explain to the technician how to find the point?
E: He has to construct the perpendicular bisectors.
T: Yes... and the technician will say to you "I do not know how to construct a perpendicular bisector".

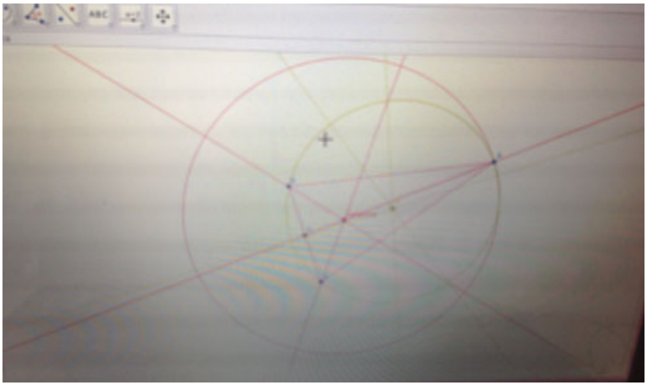


Fig. 10 Circumcentre versus incentre

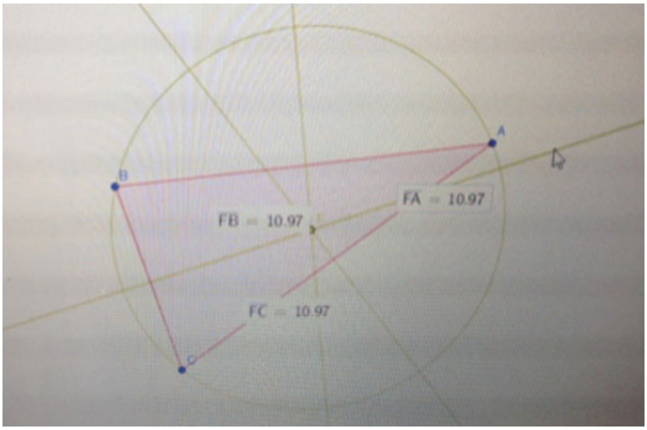


Fig. 11 The distances from the centre

E: I will say to him "Take half of each side of the park. . ."
T: And then? [. . .] What kind of point is the centre?
E: It has the same distance from A, B and C [the vertices], because if the circumference centred there passes through A, B and C. . . then the distance is the same.
V: It is perfect.
E: Let's ask GeoGebra. . . [uses the distance tool to verify if the point is equidistant from the vertices. Figure 11]
T: Can we say to the technician how to construct the point?
E: With the perpendicular bisector tool.
T: But does he have this tool?
V: No.[. . .]
E: Walk away from the sides of the park, perpendicularly, starting in the midpoint.

The students explored the problem from a dynamic point of view. Valentina suggested to draw a lot of different triangles, but Edoardo immediately replied that

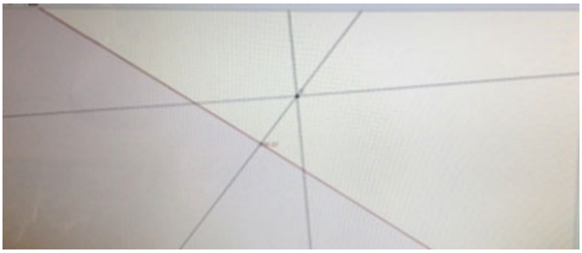


Fig. 12 Obtuse angled triangle: the circumcentre is outside (zoom tool)

they could use the dragging tool, he drew a generic triangle and moved the vertices around. They reproduced the same construction for the circumcentre and they noticed that, dragging the triangle, there were some situations in which this point seemed not so “beautiful” as before: the streetlamp was outside the park and the circumference was big. They explored with GeoGebra in order to find what kind of triangle was it and they argued it was the obtuse angled triangle. They dragged the triangle unless it was “less obtuse angled” (measuring the angle with GeoGebra) and they verified with the zoom tool that the circumcentre was still outside (Fig.12).

At first they recognised that, when the circumcentre is outside the park, the streetlamp needs to be taller in order to light up the whole area. The teacher asked if there were other problems in putting the streetlamp outside and then they decided that it would be not suitable to have it outside the park, then they moved back to the incentre for the obtuse angled triangle.

Through this activity the teacher became aware of some aspects of their students she never observed before: Edoardo, who has some difficulties with calculations, procedures and sequential activities, showed wide intuition and a great accuracy in the geometrical construction, while Valentina became more self-confident, in particular facing problems, instead of being “afraid” of problems such in previous experiences, and solved the task with determination.

Elisa’s students, used to laboratory and discussion, were able to discover by themselves that, for instance, the three angle bisectors of a triangle meet in a unique point, that, in a generic triangle, this point is not the same as the intersection of perpendicular bisectors or medians or altitudes and that the circumcentre is equidistant from the vertices. Only at the end of the activity, during the institutionalization discussion, the teacher gave the “names” to these points and formalized definitions and properties.

Comment About the Activity Experienced

The technology represents a key element of this teaching experiment. Technology is involved in the activity with the use of a DGS – GeoGebra – to explore the problem. GeoGebra has the power, as others DGS, of being dynamic, so the students can manipulate dynamically the shapes they constructed by dragging them, they can also modify the shape (enlarge, restrict, etc.) keeping unchanged the construction protocol.

The manipulation in this activity occurred twice, the first time was a concrete action with materials, while the second was a construction and dragging activity carried out with the software. Within the first part, students focus the problem and try to find a solution that will be confirmed, rejected or modified by the observation of the dynamic situation represented with technology.

The integration between "basic" materials and GeoGebra helped students to construct knowledge, and the dynamic use of GeoGebra gave students space to explore, conjecture and argue. One of the added values of this kind of activity is the mediation of instruments and technology (think about Valentina with the flashlight on the screen). The first phase pointed out that the tools we named "poor/basic" (in the meaning of simple) are instead very "rich" elements for the comprehension of the problem. But the use of technology offers more possibilities to investigate the problem with constraints changing over time. Without technological tools the activity's solution could be very different, the dynamicity of the software helped students to emphasize the critical aspects, such as the obtuse angled triangle case and to grasp the variability of the situation over time. For instance, when they used the picture of the triangle it was not A generic triangle, but it was THE particular triangle drawn. When they draw instead a triangle with GeoGebra, it was really a generic one: using the dragging it can change, but maintaining its own properties as a triangle. Looking at the experience, we noticed that students were able to use their knowledge in a real situation, different from the one in which they have learnt it, improving their competences. Finally, they have been able to manage a collective discussion, sharing their ideas and constructing together the meanings. As teachers we noticed that open-ended problems give the possibility of discussing about various aspects, even different from those designed.

Critical Analysis of the Teachers

We tried to find some answers to the research questions analysing the data collected during the teacher-training course: written materials (the beginning questionnaire and the logbook) and also video materials (the beginning interview).

We applied the MDT model to Monica, who belonged to the teachers' community while Elisa belonged to the teacher-researchers' community and acted as a *broker* during the educational programme.

Initially, the use of GeoGebra in lower secondary school and the use of open-ended problems are external components for the teachers, as we can recognize in the following excerpt from Monica's interview:

I: Do you use technology in your class? What kind of software?

M: Although I've been teaching for many years, this is the first year I use technology in class. This year we have the Interactive White Board (IWB) in class and I also attended some courses to learn how to use GeoGebra in class. We don't have a computer lab big enough to contain all the students, so I worked in class with the IWB, showing the files and the constructions. Students downloaded the software in their personal devices and used it to solve some homework.

Also the focus on the National Curriculum was an external component:

I: Does your Annual Programme of Education follow the National Guidelines?
M: When we wrote the Annual Programme, we followed the previous year's programme. When I started to work in this school the other teachers working here before had already written the programme and I didn't change anything. We never compared the National Guidelines with our Programme. Actually the reference with the Guidelines is missing, but I know the National Guidelines and I think the Programme follows their main ideas.

The laboratorial methodology (group work and discussion) was also an external component for Monica:

I: Are you used to collective discussion? What kind of activity do you manage with collective discussion?
M: I like that students compare their ideas and reasons, but I think that in a middle school (maybe due to the age of the students) it is difficult to manage effective discussions. Students are interested, but they are not able to organize properly a discussion, they have to learn to talk one at a time and to listen to their mates. You waste a lot of time trying to manage the mess and this persuades me not to use the discussion. [...] Sometimes I use it during science lessons.
I: Are you used to group work? Do you think it is useful?
M: I never used group work with this class. They are 25 students and for reasons of time and organization I avoided it. Maybe group work is useful. I have always the problem of managing time: group work needs a lot of time.

At the beginning Monica was sceptical and worried about proposing the activity to her students due to its openness and, furthermore, because the students were very young (12 years old). But she accepted the challenge. At the end of the educational programme the National Curriculum, the use of GeoGebra in middle school classes and the laboratorial methodology became internal components in her *praxeologies* as we can notice in these excerpts from Monica's logbook.

During the activity the students seemed very interested and involved, working seriously on the task given, arguing and justifying their solutions in an accurate way. I felt very involved in this activity; they worked with interest and curiosity and this gave me a great satisfaction and an incentive to repeat in the future this kind of experience. I'm going to design other activities like this one and I will use group work for other tasks.

Elisa acted as a *broker*, being a teacher as Monica but also a member of the researchers' community (as a teacher-researcher). She discussed with Monica and the other teachers, sharing ideas and doubts, reflecting on their didactical practice. The action of *brokering* was performed by the teacher-researchers during the face-to-face sessions of the course and also through the Moodle platform with forums and discussions.

Among the *preaxeologies* of the researcher community, we choose to analyse the *praxeology* of *designing a task for the teachers*. We can recognize the four elements identified in ATD (Chevallard 1999):

Task: designing the activity for teachers and students;
Technique: finding a problem considered linked to the topics of Curriculum; opening a close-ended problem, adapting it to the aims of the project, the methodology to induce, the use of GeoGebra and the institutional constrictions;
Technology: institutional (the new curriculum), from research about exploring, conjecturing, arguing, proving, the use of mathematics laboratory and the use of GeoGebra;
Theory: research elements such as: open problem, conjecturing and arguing, mathematics laboratory, meta-didactical transposition with the related literature as background.

This *praxeology* became a *shared praxeology* when Monica, during the educational course, designed tasks for her own students, in particular Monica took part in the following year to another PLS educational programme, focused on Task design for students.

Conclusion

During the activity, students worked in two different environments: the paper and pencil environment and the technological environment. Technological tools allowed students to explore a variety of different situations simply by dragging the construction made in the specific case. With DGS they can easily represent a generic situation and then study how it changes, test the different ideas and solutions found and validate those most appropriate to their model while justifying choices. Both paper and pencil and technology are important tools for problem solving, but the real potential stands in their integration. Using only paper and pencil or only technology, students do not achieve the same results as they do when using them together. The key point is the mediation and integration of the two environments.

Furthermore, the experience was useful for teachers and students alike. Monica experienced a new approach and new *praxeologies*, improving her professionalism as a teacher, while her pupils were involved with a leading role in the activity: they have made decisions, discussed, argued and mobilized their competencies. Elisa had the opportunity of observing again her didactical practice and to reflect further upon it.

Taking part in an international project is a great opportunity for sharing ideas, methodologies, doubts and for the construction of shared *praxeologies*, that will be, from now on, a critical component of the *praxeologies* of the teachers involved in the training.

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A Framework for Failed Proving Processes in a Dynamic Geometry Environment

Madona Chartouny, Iman Osta, and Nawal Abou Raad

Abstract The study aimed to evaluate the potentialities of Dynamic Geometry Environments (DGE) in the teaching and learning of mathematical proof by analyzing students' cognitive processes while solving an open geometry problem. The study was conducted in a grade-10 math class in a Lebanese school. Data were collected through whole-class observation with analysis of paper-based data and closer observation of 12 pairs of students. The analysis focused on the mistakes that occurred at the three stages of the proving process: the construction and manipulation of the figure; the formulation of the conjecture; and the proof itself. The results suggest the development of a "framework of failed proving processes" that classifies errors by type and by explanations for the failure.

Keywords Proof • Dynamic geometry • Secondary • Failed proving processes • Conjecture elaboration

Introduction

The teaching and learning of mathematical proof has always been a challenging process for both students and teachers. Many research studies in mathematics education have been conducted to investigate proving processes and to analyze the types of proofs produced by students. (Balacheff 1988; De Villiers 2012; Hanna and De Villiers 2008; Harel and Sowder 2007). Some studies have led to epistemic and pragmatic strategies aimed at the advancement of the teaching and learning of proof in school mathematics. Harel and Sowder (ibid) identify proving as the process that an individual or a community employs to remove doubts about the truth of a statement. De Villiers (2012), however, rejects the definition of proof in terms of its verification function or any other function. He argues that "proof should

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rather be defined simply as a deductive or logical argument that shows how a particular result can be derived from other proven or assumed results; nothing more, nothing less" (p. 4). In this definition, the truth or validity of the statements, whether the premise or the conclusion, is not the main concern. Hanna and De Villiers (2008) argue that even though there are diverse definitions to proof in formal mathematics one crucial principle underlies all of them. This principle is: "To specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions" (p. 329). This main principle is at the core of proof, yet it also expands to situations that are external to mathematics and establishes a base for the reasoning of human beings.

Many researchers have analyzed the types of proofs produced by students and classified them according to the types of arguments presented. Balacheff (1988) studied students' proofs as products and provided a classification of proofs according to a continuum from empirical to deductive arguments. Additionally, he distinguished two types of justification, namely pragmatic and conceptual justifications. Pragmatic justifications are based on the use of examples, actions or showings, whereas conceptual justifications are based on abstract formulations of properties and of relationships among these properties

Marrades and Gutiérrez (2000) built on Balacheff's classification by adding sublevels to the existing categories, as well as a new category, that of failed justifications: "Failed justifications are necessary to complete the classification because the assessment of students' justification and proof skills cannot be associated only to correct solutions of problems" (p. 94). A failed justification occurs when students employ empirical or deductive strategies to solve a proof problem, yet either fail to elaborate a correct conjecture, or, in case that they do elaborate a correct conjecture, they nevertheless fail to provide any justification.

According to Harel and Sowder (1998), proving or justifying a result involves ascertaining – that is, convincing oneself, and persuading – that is, convincing others. In their 1998 study, they constructed a framework for the analysis of students' proof schemes generated from teaching experience, interviews with secondary school and college students, and the work of other researchers in the field. Proof or justification schemes were in this framework organized into three categories: externally based proof schemes, empirical proof schemes, and analytic proof schemes. Each of these categories represents a cognitive level; the classification framework is not concerned with the content of proofs or their methods.

Hoyles (1998) highlighted the common practice, in the math education literature, of presenting types of proofs in hierarchical structures where the empirical precedes the deductive. Despite the need for students to distinguish empirical proofs from deductive ones, the question remains whether empirical pragmatic proofs develop into conceptual ones, or whether links can be forged between the two types. In general, Hoyles showed that students are usually aware of the limitations of empirically-based proofs, and that they recognize that a theoretical, formal proof is needed.

Boero (2007) proposes that a reform of the way proof is being taught in the classrooms is needed:

> Old teaching models (essentially based on learning and repetition of proofs of relevant theorems as they are written in textbooks) do not fit the current needs of students and teachers. (...) Therefore entirely new approaches are needed. And these approaches must take into account the actual complexity of the subject. (p. 19).

In recent decades, educational technologies, which are an example of such novel approaches, have been introduced to the teaching and learning of proof. Educational technologies have been made widely accessible to schools, and research on new technologies has demonstrated their importance for the teaching and learning of mathematics in general, and for the teaching and learning of proof, especially for argumentation methods and techniques. The mathematics education literature is indeed rich with research in this regard (e.g. Artigue 2010; Laborde and Sträßer 2010; Leung et al. 2013; Mariotti 2006).

The study presented in this chapter, and conducted in a Lebanese context, attempts to contribute to this body of research. Its aim is to investigate the role that Dynamic Geometry Software (DGS) can play in supporting students' thinking when solving geometry problems, and analyze the cognitive processes that students use, and the types of conjectures and proofs that they produce in such an environment.

Laborde (1998) introduced a distinction between objects, relations and operations belonging to the theoretical domain (denoted by T), and entities – including physical actions and opinions – belonging to the spatial-graphical domain (denoted by SG). When working on a geometrical problem, students are usually expected to give an answer belonging to the theoretical domain (Laborde 1998). Laborde adds that teachers accept that students use drawings and figures as auxiliary means; but these drawings or figures are typically not meant to be referred to in the solution. However, the solution of a geometry problem lies in both the SG and T domains, and is characterized by continuous shifts between them. The Dynamic Geometry Environment (DGE) presents the learner with a combination of the two domains since it provides diagrams whose behavior is controlled by the theory.

According to Artigue (2010), the DGE can help in generating conjectures and eventually lead to the construction of proofs, based on the dragging possibility, the instant feedback, and the dynamic figures resulting from dragging. The DGE tools are loaded with potentialities that can unite theoretical knowledge with concrete siuations in a new environment meaningful to students. One example is the possibility given to students by the dragging tool of examining a seemingly infinite number of instances of the same geometrical figure to support a certain conjecture. In addition, while dragging, students go back and forth between concrete figures and theoretical knowledge, which helps them progress from the empirical to the theoretical level.

Leung et al. (2013) suggested that a defining characteristic of DGE is the dependency among points and objects of a construction: when basic points are dragged, each dependent element moves together with the others while preserving the properties of the construction. Dragging is seen as a powerful epistemic tool that supports geometrical reasoning and, in particular, it becomes a tool capable of producing conjectures. Behind this epistemic power is an implicit assumption that connects the world of DGE to the formal axiomatic world of Euclidean geometry.

> Trying to make explicit such assumption, we state it as the following *dragging exploration principle:* During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry. (p. 458)

Graphical representations are central in the teaching of geometry as they allow the student to "see" geometric properties, to form conjectures, to experiment with and verify a given property. However, representations can also be confusing and deceiving, as they may appear to show relations among objects that are not necessarily properties of the geometric figure that the teacher intends the students to examine, which leads to wrong interpretations. For this reason, the important distinction between a drawing and a figure was introduced by Parzysz (1988) and developed later by Laborde and Capponi (1994). According to Parzysz (1988), the figure is a theoretical object defined by a descriptive text, an imaginary object, or an idea; the drawing is only the illustration of this figure. Laborde and Capponi (1994) define a figure as a theoretical construct associated with each of its possible drawings. Students interpret drawings based on their mathematical knowledge as well as on the nature of the drawings and the way they are represented. This is why and where ambiguities emerge, creating problems caused by different interpretations. It is neither easy nor evident to detach oneself from the perceived drawing in order to access the theoretical figure, as students have to distinguish the properties of the perceived drawing that correspond to the theoretical figure from the ones that are only spatio-graphical properties that cannot be used in the conjecture or proof.

In DGE teachers usually expect students to construct figures that preserve all their properties under dragging. However, Healy (2000) observed that many students worked differently. They started with constructions that verified some of the properties, and then used these constructions to find ways of obtaining the remaining properties. This strategy should not be totally rejected, since it allows the student to search for the required properties while exploring the figure. Healy thus introduced the distinction between soft and robust constructions. A robust construction is one that holds under dragging, and that has all required geometrical properties. Soft constructions are "constructions, in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student" (p. 107).

In DGE, dragging plays different roles according to the purpose of its use. In robust constructions, dragging is used as a verification tool: the correctness of the figure is verified when its properties remain invariant under dragging. While dragging for verification, students move from the general to the specific, since a multitude of drawings with the same geometrical properties is produced from the generic robust figure. On the other hand, in soft constructions dragging is used as a construction tool rather than a verification tool. The dependent property is evident when the other property is manually incorporated using dragging; thus the general can emerge from the specific.

Another role for dragging highlighted by Mariotti (2014), is the role of mediator between geometrical invariants and logical statements. In fact, dragging to elaborate a conjecture is a complex process, since it requires the interpretation of perceptual data by decomposing the image in order to identify a geometrically significant relationship between its elements and properties. For example, when dragging to search for consequences, students need to interpret the geometrical dependence between "direct invariants" and "indirect invariants" as the logical dependencies between premises and conclusions of conditional statements: direct invariants are properties given by the problem, that is, invariant properties observed between independent elements; while indirect invariants are the consequences of the properties given by the problem, or invariant properties observed between dependent elements.

The Study

In the Lebanese curriculum in effect at the present time (CERD 1997), proof writing begins at grade 7 (students at the age of 11 or 12) and continues through the intermediate school (grades 7, 8 and 9) and the secondary school (grades 10, 11 and 12). Beginning from the secondary school, i.e., grade 10 (at the age of 15 or 16), proof writing becomes more rigorous and formal. It is at this level that students are asked to formalize their thinking, link different mathematical domains in one context, choose among a variety of solution strategies what is most appropriate for the given context, and write clear and concise justifications for their solutions. However, the Lebanese curriculum neither includes explicit instruction of logic, nor does it explicitly address techniques or conditions of argumentation and proof as an independent topic of study. Students are implicitly expected to acquire the proving abilities as a by-product of their learning of geometrical theorems and properties, and through working on solutions to geometrical problems. It is assumed that if students know the properties of geometrical shapes, they should be able to develop proofs based on those properties.

Given this background, the aims of this study are: (1) to provide students with a learning environment, namely DGE, in which they can be supported in their approach to proof; and (2) to analyze the students' cognitive processes while they engage in conjecturing and proving activities. The study examines the proofs produced by the students at different stages of the proving process, including the construction and manipulation of geometric figures, as well as the formulation of the conjecture and the proof itself. These stages are, of course, neither linear nor independent. On the other hand, while most of the previous research works developed classifications of only valid proofs, the present study focuses on the failed proofs produced by the students, in order to gain a deeper understanding of their nature, at which stage they occurred, and the reasons behind them.

Method

The study took place in a tenth-grade mathematics class of a Lebanese school, and consisted of a series of geometrical problem solving sessions during which students (15–16 years old) worked using a DGS, namely GeoGebra, on solving an open geometry problem that required conjecturing and proving. The problem was taken from Olivero (2002), and involved the angle bisectors of a quadrilateral.

The study used a qualitative research method, both in the data collection and analysis. The method used consists of an analytical process of *theory-building* (Eisenhardt 1989) which was highly iterative, including going back and forth from one case to another, from building a category to looking back at the data and literature; this process allowed new elements and explanations to constantly emerge and to be continuously refined.

Data were collected through observation. The proving process is twofold: external practices and internal practices. External practices consist of what students do and say, of behaviors that can be directly observed, such as dragging, drawing, sketching on paper, talking, etc. Internal practices represent the students' reasoning processes. Students' thoughts and internal practices had to be externalized through dialogues, thus all students were set to work in pairs. Observations were conducted on two different levels: observation of the whole class and observation of specific pairs of students. All students participating in the study were introduced for the first time to GeoGebra in grade 8. Thus by grade 10, when the study was conducted, students were familiar with the basic commands and features of GeoGebra, and capable of different types of constructions and manipulations.

Whole-Class Observation During the problem-solving session, the work of the whole class was observed, i.e., 22 pairs of students. The observer targeted one pair of students at a time and took detailed notes and screenshots of all significant and interesting instances of the solving processes. In addition, paper material produced by each student was collected. The paper-based materials included sketches as well as written conjectures and proofs generated by the class. During observation and data collection, special attention was given to the interplay between the spatio-graphical field (including DGE objects, paper drawings, etc.) and the theoretical field (including geometrical properties, theorems and definitions).

Observation of Specific Pairs of Students In addition to class observation, the primary research tool used to document the details of students' proving processes was the observation of selected pairs of students. Students' interactions with each other, with the mathematical ideas of the problem, and with the computer were closely observed and recorded. Given that the research aim of this study required the analysis of students' proving processes, i.e., the production of conjectures and proofs, effective techniques were necessary for uncovering these processes. Both video and audio data were needed, since the analysis included the use of DGS and other supports such as paper and pencil, as well as the interaction between the students in each pair. The work of 12 pairs of middle and high achieving students was videotaped. The selection of pairs of students to observe was based on four criteria: Selected students were (1) used to working in pairs, (2) capable of producing conjectures and proofs, consequently middle to high achievers, (3) talkative – because the analysis was based on what is "visible" from the proving processes, and (4) willing to be video-taped during class.

The problem-solving sessionstarted with the teacher introducing the problem. Then the students were left alone to work on the problem for 60 min. Students were asked to write their conjectures and prove them. The teacher did not intervene. The main goal of these sessions was to access students' basic and essential ideas involved in their proving processes, during their interaction with DGS, and while moving back and forth between the elaboration of conjectures and their justification and formalization.

The Problem

In order to observe the proving processes there was a need to find a context where they happen naturally and can be easily and authentically observed. The choice of the mathematical context, i.e., the open proof problem was crucial. In previous studies (Leung 2012; Mariotti 2012), open proof problems affirmed themselves to be adequate contexts for observing proving processes, because they tend to allow the observation of the whole process, from exploring to conjecturing and proving.

The choice of the problem to be used in this study was based on the following theoretical and pragmatic considerations: (1) The chosen problem is in accordance with the definitions of open problems given by Charnay (1992) and Mogetta et al. (1999). The statement is short and does not suggest any particular solution method. It cannot be reduced to the execution of a set of routine procedures that students might have memorized by heart; (2) The chosen problem is a type 2 problem according to the classification of Laborde (2001). That is, the nature of the problem is not changed by the DGE which simply acts as a visual amplifier facilitating the mathematical task, i.e., exploration, identification of properties and analysis. This type of problem can be used as a research tool for investigating students' conceptions. It acts as a window on students' ideas and understandings; (3) The chosen problem lies in a conceptual domain familiar to the students; its difficulty does not reside in the understanding of the problem (Charnay 1992). The students participating in the study have a good mastery of the mathematics needed for the solution of the problem; they were taught these mathematical concepts in previous grades and are now capable of employing them skillfully.

Statement of the Problem (Previously Used by Olivero 2002)

1. Let ABCD be a quadrilateral. Consider the bisectors of its internal angles and the intersection points H, K, L, and M of pairs of consecutive bisectors.
2. Drag ABCD, considering different configurations, and explore how HKLM changes in relation to ABCD.
3. Write down conjectures and prove them.

Type of Tasks Evoked by the Problem

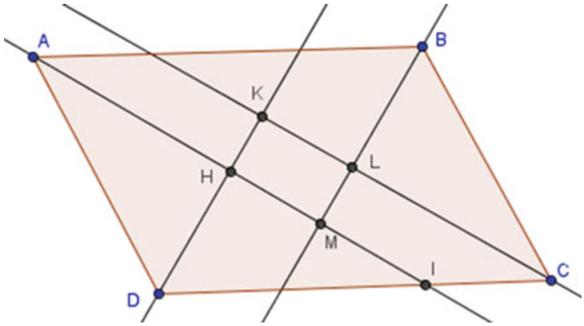
The problem was chosen as it evokes the implementation of different types of tasks and the use of various DGE tools:

- Construction and exploration tasks: Students need to construct, draw or drag different types of quadrilaterals to explore the nature of HKLM in relation to the nature of ABCD. It is expected that the figures will be soft constructions using *Polygon* or *Points* and *Segments* as they are easily transformed into different types of quadrilaterals ABCD. However, the question remains whether students might use robust figures, to which end might they be used, and how their use could affect the proving process.
- Conjecturing tasks: This problem is characterized by the richness of potential conjectures. There are more than one or a couple of correct conjectures that students can develop. They are free to explore any type of quadrilateral such as special and regular quadrilaterals, to group different cases together such as the cases of the square and rhombus, to focus on a specific quadrilateral and to investigate its types, such as the trapezoid, or to explore any other case they desire to investigate.
- Proving tasks: While some proofs are straightforward, necessitating the use of only one property, others are more complex. Students need to recall the properties of special quadrilaterals, congruent triangles, bisectors of supplementary angles, or parallelism and properties of corresponding angles. They also need to identify which angles, sides or triangles they want to use and isolate them visually or by using DGE options such as coloring or marking angles and segments.

Solution of the Problem

To facilitate the description of solutions in the rest of the chapter, the Correct Conjectures are coded consecutively **CC1, CC2,** . . .; the proof(s) are coded similarly to the corresponding conjecture i.e., **P1** is the proof of the conjecture **CC1**, **P3** is the proof of **CC3**; letters are added to the number of the proof to represent

Fig. 1 The case where ABCD is a parallelogram



different proofs for the same conjecture i.e., **P1a** and **P1b** are two possible proofs for the conjecture **CC1.** The code **IC** is used for Incorrect Conjectures.

As mentioned above, the problem has many solutions, since there are different types of quadrilaterals that can be investigated, conjectured and proven. The cases that were most commonly investigated are the following:

CC1. If ABCD is a parallelogram, then HKLM is a rectangle (Fig. 1).

P1a. In a parallelogram the consecutive angles are supplementary; in particular $\widehat{ADC}+\widehat{DCB}=180°$; also in the triangle KDC, $\widehat{KDC}+\widehat{KCD}=\frac{1}{2}(\widehat{ADC}+\widehat{DCB})=90°$ leaving 90° to $\widehat{DKC}$. The same proof should be applied to two additional angles chosen from $\widehat{KHM}$, $\widehat{HML}$ and $\widehat{KLM}$ thus making HKLM a rectangle.

P1b. The same conjecture could also be proven using corresponding angles:

- Consider I the intersection point between (DC) and (AM), the bisector of $\widehat{BAD}$.
  - $\widehat{BAI}$ = ½ $\widehat{BAD}$ so $\widehat{AID}$ = ½ $\widehat{BAD}$ ($\widehat{BAI}$ and $\widehat{AID}$ alternate interior angles)
  - $\widehat{KCD}$ = ½ $\widehat{BCD}$ ((CK) bisector of $\widehat{BCD}$)
  - But $\widehat{BAD}$ = $\widehat{BCD}$ (opposite angles in a parallelogram)

  Therefore, the two angles $\widehat{AID}$ and $\widehat{KCD}$ are equal and have the position of corresponding angles, so (KC) is parallel to (AM).
- Similarly, (DK) is parallel to (BM) thus HKLM is a parallelogram.
- In addition, $\widehat{DKC}=90°$ (same proof as in P1a) thus HKLM is a rectangle.

CC2. If ABCD is a rectangle, then HKLM is a square (Fig. 2).

P2. The proof will be divided into two phases: showing that HKLM is a rectangle, then showing that it's a square.

- Showing that HKLM is a rectangle by showing that it has three right angles:
  - $\widehat{AMB}=180°-(\widehat{MAB}+\widehat{MBA})=180°-(45°+45°)=90°$.
  - Similarly, $\widehat{DKC}=90°$
  - $\widehat{KLM}$ or $\widehat{KHM}=90°$ (using triangles BLC or AHD respectively to show $\widehat{BLC}$ or $\widehat{AHD}=90°$ then vertically opposite angles).

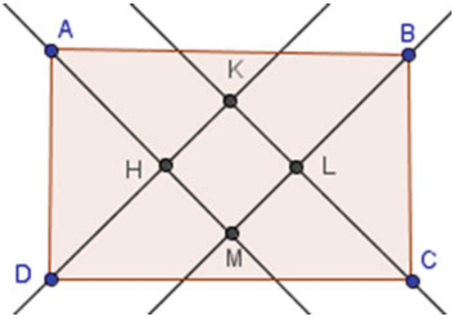


Fig. 2 The case where ABCD is a rectangle

The first phase could be proven using the same proof as **P1b, as ABCD is a parallelogram**.

- The second phase requires the use of congruent triangles to show consecutive equal sides, as follows:
  - KC = KL + LC and KD = KH + HD
  - KC = KD (KCD right isosceles triangle at K with base angles of 45°)
  - LC = HD (BLC and AHD are congruent triangles)

  Thus KH = KL, and HKLM is a square.

CC3. If ABCD is a rhombus then H, K, L, and M coincide (Fig. 3).

P3. This case requires the instantiation of only one property, namely, the bisectors of a rhombus are also its diagonals, so they intersect at one point; thus H, K, L and M are coincident.

CC4. If ABCD is a square, then H, K, L, and M coincide (Fig. 4).

P4. Since the square is a special rhombus, this case requires the same proof **P3**.

Description of Students' Work

The following section presents the work of four of the observed student pairs, which were found to be most interesting and significant for the purpose of this study. The description focuses on the important moments in students' work and prepares the ground for the detailed analysis presented in the subsequent section. The four observations are named consecutively O1, O2, O3 and O4.

O1. Kevin and Sam

Kevin and Sam explored the case of the random quadrilateral, i.e. a quadrilateral that is not a trapezoid, a kite, or a parallelogram, and four types of special quadrilaterals. For each case, they drew a new figure by placing the vertices of

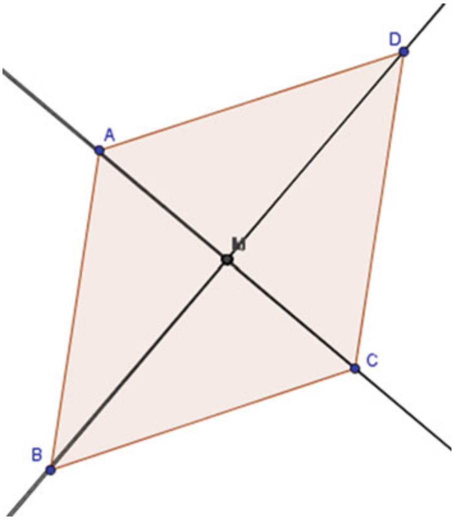


Fig. 3 The case where ABCD is a rhombus

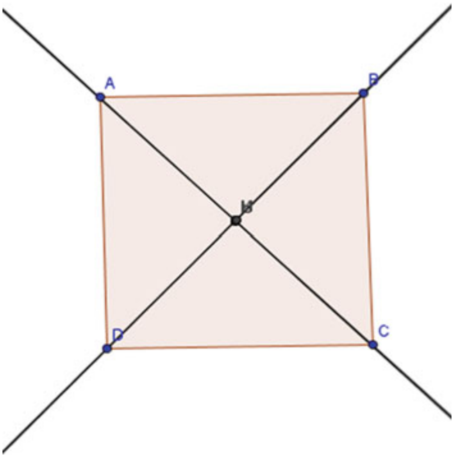


Fig. 4 The case where ABCD is a square

the quadrilateral in the form of the intended shape based on perceptual approximation.

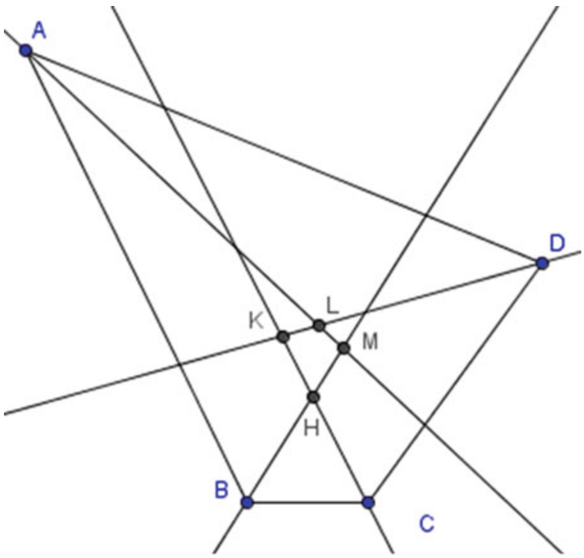
IC1. If ABCD is a random quadrilateral then HKLM is also a random quadrilateral

Kevin and Sam drew a random quadrilateral using the *Segment* tool (Fig. 5). Without dragging or doing any manipulation, they directly wrote the conjecture IC1. They did not attempt any proof.

This conjecture is considered as an empty conjecture. A quadrilateral cannot be proven to be "random" since random quadrilaterals are not defined according to specific characteristics or necessary geometric properties.

CC1. If ABCD is a square then H, K, L and M coincide

Fig. 5 **O1-**When ABCD is a random quadrilateral



Students used the *Regular Polygon* tool to construct the square. Then, they developed the conjecture **CC1** but did not attempt to find a proof.

CC2. If ABCD is a rectangle then HKLM is a square

To test the case of the rectangle, Kevin and Sam made a new drawing by placing the vertices to form a rectangle based on visual approximation by means of the *Polygon* tool (Fig. 6).

Kevin: I think it's the same as the square
Sam: Mmmmm why?
Kevin: both have angles of 90°
Sam: Oh yes, you're right.

When they saw that H, K, L and M formed a square they doubted the correctness of the drawing. After some reflection, Kevin accepted the graphical result since he was able to find part of a theoretical support: he noticed that in triangle ABM, $\widehat{MAB} = \widehat{MBA} = 45°$ thus $\widehat{AMB} = 90°$, and formulated the conjecture CC2.

However, the proof they developed was incomplete, mainly because they used "(AB) parallel to (HL)" without proving it first:

- *In triangles ABM and MLH*
  - $\widehat{AMB} = \widehat{HML} = 90°$ (vertical angles)
  - $\widehat{MHL} = \widehat{HLM} = 45°$ alternate interior angles with $\widehat{ABM}$ and $\widehat{MAB}$ ((AB) parallel to (HL))
    Thus HLM is right isosceles and MH = ML
- *Having 2 consecutive equal sides and 4 right angles then HKLM is a square.*

CC3. If ABCD is a parallelogram then HKLM is a rectangle

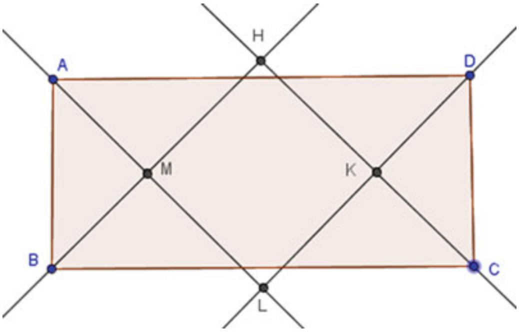


Fig. 6 **O1-**When ABCD is a rectangle

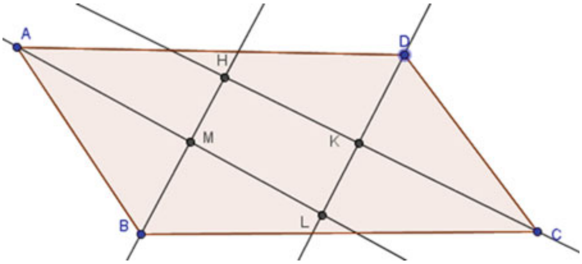


Fig. 7 **O1-**When ABCD is a parallelogram

To test the case of the parallelogram, Sam made a new soft drawing also based on visual approximation (Fig. 7). Kevin and Sam developed the conjecture **CC3** and proved it as follows:

- *In the triangle KDC,*
  $\widehat{KDC} + \widehat{KCD} = \frac{1}{2}(\widehat{ADC} + \widehat{DCB}) = \frac{1}{2}.180° = 90°$
  We deduce that $\widehat{DKC} = 180° - 90° = 90°$
- *Similarly* $\widehat{AMB} = 90°$.
- *Thus HKLM is a rectangle.*

However, they did not mention explicitly the fact that $\widehat{HKL}$ and $\widehat{HML}$ are 90°, as they are vertically opposite to $\widehat{DKC}$ and $\widehat{AMB}$ respectively. Also, they showed only two right angles.

IC2. If ABCD is a right trapezoid then HKLM is a rectangle

Kevin and Sam were able to prove that $\widehat{HML}$ and $\widehat{HKL}$are right angles (Fig. 8):

- *In triangle ABM,* $\widehat{MAB} + \widehat{MBA} = ½\,\widehat{DAB} + ½\widehat{ABC} = 90°$ *thus* $\widehat{AMB} = 90°$ *which implies that* $\widehat{HML} = 90°$.
- *In triangle DKC,* $½(\widehat{ADC} + \widehat{DCB}) = ½•180° = 90°$ *thus* $\widehat{DKC} = 90°$ *which implies that* $\widehat{HKL} = 90°$.

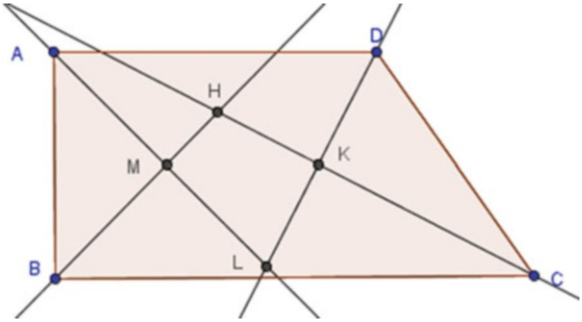


Fig. 8 O1-When ABCD is a right trapezoid

They were not able to go further and prove a third angle right. Nevertheless, Kevin insisted that HKLM should be a rectangle, despite the fact that the drawing shows it not to be a rectangle. When writing the proof, they only stated the previous two right angles $\widehat{HML}$ and $\widehat{HKL}$ but concluded that HKLM is a rectangle having four right angles.

O2. Eric and Vicky

Eric and Vicky started their work by developing three conjectures consecutively and then attempting to prove them. At the end, they developed a fourth conjecture and attempted to prove that one as well.

Eric and Vicky drew the quadrilateral ABCD using the *Polygon* tool and then throughout the session they dragged its vertices to form the intended shape, each time based on perceptual approximation. They stopped at each figure and observed how HKLM varied as a result of the changes effected to ABCD. They developed three correct conjectures for the cases where ABCD is a rectangle, a square and a parallelogram. When Vicky pointed out that they need to prove these conjectures, the conjecturing phase was interrupted and they started working on the proofs.

Following are the conjectures developed at the beginning of the session.

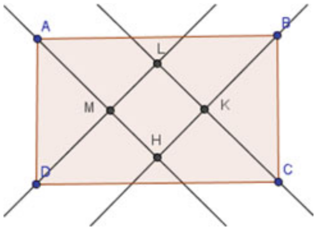
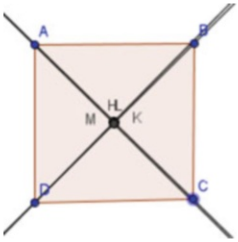
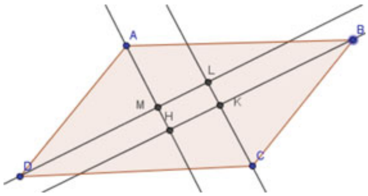
During the proving phase, Eric and Vicky worked on four different conjectures, two of which are correct and two are incorrect:

CC1. If ABCD is a rectangle then HKLM is a square

Eric and Vicky dragged A, B, C and D to form a rectangle (see figure in Table 1). They showed that in triangle ADM, $\widehat{MAD} = \widehat{MDA} = 45°$ thus $\widehat{AMD} = 90°$; similarly $\widehat{BKC} = 90°$ and $\widehat{DLC} = 90°$ thus HKLM is a rectangle. They tried using congruent triangles to further show that it's a square but failed to isolate the triangles needed for the proof.

CC2. If ABCD is a square then H, K, L and M coincide

Table 1 O2. The first three conjectures

	**CC1**. If ABCD is a rectangle then HKLM is a square.
	**CC2**. If ABCD is a square then H, K, L, and M coincide.
	**CC3**. If ABCD is a parallelogram then HKLM is a rectangle.

Eric and Vicky Eric dragged A, B, C and D to form a square (see figure in Table 1) and developed a deductive proof by saying that the angle bisectors in a square are also its diagonals, which intersect at a single point.

IC1. If ABCD is a parallelogram then H, K, L, and M coincide

Eric dragged A, B, C and D to form a parallelogram which happened to be a rhombus in which H, K, L, and M coincided (Fig. 9). Eric and Vicky thought that ABCD was only a parallelogram and elaborated the conjecture. They did not notice that this new conjecture contradicted the conjecture **CC3** previously elaborated in the conjecturing phase. Since the conjecture was incorrect it led to a wrong proof as Eric and Vicky argued that in a parallelogram the angle bisectors are also the diagonals which intersect at one point.

IC2. If ABCD is a rhombus then HKLM is a square

Eric and Vicky dragged to form a rhombus ABCD but the sides were not exactly equal, thus H, K, L and M did not coincide (Fig. 10). When they saw that points H, K, L, and M formed a small square (although in their drawing HKLM looked more like a rectangle), they generated another wrong conjecture: "If ABCD is a rhombus then HKLM is a square". They only showed that HKLM is a rectangle by stating that in a rhombus the diagonals which are also the angle bisectors are perpendicular, so the angles are right making it a rectangle, which is a failed proof.

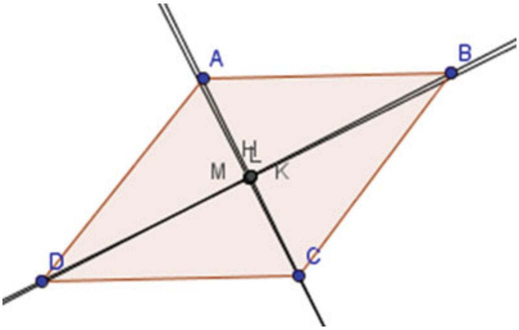


Fig. 9 O2-When ABCD is a parallelogram

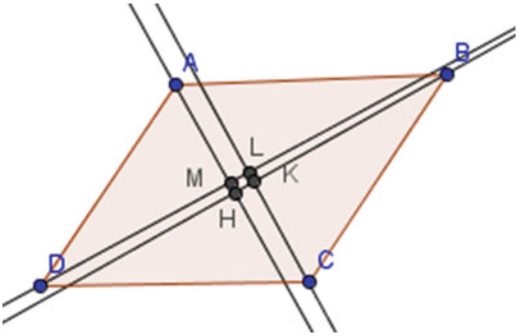


Fig. 10 O2-The parallelogram ABCD mistaken for a rhombus

O3. Matt and Jessica

The work of Matt and Jessica was divided into two separate phases: a conjecturing phase followed by a proving phase. Following are the conjectures developed at the beginning of the session.

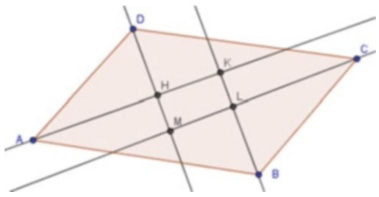
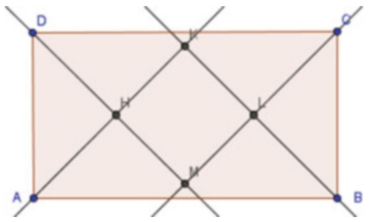
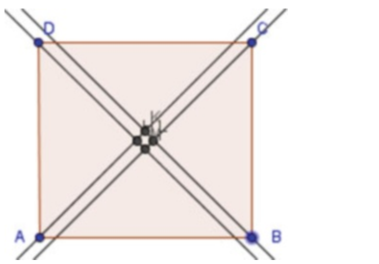
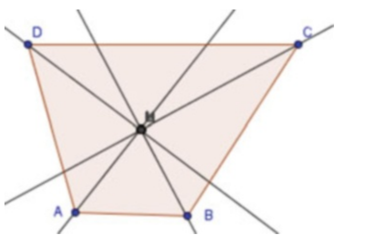
In the proving phase, Matt and Jessica worked on proving the first two conjectures:

CC1. If ABCD is a parallelogram then HKLM is a rectangle (see Table 2)

- *(AK)//(CM) since they are two lines coming from two equal and opposite angles. Similarly (DM)//(BK).*
- *In the triangle MDC,* $\widehat{MDC} + \widehat{MCD} = \frac{1}{2}(\widehat{ADC} + \widehat{BCD}) = \frac{1}{2} \bullet 180^\circ = 90^\circ$ *thus* $\widehat{DMC} = 90^\circ$.
- *Thus HKLM is a rectangle being a parallelogram with one right angle.*

Matt and Jessica showed that the opposite sides are parallel and that there is one right angle. However the property used to show the parallel sides was incorrect.

Table 2 O3. The first four conjectures

	**CC1**. If ABCD is a parallelogram then HKLM is a rectangle.
	**CC2**. If ABCD is a rectangle then HKLM is a square.
	**IC1**. If ABCD is a square then HKLM is a square.
	**IC2**. If ABCD is a trapezoid then H, K, L and M coincide.

CC2. If ABCD is a rectangle then HKLM is a square.

Matt and Jessica considered a new drawing which happened to be a particular case where AB = 2 BC thus M and K were the midpoints of [AB] and [DC] respectively (Fig. 11).

The proof was as follows:

- *HKLM is a rectangle since it has 4 right angles:*
  - *In the triangle MDC,* $\widehat{MDC} + \widehat{MCD} = \frac{1}{2}\ \widehat{ADC} + \frac{1}{2}\widehat{BCD}\ = 90°$ *thus* $\widehat{DMC} = 90°$. *Similarly* $\widehat{AKB}\ = 90°$.
  - *In the triangle ADH,* $\widehat{ADH} + \widehat{HAD}\ = \frac{1}{2}\widehat{ADC} + \frac{1}{2}\widehat{BAD} = 90°$ *thus* $\widehat{DHA} = 90°$ *which implies that* $\widehat{KHM}\ = 90°$. *Similarly* $\widehat{KLM}\ = 90°$.

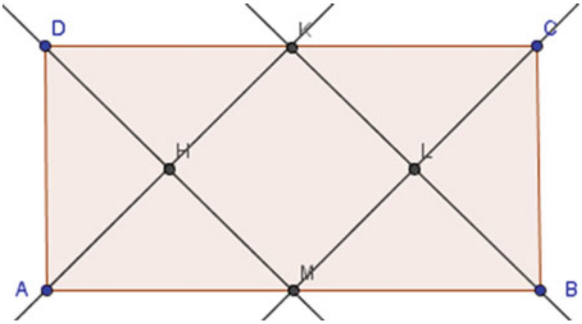


Fig. 11 O3-Special case drawing for the rectangle ABCD

- *HKLM is a square since it is a rectangle with two consecutive equal sides: We consider the two triangles DHK and AHM:*
  - $\widehat{DHK} = \widehat{AHM} = 90°$.
  - $DH = AH$ (ADH right isosceles triangle)
  - $\widehat{HDK} = \widehat{HAM} = 45°$
    Therefore $HK = HM$.

The proof started correctly; however, in the second phase, when showing that the triangles DHK and AHM are congruent, the students considered $\widehat{HDK} = \widehat{HAM} = 45°$, which worked only because M and K happened to belong to [AB] and [DC] respectively. Because the drawing represents a special case of the figure, the proof is not generic.

O4. Tom and Mary

Tom and Mary started their exploration with the case of the square ABCD. They directly observed that H, K, L, and M are coincident, since the angle bisectors in a square are also its diagonals, which intersect at a single point. Thus the first conjecture was **CC1.** If ABCD is a square then H, K, L and M coincide.

Then they considered the case of ABCD being a right trapezoid (Fig. 12). As they were not able to determine the nature of HKLM at first sight, they measured the angles of quadrilateral HKLM. Based on those measures they developed the conjecture **IC1.** If ABCD is a right trapezoid then HKLM has two right angles and one angle of 60°. They proved the two right angles as follows:

- *In the triangle KDC,* $\widehat{CKD} = 180° - (\frac{1}{2}\widehat{ADC} + \frac{1}{2}\widehat{BCD}) = 90°$ *thus* $\widehat{HKL} = 90°$.
- *In the triangle MBA,* $\widehat{BMA} = 180° - \frac{1}{2}(\widehat{ADC} + \widehat{BCD}) = 180° - \frac{1}{2}.180° = 90°$ *thus* $\widehat{HML} = 90°$.

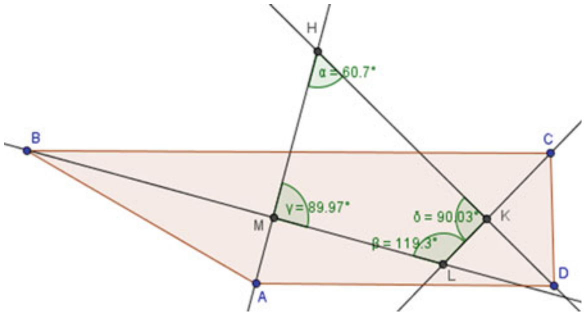


Fig. 12 O4-Displaying the measures of the angles of HKLM when ABCD is a right trapezoid

They tried, in vain, to prove that $\widehat{MHK} = 60°$. They did not realize that the measure of $\widehat{MHK}$ is a spatio-graphic one relative only to this particular instance of the figure.

Analysis of Students' Proving Processes

Building on the four previous cases, it can be seen that a majority of the proofs that the students developed were flawed. Tracing back through the mistakes, we can see that they originated at different moments of the proving process: some mistakes were found at the graphical level, which lead to both a failed conjecture and a failed proof; other mistakes were found at the conjecturing level, i.e., the figure was correct but the subsequent conjecture was not, leading to a failed proof; while other mistakes were at the theoretical level, i.e., students failed to find a correct theoretical support for their conjecture.

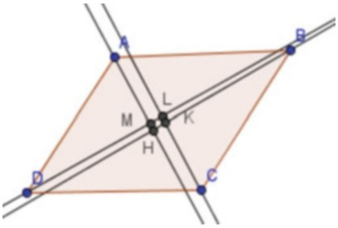
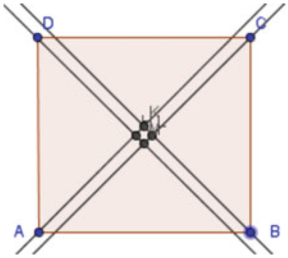
We developed an associated typology of failed proving processes with three main types, namely: *Failed Construction, Failed Conjecture* and *Failed Proof*; each type can be elaborated with particular sub-types illustrated in the following sections. The purpose of this typology is not to define a totally self-contained set of categories, but rather to make it easier for a teacher or researcher to undertake an analysis of different possible mistakes when assigning proving tasks within DGE.

Failed Construction

This first type indicates that the mistake originated from the figure, causing the subsequent conjecture and proof to be incorrect. The following three types of *Failed Construction* were identified, based on the analysis of the students' work.

Type 1 We consider the following two examples (see Table 3) taken from O2 and O3 respectively.

Table 3 First type of failed construction

Case	O2 – Eric and Vicky	O3 – Matt and Jessica
Figure		
Conjecture	**IC2**. If ABCD is a rhombus then HKLM is a square.	**IC1**. If ABCD is a square then HKLM is a square.
Comments	The mistakes found in both conjectures originated at the graphical level since both drawings were inaccurate:	
	The sides of the intended rhombus ABCD were not exactly equal; thus instead of coinciding, H, K, L and M formed a small rectangle seen as square by the students.	The sides of the intended square ABCD were not exactly equal; thus instead of coinciding, H, K, L and M formed a small square.

Therefore, the first type of *Failed Construction* can be named *Inaccurate Drawing*. An *Inaccurate Drawing* results from a construction which is based on visual approximation and does not incorporate the use of any DGE tool that can validate it. An *Inaccurate Drawing* can create the illusion of elements that do not exist in reality or properties of the figure which are incorrect.

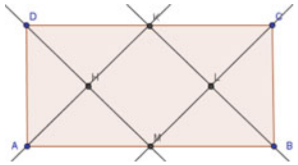
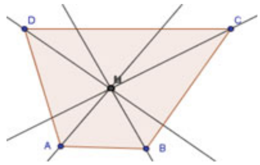
Type 2 A second type of *Failed Construction* (see Table 4) was observed twice in **O3**.

This type of *Failed Construction* can be named *Special Case Drawing*. A *Special Case Drawing* is a correct drawing but it incorporates (on purpose or not) extra properties. Thus the subsequent conjecture and proof are applicable only to this special case and cannot be generalized.

Type 3 A third type of a *Failed Construction* was seen in the work of a pair from the class who drew the figure shown in Fig. 13. The conjecture that they developed was: “If ABCD is a parallelogram then HKLM is a right trapezoid”. Although the conjecture was consistent with the figure but it was incorrect because the figure in itself was incorrect: the angle bisectors were not constructed using the *Angle Bisector* tool; instead they were constructed through using *Line through Two Points* and adjusted based on visual approximation, thus a rectangle was not formed.

This type of *Failed Construction* can be named *Incorrect Construction*. An *Incorrect Construction* is caused by the misuse or non-use of the proper tools available in DGE. Instead of a formal axiomatic construction protocol, students might construct the figure based on visual approximation and/or use some tools out of context, which leads to an incorrect construction.

Table 4 Second type of failed construction

Figure		
Conjecture	**CC2**. If ABCD is a rectangle then HKLM is a square.	**IC2**. If ABCD is a trapezoid then H, K, L and M coincide.
Comments	Both drawings represent special cases of the figure. The observations made based on these drawings do not hold for all other instances of the figure; therefore the conjecture and/or proof are not generic.	
	Although the conjecture is correct, the proof was based on the fact that $\widehat{HDK} = \widehat{HAM} = 45°$ which is specific to this case of the figure where the length of ABCD was double its width. If the rectangle was not constructed with this special relation the points K and H would not belong to the sides of ABCD and the angles $\widehat{HDK}$ and $\widehat{HAM}$ would not be not 45°.	The conjecture itself is incorrect since in a different instance of this fig. H, K, L and M will not coincide

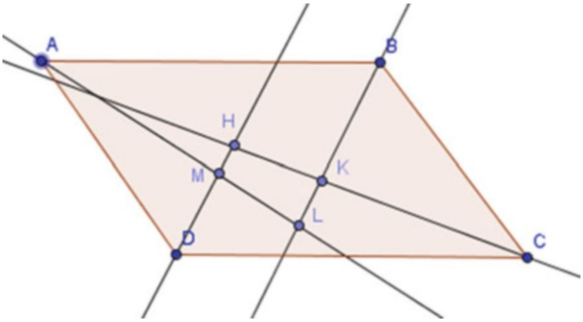


Fig. 13 The angle bisectors incorrectly constructed

Failed Conjecture

The second type of failed proving processes is *Failed Conjecture*. Three types of *Failed Conjecture* were identified, based on the analysis of the students' work.

Type 1 In **O1**, Kevin and Sam dragged the vertices of ABCD to form a right trapezoid (Fig. 14). Even though it was clear in the drawing that HKLM is not a rectangle, Kevin insisted that HKLM should be a rectangle and elaborated **IC2.** If ABCD is a right trapezoid then HKLM is a rectangle.

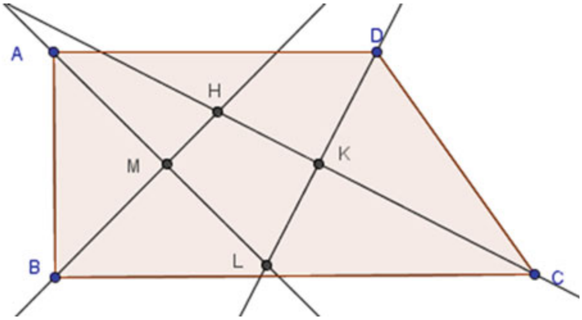


Fig. 14 Conjecturing that HKLM is a rectangle although in the figure it is not

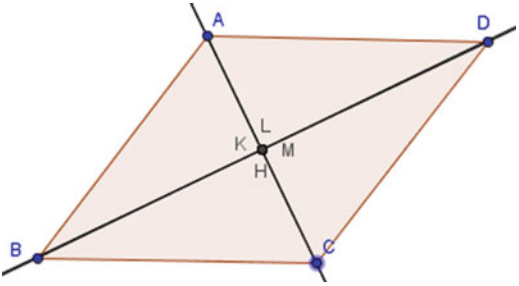


Fig. 15 The rhombus ABCD mistaken for a parallelogram

Another example of such a conjecture is taken from **O2.** Eric and Vicky dragged ABCD to form a parallelogram but without noticing that they formed a rhombus (Fig. 15). They saw that H, K, L, and M coincided and thus formulated the following conjecture: “If ABCD is a parallelogram then H, K, L, and M coincide”; whereas the correct conjecture for the case of the parallelogram in general is: “If ABCD is a parallelogram then HKLM is a rectangle”. In the particular case where the parallelogram is a rhombus, the vertices of the rectangle HKLM coincide. As a proof, Eric suggested that the bisectors of the angles of a parallelogram are also the diagonals which intersect at one point.

The figure at hand is a rhombus but Eric and Vicky thought it was a parallelogram; thus the constructed figure is correct but the elaborated conjecture did not accurately describe it, which resulted in a failed conjecture and a failed proof.

This type of *Failed Conjecture* can be named *Conjecture-Figure Inconsistency*. It occurs when the figure is correct but the students generate a conjecture that does not properly reflect its properties. They fail in recognizing all the invariants of the figure that lead to the desired conclusion since they focus on those recognized at first sight and perceived to be right.

Type 2 In **O4.** Tom and Mary developed **IC1.** If ABCD is a right trapezoid then HKLM has two right angles and one angle of 60° (Fig. 16). They based their conjectures on the measures they took for the angles of HKLM in a specific instance of the figure. They showed that $\widehat{HKL} = 90° \textit{ and } \widehat{HML} = 90°$. They did not realize

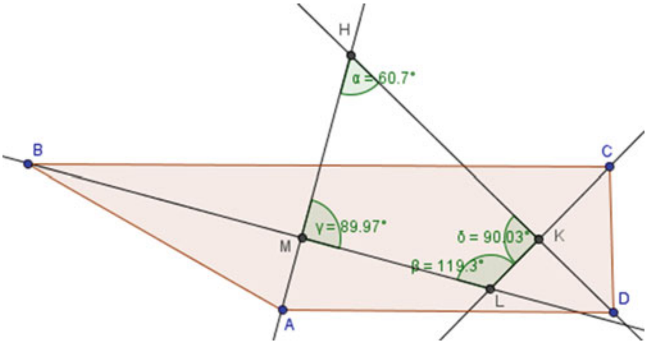


Fig. 16 Considering the measures of the angles of HKLM

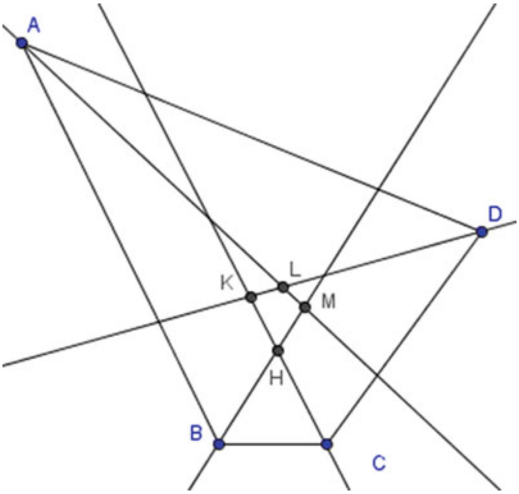


Fig. 17 When ABCD is a random quadrilateral

that the measures of these two angles are true in all instances but the measure of $\widehat{MHK}$ is a spatio-graphical one relative only to this particular instance of the figure.

This type of *Failed Conjecture* can be named *Spatio-graphical Conjecture*. A *Spatio-graphical Conjecture* occurs when students analyze geometric figures on the basis of their appearance and the visual transformations that they perform on the drawings. They do not take into consideration the geometrical relational properties of the figure.

Type 3 In **O1**, Kevin and Sam drew a random quadrilateral (Fig. 17) and developed the conjecture: "If ABCD is a random quadrilateral then HKLM is also a random quadrilateral".

This type of *Failed Conjecture* can be named *Empty Conjecture* since the premise does not provide any properties to work with; it is rather defined by the absence of properties; similarly for the conclusion, a quadrilateral cannot be proven to be "random".

Failed Proof

Building a proof is, in fact, a complex process as it involves locating the information needed to apply a property or a theorem, selecting from among several known rules, and selecting from among several theories that provide justification for the chosen rule. Four types of *Failed Proof* were identified, based on the analysis of the students' work.

Type 1 In **O2**, Eric and Vicky worked on proving "If ABCD is a rectangle then HKLM is a square". First, they showed that HKLM is a rectangle. Then they tried using congruent triangles to further show that it's a square but failed to isolate the triangles needed for the proof.

This type of *Failed Proof* can be named *Incomplete Proof*. An *Incomplete Proof* occurs when students have a road map for their proof, start part of it but interrupt their proof due to obstacles in finding theoretical justifications or simply due to time constraints. The students are aware that the proof is incomplete and do not assume the conjecture to be successfully proven.

Type 2 In **O3**, Matt and Jessica worked on proving "If ABCD is a parallelogram then HKLM is a rectangle". They showed that the opposite sides are parallel, and that the quadrilateral has one right angle. However, the property used to show the parallel sides was incorrect: *(AK)//(CM) since they are two lines coming from two equal and opposite angles*. Because the figure (Fig. 18) was strongly convincing, the students invented a property to fit what they saw as true on the screen.

This type of *Failed Proof* can be named *Fabricated Property*. It occurs when students observe in DGE a relationship among certain objects of the figure and invent a property that fits. Instead of analyzing the relationships observed in the

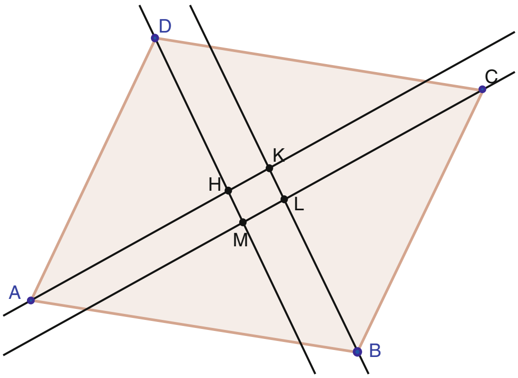


Fig. 18 Fabricating a property to justify that (AK)//(CM).

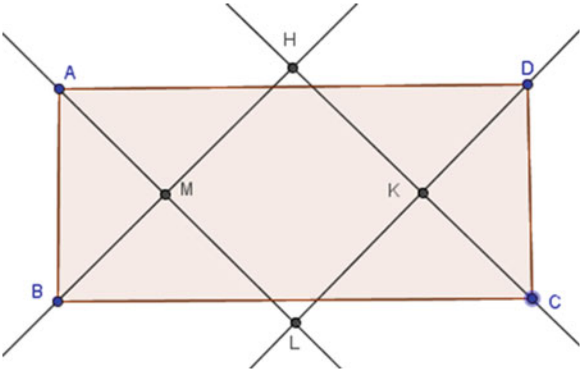


Fig. 19 Assuming a property of the figure as given

spatio-graphical field according to the properties of the theoretical field, the theoretical field is manipulated to fit the relationships observed in the spatio-graphical field.

Type 3 In **O1**, Kevin and Sam were proving "If ABCD is a rectangle then HKLM is a square" (Fig. 19). In the proof they assumed that (AB) is parallel to (HL) without proving it first:

This type of *Failed Proof* can be named *Assumed Hypothesis*. Given that the spatio-graphical field is a powerful influence on students' minds, students might confuse the hypothesis of the problem with the properties revealed by the dynamic figure, and thus assume as hypothesis, a property that needs to be proven.

Type 4 In **O1**, Kevin and Sam were proving the conjecture "If ABCD is a parallelogram then HKLM is a rectangle". They showed that HKLM has two right angles and deduced that it is a rectangle. They did not realize that the arguments that they presented to show that HKLM is a rectangle are necessary but insufficient since three right angles were needed to show that it is a rectangle.

This type of *Failed Proof* can be named *Overreached Conclusion*. An *Overreached Conclusion* occurs when students are unable to identify all the necessary and sufficient conditions for a property to be true. The reached conclusion might be true but it is not supported with all the necessary arguments.

Discussion

Interplay Between the Spatio-Graphical Field and the Theoretical Field

According to Laborde (1998), the solution of a geometry problem lies in both the Spatio-Graphical (SG) and Theoretical (T) domains, and is characterized by continuous shifts between them. Students work at three different levels while solving a problem: at a spatio-graphical level, when they observe spatio-graphical invariants; at a theoretical level, when they use definitions and theorems; and at a combination of the spatio-graphical and the theoretical level, when they understand and justify spatial observations in theoretical terms. The mistakes and difficulties identified in the proving process can be better understood in light of this distinction: since students are perceived to work at three different levels (SG, T, or SG and T), then each mistake can be localized in one of these levels and analyzed accordingly.

From the results of the analysis of students' work, the only mistake that occurred at the spatio-graphical level is the *Incorrect Construction* resulting from a purely technical obstacle related to knowing the tools available in DGE the required inputs, and the generated outputs.

The mistakes that occurred at the theoretical level are: *Empty Conjecture, Incomplete Proof* and *Overreached Conclusion*. These mistakes are purely theoretical since students were either unaware of the problem in formulating a conjecture having a premise and/or conclusion too general to be proven (the case of the empty conjecture), or they did not find the necessary and sufficient theoretical support for the conjecture (case of the incomplete proof and overreached conclusion)

The remaining observable mistakes in this study, i.e., *Inaccurate Drawing, Special-Case Drawing, Spatio-graphical Conjecture, Conjecture-Figure Inconsistency, Assumed Hypothesis* and *Fabricated Property*, occurred at the spatio-graphical and theoretical levels. These mistakes were the ones most frequently observed in students' work, and indeed these are widely discussed in the literature. According to Laborde (1998), the use of diagrams is usually a tedious task for students, because they can be interpreted on two different and ambiguous levels by students. On the one hand, they refer to theoretical objects defined by axioms, properties and theorems, while on the other hand, students are drawn to engage in a purely empirical and perceptual activity because of the strong graphical and spatial properties that diagrams provide. When dealing with a geometrical figure, students always find it hard to distinguish between what you are allowed to read and say, what you are allowed to read without saying, and what you are not allowed to read.

The main obstacle behind the *Inaccurate Drawing, Special-Case Drawing* and *Spatio-graphical Conjecture* is the confusion between drawing and figure. Laborde and Capponi (1994) defined a figure as a theoretical construct associated with all its possible drawings. The drawings are interpreted based on the mathematical knowledge of the student and on the nature of the drawing and the way it is represented. This is why and where most students face ambiguities that create interpretation problems. It is not easy for them to detach themselves from the drawing in order to access the figure, as they have to distinguish the properties of the drawing which correspond to the figure from the ones that are only spatial, perceived properties that cannot be used in the conjecture or proof. According to Hölzl (2001), DGE makes it easier for students to determine which geometrical properties of the figure can be "read", since they are those that hold under dragging; the correct constructions are those that preserve these properties under dragging. In particular, the appropriation of dragging is what should allow the students to distinguish between a drawing and a figure and thereby facilitate the transition between them.

The reason behind the *Conjecture-Figure Inconsistency* errors/difficulties can be interpreted using Mariotti's (2006) work; she argues that in order to generate conjectures, the student has to interpret the motion dependency observed through dragging in terms of the concomitant, logical dependency between what will become the premise, and the conclusion of the statement of a conjecture. It is not given that students are capable of transforming perceptual data into a conditional relationship. In fact, it is a task which is not at all trivial.

An *Assumed Hypothesis* occurs when students confuse the direct and indirect invariants identified by Mariotti (2014). The direct invariants are the invariant properties given by the problem possible to be used in a hypothesis. The indirect invariants are invariant properties observed as a consequence of the relationship between direct invariants, and therefore are not part of the hypothesis; they need to be proven.

The obstacles that the students who participated in this study faced in the case of *Fabricated Property* were also observed by Duval (1994), who underlines that students tend to over-trust the shapes and properties they recognize at first sight; the first look of the figure seems to exclude a mathematical look at this figure. In addition, students did not seem capable of identifying the solution elements that they could read in the figure, because this required them to focus on specific parts more than others.

Instructional Strategies for Remediation

The difficulty identified in the proving process and occurring at the spatio-graphical level, i.e., *Incorrect Construction*, may be remediated in the technological field as students practice the use of DGE tools, each one according to the way it was designed and to the purpose behind its use. The difficulties that occurred at the theoretical level, (*Empty Conjecture, Incomplete Proof* and *Overreached Conclusion*), may be remediated in the geometrical field, outside DGE. Students need to learn and practice the content of a proof, i.e., the required theorems and properties, as well as the structure of a proof, i.e., the technique of writing a logical chain of deductive arguments. As for the remaining mistakes, namely the ones that occurred at the spatio-graphical and theoretical levels, they may be remediated by instructional strategies on the nature of geometry in a DGE, thus at a combination of both fields. Since the nature of geometry is fundamentally changed in a DGE (Laborde and Sträßer 2010; Mariotti 2012), students need to be aware of the rules and strategies inherent to DGE in order to be well-prepared to solve problems in this environment. Instructional interventions should focus on:

- Introducing the difference between a drawing and a figure, which is overlooked in the paper-and-pencil environment but essential to DGE.
- Familiarizing the students with the differences between soft and robust constructions, and the optimal use of each type; this distinction is nonexistent in the paper-and-pencil environment.
- Defining and experimenting with invariants under dragging, and highlighting the difference between direct and indirect invariants.
- Scaffolding the students in discovering the different roles that can be played by the dragging tool, such as a construction tool and a verification tool.

Conclusion

In terms of the framework of types of justifications developed by Marrades and Gutierrez (2000), students in this study elaborated two types of justifications: *Deductive justifications by structural thought experiment*, and *failed justifications*. However, not all failed justifications were comparable, since students faced obstacles in the figure, or in the conjecture or in the proof itself; therefore, for a more accurate description, we prefer to adopt a more general term, "failed proving processes", rather than "failed justifications"; this leads to a wider analytic framework. The analysis of participating students in this study led to a framework of failed proving processes, one that was not possible without the DGE. It consists of three main categories: *Failed Construction, Failed Conjecture*, and *Failed Proof*. *Failed Figure* means that the mistake was at the graphical level; the figure was either incorrect, inaccurate or represented a special case. *Failed Conjecture* designates that the mistake was found in the conjecture, which was based on spatial properties, inconsistent with the figure or an empty conjecture. The last type was *Failed Proof* indicating a mistake in the proof; the proof was either incomplete, used a fabricated property, assumed a property observed in the figure as hypothesis, or overreached the conclusion by not presenting sufficient arguments.

The purpose of this framework is not to define a totally self-contained set of categories, but rather to make it easier for teachers and researchers to anticipate and to undertake an analysis of different possible mistakes as they design or analyze proving tasks within DGE. Therefore, when teachers become aware of the difficulties that students might confront in DGE, and of the resulting mistakes, they would be better equipped to teach, warn, and guide students. The framework of failed proving processes developed in this study can be validated in further research and extended if new categories emerge. In another study, a teaching sequence based on the suggestions of instructional strategies for remediating the difficulties faced in the proving process can be developed, implemented in a classroom, and its efficiency evaluated. Other research can conduct a comparative study to analyze whether these mistakes would present themselves differently in the paper-and-pencil environment and in which way.

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Disclosing the “Ræmotionality” of a Mathematics Teacher Using Technology in Her Classroom Activity

Marina De Simone

Abstract In this chapter, I will focus on the relation between affect and technology during the classroom activity of a mathematics teacher. This constitutes a first approach for developing a new aspect of my PhD thesis where, in general, I tried to bring together cognitive and affective dimensions in the classroom behaviour of mathematics teachers, often considered separately. In particular, in this paper, I will focus on the practice of a teacher who routinely uses digital technologies in her mathematical activity, showing how her expectations on the use of technology are actually reflected in her classroom experiences and how these expectations inform us about the reasons of their actions.

Keywords Technology • Mathematics teaching • Emotional orientation • Linear equations

Introduction

Over the last two decades, mathematics education research has increasingly focused on the role of digital technologies in teaching/learning processes (Artigue 2007, 2010; Clark-Wilson et al. 2015; Gueudet et al. 2013). Many studies have documented how the use of ICT can enhance students’ learning (Artigue 2013; Buckingham 2013; Clark-Wilson et al. 2013). As a result, curriculum documents and professional development programmes commonly encourage teachers to employ technology in their practice. However, teachers using technology have to cope with factors of a different nature than they are used to. In particular, as presented in this chapter, I will discuss how cognitive and the affective factors are unavoidably intertwined in the practice of a teacher who uses digital technologies in her classroom practice. In fact, the teacher decides to employ technology not only on a rational level, but also on an affective one, because she has expectations toward students’ learning, and toward integrating ICT into her practice.

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Looking at this teacher's expectations for the use of technology, I could also infer numerous reasons for the decisions she makes in her classroom in general.

The core of my research is the decision-making of the teacher, because, as much research in mathematics education has consistently highlighted, decision-making has a very crucial role in teaching activity. For example, Bishop pointed out that decision-making is "at the heart of the teaching process" (Bishop 1976, p. 42).

From a theoretical point of view, I rely on the notion of "emotional orientation" (Brown and Reid 2006), locally analysing teacher's decisions drawing on the philosophical theory of rationality (Habermas 1998). In particular, I employ the neologism "ræmotionality" (De Simone 2015), which refers to the rationality and the emotions of the teacher as a *unicum*, that is, as a unique example or specimen.

Entering in the structure of this chapter: in the first section, I illustrate the theoretical perspective that contextualizes my work, explaining also the analytical tool I chose for analysing my data; in the second section, I present qualitative data analyses of five excerpts of the activity of a teacher, Silvia, who uses two kinds of technology while explaining linear equations (GeoGebra and two Java applets); in the third section, I make some concluding remarks, highlighting both points common to the two different types of technology, and how my theoretical framework allows me to make an in-depth analysis of a teacher who uses digital technologies in her practice. I would like to underline that my research is a qualitative study in which I attempt to construct theoretical concepts for analysing particular case studies. These theoretical concepts might be applicable to other cases, without the presumption of generalization.

Linear equations is a mathematical topic that is very interesting to analyse in terms of the coordination among different representation registers, especially using digital technologies. Thus, I choose to develop the analysis about examples concerning linear equations to study how multi-representations influence and intervene in the affective and rational decisions within the mathematics activity.

Theoretical Perspective and Methodology

As already anticipated in the introduction of the paper, my research interest is the study of the intertwinement between the emotional and the rational aspects in the decision-making processes of a mathematics teacher, who uses digital technologies in her practice.

In the mathematics education literature, several authors have focused on the decision-making of the teacher in classroom. For example, Schoenfeld (2010) offered a model for describing the decision-making of teachers according to three different elements: "their knowledge and other intellectual, social, and material resources; their goals; and their orientations (their beliefs, values, and preferences)" (Schoenfeld 2011, p. 1). As he pointed out, these three aspects are deeply related, and the third one, orientation, heavily affects the other two.

In my work described here, I combined two different theoretical perspectives: the philosophical speculation offered by Habermas (1998) and the concept of "emotional orientation" developed by Brown and Reid (2006). The integration of these two theories has produced a new possible theoretical lens, which I have called "ræmotionality", through which I tried to bring together the rationality and emotions of the mathematics teacher, often considered separately.

The Habermasian philosophical speculation has been re-elaborated and adjusted to mathematics education by many researchers (see, e.g., Boero and Planas 2014). Habermas centred his theory on the concept of discursive rationality proper of a rational being (e.g., the mathematics teacher) involved in a discursive activity. He explains that discursive rationality is constituted by three different components: the epistemic rationality, the teleological rationality, and the communicative rationality. These three components of rationality are always present and intertwined in the discursive activity of a rational being. In particular, we face an epistemic rationality when we can simultaneously give an account of the justification of the knowledge at play, the teleological rationality surfaces when "the actor has achieved this result on the basis of the deliberately selected and implemented means" (Habermas 1998, p. 313), and the communicative rationality "is expressed in the unifying force of speech oriented toward reaching understanding" (Habermas 1998, p. 315).

Within the mathematics-related affect research, Brown and Reid (2006) proposed the notion of emotional orientation in order to study the decision-making processes both of the teacher and the students. As the words themselves suggest, the "orientation" in a teacher's decision-making processes is "emotional", that is, affected by emotions in particular ways. Hence, this concept allows me to speak of the interconnection between rationality and emotion. For operationalizing the notion of emotional orientation, I propose an adaptation of the concept of the "emotional orientation" of a teacher in terms of her "set of expectations": the term "expectation" is connected to her "emotions of being right" when she uses specific criteria for accepting an explanation from the class rather than other ones (Ferrara and De Simone 2014).

My research questions are twofold: How does the use of emotional orientation help me understand the decisions of teachers relating to the use of technology, and thus complement the Habermasian rationality framework? And, How does ræmotionality help me understand why teachers make certain choices in their teaching with digital technologies and not others?

This chapter focuses on the work of an Italian teacher-researcher,[1] Silvia, while she explains linear equations in her grade 9 classroom, in a scientifically-oriented secondary school in the Piedmont region of Italy. The teacher was first interviewed, and the interview was transcribed for analysis. Teacher's usual lessons in the classroom were also videotaped. All voice and bodily movements during the classroom activities were recorded. The videos were transcribed for data analysis.

[1]In Italy, the teacher-researcher is a teacher of the school who participates to the research carried out within the academic research group.

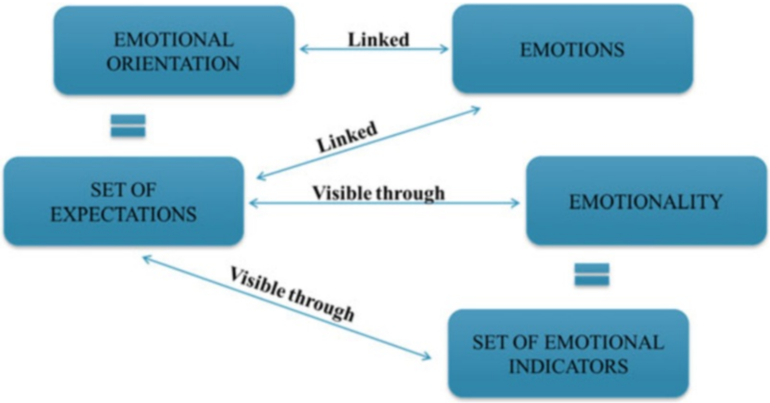


Fig. 1 Diagram about relations of different concepts

Concerning the structure of the analysis, I first considered the *a-priori* interview and, from what Silvia explicitly described to me, I was able to identify some of her expectations for the use of technology in terms of what she hoped for her students. At this stage these expectations were only potential, because they were not yet driving the action of Silvia in her classroom. Hence, I also looked at what actually happened in the classroom, in order to see if there was a correspondence between what the teacher stated *a-priori* and how she actually behaved in classroom. For determining this correspondence or non-correspondence, I looked at "emotional indicators", namely the gestures, facial expressions, word emphases, repetitions, rhetorical questions, pauses, the tone of voice, and so on. These emotional indicators informed me on the emotionality of the teacher, where the term "emotionality" is defined "in terms of *behaviours* that are *observable* and theoretically *linked to* the (hypothetical) underlying emotion" (Reber et al. 1995). Hence, the expectations of the teacher became visible through her emotionality. In this way, I outlined the emotional orientation for the teacher, intended as the set of her expectations. In the diagram below (Fig. 1), the relations among these different concepts are schematised.

Following the initial emotional schematization, I went deeper into the lessons of the teacher, in order to identify the intertwinement between her rationality and emotionality. In particular, I looked at her decisions related to the use of ICT, through the three components of rationality (epistemic, teleological and communicative); simultaneously looking at the emotional indicators and expressions of her expectations, I was able to say something about why she made linkedsome decisions and not others.

In particular, the emotionality will be always intertwined with the rationality of the teacher. For this reason, I describe the emotionality of the teacher using the adjectives of the Habermasian rationality, epistemic emotionality, of teleological emotionality and of communicative emotionality (De Simone 2015): these Habermasian adjectives constitute the different components of ræmotionality. For example, the *epistemic emotionality* surfaces when the teacher decides to draw on the properties of equations via the model of a virtual scale, and, at the same time, says, with a blaring tone of voice, the expressions “to put on the scale” and “to take away from the scale”, manipulating the virtual scale. This way, it is not only which kind of knowledge she chooses to consider (properties of equations, epistemic rationality), but why she decides to focus on it in that way (through the virtual scale). The reason is connected to her expectation that using the virtual scale will help students understand the meaning of the properties of equations, by having the possibility of visually manipulating the scale. This expectation is made visible through the tone of voice of the words “to put on the scale” and “to take away from the scale”. In other words, the *epistemic emotionality* is related to why the teacher uses that specific justification of the knowledge at play.

The *teleological emotionality* could be highlighted when, for example, the teacher decides to explain equations with the graph option of Geogebra for geometrically interpreting the solution and, simultaneously, with the highest pitch of her tone of voice, repeats many times the verb “to see”, pointing to different elements on the graph. In addition to the action the teacher undertakes to accomplish a goal, namely interpreting geometrically the resolution of a linear equation, we can also observe that she expects GeoGebra to help students to reason about equations, “seeing” through the graphical register of GeoGebra. This expectation is made visible through the tone of voice of the verb “to see”, while gesturing on the graph. Thus the *teleological emotionality* is related to why the teacher makes these actions to achieve a particular goal.

The *communicative emotionality* surfaces, for example, when the teacher has an insistent rhythm to her voice, as she directs the class to look at what happens both on the graph and on the “Algebra view” of GeoGebra. In this instance, there is not just the matter of her speech oriented towards reaching understanding within the classroom, but also the question of why she decides to communicate with an insistent rhythm. Her reason for this repetition is connected to her expectation that students are facilitated to connect different registers of representations through the use of technology. Hence, the *communicative emotionality* is related to why the teacher uses a particular type of speech during her discursive activity in the classroom.

For pragmatic necessities of analysis, these three types of emotionality could appear separated. Nevertheless, it is important to stress that they are always intertwined and present in the discursive activity of the teacher.

Data Analysis

From what Silvia explicitly described during the *a-priori* interview, I detected different expectations of the teacher, mostly concerning the role of technology. These expectations are actually reflected in her classroom activity and contribute to shaping her emotional orientation.

I will show five examples of Silvia's activity in the context of linear equations for shaping a convincing and representative outline of Silvia's ræmotionality in using two types of technology: first GeoGebra and then a Java applet.

Firstly, I will quote the passages of the interview from which I identified her expectations involved in these examples. Then, I will analyse her ræmotionality, looking at both the decisions of the teacher and, simultaneously, at the emotional indicators, expressions of her expectations. This way I can reveal the different components of her ræmotionality, and thus explain why Silvia takes those decisions and not others.

During the analysis of teacher's activity on linear equations, three different treatments of graphical representations developed by Duval in 1988 also emerge. In particular, Duval speaks of the "*démarche de pointage*", the "*démarche d'extension*" and the "*démarche d'interprétation globale*". The first approach, *dé marche de pointage*, concerns the focus on particular points of the graph. For example, it is related to the drawing of the graph of a first grade equation or to the reading of the coordinates of an interesting point of a graph. The "*démarche d'extension*" concerns the imagination of a set of infinitely potential points that have a particular property. The "*démarche d'interpretation globale*" is related to the association between what happens on the graph and on the algebraic representation of the graph.

GeoGebra

First Example

This example comes after two lessons in which Silvia had introduced the concept of equation as a mathematical statement that two expressions are equal. The solution of an equation is the value that, when substituted for the unknown, makes the equation a true statement. Then, in this excerpt, she begins to work on the dynamic geometric software GeoGebra for introducing the solution to an equation from a geometrical point of view. In this example, two of Silvia's expectations that I have extracted from the *a-priori* interview are involved. I quote the passages of the interview that allowed me to identify them.

In the middle of the *a-priori* interview, Silvia presents her way of introducing linear equations:

> I introduce linear equations through an activity of M@at.abel (M@at.abel is an Italian teacher education programme for in-service mathematics teacher supported by the Ministry of Education). In particular, we consider a pseudo-real situation of a boy who walks with constant velocity and we ask, knowing the velocity, how many kilometers he covers while the time passes. "How many kilometers while the time passes" is a linear function, then, on GeoGebra, we consider a table and we start to see after how much time he will cover 300m and then we go to see [she mimes the solution on the graph] the answer on the graph of GeoGebra. We start from that for talking of equations because, after we have the straight line [she mimes the straight line], we can read on the graph of GeoGebra and then we have the intersection between the oblique straight line that represents the velocity and the horizontal straight line that represents, for example, 300m. Then we are able to see the

> intersection point as the solution of an equation. Always working on the graph of GeoGebra, I try to highlight that if I translate the graph up or down [she mimes the translation] the solution is simply translated up or down. Hence, I can add or subtract the same term to both sides of the equation and I will obtain the same solution.

Hence, it can be plausible to think that Silvia has ***the expectation that students learn "to see" through the graphic register of GeoGebra in order to reason (think of) about the equations.*** I used the verb "to see" with the quotation marks, because, during the interview, Silvia herself used this verb for referring to the discussion of the equations.

Then, from the following of the interview, I infer that Silvia has ***the expectation that the use of GeoGebra helps students to pass from one representation register to another***. Indeed, as I am going to show, Silvia believes that GeoGebra could facilitate students in coordinating the different registers, stressing that, when she passes from one register to the other, they are all equivalent ways to speak about the same thing:

> Using GeoGebra, I expected that students are able to intertwine the graph, Algebra, numbers and words not just for the equations, but for all of the mathematical concepts. I believe that this is the power of GeoGebra. For example, yesterday, with my 11 grade classroom, when I spoke of the definition of the arithmetic progression, I remained astonished because they were able to see the graph of a straight line: we were in the classroom, within a totally numerical environment and they were quickly passed from the numbers to the graph without problems and in a fruitful way. I believe that this is a very important added value. I hope that GeoGebra helps students in thinking and searching for counterexamples even when they don't have GeoGebra at their disposal. In particular, for the equation I expected that they see the deep link among equations, inequalities, zero of a function: they are all equivalent different ways to speak of the same thing.

From the interviews' excerpts, there are initial hints of the fact that the teacher considers GeoGebra to be a useful tool for fostering students' imagination. In fact, firstly, GeoGebra allows students to actually see things, when they directly work with it. As time goes on, this thing gives an "added value" to the use of the software, namely the fact that GeoGebra supports students' imagination, even when they don't have it at their disposal.

The two expectations are actually reflected in her classroom activity. Indeed, I am going to make the analysis of two excerpts in which these expectations are present, surfacing the ræmotionality of the teacher referring to the use of GeoGebra. The first one refers to the discussion of a graphical solution for an equation. The class is working on the already quoted M@t.abel activity. They have to solve the equation $1/5x + 1/2 = 8$. The solution of it is the time Luca uses to cover 8 km, starting from 500 m from the starting line. Hence, knowing that he uses 15 min to cover 3 km, namely his velocity is 1/5 km/min, the time he uses is the solution of $1/5x + 1/2 = 8$.

Listen to Silvia as she explains the task to the classroom:

#1 T: In the activity we prepared a slider k that varies from −15 to 15 and (*blaring tone of voice*) then we considered two equations. [...] The equation previously solved was $1/5x + 1/2 = 8$. To solve this equation

Fig. 2 Pointing the abscissa of the intersection point (Here in after, the description of the rest of the figures is included in the correspondent analysis)

we have already said that, actually, we could work on two different (*pronouncing*) functions, precisely on two (*pronouncing*) straight lines: one was this straight line (*she draws on GeoGebra the function y* = *1/5x* +*1/2*) (*pause and she looks at the screen*) and the other one was y = 8. Then, you have told me, if you remember, that the solution to the equation was the intersection point between these two straight lines. (*rhetorical question*) Do you remember it? There was Elena who said "(*highest pitch*) I go to see where I intersect and then I read the solution". Actually, the solution we have to read is not on the y-axis, but it is on the x-axis, because it is the value of x that is of interest to us as solution and (*blaring tone of voice*) then the fact of asking to draw the perpendicular line to the x-axis passing through A served simply to say that (*highest pitch*) I can go to read the solution. I can go to read the solution here (*she stands up and she goes on the screen, pointing to the abscissa of the intersection point and then she looks at the class as in* Fig. 2).

(*highest pitch*) Going to read this number or (*speeding up*) given that I cannot be sure of the value of this number because GeoGebra has limits, I can read it here (*she points to the "Algebra view" of GeoGebra*). In the "Algebra view" the point A has coordinates 37.5 and 8 and, then, the solution of the equation is the number 37.5. If instead of x I put 37.5 the two straight lines intersect and they have the same value (Fig. 3), ok? (*she nods and she returns to the pc*).

Then we were asked to draw another two straight lines: one is y = 0.2x +0.5+k (Fig. 4).

In this case, k is equal to 1 and we see that GeoGebra writes (*pointing to the "Algebra view"*) y = 0.2x + 1.5. Why 1.5? (*pause: a student tries to say something but he does not finish the sentence*) because k

#2 S1: It is 1

#3 T: It is 1and, then, 1 plus 0.5 is 1.5, it has already calculated (*referring to GeoGebra*). The other straight line is y = 8+k (*blaring tone of voice and*

Fig. 3 The two straight lines intersect

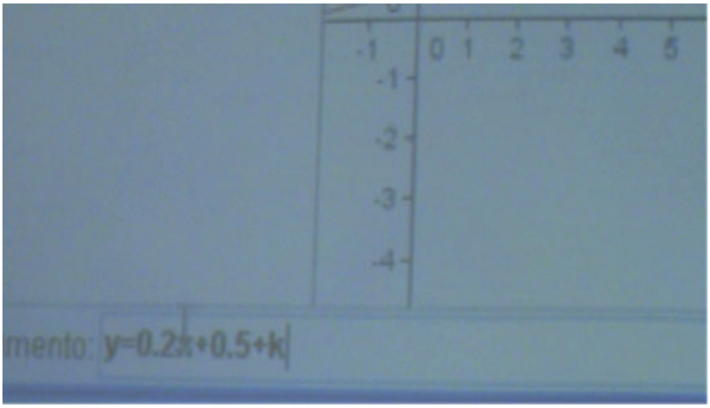


Fig. 4 Another straight line

she looks at the class) if I write y = 8+k, in the "Algebra view" it will write 8 + k? (*facial expression in* Fig. 4, *long pause and she continues to look at the class*) (*smiling*) I don't hear answers (*she looks at the class smiling*).

#4 Ss: Not

#5 T: Not, what will it write? (*same facial expression as in* Fig. 5)

#6 Ss: 9

Discussion

Silvia works with GeoGebra in order to talk about the graphical solution of an equation. She has already introduced it in the previous lesson, in fact the teacher

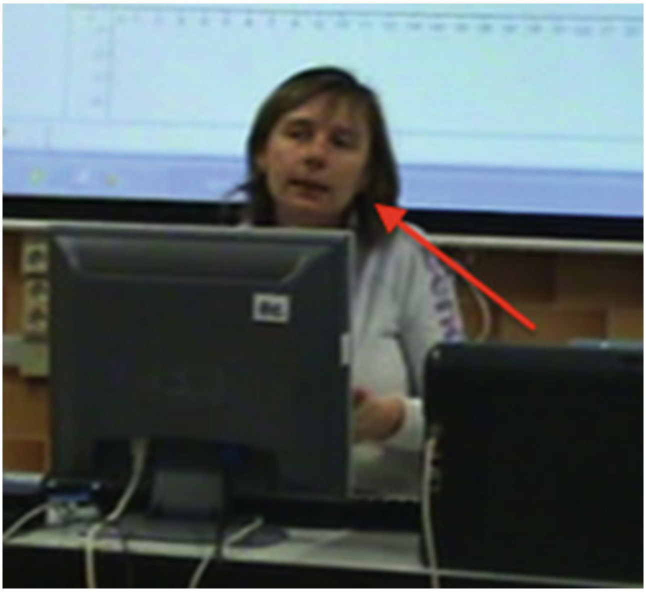


Fig. 5 Facial expression

reminds students that they could work on two straight lines: y = 1/5x + 1/2 and y = 8. She pronounces both "functions" and "straight lines" (#1), in order to recalling that the solution of the equation is related to the intersection point between them. Hence, she is working on the global interpretation of the straight lines (*dé marche d'interpretation globale*) for focusing on the intersection point between these two functions (*démarche de pointage*).

Moreover, she "plays" with two representations of the line: a geometrical representation of the straight line and the graphical representation of a function. Silvia seems quite sure that the students remember this, indeed, she rhetorically asks "Do you remember it?". Then, she recalls what a student said in the previous lesson (#1). Furthermore, she clarifies that they have constructed the perpendicular line to the x−axis passing through A, because the solution of the equation can be "read" on the x−axis (*démarche de pointage*). Silvia accompanies this justification with a blaring tone of voice (#1), for stressing it as much as possible. She repeats this fact looking at the class for feedback and pointing to the solution (see Fig. 2). She uses many times the expressions "to read the solution" and "to go to see" on the graph.

Then, stressing that GeoGebra has limits, she invites students to read the solution, not directly on the graph, but on the "Algebra view" of it. This is interesting: she is saying that it has limits if we look visually, but not if we look numerically. From a didactical point of view, the teacher would like to present the calculus as something more rigorous than visual representation. From the other side, she wants to pass on the "Algebra view", speeding up, because she surely has the expectation that students reason on equations visually on the graph; however, she needs students to coordinate the registers for institutionalizing the mathematics at play. In terms of Duval’s theory, this represents a global vision in which the teacher stresses the association between the graphical and the numerical solution (*demarche globale*).

After that, she continues with the activity in which they are requested to draw another couple of straight lines depending on k: $y = 0.2x + 0.5 + k$ and $y = 8 + k$. The teacher highlights that GeoGebra gives automatically the value of k to the first function ($y = 0.2x + 0.5 + k$), and then she seems to want from the students the response for the second equation ($y = 8 + k$). In fact, after asking, with a blaring tone of voice, what happens for $y = 8 + k$, she pauses, as in Fig. 5. The blaring tone of voice is probably intended to reveal how GeoGebra directly makes the addition both on the Algebra view and on the graph, according to the value of k. Then, she smiles when she says that she isn’t hearing any answers, perhaps to keep the mood light (#3).

Hence the *teleological emotionality* of Silvia is constituted by considering the two straight lines and their intersection point to find the solution (rational key). Moreover, the teleological involves the fact that she is expecting that students are used to “seeing” through the graphic register in order to find the solution. This emotional key is revealed, for example, by the fact that she often says, increasing her tone of voice, “to read the solution” and “to go to see” on the graph (#1, #2), possibly to draw the attention of the class to these important aspects of her lesson. This emotional aspect is shown also by the rhetorical question, “Do you remember it?” in #1. Finally, it is reveled by her gesture in Fig. 3 in which she mimes the intersection between the straight lines.

The *epistemic emotionality* of the teacher is, from one side, the geometrical interpretation of the solution of an equation, accompanied by how the software works (rational key). From the other side, it is related to her expectation that students know how to pass from one register of representation to another one (emotional key). This is strictly related to being able to “see” through the graphic register to reason about equations. For example, she hopes that students recognize the solution on the graph, pointing to it and maintaining a certain facial expression (Fig. 1), and seeking feedback from the class. Then, in #1, from her increasing velocity of speaking, her need to quickly pass from the graphical register to the algebraic one for formalizing the calculation becomes clearly visible. Moreover, she hopes that students are able to link how the algebraic and graphic registers of GeoGebra work together. In fact, after asking with the blaring tone of voice what happens for $y = 8+k$, she pauses with a facial expression as in Fig. 5. It is quite clear that the teacher is expecting an intervention from the students. Moreover, her smiling probably communicates a desire for more participation from the class (#3). Actually, this attitude did trigger several comments from the students.

Her speech is full of emotional hues because she has certain hopes and needs in relation to her students. She changes her tone of voice to emphasize what she is saying, such that students understand the importance of it. Especially in this part of the lesson, she seems like a soloist, because she speaks exclusively for most of the time. Because of this, her pausing is meaningful: when she stops, she has the need

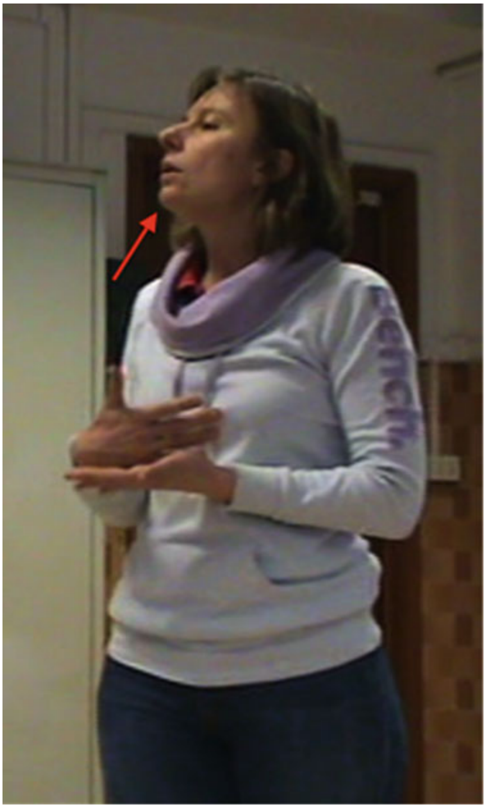


Fig. 6 Gesture moving her hand up an down for the translation

for students to speak. Being that her discourse is not neutral, we can speak of the *communicative emotionality* of Silvia.

Second Example

After discussing the graphical solution for an equation, Silvia continues to work on GeoGebra, because of her expectation that GeoGebra can help students to pass more easily from one representation register to another; and she aims to link the vertical translation of the straight lines to the concept of equivalent equations from an algebraic point of view:

#1 T: (*pronouncing*) What are we doing (Fig. 6: *gesture moving her hand up and down for the translation*)?

#2 S3: Equivalent equations

#3 T: (*repeating and nodding*) We are constructing many equivalent equations. You remember that in the previous lesson we have said that we have equivalent equations (*same gesture as in* Fig. 6), namely equations written (*pronouncing*) in a different way, but that they have (pronouncing) always (pausing) the same result. (*Highest pitch*) Do we

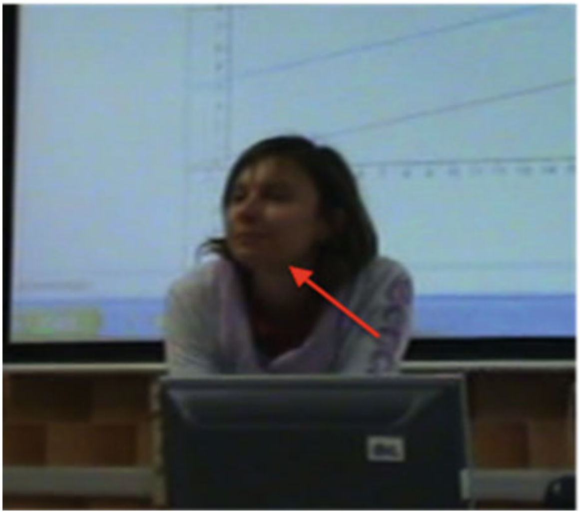


Fig. 7 Waiting for an answer

have equivalent equations just for k = 7.5, for k = 3 (*speeding up*) that are the equations we have seen? or do we have equivalent equations for many values of k (*she returns to the pc and she moves k, looking at the class and smiling waiting for an answer*, Fig. 7)?

#4 Ss: Many

#5 T: For many or for each value of k (*she continuously moves k*)?

#6 Ss: For all of them

#7 T: For each value of k. For each value of k I obtain however equivalent equations. The filling of the table was just to write equivalent equations. For example, when I write 0.2x + 1.5, what value has k to have 1.5? (*pause and she lifts up her chin*)

Confusion in the classroom

#8 S2: 1

#9 T: 1. Then, If I give the value 1 (*she returns on GeoGebra to put k equal to 1*) I see that the equation is (*pointing*) 0.2x + 1.5 = 9. (*Pronouncing*) What happened to the sides of the equations? What did we do to the sides of the equation (*she lifts up her chin*, Fig. 8)?

10 S1: We have added 1

#11 T: We have added 1 (*pausing*)

#12 S1: To both sides

#13 T: (*smirking*) We have added 1 to both sides. In the previous lesson, we have said that the first principle of equivalence said us that we could add the same number to both sides and that the result of the equation does not change, ok? then I could add or subtract the same number to both terms and have (*pronouncing*) always equivalent equations. Then, what does it mean (*returning on the "Algebra view"*)? It means that I can add to both sides (*moving k*), see that the blue straight lines have the

Fig. 8 Facial expression

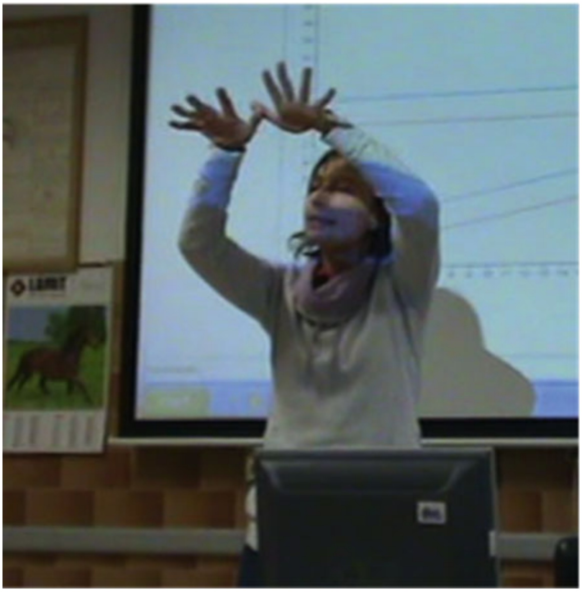


Fig. 9 A mime of the translation

same movement, they have the same translations (*she mimes the translation moving hands up and down:* Fig. 9), namely they have exactly the same movement, then we add or subtract to both sides exactly the same quantity, our result doesn't change. If I wanted to

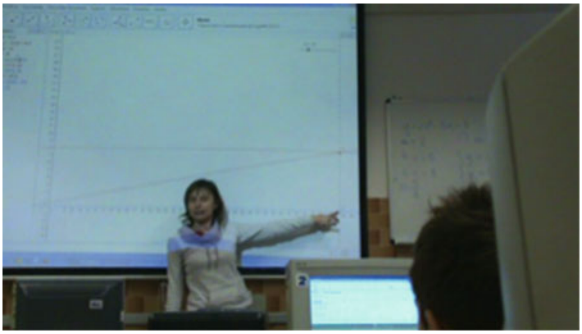


Fig. 10 Gesture along the x-axis

Fig. 11 Gesture to accompany the pronouncing

obtain the result of the equation, I would take k, I would do such that B coincide (*pronouncing*) exactly with the x-axis (*she is doing it on GeoGebra*). To let coincide B exactly with the x axis, what value I have to give to k?

#14 S11: −8

#15 T: −8. If I give −8 to k, what happens is that B belongs to the x axis (Fig. 10). The second side of the equation (*pronouncing*) takes the value 0. The first side of our equation has a certain expression and I, actually, go (*pronouncing*) to see where the blue equation intersects the x-axis (Fig. 11). I go to find what it is called the (*pronouncing*) zero of function (*gesture to accompany the pronouncing*) because it is the point in which the straight line touches the x axis, ok?

Discussion

At the beginning, Silvia explicitly asks to her class what they were doing in the previous part of the lesson (#1). This action comes along with her pronouncing and her posture of waiting (see Fig. 6) for having as much as possible the attention of the entire class, because they are about to construct an important link. Then, satisfied, she repeats, nodding, what a student answers (#3), remembering the definition of equivalent equations. She pronounces “in a different way” and “always” (#3) in order to focus the attention of students on these two crucial words of the definition.

Moreover, returning to GeoGebra, Silvia asks for how many values of k they can have equivalent equations. This question comes along with an increasing volume of voice and her emblematic posture in Fig. 6, in which she seems quite satisfied that students are able to answer. Actually, while Silvia moves the slider k, students become aware that they can have equivalent equations for infinite values of k, because the number of straight lines that could be potentially constructed moving k is also infinite.

The teacher exploits the opportunity of imaging something, without actually seeing it. In fact, she moves the slider k continuously, and this allows students to realise that they could potentially construct infinitely many straight lines, even if they cannot visualize all of them on the screen. Another interesting thing is that in #4 students just answer “many”, but then after seeing the teacher continuously moving the slider, say all together “for all of them” (#6). This has an important effect on the algebraic point of view: in fact, they become aware of the infinity of the straight lines on the number of equivalent equations.

In this brief moment, constructing in their minds the straight lines they could have sliding k, they are at the level of the “*démarche d’extension*” à la Duval that draws on infinite sets of potentially equivalent straight lines.

After, Silvia explains how the first principle works,[2] showing that if k is 1, GeoGebra automatically adds 1 on both sides (#7, #9). She accompanies this discussion with many questions for her students, pauses and facial expressions with the chin up (#7, #9). It is quite clear that she is waiting for answers from the class. This is also indicated by her smirk in #13 when a student says that they have added 1 to both sides. Then, she repeats what the first principle says, again pronouncing “always” (#13). In terms of what happens on the graph, she highlights that the straight lines are translated of the same value, hence the result doesn’t change. To explain what happens she uses a specific example: adding 0 to both sides. In fact, she invites her students to move the intersection point of the straight lines on the x-axis. She stresses this fact by pronouncing “exactly with the x-axis” (#13). At this moment, *exploiting the “démarche globale” that links the graphical and the numerical aspects*, Silvia introduces the concept of the zero of a function, stating both “zero of the function” and “because it is the point in which the straight line touches the x axis” (#15).

Hence, her *teleological emotionality* involves both the construction of the link between the concept of equivalent equations and the translation of the straight lines

[2]She speaks of the properties of the equations, that in Italy, we call “first principle of equivalence” and “second principle of equivalence”. The former says that adding or subtracting the same quantity to both sides of an equation produces an equivalent equation. The latter says that multiplying or dividing by a quantity ($\neq 0$) both sides of an equation produces an equivalent equation.

(rational key), and her expectation that, with GeoGebra, students are able to link these two different representation registers. Moreover, she hopes that it is not too difficult for students to see the translations of the straight lines as adding a quantity to both sides of the equations (emotional key). In fact, she wants to link the translation of the straight lines to the properties of equations they have already seen from an algebraic point of view. This emotional counterpart is revealed, for example, by her pausing as in Fig. 8, her satisfaction after the answer of a student, her pronouncing key-words (#1, #3), her gesture in Fig. 9 moving the hands up and down for miming what happens to the straight lines and, at the same time, for linking the movement the operations of addition and subtraction on the algebraic expressions.

She justifies how the first principle of equivalence functions using how GeoGebra works and she makes the specific example of adding 0 to both terms of the equation (rational key). At the same time, she is expecting that students are able to connect the algebraic register to the graphical one (emotional key). In particular, Silvia introduces the zero of the function as the intersection point of the straight line with the x-axis. The emotional key is revealed by pronouncing several times "exactly with the x-axis" and by her gesture to recall what they have already done with the scale. These two intertwined aspects form the *epistemic emotionality* of the teacher.

During all the activity, Silvia, being emotionally involved, cannot have a "plain" discourse, indeed she uses the language of the body, gestures, change of the tone of voice and so on. For this reason, it can be always highlighted the *communicative emotionality* of Silvia. With a pure rationality I could keep track just of what she is saying, but, looking at the emotional timbres, I could also say something about the fact that she hopes and needs in using the technology with her students.

Java Applets

In the *a-priori* interview, she anticipates that she will use two Java applets featuring virtual scales for reinforcing the meaning of the principles of equivalence. They have already experienced them from both the algebraic and the geometrical points of view.

From a mathematical point of view, through these scales, we can represent and solve simple linear equations. After having shaped the given equation using the unit-blocks and the x-boxes (whose weight is unknown) on each side of the scale, we can make arithmetical operations. The aim is to have one x-box on one side and the number of unit-blocks on the other one, from which we can read the solution. The user chooses the operation to be performed, and after each operation the new equation is updated so that both the original equation and the latest equivalent form are seen together. The first scale only allows working with positive whole numbers as coefficients, while the second one has been designed to work with negative numbers. In fact, in the latter, there are negative values for the unit-blocks and the x-boxes, represented by red balloons for thinking of something that lifts, conveying the idea of subtraction. One of the key ideas that should be highlighted is that no operation can be performed on just one side of the equation. Moreover, the virtual scale is a useful tool for visualizing the addition and subtraction of quantities (first principle of equivalence), but it is more obscure concerning the meaning of division and multiplication in terms of the scale. This limit will also surface in Silvia's lessons.

In the interview piece I'm going to quote, Silvia speaks about the metaphor of the virtual scale. I detected her ***expectation that students understand the meaning of the principles of equivalence through the direct manipulation of the virtual scale, drawing upon their arithmetical or algebraic knowledge for bypassing the technology's limit***:

> I will use the virtual scale for underlining the principles of equivalence, using a tool that students know [. . .] I stress the first principle of equivalence with another register, different from the words, namely the fact that if I add or subtract something from one side of the scale, I have to add or subtract the same thing from the other side of it [. . .] I will use it after doing the principles of equivalence first with GeoGebra and, then, from an algebraic point of view. Hence, if someone has not still got them, could catch their meaning through these scales [. . .] There are balloons for the negative values and this is again good for me, because it is not only an idea of adding, but also that of subtracting. In conclusion, the use of the virtual scale is an added support for understanding the first principle of equivalence. The choice of technology [. . .] is because all of the students can use the scale and this is an important value [. . .] it is one thing to use it and quite another to see someone else uses it; second, the students have the web link of the scales and they could use it also at home. Then, the most important thing is that I can physically manipulate it without seeing other people doing that. For example, through moving to the trash the small cubes or using the arithmetical operations, the students can physically see what happens, also seeing the direct result of the operation. On the contrary, when you are working from an algebraic point of view, removing 5 from both sides doesn't allow you to see an immediate effect. I just see a number that changes, while with the scale I see the scale that goes up and down: they are things for grasping the meaning of the principle of equivalence. For the second principle of equivalence the thing is more complicated: you cannot see the division with the metaphor of the scale, namely we cannot physically do the division on it. This is a scale limit, in fact it's interesting that the positive values are the small cubes, while the negative ones are the balloons. We do the division in a numerical way, it's not a visual thing. [. . .] From what I can see on the video, dividing means having a lower number of cubes, but that's all.

Third Example

In this example, she uses the virtual scale that works just with positive integers numbers:

#1 T: in this scale there is already written (*pronouncing*) an equation. Then, the equation that is written involves (*pronouncing*) to put (*she mimes the first member*, Fig. 12) on the first side the things on the first plate and (*she mimes the second member*, Fig. 13) in the second side the things on the second plate. For example, (*she looks at the screen of a student*), here there is written 3x+1, then it means that I put three small cubes (*she mimes again the first member*, Fig. 14) that correspond to the x, (*she is speaking with the same student*), (*with a sparky tone of voice and pronouncing*) take

Fig. 12 A mime of the first member of the equation

Fig. 13 A mime of the second member of the equation

Fig. 14 Another mime of the first member of the equation

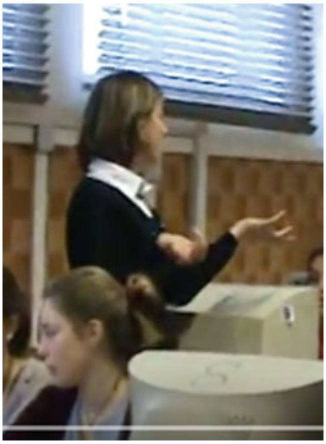


Fig. 15 Mime of the balance plate

Fig. 16 Mime of the balance

them and put them with the mouse, (*she remains as in posture of* Fig. 14) one, two, three and then one, namely one small cube. Then, on the other side (*she mimes the second plate*, Fig. 15) there are four, then (*pronouncing*) let's put 4 small cubes. When we have put four small cubes and even when we have put 4 of them, the scale remains in equilibrium (*she mimes the balance*, Fig. 16). The small cubes can be carried from one side to the other, obviously if I have a pan balance (*she mimes the balance again*, Fig. 16) in equilibrium, if I take a small cube from the left plate and I move it on the right plate, the scale will not continue of remaining in equilibrium. Actually, (*rhetorical question*) what happens? It happens that the right plate will weigh more than the left plate, then the scale will become in perfect unbalance (*she mimes the unbalance*, Fig. 17). For arriving to the equilibrium, what has to happen to the small cubes whose values are equal to 1?

#2 Ss: I have to remove it also from the other side.

Fig. 17 Mime of the unbalance

#3 T: exactly as we said above (*highest pitch*) for the equations, the first principle of equivalence says us that we can remove a small cube from one side and another small cube from the other side. If someone is lucky in "gaming" (*the applet on her computer does not work*), removing a small cube is possible because there is a trash and I can take the small cube and move it to the trash. If I move to the trash one small cube just from one side, the scale becomes unbalanced, but if I remove a small cube also from the other one, the scale is again in equilibrium. Then I also have some x, but if I remove one x, then I have to remove also from the other side the same value corresponding to x. Then I finished and I solved the equation when on one of the two plates I have one x and on the other plate I have just small cubes valuing 1, such that I can say which is the equivalent of the value of x.

Discussion

In the above excerpt, she is presenting the virtual scale in order to see the first principle of equivalence for the equations through the metaphor of the virtual scale (rational key). In particular, firstly, she explains how to set up the equation on the virtual scale (#1) and, then, she introduces how this Java applet works: it is possible to discuss the balance or the unbalance of the virtual scale moving the small cubes on the plates to the "Trash" or not (a button offered by the applet). She mimes the balance (Fig. 16) and the unbalance (Fig. 17) of the virtual scale when she is thinking of what happens if she removes "a small cube that values 1" just from "one side" of the scale (#1), without students actually setting up the equation on the applet. Evidently, in this case, the scale would be unbalanced. It is very clear in this moment that Silvia is again drawing on the fact that the virtual scale could also foster students' imaginations, without their actually acting on it. Moreover, from her gesturing in #1, it is also visible that Silvia is exploiting the potential of using a virtual scale: there is an intrinsic dynamism in the use of it (adding, moving to the trash, the balance, the unbalance) that constitutes an added value with respect to a static scale drawn on the blackboard, upon which students cannot practically work without visualizing the results of their actions on it.

Hence, we can deduce from the excerpt Silvia's expectation that students understand the meaning of the first principle of equivalence directly manipulating on the virtual scale and that students are able to translate into mathematical expression what is played at the metaphor level (emotional key). This emotional key is revealed by her use of numerous verbs that refer to the physical action on the scale: for example, she pronounces the verb "to put" as the action of student is actually happening ("put them with the mouse", "to put… on the first plate", "let's put 4 small cubes"); she invites a students using a sparky tone of voice to "take them (the small cubes) and put them with the mouse" as she actually places the cubes on the plate with her hands; the gestures of miming the balance and the unbalance that accompany the dynamism of the actions of carrying "from one side to the other", of "take away a small cube from the left plate and I move it on the right plate", for referring to the unbalance. Furthermore, the rhetorical question "What happens?": it is very immediate for the students to visualize that removing one weigh from one plate of a scale in balance provokes its unbalance, and that for returning to a state of equilibrium, it is sufficient to remove the same weight from the other plate. Hence, the *teleological emotionality* of Silvia is demonstrated here, because she not only invokes the action of considering the virtual scale for seeing in another way the principle of equivalence for the equations, but also indicates her hope that they become aware of the meaning of the first principle: she communicates implicitly via her lesson that each of them can directly act themselves on a virtual scale, immediately visualizing the results of their actions on the scale.

Furthermore, she justifies the balance and the unbalance of the virtual scale through making this model analogous to the algebraic form of the principle of equivalence, by stressing that there are two equivalent manners for explaining how the first principle of equivalence works (rational key). This rational key is accompanied by the fact that she expects again that it is useful to directly act on the virtual scale (emotional key). This emotional key is revealed in this excerpt by her exactly repeating for two times, at the end and at the beginning, the same words, explaining what happens when moving the small cubes on the scale in terms of the equilibrium or unbalance of the balance. Moreover, due to the fact that the applet sometimes could not work, she considers "lucky" the students who can use the applet, for actually becoming aware of the meaning of what they are doing. These latter emotional and rational keys shape together Silvia's *epistemic emotionality*, because it is not only the matter of constructing the equivalence between the algebraic form of the principle and the metaphor of the scale, but also her expectations for students become themselves aware of it through using the virtual scale, that create the possibility of physically manipulating the scale.

The prosody, the gestures, and the repetitions highlighted above, sketch the Silvia's emotional engagement, and accompany her verbal communication. This

Fig. 18 Miming the lifting

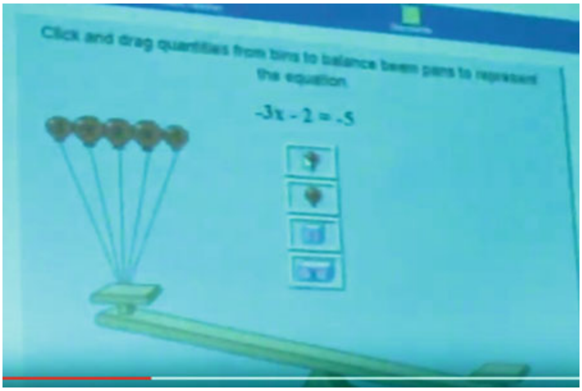


Fig. 19 Balloons that value -5

emotional involvement is unavoidable during her decision-making processes; it is for this reason that I speak of Silvia's *communicative emotionality*.

Fifth Example

In this example, the teacher presents the virtual scale that also works with negative numbers:

#1 T: This is an applet conceived for working with negative numbers (*she changes again the problem*) –3x–2 = −5, the principle is always the same, but now I have to remove (*pronouncing –3x and at the same time she raises one hand for miming the lifting*, Fig. 18) –3x, then I have to lift, then I can do –3x with some balloons that are –x and I can do −2 with some balloons that are −2 (Fig. 19) and I can do −5 with some balloons that value −5. Now if I have to solve this thing (*another time*

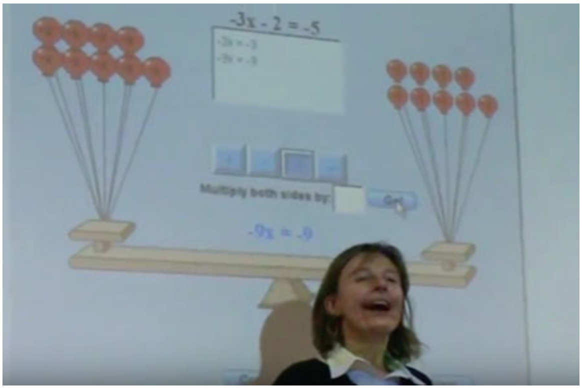


Fig. 20 Smiling: they tripled, right?

she is detaching from the screen) what can I do? (*pause and she looks at the class*)

#2 S3: Adding 2

#3 T: (nodding) Adding 2, if I add 2 to both sides, what happens? From one side 2 balloons explode and from the other side it happens the same. What can I still do?

#4 S11: Removing 3

#5 S17: Adding 3

#6 T: Then adding 3 and then multiplying by 3. If I multiply by 3 what happens? (the applet gives as result $–9x = -9$ and she smiles)?

#7 S4: They multiply each others

#8 T: (smiling, Fig. 20) They triple, right? The balloons triple. Then, for coming back to the previous thing, what can I do?

#9 Ss: Sividing by 3 (*the applet gives* $–3x = -3$)

#10 S6: And then adding 3 for knowing the value of x

#11 T: If I add 3 (*she writes 3, but she does not press on "Enter" and she asks at the class*) what happens on the left?

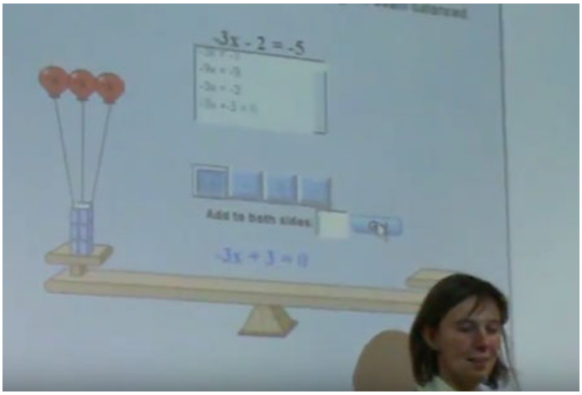
#12 S6: 3 balloons will explode

#13 T: (*nodding*) Three balloons and on the other plate?

#14 S5: There are no more balloons

#15 T: There are no more balloons then what will happen? (*She smiles and then she pushes on the "Enter" button and 3 small cubes of value 1 are created on the left plate*, Fig. 21). Here (*she refers to the right plate*) 3 balloons are exploded, while there (*she refers to the left plate*) the three units that we added are arrived. The equation is become this one (*she points to the rectangle of the equivalent equations*), namely $–3x+3=0$, 0 because $-3+3$ is 0 and then it becomes 0. Now, what is happening? (*again she detaches from the computer and she is going towards the class and she pauses*) I want the value of x, actually (*she turns towards the equation on the scale*) here I have some balloons with –x, how can I know the value of x? I want have some x and not –x.

Fig. 21 Three new balloons on the right plate



Noise in the classroom

#16 S4: Divided by –x

#17 S9: You can explode the balloons

#18 T: How can we explode the balloons?

#19 S5: +3x

#20 T: (*nodding*) +3x, because it is dangerous divided by –x, because we don't know what is x, x could also be 0. If I do +3, what happens?

S8: it is –3x+6=6

#21 T: (*she returns on the screen and she points to the plates*) On one side I will have 3 units (*she mimes them on the virtual plate of the scale*).

On the other side I already have 3 of them and I will have three more. However, I want to explode the balloons, then I add 3x, because in this way if I add 3x, what happens? On one side the 3 balloons explode (*she points to the left plate of the scale*) and on this side (*right plate*) it comes 3 values of x and now I have to know the value of x, how can do it? (*She detaches from the computer and she looks at the class*)

#22 Ss: Divided by 3

#23 T: Divided by 3 and I obtain that 1 is equal to x, so which will be the solution?

#24 Ss: x equal to 1

#25 T: Not, the number 1 because yesterday we have said that the solution was the number that makes true our sentence. If I say x equal to 1 actually I'm saying the sentence, I say that (*pronouncing*) x is equal to 1, I'm not saying the solution, and it is exactly the same thing writing (*she points to the algebraic expression on the screen*, Fig. 22) 1 equal to x or x equal to 1: it is the same equality.

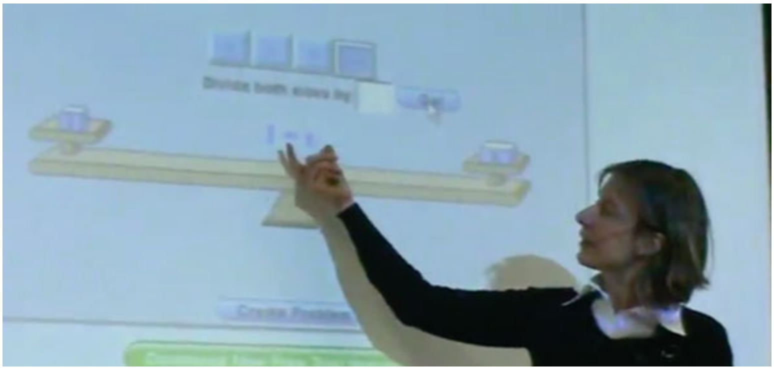


Fig. 22 Pointing the algebraic expression on the screen

Discussion

Silvia has explained that another Java applet exists which is designed to use a scale that also works with negative numbers. In the above excerpt she wants to solve the equation $-3x-2 = -5$, applying the principle of equivalences (rational key). She uses this particular virtual scale because she has the expectation that students can better understand the meaning of working with negative numbers through its modelling of the important principles. For example, she speaks of the monomial $-3x$ (#1): actually, in the scale, the balloons could convey the sense of negative values, because there is something that "lifts" the plate of the scale (emotional key). The emotional key, for example, is evident from her pronouncing $-3x$, for focusing the attention of the students on the fact that they are going to speak of this $-3x$, because it has an important role in this context. Moreover, the expectation of better understanding the meaning of $-3x$ through the use of the virtual scale is visible through her gesture of "lifting" in relation to the balloons (see Fig. 18), for better clarify this as something that has a negative weight.

In this brief passage we can also note the *teleological emotionality* of Silvia, because we find both actions for solving an equation and also her expectation that doing this with the virtual scale can assist students in better anchoring the meaning of operating with negative numbers to something that is very intuitive (such as the metaphor of the balloons). However, she detaches from the screen of the computer when she asks her students how to solve the equation $-3x-2 = -5$. It seems that she is on the algebraic level of solving the equation, not interpreting the resolution of it on the level of the metaphor of the scale (#1). Actually, one student answers on the same algebraic level (#2: "adding 2"), not referring to the scale. The teacher quickly returns to the scale for seeing the effects of having added 2 in terms of the balloons that are exploded. The interesting thing is that she uses "sides" instead of "plates". This is another hint of the fact they, firstly, worked on the numerical level. Again they return to the numerical level, in fact they arrive at the equation $-9x = -9$, working on the algebraic manipulation level. But again, she wants to focus the attention of the students on the effects of this operation in terms of the virtual scale, interpreting the multiplying by 3 as a tripling of the balloons. She smiles (see Fig. 20), perhaps because she is in an uncomfortable situation: actually, operating on the scale, they have just complicated the equation (# 8). Then, she actually wants to return to the previous equivalent equation, $-3x = -3$ (# 8). So she asks the students what they have to do on the mathematical level. After her question,"what happens on the left?" (#11), a student returns to the level of the metaphor, answering "3 balloons explode" (# 12). It is interesting that, yet again, Silvia has used the opportunity given by the applet of "reasoning without seeing". In fact, she poses this question in #11, but she does not press "Enter", expecting instead feedback from the class. Actually S6 answers on the level of imagination, thinking of what the applet would do. Silvia explains in response that, on the right plate, there are no more balloons, because $-3+3=0$ (she justifies it on the numerical level), while on the left plate, 3 small cubes appear; the equation becomes $-3x+3=0$, and she points to the rectangle in which the applet writes all the equivalent equations constructed during the resolution. Again, she smiles – probably because she knows that it is difficult to justify the things just remaining on the metaphor of the scale – and, at this point, she is considering the mathematical level (# 15). Concerning the emotional involvement, she actually detaches from the screen; but after, she turns towards the virtual scale. The interesting thing is that although one student answers on the algebraic level (#16: "divided by –x"), the other one would like to answer on the metaphor level (#18: "you can explode the balloons"). The teacher chooses to go more deeply into the metaphor level, asking how to explode the balloons. A student proposes to divide by 3x, but Silvia explains that would be dangerous because they don't know the value of x (# 20). Instead, she justifies the impossibility of adding 3x, from an algebraic point of view: actually it cannot be explained through the metaphor of the scale. This intertwining between the algebraic manipulation and the effects of it in terms of the metaphor of the scale continues, until the finding of the solution, namely the number 1. In particular, she explains that the solution is not $x=1$, because this is still a sentence with the verb "to be equal to", namely an equation, returning to the mathematical definition of equation. It seems that she is reviewing all the passages of solving the equation from the algebraic point of view, because it is not very simple directly working on the scale with the balloons and the small cube. But, after, she always returns to the metaphor for highlighting the results of what they did in terms of the balloons, that is, for linking the arithmetical operation to a possible meaning of it. There is a dialectic between the algebraic level and the metaphor level. The technology and Algebra complement each other: where the former does not explain, the latter intervenes, and where the latter lacks an actual meaning the former intervenes to make visible the results of the algebraic manipulation. In summary, in addition to the justifications of the passages within the resolution of the equation (rational key), the teacher expects that the virtual scale will help students to anchor the effects of their algebraic manipulations in terms of the scale metaphor, drawing upon their mathematical knowledge when it is not clearly evident what to do, simply by using technology (emotional key). This emotional key is demonstrated by her unconscious detaching from the screen of the computer when she clarifies what happens on the mathematical level, and again by her returning to the screen when trying to reinterpret it in terms of the metaphor. Moreover, she often smiles when she is uncomfortable, in becoming aware that reasoning on the level of technology has just complicated the problem. When she is uncomfortable in this way, she attempts to fix things by referring to the mathematics knowledge. Hence, Silvia's *epistemic emotionality* is revealed in the dialectics between the two levels.

Finally, her communication is both oriented towards reaching understanding by the class and also full of eloquent facial expressions, gestures and specific tone of voice; students grasp meanings conveyed by the combinations of these, and answer her questions, also influenced by how she is speaking. In other words, students understand what she is expecting from them because she cannot hide her hopes and needs. For this reason I further highlight Silvia's *communicative emotionality*.

Conclusion

As already described in the introduction to this chapter, my work concerns the analysis of a case study. For this reason, I cannot infer general conclusions. I show instead how the theoretical perspective enables a detailed analysis of the behaviour of the teacher. In particular, through the concept of emotional orientation, intended as the set of the expectations of the teacher, it is possible to outline the teacher's ræmotionality. As demonstrated in the data analysis, the components of the ræmotionality of the teacher (the epistemic emotionality, the teleological emotionality and the communicative emotionality) are always related to why she decides to put into play that specific knowledge related to the technology; to why she chooses to act in a specific manner for achieving a particular goal connecting to the use of digital technologies; and to why she speaks in that way for reaching understanding within the classroom. In this sense, complementing Habermasian rationality with the affective dimension allows me to detect reasons for the decisions of the teacher that occur in the moment.

Moreover, in the data analysis, I considered the use of two kinds of technology: GeoGebra and the applet virtual scale. Even if they are very different, there is at least one common point between them: the fact that both of them "feed" the creativity and the imagination of students. For example, in the case of GeoGebra, students see just some straight lines moving up and down, because of the constraints of the screen. But, thanks to the dynamic quality of the technological environment, and the illusion of "continuity" generated by the software, students can immediately imagine that they could potentially construct infinitely many straight lines, obtaining the same abscissa of the intersection point. This has an important effect from an algebraic point of view: in fact, the dynamic continuity reflects possibility of infinitely many different equivalent equations we can construct from an algebraic point of view. Hence, the fact that technology helps students in training their imagination facilitates also the coordination among different registers of representation. Furthermore, during the *a-priori* interview, Silvia declares that GeoGebra supports the imagination of students even when they don't have the technology at their disposal.

Concerning the use of the Java applet, very often the teacher and also the students imagine what might happen in terms of the metaphor evoked by the virtual scale, without actually acting on it. For example, in the third example, Silvia discusses the equilibrium and the unbalance of the scale using only one's imagination (#1), and the students do the same (#2). Moreover, Silvia asks students what the virtual scale is going to do, without press the button "Enter" for actually see the effect of the operation on it.

These concluding remarks evolved out of my analysis of the different excerpts I share above. I was curious to know if they were merely my conjectures or not. For this reason, I again interviewed Silvia, asking her a direct question about her expectation for technology in general. Look at what she answered:

> "In the last GeoGebra Day,[3] Barzel said that the technology can be used as a white box or as a black box: as the former when I know the theory and I want to know if it works; as the latter when I want to make conjectures, counterexamples. I am faced in this discourse [. . .] The technology helps you in introducing a difficult topic, because it allows you to see the things. Through the technology, it is clear the link among the different registers of representation. It is so much evident that you have not to explain things [. . .] For example, seeing the graphs in movement and making conjectures on them helps very much the imagination. It helps me also when I don't have the technology at my disposal, because in my head I can imagine the graph, the formula, the table of the points and so on very easily, having behind the support of the technology. Then, the technology helps you to imagine "what it would be, if". The technology does not flatten the thinking level, but it increases it."

Silvia describes her ***expectation that the use of technology in general fosters students' imagination for constructing mathematical concepts***. This is for me a way to "close the circle". From the *a-priori* interview, I identified the expectations on GeoGebra and the Java applet in a separated way; but after the analyses I became aware of the fact that I could infer something more generally on the use of technology. Actually there exists a deep link between the reasons of using different technologies. Moreover, it also becomes clearer why Silvia uses technology as often as she does in her classroom activity.

After having identified teacher's expectations on the use of digital technologies within the mathematics classroom, it is interesting to investigate if these expectations are really translated into student learning. During the experimentation, there are different episodes in which it is possible to verify such actual transfer from teacher's expectations to students' learning, but another more precise study would be necessary to pursue this related set of questions. I look forward to sharing my analysis as a future development of my research.

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[3]During the last GeoGebra Day (in the city of Torino in 2015), Barbel Barzel has made a conference, from the title: "From the value of teaching mathematics with technology: discovering, conceptualising, modelling".

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Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA

Fernando Hitt, Carlos Cortés, and Mireille Saboya

Abstract In the transition from arithmetic to algebra and in light of the disjunction between the natural and symbolic approach to algebra and the choice of a natural way of learning, this paper discusses the development of a cognitive control structure in pupils when they are faced with a mathematical task. Researchers sought to develop, in novice pupils in both Quebec (12–13 years old) and Mexico (14–15 years old), an arithmetic-algebraic thinking structure that would promote mathematics competencies in a method based on collaborative learning, scientific debate and self-reflection (ACODESA, acronym which comes from the French abbreviation of *Apprentissage collaboratif, Débat scientifique, Autoréflexion*), and immersed in an activity theory approach. This paper promotes the equal use of both paper and pencil and technology in order to solve a mathematical task in a sociocultural and technological environment.

Keywords Arithmetico-algebraic thinking • ACODESA • Collaborative learning • Technology • Polygonal numbers

Introduction

Over the course of the last century, the mathematics curriculum took arithmetic as a proper subject for study at primary school level education, and algebra as a proper subject for secondary school level. This dissociation influenced research in mathematics education, which, in turn, reverberated through the academic programs implemented. Examining the work of psychologists before the 1940s, Brownell (1942) noted that psychologists used "puzzles" in the study of intelligence to

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analyse processes related to the “insight” involved when individuals solved such problems. Brownell proposed a radical change focusing on the study of the resolution of the arithmetic verbal problems used in textbooks.

Brownell’s work (Ibid.) attracted the attention of psychologists and educators interested in studying the skills involved in solving arithmetic word problems as a means of understanding the phenomena linked to learning mathematics.

The experience of solving puzzle type problems (where, for example, from 9 matches one is required to build four equilateral triangles, and then, from 6 matches, one is asked to build the same number of equilateral triangles) gave rise to the analysis of solving problems that had single or multiple solutions. From a psychological point of view, these factors led, in the case of a problem with one solution, to an analysis of convergent thinking linked to direct efforts towards achieving a goal. In the case of problems with multiple solutions, this led to an analysis of both divergent thinking (Guilford 1967) and creativity (Bear 1993). In fact, Guilford’s model (Ibid.) stressed the importance of developing divergent before convergent thinking. Gradually, Brownell’s approach led to an extensive research strand focusing on the phenomena related to the resolution of arithmetic problems in primary school and the development of arithmetic thinking.

What Is Arithmetic Thinking?

Brownell’s characterisation (Ibid.) of exercises, problems and puzzles encouraged psychologists and mathematics teachers to focus their research on the study of arithmetic problem solving. Polya (1945) expanded problem solving to other levels of education, thus promoting the emergence of a new paradigm. Some mathematics educators followed this trend and contributed their own new theoretical approaches (RME through the influence of Freudenthal; Mason et al. 1982; Schoenfeld 1985; Santos-Trigo 2010). Returning to primary school level, for example, Vergnaud’s work (1990) on solving arithmetic problems led to the identification of both arithmetic in problem solving and conceptualisation in primary school, and led to the theoretical approach related to “conceptual fields”.

Similar approaches led to some research products in order to characterise arithmetic thinking. For example, Verschaffel and De Corte (1996), taking into account the research conducted in the 1990s, propose arithmetic thinking related to: (a) number concepts and number sense; (b) the meaning of arithmetic operations; (c) control of basic arithmetic facts; (d) mental and written arithmetic; and, (e) word problems using digital literacy and arithmetic skills.

While progress was made in the study of learning problems linked to the resolution of arithmetic problems, research continued toward an understanding of the problems related to learning algebra (Booth 1988). The notion of variable began to be studied (Sutherland 1993), thus promoting investigation into the learning of covariation between variables (Carlson 2002) and the identification of the role of the variable as an unknown, as a large number, and as a variation between variables from a functional point of view (Trigueros and Ursini 2008).

Given the organisation of the curriculum, which designated arithmetic for primary school and algebra for secondary school, researchers began to talk about the problems related to the transition from one level to the other. At the same time, the emergence of the notion of epistemological obstacle in the French school (Brousseau 1976/1983) possibly reinforced this idea of a "break" between arithmetic and algebra. Vergnaud (1988) points out that the transition from arithmetic to algebra is linked to an epistemological obstacle. Other approaches, related to the notion of the unknown in solving linear equations, led to the notion of a "cut" between arithmetic thinking and algebraic thinking (Filloy and Rojano 1989) or even the cognitive obstacle (Herscovics and Linchevski 1994). These studies announced the need to characterise algebraic thinking.

What Is Algebraic Thinking?

As mentioned in the previous section, the research paradigm linked to the "thinking break" between arithmetic and algebra was essential for the characterisation of algebraic thinking. Under this paradigm, Kieran (2007) characterises algebraic thinking using a model called GTG_m: Generational algebraic activities involve the forming of expressions and equations; Transformational activities such as factoring, expanding, and substituting; and, Global/meta-level mathematical activities such as problem solving and modelling. An analysis of this model reveals that, in the past, much of the secondary school level research focused on the teaching of algebra in section G and T of Kieran's model. It is likely that the G_m section is linked to the Freudenthal School's research results regarding realistic mathematics, at the heart of which approach is mathematical modelling.

Visual Aspects in Curricular Change in Mathematics

In the early 1990s, an important curricular change in the field of mathematics began. The visual aspects were highlighted in curriculum changes, promoting displays of the mathematical aspects. A clear example can be seen in the US Standards (NCTM 2000). In this context, geometric aspects in problem solving began to be included in algebra. It was explicitly important to approach a concept through the use of different representations of that concept. From a curricular standpoint as well as from a general standpoint related to research in mathematics education, mathematical visualisation has attracted the attention of researchers. These changes began from a curricular perspective, with a new approach to teaching algebra and the promotion of a geometric-algebraic approach to algebra. Progressing along this research line, for example, Zimmermann and Cunningham (1991) begin the preface of their book with the question: What is visualisation in mathematics? This study explicitly referred to an important role in the production of external representations:

> Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding. (p. 3)

Technology influenced enormously in these changes. Graphical representations that caused major programming problems were resolved, thus giving rise to the production of computer software and enabling an approach to mathematics from the multiple representations user standpoint.

Early Algebra and the Emergence of a New Paradigm

While the previous section discussed the "rupture" approach to characterise arithmetic thinking and algebraic thinking, little by little other research programs arose, which were initially tied to the idea of the "generalization of arithmetic" (see Mason 1996; and Lee 1996). Along these lines, Radford (1996) comments how these authors stressed an approach to the learning problem regarding "algebra as a generalised arithmetic", and goes on to discuss the role of the unknown and the equation:

> The above discussion suggests that the algebraic concepts of *unknowns* and *equations* appear to be intrinsically bound to the problem-solving approach, and that the concepts of *variable* and *formula* appear to be intrinsically bound to the pattern generalization approach. Thus generalization and problem solving approaches appear to be mutual complementary fields in teaching algebra. How can we connect these approaches in the classroom? I think this is an open question (p. 111).

From this perspective, a new paradigm was born. Kaput (1995, 2000) proposes a research program under the following guidelines, with the first two at the heart of the learning of algebra and the other three completing this learning:

1. (Kernel) Algebra as a generalization and formalization of patterns and constraints, with, especially, but not exclusively, Algebra as Generalized Arithmetic Reasoning and Algebra as Generalized Quantitative Reasoning
2. Kernel) Algebra as syntactically guided manipulations of formalism
3. (Topic-strand) Algebra as the study of structures and systems, abstracted from computations and relations
4. (Topic-strand) Algebra as the study of functions, relationships and joint variation
5. (Language aspect) Algebra as a cluster of (a) modelling and (b) Phenomena-controlling languages. (2000, p. 3)

Kaput called this research program *Algebrafying the K-12 Curriculum* (2000), while Carpenter et al. (2003, 2005) initiated the *Early algebra research project* in 1996. This new paradigm formed part of research programs in the twenty-first century. Thus, the Early Algebra movement, in which Carpenter and Kaput played an important early role, is well situated in the USA. The book *Early Algebraization*, edited by Cai et Knuth (2011), shows the progress of research in that area in other countries. In this book, one can appreciate a division between the "enthusiastic" and "cautious" researchers with regard to the Early Algebra movement.

Among the enthusiasts are Blanton and Kaput (2011), as are Britt and Irwin (2011, p. 139), who even criticised Filloy and Rojano's approach by highlighting Carraher et al. (2006) with regard to their Early Algebra proposal. Similarly, Schliemann et al. (2012) show how algebraic notation can be introduced in elementary school in order to develop mathematical content, stating: "The 5th grade lessons focused on algebraic notation for representing word problems, leading to linear equations with a single variable or with variables on both sides of the equal sign." (p. 115).

Among the cautious, are Cooper and Warren (2011), who argue that:

> The results have shown the negative effect of closure on generalisation in symbolic representations, the predominance of single variance generalisation over covariant generalisation in tabular representations, and the reduced ability to readily identify commonalities and relationships in enactive and iconic representations. (p. 187)

In this regard, Radford (2011, p. 304) states that: "... the idea of introducing algebra in the early years remains clouded by the lack of clear distinction between what is arithmetic and what is algebraic". On this, the debate remains open, for example, Lins and Kaput (2004) characterising the movement as:

> ...algebrafied elementary mathematics would empower students, particularly by fostering a greater degree of generality in their thinking and an increased ability to communicate that generality. (p. 58)

As spokespersons for the Early Algebra working group at ICMI 12th (Lins and Kaput 2004), they openly criticised past generations, whose results were exclusively related to "sad histories", and specified that, in contrast, the Early Algebra movement presents research results linked to "happy stories" regarding the experience of learning algebra content.

The Third Excluded Strikes Back!

In light of the research results described above, this study approached the problems of learning algebra by introducing new variables that could not be left out of the discussion. As new theoretical approaches about learning algebra are born, so are different general learning paradigms. Research in the last century was strongly cognitivist, with Harel et al. (2006), surprised to learn that, in the PME studies on the period of 1995–2005, most of the investigations related to Advanced Mathematical Thinking were much more cognitive and less socio-constructivist or sociocultural. While communication in the mathematics classroom emerges as an essential element, the literature begins to show that researchers are inclined towards a socio-constructivist or sociocultural approach.

Our Theoretical Approach to an Introduction to Learning Algebra

The research objectives for this study are founded on a cultural approach, which takes into account the theory of activity and which views communication in the classroom as essential. The work of Engeström (1999) is taken as a culturally unifying approach, as advocated by Vygostky (1962), incorporating Leontiev's (1978) activity theory, in which communication is an essential element in the building of knowledge as described by Voloshinov (1973). Our approach to Radford's processes of signification, is immersed in a mathematics classroom teaching method named ACODESA (see Hitt 2007; Hitt and González-Martín 2015; Hitt et al. 2017), that also take into account a self-reflection component.

An analysis of the literature on the followers of the Early Algebra movement shows that some research is aimed at building a "Fast Track" from arithmetic to algebra. This study posits that a functional approach to algebra should be followed, such as that developed in both Passaro (2009) and Hitt and González-Martín (2015). This study concurs with some followers of Early Algebra, in that the use of patterns is able to generate generalisation processes in pupils, and, thus, proposes, in the context of the use of patterns, the following:

Generalisation. Construction of the subsequent term in a series when the previous terms are provided. Construction of an intermediate term when the previous and subsequent terms are provided. Construction of a term when the term in the series is a "large number" and when the first terms of the series have been provided. Construction processes for "any term from the series."

This study considers generalisation in the context of a pattern, where, rather than as a way of moving quickly from arithmetic to algebra, it is an element used to integrate into the pupils' mathematical structure. This will enable the pupils to develop the skills of prediction, argumentation and validation (Saboya et al. 2015), and will assist them in their transition from arithmetic to algebra and ***vice versa***. Indeed, a research program is proposed here that would develop ***arithmetic-algebraic thinking*** within a sociocultural context of knowledge construction.

As described above, this chapter, seeks to make a modest contribution, in that it represents the beginning of a research program. Our research is focused on specific content related to the construction of polygonal numbers in a sociocultural environment within Engeström's post-Vygotskian model (1999), and takes into account the results of those post-Vygotskian authors considered as comprising the fifth generation, such as Nardi (1997):

> The object of activity theory is to understand the unity of consciousness and activity. Activity theory incorporates strong notions of intentionality, history, mediation, collaboration and development in constructing consciousness. (p. 4)

As the use of technology in the learning of mathematics is a variable yet to be mentioned here, Mariotti's (2012) work on the role of artefacts as mediators in a learning process is integral to the inclusion of technology in a sociocultural environment.

The use of patterns, and especially the construct of generalisation, is related to mathematical visualisation. Visualisation, as mentioned by Duval (2002), is different from perception. In our case, then, the following applies:

Visualisation. Considering perception as something created by the individual – a "transparent" mental image depicting the situation with which s/he are faced – visualisation requires the transformation of representations associated with the task at hand, and the ability to articulate other representations that emerge in pupils' resolution processes, as associated with the task.

This study is not only interested in institutional representations (which can be associated with a register of representations, as described by Duval 1995). It is also concerned with the non-institutional semiotic representations that can be produced in a visualisation process (diSessa et al. 1991; Hitt 2013; Hitt and González-Martín 2015; Mariotti 2012) and which emerge in a semiotic process of signification (Radford 2003) when pupils follow a process of resolving a mathematical activity immersed in a technological setting.

Institutional representation. Representation found in textbooks, computer screens or those used by the mathematics teacher.

Non-institutional representation. Representation produced by pupils, as linked to actions undertaken in a process of resolving a non-routine activity different of the institutional representation.

Since the method proposed here is related to polygonal numbers and the use of technology, ideas related to the construction of polygonal numbers that date back to the time of the Greeks were considered here. Furthermore, including technology as one of the variables led to the inclusion of Healy and Sutherland (1990) and Hitt (1994), who, in Excel environments and Excel and LOGO environments, respectively, conducted investigations into the construction of polygonal numbers by secondary school and pre-service teachers respectively.

Healy and Sutherland (Ibid.) mention that the result obtained by those secondary level pupils (in the Excel environment) that expresses a relationship linked to the calculation of a triangular number "***n***", "***trig. Δn = na before + position***", is a non-institutional representation linked to a process of iteration. Hitt (ibid.) criticises these results, indicating that activities in an exclusively Excel environment provoke "an anchor" which does not allow them to switch to a classical algebraic context. Hitt (Ibid.) aimed to combine working with paper and pencil with the use of an applet constructed using the LOGO program. In light of these results and considering new theoretical and curricular contributions, both approaches are of contemporary importance, in that they generate diversified thinking for the production of non-institutional representations and iteration processes. Secondly, they enable the careful design of activities that promote the use of paper and pencil, while also fostering the evolution of the non-institutional representations that emerge at the initial stage into institutional representations within meaningful processes (a broader discussion on task design is provided in chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers" of this volume).

Methodology

Our research was developed within two populations, with one group from Quebec comprising 13 first grade secondary school pupils (aged 12–13 years old), and the other from Mexico, which consisted of 14 third year secondary school pupils (aged 14–15 year-old). Pupils agreed voluntarily to take part in the experiment, which aimed to gain insight into the problem, as occurring in the two populations individually, rather than comparing results.

- The Quebec experiment used Excel and an applet called POLY (see below), which had been designed exclusively for this activity (Cortés and Hitt 2012). Two researchers, known here as R_1 and R_2, developed the teaching experiment in a sociocultural setting. Two cameras and several voice recorders were used in this experiment.
- The Mexican experiment used a calculator (TI-Nspire) instead of Excel and the POLY applet. The activities were developed by one teacher, known here as P_1, and another researcher, known here as R_3. One camera was used in this experiment.

This study adheres to a teaching method known as ACODESA is divided into 5 steps (fully explained in chapter "Task Design in a Paper and Pencil and Technological Environment to Promote Inclusive Learning: An Example with Polygonal Numbers"):

- Individual work: production of official and non-official representations related to the task.
- Teamwork on the same task. Process of prediction, argumentation and validation.
- Debate (could become scientific debate). Process of argumentation and validation.
- Self-reflection (individual work in a process of reconstruction)
- Process of institutionalisation.

Engeström's (Ibid.) model was used to organise the ACODESA steps, taking into account a sociocultural learning setting. In previous research undertaken by these authors (see Hitt 2007; Hitt 2011; and, Hitt and González-Martín 2015), the

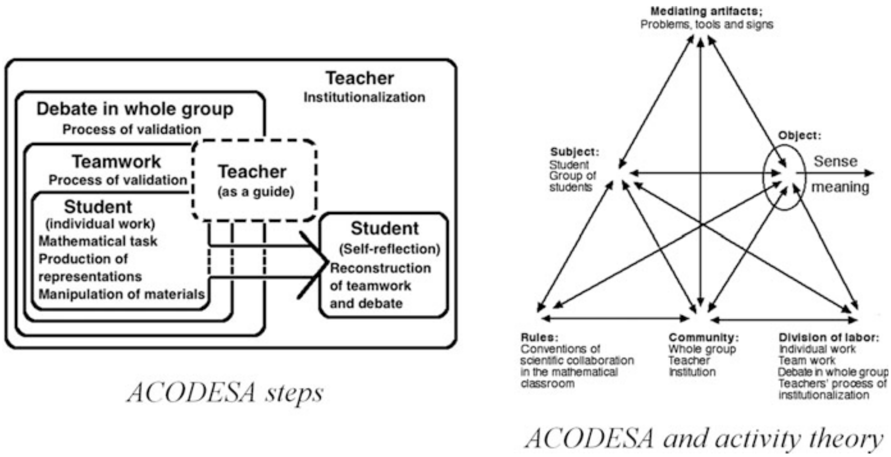


Fig. 1 ACODESA, as immersed in activity theory in line with Engeström's model

self-reflection step was implemented immediately after plenary discussion. Due to problems of knowledge retention (Hitt and González-Martín idem; Karsenty 2003; Thompson 2002), for this experiment we decided that for the self-reflection step, it would be interesting to implemented 45 days after the plenary debate.

The two first activities were implemented as an introductory activity in order to remind pupils of some of the Excel commands and provide a historical approach to polygonal numbers. The ACODESA method was implemented after the two previous activities.

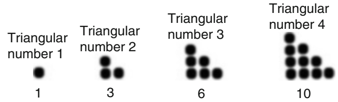
1. Resolution of two arithmetic word problems in a paper and pencil setting and a plenary discussion about how, according to the population, to solve them with either Excel or a calculator. This was implemented to remind pupils how to use Excel or a calculator.
2. Introduction to polygonal numbers from a historical point of view.
3. Invitation to the populations to solve the activity in line with the ACODESA characteristics shown in Fig. 1.

The tasks were used in both countries with only a few changes.

The second part of the activity was designed to work with Excel or CAS and to be verified with the POLY applet.

Analysis of the Quebec Results

The first introductory part of the session comprised the individual resolution of the two word arithmetic problems using paper and pencil, and a plenary discussion about how to solve the same problems using Excel. Researcher R_1 conducted the



1) Look carefully at these numbers. What is the fifth triangular number? Make a representation. Explain how you proceeded.
2) In your opinion, how are the triangular numbers constructed? What do you observe?
3) What is the 11^{th} triangular number? Explain how you found it.
4) You have to write a SHORT email to a friend describing how to calculate the triangular number 83. Describe what you would write. YOU DO NOT HAVE TO DO THE CALCULATIONS!
5) And, how do you calculate any triangular number (we still want a SHORT message here).

Fig. 2 First five questions in a paper and pencil environment

Develop the same ideas as in the previous section, but this time using Excel (or a calculator). Here is what we are requesting of you:

	A	B	C	D	E	F	G
1	Nombres polygonaux						
2	Position	1	2	3	4	5	
3	Triangulaire	1					
4							
5							

What do you do to find the 6^{th}, 7^{th}, and 8^{th} triangular numbers?

Is it possible to calculate the 30^{th} triangular number, the 83^{rd} triangular number, and the 120^{th} triangular number?

How do you do this? Explain

What kind of limitations and possibilities do you encounter when calculating under this approach?

Please show the operations you must undertake when calculating a polygonal number.

Fig. 3 Second part of the activity

plenary discussion, immediately after which Researcher R_2 conducted a short historical introduction to polygonal numbers and then initiated the first part of the ACODESA activity related to polygonal numbers. In this first step, individual work was required, as was work in a pencil and paper environment.

Once the pupils had undertaken the first individual explorations, R_2 organised the teamwork, with Team G_1 comprising three girls, Team G_2 comprising 3 girls, Team G_3 comprising 3 boys and a girl, and Team G_4 comprising a boy and two girls. Only one computer was permitted for each team.

Teamwork and the First Results

After pupils exchanged ideas, R_2 requested a plenary discussion, asking each team to present its findings on how to calculate the 11th Triangular number (T_{11}). Three teams (G_1, G_2 and G_4) presented their findings, while members of Team G_3 mentioned that their strategy was similar to the first team (see Fig. 3).

An initial and surprising outcome was the emergence of three different strategies. Their initial production (see Fig. 3) indicates that the pupils have undertaken a process of visualisation. They realised that it is possible to move along the diagonal, adding balls progressively, while one can add to the number of balls along the diagonal in an arithmetic progression. Pupils presented the first three iconic figures with their respective values and a process of generalisation in order to calculate T_{11} (see Fig. 4).

It seems that these pupils have undertaken a visualisation process in order to construct a general numerical progression. The action of adding balls along on the diagonal is transformed by adding the number of balls to the arithmetic progression, thus abandoning the iconic representation used to calculate the 3rd triangular number.

Team G_2 presented the results of their calculation of T_{11} with a single figure, indicating that, in the first column, one should place 11 balls, and then reduce the number of balls in the next column by one (10) in order to reach, at the end, only one ball, imagining the 1st column with 11 balls, the next with 10, and so on (see Fig. 5). An initial process of visualisation and generalisation is then made with only one drawing, thus inducing a numerical process which is inverted, as compared to that presented by the first team in Fig. 5. Team G_2's visualisation process is more compact than Team G_1, in that the team members made a direct iconic representation of T_{11}, expressing their process arithmetically. This example is generic in

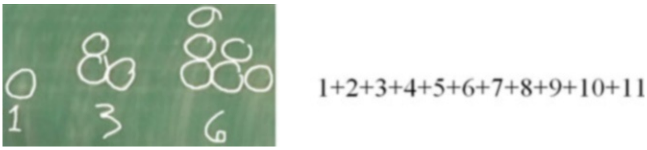


Fig. 4 The representation used by Team G_1

Fig. 5 The representation used by Team G_2

Fig. 6 The representations used by Team G_4 to calculate T_{11}

that they were able to represent any triangular number under this visual representation.

Team G_3 mentioned that they had a similar approach to Team G_1. A boy from Team G_4 (named G_4-1 hereafter) then approached the blackboard. From his first representation onwards, this pupil substituted the iconic ball-based representation for a more practical one, explaining that whenever one passed from one triangular number to the next, one had to add the appropriate number (see Fig. 6, below and left).

While giving his explanation, he suddenly changed the strategy without discussing this with his team members. He changed the representation he was using to calculate T_{11} for one which enabled him to construct both an iterative process to calculate T_{11} and a generic algorithm for any Triangular number (see Fig. 6). It seems that, through a process of signification (Radford 2003), the pupil was constructing a sign that enabled him to arrive at an iterative process for the calculation of triangular numbers.

An analysis of the pupils' written productions reveals that there were pupils in each team who undertook iconic calculations solely counting ball by ball. One female member of Team G_3 said nothing in response to the "leader" of the group indicating that they had done something similar to Team G_1, when in fact she had actually done something similar to Team G_2.

Process of Generalisation

R_2 then requested that teamwork continue, approaching team G_4 and mentioning that, when undertaking a calculation, they should show their working. A female member of Team G4 (known as G_4-2) interjected by saying that she did not understand how to calculate T_{83}, thus initiating a dialogue between G_4 and R_2, with G_4-1 and G_4-2 mainly involved in the discussion.

R_2 You must calculate it ... and show what you did.
(...)
G_4-2 I do not understand.
G_4-2 I do not understand.
R_2 Well, here we do not tell you the number of....
G_4-1 Is this number related to the diagonal?

G_4-2 The number on the side?
(...)
G_4-1 I take 83 on the side or on the diagonal and then you can count 1, 2, and 3 up to 83.
R_2 It's interesting, you have two different strategies.

In this excerpt, trying to assimilate that which was presented to the whole group by her teammate G_4-1, G_4-2 ceases to refer to the iterative process, instead only associating the number of balls, either vertically, at the base, or on the diagonal, and, thus, jumping from one triangular number to the next.

Once the pupils had worked in teams, R_2 again requested a process of recapitulation in a large group discussion, asking pupils how they would perform the calculation of the triangular number T_{83} (Fig. 7).

This extract is extremely important to the research conducted in this study. Pupils proposed a calculation of T_{83} that is identical to that proposed by Team G_1 for the triangular number T_{11} – namely $T_{83} = 1 + 2 + 3 + \cdots + 83$. R_2 tried to verbalise the calculation in terms of a generalisation for any triangular number. Pupils had no difficulty with this kind of process of generalisation. The symbolic process was executed naturally, with the assignation of a variable seeming not to disturb pupils at all. Even when the researcher proposed the use of a heart (♥) as a variable, this did not appear to disrupt the pupils in any way.

While, right up until this point, it is possible to say that pupils have been following various processes of generalisation, the question remains as to who, precisely, undertook this process. Throughout this process, pupils generally seemed to show that there was consensus. How stable was the pupils' knowledge as it emerged from a process of communication in the mathematics class? Can these pupils retain these results in the future?

Once this part of the activity had been completed, R_2 asked the pupils to return to work in teams, suddenly announcing "I can calculate any triangular number with three operations. Can you?" This was a question that resonated with some pupils, as described below.

Generalisation and Emergence of the Concept of the Variable

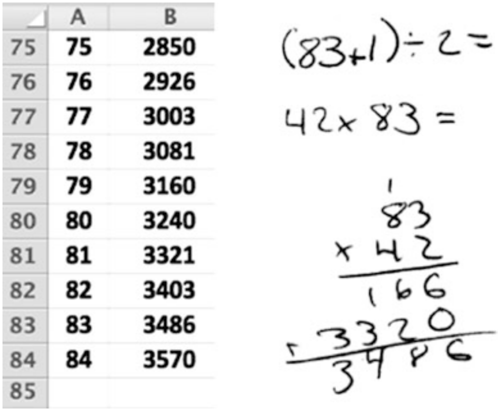
At this stage, pupils could use Excel in order to continue the activity. The idea of using a single computer introduced an unanticipated variable. The owner of the computer *determined the user*. For example, in Team G_3, the owner of the computer (a boy) was the only user. This reminded researchers of Hoyles' (1988) recommendation that attention should be paid to the constitution of a team when mixing boys and girls in a computational setting.

Once teamwork had commenced, pupil G_4-1 called R_1 over, saying that his group had formed a strategy to calculate any triangular number. He mentioned that

Dialogue between the researcher and pupils	Interpretation
Pupil 1: Uh ... you have to add up all the numbers from 1 to 80 for the triangular number of 80 ... er from 1 to 83 for the triangular number of 83.	Addition of 1 + 2 + 3 + ... + 83
R_2: Yes, but if you give me a number, which can change, it can always change. What kind of operation do I have to do?	R_2 tried to promote a generalisation.
G_4-1: You added together 1 + 2 + 3 + 4 +5 + 6 etc., until you arrive at your number. Ben! Then the answer is the ... your answer is the triangular number.	Using words, the pupils could describe the last number "until you arrive at your number."
R_2 : Ok. So there I would do 1 + 2 + 3 + ...	R_2 repeated the question, but continued to say "until your number."
G_4- 1: ... 4 + 5 + 6 ...	
R_2 : until my number.	
Pupil 1: Etc, until your number. You add up all this and it gives you your triangular number.	
R_2 : How do I write my number I do not know?	As there had been no change, R_2 directly asked "how do I write the number I do not know?"
G_4- 1: Question Mark!	
R_2 : Question Mark? Do you all agree? Yes? That's going to be my number I do not know?	Here pupils have shown that they have mastered the situation, suggesting several symbolisms.
G_4- 1: + x.	
Pupil 3: Yes + x.	
R_2 : x? Do I put something else? Yes, do I? (Point to a pupil)	
Pupil 4: Any letter.	
R_2 : Any letter, yes. There? A heart? Can we put a heart?	Even the ♥(heart) proposed by R_2 did not bother the pupils.
G_4- 1: You can put anything that is not a number.	

Fig. 7 Dialogue between R_2 and the group in a plenary session

their strategy involved taking any triangular number, for example 101, adding 1, dividing by two and multiplying the result by 101. R_1 told them to use another number, such as100. Then, G_4-1, stating that he would add one to 100 and divide the result by two, *suddenly stopped*, turned to his companion (G_4-2) and asked whether it made sense to get a decimal number when dividing by two. R_1 suggested that they discuss their strategy again and use the Poly applet to verify their results. R_1 remarked that the team had used Excel to calculate the triangular numbers up to T_{84} in a column and that they had the operation (indicated in Fig. 8) in their workbook.



	A	B
75	75	2850
76	76	2926
77	77	3003
78	78	3081
79	79	3160
80	80	3240
81	81	3321
82	82	3403
83	83	3486
84	84	3570
85		

Fig. 8 Technology and paper and pencil – the construction of a strategy

This is another key episode in this research. Pupils already had an algorithm to calculate T_{83}, which was to add $1+2+3+\cdots+83$. It was G_4-1who put the iterative algorithm on the board (see Fig. 6), using it with Excel to calculate up to T_{84}. The hypothesis explored in this study is that, using T_{83} and T_{84}, the pupil built his *new* algorithm.

Both the design of the activity and the pupils' processes show the possibility of reconciling work with pencil and paper and technology. In addition, and very importantly, the team destabilization generated by R_1's question about using an even triangular number (T_{100}) caused the team to reflect on that which is expressed by the use of the letter "x" in the arithmetic operations (see Fig. 9). It seems that pupil G_4-1 had written 100 + 1 divided by two ("no matter what you get"), in this case $\boldsymbol{x}$, the result ($\boldsymbol{x}$) must be multiplied by 100. Finally, the pupils used the POLY applet to ensure that the conjecture obtained might work with other triangular numbers.

The POLY applet is able to show a series of polygonal numbers or to give a specific polygonal number (as in Fig. 10). Due to screen limitation problems, if a polygonal number is too big, the applet can only provide the numeric result and thus excludes the figure.

Discovery of an Algebraic Expression for Calculating any Polygonal Number

The session was ending and R_2 called for a plenary discussion. G_4-1 asked to present what his team had found. G_4-1 exemplified their strategy with T_{46} and explained their algorithm. After R_2 explicitly asked him to explain how they had discovered their strategy, he repeated the algorithm, but did not mention how the discovery was made. It seemed that he did not understand R_2's question.

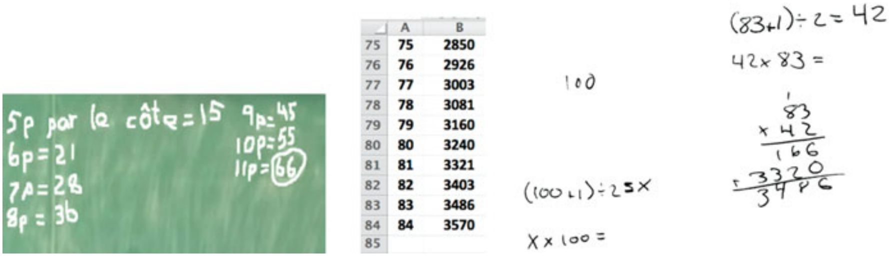


Fig. 9 Construction of a general strategy to calculate any triangular number

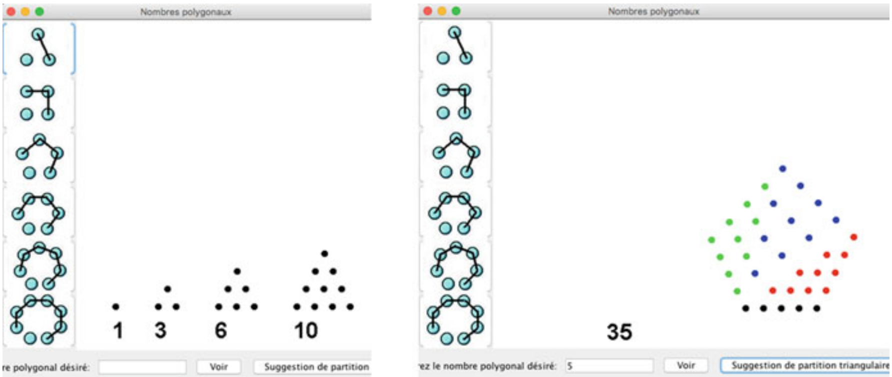
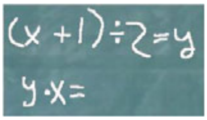


Fig. 10 Examples using POLY (series of triangular numbers and the fifth pentagonal number, including partition into triangles if required)



R_2 : x + 1, ***x***... So it is your triangular number? [Trying to interpret what the pupil wrote]

G4- 1: ***x*** is not my triangular number, it is my base number, plus one, divided by two, it's going to give ***y***. ***y*** times ***x*** gives the triangular number.

Fig. 11 Symbolising in a communication process

When the bell rang and R_2 had finished the session, a girl's voice, almost drowned out by the noise made by the pupils, indicated that she would have liked to know how the three operations could be used to calculate any triangular number. R_2 mentioned that there was no time for that explanation and that she would show it to her later. However, as G_4-1 mentioned that he knew this, R_1 and R_2 asked him to write it on the blackboard, even though the entire class was on their feet and ready to leave the classroom, whereupon G_4-1 wrote the following algebraic expression (see Fig. 11).

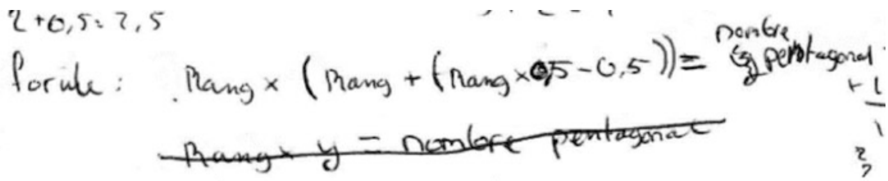
The researchers reacted very positively to this development at the end of this stage of ACODESA, deciding at that time to interview G_4-1 to obtain more information about the process of constructing the algebraic expression. The interview provided a few elements of which researchers were already aware. While G_4-1 insisted that it was through the POLY applet that he had come to discover the formula, both pupil productions (Fig. 9) and R_1's discussion with Team G_4 seem to indicate that the discovery took place while working with either Excel and pencil and paper, with Poly allowing them to check their conjecture.

Self-Reflection Phase Without Technology. What Happened 45 Days Later?

Aware of the problem with student retention of the mathematics that they learn and also aware, thus, that "consensus is ephemeral", the authors decided to make the self-reflection phase as different as possible to that which had had been undertaken in other experiments. Also, given that a talented pupil had been discovered in the sample, it was decided that an additional challenge would be added to the self-reflection activity (which, while generally the same, excludes technology) exclusively for him. So, in addition to the reconstruction process related to triangular numbers, he was asked to work with pentagonal numbers, something which was not dealt with in the classroom experiment.

It seems that G_4-1 did not pay attention to the examples given about triangular numbers, as he wrote that he already knew the formula to calculate any triangular number. ***He applied a wrong formula*** and did not check his results against the examples provided. However, and to the researchers' surprise, in a paper and pencil task (the use of technology is not allowed in this phase) that followed a similar process of finding relationships among the first four examples provided for pentagonal numbers, he constructed both the fifth pentagonal number and a general expression that allowed him to calculate any pentagonal number.

The results obtained are presented below. The data was collected on an individual basis before the teamwork and self-reflection phase 45 days later, with only eight pupils sampled from the other phases, plus others that could not be taken into account. Pupils 1–13 (the last being G_4-1) were identified in order to observe progress and setbacks.



$$rang \times (rang + (rang \times 0,5 - 0,5)) = nombre\ pentagonal$$

Fig. 12 G_4-1′s production in the self-reflection phase

Table 1 Results from the ACODESA self-reflection phase after 45 days

I. Anchor to the drawing	II. Drawing + addition	III. Abandoning drawing + addition	IV. Abandoning drawing + another strategy	"Algebraic expression", triangular or pentagonal numbers
1, 2	3, 4, 5, 6, 7	8, 9, 10, 11	12	13
45 days without technology				
4, 8			1, 5, 7, 12	11, 13

Table 1 shows two setbacks and four advances, not counting pupil G_4-1, who used an incorrect algebraic expression to calculate the triangular numbers. The female pupil (subject 11) followed her teammates from Team G_3 passively.

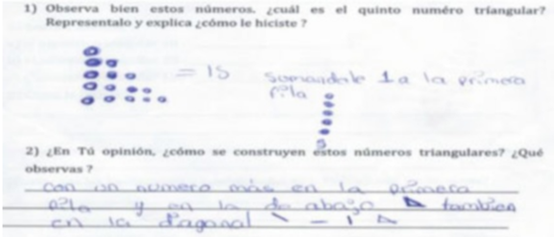
While her strategies were different, she did not discuss them with her teammates. The owner of the computer in team G_3 became the leader, which corresponds to Sela and Zaslavsky's research (2007) with four people working together. This male pupil, when using the computer, showed his colleagues various things not related to the task, thus creating a situation unrelated to the requested task. Teams G_1 and G_2 were more homogeneous and presented more balanced participation, with both teams composed of girls. The computer owner from team G_4 was strongly committed to the task and adapted very quickly to the rhythm of his colleagues, with one of the setbacks for the team posed by G_4-2 (G_4-3 was not present during the self-reflection phase). More careful study is required to analyse the role of technology in sociocultural learning. In fact, this, bearing in mind Hoyles (1988) on *Girls and computers* and the results reported by Sela and Zaslavsky (2007), leads to the realisation of the importance of creating teams consisting of a maximum of two or three subjects, and of trying to balance the use of technology in each team.

Experiment Conducted in Mexico

The experiment conducted in Mexico proceeded as follows. Once the initial problems were resolved in order to introduce pupils to the TI-Nspire calculator, the teacher asked the pupils to work on the first five questions individually (see Fig. 2). Four teams with three pupils and one team with two pupils were formed, and in which each pupil had a calculator (TI-Nspire).

Individual Work

Two types of strategies emerged from the individual work – one linked to the drawings as shown in the examples, and the other to the formation of a table of values. Again, spontaneous representations linked to functional representations appeared in the communication process (Fig. 13).



1. Adding 1 to the first range.
2. With 1 added to the first range and both the line at the base of the ◣ and the diagonal \ – | ◣ .

Fig. 13 Spontaneous representations used by one pupil

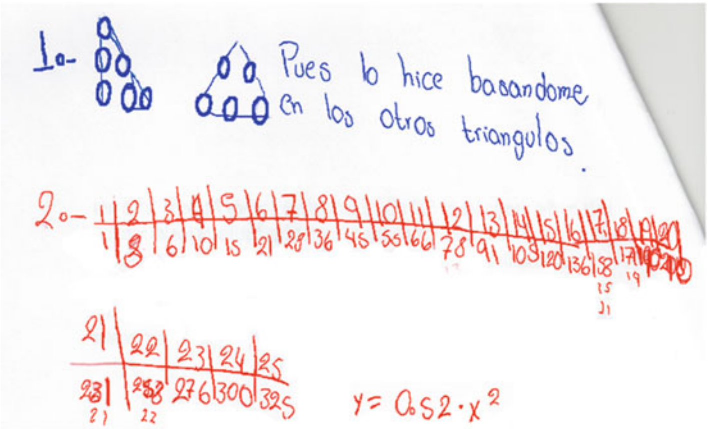


Fig. 14 Karla's individual work and teamwork

Teamwork

One team made up of Diana, Karla and Omar underwent a reconciliation of strategies, with Diana and Karla making the calculation by counting balls from the drawings, while Omar used a table of values. Through this strategy integration process, Diana and Karla left their strategy and decided to use the table of values.

The pupils' individual productions and the film of the plenary session are the only evidence of the individual work carried out by the subjects. A useful research technique was that pupils were asked to write with red ink when working in teams (see Fig. 14).

There is no evidence of how the formula was obtained. The formula appeared in Karla's productions and was written in red ink during the teamwork phase. This shows that, in her team, she adopted the table of values and proposed an algebraic expression. At this secondary level pupils had already learned about the notation of variables, and can clearly be seen to use the variables x and y. The formula enables pupils to estimate the calculation of polygonal numbers. Our hypothesis is that having used y = $x^2/2$ (related to ***base* * *height/*2**) and having realised that it did not work with the examples given in the table, these pupils decided to approximate the results, thus giving $\boldsymbol{y = 0.52 * x^2}$. They did not present their proposal in the large group discussion.

A surprising result is that many Mexican pupils participating in this research immediately associated the triangular arrangement of the triangular numbers, ***base* * *height/*2**, with the calculation of the area of a triangle, thinking that this was the algebraic expression required.

This was the first thing that emerged during the large group presentation, having been captured from the beginning of the pupil discussion.

Plenary Discussion

Monica, facing the blackboard, gave as example the T_8 (see Fig. 15), describing the following as necessary in order to calculate it.

R3 intervened, saying that, at this point, it was necessary to review her theory. She was then interrupted by the pupil Rob.

Rob ... But... this, the triangular number 8 would be 36 and not 32 – then it cannot be.

R_3 Please come to the blackboard Rob.

Rob This is number 6 [pointing to the figure just produced by María]. We apply the formula that says the base multiplied by height would be 6 by 6 giving 36 divided by 2, giving us 18, while the triangular number would be 21. This, therefore, is not the formula [points to the ***base* * *height/*2** formula].

P_1 What is the difference between. .. if you apply the formula that tells you, apply the formula. Write it there.

María [Writes the formula].

P_1 In this case, what is the basis?

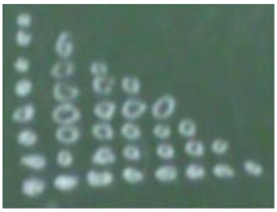
Rob [Writing] 6 * 6/2 is 18 and there is the triangular number.

P_1 What is the triangular number?

Pupils 21 [answering chorus]

Rob The difference would be 3.

P_1 The difference would be...?



Monica: To calculate the area of the triangle would require 8 times 8, giving 64, which, divided by 2, is 32...

Another pupil: [a girl is heard addressing her classmate in a very low voice] But no, that would be 36!

Fig. 15 Monica presenting the calculation of T_8

Rob 3.
Rob calculates the Triangular 8, and, typing the formula, obtains 32.
P_1 And what is it for the triangular number 8?
Rob It's 36.
P_1 The 8?

Rob starts counting the balls for the figure that was already on the blackboard, and says that it is 36.

Pupils The difference is 4.

Rob realizes that the formula does not work and that he has shown a counter-example (similar to the comment made by the unidentified girl). However, so far he is not able to build the exact formula for triangular numbers.

EUREKA! Rectification of the Formula in a Scientific Debate

In the midst of the discussion, a surprised voice is heard, saying, "and from 8 it is four, then it would be half."

Gaby Half of 8 is 4, and 4 is what is missing from 32 to 36 in the formula, then we have to put the base multiplied by height divided by two more.... [PAUSE] plus the half of [PAUSE] plus the half of the triangular number, half [PAUSE] half of the base.

While speaking, Gaby paused several times while she completed the transformation of her numerical idea into a geometric-algebraic idea. It is clear that the control element was provided by the arithmetic relationship and the transformation from that into a geometric relationship. However, something else occurred in the process of communication when Gaby was verbalising what she was thinking: there was a process of deduction. At this very moment, the pupils were rejecting their initial conjecture in favour of a new one, using the arguments to refute the conjecture.

P1 Write it!
Pupil ***I think that we, all together, are arriving at something, not alone!***

The sociocultural construction of knowledge has occurred at this stage of scientific debate, in accordance with ACODESA. The pupil openly expresses the co-construction generated through the debate.

Gaby goes to the blackboard to write the idea that she had just thought of.

Gaby How do I represent half the base?

Interestingly, at this point, Gaby has difficulties in transforming the geometric argument "half the base" into algebraic terms:

R_3 . . . One half, or that in half.

Gaby writes $\frac{b \times h}{2} + \frac{1}{2}$.

Many pupils worked together to undertake the final writing activity. In this process, Gaby was guided by her colleagues because she did not fully understand the process of algebrafying “half the base”. Finally, she wrote: $\frac{b \times h}{2} + \frac{b}{2}$

P_1 And how do you represent the triangular number in that formula you are writing there? How do you represent the triangular number?
Pupils .... The base or the height...

Gaby writes the formula $\frac{b \times h}{2} + \frac{b}{2}$.

P_1 Precisely, what she said, the height is the same as the...
Pupils The base.
P1 Replace it and do not write the height.
Pupil It would be base times base.

Gaby misspelled the formula and was corrected by her peers, after which she wrote: $\frac{b \times b}{2} + \frac{b}{2}$.

P_1 Ok then, base times base is what?
Pupils Base squared.

Gaby writes finally the formula $\frac{b^2}{2} + \frac{b}{2}$.

P_1 Do you think that is the formula? Verify it with the triangular number 15
Gaby Do I have to count the balls in a drawing?
P_1 No, no, no you have already got the formula!

Self-Reflection Phase Without Technology. What Happened 30 Days Later?

This phase, referred to here as self-reflection without technology, comprised a questionnaire (with slight modifications) similar to that completed by the participants 30 days previously. Only 10 of the 14 pupils participated, with the main idea at this stage being a reconstruction of what had been undertaken in the classroom.

The questionnaire for the self-reflection phase had three questions:

1st Question Calculate the 27th triangular number.
2nd Question Write the formula for calculating any triangular number.
3rd Question Using your formula, calculate the 313th triangular number.

The results are as follows: from the ten pupils, two continued the process of “drawing balls and counting”, while four of the ten rebuilt a similar expression related to the area of a triangle – b $\times$ h/2. A pupil was able to reconstruct the formula, but mistook the result of calculating T_{27} by finding a triangular number to provide 27 as a result, with the closest being $T_7 = 28$. It is possible that he made a mistake when counting the balls. He followed the same strategy when calculating T_{313}. Among those who were able to reconstruct the right formula were Alejandra, Omar and Rob.

Conclusions

These results reveal the importance of building arithmetic-algebraic thinking in order to support algebraic thinking. The experiment conducted in Quebec with 1st year secondary pupils (11–12 years old) revealed the following:

- It was performed in a sociocultural environment, with a gradual construction of the concept of the variable, and patterns built from the visual work.
- The strategy consisted of visualisation processes that related drawings, arithmetic addition series, iterations and formulas. Pupils used natural language with letters representing variables.
- The validation process was supported by the use of technology.
- The availability of a device on the table, shows that its use is delicate (Hoyles 1988). In the case of one team, it was the owner of the computer who exclusively used it. In the team with a boy and two girls, the boy mostly used his computer, the girls used it when he was at the blackboard.
- Even though there was more progress than setbacks in the self-reflection phase, the results show that concluding that consensus had occurred should be undertaken with caution.

The experiment conducted in Mexico with 9th grade secondary pupils (14–15 year old) revealed the following:

- It was performed around visualising a process related to the area of a triangle, with use of variables to represent the variation (x, y, b, h).
- The validation process rested more on visual configurations.
- The technology was not widely used by the pupils. Plenary discussion and co-construction attracted the pupils' attention.
- Again, it showed that "*consensus is ephemeral*", with only four out of ten able to rebuild the formula and one of them mistaking the number of the triangular with the result.

A surprising fact is that the institutional representation ***n (n* + *1)/2*** did not emerge in neither of the two populations. This demonstrates the importance of pupils' spontaneous representations in the construction of mathematical concepts (Hitt 2013; Hitt and González-Martín 2015). This reveals that evolution of spontaneous representations is important in a signification process. During the institutional stage under the ACODESA model, the teacher must collect different ideas and productions and relate them to the institutional representations.

Our research is taking into account the importance made in the 40s when psychologists payed attention to the importance to moving from analysis of pupils' performances when solving puzzles to the analysis of pupils' problem solving activity (see Brownell 1947). Our approach, in this technological era goes from an arithmetic context to an algebraic one, in a natural way using technology as a tool in a process of generalization and, in a sociocultural context of learning. In this way, pupils construct arithmetic relations (product of this is what is shown in Fig. 9) that permit them to control their process of generalization to an algebraic context in a milieu of creativity and autonomy (see Fig. 12). The results are showing the importance to promote the production of spontaneous representations and conversions among them even if they are not the institutional representations. This contrast directly with Kirshner's approach (2000) concerning his ideas of exercises, probes and puzzles; and, about his restricted approach to learning algebra focusing on the algebraic register about visually salient rules (Kirshner 2004). Our research takes as central a task-design where the but is related to enchained tasks, to be solved by the pupils in a sociocultural milieu, and the teacher role is to promote students' reflexion and productions of spontaneous representations, not only the algebraic one.

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L-System Fractals as Geometric Patterns: A Case Study

Anna Alfieri

Abstract Digital technologies are impacting all aspects of personal, social and professional life by now, spreading out at an incredible speed. We should take into account all these changes in the teaching and learning processes of mathematics, implying new challenges and responsibilities. In this paper, the role of technology in a mathematics education activity is analysed into two aspects: in terms of its operational features for presenting mathematical content, to research for information on the web, to work in an e-learning environment with students, and then for enhancing cognitive learning processes in the student, through the digital manipulation of geometric objects. The context is the L-system fractal theory.

Keywords L-system fractals • Technology • Education

Introduction

New technologies strongly influence our social and professional lifestyle, becoming necessary in our communication, as considerably as in our relationships with authorities and Institution. They also affect the learning process and teacher's work inside and outside school, contributing to an evolution of the relationship between teachers and students, and creating new challenges and responsibilities.

Many questions arise regarding the impact of technology on mathematics education. Artigue (2013) highlights some crucial questions that have been shared and discussed within our community:

> How to avoid an increasing divorce between social practices and school practices, with the resulting negative effects on the image of the discipline and the students' motivation for engaging in its study? How to benefit in mathematics education from the incredible amount of accessible information and resources? How to benefit from the changing modes of social communication and Internet facilities for creating and supporting communities of mathematics teachers, and definitively break with the vision of the teacher as a solitary worker? (p. 5)

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The issues introduced by Artigue (Ibid.) led me to renew the mathematics education methodology through ICT. In fact, the most relevant aims in my educational activities are:

1. Developing a mathematics education open to the external world, remaining anchored to the epistemology of the discipline;
2. Giving students more autonomy in geometric knowledge;
3. Using new technologies to better address the specific needs students may have.

From Solitary Teaching to a Community Mathematics Project

In this evolving technological context, Artigue (Ibid.) highlights:

> Learning mathematics is not achieved just by accessing mathematical information or even direct answers on the Internet. Accepting to live in the digital era is accepting to have this flux of information entering the school world. (p. 5)

This "flux of information" also leads to discover countless technological tools that teachers can test with their students in the classroom, even if often the teacher is left alone in this researching process.

For this reason, many teachers are looking for training course to introduce new technologies in their learning and teaching processes; they need to find community with other teachers to share experiences, to exchange results, and to know the issues that have been addressed by other teachers in similar educational contexts.

In Italy, for example, the national M@t.abel project is a very important educational program, involving several mathematics teachers. The M@t.abel project, performed by INDIRE (Istituto Nazionale Documentazione, Innovazione, Ricerca Educativa), introduces teachers to mathematical training through examples of real class activities. The teachers work together in a technological platform and are included in virtual classes, managed by a pool of tutors (including the author of this chapter), in order to discuss and share their experiences cooperatively (Arzarello et al. 2006). M@t.abel is an important national activity that turns solitary teachers into members of "communities of practice" (Wenger 2000).

Another interesting national project is Matematica & Realtà (M&R), managed by the department of mathematics at the University of Perugia. This project deals with making proposals to develop innovative and educational connections between mathematics and the real world (Brandi and Salvadori 2004). The Liceo Scientifico "Luigi Siciliani" in Catanzaro (Italy) has been taking part in the M&R project for 8 years with a pool of teachers (me included).

The central focus of M&R is mathematical models, defined as a result of a rationalization and abstraction process, which allows teachers and students together to analyze the problem, to describe it, and to create a representation with the universal symbolic language of mathematics (Ibidem).

The M&R project employs a variety of activities, including: mathematical modelling and best maths presentation competitions. A central feature of the project is the creations of mathematical laboratories, organized in each participating school; in these laboratories, the teacher proposes different themes, related to M&R, depending on the age of the pupils and in connection to compulsory mathematical content. For instance:

1. Linear mathematical models of reality (for 15–16 years old students);
2. Iterative functions and fractal geometry (for 16–17 years old students);
3. Nonlinear mathematical (exponential and trigonometric) models of reality (for 17–18 years old students).

In Liceo, many different mathematical laboratories have taken place since 2006, attended by approximately 1000 students. The M&R project is one of the most important extra-time activities in the school to entail additional credit to the students.

The Role of Technology: From Classroom to Cloud-Learning

My experience in the project concerns geometry, in particular fractal geometry, geometric properties of fractal figures, and geometric transformations. In this chapter, I am going to describe the educational activity concerning Lindenmayer Fractals (or L-system fractals, as they are commonly called), in order to examine the role of new technologies in this specific teaching and learning process. Technological tools can have several educational functionalities, three of which are suggested by Drijvers (2012):

1. The tool function for doing mathematics, which refers to outsourcing work that could also be done by hand,
2. the function of learning environment for practicing skills,
3. the function of learning environment for fostering the development of conceptual understanding. (p. 3)

In the M&R activity, the teaching and learning of fractal geometry have dealt with through several stages; in each of these, technology plays a relevant role characterized by a significant use of ICT. The different "functionalities" allow me to split the teaching activities into following steps.

Working Steps

The educational activity in the L-Fractal M&R project has three different steps:

Table 1 Connection among steps of activity, role of technology and tools

Step activity	Role of technology	Tools
Fractals class	Presenting fractal theory	Interactive with e-board, laptop
Fractals webquest	Researching for data and information	Internet
Fractals cloud learning	Working and sharing in virtual environment	Google doc, Google drive, PowerPoint, Paint

1. *Fractals class*, an extra-time course about fractal geometry;
2. *Fractals webquest*, the design of a final project about L-system fractal theory;
3. *Fractals cloud learning*, a methodology used to work online.

In my experience, each of these steps is related to one of the three different facets of using technology and its tools. *In Fractals class* technology is used to present geometrical objects and their properties; *in fractals webquest* it is employed for researching information on the web; and *in fractals cloud learning* it is used to share and to work with students in an e-learning environment (see Table 1).

These connections are explained in the following sections. Actually, these different types of the use of technology are not distinct and are deeply intertwined. However, in the design of the activities, the use of a specific facet of technology has characterized the corresponding phase more than any other.

Fractals Class: Technology for Learning by Presenting

Fractals class is a basic course about fractal geometry, which I have been developing in my school since 2006. It requires 16 h (2 hour per week in a single lesson). Many 16–17 years old students choose to attend the course voluntarily (see Fig.1).

The principal aims of the course are:

1. Modelling the world around us using affine transformations;
2. Building the most famous fractals (Sierpinski's Triangle, Koch's Snowflake, etc.);
3. Plotting fractals with software tools;
4. Making conjectures and simulations using free software;
5. Mixing traditional teaching and new technologies.

The geometrical content, dealt with in the fractals class, includes:

1. Geometric transformations and matrices (composition of geometric transformations), the inverse of a geometric transformation, affine transformations (rotation, contraction, translation);
2. Iterated function system, codes of fractals, evolution of an iterative process of figures patterns and attractors;
3. Fractal properties.

Fig. 1 Fractals class: during the lesson

During this step, the teacher works using the following educational strategies:

1. *Interactive lesson and practice*

The teacher explains the basic concepts of fractal theory: affine transformations and the most famous fractal codes. Then she enhances the learning process through some exercises. At this stage, the student is encouraged to work alone.

2. *Cooperative learning*

The teacher arranges students in small groups to work together, in order to improving communication and cooperation. Cooperative learning fosters their participation and their involvement without inhibition, promoting skills development of the students. Peer-to-peer communication multiplies the value of the educational message and increases its effectiveness: sending and receiving communications spread throughout the network of students, and are not limited to the first sender.

In this step, the presentation of fractal theory is simplified by using interactive whiteboards (IWB) during the lessons. Some aspects of direct teaching, such as explaining, modelling, directing and instructing, are facilitated by the IWB, or more specifically, the software accessed via a large screen presentation device. (Wood and Ashfield 2007). The quality and clarity of multimedia resources may offer enhanced visual material for presenting to a large audience, and the teacher is able to move between varieties of electronic resources, with greater speed in comparison to non-electronic tools. In this step, technologies, such as the IWB, may change the pedagogic practice to make easier the learning process of several mathematical contents.

Fractals WebQuests: Technology for Learning by Searching For

At the conclusion of the fractals class, the teacher offers an advanced course about L-system fractals. This is a new step of the activity, called *fractals webquest*. The teacher assigns a final project work concerning L-system fractals which would enable students who complete the assignment to participate in the annual M&R competition, *"Best Maths Presentation*", a special feature of the yearly M&R National Congress at the University of Perugia. Each year, there are many groups of students, from the fractals class, who wish to experience this unique educational opportunity.

During this step, the role of the teacher is to support students in cultivating the critical skills, necessary for appropriately utilizing media tools inside and outside of school. Living digitally means that, when a question springs up or a problem arises, we already have instant access to relevant information or even the correct answer. For this reason, the student must learn to consciously explore and use this flux of information (Fig. 2).

The webquest educational approach has the primary goal of discovering additional information on a specific topic, and to create a presentation, using the collected data. Bernie Dodge at the University of San Diego (USA) brought international recognition to webquest pedagogy in the mid-1990s, defining webquests as:

> An inquiry-oriented activity in which most or all of the information used by learners is drawn from the Web. WebQuests are designed to use learners' time well, to focus on using information rather than looking for it, and to support learners' thinking at the levels of analysis, synthesis and evaluation. (Dodge 2001, p. 6)

The Internet is a chaotic space, in which anything or nothing can be found. The network carries a plethora of information, presents endless realities, news, and experiences; the Internet lives on exchanges, enriching overall products with multiple subscribers.

This chaotic knowledge must be deciphered, selected, structured; otherwise, without the skills of artfully curating what you find and search could seem sterile and superficial. The webquest helps students to avoid getting lost in the network and to use their time efficiently. In order to achieve such efficiency and clarity of purpose, a good webquests contains at least these accompanying parts (Dogde 2001):

1. An **introduction** that sets the stage and provides some background information.
2. A **task** that is doable and interesting.
3. A **description** of the process, the learners should get through in carrying out the task.
4. A **conclusion** that brings closure to the quest.

The strategy employed during this step of my activity is based on two kinds of webquests:

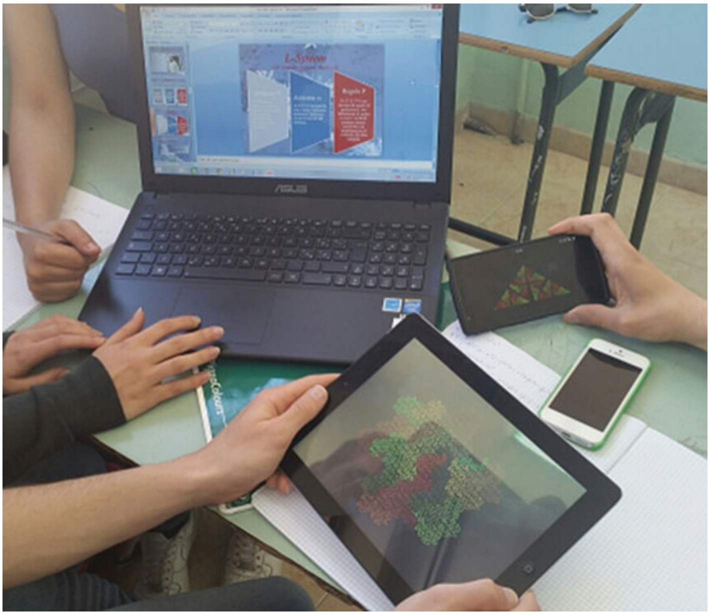

Fig. 2 Fractals Webquest: During the Lesson

1. *Short Term WebQuests*.

It consists of two lessons to gather data and to elaborate their structure. The project work about L-system fractals is divided into four different parts:

1. Fractals in general (what a fractal is and the most important properties of fractals);
2. Mathematical content (affine transformations and matrices);
3. L-system fractals (theoretical definitions about L-system fractals and their codes);
4. Logo (or Turtle) language and free software for plotting images of fractals.

2. *Longer Term WebQuests*.

At this phase, the students collect materials about L-sytem fractals from the Web. Each student has to analyse, study and summarize the documents respecting the sequence of previus points in *Short WebQuest*, without the teacher's help. After completing this part, the learner is able to create an original presentation about L-system fractals, using the material downloaded from Internet.

The role of the teacher, in this step, is to guide students' choices, and to introduce a timeline.

In this activity, students acquire the ability to look for information on the web or in data sources, including web documents, databases and freeware software, to select the most relevant parts, and to apply the most suitable results of their search in developing their topic.

Fractals Cloud-Learning: Technology for Learning by Sharing

After fractals class and the fractals webquests, the teacher works with the groups of students to prepare multimedia presentations for the national M&R congress. *Fractals cloud-learning* is the name of this step.

New technologies enable people to customize the working/learning environment using a range of instruments to match personal interests and demands. This is the reason why it has been explored the educational potential of *'cloud computing'*, in our activities. Web 2.0 tools offer the opportunity to interact and to cooperate with one another in a social media dialogue as creators of user-generated content in a virtual community, in contrast with websites that limit users to a passive view of the content (Despotović-Zrakić at all. 2013; Katz 2009). Examples of Web 2.0 technologies include social networking sites, blogs, wikis, video-sharing sites. Web 2.0 or cloud-based technologies support that trend begins with the emergence of the Internet: a shift away from large organizational control of the instructional function toward the individual user.

These emerging technologies, not necessarily created for higher education, support, require individual creativity and autonomy, and foster the growing trend towards user-generated content and knowledge, in a way that many institutionally developed products do not.

They also have the potential to promote sharing, openness, transparency and collective knowledge construction. A part of their proliferation can be ascribed to the low-cost instructional innovation; the emerging technologies enable, along with ease of utilization in a higher education environment of shrinking budgets and increased competition for information technology budget.

In my educational activity, Google Docs® and Google Drive® (see Fig. 3) are applied and used as a part of an e-learning environment, where students and teacher toghether can exchange and share ideas and information and they can work in a synchronous way – despite not being in the same classroom – in order to achieve the final version of the multimedial presentation about L-system fractals.

Google Drive® is used as storage space for students' files, while Google Docs® is used as a learning and teaching tool for working between teacher and students simultaneously. About the role of technological tools, in general, Railean (2012) claims that:

> The role of these tools for teachers is to provide a learning environment for team work as a need for each child in order to develop self-regulated skills. Imitation, cooperation, confrontation, discussions and sharing are all part of the development of the individual and his or her socialization. These tools play an important role in their cognitive, affective activities. (p. 22)

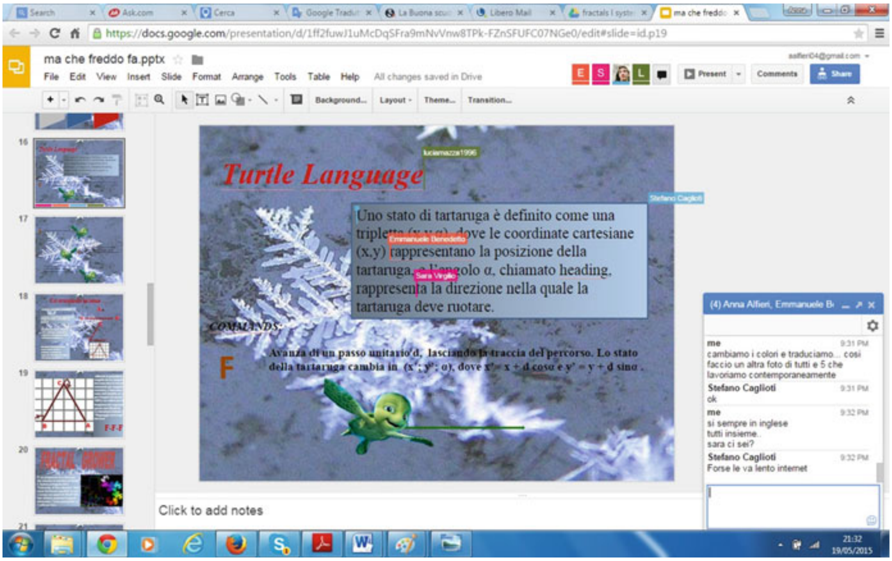

Fig. 3 Fractals cloud-Learning: a discussion board in Google Doc®

The advantages, in addition, of using Google Docs® are:

1. Many people can work at the same time on the same document and everyone can see people's changes as they make them, and every change is saved automatically.
2. Everyone can also propose changes directly by suggesting an edit without editing the text. These suggestions won't change the original text until the document owner approves them.
3. Everyone can collaborate in real time over chat, too. If more than one person has the document open, he just has to click to open a group chat. Instant feedback is possible without leaving the document.

In Google Docs®, the teacher acts more systematically as advisor, guide and supervisor, as well as a provider of the frameworks in the learning process of her students.

The Role of Technology in the Cognitive Learning Process During the Project

During this educational activity, I recognize another important role of technology in the knowledge of Geometry: the use of ICT enhances cognitive learning processes of the students and allows themselves to discover in autonomy some geometrical properties. In fact, during the development of the L-fractal theory, I observe a dialectic between technology and the construction of the geometric content: there are some moments in which technology supportes the comprehension of theory and there are others in which the use of technology transforms theoretical properties in a repetitive training of procedures, these must be reduced. In order to discuss this aspect, a summary of the L-system fractals is necessary.

What an L-System Fractal Is

L-system fractal geometry regards a mathematical theory of plant development. Fractals are geometrical figures, characterized by unlimited repetition of the same shape on lower sequence. Fractal's proprieties are: self-similarity; scaling laws and non integer dimension (Mandelbrot 1977; Gowers 2004).

Aristid Lindenmayer (1925–1989) was a Hungarian biologist who created a formal language called Lindenmayer System or L-system to generate fractals.

The central concept of L-system is the rewriting process, that is a technique for defining complex objects by successively reproducing parts of a simple initial object using a set of rewriting rules or productions (Prusinkiewicz and Lindenmayer 1990).

In L-system, a string can be defined as an ordered triplet $\mathbf{G} = (\mathbf{V}, \boldsymbol{\omega}, \mathbf{P})$ in which:

1. **V** is a finite set of symbols called "alphabet";
2. $\boldsymbol{\omega} \in \mathbf{V}^{+}$ is a non-empty word called axiom ($\mathbf{V}^{+}$ is the set of all non-empty words over V);
3. **P** is a finite set of production**:** $\mathbf{P} \subset \mathbf{V} \times \mathbf{V}^{*}$, $\mathbf{V}^{*}$ is the set of all words over V. **P** defines how the variables can be replaced with combinations of constants and other variables. A production $(\mathbf{a}, \boldsymbol{\omega}) \in \mathbf{P}$ is written as $\mathbf{a} \rightarrow \boldsymbol{\omega}$**.** The letter **a** and the word $\boldsymbol{\omega}$ are called the predecessor and the successor of this production, respectively. It is assumed that for any letter $\mathbf{a} \in \mathbf{V}$, there is at least one word $\boldsymbol{\omega} \in \mathbf{V}^{*}$ such that $\mathbf{a} \rightarrow \boldsymbol{\omega}$.

The Connection Between L-System Fractals and Technology Through Productions of Students

Many software of computer graphics, based on L-system theory, are available on the Web, they allow us realistic visualization of plant structures and their development processes. During the activities, the generated fractal images fascinate the students for their colors and shapes, similar to real plants, but the use of technology, for creating these patterns, supportes them, especially, in reasoning and solving some problems, in developing of their curiosity. According to the position of NCTM (2011) (National Council of Teachers of Mathematics):

> It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication. Effective teachers optimize the potential of technology to develop students' understanding, stimulate their interest, and increase their proficiency in mathematics. When teachers use technology strategically, they can provide greater access to mathematics for all students. (p. 1)

In the Italian educational program, regarding high school mathematics curriculum, there are the same goals: problem solving ability, creative thinking and logical thinking to enhance mathematical ability of modelling reality. For achieving these purposes, during my educational activity, the following theoretical issues from L-system theory are proposed and discussed with students:

(a) *Converting logical and formal rules of L-system language in geometrical patterns: making fractals;*
(b) *Proving that if the starting figure changes, the fractal does not: fractals as attractors;*
(c) *Modelling reality: fractals as geometric patterns.*

Every issue is the expression of a specific goal and for everyone the technology plays a relevant role, that I summarize in the table below (see Table 2). Goals, issues and contents are connected and solved by a digital manipulating of geometric objects. Actually, it is difficult to separate the single goal from the single content and single issue; they are interconnected in fractal theory. This is a necessary attempt to analyse the "dialectic" relation between the use of technology and the comprehension of L-system fractal theory, at this school level.

The issues (a), (b), (c) will be discussed separately in order to document what happens during the activity and to describe the role of technology in the comprehension of fractals. For the description, I am going to use many images from two final multimedia presentations, made by students during the M&R activities:

1. "*To make a tree… it takes an L-system fractal*"
2. "*Fractal snowfall in Catanzaro*"

Both works were presented at M&R National Congress in Perugia in 2012 and 2014, the first topic was also presented at European Mathematical Congress for Students in Gothenburg.

Both works are an example of mixing affine transformations and L-system fractals for modelling real world: in the first case the world of trees and in the second one the world of snowflakes.

(a) *Converting logical and formal rules of L-system language in geometrical patterns: making fractals.*

In this issue, the formal language and the geometric patterns are compared: the passage from the numerical codes and formal rules of the L-system fractal to the visualization of the geometric patterns is allowed by technology.

In fact, one of the geometric system that computer graphics used for the L-system's generation is called Turtle Geometry.

Table 2 Connection among goals, content, issues and solution by technology

Goals	L-system fractals contents	Issues	Solution by technology
Logical thinking	String, **ω** axiom, **p** production, turtle-language	Converting logical and formal rules in graphical patterns	Computing codes in graphical patterns
Problem solving ability	Affine transformation, contraction, rotation, translation	Proving that the fractal does not change, if the starting figure changes	Applying new codes in geometric transformations
Creative thinking	Logo-language	Modelling reality by fractals	Plotting graphical patterns

A state of the turtle is defined as a triplet (x, y, α) where the Cartesian coordinates (x, y) represent its position, and the angle α, called the heading, is the direction the turtle is facing in.

Given step size *d* and the angle increment *δ*, the turtle can respond to commands represented by the following symbols:

1. **F** (it moves forward a step of length *d* the state changes to ($x^{'} = x + d \cdot \cos\alpha$, $y^{'} = y + d \cdot \sin\alpha, \alpha$) a line segment between points (x, y) and (x', y') is drawn);
2. **f** (it moves forward a step of length *d* without drawing a line);
3. **+** (it turns left by angle δ, the state changes to $(x, y, \alpha + \delta)$);
4. **–** (It turns right by angle δ, the state changes to $(x, y, \alpha - \delta)$).

Given a string *ν*, the initial state of the turtle (x_0, y_0, α_0) and fixed parameters *d* and *δ*, the *turtle interpretation* of *ν* is the figure drawn by the turtle in response to the string *ν*. Specifically, this method can be applied to interpret strings which are generated by the L-system (Prusinkiewicz, 1999). This language has been used to generate many fractal figures, during the project work, with students, like into following examples.

Example 1: Quadratic Koch

1. Axiom **ω: F − F − F – F**, start angle 0°, turn angle 90° (it corresponds to the initiator or starter figure of the fractal) (Fig.4a);
2. Production **p: F → F − F + F + FF − F − F + F** (it corresponds to the generator of the fractal) (Fig.4b);
3. Quadratic Koch island at the fifth generation (Fig.4c).

Example 2: A snowflake

1. Axiom **ω: F--F--F--F--F**, start angle 18°, turn angle 36° (initiator or starter figure of the fractal) (Fig.5a);
2. Production **p: F=F--F--F-----F+F--F** (generator of the fractal) (Fig.5b);
3. Snowflake at the fifth generation (Fig.5c).

In order to manipulate geometric objects and to plot the fractal figures, the students use a free software, called Fractal Grower. It is a Java program (http://www.cs.unm.edu/) created by the University of New Mexico.

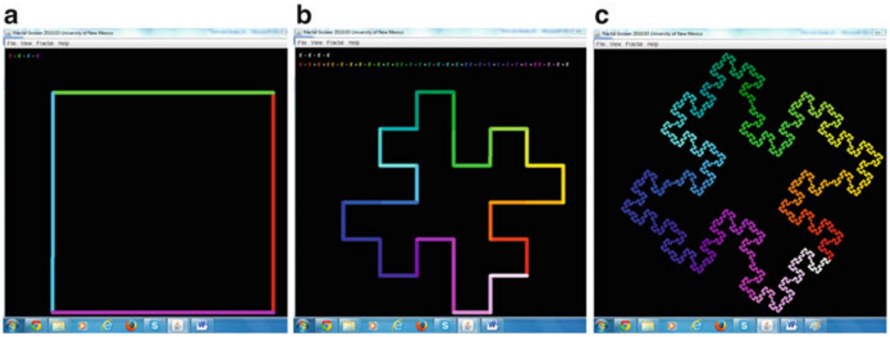

Fig. 4 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A quadratic Koch island

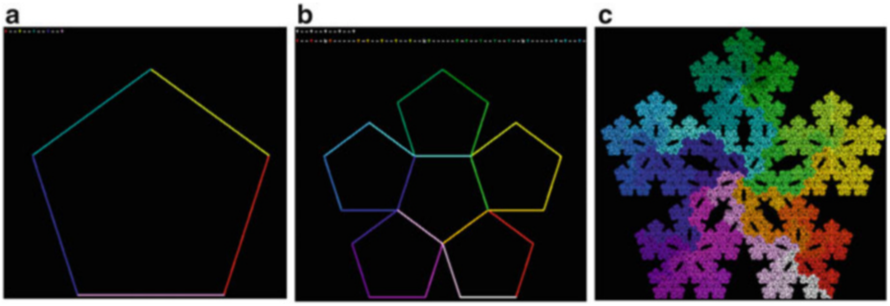

Fig. 5 (**a**) Axiom or Initiator. (**b**) Production, Generator. (**c**) A snowflake

The software allows students to visualize immediately the codes of fractals in figures and realize if they are corrected or not. Visualization is very significant aspect in learning geometric process. According to Duval (2000), three kinds of cognitive processes involved in it can be distinguished: visualization processes, construction processes with tools, and reasoning. Duval (2000) also analyses the role of visualization in the solution processes of a geometry problem and distinguishes several approaches to a diagram in geometry:

> An immediate perceptual approach that may be an obstacle for the geometric interpretation of the diagram, an operative approach that is used for identifying sub-configurations useful for solving the problem and a discursive approach that is related to the statement describing the givens of the problem. (p. 64)

The construction of fractals, in its visualizazion aspect, is a geometric problem solved by a "discursive approach". The students, in fact, understand and convert the affine transformations in the codes of fractals and transform them in images successively. Only the use of technological tools can help students to solve and to interpret these steps correctly. Besides they make some conjectures about geometrical properties of fractal figures. For example, changing the start angle or the turn angle or the turtle code, they are able to verify new hypotheses and visualize their solutions quickly. In this case, the visualization improves the theoretical knowledge of formal languages in order to turn them into geometric information and vice versa. In addition, I would highlight that technology has a role as more "reorganizer" than an "amplifier" (Dörfler 1993; Pea 1987). Fuglestad (2007) describes these functionalities by:

> The amplifier metaphor means doing the same as before, more efficiently, but not fundamentally changing the objects and tasks we work on, whereas seeing ICT tools as organizers implies fundamental changes in objects to work on, and the way we work. For example in using a graph plotting program as an amplifier the software produces quickly the graph as the end product, whereas seen as a reorganizer the function graph itself is seen as a new object which can be manipulated either directly or by setting parameters. (p. 250)

The features of the software used in the activity allowed us to manipulate geometric objects directly, setting codes and formal language and experiencing new in order to reduce the moment of repetitive application of the same geometric rules and to understand better the fractal theory.

(b) *Proving that if the starting figure changes, the fractal does not: fractal as an attractor*

In the recursive process of fractals, the rules of production are more important than an axiom. The starting figure is not decisive for the fractal; it could be a triangle or a square, in any case, after a few iterations, the figure converges to the same fractal as an attractor. This important feature of fractal is not evident to the student, unless he uses technology like in Examples 3 and 4.

Example 3: A triangle as starting figure

1. Axiom **ω: F++F++F**, start angle: 90°, turn angle 60° (a triangle corresponds as initiator or starting figure of the fractal) (Fig. 6a).
2. Production **p: F=F-F++F-F** (it corresponds to the generator of the fractal) (Fig. 6b).
3. Koch fractal at sixth generation (Fig. 6c).

Example 4:A square as starting figure

If the students change the starting figure, using a square:

axiom **ω: f++f++f++f**, start angle: 0, turn angle 45°, with the same production and same number of iterations, they obtain the same attractor (see Fig. 7a–c).

Through the manipulation of geometric objects graphically (translating, turning or reducing them), the students transform numerical codes in dynamic figures and find out some properties of geometric objects in fractal theory. In fact, the technology offers them the opportunity to learn and explore the fractal geometry in autonomy.

(c) *Modelling reality: fractals as geometric patterns.*

Modelling real world means to study shapes and patterns, to discover similarities and differences among objects, to analyse the components of a form and to recognize their different representations (Barnsley 1993; Edgar 2008; Prusinkiewicz and Lindenmayer 1990; Steen 1990). According to Steen (ibid.):

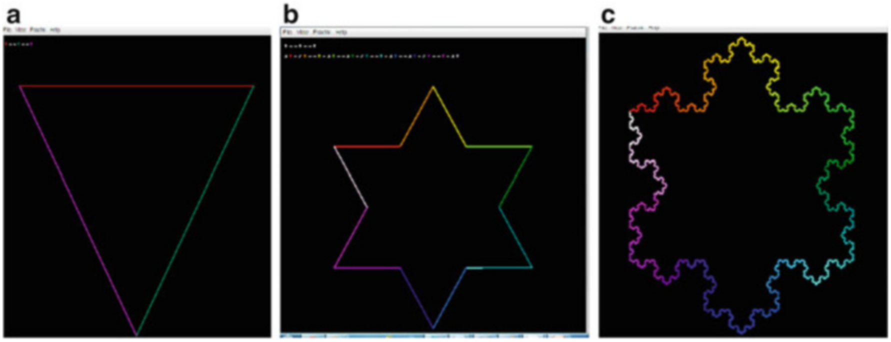

Fig. 6 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A snowflake

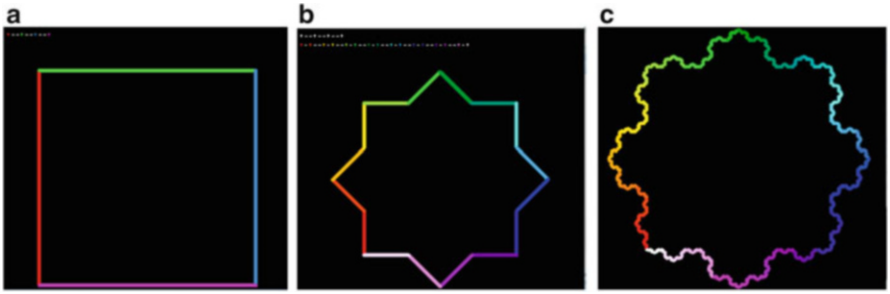

Fig. 7 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A snowflake

> Patterns are evident in the simple repetition of a sound, a motion, or a geometric figure, as in the intricate assemblies of molecules into crystals, of cells into higher forms of life, or in countless other examples of organizational hierarchies. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels. (p. 139)

For describing the complexity of the nature, we need to enforce the turtle language by adding new symbols (Prusinkiewicz 1999):

[(It pushes the current state of the turtle onto a pushdown stack. The information saved on the stack contains the turtle's position and orientation);
] (It pops a state from the stack and make it the current state of the turtle. No line is drawn, although in general the position of the turtle changes);
! (It branches out smaller pattern in the same).

Trees and snowflakes representations generated by the students are very similar to real shapes, as reported in the Examples 5, 6, and 7:

Example 5
Axiom: **f** (line as initiator), start angle: 0° turn angle: 20°
Production: **f** =**f [+f] f[-f][f]** (as generator) (see Fig. 8)

Fig. 8 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A tree. (**d**) A fractal tree branch at fifth generation

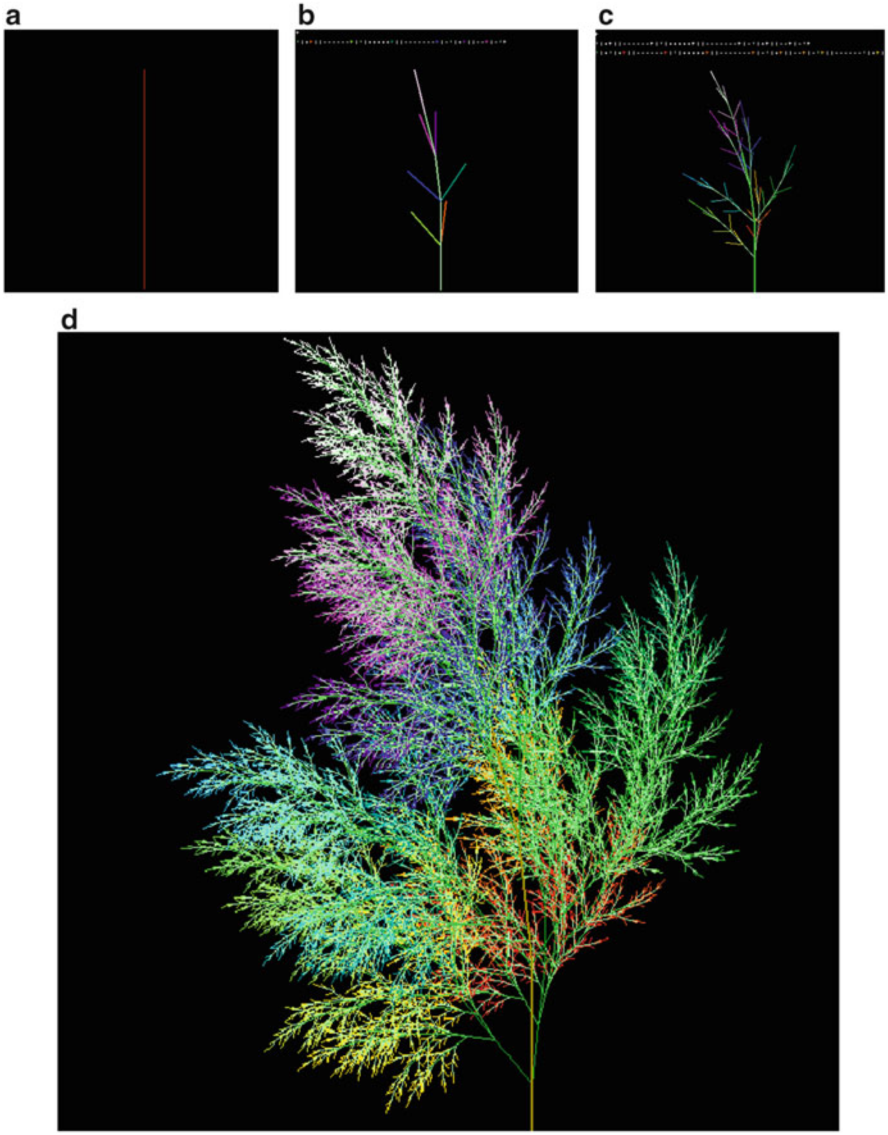

Fig. 9 (**a**) Axiom or initiator. (**b**) Production, generator. (**c**) A tree. (**d**) A fractal tree at fifth generation

Example 6
Axiom: **f** (line as initiator), start angle: 0° turn angle: 7°

Production: **f** = **![+f][------f]![+++++f][-------f]-![+f][--f]-!f** (as generator)(see Fig. 9)

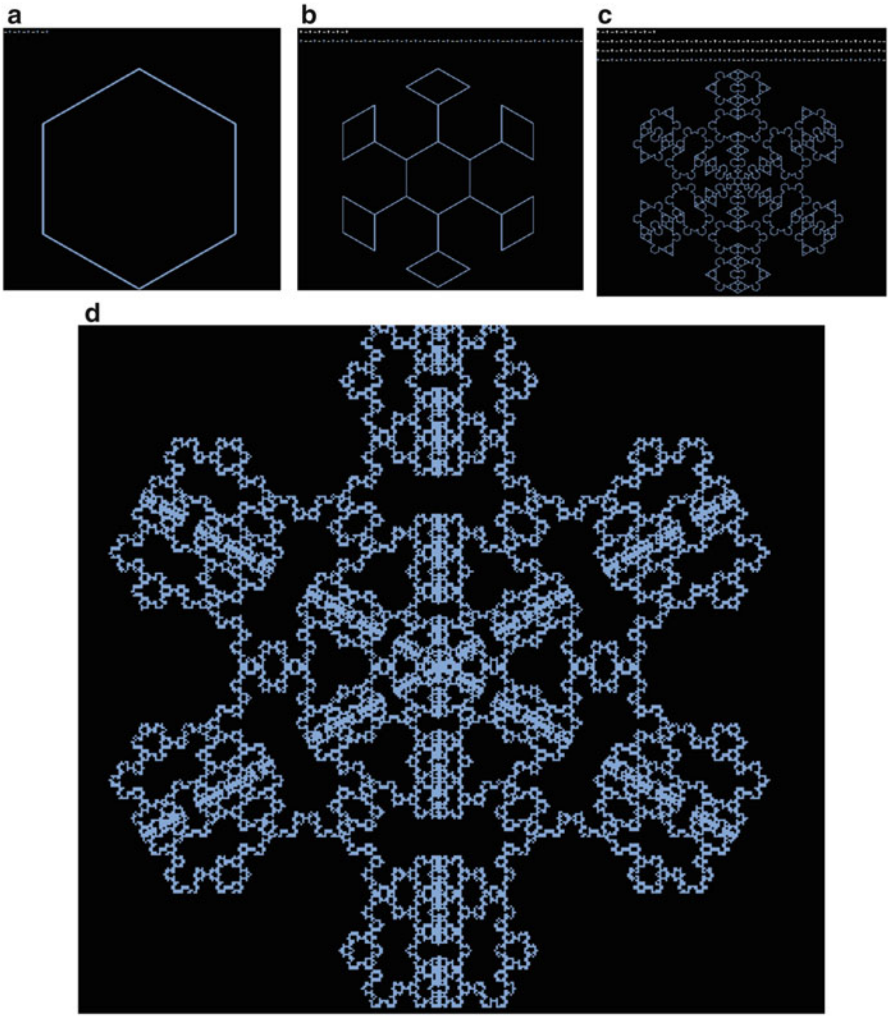

Fig. 10 (**a**) Axiom or Initiator. (**b**) Production, generator. (**c**) A snowflake. (**d**) A fractal snowflake at fifth generation

Example 7

Axiom: **ω:f+f+f+f+f+f** (initiator), start angle: 0°, turn angle: 60°

Production: **f=f++f-f-f-f-f++f** (as generator) (see Fig. 10)

Some links of the videos about L-system fractals, made by students and posted on youtube.com, are reported below:

https://youtu.be/1tNfXrp2JXI
https://youtu.be/EYCHgLr3YhY
https://youtu.be/4E7ECRwEbu0
https://youtu.be/UXWDpmlc2-k
https://youtu.be/DQTf-QOxmzA

In my activity, ICT are essential in order to plot fractals with many iterative applications, to manipulate fractal codes for creating different figure, to make conjectures for applying affine transformations to the figures. These applications increase the sense of independence and autonomy in the students and stimulate their creativity and imagination.

Conclusion

The educational experience described in this chapter, based on L-system fractals and ICT is extremely rewarding. In fact, discovering mathematics in real life through the study of geometric shapes, increasing students' geometric knowledge, and also showing them that there are new educational approaches to geometry mediated by new technologies are the principal aims achieved at the end of this educational activity.

Web 2.0 tools are significant in my activities with students, for:

1. Introducing theoretical content about L-system fractals, an uncommon topic at this school level;
2. Turning formal language into geometric figures and analysing fractal properties;
3. Working in e-learning and teaching spaces (using of Google Docs®).

The students are able to make conjectures, to create fractals and to model the real world through geometric shapes. At the beginning, they study the theoretical content and then check their results by multimedia tools.

The outcomes achieved during these activities are: strengthening mathematics skills (geometric transformations, iterative processes and functions) in the sense of the definition given by Programme for International Student Assessment (PISA):

> Mathematical skill is the ability of an individual to identify and understand the role that mathematics plays in the real world, to operate based assessments and to use mathematics and confront it in ways that meet the needs of the life of that individual as citizen exercising a constructive role, committed and based on the reflection.

The educational activity, applied in this project, features an inclusive approach between theacher and student, among students, it stimulates students' curiosity, it fosters the knowledge of geometrical properties in autonomy and it enhances their sense of self-confidence. In addition, the teacher also gains from this experience:

– Discovering new educational approaches with students in order to make mathematical content interesting and more accessible;
– Experimenting with new methodology to renew his/her own teaching of mathematical topics.

New challenges are still to be experienced regarding the educational approach described in this chapter, for example to extend it to more mathematical topics and above all to involve the whole class, not only those students who voluntarily participate in any optional activity.

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http://www.gcflearnfree.org/googledriveanddocs/1
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Learning and Technology? Technology and Learning? A Commentary

Peter Appelbaum

Abstract Should we or can we understand learning better by working with technology, and then better support learning through technology? Research that compares learning with "low-tech" technologies (paper and pencil, models and metaphors) with learning with "high-tech" technologies (calculators, dynamic software environments, touch-interactive devices) promises contributions to this question. This commentary argues that learning uses technology while technology uses learning, as demonstrated by the studies in this section of Sourcebook. Researchers in these chapters resist a common tendency to conceive of technology outside of humanity, and in this way offer models of richly informed by the co-construction of humans and their technologies.

Keywords Technology • Curriculum • Reseach questions • Professional development

Can we understand learning better, and then provoke more, better, deeper, or a different sort of learning, by working with technology? How can teaching with technology help us to better learn about learning? Does it help to compare learning with "low-tech" technologies – paper and pencil, models and metaphors – with learning that transpires along with the use of "high-tech" technologies – efficient devices such as calculators, dynamic software environments, touch-interactive devices? If so, how/what/when? The chapters in Part II lead us into such questions by providing a variety of examples of classroom-based research where the collected and analyzed data focuses primarily on learners. The studies described in this part also offer opportunities to compare across different types of boundaries: the boundary of national, cultural context (chapter "Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive and Epistemological Implications for Improving Geometric Thinking" by Bairral, Arzarello, and Assis, between Italy and Brazil; chapter "Integrating Arithmetic and Algebra in a Collaborative Learn ing and Computational Environment Using ACODESA" by Hitt, Cortés, and

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Saboya, between Quebec and Mexico), the boundary between low-tech and high-tech forms of technology (chapter "Graphs in Primary School: Playing with Technology" by Ferrarello, between paper and pencil and graphing calculators; chapter "Pocket Calculator as an Experimental Milieu: Emblematic Tasks and Activities" by Floris, between activities with and without calculators; chapter "The Street Lamp Problem: Technologies and Meaningful Situations in Class" by Gentile and Mattei, among paper and pencil, pictures and simulations, and dynamic software; chapter "A Framework for Failed Proving Processes in a Dynamic Geometry Environment", by Chartouny, Osta, and Raad, between paper and pencil and dynamic software; chapter "Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA" by Hitt, Cortés, and Saboya, between paper and pencil and calculators; and chapter "L-System Fractals as Geometric Patterns: A Case Study", by Alfieri, among paper and pencil, webquests, and technology-based presentations), the boundary between planned and implemented curriculum (chapter "Disclosing the "Ræmotionality" of a Mathematics Teacher Using Technology in Her Classroom Activity", by De Simone; chapter "L-System Fractals as Geometric Patterns: A Case Study", by Alfieri), the boundary between types of interactivity of touch devices (chapter "Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive and Epistemological Implications for Improving Geometric Thinking" by Bairral, Arzarello, and Assis), the boundary between required and voluntary use of technology to pursue mathematical investigations (chapter "L-System Fractals as Geometric Patterns: A Case Study", by Alfieri), and boundaries across different stages of lessons (chapter "Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive and Epistemological Implications for Improving Geometric Thinking" by Bairral, Arzarello, and Assis; chapter "Disclosing the "Ræmotionality" of a Mathematics Teacher Using Technology in Her Classroom Activity", by De Simone; chapter "Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA" by Hitt, Cortés, and Saboya; chapter "L-System Fractals as Geometric Patterns: A Case Study", by Alfieri).

One boundary left under-theorized by this collection is that between what we mean or might mean by "learning," and what we mean or might mean by "technology." The orientation of this collection, as with much research in mathematics education, frames the research around the following sorts of questions:

- Can we tame technology so that it promotes learning rather than discourages it from taking place?
- Can we understand technology better and then make more profoundly useful technologies, or use technologies in cleverer ways, if we study technologies through how learners interact with them?
- How can learning with technology help us to better comprehend technologies, and how to classify, categorize, exploit, and control them?

Questions that I would like use to further consider as we reflect on the implications of the research reported in this collection include:

- What could it possibly mean to learn without technology? Or to have technology without learning?
- Are we inadvertently trying to tame learning so that it promotes appropriate rather than inappropriate technologies from taking hold in learning environments?
- How does what is considered technology influence what we say about technology?
- How does what one considers learning influence what we say about technology?

And I will conclude with what I see as the strengths of this collections for responding to implications of the learning-technology dichotomy for how we understand both learning and technology: when technology and learning are assumed prior to research to be distinct categories of things that can be defined, studied, and then brought into interaction, we find that technology in such research is already constructed, and hence reality is already predetermined to be created out of technology which is outside of human beings and humans "learning." Technology in this way is an add-on to a more fundamental notion of learning (and teaching/learning of course); this creates a sort of hierarchy, one way or the other, between learning with and without what is considered technology.

Can We Tame Technology?

A careful reading of the chapters in this part suggests that it might be possible that technology can be tamed and domesticated in order to nourish and support learning. My impression is that one approach to such domestication is to first of all keep all technology out of the classroom, and then to let it enter under careful scrutiny and in limited ways. By this I mean that learners who live in a technological-device-rich world outside of school are described as existing in a different world in the classroom. This is consistent with my own experience of many school mathematics learning environments in numerous countries: I rarely see learners with mobile phones, televisions, tablets, music players, fitness devices, and so on, coming together in classrooms in ways that are similar to how they interact outside of school, multitasking with entertainment, school assignments, social networking, and so on, all at the same time. What we see in such classrooms are a sort of sterile, technology-free environment into which calculators, tablets, dynamic software, or Internet exploration is introduced for examination and study. Technology becomes an intervention in the learning experience, analogous to a medical study of a drug for its effects on test subjects. Of course, it is questionable whether one can really establish a technology-pure environment, given that humans are technological beings. Technology as a tool that humans create in order to help them accomplish something far more challenging or impossible without the tool is a matter of semantics: pencil and paper is a technology for communication and learning as much as a touchscreen device; language is a tool for thinking and making sense of the world around us, for communication, for reflection, and so on; mathematics as a field of knowledge if a kind of technology. Still, there is something to be said for electronic technologies, such as calculators, touchscreen devices, dynamic software, and social networking, and that is what we are looking at in this part of this volume.

Once we domesticate technology it seems possible to imagine selective and controlled design, hybridization, and specialization, similar to the ways in which humans have carefully bred forms of livestock and plant life over generations and millennia to maximize perceived benefits. So we can and should ask, "Can we understand technology better and then make more profoundly useful technologies, or use technologies in cleverer ways, if we study technologies through how learners interact with them?" And the research collected here provides a strong "yes" in response. For example, calculators, spreadsheets, and dynamic software environments speed up the process of collecting specific cases of mathematical phenomena – values, variables, functions, shapes, locations along a graph, or properties of geometric constructions. Once this process of collecting specific cases is sped up enough, it is possible for learners to more efficiently consider at once the collection of cases rather than individual instances. This quite naturally supports a focus on collections of cases and generalizations, and in this way promotes a kind of learning that values consistent attempts to generalize, and to study the process of generalization itself as a learner. While this does not change the nature of learning – mathematics educators have always valued processes of generalization, it suggests that technology can indeed enhance the likelihood that learners can have more experiences with generalizing more often. In fact, the research collected here strongly supports the notion that dynamic software in particular models in its very form the idea of generalization, making dynamic software significantly rich in potential for promotion of a disposition to generalize.

Nevertheless, we should be cautious to assume that technology forces generalization. Similar dreams have been provoked by educational tools throughout the centuries – whether the tools are pictures drawn on the sand, blocks grouped into tens of tens, pegs arranged in arrays on boards, or circles cut into various numbers of equal pieces. Adults who already understand mathematical concepts "see" the mathematics in these models of mathematical concepts, since the models were so nicely constructed to represent the concepts themselves. So a picture in the sand, base-ten blocks used to model the decimal system of numbers, geoboards upon which rubber bands can create shapes, and fraction circles, appears to the adults to scream the mathematics concepts at a high volume. What researchers have found is that, yes, these models can often be helpful in learning environments, but they can also simply reproduce the same issues with learning and teaching, since the learners do not bring the concept fully formed to their experiences of the materials. Instead, learners often mis-learn or un-learn concepts with the models, or mindlessly attempt to follow procedures for manipulating the models with no concurrent development of understanding. So it is not automatic that technologies that speed up the creation of specific cases, or which rapidly generate seemingly infinitely many examples of a geometric construction following certain properties, will force learners to generalize; it is rather those who approach a task with a propensity to generalize will readily grasp that generalization is indeed possible in such situations. What we can say based on these chapters is that learners who have been enculturated prior to experiences with technology to pursue generalization can take advantage of technologies that support generalization, and are likely to do so.

Can Learning with Technology Help Us to Better Comprehend Technologies, and How to Classify, Categorize, Exploit, and Control Them?

Here too we can use the research reported in this part to say yes. We have examples of researchers who have designed interventions to study differences that they have perceived between low-tech paper and pencil and dynamic software, drawing pictures and using calculators, different kinds of touchscreen devices, and software versus internet-based investigations. In each case, it is possible to identify important differences among the forms of technology. And in each case, a teacher can use these differences in their planning to design encounters that exploit the differences. On the one hand, it is worth pointing out that, on a certain level of analysis, there is little change in pedagogical techniques: if learners are engaging with paper and pencil, drawings, constructions on a screen, calculators, or dragging images on a touchscreen to create transformations, the critical pedagogical method is to facilitate reflection upon changes that can be observed when one looks at the changes that one is able to make happen through interaction with the mathematical objects. This can take the form of individual thinking, small group or large group discussion, reports by individuals or small groups to an audience (either the rest of the class or people outside of the class), or written reports for a particular audience (a private journal, a letter to the teacher, a video posted online, or an interactive online presentation).

I note two important points thanks to the researchers collected in this part. First, new technology requires educators to translate what they know and believe into new contexts, and this process alone might be the most valuable aspect of technologies for learning, because it provides ways for the teachers to promote what they value in the learning experience. Sometimes the learners surprise us in how they interact with the mathematics or with the technology, and in the process, clarify through their successes in achieving our goals for them whether or not our values for learning are consistent with our expectations. De Simone makes this paramount in her research on ræmotionality in chapter "Disclosing the "Ræmotionality" of a Mathematics Teacher Using Technology in Her Classroom Activity". In this study we clearly see how a teacher's hopes, dreams and fears for her students are enacted in her decisions about technology in her classroom; it is in this study in particular that we can most easily consider as well how learning and technology are difficult to unravel from each other, because the learning experience influences how the technology is defined, experienced and exploited, or not; at the same time, we see in the discussion with teacher Silvia how the technology influences the learning experience, because it is enacted in the classroom in accordance with her emotional investments for her students.

But we also see the value of translation in the other chapters in this part. Chartouny, Osta, and Raad bring their interest in cognitive processes for the development of proof into the dynamic geometry world in chapter "A Framework for Failed Proving Processes in a Dynamic Geometry Environment". Because of their interest in stages of proof sophistication, they looked for this in their work with geometry learners. In the process, they highlighted important ways that teachers can use interaction with dynamic software to carry out ongoing assessment of specific kinds of misunderstanding within instructional experiences; in this way they suggest how teachers can plan for embedding such assessment within instruction, simply by interacting with learners as they are exploring open problems. Similarly, in chapter "The Street Lamp Problem: Technologies and Meaningful Situations in Class", Gentile and Mattei clarify the relationships among conjectures, exploration of cases, and argumentation through their own use of dynamic geometry environments with learners. Hitt, Cortés, and Saboya translate an interest in algebraic thinking as emerging from investigations into number relationships chapter "Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA"; Floris also demonstrates this aspect in chapter "Pocket Calculator as an Experimental Milieu: Emblematic Tasks and Activities", as do Ferrarello in chapter "Graphs in Primary School: Playing with Technology" with graphing calculators and Alfieri in chapter "L-System Fractals as Geometric Patterns: A Case Study" with explorations designed to help students appreciate that the reproduction rule is more important in determining a fractal than its seed shape.

It seems that we are still at an early stage of classifying technologies. So far we have paper and pencil and other picture-creating tools; calculators and spreadsheets and other similar tools for carrying out repetitive calculations – whether arithmetical, tabular, or creating a static graph; dynamic environments, in which it is easy to drag and change elements while holding others static; social networking, incorporating extensive opportunities to research what others have already done; and touch-devices, in which the ways in which one interacts with the screen might, according to Bairral, Arzarello, and Assis in chapter "Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive and Epistemological Implications for Improving Geometric Thinking", have significant impacts on the epistemologies that are carried through the learning experience. Sensorial process, motion and manipulation on-screen take an important cognitive role in this research; in their movement into existence, in which they become objects of thought and consciousness, geometric concepts are endowed with particular determinations, which are in turn actualized in sensuous, multimodal and material activity. On the other end of the experiential dimension, we might have the webquest activities described by Alfieri in chapter "L-System Fractals as Geometric Patterns: A Case Study". Because the students volunteered to pursue these beyond what is ordinarily expected of learners, and for an outside audience as part of a regional presentation competition, the learners carried their own expectations for learning into the experience, rather than being manipulated by the technological encounter into a particular form of thinking or learning.

Learning with/out/for Technology?

I want to return to the image evoked earlier, of a technologically pure environment into which we inject technology as an intervention. Could we have learning without some form of technology? This rhetorical question makes it clear that humans do not really exist independent of technologies. The history of humanity is one of co-evolving with our technologies, in ways similar to other animals with which we share our planet (including birds, mammals, dolphins, octopi, etc.). So it is important to tease out what the research highlighted in this section on learning can help us to think about in this broader context. Technology runs the risk of being understood as a prosthetic device – an enhancement either for the teacher to more powerfully trigger learning, or for the learner to more powerfully see, hear, and sense in general, the mathematics (Haraway 1991). This has positives and negatives in terms of giving superpowers – in terms of what we can accomplish in a given amount of time, what we can perceive in one glance, what we can produce, and so on; but also in the process creating super-power-related weaknesses (for example, in the rush to generalize about functions as exponents change, we overlook the nuances of a change in constants; or, in the rush to find patterns in shaped numbers, we no longer see patterns within the same shapes of numbers of objects; or, in our attempts to explore geometric relationships, we rely on argumentative fallacies as in chapter "A Framework for Failed Proving Processes in a Dynamic Geometry Environment"). Sometimes educators want to slow down rather than speed up processes, because the volume of information is overwhelming for the learner. At other times, technology narrows our focus too much, or not enough. In these instances, technologies understood as enhancing powers of perception turn into disabilities of oversensitivity to too many stimuli (Appelbaum 2007).

We sometimes inadvertently limit "technologies" to devices outside of people. What about language, knowledge, and specific terrains of subject expertise such as mathematics itself, as technologies? (Keitel et al. 1993) When we reduce technologies to tools outside of people, we also reduce learning to perception, and we exploit metaphors of perception (often reduced to an ideology of vision) to describe learning; in these ways we miss out on other forms of learning not captured by perceptual discourse. How might we help learners feel, taste, smell, and otherwise experience mathematics? Or, more fully, are their ways of comprehending learning outside of the metaphors of seeing and touching mathematical objects and mathematical relationships? What else is inherent to mathematics not captured by our ways of thinking about technology? (Appelbaum 2007).

I suppose we could say that the co-evolving humans and technologies are both influencing each other, so that our notions of mathematics, technology, and learning are all buttressing each other (Puech 2016). But there are also ways in which mathematics education might benefit from challenges to our natural ways of thinking. What if we assume that technology and learning are inseparable concepts? The technology-learning nexus, if you will, collapses all distinctions between technology and learning. What this means is learning is a technology; and technologies are a form or a crystallization of or a promise of certain “learnings”. Learning leads to acting with technologies to continue learning; technologies provoke learning how to further use the technologies to further learn to yet further use technologies. There is circularity to the overlapping and mutually defining nexus of learning and/of/with/through/for technology. The critical point is that there is no learner without the presence of some kind of technology, and no technology without a learner using the technology. Technology-learning is a collection of characteristics that are essential to mathematics education. We can similarly state what might seem obvious but which is lost in its obviousness when we try to come up with fancy research-based “results”: the learning-technology nexus is at once both personal and social, in that it is apparent in private, intimate and personal moments, both in solo explorations, and in experiences that are emotional and raemotional, and also in group activities, such as those that take place in classrooms and in small and large numbers of teachers, learners, and audiences.

None of the researchers collected here have attempted the folly of trying to isolate learning or technology outside of a culturally rich and institutionally defined form of learning-technology. That way of approach is nothing more than a trap where technology ironically becomes a form of taming learning, capturing learning in the grip of technological constraints. We might want to proceed with caution, and ask, “Who/what are we serving when we carry out research with technology?” Are we merely serving a technology outside of ourselves when looking for reasons to value the technology? This is occurring when we translate our values for mathematics into our research and desperately search for them in the learning/technology context that we have created. This is present when we introduce technologies as prosthetic power enhancers in an otherwise pedagogically dead classroom. In contrast, we are critically examining our learning-technology nexus when we explore with the technology at hand what might be possible in terms of the values that we hold for the mathematics that is being learned. Is the technology supporting the learning that is supporting further use of the technology to support learning? Do we have evidence of this circularity? When educators seek a pure idea of learning and/or a pure idea of technology independent of time, place, culture, or institutional context, I believe they misunderstand the nature of the learning-technology co-evolution that characterizes humans who are learning mathematics and creating technologies for learning mathematics, and using things at hand as technologies to learning mathematics, and in turn structuring learning to be grounded in technologies.

It is important to clarify whether the learning-technology experience is technology-driven or curriculum-driven (Bromley 1997). When confronted by a teaching/research project, we can ask:

- How did this project come about?
- Why is this initiative taking place?

If an answer to one of these questions is to insert some cool technology into a learning environment, then the project is technology-driven and is likely to lead to little significant change in learning; I would expect more of the same, as in digitized forms of exercises that could easily have been accomplished through paper and pencil worksheets, digital collection of data on student performance rather than learning, or soon-outdated equipment purchased for a large sum of money. On the other hand, if the responses to these questions involve discussion of forms of social interaction tied to goals that educators have for learning, then there is great potential that learning and technology are together carrying social values that are crucial to educational transformation. In the projects described in Part II of this volume, we have seven examples of learning-technology that are curriculum-driven rather than technology-driven and therefore demonstrate powerful forms of social transformation: Bairral, Arzarello, and Assis (chapter "Domains of Manipulation in Touchscreen Devices and Some Didactic, Cognitive and Epistemological Implica tions for Improving Geometric Thinking") create opportunities for the learner to be active sculptures of geometric objects and their transformation using GeoGebra and Geometric Constructor software; the secondary school students become active strategists whose gestures touching the screen physically drag through multiple cases of a possible construction. Ferrarello (chapter "Graphs in Primary School: Playing with Technology") establishes primary school students as explorers in a mathematics laboratory, who exploit technology to efficiently gather observations so that their comparisons can carefully test their own conjectures. Learners in the contribution from Floris (chapter "Pocket Calculator as an Experimental Milieu: Emblematic Tasks and Activities") establish learning milieus through anticipating actions that they then carry out, in the ongoing negotiation of the didactic contract of the classroom. Whether expectations are confirmed or met with surprising feedback from the technology, the important component of the learning-technology nexus is the ongoing construction of the possibility for "adidacticity", specifically, something to learn as an inescapable characteristic of the learning milieu.

Gentile and Mattei (chapter "The Street Lamp Problem: Technologies and Meaningful Situations in Class") raise the question of situations posed by the teacher in the classroom; the learners in this study represent the situation with GoGebra, and in the process become what the teacher describes as.

> . . .very interested and involved, working seriously on the task given, arguing and justifying their solutions in an accurate way. I felt very involved in this activity; they worked with interest and curiosity. . . (Gentile and Mattei, this volume, p. 208)

This in turn led the teacher to describe herself as transformed by the observations of her students: ". . . this gave me a great satisfaction and an incentive to repeat in the future this kind of experience." In this case, designing a social learning context in which technology is used to translate an open situation into a representation changed the forms of participation and relationships among the teacher and the learners. Similar changes in the adults are discussed in chapter "A Framework for Failed Proving Processes in a Dynamic Geometry Environment" by Chartouny, Osta, and Raad; once the teachers identify three stages of the proving process – the construction and manipulation of the figure; the formulation of the conjecture; and the proof itself – they become students of the learners, understanding the kinds of (perhaps faulty) reasoning that learners often employ, and how these kinds of reasoning can be the result of the interaction with the technology that is meant to help them learn. What we see here is a nuanced comprehension of learning with technology and technology with learning, a relationship with learners that recognizes the need for a nonlinear path toward a teacher's objectives. What learners will create as products of their learning is not always mathematically perfect; instead, forms of proving that would be labeled failures by a seasoned mathematician are evidence of learners doing just what they should be doing: learning.

Such changes in the teacher in response to the changed learning-technology world that is created by the research project are echoed in chapters "Disclosing the "Ræmotionality" of a Mathematics Teacher Using Technology in Her Classroom Activity", "Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA" and "L-System Fractals as Geo metric Patterns: A Case Study": De Simone (chapter "Disclosing the "Ræmotionality" of a Mathematics Teacher Using Technology in Her Classroom Activity") describes how a teacher thinks with potential integration of technology about how to make her dreams for her students to be more possible. As she experiments with GeoGebra and Java applets, she becomes increasingly creative in the ways that she can make it possible for learners to use their imagination to construct mathematical concepts; as students demonstrate that the dynamic software does in fact support their imagination in such ways, the teacher relies more and more on dynamic software to validate the students as constructors of mathematical concepts. Hitt, Cortés, and Saboya (chapter "Integrating Arithmetic and Algebra in a Collaborative Learning and Computational Environment Using ACODESA") place the learning-technology nexus in support of collaborative learning, scientific debate and self-reflection, a pedagogical approach that has come to be known as ACODESA; they noted, for example, how spontaneous representations impacts on three different forms of social action – individual work, teamwork and large group discussion. Finally, Alfieri (chapter "L-System Fractals as Geometric Patterns: A Case Study") turns her learners into special members of a mathematical community who self-select to pursue further investigations into interesting mathematics beyond the regular curriculum; as they pursue mathematics with technology, and as they readily make use of technology because they are learning, they become members of a new community of mathematics learners that interacts with students in other schools, and adults who are interested in what they can learn from these students.

What I find missing from this collection – and surely any small number of studies cannot reach all relevant areas of research – is attention to how the learning-technology nexus created differential curriculum-driven opportunities for different learners. Who was best served by the various forms of learning-technology that unfolded in these contexts? (Bowers 2001; Leigh 2002) Social class, ethnicity, immigration status, or other important learner communities means different opportunities in the same learning-technology world that is established in a school or classroom. What we find in these chapters is an assumed, normalized learner who interacts with a generic teacher. So I ask us to strive for further analysis in this direction.

What the research in Part II shares nevertheless is the pursuit of complexification worthy of mathematics education practice and theory, rather than an empty but rational simplicity. Learning uses technology while technology uses learning in each of these studies: I see this positive circularity in each of these chapters, and for this reason alone I applaud my colleagues, and thank them for the chance to contribute this commentary, and thus to share in their important work. Here in these chapters are researchers strongly resisting the pull to conceive of technology outside of humanity; here are researchers critically embedding themselves in the learning-technology nexus, and reflecting on that process.

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Part III
Communication and Information:
A. Communication Inside and Outside the Classroom

e-Mathematics Engineering for Effective Learning

Giovannina Albano

Abstract Education in digital age requires a strong focus on "engineering of learning", as an emergent field of study and design of effective innovative learning experiences and environments. In this respect, the chapter addresses mathematics education in e-learning settings according to a tetrahedron model, as extension of the classical didactical triangle, to which adds a fourth vertex: the 'author' (A). The introduction of the vertex Author is due to our view that full exploitation of e-environment and its integration with results from research in mathematics education requires properly designed didactical interventions, based on a scientific approach, such as Didactic Engineering. Moreover, such exploitation should consider the centrality of the student, which means that the vertex-positions of the tetrahedron can be assumed also by the student along the learning process. We discuss the didactic engineering work from the perspectives of the tetrahedron faces and taking into account the dynamicity of the vertex-positions.

Keywords e-learning • Didactic engineering • Didactical tetrahedron • Mathematics education

Introduction

This paper concerns the learning/teaching process in the context of e-learning. Many definitions have been given to the term 'e-learning' and sometimes it has been used as synonymous to 'online learning' or 'distance learning' or 'web-based learning'. Lately 'e-learning' has also been referred to 'communication' and 'connectivity' issues. Thus, first of all, we want to make the boundaries of our context precise. Then, we will draw our attention to the teaching/learning process which occurs in a technology-enhanced environment consisting in an online teaching platform added with facilities of the Web2.0 and (eventually) with online mathematical software. From now on, we will refer to it as 'e-environment', and in general, we will put the prefix 'e-' to a word meaning that such word refers to

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e-environment (for instance, 'e-tool' is a tool included in the e-environment). So, in this paper, e-learning means teaching/learning process in an e-environment.

Looking at the experiences in e-learning applied to various domain, including mathematics (Borba and Llinares 2012; Garcia Peñalvo 2008; Juan et al. 2012a, b), we can individuate two main strands. One is guided by an exploration of the teaching and learning opportunities offered by the e-learning platform, which has characterized the early studies, where great attention has been devoted to resources (namely learning objects) that exploited the interactive features of the e-tools. The other one has drawn attention to learning activities that integrated the delivery of some resources within a more complex structure of engagement of the students by means of cooperative and collaborative features of the platforms such as forum and wiki.

What we observe in any case is that the researchers have devoted great effort to the design of the resources or learning experiences in a "Design Research" (Kelly et al. 2008) view. This means that they move along a path consisting in planning a set of resources or activities, experiment their potential in order to support learning and analyse the data that can give rise to theories.

We suppose that such approach has been greatly affected by the focus on the technologies rather than on the learning process. For some part, it is due to historical and social constraints since e-tools came before than their didactic consideration and conveyed certain – perhaps unconscious – traditionalistic view of teaching of their high-tech designers (Chevallard and Ladage 2008):

> Much to the contrary, a sound view of didactic engineering calls for the reverse: didactic functions, not structures, must be considered first. (p. 168)

Thus we are firmly convinced that now we need a research methodology allowing us to integrate results from the mathematics education research into learning process in the new global environment empowered by technologies (in their continual and fast change) and to validate the design. The use of a methodology should yield robust teaching/learning proposals (also in terms of strategies) that can be shared and reproduced in the teaching/learning community. In this respect, we claim that *didactic engineering* could be a good research and development tool.

The paper has been structured as follows. After a brief presentation of the Didactic Engineering approach and its relation with the Design Research, we discuss a tetrahedron model of the teaching/learning process in e-environment, that add to the classical vertices a new one, the Author. Then we discuss how each face of the tetrahedron is related to phases of the didactic engineering approach and which the role of the Author is.

Didactic Engineering and Design Research

In the recent years a substantial debate on how the two approaches of "Didactic Engineering" (DE) and "Design Research" (DR) relates each another is going on (Artigue 2015; Godino et al. 2013; Margolinas and Drijvers 2015).

Didactic Engineering (Artigue 1992, 1994, 2009) emerged in the France context of mathematics education in the early eighties to indicate an approach to didactical work in mathematics similar to the engineer's work. This implies that mathematics didacticians should act like engineers, who in order to realise a fixed (learning) project rely on the (disciplinary and didactical) scientific knowledge of the domain, accepting to be submitted to a scientific testing. At the same time they are forced to manage more complex objects than those of science and to face problems not yet tackled by science.

It is worthwhile to recall that Didactic Engineering has become a polysemic notion, that can be development-oriented or research-oriented as pointed out by Artigue (1994):

> designating both productions for teaching derived or based on research and a specific research methodology based on classroom experimentations. (p. 30)

Thus it is conceived as a practice of controlled theory-based intervention, consisting in the design of teaching sequences, the setting up of didactical tools and resources organised and structured in a period of time coherently to the aims of reaching fixed learning objectives, their monitoring and evaluation. The results allow to test theory-based hypotheses and to produce teaching resources scientifically validated.

It is also referred to as a research methodology of qualitative type, in which an essential role is played by the "didactic realisations in classroom" as investigation practises of the elaborate theoretical hypotheses.

At the beginning the base-theory for Didactic Engineering was the Theory of Didactic Situations (TDS) (Brousseau 1997), which contributed to shape the structure of the design (methodology), consisting in the following phases: (1) preliminary phase consisting in epistemological, cognitive and didactical analysis of the mathematical knowledge to be taught; (2) design and *a priori* analysis of the teaching/learning situations; (3) implementation and experimentation; (4) *a posteriori* analysis and validation.

Comparing Didactic Engineering with Design Research, Godino et al. (2013) argued that what mainly distinguishes them is that Didactic Engineering is grounded on a theory (such as Theory of Didactic Situations) whilst Design Research does not. In Didactic Engineering the underlying theory guides the *a priori* analysis and the consequent design and the expected results, whilst in Design Research the design is supported by various interpretive frameworks and theories emerge from the data. This latter thing justifies the foreseen internal validation step in Didactic Engineering, whilst Design Research does not. Anyway, Didactic Engineering and Design Research face similar paradigmatic questions, concerning the improvement of learning in context by instructional design intertwined with educational research. Godino and colleagues conclude that Didactic Engineering can be seen an instantiation of Design Research. Changing the base-theory, they envisage various Didactic Engineering differing in didactic design. Thus they introduce a new notation, as reported in the following:

> We introduce the notation DE (TSD) to indicate the dependence of the "Didactic engineering" from the Theory of Didactic Situations. This will help to express possible generalizations of didactic engineering changing the base-theory used to support the instructional design. (p. 6)

So we can have DE (ATD), DE (SM), DE (IA) according respectively to the Anthropological Theory of Didactic (Chevallard 1992, 2006), the Theory of Semiotic Mediation (Bartolini and Mariotti 2008), the Instrumental Approach (Drijvers et al. 2010a; Drijvers and Trouche 2008; Trouche 2004), and so many others. We also can have a DE based on a network of theories (Bikner-Ahsbahs 2010), that can provide complementary insights and, hence, deepens analysis and understanding of some didactical phenomena.

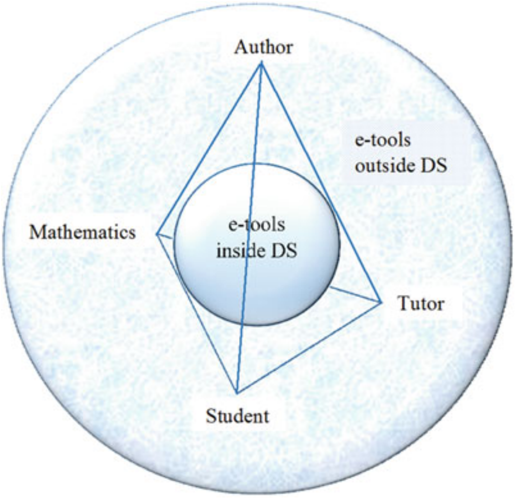
Considering a didactic engineering approach according to various based-theories seems us a very interesting idea because it allows researchers to define a unified methodology to construct a wide range of proposals for learning experiences with two rewards in our view. On one hand, they are designed in the frame of a reference theory, and tested and validated according to scientific criteria. On the other hand, they do not assume a one-sided insight of learning, without referring to one learning theory only, so keeping away from the original drawback of the use of e-learning environments.

The Tetrahedron Model

Following Albano et al. (2013), we assume that the *didactic system* (DS) concerning the teaching/learning process in e-environments can be modelled in a systemic way by a tetrahedron (see Fig. 1). It includes the main classical entities: some mathematical knowledge, that is Mathematics (M); someone who is expected to learn M, that is the Student (S); someone who is supposed to officially help S to learn M, that is the Tutor (T). A further character has been introduced, the Author (A), consisting in a team who is in charge of planning, developing and managing the *didactic organization*.

The addition of the new vertex A has created three new triangular faces, besides the classical didactic triangle. The function of the tetrahedron is not explicative or descriptive of educational experience, but mainly methodological: each vertex of the system can be used as the observer's position that looks at the inter-relationships within the face generated by the other three vertices, though none of the elements involved can be completely separated from the others.

Fig. 1 The tetrahedron model



We want to underline that technology has brought the Student to have a central role in the learning process and to look for a larger topos, not only restricted to someone who learn, but enlarged as someone who can be engaged in other didactic functions (Chevallard and Ladage 2008). This means, in our systemic view, that the vertices of the tetrahedron are not static figures, but we consider them just as *position*, that is as a system's element (for deepening discussion on teacher's position, see chapter "The Professional Development of Mathematics Teachers: Generality and Specificity" this volume), that can be played also by Student in some situations along the learning process. Transforming the didactic triangle into a didactic tetrahedron through the addition of the Author lets more evident the crucial role of the students, since looking at the tetrahedron faces allows to image, design, observe learning situations where students can play a position different from S. For instance, a student can be in the position Author when she creates resources that modify the *milieu* used to reach the knowledge M. Analogously, a student is in the position Tutor when she is expected or required to officially support some other students, that is she is no more in a symmetric relationship among peers but she assumes an asymmetric role.

The introduction of the vertex Author is due to our view that full exploitation of e-environment and its integration with results from research in mathematics education require properly designed didactical interventions, based on a scientific approach both in terms of *didactical transposition* (Chevallard 1989) and in terms of *didactic engineering*. The vertex Author has a key role in the design and in the validation of the teaching/learning resources and situations, with both contents and activities empowered by technology features. Differing from the classical teaching, the didactic engineering should take into account the chance for students to move in other vertex-positions of the tetrahedron. This adds a dimension of dynamicity where learning does not come out from the fruition of a ready-made product but it is the outcome of a construction where students have chance to carry out didactic functions, suitably supported, guided and supervised, impacting on the didactic organization itself.

Whilst Author, although is a team including in herself various competencies, from pedagogical, to disciplinary, to technological ones, can be thought as a collective subject, that is a single entity, we can think at Student and at Tutor both as a single entity and as a community. The first one is a community of "peers" who are engaged in learning. The second one refers to all the persons dedicated to facilitate student learning – in face-to-face or online settings, from the traditional teacher and any other educational figure (coach, advisor, and so on). We refer to them as singles or as community depending on the teaching strategy adopted, such as individual learning versus group learning, or on the size of the learning group to be supported which can require more than one tutor. Moreover, if Author and Tutor are not the same person, they should anyway intensely interact to succeed in their roles.

In our approach, technology is both outside and inside the teaching/learning system, thus we represent it as a couple of spheres: one outside and one inside.

The external sphere refers to the immersion of humans in a technology-enriched world. It contains all kind of e-tools (for a flavour, see the list presented in Chevallard and Ladage 2008), which permeate our life in all its aspects and our identity too (some years ago, I was astonished when a friend said to me: "*you do not exist if you do not exist on Facebook*"). There is no doubt that nowadays we cannot conceive to being human without being connected to and interacting with the global virtual word created by the web, as noted in Borba (2012).

Although technology is everywhere and affects our life with respect to what we are and what we do, we want to distinguish the intention of use of some technologies, related to the *didactic system*, represented by the tetrahedron.

Thus we introduce an internal sphere. It refers to the use of particular e-tools with an explicit *didactic intent*, that means that they are exploited with the intention to teach in a specific educational context (Chevallard 1989). It is worthwhile to note that the e-tools have not been invented for educational purposes but, more probably, for business, and then their meaningful didactic use is not granted, but it has to be attained (Borba et al. 2013; Chevallard and Ladage 2008).

According to Arzarello et al. (2012), now we assist to a shift from visible to invisible technologies, putting their add-value in their particular use with respect to learning purposes.

This is why we do not consider technology as a new vertex in the model, as others in literature do (Rezat and Sträßer 2012; Ruthven 2012; Tchoshanov 2013), but as a sphere which is into the background.

Due to nature itself of Author, we can assume a certain use of means from the external sphere in order to support group work. It is quite natural that their use comes out spontaneously (for instance, think of the use of Dropbox or GoogleDocs for constructing shared documents). Some kind of natural interactions within the external sphere can occur inside the communities of Student and Tutor (for instance, the use of Facebook group as a notice board in order to share and keep up-to-date on organizational information about a course). According to the tetrahedron model, when interactions inside Student or between Student and Tutor are guided by didactic strategies they will be frame in the internal sphere.

The tangent point of internal sphere with the tetrahedron addresses the add-value that technologies can give to the relations among the actors of the face in the learning process, which, according to Arzarello et al. (2012), is:

> based on the methodology, or in their use as a tool to support well-defined teaching strategies, focused on the learning process of students or teachers engaged in professional training. (p. 3)

In this respect, we are firmly convinced that it is more and more appropriate the vision of mathematics education as a 'design science' (Wittmann 1995), which should combine and relate to one another the design of "artificial objects" – moving from teaching units (ibid, p. 362) to e-activities when they make use of e-environment – and empirical research. This science should produce proposals for the classroom, re-usable in other situations, not only test them (Lesh and Sriraman 2010) and this should hold for online classroom in e-environments (Borba 2012).

An Overview of the Tetrahedron Vertices

In this section, we are going to define the new vertex and to specify some characteristics of the other vertices that differ from the actors in the traditional didactic system.

The Author is a collective subject with different professional skills, including educational expert in relation to the knowledge domain, instructional design and management, ICT and pedagogical/sociological expertise. This subject acts as "scriptwriter" of didactical experiences mediated by technology.

The importance of teamwork inside the collective author is noted by Borba (2013), which identify as weakness of Internet the low design and pedagogical quality of online interactive mathematics contents. This can be avoided by the simultaneous work of a variety of experts, such as mathematics educators and human-computer designers, who can take into account and integrate both didactic objectives and interface design principles. Also Schoenfeld (2009) hopes for a synergy between educational researchers and educational designers. The richness of figure Author allows to create a variegated scenario of pedagogical expectations concerning knowledge, of professional or ideological beliefs, of implicit philosophies that supply an enrichment of the e-environment and carry out robust and well-engineered products to made available to the targeted learners. We consider that the comparison, the discussion, the thoughts that can occur among the different experts above, with assorted expertise and experiences, affect the decisions about resources and methodologies to be implemented with respect to a fixed didactic goal, and achieve in such way the construction of a rich and deep product.

Besides Author, the didactic system should include someone who teaches. We prefer to use the word "tutor" instead of "teacher" for making evident the change of the role of the teacher in an e-environment. She is the person who supports student learning and facilitates interaction between learners, and sometimes completely disappears (Arzarello et al. 2012) in the so-called self-directed learning or learning without teaching. In the framework of instrumental orchestration, we can say that the tutor is also the one in charge of the *didactical performance* (Drijvers et al. 2010a):

> A *didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals. (p. 215)

Conversely, the didactic system should include someone who learns: the Student, who also refers to those involved in continuing education program. What seems as important is the 'active' role she has in an e-environment. On the one hand, we have the shift from the centrality of the teacher to the centrality of the learner, which has in her hands the control of the learning process (Arzarello et al. 2012). On the other hand, learners change their role, becoming active in constructing knowledge too (Jahnke et al. 2014).

Finally, all the previous figures move around the element that gives reason of the didactic system, that is Mathematics Knowledge. Although Mathematics mainly refers to the academic mathematical knowledge, resulting from the scientific research, with its specific structural, methodological, historical and cultural characteristics, we cannot ignore that its didactic transposition is no more under the umbrella of a certain certification through official academic or school channels. With the advent of Web2.0 and of the communication infrastructures growth, it has become prerogative of anybody due to the availability of authoring and sharing tools (Borba et al. 2013).

An Overview of the Relations Inside the Tetrahedron

As seen in Albano et al. (2013), despite the fact that the didactical tetrahedron represents a whole, looking at each face allows us to consider the interactions of three characteristics at once and to think of a range of configurations of the teaching/learning process in e-environment.

The Author-Mathematics-Student face refers to a configuration where learning comes out from the interactions of the learner with the mathematics knowledge, which has been transposed in some digital resources by the Author or that she herself can author. No support of any Tutor is planned. Thus this face may refer to a configuration of self-directed learning.

This requires the learner to have an attitude of personal responsibility of her learning and a good level of the self-regulation competency. She is supposed to be able to control her own process of acquisition, elaboration and exploitation of her knowledge in relation to the learning goals, self-posed, to the awareness of her knowings, to the attitude towards failure in the past and expectations for the future (Schunk 1990; Zimmerman 1990). It is worthwhile to note that the lack of such competency, in terms of bad management of personal knowings and negative attitudes, seems the key of repeated unsuccessful behaviours (Zan 2000).

Looking at Author-Mathematics-Student does not mean that the whole learning process we are dealing with is in the previous configuration. More generally, it can also refer to the possibility for the learner of autonomously and freely moving, of choosing, designing and managing her learning in the e-environment. Self-regulation capability requires that Student should be free to construct dynamically her own learning path. In this view, the role of the Author is not just who edits some digital resources, but she becomes an arranger of technology-enhanced contexts where the resources can foster the achievement of a precise purpose.

Besides, Student should be also free to construct her own resources, realizing her own transposition and being herself part of the Author. Students do not restrict themselves to receive and elaborate objects (such as in the case of the book), but they can produce new ones from scratch or starting from those available. For instance, students can rewrite learning objects, traditionally viewed as read-only, by adding personal annotations (not only text-based) and share the so constructed new objects with peers (Borba et al. 2013). Further examples can be found in the frame of the narrative approach to mathematics learning, based on the educational use of story problems (Zan 2011) and of digital storytelling (Robin 2008): students are involved in the creation of digital story problems about real-life situations whose solution needs to apply some mathematics, giving rise to complex interactive resources (Gould and Schmidt 2010). The resources created by the students can be validated by the tutor or not, before being shared, according to the settings of the context in which they arise.

The Mathematics-Student-Tutor face refers to a learning configuration that requires the Tutor's interventions so that learning comes out by the interactions among students, tutor and mathematics. Interactions can occur between tutor and student or group of students, or among students, or among students and resources and e-tools. The tutor's mediation depends on the base-theory that has guided the design of the learning activity, and in an individual or cooperative/collaborative setting. Due to the availability of technological tools allowing online monitoring and recording of the learner's work, the tutor can support students' interaction by means of the *Spot-and-show* orchestration. She can access student work, identify interesting pieces and deliberately use them to set up students' interactions – for instance, ask for explaining the reasoning or providing reactions or feedback on the work (Drijvers et al. 2010a).

In any case, we can say that this face is characterized by the communication and we can frame learning in the so called *discoursive* approach (Sfard 2001).

The features of the new communication means and the various contexts in which mathematical communication occur highlight the importance of some characteristics of the mathematical discourse: not only multisemioticity, but also multimodality and multivariety, which are crucial for developing competence in mathematics (Albano and Ferrari 2013; Arzarello 2006; Duval 2006, Way 2014).

The Author-Mathematics-Tutor face refers to the relation of the author and the tutor with mathematics, consisting in the didactic transposition, that is in the design of the learning situations in the e-environment. Here the Author focuses her efforts on the configuration and exploitation of suitable e-tools and on the analysis, the design and the implementation of educational resources and activities (with or without tutor). It is well known that face-to-face teaching methods cannot be simply 'transferred' in e-environment, but they need to be revised and modified in order to integrate technology and pedagogy and take really advantage of the e-tools (Kahiigi et al. 2008). Moreover, we should observe that even problems trivial in paper and pencil environment can assume new light in e-environment, because of e-tools constraints (Borba 2012).

Didactic transposition should also take into account the shift from text-based communication to multimodal communication. Mathematical resources should combine various semiotic systems (text, graphs and figures, symbols) but also interactivity, animation, videos because it is what young people experience of in their ordinary life and thus it is what they expect in learning context (Borba et al. 2013).

The interactions between the author and the tutor is very important since the tutor can report her own feedback of usage of the didactical proposals. Such feedbacks contribute to a continuous re-design in order to make the proposals able to produce the desired learning. This means that the educational resources and activities are not static objects but they evolve according to the practices' feedback.

Finally, it is important to point out that the didactical proposals, related to certain goals, should be as many and various as possible, according to various approaches and methods, in order to support various students' needs, profiles and preferences, which in the constructivist perspective is viewed being the driver of mathematical learning (Balacheff and Sutherland 1999).

The Author-Student-Tutor face mainly refers to the relations of Tutor with the digital resources authored by Author on one hand and with Student on the other hand. Here we can frame the *documentational genesis* (Gueudet and Trouche 2009): Tutor appropriates and reshapes resources initially made available by Author and builds schemes of their utilization, for a given class of situations, across a variety of contexts, giving rise to the so called *document*. Such product becomes a new resource and it can be involved in a further process of documentational genesis, producing new document. The tutor arranges the documents she generated.

Finally, we note that the relation of Author with Tutor and Student can allow to collect students' and tutors' feedbacks about the resources initially authored by Author and to adjust/refine them in profitable way.

Didactic Engineering Work According to the Tetrahedron Model

In this section, we analyse, according to the systemic view given by the tetrahedron model, the phases of DE (TDS), previously listed, that are: (1) epistemological, cognitive, didactical analysis of the mathematical knowledge to be taught, (2) design and *a priori* analysis of the teaching/learning situations, (3) implementation and experimentation, (4) *a posteriori* analysis and validation.

In particular, we try to exploit the methodological function of the systemic view given by the tetrahedron, looking at the didactic engineering work as we are observing it from each vertex and thus in terms of the opposite face and of the relations among the related elements. This means that in what follows, said X, Y and Z any triplet of vertices of the tetrahedron, we are guided by the question: *assuming the perspective of the face X-Y-Z, what contribution can be given to the didactic engineering work?*

It is worthwhile to remark that the relations and processes inside the face can be affected by the fact that the vertices are not static figures but are considered as positions that can be assumed also by the students. Therefore the didactic engineering work, which can be seen as linear (although back and forth paths can be foreseen among the design and the validation steps), benefits from the systemic tetrahedron model as, conversely to its origin, it is no more teacher's prerogative but it conveys the relations among various positions. Thinking at each of the didactic engineering phases from the various perspectives given by the different faces of the tetrahedron allows to actually put the student in the centre of the whole teaching/learning process, as she can participate in such engineering work.

Author-Mathematics-Tutor: Preparing Mathematics for Students

We consider this face as the most important for the development of teaching products, as the relations among the vertices evoke a process of preparation of mathematics for the students, meaning not only contents by also learning activities that allow the students to reach their own mathematical knowing (referring to an individual's knowledge).

Looking at DE, the first two phases consists in a conceptual work that guides hypotheses concerning student's learning and consequently choices on the design. This conceptual work is constantly referred along the whole engineering effort, in a back and forth path as long as the learning situation is implemented and experimented, in order to revise the choices made.

Generalizing to learning activities what Artigue (1994) says about the Didactic Engineering applied to the teaching contents, the DE work starts with a preliminary phase, which is charge of the Author. It consists in outlining the epistemology (ideas on knowledge and learning) of the "new" element of teaching/learning practice to be designed, starting from examining what already exists and its drawbacks.

Then *a priori* analysis along three dimensions should be carried out: the *epistemological*, related to the characteristics and the way of functioning of the mathematical knowledge; *cognitive*, related to the Student targeted; *didactical*, related to the specificity of the educational system where the teaching/learning process occurs.

The face Author-Mathematics-Tutor seems to contribute mainly in the first dimension, as it concerns mathematics per se. The Author, by means of the internal competence as domain expert, can analyse the feature of the knowledge at stake, going in more details on the specific subjects considered.

The interaction with the Tutor allows to give a certain contribute to the other dimensions too. In fact, as the cognitive dimension takes into account the features of the learner, the analysis can benefits from the teaching experience of the Tutor. She can devise to consider the characteristics of a *generic* learner to whom the knowledge at stake is foreseen to be taught. Analogously, the Tutor can contribute to didactical analysis a certain configuration of the educational system, according to her experience.

After the analysis is completed, the phase of design starts, guided by the outcomes of the previous conceptual work. Here, the DE work involves a certain number of choices, at two levels: macrodidactic or global one, affecting the whole design; microdidactic or local one, guiding the organization of a specific session of a learning activity.

According to Theory of Didactic Situations, the design should include the definition of a *milieu* and of a learning situation, as the ideal model of the system relations among Student, Tutor and the *milieu*, that we call *Learning Situation Model* (LSM). The milieu was originally defined as the set of anything is acting on the student or the student is acting on (Brousseau 1997). Its function appears extremely important as student's knowing comes out as personal answers to the constraints of the milieu rather than to the teacher's expectations. Therefore, it consists in a set of resources to be used in order to grasp the knowledge to be learnt.

From the technological point of view, the definition of the milieu requires on one hand the use of digital resource and on the other hand the configuration of the e-tools to be used, at least in terms of the features of such tools needed to the situation performs, producing such a way a particular instantiation of e-environment.

It is worthwhile to note that the characteristics of the digital resources, shaping mathematical contents, initially designed by Author, consists in their openness for re-design can be designed by Tutor (actually assuming the Author position), both individually and collectively (Pepin et al. 2014).

Generally speaking, we note two aspects of the e-tools. The first concerns the fact that the choice of the e-tools that students are required to use is not neutral with respect to envisaging the learning situations, as seen in Borba (2012). The second one affects the functioning of the e-tools inside the milieu: they can act both in antagonistic and in cooperative way (Drijvers et al. 2010b).

Author-Student-Tutor: Refining Mathematics for Students

Looking at DE work from the Author-Student-Tutor, generally we can say that it contributes wherever we refer to concrete cases and characteristics for each of the positions, not just hypothetical categories or models.

Therefore, at level of the *a priori analysis*, this face contributes to the cognitive and didactical dimensions. In fact, the Tutor and the Author can make more precise the analysis along these dimensions considering specific characteristics of the learners engaged and of the educational system in which the learners are framed (that can depend on social or demographic issues).

The reference to real cases allows the Author to make actual choices of specific e-tools (for instance, in case of a teaching platform, Moodle versus IWT). This leads the Learning Situation Model to be more concrete and thus to go towards an implementation of Learning Situation Model, that we call *Learning Situation Instantiation* (LMI), which will be actually experimented.

Actually, the Tutor and the Author realize an instantiation of the design made in Author-Mathematics-Tutor face, taking into account the target Student and the context where the Learning Situation Instantiation takes place. Hence, some choices previously outlined in the Learning Situation Model are made more precise and the e-environment is definitely set up, choosing the actual e-tools to be used and defining the modality of their use in order to obtain the features foreseen in the design. Analogously for other elements of the milieu, generated for instance by the didactical transposition: in order to set up the milieu, the Tutor can choose resources, previously designed on Author-Mathematics-Tutor face, and she can re-design them because of the customization to the needs of the Student. This way she continues the design phase.

From the technological viewpoint, the e-environment has made ready for the Learning Situation Instantiation, thus e-tools satisfying the Learning Situation Model requirements are set and arranged as well as the way of using concretely according to what defined in the Model.

Finally, we also note that, in DE view, this face realizes the back and forth process of continuous comparison of what is expected in the design and what actually occurs, allowing suitable reciprocal adjustments of Learning Situation Model and Instantiation. In fact, the Author, getting in touch with the Tutor and the Student, can collect feedbacks from both of them (also exploiting the technological empowerment) and refine the design as well as the implementation according to what emerges from the data analysis.

Author-Mathematics-Student: Mathematics Construction Without Official Help

Retracing the DE work from the perspective of this face put emphasis on the possible absence of the Tutor position in the learning process.

From the design point of view, it implies to devise, if the case, piece of the learning situation, for which no support from the tutor is expected. Knowings come from various kind of engagement of the Student with Mathematics: (a) Student can interact with the milieu, suitably set up by the Author, and Mathematics knowing comes out by the interaction; (b) Student can assume the Author position, authoring herself mathematical resources, both starting from scratch and modifying and personalizing resources already created by the Author; (c) Student can work together with peers in cooperative and collaborative activities occurring in the learning situation.

In all the previous cases, the absence of Tutor does not mean that Student does not receive any help, but that it can come implicitly from the engagement in the situation (i.e. the responses of the milieu, the feedback of peers, etc.).

Moving along the DE work, here we can observe the experimentation of the above cases and thus collect the data for the a posteriori analysis: in fact, the Author can benefits from direct or indirect observations (for instance, students' products/ protocols or log files available in the e-environment) of what happens between Student and Mathematics and can use these data for adjusting the design of the learning situation.

Mathematics-Student-Tutor: Mathematics Construction Officially Supported

Differing from the previous perspective, looking at DE work from this face emphasizes the a-symmetric relation Student-Tutor with respect to knowledge construction.

The design should take care of what concerns in particular the didactic part of the learning situation, where Mathematics knowing comes out from the interaction between the Students and the milieu, made viable by the Tutor. Thus, a troublesome issue of this face is the didactic contract, consisting in specific behaviour of the teacher expected from the learner and the behaviour of the learner expected from the teacher (Brousseau 1997). In e-environment, it can be affected by more than one variable. A key variable depends on the role of the Tutor, who can or cannot be the same person in charge of the Student's learning assessment (for instance, this is certainly the case of distance online courses). In the latter case, the traditional a-symmetric situation between teacher and learner is not so stressed and the *didactic contract* between Tutor and Student is remarkable modified because it is no more communicated by the assessment phase. Moreover, we highlight that the contemporary interaction of Student with the Tutor and with the digital resources of the milieu, by which the Author shaped Mathematics, can bring to appear two different didactic contracts, one between Student and the digital resources and one between Student and Tutor, which can conflict each other (Cazes et al. 2006).

A further key issue in this face concerns the dealings between the Tutor and the Student, at peer level too. In fact, we want to warn on a misconception that can arise from the familiarity of the digital natives with the social networks. One can be led to confuse their habit of continuous online social interactions with a natural disposal to forms of collaborative learning. Although there can be found some cases of positive correlation between collaborative affordances of e-environment and learning, some other studies show the importance of deepening the issue of collaboration, even in the phases of analysis and design (Borba et al. 2013).

Moving along the DE work, also in this case we can observe the experimentation and gather data for the a posteriori analysis, also exploiting the direct involvement of Tutor.

A Case Study

In this section we want to discuss a case study as an instantiation of the proposed theory. We have used the DE approach according to the tetrahedron model in a scholar experience that was piloted in Italy in blended courses at the Universities of Salerno and of Piemonte Orientale (Albano and Ferrari 2008).

As already noted, the tetrahedron model gives a systemic view; consequently the starting point of a teaching/learning process does not lie on a fixed face. In our case study, the insight for the experience we are going to describe come in Mathematics-Student-Tutor face of the tetrahedron. The position of Student was taken by the first-year engineering students attending a trimester intensive module in mathematics which concerns topics from linear algebra and calculus. Being in the position of Tutor, we should face the evidence of the Student belief that, especially in those contexts where mathematics has seen as a “service domain”, instrumental approach is enough and a more in-depth understanding was unnecessary, despite the failures at the exams. As the assessment focused on conceptual understanding, the same belief was exactly the reason why they failed. We decided to investigate more but, asking the students why they failed, someone said “because of strange and unexpected questions”, referring to traditional examination questions which aim to explore if the student has understood some theorem’s statement or proof. Examples can be questions like the following: “what theorem guarantees the validity of this passage?” or “what means this expression?” or “where did you use this hypothesis in the context of the proof?” It was just the previous someone’s answer to launch the DE work, whose outcome is the teaching/learning experience of our case study.

The initial underlying idea was to elaborate learning situations aiming to foster the students to face topics in a more critical way and to change their attitude from rote learning to critical learning.

Therefore we start with the preliminary phase of DE. Remaining in the perspective of the Mathematics-Student-Tutor face and looking at their interaction, we contribute, in the position of Tutor, to the didactical and cognitive dimensions of the *a priori* analysis.

Concerning the first one, the analysis stresses the specificity of the Italian University system, mainly transmission-based traditional lectures, attended by more than one hundred of people, especially in Faculty such as Engineering. This is true also for the exercises sessions, which remain transmission-based (the teacher shows some solving techniques for typical exercises). In our case study, the course consists in 90 h face-to-face lectures (60 devoted to theory and 30 to solving procedures) and it deals with a large amount of mathematical concepts from basic linear algebra and calculus II. No institutional space is devoted to practice some problematic approach, such as posing questions, although the latter is needed for being successful in assessment. The students are supposed to enter University being endowed with such competency, but the analysis shows that it was true no more. Then it is clear the impact on the exam results and on the students' feeling of "strange and unexpected questions".

Concerning the second dimension, the analysis highlights the Student difficulties during oral discussion of theorems' statement and proof within the exam session, whose roots can be found in what Paul (1990) well expresses:

> What students often learn well – that school is a place to repeat back what the teacher or textbook said and to follow the correct steps in the correct order to get the correct answer – blocks them from thinking seriously about what they learn. (p. 808)

This is a transversal difficulty, not strictly specific of topics at stake, but related to the way learning is approached by students whose many drawbacks conflict with the undergraduate learning approach requiring to grasp complex knowledge going-in-depth and critically thinking.

To complete the *a priori* analysis, we move on the Author-Mathematics-Tutor in order to focus on the epistemological dimension. To this aim, we need to co-opt in the Author position, besides ourselves, other colleagues experts in mathematics and in its education, with especially regards to the subjects we are interested in, and also to take into account the existing literature too. The analysis puts emphasis on some features of the linear algebra and calculus at stake in the given course. For instance, a higher abstraction and complexity of the mathematical objects in linear algebra (the *n*-dimensional space), a huge usage of symbolic representations (letters instead of numbers), the management of and the dealing with generic objects (reasoning on the basis of the properties of the object and not of its instantiation), the need of coordinating various semiotic representations (algebraic and geometric ones), and so on.

Them we start the design of a new learning situation and the previous analysis guides the didactic organization. We adopt to integrate the face-to-face lectures with on-line time restricted activities to be performed throughout the course.

We aim to design a Learning Situation Model (LSM) that can simulate the oral dissertation of a theorem and its proof during examination. As the analysis highlighted that the students lacked the practice of posing questions in order to understand a topic, we decide, as pedagogical choice at global level (that is concerning the whole design), to frame the situation in cooperative learning (Dillenbourg et al. 1996), in particular in a role-play setting. The envisaged LSM requires the student to play subsequently three roles: the teacher who makes questions, teacher aiming to evaluate student's learning concerning a given topic; the student who gives answers to the teacher aiming to prove her understanding of the topic; the teacher who assesses the student's answers with respect to learning outcomes. The pedagogical choice done suggest to look at this part of design in Author-Mathematics-Student perspective, as no official help is foreseen. Thus, we fine-tune the design fixing some further global choices. The LSM consists in three consecutive tasks corresponding to the roles described above (from now on, we call 'round' an occurrence of all the three tasks). For each task the students are required to produce resources to be used by peers at the next task: in the first, the student make some questions as if she has to assess someone other's learning outcomes; in the second step, the student gives answers to the questions posed by a peer; in the third step, the student checks the correctness of the outcomes (both question and answer) of two peers. All the students play contemporarily the same role, i.e. all of them performed individually the same task at the same time, preferably addressing different topics. At the end, they go to the next task. Once a round finished, a new one can start. The model devises that for each task the devolution is activated by explicitly asking the student to act in the role at stake.

We also define the related global milieu, which consists in a teaching platform, equipped with the following e-tools needed to implement the design: (i) a means that allows implementing cooperative and time-restricted activities; (ii) a repository for sharing resources; (iii) communication tools for interactions among participants. We want to note that the local milieu, i.e. the resources needed during the activities, is not made ready from the Author, but it is supposed to be constructed gradually and for each task it consists in the products of the students themselves playing the previous roles. In this respect, the student assumes the Author position.

Moving to the Mathematics-Student-Tutor face, we think if some official help for the students could be devised. We decide that the Tutor did not intervene during the tasks, leaving them in the context of peer-to-peer interactions, but she gives comments, suggestions or corrections only at the end of the round. This way the Student can benefit from the implicit help of peers playing the third role and form the explicit Tutor's help improving the products of next round, both in terms of the Mathematics and of the methodology used.

At this point, the Learning Situation Model has been completed and we require to move towards the Learning Situation Instantiation, in order to actual implement the situation.

To this aim, in the perspective of the Author-Mathematics-Tutor face, we need to make concrete the devised global milieu. Co-opting and exploiting technological experts within the Author position, we choose the teaching platform and the e-tools to be used. In our case study, we select the platform IWT, available at the University of Salerno. The role-play has been implemented using at each step the

IWT module 'homework' that allows the students to submit their work. Then the Tutor, acting as technical support, is in charge of receiving the submissions in each step and then randomly distributing them to the students as resources for the next step. Further, a shared repository, with access restricted only to students involved in the activities, has been set up in order to allow the Tutor, at the end of each round, to make available the products of the students suitably annotated.

We note that, as the global milieu is affected by the technology at disposal, we can have various implementation of the learning situation. In fact, the same situation has been implemented at the University of Piemonte Orientale using the platform Moodle and the 'workshop' module, which overcome the Tutor technical support.

Let us consider now the local milieu that is the resources needed for the Learning Situation Instantiation. According to model we defined, the setting of the local milieu can be seen as an ongoing process in the Author-Mathematics-Student perspective. In fact, as the first task do not foresee to deliver any resource specifically designed, the student can freely surf among mathematical resources, concerning the mathematics theorem she has to address, available inside or outside the e-environment (learning objects in the platform, various resources on the web, to her notes during face-to-face lectures etc.). On the contrary, the second and the third tasks devises the delivery of a specific resource, consisting in a product randomly chosen among the ones submitted by the students in the previous sessions. Thus the Student contributed to set up the local milieu, assuming the Author position, as she is required to author resources, completely from scratch in the first session or starting from a peer-authored one of the other two sessions.

Once completed the implementation phase, we experiment the Learning Situation Instantiation. In order to do this, we refined some global and local choices, listed below, taking into account the specificity of Student and of our didactic context:

(a) the duration of each task and of a round of the learning situation: we fixed in 2 day the time needed to the student in order to perform each task; then one day was foreseen to technically distribute the submitting products in order to begin the next task; thus, each round lasted 9 days;
(b) the number of rounds to experiment along the blended course: according to the duration of the face-to-face course and of each round, and taking into account that the experimentation did not start at beginning of the course, we chose to make three rounds;
(c) the mathematical contents to work on: we selected a list of 19 theorems, 11 from the linear algebra and 8 from calculus, consisting in the main theorems whose understanding, in terms of statement and proof, was required to the student in assessment phase;
(d) the partition of the previous list into three sub-lists, each of them to be used in one round; we did not made this partition a priori, but the sub-lists were determined according to the flow of the face-to-face lectures.

Therefore the experimentation starts assigning to each student one of the theorem from the first sub-list (for instance, Steinitz lemma). The task of the first activity requires "*Prepare a file containing four questions which you consider useful to verify that a student has understood the claim and the proof of the Steinitz lemma, as you are a teacher who wants to assess a student's learning about such topic*.". For the next task, each student receives one the previous output file and she is required to "*Answer to the questions contained in the attached file as they are the questions posed by a teacher during an examination and you want to prove that you grasp knowledge about the theorem at stake showing to know how to answer*". The last task expects: "*The attached file contains some questions and answers concerning a given theorem. Correct as you are a teacher during an examination who wants to assess both the questions and the answers with respect the given theorem*.". For the *a posteriori* analysis and validation, we look at the students' engagement from the Author-Matematics-Student face, and at the Tutor's revision after the round finished, from the perspective of Mathematics-Student-Tutor. We conducted it using mainly qualitative methodology, examining at cognitive level the protocols produced by the students and interviewing at affective level some of them regarding the roles played (for more extensive reading, see Albano et al. 2007; Albano and Pierri 2014). The analysis highlighted the add-value of the first task both at cognitive and affective levels, with rich products, addressing thinking and reasoning mathematically, and communication and representation, and leading students to go-in-depth facing a topic.

The analysis also highlighted a drawback concerning the didactic contract caused by technological constraints of the specific e-tool, homework, that need a mandatory score to the learner's product. This brought to refine the implementation in order to focus on the formative assessment, avoiding the drawback.

Conclusions

In the last years, the emergence of coming back to the centrality of the methodology with respect to the technology asks for the mathematics educators to be much more 'design scientists' in their use of e-environment for supporting learning.

The need of combine technology features with well-known theory in mathematics education strongly requires the design of learning situations based on a scientific approach. We assume that a didactic engineering approach should be proper. In this paper, we have looked at the didactic engineering work from the systemic perspective given by a didactic tetrahedron. This latter allows having a systemic view of the learning situations to be designed and validated, taking into account new characteristics of the learning process in a pervasive technology-enriched world.

The didactic tetrahedron makes evident a new figure, the Author, which wants to draw attention to the necessary synergy among technological and educational experts in order to balance and full exploit the benefits of tools and methodologies.

Finally, we underline that the vertices of the tetrahedron refer no more only to static figures but to "dynamic position", as learners can play the role of Author (creating resources) and of Tutor (coaching their peers) during their learning process.

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Learning Paths and Teaching Bridges: The Emergent Mathematics Classroom within the Open System of a Globalised Virtual Social Network

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Abstract In this chapter, we adopt a systemic approach to the phenomenology of the emergent 'connected' mathematics classroom, in order to investigate the views of primary school teachers, principals and school advisors about mathematics and social network sites (SNS) across and within two interrelated systems: the scientific disciplines and the school unit (including the symbolic/normative level, pragmatic representations of the school practices, and the personal desired/intentioned actions). This inter-systemic, tri-focussed perspective allows the meaningful re-approach of the emergent classroom and is operationalised with a questionnaire, which constitutes a pragmatic diagnostic-hermeneutic-research tool informing the decision-making of educators and policy makers.

Keywords System • Complexity • Social networking sites • Social networks • Mathematics discipline • School mathematics

Mathematics and Mathematics Education in a Globalised Social Network

Prelude

The current educational projects are influenced by two interacting factors: the globalisation (a broader, relatively stable factor) and the economic crisis (a recent, strong regulating factor). These two factors seem to temporally coincide with the rise of the social networking websites. This is further

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complicated in countries like Greece, which consists of areas that are economically inefficient (islands, mountains etc.) for the implementation of traditional educational means, thus putting the pressure on the educational system to efficiently utilise the 'distance' learning technologies. In this study, we introduce a systemic approach to the complex phenomenology of mathematics education as experienced by the educational protagonists who act and interact the emergent expanded mathematics classroom of a globalised social network. Drawing upon a co-developed theoretical-methodological framework, we utilise a comprehensive research tool comprised of a questionnaire, quantitative analyses and a hybrid diagrammatic-symbolic representation of the obtained results, in order to investigate and to effectively communicate the complexity of the co-existing multiple constructions about the complex phenomena as perceived by the primary educational protagonists, including the teachers, the school principals and the school advisors.

A Complex Globalised Social Network

The increased complexity of the school social network constitutes a challenge for the educational protagonists. The school principals are required to function within a transformed administrative, social and pedagogical space. The teachers are expected through their teaching to pedagogically interact within multiple coexisting realities related to social realities of varied roles and relationships (including the students' psychological and family realities). Within this complex school reality (internal and external), the school advisors are required to communicate and to support the implementation of the official educational policy as described by the Ministry of Education. Hence, it is reasonable to investigate the diverse views that the educational protagonists hold about the role of the social networking sites in the school space-time. Through the study of the development of this diversity, we argue that it becomes feasible to provide valid and reliable means for identifying and tracing the transformations of the complexity of the educational situation.

Mathematics seems to lie at the heart of these transformations. On the one hand, mathematics as a discipline has been linked with the notion of universality and through its applications is at the heart of various technological advances, including the means for communicating, designing and constructing the structure and the functions of the contemporary network reality. Branches of mathematics have been applied to and/or have been developed for the virtual social networks. On the other hand, mathematics as school course is at the crux of the contemporary curricula, constituting a large part of the allocated school hours and, importantly, of the evaluation of the educational outcomes of the school unit locally, nationally and internationally. In both the research and the learning space, mathematics does not function in 'vacuum', in isolation. On the contrary, the meaning and essence of mathematics is crucially redefined within the space that is settled each time, through the continuous interdisciplinary interactions and exchanges with technology and other disciplines. Hence, it is sensible to investigate the role of the social networking sites in the school unit, focussing on mathematics with the purpose to identify the interactions within such networks for a crucial for the educational system course.

The social media are increasingly prominent in the modern societies, gathering the interest of scientists of various disciplines (Christakis and Fowler 2009). A social networking site is an online structure that allows the formation of social networks within the reconceptualised virtual social space. The social networking sites have been developed and have been utilised 'outside' the classroom and the school unit, in the sense that although the educational protagonists (including teachers, principals, students, parents) interact within such sites, these interactions are in most cases not explicitly related with the class and outside the school control (both interest and authority), thus constructing new learning space-time. Hence, there appear to co-exist isolated learning paths, which are also incongruent with the school main learning 'highway'.

Bridging a Fragmented Complex Educational Reality

It is argued that this fragmented reality, which characterises the emergence of the "connected" mathematics classroom, needs to be addressed through the construction of appropriate teaching bridges. In line with this perspective, mathematics teachers explore the educational possibilities of social media by utilising virtual social networks of general purpose, such as Twitter and Facebook (Borovoy 2013; Sheehy 2012) or of educational focus, such as the MathForum (mathforum.org) or Learnist (Learni.st). Moreover, the Massive Open Online Courses (MOOCs) appear to be a divisive topic (Anderson 2013; Yardi 2012), stressing advantages (such as ease of access) and disadvantages (such multifaceted validity and reliability issues). Furthermore, mathematics educators have investigated the implementation of social media in mathematics teaching and learning, including studies about Facebook and the MathForum (Baya'a and Daher 2012; Renninger and Shumar 2002). In addition, a growing body of research discusses the role of social networking sites in the broader psychosocial development of young students (focussing on adolescents; Shapiro and Margolin 2014), whilst the research on younger students seems to focus mainly on cyberbullying (Monks et al. 2012).

The results of these recent studies have revealed the complexity of the networking structures and the diversity in the ways that the teachers choose to utilise them in the pedagogical practices. Pedagogical, epistemological and ontological issues are at the crux of the problematic about the social networks, which may complement the existing structures and/or constitute apparently autonomous new structures.

Nevertheless, existing research projects appear to concentrate only in partial aspects of the phenomenon (especially with respect to younger students), projecting the existing complexity to an over-simplified space, thus conflating the complex functions and relationships of the context, the environment within which the educational experience occurs. Within such a simplified perspective, the new structures may indeed appear to be autonomous. For example, many of the learning difficulties about the links between the notions of area and perimeter that are observed in the traditional paper-and-pencil teaching environment seem to be resolved through the use of ICT (for example, Dynamic Geometry Environments, DGEs), since the software allows for the relatively easily performed variations of lengths, thus allowing for their invariant relations (area, perimeter) to be visible and observable. Nevertheless, the mathematics in the traditional context and the mathematics in the dynamic graphic software environment essentially constitute two mathematics worlds (cf Tall 2013). It is argued that by addressing the learning problems in one mathematics world through the means of the other mathematics world, the role of the transitions between the two contexts and the two mathematics that functions in each context is set at the crux of the didactical situation. This can be linked with Douady's (1984) *jeu des cadres* theory concerning the dialectic between the tool and the object that occurs during the change of framework in mathematics teaching (for example between the algebraic and the geometric framework). According to Duval (2002), a *framework* is a set of concepts that may be organised in a theoretic progression (such as a branch of mathematics), whilst a *register* is a semiotic system that produces different types of representations linked with different cognitive functions. However, the under consideration situation is at the same time completely different and complex, since the discussed changes are not just changes of register, involving deeper techno-epistemological changes of framework. Following these, it is argued the attempted teaching bridges should consider the existing complexity concentrating in linking the various levels of mathematics, frameworks and contexts that coexist in the emerging learning paths.

However, the interactions of the protagonists, the environments within which they happen (virtual or nor), the complex educational totality seem to redefine the didactical relationship and the didactical contract (Brousseau 1997), thus rendering important to use plural tense in the usually implicit narratives (in line with the plurality of the experienced realities). Within the transformed network, the didactical relationships are defined in a perceived intersection of usually incongruent constructions about relationships of power, regulations, desires, and structures. It is argued that by viewing the school unit as an open complex system (Bertalanffy 1968), this apparent contradiction of 'intersected incongruences' may be didactically resolved, allowing for the explicit valid mapping of the implicit complex relationships and, crucially, of the meanings and of the narratives linked with the diverse realities.

Consequently, in this study we consider the school unit as a complex open system, to investigate mathematics education and social media in the expanded school network as perceived by its educational protagonists (teachers, principals, school advisors).

The School Unit as an Open System: A Tri-focussed, Inter-systemic Approach

We address these complex phenomena within the school unit by adopting a 'soft' *systems theory* perspective (Ibid.) to investigate the interacting and interrelating views which inform teaching practices. A system is an integrated whole, serving a specific goal with varying openness to its environment (Ibid.). The interrelationships amongst the parts of a system alter their purpose and utility, providing the system with 'non-summativity' (structurally and functionally superseding its parts and their properties). Educators have discussed issues linked with the notions of system and complexity (Bouvier et al. 2010; Chen and Stroup 1993; Davis and Sumara 2006; Davis et al. 2008). Specifically, the discussion has included the implementation of systemic ideas with respect to: mathematics learning and learning systems (Davis and Simmt 2003; Wittmann 2001), school improvement and curriculum (Begg 2003; English 2008; Thornton et al. 2007), and the importance of teaching the notions related with complex systems to students (Jacobson and Wilensky 2006; Kalavasis et al. 2010).

The systemic perspective allows our focusing on the links amongst the system components, thus facilitating a qualitative shift: *from* investigating views about mathematics *to* investigating views about mathematics within the system of consideration. Drawing upon a systemic perspective, Moutsios-Rentzos and Kalavasis (2012, 2013) introduced a research framework to investigate the views of the educational protagonists about mathematics. According to this theoretical and methodological framework, the views about mathematics can be investigated within two interrelated systems: a) the *system of all disciplines*, and b) the *school system*. Within the system of all disciplines the views about mathematics are investigated in comparison with the other disciplines, allowing the identification of 'special' to mathematics characteristics, thus providing a relational perspective about the epistemic views about mathematics. Regarding the school system, each of the protagonists assumes at least three roles: What the protagonist is expected to do; What the protagonist actually does; What the protagonist would choose to do. Thus, the school system can be viewed through the lenses of three foci: (i) the *symbolic/normative* (the perceived official regulations), (ii) the *pragmatic representations* (the perceived current state of school practices), and (iii) the *desired/intentioned actions* (the personal hypothetical actions, assuming the power to implement them). We theorise that each of these foci constitutes a sub-system of views (cf 'belief systems'; Green 1971) as it is affected by the broader constructions to which each focus is concentrated. For example, the pragmatic representations focus about mathematics in the school unit is settled within the broader system of the pragmatic representations that constitute the individual's pragmatic representation construction, whilst the desired/intentioned actions focus is affected by the broader desired/intentioned actions that an individual may hold.

The proposed framework is diagrammatically outlined in Fig. 1, where the hexagons represent the two systems, the ellipses sub-systems of a system, the

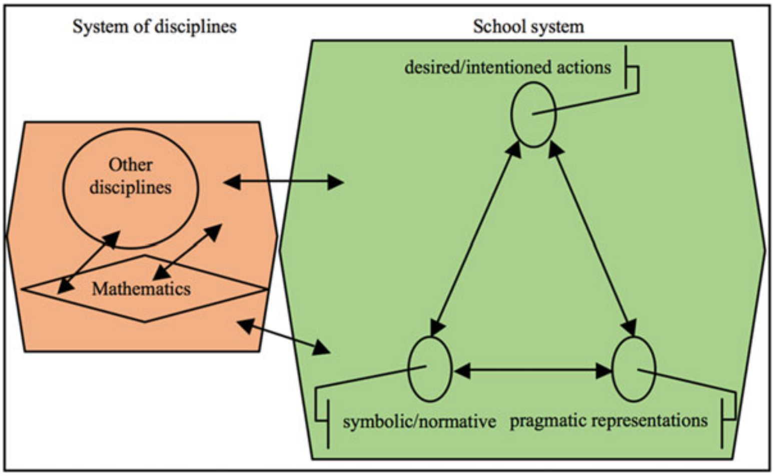

Fig. 1 An inter-systemic, multi-focussed approach to views about mathematics: systems, sub-systems, elements, interactions

rhombi represent elements of a system and the left right arrows represent inter-systemic and intra-systemic interactions.

In this project, we focus on the current Greek school reality, which is characterised by major changes, due to the attempted transformation towards a decentralised system, within which the school units would be pedagogically and administratively semi-autonomous. Moreover, the varied knowledge and usage of these networks increases the complexity of the school social reality within which the school principals are required to investigate and undertake new roles of elevated responsibility, while the teachers are required to act and interact in a reality with observable variance about the meaning of the pedagogical roles and about the nature of learning and the taught knowledge. Consequently, we attempt to map the views of primary school teachers and principals about the links of mathematics with globalisation and the virtual social networks.

Methods and Procedures

Setting the Scene: Teachers, Principals and School Advisors

In Greece the primary and secondary educational system consists of Dimotiko (6–12 years old), Gymnasio (13–15 years old) and Lykeio (16–18 years old). In this study we concentrate in the Dimotiko. In all six grades of Dimotiko the students are taught mathematics 4 h per week. The same teacher teaches most of the courses each year (except for music, arts, etc.) In each school a principal and a vice principal are responsible for the pedagogical and administrative school functions.

Usually the principals and vice principals are more experienced teachers who usually obtain additional postgraduate degrees (pedagogical and/or administrative). The schools of an educational district are within the responsibility of a school advisor. The role of the school advisor is complex, including the scientific and pedagogical guidance of both teachers and principals in line with the official educational policy as described by the Ministry of Education, as well as the evaluation of the educational outcomes of a school unit. Moreover, the school advisors support, supervise and evaluate the implementation of educational innovations and ICT in education, discern its weak points and suggest solutions.

Hence, for the purposes of this study we included in our sample teachers, principals (and the vice principals) and school advisors, in order to more validly capture the realities that interact within a school unit (Moutsios-Rentzos and Kalavasis 2015): the *micro level reality* (the class; teachers), the *meso level reality* (the school unit; principals and vice-principals), and the *macro level reality* (school district; school advisors).

Sample

Overall, 169 teachers, principals (and vice-principals) and school advisors participated in the study (N=169). All the participants had a 'Ptychio' in Education (a 4-year, bachelor-equivalent university degree) or a Ptychio-equivalent qualification. Their demographics and their educational studies, training and experience are outlined in Table 1.

The Instrument

The aforementioned approach has been operationalised through a four-part questionnaire in accordance with the proposed research framework (Moutsios-Rentzos and Kalavasis 2012, 2013), as implemented in previous research projects (Moutsios-Rentzos et al. Moutsios-Rentzos et al. 2012a, b). The first part included 8 closed items investigating views about mathematics within the system of all disciplines with respect to:

- *everyday life* (5 items)
  - *usefulness* ("...mathematics in the era of globalisation is more useful in everyday life in comparison with other disciplines?")
  - *importance* ("...mathematics in the era of globalisation is more important in everyday life in comparison with other disciplines?")
  - *problem-solving* ("...mathematics is more useful in our dealing with everyday problems in comparison with other disciplines?")

Table 1 The participants of the study

		School advisors		Principals		Teachers	
Total		30[a]	17.8%[b]	31	18.3%	108	63.9%
Gender	Female	16	57.1%	12	40.0%	81	79.4%
	Male	12	42.9%	18	60.0%	21	20.6%
Age	<=33	1	4.0%	0	0.0%	39	38.2%
	<=45	2	8.0%	4	13.3%	35	34.3%
	<=57	22	88.0%	26	86.7%	28	27.5%
	>57	0	0.0%	0	0.0%	0	0.0%
Postgraduate administrative diploma		20	100.0%	13	86.7%	7	20.6%
Postgraduate pedagogical diploma		23	100.0%	20	95.2%	35	71.4%
Master's		26	100.0%	15	83.3%	24	47.1%
PhD		19	90.5%	3	30.0%	1	3.2%
Experience (in years)							
Teaching		23(6), [4,31], 23 [c]		25(6), [10,35], 26		13(8), [0,28], 11	
Administrative		6(5), [0,22], 4		6(4), [1,16], 5		0(1), [0,5], 0	
Current		7(6), [2,30], 6		5(4), [1,20], 4		11(8), [0,27], 9	

[a]Frequency
[b]Valid percent
[c]*M*(*SD*), [*min,max*], *Mdn*

- *complex problem-solving* ("…mathematics become more useful in everyday life in comparison with other disciplines, as the complexity of the problems we deal with increases?")
- *natural world* ("…mathematics has weaker relationship with the real world in comparison with other disciplines?")

* *professional success* ("…mathematics school success is more than other disciplines positively linked with professional success in the era of globalisation?")
* *teaching necessity* ("…mathematics in the contemporary society require more than other disciplines systematic teaching in order to be learned?")
* *logic* ("…mathematics in the era of globalisation more than other disciplines promotes the development of logical reasoning?").

Each topic is expressed in way that explicitly asks the participants to provide an answer that compares mathematics with the other disciplines. For example, "Do you think that mathematics is more useful in comparison with other disciplines?". Moreover, the format of the participants' answers are two-faceted: first, a 'Yes/No' dichotomy indicating their agreement or disagreement, followed by a 4-point Likert scale to identify their degree of agreement or disagreement. In this way, we intend to first obtain a positive or negative reaction and then the degree of this reaction (which psychologically is not feasible with a simple Likert item). In Fig. 2, a sample item of the first part of the questionnaire is presented.

Do you think that mathematics more than other disciplines promotes the development of logical reasoning?	Y	1	2	3	4
	N	1	2	3	4

Fig. 2 A sample of the items included in first part the questionnaire (system of all disciplines)

According to your opinion, the official regulations demand the use of social networking sites with the purpose for the students to gain deeper understanding about mathematics?	Y	1	2	3	4
	N	1	2	3	4
Do you think that in reality in schools social networking sites are used with the purpose for the students to gain deeper understanding about mathematics?	Y	1	2	3	4
	N	1	2	3	4
As an educator and assuming you had the power, would you promote the use of social networking sites with the purpose for the students to gain deeper understanding about mathematics?	Y	1	2	3	4
	N	1	2	3	4

Fig. 3 A sample of the triplets of items included in the second part of the questionnaire (system of school courses; underlined the introductory part of the triplet)

The second part consists of 9 triplets of closed items realising our tri-focussed approach: *S*/*N* (‘symbolic/normative’) – *PR* (‘pragmatic representations’) – *D*/*IA* (‘desired/intentioned actions’). Two areas were investigated: *didactics of mathematics* (four topics) and *general didactics* (five topics). Each topic of each area is investigated through a triplet following a specific pattern. The introductory phrases of the three items is differentiated in line with each of the three foci of our approach. The rest of the three items of the triplet is common for each triplet indicating the content the topic. Thus, all triplets consist of three items investigating the same topic (the common to the triplet part of all items) from three perspectives (indicated by the introductory part of the items). In Fig. 3, a sample triplet for a topic is presented. Moreover, the response format is the same as in the first part of the questionnaire (a ‘Yes/No’ dichotomy and a 4-point Likert scale).

The investigated topics addressed both didactics of mathematics and general didactics (in parentheses the topic –the common part– of each triplet):

- *didactics of mathematics* (four topics-triplets)
  - *presentation* (“. . .use of social networking sites with the purpose for the teachers to provide a more comprehensive mathematics teaching”)
  - *understanding* (“. . .use of social networking sites with the purpose for the students to gain deeper understanding about mathematics”)
  - *goals* (“. . .use of social networking sites because they are incompatible with the goals of mathematics teaching”)
  - *democratic access* (“. . .use of social networking sites because they disturb the class balance and the democratic access in mathematics education”)
- *general didactics* (five topics-triplets)
  - *quality of teaching* (“. . .social networking sites with the purpose to improve the quality of teaching”)

- *communication* ("…use of social networking sites with the purpose to improve the students' communication")
- *coherence and cooperation* ("…use of social networking sites because they disturb the students' coherence and cooperation")
- *control* ("…use of social networking sites because they hinder the control of the class")
- *professional development* ("…use of social networking sites for the teachers' professional development")

In the third part of the questionnaire the participants are given the opportunity to comment on the aforementioned issues through an open question.

The fourth and final part included questions about the participants' background, investigating their age, education, working experience (teaching and administrative) and professional development. In this part of the questionnaire, the participants were asked about their knowledge and their use of social media, as well as about their perceptions of the students' use of social networking sites (SNS).

Analyses

The purpose of this study is to trace the emergent mathematics classroom and to underline the need for constructing teaching bridges, in order to attempt to support equilibrium of the expanded classroom towards inclusivity of the diverse learning paths of the students. In this chapter, we focus only on the quantitative part of the questionnaire with the purpose to investigate the potential research, theoretical and, crucially, pragmatic value of the introduced quantitative instrument in facilitating these purposes.

The quantitative analyses were conducted with IBM SPSS Statistics 22. The participants' answers in each item were scored as follows: for each response, we note '+1' or '–1' respectively for a 'Yes' or 'No' and subsequently we calculate the *intensity* for each participant's answer on an item as the product of '+1' or '–1' times the degree of agreement/disagreement as identified in the questionnaire ('1' to '4'). For example, if someone answered 'Yes' and '3' on an item this resulted to an intensity of '+3'. Accordingly, 'No' and '2' resulted in '–2'. The reported analyses were conducted with the intensities of the participants' answers. The statistically significant Kolmogorov-Smirnov and Shapiro-Wilk tests, as well as the visual inspection of the P-P plots suggested the non-normality of the collected data. Consequently, for the comparisons of two groups the Mann-Whitney U tests were employed. In order to identify statistically significant intra-population differences amongst the three foci ('symbolic/normative' – 'pragmatic representations' – 'desired/intentioned actions', Friedman's ANOVAs were conducted, whilst for inter-population (school advisors, principals, teachers) differences Kruskal-Wallis tests and MANOVAs were conducted. Statistical significance was considered at $P < 0.05$.

Results

Knowledge and Use of Social Media

First, we briefly present the participants' social media and internet knowledge and habits (see Table 2). Of the proposed social networking sites (SNS) only five appeared to be used by more than 10% of the participants with Facebook dominating their preferences; in descending order: Facebook (60.6%), LinkedIn (19.7%), Academia (17.5%), Twitter (16.3%), Pinterest (10.1%). Thus, the rest of the discussion will concentrate in these social networking sites.

It appears that Facebook is the most preferred SNS (60.6%). Although, Facebook is strongly preferred by all three protagonists (school advisors 53.6%, principals 56.7%, teachers 63.7%). it appears that only for the school advisors LinkedIn (61.5%) is slightly preferred to Facebook (53.6%). Moreover, though the percentages of the protagonists' Facebook accounts are comparable, they differ in their *frequency* of accessing them, with 42.9% of teachers accessing them everyday, in contrast with 28.6% of the principals and with only 14.3% of the school advisors.

Table 2 The participants' most preferred social networking sites (SNS)

		School advisors		Principals		Teachers		Total
Facebook		15[a]	53.6%[b]	17	56.7%	65	63.7%	60.6%
Usage	Less frequently	9	64.3%	6	42.9%	16	25.4%	34.1%
	3–4 times per week	3	21.4%	4	28.6%	20	31.7%	29.7%
	Every day	2	14.3%	4	28.6%	27	42.9%	19.5%
LinkedIn		16	61.5%	2	10.0%	6	7.9%	19.7%
Usage	Less frequently	1	6.7%	2	100.0%	4	57.1%	29.2%
	3–4 times per week	2	13.3%	0	0.0%	2	28.6%	16.7%
	Every day	12	80.0%	0	0.0%	1	14.3%	54.2%
Academia		8	36.4%	4	19.0%	9	11.7%	17.5%
Usage	Less frequently	1	12.5%	2	66.7%	2	20.0%	23.8%
	3–4 times per week	2	25.0%	0	0.0%	5	50.0%	33.3%
	Every day	5	62.5%	1	33.3%	3	30.0%	42.9%
Twitter		7	30.4%	4	19.0%	9	11.4%	16.3%
Usage	Less frequently	1	16.7%	2	50.0%	5	50.0%	40.0%
	3–4 times per week	0	0.0%	1	25.0%	1	10.0%	10.0%
	Every day	5	83.3%	1	25.0%	4	40.0%	50.0%
Pinterest		5	21.7%	2	9.5%	5	6.7%	10.1%
Usage	Less frequently	2	40.0%	0	0.0%	2	33.3%	30.8%
	3–4 times per week	1	20.0%	0	0.0%	3	50.0%	30.8%
	Every day	2	40.0%	2	100.0%	1	16.7%	38.5%
SNS usage	Personal	13	76.5%	17	63.0%	64	70.3%	69.6%
	School	11	73.3%	9	40.9%	36	48.0%	50.0%

[a]Frequency
[b]Valid percent

The school advisors and principals have Facebook, but they don't really use it. The school advisors' SNS of choice is LinkedIn, with more than 80% of the account holders access it everyday. Furthermore, the second and third SNS of choice (Academia & LinkedIn) appear to be mainly preferred by the school advisors and the principals and not by the teachers. Finally, the principals and the teachers use SNS mainly for personal purposes and considerably less for school purposes, in contrast with the school advisors who seem to equally use SNS for personal and school purposes. In Table 2, the participants' use and usage frequency of SNS is outlined.

Considering the participants' *perceived* computer and internet expertise (see Table 3), the protagonists appear to be relatively comfortable with these technologies with the school advisors and the principals appearing to be more certain about their knowledge: 96.6% of the school advisors appear to think of themselves as

Table 3 The participants' perceptions about computer and internet expertise

		School advisor		Principals		Teachers	
PC expertise	Very low	0	0.0%	0	0.0%	3	2.8%
	Low	1	3.4%	0	0.0%	3	2.8%
	Moderate	0	0.0%	7	22.6%	33	30.8%
	High	18	62.1%	18	58.1%	54	50.5%
	Very high	10	34.5%	6	19.4%	14	13.1%
	Total [a]	*M*= 4.28, *Mdn*= 4		*M*= 3.97, *Mdn*= 4		*M*= 3.68, *Mdn*= 4	
Internet expertise	Very low	0	0.0%	0	0.0%	4	3.8%
	Low	1	3.6%	0	0.0%	4	3.8%
	Moderate	1	3.6%	6	19.4%	28	26.9%
	High	17	60.7%	20	64.5%	52	50.0%
	Very high	9	32.1%	5	16.1%	16	15.4%
	Total[a]	*M*= 4.21, *Mdn*= 4		*M*= 3.97, *Mdn*= 4		*M*= 3.69, *Mdn*= 4	
Internet security	Very low	0	0.0%	2	6.5%	4	3.8%
	Low	2	7.1%	4	12.9%	22	21.2%
	Moderate	9	32.1%	15	48.4%	49	47.1%
	High	14	50.0%	10	32.3%	25	24.0%
	Very high	3	10.7%	0	0.0%	4	3.8%
	Total[a]	*M*= 3.64, *Mdn*= 4		*M*= 3.06, *Mdn*= 3		*M*= 3.03, *Mdn*= 3	
Students' SNS usage	<10%	1	3.7%	6	19.4%	10	9.6%
	1 out of 4	4	14.8%	6	19.4%	20	19.2%
	Half	13	48.1%	8	25.8%	36	34.6%
	3 out of 4	6	22.2%	10	32.3%	29	27.9%
	>90%	3	11.1%	1	3.2%	9	8.7%
	Total[a]	*M*= 3.22, *Mdn*= 3		*M*= 2.81, *Mdn*= 3		*M*= 3.07, *Mdn*= 3	

[a]Values may range from '1' (very low) to "5 (very high)]

experts in using the internet, followed by 77.5% of the principals and the 63.6% of the teachers; 92.8% of the school advisors appear to think of themselves as experts in using the internet, followed by 80.6% of the principals and the 65.4% of the teachers. A notable difference concerns ‘internet security’, with 60.7% of the School advisors appear to think of themselves as safe in the internet (‘high’ or ‘very high’), contrasting the considerably lower 32.3% of the principals and the 27.8% of the teachers.

Finally, we asked the protagonists about their representations with respect to the students’ use of SNS in order to obtain a measure of the participants’ perceived degree of the SNS permeability in the school reality. The results revealed that similar percent of the protagonists (36.6% of the teachers, 38.8% of the principals and 33.3% of the school advisors) appear to think that more than 25% of the students have an SNS account. These perceptions appear to be in stark contrast with recent research conducted by the Hellenic Police (presented in the 4th Conference for Safe Internet Surfing, February 13, 2015) suggesting that 85.5% of the primary school students have an SNS account (including Facebook 36.7%, Instagram 29.8% and Twitter 9.2%). This contrast is further amplified considering that 38.8% of the principals and 28% of the teachers think that less than 25% of the students have an SNS account.

Mathematics as a Discipline Within the System of All Disciplines

Regarding the system of disciplines, no statistically significant differences were found with respect to either the participants’ gender or their educational role (school advisor, principal, teacher). Thus, it is reasonable to discuss the sample as a whole.

In Table 4 the participants’ views about mathematics within the system of all disciplines are outlined. Bearing in mind that the hypothetical range is [−4,+4], we focused on the scores of absolute value of mean more than 1 as an indication of a relatively clear positive or negative perspective on a topic (noted in bold in Table 4). It was revealed that the participants think that:

(a) considering everyday life, mathematics is more useful, important and helpful in everyday problem-solving compared to the other disciplines, whilst mathematics is not considered to have weaker links with the natural world than the other disciplines,
(b) learning mathematics needs more systematics teaching compared to the other disciplines, and
(c) mathematics promotes the learners’ development of logical reasoning.

Hence, the epistemic views that the educational protagonists hold appear to describe mathematics to be intertwined with everyday life, having direct applications to everyday life and promoting logic, whilst it needs systematic teaching in

Table 4 Mathematics as a discipline within the system of all disciplines

	School Advisors		Principals		Teachers		Whole	
	M[a]	*Mdn*[a]	*M*	*Mdn*	*M*	*Mdn*	*M*	*Mdn*
Everyday life								
Usefulness	1.14	2	1.70	3	1.10	2	**1.21**[b]	2
Importance	1.31	2	1.26	2	1.01	2	**1.11**	2
Problem-solving	0.50	2	1.47	2	0.92	2	0.94	2
Complex problem-solving	0.66	2	1.45	3	0.27	1	0.55	2
Natural world	−1.47	−2	−1.84	−2	−1.82	−2	**−1.76**	−2
Professional success	0.10	0	1.50	2	0.26	1	0.45	2
Teaching necessity	1.20	2	1.06	2	1.74	3	**1.52**	2
Logic	2.03	3	1.77	3	2.17	3	**2.07**	3

[a]Values may range from '−4' (maximum disagreement) to '+4' (maximum agreement)
[b]Absolute value of mean score greater than 1 in bold

order to learned; we emphasise that these views are professed *considering mathematics in comparison to the other disciplines.*

Mathematics as a Course Within the School System

In Table 5 the participants' views about mathematics within the school system are outlined focussed on the views that each protagonist holds (intra-role comparisons) about the three foci considered in our framework (Symbolic/Normative, Pragmatic Representations, Desired/Intentioned Actions). First, it is noted that all the intra-role comparisons were found to be statistically significant, except for the topic 'goals' of the school advisors and the topic 'control' of the principals. We posit that this supports our framework, as this statistically significant difference suggests that the professed views on a topic indeed has at least one aspect that is different enough than the other two.

Moreover, in order to gain deeper understanding about the observed phenomena, the identified results were qualitatively recoded in line with the previous section: ↓ ('negative') when the mean value is less than −1, ↑ ('positive')when it is more than +1, and □ ('neutral') when it is between −1 and +1. Each of the protagonists appeared to perceive 'symbolic/normative' to be in the same direction or in one direction and 'neutral' with 'pragmatic representation'. Considering the 'desired/intentioned actions', protagonists' constructions appeared to be on the positive direction (regardless the positive or negative phrasing of the triplet) with respect to the inclusion of the social media both in didactics of mathematics and in general didactics. This is especially true for the school advisors who seem to hold the strongest intention of including social networking sites in the everyday teaching. Furthermore, it seems that the intra-role comparisons do not reveal qualitative differences between the didactics of mathematics triplets with general didactics triplets.

Table 5 Mathematics as a course within the school system (intra-role and inter-role comparisons)

		$M_{SA}^{b,e}$	P^{d}	$M_{Pr}^{b,e}$	P^{d}	$M_{Te}^{b,e}$	P^{d}	Trends (SA, Pr, Ta)	F(*df*, error *df*)[c, g]	P^{d}
Didactics of mathematics										
Presentation (+)[f]	S/N	−0.62	<**0.001**	−0.48	<**0.001**	−0.32	<**0.001**	□ □ □	F(6324)= 1.889	0.082
	PR	−0.86		−1.48		−0.40		□ ↓ □		
	D/IA	2.45		1.16		1.85		↑ ↑ ↑		
Understanding (+)	S/N	−1.07	<**0.001**	−0.07	<**0.001**	0.28	<**0.001**	↓ □ □	F(6320)= 3.123	**0.005**
	PR	−0.83		−0.74		−0.31		□ □ □		
	D/IA	2.77		1.45		1.88		↑ ↑ ↑		
Goals (−)[e]	S/N	−0.90	**0.003**	−0.63	**0.346**	−0.71	**0.002**	□ □ □	F(6310)= 1.844	0.090
	PR	−1.13		−0.31		−1.30		↓ □ ↓		
	D/IA	−2.03		−0.94		−1.50		↓ □ ↓		
Democratic access (−)	S/N	−1.52	0.158	−1.45	**0.025**	−1.60	**0.014**	↓ ↓ ↓	F(6312)= 0.609	0.723
	PR	−1.62		−1.35		−1.23		↓ ↓ ↓		
	D/IA	−2.07		−1.94		−1.84		↓ ↓ ↓		
General didactics										
Quality of teaching (+)	S/N	−0.93	<**0.001**	0.40	<**0.001**	0.65	<**0.001**	↓ □ □	F(6324)= 2.796	**0.011**
	PR	−0.21		0.00		−0.19		□ □ □		
	D/IA	2.60		1.90		2.18		↑ ↑ ↑		
Communication (+)	S/N	−0.25	=**0.001**	−0.16	<**0.001**	0.25	<**0.001**	□ □ □	F(6316)=1.133	0.343
	PR	0.11		0.29		−0.07		□ □ □		
	D/IA	2.37		2.10		1.58		↑ ↑ ↑		

(continued)

Table 5 (continued)

		$M_{SA}^{b,e}$	P^d	$M_{Pr}^{b,e}$	P^d	$M_{Te}^{b,e}$	P^d	Trends (SA, Pr, Ta)	F(*df*, error *df*)[c, g]	P^d
Coherence and cooperation (−)	S/N	−1.68	**0.004**	−1.30	0.056	−1.42	<**0.001**	↓ ↓ ↓	F(6316)= 0.608	0.724
	PR	−1.21		−1.55		−1.18		↓ ↓ ↓		
	D/IA	−2.25		−1.90		−1.86		↓ ↓ ↓		
Control (−)	S/N	−0.57	<**0.001**	−1.40	0.282	−1.38	**0.004**	□ ↓ ↓	F(6314)= 1.223	0.294
	PR	−0.45		−0.87		−0.79		□ □ □		
	D/IA	−2.32		−1.81		−1.77		↓ ↓ ↓		
Professional development (+)	S/N	1.04	<**0.001**	0.74	<**0.001**	0.40	<**0.001**	↑ □ □	F(6318)= 0.970	0.446
	PR	1.18		0.35		0.18		↑ □ □		
	D/IA	3.21		2.45		2.55		↑ ↑ ↑		

[a]Symbolic/Normative, Pragmatic Representations, Desired/Intentioned Actions
[b]values may range from '−4' (maximum disagreement) to '+4' (maximum agreement)
[c]Kruskal-Wallis H test
[d]Statistically significant differences in bold
[e]'SA' school advisors, 'Pr' Principals, 'Te' Teachers
[f](+) indicates that the phrasing of the item is on the positive, whilst (−) indicates that the phrasing is on the negative
[g]MANOVA

The aforementioned findings and the inter-role reported trends in Table 5 may also serve as an introduction to the inter-role statistical comparisons. It appears that there the identified qualitative topics follow similar patterns for all the protagonists for each topic-triplet. Nevertheless, bearing in mind the variety in the intensity of the trends, it is crucial to conduct inter-role comparisons, in order to identify whether statistically significant differences amongst the three roles-protagonists exist or not. The results of the inter-role analyses suggested that the three protagonists appeared to statistically significantly differ in their constructions with respect to only two topics: 'understanding' (didactics of mathematics) and 'quality of teaching' (general didactics).

In order to gain deeper understanding of both the intra-role and the inter-role findings, we introduce a mixed symbolic (numerical)-diagrammatical representation of the complexity of the views that the school advisors, the principals and the teachers hold about social media within the school unit system in the topic 'understanding' and in the topic 'quality of teaching' (see Fig. 4).

It is posited that the proposed representation allows for the coexistence of inter-role and intra-role comparisons, as well as for the combination of the *wholistic* advantages of a diagrammatic representation (which are in line with the systemic perspective of this study) with the *analytic* tools of a symbolic representation. For example, for the 'understanding' topic, a wholistic view of the triangular space of each protagonist is clearly skewed in comparison with the hypothetical 'neutral' towards the positive of the desired/intentioned actions axis. A closer look reveals that the school advisors' 'triangle' differs from the other two in that the symbolic/normative vertex is on the negative part of the axes ('within' the neutral triangle). Thus, it can be argued that all the protagonists are willing to incorporate SNS in the teaching of mathematics for the purpose of improving the students' understanding and they hold neutral views about what actually happens in schools. Nevertheless, the school advisors seem to hold stronger negative views about whether the official regulations promote such practices, with the other protagonists holding a very weak positive perception.

These results can be also analytically approached through the reported means and the statistical tests included in the Fig. 4 (the numbers are obtained from analyses reported in the respective 'understanding' and 'quality of teaching' in Table 5). Thus, this information may be especially useful for policy makers who need a tool to wholistically and analytically identify the perceived realities of the protagonists of the educational system.

Consequently, the conducted analyses suggest that all the protagonists seem to be willing to include the use of SNS in their teaching (mathematics and general). Considering that their SNS experience is relatively small (mainly Facebook and LinkedIn) and that both their pragmatic and their symbolic constructions are almost neutral, it is posited that the protagonists experience the multifaceted didactical potential for the SNS, but they don't experience the pragmatic ways of realising their use, nor do they experience a positive support from the official regulations. The latter negative view is stronger in some cases for the school advisors, which is especially interesting considering that the official regulations explicitly expect the school advisors to support the implementation of new technologies in education.

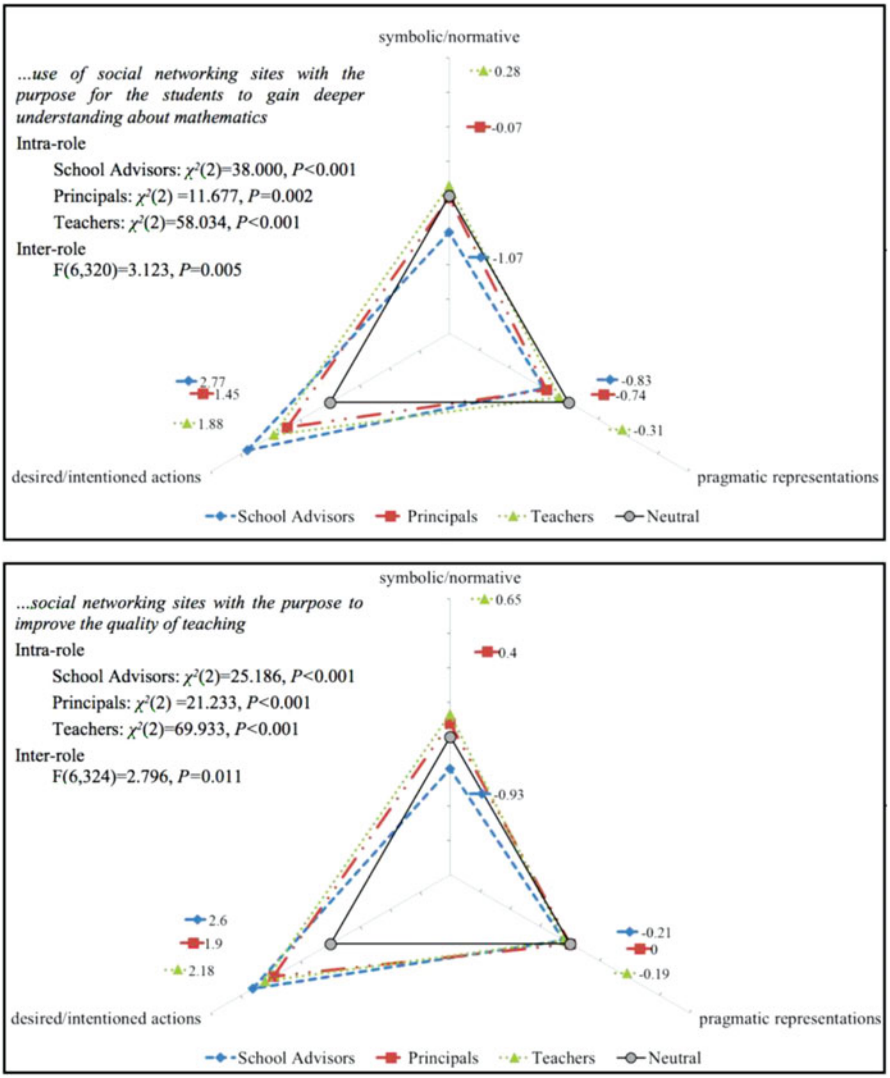

Fig. 4 Mathematics within the school unit system (both inter- and intra-role comparisons): 'understanding' (*above*) and 'quality of teaching' (*below*)

Discussion and Concluding Remarks

The aforementioned results revealed a complex educational reality that constitutes the emergent mathematics classroom with respect to social networking sites (SNS). First, in terms of SNS and broader internet experience and perception, the protagonists appear to be familiar with the 'new' technologies. They consider themselves relatively experienced with computers and internet. They use SNS for both school and personal purposes. Facebook appears to be the SNS that dominates their SNS accounts. Nevertheless, in contrast with the teachers and the principals, the school advisors seem to have Facebook accounts, but the accounts they actually use are their LinkedIn and Academia accounts. Thus, considering the professional-academic status of these SNS and the more personal-informal status of Facebook, is may be inferred that there is a clear qualitative difference in the role of SNS for the three protagonists: the school advisors see SNS as professional tool, whilst the teachers and the principals consider them to belong more to the informal, presumably indirectly linked with (or even disjointed from) the school reality. The perceived connected reality concludes with the educators' perceptions about the students' use of SNS. Their perceptions appear not to be in line with the reality identified by relevant research: all three protagonists seem to considerably underestimate the infiltration of SNS in the students' lives. This also has qualitative aspects, as the educators appear not to have accounts in (or even to know about) SNS that appear to be important for the students (such as Instagram, which is reported to be a close second to Facebook choice for the young students). Consequently, the educators seem to know about SNS, they use SNS for personal and professional purposes (though the school advisors seem to prefer more 'professional 'SNS), but they don't appear to perceive their reality as shared with their students. Thus, it is posited that communicating this common (though not shared) reality to all the mathematics classroom protagonists (including students, educators, parents) is a crucial first step with a pragmatic reference, in order for the emergent connected classroom to be perceived as such.

Considering the system of disciplines, it seems that mathematics is considered by all three protagonists to hold a special place in comparison with other disciplines: both in everyday life (by perceived as being relatively more important, more useful and not more disjointed from the natural world) and in reasoning (by promoting logic to a greater degree). At the same time, these views coexist with the perception that mathematics more than other disciplines requires systematic teaching in order to be learned. Thus, all three protagonists seem to share a special with a pragmatic reference view of mathematics, which nevertheless may require systematics teaching in order to become part of the learners' reality. The learning paths of mathematics appear to be special with pragmatic comparative benefits, requiring appropriate teaching bridges to be walked.

In which ways (if any) the aforementioned SNS realities and epistemic views about mathematics seem to be linked with SNS and mathematics as a school course? We investigated the three protagonists' views with respect to three foci (symbolic/normative, pragmatic representations, desired/intentioned actions) and to both didactics of mathematics and general didactics (in order to identify intra-didactics differences). First, the results of the conducted analyses revealed that the participants of the study are willing to incorporate SNS in the school teaching for the didactical gains special to mathematics and broader didactical gains. This is contrasted with the negative to neutral perception of a symbolic/normative and pragmatic SNS reality. This contrast in almost all cases was found to be statistically significant except for three topics: 'democratic access' (school advisors), 'control' (principals), 'coherence and cooperation' (principals). In these cases, the desired/intentioned actions were found to being line with the both the symbolic/normative and the pragmatic representations. Bearing in mind the negative phrasing of these triplets, it appears that in the abovementioned instances the participants' view that this topic is not relevant with the employment SNS in the school teaching (mathematics or general) in the sense that they don't view that these aspects are one of the reasons that SNS are not required to be employed, or that SNS are not actually employed, or even that SNS would be employed.

Notwithstanding these exceptions, the general trend appears to be relatively clear for all protagonists: positive stance towards to the hypothetical inclusion of SNS in the school classroom (even more positive for the school advisors) and negative to neutral constructions about the school reality and the official regulations reality. The educators' positive stance is in accordance with the students' positive stance as reported in a recent study with primary school students (Stamatis 2013). On the other hand, the negative-neutral constructions about the official regulations appear to be interesting especially for the school advisors since the official descriptions about their responsibilities explicitly mention the support of educational innovations and ICT. Could it be that SNS are not interpreted by the school advisors as being within the official meaning space of "educational innovations and ICT"? Or that broader normative/symbolic restrictions favouring the protection of the students' personal data may implicitly 'override' the identified constructions?

Moreover, the identified trends appeared to be similar about both the didactics of mathematics and the general didactics. This indicates an inconsistency with the aforementioned epistemic views about mathematics as being special in comparison with other disciplines.

Following these, considering the discerned learning paths about mathematics as a school course, it appears that the teaching bridges should concentrate in communicating amongst the school protagonists (including the educators and the students) that, on the one hand, they hold similar perspective about SNS and, on the other, that these perspectives are not in contrast with the existing official regulations. Moreover, the teaching bridges should attempt to make inter-systemic links, in the sense of utilising the comparative positive epistemic views about the pragmatic everyday benefits of mathematics as a discipline and the necessity of its systematic teaching, with the intentions of incorporating SNS in everyday teaching. Towards this direction, the growing body of mathematics education research should be a crucial element of these bridges, providing the scientific rationale and scientific ways of including SNS in everyday teaching practices (compatible with the protection of personal data). It is posited that the emergent classroom and the diverse learning paths exist in a technological-methodological-pedagogical environment and the attempted teaching bridges should acknowledge this complexity.

In accordance with this discussion, the introduced instrument appears to constitute a multidimensional valuable tool. It helps in identifying the learning paths space and in efficiently and comprehensively communicating (through a hybrid diagrammatic-symbolic representation) the constructions held by various protagonists. By superimposing the representations views spaces of the protagonists, we argue that the convergences and the divergences are becoming transparent to the researcher, the educator, the policy maker. At the same time, the space that is amenable to change as well as the engineering required in order for this change to be accomplished may become clearer. For example, in Fig. 4, the stronger negative symbolic/normative views of the school advisors may be the first engineering step: the policy makers' communicating to the school advisors that the SNS are within the meaning space of the official regulations and that the mathematics educations research suggests ways of accomplishing this. In this way, a domino positive effect may result to a new equilibrium of the school system in which the emergent mathematics classroom is connected, incorporating the tendency to be open to the inclusivity of the emergent diverse learning paths.

In conclusion, the inter-systemic, tri-focussed approach adopted in this study helped in more validly identifying the views and practices of the Greek teachers, principals and school advisors. Though all three protagonists wish to incorporate social media in their teaching, their views of the regulations and of the school reality are not in line with their intended actions. Moreover, they appear to consider that the students are not as engaged with such networks as the teachers think. Though in this study we partially investigated the considered system and its environment, we argue that this methodological choice enabled our collecting quantitative data with a qualitative-like complexity, thus combining the pragmatic advantages of a questionnaire with the validity benefits of a qualitative approach. Moreover, the proposed research framework is paired with a representational tool that allows the comprehensive and efficient communication of the results of the conducted analyses. The proposed research framework draws upon the diverse points of view as means for accessing the existing complexity, rather than as obstacles for the understanding of the transformation of both school teaching practices and mathematics education research. Diverse learning paths are viewed as teaching bridges opportunities that further weave the collective web of knowledge of the mathematics classroom, situated within the deep and fundamental connections of mathematics with the open system of the globalised virtual social networks.

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e-Collaborative Forums as Mediators When Solving Algebraic Problems

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Abstract In this chapter, we analyze Students when solving classic algebraic problems in a collaborative way, by using electronic forums to see the power of such a tool in a learning process. We used a task as an example to show the interactions appearing when using electronic forums as mediators on the reflective process of co-constructing algebraic ideas. It is found that the highest profile students not only participate actively in the task but they introduce more mathematical meaningful issues. Qualitative analysis shows that generalization methods are close to which it is regularly presented in face-to-face classrooms but reflection spontaneously emerges as a need for revealing the importance of exchanging the representations.

Keywords E-forums • Algebra • Problems • Collaborative learning • Co-construction

Introduction

According to Coll (2007) various studies show that the effective capacity of ICT to transform the dynamics of work of teachers and students in schools and the processes of teaching and learning in the classroom is, in general, far below the transformer and innovative potential, usually attributed to them. In this chapter, we assume that collaborative tools can act as powerful mediators to overcome numerical traditional strategies for algebraic problem solving. New approaches to algebra are related to the use of generalized properties by using technologies, but we introduce a reflection about the cooperative interactive perspective.

Kieran and Filloy (1989) confirm the absence of algebraic methods as a common denominator in the responses of the students to the resolution of verbal algebraic

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problems, and they point out the difficulties implicit in the syntactic algebraic methods as a possible cause of it. This interpretation leads teachers to the use of methods focused on the development of techniques in the teaching and learning of algebra, paying scant relevance to the speech of the students. It seems difficult to place in the foreground the interactive and reflective nature of mathematics education, in which the importance is attached to the content that is learned, but also the manner in which the students learn.

In such a framework, our main hypothesis is that the forum of conversation, as a Computer-mediated Communication (CMC from now), can turn out to be a useful instrument to solve, reflect and discuss problems jointly. This research assumes the hypothesis that in computer-supported collaborative learning contexts, all participants are potential sources of educational influence (The challenge is for teachers to create learning environments that help students to make the transition from basic to high-order skills. In this new scenario, the focus is not on teachers but the process of how the students learn. In such a framework, the use of ICT in first Secondary Schools will entail the gradual disappearance of the limitations of space and time, which will result in a transition toward a usual student-centered model based on cooperative work.

Recent research focused to analyze the impact of information on the evolution of activity across the forums, and the relationship with the value that participants assigned to the forums and the usefulness of the forums in the learning process (Coll et al. 2015). Online discussion forums are web-based communities that allow users to share ideas, post problems, comment on posts by other users and obtain feedback. Previous research experiences use forums to promote debate and thinking in prospective teachers learning (Bairral and Giménez 2004; Bairral and Powell 2013), but less use is devoted to a common problem solving activity with young students (Murillo and Marcos 2011). Though students are more and more confident in their technical abilities for online communication, we know that their online experiences do not generally require them to use dialogue as a way to explore, expand, and drill down into problem solving issues significantly (Jonassen 2002).

In this chapter, our aim is to present how a group of eighteen students of Junior High School (13–14 years old) interact when they solve a classic algebraic problem in a collaborative ICT environment. By analyzing such a task, we explain some benefits and possibilities of an online algebraic problem solving teaching issues, focusing on the implications of forum conversations as a mediator for reflective interactions. The activity leads to the development of generalization and symbolism processes. The “listening” of the ideas and reasoning of the students allows the planning of activities in order to promote individual learning and the learning of the group. It also contributes to claim for making more observable and interactive the space of the communication in the borders among inside and outside the classroom.

The condition of transformation agents assigned to the ICT is worth to be taken into account for conceiving deliberate interventions to change the pedagogical models, the practices in the classroom, and the curricular contents in educative systems in order to lead the students towards a significant and satisfactory learning (Rojano 2003, p. 138). We tried to show a real challenge to introduce dynamic communicative interaction of student/student and student /teacher. The teacher acts as the designer and manager, allowing students to explore, to construct and to express their learning.

Conversation and Productive Meanings

We assume that running online discussion forums enable the application of constructivist learning issues for problem solving (Jonassen 2002, p. 7) and we also understand that a collaborative script is essentially a set of instructions regarding how students should form groups, interact with one another, collaborate and use the resources made available by the environment to tackle the learning tasks in a collaborative way, then the key lies in the point of reference that is chosen when formulating these instructions, such that they fulfill their purpose as effectively as possible (Engel et al. 2013). In math class, "the individual construction of meanings takes place in interaction with the culture of the class while at the same time contributes to the constitution of this culture." (Cobb and Bauersfeld 1995, p. 9, cited in Sierpinska and Lerman 1996, p. 15).

The conversation is one of the most important means of communication in all over the world. The writing allows the evolution from the natural language used by the students to numerical and symbolic expressions corresponding to the mathematical language. In this way, the mathematical learning connects with the emotional perceptions and experiences of the partners. In our assumption, teachers have to guide the pedagogical setting towards situations in which relevant aspects are discussed, such as posing questions related to the critical analysis of contexts or the necessity for the generation of new and useful information to promote attention (Ainley and Luntley 2007).

On-line forums and blogs have also been recognized as fertile ground for meaning product discussion in Mathematics Education. Some recent researches analyze forum discourses with preservice teachers or in training teachers courses (Bairral and Giménez 2004), and also collaborative problem solving activities with future teachers (Bairral and Powell 2013). But a few researches focus on what kind of behavior and strategies appear when young students solve algebraic problems in an asynchronous way.

Generally speaking, the use of the Internet in a student-centered model has a great deal of potential strengths: (1) computer networking facilitates the implementation of cooperative learning overcoming the relation human-media (Borba and Vilareal 2005); (2) promotes articulated communication by compelling the students them to state their needs in a concise and highly articulate way (Royo and Giménez 2008); (3) Asynchronous web-based forums give students time to reflect on issues before they add their own contribution (Bairral and Powell 2013); (4) accommodate the potential for e-tutors and e-learners to engage in continuing tutorials, rich in dialogues and reflections, and generate processes of meaning construction and knowledge advancement (Rowntree 1997); (5) gives specific opportunities for co-construction of learning (Royo 2012) In particular, using forums allows students to become the center of their own learning (Jonassen 2002). In fact, the aim is to facilitate the conversation between the students and between the students and the professor; encourage the talk about mathematics, using the natural language to express mathematical ideas, discovering progressively the usefulness of the application and use of mathematical language. The oral conversation enjoys some irreplaceable conditions in this educational level. Nevertheless, the electronic forums allow increase the participation and communication of everybody, and facilitates teacher guidance through reading and observing the thoughts and beliefs of the students.

Participation in our analysis of reflective interactions should be considered as something that improve or restrict mathematics development (Cobb et al. 1997, p. 272). When analyzing interactions we want to understand if the media facilitate mathematical discourse and scaffolding by providing direct instruction (Anderson 2004). We assume some previous research results about the use of electronic forums with geometry problems (Murillo and Marcos 2011) that the use of the forums of conversation in a digital environment, used to jointly investigate algebraic problem-solving strategies, create favorable conditions (a) so that the process of problem solving promote reflection and communication of ideas among the students; (b) for influencing changes affecting the teaching role and the relations that are set out in the classroom; (c) individual growing and collective development of objects and processes in the topic. Even e-activities based on group work must be properly structured to avoid the free rider effect (Jonassen 2002), we decide to use the forums in a completely free way, assuming that students know about using collaborative problem solving from previous experiences.

For analyzing the educational influence of interactions on electronic environments we consider two dimensions: the academic task management and the management of the meanings. It is considered the construct called educational profile, which relates quantitative contributions data by accessibility, participation and connectivity criteria (Coll et al. 2013). We also assume that content analysis helps to categorize students' interventions in algebraic settings. Many cognitive results are not in this paper, but fully described and justified in Royo (2012). It was also considered that the inscriptions of individuals working online in a small-group or team provide observers, who must interpret meanings constituted in the contributions, as evidences of individual and collective thinking (Bairral and Powell 2013).

The Role of Designing Processes

The task presented in this paper, belongs to a part of a wider research in which several problem solving tasks were conducted, analyzed and redesigned through the application of a Design-Based Research methodology (Gravemeijer 2002; Royo 2012) applied in order to improve the procedures for teaching and learning algebra. For the design of the learning environment (Murillo and Marcos 2011), we used the

Fig. 1 Presentation page (*left*) and contributions of the forum (*right*)

Moodle-platform provided by the School. This paper analyzes just a simple problem, and correspondent conversation forums on the virtual environment Moodle, which was new, both for teacher and students at that moment (see Fig. 1). The experiment presented in this paper was developed during 2 months, the year 2008/09 and repeated in 2010 and 2011.

In order to see an example of the forums, we focus on the classical following problem: ***"Given n dots, how many segments do we need to unite them in pairs?"*** We choose this problem, because is the first one in the global project, and means a classic in the work of generalization [in the resolution, we expect students arrive to understand the meaning of $n*(n-1)/2$]. Wiki spaces were also used to introduce more final reflections, not analyzed here.

Such a problem allows a wide variety of representations and the development of inductive processes, starting with particular cases. Oral conversation occurred in the classroom at the same time that they did contributions to the forums. In addition to using computers, students had paper and other material written or manipulative aids as instruments of work to look for strategies for resolution of the problems. The use of the forums in situation of non-attendance and outside the assigned hours (from home, library...) was optional.

The participants communicated individually among themselves simulating to be at home in a computer room, and combining it with group of four discussions in the regular face-to-face classroom. At the end of the intervention period in forums, students carried out a written test and answered a survey for the evaluation of the use of electronic conversation forums. The collected data for our study is constituted by registered dialogues on the forum, and also audiovisual records of some moments of the session; direct records of the diaries of the students in the Moodle platform; record direct from the journal of the teacher in the Moodle platform.

We have analyzed the dialogues of the forums by applying descriptive methods to its development during the sessions, including aspects of activities that have had impact on them. We think that it should lead the mathematical reasoning through observation of individual cases, guess, check and argumentation, since thus prepares the task of orientation of the process of generalization, one of the main ways of introduction of the algebra. To follow our aims, student's interactions had been analyzed by using educational profile using e-accessibility and e- connectivity categories. Content analysis is also presented next by identifying algebraic contributions in the task, and quantifying the use of problem solving strategies in terms of applying and answering, grounding, or interpreting other's contributions.

The Interactional Situation

To see the results of our first structural interaction analysis of contributions within the DBR process, we selected and adapted a set of e-communication indicators (following Coll et al. 2009).

Expected profile considers the following categories: (a) accessibility, by computing individual average of entries being >1 (**MDIE)** and individual entries $\geq 1/n$ (**IIE)**; (b) participation by observing individual contributions in front of readings being $\leq 0{,}5$ (**IIRCL)** and individual contributions in the total being $\geq 1/n$ (**IIC)** ; (c) connectivity by seeing number of messages received out of the whole $\geq 1/n$ (**IIMR)**, and number of messages received over sent being almost 1 (**IIR).** In the Table 1, we see the results of couples or individuals found to each of the column categories just above explained.

We also separate in four parts according how many criteria are satisfied. It's also considered collective accessibility being everyday average of entries $>n$; tax of contributions being $\leq 0{,}5$ not written in the table. The teacher is also included in the first group.

We found similar tables for all the different tasks to confirm that the profile of interactions is not always the same.

About the Content

To describe methodologically the type and amount of contributions, we used three different categories: (S) social, (D) dynamics posing questions, and (E) explanations. For the mathematical strategies used, we considered the following categories (according Mason et al. 1985): (V) Students verbally explicit relationships between the data of the problem. (PA) Arithmetic procedures are used to express relationships between the data of the problem. (LlS) Symbolic language is used to express relationships. (RP) Records a pattern or regularity, preferably using symbolic language in which formula appears in accordance with symbolic expressions, including ways to iterative and recursive procedures. The category (PVF) is used when students prove the validity of the formulas used. Such categories are used in Table 1, and also when they appear during the dialogues.

Table 1 Profile and interaction characteristics found in the problem analyzed

	N Dot problem (individual indicators)					
Participants	Access		Participation		Connectivity	
Students + Teacher	MDIE (>1?)	IIE (≥1/n)	IIRCL (≤ 0,5)	IIC (≥1/n)	IIMR (≥1/n)	IIR (≈1)
St 1_E14	**20**	**0,23**	**0,17**	**0,21**	**0,26**	**0,83**
St 6_St 9	**6**	**0,07**	**0,24**	**0,09**	**0,13**	**1,00**
St 8	**12,6**	**0,15**	**0,11**	**0,09**	**0,12**	**0,90**
Teacher	**5,57**	**0,07**	**0,21**	**0,07**	**0,13**	**1,25**
Profile A satisfying all the criteria with expected indicators (highest profile)						
St 5	**8**	**0,09**	**0,11**	0,05	**0,08**	**1,00**
Profile B satisfying five criteria						
St 11	**5,71**	**0,07**	**0,20**	**0,07**	0,01	0,13
Profile C satisfying four criteria						
St 2	**5,86**	**0,07**	**0,12**	0,04	0,01	0,20
St 12	**1,71**	0,02	**0,33**	0,03	**0,05**	**1,00**
St 15	**4,86**	**0,06**	**0,09**	0,03	0,03	0,67
St 16	**6,43**	**0,08**	**0,13**	0,05	0,01	0,17
Profile D satisfying three criteria						

In Table 2, we present part of the data research table in this task of n dots we analyze here. It is possible to see the categories according the type of strategies above explained and codified as we see in the dialogue. We also see the contributions not related to algebra but including possible clarification issues (a); and related to algebra over a colleague contribution (b).

We have classified the content contributions into groups of students depending on whether its focus is the algebraic content or the relationship with the other participants (see Table 1, above), and we see that it changes according tasks (Royo 2012). We found that in the electronic forums, students passed through the access/motivation and online socialization stages to the information algebraic exchange stage in a few days during the analyzed forum as we can see next when describing the interactions. An interaction scheme shows three main nodes for this task, but a lot of common interventions (explained in detail in Royo 2012, p. 212). The profile A means the highest level of interaction, and F is the lowest.

After doing all codifications of students' writings, when some strategies and categories appeared, some similar tables were constructed to identify what happens in each task.

Finally, to see the influence of educational profile on the final results, we observe how the students in highest profiles evolve in some algebraic meanings. Analyzing the results from all the tables associated to the different tasks and final test, we observed that the categories corresponding to the higher level of algebraic content are related to the contributions of higher educative profile students. Just the Student 16 is different, perhaps his comments are not enough understood by the colleagues.

Table 2 Part of classification of content contributions in the task, according to different students and profiles

Problem "n dots"		Type contributions								Total	a	b
Participants	Profile	S	D	V	PA	E	LIS	RP	PVF			
St 1_St 14	A		4	1		9	1	4	1	**24**	7	17
St 2	C	2	1			1	1			**5**	1	4
St 3	F	1	2					1		**4**	1	3
St 4	E	6	3		1					**10**	1	9
St 5	B	2				2	1		1	**6**	2	4
Teacher	A		2		1	5				**8**	1	7
Total group		**34**	**28**	**1**	**6**	**26**	**9**	**9**	**3**	**115**	28	87

Qualitative Findings

Next, we describe some dialogues of this task, to see how the dialogues introduce problem solving issues among interactions and how they were codified and associated to each contributor. In such an explanation, we focus on how the students use formal language for describing the solution.

In their first interventions, the students showed that teacher guidance was still desirable.

Student 1 Pili, which number is n???
Student 2 We can unite nothing because we don't know how many dots we have.
Teacher Well, maybe we can start with a few dots... How many dots you want to start? (AssumeV)

The second part of the Forum starts from a contribution in which St 6 and St 9 vary referential perspective to propose a new strategy to find the sum of n consecutive natural numbers (PA). The teacher proposal offered some security when attempting to begin to draw diagrams with some groups of dots. Therefore, she decided to make a new proposal:

Teacher You have been drawing dots. You have tried with 2 dots, with 8 dots... Is it possible that you make a table with the results?

Then, many students follow the suggestion, and ordered the data in a table (PA). Others simply accept. This action and segments facilitated the passage from the particularization to the generalization, a process that each student understands as a compulsory proposal. As usually in face-to-face classrooms, the development of tables gave rise to the emergence of recursion strategies.

Dots	Segments
1	*0*
2	*1*
3	*3*
4	*6*
5	*10*
6	*15*

Student 3 says that "I have found this: the difference between 1 and 2 is one. Between 2 and 3, it is 2. Between 3 and 4, it is 3. And so on… *a* is the number of segments a = n – 1 + a of previous number." (PA)

This kind of contributions gave rise to the expression of diverse ideas, request for clarification and comments. It is interesting to observe the last expression of the written contribution above ("a = n − 1 + a, of the previous number"), which are not valued as wrong but as a step in the use of algebraic language (LIS). The teacher-researcher wrote in her research diary some comments on content analysis, using different categories: "They have decreased the contributions posedquestions (D). They have increasedthe socialcontributions (S) and explanations(E). In termsof strategies, they appear for the first time contributions of registration pattern (RP) and evidenceof the validity of formulas (PAV). Communication of cognitive elements appearing in some interventions mixed with emotional and affective elements. As an example, St.13 participates for the first time, after having refused initially to use the Forum. The teacher had been talking with him individually after the first session".

Finally, the students also commented in their written forum reports that they thought the forum allowing active processing, mediated contribution, and a better understanding and control of the problem solving process. In order to facilitate the communication of these expressions, the teacher suggested an oral discussion and the use of the whiteboard to represent the situation jointly developed (see Fig. 2).

At the end of the oral discussion, in the forums, new representations appeared: The input to the forum continued. The students kept their communication by requesting and offering explanationsand help, or exchanging their findings:

Student 6 [to Student 7] Is it possible that I understand a thing that you don't? ¬ ¬"The question is to catch any number, which will be "n", then you must split between 2 multiply by the result of n minus 1. (LIS+ RP)

Student 8 Albert is right. I checked it out. The formula is n:2 (n−1) (LIS+RP) From here, a discussion arose about the equivalence between the expressions **"(n−1), n:2" and "n+(n−1)/2 * n".**

Student 4 We can do both things and it goes well. (Example of agreement)

Some students proposed the use of known resources:

Student 9 We can take the geoboard and go testing with dots and segments http://nlvm.usu.edu/en/nav/frames_asid_279_g_4_t_3.html?open=activities&hidepanel=true&from=vlibrary.html (non cognitive comment)

Some other students sought convincing explanations to the formula. Although this explanation had already been found a week earlier, the attribution and appropriation of meaning has to be performed individually, and each student needs to perform an individual process:

Student 10 Why the formula works: It is because each dot matches with another one, but not with itself. So, you have to subtract 1 from the starting number: ·(n−1). The result is divided by 2 because the dots are listed only once. (RP)

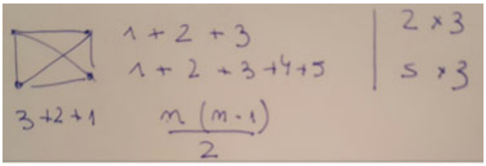

Fig. 2 From the particular cases to the generalization

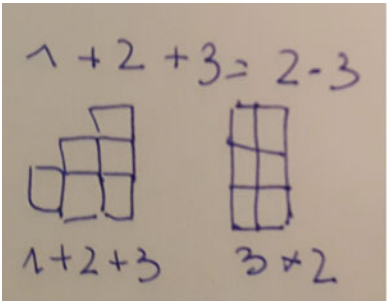

Fig. 3 Representing 1+2+3 as a drawing

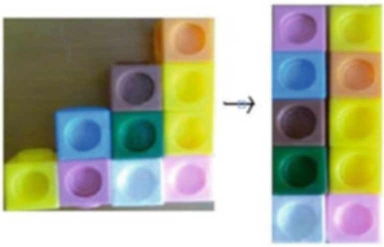

Fig. 4 Representing 1+2+3+4 as a rectangle and as a half of a square using manipulative material

Meanwhile, new representations emerged:

Student 11 Me and [Alumn.12] have discovered: When n= 5 dots: (5−1) + (5−2) + (5−3) + (5−4) = 4 +3 +2 + 1= 10 segments (we are still investigating)

This latest contribution led to the introduction of manipulative material (polycubes) to encourage representations (see Figs. 3 and 4) and oral discussion about them:

Teacher Which relationship do you see in 1+2+3+4+5 in this new context (polycubes)?
[Students It's the same].
[Teacher Can you find a formula for it?
[Students Yes, It is the sum of the "squares"] (PA)

After the work with material and some discussion, the students returned to the forum. New representations and relationships appeared, as we can observe by Student 13 comment.

Student 13 Another strategy: "5 dots and n=4: 1+2+3+4 = (1+4) + (2+3)= 5+5= 10, and that's equal to: (1 +4) • 4/2= 5 •2= 10 (1+n) ·n/2" (LIS)

Interactions and Technological Tools

The value of technological tools such as virtual learning environments is not to replace the role of the teacher, but enhance the distributed teaching presence, creating a context that promotes the understanding and development of growing significant algebraic knowledge. In our study we introduced a theoretical tool called "educational profile influence" that served to explain how evolve the interactions in each task as it is in the example. Content categories (as we see in our example in Table 2) also were helpful to analyze if the interactions are focused on certain aspects of problem solving activity. Such tools provide the possibility to understand how the interactions relate the educational profile with algebraic content issues and strategies. The use of different kind of semiotic tools is also evident. The students need to relate sentences in which not only regular language is used, but manipulative materials, and algebraic reasoning. For relating a generalization mode, they seem to need such multisemiotic tools (Albano and Ferrari 2013).

If we look not only the example, but the global amount of results and problems, we observe that electronic forums act as agents of change that affected the teaching role and in the relationships and interactions established in the classroom. In particular, highest profile students correspond to the highest problem solving contributions. We also found that in almost all the tasks, electronic forums enable individual and collective construction of objects and processes in the learning of algebra (Royo 2012), and improve generalization attitudes, similar to face-to-face conversations. Interlocution interactions yield different outcomes and influence the development of mathematical ideas and reasoning in diverse ways. Its use allows: (a) to facilitate guided construction of objects and processes in the learning of algebra (as generalized properties or inquiry methods), encouraging ideas partially developed and without rushing off to get results; (b) to promote the cognitive and linguistic capacities of the students, encouraging them to reflect on what they learn and to express what they know; (c) to develop the communication of mathematical ideas. (d) to develop an increasing use of formal language from problematic situations. (e) to stimulate the ability to share and compare ideas; (f) to stimulate joint construction of meanings; (f) to set the thread around which other activities are developed that also become part of the teaching and learning process. Some of these results are consistent to which were considered in other geometrical problem solving e-studies (Murillo and Marcos 2011).

It can be expected that all students participate in forums significantly, although as with all educational action/profile, some cases require guidance or teaching intervention to adjust the conditions of participation. In didactic programming it may be convenient to include participation in forums such as evaluation activity.

The use of the forums in the joint resolution of algebraic problems facilitated reflection during time, access to different points of view or contributions, review of what you try to communicate, raise questions and request or offer help or clarification.

Qualitative analysis shows that generalization methods are close to which it's regularly presented in face-to-face classrooms and spontaneously emerges a need for revealing the importance of exchanging the representations not always usual in face-to-face classrooms. These features have also been evaluated positively by the students and the teacher. In addition, the students read messages from other participants even when they have not spoken with contributions. At the end of the forum, it is important to provide a new step to be seen to be appreciated transfer programming made.

Final Reflections

Cognitive and content analysis was not fully explained in this paper, but we observed that almost all students have participated in the forums in a significant way introducing algebraic contributions (Royo 2012). Although it is not usual, if any student is not involved in a forum with interventions, you would expect that their participation consist of reading the contributions of others. When it doesn't, there are external causes that should be considered (absence. . .). Or maybe there are grounds which require teacher intervention. We have seen that at the start of the forums the largest number of contributions raise questions or request clarification, but later on, there are more and more algebraic features present during the conversation.

Interactional data corroborate the findings of authors who point out that these instruments allow reflection during the time necessary, giving opportunities to have access to different points of view or contributions of the members of the group, or review what you try to communicate before sending it. They combine features of spoken and written discourse which can facilitate collective learning (Murillo and Marcos 2011).

Comparing the results presented here with the global research study, we have seen that with 13–14-years old students, the use of electronic forums has been compatible and effective when combined with face-to-face classroom methods. This leads to question the asynchronous use that is usually given to these tools. It makes emerging features of the electronic speeches while synchronous (DES) and asynchronous (DEA) which are combined in a peculiar way offering features that should be considered: If treated properly in the school context, they allow overcoming some difficult or limiting aspects of one and other speeches when performed separately, as it was observed in other school subjects.

According to global data for all the problems during the broader study (Royo 2012), the number of contributions with a focus on the algebraic content is significantly less than the contributions with a focus on the relationship with other participants. However, the ratio varies significantly depending on the problem that it was discussed in the Forum. We found enough evidences to tell that the interlocutions observed support the reflective development of the participants' mathematical ideas and reasoning (Bairral and Powell 2013) sharing multiple unexpected representations as manipulatives (not related to web environment, and usually presented inside the task). Because of lack of space, we don't explain here the results of the achieved scaffolding process built in the environment.

The results indicated that individual information had a significant impact on participation in subsequent forums. This study also present an example contributing to describe some values of using technological resources for analyzing the role of interactions in teaching practice for teacher training preparation of future Secondary Mathematics teachers allows promoting linguistic capabilities of the students and improving communication of mathematical ideas. Such benefits are strongly linked to mathematics features such as multisemioticity and multivariety that are well supported by technological tools when suitably didactically planned (Albano and Ferrari 2013).

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Part IV
Communication and Information: B. Information's Tools, to Inform Oneself and to Inform Others

Problems Promoting the Devolution of the Process of Mathematisation: An Example in Number Theory and a Realistic Fiction

Gilles Aldon, Viviane Durand-Guerrier, and Benoit Ray

Abstract Modelling is a complex and difficult process when studying phenomena from real-life situations. However, it is an important part of the work of mathematicians that needs to be addressed in the teaching of mathematics. We have studied two different types of problems, the first within mathematics and the second in the form of "realistic fictions", that is to say, a problem inserted into a fictional context allowing a modelling process. We hypothesize that the solving of both of these types of problem requires constructing meta-mathematical skills as well as using mathematical knowledge. In both cases, we stress the different roles of technology as a medium of communication and in providing a dynamic environment.

Keywords Problem solving • Modelling • Realistic fictions • Dynamic

Introduction

For many years, two French research teams (DREAM[2] and Resco[3]) have been working together in Lyon and Montpellier on the introduction of research problems in the math class. This work draws on the overall work developed around "open problems" in teaching and learning that have been studied for at least 30 years in Lyon (Aldon et al. 2010; Arsac et al. 1988; Arsac and Mante 2007) and also on research developed around the experimental dimension of mathematics (Dias and Durand-Guerrier 2005). The collaborative dimension based on exchanges between

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classes has been developed in Montpellier, where classes work on a problem posed in a non-mathematical form and exchange questions, ideas, procedures and conjectures via a platform. Following numerous didactic works conducted on problem solving, both in France and abroad, among them Polya (1945); Peix and Tisseron (1998) and Schoenfeld (1994), the French Board of Education highly recommended including problem solving in the mathematical learning process, as the heart of mathematical activity. Problems arising from other disciplinary areas or from everyday life are promoted in order to help students make sense of the mathematics under study. In this context, and taking note of the difficulty for teachers to implement such situations in the classroom, we have developed didactic situations (Brousseau 1998) and have offered training to support teachers in setting up classroom problem-solving activities (Aldon and Durand-Guerrier 2009). One of the challenges we want to face is to provide a real work of modelling, that is to say, the devolution of the choice of mathematical tools that can be used in the process of problem solving. This modelling work can be "horizontal", that is to say within mathematics where students have the choice and the responsibility of the mathematical knowledge they want to use, or "vertical" when the first phase of modelling comes from the mathematisation of a non-mathematical situation. In both cases we hypothesized that students would work not only on heuristics but also on the mathematical knowledge relevant to the mathematical situations. Two different situations were studied: firstly, situations where questioning is internal to mathematics (Gardes 2013) and secondly, situations stated in a non-mathematical form, leaving the responsibility of modelling to students prior to the phase of resolution. In this chapter, we will discuss this mathematisation process in two situations: the first one comes from the EXPRIME cd (Aldon et al. 2010) whereas the second one is an example of realistic fiction that has been developed within the framework of collaborative research (Ray 2013). In both cases, the experimental part of mathematics plays an important role in the construction of mathematical knowledge, and technology is used in different perspectives: as a tool facilitating experience and allowing patterns to be discovered and also as a means of communication and exchange. The aim of this chapter is to demonstrate how various aspects of ICT can facilitate the mediation of problems in the teaching and learning of mathematics. More precisely, in the first example we will show how new representations of mathematical objects using technology can provide different ways of experimentation; while in the second example, we will stress the communication properties of technology both in the devolution of didactic situations and in the emergence of communities around the process of collaborative problem solving.

The Experimental Dimension of Mathematics

As Polya (1945) noticed:

> Studying the methods of solving problems, we perceive another face of mathematics. Yes, mathematics has two faces; it is the rigorous science of Euclid, but it is also something else. Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears to be an experimental, inductive science. (p. VII)

The experimental face of mathematics that Polya quoted must be clarified. The notion of experiment can be viewed both in the field of philosophy of sciences and the philosophy of knowledge. The subjectivity of sense perception is an obstacle to the interpretation of experiments in all sciences and particularly in mathematics, due to the fact that experiments are carried out on representations of mathematical objects and not on the objects themselves. The two examples shown in Figs. 1 and 2

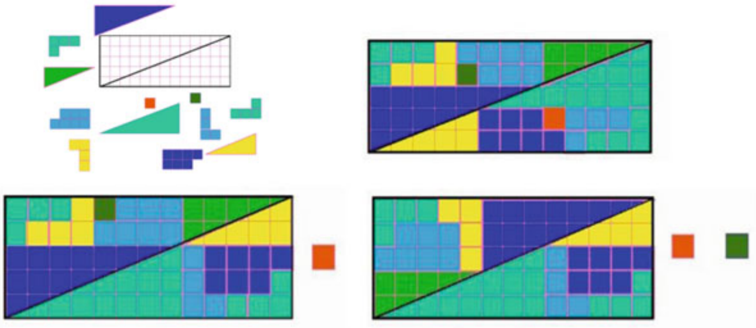

Fig. 1 (**a–d**) A puzzle from http://irem-fpb.univ-lyon1.fr/feuillesprobleme/feuille4/enonces/Deuxcarres/indexcarres.html (accessed 27 Dec 2015)

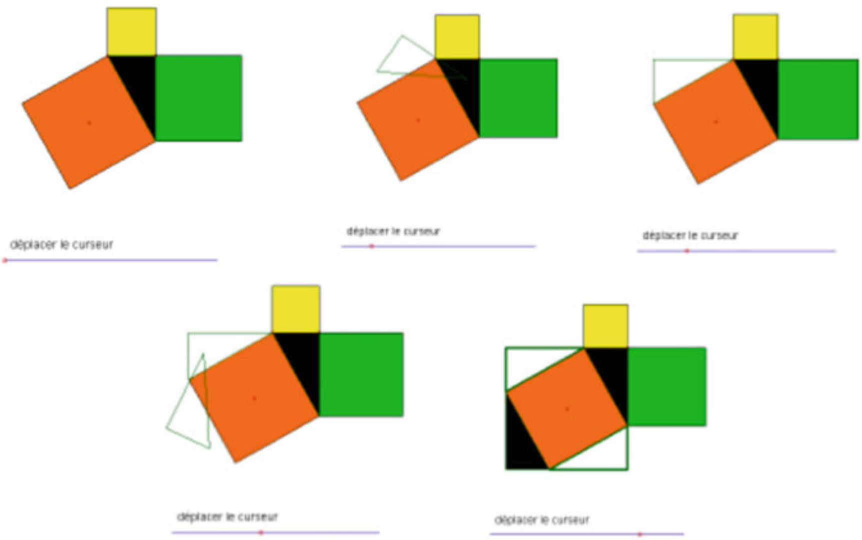

Fig. 2 Proof of a Pythagorean theorem

illustrate this. In the first example, the puzzle is not rigorously completed in Fig. 1c, d but visual perception does not enable us to see the holes. The pieces are moved freely and the animation does not give any clues to solving the apparent paradox. In the second example, the animation becomes clear proof of the Pythagorean theorem if, and only if, it is possible to link the visual perception to the mathematical properties of objects, say rotations and their property of conservation of distance and area and the algebraic identity $(a + b)^2 = a^2 + b^2 + 2ab$.

Without going into an exhaustive description of the role of experience in science, the links between theory and experience have always led to consider the relationship between sensitivity and theoretical formalization in a particular language. Kuhn (1962) argues that each theory carries the interpretations of the terms it uses so that the same experience and results may lead to different interpretations according to the underlying theoretical assumptions:

> My remarks on incommensurability and its consequences for scientists debating the choice between successive theories, In Sections X and XIII have argued that the parties to such debates inevitably see differently certain of the experimental or observational situations to which both have recourse. Since the vocabularies in which they discuss such situations consist, however, predominantly of the same terms, they must be attaching some of those terms to nature differently, and their communication is inevitably only partial. As a result, the superiority of one theory to another is something that cannot be proved in the debate. (Ibid., p. 198)

Observation and manipulation by linking the action (the relationship to the perceptible world) and reflection (the relationship with the theoretical world) are a foundation of experience that need to be transposed both towards mathematics and teaching and learning of mathematics. Thus the mathematical objects, objects of experience, can be perceived dialectically in a perceptible view by direct manipulation of some of their representations and in a theoretical view by their relationships within abstract structures. Therefore the manipulation of mathematical objects depends on an appropriation of systems of signs to make the objects familiar, that is to say controllable in their relationship to the underlying theories. In that sense, technology brings new representations and therefore new possibilities of experience but also new difficulties to connect with other representations of the same object.

Starting from these epistemological considerations, the aim of our work is to study the possible links between doing maths and have learners do maths in a didactic perspective as well as showing the role of technology in this process. The next sections will present two different examples that illustrate these considerations and show the place of technology in the process of problem solving.

Principles of the Collaborative Resolution of Problems

Over the last 15 years, the Resco team in Montpellier (Resco 2014) has been developing an innovative frame for developing collaborative research involving a network of volunteer teachers who engage their students in a problem-solving session running over 5 weeks. Some teachers are involved through an interdepartmental teacher training session; others join the research on their own, generally after having taken part in a previous training session. In this frame, technology plays a role of communication and information by providing a virtual space for exchanging and sharing.

In the first 2 weeks, a game of questions and answers between two or three classes aims to enlighten the necessity of making choices in order to engage in a mathematical exploration of the problem, a point which is generally hidden in modelling activities in class. The questions and answers are exchanged via the platform. This corresponds to the horizontal mathematisation[1] (Herskowitz et al. 1996):

> ... that has to do with establishing a relation between non-mathematical situations and mathematical ideas (metaphorically, this is like building a bridge between the two). (p. 117)

The Resco team then works on this material in order to provide a 'prompt[2]' which is sent to students by one of the team members who is a university lecturer. This prompt takes into consideration the *a priori* analysis of the group and the questions and answers that have been posted on the platform; it makes explicit the choices that will allow a mathematical investigation of a common problem. The students then engage in this mathematical exploration (corresponding more or less to a vertical mathematisation[3]) for 2 weeks and send their ideas, procedures, arguments, proof, and a complete or partial solution to the other two groups. The last week is devoted to a synthesis taking into consideration the various contributions and the mathematical and didactical analyses produced by the team.

The specific organization of this collaborative session and our desire to engage students in a process of mathematisation entails various constraints on the choice of the problem, which led us to propose a specific kind of problem that we have called "*realistic fictions*". We present below the various aspects of this innovation using the example presented during the academic year 2009–2010.

An Example of Elementary Number Theory

This example has been developed in Aldon et al. (2010). The mathematical situation is the following: "Find all whole numbers which are the sum of two or more consecutive positive integers." (n.p.).

[1]The importance of mathematisation was enlightened by Freudenthal (1973); the distinction between horizontal and vertical mathematisation was first introduced by Treffers (1978), see Menon (2013).

[2]We call 'prompt' a letter sent by the university lecturer to students in order to redirect their researches.

[3]The vertical mathematisation is an activity in which mathematical elements are put together, structured, organized, developed etc. into other elements, generally in a more formal or abstract form than the original (Herskowitz et al. 1996, p. 117).

An objective of the work is to identify the mathematical knowledge that may be called upon at different class levels where students, using their own knowledge, enter into a mathematical research situation. The interaction between this knowledge, the results of experiences, in the sense given in the previous section, and the relationship to the theory allows the construction of new knowledge. In the following paragraph we analyse both the mathematical and the didactical situations in the light of the use of technology. This situation has been experimented repeatedly but the examples shown in this chapter come from two experiments: the first in two classes of 14–15 year-old students (Grade 9) in a secondary school in the suburb of Lyon and the second in a class of 16–17 year-old students (Grade 11) in a scientific class at a high school in Lyon.

The Mathematical Situation

A numerical experience that can be conducted using a pencil and paper or a calculator leads to conjecture that all positive numbers except the power of 2 can be reached. It could be of interest to program this research using programming language. For example, using Python, the following functions give the results shown in Table 1.

Table 1 Programming the experiments

def sum(a,n):
r=a
for i in range(n-1):
r=r+i+a+1
return r
	The function *sum* computes the sum of *n* consecutive integers beginning with *a*.

	Example: sum(2,5) returns 20
	2 + 3 + 4 + 5 + 6 = 20

def test(p):
for i in range(1,p/2
+2):
for k in range(2,p/2
+2):
if p==sum(i,k):
return i,k
return 'false'
	The function *test* searches an additive decomposition with consecutive integers of the input number *p*. It returns false when it does find such a decomposition.

for i in range(2100): print i,test(i)	2 false
	3 (1, 2) $\rightarrow$ 3 = 1 + 2
	4 false
	5 (2, 2) $\rightarrow$ 5 = 2 + 3
	6 (1, 3) $\rightarrow$ 6 = 1 + 2 +3
	7 (3, 2) $\rightarrow$ 7 = 3 + 4
	8 false
	9 (2, 3) $\rightarrow$ 9 = 2 + 3 + 4
	...

It is interesting to note that this computing allows us to conjecture the results (the program returns “false” in front of the power of 2), but does not give us any clues on how to reach mathematical proof because the algorithm is the translation in a particular language (here Python) of the mathematical wording of the problem. The technological tool allows us to compute more quickly than by hand, but does not bring us any further advantages. This approach is very fruitful in finding the conjecture but insufficient to prove it. Mathematisation of the problem goes through the choice of other tools. An algebraic resolution seeking to build an explicit formula of the sum of two consecutive integers, three consecutive integers, etc. leads to experimentally finding the pattern of these sums and reciprocally to proving that all numbers of this form are reached. For example, with two consecutive numbers, all odd numbers are reached:

$$n + (n + 1) = 2n + 1 \text{ is odd}$$

if p is odd, it exists a unique n such that $p = 2n + 1 = n + (n + 1)$.

With three consecutive whole numbers, all multiples of 3 are reached:

$$n - 1 + n + (n + 1) = 3n$$

Conversely, if n is a multiple of 3, then $n = 3p = (p - 1) + p + (p + 1)$

Becoming involved in this strategy leads us to identify sub-problems, to pose the problem of formal proof, including the converse, but also to observe patterns and invariants. The mathematical knowledge used is multiple: the algebraisation of the problem, the decomposition of integers, the concept of multiple and divisor and their characterization with algebraic formulae. This knowledge, which is generated by the interaction between experience and interpretation of the results to theory, is sufficient to build proof of the conjecture, as long as the Gauss formula of the sum of the first positive integers is known.

This mathematical proof can be given taking into account the sum of the n first positive integers:

$$\mathrm{S_n} = 1 + \ldots + n = n(n + 1)/2 \tag{1}$$

Let N be a natural number, we search a and b such that:

$$\mathrm{N} = \mathrm{S_{a+b-1}} - \mathrm{S_{a-1}} \tag{2}$$

That is to say:

$$2N = (a + b)^2 - a - b - a^2 + a = b(2a + b - 1) \tag{3}$$

We can consider the parity of b.

If b is even: $2a + b - 1$ is odd
If b is odd: $2a + b - 1$ is even

Thus the two numbers b and $2a + b - 1$ do not have the same parity and as their product is even ($2N$), N has an odd prime factor and cannot be a power of 2.

Conversely, $2N$ is the product of an odd number i and an even number p and

$$2N = b(2a + b - 1) \tag{4}$$

$$\text{if } i < p \text{ then } b = i \text{ and } p = 2a + b - 1 \text{ that is to say } a = (p - b + 1)/2 \tag{5}$$

$$\text{if } i > p \text{ then } b = p \text{ and } i = 2a + b - 1 \text{ that is to say } a = (i - b + 1)/2 \tag{6}$$

The conjecture is proven and this proof provides us with a practical process to determine a and b such that $N = a + (a + 1) + (a + 2) + \ldots + (a + b - 1)$.

For example, $20 = 2 + 3 + 4 + 5 + 6$

$$40 = 5 \times 8 \text{ and } 5 < 8 \text{ then } b = 5 \text{ and } a = 2$$

It is of interest to note that we are not stating that the solution found with this method is unique and other problems could emerge from this one when trying to find the number of additive decompositions of a number. We will discuss later how technology can help to pose this new problem, extending the field of research in the elementary theory of numbers.

The Didactical Situation

Using problems to teach mathematics requires paying special attention to the situation, i.e. the milieu to which the students will be confronted. The Didactic Situations of Problem Solving are situations in the sense of Brousseau (1997), but they are situations that do not involve specific knowledge and thus stand out from the “fundamental situations” of the Theory of Didactical Situations. They are also connected to “Problem Solving” in that the enrolment of students leads them to discover a small part of mathematics and to use and develop heuristics and meta-mathematical skills; but these situations are built on the idea of construction of knowledge. Consequently, the experimental dimension, as described above, is also an essential component of these situations. The links between the mathematical objects involved in the mathematical situation are built through the experiences and the reflections on the results of the experience using concrete artefacts or naturalized mathematical objects that are present in the milieu of the situation. In particular, available technological tools are part of this milieu and give students an opportunity to experiment in a different way to what they could do using a pencil and paper. Technology brings a new approach and even if it does not lead to mathematical proof, it allows a better understanding of mathematical situations by providing the opportunity of rephrasing the problem in another system of signs, as shown in Table 1.

In the case of the given problem, it is interesting to note that in the different experiments, the students’ research led to different results and conjectures.

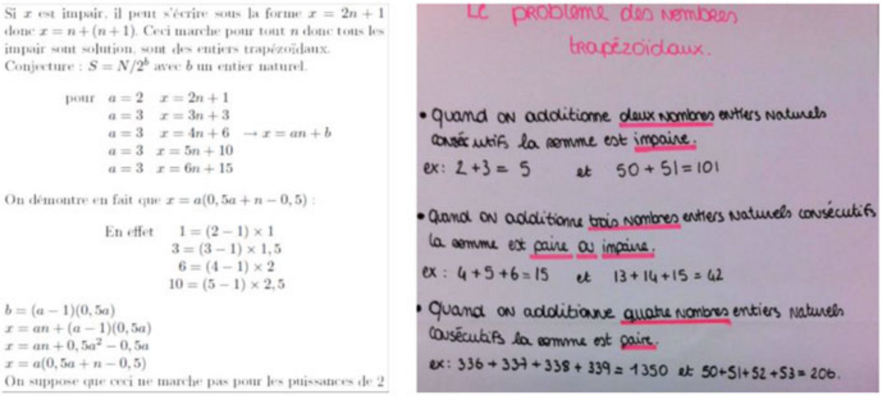

Fig. 3 Poster of Grade 11 students (*left*) and Grade 9 students (*right*)

Nevertheless, the analysis of the problem is robust enough to highlight some mathematical knowledge, allowing an institutionalization of knowledge linked to the aims of the curriculum, as shown in Fig. 3 with a *facsimile* of the Grade 11 students' poster and the original poster of Grade 9 students. Even if the results of the Grade 9 students are less developed, we can see the same strategy, i.e. the study of the properties of the sum of two, three, . . ., consecutive integers.

In this problem, and due to the fact that technology does not give us any clues to the mathematical proof, students use technology as an external tool facilitating computation. It generates debate between students as shown in the short dialogue where G (a girl) is using her calculator to compute the powers of 2 and B (a boy) is doing the calculation from memory:

G (using her calculator): 2 to the power 5 is... 32... Yes, 2 to the power 7 is 128
B: 256, 512, 1024, 2048...
G: How can you compute so quickly?
B: Well, I do times 2
G: Ah, yes, not bad!

In this case, using the calculator hides the recursive definition of the power of 2. In that small excerpt G is using the possibilities of her calculator making reference to the iterative definition of the power: $a^n = a \times a \times \ldots \times a$ (n times) while B is using the recursive definition of the power: $a^n = a^{n-1} \times a$.

An in-depth analysis of the possibilities given by the spreadsheet shows that it is not only possible to establish the conjecture as long as the Gauss formula is known (Fig. 4) but also to find all decompositions of an integer into the sum of consecutive integers. For example, the number 90 has 5 decompositions:

90 = 2+3+...+13 = 6+7+...14 = 16+17+...+20 = 21+22+23+24 = 29+30+31, which is exactly the number of odd divisors of 90 different from 1. The solution given by the proof is here 90 = 21 + 22 + 23 + 24.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
2	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	84	87	**90**	93
4	10	14	18	22	26	30	34	38	42	46	50	54	58	62	66	70	74	78	82	86	**90**	94	98	102	106	110	114	118	122	126
5	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	**90**	95	100	105	110	115	120	125	130	135	140	145	150	155	160
6	21	27	33	39	45	51	57	63	69	75	81	87	93	99	105	111	117	123	129	135	141	147	153	159	165	171	177	183	189	195
7	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175	182	189	196	203	210	217	224	231
8	36	44	52	60	68	76	84	92	100	108	116	124	132	140	148	156	164	172	180	188	196	204	212	220	228	236	244	252	260	268
9	45	54	63	72	81	**90**	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225	234	243	252	261	270	279	288	297	306
10	55	65	75	85	95	105	115	125	135	145	155	165	175	185	195	205	215	225	235	245	255	265	275	285	295	305	315	325	335	345
11	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275	286	297	308	319	330	341	352	363	374	385
12	78	**90**	102	114	126	138	150	162	174	186	198	210	222	234	246	258	270	282	294	306	318	330	342	354	366	378	390	402	414	426
13	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325	338	351	364	377	390	403	416	429	442	455	468
14	105	119	133	147	161	175	189	203	217	231	245	259	273	287	301	315	329	343	357	371	385	399	413	427	441	455	469	483	497	511
15	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375	390	405	420	435	450	465	480	495	510	525	540	555
16	136	152	168	184	200	216	232	248	264	280	296	312	328	344	360	376	392	408	424	440	456	472	488	504	520	536	552	568	584	600
17	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408	425	442	459	476	493	510	527	544	561	578	595	612	629	646
18	171	189	207	225	243	261	279	297	315	333	351	369	387	405	423	441	459	477	495	513	531	549	567	585	603	621	639	657	675	693
19	190	209	228	247	266	285	304	323	342	361	380	399	418	437	456	475	494	513	532	551	570	589	608	627	646	665	684	703	722	741
20	210	230	250	270	290	310	330	350	370	390	410	430	450	470	490	510	530	550	570	590	610	630	650	670	690	710	730	750	770	790
21	231	252	273	294	315	336	357	378	399	420	441	462	483	504	525	546	567	588	609	630	651	672	693	714	735	756	777	798	819	840

Fig. 4 The first line gives the starting point of the sum, the first column gives the number of terms

In this case the spreadsheet, and more generally technology, appears as a means of development in the resolution of the problem as well as allowing an application of the algebraic formula in a different context: from $S = a(2n + a - 1)/2$ where a is the initial number of the sum and n the number of terms to "=$A4*(2*C$1+$A4–1)/2" which is the translation in the language of the spreadsheet.

It appears that using technology to research a problem depends on different factors, coming from the mathematical situation itself as well as the didactical situation. In the case of the problem, exploration of the situation using technology quickly provides conjectures but takes the mathematical proof away. It is only by alternating between the use of technology and the mathematical theory that clues leading to proof can be found. In this case, technology is an exploratory tool that enables the problem to be extended, bringing new questions that add to the richness of the mathematical situation.

On the other hand, the didactical situation, by offering a milieu, can either promote the use of technology, or not. It is the responsibility of the teacher to include technology in the material milieu but also to institutionalize its use in relation to the related mathematical content.

An Example of a Session of Collaborative Resolution of a Problem

In this section we present the collaborative resolution of a problem innovation using an example from the 2009–2010 session. We describe and exemplify: first the main features of what we have called "realistic fiction" and second the different phases of the session. In both cases, we motivate our choices by taking into account our aims and the specific constraints induced by the organization of the session, both being closely intertwined.

An Example of a Realistic Fiction: The Artist's Problem

A *realistic fiction* is a contextualized problem with the following characteristics:

(a) The situation should appear *a priori* as a non-mathematical one; i.e. the mathematics that could be involved are not immediately visible for a novice.
(b) The context of the situation is fictional, but realistic, that means that it could be interpreted in reference to ordinary life.
(c) In order to be able to engage in a mathematical investigation on this situation, it is necessary to first engage in a process of horizontal mathematisation.
(d) The horizontal mathematisation can lead to various mathematical problems, depending on the choices that have been made.

We illustrate these characteristics with the example of the Artist's Problem that was proposed to 38 middle and high school classes in the 2009–2010 academic year. This problem was elaborated as a realistic contextualization of the following classical problem (Fig. 5).

In order to reach a satisfactory realistic fiction in accordance with the above criteria, several contexts were considered and discussed in the Resco team (puzzle, logo, video game etc.). Finally, we retained the idea of the realization of a contemporaneous work of art (Fig. 6).

Characteristics a, b and c of a realistic fiction are satisfied: the situation appears *a priori* as a non-mathematical one; the context is fictional but realistic; mathematisation of the problem is likely to provide a general answer to the artist. This third characteristic is an illustration of the predictive role of mathematics that permits an anticipation of results of actions not yet done.

Concerning the 4th characteristic, we hypothesised that the various possible choices for mathematizing the problem will favour rich exchanges during the two sessions devoted to the game of "questions and answers". This was confirmed by the data collected in the classes (see next section).

Placing n distinct points on the circumference of a circle. How many parts do we determine within this circle by drawing all possible strings through pairs of these points?

Fig. 5 Text of the underlying mathematical problem

A contemporaneous artist is willing to create a work on a round support, by planting nails around the circumference and pulling strings between the nails. He proposed painting each zone in a different colour. How many colours will he need?

Fig. 6 The artist's problem

Fig. 7 Influence of the position of the nails

The problem of the artist leads to the underlying problem of *the number of areas in a disk* whenever the following choices are made: the support is a disk; the size of the other objects is not taken into consideration (nails are modelled by points; stretched strings are modelled by straight lines); nails are points on the circumference of the disk; each nail is linked to all the other nails: an area is a surface defined by cords and/or the side of the disk, and not crossed by others cords (i.e. there is no other zone inside): the number of colours is exactly the number of zones. Once these choices are made, several different problems remain, according to the choices made concerning the position of the nails (regular or not, allowing multiple intersection points or not) (Fig. 7).

The Game of Questions and Answers

During this phase, students use a platform to communicate with their peers. The platform is a private part of a site with free access. This ensures the confidentiality of the exchanges: the research of some classes could be parasitized by the publication of student work on the same subject. At the middle school (Grades 6 to 9), most of the questions sent to the other classes at the end of the first session concerned (in decreasing order of occurrence) the size of the involved objects, their position, their number and their shape (sometimes with aesthetic considerations), as in the following example (Fig. 8).

At this level, the process of vertical mathematisation is not yet engaged, contrary to what could be observed at a more advanced level, as in the following example (Fig. 9): the number of nails is kept as a relevant variable of the problem and the size of the objects as a non-relevant one.

A detailed analysis of the students' exchanges is provided in Ray (2013), which concludes that the devolution of horizontal mathematisation occurred in the classes during the two first sessions: students' discussion on the way of representing the

- What is the size of the support? What is the area of the circle? Of its diameter? Is it a disk? What is it made of?
- What is the size of the nails? How many nails are there? How much space between the nails?
- How many strings are there? How are the strings arranged? Are the strings stretched? Do several strings leave from a single nail?

Fig. 8 Questions from a middle school class (Grade 9) (We translate from French.)

- The number of nails is a variable; we wish to study its influence on the problem.
- It is not necessary to know the dimensions of the support: indeed, we seek to know the number of zones and not the size of each area.
- By increasing the area of the support, we increase the size of areas but the number of zones remains unchanged. In this problem, the dimensions of the support have no influence on the required number.

Fig. 9 Answers of a high school class (Grade 12)

different objects implicitly denotes elements for the elaboration of "candidate-models". In order to engage the students in a common mathematical problem, it is necessary to set some essential choices. For this purpose, a prompt signed by a researcher is sent to all the students via their teachers.

The Prompt Sent to Students

The platform is also at this stage of the process a tool allowing the communication between actors. For the team the fact that all discussions are available on the platform is very useful for understanding the questions that the problem pose, the main ideas developed during the discussions, and so on. Apart the text of the problem, the prompt is the only scientific contribution from outside the students' community; it is elaborated taking into account the exchanges during the first two sessions of the questions and answers game. Indeed, it aims to enhance the research process and as a consequence, it must be understood by the students as setting in a non-artificial manner some aspects of the problem that they raised in their questions. For this problem, at every school level, students, sometimes after long debates, agree with nearly the same choices for mathematizing the objects. The prompt (Fig. 10) consisted in validating these choices and stating that the expected answer was on the *maximal* number of colours.

This prompt provides answers to the most frequent questions and to those that require a joint decision:

– The first point allows discussion on questions like "Do different tones of a same colour make up different colours?"; indeed, a lack of response might interfere with the result of the research,

We wish to give a precise answer to the Artist in order to help him to make choices for the realisation of his work of art. So we intend to provide a mathematical treatment of the Artist's problem For this purpose, I propose that you take into consideration that: 1. The number of colours is the number of zones 2. We are seeking a general solution, that is to say we're looking for the maximum number of zones according to the number of nails 3. The support of the work of art is a disk and the nails are distributed over its circumference. 4. The support size is sufficient so that one can disregard the size of the nails and the thickness of the threads. Therefore, the nails are assimilated to points, and the stretched threads to straight lines.

Fig. 10 Excerpt from the prompt*relance* (the full text in French is in Annex 2)

- The second point confirms that the nails are not necessarily arranged in a regular manner and that the multiple intersection points are avoided (otherwise there is no functional relationship between the number of nails and the number of colours),
- The third point eliminates the possibilities of various work supports mentioned by the students: spherical, hemispherical, cylindrical, rings, etc.,
- The last point consolidates most students in their choices of mathematisation of the problem.

Based on the students' exchanges, and respecting the choices made by the majority, the prompt engages students with the same mathematical problem, without inducing any particular method of resolution.

Realistic Fictions and Collaborative Problem-Solving Innovation: Additional Constraints

The contextualization of the realistic fiction should guarantee that the students do not immediately identify the underlying mathematical problem: it must offer the possibility of conducting a real horizontal mathematisation pondering the dialectical relationship between real objects and mathematical objects. In other words, we distinguish "contextualization" and "dressing". The artist's problem is not a simple dressing of the initial problem: the rich set of questions and answers discussed earlier and the different possible mathematisation choices prove it.

One of the principles of collaborative problem solving is to propose the same problem to students early in the middle school up to the end of high school. Moreover, as part of a research of several hours over 5 weeks, the mathematical problem itself must be sufficiently consistent.

To elaborate the Artist's Problem, the Resco Group relied on the work of the research group EXPRIME (IREM Lyon - INRP) on the underlying mathematical problem (i.e. *the number of areas in a disk)*: the many experiments conducted in middle and high schools had showed its relevance to all levels of education and the richness of exploration and resolution strategies; it contains interesting sub-problems and each student can engage in research. This makes it a good candidate for elaborating a realistic fiction: indeed, it guarantees the possibility to have consistent work not only for horizontal mathematisation, but also for vertical mathematisation after the prompt.

The Role of the Platform in Resco

The role of the Internet platform is crucial in the collaborative research innovation, far beyond the technological aspects. Three main aspects show the importance of technology as a medium of information and communication:

– First, it is the support for the exchanges in the classes' network and has an impact on the devolution, the didactical contract and the milieu (Brousseau 1997, 1998; Brousseau and Warfield 2014); in that sense, technology plays a role of facilitator in the introduction of the problem within the classes and forces students to discuss directly with their peers. Therefore, the problem is no longer the teacher's problem but becomes internal to the students' community.
– Second, it becomes a tool for facilitating communication within students and teachers communities and allows the appearance of communities of practice (Wenger 1998): for students with a common goal in solving the problem and for teachers with a mathematics teaching goal through problem solving.
– Third, it provides the research team with data for elaborating the prompt and also for analysing the process of communication by providing a record of the students' exchanges.

In this experiment, the properties of communication, sharing and providing information facilitate the teaching organization within each class as well as between classes.

A Support for Exchange Between Classes

The classes involved in a collaborative problem-solving session work separately for five sessions, but share with the two other classes of their team, sending and receiving documents (texts, figures, diagrams, spreadsheet files or dynamic geometry files) via their teachers who are responsible for the technical aspect of the exchanges via the platform. The teacher does not moderate the exchanges (questions, answers, resolution procedures, protocols, partial results correct or not), so that students are responsible for what they send during the session. It is only at the last step that the teacher, together with the students, comments on what has been done, what has been proved, what remains as conjecture, which mathematical notions had been involved and reworked during the session.

The chosen methods of communication make it essential to produce a structured written document: writing a summary of the work before each post on the platform allows to clarify and refine ideas which expressed orally might sometimes only remain in a draft form. Also, it highlights for students the need for rigour required by argumentative writing and provides an opportunity to use and sometimes to develop relevant alternative representations when the text becomes insufficient to express ideas or to conduct reasoning.

Impact on the Devolution, the Didactical Contract and the Milieu

During the collaborative research session, students are invited to correspond with other classes; they decide by themselves which content will be sent to the two other classes; the teacher's role in the first two phases is mostly to organize the small groups work and the collective summary and to post it on the platform. Thus, in the first phase, the students ask questions to their peers and not to their teacher, and in the second phase they answer questions from other students: we assume that it is a factor promoting the devolution of the horizontal mathematisation of the problem. Moreover, in the following phases, the teacher does not control the correctness nor the validity of the statements and reasoning sent to the two other classes via the platform. We assume that this contributes to modifying the didactical contract, letting students take the responsibility of their assertions, conjectures or proof (Brousseau and Warfield 2014). This claim is supported by classroom observations over more than 10 years showing that the documents are often very strongly discussed before being sent.

Thus, the platform can be seen as part of the *didactical milieu* that plays an important role in the *didactical contract* (Brousseau 1997, 1998): letting students take the responsibility of the scientific content of their posts and consequently improve their involvement in the research.

A Crucial Role for the Resco Team

By collecting the questions and answers posted on the platform during the two first sessions devoted to horizontal mathematisation of the problem, the members of the Resco team have access in real time to the reflections of the classes engaged in the session. It plays an irreplaceable role in the elaboration of the prompt which, as we have seen, explicitly takes the students' work into account.

Furthermore, the platform helps the Resco team regulating the organization of the classes: groups of two or three classes may have to be reconfigured (temporary participation problem, incompatibility levels, dropping out, etc.).

Last but not least, the platform is also a record of the students' work: the exchanges reflect the ambition of the collaborative research innovation; they are stored in a sustainable manner and can be used for academic research or for teacher training, as well for inter-departmental teachers taking classes in the session, and also for prospective teachers following a Masters degree program.

The Communities of Practice

A collaborative problem solving session can involve up to more than 70 classes mainly from the Montpellier area, but also from other areas in France and other countries (i.e. Morocco, Canada). The platform appears as the unifying link of three communities of practice: the students, the teachers and the Resco team, who manages the entire process (Sauter et al. 2008).

The Resco Team

The Resco team is a research group from the IREM de Montpellier who has been working throughout the whole year, every year since 2000 on the various tasks required by the collaborative research innovation: development and *a priori* analysis of the annual problem, development and animation of the inter-departmental teacher training which is associated with each annual session, management and regulation of the exchanges on the platform, elaboration of the prompt, *a posteriori* analysis, assessment of the mathematical notions worked out during the session, design of resources relying on the data posted on the platform, and on all the documents elaborated during the session by the team.

The Teachers

The inter-departmental teacher-training course associated with the session allows teachers who are not yet promoting research problems in their class to make a first attempt with a classical research problem (i.e. open problems, Arsac and Mante 2007) before engaging their classes in the collaborative research session. Following this first experience, some teachers regularly renew their engagement. Furthermore, the presentation of the innovation in various national and international conferences, the publication of papers in various journals, and the public part of the platform attract teachers from all over France and abroad. Thus, the community of teachers involved in Resco is increasing steadily, contributing to the dissemination of innovative practices recommended by official standards in France and in many other countries.

The Students

Student motivation can be enhanced when the teacher shows them the platform. Indeed, we hypothesize that students who are placed in a position of researchers (exchanging questions, answers, conjecture, reasoning or proof) have a more objective view of the community they belong to.

Conclusion

The two types of situation we have presented correspond to two complementary approaches of the game played in the confrontation of a problem of mathematics or a problem that can becomes a mathematics problem through modelling. In both cases, the choice of the problems and the organization of student work are carefully controlled to allow the devolution of the mathematisation process. The experiments conducted with both the problems of EXPRIME and the realistic fictions developed by Resco show that the relation of students to mathematical knowledge is evolving to be closer to what is recommended by the institution on problem solving, in line with the results from international research. It is of interest to note that in both cases technology plays an important role but in different ways: providing a dynamic environment in the first example, enabling the addition of representations of mathematical objects on which it is possible to experiment, and providing a communication tool facilitating the devolution of a situation, as well as the emergence of communities of practice. To extend this work, research questions still arise, such as: what does it mean exactly to place students in the position of a researcher? How would extending our results on the characteristics of *didactical milieu* favouring authentic mathematical work compare to other didactical organizations, in particular in cases where technology tools are included in the milieu of the situation? Do these research problems, which develop a genuine form of acquisition of mathematical knowledge, foster students' improvement in other aspects of mathematical activity? How do students transfer the skills and knowledge developed in such types of situations to other frameworks?

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A Classroom Activity to Work with Real Data and Diverse Strategies in Order to Build Models with the Help of the Computer

Marta Ginovart

Abstract Dealing with pseudo-mechanistic models, which are continuous and empirical models where the parameters involved have a meaning according to the context where they are applied, has an added value. The aim of this study was to design a set of tasks to be performed with the help of the computer and implement them in the classroom in order to investigate a real data set with empirical models and pseudo-mechanistic models. A framework showing different strategies to tackle these data, and how they generate a variety of plausible responses to the problem, was configured. The sequence and structure of these tasks jointly with the help of appropriate computer resources, according to the students' perceptions, enhanced the understanding of the construction and use of these models.

Keywords Population growth • Real data • Empirical model • Pseudo-mechanistic model • Sigmoid functions

Introduction

In the twenty-first century, computation is more than an assistant support of scientific activity. It is actually changing the fundamental way that science is practiced and also how this it is being learnt and taught (Shiflet and Shiflet 2014).

Computation allows us to obtain and analyze big data, consider and solve problems inaccessible until now, build sophisticated models, visualize phenomena, and conduct experiments which would be difficult or impossible in laboratories, among other options. Teaching and learning mathematics in any context should promote the development of thinking and the possibility of an appropriate use of the technology available nowadays. Processes such as developing curiosity, critical thinking, reasoning, as well as developing modes of verification, refutation and deduction should be found in the activities proposed to our students in classroom, and for some of these processes the help of a computer can be very valuable. In this context, for instance, the analysis of real data and the building of different kinds of

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models for these data would be a good opportunity to train students in those processes. In the teaching of applied mathematics, it would be desirable to design and develop profitable strategies to tackle observed data and models that require thinking at multiple levels of abstraction or understanding. The potential of the software present in the majority of computers in our schools and universities cannot remain unexplored and unexploited when activities related to quantitative modelling and numerical methods are carried out in the classroom. The computer must be seen as a convenient work "companion" and an attractive resource, and never as an obstacle to learning mathematics. In this line, teachers need to have (or develop) suitable knowledge and competences in digital technologies and how to use them in convenient activities, otherwise their teaching will not be so effective (Bennison and Goos 2010; Prado and Lobo da Costa 2015).

It is widely accepted that mathematical thinking arises and develops in a complex interplay of languages and representations. There is a relatively new term or idea that is "Computational Thinking" (Papert 1996; Wing 2006), that although its definition is still under discussion, it appears to focus on computer science concepts in relation to processes of problem solving such as: pattern recognition, pattern generation, abstraction (composition, de-composition, generalization and specialization), modelling, algorithm design (sequence, iteration and selection), data analysis, and visualization (Caspersen and Nowack 2014). Thus, it is also accepted that "Computational Thinking" can be envisaged as a fundamental skill for everyone, and in particular, very attractive for anyone involved in teaching and learning mathematics.

In the context of the Millennium Mathematics Project (http://mmp.maths.org/) can be found the NRICH website (http://nrich.maths.org/) containing a list with some characteristics that make a task "rich", highlighting the fact that it is the way in which the task is planned and used in the classroom that makes it "rich". Some of those characteristics can be common to the way in which "Computational Thinking" can be practiced.

In the field of mathematical biology, the ability to design classroom activities that encompass quantitative modelling with mathematical concepts and tools to deal with biosystems is much appreciated (de Vries et al. 2006). In addition, these activities can justify and give room for the introduction and workout of complementary computational methods (Ginovart 2014). Mechanistic (or heuristic) models are those whose development comes from the understanding of the underlying biochemical or biological processes governing population phenomena and their parameters have biological meaning. Meanwhile, empirical models are mathematical functions simply describing observations of the phenomenon. Taking into account that the majority of the models with more tradition in the mathematical curriculum to represent temporal evolutions of populations are continuous models in the class of empirical models, it has added value to deal with these models but those in which the parameters involved are claimed to have a biological meaning. These types of models are called pseudo-mechanistic models (Buchanan et al. 1997; Perez-Rodriguez 2014; Zwietering et al. 1990). It is a challenge to link mathematical tools and concepts with biological ideas, and also a chance to use the help that computers can provide in these modeling processes.

The aim of this study was to design a set of "rich" tasks to be performed with the help of the computer and implement them in the classroom in order to investigate a real data set (a temporal evolution of a microbial population grown in a specific environment) to deal with different kinds of models. One of the main purposes in the designing of this set of tasks was to configure a computing framework showing different strategies for dealing with the data and software and also, how each of these approaches could generate a variety of plausible responses to the problem in hand. With all this, the specific objectives of this study were:

(a) To know which prior knowledge of the students could be used in order to build models describing the growth of populations and which accessible computer resources, already familiar to the students, could be used to perform the analysis of a real data set in this particular context.
(b) To design and implement in the classroom a set of tasks grouped into different modules to be developed in several computing environments, using spreadsheets, mathematical programs or statistical software, with procedures or actions that enable the analysis of a real data set in order to obtain both empirical models and pseudo-mechanistic models for the representation of this data.
(c) To collect and analyze the students' perceptions in order to discover whether the sequence and structure of these tasks enhanced the students' understanding of the construction and use of growth models in a situation of interest in their academic background.

Material and Methods

The participants in this study were a group of 50, third-year students of a Bachelor's degree in the field of Biosystems Engineering at the Universitat Politècnica de Catalunya - BarcelonaTech (UPC) (Spain). The designed activities were carried out in the context of the compulsory subject "Programming and problem solving in engineering" in the sixth semester of the third year of this degree. The prior coursework for these students (during the first and second years) was related to the following compulsory subjects: Mathematics I and II, Physics I and II, Chemistry I and II, General Biology, Microbiology and Statistics, among others. In particular, the students had nearly completed 12 ECTS (European Credit Transfer System) in mathematics, with linear algebra, differential and integral calculus with one variable, ordinary differential equations, and a brief introduction to calculus of several variables and numerical methods. The previous preparation guarantees a solid knowledge of some biosystems (microbial systems, in particular) and basic mathematical concepts and tools.

Table 1 shows the data to be analyzed and used to build empirical models and pseudo-mechanistic models. A set of 17 observations corresponding to the size of a population, number of microbes, grown in a liquid medium of 1 mL with an initial

Table 1 The experimental data to be analyzed

Time (h)	Number of microbes
0	145349
3	146217
6	139333
9	143620
12	168557
15	287768
18	972270
21	2996236
24	4444266
27	5953756
30	7245644
33	7614686
36	8187928
39	10214427
42	11842517
45	13650985
48	12837014

quantity of sugar and no further addition of nutrient during a period of 48 h (a batch culture) is the data to be analyzed.

The collection of sequential tasks given to the students to meet the description of this data was managed through the virtual campus Atenea, the support platform for teaching and learning utilized by the university. Each student had a computer with access to spreadsheets (e.g. Excel and Open Office), mathematical programs (e.g. Maple) and statistical software (e.g. Minitab and R) used in the previous subjects of Mathematics I, Mathematics II and Statistics, and with a free connexion to Internet. The technological support and the computational tools were not chosen specifically for this activity, but were tools that had been used before by the students in other subjects. Therefore, these computational resources had already participated in previous learning processes, and consequently, they could not be in themselves obstacles for the correct performance of the tasks. A basic knowledge of these resources is theoretically guaranteed, and at this juncture, they must serve as a means to deal with the activities. They are used to help in the development of curiosity (with prompt explorations) and to facilitate the application of critical thinking (testing results), as well as to practice modes of verification, refutation and deduction based on graphical outputs, numerical results and algebraic manipulations.

The activities were designed to be carried out individually as it is desirable that during these lab sessions the computer and the student were two elements inter-connected or linked to produce outputs. The practice and the ability required for the achievement of these final outputs needed to be trained. The computational level of this group of students was very far from being homogenous. There was a great variety in the level of skills among the students (maybe due to their previous activities or their personal preferences) and therefore, the rhythm in the execution of the tasks proposed is expected to be different. On the other hand, previous experience in this type of computational activity (Ginovart 2014) showed that if students work in a team or in pairs, it is easy to “watch” rather than “do”. In the case of students working alone, for those who are not sufficiently computer literate, an extra effort must be made. One of the purposes added to the objectives previously mentioned is to guarantee that each one of the students tackles their own tasks with an active and personal use of the computational resources. However, in lab sessions and with small groups of students, comments, suggestions and interactions with the teacher were always held whenever required or considered convenient. Also, intercommunications peer side-by-side are permitted as long as each one performs their own activity on their own computer.

Students’ responses regarding the distinct methodologies applied sequentially for the analysis and modelling of the population data were collected via commented spreadsheets, outputs of mathematical or statistical software, open-ended questionnaires, and face-to face dialogues during the development of the sessions in the computer lab. The students’ perceptions regarding the set of tasks conducted were explicitly asked for and collected at the end of the activity. All the material required was prepared during four sessions of two hours each in the computer lab, plus possible extra dedication at home if necessary.

At the beginning of the opening session, before reading the first part of the guide to the activity, some preliminary questions were answered as an initial assignment, so that the students had the possibility to reflect on what they had studied and learned up till then, as well as what to apply to solve the problem in hand. These preliminary questions were as follows:

“Taking into account the set of observed data in Table 1,

- Which strategies or methodologies that you already know can be used to analyze these data?
- Which types of functions or models can be adjusted?
- Which computer resources can be used for this purpose?”

The Outline of the Set of Tasks to Be Performed with the Help of the Computer

The activity was divided into a set of tasks grouped in seven Parts (A, B, . . ., G) with a reflection at the end (Part H), facilitating the organization of the work to be performed by the students. The different parts designed to investigate, describe and model the microbial growth reported in Table 1 are presented below, in a format that reflects the documentation delivered to the students.

Part A: An Exploratory Analysis of the Data

In this part you will have to perform an exploratory analysis of the data and decide the best way to represent it, carrying out, if necessary, nonlinear transformations of the data. To begin the activity you can work directly with a spreadsheet, or, if you believe it convenient, you can also work with any statistical package.

Part B: Polynomial Functions and Empirical Models

In this part you will have to deal with polynomial functions to describe or fit the data highlighting some advantages and disadvantages of this type of approach. What information can you extract from the value of the parameters involved in this type of empirical model (the coefficients of the polynomial)?

Part C: Linear Functions and a Very Simple Pseudo-mechanistic Model

In this part you will have to identify the three principal phases in the temporal evolution of the population and to use straight lines to describe each of these phases.

The succession of phases of the temporal evolution that represent this set of data can be distinguished because they are characterized by variations of the growth rate of the population: first the lag phase with growth rate null, second the exponential phase with a constant rate, and third the stationary phase with no clear growth. To complete this part of the activity you are required to read the paper of Buchanan and coauthors (1997), where the three-phase linear model is presented, in order to illustrate how your own work performed matches with a part of the content of this scientific paper. At the same time this highlights the role that a simple pseudo-mechanistic model has in the microbial application context.

From now on, use the data obtained by applying the logarithmic transformation in base 10. You will work with LOG10(N_t), where N_t is the number of microorganisms corresponding to the observation in time t. Represent graphically this transformed data set. What do you observe? Build the corresponding graph of the variation in growth for this population, i.e. at each time instant t represents LOG10(N_t)-LOG10(N_{t-1}). With this plot of variations, can you identify different phases or stages for this temporal evolution? Use linear functions or straight lines to describe the three principal phases: lag phase, exponential phase and stationary phase. Can you assign biological meaning to the parameters involved in this mathematical description? Which meanings?

Part D: The Discrete Logistic Model and Its Step-by-Step Construction

In this part you will have to build, step-by-step, the discrete logistic model by means of a set of calculations and linear estimations with the transformed data, and compare the built model with the observed data with a simulation on a spreadsheet in a simple way.

Given that the number of microbes are now expressed in logarithmic units in base 10, define the M_t variable to simplify the notation as M_t=LOG10(N_t) (or as the original value N_t) with which you will work for the construction of a discrete logistic model. A good start for the construction of this type of model is to focus attention on the equation "Future value - Present value = Change". In terms of the new variable this equation can be written as:

$$\Delta M_t = M_{t+1} - M_t \tag{1}$$

The purpose of carrying out an iterative process, step-by-step and from an initial value M_0, is to find an approach to this ΔM_t that can reproduce experimental data reasonably well. Therefore, the question is how to intuit some approximations for these observed changes. Check the graphs of the observations and variations in growth that you have already built. You can guess that there is an exponential growth at the beginning of evolution (fast-growing, using the available nutrient and space), but as we approach a value K (maximum capacity that the system can support, the final size of the population with the nutrient consumed or the space occupied) this growth decreases or slows down, and finally, the growth essentially stops. With this idea in mind you can test this expression:

$$\Delta M_t = M_{t+1} - M_t = \mathrm{r}M_t(\mathrm{K} - M_t) \tag{2}$$

To explore how to find the values of r and K, draw a scatterplot, placing $M_{t+1}-M_t$ on the vertical axis and M_t (K$-M_t$) on the horizontal axis, and use a linear regression without intercept for these observed points to find out the value of r (the slope of the line of best fit). Try with different values for the constant K, and see which values are obtained for r, and choose the best values to build the following expression (a discrete logistic model):

$$M_{t+1} = M_t + \mathrm{r}M_t(\mathrm{K} - M_t) \tag{3}$$

Graph the observed values and the simulated values in the same plot. What can you say? How can you appreciate the goodness of fit of the different models obtained with the different values of r and K tested before? What do you think you should do to find the best possible model of this type?

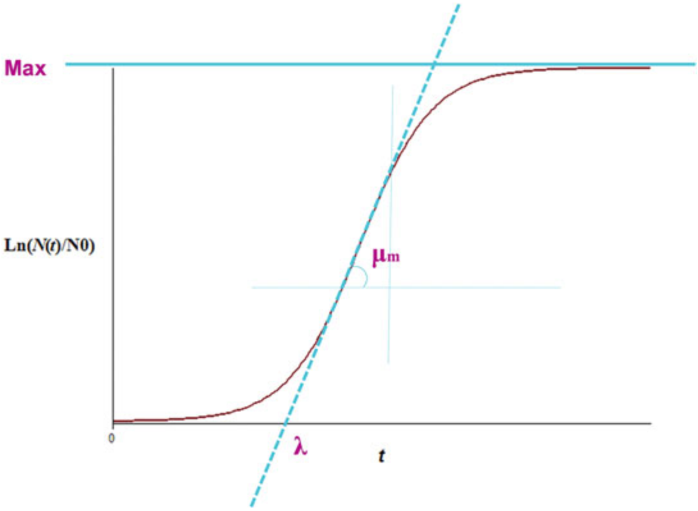

Fig. 1 Generic growth curve with the identification of parameters with biological significance: maximum specific growth rate (μ_m), lag time (λ), and final value that can be achieved (Max)

Part E: Sigmoid Functions and the Continuous Logistic Model

In this part you will deal with the family of continuous mathematical models known as sigmoid functions, that is, functions with an "S" shape (Fig. 1). The continuous logistic model is one of them and is well-known in many academic contexts. Reparameterizations of these models, that is, modified models with different but equivalent expressions in their formulations, can be performed.

The model called logistic assumes that the growth rate is proportional to both the population itself and the quantity lacking to reach the sustainable maximum population. It is possible to write this model like an ordinary differential equation, as follows:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{4}$$

where r is defined as the intrinsic growth rate and K is the carrying capacity which is the maximum value that N can achieve. For $N \ll K$, $\frac{N}{K}$ is close to 0, and therefore $\left(1 - \frac{N}{K}\right)$ is close to 1, which makes $\frac{dN}{dt} \cong rN$ and thus, the growth rate is given by the Malthus model (exponential model), whereas when N tends to K, $\frac{N}{K} \cong 1$ and therefore $\left(1 - \frac{N}{K}\right)$ is very small, so then $\frac{dN}{dt} \cong 0$.

Firstly, rewrite Eq. 3 in the form $M_{t+1} = a\,M_t\,\left(1 - \frac{M_t}{b}\right)$, identifying the relationships of a and b with r and K, in order to note that the equation found for the previous discrete model corresponds to the pattern of the equation for the continuous model (Eq. 5), so both are known by the name of logistic models, discrete or continuous. Solve this ordinary differential equation (Eq. 5) and verify that there are different ways to write the solution in terms of the initial population, r and K. Plot diverse solutions to this equation by setting different values for the initial population and the parameters r and K. What is the effect of these parameters on the shape of the curves drawn?

Return now to the raw data in Table 1 (it is also possible to use the transformed data) to fit a logistic function using Solver, a useful complement to be found in the spreadsheet Excel for optimization problems. Work with the column of time (hours) and the experimental data (number of microbes) and build a new column with the data generated by a logistic function with preliminary or initial values for the parameters involved in its expression to start. Fitting this function to the experimental data means getting the best possible values for the parameters involved, i.e. those that would minimize the error or discrepancy between data observed and values predicted by this logistic function. In the spreadsheet add a new column with the absolute value of the difference between observed and predicted pairs of values for each sample time, and calculate the sum of all these discrepancies. In a range of cells on the spreadsheet set down the values of the parameters that will be collected by the logistic function, and which will be changing until the achievement of a combination of values that minimize the sum of the discrepancies (according to the iterative algorithms implemented in this Solver software). The approximation for the experimental data can be investigated with a graphical representation combining observed values and predicted values. Prior to using Solver, what do you get when trying various parameter settings? Using Solver, what are the values obtained for the set of parameters involved? How do you assess this fitting?

Part F: The Reparameterized Gompertz Function to Generate a Pseudo-mechanistic Model

In this part you will have to deal with another sigmoid function called Gompertz and obtain the reparameterized or modified Gompertz model which is to be applied in this context of growth models. The identification of the meaning of the parameters involved in its definition is convenient and necessary for its reparameterization, allowing you to obtain parameters with a clearer biological meaning, and thus, achieving a pseudo-mechanistic model utilized in the field of predictive microbiology. The papers of Buchanan et al. (1997) and Zwietering et al. (1990) inform you on the use of different models to describe microbial population growing in batch cultures.

There are other sigmoid functions distinct from the logistic functions. For example, the solution of the following ordinary differential equation:

$$x'(t) = \frac{dx}{dt} = r \cdot x(t) \cdot \left(\mathrm{Ln}\left(\frac{K}{x(t)} \right) \right) \tag{5}$$

is the Gompertz function, and it includes the same parameters as those appearing in the logistic equation (the intrinsic growth rate r and the carrying capacity K). It aims to describe the same kind of temporal evolution with "S" shape, when $x(t)$ is close to K, the ratio $K/x(t)$ is close to 1, so that the natural logarithm is close to 0. Find the solution to this ordinary differential equation (Eq. 6) and check that this solution can be expressed as follows:

$$x(t) = A \cdot \exp(-\exp(B - C \cdot t)) = Ae^{-e^{B-C \cdot t}} \tag{6}$$

where A, B and C are the parameters of the Gompertz function. Graph this function for different sets of values A, B and C to justify that it can also be used to describe the growth of the population at hand. From now on, work with the natural logarithm of the relative size of the population, i.e. consider the new variable $y(t)$, a transformation of $N(t)$ where N_0 is the size of initial population, as follows:

$$y(t) = Ln\left(\frac{N(t)}{N_0} \right) \tag{7}$$

which means that you can manage a generic curve of the type represented in Fig. 1. The three principal phases of the microbial growth can be described by three significant parameters:

(i) μ_m, the maximum specific growth rate which is defined as the tangent at the inflection point of the curve.
(ii) λ, the lag time which is defined as the value of the intersection of this tangent with the horizontal axis.
(iii) Max, the asymptote determined by the maximum value that the variable can reach.

Rewrite the Gompertz function replacing the parameters A, B, and C given in Eq. 7, which have no biological meaning, with the parameters μ_m, λ, and Max which have biological meaning. Outline and implement in your report the steps to take in order to obtain the reparameterized Gompertz function.

Part G: The Options of a Relatively New Free Software to Deal with Growth Models

In this part you will have to explore the options that a free software environment for statistical computing and graphics R has, and in particular, to deal with the packages developed specially to manage experimental data of growth curves (https://www.r-project.org/about.html).

R includes an integrated collection of intermediate tools for statistical data analysis, and a well-developed, simple and effective programming language among others. In recent years, packages in R for dealing with biological modeling have been developed, particularly for use in primary growth models in microbial context, for instance, "nlsMicrobio"[1]. These models describe the evolution of the decimal logarithm of the microbial count (LOG10N) as a function of the time (t). Among the different models there is the three-phase linear model proposed by Buchanan et al. (1997) and the modified Gompertz model introduced by Gibson et al. (1988) and reparameterized by Zwietering et al. (1990). Review the documentation that you will find in the virtual campus Atenea that will facilitate the use of the R software and the "nlsMicrobio" package to fit the experimental data with the two models already known to you: the Buchanan model and the reparameterized or modified Gompertz model. What outputs are provided by this R package when it runs with the experimental data? What additional information does R provide?

Compare the results of the three-phase linear model you got with the spreadsheet in Part C with the results provided by R with the Buchanan model option. If you use the output of R which gives you the punctual estimation of the parameters and their confidence intervals, what can you say about this? In relation to R's outputs from the reparameterized or modified Gompertz model, what can you say? Compare the two expressions for this model, one obtained from Part F and the other given by the "nlsMicrobio" package, can you say that they are the same function. From your perspective, what are the advantages of this modeling with R with respect to the different approaches developed in previous tasks? Is there any disadvantage? If so, what?

Part H: A Final Reflection to Organize the Central Ideas

If you have to analyze another set of real growth data,

- Now, what would your answers be to the preliminary questions posted before the start of this activity?
- What would your priority be in the set of tasks to carry out the modelling of the new data set?

Results and Discussion

Students' responses regarding the analyses of the data and the various growth models were prepared individually and collected by means of file texts, giving answers to the questions posed, spreadsheets with comments, and outputs of

[1]http://cran.r-project.org/web/packages/nlsMicrobio/nlsMicrobio.pdf

mathematical and statistical programs inserted in an explicative text. In addition, interviews and personal communication during sessions in the computer lab made it possible to collect further evidence on the development of the activity in the classroom. The extension and diversity of the results obtained by the students were remarkable. The majority of the results involved in the designed tasks is presented in this section, accompanied by some computer screenshots to illustrate them.

At first, the answers to the preliminary questions before beginning the activity were rather disappointing and poor. The majority of these responses were about the use of a spreadsheet to achieve a graphical representation, but no references to the use of mathematical programs like Maple or statistical programs like R were made, although these programs had been used in previous mathematical and statistical subjects during the two preceding years. Undoubtedly, it was also evident that no relationships or links between the prior knowledge of microbiology and different phases of the population growth were made or identified, in spite of the fact that these phases can be linked to numerical derivatives of the data by means of growth rates. The possibility of connecting the two disciplines involved in the modelling process, microbiology and mathematics, did not appear in those initial students' answers. Only approximately 10% of the students mentioned the fact that the transformation of the data by means of the logarithm function could be helpful in this microbial context, due to the magnitude of the numbers and the rate of growth in population. It was really discouraging to see the lack of association with other subjects in the students' answers. Maybe, this is not really surprising in teaching in general, taking into account that subjects and teachers have their own specific areas of knowledge, and on very few occasions do they allow interference or collaboration in sharing activities that involve diverse fields simultaneously.

Regarding the tasks performed in Part A, the students completed an exploratory analysis of the data. They verified that neither linear nor exponential growth was observed for this population. They decided the best way to represent the data was by taking into account the nature of these observations and identifying the way to present data in a microbiology field. The graphical representation of the original data at different scales and through some nonlinear transformations were options examined by the students, mainly logarithm transformations with different bases were one of the strategies most tested by students on the spreadsheet. At the beginning of the temporal evolution the original data had values of about 10^5, and values of around 10^7 were reached at the end of the evolution, so the magnitude of the range to be represented was considerable (Fig. 2). The increases (in absolute values) from one time to another showed very different magnitudes depending on the sample (Fig. 2). At this point, the students appreciated why, in the context of microbial communities with huge numbers of individuals, the populations are usually expressed in base-10 logarithmic units.

They also appreciated the "options" that a spreadsheet like Excel has in order to modify and alter graphical representations, as well as the easy way to manipulate the appearance of the plots.

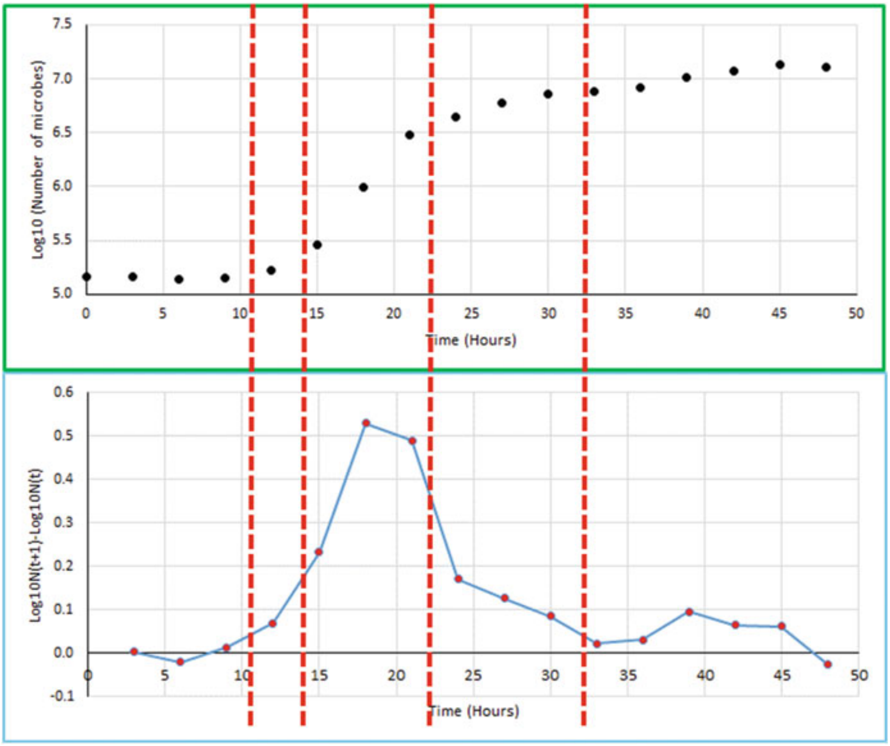

Fig. 2 Temporal evolution of the base10 logarithm of the number of microbes (*upper plot*) and the corresponding variations observed in each of the time samples (*lower plot*). *Vertical lines* indicate approximately the five consecutive phases with changes in the population growth rate

In Part B, the fittings with polynomials of diverse degrees to describe the original data were examined and discussed by students, focusing the discussion on the advantages of this empirical modelling and emphasizing some of the disadvantages. Although statistical software could have been used to deal with these fittings, all students used the "trendline options" of Excel to obtain the plots with the original data and the graphical representation of the fitted polynomials with their corresponding equations and R^2 values (Fig. 3).

In particular, the concept of the model or what a model should be, together with its purpose was debated, and the conclusion in this part was that this type of model (polynomial functions) did not exemplify it very well. Although the goodness of fit assessed by the value of the coefficient of determination was really notable in all cases (with R^2 higher than 0.9), the fitted polynomials showed negative values in the range of time used to describe the population growth, and their coefficients had no meaning and they changed from one polynomial to another without any sense.

Students realized what it was like to work with these empirical models and their limitations in a real application context, and applauded the option of the pseudo-mechanistic models. The coefficients of these fitted polynomials showed no

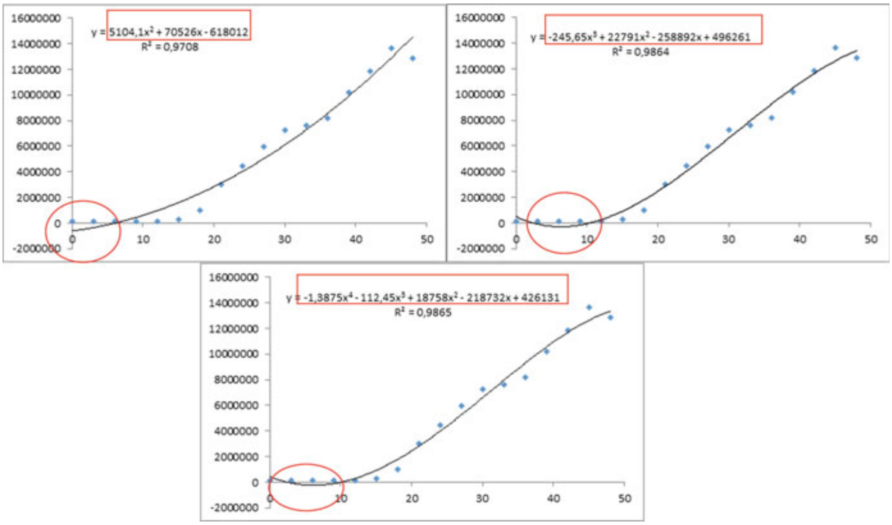

Fig. 3 The observed data of Table 1 (*points*) and the three fitted polynomials (*continuous lines*) with the output options provided by the spreadsheet Excel taken from students' work

opportunity to incorporate any of the ideas behind the microbial growth phenomenon studied: a population located in a new environment with nutrient and using the energy resources found in it, thus enabling the reproduction of individuals and the corresponding increase in size of the population, and the uptake of nutrient from the environment without its replacement, leading to an unfavorable situation, in which it is no longer possible for the population to continue to grow. The inability of these empirical models to explain and represent the biological phenomenon made it evident that they were not appropriate and these mathematical expressions had lost all sense in the modelling of this phenomenon.

In Part C of the activity, the use of the logarithm transformation on the number of microbes observed over time conjointly with the increases observed in each sampled time allowed students to identify various growth phases occurring during the temporal evolution of the population (Fig. 2). The students identified those phases connecting the knowledge acquired previously in microbiological subjects (Monod 1949), and interpreting and linking mathematical concepts on numerical derivatives.

In microbial populations developing in a batch (closed) culture, a succession of phases characterized by changes in the growth rate can be identified as Fig. 2 shows: (i) adaptation period or "lag phase" with zero growth rate, (ii) acceleration phase with increasing growth rate, (iii) exponential phase or "log phase" with positive growth rate, which remains constant and keeps maximum value, (iv) retardation phase with diminishing growth rate, and (v) stationary phase with growth rate (approximately) zero

A first approach for the construction of a simple model, able to pick up the main features or major trends in this type of growth goes through: (i) the consideration or

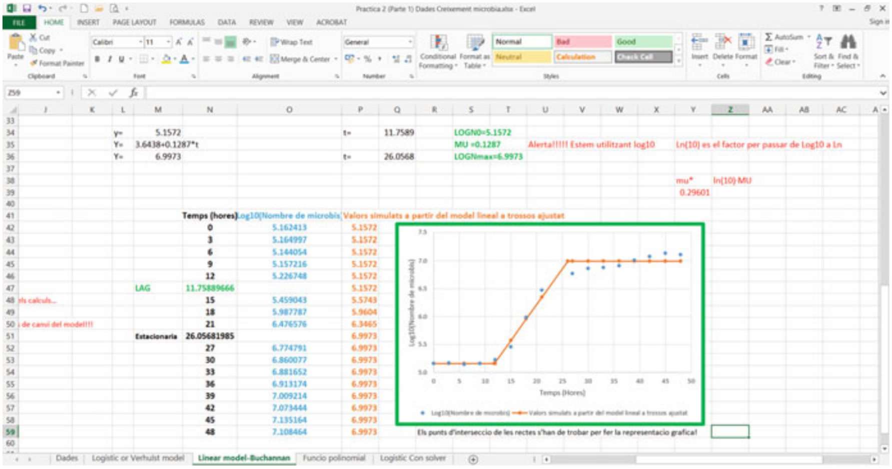

Fig. 4 Screenshot of a spreadsheet with the implementation of the tasks leading to the attainment of a piecewise linear function from data of the Table 1, a simple pseudo-mechanistic model known as the three-phase linear fitting model taken from students' work

recognition of only the three principal phases of the growth, that is, lag, exponential and stationary phases, and (ii) the use of linear functions for its formulation. The students studied this approach with the spreadsheet using linear regressions to describe each of these three phases which were characterized by significant variations in growth: first, the lag phase with zero growth rate, after that the exponential phase with approximately constant growth rate, and finally stationary phase with no growth, as Fig. 4 shows.

Again, handling data with Excel was quite intuitive and simple, and because of its grid nature, the organization and preparation of the data for those pieces of linear functions required was made in an easy and direct way. This facilitated the repetition of successive linear regressions with more or less data, entering and removing points, and comparing results until a convenient one was achieved (Fig. 4).

Reading the companion document of Buchanan et al. (1997) to which the students had access, illustrated and revealed that this piecewise linear model built by the students was in tune with one of the possible solutions that researchers accepted in the environment of predictive microbiology.

The three-phase linear model was the first pseudo-mechanistic model obtained, as in its formulation, parameters with biological meaning were recognized immediately (Fig. 4), such as the logarithm of the initial population and the final population (the first and third straight lines), the duration of the lag phase (intersection of the first straight line with the second), the maximum growth rate (slope of the second line), and the start of the stationary phase (intersection of the second straight line with the third). They realized that the achievement of the parameter estimations obtained from these calculations was slightly conditioned by the selection of the observed points in the regression equations, somewhat subjective for few points, but leading to acceptable responses in all cases. Students were able to recognize their own work on a spreadsheet (Fig. 4) as a simple pseudo-mechanistic modelling choice but very well placed in a broader context of interest in biotechnology and predictive microbiology.

Students tested that when a natural process exhibits a progression from small beginnings that accelerates and approaches a climax over time, a sigmoid function, a mathematical function having an “S” shape as shown in Fig. 1, can be used for its description. In relation to the use of the logistic function in parts D and E, two screenshots (Figs. 5 and 6) illustrate the work performed by students on the spreadsheet Excel. Both the step-by-step construction of the discrete logistic function through a set of calculations and linear fittings of the data handled supporting the idea of a discrete simulation (Fig. 5), and the fitting of the continuous logistic function with the Solver option (Fig. 6), enable the introduction of the ideas and concepts that underlie the construction of a logistic model.

Special attention was paid to the conditions that generate or delimit the logistic model and how these influenced its adequacy to the problem in hand. It was found that the discrete logistic function was reasonably adequate in order to represent the raw data (Fig. 5). Regarding the use of continuous logistic function, it was observed that for the data transformed by the logarithm it was not possible to reach a good fitting, whereas when Solver worked with the raw data the fitting achieved was considerably better (Fig. 6).

The use of another function of the sigmoid family different from the logistic one, the Gompertz function, jointly with the implementation of the tasks for the assignment of biological meaning to the parameters involved in this new function (Part F) was one of the main stimulating parts of the activity for diverse reasons. The study of the first derivative and second derivative of the Gompertz function (Eq. 7) allowed students to identify the role played by the parameters involved in its expression, which, followed by a suitable guided reassignment of parameters (Part F), enabled students to obtain a new expression for this function. An expression with parameters that had a biological meaning linked to the context of microbial population growth was, therefore, a pseudo-mechanistic model. Hence, the pseudo-mechanistic modified Gompertz model was achieved by the students.

Maple computation engine combines high-performance numerical computations with symbolic capabilities, which offer many advantages for mathematical education. It allows you to use symbolic parameters, eliminate the need to manipulate algebraic expressions by hand, avoiding mistakes and ensuring correct calculations. It allows you to differentiate functions, evaluate expressions, solve equations, and expand, factor or simplify expressions, among other actions. The screenshots of the sequence of the steps performed by the students with the mathematical software Maple to solve this reparameterization are presented as follows:

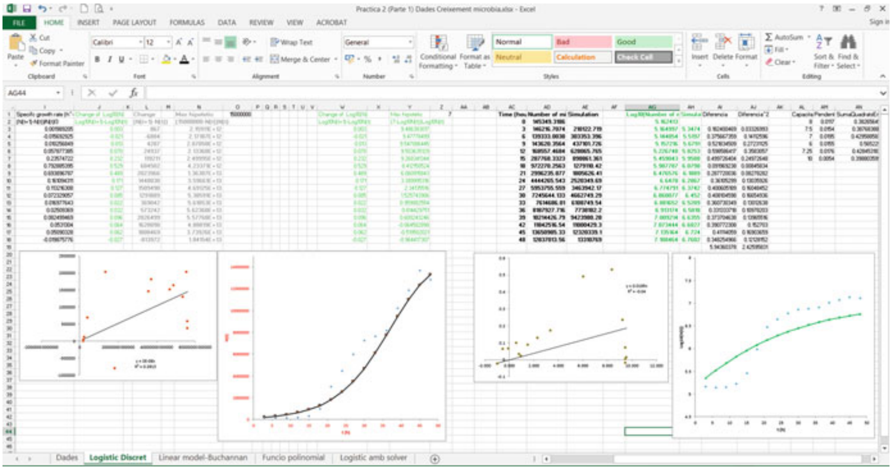

Fig. 5 Screenshot of an Excel spreadsheet with the implementation of the tasks leading to the adjusted data and simulated data with a discrete logistic model taken from students' work

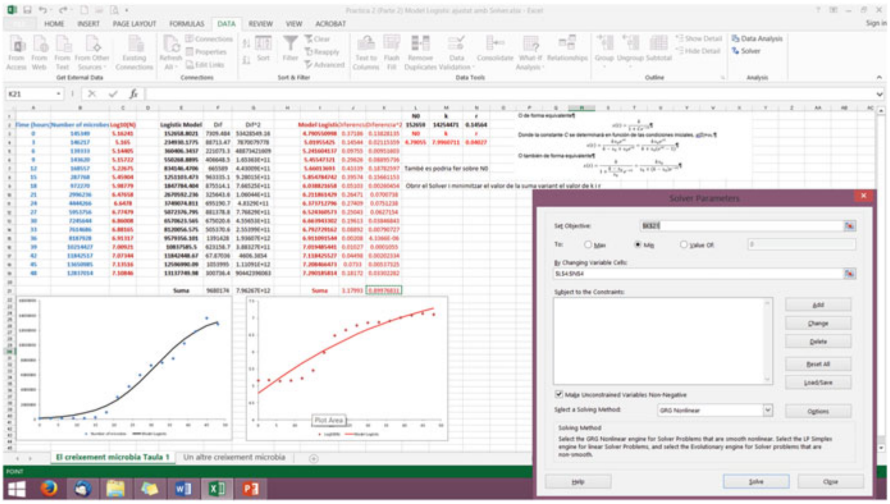

Fig. 6 Screenshot of a spreadsheet with the implementation of the tasks leading to the continuous logistic fitting model with the Solver option of Excel taken from students' work

$$> \; y(t) := A \cdot \mathrm{e}^{-\mathrm{e}^{(B - C \cdot t)}}$$

$$y := t \rightarrow A\, \mathrm{e}^{-\mathrm{e}^{B - C\, t}}$$

$> \frac{d}{dt} y(t)$

$$A\,C\,e^{-C\,t+B}\,e^{-e^{-C\,t+B}}$$

$> \frac{d}{dt}\frac{d}{dt} y(t)$

$$-A\,C^2\,e^{-C\,t+B}\,e^{-e^{-C\,t+B}} + A\,C^2\left(e^{-C\,t+B}\right)^2 e^{-e^{-C\,t+B}}$$

$> solve\left(\frac{d}{dt}\frac{d}{dt} y(t) = 0,\ t\right)$

$$\frac{B}{C}$$

$> eval\left(\frac{d}{dt} y(t),\ t = \frac{B}{C}\right)$

$$A\,C\,e^{-1}$$

$> \boldsymbol{mumax = A\,C\,e^{-1}}$

$$mumax = A\,C\,e^{-1}$$

$> solve\left(mumax = A\,C\,e^{-1},\ C\right)$

$$\frac{mumax}{A\,e^{-1}}$$

$> \boldsymbol{C := \frac{mumax}{e^{-1}\,A}}$

$$C := \frac{mumax}{A\,e^{-1}}$$

$$> \; 0 = y\left(\frac{B}{C}\right) + mumax\cdot\left(t - \frac{B}{C}\right)$$

$$0 = A\,\mathrm{e}^{-1} + mumax\left(t - \frac{B\,A\,\mathrm{e}^{-1}}{mumax}\right)$$

$$> \; solve\left(0 = y\left(\frac{B}{C}\right) + mumax\cdot\left(t - \frac{B}{C}\right), t\right)$$

$$\frac{A\,\mathrm{e}^{-1}\,(B-1)}{mumax}$$

$$> \; \boldsymbol{lag = \frac{\mathrm{e}^{-1}\,A\,(B-1)}{mumax}}$$

$$lag = \frac{A\,\mathrm{e}^{-1}\,(B-1)}{mumax}$$

$$> \; solve\left(lag = \frac{\mathrm{e}^{-1}\,A\,(B-1)}{mumax}, B\right)$$

$$\frac{A\,\mathrm{e}^{-1} + lag\; mumax}{A\,\mathrm{e}^{-1}}$$

$$> \; \boldsymbol{B := \frac{\mathrm{e}^{-1}\,A + lag\; mumax}{\mathrm{e}^{-1}\,A}}$$

$$B := \frac{A\,\mathrm{e}^{-1} + lag\; mumax}{A\,\mathrm{e}^{-1}}$$

$$> \; simplify\left(\frac{\mathrm{e}^{-1}\,A + lag\; mumax}{\mathrm{e}^{-1}\,A}\right)$$

$$\frac{lag\; mumax\;\mathrm{e} + A}{A}$$

$$> \; B := \frac{lag\; mumax\;\mathrm{e} + A}{A}$$

$$B := \frac{lag\; mumax\;\mathrm{e} + A}{A}$$

> **$A := Max;$**

$$A := Max$$

> $y(t)$

$$y := t \rightarrow Max\, e^{-e^{\frac{-mumax\,(t - lag)\,e + Max}{Max}}}$$

Now, considering that $y(t) = Ln\left(\frac{N(t)}{N_0}\right)$, $Max = Ln\left(\frac{N_{Max}}{N_0}\right)$, $mumax = \mu_{Max}$, $lag = \lambda$, the students obtained:

$$Ln\left(\frac{N(t)}{N_0}\right) = Ln\left(\frac{N_{Max}}{N_0}\right) e^{-e^{\left(\frac{-\mu_{max}(t-\lambda)e}{Ln\left(\frac{N_{Max}}{N_0}\right)}+1\right)}} \tag{8}$$

$$Ln(N(t)) = Ln(N_0) + (Ln(N_{Max}) - Ln(N_0))e^{-e^{\left(\frac{\mu_{max}(\lambda-t)e}{Ln(N_{Max})-Ln(N_0)}+1\right)}} \tag{9}$$

which is a formulation of the pseudo-mechanistic model for the population growth with the Gompertz function.

Part G proposed the use of R and one of its packages ("nlsMicrobio") dealing with growth models. The modified Gompertz model was used with the set of data (Table 1) according to the instruction detailed in this package. Some of the results achieved with R are shown in Fig. 7. It is a graphical representation of the fitting data with the modified Gompertz function and the punctual estimations of the parameters involved in its formulation with some statistical complements, which were commented on by students (connecting with their prior knowledge of statistics). In order to recognize the formula of R's output as the formula obtained by the students in the previous task, a change of base for the logarithmic transformation must be made ($\mathrm{Ln}(x) = \mathrm{LOG10}(x)\,\mathrm{Ln}(10)$). Once more students were applying prior knowledge acquired in previous mathematical subjects.

Although the students needed to be guided by a sample of the program to be executed in order to resolve this task, the capabilities and potential of this free programing environment were recognized and widely praised. They agreed that the use of R was justified for the grand statistical options, in general, and the specificity of their packages (very convenient for certain problems). Nevertheless, for the majority of the students the skills required to deal with this software were perceived as too advanced and scarcely accessible in the near future without help.

Regarding the students' learning both in mathematics and biology, it is important to highlight the relevance of this co-disciplinary approach, where the tasks

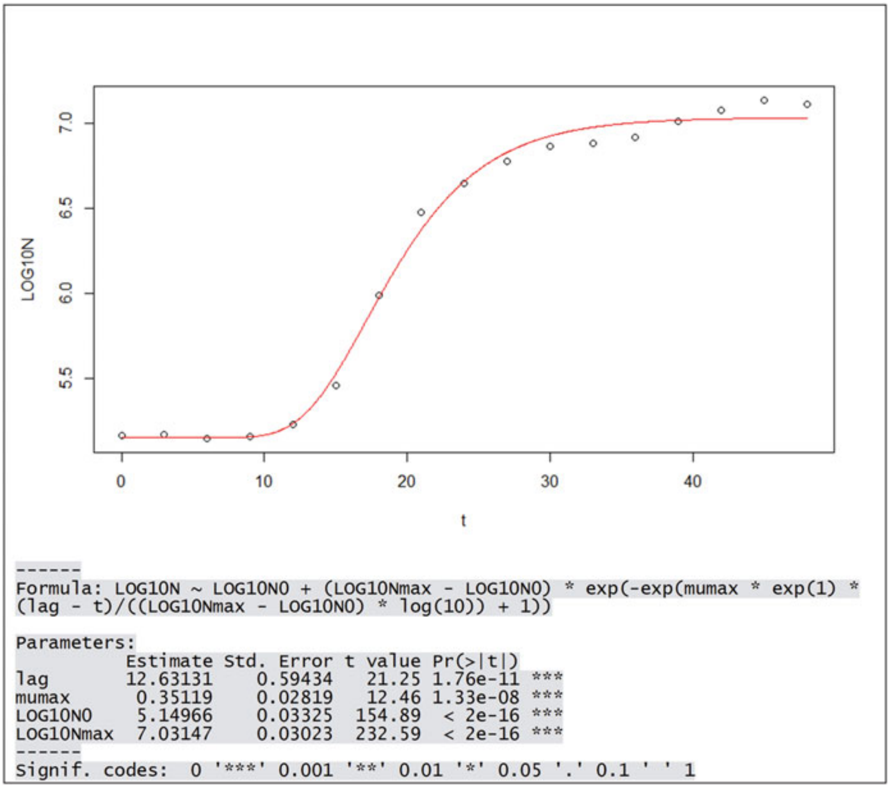

Fig. 7 Screenshot of a part of the output produced by the “nlsMicrobio” package of R when the reparameterized or modified Gompertz model is fitted to the data of Table 1

enhance the mathematical knowledge of the students and in parallel strengthen their biological skills. The competence of the students was rather varied, consequently some of them needed more dedication than others to achieve the same tasks. For instance, at the beginning of the activity, more than 50% of the students did not have clear ideas about the parameters describing the different growth phases of a closed bacterial culture in liquid medium. However, by the end of the activity, and because, simultaneously, they had been using “microbial definitions” in terms of growth processes and “mathematical definitions” with graphical representations of functions characterized by a set of parameters, all the students had consolidated those concepts. The idea of the first derivative was involved in the definition of the growth phases and in the transitions from one to another, and the second derivative was involved in the maximum specific growth rate definition. The use of asymptotes and the notion of limit had a clear role in the temporal evolution of this bacterial population. It is undoubtable that the creation of mathematical-biological models requires cooperation between biologists and mathematicians, or at least, mathematicians interested in biology or biologists with interest in mathematics.

Although the design of the tasks was really prepared to force the students to use biological reflections in the assessments of the mathematical models, not all students were able to understand this at first. Some of them needed additional indications to finish those reflections successfully, but others captured rapidly the reasons for the sequence of the tasks and the mathematical-biological ideas behind them. The majority of students finished the activity with a very good perception of the different models explored, accompanied with critical reflections on them (mainly provoked by the questions formulated regarding their advantages and disadvantages). Successful interdisciplinary teaching requires new materials and approaches, and the technology available nowadays provides excellent opportunities for elaborating attractive proposals.

The Perceptions of the Students of This Activity

The final reflection proposed (Part H) in order to revise and organize the principal ideas of the whole activity provided very positive answers from the students. All the students responded that after this activity their ideas of how to perform the modelling of observed growth data had significantly improved due to the various approaches practiced. They recognized that they knew the majority of the contents from other subjects, but never before then had they had the opportunity to put them together and connect them with a real application of interest for their studies. The most common answers were in the sense that now they had an example of reference with options and alternatives, and that they were aware that the different modelling methodologies were illustrative of plausible approaches depending on the answers they wanted to achieve. Thus, the priority in the set of tasks for the description of experimental data should be associated with the research questions formulated at the beginning of the study. At the same time, the help provided by computer resources made the exploration and testing of diverse alternatives much easier and quicker. They expressed the conviction that without the help of a computer and its resources nothing, or very few and simple achievements, could have been accomplished in this context. They were aware of the power of the technology to deal with the resolution of these tasks, and its indisputable role for their consecution. At the same time, they confirmed that many diverse computational resources available in personal computers were unknown to them and unexplored in the way carried out during this activity. Probably, till that moment, the use of computational support had been framed in a more specific and limited context. They recognized that the use they had had of these computational tools or programs in previous subjects was not enough to tackle this modelling activity with autonomy. They agreed that the detailed guide to the development of the activity was essential for the achievement of the final results.

The following are some of the students' comments, which are illustrative of the whole set of responses, when answering the question: "Please, indicate any positive aspect of the activity performed":

- Tracking the activity in parts I think is good and helps you to get a pretty solid idea of the different ways to build a model from the same data and compare these models.
- One of the positive aspects of the activity is the use of software tools for processing data, being the Solver option in the Excel spreadsheet and "nlsMicrobio" package in the R project the two new or unknown computer resources much appreciated.
- The previous analysis of the data that has been done with spreadsheets helped me a lot. I think the section corresponding to the use of R is a higher level because you have to understand the program and run it properly. You should also have a very clear idea of what you are analyzing and what you want to achieve, because R gives you more information (graphs and tables) than what is absolutely necessary for the analysis or which for me is not completely understandable.
- It is important to learn to manage helpful data processing programs, and this practice gives us an idea of how to do this. Plot some functions using Maple and compare the differences between Excel Solver and the R program has helped me to better understand the data analysis and modelling.
- On my part, the application of different ways of analyzing the same data, trying to discover which is the best fit for the data, testing what happened when increasing or decreasing the parameters of functions, was a good idea. So, the whole process of going slowly and answering the questions has helped me learn a way to try using some experimental data, and also the fact that you can apply many models and some of them will adjust better or worse. Nevertheless, it is really important to try to understand the situation and not just end up having results or numerical values of certain parameters, for example. If you have not made the effort to think about the data, may not be able to interpret these results correctly.
- One positive aspect of this activity could be the use of mathematical programs such as Maple, because it is a very useful when solving differential equations, representing data, drawing graphs of functions, and manipulating expressions.
- It helps you understand that in order to solve the problem with the data available you can choose different methods. My opinion is that the practice has been very good. The fact that it is structured in straightforward steps makes it possible to follow without getting lost, while improving learning. At the same time it encourages an explanation in steps. It teaches us that every experiment fits a certain model and goes through a series of steps to obtain the best result.
- I learned various methods to address a situation that may recur in my studies such as the growth of microorganisms, and also the fact that what I did can be extrapolated to other populations.
- I found it very positive to understand the fundamentals of a solid growth model used in the R program.

Regarding the question: "Please, indicate any negative or improvable aspects of the activity performed", about 60% of students asked for a preliminary training session in the following way: "I think it would be necessary to have some explanations or recall of the computer resources used in previous years" or "I think prior oral explanations would have been helpful to refresh some mathematical concepts and programming". Although they recognized that these programs were used in previous subjects, they felt they did not have enough confidence and ability when managing the programs. This was somewhat surprising due to the generalized use of computer resources in several subjects during their academic training. Consequently, this fact may be contextualized as the evidence that these students needed reinforcement in an autonomous but non-guided use of these computer resources.

The potential of the programs available in most computers, or those freely available from the Internet, should not remain unexplored and untapped when exercises related to quantitative models and numerical methods are performed in the classroom. This kind of training activity can be very beneficial and should be incremented in classroom nowadays taking into account the level of technological development of our world.

Only very few students (no more than six or seven) suggested as an improvement or extension of this activity the possibility of building other types of models that they remembered from specific sessions carried out in a previous mathematical subject, the computational models called individual-based models or agent-based models (Ginovart 2014). They recalled the ideas presented and practiced in the introduction to these computational models, in which the individuals, in this case microbes, that make up the system (population growing in a specific environment) are treated as autonomous and discrete entities (Railsback and Grimm 2012; Wilensky and Rand 2015). These students remembered well those sessions where they discovered the individual-based models (Ginovart 2014) with the help provided by the multi-agent programmable modelling environment called NetLogo, a free tool accessible on the Web (Wilensky 1999). As soon as the students had been trained in the use of this programmable modelling environment, they would be able to pass from the level of being users of simulators already prepared in this platform to the level of developing and implementing their own simple models, constructing their own microbial simulators, discrete and stochastic models of microbial populations (Ginovart et al. 2002; Hellweger and Bucci 2009). The use of this new modelling methodology would be an innovative approach in academia in this context, and thanks to being able to use a platform as well prepared as NetLogo. This type of task would be attainable and feasible for the near future. An approach where NetLogo could be used would encourage scientific (and mathematical) thinking across disciplines and would be an attractive initiative for teaching and learning with technology (see, for instance, Gammack 2015 or Levy and Wilensky 2011).

Innovative practices in modelling education research and teaching, pedagogical issues involved in these actions, research into (or evaluation of) this teaching practices, assessments in schools and universities of modelling tasks, and applicability of the modelling at different levels of schooling and in tertiary education are topics that require the attention of the mathematical community (Stillman et al. 2015). In relation to the influence of the technology in the mathematical education (Aldon 2015; Hitt 2015) and, in particular, of the computer in mathematical modelling, it is evident that further studies and more experience must be accumulated and analyzed for the emergence of new forms of activity in classrooms. Modelling activities can be reorganized, adjusted or extended according to the software available nowadays more easily than in previous years, since the benefits offered by computers are every day more versatile and powerful.

Conclusion

An amusing context for exploring mathematical ideas and developing mathematical skills is biology, where an assortment of models can be used in order to analyze and understand phenomena and to design and construct instruments that make a virtual "experimentation" possible, improving in an iterative way our representation of reality. With the help of the computer resources found in any academic framework, the set of results that the students have obtained, analyzed and discussed during the development of the activity designed, in connection with prior knowledge of mathematics and biology, were: (i) Mathematical transformations performed on the observed data and their graphical representations, (ii) Definitions and manipulations of mathematical functions to construct and formulate diverse models, (iii) Calculations or estimations of parameters involved in various types of models, and (iv) Identification and assignment of biological meaning to model parameters. The students perceived a significant learning scenario for the real world, with modeling methodologies used in the field of predictive microbiology and biotechnological applications connected to their Bachelor's degree.

The activity is made up of a set of tasks that after their implementation in the classroom can be classified as a set of "rich" tasks taking into account that:

(a) Students were placed at an interesting starting point with initial intriguing findings providing opportunities for the initial success and possibilities to go on encouraging students to develop their confidence, independence and freelance work.
(b) The students had the opportunity to reflect on issues and ideas presented previously in other subjects but, up to then, they had not been worked on together, establishing connections and relationships verified in the modeling context and deepening their comprehension.
(c) The actions executed stimulated creativity and imaginative applications of mathematical and biological knowledge that students already possessed in order to build new models, starting with initial and not complicated approximations and relatively simple models to go deeper into the mathematical understanding of more sophisticated models, where the help of the computer and its resources were revealed as necessary and indispensable.

It was rather surprising to find that although the computer resources used (the spreadsheet Excel, the mathematical program Maple and the statistical software R) were known to the students and used in former compulsory subjects, some initial difficulties in their use were discovered, and consequently the activity strengthened and expanded their digital skills. The ability to use computers undoubtedly helps students with their education in general, and in particular, in mathematical applied contexts like this one. The computer resources cannot in any way be overlooked or undervalued if you want to tackle problems in a real context (and not in an artificial or imaginary context prepared with "nice" or "simple" values or results). The use of computer resources made it possible to: (i) repeat calculations as many times as necessary, (ii) try multiple choices of values to see what would happen, (iii) check the achieved results regardless of whether they were right or not in the first election or completion, and (iv) visualize and correct mistakes, boost confidence in individual work. The opportunity of quick repeating or, automatically undoing and redoing, if not done accurately the first time, allowed students to gain assurance in the whole process of modeling. It was, in some ways, a learning and training activity where they were able to perform "virtual" experiments with mathematical functions and models. These computer-aided tasks allowed students to enhance their biological and mathematical knowledge.

The fact that the students had the opportunity to inspect the problem from several perspectives using a variety of resources was positively highlighted. These perspectives led to different solutions, but all meaningful and acceptable in the context of the study, which was much appreciated by them (and even surprising to some). In academia, and perhaps due to the necessity of assessing the answers of the students as correct or incorrect "in the day to day", there is an excess of issues that lead to only one correct solution, and students are better trained to get "the correct answer" than to get "plausible and right answers".

The use of the computer and its resources were justified and integrated efficiently during the entire activity, training and improving the digital literacy of the students, that is, the general ability to use computers to tackle real problems. The modelling process, in general, provides a way that connects the academic world to the real world, where students can apply mathematical contents in order to solve problems in a variety of situations. It is evident that the incursion of advanced computer resources facilitates new ways of doing, thinking about, and applying mathematics to other scientific subjects. The workflow configured in these sets of tasks could be readapted successfully to study other phenomena or processes, and some of the tasks could be used at other academic levels apart from tertiary education.

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Communication Inside and Outside the Classroom: A Commentary

Corinne Hahn

Abstract There is no doubt that technology facilitates access to information and induces new modes of communication. No doubt also that we should go beyond the technical aspects of the tool and explore how it can change classroom practice, pedagogical activities and the role of actors. This is what the five chapters in this Section explore.

Keywords Communication • Classroom practice • Pedagogical activities

There is no doubt that technology facilitates access to information and induces new modes of communication. No doubt also that we should go beyond the technical aspects of the tool and explore how it can change classroom practice, pedagogical activities and the role of actors. This is what the five chapters in this Section explore. Their work was conducted in very different contexts: primary education in Greece (Moutsios-Rentzos, Kalavasis, and Sofos), junior highschool in France (Aldon, Durand-Guerrier, and Ray) and Spain (Royo, Coll, and Giménez), higher education in Italy (Albano) and Spain (Ginovart).

Most authors focus their work on the meso dimension by describing innovative activities in the classroom based on technology. Their work draws on different forms of didactic engineering (Godino et al. 2013), forms that are often combined. They each offer a personal and innovative reading of the question of activity in the classroom. Brousseau's theory of didactic situations is at the heart of work of Albano and Aldon, Durand-Guerrier, and Ray, combined with issues related to problem solving or to design-based research based on Realistic Mathematics Education. Ginovart combines problem solving with computational thinking, while Royo, Call, and Giménez, adopt a design approach as they analyse the effects of a learning environment over several cycles of experimentation and redesign of the task proposed to students.

If the meso approach is predominant, one chapter (Moutsios-Rentzos, Kalavasis, and Sofos) is positioned at the overall level of the education system of a country,

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Greece. Their systemic approach leads these authors to adopt a quantitative methodology based on data collection by questionnaire, in order to investigate the views of primary school teachers and principals about mathematics, globalisation and social networks. The authors of the other chapters study the effects of their pedagogical device, conducting a systematic analysis of observed interactions (Royo, Coll, and Giménez), students' perceptions (Ginovart) or protocols implemented by students (Albano and Aldon, Durand-Guerrier, and Ray).

Beyond this brief comparative overview, I would like in this commentary to focus on three interrelated issues that emerged from my personal reading of the five chapters.

Questioning the Notion of Problem

Fabre (1997) explains that today any pedagogy proposes to confront students with tasks that can be grouped under the general heading of "problem". Indeed this word refers to three distinct but complementary semantic registers: project, obstacle and saliency. The concentration on problems, of course particularly strong in mathematics (Gellert and Hahn 2015), falls into two very different epistemological traditions. On one hand it relies to the Cartesian tradition, which focuses on solving problems, and, on the other hand, to a constructivist epistemology that takes into account the domains of the problems. The second epistemology emphasizes the construction of problems and not just their resolution. The use of technology should facilitate this construction, not only by opening up access to information but also by facilitating collaborative work and changing the nature of interactions in classrooms (Royo, Coll, and Giménez). It therefore suggests a different type of activities based on problems: activities that will integrate the change of roles and the change of frameworks. It does not mean only changing frameworks within mathematics (Douady 1986) but also to help students to cross boundaries between contexts and practices. Albano formalizes this issue of change of role by proposing to replace the didactic triangle by a tetrahedron. The teacher's role would be split into two complementary roles: author and tutor. The three roles of tutor, writer and student are thought of as both individual and collective. Indeed, the construction of the problem requires different expertise – educational, disciplinary, technological – and participation can take many forms. Students are encouraged to also adopt the author and tutor roles. It means that peer tutoring is favored but also that the student is involved in the construction of the problem. I am convinced that this is a crucial issue. Technology enriches the problem and the questioning, as highlighted by Aldon, Durand-Guerrier, and Ray, Royo, Coll, and Giménez, and Ginovart, and allows a collective construction of the problem (Royo, Coll, and Giménez). According to Fabre (1997), we should adopt a pedagogy that leads students to pose and build the problem themselves. Technology can help as it facilitates communication inside and outside of classrooms: these tools support questioning and thus students are able to co-construct questions and solutions. This is a common feature in the studies presented by Albano, Aldon, Durand-Guerrier, and Ray, Royo, Coll, and Giménez, and Ginovart. Albano led the students to collaboratively produce questions, then answers (to questions asked by other students), and to evaluate responses (provided by others). She observes that the phase where students produce questions, thus moving them to an author position, was the richest. It is a way for them to access problematizing in a context that is purely mathematical.

Questioning the Motion of Knowledge

Fabre claims that knowledge has three dimensions: historical, systematic and practical. Knowledge is part of a history, a culture, it is integrated into a structured body and it also has a use value. For example, "Thales theorem refers to an engineering method for measuring the pyramids, to its formalization in Euclid system and to its involvement in issues that have nothing to do with the measurement of inaccessible objects" (Fabre 2011, p. 126). Fabre adds that it is essential to organize a circular relationship between the three dimensions of knowledge. How can technology support this?

The authors of this part of the book show that, beyond the technical contribution, the tool – platform (Albano, Ginovart, Aldon, Durand-Guerrier, and Ray) or forum (Royo, Coll, and Giménez) – plays a mediating role. Royo, Coll, and Giménez shows that the use of the forum has an influence on the development of mathematical ideas. Many works on technology consider the cultural and the practical dimensions but seem sometimes to forget the systematic dimension. I think a great strength of the experiments described in these chapters is that the authors never lose sight of this dimension. Albano presents the results of a device whose purpose is to facilitate understanding algebra and calculus theorems. Ginovart studies the implementation of increasingly complex modelling methods based on different types of function (polynomial, logistic, sigmoid). Royo, Coll, and Giménez and Aldon, Durand-Guerrier, and Ray explore how students are driven to use algebra. They explain that technology allows them to build conjectures but not to prove the result obtained by calculating: the student is therefore led to use algebra.

According to a situated perspective, the construction of knowledge is due to the integration in a community of practice and the construction of the learner's identity. Boaler (2002) explains that to truly integrate disciplinary knowledge, students must develop a special relationship with the discipline and to project themselves as learners of it. Only then, can they implement the school's knowledge in other environments. What will be the effect of technological devices on the learners' identity dynamics? How does the change of role facilitate the construction of disciplinary knowledge? This change of role is present in all devices that are described: The student takes on the teacher's role through the two dimensions of tutor and author (Albano), or a researcher's role (Aldon, Durand-Guerrier, and Ray), or the role of a prospective professional (Ginovart). Ginovart's experiment shows that students do not spontaneously use tools learned with different school subjects (e.g., Excel, Maple, R) when they solve a problem in Biology. And yet to become a competent biologist, students need to combine their knowledge from different classes. The observations made by the authors seem to indicate in each case that these role changes have had a positive effect on construction of knowledge.

Questioning the Notion of Activity and Device

Communication takes place through devices leading the students to engage in activities to which technology can give specific forms. For example, by enabling collaborative work through a specific platform (Aldon, Durand-Guerrier, and Ray) or the use of forums (Royo, Coll, and Giménez). Because it achieves calculations faster and easier, technology facilitates problem solving but can sometimes make access to theory harder – as observed by Aldon and colleagues. For this reason, their device alternates the use of technology with periods of immersion in mathematical theory.

In a socio-cultural framework, building of knowledge and identity happens through participation in an activity. The French tradition, centred on the individual, distinguishes between activity –that which is required−, and task –that which is actually performed by the individual. Boudjaoui and Leclercq, drawing on Rabardel and Pastre (2005), see a device as an artifact that users turn into an instrument by putting part of themselves into it (Boudjaoui and Leclercq 2014, p. 26). It is obviously interesting to study affordances (Gibson 1977), and situations where learners behave as anticipated. But it is also often interesting to study cases of catachresis (Clot 1997), situations where learners turn away from the intended use. This should offer an opportunity to open a window onto how students give meaning to the activity and therefore, if one refers to Deleuze (1969), mobilize the three dimensions of sense making: "signification" ("the subject's relation to his actions"), "manifestation" ("relation to the concept") and reference ("relation to the world"). In fact, we have little information on possible catachresis which often led the authors to redefine their activities at successive stages of the design approach.

Beyond their innovative aspect which is likely to facilitate student motivation, the devices described by the authors also aim to develop intrinsic motivation related to students' personal questioning. This intrinsic motivation is too often undervalued as compared to the extrinsic motivation linked to the tool (Fabre 2011). On the contrary, this part of the book demonstrates how mathematical knowledge can emerge from problems in carefully designed activities using technology. This can be achieved by students taking on the role of the author of assessment questions (Albano), having to prove a result obtained through a technological tool (Aldon, Durand-Guerrier, and Ray), interacting via a forum (Royo, Coll, and Giménez), or using different tools related to various forms of expertise (Ginovart).

In an environment that frees itself of the classroom walls through technology, new relationships between knowledge, problem and activity have to be considered. This question seems to me at the heart of the papers presented in this section. The Moutsios-Rentzos, Kalavasis, and Sofos study in Greece suggests that teachers, principals and school advisors are ready to use educational innovations based on ICT. The results of the experiments analyzed by Royo, Coll, and Giménez, Aldon, Durand-Guerrier, and Ray, Ginovart, and Albano show positive effects on learning and engagement in mathematical tasks. Nevertheless, we need to analyze the effects of these devices in the longer term. What are the effects on students' behavior and the learning of mathematics in more traditional activities? What are the effects on the implementation of mathematical knowledge in out-of-school life? On students' career choices? These kinds of longitudinal studies are more complex to implement but important for those who want to measure the effects of these innovations at the micro, meso and also macro levels.

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Part V
Technology and Teachers' Professional Development

A Study on Statistical Technological and Pedagogical Content Knowledge on an Innovative Course on Quantitative Research Methods

Ana Serradó Bayés, Maria Meletiou-Mavrotheris, and Efi Paparistodemou

Abstract This chapter is part of a main study, which aimed to (a) analyse the affordances of a Quantitative Research Methods course towards developing students' Statistical Technological and Pedagogical Content Knowledge (STPACK), and (b) apply the STPACK model to investigate its effects in graduate Educational Studies. In particular, the chapter provides an example of raising teachers' awareness of statistical content and pedagogy about models and modelling through exploiting the model building affordances provided by a technological learning environment like TinkerPlots2® (Konold and Miller 2011). The model was applied in a Quantitative Educational Research Methods course with nineteen (n=19) Cypriot participants with different academic backgrounds.

Keywords Statistics • Technology • Pedagogical content knowledge • Quantitative educational research methods

Introduction

Educational research is a systematic process of inquiry and exploration of issues and problems in the field of education, with an aim of finding possible solutions. Thus, a Quantitative Research Methods Course targeting pre-service and/or in-service teachers should not only aim at providing participants with the knowledge and skills required to conduct quantitative research. Rather, such a course

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ought to acquaint teachers with the whole complexity of experimental research methodology, with the aim of encouraging the development of a "teacher as a researcher" mindset. However, students attending the course tend to experience negative attitudes due to the unfamiliarity and difficulty of statistical concepts and content, and the difficulties in linking theory with practice (Murtonen and Lehtinen 2003).

Considering the difficulties of concepts and content in Statistics learning, we have to distinguish between descriptive and inferential statistics. Descriptive statistics, which is devoted to the organization, summarization, and presentation of data, seems to be understandable by most students taking introductory courses. In contrast, inferential statistics is intended to reach conclusions that extend beyond the immediate data. This extension to the data analysis that promotes the transition from descriptive to inferential statistics is a known area of difficulties (e.g. Rubin et al. 2006).

With the aim of facilitating this transition, and metaphorically constructing a bridge between descriptive and inferential statistics, researchers, curriculum designers and tertiary teachers have begun to consider why and how modelling can help students to reason about formal and informal statistical inference. Some of the opportunities that models and modelling can provide to teaching statistics have been explored by researchers, including the following: (a) providing the basis for introducing estimation and hypothesis testing (Garfield and Ben-Zvi 2008); (b) fostering students' statistical thinking (Wild and Pfannkuch 1999); (c) steering probability learning (Batanero et al. 2005); (d) providing a choice of whether to access real world data (Graham 2006); (e) using technological tools to integrate exploratory data analysis approaches and probabilistic models through simulations and visualization (Eichler and Vogel 2014); and (f) developing new learning theories and proposed learning progressions to inform future standards and curriculum efforts in mathematics and science education (Lee 2013).

With all these opportunities in mind, and the adoption of a continuous improvement iterative model to progress on giving a more global view of statistical solving process as an organizer of the whole curriculum (Serradó et al. 2013), we aimed in this study to gain insights on how to analyse the needs of a Quantitative Research Methods course that promotes the development of teachers' Statistical Technological and Pedagogical Content Knowledge (STPACK) (Serradó et al. 2014). Furthermore, we examined how to innovate the design of the course, by putting models and modelling at the core of the curriculum.

The study, adopted an informal, data-driven approach to statistical inference using the dynamic statistic software TinkerPlots2® (Konold and Miller 2005) as an investigative tool. It sought to answer the following questions:

- How do teachers develop their awareness of statistical content and pedagogy about models and modelling through the model building affordances provided by a technological learning environment, like TinkerPlots2®?
- How does the modelling process integrate the elements of the STPACK in a Quantitative Research Methods course?

The pedagogical framework guiding this course is the Statistical Technological Pedagogical Content Knowledge (STPACK) (Serradó et al. 2014), a model for statistics teachers' professional development built based on the Technological Pedagogical Content Knowledge (TPACK) (Mishra and Koelher 2006), and Technological-Pedagogical Statistical Knowledge (TPSK) (Lee and Hollebrands 2011).

In this chapter, firstly we theoretically analyse the elements that highlight the need to integrate STPACK from a modelling perspective. Secondly, we apply STPACK to analyse the design of Model-Eliciting Activities (MEAs), with the aim of distinguishing the different processes of probabilistic modelling, contrasted with mathematical modelling processes, uses of technology, and students' productions about their interactions and reflections with the Dynamic Data Exploration software TinkerPlots2®. Thirdly, we discuss how these model-eliciting activities promote the development of different elements of STPACK. And finally, we conclude by describing how we envision the model-eliciting activities being designed in a Quantitative Research Methods course in Educational Studies, in order to promote integrated learning of models and modelling.

Models and Modelling: Mathematical, Statistical or Stochastic Pedagogical Content Knowledge

Statistics can be conceived as a process different from mathematics for dealing with variability in data. At the same time, stochastics can be regarded as a sub-domain of mathematics comprising probability and statistics. The first conception perceives probability as a domain of mathematics that enriches the subject, becoming an essential tool in applied mathematics and mathematics modelling (Franklin et al. 2005). Meanwhile, in a stochastic approach probability has not only the role of a "servant" of statistics, but also is the mathematical branch that models nondeterministic relationships, random phenomena, and decisions under uncertainty (Burril and Biehler 2011).

Using the language of models and modelling, as summarized by Blum (2015), this epistemological distinction can be categorized as *"descriptive models"* and *"normative models"* (Blum 2015). We think that this distinction does not suffice, and that we should also consider *"relational models"*. Their purpose should be the relation or connection between different models in the intra-mathematical or extra-mathematical field.

Furthermore, we consider that the differences between mathematics, statistics and stochastics are highly complex, and can only be understood through the lenses of the specific differences between: determinism versus non-determinism, certainty versus uncertainty, objectivism versus subjectivism, etc. It is not our aim to present a dualistic vision of both fields of knowledge; it is to have a praxeologic conception of human action that considers the specificity and generality that highlight the opportunities that models and modelling provide for understanding and clarifying this complexity. However, the understanding of these complexities could help the clarification of the Statistical or Stochastic Pedagogical Content Knowledge.

In the field of mathematics, Blum (2015) summarises the analysis done by different researchers about the categories and items describing the Pedagogical Content Knowledge (PCK) for modelling, and highlights the existence of four dimensions: (1) a theoretical dimension, (2) a task dimension, (3) an instructional dimension, and (4) a diagnostic dimension. We consider that these four dimensions are not adequate to describe the needs of the Statistical Pedagogical Content Knowledge (SPCK) for modelling. Primarily, because we think that it is also necessary to clarify how the different perceptions about the nature of statistics and probability configure different pedagogical views of the descriptive, normative or relational models. Secondly, we contemplate the need to integrate the technological pedagogical and content knowledge of models and modelling.

We are going to begin the analysis of the differences between the mathematical, statistical and stochastic pedagogical content knowledge through the identification of the theoretical dimension of the modelling cycles. Different schemas for the modelling cycles, each with specific strengths and weaknesses depending on their purposes, have been described in the mathematical literature of modelling (Blum 2015). Among them, we distinguish the seven-step-process of Blum and Leiss (2007): (1) constructing, (2) simplifying/structuring, (3) mathematizing, (4) working mathematically, (5) interpreting, (6) validating, and (7) exposing.

Different claims can be made from a statistical and stochastic point of view to point out the weakness of this scheme. On one hand, the transition from observational to theoretical concepts of probability, as stated by Von Mises (1964) *"cannot be completely mathematized. It is not a logical conclusion but rather a choice, which, one believes, will stand up in the face of new observations"* (p. 45). Moreover, these theoretical concepts can be analysed from different approaches: classical, frequentist and subjectivist.

On the other hand, Blum and Leiss's (2007) seven steps can be considered as formal expressions of the investigative cycle. We believe that working only from an investigative perspective is not adequate for developing mathematical, statistical and probabilistic thinking. Besides, these investigative processes should grow at the same compass of the interrogative cycle (Wild and Pfannkuch 1999), helping to develop the process of hypothesis generation. The maturation of the different perspectives on teaching statistics and probability has had repercussions on the conception of the modelling process.

Knowledge for Teaching and Learning Statistics and Probabilistic Modelling

The first attempts to integrate the interrogative and investigative cycle were made by Engel (2002) when describing a five-step process for introducing future secondary teachers to applied mathematics and modelling: (1) introduction of a "real-world" problem involving some data analysis activity to experience the dynamics of the phenomena; (2) building of a simulation model in a technological environment; (3) generation and analysis of data, including simulation-based inferences; (4) critical evolution of conclusions and reflection on the impact of our assumptions; and (5) mathematical analysis based on probability and mathematical statistics.

Additionally, Chaput et al. (2011) describe three stages in the modelling process from a teaching perspective: (1) pseudo-concrete model (putting empirical observations into a working model); (2) mathematization and formalization (translating working hypotheses into model hypotheses to design probability model); and (3) validation and interpretation in context (checking fit of a probability model to data).

If instead of analysing the modelling process from the teaching perspective, we consider the learner and his/her learning process, which is dependent upon the level of the learner, we can distinguish between data-driven and theory-driven approaches. In *data-driven approaches*, students experience models for which there is no theoretical probability model or the presumed theoretical model is inadequate. The introduction to modelling is done through measurement activities that help learners to reflect on the variability of the data, getting a sense of the measurement distribution, and appreciating the types of measurement errors. This, in turn, can lead them to construct the model observed measurement as an initial theoretical view of the real world system for their probability model, fit the probability model to the data, check whether the model is an adequate model of the real world or not, and then adjust the model until obtaining a working model that adequately reflects the actual data distribution (Konold and Kazak 2008).

Meanwhile, in a *theory-driven approach*, both the problem and "data" are given, from which students are able to recognise the underlying theoretical model, proposed by the teacher. Students use the model based on its "goodness of fit" to predict future outcomes in the real world system, although it is not built or tested a priori. In this case, students work within the enclosed world of the model, asking questions about the model, looking at the consequences of the model, and using the model to choose between different actions to improve a situation (Borovcnik and Kapadia 2011).

Attending to students' learning, Pfannkuch and Zledins (2014) argue that independent experiences using data-driven or theory-driven approaches are not enough. Students need to also appreciate the circularity between theory-driven and data-driven probability modelling (Pfannkuch and Zledins 2014), and understand that modelling is an iterative cycle, which leads to more insights in a step-by-step manner (Borovcnik and Kapadia 2011).

Again arise the complexity of establishing the limits between reality, model of reality, and theory. And in this case, probability serves to model reality and impose a specific structure upon it.

Pedagogical Epistemic Components of Probabilistic Modelling Approaches

Restricting now the complexity of models and modelling to the transfer between the empirical world (of data) and the theoretical world (of probabilities), Eichler and Vogel (2014) illustrate three modelling perspectives regarding the classical, frequentist and subjectivist approach to probability. The modelling structure with regard to the *classical approach* is conceived as a unidirectional way between the theoretical world, where the theoretical model is built, and the empirical world, where the model is validated. The modelling structure with regard to the *frequentist approach* is bidirectional. Beginning in the empirical world, where the analysis of the available empirical data and the detection of patterns occur, then building a theoretical model base, and finally returning to the empirical world to validate the model. For the modelling structure with regard to the *subjectivist approach*, iterative cycles of getting information in the empirical world and re-building a theoretical model are suggested. The three modelling structures begin with the definition and structuring of the problem at hand.

Considering the different situations that could be regarded concerning approaches of probability measurement and the different roles played by data during the modelling process, Eichler and Vogel (2014) differentiate three kinds of problem situations: (a) *virtual problem situations* which contain all necessary information and represent a stochastic concept; (b) *virtual real world problem situations* which demand analysing a situation's context that is more "authentic" and provide a "narrative anchor"; and (c) *real world problem situations* that include the aim to reproduce real societal problems.

The differentiation of problem solving is a key for deciding the kind of modelling to be done in coherence with the probabilistic approach. In this case, the problem and the probabilistic approach selected are conceived as the lens through which to structure the modelling process from reality. Furthermore, different epistemological perspectives confront reality and mathematical objects, and conceive the problems as a lens through which to view reality (Serradó and Gellert 2015). For example, when Borovcnik and Kapadia (2011) argue that a problem posed to students has to be clear-cut, with no ambiguities involved –neither about the context nor about the questions, they are emphasising the role that virtual problem situations in mathematical reality have in a theory-driven approach to modelling. Meanwhile, data-driven approaches tend to work in either virtual real world problem situations or real world problem situations. Furthermore, the structuring of sequences of different kinds of problems organized as tasks with or without a common context, allow exploring the relationship between both approaches (Konold and Kazak 2008).

Nonetheless, determining desired characteristics of the tasks to promote mathematical, statistical and probabilistic modelling still remains an open question for research. In the field of mathematics, there are many rich teaching/learning environments aimed at modelling (Blum 2015), including Lesh's Model Eliciting Activities (MEAs) (Lesh and Doerr 2003). In the current study, several MEAs were used in our Quantitative Research Methods course, with the goal of developing a conceptual tool that goes beyond being useful for some specific purpose in a given situation, to being reusable in other similar situations. In MEAs, difficulties can arise when students are asked to make symbolic descriptions of meaningful situations (Lesh and Doerr 2003). In general, students can face difficulties when advancing between the different steps of the modelling process (Blum 2015). In the case of the field of Statistics and Probability, the difficulties that can emerge during the teaching and learning process have been largely investigated (see for example Batanero et al. 2005; Ben-Zvi et al. 2015; Borovcnik and Kapadia 2014; Serradó 2015a; Watson 2005).

Pedagogic Cognitive Component: Obstacles on a Probabilistic Modelling Process

In this section, we classify these difficulties using the language of obstacles, introduced by Brousseau (1997), who identified three types of obstacles: epistemological, didactical and ontogenic. We consider those obstacles, which can theoretically emerge in a probabilistic modelling process (Serradó 2015b).

The *epistemological obstacles* are usually identified from a historical analysis, since they coincide with difficulties that arose in the development of the subject (Brousseau 1997). We can distinguish four epistemological obstacles in a frequentist approach: (a) considering probability as the expected value instead of the theoretical value; (b) conceptualizing the convergence on probability when establishing the Law(s) of Large Numbers; (c) considering individual events instead of an aggregate view of data to conceptualize the probability distribution; and (d) failing to distinguish between experimental value, modelled value and theoretical value of the probability of an outcome.

The *didactical obstacles* are related to the way a topic is taught (Brousseau 1997). In this chapter, we consider those didactical obstacles that arise from the selection of models during the teaching and learning process. Those obstacles are: (a) confusion of a model with reality, instead of considering models as an approximation of reality; (b) failure to recognize that the purpose of the model is to analyse its "goodness" in relation to its accuracy in representing the real world; (c) failure to recognize that the purpose of the model is to predict future outcomes in the given world system; and (d) lack of circularity between the theory-driven and the data-driven probability modelling approach.

The *ontogenic obstacles* are related to learners' cognitive development due to lack of prior knowledge, observation conditions, and developmental limitations: (a) lack of prior knowledge can lead to biases, misconceptions and fallacies related to the consideration of a frequentist probabilistic approach; (b) obstacles related to observation conditions, can appear when students have to construct complex models of compound experiments based upon simpler ones, and the credence of deterministic behaviour of the events; (c) inadequate developmental maturation of cognitive structures thinking; and (d) inadequate and encompassed maturation of three knowledge structures: distribution, sampling, and variability.

In a data-driven approach, this encompassed maturation causes students to overrely on sample representativeness, believing that a random sample has to be representative of the population, and that it is not randomness but some other mechanisms that have caused sampling variability. Meanwhile, in the theory-driven approach for modelling, ontogenic obstacles can emerge if students do not encompass and integrate three notions: (a) variability of results when repeating an experiment; (b) stability of frequencies of observed outcomes; and (c) relation between the value of the limit of frequencies, the distribution of possible outcomes and the theoretical value of the probability.

Models and Modelling: Use of Technological Tools

The affordances of technological tools have improved the situation, helping to surpass some of the aforementioned obstacles. As pointed out by Blum (2015), a lot of case studies show that digital technologies can be used as powerful tools for modelling activities, and for extending the modelling cycle by adding a third world: the technological world (Greefrath et al. 2011). Furthermore, existing research shows that effective teacher preparation is an important factor for successful integration and sustainability of ICT in education (BECTA 2004; Davis et al. 2009; Hennessy et al. 2007). In the case of models and modelling, the integration of the two approaches, data-driven and theory-driven, allows using technology for different purposes: accessing meaningful data, exploring data, and conducting data simulations.

Accessing Meaningful Data

Chance et al. (2007) and Gould (2010) describe the benefits afforded by technological tools for teaching statistics when accessing large, messy, real world data. These tools give students the opportunity to engage in the first steps of the modelling process when adopting a data-driven approach within the real context, and to appreciate that the purpose behind using statistics is to make sense of questions that arise from context. In quantitative approaches, these practices should include developing good searching habits, learning what types of search engines to use for certain purposes, and which websites and organizations are posting trustworthy data and information. Also, to improve the capacity to process complex multivariate data, presented in different formats (table, graph on website, within a PDF file, downloaded as XLS, CVS, TXT, etc.), and often aggregated in many different ways that make it more or less useful (Lee 2013). In fact, this quantitative reasoning consisting in the practice of finding, trusting and processing data into a form can add meaning to developing hands-on tasks. Nevertheless, meaningful hand-on tasks should lead to a habit of question posing and the promotion of exploratory data analysis.

Exploratory Data Analysis

Technologies such as Fathom®, TinkerPlots2® and Probability Explorer® allow the dynamic control over data – meaning that as data changes, representations of that data dynamically update. These technological tools are used as *amplifiers*, giving an opportunity to focus on conceptual understanding and time to engage in exploratory data analysis. They are also used as *reorganisers*, helping to reorganise and change a statistical conception of students or teachers (Lee and Hollebrands 2011). Several researchers have been exploiting the affordances provided by these tools for promoting learners' ability to reason about data-based inferences, with highly encouraging results in primary, secondary and tertiary education (e.g. Meletiou-Mavrotheris et al. 2015). Furthermore, working with dynamic visual representations of data can help students to develop mental models of possible relationships between multiple variables (Nicholson et al. 2010).

Working with a variety of digital tools, modelling practices involve sense making of real world phenomena and defining attributes of the phenomena that can be measured in some way. Once a model is designed, digital environments afford the collection of data from the model that can be used to test the model by comparing model-generated data to real-world data. That can be considered a particular aim of simulation.

Simulating Data

Eichler and Vogel (2014) identify three different roles of simulation as a tool: (a) explore a model that supports learners' prediction of future frequencies on such a well-known model to the estimation of a frequentist probability based on relative frequencies concerning a (large) data sample; (b) develop an unknown model approximately producing computer-based virtual data that could support students by assuring them of their theoretical considerations before, during and after the modelling process; and (c) represent data generation that might facilitate the comprehension of a problem's structure concerning a subjectivist approach, and build adequate mental models of the problem domain.

The flexibility of many simulation tools allows the learner to: (a) using algorithms and models to input the properties of a theoretical distribution that would control the pseudorandom number generation; (b) controlling parameters such as sample size; and (c) displaying graphs generated in real time (Lee and Hollebrands 2011). However, sole knowledge of the uses and flexibilities of digital tools does not suffice for facilitating teachers' professional development on Quantitative Research Methods.

Statistical Technological Pedagogical Content Knowledge for Models and Modelling

Summing up, we consider the existence of three approaches (descriptive, normative and relational), which model the relationships between the mathematical, statistical, stochastic, and probabilistic content knowledge. From a pedagogical point of view, we maintain the need of encompassing the statistical investigative process, identified in a "grosso modo" with the modelling process, and the interrogative cycle, which should help to develop the process of hypothesis generation. We understand this process as a process of maturation of the thinking of the learner. From the point of view of the learner, students can experience data-driven and theory-driven approaches to modelling or the circularity between them. The complexity of this circularity can be understood through the analysis of the transfer between the empirical world (data) and the theoretical world (probabilities), where we can find three modelling approaches: classical, frequentist and subjectivist. In the three approaches, data plays different roles during the modelling process, that allow us to identify three levels of problem situations: virtual problem situations, virtual real world problem situations, or real world problem situations.

We think that MEAs, based on these three kinds of situations, can promote mathematical, statistical and probabilistic modelling environments. We do acknowledge the fact that when students get involved in solving such MEAs, epistemological, didactical and ontogenic obstacles might emerge. Nevertheless, the affordances of appropriate technological tools can help to surpass some of these obstacles. Those technological tools are used in data-driven and theory-driven modelling approaches to access meaningful data, explore data, and conduct data simulations. From a cognitive point of view, such technological tools may act as amplifiers or reorganisers of the teachers' and students' conceptions. The following figure (see Fig. 1) summarises the possible uses of the technological affordances

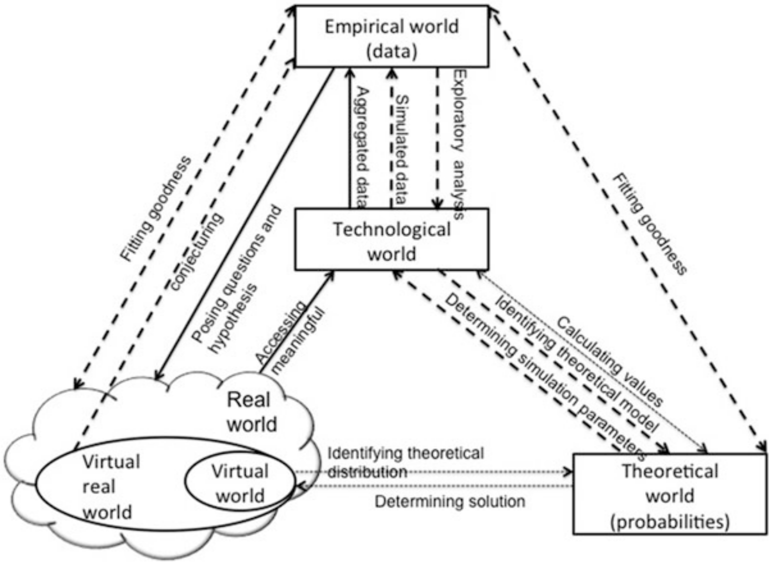

Fig. 1 The four world modelling processes

when transferring from the empirical and theoretical world, considering the different roles that data plays in the modelling process.

In coherence with previous proposals of Lee and Hollebrands (2011) and Serradó et al. (2014), and in order to help teachers develop their STPACK and improve their knowledge and skills in applied research methods, we have designed the MEAs used in the current study with three levels of questions. Firstly, we included in the MEAs questions that engage teachers in statistical thinking as doers of statistics, using technology as a tool. Secondly, we also encompassed questions that provide teachers with opportunities to reflect on the uses of the technological tools to improve their own learning. And, finally we included questions that require teachers to consider the pedagogical issues related to the modelling process.

The case study that we present in this chapter aims to provide an example of how to integrally develop teachers' understandings of statistical content, technology, and pedagogy. In particular, the course aims to prepare teachers to scaffold and extend students' reasoning about models and modelling. Furthermore, it aims to improve their understanding of the similarities and/or differences between mathematical modelling and statistical modelling through the model building affordances provided by a technological learning environment like TinkerPlots2®.

In this section, we present an integrated analysis of the affordances that technology could provide in a course devoted to building expertise in quantitative educational research methods that has models and modelling at its core. In particular, we conceive STPACK as expertise knowledge of the ways to facilitate students' learning of different statistical concepts through appropriate pedagogy and technology. In this chapter, we extend the previously developed Statistical

Technological Pedagogical and Content Knowledge (STPACK) items, published in Serradó et al. (2014):

STPACK1: Understanding of students' learning (reasoning and thinking) of statistical ideas through the use of ICT tools for models and modelling, and reasoning on the difficulties and obstacles (epistemological, didactical and ontogenic) in a modelling process.

STPACK2: A critical stance towards the use and evaluation of MEAs curricula materials for teaching and learning models and modelling with technology.

STPACK3: Conceptions of how technological tools and representations support models and modelling: providing access to meaningful data, acting as amplifiers and reorganisers in an exploratory data analysis, and enabling data simulations.

STPACK4: Instructional strategies for developing models and modelling process, distinguishing their purpose: relational, normative and descriptive; classical, frequentist and subjective approaches; and pseudo-models, data-driven and theory-driven approaches.

STPACK5: Distinguishing the role of problems in the modelling process: virtual problem situations, virtual real world problem situations and real world problem situations.

STPACK6: Design and development of complete learning scenarios based on models and modelling using ICT tools.

Methodology

The teaching experiment adopted a non-conventional approach to the teaching of the Quantitative Research Methods course, which put models and modelling at the core of the curriculum. In designing the teaching experiment, we ensured that our intervention covered the set curriculum included in the course syllabus. However, we expanded the curriculum by including, throughout the semester, activities that aimed at raising students' awareness of models and modelling, and of their usefulness in research settings involving statistical investigations.

The study presented here had three objectives: (a) extending the previous Statistical Technological Pedagogical and Content Knowledge framework (STPACK) to integrate models and modelling in Inferential Statistical Reasoning in a Quantitative Research Methods course in Educational Studies; (b) applying this theoretical framework to a case study of the STPACK curricular dimensions for models and modelling to the analysis of two MEAs activities; and (c) analysing students' productions about their interactions and reflections with the Dynamic Data Exploration software TinkerPlots2®.

Case Study on the Model Eliciting Activities

This case study consisted of a content analysis of the STPACK curricular dimensions of models and modelling related to two selected model-eliciting activities: "Helper or Hinderer" and "How many tickets to sell?" The activities were carefully designed to support but, at the same time, also explore students' evolving understandings of fundamental ideas related to statistical inference in context, as they engage in models and modelling for simulating data and evaluating their research claims and hypotheses.

The "Helper or Hinderer" MEA is based on an actual research study reported in a November 2007 issue of *Nature* (Hamlin et al. 2007), in which researchers investigated whether infants take into account an individual's actions towards others in evaluating that individual as appealing or aversive. Sixteen (n=16) 10-months-old infants were shown a "climber" character that could not make it up a hill in two tries. Then they were alternately shown two scenarios for the climber's next try: a scenario where the climber was pushed to the top of the hill by another character ("helper"), and a scenario where the climber was pushed back down the hill by another character ("hinderer"). This was repeated several times. Then the infant was presented with both characters and was asked to pick one to play with. The researchers found that 14 of the 16 infants participating in the study chose the helper over the hinderer. This research was converted into a MEA (http://www.tc.umn.edu/~catalst/materials.php), connecting chance models and simulation in the CATALST project, and was adapted by Meletiou-Mavrotheris and Appiou-Nikiforou (2015) using model and modelling to support reasoning about statistical inference.

Meanwhile, the "How many tickets to sell?" MEA is based on the following fictitious scenario: "*Air Zland has found that on average 2.9% of the passengers that have booked tickets on its main domestic routes fail to show up for departure. It responds by overbooking flights. The Airbus A320, used on these routes, has 171 seats. How many extra tickets can Air Zland sell without upsetting passengers who do show up at the terminal too often?"* This MEA was initially designed to promote practice in the use of the Poisson distribution as an approximation to the Binomial Distribution (http://www.umass.edu/wsp/resources/poisson/proble-ms.html). Then, the problem was adapted (http://new.cen-susatschool.org.nz/resource/using-tinkerplots-for-probability-modelling/) with the aim of connecting experimental and theoretical approaches to probability. And, finally the scenario was readapted to analyse how technology was used (Meletiou-Mavrotheris et al. 2015).

Each of the two MEAs, composed of the word problem and a number of didactical questions to work with, was codified four times. Firstly, identifying and codifying the different actions of the modelling process. Secondly, identifying in which world each action was performed: real, virtual real, empirical, theoretical or technological world. Thirdly, for those actions in the technological world, categorizing the purpose of the use of technology: access to meaningful data, exploratory analysis, visualization and simulation. And, finally for each purpose of the technology established, determining the flexibility in using it.

Case Study on Uses of Dynamic Data Exploration Software TinkerPlots2®

The teaching experiment took place in a Quantitative Research Methods course targeting graduate students enrolled in an M.A. in Educational Studies program offered at European University Cyprus. The course content and structure was such that it encouraged “statistical enculturation”. Statistical thinking was presented as a synthesis of statistical knowledge, context knowledge, and the information in the data in order to produce implications and insights, and to test and refine conjectures. Probability was not presented as a modelling tool. Probability distributions were presented as models based on some assumptions which, when changed, might lead to changes in the distributions. The emphasis was not on teaching their formal properties, but on helping students understand why and where one could use these probability distributions to model a certain phenomenon, and in what ways this is useful. In this process of statistical enculturation, the use of the software Tinkerplots2® was instrumental in the individual evolution of students’ use and reasoning about the exploration and simulation affordances. Giving each student the challenge to use any of the Tinkerplots2® affordances when engaged in the open-ended MEAs activities.

The course began on the first week of October 2014 and was completed at the end of January 2015. The second author was the course instructor. There were nineteen (n=19) students enrolled in the course. Participants were either pre-service or in-service teachers, who were characterized by diversity in a number of parameters including age, and professional and academic background, summarized the table (see Table 1).

Their age ranged from 23 to 42. Some had an academic background in primary education, while the rest were secondary school teachers in different domains (languages, humanities, natural sciences, physical sciences etc.). Students also had a varied background in statistics. Most of the older participants (aged over 30) had very limited exposure to statistics prior to the course and had never formally studied the subject, while the younger ones had typically completed a statistics course while at college. However, even those students who had formally studied statistics in the past had attended traditional lecture-based statistics courses that made minimal use of technology. Thus, upon entering the course, almost all of the students had very weak statistical reasoning and a tendency to focus on the procedural aspects of statistics.

The case study used classroom observation, videotaping, interviews of selected students, and student work samples to investigate learners’ interactions with TinkerPlots2®, and to document the different ways in which students’ engagement

Table 1 Sample characteristics

Gender	
Male	*26% (n=5)*
Female	*74% (n=14)*
Age	
Under 30	*58% (n=11)*
30–40	*26% (n=5)*
Over 40	*16% (n=3)*
Professional Status	
Pre-service teacher	*53% (n=10)*
In-service teacher	*47% (n=9)*

with data modelling activities influenced their understanding of key ideas related to inferential statistics. The collected data were transcribed (interviews and videotaped episodes), coded, and analysed to guide the investigation on the impact of the intervention on participants' learning. For the purpose of analysis, we did not use an analytical framework with predetermined categories. What we instead did was to identify, through careful reviewing of the transcripts, student work samples, and other data collected during the course, recurring themes or patterns in the data.

Based on this process, the collected data was eventually codified to identify the following narratives about the uses of the Dynamical Software TinkerPlots2®: access to meaningful data, data exploration, visualization and simulation, and metacognitive use of technology as an amplifier or a reorganiser of statistical knowledge.

To increase the reliability of the findings, the activities were analyzed and categorized by all three researchers. Inter-rater discrepancies were resolved through discussion.

Results

In this section, we present the results corresponding to the modelling processes and their relation to the four world interaction of both case studies related to the "Helper and hinderer" and "How many tickets to sell?" Model Eliciting Activities (MEAs).

Four World Modelling Cycle

The content analysis about the case study on the MEAs has allowed identifying a four world modelling cycle, with differences between each task. The *"Helper or hinderer"* MEA processes related to the four worlds are summarised in the figure (see Fig. 2), and the processes related to the modelling cycle are described below.

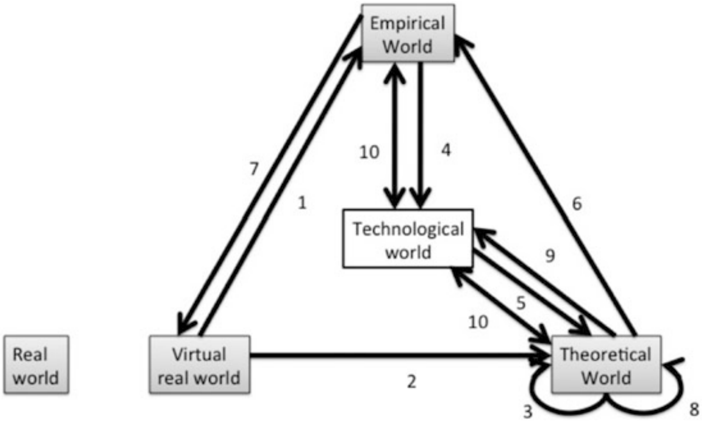

Fig. 2 Four world modelling cycle of "Helper or hinderer" MEAs

1. Introduce the "virtual real world" through a detailed reading of the word problem and ask students to make conjectures about the answer to the research question.
2. Put empirical observations into a working model through reflection on the evidence of rejecting the suggested theoretical chance model.
3. Do a mathematization of the null model by identifying all the parameters needed that are fully described in the activity.
4. Build a simulation model in the TinkerPlot2 technological environment based on the theoretical analysis.
5. Conduct mathematical analysis of the modelled distribution displayed to draw conclusions experimentally and theoretically.
6. Check whether the model is adequate or not with different large numbers of simulated outcomes using TinkerPlots2®.
7. Predict future outcomes in the virtual world system to answer the question posed in the word problem.
8. Mathematize the probabilistic model through making assumptions about the distribution.
9. Generate the theoretical probabilistic value using Tinkerplot2.
10. Compare the probability theoretical value with the sampling experimental value modelled, using the results obtained in the technological world.

The *"How many tickets to sell?"* MEA processes are summarised in the figure (see Fig. 3), and described below:

1. Reason on the real virtual world through a detailed reading of the word problem and on the real world system through drawing upon students' real world knowledge and experiences regarding airlines' practices in overbooking flights.
2. Build a pseudo-concrete model (putting empirical observations into a working model).

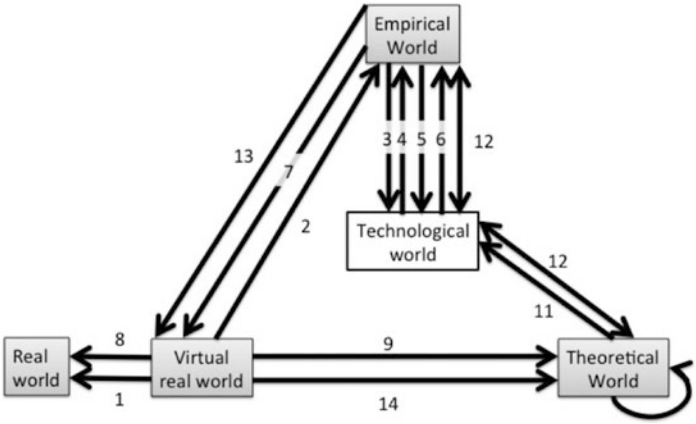

Fig. 3 The four world modelling cycle for "How many tickets to sell?" MEA

3. Build a simulation model in Tinkerplot2 through reasoning on some open questions about the parameters to analyse.
4. Readjust the model (without expression of the conditions for the new parameters).
5. Run the model with those new parameters readjusted.
6. Describe the sampling distribution and analyse how it fits to the generated data.
7. Predict future outcomes into the "virtual world system" to answer the question posed in the word problem.
8. Reason in the "real world system" to describe the consequences of this decision.
9. Recognise the underlying theoretical model.
10. Design the probability model recognised in the theoretical world.
11. Generate the probabilistic theoretical value using Tinkerplot2.
12. Compare the probabilistic distribution with the sampling modelled distribution.
13. Reaffirm the predictions in the "virtual real world" system.

On the description of the modelling cycle, we have presented different uses of TinkerPlot2, which we will next describe in more depth.

Uses of TinkerPlot2® in the Modelling Process

Both activities proposed have been categorized as "virtual real world" problems, because they have all the needed information to solve the problem and students do not need to access or organize meaningful real data. Consequently, to obtain data students have to generate it using TinkerPlots2® as a simulation tool. We can find differences in this simulation process, due to the proposed modelling cycle established in each MEA.

Fig. 4 Outcomes of single sample simulations, under the null model, of the "Helper or Hinderer" MEA

In the "Helper or hinderer" activity, three uses of the technological tool are suggested. Firstly, building a simulation model in the TinkerPlot2® technological environment under the null model based on the theoretical analysis (see Fig. 4).

As a result of the interaction with technology, students in our study reasoned: *"Results vary but 14 or more infants out of 16 selecting the "helper" character almost never occurs. Thus, it seems that infants really have a preference and are not just picking a toy at random"*. Their reasoning leads us to think that they have used the simulation tool to explore the theoretical well-known null model to the estimation of the frequentist probability based on relative frequencies of a data sample. They also have reckoned upon the flexibility of the technological tool to display different graphs in real time.

Secondly, acknowledging the need to repeat the experiment a very large number of trials, students chose the "Collect Statistics" feature of TinkerPlots2® to keep track of the number of students picking the "Helper" toy each time. Students were able to test whether the model is adequate or not with repeated, large samples of simulated outcomes. In the figure (see Fig. 5) for example, we see the distribution of sample statistics drawn by a group of students who repeated the experiment 2000 times.

Students were able to develop an unknown model producing computer-based virtual data that could support their theoretical considerations after the modelling process. As expressed by these students: *"a number as high as 14 or higher is very rare since almost all numbers in the graph are smaller than 14"* or *"the distribution of the collected statistics is normal and its center is close to 8 which is the mean expected value if we assume no real preference among infants."*

Next, students were asked to generate the theoretical probabilistic value. They used the properties of the binomial distribution to determine the theoretical probability of at least 14 out of 16 infants randomly choosing the "helper" toy under the chance model. Finally, students were asked to compare the theoretical probabilistic value with the experimental modelled value. Although no extra comparison between the sampling modelled distribution and the theoretical one was asked, that would help to fit the goodness of the model.

In the "How many tickets to sell? MEA, three different uses of the technological tool were made. Firstly, students were asked to build a simulation model in

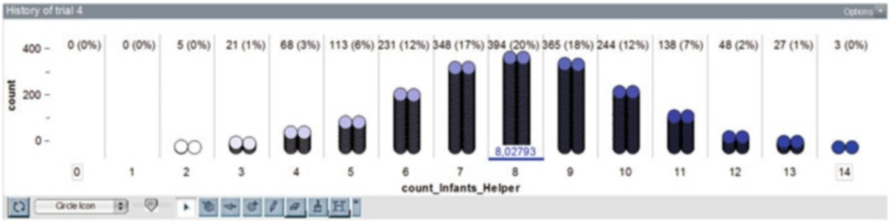

Fig. 5 Histogram of sample statistics for the "Helper or hinderer" MEA- A measures collection of 2000 samples

TinkerPlot2. Students ran TinkerPlot2 to obtain different cases in order to understand if they were working with a chance model, considering also the null model (see Fig. 6).

We consider that the purpose of using the technological tool was to represent data generation that might facilitate the comprehension of the problem's structure and building an adequate chance mental model of the problem domain (*"the data changed each time, as there is the concept of chance"*).

Secondly, students were asked to readjust the data through repeating the experiment a large number of times and to draw the resulting distribution of sample statistics. This was done for the scenario in which AirZland sold five extra tickets (see Fig. 7) and four extra tickets (see Fig. 8).

In this case, students used the simulation tool to develop an unknown model approximately producing computer-based virtual data. The comparison of both histograms led them to develop their theoretical considerations during and after the modelling process: *"Although the possibility of 5 passengers not showing up has the highest chance, when looking at 176 seats there is a 43% of chance of some passengers not getting a seat [. . .]. On the other hand, when booking 175 tickets rather than 176, the chance of having passengers who are unable to get a seat due to overbooking is reduced from 43% to 21%"*. Students were able to analyse the flexibility of the simulation tool when controlling the sample size and the number of tickets.

Thirdly, students were then asked to generate the theoretical probabilistic value. They used the properties of the binomial distribution to determine the theoretical probability for 176 seats. Finally, students were asked to compare the theoretical probabilistic value with the experimental modelled value. Although they were not explicitly requested to make any additional comparison between the sampling modelled distribution and the theoretical one, doing so would help them in testing the goodness of the model. Furthermore, students were able to analyse the flexibility of TinkerPlots2® to input the properties of a theoretical distribution (Meletiou-Mavrotheris et al. 2015).

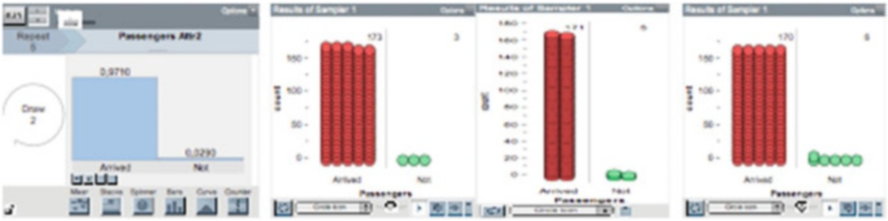

Fig. 6 Outcomes of single sample simulations, under the null model, of the "How many tickets to sell?" MEA

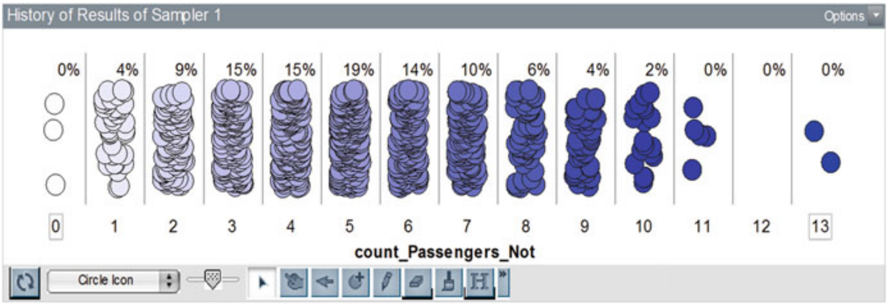

Fig. 7 Outcomes of 1000 simulations for the 176 of "How may tickets to sell?" MEA

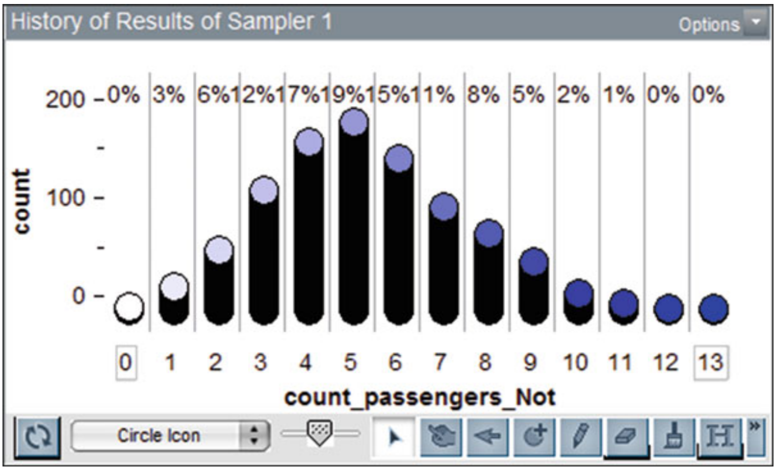

Fig. 8 Outcomes of 1000 simulations for 175 tickets of the "How many tickets to sell?" MEA

Metacognitive Use of the Technology: Amplifier and Reorganiser

We can observe differences in the use of technology between the “Helper or Hinderer” and the “How many tickets to sell?” MEAs. The “Helper or Hinderer” MEA proposal for interaction with technology acted basically as an amplifier, providing students with the opportunity to focus on conceptual understanding and time to engage in exploratory analysis due to the data generated through simulation. Participants improved their conceptual understanding of the null model (*“Based on our analysis, there is strong evidence against the null model”)* or the normal distribution *(“the distribution of the collected statistics is normal and its centre is close to 8 which is the mean expected value if we assume no real preference among infants)*. However, they argued about the possibilities of becoming a reorganiser of the statistical conception for the researchers of the real world experiment: *“The experiments showed that infants are not making their selection based on chance, but tend to choose a helper toy. The reasons for this tendency warrant further investigation by researchers”*.

Meanwhile, the “How many tickets to sell?” MEA proposal for interaction with the technology acted both as an amplifier and a reorganiser of knowledge. As in the case of the “Helper or hinderer” MEA, the “How many tickets to sell?” MEA provided students with the opportunity to focus on conceptual understanding. The simulations helped them *“to understand these theoretical probabilities”*, and the sample distributions: *“It [the software] simulated more that 3000 flights in few minutes, so we got a lot of data and we understood how the distributions works”*. The flexibility of TinkerPlot2 in generating different graphs proved particularly helpful: *“I saw how graphs work. I had direct access to data and I could make changes and see how these changes influenced the whole graphs. That made things clear”*.

The software also acted as a reorganiser of knowledge, contributing to the change of some students’ statistical conceptions. For example, Tinkerplots2® helped improve students’ conceptions about randomness (*“It was useful for me because I understood how a random procedure with large numbers can work. This is very important for understanding randomness”)*, or conceptions about why to use a particular theory *(“The software helped me to understand generally how a distribution works in theory. Working with TinkerPlots for simulating and analysing our data, was when I understood why we used the particular theory”)*.

Of particular interest is the case of one student (n=1) who used TinkerPlots2® only to understand the existence of a chance model, and then to reorganise his understanding of the problem *(I worked with TinkerPlots only for making graphs. I didn’t use it for anything else. I understood the distribution, so I didn’t need to do anything for solving the problem)*. So, we can say that he did not complete all the modelling cycle. His modelling cycle comprised of processes 1, 2, 3, 4, 9, 10 of Fig. 2.

Nevertheless, there were four students (n=4) who argued that they did not use the TinkerPlot2 affordances. One of the teams reasoned that they did not use the technology, because they recognised the problem as a traditional word problem on statistical inference (*"The software did not help us at all. We realised at a first glance that we had a binomial distribution. Here we had 'arrivals" and 'no-arrivals' and independence=0,029. So, we used the formulas and we solved the problem")*. When inquired about their traditional conceptions of solving a statistical inferential problem, one of the team members answered: *"I believe that the important thing is to understand the distribution and use the right formula. In that case, it wasn't useful for us to run the data in the software, as the result was found directly by using the formula"*. Our interpretation of this is that the student was identifying solving the problem with an incomplete theory-driven modelling approach, corresponding to processes 1, 9 and 13 of the modelling cycle of Fig. 3.

Trying to understand why these four students did not complete all four worlds of the modelling cycle, leads to a consideration of their prior Statistical Content Knowledge, which we are going to discuss in the next section.

Discussion and Implications

The discussion is organised based on an analysis of how the use of the TinkerPlots2 ® technological tool, and its affordances for metacognition gave our study participants with insights for improving their STPACK. We have opted for an isomorphic approach to designing a professional learning environment, in which pre-service and in-service teachers taking the Quantitative Research Methods course were asked to apply the same knowledge, strategies and competences that they should require for their students (Serradó et al. 2014). We believe that this isomorphic approach has given participants the opportunity to *understand students' learning (reasoning and thinking) of statistical ideas through the use of ICT tools for models and modelling, and reasoning on the difficulties and obstacles in the modelling process (STPACK1).*

We can interpret that the development of the two MEAS, in general, and participants' engagement in responding to reflective questions in particular (e.g. *"Compare your answer with those of other students in your class and reason why you have different answers"* or *"To what extend has TinkerPlots helped you (or not) to construct the theoretical probabilistic distribution?")*, allowed participants to become aware of the amplification and reorganiser role of technology.

Participants have used TinkerPlot2 as an amplifier to focus on conceptual understanding when engaging in exploratory data analysis of simulated data, which has contributed towards changing their statistical conceptions about randomness, and about the rationale behind use of a particular theory. We agree with Lee and Hollebrands (2011) regarding the possibilities of technological devices in general, and TinkerPlots2® in particular, to act as amplifiers and reorganisers of knowledge in an exploratory data analysis setting. But, we consider that these possibilities have to be extended to the opportunity that simulation process and tools give to explore a model that supports learner estimation. We conclude that this use of technology has given teachers the chance to develop *conceptions of how technological tools and representations support models and modelling in exploratory data analysis, and reason about the role of simulations as an amplifier and reorganiser tool (STPACK3)*.

We consider that those students that do not go through the four world modelling cycle (reducing the cycle to the processes 1, 9 and 13 of the Fig. 3) tend to have a deterministic traditional view of solving statistical problems and a constricted vision of the theory-driven modelling approach (Borovcnik and Kapadia 2011). In the problem solving process, students have identified a unique possible distribution of outcomes that will provide the theoretical value of the probability. On this identification, we consider that has emerged an ontogenic obstacle due to a non-encompassed integration of the notions of distribution and probability. We concur with Borovcnik and Kapadia (2011), when affirming that a deterministic view of the problem solving is an obstacle that constricts the vision of what a theory-driven modelling approach actually means. And, according to Serradó et al. (2014), a didactical obstacle have emerged when the students had not recognised that the purpose of the model is to analyse its “goodness” in relation to its accuracy in representing the real world.

A didactical interpretation of the causes of these ontogenic and didactical obstacles can be discussed through a careful analysis of the four world modelling cycle (see Fig. 2 and 3). In each MEA, the word problem provides all the needed information to solve the problem, and the clues to understand that it is a null model. In some sense, although the MEAs are designed to promote the circularity between the virtual, experimental, theoretical and technological worlds, this circularity can be avoided if the aim of modelling is incorrectly understood as a statistical solving problem process.

On this lifting from the virtual world to the theoretical world, and finally the return to the virtual world, two didactical obstacles can emerge. From a procedural point of view, a didactical obstacle can emerge due to the lack of circularity between the theory-driven and data-driven probability modelling approach. Meanwhile, if we consider a contextual point of view, another didactical obstacle can emerge due to the selection of only virtual world problems and the exploration of only experimental data obtained from the TinkerPlots2® simulation tool –confusing a model with reality instead of considering a model as an approximation of reality.

However, if we consider a positive treatment of the obstacles as a didactical tool for professional development, we can improve the Model-Eliciting Activities (Lesh and Doerr 2003) through including reflective questions such as the following: Considering the difficulties that you have encountered during the realization of the MEA, which obstacles do you think students can have? Responding to such questions could give pre-service and in-service teachers the opportunity to develop a *critical stance towards the use and evaluation of MEAs as curricular materials for teaching and learning models and modelling with technology (STPACK2)*.

The MEAs proposed during the course, and particularly those analysed in this chapter, are "virtual real world problem situations". Although both have been categorised as "virtual real world problem situations", they are categorized as virtual for different reasons. We consider the "Helper or hinderer" MEA virtual because it has used real research data (Greefrath et al. 2011) to construct the word problem (Garfield et al. 2010). Meanwhile, the "How many tickets to sell?" MEA can be categorized as a virtual real world problem, because processes 1 and 8 facilitate the connection between the virtual real and the real world.

Despite the attempt to connect the virtual real world and the real world, there are no proposals for a bidirectional connection. This lack of bidirectionality can constrict teachers' development in two respects. On the one hand, teachers do not have the opportunity to work with real world problems and to understand the differences between the virtual problem situations, the virtual real world problem situations, and the real world problem situations. Although both MEA have provided students with the possibility to reflect on the role that problems have in the modelling process, they did not completely get the opportunity of *distinguishing the role that each kind of problem (virtual, real virtual or real) has in the modelling process (STPACK5).*

On the other hand, the fact that the proposed activities are virtual real world problems restricts the possibility of accessing meaningful real world data and conducting exploratory data analysis, to completely develop a data-driven modelling cycle. In this case, constricting the *conceptions of how technological tools and representations support models and modelling through accessing meaningful data and developing an exploratory data analysis (STPCK3).*

Each MEA has excluded different processes, considered crucial in a data-driven modelling cycle (Konold and Kazak 2008): measurement activities, appreciation of the types of measurement errors, and fitting of the probability model to the data. Neither the students who went through the complete modelling cycle of Figs. 1 or 2, nor the students that skipped some processes applying an incomplete theory-driven modelling approach, fitted the probability model to the data. We consider that this omission could become an obstacle to truly developing the data-driven or theory-driven model cycle. But, we can surpass this obstacle by changing the word "compare" by "fit" in Fig. 2 process 10 and Fig. 3 process 12. This change, as proposed by Pfannkuch and Zledins (2014), can improve the analysis of the "goodness" of the model.

Furthermore, Pfannkuch and Zledins (2014) suggest improving the modelling process through providing circularity between the data-driven and the theory-driven modelling approach. As it can be seen in Figs. 2 and 3, each MEA provides some kind of circularity between the data-driven and the theory-driven approach, however differences between them can be observed. The "Helper or hinderer" MEA begins the circularity through a theoretical analysis and the use of simulation tools to explore the theoretical well-known null model through an estimation of the frequentist probability based on relative frequencies of sample data. By contrast, the "How many tickets to sell?" MEA begins the circularity through using the simulating tool to develop an unknown model producing computer-based virtual data. Then representation of the generated data could facilitate the comprehension of the underlying structure. And, finally students could identify the theoretical model in hand. The integration of both activities in the course provided students with the possibility of becoming familiarized with the three goals of the simulation in a probabilistic modelling process (Eichler and Vogel 2014).

Eichler and Vogel (2014) proposed a two-world modelling cycle, with less or more complexity, between the empirical and theoretical world. They also reasoned about the opportunities that this two-world model gives to teachers' professional development and the need of integrating the technological knowledge with pedagogical content knowledge. We consider that this two-world modelling cycle is restricted, because it does not integrate technology. Meanwhile, the three-world modelling cycle proposed by Greefrath et al. (2011) that integrates the technology world, does not integrate the theoretical world. And, we consider it crucial to integrate the theoretical world, because probabilistic theoretical models can be obtained not only from the experimental-frequentist approach but also from other approaches (e.g. classical or subjective). This controversial view of probability aims at carefully considering the mathematization process in a probabilistic modelling approach, which can be quite different from those proposed by classical mathematical modelling approaches (Blum 2015). Thus, we consider that both the two-world, and the three-world modelling cycle have serious limitations. Rather, we propose the adoption of a four-world modelling cycle, which integrates the real, experimental, theoretical and technological worlds.

Furthermore, we consider that promoting students' participation in different MEAs integrating the four-world model, as the ones developed in the Quantitative Research Method course presented here, can provide learners with the *instructional strategies for developing models and modelling processes and distinguishing between their purposes (STPACK5)*. The integration of STPACK5 is constricted by the fact that the model eliciting activities presented are situated in a virtual real world instead of the real world. To improve this situation, we propose, from a cultural and sociological perspective, to incorporate MEAs that provide students with an open question on the educational context to completely develop the four-world modelling process.

Conclusion

To sum up, we have theoretically analysed the differences between mathematical, statistical and stochastical models in relation to the processes in the modelling cycle, and the interactions with the real, empirical, theoretical and technological world. Our case study of a Quantitative Research Methods course in M.A. Education Studies, has led us to the conclusion that the development of probabilistic modelling in such a context needs to integrate the four worlds, providing circularity between them.

This circularity can help teachers participating in the course to surpass some didactical and ontogenic obstacles, and to develop the required knowledge about models and the modelling process. Tackling these obstacles means improving the two model-eliciting activities analysed and presented in this chapter. We suggest improving the construction of the activities to promote the analysis of the "goodness" of the model and introducing other reflective questions related to obstacles in the modelling process.

We believe that reflective questions could help pre-service and in-service teachers participating in a quantitative educational research methods course develop and integrate their Statistical Technological Pedagogical and Content Knowledge. But, the development of STPACK is still constricted by the fact that all the MEAs are proposed in the virtual real world. To improve this situation, we propose the incorporation of MEAs that pose open questions related to an educational context, and require learners to experience the whole four-world modelling process. Such opportunities can help pre-service and in-service teachers develop their STPACK, building the processes and attitudes of a "teacher as a researcher".

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The Professional Development of Mathematics Teachers: Generality and Specificity

Maria Polo

Abstract The aim of this work is to present a study concerning primary and secondary in-service teacher professional development in the perspective of carrying out some innovations related to usual practice. The first results show that one major difference among the teachers groups involved is represented by the voluntary nature as well as by the training duration; the latter would permit the overcoming of a natural resistance to the change. We claim, as a base of our work, the need and the pertinence of a debate on the necessity of comparative studies concerning the teacher's position according to the results of the pedagogical, psychological and sociological research on the role of the teacher.

Keywords Professional development • In-service mathematics teachers • Disciplinary didactic • Pedagogy • ICT

The Professional Development of the Teacher: Preliminary Questions and Problems

The professional development of the teacher, in particular of in-service mathematics teachers, has always been a need in human societies. Nowadays, it takes on a new challenge since it has to face the complex globalization processes. Defining the knowledge and competence that are to be considered for mathematics teachers professional development is one of the core issues both for researchers in mathematics education and for Institutions involved in teacher education. The reflection on knowledge and methods is essential also with reference to the professional development systems. In institutional teacher education systems, issues on methods are often relegated to the fields of pedagogy, psychology and sociology, while debate on content and knowledge to be taught is often tackled marginally or without connections with the results of researches in the field of disciplinary didactics. The dialectic needed between the different research fields (for example between

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disciplinary didactics and pedagogy) and the distinction between knowing, knowledge and competence are still open issues.

In this work, we begin from the assumption that, in order to deal with methods and contents of teacher training, some didactic-disciplinary peculiarities concerning teacher's knowledge are to be claimed. We will show the results of a study on some primary and secondary school in-service teacher training programs[1]. De facto, in-service teacher professional development, like all contexts of education, is subject to deep changes because of the present rapid evolution of modes and means of communication. For example, at the institutional level of the Italian Ministry, in-service training and innovation diffusion are carried out by e-Learning with a blended system, not just for mathematics teachers but for all disciplines. Distance interaction is, therefore, added to the previous usual conditions of training systems.

Another open issue in different research fields is trying to identify which conditions and specific aspects of mathematics teacher training change according to the conditions of training practices and how they influence professional development itself. In this respect, Clark-Wilson et al. (2014) face the issue of teacher initial and in-service training in relation to the use of technologies in mathematics teaching and learning. They highlight the existence of shared theories, such as instrumental orchestration, instrumental distance and double instrumental genesis, which, in our opinion too, are patrimony of the international research community of mathematics education. Their work, however, confirms the difficulty in retracing a complete overview of the practices that the teacher applies when using digital technologies in the classroom.

With reference to training systems, a recent study by Arzarello et al. (2014) proposed a model for teacher development called Meta-Didactical Transposition. This model complements the one by and Bass (2003), Ball, Hill and Bass (2005), Ball, Thames and Phelps (2008) adding to it the approach of the anthropological theory by Chevallard (1999). Moreover, Arzarello et al. (2014) address some features of the influence of institutional aspects connected to the new training systems both in presence and at distance. In our study we deal with these aspects only marginally (and we do not examine the role played by specific tools and methods on the professional development of the teacher). We focus our attention on teacher training content. Already at the end of the 80s, Shulman faced this issue and, when characterizing teaching professional knowledge, he identified, among the others, the "General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter" (Shulman 1987, p. 8). In our work we discuss the distinction between general knowledge and specific knowledge of the teacher,

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[1]In the scope of the activities of the "Centre of Research and Experimentation in Mathematics Education" (Centro di Ricerca e Sperimentazione dell'Educazione Matematica - CRSEM http://cli.sc.unica.it/crsem/) established in 1980 at the Department of Mathematics and Information Technology of the University of Cagliari.

which is already evident in the above mentioned definition given by Shulman. Therefore, we ask ourselves: why does pedagogic knowledge, such as the classroom organization and management, only apparently go beyond the scope of the discipline? What specifically distinguishes the classroom organization and management if the teacher involved is the mathematics or languages or science teacher?

The themes of the above questions have been investigated in recent research. The 15th ICMI study on teacher education (Even and Ball 2009) and the four volumes of the *International Handbook of Mathematics Teacher Education* (Wood 2008) concern research on teacher education and teacher professional development. They focus on identifying the knowledge that is necessary for teaching mathematics. Our work agrees with this research asserting that this kind of knowledge consists of three main components, interrelated to each other: knowledge about mathematics content, general pedagogical knowledge, and mathematical-didactical knowledge.

Purpose and Theoretical Approaches of This Study

The study of the relationship between school training and education, teacher professional development and research is one of the core issues to satisfy the expectations that globalization imposes. This is important not only in the field of Mathematics Didactics, but more broadly in the scope of the construction and development of citizens' knowledge and skills. Through the study of a case in professional development of in-service mathematics teachers, the purpose of this survey is to raise the issue of the specific or generic character of content and methods in the systems of teacher professional development.

A study realized in the pedagogic-psychologic field identified the reflection process as one of the fundamental aspects for teachers' professional development. This is the ALACT model (Korthagen 2001; Korthagen and Vasalos 2005), describing five phases in the reflection process: phase 1 (Action) is a teacher's experience (for example, a discipline problem in a teacher's classroom); phase 2 (Looking back to the action) is the turn back or boomerang phase; phase 3 (Awareness of essential aspects) is becoming conscious of the essential aspects; phase 4 (Creating alternative methods of action) is the creation of alternative action methods; phase 5 (Trial of alternative method) is a test.

We will come back to this issue by comparing our results with those reported in Korthagen (2001) and Korthagen and Vasalos (2005) that mainly refer to pre-service teachers. Our study is realized in the field of Mathematics Education but it analyses the case of in-service teacher professional development. We will recall and develop the results of a work presented at the CIEAEM65 meeting, in the light of some studies now analysed on mathematics teachers' development.

Our starting point is the idea that different issues concerning teacher professional development cannot be examined without a preliminary reference to a shared paradigm definition of teacher work. The teacher is considered as an element of the *didactic system*. Identifying specific contents and methods for mathematics teacher education, therefore, requires having already detected both *specific* or *generic* forms of competence and knowledge related to the *teacher's position*[2]. *Specific* competence and knowledge should characterize mathematics teaching while *generic* competence and knowledge should, on the contrary, characterize any disciplinary teaching in a formal learning context. Our position is based on the description of the *teacher's position* as an element of the didactic system and on some studies concerning the analysis of the teacher practice (Assude et al. 2007; Lai and Polo 2012; Polo 2002, 2008; Polo et al. 2008; Robert 2007; Sensevy et al. 2000).

Learning is a social process of negotiation of meanings within a classroom environment (Cobb 1997; Polo et al. 2008; Vygotskij 1990), and to understand this process it is necessary to understand the global character of the social-dialogue interactions that are typical of the teaching/learning process. The Theory of Didactic Situations (Brousseau 1986) and the Theory of Didactical Transposition (Chevallard 1985) together provide one of the possible models of description of this process: they assert the need and possibility to study the mutual relationships that characterize the *didactic system Teacher-Student-Knowing-Learning Environment*[3]. According to the Theory of Situations, the Didactic Contract regulates the interaction between devolution and institutionalization processes. The evolution of teaching and learning is determined by a continuous alternation of didactic contract breach and renegotiation. When analysing activities in the classroom by means of this model, one has to consider that different types of didactic situations coexist and evolve according to the different kinds of knowing that intervene in a specific activity. To analyse how, in the different activities carried out in the classroom, the teacher can influence the development of the "cultural history of the classroom" (Radford and Demers 2006) and the learning environment (but also be influenced by the environment itself), it is necessary to define the different kinds of knowledge that the teacher applies and uses (even unconsciously) in the different phases of the teaching/learning process.

Teaching Education: Generality and Specificity

Recent developments led to the definition of mathematical knowledge for teaching (MKT). Ball et al. (2005) defined MKT as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching". In

[2]We distinguish between the teacher's and student's position to indicate the system's element and not the institutional roles. In the analysis of teaching practices (also in relation to multimedia learning environments) who undertakes the teacher's position can be a peer, a tutor or a specific practice, also in ITC or e-learning. With regards to this, read the chapter by Albano G. (2017) in this volume.

[3]"Milieu" in the terminology introduced by Brousseau (1986).

particular, they have highlighted "the daily tasks in which teachers engage, and the responsibilities they have to teach mathematics, both inside and outside the classroom". Ball et al. (2008) state that:

> It is rooted in attention to the demands of practice to consider what mathematics arises in the work that teachers do. Our work tests these ideas by developing instruments to measure this knowledge, by using the results to inform our understanding of a map of teacher content knowledge, and by tying this knowledge to its use in practice. That there is a domain of content knowledge unique to the work of teaching is a hypothesis that has already developed. However, the notion of specialized content knowledge is in need of further work in order to understand the most important dimensions of teachers' professional knowledge. Doing so with care promises to have significant implications for understanding teaching and for improving the content preparation of teachers. (p. 405)

We agree with the fact that a notion of *specialized content knowledge* is needed. Moreover, recalling the model by Ball et al. (2008), we state that it is possible to identify a specific knowledge of the teacher which is different from the Pedagogical Content Knowledge (PCK), but also from the Subject Matter Knowledge (SMK). We refer here to the Didactical Content Knowledge (DCK), which is the didactic-disciplinary knowledge. This kind of knowledge is strictly connected to the content of the learned object and it is at the base of the teacher's decisions before, during and after classroom activities.

We highlighted (Polo 2008) a possible distinction in the nature of decisions on which the teacher's choices, actions and activities are based within and outside the classroom. Pedagogic or psychological decisions concern the personal relationship of the teacher with students: these decisions, and the knowledge connected to them, have a generic character since they concern teaching and learning at school in general. Didactic decisions, on the contrary, concern the teacher's relationship with mathematics and mathematics activities at school. They are connected to knowing and therefore have a specific character: they concern mathematics teaching in particular and the specific knowledge that characterizes mathematics teachers.

We agree with Chevallard (1999), who maintains that the classic approach of Mathematics Education has ignored some *general* aspects of the organization in a *specific* type of didactic system. If choices are made (consciously or unconsciously) at level of a *general* organization, problems in the study of a *specific* mathematic organization remain misplaced. In our experience as well, we have observed that aspects considered as related to *general* choices are often neglected in the scope of mathematics education.

Our work integrates itself in this line of studies and analysis in mathematics didactics, which examines the aspects connected to mathematics and didactics organization. However, we assume that comparative and integrated studies are needed as well to identify the specificity and generality of the teacher's competence and knowledge. In our opinion, it is indeed fundamental to distinguish between didactic variables and pedagogic-relational variables; this is necessary if we want to deeply understand the specificity and generality that can characterize the development of the teacher's professional knowledge.

In several fields of research and in some debates within training Institutions, detecting an efficient teaching method appears to be a goal. Some of the same surveys, however, assert that using a specific teaching method does not necessarily lead to effective learning. The use of the same method (frontal lessons, didactic materials, group discussion) by several teachers may actually lead to different results in students' learning. This was also confirmed by some surveys that have examined teachers using the same class method but with different teaching styles, thanks to which they present didactic activities with different personal practices. These results allow us to state that a specific methodology affects students' motivational, operational and relational behaviours more than the teacher's specific knowledge and competence. This statement is confirmed by several studies (Lai 2003; Lai and Polo 2002) realized within local and national projects focused on dropouts. In one of these cases, we could observe this phenomenon also in activities concerning not only Mathematics teaching/learning but also other disciplines, such as Italian language.

As an example, we report here the case of one of the activities of Italian language teaching, aimed at introducing the content "compound words". Students, divided into groups, worked according to the task assigned by the teacher for the construction of "meaningful words starting from the term *sea*". The working atmosphere among students was positive and collaborative and the teacher had the role of an observer. In one of the groups, students wrote pertinent sentences instead of compound words: calm sea, blue sea, etc. On the contrary, the teacher expected terms like seaquake, sea storm, sea landing[4]. While moving around the groups, the teacher (who decided not to help the students on the correct answers) did not notice the difficulties of the students and did not intervene with relevant questions that could have restarted the 'devolution' (in Brousseau's sense). During the collective discussion therefore, the teacher was obliged to give the correct answer, due to the wrong intervention of the students. In this case, the relational dynamics were positive, but the fact that the teacher did not foresee this kind of event did not allow him/her to intervene in a relevant way with respect to the new knowledge to be learnt.

The possibility to identify and distinguish *didactic* and *pedagogic* knowledge concerning the teacher's position is important. In this respect, the above example shows how a classroom activity defined by the teacher as "well managed" (because students have actively taken part in the working groups) could turn out to be not consistent with the *mathematical and didactic organization* and therefore sometimes could also be ineffective for learning.

On the contrary, a suitable mathematical and didactic organization (based on an a priori in-depth analysis of knowledge at stake) allows the teacher to better manage his/her own pedagogic and didactic decisions. We define the first decisions as *general* since they concern the pedagogic and psychological relationship regulating

[4]These examples are more pertinent in Italian since in this language these words are a single term (maremoto, mareggiata, ammaraggio).

the communication between teachers and students. The second kind of decisions is defined as *specific*, because they are related to the mathematical and didactic organization, therefore regulating the *didactic relationship* between teachers and students.

Training Content and Methods

In the training experiences for in-service teachers that we have realized, the lack or shortage of training on a specific area of mathematics content often remains latent. Teachers highlight the need for training in specific mathematics content only if that content was not part of the initial education[5]. In this sense, the case of ICT introduction is emblematic. When studying these aspects, Assude and Loisy (2008) distinguish between the technical, didactic and pedagogic nature of teachers' competences. This is in accord with our distinction and separation between the knowledge concerning the personal relationship of a teacher with a student (pedagogic nature of the teacher's competences) and the knowledge concerning the didactic relationship. This didactic relationship is generated and structured by a specific discipline. The teacher's technical competence has to be considered if the objective of teacher training is the use of ICT.

With reference to the identification of the teacher's knowledge in this field, Ruthven (2014) examines the three following frameworks for analysing relevant expertise on the part of the teacher, and explores commonalities, complementarities and contrasts between them: the Technological, Pedagogical and Content Knowledge (TPACK – see Fig. 1), the Instrumental Orchestration and the Structuring Features of Classroom Practice. In particular, Ruthven (2014), referring to TPACK, asserts that:

> These problems [that arise in the course of teaching a particular topic] raise considerations both of content and pedagogy, and solutions to them are typically not reducible to the logic of either knowledge domain alone. Moreover, while solutions to such teaching problems may become crystallised as stable professional knowledge, they may equally be subject to continuing adaptation and refinement, and they will vary between teachers and across teaching settings. Finally, for reasons both of ecological adaptation and cognitive economy, such knowledge is typically organised around prototypical teaching situations. For these reasons, the subsequent development of this line of work has been criticised for an unproductive focus on a logical demarcation of types of teacher knowledge rather than on its functional organization. (p. 374)

In our study, we focus our attention on the teacher's knowledge concerning the different functional organisations of the teaching/learning process. To this purpose, we have used and adapted the results by Comiti et al. (1995), which proposed a classification of the teacher's knowledge in connection to the different levels of

[5]In the group G1 of teachers under training who worked with us, the same phenomenon appeared in the case of training on astronomy contents, which were not part of their background.

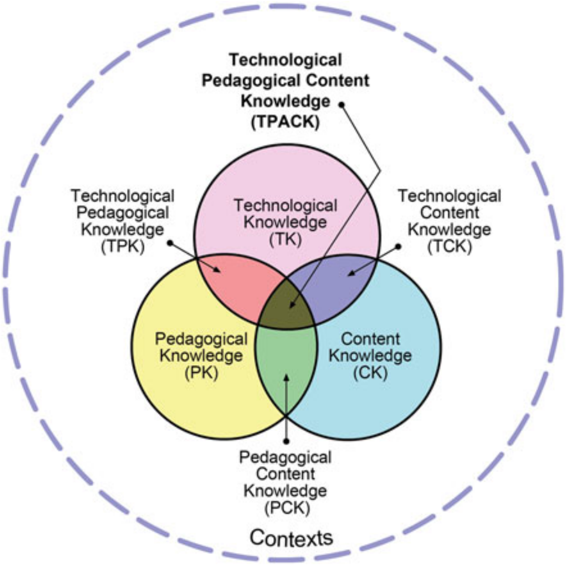

Fig. 1 Diagram metaphor for the TPACK model as shown at http://tpack.org

Fig. 2 Different levels of didactic situations

M_3	P_3	S_3
M_2	P_2	S_2
M_1	P_1	S_1
M_0	P_0	S_0
M_{-1}	P_{-1}	S_{-1}

didactic situations. The model was analysed in a case study on networking of theories by Artigue et al. (2014). We recall this model since it describes and allows predicting the teacher's knowledge in the different phases of his/her work inside and outside the classroom.

In the following table (see Fig. 2) by Comiti et al. (1995), by $\boldsymbol{P_i}$ we indicate the teacher's position with reference to the different didactic situations $\boldsymbol{S_i}$, while by $\boldsymbol{M_i}$ we mean the conditions referring to the learning environment concerning the "knowledge" which is the teaching/learning object[6].

The model developed by Comiti et al. (1995), reused by Lai and Polo (2002), Artigue et al. (2002), and then modified by Margolinas (2004) allows us to describe

[6]Knowing concerning students in $\boldsymbol{E_i}$ position completes the model, presented by Comiti et al. (1995).

and interpret didactic practices in terms of student's intervention (or non-intervention) *resonance* with *the teacher's project* (which is what the teacher established as his/her teaching objective with reference to a specific knowledge).

By *resonance* we mean the student's intervention that the teacher gathers and recalls totally, or that he/she gathers and recalls only partially or that he/she does not gather at all.

In the classroom activity (levels S_0 and S_{-1}), the *teacher's project* meets the "student's project" or clashes with it. By student's project we mean the set of behaviours that students implement according to the compliance/non-compliance of a specific activity with their expectations in that specific moment of the classroom life.

Creating a model in terms of teacher's knowledge, as follows, highlights the different levels of knowing that are simultaneously present in planning and managing the didactic activities in the classroom. The different levels alternate with each other as in a circular sequence and in different forms according to the teacher's reflection on practice. We indicate by $\boldsymbol{K_i}$ the knowing of the teacher in $\boldsymbol{P_i}$ position and concerning the corresponding $\mathbf{S}_i$ situation.

At Level of Noosphere[7] (S_3)

K_3: This is a kind of knowing that implies a teaching project (for example: knowledge, beliefs, representations of a discipline, of the learning and teaching processes, etc.). This knowing concerns the general choices and the explanation for these choices according to the lesson sequence that was elaborated or has to be elaborated for a certain subject.

At Level of Activities or Sequence of Activities Construction (S_2)

K_2: This is a kind of knowledge concerning the teaching/learning situation on a specific subject, classified in a sequence and related to past or present situations of that subject teaching in a definite class or school specialization.

At Planning Level of Activities Management Prevision (S_1)

K_1: The teacher is in a position where he/she takes decisions (providing or not providing a certain definition, an example, an answer, asking or not asking a

[7]The term *noosphere* is used here in the sense of Chevallard (1985). In this context it indicates, more precisely, a teacher's certainties on a cultural, epistemological and institutional level on a certain mathematics subject.

question, etc.) on which to lay the foundations of the institutionalization process of certain knowledge.

K_1 knowledge is a kind of global knowledge concerning common knowing and difficulties of students in a particular class, in a specific period of the school year related to a specific subject or to other subjects connected to it. It concerns the prevision of possible modes of activities' fulfilment and the different variables that can influence this process, the evaluation of their role and the effects that they can have on the learning progression as well as on the following activities to be carried out.

At Level of Activities Realization and Management (S_0)

This level and the following correspond to the phases of work or reflection for students.

K_0 knowing concerns the interpretation and/or representation of students' difficulties and of their causes or of their raw answers, concerning the knowledge at stake in the activity.

At Level of A-didactic Situations[8] (S_{-1})

In the a-didactic situation, position P_{-1} does not correspond to "teacher's silence or absence", but to a suspension of the conclusion for what concerns the knowledge to be learned. A teacher's knowing allows him/her to intervene or not with reference to other knowledge involved in a given activity that, compared to this kind of knowledge, does not have the character of a-didactic situation. Not all activities and phases of a didactic course on a given subject have the character of a-didactic situation with reference to the knowledge at stake. K_{-1} knowing concerns: interpretation of (and decision making on) specific questions students may ask and the related answers concerning the knowledge which is object of the activities (or other kinds of knowledge) but also interpretation of the possible causes of students' behaviours; mastery of the decisions taken, the questions asked and the answers given during the activities of the whole didactic course. This level concerns the micro-decisions taken on the ground, in a conscious or less conscious way during the class activities, but also the following consideration of the activities' results.

This non-hierarchical kind of knowledge that focuses on a specific knowing, operates at different levels and moments in the teaching/learning process. During our experience of in-service teacher training, we have tried to develop the different kinds of knowledge by means of different didactic organizations of this training. In the different didactic organisations we have developed, over time, teachers' K_1 knowledge concerning S_1 to S_3 levels.

[8]In Brousseau's sense.

As to the methodology used by trainers during our training activities, we have mainly implemented the accompanying method (Assude and Grugeon 2003). In other words, we have planned and tested some classroom activities during which the trainer was an active support in some phases of the work with pupils (levels S_0 and S_1). In some cases, in the different contexts of training (in particular with teachers who were already expert experimenters) we have applied the method that recalls the characteristics of action research. In this case we have worked with teachers in reflecting on and analysing the activities chosen and proposed by the trainer. In the case of more long term training devices we have worked at the construction of activities and, in particular, at the a priori analysis of the possible classroom implementations and we reflected on the results of the implemented activities (in this case we had involved all levels from S_3 to S_1).

Case Study: In-service Training

Teachers' in-service training in Italy has recently been managed at an institutional level by an agency (INDIRE[9]) that implemented distance training projects managed at regional level through activities carried out in person or on an interactive platform, synchronous or deferred. For mathematics, the national project of in-service training (m@t.abel) for teachers of primary and secondary school – mainly from the sixth to the tenth year of school – works in this direction. At present, INDIRE, as a support for the reform realization, makes available for teachers resources and documents[10] that are the result of important inter-institutional work that has transposed for training some results of research in Mathematics Didactics. The training system is managed at the local level by the Regional School Office (USR), which is a regional ministerial institution. As the USR proposed, for the m@t.abel project teachers were trained at the national level and then have afterwards taken on the role of tutors. Each tutor manages a network of schools where teachers give their support voluntarily to study all available documents, since they are precious resources and drafts for experimental activities to be implemented in the classroom during the school year.

Thanks to the experience of tutors and teachers taking part in these professional development activities we could observe a certain level of criticality in the implementation of these kinds of activities, both nationally and locally. Teachers mainly highlight the difficulties to manage the interactions between activities in presence and at distance, but also the lack of institutional acknowledgment for teachers contributing as participants. This limits the large-scale diffusion of this training experience among mathematics teachers.

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[9] http://www.indire.it/

[10] http://risorsedocentipon.indire.it/offerta_formativa/f/index.php?action=home&area_t=f&d_ambiente=7

Since 2006, the m@t.abel project, both in its methodology and its content, is well in line with the latest themes stressed by the current educational research in the international framework. M@t.abel puts into practice, in the territory, the creation and, possibly, the consolidation of several communities of practice introduced by Lave and Wenger (1998). Since 2009, the project provided for a systemic monitoring by INDIRE (A. A. 2009b). In 2012, the m@t.abel project was integrated in the PON training projects for teachers. These projects were part of the activities organized by the National Operational Program 2007–2013, which aimed at training in-service teachers in primary and secondary school. Actions concerned competence updates with reference to Italian language, Mathematics, Foreign Languages and Science didactics. As specified in the guidelines[11] English version:

> (. . .) the m@t.abel PON project broadens and intensifies the training in the "national" m@t.abel, and it follows that the training system is similar (. . .). It should be specified, however, that the training is structured over a wider time span that covers most of the school year (October-May); consequently the Technical-scientific Committee has developed a more detailed script of [various phases] the course.

1. **Initial in-person training**
    - the training starts with a 4 h in-person meeting in which the training, objectives, conceptual nodes and methodology are presented;
    - 6 in-person meetings in which the didactic paths are analyzed and the course members choose four activities, one for each thematic Nucleus, to experiment in the classroom with the students;
    - the last in-person meeting, 4 h long, provides a general reflection on the training and in-class experimentation.
2. **Online training**
    - analysis of materials;
    - participation in online activities arranged by the Tutor (e.g. synchronous laboratories);
    - independent activities.
3. **In-class experimentation**
    (. . .)
4. **Independent study**
    - thinking about and analyzing the training;
    - writing a logbook entry for each activity experimented.

(. . .) The project is deeply marked by the use of technologies, both in terms of teaching in the classroom and virtual teaching. (pp. 14–43)

[11] http://mediarepository.indire.it/iko/uploads/allegati/M7PWITOE.pdf (A.A. 2012)

In this project teachers have to describe, in logbooks, the main conceptual points of the activity, its phases and the implemented methodology, the reactions and difficulties of students, their reflections on the trends and the final evaluation both of students and the activity. The results of logbooks analysis (that is the direct opinion of trained teachers) has shown that the impact on common practices of innovation and change does not reach 40%:

> In about one fourth of logbooks (25.29%) teachers declare they have modified their didactic planning and behaviour towards the discipline with regards to the working unit carried out in the classroom. This does not happen for 26.56% of the experimented didactic units in which a significant impact on the usual teaching practice is not observed. In about half of the logbooks (48,15%), on the contrary, teachers‘ answers were evaluated – in a codification phase – as non-defined since they were missing (9.45%) or not clearly definable since:
>
> – they underline the strengthening of teaching methodologies usually implemented and this does not represent, therefore, a significant change due to the experimented activities (13,28%)
> – they underline the usefulness that the proposed working unit has for students but they do not explain the didactic planning implemented (25.42%). (A. A. 2010, p. 36)

The PON 2007–2013 Monitoring Report[12] is based on dimensions structured in fields and related criteria (of Adequacy – Effectiveness – Efficiency – Satisfaction – Accessibility). We quote, for illustrative purpose, those related to ICT but we refer to the original document for the details of results.

STRUCTURE AND EFFECTIVENESS OF THE TRAINING

– Effectiveness of the blended model
– Effectiveness of the single phases and usefulness of the instruments according to the reflection capacity of the trained teacher
– Usefulness of the training materials (for the teacher) and of the didactic materials (for students)
– (. . .)
– Composition of the virtual classrooms (. . .)

ENVIRONMENT FOR DIGITAL COLLABORATION

– Participation in the online activities
– Use regularity, usefulness, instruments usability
– Possible difficulties

We summarise here after the Monitoring Report results concerning the above mentioned dimensions:

(. . .) the alternation between in-presence and at distance didactics met the needs of the majority of the subjects interviewed since, according to 56.7% of them, the time articulation of the two macro-phases of the training was adequate. (. . .) Experimentation in the classroom is (. . .) the aspect that better motivated trainees.

[12] http://formazionedocentipon.indire.it/wp-content/uploads/2014/12/GDA_R1-Report-INDIRE-rev.-10.07.2014_lm_def.pdf.

Among the seven instrumental supports provided in the platform (Calendar and News, Forum, written Chat, materials sharing, synchronous Laboratory, Blog/Wiki, email to the working group) the best used are "Calendar and News" (39.2% of people interviewed assert they have always used them) and the area of materials sharing. (. . .) On the contrary, a negative result is registered by the blog/wiki: 41.8% of interviewed people state they have never used it, while 18.8% of the subjects declare they have used it rarely. According to 85.4% of trainees (. . .), the most useful instrument in the online platform is the area of materials sharing.

In relation to the informatics technologies approach, finally, we can add that from the monitoring data we can deduce the existence of a rather diffused tendency to use the traditional means of communication through the computer: e-mails, messages in the forum, file exchanges are basic operations useful to improve interpersonal communication, making it faster. (. . .) Although we can talk about inclination to collaboration and exchange, a school where concepts like user generated content, collective intelligence and distributed knowledge seems to be still far away. (Free translation pp. 14–68)

In the current national panorama of in-service teachers, PON is the only initiative of the Ministry. Other opportunities for training of teachers of mathematics and other disciplines[13] can be achieved also as an initiative of a single school or of a network of schools, teachers' associations or other institutions but they have an episodic and local character.

The Contexts of the Case Study on In-service Training

To try to identify general and specific characteristics of teachers' professional training, in our survey we have compared two groups of teachers:

- G1 teachers from primary school to the second year of college voluntarily occupied in training activities, carried out in presence and aiming at innovating the mathematics official curriculum;
- G2 teachers from primary school to the second year of college occupied in activities of institutional training often imposed at collegial level and managed in presence and/or at distance on the interactive platform.

The first group is composed of 56 teachers from primary school, among which 34 are occupied in a 4-day course without classroom experimentation activities. The activities concerned innovative courses making use of software such as GeoGebra. The other teachers participated in a training course with monthly meetings and carried out experimental work in the classroom during the school years from 2010 to 2012.

[13] http://www.indire.it/progetto/formazione-disciplinare-docenti-pon/

The second group is composed of about 90 teachers, 10 of whom were occupied in e-learning activities on the platform. The activity was carried out in 4 meetings in the school year 2008/2009. In the years 2010/2012, to support the four meetings, we have experimented with an interaction on the platform and the classroom work monitoring. Also in this group are ICT instruments that are sometimes part of the didactic organisation of experimental activities and have had interdisciplinary character. In these cases, the mathematics teacher has interacted with the IT teacher.

In the experiences with both groups, teachers were involved in an individual or group work aiming at the construction, analysis, experimentation of and reflection on at least one activity implemented in the classroom. The results that we describe in the following paragraph are based on the analysis of memorandums from the training meetings, interviews and end-of-course questionnaires.

An Experience of Integration Activity in Usual Practice

This training course had the objective to establish a practice community able to interact also in the platform: this objective, however, was not realized. As in the case of some results of the PON projects, if there is no need, teachers prefer experience exchanges and in-presence discussions. The platform was used as repository and documents exchange. Communication, including at the organization level, was carried out by e-mail through a mailing list managed by the project supervisor. During the two years of training, two initiatives of trained teachers turned out to be very important: they aimed at inserting experimental activities into their usual practice.

In the first initiative, coordinated by the teacher who was the representative for promotion of the Institute and for Orientation/re-Orientation, the school introduction to pupils of the previous level was organized by means of permanent laboratories where students (as peer tutors) have taken part directly involving students of the secondary school.

The second initiative was significant for the inter-disciplinary character by means of IT[14]: the activity started from one of the queries on triangle construction with manipulable materials used in experimental activities (its objective is discovering the triangular irregularity). The students have created an Excel file that was, afterwards, perfected to be used by a generic user as a didactic game during an educational event. The students themselves managed the exhibit inviting visitors to play with the game they had realized (Fig. 3).

[14]I thank G. Deiana, M.G. Sciabica, I. Arthemalle (who have developed the activities) and teacher S. Deplano (trainer during the Course).

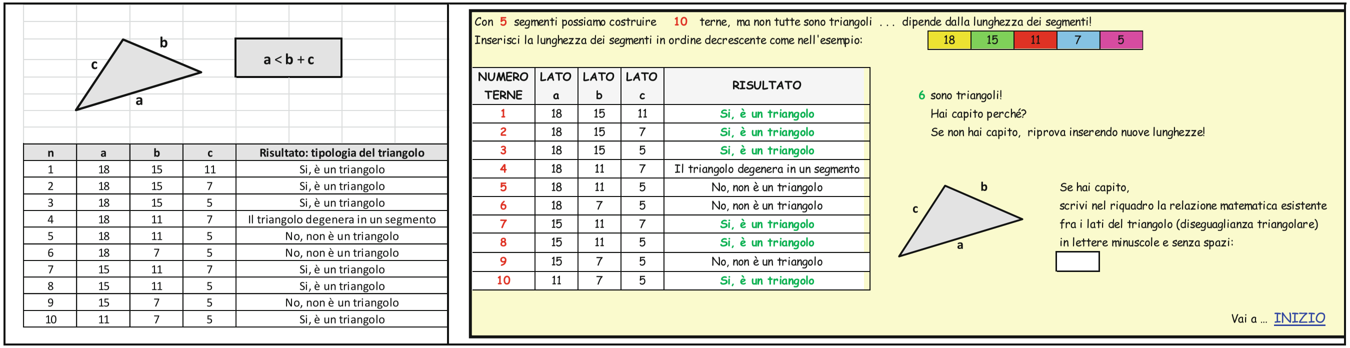

Fig. 3 From the Excel file to the game construction

An Experience with the Teachers from Primary School

We will report here a synthesis of the in-presence development activities on interdisciplinary paths on mathematics-astronomy realized with a group of teachers from primary school. This experience was realized with five teachers during two school years and under the scientific responsibility and project management of S. Lai, in the scope of the CRSEM training and research activities. In this training, the researcher/trainer worked with teachers by using a methodology of research-action and accompaniment in the experimental phases with students (Lai and Polo 2012).

Teachers learned about both astronomy content, with an experimental approach, and how to realize a priori analysis in the classroom. In the phases of a priori analysis, achieved with the trainer and in collaboration with the other teachers (or personally by each teacher for his/her own classroom), some guidelines were used with reference to the different kinds of knowledge to be developed according to the various aspects and tasks of conception, planning, prevision, management and analysis of the experimental activity (respectively levels P_3, P_2, P_0 and P_{-1} of the model on the teacher's position).

Some aspects of the training process are expressed by the experience of one of the teachers of the first group[15], who describes her experience and some work phases as follows. In his/her report, this teacher clearly describes both the methodology and content of the training and, in particular, he/she highlights the importance of the assigned task "from the teacher's point of view".

> I took part in a training course on astronomy and mathematics for a fourth class of primary school.
>
> – from the **teacher**'s point of view, since he/she is involved both in the contents connected to astronomy (with the need to propose ideas and show her knowing about the studied and simulated phenomenon, with the trainer's guide in the observation phases to be implemented in the class) and in the didactical transposition of the knowledge (a-priori analysis, activity preparation, management of the activity and reflection on the congruence between previsions and effective implementation, analysis of the possible causes of unexpected events, conscious or unconscious changes in the activities management in comparison with the planning phase previsions).

Table 1 shows this teacher's remarks in the planning phase (at levels P_2 and P_1) of activities implemented with students. In this phase, the teacher explains his/her own actions, the specific mathematic or astronomy knowledge at stake in the didactic activity to be implemented, and the tasks and actions of the students at level of the prevision, realization and management of the activities in the classroom.

[15] I thank M. Alberti, who accepted to describe here training experience also personally during the presentation of this work during the CIEAEM65. A synthesis of the activities realized in the third, fourth and fifth classes of the primary school can be found on the CRSEM website.

Table 1 Length of the shadow in relation to the height of the Sun (Expected time: 2 h)

Teacher's actions	Knowledge at stake	Actions / tasks of the student
The teacher proposes to measure the shadow of a pupil at 9.00 am and at 12.00pm on the same day. In the classroom, he/she asks: "Look at the marked shadows, what can you see?"	Length of the shadows (change). Direction of the shadows.	Students trace the cut-out over and observe the different length of the shadow at 9.00am and 12:00pm.
The teacher asks the pupils to make assumptions on the different length of the shadows at different times: "How is it that the child's shadow changes length according to the time?" (work in pairs)	Length of the shadow in relation to the height of the sun and to the horizon (in astronomy: the plan materialized by the vertical plumb line).	Some students will say that the shadow is shorter at 12.00 pm because the sun is in a higher position than at 9.00am.

In the last phase of the course, training is structured in a reflection on the activities implementation in the classroom compared to previsions and on a debate with trainers who, according to the case, have managed or only observed the experimental activities. This last training phase mainly focuses on the reflection on unexpected events and on how the teacher faced them, on the level of consciousness with which choices were made in the classroom.

Results Discussion

First Results of the Case Study on In-service Training

If we consider the different conditions of the two groups of teachers trained (at distance/in presence and voluntary/institutional choice) there are not significant differences concerning the opinions that teachers expressed in the questionnaire and during the debates. The importance of homology and accompaniment training strategies (Assude and Grugeon 2003), as well as the debates with colleagues and trainers, are considered as strong points in the training experience. The lack of recognition of the training course at institutional level, moreover, was highlighted as a negative point of training in both groups of teachers.

The results confirm those expressed by the opinions of teachers involved in the m@t.abel national project, as stated in the summary document of the Project (A. A. 2009a, b) and in the PON 2007_2013 Monitoring Report.

The lack of initial education on specific knowledge is often latent as it is a training need for teachers: this need is not explained or expressed except as the kinds of knowledge that are not formally requested in the university training curriculum (like astronomy for teachers in primary school). Also the summary document of the National Project confirms that this difficulty remains latent. This document (A. A. 2009a) shows that teachers mainly chose experimental activities concerning "familiar" notions or knowledge and on which they have deeper knowing.

In the time range of one school year, the majority of trained teachers _ both in the National Project m@a.abel and in the G2 group nearly all teachers in institutional training - underline the fact that it is not possible or it is very difficult to add experimental activities in the real curriculum. Therefore these activities are juxtaposed to common lessons. In the case of teachers of G1 group on *voluntary training*, a change was generated only after three or more training years and not for all teachers. All teachers who took part in innovative experiences, in particular those employing an inquiry-based learning, show that difficulties in time management of these activities is one of the main reasons of change resistance. At the same time, a nearly general consensus confirms the efficiency of this innovation, as we can observe in the following experience of one of the teachers of the first group:

> in my opinion... to discover the interest in constructing and manipulating objects with the purpose of mastering and becoming aware of the mathematical conclusions involved not only students but also teachers. And, moreover, the experience attracted also students who appear not to be gifted for mathematics.

In our study we have analysed the changes in the teachers' behaviours in the two groups G1 and G2 with regards to the professional competence in planning activities, to the prevision of results but also to the concept of mathematics and of the student's role. The most significant differences depended on the voluntary or non-voluntary character of training and on its time duration. In other words, in our experience non-imposed participation in training and experimentation activities as well as training duration (at least two school years) turned out to be the most pertinent and efficient conditions or variables for change implementation and professional development. In our sample of 56 teachers in group G1, only 12 of them modified their usual practices. For all the others, including members of group G2, the innovative practice turns out to be an interesting but occasional modification of their usual working practice so far. In particular, training duration, which is connected to the times and experimental modalities of the development practices, would allow overcoming the natural resistance (Artigue 2012) to change of trained teachers. These modalities would actually allow a significant change in the relationship teachers have with mathematics, learning and teaching.

Efficient Practices for Teachers' Professional Development

Answering questions connected specifically or generally to mathematics teaching in order to create efficient training practices also means having answers to questions like the following: how can we translate in terms of teacher's action – in particular mathematics teachers (for example in terms of technique, technology or theory according to the ATD, Chevallard 1999) – the role of *reflective teacher* and *learning facilitator* that some pedagogical studies propose? What would be specific about mathematics teaching, or language teaching, to be a reflective teacher and learning facilitator?

At present, there is no general consensus about which could be the training practices that better fit purposes like these or similar others. Arzarello at al. (2014), whose main subject was the research on training, defined in terms of ADT (Chevallard 1999) a teachers training model, Here is an extract of the authors' position:

> The complexity arising from the intertwining of the processes involved during a teacher education program has led us to introduce a descriptive and interpretative model, which considers some of the main variables in teacher education (the community of teachers, the researchers, the role of the institutions), and accounts for their mutual relationships and evolution over time. We call the overall resulting process *Meta-didactical Transposition*. We offer the model as a tool for studying the complexity of teacher education as a research problem that involves a transposition from the practice of research to that of teaching. the model's potential with respect to current research in the field. [. . .] Meta-didactical Transposition consists of a dynamic process through which, thanks to the dialectical interactions between two communities, both the didactic praxeologies of the community of researchers and of the teachers' community change within the institutional environment in which the two communities reside. This dialectical interaction leads to the development of a shared praxeology, which represents the core of our model. One of the main results of the dialectical interaction is the teachers' development of both a new awareness (on the cultural level) and new competences (on the methodological-didactical level, i.e. that of teaching practice), (...). Therefore, the term "meta-didactical" refers to the fact that important issues related to the didactical transposition of knowledge are faced at a meta-level. [. . .] (Arzarello et al. 2014, pp. 348_355).

The results by Arzarello refer in their work to in-service teachers. But also the ALACT model, that derives from initial teacher training, underlines the importance of the teacher's awareness and the need for reflection in this respect. The ALACT model shows on which contents reflection can focus: Environment, Behaviour, Competence, Convictions, Professional Identity, and Mission; and the conclusions of these authors are as follows:

> Recent studies in positive psychology support the beneficial effects of the view of human growth underlying the core reflection approach. (. . .) ultimately teachers can learn to activate the process of core reflection during their teaching, and in this way to make contact with the core qualities which are of importance at that particular moment.
>
> Good teaching, in our view, is characterized by a proper balance between the various levels (. . .). Finally, directing attention to core reflection during their professional preparation can help prospective teachers to become more aware of the core qualities of their pupils, so that they will be better able to guide these children in their learning, and help them mobilize their core qualities, in school and in their future lives. (Korthagen and Vasalos 2005, pp. 67_68)

The two studies analysed express from different points of view their own description of appropriate professional development but it is not clear if and how this could or should have a specific character for the professional development of mathematics teachers. The question concerning *generality/specificity* of teachers' training remains an open issue, the answer to which the studies and theoretical references considered here certainly do not represent a complete overview of existing studies. But for the moment, our study has come to the same conclusions. We have elaborated and implemented a descriptive model of the teacher's position that lays the foundations for the construction of training courses but which appears unable to identify the specificity in mathematics teaching.

In our research we have used and modified a model introduced in Comiti et al. (1995) by transposing the different kinds of knowledge that characterize the *teacher's position* in content, instruments and support of the experimental training practices.

We have worked to develop the different kinds of knowledge related to five different conditions of the *teacher's position*, that we have transposed into tasks assigned to teachers during training meetings, synthetically summarized below.

Trained teachers received from their trainers (or prepared by themselves) the planning of the activities to be implemented according to some guidelines that involved tasks from P_3 to P_1 before classroom implementation. During the activity implementation in the classroom (positions P_0 and P_{-1}) the teacher and the trainer also had observation functions. After the activity implementation, during a reflective discussion between teachers and their trainers (P_{-2}), they analysed the events which occurred in the classroom comparing them with the a priori forecasts (position P_1). For each position we sum up the guidelines that describe the task that was assigned to teachers during their training.

Position P_3. Epistemological Aspects of S[16] Knowledge Transposition

Importance of ***S*** knowledge in mathematics and in its relations with other disciplines. Analysis/study of historical aspects of ***S*** knowledge development. Analysis/study of ***S*** knowledge positioning in the school programs. Analysis/study/positioning of ***S*** knowledge in the teacher's yearly (or cyclical) program. Definition of the educational objectives or of the general didactic objectives.

Position P_2. Didactic Course Planning

Objective of the course (organization of units or modules divided into several lessons and/or activities). To characterize the type of situation with reference to (at least) one **S** knowledge, by using different categories: Approach to **S** (phase/s of devolution of **S** knowledge – phase/s of conclusion with respect to **S**), Institutionalization of **S**, Support/Recovery, Examination/Evaluation of **S**.

To describe the scenario of planned activities: teacher's tasks with respect to **S** and to other $\mathbf{S}_i$ concerning the same scenario (situation conditions-variables). Analysis of answers and possible strategies. Student's tasks with respect to **S** and

[16] **S** stands for a given knowledge.

other $\mathbf{S}_i$ pertinent to the scenario. Questions/answers by student expected/unexpected by teacher with respect to the planned scenario.

Position $\mathbf{P_1}$. Prevision of the course implementation (with reference to the scenario of one or more lessons/activities described in details at point $\mathbf{P}_2$). To establish how the teacher intervenes/does not intervenes on the base of expected/unexpected answers: teacher's behaviours (with reference to his/her own project) in the interaction with students' answers.

Position $\mathbf{P_0}$. Course implementation in the classroom (with reference to the scenario of one or more lessons/activities described in details at point $\mathbf{P}_2$). Implementation of the practices to control the activities management – course implementation monitoring. To manage during the activities implementation or to analyse the "resonance" of the student's intervention with respect to the teacher's previsions.

Position $\mathbf{P_{-1}}$. A-didactic Situation with Respect to at Least One S Knowledge and to Decision Making or Ongoing Reflections

We considered position P_{-1} as Comiti et al. (1995) did in the original model: $_{-}1$ indicates the a-didactic situation in the sense shown by Brousseau (1986). Moreover we prefer this denotation (and not $\mathbf{P}_4$, for example) because in position $\mathbf{P}_{-1}$ there can be an implicit and unconscious level both for the teacher and the student.

The model can be graphically represented as in the following pentagonal shape (see Fig. 4) to explain how these positions do not have a hierarchic or sequential character but an essentially systemic-relational character.

At the end of our work we would like to talk about the possibility to recall from the model by Comiti et al. (1995) also a level that characterizes $\mathbf{P}_{-2}$ position and according to which the pattern would take the following hexagonal shape (Fig. 5), where we indicated the mutual possible relations between the different $\mathbf{P_i}$ positions as considered theoretically existing.

P_{-2} position is typical of a reflective teacher in different phases from the practice on-going in the classroom or from activities planning. This level describes the teacher's position, situation and knowledge concerning the a-posteriori reflection on the implemented activity or on a past experience.

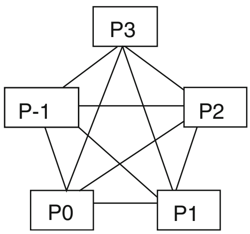

Fig. 4 Model 1 of *teacher's position*

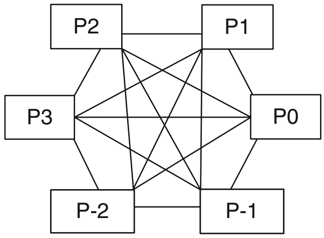

Fig. 5 Model 2 of *teacher's position*

We have tried to summarize in this model the specific features of the knowledge and decision making practices that characterize the *mathematics teacher's position* from the point of view of Mathematics Didactics in terms of Systemic Theory. Differently from what we could expect, if we exclude the specific character of "mathematics" knowledge, none of the positions seems characteristic of mathematics teaching but rather general. The issue concerning generality/specificity of teachers' training remains an open issue, to answer to which the studies and theoretical references considered here certainly do not represent a complete overview. But for the moment, our study has come to the same conclusions. Comparative studies to observe the two dynamics of interaction between *Teacher – Student – Environment* are needed: the didactic-relational one, in which knowledge is involved, and the pedagogic-relational one, which identifies and interprets (if they exist) the *specific decision* making practices of mathematics teaching and analyses of if and how these interpretations agree with studies in other scopes of research.

To show this need to distinguish between didactic knowledge and relational-pedagogic knowledge for teaching we used and adapted the model by Ball et al. (2008) (Fig. 6). In the central column we inserted the teacher's knowledge which is specific to the didactic relationship, defining it as DCK. In this scope we consider KCS and KCT, that Ball's model included in the field of PDK. The three columns could be integrated with the model used by Ruthven (2014) (see Fig. 1) of Technological, Pedagogical and Content Knowledge (TPACK) when using Technology in Mathematics Education. We also inserted KDTC, which is the knowledge referred to the process of didactic transposition and that characterizes $\mathbf{P}_i$ teacher-position and according to which the pattern would take the hexagonal shape (see Fig. 5).

We are planning to carry out a new research in the future to better define, in relation to the different contexts and to the accompaniment period in the experimental phase, the specific character of mathematics teacher training, compared to the more general aspects of professional training of teachers of any subject. A comparison between training contexts for teachers of any subject is therefore needed.

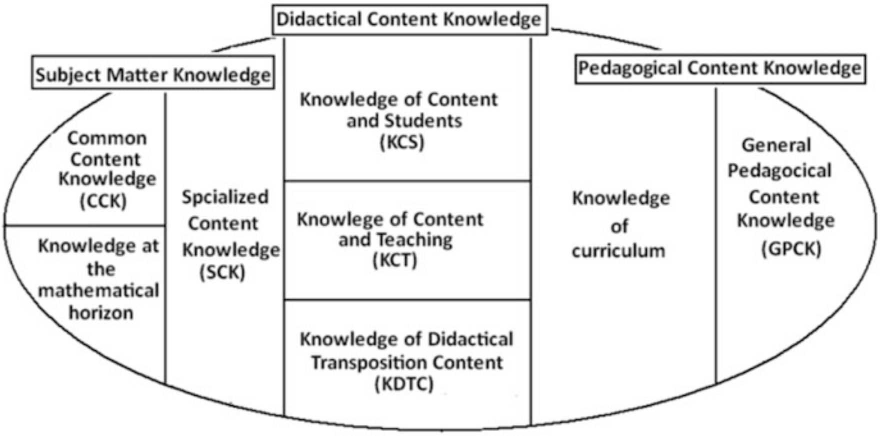

Fig. 6 Adaptation of the model by Ball et al. (2008)

Conclusions

If we accept that competence, which is object of both pre-service and in-service training for teachers, cannot be reduced to a juxtaposition of knowledge in pedagogy and psychology with knowledge in a specific discipline, this means that specific knowledge and competence for mathematics teacher training still need to be studied and this requires a comparative study of different research sectors.

As far as the connections between pre-service and in-service training are concerned, in the CIEAEM65 work group, participants agreed on the need to overcome the episodic nature of in-service training not only at local but also at national and transnational levels. The group also underlined the need to have an institutional integration between pre-service and in-service training for teachers: this integration, indeed, is not accomplished in any of the countries of teachers participating in the work group.

We finally want to add an element to the practices established in all countries for pre-service training: the presence or absence of a final evaluation of the trained subject. This is actually not specific to mathematics teacher training but, in our opinion, it is a fundamental variable to identify and better understand general content and practices for teacher training. In the case of pre-service training, indeed, at institutional level an evaluation is established: a teacher is not only a trainer or tutor but has the institutional task to define the success or not of the trained subject; this fundamental variable gives a character of didactic system to the actors involved, who meet all the characteristics of the Teacher-Student position. In the second case of in-service teachers, normally, the trainer does not evaluate the trainee: the evaluation of the training course is organized differently and this deeply changes the nature of the on-going didactic system. This particular variable reinforces the need for an efficient shaping of the *teacher's position* in relation to the possible characters of generality and specificity induced by teaching one discipline rather than another.

At present we observe a lack of integration of the different theoretical models used to describe the teacher's position and obtained as a result of the research in Mathematics Didactics. Moreover, we also observe a lack of theoretical interaction between the field of Mathematics and those of general Didactics, Pedagogy and Psychology. In our opinion, these deficiencies compromise the possibility (but also the need and relevance) of a definition of the generic and specific characters of knowledge and competences that are useful to define an efficient and up-to-date professional development for teachers.

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Integration of Digital Technologies in Mathematics Teacher Education: The Reconstruction Process of Previous Trigonometrical Knowledge

Nielce Meneguelo Lobo da Costa, Maria Elisa Esteves Lopes Galvão, and Maria Elisabette Brisola Brito Prado

Abstract This chapter presents an analysis of two case studies of use of technology in prospective and inservice teachers' education. Both cases derive from researches supervised by the authors. The purpose of this chapter is both to discuss and analyze the process of teachers' education focused on the integration of technological resources, and to understand how the reconstruction of previous trigonometric concepts occurs within groups of teachers. In addition, Mishra and Khoeler on the integrative perspective of the Technological Pedagogical Content Knowledge (TPACK) together with the Rabardel instrumentation theory and Imbernón's researches on teachers' education set the theoretical framework. Data were collected through questionnaires, digital files, and audio and video recordings. The interpretative analysis pointed out some circumstances that emphasized the reconstruction of previous professional knowledge and highlighted the differences, similarities, challenges and opportunities related to initial and continued education.

Keywords Technological resources • Periodic functions • Dynamic geometric software • TPACK

Introduction

The debate on the use of digital technology within the context of education together with the urgency for promoting changes in Mathematics teaching has been a recurring talk in our country. We understand that in this process of changes, the teacher plays an essential role. However, what draws to our attention is the complexity of concerns involving the process of teachers' education in a way to support them in promoting such changes. Presently, to integrate any innovation resulting from the advancement in Science and Technology to school practices is a

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great challenge to teachers' education, both in the initial as well as in the continuing education.

Most universities in Brazil have simplified their Mathematics curricula for undergraduate courses. Furthermore, some surveys e.g. Almeida and Valente (2011), Maltempi (2008), Prado and Lobo da Costa (2015), have found that the outcome of such initial education only qualifies a teacher to just reproduce Mathematics teachings aimed at procedures and techniques for solving mathematics exercises. The teacher, while in his/her initial education, has few opportunities to gather knowledge on the pedagogical use of information technology. Even if the prospective teacher is someone included in the digital culture, to know how to use such technology is not enough to sufficiently integrate it into teaching practices. This context has impaired the development of both content knowledge and Mathematics teaching. Such fact directly impacts on the teacher's practice, jeopardizing the quality of Mathematics teaching at Basic Education level.

Brazilian Government programs such as the "Institutional Scholarship for Teaching Initiation Program – PIBID",[1] the "Degree Consolidation Program – PRODOCENCIA,[2] the "Observatory of Education – OBEDUC[3] aim at supporting preservice and inservice teachers. Other initiatives developed by either public or private school systems also offer continuing education programs both to address the shortcomings of initial training and to promote broadening and deepening Basic Education teachers' knowledge. Continuing education, according to Imbernón (2009, 2010), may enable life-long learning, so essential to Education professionals. This education process should take into account the characteristics of the technological society as well as Science advancements, which will demand from any and every professional their engagement in permanent learning.

The teacher education faces new challenges posed by digital technologies within the school context. Within the scope of initial education, the pedagogic use of digital technologies is still limited where Mathematics is concerned; in general, such component appears more often in specific technology courses. For inservice teachers, the groundwork for the actual practical use of technological resources has been gained through continuing education. Projects[4] for implementing the use of computers, laptops and tablets in Basic Level Schools have been developed and extended in partnership with researchers from various universities around the country. Alongside with the implementation process, continuing education programs were launched with the intent of integrating information technology to school practices. Thus, continuing education for the pedagogic use of digital technology has also become a matter of research. Some methodological innovations in the educator's practice with the use of technology resources were noticed.

[1]More details on http://www.capes.gov.br/educacao-basica/capespibid

[2]More details on: http://www.capes.gov.br/educacao-basica/prodocencia

[3]More details on http://www.capes.gov.br/educacao-basica/observatorio-da-educacao

[4]Projects such as: EDUCOM, PRONINFE, PROINFO INTEGRADO, UCA, PROUCA, Educação Digital.

However, these occur at random and most of the times detached from curricular contents. Research such as Prado and Valente (2003) reveal that the reconstruction of practices promoted by continuing education is a complex task. The difficulty is mainly associated with the integration of the use of resources into the specific contents of each field of knowledge. In addition to learning how to operate the technology, the teacher has to understand the pedagogic implications of teaching and learning by way of a new format. To that, the educator will have to renew his/her knowledge reasoning and representation.

Concerning the Mathematics educator, the use of a variety of digital technology resources is the subject of debate in many continuing education programs that explore specific education software such as: Cabri géomètre, Winplot, Geogebra, and others. However, the process for appropriating the pedagogic use of specific software focused on Mathematics teaching and learning requires the reconstruction of previous integrated knowledge, under the TPACK perspective advocated by Mishra and Koehler (2006) and by Koehler and Mishra (2009).

Focusing on the integration of digital technology into the teaching and learning processes, this chapter analyses two researches (Master's and Doctoral's Degrees) supervised by the Authors. The first approaches the prospective teacher education and the latter the continuing education integrated to the Education Observatory Program (Programa Observatório da Educação).[5] Both deal with trigonometric functions and explore aspects and resources from dynamic geometry software (GDS). Miashiro (2013) works in the context of prospective mathematics teachers' education, and Poloni (2015) in a continuing education project. In addition, they present situations favoring the instrumental genesis, as understood by Rabardel (1995), and development of TPACK as understood by Mishra and Koehler (2006). The two episodes consider the use of technology under an integrating perspective of knowledge in order to understand how the reconstruction of previously understood trigonometrical concepts occurs both in preservice and continuous teachers' education.

Theoretical Foundation

The evolution of digital technology has instigated researchers from different fields of knowledge to realize its potential regarding the processes of teaching and learning. This understanding is critical to guide teachers' education proposals, since discoveries in several studies show the teacher's difficulty in making pedagogical use of computing resources integrated into curricula (Almeida and Valente 2011; Bittar 2010; Borba and Penteado 2001; Prado 2005). Said difficulty can be

[5]This funding Program is an initiative of the Brazilian Government which purpose is to improve the teaching and learning processes in public schools around the country, and is developed in partnership with certain universities.

understood by taking into consideration that both future and acting teachers' references from pedagogical practices have been built without the presence of digital technologies in their trajectory of training and/or professional experience. On the other hand, the pedagogical use of digital technology demands different skills that are necessary for the mathematics teacher to "reason with", "to create" and "to teach with" technology. Teach, not just place it in the classroom, but integrating it properly and exploring what it brings to Mathematics teaching and learning (Lobo da Costa and Prado 2015).

The digital technology usage for teaching Mathematics must provide the student with skills on how to build concepts and to that it is necessary to give him/her conditions to formulate hypotheses, test and externalize conjectures that support the structure of thinking, the resolution of problems and the understanding of concepts. The provision of this kind of support takes for grant the teacher knowledge of the specific characteristics of each selected digital technology, whether software, simulators, learning objects, programming languages, and others that need to be linked to a particular field of Mathematics. For a mathematics teacher there is certainly a need to know, for each field of Mathematics, the possibilities and limitations of any available educational software. To explore the educational potential of a computer software it is necessary to know its structure in order to create activities and develop teaching strategies that can lead the student to experience founding ideas of Mathematics and rich situations for learning. These are situations that favor the construction of knowledge by the student.

Therefore, to practice basic mathematics education by integrating digital technologies into the curriculum it is necessary for the teacher to build a new type of knowledge generated by the integration of three different subject areas: mathematics, technological and pedagogical, as we find in Mishra and Koehler (2006) and by Koehler and Mishra (2009) model of the TPACK.

The TPACK (Technological Pedagogical Content Knowledge) includes the teacher understanding of how to represent concepts using technologies; pedagogically address the use of technological resources to teach, and constructively promote the student learning on the curricular concepts – in this case, Mathematics concepts. It is such integration of technological, pedagogical and content knowledge that enables the use of digital technology as a new form of representation of thinking.

In Fig. 1, the Technological Content Knowledge (TCK) refers to the interplay meaning between technology and content, the actual mathematical content at matter. Technology, even though having limited types of representations, introduces new possibilities of varied performances, as well as greater conversion flexibility between them. Therefore, it is necessary that teachers know how to explore different possibilities of representing certain mathematical concepts by means of technological resources. The Technological Pedagogical Knowledge (TPK) refers to the knowledge over the pedagogical implications involving the teaching and learning education process by means of technology. The Pedagogical Content Knowledge (PCK) refers to the knowledge over the strategies that might be more appropriate for teaching and promoting students' understanding of the

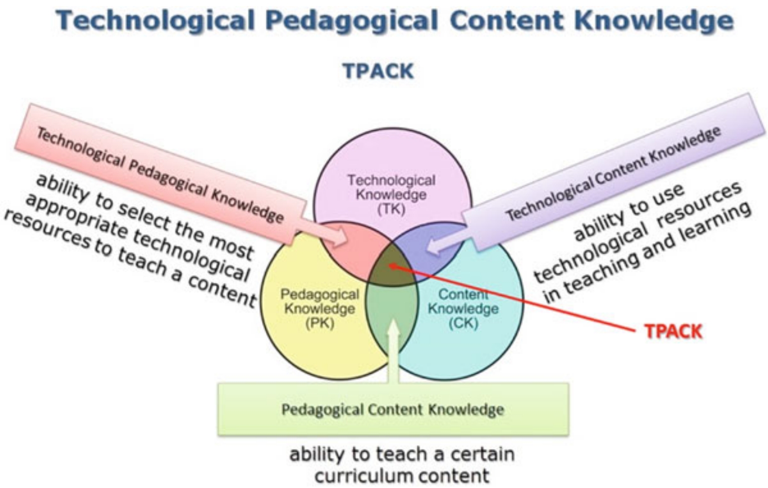

Fig. 1 TPACK – Intersection of knowledge (Source: Adapted from Koehler and Mishra 2009, p. 63)

content. It is by the intersection of these three domains that the TPACK knowledge is originated. It is the knowledge that needs to be built and mobilized by the teacher to teach Mathematics by means of technology.

Researches indicate that knowledge building goes through processes of appropriation. Rabardel (1995) proposes the instrumentation theory, which translates the idea that a computer artifact becomes an instrument to an individual through the instrumentalization and instrumentation processes. To the author, an artifact is understood as any item, either material or symbolic, effected for a firm purpose. A device changes into an instrument for an individual insofar it is used and manipulated by this individual during his/her activities.

The process of building an instrument from the individual's actions over the artifact is progressive. When starting to use the artifact, the person builds his/her own schemes of usage and, from that his/her own mental schemes start to enhance. Therefore, an instrument includes the artifact enhanced by the schemes of usage developed by the individual. This progressive over-time process of activity by which the artifact is transformed into an instrument for the individual own use and for a determined purpose, is called Instrumental Genesis.

The instrumentalization and the instrumentation are two dimensions that co-participate in the instrumental genesis process. The instrumentalization process refers to the adjustment of the instrument by the individual for specific usage. The individual, through his/her knowledge and mode of work, acts towards the artifact gathering awareness of its possibilities and limitations, thus using it in a more particular way. The instrumentation is the process by which the artifact's potentialities and restrictions control de action, i.e., it is how the instrument shapes the strategies and knowledge of the individual.

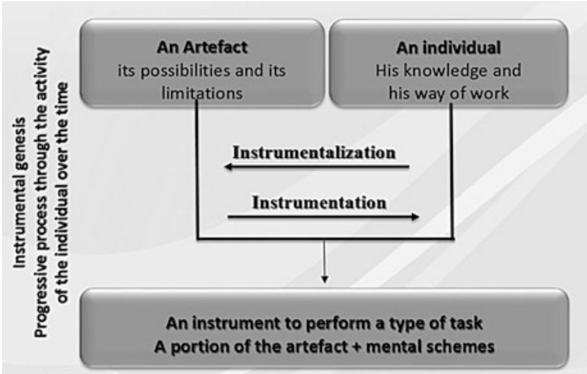

Fig. 2 Instrumental genesis as a combination of two processes (Source: Adapted from Trouche 2007, p. 22)

The instrumental genesis building process is complex; it demands time and depends on the characteristics (potentialities and limitations) of the artifact as well as the individual's mode of work and knowledge. It is from the instrumentalization and instrumentation processes that the individual makes use of the artifact, developing mental schemes in order to change the artifact into an instrument for a purpose (Trouche 2007), see Fig. 2.

One of the aspects of the TPACK construction process is that it relies on the ownership and instrumentation of the technological resource, which should make easier the reconstruction of teachers' pedagogic practice. Therefore, we believe it is essential to promote the instrumentalization and instrumentation of teachers' processes within the education progressions, as per Rabardel (1995), in order to support the integration of the Technological Pedagogical Content Knowledge (TPACK). When considering the teacher, this process of instrumental genesis is part of the construction of the TPACK, since in it, other types of knowledge besides the technological, is mobilized and amalgamated, as shown in Fig. 1.

A double instrumental genesis is required for mathematics teachers to teach with technology (Tapan 2006), which includes the *Technique Instrumentation*- building tools for mathematical tasks and the *Didactic Instrumentation* – building tools to teach Mathematics for the development of professional knowledge for teaching.

The researches discussed here are based on Mishra and Koehler (2006) studies on the integration of Technological Pedagogical Content Knowledge (TPACK). Rabardel's Instrumentation Theory regarding the construction of this knowledge and Tapan's (2006) considerations were also taken in, since it was assumed by the authors in this analysis that the understanding of the instrumental genesis process can guide the development of the strategies for teachers' education, supporting teachers while building up the TPACK.

The Research

This study was conducted under a qualitative methodology taking into consideration teachers' preservice and continuing education. Both processes have focused on the use of digital technology for teaching trigonometry, specifically with the use of the Dynamic Geometry Software (DGS). One group was formed by prospective teachers (second year of Mathematics Degree) and the other by teachers working in Basic Education (teaching 11–18 year students). Data were collected through questionnaires, teachers' activities protocols, digital files, meetings and audio and video recordings.

The preservice education process was conducted by Miashiro (2013) and consisted of nine meetings of 3 h each, with the participation of nine prospective teachers in the second year of Mathematics degree, in order to verify the contributions of a teaching strategy based on a combination of an experimental context with a computational context, for the construction of mathematical content knowledge, focusing on Trigonometry. Brazilian Government recommendations from the National Curriculum Parameters (Brasil 1998); data and analyses from related Brazilian researches, and content approaches found in many textbooks are regarded as subsidies for the activities plan. A diagnostic activity to investigate future teachers' content knowledge about trigonometric ratios and applications of these reasons was also held.

From these analyses, the education process activities were planned in order to consider possibilities to act on the transition from trigonometric ratios to trigonometric functions. Static resources such as paper and pencil or physical material were used. Dynamic resources were also exploited to enable future teachers to interact with trigonometric ratios, trigonometric circles and trigonometric functions, and realize characteristics such as periodicity of trigonometric functions. This choice is aimed at giving the participants the opportunity to be acquainted and debate on the pedagogical possibilities of the many available resources, either in the form of artifacts or instruments. The construction of models by means of technology leads to a "dual-mode artifact," as in Maschietto and Sourty-Lavergne (Maschietto and Soury-Lavergne 2013), that is, a digital model combined with a physical artifact. This construction may lead to a deeper understanding of the physical item as well as of the advantages and limitations of the technological resource and adjustments needed. The activities mobilized resources in an integrated manner for the reconstruction of content knowledge and as pedagogical alternatives to be used in the classroom, for the construction of the Technological Pedagogical Content Knowledge (TPACK) with regard to such contents.

The continuing education process documented by Poloni (2015) is aimed at exploring and discussing resources for teaching trigonometry in high school, in order to support the expansion of teachers' professional knowledge. The inservice education process assembled seven teachers, the research subjects, who participated in the "Trigonometry Topics" course of ten meetings of 3 h each. The teaching resources used throughout the continuous education process were both analogical and digital. A Dynamic Geometry Software (DGS), the GeoGebra software, was used to discuss with teachers the learning possibilities through investigation. Earlier in this education process a diagnosis was made to identify the teachers profile, their knowledge on Trigonometry beyond their computer skills and their teaching practice. We emphasize that these teachers teach trigonometry to their students, though none of them use Dynamic Geometry Software for teaching.

Episodes, linkage and relationship established between them are as follows.

Episode 1 – Exploring the Concept of Periodic Function

At the beginning of the preservice education process a diagnosis was performed to identify the content knowledge of future teachers on the trigonometrical concepts in relation to, for example, the trigonometrical relationships in the right triangle, the radian angle measures, and the understanding of the trigonometric circle, besides various aspects related to the definition and periodicity of trigonometric functions. The decision to start discussing the trigonometry in the right triangle was based on this diagnosis, which revealed the need to discuss the trigonometric ratios and their applications with the future teachers.

As the prospective teachers are referenced to the mathematical content developed in high school, exploitation of these contents was first made with the use of specific resources as a preparation to a later introduction of digital resources. Preliminarily, two meetings were held for the use of Dynamic Geometry Software aimed at familiarizing future teachers with the resources available.

Subsequently to identifying potential difficulties, the education process was reorganized through the development of proposals for activities and discussions over the trigonometrical concepts involved. The construction of the main concepts of trigonometry was reviewed and discussed using concrete materials named Objects of Experimental Environment as shown in Fig. 3, the use of which was taken in parallel with activities through the Dynamic Geometry Software (DGS). The similarity of the triangles in Fig. 3 was employed to conduct the study of the trigonometric ratios using the software, as shown in Figs. 4 and 5 for comparison with the corresponding record on paper and pencil.

The discussion of the content through dissimilar resources made possible the broadening of the future teachers' experience, helping them realize that a concept can be approached using various resources and that diversity can enhance the learning environment and contribute to a better understanding of the content knowledge as well as of the pedagogical possibilities for the use of these resources.

The discs in Fig. 3 were used to introduce the radian angle measure, meaning the ratios between the measures of the arcs and the radius of the circles. This was followed by an investigative proposal on these measurements using the software, as shown in Fig. 6. They were asked to compare the ratios in the experimental and digital cases to introduce the angle measure in radians.

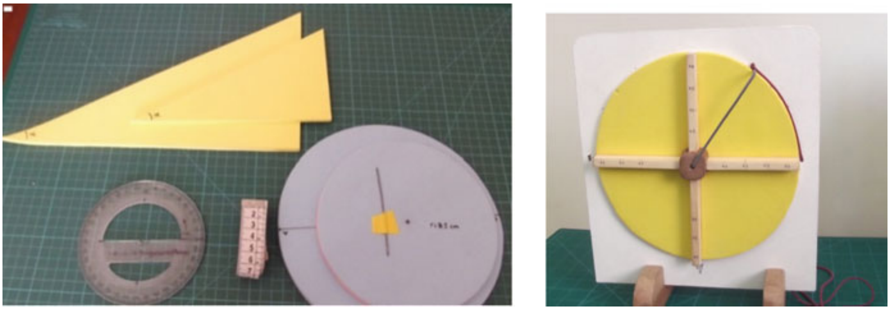

Fig. 3 Objects of experimental environment (Source: Miashiro 2013, pp. 88–89)

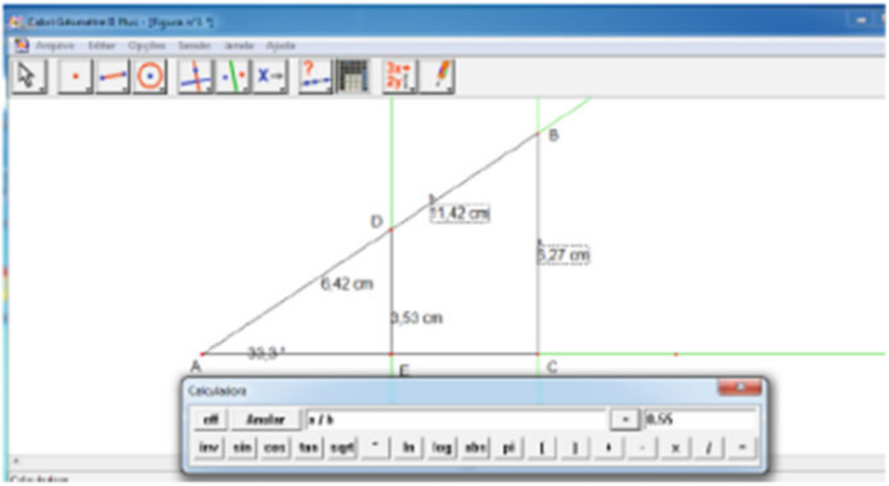

Fig. 4 File built to explore the sin A (Source: Miashiro 2013, p. 103)

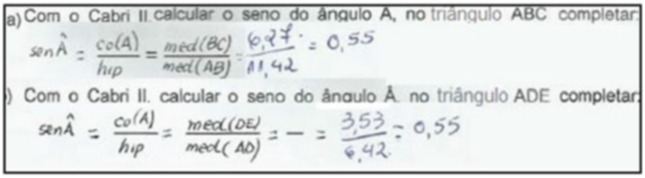

Fig. 5 Transcript of a prospective teacher to solve activity (Source: Miashiro 2013, p. 102)

The discussion on the radian angle measure was driven by the participants' difficulties identified in the diagnosis as well as the problems in their practice found by the researcher, through textbooks that adopt the same notation x to represent the variation of the angle in the trigonometric function sin$x$ to mark the abscissa in the Cartesian system associated with the trigonometric circle, as noted in Fig. 7.

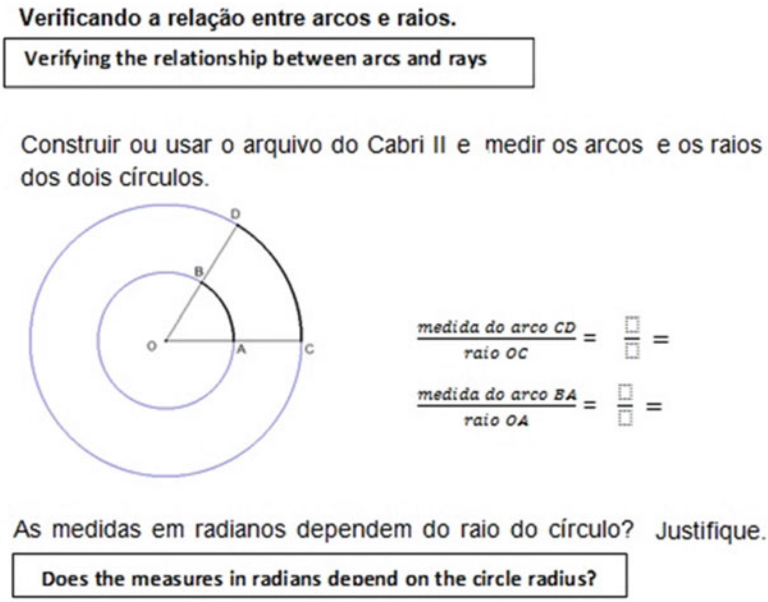

Fig. 6 Introduction to radian angle measures (Source: Adapted from Miashiro 2013, p. 111)

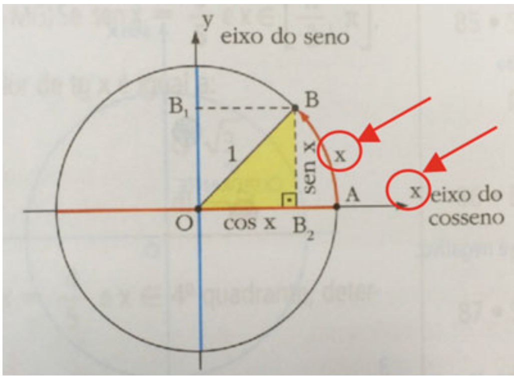

Fig. 7 Functions in trigonometric circle (Source: Xavier and Barreto 2008, p. 309)

During the training process the prospective teachers' attention was drawn to the issue of terminology. Teacher must choose very well the symbols to be used to prevent misunderstandings such as using the same letter (x) to represent different meanings. At this point it seems important to draw the attention of the future teacher to the pedagogical aspects that must be considered in the teaching process.

A sketch graph of a periodic function can be used to address a context based on the real world. A model was developed during the education process to simulate the motion of a Ferris wheel, adapted from the book "Functions Modelling Change: a preparation for calculus" (Connally et al. 1998). The model suggested the Ferris

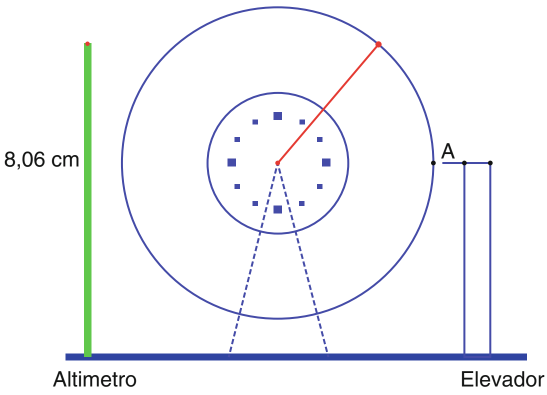

Fig. 8 Ferris wheel constructed in the software (Source: Miashiro 2013, p. 121)

wheel located in London. The Ferris wheel built with the Dynamic Geometry Software was prepared to allow the measurement of the height of a point on the circle at intervals of 5 min, and to display on the left side of the figure, above the altimeter word, a measure in centimeters in the range of 1–9 cm. The point on the circle representing the position of a passenger could be moved with the "hand" tool (see Fig. 8).

The collected data was organized as a table, producing a sketch of the graph of the function describing the variation of heights, following a guided discussion on the periodicity related to the Ferris wheel motion (Figs. 9 and 10).

The work refers to a real situation and the construction of its model, in which two conversions of registers of semiotic representations occurred, according to Duval (2006); it was possible to accompany, in many of the future teachers, the process of instrumental genesis, especially when they proposed didactic adaptations to the activity, as seen in the paragraph below.

After these investigations with objects and paper and pencil constructions, it was clear that the prospective teachers expanded their specific content knowledge and also experienced situations that have led them to ponder over varied possibilities of approaching this content in their future classroom teaching. They may take these experiences as a reference in their future classroom. From the trigonometric circle explorations with the DGS they began their instrumentalization process and the continued exploitation of periodicity using the Ferris wheel model. In this case, an attempt to assist with the instrumental genesis takes place in the initial education context. It is important to notice that this model allows for the investigation of the behavior of a periodic function, in which time is the independent variable. One of the difficulties encountered at introducing the sine function is connected with the fact that, in this case, the new variable is the measure of the angle in radians. The exploration of the trigonometric circle below was used for the work in this transition.

For the introduction of the sine function one of the experimental environment objects (called "trigonometric circle") was taken up, so that the participants could work in the construction of the sine function using the vertical projections with the

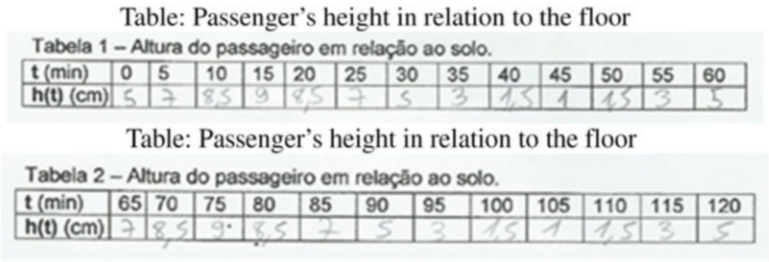

Table: Passenger's height in relation to the floor

Tabela 1 – Altura do passageiro em relação ao solo.

t (min)	0	5	10	15	20	25	30	35	40	45	50	55	60
h(t) (cm)	5	7	8,5	9	8,5	7	5	3	1,5	1	1,5	3	5

Table: Passenger's height in relation to the floor

Tabela 2 – Altura do passageiro em relação ao solo.

t (min)	65	70	75	80	85	90	95	100	105	110	115	120
h(t) (cm)	7	8,5	9	8,5	7	5	3	1,5	1	1,5	3	5

Fig. 9 Manuscript of a prospective teacher to solve activity (Source: Miashiro 2013, p. 123)

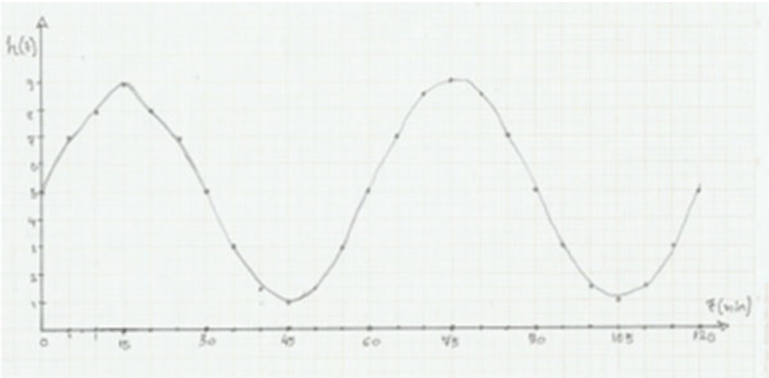

Fig. 10 Transcript of a prospective teacher to solve activity (Source: Miashiro 2013, p. 123)

aid of a light (Fig. 11) plus the information previously retrieved on calculations related to sine values, cosine and tangent in right triangles whose acute angles measure 30° or 45°.

The function graph was drafted from the set of the table (Fig. 12).

As a result, the exploitation started to occur in the DGS environment; similar discussions guided the activity using the software for the set of a table of values for the sine function and its corresponding graphic representation (Fig. 13).

The work was completed with the construction of a model for the sine function with the use of software resources (see Fig. 14). The familiarization of the participants with the software was supported by Baldin and Villagra (2002), where a chapter is dedicated to the use of Cabri for the study of Trigonometry. Mishiaro worked with resources of measurements transfer and the possibility of measuring the angle in radian for the proposal of the sine function graph construction. During the construction, the difficulties faced and the resources used were discussed step-by-step with the participants. Afterwards, a complete file enabled everybody to work with the same digital model.

Considering the performance of students in the final evaluation, we concluded that the contributions of activities for reconstruction of the participants' knowledge were important as they all managed to build the sine table. Two of these students

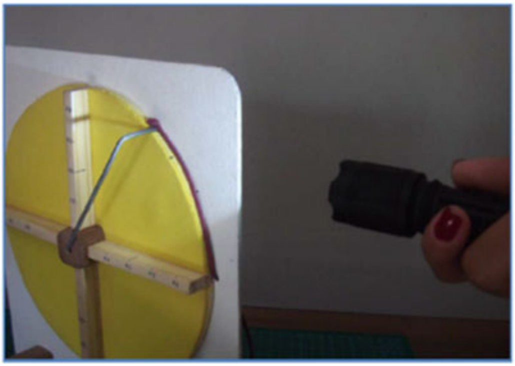

Fig. 11 The trigonometric circle artifact and the orthogonal projection of a point on the vertical axis (Source: Miashiro 2013, p. 90)

x	sen x
0°	0
$\frac{\pi}{6}$ (30°)	0,50
$\frac{\pi}{3}$ (60°)	0,86
$\frac{\pi}{2}$ (90°)	1
$\frac{2\pi}{3}$ (120°)	-0 86
$\frac{5\pi}{6}$ (150°)	0 50
π (180°)	0
$\frac{7\pi}{6}$ (210°)	-0,50
$\frac{4\pi}{3}$ (240°)	-0 86
$\frac{3\pi}{2}$ (270°)	-1
$\frac{5\pi}{3}$ (300°)	-0,86
$\frac{11\pi}{6}$ (330°)	-0,50
2π (360°)	0

Tabela 1- Valores do seno.

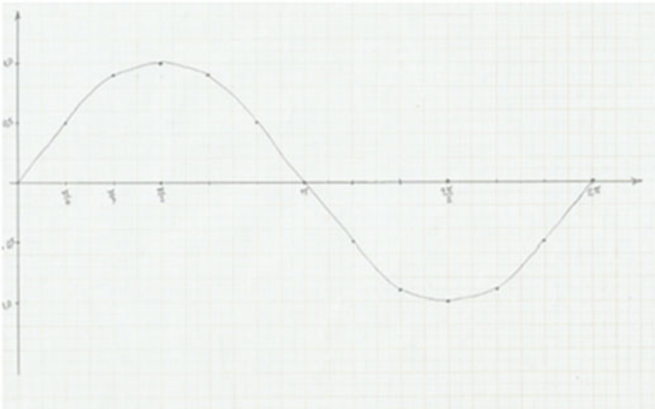

Fig. 12 Notes by a prospective teacher for construction of a graph (Source: Miashiro 2013, p. 132)

Construção do gráfico da função seno.

- Construir a figura 3 com o Cabri, colocar as coordenadas no ponto localizado na extremidade do arco x (apagar a abscissa), e com o "ponteiro" arrastar esse ponto de 30° em 30°, para preencher a tabela com os respectivos valores desta função.

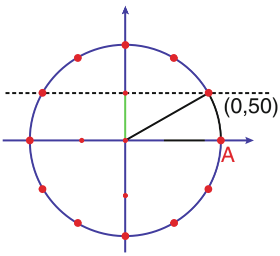

Fig. 13 Study of the sine function in the software (Source: Miashiro 2013, p. 131)

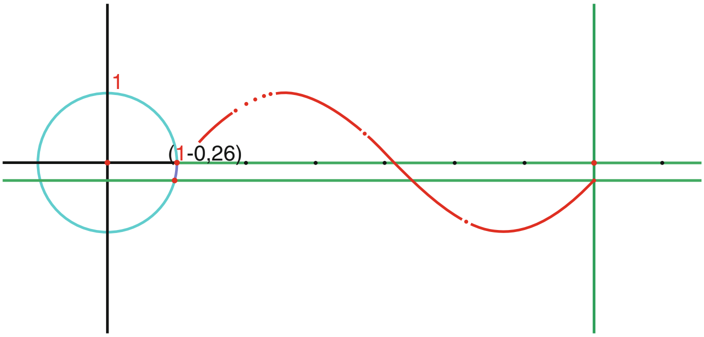

Fig. 14 Study of the sine function with the software (Source: Miashiro 2013, p. 134)

demonstrated to master the concepts discussed in the sequence of activities, building the function graph sine to perfection.

Among the contributions from this teaching strategy for learning the concepts existing in the transition from the trigonometric ratios to the trigonometric functions, subject of the interventions, it is highlighted:

- The perception of trigonometric ratios has emerged due to the similarity between right triangles.
- Learning from the calculation of the measures in radians.
- The rediscovery of the number π.
- The ability to build a table and a graph from the context of a periodic function as yet unidentified.
- The understanding of the construction of a table and graph of the sine function.

The dynamic software and the resources for measurement and tracking led to strengthening the understanding of ratios, the angle in radian measurement and the discussions on the transfers of discrete/continuous and time/variation of the angle in radians occurring when constructing trigonometric functions graphs from tables and modelling a real example. Experiencing resources of a varied nature used in an integral format was very enriching to those prospective teachers coming to university with little knowledge of Trigonometry. The use of technology was encouraging and soon allied with the process of understanding knowledge.

In fact, the education process contributes to the reconstruction of previous content knowledge and pedagogical alternatives to classroom teaching with the aid of experimental objects and software activities, bringing a new approach to Technological Content Knowledge regarding trigonometrical concepts. In this mathematical knowledge building process it was important the diversity of resources used to broaden the scope of possibilities for teaching. Particularly in relation to technological resources, these determined the integration with the mathematical content from the perspective of Technological Content Knowledge (TCK).

Episode 2 – Continuing Education: From Circle to Trigonometric Functions

We will highlight some characteristics of the process of continuing education and detail an episode on the exploitation of trigonometric functions.

The course "Trigonometry Topics" began with the application of a diagnostic questionnaire, from which we identified teachers' expectations to discuss trigonometry in trigonometric circle and the specific high school leads such as trigonometry in right triangle. The calculation of inaccessible distances, which is commonly used in classroom to rescue what students know about trigonometry in right triangle, was chosen as the starting point to address trigonometry in the circle. Thus, a course of ten meetings was held, focused on the subjects: (1) Construction of a sine and cosine table and of a radian angle measure table with paper and pencil as well as with the GeoGebra DGS; (2) Arc or angle measures domino game and catch-lot game for measuring conversions from degrees to radians; (3) Construction of a trigonometric circle with compass and the DGS; (4) Construction of a trigonometric circle with the DGS displaying the sine and the cosine in the Cartesian axis and a trigonometric circle displaying symmetrical arcs; (5) DGS programming for the construction of trigonometric functions; (6) GeoGebra investigative activity; (7) Trigonometric functions bingo game; (8) Planning classroom activities; (9) Group discussion on the prepared activities, and (10) Evaluation of the program. Each meeting was planned from the feedback from the previous one.

The episode at matter in this text involves the meetings that addressed topics (3), (4) and (5). Initially the inservice teachers developed a diagnosis activity from topic (3) in order to inform the trainers about the aspects of their practice, investigating trigonometrical and pedagogical content knowledge. The responses supported the preparation of the activities to be developed in the coming meetings of the program (Fig. 15).

It was followed by an activity to build the trigonometric circle in paper and pencil using a ruler and compass. The subjects positioned in the trigonometric circle notable arcs such as $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their symmetrical arcs. The figures above show some of the material produced by the research subjects.

The teachers' strategies to build the circle were explained in the following dialog:

Teacher RG: I will make it with a compass, tracing the perpendicular . . . I love doing these things.

Trainer: How can you be sure that these lines you drew are perpendicular?

Teacher RG: I will make the bisector with the compass**.** [time to trace the bisector] See? And now I will draw bisectors to get the angles *45*°, *90*°, *135*° and so on until *360*° that coincides with angle *0*°.

Analyzing this dialogue, we find that **Teacher RG** mobilized content knowledge and demonstrated ability to make the necessary constructions, guided by the known

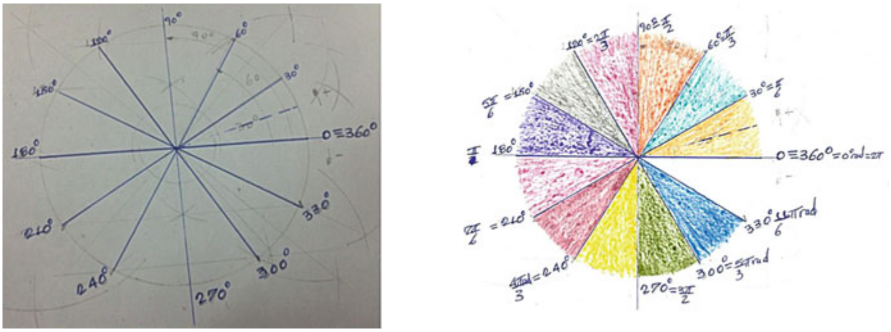

Fig. 15 Location of arcs in two screen of participating teachers (Source: Poloni 2015, p. 196)

properties of geometric figures. To do this he used a ruler and a compass as instruments to represent his content knowledge in plane geometry.

Thereafter, the teachers repeated the construction of the arcs corresponding to $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their symmetrical arcs, this time using the DGS GeoGebra. These activities intended to identify the technological content knowledge of inservice teachers (Mishra and Koehler 2006).

The screen in the software with the arcs and trigonometric circle construction in the four quarters is shown below (Fig. 16).

We noticed that the above construction requires mathematical and technological knowledge in order to draw perpendicular lines, parallel lines, segments and the actual trigonometric circle. Teachers needed to determine the position of the arcs type $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$, etc.

Teacher RG carried out the conversion from degrees to radians and then located the arcs in the circle; teacher CP, who was sitting close, showed her strategy for the arcs:

Teacher RG: I can't believe it! What for do I have to memorize $\frac{4\pi}{3}$?

Teacher CP: No RG! Look! I divided the circle into eight parts. Here is $\frac{\pi}{4}$, here is $\frac{2\pi}{4}$ which is also $\frac{\pi}{2}$ when simplified. Then comes $\frac{3\pi}{4}$ and $\frac{4\pi}{4}$ or π when simplified.

Teacher RG: Ah! OK. This is also possible with others arcs, following the same reasoning. It goes faster!

Teacher CL: It's easier. How great!

Teacher MC: It is also possible to think in degrees in the same way.

[CL teachers, RG and CP began to tell the arcs in radians.]

The analysis of this dialogue, combined with the analysis of the recordings of the meetings, shows signs of expansion of mathematical content knowledge for

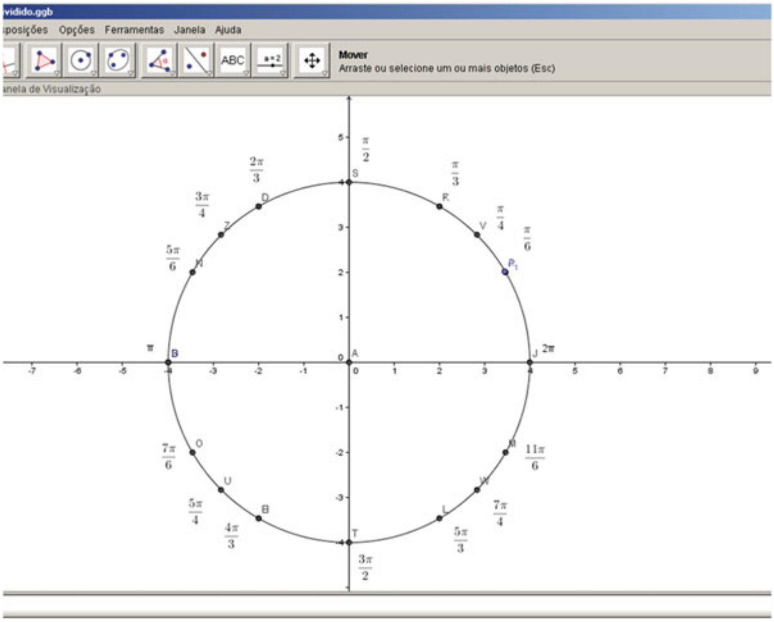

Fig. 16 Arcs located on the screen (Source: Poloni 2015, p. 274)

Teacher RG and Teacher CL; both of them, when interacting with Teacher CP, made explicit a new line of thinking among them.

When the inservice teachers finished the arcs constructions, the trainers discussed the DGS construction strategies with the whole group.

One of the reports follows:

Teacher CL: [CL teachers, RG and CP began to tell the arcs in radians.]I did like the compass activity. I designed the unit circle and the axis and got *90°*, *180°*, *270°* and *360°* angles, then I traced the bisectors and got arcs of *45°*, *135°*, *225°* and *315°*. After that I took the radius and found the *60°*, *120°*, *240°* angles and then I did the family of these bisectors to find the arcs of the *30°* family.

From this dialogue, the broadening of technological content knowledge is clear, because the teachers used their knowledge of geometric constructions in the context of paper and pencil in order to choose and use the DGS tools properly.

In the discussion on ways to conduct this activity when applied to students, Professor MC stated:

Teacher MC: I would ask my students to build the trigonometric circle in the GeoGebra with radius 1. I would explain the concepts of radius, diameter and center of the circle. Then I would tell them to score arc 90° and draw the bisector, they mark arc 45° ...

Teacher RG: They never know what the bisector is ... would have to explain again

Teacher MC: Another concept to review: bisector ... but coming back ... with the radius measure they would do *arc 60* and again, by bisecting, they would do *arc 30*. Then go on to just do the triangles and, by symmetry, move to other quarters.

We notice by this dialogue that MC and RG teachers mobilized their technological pedagogical content knowledge when referring to what is needed to plan in order to develop the activity in their classrooms and what is necessary to teach their students, from the Mathematics point of view and with the technological tools available in the software.

Some teachers using DGS tools for these constructions revealed lack of technological content knowledge (Mishra and Koehler 2006), even knowing the geometry involved, as shown in the dialogue below.

Teacher CP: I want to bisect the *90°*, where do I have to click?

Trainer explains where CP should click.

Teacher RA: The bisect I can do, but how do I divide the circumference by using the radius as the opening?

Trainer explains the compass tool.

Teacher RA: Ah! Got it. It's easy, isn't it? It's like on the paper... the same reasoning.

The dialog above shows the moment when Teacher RA realizes the similarity between the reasoning made in the construction with ruler and compass and the one to be done with the DGS software. It was then realized that the teacher was reasoning over the software tools showing his/her instrumentation process.

At the meeting in which item (4) of the course was developed, the teachers used the DGS GeoGebra to construct the trigonometric circle and the segments corresponding to the sine and cosine to a generic arc, as can be seen in the figure below (Fig. 17).

As we understand it, this experience helped teachers to review their forms of exploring together with their students the sine and cosine on the trigonometric circle, particularly for trigonometric projections in the vertical and horizontal axis concerning the fundamental relations of trigonometry.

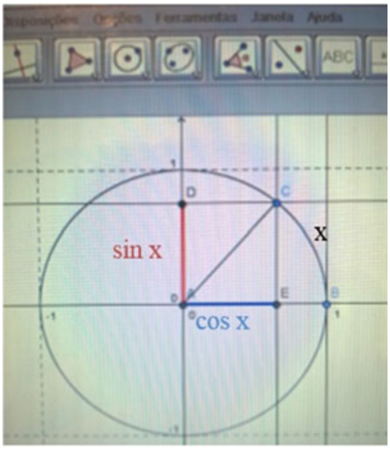

Fig. 17 Segments corresponding to the sine and cosine of a generic arc in the circle (Source: Poloni 2015, p. 204)

While building the projections of a generic arc on the axis to have sine and cosine placed in, dialogues such as the following highlighted the difficulties of using the artifact, revealing who was in the process of instrumentalization:

Teacher MC: The projection is always perpendicular ... I'll have to use perpendicular lines to make this construction.
Teacher RA: I know how to do it with the compass... but here, I don't. . .
Teacher RO: Neither do I.

We noticed that to develop this new activity other parts of the artifact would need to be triggered, which requires the individual construction of new mental schemes, as underlined by Trouche (2007).

These difficulties were gradually overcome during the meetings, explaining the continuous and progressive process of instrumentation of the subjects, in Rabardel's view.

The next activity addressed the construction of a trigonometric circle displaying the symmetries of arcs. This activity was chosen after identifying that teachers always used the formulas (180 - α) (α - 180) and (360 - α) to effect the reduction of arcs to the first quarter.

We discussed the sine and the cosine values as shown on the screen. Based on the constructions performed with the software, it was possible to show that the sine of symmetrical arcs in relation to the vertical axis are equal in magnitude, and likewise for the cosine. The following figure (Fig. 18) shows a sample file built in this activity.

The aim of the activity was to support the broadening of technological content knowledge as understood by Mishra and Koehler (2006) through the instrumentalization of the subjects in dealing with the DGS.

As regards to sine and cosine values of arcs not located in the first quarter, the dialogue below took place:

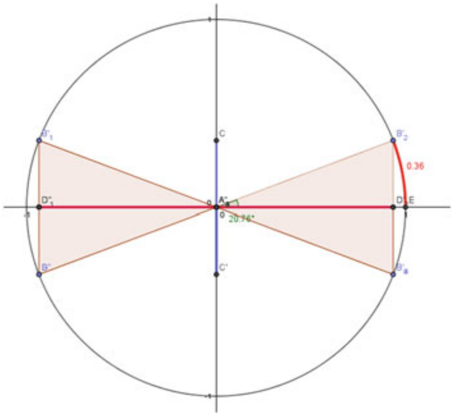

Fig. 18 Arcs in the trigonometric circle (Source: Adapted from Poloni 2015, p 155)

Trainer: And why is *cos(135°)* negative?
Teacher RA: No way! A distance cannot be negative!
Trainer: We tell the students that $cos(135°) = -cos(45°)$.
Teacher RA: But it is here. .. because of the side.

[Teacher RA showed the segment on the x axis that is to the left of the origin.]

We noticed that Teacher RA pointed at the figure constructed in GeoGebra for the cos135° segment to give his answer, demonstrating technological knowledge.

In the discussion we figured out the sine and cosine values. The dialogues were as follows:

Trainer: Why the sine of this arc in the first quarter has the same value as the sine of this second quarter arc?
Teacher RG: Because they have the same length in the axis of the sinus.
Teacher RO: "The projection is the same, so they have the same measure."
Trainer: And the sign? We say that the *cos(135°)* is equal to – *cos(45°)*, why?
Teacher RG: It is the same triangle that is turning.

These statements showed that the construction and operation supported by the figure in the GeoGebra, may favor the establishment of conclusions about the sine and cosine values of the arcs at matter.

The next dialog also supports this finding.

Trainer: Can the visualization of the sine and cosine of symmetrical arcs in the trigonometric circle through the GeoGebra lead the student to understanding that $sin(30°) = sin(150°)$, for example?
Trainer: And to reduce the first quarter? $\pi - x$, $x - \pi$ and $2\pi - x$, how is it? You have to know it by heart? [Laughs].

Teacher RG: No. You can see this construction. It is impossible not to see it. And I always told students to memorize it ...

This dialogue shows the perception of Teacher RG that the use of this technology can assist students in the construction of knowledge. When he said "*No.* [referring to not having to memorize the formulas] *You can see this construction. It is impossible not to see it*"we noticed that he made a connection between what he experienced and understood and what it will happen to his students. Then he said: "*And I always told students to memorize it...*", hence we understood that there was an awareness of his own teaching practice.

We consider that there was, in this case, an expansion of the pedagogical content knowledge. We understand that the discussion on the symmetries that may be found in the circle led teachers to reflect on how to explore them with the students, eliminating the need to memorize arcs reduction formulas to the first quarter.

Ending the above discussions, we proposed topic (5) to the teachers, referring to a construction activity of the function y = sin x in the DGS GeoGebra environment, followed by the analysis of the characteristics of this function, such as being limited (image from −1 to 1) and with periodicity 2π. Conversions of registers (Duval 2006) were presented (algebraic to graphic), and the teachers built usage schemes and began to realize what kind of strategy was possible to be developed in the software, i.e., the way to model the situation under study in the software.

The *f(x) = sin(x)* graph can be built in the Geogebra Algebra window using "Input" *y = sin(x)*, and so the graph is plotted on the screen (see Fig. 19). However, one of the inservice teachers said they would like to trace the trigonometric circle to show the projection of the arc on the vertical axis and plot the graph (see Fig. 20). In this way, when moving a point P in the circle, it was possible to "see" the corresponding point moving in the graph.

To build the function graph, as required by this teacher during the meeting, a strategy was found: to associate each value *x* on the arc in the circle to its respective *sin(x)* value. Geogebra does not come with the resource of arc measure transfer to a segment building. Such resource can be found in the *Cabri* and was used in the previous episode for the construction of the sine function graph.

We emphasize that this is a difficult task that demands understanding the concept of angle measure in radians. Moreover, it is necessary to represent this concept taking into consideration the limitations and specificities of the software resources. It has to do, therefore, with the Technological Content Knowledge (TCK).

For this reason, it was necessary to lead the teacher to conclude that any point of the graph of this function is as follows: P = (*α*, *sin(α)*), where α is the measure of the arc. This means that knowing these features of the software requires mathematical and technological knowledge. We highlight that in the trigonometric circle the notation P (*cos(x), sin(x)*) was adopted.

The inservice teachers need to know the syntax of programming and how to represent a generic point of this function in that syntax.

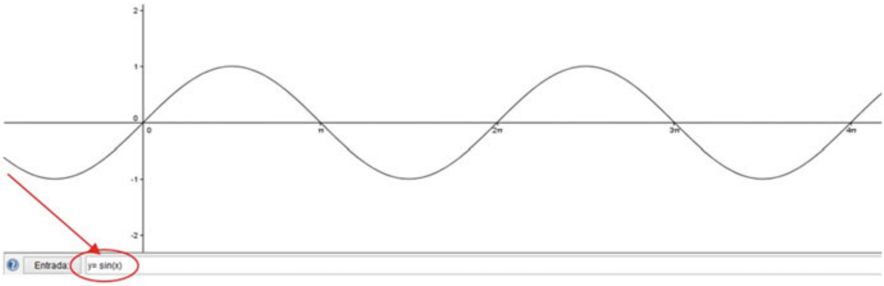

Fig. 19 Screen with the function graph $y = sin(x)$

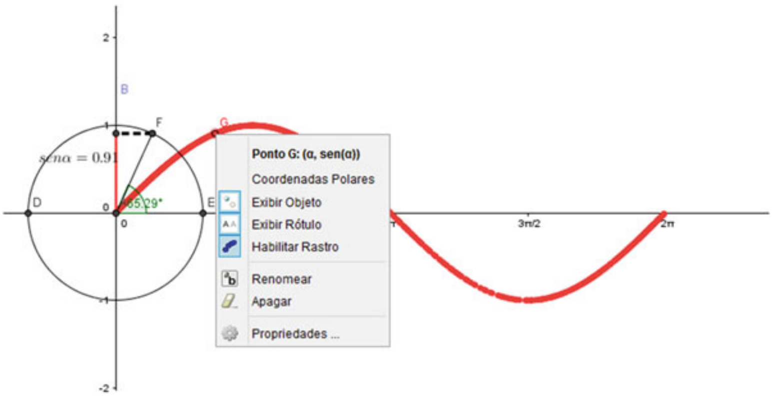

Fig. 20 Trigonometric circle and a period of the function $f(\alpha) = sin(\alpha)$ built with the trace tool

We noticed in this episode, a moment in which it was possible for the trainer see the development of the instrumental genesis teachers, as well as the possibilities and limitations of the software that conditioned the actions of the participants.

After the generalization of the coordinates of the point of function $f(x) = sin(x)$, the construction of graphics for $f(x) = cos(x)$ using the software was immediate.

Figure 21 shows Teacher RG computer screen with the circle and the functions $y = sin(x)$ and $y = cos(x)$ appearing together in the same period 2π.

After developing these activities and learning how to express the coordinates of a generic point of the graph, the teachers began to investigate how to generalize points of other trigonometric functions. This situation occurred especially because teachers looked at each other's computers screens (for example, Teacher RG screen with the functions $y = sin(x)$ and $y = cos(x)$). The following figure (Fig. 22) displays graphs built by Teachers RO and Teacher CI with programming and trace tools.

To the extent that the inservice teachers were experiencing the programming activity of these new functions features, we noticed that they began to feel the empowerment that occurs by the act of "teaching the computer" as per Papert's constructionism (Papert 1980).

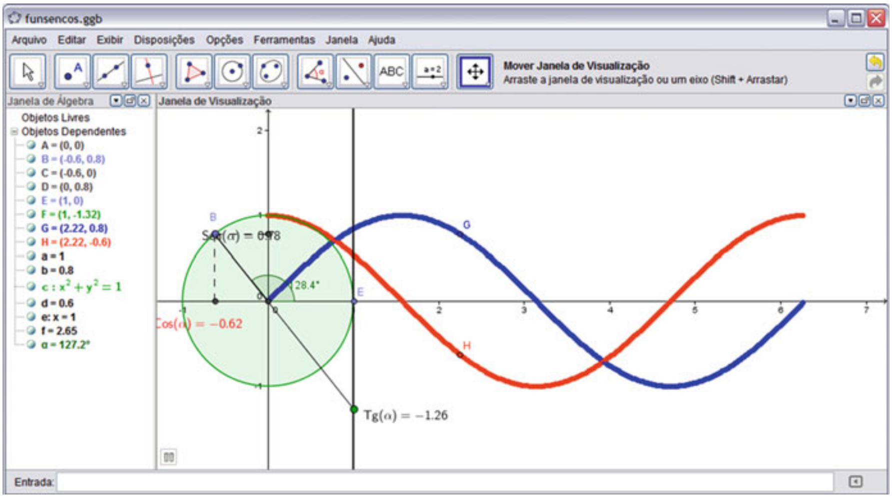

Fig. 21 Screen with a period of functions *f(x)* = *sin(x)* and *g(x)* = *cos(x)* (Source: Poloni 2015, p. 158)

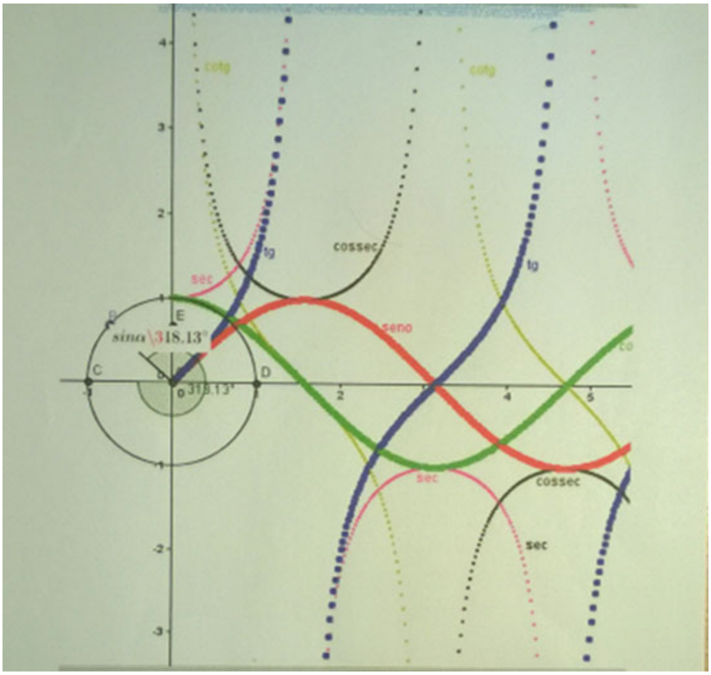

Fig. 22 Screen with graphs of trigonometric functions built by programming and Trace tool (Source: Poloni 2015, p. 157)

In the current situation the inservice teachers had two possible strategies to build the trigonometric functions: to use the algebra window and enter the algebraic expression of functions (varying parameters **a** and **b**, obtaining the graphs of these functions with real domain, or to use programming to connect the trigonometrical circle, although limited to an investigation of the function to the domain 2π.

Once the graphs were constructed, the strategy used in the DGS was discussed with the whole group (programming and tracing); it was possible to obtain graphs of a period if the functions were $f(x) = sin(x)$ and $g(x) = cos(x)$. Again, the DGS features conditioned the actions.

In addition, the participants were able to construct graphs of trigonometric functions involving sine such as $f(x) = a + b\ sin(x)$. From these constructions it was possible to vary the values of the parameters ***a*** and ***b*** and to identify the changes in the graphs of functions, articulating an algebraic expression for each function with the corresponding graph. The periodicity analysis and processing, for example, changing function $y = sin(x)$ for the family of type $y = a + b\ sin(wx + c)$ with a, b and c being any real number, and w a real positive number, ended up the discussions for this meeting.

This activity enabled them to start programming with the GeoGebra software and to engage in discussions with their students over the variations of parameters in sine and cosine functions, including the impact caused by these on the graph of each function.

The activities highlighted the expansion of technological and pedagogical content knowledge of the teachers (Mishra and Koehler 2006). Particularly, in this research documented by Poloni (2015), the GeoGebra software was used to discuss, with teachers, the learning possibilities through investigation.

In interviews with the participants, we noticed, for example, that teacher RG held the DGS efficient both concerning the teaching process and the learning of trigonometry. In his words:

Trainer 1: What has changed in the education process in your professional practice?

Teacher RG: Every training process adds something [...] The Trigonometry made me revisit many concepts ... [...]it made me take a fresh look into the content, and I really did it.

Trainer 1: What is this new look?

Teacher RG: It's a new understanding ... a deeper view of the content. This improves the classes. [...] [...]

From Professor RG's statements, we gather that the education process helped him to improve his trigonometry knowledge driving him into a deeper understanding of this content, which, in his case, will impact his classes positively.

To conclude this analysis, we pointed out the results recommending a continuing education on the Trigonometry subject, supported by the use of technological resources for teaching Mathematics. This can help the broadening of teachers' professional knowledge. Activities involving technology usage for teaching aroused discussions that unleashed ideas about classroom practices, mediations made by trainers, and their own mathematical content.

The following aspects of training favored the expansion of teachers' professional knowledge: the non-traditional design training; the professional attitude of inservice teachers; the mathematical content of interest to inservice teachers; the trainers' mediation and intention to take discussions beyond mediations and resources for teaching, through the education process. The research revealed that this kind of continuing education can be a variation when considering high school teachers' needs to teach trigonometry.

Conclusion

This study presents an analysis of two episodes from Miashiro and Poloni's research, involving the use of technology under an integrative perspective of knowledge in order to understand how the reconstruction of trigonometric concepts occurs both in preservice and continuous teachers' education.

We point out that, although the scope of continuing education is quite different from that of initial education, there have been employed, in both education processes, a varied range of possible resources for investigating the basic concepts of trigonometry. It was done so in order to improve the subjects' content knowledge, in addition to supplementing their pedagogical practices, that is, improving their pedagogical technological knowledge.

In both episodes it was clear that each participant had the opportunity to develop a process of technological appropriation through instrumentalization and instrumentation. We observed that the participants experienced new learning situations. These experiences involved mathematical content (trigonometry), a component usually dealt with by the participant teachers in their daily work in the classroom.

Furthermore, in both programs, the trainer's strategy to provide other technological resources (ruler, compass, paper pencil, wooden models, etc.) was quite adequate, as such resources were closer to their previous learning experiences. This worked as a starting point so that they could comfortably explore possible forms of knowledge representation and then move on to new discoveries using the software features (DGS).

In this aspect, the possession of Technological Content Knowledge (TCK) had developed quite similarly for both preservice and inservice teachers. However, the integration of the Pedagogical Content Knowledge (PCK) that consolidates the model TPACK worked differently in the initial and in the continuing trainings. In this research the development of this kind of knowledge, such as the TPACK, could be observed only with the inservice teachers. In this case, we realized that to preservice teachers the learning process was mainly focused on the mathematical content and on the technological knowledge. To inservice teachers otherwise, the learning process was focused on how to use technology in their Mathematics classes.

For this reason, this study prompted us to think how to enable the preservice teachers' education to prepare future teachers to build knowledge under the TPACK perspective. We think that one possibility would be to develop new strategies that should lead preservice learning to also focus on the pedagogical aspects, i.e., how to teach Mathematics to their future students using technology.

To this, we highlight the studies of some researchers, for example, Llinares (2008), who tried using real-situation videos in classroom with the objective of practicing and using them as objects of analysis and discussions, including making theoretical contributions to future Mathematics teachers' education. This teachers' education strategy that seeks to approximate real situations to classroom practices meets the guidelines of programs promoted by the Brazilian Ministry of Education, such as the aforementioned PIBID, PRODOCENCIA and Obeduc, that aim to improve the initial and continuing teachers' education as well as students' basic education.

We have indeed understood that teachers' education – initial and continuing, has encouraged the learning under an integrative perspective of knowledge: mathematical content, technology and pedagogical aspects – combined with practice. It would indeed contribute to the reconstruction of knowledge and professional development. Concerning this teachers' education approach, once geared towards professional development, the teacher will see the extreme need to take a proactive stance to learn and also be included in the technological and scientific advancements that bring new demands and implications to education, getting ready to a professional future in the twenty-first Century.

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Formative Assessment and Technology: Reflections Developed Through the Collaboration Between Teachers and Researchers

Gilles Aldon, Annalisa Cusi, Francesca Morselli, Monica Panero, and Cristina Sabena

Abstract In this work, we present the analysis of the ways in which formative assessment processes can be developed, by the teacher and the students, thanks to the support given by technology. The analysis is carried out focusing on two case studies developed in France and Italy within the European Project FaSMEd, with two main aims: (1) highlighting how the different functionalities of technology could support formative assessment strategies at the teacher's, the students' and the peers' levels; and (2) characterising the dynamics that intervene within programs involving a strict collaboration between teachers and researchers. Through the analysis of the two case studies we discuss, on one side, the effectiveness of the adopted theoretical tools, and, on the other side, the contribution, in terms of professional development, of the collaborative work developed within the project.

Keywords Formative assessment • Meta didactical transposition • Feedback • Design based research • Information and communication technology

Introduction

In this chapter we analyse the role played by technology in supporting the formative assessment (FA) process, referring to some examples from the case studies developed in France and Italy within the European Project FaSMEd (Improving progress

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for lower achievers through Formative Assessment in Science and Mathematics Education).

The FaSMEd project aims at investigating the role of technologically enhanced FA methods in raising the attainment levels of low-achieving students. It relies on the hypothesis that a technological environment can potentially support both the students and the teachers in getting information about students' achievement in real-time. Indeed, since connectivity and feedback are enhanced, students can use data from technology for informing their learning trajectories, and teachers have the possibility of collecting data from students, making more timely formative interpretations and informing their future teaching.

In line with this hypothesis, the project investigates: (a) students' use of FA data to inform their learning trajectories; (b) teachers' ways of processing FA data from students using a range of technologies; (c) teachers' ways of using these data to inform their future teaching; and (d) the role played by technology, as a learning tool, in enabling the teachers to become more informed about student understanding.

The research is based on successive cycles of design, observation, analysis and redesign of classroom sequences (Swan 2014) in order to produce and feed into the *toolkit*, a set of curriculum materials and methods for pedagogical intervention aimed at supporting the development of practice.

The teachers engaged in the project are involved in the different phases of design, implementation, analysis and subsequent redesign and adaptation of the toolkit. Besides FA, technology and low-achievers, the FaSMEd project addresses also the issue of teacher professional development, as a consequence of the collaboration between teachers and researchers.

Some studies highlighted the considerable amount of time for teachers to change their beliefs about teaching and learning, classroom culture and the teacher's role (Foshayla and Bellman 2012) and their ways of being so that FA with technology becomes an integral part of their practice (Feldman and Capobianco 2008).

Our hypothesis (FaSMEd DOW, p. 8 PART B[1]) is that, to enable teachers to incorporate elements of the new pedagogical model into their everyday teaching, it is important to give them the opportunity to engage in a process of development where they can reflect on and contrast their experience in using this approach. The teachers are therefore engaged as practitioner researchers since they undertake inquiry-based practice, strictly collaborate with researchers, and engage seriously with reflective developmental practice.

Considering the double concern of the project this chapter aims at (1) highlighting how the different functionalities of technology could support formative assessment strategies at the teacher's, the students' and the peers' levels; and (2) characterising the dynamics that intervene within programs involving a strict collaboration between teachers and researchers. We present the theoretical

[1]FaSMEd: Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education. Annex I, Description of Work.

framework, which is constituted by two main poles: one frames the role of technology in supporting FA, and the other provides tools to analyse the dynamics within a teachers-researchers collaboration. Then we will present two case studies, from teaching experiments carried out in France (Lyon) and in Italy (Torino). The presentation of the case studies will include methodological aspects as well as data analysis from the classroom. We will conclude proposing some reflections arisen from the comparison of our case studies.

Theoretical Framework

FA and the Role of Technology

In the FaSMEd Project, FA is intended as a method of teaching where.

> [...] evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited (Black and Wiliam 2009, p. 7).

Such learning evidences can be collected, interpreted and exploited in different moments of the learning process and with different purposes. In particular, Wiliam and Thompson (2007, adapted from Ramaprasad 1983) focus on three central processes in learning and teaching: (a) Establishing where learners are in their learning; (b) Establishing where learners are going; (c) Establishing how to get there.

Different agents are involved in these three processes: the teacher, the learners and their peers. Black and Wiliam (2009) conceptualise FA as consisting of five key strategies, that could be activated by agents:

1. Clarifying and sharing learning intentions and criteria for success;
2. Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding;
3. Providing feedback that moves learners forward;
4. Activating students as instructional resources for one another;
5. Activating students as the owners of their own learning.

Effective feedback from the different involved agents plays a central role in FA. According to Hattie and Temperley (2007), there are four major levels, and the level at which feedback is produced influences its effectiveness. They distinguish between:

1. feedback about the task, which includes feedback about how well a task is being accomplished or performed;
2. feedback about the processing of the task, which concerns the processes underlying tasks or relating and extending tasks;

3. feedback about self-regulation, which addresses the way students monitor, direct, and regulate actions toward the learning goal;
4. feedback about the self as a person, which expresses positive (and sometimes negative) evaluations and affect about the student.

Recently, research has started to investigate the different ways in which technology could support FA. Quellmalz and colleagues (2012), for example, stresses that technology enables the assessment of those aspects of cognition and performance that are complex and dynamic, through rich and authentic contexts, interactive and dynamic responses, individualized feedback and coaching, and diagnostic progress reporting.

Referring specifically to *connected classroom technologies*,[2] further aspects are pointed out as features that make them effective tools for FA:

1. they may enable the teachers to monitor students' incremental progress and keep them oriented on the path to deep conceptual understanding, providing appropriate remediation to address student needs (Irving 2006; Shirley et al. 2011);
2. they may support positive students' thinking habits, such as arguing for their point of view (Roschelle et al. 2007), creating immersive learning environments that highlight problem-solving processes (Looney 2010) and giving powerful clues to what students are doing, thinking, and understanding (Roschelle et al. 2004);
3. they may enable most or all of the students contribute to the activities and work toward the classroom performance, taking a more active role in the discussions (Roschelle and Pea 2002; Shirley et al. 2011);
4. they may provide students with immediate private feedback, encouraging them to reflect and monitor their own progress (Looney 2010; Roschelle et al. 2007);
5. they may enable to carry out a multi-level analyses of patterns of interactions and outcomes thanks to their potential to instrument the learning space to collect the content of students' interaction over longer timespans and over multiple sets of classroom participants (Roschelle and Pea 2002).

Together with the other researchers involved in the FaSMEd project, we started developing a three-dimensional framework[3] that extends Black and Wiliam's (2009) model to include the use of technology in FA processes. The new model takes into account three main dimensions:

- the five FA key-strategies,
- the three main agents (teacher, student, peers/group),

[2]"Connected classroom technology" refers to a networked system of personal computers or handheld devices specifically designed to be used in a classroom for interactive teaching and learning (Irving 2006).

[3]The framework was initially developed during a meeting in Essen (Germany) in July 2015 by some of the FaSMEd partners from Italy, United Kingdom, and Germany. It was also presented at ECER 2015, in Budapest, during the symposium "Formative Assessment in Science and Mathematics Education".

- the functionalities through which technology can support the three agents in developing the FA strategies. Within the third dimension (functionalities of technology), that we have introduced, we distinguish three main categories, according to the different uses of technology for FA within the FaSMEd project:
  - *Sending and displaying*: when technology is used to support communication among the agents of FA processes and to activate fruitful discussions. For example: sending questions and answers, sending messages, sending files, displaying and sharing screens to the whole class or to specific students, sharing students' worksheets.
  - *Processing and analysing*: includes all the functionalities that support the processing and the analysis of the data collected during the lessons, such as the statistics of students' answers to polls or questionnaires, the feedbacks given directly by the technology to the students, the tracking of students' learning paths.
  - *Providing an interactive environment*: those functionalities of technology that enable to create a shared interactive environment within which students can work individually or collaboratively on a task or a learning environment where mathematical/scientific contents could be explored.

The following chart outlines the FaSMEd three-dimensional framework, where the three dimensions described above are represented along the axes of a three-dimensional Cartesian reference (see Fig. 1).

In the analysis of our examples, we will use this three-dimensional framework to describe and characterize the use of the technology to support FA in classrooms. In the discussion of each example we will also use other theoretical references introduced in this section (i.e. the levels of feedback) to go into detail in the analysis.

Meta-Didactical Transposition

In tune with the model of design-based research (Shavelson et al. 2003) we take into account the existing state of teachers' pedagogical knowledge relatively to formative assessment and use of technology and the evolution of this knowledge as a result of the collaboration between teachers and researchers. In order to analyse the complex dynamics involved in these processes, we refer to the model of the Meta-Didactical Transposition (MDT) developed by Arzarello and colleagues (2014).

The MDT model is based on the Anthropological Theory of the Didactic (ATD, Chevallard 1985, 1999) in which every human activity within an institution can be described by a *praxeology*. A praxeology consists of the *praxis*, the *"how to"*, the techniques allowing to solve tasks that can be classified into types of tasks, and the *logos*, that is to say the discourses on these techniques bringing their justification in reference to a certain theory. A praxeology is therefore the given quadruplet type of task, techniques, justification of the techniques and theory developed within

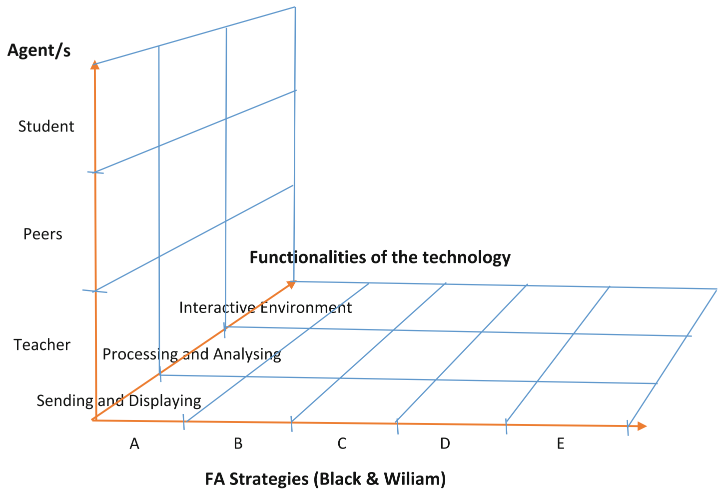

Fig. 1 Chart of the FaSMEd three-dimensional framework: (*A*) Clarifying and sharing learning intentions and criteria for success; (*B*) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (*C*) Providing feedback that moves learners forward; (*D*) Activating students as instructional resources for one another; (*E*) Activating students as the owners of their own learning

specific institutions. In line with this perspective, the MDT-model considers teacher education as a human activity that takes place in an (or several) institution(s).

During the work with teachers, the research team develops tasks and techniques in order to solve these tasks at a didactical level but also builds justifications of these techniques at a teaching level of reflection. These praxeologies are built at a meta-level justifying the term of meta-didactical praxeologies. But they are also built and discussed for applications in the classroom that is to say at the didactical level. In this way, two dialectics are generated. A *first dialectic* is developed at a didactical level in the classroom between students, teacher and knowledge. A *second dialectic* is developed at a meta-didactical level in the interaction between teachers and researchers relatively to the interpretation of the first dialectic.

> Typically the second (meta-didactical) dialectic arises from a contrast/comparison between the researchers' praxeologies and the teachers' praxeologies and the first dialectic engenders the second one as an outcome of a suitable meta-didactical trajectory, which is designed by the researchers. It is through this double dialectic that teachers' and researchers develop a shared praxeology. (Aldon et al. 2013, p.104)

In the FaSMEd project, the general discussion about FA at a meta-level, for example using the three-dimensional model, nurtures the construction of activities for the classroom and their implementation in the classrooms give feedback that makes the model evolve and participate to the teachers' professional development. During the FaSMEd experimental phases (task design, *aposteriori* reflections, etc.), ideally researchers share research results and innovative inputs, while teachers mainly offer their professional knowledge and the pragmatic justifications of these practices. In this way, during the MDT process, the researchers' praxeologies encounter the teachers' ones, and it may happen that components of praxeologies that were *external* to a certain community become gradually *internal*, within a praxeology *shared* by the two communities. This process, called internalisation, can be iterated in a cyclic way.

Case Studies

FaSMEd in France: One Example and Its Analysis

Amongst all studies done in the context of FaSMEd in France, we choose to highlight one particular sequence in a grade 9 class where all students are equipped with tablets that are connected within a network.

Our methodological choice consists in giving to teachers the responsibility for designing lessons. We ask them to fill in a grid concerning important points to reflect upon before and after an observed lesson. We collect this information in order to contextualise what is happening in the classroom when we are about to visit it. While observing lessons, we collect videos, pictures, audios, teacher's report and notes. We also have intermediary discussions and meetings with the teacher, and final interviews with the teacher and with the students. The object of our analysis are the dynamics within the FaSMEd three-dimensional model (see above), which represent the agents' different uses of the technology for enhancing the process of formative assessment.

In order to account for the mathematical knowledge and competences at stake, we refer to the *Theory of Didactic Situation* (TDS, Brousseau and Balacheff 1998). Starting from Brousseau's didactic triangle to interpret the mutual relationships between teacher, student and knowledge, we consider its tridimensionalisation. Indeed, with the introduction of the technological dimension in the classroom, this triangle becomes a didactic tetrahedron, where technology represents a new vertex establishing relationships with the other didactic actors of the mathematical situation and getting a part of the *milieu* that teachers and students have to cope with. In the tetrahedron, the edges connect the vertices in order to describe and to analyse the mutual relationships between the four actors of the didactic situation: student, teacher, knowledge, technology. A face of the tetrahedron represents instead a point of view on the situation taking into account the relationships of three of the four actors. On the contrary, in the FaSMEd three-dimensional model, the axes correspond to agents, FA strategies and functionalities of technology. Thus, the relationships agents-technology or agents-knowledge can be found within the three-dimensional model, when for example an agent uses technology or mobilises knowledge in a particular moment of the FA process. For example, in Fig. 2, we can analyse the mathematical situation in which the teacher is using technology for processing and analysing data from the classroom, in order to

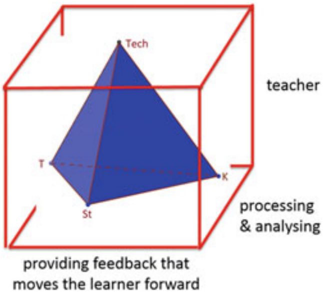

Fig. 2 The didactic situation within a particular cuboid of the FaSMEd three-dimensional model

provide feedback to students. We are interested in the evolution of the didactic situation described by the tetrahedrons within the cuboids.

In this grade 9th class and for networking tablets, the teacher (Thomas) uses the NetSupport School software that allows classroom monitoring, management, orchestration and collaboration. In addition, Thomas decides to use MapleTA that is an online testing and assessment system designed especially for courses involving mathematics. The classroom is also equipped with an IWB. From a didactical point of view, the use of such technologies for formative assessment in his classroom is completely new for the teacher. These aspects of professional development will be further analysed below under the lens of the Meta-Didactical Transposition.

The case study leans on a sequence about linear functions, where the following competences are to be acquired, according to the different representations of functions.

(a) Calculating and detecting images.
(b) Calculating and detecting inverse images.
(c) Recognising a linear function.
(d) Shifting from the graphical frame to the algebraic frame and vice versa.

Thomas decides to create a sequence of questionnaires around these four competences, using MapleTA. Following a typical Thomas' teaching sequence with MapleTA, we propose to analyse three specific episodes taken from our observations and referred to the third quiz proposed by Thomas to the students during this learning sequence about linear functions. The first moment concerns a student taking the quiz and the teacher declaring his potential use of the class' results. In the second episode, the teacher comments the quiz results of a student and, during the third excerpt, the teacher comments the whole set of the class' results.

First Episode

Mor is working on Question 3 concerning the competence (a): calculating and detecting images (see Fig. 3). Formulated in the graphical register of representation, the question is: "The curve below represents a linear function. The image of 9 is -2. True/False."

Mor is a low-achiever, with high difficulties in mathematics. Since he is working alone on the mathematical task, he is *activated as the owner of his own learning* (E).

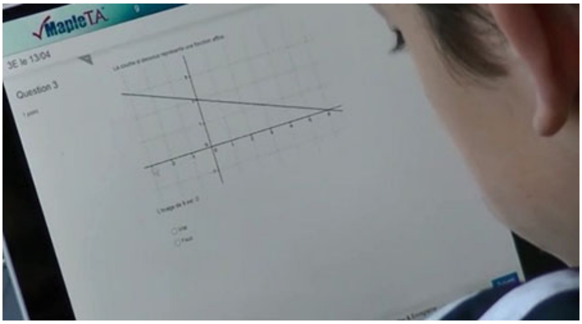

Fig. 3 Mor is reading Question 3 on his tablet on the platform MapleTA

He is reading the task given by the teacher on MapleTA. Therefore, the technology is used with the functionality of 'sending and displaying'. The student faces the didactic situation devolved by the teacher, represented by the tetrahedron in Fig. 4.

After a while, Mor copies the question, leaves MapleTA, and pastes the question on the interactive environment of his tablet (OneNote) in order to work on the given graphical representation using his previous experience on similar exercises. On his screen, indeed, we can see a previously solved exercise that is very similar to the new one (Fig. 5a). Mor starts using the same graphical technique (Fig. 5b), mobilising his knowledge as a reflexive student. He has transformed the didactic situation into an a-didactic situation represented by the face Student-Knowledge-Technology of the tetrahedron, where he acts on a reacting *milieu*.

Confronted to the *milieu*, composed of the task, the technology and the previous knowledge, Mor reflects on the mathematical situation mobilising his knowledge and finally submits his answer, sending it back to the teacher through the MapleTA. At the student's level, there is a shift between different functionalities of technology (Fig. 6), and the devolution of the mathematical situation by the teacher is at the base of this dynamics.

The back and forth between the two cuboids in Fig. 6, related to the different functionalities of technology, is the starting point of a more extended dynamics:

- towards other FA strategies;
- towards the peers and the teacher;
- towards other functionalities of technology.

Moving on to the teacher's level, he is collecting the students' answers and will use MapleTA for 'processing and analysing' such data. To this purpose, during the lesson, the teacher declares his potential FA strategies depending on the students' responses.

> Thomas (to Ant, an other student): I don't know if I'm going to take it into account or not. The idea is that I would like to mark it. If I realise that it doesn't work... I don't know... I'm going to see what's going on... At least I'll know that you don't succeed here. You can skip it if you don't know what to do.

The students' results are feedback for the teacher, who will process and analyse these data. Depending on the students' performances, he may adapt his teaching, for

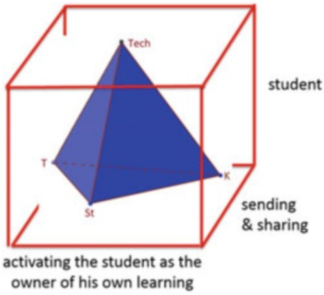

Fig. 4 The student is activated as owner of his own learning in the process of sending and displaying started with MapleTA by the teacher

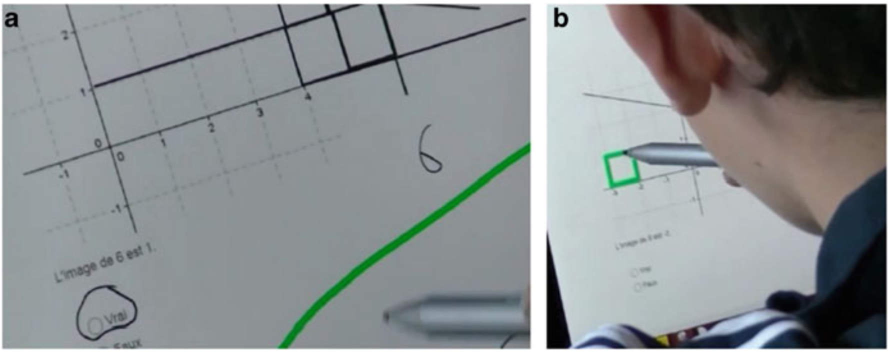

Fig. 5 (**a**, **b**) Mor working on OneNote

example by choosing the FA strategy of providing feedback to students (Fig. 7). Through his words, Thomas gives to Ant a '*feedback about the task*', by saying "At least I'll know that you don't succeed here".

Second Episode

Teacher's feedback can be made on the spot, like in the second transcription that we propose to analyse. Rom has completed his quiz, submitted his answers and got a '*feedback about the task*' from MapleTA: "good answer" or "wrong answer". Then he calls Thomas in order to have further explanations.

> Thomas: The first one is right, the second one is false, the third one is right, and the fourth one is false. Finally, I consider that you were right on the two that are easier to explain and you got false on the two that require more mathematical work. That's normal. I consider your result as normal.

Both the teacher and the student benefit from the feedback in this episode. The student gets a '*feedback about the processing of the task*' and also on his global performance according to the teacher's norm. The teacher, who analyses this quiz result on the spot and considers it as normal, gets information about the student's achievement.

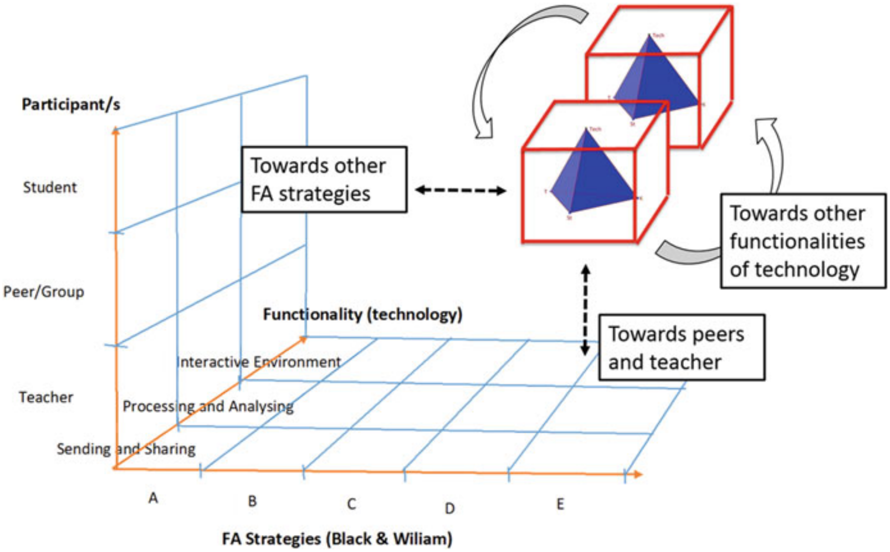

Fig. 6 Dynamics within the three-dimensional model at the student's level between the functionalities "Sending and Displaying" and "Providing an interactive environment" are the starting point of other dynamics

Third Episode

Teacher's feedback can be made after reflection and this is the case of the third transcription. When all students have completed the quiz, Thomas leads the correction of the questions with the whole classroom using NetSupport School. The lesson after, he proposes a global lecture of the class' results at the three quiz and analyses the whole set of answers stored by MapleTA, by showing them at the IWB and commenting them with the students. In this way, he provides feedback for the whole class on the attained mathematical competences.

> Thomas: [Here are your results] on several trials. What we can see is that in calculating images you reached 0.778. What does it mean? [. . .] about 8 successful students over 10, here. There we had 6 over 10, then 8 over 10. So we are good in calculating images. [. . .] I'm not going further. However, we'll come back on determining the expression of a linear function. 0.1, you see 0.1, 0.3, and here we went down at 0.2. [. . .] I would like to get to realise if I succeed in teaching you two or three things last time, so we are going to work again on these two questions. Open Maple TA, and answer the two questions of the day. Let's go.

Thomas analyses the class' results and he *clarifies the learning intentions and criteria for success*. He has worked again with the students on the required competences during the correction phase, and now he wants to test again the competences revealed as not achieved by the analysis of the results, namely competences (b) and (d). Thus, he *engineers other learning tasks* on MapleTA. Two new questions are properly prepared and sent to students as a result of this dynamics (Fig. 8). From Thomas' words, we can observe that, as he expected in the

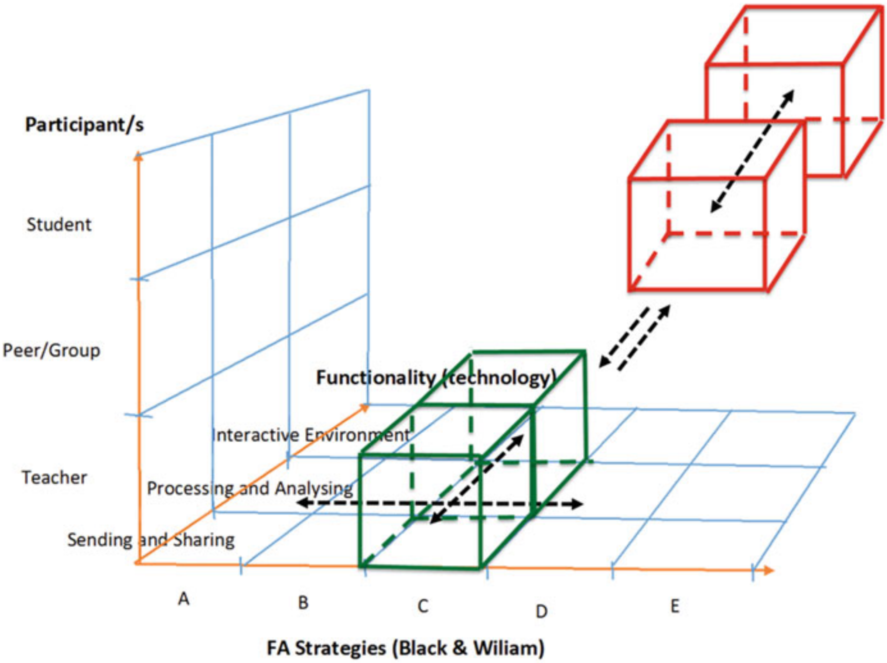

Fig. 7 Dynamics between the individual students and the teacher are the driving force of the dynamics at the teacher's level between the functionalities "Sending and Displaying" and "Processing and Analysing" for providing feedback to students

first episode, he has adapted his teaching depending on students' progressive achievement.

More generally, relatively to FA strategies, Thomas orchestrates the different functionalities of technology in direction of individual students, of the whole classroom or even of himself. Indeed, moving from 'processing and analysing' data to 'sending and displaying' results or new learning tasks allows him to choose the most powerful FA strategy according to students' mastering of the competences at stake.

FaSMEd in Italy: One Example and Its Analysis

In Italy the FaSMEd Project involves 19 teachers from three different clusters of schools located in the North-West of Italy. 12 of them work in primary school (grades 4–5) and the other 7 in lower secondary school (grades 6–7). All the teachers work on the same mathematical topic: functions and their different representations (symbolic representation, tables, graphs).

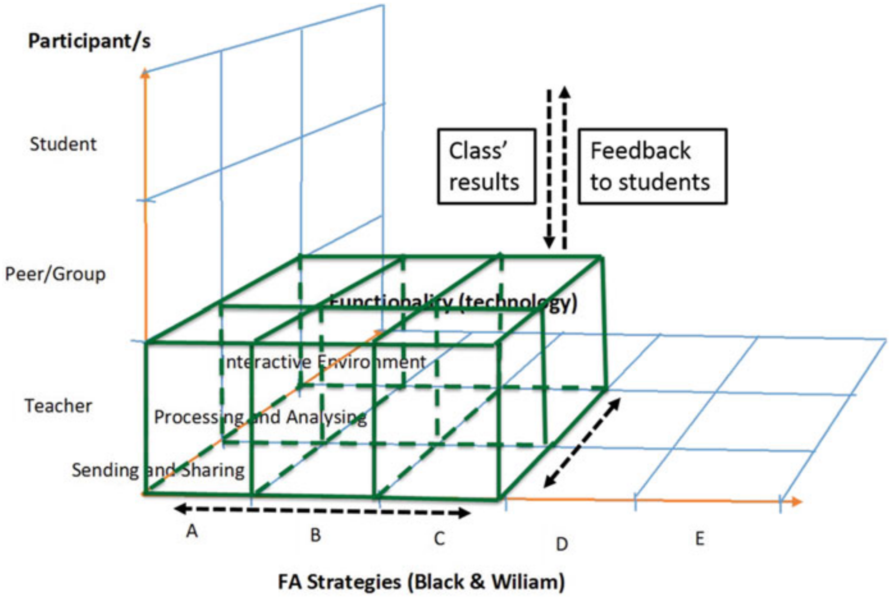

Fig. 8 Dynamics at the teacher's level to provide feedback to the whole class

We believe that low achievement is linked not only to a lack of basic competences, but also to metacognitive factors. For this reason, during class activities, it is important to make students: (a) develop ongoing reflections on the teaching-learning processes; (b) make their thinking visible (Collins et al. 1989) and share it with the teacher and the classmates.

Starting from these assumptions, we chose to exploit IDM-TClass, a *connected classroom technology*, because it enables both to share the students' ongoing and final productions, and to collect their opinions during and after the activities. Specifically, it allows the teacher to: (a) show, to one or more students, the teacher's screen and also other students' screens; (b) distribute documents to students and collect documents from the students' tablets; (c) create different kinds of tests and have a real-time visualization of the correct and the wrong answers; (d) create instant polls and immediately show their results to the whole class.

Each school has been provided with tablets for the students (who work in pairs), computers for the teachers and, where the interactive whiteboard was not available, a data projector to display students' written productions. The students' tablets are connected with the teachers' laptop through the IDM-TClass software. IDM-TClass was integrated within a set of activities coming from different sources. Among them, the ArAl Units, which are models of sequences of didactic paths developed within the project "ArAl – Arithmetic pathways towards favouring pre-algebraic thinking" (Cusi et al. 2011).

For each lesson carried out in the classes, we have prepared a set of different worksheets, aimed at:

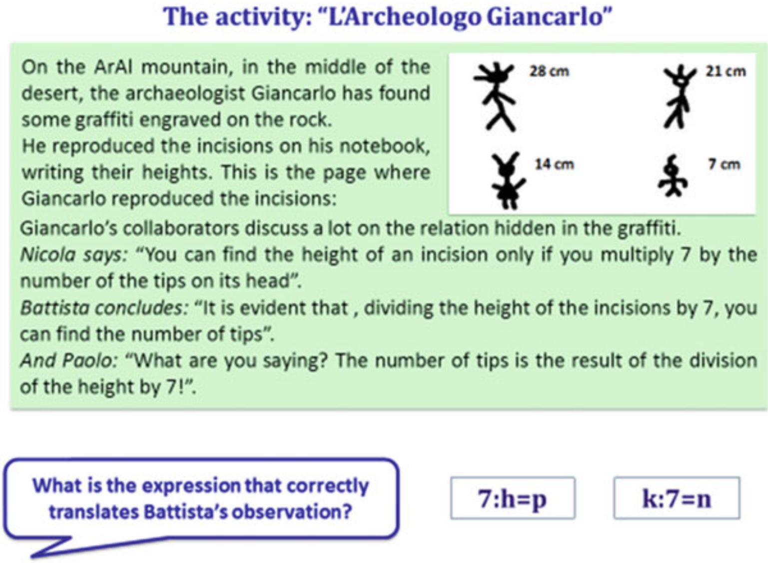

Fig. 9 The worksheet provided to students

- supporting the students in the verbalisation and the representation of the relations introduced within the lesson;
- enabling the students to compare and discuss their answers;
- making the students reflect at both the cognitive and metacognitive level.

In this paragraph we are going to analyse an excerpt from a class discussion, which was chosen because of the richness in the levels of feedbacks provided and in the FA strategies that are activated. The discussion refers to the following worksheet (Fig. 9).

We remark that this is not the first worksheet on this problem. During the previous lesson, students were asked to discuss and compare Nicola, Battista and Paolo's statements and to interpret a symbolic expression ($7 \times n = k$) proposed by a fictive student from another class with reference to the problem.

During the lesson reported in this example, 5th grade students are asked to answer the question through a poll. The exploited functionality of the technology is 'processing and analysing', because the technological tool collects all the students' answers and processes them, displaying an analytical as well as a synthetic overview (bar chart) to the teacher. We (the teacher and the researchers) decided not to provide an immediate correction.

When all the students answer the question, the teacher (Monica) shares with them her screen, where the bar chart and the list of students' answers are displayed (Fig. 10).

The worksheet is also projected on the interactive whiteboard, next to the poll.

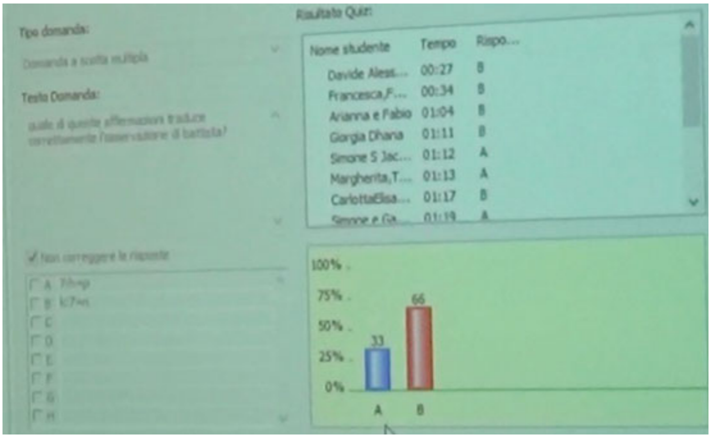

Fig. 10 The teacher's screen shared with the students

Monica observes that the 33% of the students chose the answer *7:h* = *p*, while the 66% of the students chose the answer *k:7* = *n*. Monica reads the list of the couples of students and the corresponding answer.

Monica chooses not to tell to the students what the right answer is, and asks to the different pairs to explain why they chose a specific answer. The class discusses on the possible strategies that could be used to identify the correct expression, in case the only reading of Battista's observation is not enough. The students are invited to check if the number of tips and the height of every incision verify the two expressions. Some students are asked to substitute, in the two expressions, the different values connected to each incision (4,28; 3,21; 2,14; 1,7). One of them observes that she discarded expression A because the result of the division 7:28 is not 4.

Alice, softly, says that *7:7* = *1*. Monica asks her to explain what she means. We report the related excerpt:

1. Monica (*to Alice*): "What were you saying?"
2. Alice: "I was saying that, for example, the figure, the one on the bottom right, is 7 cm, so *7:7* is *1*, therefore the result is not a decimal number, while with the others (*the other figures*) it is" (*the result is a decimal number*).
Monica focuses on Alice's observation and states that the chosen expression should represent all the incisions, not only the first one. Lisa and Nicolò ask if they can change their mind.
9. Monica: "Have you changed your mind? That is, Lisa, you chose answer A, but now you have changed your mind. Why?"
10. Lisa: "Ahem … *7* is only that figure. While, if you divide the height by 7, you mean all the figures."
Another student declares that, although h in Italy always stands for the height, in the expression "7:h=p" h does not refer to the height.
14. Monica: "It does not refer to the height. Is it right, Lisa?"
15. Nicolò (*raising his hand*): "Monica, because h refers only to one (*height*), while k…"

(continued)

16. Lisa: "Both (*the letters*) . . . (*Nicolò is speaking*). . .no, wait! (*to Nicolò*)"
17. Monica: "One at a time"
18. Lisa: "Both the letters are always the height, but h is only for one (*height*) . . . only for this one (*Lisa goes near the interactive whiteboard to indicate the incision 7 cm height*), while k is valid for all (*the incisions*)."
19. Monica: "k is valid for every incision. (*Stefano is raising his hand*) Stefano?"
20. Stefano: "The first expression . . . No, I mean: the second expression is more correct than the first. Battista says . . . where is it? (*Stefano is trying to find Battista's statement*) 'It is evident that dividing by *7*'. It is 'Dividing by *7*', not 'dividing the height'. . . that is . . .
21. Monica: "Dividing 7 by ...the height"
Dialogue between Monica and Amalia, who observes that Lisa's interpretation of the two expressions is right and declares that, after having listened what Lisa and Nicolò said, she realised that the expression could be interpreted in different ways. Nicolò asks to intervene.
36. Nicolò: "Monica, in the first statement (*he is referring to the first expression*) 7 is divided by the height. Instead, in the second (*expression*) the height is divided by 7!"
37. Monica: "Very good! So . . . Many times, I realised that many times it is not the same thing. It is necessary to pay attention. It is necessary to think very carefully to what is written. Exchanging, inverting the numbers is not the same thing."
Monica comments on the usefulness of this kind of activity, stressing on the reflections developed by the students during this discussion.
39. Amalia: "Last time, you (*Monica*) asked us if this thing, this kind of work, was useful. In my opinion, many (*students*) have changed their mind because they said 'Look, now, thanks to these explanations and to all of these . . . thanks to all of these explanations, I understood what I have to do."
40. Monica: "Lisa, was this activity really useful for you?"
41. Lisa: "Yes"
42. Monica: "Why?"
43. Lisa: "Ahem, because I have never done this kind of work that . . . together . . . even if I made a mistake, because I chose A, the fact that the others chose B and explained their motivations 'oppenned' my mind, it opened my mind."
44. Monica: " 'It oppenned your mind', it is ok!"
45. Nicolò: "Monica, it made us understand that first you have to reason, then you can choose."

The process *'establishing where the learners are in their learning'* is central in this lesson: the discussion is planned in order to support the students in making the motivations of their choices explicit. This enables to highlight erroneous ways of reasoning and incomplete explanations, but also to highlight the evolution of students' reasoning, together with the way in which it is influenced by the other students' interventions.

Our analysis of this excerpt will focus on Nicolò and Lisa, two low-achievers.

We may say that Lisa and Nicolò are *activated as owners of their own learning* during the discussion: they ask to correct their initial answers, effectively motivating their new choice (from line 10). The teacher's conduction of the discussion fosters the activation of this strategy by the two students: she is another fundamental agent within this process. The following diagram (Fig. 11) represents the two cuboids that could be, therefore, highlighted within the FaSMEd three-dimensional model at the teacher's level and at the students' level.

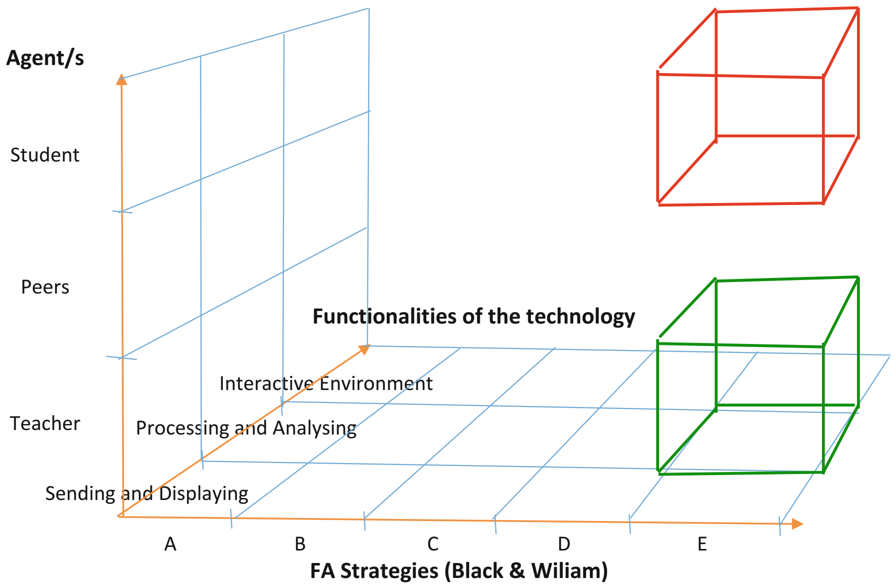

Fig. 11 The activation of the FA strategy "Activating students as owners of their own learning", at the students' and teacher's levels, through the "Processing and Analysing" functionality

There are also evidences of the *activation of students as instructional resources for one another*. Lisa (lines 10 and 18), for example, refers to Alice's intervention (line 2) and elaborates it to start developing her own argumentation. Also Nicolò (line 36) refers to Stefano's intervention (line 20) and elaborates it. Again, we must stress that the teacher is another fundamental agent in this process, because her conduction of the discussion fosters the activation of this strategy at the peers' level. The following diagram (Fig. 12) represents the two cuboids that could be highlighted within the FaSMEd three-dimensional model at the teacher's level and at the peers' level.

Another process that is central in this lesson is *'establishing what needs to be done to get them there'*: the teacher intervenes to highlight the most effective ways of reading symbolic expressions and of identifying the one that better represents the involved relations, providing also guidance on how to read the tasks and the texts of the problems (line 37). During the discussion, some interventions are also focused on the positive effects of students' deep involvement in the activities.

It is possible to highlight feedback related to three of the four levels proposed by Hattie and Temperley (2007).

Students' explanations of the reasoning on which their choice was based represent an example of *'feedback about the task'*, which is given *among peers*, because of the different levels of effectiveness of these explanations. For example, Stefano's intervention (line 20), which highlights that the expression 7:h=p does not represent Battista's sentence because, in the expression, 7 is divided by the height and

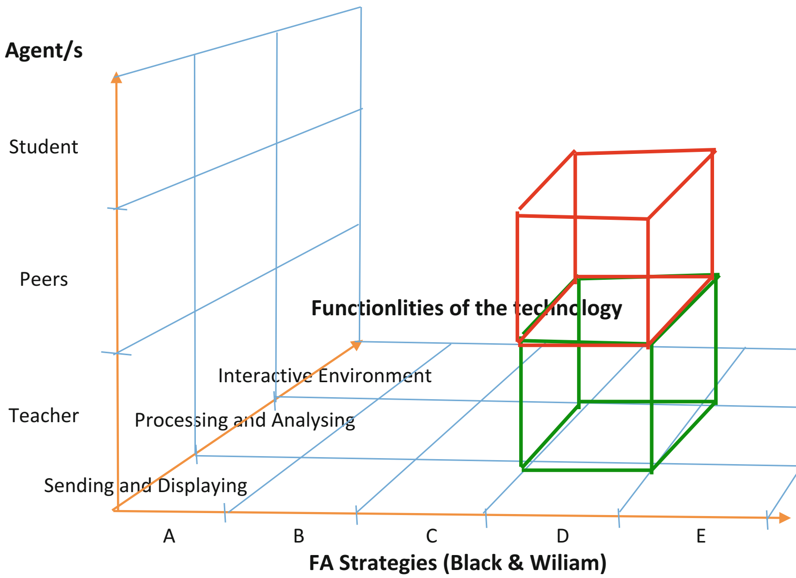

Fig. 12 The activation of the FA strategy "Activating students as instructional resources for one another", at the peers' and teacher's levels, through the "Processing and Analysing" functionality

not vice-versa, represents a feedback for Nicolò, who refers to Stefano's statement, clarifying it in an effective way (line 36).

Monica's meta-level intervention in line 37 aims at sharing criteria to correctly identify the expressions that better represent specific relations among quantities: it can be interpreted as '*feedback about the processing of the task*'. This is also an example of the teacher's exploitation of feedback from the students, because Nicolò's statement (line 36) provides Monica the opportunity to discuss the importance of a careful interpretation of symbolic expressions (line 37). Another example of this kind of feedback is Alice's intervention (line 2), which introduces the special case of the 7 cm figure, enabling Lisa to understand her mistake and ask to change her answer, proposing motivations (line 10, line 18) that clearly refer to Alice's observation.

The interventions that refer to the importance of listening to each other and of actively participating to class discussions (line 39, line 43, line 45) could be interpreted as '*feedback about self-regulation*'. Amalia's statement (line 39), in fact, provides Monica with the opportunity to discuss on the support given by the discussion with the classmates, asking to Lisa if the activity was useful for her (line 40). It also represents a feedback for Lisa, who proposes a meaningful reflection on the positive effects of the discussion in supporting her understanding of the problem (line 43).

We can therefore highlight, again, two cuboids, within the FaSMEd three-dimensional model (Fig. 13), related to the "Providing feedback that moves learners forward" FA strategy at the teacher's level and at the peers' level).

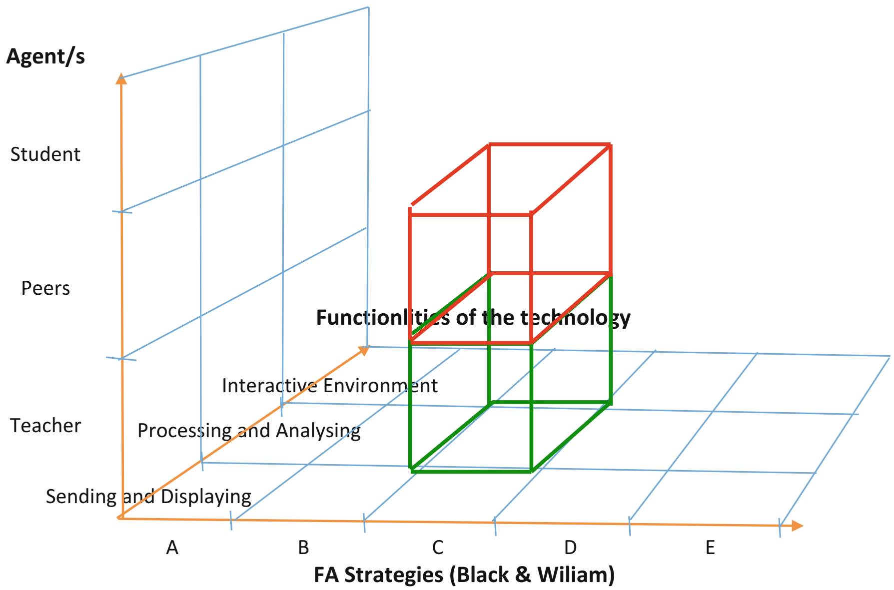

Fig. 13 The activation of the FA strategy "Providing feedback that moves learners forward", at the peers' and teacher's levels, through the "Processing and Analysing" functionality

As already stressed, starting from the poll, Monica has planned a rich discussion, that enabled the activation of different FA strategies by the different participants. This observation enables to highlight another cuboid within the FaSMEd three-dimensional model (Fig. 14), referring to the FA strategy "Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding".

The technology plays an important role in supporting the agents involved in these processes, in particular in providing feedback to each other. First of all, the software elaboration of the data and the graphical representation of the results of the poll give the teacher the chance to ask for the interpretation of these results and to plan the order of students' interventions during the discussion (Monica decides to start the discussion involving firstly those who have given the wrong answer).

The teacher's choice of not providing students with an immediate automatic correction of their answers may represent a support for students at different levels: (a) it enables to focus on the explanations of the answers, more than on the identification of the correct answer; (b) it pushes the students to motivate their answers; (c) at affective level, the lack of a written evaluation ensures that the students do not feel worried when they comment upon their choices.

Finally the time given to students to choose their answer (in this case, all students answered before the allowed time of 6 min) enables them to reflect, in pairs, on the motivations on which their choice is based. The moment that precedes the answer to the poll is, therefore, preparatory to the subsequent discussion.

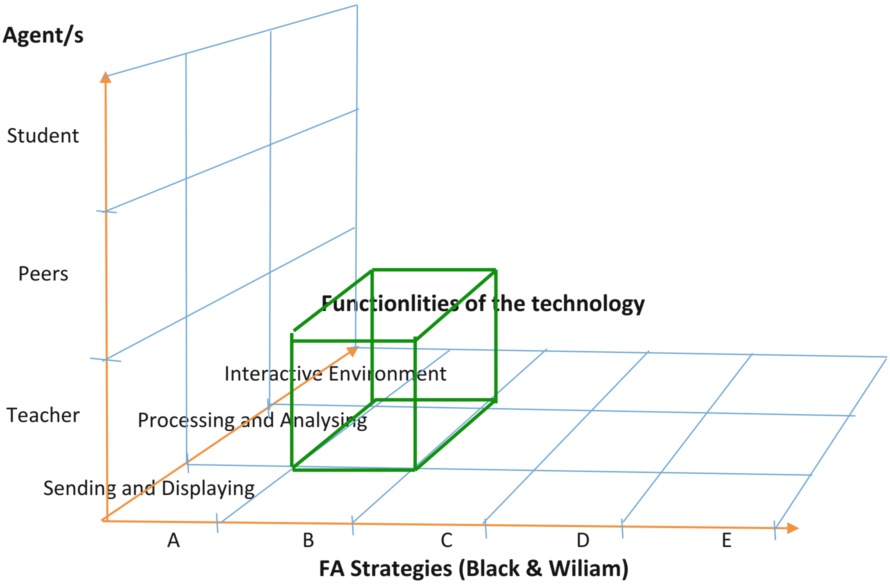

Fig. 14 The activation, by the teacher, of the FA strategy 'Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding', through the 'Processing and Analysing' functionality

The effective role played by the chosen functionality of the technology could be also highlighted if we refer to the FaSMEd three-dimensional model to develop a global analysis of the example. The representation of all the cuboids highlighted looking at micro-episodes within our example (Fig. 15), indeed, points out the active involvement of the three agents and the activation of a wide range of FA strategies, drawing attention to the complex dynamics activated through the support of the 'Processing and Analysing' functionality of the technology.

FA with Technology and Professional Development

The framework of the MDT, where the interactions between teachers and researchers allow interpretation, at a meta-didactical level, what happened at a didactical level, completes the analysis and illustrates the contribution of the collaborative work both in the design of lessons and for professional development. In the next sections, we present the point of view of teachers through the lens of the MDT model.

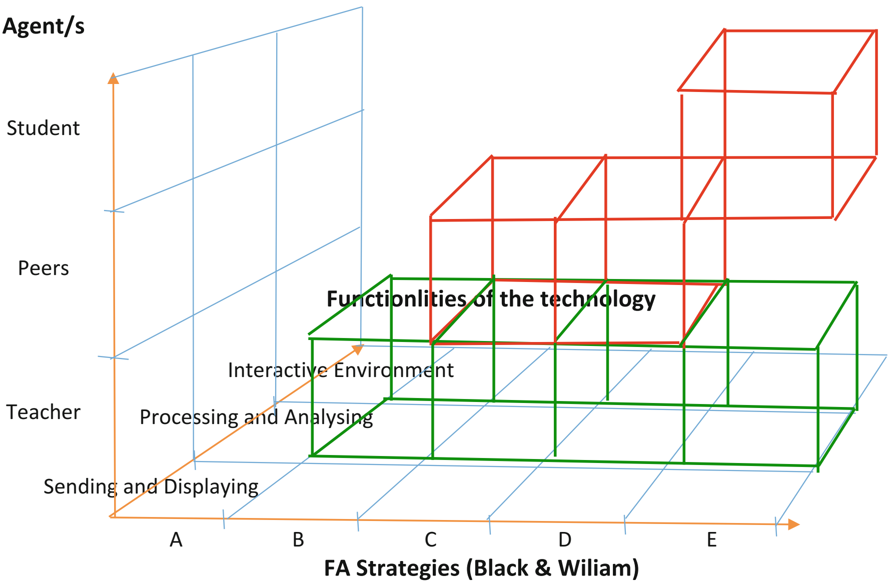

Fig. 15 The use of the FaSMEd three-dimensional model for a global analysis of the example

FA and PD Within the FaSMEd Project in France

One of the main concepts of the MDT is the internalisation phenomenon, which has been transversal in the collaborative work between teachers and researchers throughout the project. Particularly, the concept of FA, as collecting data, analysing them, providing feedback to students and therefore adapting teaching according to students' achievement, could be seen as an external component for teachers at the beginning of the project. It has become internal in the observed teachers' practices through the use of technology. Also the use of technology can be considered as an external component becoming internal as a consequence of the role played by the institutional dimension of the project. The process of internalisation leans on the double dialectic where the meta-level consists in conceptualising the FA process, while the didactic level consists in implementing this concept in the classroom practices. In order to illustrate the double dialectic, we will refer to the interviews done with Thomas as well as excerpts of the logbook he filled in all year long.

After Thomas' lessons on linear functions, we interviewed him starting from an analysis of what happened in the classroom and going to more general questions about the way he considers his teaching.

Researcher: "We were in the classroom yesterday when you asked the four fundamental questions relatively to the linear functions [. . .] then you interpreted the results, you showed which questions were well-done or not so well-done and you proposed again some questions..."
Thomas: "Where I simplified didactic variables."
Researcher: "Yes and it's very interesting because..."

Thomas: “To come back to the mathematical notion.”

Researcher: “Yes, to the notion, without the trouble of calculating the sum of fractions and other things...”

[. . .]

Thomas: “On the third quiz we saw the students’ results decreasing that was due to two things: the difficulty of calculation with negative numbers and fractions, but also because I made a bad parametrization of the second quiz and the students could do several trials, which makes statistics increase [. . .] this progress was not reliable. Instead, the results of today will give a good representation of the evolution of students’ knowledge.”

In this excerpt, Thomas remains on the didactic level and analyses the difficulties his students encountered when they took the quiz. He justifies his didactic choices and explains why he eliminates troubles coming from calculation difficulties. In front of each type of task, he takes into account the techniques that students have to know and gives the justification of these choices, completing his praxeology at a didactical level. However, during the interview, interacting with researchers, Thomas moves on to a meta-didactical level when he speaks of the evolution of his FA practices with technology:

Researcher: “You see an evolution but also you see that some students do not succeed yet. Do you plan to do something and how do you plan to continue relatively to this, to the results that you see in the classroom?”

Thomas: “To do something... From the beginning of the experimentation, it’s much easier with NetSupport. Globally, to give a simple answer... When I use NetSupport, I can intervene individually. . . I intervene directly with some students and I explain again [. . .] I deal with difficulties, perhaps not of all, but I treat answers individually and now when I use Maple TA, I don’t personalize. . . depending on the statistical results that I get, I decide to give a feedback to the whole class or not.”

Thomas has built a technique within a meta-didactical praxeology including technology as a tool allowing to personalize or to redirect his teaching. His reflections at a meta-didactical level are transposed into the *logos* of his didactical praxeologies, becoming the FA principles that justify his FA strategies. This is an example of a dialectic developed at a meta-didactical level that feeds into the dialectic at a didactical level within the MDT process. Thomas justifies his didactical praxeologies, by including the fundamental principles of FA.

Thomas also points out the role of the institutions in his FA practices showing that the global institution of the school does not give time to support low achievers but in the local institution of the classroom, he organizes some special work sessions with students. The particularity is that Thomas leaves to students the responsibility for participating in these supplementary lessons, which contributes to the engagement of students in their own learning. Students’ results in the process of FA may influence their decision to participate.

Moreover, as we can read in the teacher’s logbook, when he speaks about his praxeologies, he notices some changes that have become stable regarding FA.

After several experiments, my uses are stable and centred on formative assessment as well as the modification of the students’ status of writing.

Formative assessment: use of NetSupport School and MapleTA.

Valorisation and exploitation of students' writing (storing data with NetSupport School and processing with the whole class thanks to the IWB)

The use of software specifically related to mathematics (DGS, Spreadsheets, . . .) is now regular.

We can interpret Thomas' words as emblematic of an advanced state of the internalisation process concerning the component of FA. Technology emerges as a strong institutional component of his practices that entails teaching modifications relatively to FA. We can detect signs of a shared praxeology in the vocabulary used by Thomas to describe his teaching practices. This shared praxeology as well as the concepts of FA and the functionalities of technology that are stressed by Thomas can be considered as a first result of the whole MDT process. In a general meeting at the end of the year with other teachers involved in the project, Thomas declares: "I had the pleasure of discovering formative assessment. [. . .] I will never go back".

FA and PD Within the FaSMEd Project in Italy

The teachers collaborating with the Italian team are interviewed after each lesson carried out within the FaSMEd Project. They are asked to reflect on the lesson focusing on:

(a) the most effective/most problematic moments during the lesson;
(b) the most striking students' interventions, in relation to the given feedback;
(c) the effectiveness of the support given by the technology in relation to FA;
(d) the support given by the technology to low-achievers.

In this paragraph, we will analyse the reflections carried out by Monica during her interview on the lesson documented in the previous section. We focus, in particular, on Monica's praxeology related to the *task* of developing FA in her class. The aim is to highlight the evolution of her praxeologies during the FaSMEd Project, occurred in the interaction with the researchers' team.

During the interview, Monica focuses on two main aspects: (1) the surprising effective participation of Lisa and Nicolò during the discussion; (2) the functionalities of the technology that better support the students and, in particular, low-achievers.

As regards point (1), Monica stresses that she is really surprised by Lisa and Nicolò's interventions during the discussion:

> Monica: "The ones that surprised me more are Lisa and Nicolò. In particular Nicolò, because Lisa is a more logical girl, and if she is focused when doing mathematics, she produces more [than him]. On the contrary, Nicolò faces great difficulties in understanding, even to understand the text of a task. He is the typical student with great difficulties in understanding. The fact that today he managed to tell those things has been a surprise."

In Monica's opinion, Nicolò and Lisa were able to autonomously explain the decision of changing their answers thanks to the discussion activated starting from the screen sharing, where students' answers were displayed. She identified it as one of the most effective moments during the lesson.

As regards point (2), Monica stresses that:

> Monica: "…the fact of using this technology and getting the image on the screen allows you, in the meantime, to keep it always present – and when you work in paper and pencil environment, you don't have it, or you get it only on the spot. Then, when the students write and try to answer, many times they make mistakes, they scribble, and so their paper is not clean anymore. Having the possibility, with the tablets, also to delete and get all the screen clean again, so to be able to start again, in my opinion, creates also a mental order not possible in paper and pencil, where, when you write, you cannot delete anymore, you get tired and you lose your focus a lot.
>
> …the possibility of showing the solutions on the screen, the rapidity in being able to see things: they send you their solutions and you can show them to the others. And seeing is not the same as only listening to. In a lesson with paper and pencil worksheets you can listen to all answers and report them on the blackboard, but it takes more time. It takes too much time and the children get lost."

On one side, Monica focused on a functionality of the technology that can be included in the 'sending and displaying' category: in fact, she stresses again that the 'displaying of the screens' is efficient because it enables the students to see, on the interactive whiteboard, the other students' productions, reflecting on them without having to imagine and/or memorise them. On the other side, she focused on another component of the technology: the tablets. She specifically focused on the possibility, given by tablets, of deleting mistakes. In Monica's opinion it supports, in particular, those students (like Lisa and Nicolò, for example) who tend to be not concentrated.

Monica is part of a group of teachers that are involved also in another long-term regional project, called AVIMES,[4] aimed at fostering FA in school. Monica and her colleagues are used to collecting students' written productions during class discussions to share and discuss them with the class. A vision of FA in line with our perspective was, therefore, an internal component of Monica's praxeology also before her participation to FaSMEd. This information enables us to characterize Monica's initial praxeology when she started her experience within the FaSMEd Project: (a) the didactical techniques used by Monica to develop FA in her classes include the sharing of students' productions and collective discussions to make the students compare their works and reflect on them; (b) the theories/justifications of the techniques developed in AVIMES. Digital technologies are external components to this praxeology.

In her interview, Monica compared her way of working during the AVIMES lessons with the new way of working with the support of the digital technology. She stressed the rapidity in which the technology enables collecting, displaying and sharing the students' answers and the consequent saving of time:

[4]AVIMES (http://www.avimes.it/who%20are%20we.htm) is a Regional Project focused on research, innovation and professional development in the field of school self-evaluation.

> Monica: "The fact of having always in real time, on the screen, the various solutions... We, with AVIMES, were used to working in this way: collecting the students' solutions, transcribing what they say and what you say... but it takes a longer time, because you collect them [through the audio-recording], write them down, and then report them. This is a loss of time. With the technology, you can get immediately all projected on the screen and the students' attention and concentration keeps longer."

When asked if she would change something in the lesson, Monica introduced also a reflection on her way of working with this technology. Monica, as many other colleagues, who declare to have faced some difficulties because they were not used to digital technologies during their lessons, seems recognising the effectiveness of the use of a connected classroom technology in fostering FA in her class:

> Monica: "No, currently I would not change anything. I'm a little getting used to this. Maybe I could get better results as I were more able to use the computer; if I were quicker I could do something more. Being more able in getting, opening, and closing files, maybe also the students' attention would be greater. A great part depends also on the teacher carrying out the lesson: results do not depend only on technology, but also on me, and maybe before the end of the project I can do something more."

We can therefore highlight an evolution of Monica's praxeology: digital technology, which was external, is slowly becoming an internal component. In fact, Monica recognizes the role it plays, integrated with the other techniques that were already part of her praxeology, in supporting FA in her classes. This evolution is still in progress. The evolution of the *logos* components of the praxeology, does not go at the same pace as the evolution of the technique. In fact, we can notice that Monica does not explicitly refer to the theoretical frame that support the approach developed within FaSMEd. The appropriation of this frame will require further discussion with the teachers and activities specifically devoted to a real sharing of the theoretical basis of this work.

Conclusion

This paper had a twofold aim: (1) highlighting how the different functionalities of technology could enable the enactment of FA strategies at the teacher's, the students' and the peers' levels; and (2) characterising the dynamics that intervene within programs involving a strict collaboration between teachers and researchers.

Concerning the first aim, the analysis of the selected episodes shows clearly the contribution of technology as a medium facilitating the different FA strategies but also the dynamics between these strategies. In particular, the Italian case study points out how the FA strategies and the different levels of feedback emerge in the classroom interactions. The French example highlighted different moments in a sequence, providing a dynamic view of the FA process. The FaSMEd three-dimensional framework enabled us to describe and to analyse the FA lessons from both static and dynamic perspective, considering both the teacher and the student viewpoints. And particularly this model points out the role of technology as a facilitator of the whole process.

Our analysis has highlighted the interrelations between the different functionalities of technology and the FA strategies. The possibility given by technology to store data and the ease to come back to these data is an important functionality that teachers can use to enhance their teaching strategies. As recognised also by the teachers in the interviews, technology is not at the base of FA, but appears as an essential tool to improve the effects of FA for students and for teachers as well. For example, it is possible to *clarify learning intentions and criteria for success* also without technology but technology supports the teacher in making these intentions explicit and to share them with the students, at both individual and class level. Concerning students, the strategies of *activating students as the owners of their own learning* and *activating students as instructional resources for one another* appear as the core of FA, since they enable the active involvement of all the agents (teacher, students, peer/group) within the FA process.

Moreover, the analysis enabled to highlight interrelations between different FA strategies: for instance, *engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding* is facilitated by giving the opportunity of *providing feedback that moves learners forward* on the spot as well as after reflection.

As regards the second aim, the teachers-researchers teams engaged in a process of design-based research benefit of a professional development based on the collaborative work. The Meta-Didactical Transposition framework can show the evolution of beliefs about FA with technology and contribute to the understanding of the meta-level of reflection necessary for a daily use of FA strategies in the classroom. The double dialectic and the internalization of components had a direct impact on the involved teachers but more generally their analysis give information that can be useful for in-training sessions.

The importance of teachers as guides in FA lessons with technology has essential consequences on their practices in terms of professional development. When we began the project, most of the involved teachers stated that FA was present in their practices. However, most of the time, FA was not developed over time and appeared occasionally in the classroom more as a reassuring method than as a teaching strategy. Considering technology as a tool enabling to enhance teaching strategies including FA is surely an important issue of the next years, regarding teachers' professional development.

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Teaching Intriguing Geometric Loci with DGS

Daniela Ferrarello, Maria Flavia Mammana, Mario Pennisi, and Eugenia Taranto

Abstract In this chapter we present an experimental activity conducted with eight teachers in several high-schools in the South of Italy. The activity deals with the study of some geometric loci and is based on the use of a Dynamic Geometry System (DGS) on the one hand and on properties of geometric transformations on the other hand. Here we explain the motivations at the base of our activity and show the contents, the modalities of building up the activity and the results of the teaching experiment.

Keywords Geometric loci • Geometric transformations • TPACK • DGS • Teachers and students' laboratory

Introduction

The concept of locus is usually introduced in school studying the perpendicular bisector of a segment, the circumference first and the other conics afterwards. It remains a quite hard topic to understand for students (Pech 2012), and teachers quite often do not go into the details of it. The idea of our activity is to present new geometric loci so to deepen the concept and to be sure that students internalize it correctly. We carried out this activity with 210 students in several high schools in Sicily (grade 10-11-12) and eight teachers that have collaborated with the researchers (the authors and two more high school teachers). The whole activity is centred on the concept of geometric locus: several loci are introduced and some properties are explored and proved. In particular, we based the activity on the use of a DGS (Geogebra) on the one hand and some properties of the geometric transformations on the other. The software helps students in the "investigation" process, the geometrical transformations in the proving part.

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The chapter is divided into four parts: in the first the theoretical framework is presented, the TPACK (Technological, Pedagogical and Content Knowledge) model; the second describes the contents of our activity; the third is related to the teaching experiment; the fourth shows the results of the experimentation.

Theoretical Framework

Koehler and Mishra (2008) highlight, in the TPACK framework, the interplay of the three components, Technological, Pedagogical and Content Knowledge, in the learning and teaching process (Fig. 1). "Good teaching is not simply adding technology to the existing teaching and content domain. Rather, the introduction of technology causes the representation of new concepts and requires the development of sensitivity to the dynamic and transactional relationship between all three components suggested by the TPACK framework" (Koehler and Mishra 2005, p. 134). The interplay of the three components results into the seven sets described in (Koehler and Mishra 2009), namely: Technological Knowledge, Pedagogical Knowledge, Content Knowledge, Technological Pedagogical Knowledge,

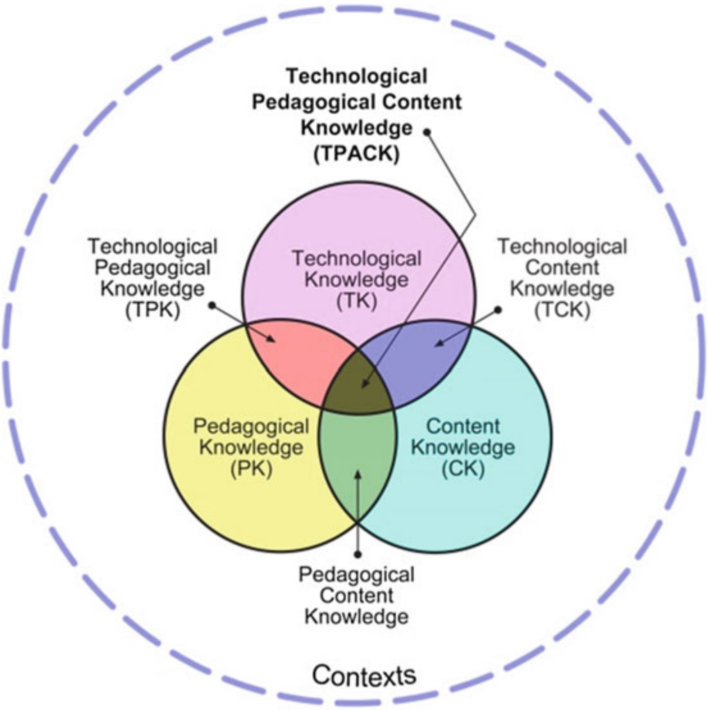


Fig. 1 TPACK image (from http://tpack.org/)

Technological Content Knowledge, Pedagogical Content Knowledge, Technological Pedagogical Content Knowledge.

The activity we present fits this framework, conjugated as it follows:

As for the Technological Knowledge (TK) we use the DGS to discover, conjecture and verify properties. First of all, many loci involved in the activity are quite difficult to be drawn in the blackboard, despite the teacher can be precise. Instead, by using technology, students have well-drawn figures and they have the possibility to discover a property by observing pictures in which the property really holds (at least with a very small approximation).

Second, the use of the “Dragging” mode in the DGS is very useful to understand that the discovered property not only holds for that particular picture you draw but it holds for all the pictures with the same features, so, by dragging, students are able to generalize and they conjecture the observed property.

Third, the software raises up the students from the effort of calculations, giving them the opportunity to reflect more deeply on the concept: to verify a property students do not get lost in calculations that are immediately made, instead.

Finally, technology for students is fun and they can’t wait to use it, so a part of the good role of technology is given by the enthusiasm that students lavish in facing a not so easy task.

The proof, instead, is made by paper and pencil: students are guided by worksheets that give them a space and a time to think over the “why” that property holds, in such a way the topic is mastered.

As for the Pedagogical Knowledge (PK) we refer to the Zone of Proximal Development (ZPD) of Vygotskij, Learning by doing of Dewey and Enactivism, all within a “mathematics laboratory,” (Anichini et al. 2004). The laboratorial activity enables students to “get their hands dirty” (Dewey 1916), as Dewey proposes in his pedagogical activism, because the learner should be an active character in the learning process, interacting with the object to be known, by doing. Doing is not just a mere and blind practice, but students need to use mind together with hands. Body and mind, manipulating and thinking, are one, as the *Embodiment* theory claims (Johnson 1989; Johnson and Lakoff 1999). *Learning by doing* is completed with *Learning by thinking* that in our activity is fostered by the argumentation and proof activity, and by the discussion with classmates and teacher. Finally it is important also *Learning by loving*, i.e. to be emotionally involved in the task, in such a way learners study “not for duty, but for beauty”. Students face new problems to work on. The problem, according to Vygotskij’s theory, should be suitable for the ZPD (Vygotskij 1986), i.e. the right problem to be handled is the one that is not solvable by a single student, but solvable with the guide and assistance of more skilful peers (or teacher). The ZPD is enlarged whenever a problem is solved by collaboration and it becomes the new actual developmental level (ibid.). In such a way the individual competence of a single student has an increase. In the process of solving problems together with classmates, students are the principal actors (Rossi 2011), according to Enactivism principles: the subject of the action is not the teacher, (that is an *actor*, like the student, but also a *director*) but the student who actively takes part of the teaching/learning process “here and now”. By using the DGS and peer confronting, supported by the careful presence of the teacher, students discover, understand and master a new mathematical concept.

As for the Content Knowledge (CK) we refer to some recent studies (Ferrarello et al. 2014b) that deals with the study of some geometric loci, detailed in the "Contents" section. The choice of the topics aims to the concept of geometric locus, by means of an elegant approach tied to a frequent use of geometric transformations that simplify many proofs (Ferrarello et al. *ibid*). Since this paper is a bit general, the research team provided notes for teachers, who do not know them. These notes accompany them in the construction of educational material.

The TPACK's authors underline that these three discussed components intersect each other, generating TCK, PTK and PCK (see Fig. 1). Explanations of how these areas have been orchestrated in our work can be read in the following:

"Teachers need to master more than the subject matter they teach; they must also have a deep understanding of the manner in which the subject matter can be changed by the application of particular technologies" (Koehler and Mishra 2009, p. 65). We declined TCK intersection by means of a specific training of teachers on the DGS. We did not only explain how the artefact works, but also we made them aware of "how" the artefact can be used in the teaching practice to be effective. The PTK intersection is implemented in our activity by the way we use the Vygotskijan idea of the semiotic mediation (Vygotskij 1986): the concepts we aimed at were not immediate, but they need the mediation of an artefact to be completely understood. The digital artefact (the DGS) becomes tool whenever it participates to the "construction of mathematical meaning". In fact the DGS not only draws the locus, and constructs it into the screen but, what is more important, it plays an important role in constructing the "locus idea" in the students' minds.

Our PCK intersection perfectly fits in Shulmann (1987) theory of pedagogy and content. Shulmann provocatory starting from "He who can, does. He who cannot, teaches", finally aims to teachers that are not "those who can" or "those who cannot", but rather "those who understand", i.e. those who are able to effectively integrate pedagogy and contents. To make teachers "those who understand" it is important to give them the opportunity to build the teaching material, instead of receiving a ready package to be blindly applied in every environment. In such a way "the teacher interprets the subject matter, finds multiple ways to represent it, and adapts and tailors the instructional materials" (Koehler and Mishra 2009, p. 64).

Contents

The activity is inspired by a recent work (Ferrarello et al. 2014b) and deals with the study of some geometric loci.

We present here some of the contents of the activity. Precisely, we recall some preliminary concepts, some basic definition and properties that are used in the activity, give the general definition of the proposed problem and present all the properties studied with the students (Table 1). Then we enter the details of some cases (*n=1*, *n=2*, *a and n=3*, *a*) whose worksheets prepared for the students are presented in the following.

Preliminary Concepts: Homothetic Transformations

Let us fix a point O of the plane and a real non zero number k. We call *homothetic transformation*, or *central dilatation*, with *centre* O and *ratio k* the transformation of the plane that leaves O fixed, and associates a point A with A′, such that $\frac{OA'}{OA} = |k|$ and A′ belonging to the semi-line OA if $k>0$, or on the semi-line opposite to OA if $k<0$. When $k = 1$ the transformation is the identity, when $k = -1$ is the central symmetry centred at O.

A homothetic transformation is a bijection from the plane to itself. It maps straight lines into straight lines and segments into segments. Moreover, in a homothetic transformation with ratio *k*, the ratio of two correspondent segments is constant and is equal to $|k|$.

In a homothetic transformation the midpoints of correspondent segments are correspondent, correspondent straight lines are parallel, in particular every straight line passing through O is fixed, then its correspondent is itself.

A homothetic transformation with centre O and ratio k maps a circle with centre C and radius *r* into the circle with centre C′ and radius $|k|\ r$, being C′ the correspondent of C.

Preliminary Concepts: Nine-Point Circle of a Triangle

Consider a triangle ABC and let H and O be its orthocentre and circumcentre, respectively. Let A′B′C′ be the *medial triangle* of ABC, where A′, B′, C′ are the midpoints of the sides BC, AC, AB, respectively. Let A*B*C* be the *orthic triangle* of ABC, where A*, B*, C* are the feet of the altitudes trough A, B, C, respectively. Let A″, B″, C″ be the *Euler points* of ABC, i.e. the midpoints of the segments AH, BH, CH, respectively (Fig. 2).

Theorem of Poncelet (1821). The nine points A′, B′, C′, A, B*, C*, A″, B″, C″ lie on a circle.*

This circle is called *nine-point circle* or *circle of Feuerbach of ABC*. Its centre N, *called* nine-point centre or *point of Feuerbach*, is the midpoint of OH and its radius is half of the circumradius of ABC.

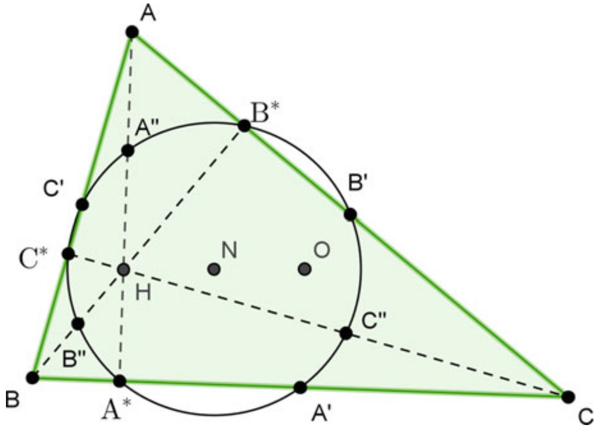


Fig. 2 Nine-point circle

Preliminary Concepts: Centroid of a Quadrilateral

Consider a convex quadrilateral ABCD and let M_1, M_2, M_3, M_4 be the midpoints of the sides AB, BC, CD, AD, respectively.

Varignon theorem (1731). The quadrilateral $M_1M_2M_3M_4$ is a parallelogram.

The quadrilateral $M_1M_2M_3M_4$ is called *Varignon parallelogram* of ABCD. The segments M_1M_3 and M_2M_4 are called *bimedians* of ABCD and their common point G is called *centroid* of ABCD. Note that the bimedians are the diagonals of the Varignon parallelogram. It follow that *the centroid G bisects the bimedians*.

Preliminary Concepts: Anticentre of a Cyclic Quadrilateral

Given a convex quadrilateral ABCD, the line through the midpoint of a side and perpendicular to his opposite side is called *maltitude*. There are four maltitudes of a quadrilateral.

In general the maltitudes of a quadrilateral are not concurrent in a point. It is possible to prove that *quadrilaterals whose maltitudes are concurrent in a point are all and only the cyclic quadrilaterals*, i.e. quadrilaterals that are inscribable in a circle.

The point H of concurrency of the maltitudes in a cyclic quadrilateral is called *anticentre*.

It is possible to prove that in a cyclic quadrilateral the anticentre is the symmetric point of the circumcentre with respect to the centroid (Fig. 3).

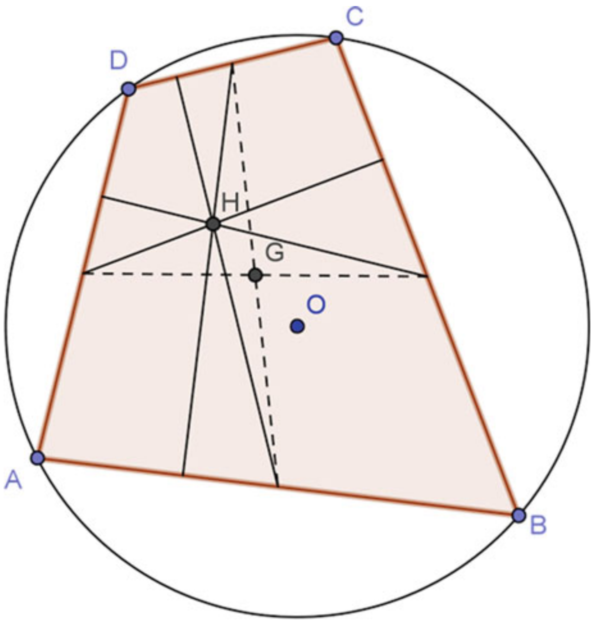


Fig. 3 Anticentre, circumcentre and centroid of ABCD

The notion of maltitude can be extended to concave or crossed quadrilaterals. The notion of anticentre can be extended to crossed cyclic quadrilaterals. Note that a concave quadrilateral cannot be cyclic.

Geometric Loci

A *geometric locus* of the plane is the set λ of all and only points P of the plane that satisfies a given property (classical examples are the circle, the perpendicular bisector of a segment). There are loci that are described by a point L that is function of a point P that moves, for example, over a line, or a circle (or more in general, a conic).

General problem:

Let γ be a circle with centre O and radius r; fix n points on γ, with $n = 1, 2, 3$ and consider a generic point P of γ; we define a point L that depends on P and study the locus λ described by L when P moves on γ.

In the following table we report all the cases we examined.

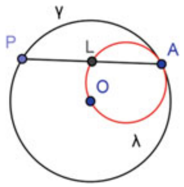
We report here the details of cases $n = 1$, $n = 2$, *a and* $n = 3$, *a*. An extended version of all the details can be found in (Taranto 2014).

Table 1 Contents

<table>
<tr>
<td>

n = 1

Let A be a fixed point of γ and P a generic point of γ. Let L be the midpoint of the segment AP. The locus λ described by L when P moves on γ is a circle with radius *r*/*2* and centre the midpoint of AO.



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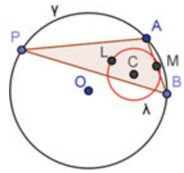
<td>

n = 2, a

Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the centroid L of ABP is the circle with radius *r*/*3* and centre the point C of the segment OM such that CM=1/3OM, where M is the midpoint of AB.



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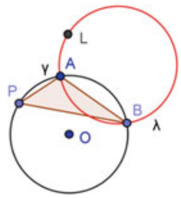
<td>

n = 2, b

Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the orthocentre L of ABP is the symmetric circle of γ with respect to AB.



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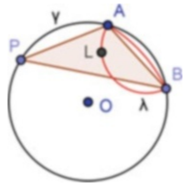
n = 2, c

Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the incentre L of ABP is formed by two arcs of the circles through A, B and with centres the common points to γ and to the perpendicular bisector of AB.



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<td>

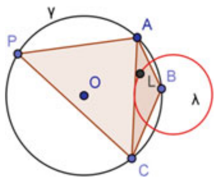
n = 2, d

Let A and B be two distinct points of γ, P be a generic point of γ. When P moves on γ, the locus λ described by the circumcentre L of ABP is one point, the centre O of γ.

</td>
<td>

n = 3, a

Let A, B and C be three distinct points of γ. Let P be a generic point of γ. When P moves on γ, the locus λ described by the anticentre L of ABCP is the nine-point circle of the triangle ABC.



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n =3, b

Let A, B and C be three distinct points of γ. Let P be a generic point of γ. When P moves on γ, the locus λ described by the centroid L of ABCP is the nine-point circle of the medial triangle of ABC.

</td>
<td>

Some more loci, Example 1

Let A and P be points of γ. The locus λ described by the orthocentre L of the triangle AOP, when P moves on γ, is a cubic with a node in O (*Strophoid*).

</td>
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</table>

(continued)

Table 1 (continued)

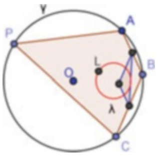
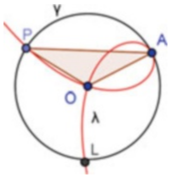
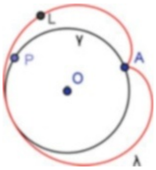
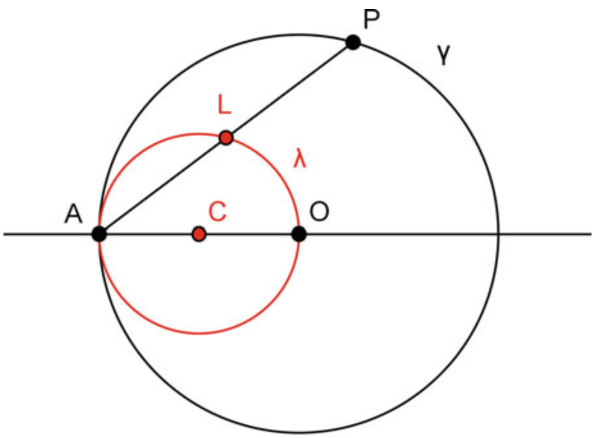
	
<u>Some more loci, Example 2</u> Let A and P be points of γ, t the tangent line to γ in P and L the foot of the perpendicular to *t* from A. The locus λ described by L when P moves on γ is a bicircular quartic with one cusp in A (*Cardioid*). 	<u>Some more loci, Example 3</u> Let AB be a diameter of γ. Let r be a perpendicular line to AB. Let P a point of γ and R the common point to the lines AP and *r*. Let L the midpoint of PR. The locus λ described by L when P moves on γ depends on *r*, but it is in general a cubic.

Fig. 4 Case *n=1*



Case n=1

Fix a point A on γ and consider a generic point P of γ. Let L be the midpoint of AP (Fig. 4).

The locus λ described by L is the circle with radius r/2 and with centre at the midpoint C of AO.

Proof Since L is the midpoint of AP, the points P and L are related as it follows: $AL = \frac{1}{2}AP$. Then the homothetic transformation with centre A and ratio 1/2 transforms P in L. When P moves on γ L describes λ, then the homothetic transformation transforms γ in λ. Therefore λ is a circle with radius r/2. Moreover, the homothetic transformation transforms the centre O of γ into the centre C of λ, then C is the midpoint of AO.

Case n=2

Fix two distinct points A and B on γ. Let P be a generic point of γ and L a point that, in the four following cases, indicates: a) the centroid, b) the orthocentre, c) the incentre, d) the circumcentre of the triangle ABP. In each of these cases we consider the locus λ described by L when P moves on γ.

We detail here only case $n = 2$, *a*.

(a) *The locus λ described by the centroid of the triangle ABP is the circle with radius r/3 and with centre the point C of the segment OM such that* $CM = \frac{1}{3}OM$, *where M is the midpoint of AB.*

Proof Since L is the centroid of the triangle ABP and PM is a median, L is the point of the segment PM such that $LM = \frac{1}{3}PM$ (Fig. 5).

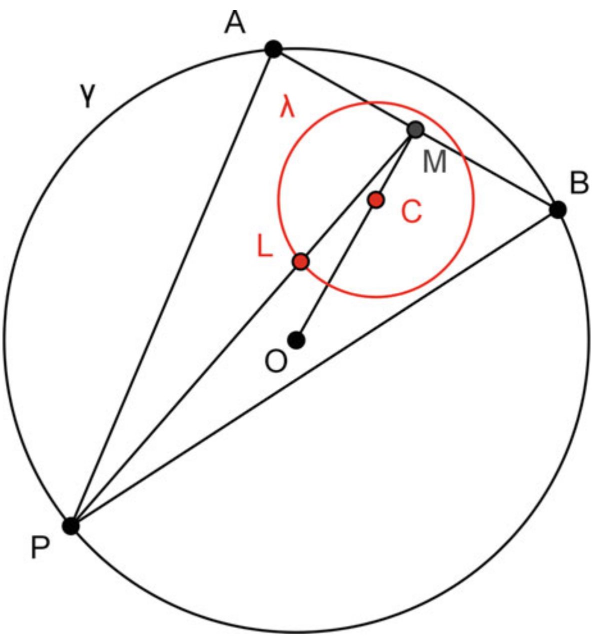


Fig. 5 Case *n=2*, centroid

It follows that the homothetic transformation with centre M and ratio 1/3 transforms γ in λ. Then the property easily holds.

Case n=3

Fix three distinct points A, B, C of γ. Let P be a generic point of γ and let L be a point that, in the two cases discussed below, indicates: a) the anticentre, b) the centroid of the quadrilateral ABCP (note that the quadrilateral can be convex or twisted). In each of these two cases we consider the locus λ generated by L when P moves on γ.

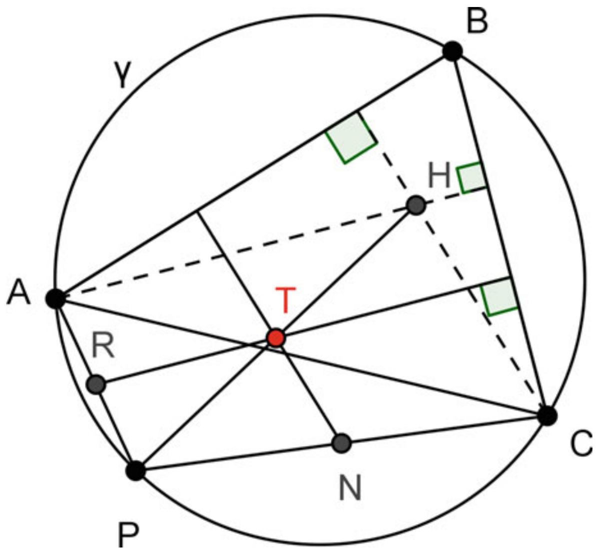
We detail here only case $n = 3$, a.

(a) *The locus* λ *described by the anticentre L of the quadrilateral ABCP is the nine-point circle of the triangle ABC.*

Proof Let us consider a generic point P of γ and let L be the anticentre of the quadrilateral ABCP. Let T be the midpoint of the segment joining P with H, orthocentre of the triangle ABC (Fig. 6).

Let R and N be the midpoints of AP and CP, respectively. In the triangle AHP the segment RT is parallel to AH, because it joins the midpoints of the sides AP and PH, so it is perpendicular to BC. It follows that the line RT is the maltitude of the quadrilateral ABCP with respect to the side BC. In a similar way, if we consider the triangle CHP, we prove that NT is the maltitude of ABCP with respect to the side AB. Then, T is the common point to two maltitudes of ABCP and it is, then, the anticentre L. Therefore L is the midpoint of HP.

Fig. 6 Case *n=3*, anticentre, 1



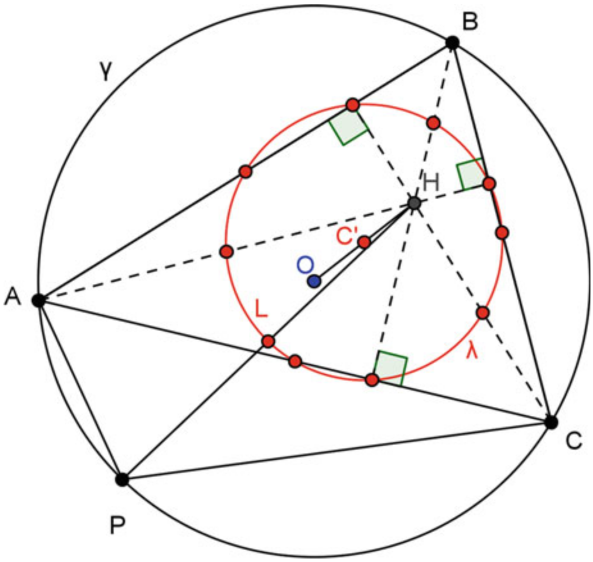


Fig. 7 Case *n=3*, anticentre, 2

It follows that the homothetic transformation with centre H and ratio 1/2 transforms γ into λ, then λ is a circle whose radius is an half of the radius of γ. Furthermore, since the centre O of γ and the centre C′ of λ are correspondent in the homothetic transformation, C′ is the midpoint of OH. It follows that C′ is the nine-point centre of the triangle ABC and λ is the nine-point circle of ABC (Fig. 7).

Teaching Experiment: Research Context and Methodology

The activity focuses on the concept of geometric locus. This concept is a transversal concept that recurs not only temporally repeatedly in the school career of a student, but that pervades all areas of mathematics, from the Euclidean geometry to the analysis, from algebra to physics. We consider very important that the concept of locus is properly understood by the students, not only because of its cross-cutting, but also because it is an example of methodological rigor, typically mathematical, good to establish the reasoning about the properties of mathematical objects.

The properties of definition of a locus is not a concept to be easily internalized: students are not always aware that they need to demonstrate not only that all points that verify a certain property belong to the locus, but also that the only points of the locus meet the property.

The idea of our activity is to present new geometric loci so to deepen the concept and to be sure that students internalize it correctly.

The experimentation involved seven secondary high schools in Sicily (5 scientific oriented high schools, a foreign language high schools and a human science high school) for a total of 210 high-school students (grade 10-11-12) and eight teachers that have collaborated with the researchers (the authors of the paper together with two more high school teachers).

The activity was carried out by using the method, widely experienced by the authors – for example in (Ferrarello et al. 2014a) – of a double laboratory (Ferrarello et al. 2013): in the first laboratory (teachers lab) teachers design teaching materials and in the second one (students lab) students benefit the produced material. This is in accordance with the "Learning by doing" of Dewey, and the Vygotskijan practical intelligence perspective, which supports the creation of mathematical concepts through a mediated relationship, through the use of artefacts. The activity is based on the idea that "the best way for students to learn is to touch and build . . . [and] for teachers the best way to learn to teach is to experience first-hand, touch and build on their own teaching materials" (Ferrarello et al. 2013, p. 469).

The activity refers to (Ferrarello et al. 2014b) has been simplified and rewritten for the teachers taking part in the activity, by the researchers in the form of "notes for teachers". These notes accompany them in the construction of educational material (11 worksheets, numbered and to use following this ascending order), built up together with the research team and the teachers.

Those who were exposed in the content section as preliminary concepts (homothetic transformations, nine-point circle of a triangle, centroid of a quadrilateral, anticentre of a cyclic quadrilateral), were topics that the teachers involved in the experiment have explained to their student before starting the activity with the worksheets.

Clearly prerequisites for the activities are the notions of Euclidean geometry to the circumference, in particular the notion of congruence, parallelism, incidence, alternate angles and corresponding criteria of congruence of triangles. The intermediate objectives are: to know a simple locus (perpendicular bisector of a segment, circle), speculate on the nature of a locus, become familiar with the software Geogebra, show that the points of the locus described meet the property enunciated.

General objectives are therefore acquiring the concept of locus and demonstrate that a figure is a locus.

During the experimentation we have asked to the teachers to fill a personal logbooks, in which they described the development of the activity in class, reporting the management for the whole task, the behaviour of students and the results, both on the motivational point of view (attitude, interest, engagement) and on the cognitive point of view (learning development).

Tutors have occasionally participated in the sessions in the classes, not with the aim of controlling the actions of their colleagues, but rather to make the students aware that the carried out activity is part of a larger project that involves both schools and the university.

At the end of the activity teachers' logbooks and students' worksheets were collected, together with a questionnaire that was given to teachers and a questionnaire given to students (see Results section).

Teachers' Laboratory

Teachers met with the research team eight times, once a week, for 2.5 h per time, in autumn. The constructions of the classroom worksheets are the heart of the teachers' lab. Eleven worksheets were written, according to the various cases described in Table 1, plus an initial worksheet on preliminary knowledge. Each one is divided into two parts: the first part guides discovery's activities with the use of the software; the second part guides the proofs (see some examples in the following section).

The researchers planned a preliminary worksheet for the teachers. The worksheets have usually two columns: in the left column an action is indicated, in the right column the action is explicitly made (Ferrarello and Mammana 2012). In such a way students are perfectly aware of what they are doing. Some parts of the worksheet presents construction of figures, exploration, and the student has to follow the instruction and verify the properties; other parts of the worksheet have to be filled by students, for example he/she has to write the conjecture or the proof of the theorem. In preparing the worksheets, particular attention was given to the ZPD related to the individual competence, with the aim of collocate the teaching experiment in that zone and to organize good hints and the metacognitive reflection. The teachers, guided and supported by researchers, build other worksheets comparing each other's ideas and practicing "first-hand" how to run a laboratory. In this way the teachers acquire skills useful for the conduction of the students lab, which has different protagonists (students), but the same methods of the Teachers lab.

Common factor to the two laboratories is the modus operandi of the mathematics laboratory (Chiappini 2007) in a collaborative learning environment, since working with peers, with an experienced guide (tutor and teacher respectively) facilitates the socialized learning in the ZPD in a positive atmosphere, as advocated in the theory of Vygotskij.

The choice of writing the worksheets with teachers is due to several fact: above all because, "The integration of technology in mathematics education is not a panacea that reduces the importance of the teacher. Rather, the teacher has to orchestrate learning. [...] To be able to do so, a process of professional development is required" (Drijvers 2012, p. 148). Furthermore, the possibility of using an artefact like the DGS lets the learners discover the properties independently. Nowadays the educational software is widespread, but not always the teachers know how to use them from an educational point of view (Doğan 2011). So, it is very useful to make teachers able to effectively manage such tools, in order to disseminate good practices of teaching (and learning). This is how we orchestrated TCK, PTK and PCK as already described.

Students' Laboratory

The experimentation with the students was carried out in spring. The worksheets built up in the teachers' laboratory gave the student the opportunity to fully use the "explore-discover-test-conjecture-proof" model. In this way the student is not a passive listener of a lecture, but an active subject in the first-hand experience of discovery.

The students' work is based on the use of a DGS (Geogebra) for the discovery of properties and the use of the geometric transformation for their proof. In this way we provide examples of applications of geometric transformations: this topic is often mistreated and relegated to a separate chapter, while it often simplifies demonstrations otherwise very long and hard to understand. Proofs made in this way allowing students to reason in a simple way, without getting lost in the calculations and focusing on the concepts.

We specify here that in our experience the software Geogebra acted as mediator between the student and the knowledge, facilitating the understanding of the concepts because of the possibilities offered by the manipulative function of dragging. The ability to dynamically change the represented objects and examine them from different points of view, has allowed the development of the activities of guided discovery would also has increased student motivation.

Students, as mentioned, work on their own and together: they work on their own because each student has a worksheet to be elaborated and they also work together, comparing ideas and insights, and having at their disposal a space (the worksheet and the working environment of the software) and a time, suitable to their own learning. So each one builds his/her own knowledge according to his/her need of time and space, in an inclusive education perspective, which fosters both "less achievers" students in motivation (see paragraph on the results) and "more achievers" students in deepening the concepts. Students work independently, but they are never left completely alone: the teacher supports, encourages, helps them, according to the Enactivism principles. Moreover, at the end of each worksheet, the teacher, by means of a final discussion with all the students, remarks the obtained results, giving strength and clarity to the concepts just discovered, so that they are clear to all. This allows students to check the accuracy and richness of the proposed solutions, their consistency and the reliability and their level of adopted generalization. This phase leads to the construction of meanings that go beyond those directly involved in the solution of the task, to enable students to know new aspects of mathematical culture, enhancing in particular, a gradual but systematic approach to theoretical thinking. In the mathematics discussion teacher has a leading role: he/she influences the discussion in a decisive way, with proper and effective interventions, because he/she has in mind both general and specific targets of the activity.

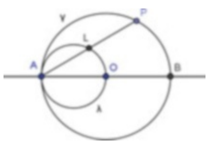
In the final discussion the students reflect together or alone on the difficulties they encountered, about what they did to overcome them, on the hints that have been decisive and what was misleading, thus developing the metacognitive awareness, which allows them to assimilate new skills and knowledge to add to those they already hold in the long-term memory.

In this context, it is very important to pose problems that are watching mathematics, school and world with a critical sense, to become a citizen who uses mathematics as an aware person.

Students' Worksheets

It follows here some of the worksheets produced with the teachers, the case *n=1*, one of the cases *n=2* and one of the cases *n=3*.

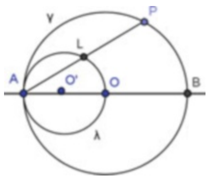
Each worksheet consists of two parts: an explorative part (Part I), where students discover the property, and a proof part (Part II) where students are leaded to the demonstration of the property they have just found. Eleven worksheets have been produced. Gradually they became more difficult because they are less guided (Table 2).

<table>
<tr><td colspan="2">Worksheet 2 – Part I</td></tr>
<tr><td colspan="2">Case $n = 1$</td></tr>
<tr><td colspan="2">It is given a circle γ with a fixd point on it.
With the help of Geogebra, let's discover the geometric locus of the mid points of the chords of a circle coming out from a fixed point of it.</td></tr>
<tr><td>Construction and exploration</td><td>With Circle with centre and radius, construct a circle with centre O and radius as you like.
Name this circle γ as it follows: right click on the circle with the mouse, select Rename, click on the small square with the letter α, then on γ and in the end on OK.
With Point on Object construct two points on γ and name them A and P.
With Segment construct the segment AP.
Construct the Midpoint or Centre of AP and call it L.
Right click on L, a mask will appear, click on Trace On, then Move P on γ.

When P moves, the chords out of A moves and the midpoint of each chord leaves its trace on the plane.
The set of the «traced» points seems a circle.
Use Edit/Undo to erase the trace.
Construct now the locus described by L when P moves on γ, with Locus and by clicking on L first and on P afterwards. Call λ this figure.
The obtained figure seems again a circle with diameter
. . .. and with centre ....</td></tr>
</table>

(continued)

Worksheet 2 – Part I	
Case *n* = *1*	
Verify	Grafically verify your considerations on the circle you have found, by using, for example, ***Circle with Centre through Point*** or ***Circle with Centre and Radius***.
Conjecture	You can then conjecture that, *Given a circle γ with centre O and a point A on it, the locus of the midpoints of the chords of γ throught A is* ………………………………………………………

Worksheet 2 – Part II
Case n = 1
Proof
Prove the proposition you have conjectured in Worksheet II – Part I. Let γ be a circle with centre O and radius r, and let A a fixed point of γ. **The locus of the midpoints of the chords of γ throught A is the circle λ with radius r′ = (1/2)r and with centre the midpoint O′ of AO.** Proof. Since L is the midpoint of AP, then the following holds for the points P and L: AL = (1/2)AP. Then, the homothetic transformation with centre A and ratio 1/2 maps P into L and then γ in ……... Moreover……………………………………………………………………………….- ………………………………………………………………………………...

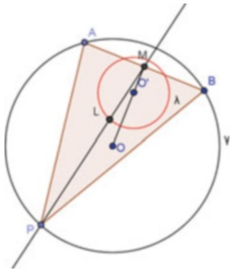


Since the circle λ is the correspondent of the circle γ through an homothetic transformation, λ contains all and only the midpoints of the chords of γ through A.

Worksheet 3 – Part I	
Case n = 2 – centroid	
It is given a circle γ with two fixd points on it, A and B. Given a third point P on the circle, with the help of *Geogebra*, let's discover the **geometric locus described by the centroid of the triangle ABP when P moves on the circle**.	
Construction	1. With ***Circle with Centre and Radius*** draw a circle with centre O and radius r (any number) and call it γ. 2. With ***Point on Object*** construct three points on γ and name them A, B and P. 3. With ***Polygon*** construct the triangle ABP. 4. Construct the ***Midpoint or Centre*** of AB and name it M, and the midpoint of another triangle edge. 5. With ***Segment*** construct the two medians, one of which is PM and the other is the one through the midpoint you have constructed. 6. With ***Intersect*** construct the centroid of the triangle and call it L. With ***Show/Hide Object*** select the two medians and then go to ***Move***.
Explore and verify	1. Right click on L, a mask appears. Click on ***Trace on***, then ***Move*** P on γ. The centroid leaves its trace that looks like a circle. Use ***Edit/Undo*** to erase the trace.

(continued)

Worksheet 3 – Part I	
Case n = 2 – centroid	
	2. Construct now the locus described by L when P moves on γ, with ***Locus*** and by clicking on L first and on P afterwards. Call λ this figure. 3. Graphically verify that λ is a circle by constructing, with ***Point on Object***, two more points on λ, different from L, and the the ***Circle through 3 Points***. 4. With ***Midpoint or centre*** construct the centre of λ and call it O′. 5. With ***Segment*** construct the radius *a* of the circle γ and the radius *b* of λ. 6. Open the ***Spreadsheet*** and by right-clicking on the two radius select ***Register on the Spreadsheet*** to import the values of *a* and *b*. Then calculate the ratio *a*/*b*. Repeat the same procedure described in points 5. and 6. to evaluate the ratio between the measures of the segments OM and O′M.
Conjecture	You can then conjecture that: *the locus described by the centroid of the triangle ABP, incribed in γ, when P moves on γ, is a circle with radius r′=(. . ../. . . .) r and with centre the point O′ on the segment OM such that* . . ..

Worksheet 3 – Part II
Case n = 2 – centroid
Proof (optional)
Prove the proposition you have conjectured in Worksheet 3 – Part I. Let γ be a circle with centre O and radius r, and let A and B two fixed points of γ. Let M be the midpoint of AB. **The locus described by the centroid of the triangle ABP, when P moves on γ, is a circle λ with radius r′= (1/3)r and with centre the point O′ on the segment OM such that O′M = (1/3)OM.**  Proof. Let us prove that, if L is the centroid of the triangle ABP, then L lies on the circle λ. Since L is the centroid of the triangle ABP and PM is one of its medians, L is the point on the segment PM such that LM = (1/3)PM. Therefore, in the homothetic transformation with centre M and ratio 1/3, L is the correspondent of P and the circle . . .. . . is the correspondent of the circle γ. Moreover the correspondent of O is O′ such that O′M = ... OM and the radius of λ is r′ = ... Since the circle . . ... is the correspondent of the circle . . . .. by an homothetic transformation, ... contains all and only the centroids of the triangles ABP.

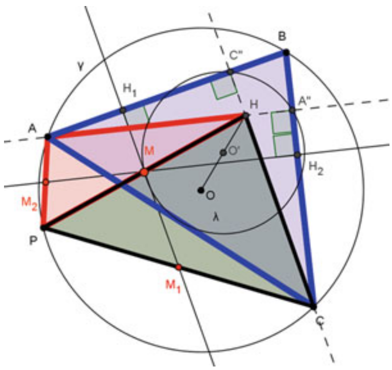
Worksheet 7 – Part I
Case n = 3 - anticentre
It is given a circle with three fixd points on it, A, B and C. Given a fourth point P on the circle, with the help of *Geogebra*, let's discover the **geometric locus described by the anticentre of the quadrilateral ABCP when P moves on the circle.**

(continued)

Worksheet 7 – Part I	
Case n = 3 - anticentre	
Construction and exploration	1. Construct a ***Circle with Centre and Radius*** (radius any number) and call it γ. 2. Construct a convex ***Polygon*** ABCP inscribed in γ. 3. Construct the malititudes of ABCP through the ***Midpoint or Centre*** M_1 of PC and through the ***Midpoint or Centre*** M_2 of AB. 4. Construct the anticentre of ABCP and call it L. 5. ***Show/Hide Object***s that you used to construct L. 6. Right click on L, click on ***Trace on***, then ***Move P*** on γ. The centroid leaves its trace that looks like a . . .. 7. Use ***Edit/Undo*** to erase the trace. 8. Construct now the ***Locus*** described by L when P moves on γ. Call λ this locus. Select the dotted style for it. 9. Construct the triangle ABC, the ***Midpoint or centre*** of its edges and its Feuerbach circle. What do you observe ? . . ..
Conjecture	You can then conjecture that: given a circle γ with centre O and radius r, *the locus described by the anticentre of the polygon ABCP, incribed in γ, when P moves on γ, is* . . ..
Explore	Construct a ***Circle with Centre and Radius.*** With ***Polygon*** construct a quadrilateral ABCD inscribed in the circle, and construct the triangles ABC, BCD, CDA, DAB. Construct the ***Midpoint or Centre*** of the edges of the triangles and the nine-point circle of each triangle. What do you observe ? . . .. Since each nine-point circle contains . . .. of the quadrilateral ABCD, then the common point of the four circles coincides with . . .. of the quadrilateral ABCD
Conjecture	You can then conjecture that : *in a cyclic quadrilateral* . . ..

Worksheet 7 – Part II
Case n = 3 – anticentre
Proof (optional)

Prove the proposition you have conjectured in Worksheet 7- Part I.
Let γ be a circle with centre O and radius r, and let A, B and C three fixed points of γ.
The locus described by the anticentre of the quadrilateral ABCP, when P moves on γ, is the nine-point circle of the triangle ABC.
Proof.



(continued)

Worksheet 7 – Part II
Case n = 3 – anticentre
We have to prove that the anticentre L of the quadrilateral ABCP lies on the nine-point circle of the triangle ABC. Let M_1 and M_2 be the midpoints of AP and PC, respectively. Consider the triangle ABC and its othocentre H, intersection point of the altitudes AA″ and CC″. Let M be the midpoint of PH. Consider the triangle PHA and its line M_2M that joins the midpoints of two of its edges. It is …………………………………….... to the third edge AH and then to the altitude AA″ of ABC. It follows that M_2M is …………………………………… to BC, it is then a ……………...…………...……… of ABCP. Consider the triangle PHC and the line M_1M that joins the midpoints of two of its edges, it is …………………………........ to the third edge CH and then to the altitude CC″ of ABC. Therefore M_1M is ……...…………………………… to AB, it is then a ……………………………...………….. of ABCP. The point M, point of intersection of the lines M_1M and M_2M coincides then with ……………...………………………….. of the quadrilateral ABCP. Consider the homothetic transformation with centre H and ratio h = 1/2. In this homothetic transformation to the correspondent of the point P is …… and the correspondent to the point O is ………………………………....……………. of HO. Call this last point O′. The correspondent of the circle γ with centre O passing through P is the circle with centre O′ passing through L. Since the centre O′ of λ is the midpoint of the segment joining the circumcentre and the orthocentre of the triangle ABC and since the radius r′ of λ is r′ = (1/2) r, then λ is ……………...………………………………………………… of the triangle ABC. Therefore lies on ……………………………………………………………………………….of the triangle ABC. It follows that when P describes γ L describes λ. Since the circle ….... is the correspondent of the circle …….. through an homothetic transformation, λ contains all and only the anticentres of the quadrilaterals ABCP.

Results

As already mentioned, at the end of the experimentation, teachers' logbooks and students' worksheets were collected; moreover a questionnaire was administered to students and teachers, to assess the level of liking of the proposal and if the targets have been achieved.

The teachers' questionnaire consisted of ten open questions, asking them to highlight remarks on the effectiveness of the teaching proposal, on the mood created in class during the activity, and some other questions about the possibility to share such a methodology with colleagues or to participate to similar activities.

The students' questionnaire, aimed at verifying if the goals of the activity were achieved and evaluate the satisfaction level, consists of two three sections:

- Overall assessment of the course;
- Self-assessment of skills that the student believes to have developed during the course;
- Personal comments.

Table 2 Students questionnaire

Overall assessment of the course	Definitely not	More no than yes	More yes than no	Definitely yes
Was the topic of the activity interesting?	2%	17%	51%	30%
Was the activity difficult?	7%	42%	36%	15%
Was your school knowledge enough to attend the activity?	0%	6%	54%	40%
Were the activity worksheets clear?	2%	10%	38%	50%
Was interesting and surprising studying loci?	4%	19%	47%	30%
Were the teachers clear?	0%	4%	34%	62%
After this course has your motivation to study mathematics increased?	6%	32%	40%	22%
Was your participation in the course active?	1%	4%	40%	55%
Was it worth to participate in the activity?	2%	16%	37%	45%
Self-assessment of skills that the student feels to have developed during the course:	**1(min)-4(max)**			
	1	**2**	**3**	**4**
Commitment	4%	6%	59%	31%
Desire to explore the topics	3%	32%	45%	20%
Availability for teamwork	1%	9%	32%	58%
Ability to speak in public	10%	28%	47%	15%
Ability to think about things	0%	15%	64%	21%
Integration in group	0%	9%	26%	65%
What would you like your math teacher would give more attention to?				
Practical and applicative aspects		56%		
Theoretical aspects		10%		
History		3%		
Relationship with other subjects		31%		

In the first section there were nine closed questions, structured using ordinal scales, with these answers: Definitely not, More no than yes, More yes than no, Definitely yes.

In the second section there were six closed questions, structured by ordinal scales at intervals whose parameters answers vary from a minimum of 1 to a maximum of 4 points.

In the third section there are seven open questions.

In the following we report some considerations arising from logbooks, teachers questionnaires and students questionnaires (sentences of the teachers or of the students are written with italic font).

As a general remark, the activity was appreciated both by teachers and students.

In particular, teachers underlined the importance to create and write the class worksheets together with a team of teachers from University (the tutors), because "*not only we focus on the targets to be reached, but also we did not overlook the* *difficulties pupils could meet in learning*"; moreover, during our meetings teachers not only mastered the use of Geogebra, but they also comprehended "*the methods and techniques to offer the activity to students and to get in such a way the best results*". Then teachers understood the interplay of "what" (CK), "how" (PK) and "by what" (TK) of the TPACK framework.

In particular, teachers learnt how to effective integrate technology in a pedagogical context, experiencing "first hand" the PTK intersection of TPACK.

All the teachers unanimously stated that it is desirable to integrate DGS in everyday teaching, because "*not only it is able to make immediately visualize the geometric locus, but also it is a valid help to verify the results obtained in analytical way*", perceiving the role of practical intelligence and semiotic mediation of tools.

The same is strengthened by their students, that said, for instance "*the use of Geogebra is useful to better understand studied topics, because by the construction you get the definition. Step by step construction of geometric loci through Geogebra surely helped to facilitate the understanding of the concept of locus*" and "*Geogebra is a very interesting program because it let me to verify with my eyes all the properties I just studied in the books*", highlighting the connection between Technology and Content (TCK).

When the student claims that by the construction with the DGS he/she gets the definition, it is clear, from the reseराchers' point of view, that the artefact was effectively used to build a mathematical meaning, so it was a tool.

As for the use of class worksheets, it came to light that "*the activity worksheets were a valid support: each student, with the help of the worksheet, succeeded to work peacefully and in autonomy*" being "*teacher of him/her self*". Students really were main actors in the construction of knowledge, as hoped for Enactivism principles. All the teachers highlighted enthusiasm in his/her own pupils, that "*faced the task with a certain level of autonomy, sometimes giving some original contribution: [. . .] the enthusiasm is due to feeling of the activity as cool and easy, almost a geometric game. A game of construction e visual verify, free by the struggle of computations*".

Moreover a collaborative way of working seems to be effective in students' learning. They stated that "*among us (classmates) there was a collaborative climate, that let us carry out the assigned task in the best way, we learned several topics in a easy way, collaborating in group*" (in a social learning mode, as the one hoped for Vygotskij).

Working in the Laboratory of Mathematics, all students tested their knowledge, used artefacts and tools, made explorations, formulated conjectures, acquired concepts and skills: "*mathematics does not belong to another planet and it is not just for someone who has some special skills, but it is accessible to all*". Some teachers pointed out that "*the activity allowed us to stimulate the interest of those pupils more fragile and less inclined to discipline, being personally committed to work, those students had fun discovering and building geometry through Geogebra*".

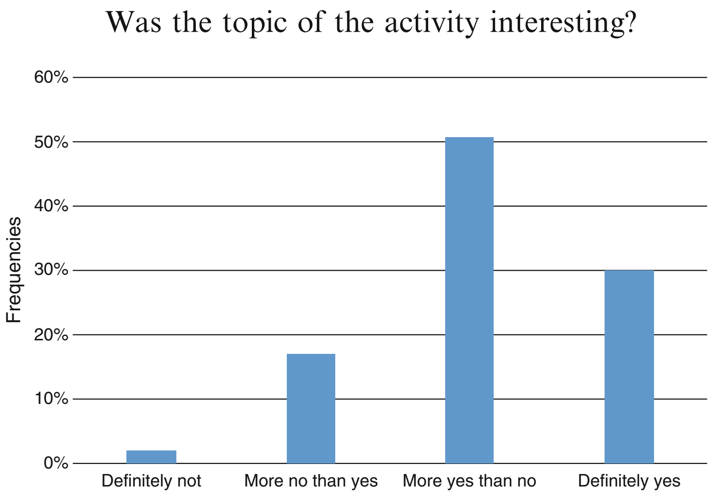


Fig. 8 Question 1

On the other hand the teacher was always ready, if necessary, to correct their conjectures with appropriate suggestions; to ask questions to make them guess something could be useful and/or necessary to discover something else; to encourage them to continue; to praise them for any significant obtained result, in the Vygotskijan perspective (ZPD and positive mood).

Students had no problems in approaching the proposed loci. Some of them even considered easy to apply the encountered properties and to use them. We should underline that worksheets are sometimes too guided for some students, but we took this choice, so that all students would feel equally involved.

Finally, students knew that they were participating to an experimental activity and that the teacher will not evaluate them. This allowed also fearful or less achievers students to feel involved, the wrong answers were not evaluated negatively, indeed those answers served as a basis to trigger collective discussions to clarify the problem: the error is seen as a resource, rather than as something to be condemned, according to the Enactivism's perspective.

The following general overview concludes the analysis of the student questionnaire, with regard to the first and the second part. An overview of the questionnaire is given in the end.

For most of the pupils involved in the activity (81%) the topics are interesting (Fig. 8), also confirmed by 95% that said they had actively participated to it (Fig. 9).

Almost all the students had found the activity is not too demanding, but they worked hard into it (Fig. 10). Students appreciated working in groups, in fact 90% of the statistical units is favourable to the work group (Fig. 11), and the interaction established between the members of each group (Fig. 12) were positively judged by more than half of the sample (91%).

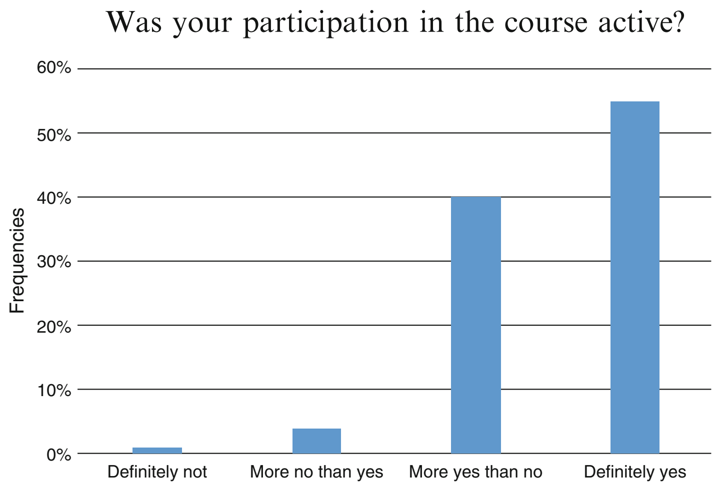


Fig. 9 Question 2

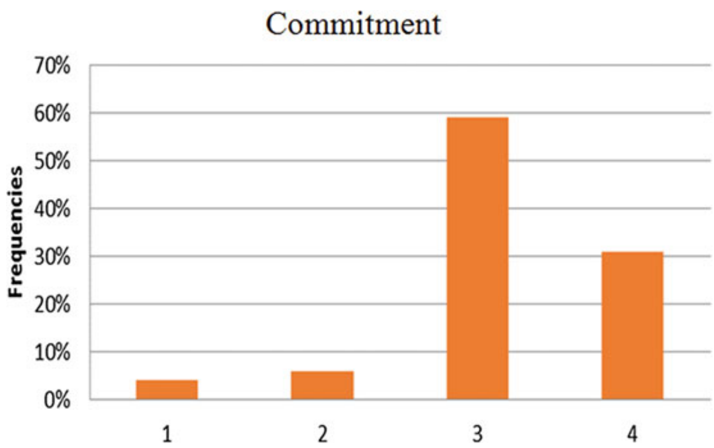


Fig. 10 Question 3

It is interesting to report the claims made by some students who confessed their disaffection to mathematics, "*I hate math, for me it's boring because I do not understand it. But I liked this experience with the computer because math seemed to me more interesting*."

In any case, the 82% of students admits that it was is worth to participate to the activity (Fig. 13); moreover some students said that such an activity allowed them to see the math from another point of view; they would like to repeat a similar experience and hope the same for the students of lower classes "*because maybe everyone could like more mathematics, if you start on the right foot*."

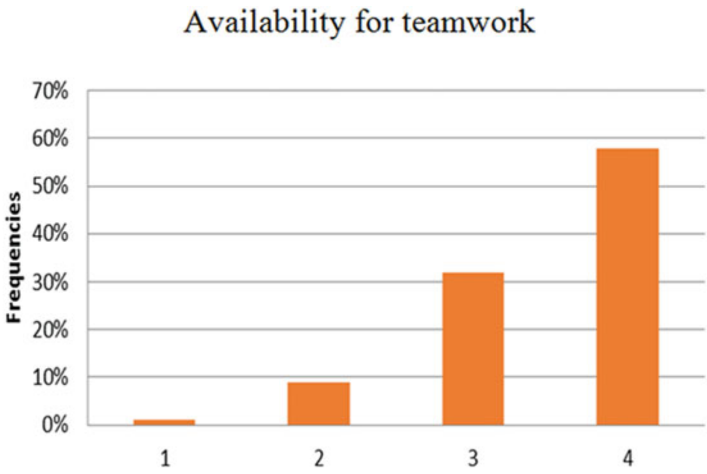


Fig. 11 Question 4

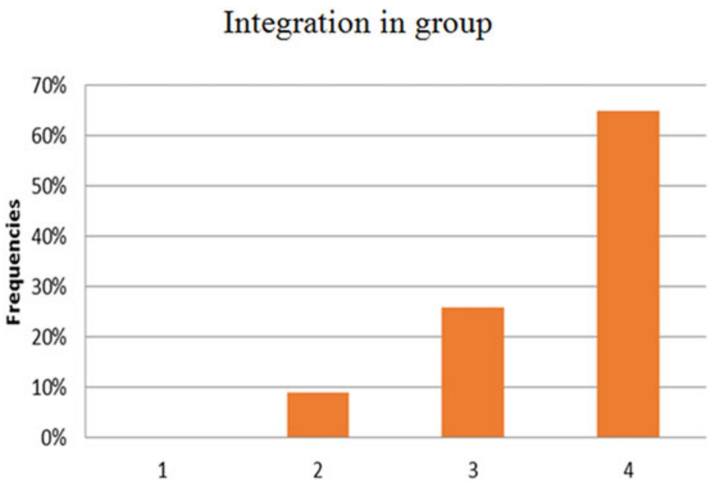


Fig. 12 Question 5

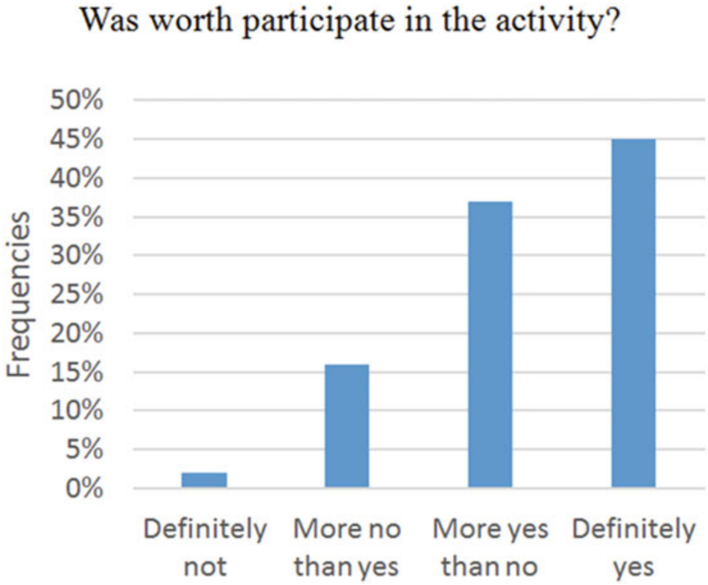


Fig. 13 Question 6

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Technology and Teachers' Professional Development: A Commentary

Gail E. FitzSimons

Abstract In this Commentary I reflect on the work done by the authors of five chapters in relation to technology-related professional development for mathematics teachers across a range of educational levels, undertaken in a variety of contextual settings, and drawing on a wealth of theoretical frameworks to support a diversity of practical approaches to implementation involving teachers, current and prospective, their students, and researchers in ongoing knowledge building and theory development. My reflections will be refracted through the three other major themes of this book which intersect in professional development: (a) technology as a tool for teaching and learning mathematics, (b) communication inside and outside the classroom, and (c) information tools, to inform oneself and to inform others.

Keywords Technology • Professional Development • Research • Practice • Theory Development

Introduction

In my experience professional development is generally considered as a non-formal educational activity for practising teachers, ranging from single sessions lasting a few hours to regular sessions over a fixed period of time. The pedagogical style usually takes the form of those with more expertise leading those with less expertise, and sessions are comprised of some form of theory accompanied by participant activity involving hands-on experience with the physical and/or virtual materials, concluding with some kind of discussion and reflection. The complexity and sophistication of the five accounts given in this section of the book illustrates just how far professional development in mathematics education has come. As amply demonstrated by the authors in this section of the book, there is value in drawing upon multiple theories to enrich the professional development experience for teachers and researchers alike.

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The introduction of generic tools such as calculators (four-function, scientific), spreadsheets, and statistical packages used in statistical quality control (e.g., *Minitab*™) into mathematics classrooms posed the problem of how to optimise their use for pedagogical purposes. Such tools, which carry the propensity to substitute for routine human operations, ideally saving time and reducing mathematical errors, also evoked (and continue to do so) widespread community fears concerning their impact on mathematics education. These technologies are also able to support manipulative and transformative actions, reducing the cognitive load of demanding and time consuming calculations. In the past, much professional development was directed to demonstration of the mechanics of technique, and showing how to optimise new pedagogical possibilities through helping in actions directed towards sense making and extending the kinds of tasks that could be offered to students, including more complex investigations. In the mathematics education sphere, people began to design research-based tools and software with a specific pedagogic emphasis designed to enrich concept development, greatly extending the potential repertoire of classroom teachers at all academic levels. At the same time a range of technologies supporting communicative actions between people have vastly enriched options for individual, peer, and group interactions within the classroom and beyond, as well as information sharing at a meta level for educational and administrative purposes. These developments have been accompanied by a range of theorisations intended to understand their impact on existing mathematics curricula and assessment and, more importantly, to improve teaching and learning outcomes in mathematics through refinement of these theories, as a consequence of case studies or other kinds of formal and informal classroom observation. Any radical change to established teaching practices, whether mandated or not, should ideally be supported by some kind of professional education for the teachers concerned – and also public education for important stakeholders such as parents. Formal or non-formal courses organised by a university or as university partnerships with groups of teachers or schools are elaborated on in all chapters in this section of the book. This book, taken as a whole, encompasses four major aspects of the growing importance of technology in mathematics education: (a) as a pedagogical tool, (b) as a means of communication, (c) as a means of information sharing about teaching and learning, and (d) professional development. Because the first three overlap and intersect with professional development in various ways I believe that it could be useful to rehearse them here.

Reflections Through the Lens of the Book Proposal

In this section I will paraphrase the contents of each of the four aspects identified above, and briefly outline their connections with professional development as reflected in the chapters included in the text.

Technology, a Tool for Teaching and Learning Mathematics (Also Pedagogy)

The proliferation of digital tools and resources, together with new and emerging related theoretical models for mathematics education, means that classroom teachers at all levels have the right to professional education which enriches their teaching repertoire in theoretically well founded ways. For many, the use of new technologies can pose problems in relation to their self-confidence with both the techniques of operating the tools themselves, and the organisation their pedagogic actions and classroom management to accommodate these resources. In addition, any negative personal beliefs and attitudes by teachers towards technological change in mathematics education may also be related to genuinely held fears such as the de-skilling of students, and these also need to be addressed. In order to ameliorate these concerns, such teachers could benefit from the opportunity to learn the techniques in supportive environments, whether at organised training sessions or at their own or local schools in small communities of practice. Along with technique, teachers also need support, initially at least, in locating, collecting, introducing and orchestrating tasks, refining them to meet their own students' particular needs, and perhaps even extending or initiating new technology-enhanced or technology-driven tasks and explorations. Importantly, the focus needs to be on the educative potential of the tasks rather than an exposé of the "amazing things" that technological tools can do. This means that task design has to be research-based, building on well founded theories of mathematics education, while recognising that mathematics-related digital technologies are an important part of the distributed mathematical knowledge available through a suite of available resources supporting learning, communication, and information sharing.

Communication Inside and Outside the Classroom (Between Students, Teachers, Researchers)

In the digital era especially, the importance of communication, within and beyond the mathematics classroom, for students, teachers, and researchers cannot be underestimated. The rapid evolution of technological resources readily available for individual and mass communication and information sharing has opened up many more possibilities in the field of mathematics education, transforming both teaching and research. It has long been accepted that tools, technological and otherwise, can shape the goals and activity of the user, whether they are aware of this or not. Communication technologies in particular impose their own ways of being used, by implicit as well as explicit means – albeit with unpredictable consequences, for example the world wide web and social media. Teachers at all levels of mathematics education have the right to professional support in making efficient use of available communication resources to use with and for their students in the classroom and beyond. At the same time, researchers and teachers also need to find effective and efficient ways to communicate and to share information about pedagogic activities which are the subject of their collaborative research. Through a variety of communication media teachers and researchers are now able to share the kind of professional relationship and interaction on curriculum, assessment, and pedagogy not previously possible. Related to this is the possibility of teachers documenting and sharing resources with each other and researchers via technological networks, as discussed below.

Information Tools, to Inform Oneself and to Inform Others (Through Connected Networks)

Mathematics education has to take into account new ways of learning through connected networks as well as new ways of teaching with an extensive documentation of teachers' and students' work, made available to other teachers and to researchers. Clearly there are problems facing the education professional sphere in enabling information to flow in different directions. The question was posed: How can researchers appropriate for themselves knowledge about teachers' experiences and, in turn, how can teachers appropriate for themselves knowledge about research results? The process of formal networking arrangements to support such documentation as described here appears to be distinct from the formal provision of professional development. In this sense connected networks offer a viable alternative, or even supplement, to organised professional development as another way to support teachers and researchers to share valuable information, particularly where there could be problems of distance, time, and money which may prevent personal participation in regular professional development programs.

Technology and Teachers' Professional Education and Development

The organisation and funding of formal initial and continuing education of teachers at all levels is politically contested, depending on the values placed on economic, social, and cultural priorities. Mathematics teacher education in particular carries, among other things, community expectations of developing a technologically literate and mathematically competent citizenry, especially among the workforce. The political injunctions, often insistent, of introducing technology into classrooms have to face the realities of teachers' current technological knowledge, beliefs, and attitudes, any or all of which may not be in accord with official expectations for a variety of reasons. (This is aside from any issues of who will fund the hardware, software, licences, and professional development – and how much.) Underlying political and community attention are widespread concerns (in Anglophone countries at least) about the quality of teachers' mathematical knowledge at all educational levels (e.g., Fried and Dreyfus 2014; Ruthven 2011). In addition, a working knowledge of at least basic statistical ideas is essential to participation in contemporary society by people from all walks of life (see, e.g., FitzSimons 1998, 2001, 2015; Kent et al. 2011), including teachers from across disciplines and levels, starting from primary school education. The widespread use and abuse of data and misconceptions about probabilistic events can have serious personal, social, community, and industrial consequences if decision makers at all levels are not well informed. Three of the five chapters in this section address issues around the development of competence and confidence in the mathematical and/or statistical knowledge of teachers, integral with their use of technology in mathematics teaching. A fourth chapter focuses instead on developing teacher awareness of formative assessment as part of teachers' pedagogical repertoires; also the potential benefits from technology-based information sharing among researchers, teachers, as well as timely feedback to students. The teachers engaged in all projects were involved variously in the different phases of design, implementation, and analysis. Two of these four chapters involved prospective and practising teachers in formal university courses designed for initial and continuing education, respectively, while the other two were classroom-focused investigations each conducted over a period of time. A fifth chapter addresses the question of the actual provision of formal mathematics teacher education and non-formal professional development, and whether they might be better informed by a coherent theoretical framework which took into account specific mathematics-didactics knowledge. Each chapter illustrates in its own way a respectful partnership and collaboration between teachers and researchers.

Bellamy (1996) offers three principles for design of educational environments involving technology which I believe are relevant to the chapters included in this commentary from the perspective of professional development and professional education:

1. Authentic activities
2. Construction of artefacts and sharing with the community
3. Collaboration between learners and with experts.

The implications here are that, first, activities should be seen as culturally relevant to the particular learners – both the teachers undertaking professional development, and the students (current and potential) of those teachers. Second, professional development activities should lead to the construction of artefacts (real or virtual) and sharing with community of teachers and researchers. For school students the goal may be, initially at least, to construct meanings for particular mathematical concepts, but may shift ultimately to using those concepts, along with technological skills, to work towards solving culturally meaningful problems at school, at work, or elsewhere. Finally, collaboration between learners and with experts can take place on two levels: the professional development level, where teachers collaborate with academics to improve pedagogic practice or to produce teacher support materials in relation to technology; and at the school level, where students work collaboratively with each other (locally or globally) and the teacher as expert.

Reflections on the Chapters

Each of the five chapters has something different and something interesting to say. In this section I am particularly interested in the processes adopted by the authors in order that other prospective researchers in this area might build on their theoretical frameworks and general research methods in the manner recommended by Bellamy's (1996) three principles, along with the mutual respect expected between professionals from different disciplinary backgrounds as they collaborate to work towards a common goal (FitzSimons 2014).

The most common theoretical framework, utilised in the chapters directly attending to developing teachers' mathematical or statistical knowledge, is that of Mishra and Koehler (2006), refined in Koehler and Mishra (2008, 2009), on the integrative perspective of the *Technological Pedagogical Content Knowledge* (TPACK). Rabardel's (1995) theory of instrumentation and subsequent developments regarding *Instrumental Genesis* and *Instrumental Orchestration* (e.g., Drijvers 2012; Tapan 2006; Trouche 2007) have also played critical influential roles. Authors of chapters "Integration of Digital Technologies in Mathematics Teacher Education: The Reconstruction Processes of Previous Trigonometrical Knowledge" and "Teaching Intriguing Geometric Loci with DGS" have devoted considerable space to explicating the theoretical aspects of TPACK which is a taxonomy based on the work of Schulman and others to identify the component parts of a complex model involving technological, pedagogical, and content knowledge for teaching; their various intersections; and also the interconnected model as a whole. Thomas and Palmer (2014) claimed that earlier work by Thomas and Hong, contemporaneous with but independent of the TPACK framework also offered a useful version, called *Pedagogical Technology Knowledge* (PTK). This model was described as (a) being specifically focused on the discipline of mathematics and mathematical knowledge for teaching, (b) integrating the theoretical base of Instrumental Genesis with a strong emphasis on the epistemic value of technology to produce mathematical objects of interest, and (c) the personal orientations of the teacher in relation to technology. In particular, they stress the critical importance of teacher confidence in using technology, and this is clearly an intended outcome of the chapters in this section, and technology related professional development in general. The chapters have all included considerable amounts of data to allow for readers' own interpretations.

The first four chapters discussed in this part of commentary reflect a sequence which I consider reflects an increasing development of theoretical aspects. However, this by no means reflects on the quality of any chapter since it is the overall professional development program set in its own particular context that matters most. The first project is a case study in the traditional sense of professional development, involving university researchers, practising teachers, and their students, with the aim of broadening and extending students' mathematical horizons, along with their teachers' pedagogical repertoires in relation to technology. This chapter and the following one both use the freely available dynamic geometry software (DGS) *GeoGebra*.

Teaching Intriguing Geometric Loci with DGS by Ferrarello, Mammana, Pennisi, and Taranto

This research involved eight practising teachers in high schools in Sicily in a teaching experiment to study geometric loci. Besides the TPACK framework, the researchers' work was also framed theoretically by the *Zone of Proximal Development* (ZPD) as well as Dewey's learning by doing, and *Enactivism* where the student takes a central role in the teaching and learning process in a dynamic interaction with their environment. The principles of ZPD were adhered to when the students were able to complete the tasks through collaboration; learning by doing involved both mind and body, complemented by learning by thinking and learning through being emotionally involved in the tasks, in accordance with Enactivism. The researchers used the method called a *double laboratory* which comprised a teachers' laboratory, where the materials were designed in collaboration with the researchers, and a students' laboratory, where the students trialled the materials produced. Teachers were asked to complete logbooks on a regular basis, and students were given worksheets to complete, followed by a class discussion at the end of each session. Both teachers and students completed a final questionnaire.

Technology as a tool for teaching and learning allowed the students to have well drawn figures and, through this, the possibility of discovering a real property. The use of the "Dragging" mode in the DGS enabled students to generalize and to make conjectures about the observed property. It also released students from arduous calculations in order to verify properties. On the other hand, students were asked to prepare proofs using paper and pencil, and guided by worksheets in order to give them the space and time to reason about a property. The technology enhanced environment allowed the students to communicate with one another within the classroom using screen images as well as through written and discussion modes, taking the students into real as well as virtual worlds. The worksheets prepared by the teachers in collaboration with the researchers have the potential to be shared through communication networks, although this was not specifically mentioned. The results of this project showed that the teachers learned how to integrate technology into their regular teaching and valued doing so. The students, having been actors in the construction of their own mathematical knowledge were enthusiastic, and enjoyed the collaborative work, guided by supportive teachers. Just as the chapter "Teaching Intriguing Geometric Loci with DGS" took teachers and students on a mathematical journey into the world of loci made possible by DGS, the chapter "Integration of Digital Technologies in Mathematics Teacher Education: The Reconstruction Processes of Previous Trigonometrical Knowledge" offered pre-service and in-service mathematics teachers the possibility of seeing school mathematical concepts of trigonometry in new ways, deepening mathematical knowledge as well as developing their pedagogical and technological confidence and skills.

Integration of Digital Technologies in Mathematics Teacher Education: The Reconstruction Process of Previous Trigonometrical Knowledge by Lobo Da Costa, Esteves Lopes Galvão, and Brisola Brito Prado

This research offered two case studies and an analysis of the use of DGS technology; one in a group of prospective teachers (2nd year of a Mathematics degree), and the other a group of in-service secondary level teachers undertaking a continuing education course at a university in Brazil. The focus of the chapter was the integration of digital technology into the teaching and learning processes, and how the reconstruction of previous trigonometric concepts occurred within the two groups. Both case studies explored aspects of trigonometric functions using *GeoGebra*, as well as familiar physical artefacts, once again incorporating both real and virtual worlds. The preservice teachers explored the concept of periodic function using concrete mathematical objects as well as an orthogonal projection of light onto a trigonometric circle to visualise and to measure, in parallel with their activities with the DGS. The in-service teachers constructed trigonometric functions integrating the DGS with mathematical artefacts and games, as well as programming the software, leading to deepen their knowledge of trigonometry.

The theoretical framework included TPACK, as well as instrumental genesis, as understood by Rabardel (1995) and Trouche (2007). In addition, a *double instrumental genesis* was required for teachers to work with technology (Tapan 2006), including *Technique Instrumentation*, building tools for mathematical tasks, and *Didactic Instrumentation*, building tools to teach mathematics for the professional development of teachers. In both groups each participant had the opportunity to develop a process of technological appropriation through instrumentation and instrumentalisation. The main difference between the two groups was found to be that the *Pedagogical Content Knowledge* (PCK) component of TPACK was only observed with the practising, in-service teachers who were focused on how to use technology with their classes, since none had previously used DGS in their teaching; whereas the preservice teachers focused mainly on the (Mathematical) Content Knowledge (CK) and the Technological Knowledge (TK) components, leading the authors to conclude that more attention should be given to pedagogical aspects for this group. Whereas the first two chapters used the TPACK theoretical framework as a major basis for designing professional development for current and future teachers in relation to mathematical content of geometry and trigonometry, the authors of the chapter "A Study on Statistical Technological and Pedagogical Content Knowledge on an Innovative Course on Quantitative Research Methods" move from deterministic modelling to focus on probabilistic modelling.

Knowledge on an Innovative Course on Quantitative Research Methods by Serradó Bayés, Meletiou-Mavrotheris, and Paparistodemou

This chapter offers a significant theoretical development as the authors both challenge and extend the TPACK model, as well as existing research on mathematical modelling, specifically in relation to statistics, and for this reason I attend to it in more detail. The study reported in this chapter drew on a larger study of a graduate Quantitative Research Methods course intended to develop *Statistical Technological and Pedagogical Content Knowledge* (STPACK), for a group of Cypriot teachers with diverse academic backgrounds.

First, the researchers extend the epistemological distinction between *descriptive* and *normative* models to include *relational* models connecting what they term intra-mathematical and extra-mathematical fields, asserting that three approaches, rather than two, are needed to model the relationships between the mathematical, statistical (including stochastic and probabilistic) content knowledge. From my own perspective as a teacher-researcher with adult, vocational, and workplace mathematics education interests, statistics education pragmatically linking both intra- and extra-mathematical fields has always been a critical facet, no matter what the educational level of the students (see, e.g., FitzSimons 1998, 2001, 2015). Investigating aspects of the realities of people's working and civic (i.e., extra-mathematical) lives not only serves to protect them from being disadvantaged at home or at work through ignorance, but also provides a sense of reality that is often missing from most mathematical questions feigning reality commonly found in school.

Second, from a pedagogical point of view, the authors maintain the need to extend the statistical and probabilistic investigative modelling processes to also encompass the interrogative cycle which they claim should help to develop the process of hypothesis generation, as part of a long-term process of maturation of the thinking of the learner. Here, learners are being offered an extended level of engagement, from working on a given problem to actually creating their own hypotheses about what might have led to that problematic situation. Hypothetical thought is an essential, albeit mostly unremarked, part of workplace activity, seeking to improve or ameliorate a situation which could ultimately have serious or high-stakes consequences (FitzSimons 2014). Recognition of this important feature brings with it an increased complexity to the teaching-learning process.

Third, in moving toward the interrogative cycle, the authors switch perspectives from teacher to student. Although students could experience both data-driven approaches where they construct a working theoretical model to reflect the data, and theory-driven approaches to modelling where they recognise a model they have been taught and working within this enclosed world, they also need to appreciate the circularity between these two, understanding modelling as an iterative cycle. The authors continue that the complexity of this circularity can be understood through the analysis of the transfer between the empirical world (data) and the theoretical world (probabilities). This leads them to identify three modelling approaches: (a) classical (unidirectional, from theory to practice to validate the model), (b) frequentist (bidirectional, from data to theory and back to validate the model), and (c) subjectivist (following iterative cycles of empirical data collection and theoretical model building). The teacher's choice of modelling approach also has consequences for which kinds of problem situation are most appropriate, consistent with the roles played by data in each kind of problem situation, namely, virtual theoretically-driven, virtual real world, and actual real world problem situations.

Finally, in order to identify how technology can assist teachers in the challenging situations faced in finding or designing and creating tasks appropriate to reflect the aims and approaches that promote the desirable kinds of modelling, the authors add a fourth, technological, world to address three purposes: accessing meaningful data, exploratory data analysis, and data simulation. To assist the reader, they offer a visual summary integrating all of the complexities discussed above. The authors conclude their theoretical section with a framework for STPACK, addressing teachers' expertise and knowledge of ways of facilitating students' learning of different statistical concepts.

The conclusions drawn in this chapter from the two reported case studies, focused on Model Eliciting Activities (MEAs) and uses of Dynamic Data Exploratory software, supported the need for circularity between the four worlds, but also recommended the inclusion of reflective questions for teachers related to the didactical and other obstacles faced by teachers and learners in the modelling process. Finally, the authors noted the need for more opportunities for teachers to experience the whole four-world modelling process, specifically in order to develop their attitudes and skills as researchers.

The chapter A Study on Statistical Technological and Pedagogical Content Knowledge on an Innovative Course on Quantitative Research Methods has taken the reader very deeply into the epistemological and pedagogical complexities of extending mathematical modelling into the statistical realm, and the authors are to be commended for their critical appraisal of existing research and their attention to detail. Clearly there is much more to be done in developing more comprehensive statistical knowledge among the population as a whole, including teachers from all disciplines – not just mathematics – and from all sectors including professional and vocational. However, from my own experience of teaching statistics to adults young and old in high level vocational laboratory technician courses as well workplace basic skills courses, the pedagogical importance of concrete hands-on experience – at least in the early stages – cannot be overlooked. It is invaluable to have students actually physically experience random behaviour (e.g., tossing coins, rolling dice), observing, counting, measuring, and collecting their own data relevant to the distribution in question and also relevant to their own real world lives at work and elsewhere. These real world, embodied kinds of activity offer students a meaningful point of reference to anchor the abstractions that follow in the complex theoretical and virtual worlds made available by technology. They also demonstrate that statistical methods of data aggregation, while useful to work with, can result in a loss of detail about individual outcomes or observations. In the education sphere, for example, aggregation of student test results at the state level is likely to ignore the very diverse contextual settings (social, cultural, economic, etc.) of particular cohorts of students, potentially leading to incorrect or over-generalisations by policy makers and the media.

Teachers from all academic backgrounds who are comfortable with statistics can provide role models for their students, especially when they exhibit the qualities of researchers themselves in working with quantifiable forms of uncertainty. The fourth chapter details a project where teachers work on a meta-level of collaboration with academics, acting as researchers in a pedagogical sense, learning more about their students and how they learn, along with seeking to continuously improve their own pedagogical repertoires. Like the chapter A Study on Statistical Technological and Pedagogical Content Knowledge on an Innovative Course on Quantitative Research Methods, it also details substantial theoretical developments.

Formative Assessment and Technology: Reflections Developed through the Collaboration Between Teachers and Researchers by Aldon, Cusi, Morselli, Panero, and Sabena

In this chapter, drawn from a large-scale study involving researchers and teachers in France and Italy, the focus has shifted from using technology to support student learning of mathematical concepts per se to technology-supported ways of using formative assessment (FA) to inform both students and their teachers about student understanding and learning trajectories. The project investigates both the effectiveness of the adopted theoretical tools in terms of enhancing connectivity and feedback, and also, in terms of professional development, the contribution made by the collaborative work between teachers and researchers within the project. An important outcome of the project was that it enabled the production of a set of curriculum materials and methods for pedagogical intervention aimed at supporting the development of teachers' practice. Based on a cyclical process, the teachers engaged in the project were involved in the different phases of design, implementation, analysis, and subsequent redesign and adaptation of the toolkit.

The project was supported by a comprehensive and complex dynamic model based on a combination of several relevant theoretical foundations. First, a three-dimensional framework was established consisting of five FA strategies, three main agents (teacher, student, peers/group), and three different functionalities through which technology could support each of the three agents in developing their FA strategies. Second, a meta level framework called the *Meta-Didactical Transposition* (MDT) was established based on Chevallard's *Anthropological Theory of the Didactic* (ATD) which asserts that every human activity within an institution can be described by a *praxeology* which comprises the task, techniques, justification of the techniques and theory developed within specific institutions: Teacher education is considered to be one such activity.

In this study, praxeologies were developed at two levels. Working with teachers, the research team developed a set of tasks, techniques, and justifications at a teaching level of reflection – a meta-didactical level. Their implementation in the classroom at the didactical level generated a first dialectic between students, teachers, and the knowledge. The interaction between teachers and researchers at the meta-didactical level in relation to interpretations of the classroom implementation, and the comparisons between researchers' and teachers' praxeologies, generated a second dialectic. It is through this second process that both groups come to develop a shared praxeology, on an iterative basis, enabling the professional learning and development of both teachers and researchers.

At the heart of the model in this chapter is mathematical knowledge and competence, and the authors turn to Brousseau's *Theory of Didactic Situation* (TDS). They build on Brousseau's didactic triangle to interpret the mutual relationships between teacher, student, and knowledge, and add a fourth vertex to accommodate the interaction of technology with all three existing relationships, implying a three-dimension didactic tetrahedron which sits within the first 3-D cuboid framework. In addition, each of the faces of the tetrahedron offers a representation of the relationship between three of the four actors. The authors use this complex model in order to study the evolution of didactical situations in a comprehensive way, through analysing the situation in the classroom as a whole and, through processing and analysing actual classroom data, to enable specific feedback to each student made possible by the technology, statically and dynamically improving the effects of formative assessment for students and teachers alike. In fact, the classroom technology of Maple TA with an IWB, in France, and IDM-TClass (integrated with a selection of activities) in Italy, had a role to play in facilitating the whole process: It supported the students through efficient FA feedback, and the teachers through providing information on individual students as well as the group as a whole. An important feature of technology was that it enabled teachers to store the data and return to it as necessary, to enhance teaching strategies. Students were constructed as owners of their own learning as well as instructional resources for others. Technology also supported the researchers in their endeavours as well as facilitating communication and collaboration between teachers and researchers to benefit both groups in the ultimate form of shared praxeologies. The authors observed that although the double dialectic and internalisation of the components of praxeology directly impacted the teachers, they did not develop FA as a teaching strategy as hoped. They concluded that more professional development would be needed to normalise FA as a regular component of the teaching process.

The Professional Development of Mathematics Teachers: Generality and Specificity by Polo

This chapter takes a reflective position to address the education and professional development of pre-service and in-service mathematics teachers primarily from the perspective that a dialectic is needed between specific disciplinary didactics and general issues of pedagogy. *Specific* forms of competence and knowledge should characterize mathematics teaching in particular while *generic* forms competence and knowledge should characterize any disciplinary teaching in a formal learning context. The author recognises that the conditions and specific aspects of mathematics teacher education change according to the conditions of training practices and asserts that it is important to understand how they influence professional development itself, both in relation to these two forms of competence as well as the position of the teacher within the overall didactic system and its dynamic interrelationships. This issue is made more pertinent with the rapid evolution of modes and means of communication made possible by technology influencing the conditions of professional development for in-service teachers (and also pre-service teachers – see Borba and Villarreal 2005) through the inclusion of distance education and blended modes of education.

Building on the work of Ball et al. (2008), among others, the author identifies three main interrelated components necessary for teaching mathematics: knowledge about mathematics content, general pedagogical knowledge, and mathematical-didactical knowledge, asserting that a notion of *specialized content knowledge* is also needed. Distinct from general Pedagogical Content Knowledge (PCK), and Subject Matter Knowledge (SMK), is Didactical Content Knowledge (DCK), specific didactic-disciplinary knowledge. DCK is strictly connected to the content of the learned object and it is at the base of the teacher's decisions before, during and after classroom activities. Because it is related to the mathematical and didactic organization, it regulates the *didactic relationship* between teachers and students. In the chapter's Fig. 6, the author represents in a detailed form the various interrelationships between DCK, SMK, and PCK, and their component parts. The chapter concludes with a call for integrated theoretical models to inform both pre-service and in-service mathematics teacher education in order to improve on current models, especially in the non-formal provision of the latter.

Conclusion

The five chapters that are the subject of this commentary have much to offer teachers and researchers alike, and I commend each team for its innovative work on behalf of the students concerned and, hopefully, students of the future. Something that concerned me recently when I was a CIEAEM Working Group co-animater was hearing a group of teachers lamenting the fact that when their government mandated changes to mathematics education there was no professional development other than what enthusiastic local teachers were able to offer on a voluntary basis. Although chapters in Part II of the book address communication and information sharing, there still seems to be a need for professional development for teachers that is different from formal online or face-to-face courses or information sharing through connected networks. One possible suggestion is based on the work begun by Ball et al. (2015) on a proof-of-concept virtual learning environment. This form of professional development uses technology as a medium to enable teachers to access and replay the material at their own convenience. Although at this stage the costs of time and money are considerable, in future it may be possible to integrate all four themes of this book with a view to access and equity in the study and improvement of mathematics teaching – to paraphrase the aim of CIEAEM – in relation to technology.

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Conclusion

It is obviously impossible to conclude when the subject of a book concerns such an upgradeable theme. Each day new technological potentialities come into existence. It is interesting to think that in the past 30 years we have seen the apparition and the disappearance of cassette tapes, floppy disks, CD-ROM to quote only the mass memory that we have used daily in our work or in our spare time. The last CIEAEM conferences have shown very interesting and convergent studies about the potentialities of software and their added values for the teaching and learning of mathematics: geometry, algebra, calculus, modeling,... In the same time, the new role and the new necessary skills of teachers have been highlighted as well as the opportunities to catch mathematical objects through different and intertwined representations. But what exactly change? Information, resources, documents are available, knowledge is everywhere, within reach of a click, answers to school questions are, most of the time, available on the web and reachable through research engines. Students in their everyday life dispose of these tools and use them widely and freely; most of the time, for their everyday life, they don't need to know because they know where the knowledge is available; or more precisely, they believe that they don't need to know because they believe that the available knowledge is understandable.

Regarding mathematics education, the main issue is more to understand how future mathematics education will evolve than looking at micro use of a particular technology that will not be present in the next years. The great ideas that cross the years have been built on transversal concepts that take into account the potentialities of technology more than local possibilities. DGS revolutionizes the teaching and learning of geometry because they offer open playground to visualize mathematical concepts in a dynamic environment. This idea of dynamic environment can be then reused in other mathematical topics and gives the opportunity to explore

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algebra, calculus, statistics, and so on independently of a particular software. Therefore, the idea can be reused in several ways allowing to display to students an enriched environment in which mathematical objects live. Teaching and learning mathematics, but also doing mathematics have been deeply transformed by this original and fantastic idea.

On another hand, the technology presented as Information and Communication Technology offers potentialities disrupting fundamentally the approach of the communication and the sharing of information. In that way, teaching mathematics as well as learning mathematics cannot be the same that they was some years ago. Knowledge is available easily, without filters, to each student and far before coming into the classroom; resources are available for teachers on every topic. What does it transform in term of mathematics education? How is it possible to transform into attractive feature what is considered as a competitor or an adversary in the field of maths education?

This book takes into account these fundamental ideas and the different authors through their communication during CIEAEM conferences and the discussion that occur, give a picture of the actual landscape of mathematical education in the technological era.

Sixto Romero shows in his comments chapter of the section “Technology, a tool for teaching mathematics” that the role of the teacher should perform works in almost unexplored domains and with the help of technology, produce creative activities enriching students’ mathematical skills.

Peter Appelbaum in the comments chapter of the section “Technology, a tool for learning mathematics” brings out the different boundaries that are and that appear when using technology. The country of mathematics education enlarges itself and grows back the boundaries in a new land that has to be explored. And the different chapters of this section are as many tentative that explain and analyze this exploration with different point of views, including in the word technology all that makes education efficient in a given world and a given society: language, knowledge, mathematics, “low technology” as well as “high technology” participate jointly to the complexity and the enhancing of mathematical education.

Corinne Hahn through her comments of the section “Communication and Information” emphasized the part played by technology in mathematical education and particularly about problem solving that appears in the different chapters of this section as an important feature of mathematics education. And finally, the important question of the knowledge at stake arises and she synthesizes the important contribution of technology in term of students’ autonomy in front of the knowledge construction. The final questions of her contribution could be employed as a research program that brings our community in the next years; particularly the issues of mathematics education out of school.

Gail E. FitzSimons comments the section related to the professional development of teachers in the digital era which is surely one important issue of mathematical education in the next years. She shows that in the different chapters of the section, the different theoretical frameworks converge to offer teachers training based on authentic activities, construction of artefacts and collaboration between learners and experts. Each of the contribution stresses the importance in teachers training to offer opportunities to become confident with technology that is surely a fundamental outcome of the different chapters.

The questions that have been at the origin of this book and that are presented in the introduction are not obviously closed but the authors, by their contributions have given elements of answers that push away the boundaries of mathematical education with technology. In the next years, we'll need to offer in the CIEAEM conferences the next steps taking into account the different actors of maths education research.

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