Chapter 5 From Pure State and Input Constraints to Mixed Constraints in Nonlinear Systems

Willem Esterhuizen and Jean Lévine

Abstract We survey the results on the problem of pure/mixed state and input constrained control, with multidimensional constraints, for finite dimensional nonlinear differential systems with focus on the so-called *admissible set* and its boundary. The admissible set is the set of initial conditions for which there exist a control and an integral curve satisfying the constraints for all time. Its boundary is made of two disjoint parts: the subset of the state constraint boundary on which there are trajectories pointing towards the interior of the admissible set or tangentially to it; and a *barrier*, namely a semipermeable surface which is constructed via a generalized minimumlike principle with nonsmooth terminal conditions. Comparisons between pure state constraints and mixed ones are presented on a series of simple academic examples.

5.1 Introduction

Though constrained systems, namely with restrictions on the control and the state, are present in many applications due to actuator limitations and obstacles, they are not generally studied on their own and are more often studied in the context of optimal control or differential games [\[8\]](#page-16-0). We focus here on a fully qualitative approach, i.e., without any optimisation framework where the aim is the construction of the set of initial conditions such that the system variables can satisfy the constraints for all time, called *admissible set*, and we show how to compute its boundary. Other approaches based on flow computation, or Lyapunov functions, or other variants, may be found in [\[1,](#page-16-1) [2,](#page-16-2) [11](#page-16-3)[–14,](#page-16-4) [16](#page-16-5)[–20\]](#page-16-6).

We first review the results of [\[6](#page-16-7)] for pure state and input constraints (Sect. [5.2\)](#page-1-0) and present a simple example of double integrator. In a second part (Sect. [5.3\)](#page-4-0), we

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N. Petit (ed.), *Feedback Stabilization of Controlled Dynamical Systems*,

review their extension to mixed constraints (see [\[7\]](#page-16-8)) and show, on the double integrator example, how mixed constraints may modify the previously presented behavior. Then another simple example of a spring system is presented in two versions with different mixed constraints and again, we compare their consequences on the respective solutions.

5.2 Recalls on Pure State and Input Constrained Systems

The material of this section is a summary of [\[6](#page-16-7)]. We consider the constrained nonlinear system $\lim_{x \to a} \int f(x, u)$, (5.1)
 \therefore *x* = *f*(*x*, *u*)*,* (5.1) *x*(*t*₀) = *x*₀, $x(t_0) = x_0$, (5.2)
 x(*t*₀) = *x*₀, (5.2)

$$
\dot{x} = f(x, u),\tag{5.1}
$$

$$
x(t_0) = x_0,\t(5.2)
$$

$$
x = f(x, u),
$$

\n
$$
x(t_0) = x_0,
$$

\n
$$
u \in U,
$$

\n(5.1)
\n(5.2)
\n(5.3)

$$
\begin{aligned}\n\dot{x} &= f(x, u), & (5.1) \\
x(t_0) &= x_0, & (5.2) \\
u &\in U, & (5.3)\n\end{aligned}
$$
\n
$$
g_i(x(t)) \le 0 \quad \forall t \in [t_0, \infty), \quad \forall i \in \{1, ..., p\} \tag{5.4}
$$

 $x(t_0) = x_0,$ (5.2)
 $u \in U,$ (5.3)
 $g_i(x(t)) \le 0 \quad \forall t \in [t_0, \infty), \quad \forall i \in \{1, ..., p\}$ (5.4)

where $x(t) \in \mathbb{R}^n$. *U* is the set of Lebesgue measurable functions from $[t_0, \infty)$ to *U*,

where *II* is a compact convex subset of where *U* is a compact convex subset of ℝ^{*m*}, and not a singleton. (*x*) \mathbb{R}^m , and not a singlet
(*x*) $\leq 0, i = 1, ..., p$

The *constraint set* is defined by

$$
G \triangleq \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, \dots, p\}
$$

The notation $g(x) \triangleq 0$ indicates that there exists an $i \in \{1, \ldots, p\}$ such that *x* satisfies *g* \in { $x \in \mathbb{R}^n$: $g_i(x) \le 0, i = 1, ..., p$ }

The notation $g(x) \triangleq 0$ indicates that there exists an $i \in \{1, ..., p\}$ such that *x* satisfies $g_i(x) = 0$ and $g_j(x) \le 0$ for all $j \in \{1, ..., p\}$, and $\mathbb{I}(x)$ denotes the set of $G \triangleq \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, ..., p\}$

The notation $g(x) \triangleq 0$ indicates that there exists an $i \in \{1, ..., p\}$ such that *x* satisfies
 $g_i(x) = 0$ and $g_j(x) \le 0$ for all $j \in \{1, ..., p\}$, and $\mathbb{I}(x)$ denotes the set of all i (resp. $g_i(x) \le 0$) for all $i \in \{1, ..., p\}$. (*x*) [≤] 0) for all *ⁱ* ∈ {1*,*…*, ^p*}. $\{1, \ldots, p\}$ such that $g_i(x) = 0$. Also, $g(x) < 0$ (resp. $g(x) \le 0$) indicates that $g_i(x) < 0$

The sets

$$
G_0 \triangleq \{x \in \mathbb{R}^n : g(x) \triangleq 0\}, \qquad G_- \triangleq \{x \in \mathbb{R}^n : g(x) < 0\}.\tag{5.5}
$$

are indeed such that $G = G_0 \cup G_$.

We further assume (see $[6]$)

- (*G*₀ \triangleq { $x \in \mathbb{R}^n$: $g(x) \triangleq 0$ }, $G_{-} \triangleq$ { $x \in \mathbb{R}^n$: $g(x) \lt 0$ }.

(A1) *f* is at least *C*² on $\mathbb{R}^n \times \tilde{U}$ where \tilde{U} in an open subset of \mathbb{R}^m , *U* ⊂ \tilde{U} .

(A2) There exists a p
- (A2) There exists a positive and finite constant *C* such that

\n
$$
\mathbb{R}^n \times \tilde{U}
$$
 where \tilde{U} in an open subset of \mathbb{R} is
\n if $X \times \tilde{U}$ where \tilde{U} in an open subset of \mathbb{R} is
\n if U such that $\sup_{u \in U} |x^T f(x, u)| \leq C(1 + \|x\|^2)$, for all x is
\n if U is a
\n if <

- (A3) The set $f(x, U)$, called the *vectogram* in [\[10](#page-16-9)], is convex for all $x \in \mathbb{R}^n$.
- (A4) For each $i = 1, ..., p, g_i$ is an at least C^2 function from \mathbb{R}^n to \mathbb{R} ,
- (A5) the set of points given by $g_i(x) = 0$ defines an $n 1$ dimensional manifold.

In the sequel we will denote by $x^{(u,x_0)}$ the solution of the differential equation [\(5.1\)](#page-1-1) 5 From Pure State and Input Constraints to Mixed Constraints ... 127

In the sequel we will denote by $x^{(u,x_0)}$ the solution of the differential equation

(5.1) with input $u \in U$ and initial condition x_0 , and by We also use the notation x^u and $x^u(t)$ when the initial condition is unambiguous or unimportant.

5.2.1 The Admissible Set

Following $[6]$ $[6]$, we define:

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 Definition 5.1 (*Admissible Set*) We say that the point $\bar{x} \in G$ is *admissible* if, and

only if there exists at least one input function $y \in \mathcal{U}$ such that (5 only if, there exists at least one input function $v \in \mathcal{U}$, such that [\(5.1\)](#page-1-1)–[\(5.4\)](#page-1-1) are sat-Following [6], we define:
Definition 5.1 (*Admissible Set*) We say that the point $\bar{x} \in G$ is *admissible* only if, there exists at least one input function $v \in U$, such that (5.1)–(5.4) isfied for $x_0 = \bar{x}$ and $u = v$ (*Admissible Set*) We say that exists at least one input function \bar{x} and $u = v$. The set of all such $\mathcal{A} \triangleq {\bar{x} \in G : \exists u \in \mathcal{U}, g(x^{(u,\bar{x})})}$ *ooint* \bar{x} ∈ *G* is *admissible* if, and *U*, such that (5.1)–(5.4) are satcalled the *admissible set*:

≤ 0, ∀*t* ∈ [*t*₀, ∞)}. (5.6) External contracts of the integral curve of the integral curve, $x_0 = \bar{x}$ and $u = v$. The set of all such \bar{x} is called the *admis* $A \triangleq {\bar{x} \in G : \exists u \in \mathcal{U}, g(x^{(u,\bar{x})}(t)) \leq 0, \forall t \in [t_0, \infty)}$
Clearly, if \bar{x} is admiss

$$
\mathcal{A} \triangleq \{ \bar{x} \in G : \exists u \in \mathcal{U}, \ g\big(x^{(u,\bar{x})}(t)\big) \le 0, \forall t \in [t_0, \infty) \}.
$$
 (5.6)

Clearly, if \bar{x} is admissible, any point of the integral curve, $x^{(\nu,\bar{x})}(t_1)$, $t_1 \in [t_0, \infty)$, with $v \in U$ as in the above definition, is also an admissible point.

We now recall from [\[6](#page-16-7)] the following results:

Proposition 5.1 *Assume that (A1)–(A4) are valid. The set is closed.*

Denote by ∂A the boundary of the admissible set and define

ume that (AI)–*(A4) are valid. The set A is closed.*
boundary of the admissible set and define

$$
[\partial \mathcal{A}]_0 = \partial \mathcal{A} \cap G_0, \quad [\partial \mathcal{A}]_- = \partial \mathcal{A} \cap G_-.
$$
 (5.7)

We indeed have $\partial A = [\partial A]_0 \cup [\partial A]_-.$

5.2.2 The Barrier

5.2.2 The Barrier
We next consider the subset $[\partial \mathcal{A}]_$ of the boundary of the admissible set. **5.2.2** The Barrier
We next consider the subset [∂A]₋ of the boundary of the admistion 5.2 The set [∂A]₋ is called the *barrier* of the set A. We next consider the subset $[\partial \mathcal{A}]_-$ of the boundary of the admis
Definition 5.2 The set $[\partial \mathcal{A}]_-$ is called the *barrier* of the set \mathcal{A} .
Still following [\[6\]](#page-16-7), $[\partial \mathcal{A}]_-$ is "fibered" by arcs of integral curv

Proposition 5.2 The set $[\partial \mathcal{A}]_-\$ is called the *barrier* of the set \mathcal{A} .
Still following [6], $[\partial \mathcal{A}]_-\$ is "fibered" by arcs of integral curves:
Proposition 5.2 *Assume that (A1)–(A4) hold. The barrier* [*Definition 5.2 The set* $[\partial \mathcal{A}]_-\$ *is called the <i>barrier* of the set \mathcal{A} .
 Still following [6], $[\partial \mathcal{A}]_-\$ is "fibered" by arcs of integral curves:
 Proposition 5.2 *Assume that* $(A1)$ – $(A4)$ *hold. The bar* **i***tail following [6],* $[\partial \mathcal{A}]_-\$ *is "fibered" by arcs of integral curves:
Proposition 5.2 <i>Assume that (A1)–(A4) hold. The barrier* $[\partial \mathcal{A}]_-\$ *is made of* $\bar{x} \in G_$ *for which there exists* $\bar{u} \in \mathcal{U}$ *an* **Proposition 5.2** Assume that (A1)–(A4) hold. The barrier $[\partial \mathcal{A}]_$ is made of points $\bar{x} \in G_$ for which there exists $\bar{u} \in \mathcal{U}$ and an arc of integral curve $x^{(\bar{u}, \bar{x})}$ entirely contained in $[\partial \mathcal{A}]_$ unti

cannot exist a trajectory penetrating the interior of A *, denoted by* $int(A)$ *, before leaving G*−*.*

The intersection of $cl([\partial \mathcal{A}]_+)$, the closure of $[\partial \mathcal{A}]_-,$ with G_0 is remarkable:

Proposition 5.3 (Ultimate Tangentiality Condition [\[6\]](#page-16-7)) *Assume that (A1)–(A5) hold a x and i.* **and i. and** *x c c and a*<sup> \bar{d} *c a*^{\bar{d}} *and consider* $\bar{x} \in [\partial \mathcal{A}]$ _{*-} and* $\bar{u} \in \mathcal{U}$ *as in Proposition [5.2,](#page-2-0) i.e., such that* $x^{(\bar{u},\bar{x})}$ *and consider* $\bar{x} \$ and consider $\bar{x} \in [\partial \mathcal{A}]$ and $\bar{u} \in \mathcal{U}$ as in Proposition 5.2, i.e., such that $x^{(\bar{u},\bar{x})}(t) \in$ The intersection of cl([∂A]₋), the closure of $[\partial A]$ ₋, with G_0 is remarkable:
Proposition 5.3 (Ultimate Tangentiality Condition [6]) Assume that $(A1)$ – $(A5)$ hold
and consider $\bar{x} \in [\partial A]$ ₋ and $\bar{u} \in U$ as *z* = *x*(*ū,̄x*) (*̄t*) ∈ ([]−) ∩ *^G*0 *for some finite time ̄^t* [≥] *^t*0 *such that* $\lceil \partial \mathcal{A} \rceil$ for all t in some time interval until it reaches G_0 . Then, there exists a point

$$
\min_{u \in U} \max_{i \in \mathbb{I}(z)} L_f g_i(z, u) = 0. \tag{5.8}
$$

z = $x^{(u,x)}(t)$ ∈ cl([∂A]_ $)$ ∩ G_0 *for some finite time* $\overline{t} \geq t_0$ *such that*
 $\min_{u \in U} \max_{i \in U} L_f g_i(z, u) = 0.$ (5.8)
 where $L_f g_i(x, u) \triangleq Dg_i(x) . f(x, u)$ *is the Lie derivative of g_i along the vector field f at the point x.* $\lim_{u \in U} \lim_{i \in I(x)} \int_{0}^{x} f(x, u) \, du$
 ere $L_f g_i(x, u) \triangleq Dg_i(x) f(x, u)$ *is the Lie derivative contention.*

Let $H(x, \lambda, u) = \lambda^T f(x, u)$ denote the Hamiltonian.

Theorem 5.1 (Minimum-like principle [\[6](#page-16-7)]) *Under the assumptions of Proposi*at the point x.

Let $H(x, \lambda, u) = \lambda^T f(x, u)$ denote the Hamiltonian.
 Theorem 5.1 (Minimum-like principle [6]) *Under the assumptions of Proposition [5.3,](#page-3-0) every integral curve* $x^{\bar{u}}$ on $[\partial A]_$ \cap cl(int(A)) *and th There* $H(x, \lambda, u) = \lambda^T f(x, u)$ *denote the Hamiltonian.*
eorem 5.1 (Minimum-like principle [6]) *Under the assumptions of Proposi-*
n 5.3, every integral curve $x^{\bar{u}}$ on $[\partial \mathcal{A}]_T \cap$ cl(int(\mathcal{A})) and the corresp *in Proposition 5.2, satisfies th*

equation ū ū

in Proposition 5.2, satisfies the following necessary condition.
\nis a (nonzero) absolutely continuous maximal solution
$$
\lambda^{\bar{u}}
$$
 to the adjoint
\n
$$
\dot{\lambda}^{\bar{u}}(t) = -\left(\frac{\partial f}{\partial x}(x^{\bar{u}}(t), \bar{u}(t))\right)^T \lambda^{\bar{u}}(t), \quad \lambda^{\bar{u}}(\bar{t}) = (Dg_{i^*}(z))^T
$$
\n(5.9)
\nHamiltonian is minimized
\n
$$
\min_{u \in U} \left\{ (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), u) \right\} = (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), \bar{u}(t)) = 0
$$
\n(5.10)

such that the Hamiltonian is minimized

$$
\text{amiltonian is minimized}
$$
\n
$$
\min_{u \in U} \left\{ (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), u) \right\} = (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), \bar{u}(t)) = 0 \tag{5.10}
$$

at every Lebesgue point t of \bar{u} *(i.e., for almost all* $t \leq \bar{t}$ *).*

In $\{ (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), u) \} = (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t), \bar{u}(t)) = 0$ *(5.10)*
 In [\(5.9\)](#page-3-1), t denotes the time at which z is reached, i.e., $x^{\bar{u}}(\bar{t}) = z$, with $z \in G_0$ *sat-*
 ing the ultimate tangentiality conditio isfying the ultimate tangentiality condition (*z*)=0*, ⁱ* ∈ (*z*)*,* min (*z, ^u*) [≜] *Lf gi*[∗] (*^z, ̄^u*(*̄t*)) = 0*.* (5.11)

$$
g_i(z) = 0, \quad i \in \mathbb{I}(z), \quad \min_{u \in U} \max_{i \in \mathbb{I}(z)} L_f g_i(z, u) \triangleq L_f g_{i^*}(z, \bar{u}(\bar{t})) = 0. \tag{5.11}
$$

We illustrate this result by the next particularly simple example (double integrator).

5.2.3 Double Integrator, Pure State Constraint

Let us consider the double integrator subjected to a pure state constraint

integrator, Pure State Constraint
double integrator subjected to a pure state constraint

$$
\dot{x}_1 = x_2
$$
, $\dot{x}_2 = u$, $|u| \le 1$, $x_1 - 1 \le 0$ (5.12)

Fig. 5.1 Admissible set and barrier for system (5.12)

Fig. 5.1 Admissible set and barrier for system (5.12)
The ultimate tangentiality condition reads $\min_{|u| \le 1} Dg(z) \cdot f(z, u) = z_2 = 0$ with $z \triangleq (z_1, z_2) = (x_1^{\bar{u}}(\bar{t}), x_2^{\bar{u}}(\bar{t})) = (1, 0), \bar{t}$ indicating the time of tangen *̇*ads $\min_{|u| \le 1} Dg(z)$
 ing the time of tar
 barrier trajectory.
 λ , $\lambda^{u}(\bar{t}) = (1,0)$

$$
\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda, \quad \lambda^{\bar{u}}(\bar{t}) = (1, 0)
$$

 $\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda, \quad \lambda^{\bar{u}}(\bar{t}) = (1, 0)$

and we deduce $\lambda_1^{\bar{u}}(t) \equiv 1$ and $\lambda_2^{\bar{u}}(t) = -t + \bar{t} > 0$ for all $t \in (-\infty, \bar{t}]$. We find that the

control is given by $\bar{u}(t) = -\text{sign}(\lambda_0(t)) = -1$. Integr $\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda$, $\lambda^{\bar{u}}(\bar{t}) = (1,0)$
and we deduce $\lambda^{\bar{u}}_1(t) \equiv 1$ and $\lambda^{\bar{u}}_2(t) = -t + \bar{t} > 0$ for all $t \in (-\infty, \bar{t}]$. We find that the
control is given by $\bar{u}(t) = -\text{sign}(\lambda_2(t)) \equiv -1$. Integratin gives the parabola-shaped barrier in Fig. [5.1.](#page-4-1)

5.3 Dynamical Control Systems with Mixed Constraints

The material of this section is borrowed from [\[7\]](#page-16-8). We now consider the following constrained nonlinear system: *x*^{*x*} this section is borrowed from [7]. We now consider the following dinear system: $\dot{x} = f(x, u)$, (5.13) *x*(*t*₀) = *x*₀, $x(t_0) = x_0$, $x(t_1) = x_0$, $x(t_$

$$
\dot{x} = f(x, u),\tag{5.13}
$$

$$
\begin{aligned}\n\dot{x} &= f(x, u), & (5.13) \\
x(t_0) &= x_0, & (5.14) \\
u &\in U, & (5.15) \\
g_i(x(t), u(t)) &\le 0 & \text{for } a.e. \ t \in [t_0, \infty) \quad i = 1, \dots, p & (5.16)\n\end{aligned}
$$

linear system:

\n
$$
\begin{aligned}\n\dot{x} &= f(x, u), \\
x(t_0) &= x_0, \\
u &\in \mathcal{U}, \\
g_i(x(t), u(t)) &\le 0 \quad \text{for a.e. } t \in [t_0, \infty) \quad i = 1, \dots, p\n\end{aligned}
$$
\n
$$
(5.15)
$$
\n(5.16)

$$
g_i(x(t), u(t)) \le 0
$$
 for *a.e.* $t \in [t_0, \infty)$ $i = 1, ..., p$ (5.16)

where $x(t) \in \mathbb{R}^n$.

As before, U is the set of Lebesgue measurable functions from $[t_0, \infty)$ to U , with *U* a given compact convex subset of ℝ*^m*, expressible as

W.

$$
U \triangleq \{u \in \mathbb{R}^m : \gamma_j(u) \le 0, j = 1, \dots, r\}
$$

with $r \ge m$, the functions γ_i being convex and of class C^2 .

Let us stress that the constraints [\(5.16\)](#page-4-2), called *mixed constraints* [\[3](#page-16-10), [9\]](#page-16-11), depend both on the state and the control. We denote by $g(x, u)$ the vector-valued function with $r \ge m$, the functions γ_j being convex and of class C^2 .
Let us stress that the constraints (5.16), called *mixed constraints* [3, 9], depend
both on the state and the control. We denote by $g(x, u)$ the vector-val mean $g_i(x, u) < 0$ (resp. $g_i(x, u) \le 0$) for all *i* and by $g(x, u) \le 0$, we mean $g_i(x, u) = 0$
for at least one *i*.
We define the following sets:
 $G \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) \le 0\}$ (5.17) \times *m*, the functions γ_j being convex and of class C^2 .

is stress that the constraints (5.16), called *mixed constraints* [3, 9], depend

the state and the control. We denote by $g(x, u)$ the vector-valued function
 for at least one *i*.

We define the following sets:

g sets:
\n
$$
G \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) \le 0\}
$$
\n
$$
\geq \{x \in G : \min_{u \in U} \max_{i \in \{1, ..., p\}} g_i(x, u) = 0\}
$$
\n(5.17)
\n
$$
G_{-} \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) < 0\}
$$
\n(5.19)

$$
G_0 \triangleq \{x \in G : \min_{u \in U} \max_{i \in \{1, ..., p\}} g_i(x, u) = 0\}
$$
(5.18)
\n
$$
G_- \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) \le 0\}
$$
(5.19)
\n
$$
U(x) \triangleq \{u \in U : g(x, u) \le 0\} \quad \forall x \in G.
$$
(5.20)

$$
G_{-} \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) < 0\} \tag{5.19}
$$

$$
U(x) \triangleq \{u \in U : g(x, u) \le 0\} \quad \forall x \in G. \tag{5.20}
$$

G₋ $\triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) < 0\}$ (5.19)
 $U(x) \triangleq \{u \in U : g(x, u) \le 0\} \quad \forall x \in G.$ (5.20)

Given a pair $(x, u) \in \mathbb{R}^n \times U$, we denote by $\mathbb{I}(x, u)$ the set of indices, possibly

nty corresponding to the "active" m empty, corresponding to the "active" mixed constraints, namely $(x, u) \in \mathbb{R}^n \times U$, we denote by $\mathbb{I}(x, u)$ the set of in onding to the "active" mixed constraints, namely $\mathbb{I}(x, u) = \{i_1, \dots, i_{s_1}\} \triangleq \{i \in \{1, \dots, p\} : g_i(x, u) = 0\}$

$$
\mathbb{I}(x, u) = \{i_1, \dots, i_{s_1}\} \triangleq \{i \in \{1, \dots, p\} : g_i(x, u) = 0\}
$$

and by $J(u)$ the set of indices, possibly empty, corresponding to the "active" input constraints: constraints: $(x, u) = \{i_1, ..., i_{s_1}\} \triangleq \{i \in \{1, ..., p\} : g_i(x, u) =$
 $\exists (u) = \{j_1, ..., j_{s_2}\} \triangleq \{j \in \{1, ..., r\} : \gamma_j(u) = 0\}.$ and by $\mathbb{J}(u)$ the set of indices, possibly empty, corresponding to the "active" input
constraints:
 $\mathbb{J}(u) = \{j_1, \dots, j_{s_2}\} \triangleq \{j \in \{1, \dots r\} : \gamma_j(u) = 0\}.$
The integer $s_1 \triangleq \#(\mathbb{J}(x, u)) \leq p$ (resp. $s_2 \triangleq \#(\mathbb{J}(u)) \$

$$
\mathbb{J}(u) = \{j_1, \dots, j_{s_2}\} \triangleq \{j \in \{1, \dots r\} : \gamma_j(u) = 0\}.
$$

constraints:
 $\mathbb{J}(u) = \{j_1, \dots, j_{s_2}\} \triangleq \{j \in \{1, \dots r\} : \gamma_j(u) = 0\}.$

The integer $s_1 \triangleq \#(\mathbb{I}(x, u)) \leq p$ (resp. $s_2 \triangleq \#(\mathbb{J}(u)) \leq r$) is the number of elements of $\mathbb{I}(x, u)$ (resp. of $\mathbb{J}(u)$). Thus, $s_1 + s_2$ r $\mathbb{J}(u) = \{j_1, \dots, j_{s_2}\}\$

The integer $s_1 \triangleq \#(\mathbb{I}(x, u)) \leq p$ (resp.
 $\mathbb{I}(x, u)$ (resp. of $\mathbb{J}(u)$). Thus, $s_1 + s_2$ is

among the $p + r$ constraints, at (x, u) .

In addition to $(\mathbb{A}1)$ – $(\mathbb{A}4)$ of the pri-The integer $s_1 \triangleq \#(\mathbb{I}(x, u)) \leq p$ (resp. $s_2 \triangleq \#(\mathbb{J}(u)) \leq r$) is the number of elen $\mathbb{I}(x, u)$ (resp. of $\mathbb{J}(u)$). Thus, $s_1 + s_2$ represents the number of "active" consamong the $p + r$ constraints, at (x, u) .
I

In addition to $(A1)$ – $(A4)$ of the previous section, we assume

(A7) The (row) vectors $(a4)$, p , th
rs
 $\int \partial g_i$

(A4) of the previous section, we assume
\n
$$
p
$$
, the mapping $u \mapsto g_i(x, u)$ is convex for all $x \in \mathbb{R}^n$.
\n
$$
\left\{\frac{\partial g_i}{\partial u}(x, u), \frac{\partial \gamma_j}{\partial u}(u) : i \in \mathbb{I}(x, u), j \in \mathbb{J}(u)\right\}
$$
\n(5.21)

are linearly independent at every $(x, u) \in \mathbb{R}^n \times U$ for which $\mathbb{I}(x, u)$ or $\mathbb{J}(u)$ is non empty.¹ We say, in this case, that the point *x* is *regular* with respect to *u* (see e.g., [\[9,](#page-16-11) [15\]](#page-16-12)). are linearly independent at every $(x, u) \in \mathbb{R}^n \times U$ for which $\mathbb{I}(x, u)$ or $\mathbb{J}(u)$ is
non empty.¹ We say, in this case, that the point *x* is *regular* with respect to *u*
(see e.g., [9, 15]).
Given $u \in \mathcal{U}$,

regular if, and only if, at each *Lebesgue* point, or shortly L-point, *t* of *u*, $x^u(t)$ is that an integral curve x^u of Eq. (5.13) denote that each *Lebesgue* point, or shortly L-poi
 $\frac{1}{1} + s_2 \le m$, with $s_1 = #(\mathbb{I}(x, u))$ and $s_2 = #(\mathbb{I}(u))$.

¹Note that this implies that $s_1 + s_2 \le m$, with $s_1 = #(\mathbb{I}(x, u))$ and $s_2 = #(\mathbb{J}(u))$.

regular in the aforementioned sense w.r.t. $u(t)$, and, if t is a point of discontinuity of *u*, $x^u(t)$ is regular in the aforementioned sense w.r.t. $u(t_$) and $u(t_$ _⊥), with $u(t_$ ^{$) \triangleq$} f From Pure State and Input Constraints to Mixed Constraints ... 131
regular in the aforementioned sense w.r.t. *u*(*t*), and, if *t* is a point of discontinuity
of *u*, *x^u*(*t*) is regular in the aforementioned sens ℝ. clear from the context, " $\forall t$ " or "for *a.e* t^{*i*}" will mean " $\forall t \in [0, \infty)$ " or "for *a.e.*

the context, " $\forall t$ " or "for *a.e.* t^{*i*}" will mean " $\forall t \in [0, \infty)$ " or "for *a.e.*
 $\forall t \in [0, \infty)$ " Note that throughout

Since system [\(5.13\)](#page-4-3) is time-invariant, the initial time t_0 may be taken as 0. When
ar from the context " $\forall t$ " or "for *a.e.t*" will mean " $\forall t \in [0, \infty]$ " or "for *a.e.* $\lim_{\tau \nearrow t, t \notin I_0} u(\tau)$ and $u(t_+) \triangleq \lim_{\tau \searrow t, t \notin I_0} u(\tau)$, I_0 being a suitable zero measure set of \mathbb{R} .

Since system (5.13) is time-invariant, the initial time t_0 may be taken as 0. When

clear from the cont Lebesgue measure.

5.3.1 The Admissible Set in the Mixed Case: Topological Properties

Definition 5.1 (*Admissible States, Mixed Case*) We say that the point $\bar{x} \in G$ is *admissible* if, and only if, there exists $v \in U$, such that [\(5.13\)](#page-4-3)–[\(5.16\)](#page-4-2) are satisfied for $x - \overline{x}$ and $u - v$. for $x_0 = \bar{x}$ and $u = v$: **Properties**
 nition 5.1 (Adm
 issible if, and or
 $v_0 = \bar{x}$ and $u = v$: **1** (*Admissible States, Mixed Case)* We say that the point $\bar{x} \in G$ is and only if, there exists $v \in U$, such that (5.13)–(5.16) are satisfied $1u = v$:
 $\mathcal{A} \triangleq {\bar{x} \in G : \exists u \in U, g(x^{(u,\bar{x})}(t), u(t)) \leq 0, \text{for } a.e. t}.$ (5.22)

$$
\mathcal{A} \triangleq \{ \bar{x} \in G : \exists u \in \mathcal{U}, \ g(x^{(u,\bar{x})}(t), u(t)) \le 0, \text{for a.e. } t \}. \tag{5.22}
$$

for $x_0 = \bar{x}$ and $u = v$:
 $A \triangleq {\bar{x} \in G : \exists u \in V, g(x^{(u,\bar{x})}(t), u(t)) \le 0, \text{ for a.e. } t}.$ (5.22)

As before, any point of the integral curve, $x^{(v,\bar{x})}(t')$, $t' \in [0, \infty)$, is also an admissible

noint point.

We assume that both A and A^C contain at least one element to discard the trivial cases $A = \emptyset$ and $A^C = \emptyset$.

We use the notations $int(S)$ (resp. $cl(S)$) (resp. $co(S)$) for the interior (resp. the closure) (resp. the closed and convex hull) of a set *S*.

Proposition 5.4 *Assume that (A1)–(A5) are valid. The set is closed.*

5.3.2 Boundary of the Admissible Set (Mixed Case)

5.3.2.1 Geometric Description of the Barrier

As before, we define the barrier as $[\partial \mathcal{A}]_-=\partial \mathcal{A}\cap G_-\mathcal{A}$.

5.3.2.1 Geometric Description of the Barrier

As before, we define the barrier as $[\partial \mathcal{A}]_-=\partial \mathcal{A}\cap G_-.$
Proposition 5.5 *Assume (A1)–(A4) and (A6) hold.* $[\partial \mathcal{A}]_-.$ *is made of points* $\bar{x} \in$ **5.3.2.1 Geometric Description of the Barrier**
 As before, we define the barrier as $[\partial \mathcal{A}]_ = \partial \mathcal{A} \cap G_-.$
 Proposition 5.5 *Assume (A1)–(A4) and (A6) hold.* $[\partial \mathcal{A}]_$ *is made of points* $\bar{x} \in G_$ *for which t in* []− *until it intersects G*0*, i.e., at a point z* ⁼ *^x*(*ū,̄x*) (*̃t*)*, for some ̃t, such that* As before, we define the barric
Proposition 5.5 Assume (A1)
 G_{-} for which there exists $\bar{u} \in$
in $[\partial \mathcal{A}]_{-}$ until it intersects G
min_{u∈*U*} max_{i=1,...,*p*} $g_i(z, u) = 0$.

The "fibered" nature of the barrier thus extends to the mixed case. Note however that G_0 is now modified: it is not defined as the set of *x* for which there exists $u \in U$
such that $g(x, u) \triangleq 0$ but is given by (5.18). Note that $\tilde{\tau}$ may be infinite in which case in $[\partial \mathcal{A}]_-$ *until it intersects* G_0 *, i.e., at a point* $z = x^{(\bar{u}, \bar{x})}(\tilde{t})$ *, for some* \tilde{t} *, such that* $\min_{u \in U} \max_{i=1,...,p} g_i(z, u) = 0$.
The "fibered" nature of the barrier thus extends to the mixed case. No

the barrier does not intersect G_0 as shown in the next double integrator with mixed constraint example constraint example.

Corollary 5.2 (Semi-permeability) *Assume (A1)–(A4) and (A6) hold. Then from and the barrier does not intersect* G_0 as shown in the next double integrator with mixed constraint example.
 Corollary 5.2 (Semi-permeability) *Assume* (*A1*)–(*A4*) and (*A6*) *hold. Then from* any point on the bo *interior of before leaving G*−*.*

5.3.2.2 Ultimate Tangentiality

5.3.2.2 Ultimate Tangentiality
We now characterize the intersection of $[\partial \mathcal{A}]_$ with G_0 at the point *z* defined in
Proposition 5.5 We define Proposition [5.5.](#page-6-0) We define tersection of $[\partial \mathcal{A}]_$ with G_0 at the point *z* defined in
 $\tilde{g}(x) \triangleq \min_{u \in U} \max_{i \in \{1, ..., p\}} g_i(x, u).$ (5.23)

$$
\tilde{g}(x) \triangleq \min_{u \in U} \max_{i \in \{1, \dots, p\}} g_i(x, u). \tag{5.23}
$$

Proposition 5.5. We define
 $\tilde{g}(x) \triangleq \min_{u \in U} \max_{i \in \{1, ..., p\}} g_i(x, u).$ (5.23)

Comparing to [\(5.18\)](#page-5-1) we readily see that $G_0 = \{x \in G : \tilde{g}(x) = 0\}$. According to a

result of Danskin [51] \tilde{g} is locally Linschitz and th result of Danskin [\[5\]](#page-16-13), *^g̃* is locally Lipschitz and thus absolutely continuous and almost everywhere differentiable, on every open and bounded subset of ℝ*ⁿ*.

We now recall basic notions from nonsmooth analysis [\[4](#page-16-14)] that are used in the *s*t proposition. Consider $h : \mathbb{R}^n \to \mathbb{R}$ Lipschitz near a given point $x \in \mathbb{R}^n$. The *eralized directional derivative* of *h* at *x* next proposition. Consider $h : \mathbb{R}^n \to \mathbb{R}$ Lipschitz near a given point $x \in \mathbb{R}^n$. The

generalized directional derivative of *h* at *x* in the direction *v* is defined as follows:
\n
$$
h^{0}(x; v) \triangleq \lim_{y \to x, t \to 0^{+}} \frac{h(y + tv) - h(y)}{t}.
$$
\n(5.24)

We also need to introduce the *generalized gradient* of *h* at *x*, labeled $\partial h(x)$. It is

We also need to introduce the *generalized gradient* of *h* at *x*, labeled $\partial h(x)$. It is

IL-known that the generalized gradien well-known that the generalized gradient of a locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$
is the compact and convex set is the compact and convex set \overline{a} *alized gradient* of *h* at *x*, labeled $\partial h(x)$. It is
 x t of a locally Lipschitz function *h* : ℝ^{*n*} → ℝ
 \vdots ∴ $x_i \rightarrow x, x_i \notin \Omega_1 \cup \Omega_2$ (5.25)

$$
\partial h(x) = \text{co}\{\lim_{i \to \infty} Dh(x_i) : x_i \to x, x_i \notin \Omega_1 \cup \Omega_2\}
$$
(5.25)

is the compact and convex set
 $\partial h(x) = \text{co}\{\lim_{i \to \infty} Dh(x_i) : x_i \to x, x_i \notin \Omega_1 \cup \Omega_2\}$ (5.25)

where $Dh(x)$ denotes the row vector $Dh(x)$ at *x*, Ω_1 is a zero measure set where *h* is

nondifferentiable and Ω_2 is an ar nondifferentiable and Ω_2 is an arbitrary zero measure set. Q_1 is a zero measure set where *h* is
asure set.
ectional derivative and the general-
 ζv . (5.26)

The relationship between the generalized directional derivative and the general-
d gradient is given by
 $h^0(x; v) = \max_{\xi \in \partial h(x)} \xi v.$ (5.26) ized gradient is given by

$$
h^{0}(x; v) = \max_{\xi \in \partial h(x)} \xi v.
$$
 (5.26)

Proposition 5.6 (Ultimate Generalized Tangentiality Condition [\[7\]](#page-16-8)) *Assume* $(A1)$ – $(A4)$ and $(A6)$ – $(A7)$ hold. Consider \bar{x} ∈ $[\partial \mathcal{A}]$ _– and \bar{u} ∈ \hat{U} as in Proposi*tion* $h^0(x; v) = \max_{\xi \in \partial h(x)} \xi v.$ (5.26)
 Proposition 5.6 (Ultimate Generalized Tangentiality Condition [7]) Assume

(*A1*)–(*A4*) and (*A6*)–(*A7*) hold. Consider $\bar{x} \in [\partial \mathcal{A}]_-\text{ and } \bar{u} \in \mathcal{U}$ as in Propositio *some time interval until it reaches G*ed Tangentiality Condition [7]) Assume

ider $\bar{x} \in [\partial A]_+$ and $\bar{u} \in U$ as in Proposi-
 curve $x^{(\bar{u},\bar{x})}(t)$ remains in $[\partial A]_+$ for all t in
 *i*₀ at some finite time $\bar{t} \ge 0$. Then, the point *Proposition 5.6 (Ultimate Generaliz (<i>A1*)–(*A4*) and (*A6*)–(*A7*) hold. Constion 5.5, i.e., such that the integral α some time interval until it reaches G $z = x^{(\bar{u}, \bar{x})}(\bar{t}) \in \text{cl}([\partial \mathcal{A}]_{-}) \cap G_0$, satisfies *e*, such that the integrificity e , such that the integrificity \in cl($[\partial \mathcal{A}]$) ∩ G_0 , satis $max_{\xi \in \partial \tilde{g}(z)} \xi f(z, \bar{u}(\bar{t})) = min_{v \in U(z)}$ *f*(*z) f*(*z) f*(*z) f*(*z) f*(*z*^{*i*}) *f*(*z*^{*i*})*x*₁*j*(*z*^{*i*})*x*₃*j*(*f*) *femains in* [*dA*]_{*-*} *for all <i>t in in fal*]_{*j*}(*f*(*z*_{*i*})</sub>*f*(*z*_{*i*})*i*) \cap *G*₀*, satisf*

$$
i\bar{x}(\bar{t}) = \text{cl}([\partial \mathcal{A}]_-) \cap G_0, \text{ satisfies}
$$
\n
$$
0 = \max_{\xi \in \partial \bar{g}(\xi)} \xi f(z, \bar{u}(\bar{t})) = \min_{v \in U(\xi)} \max_{\xi \in \partial \bar{g}(\xi)} \xi f(z, v) = \max_{\xi \in \partial \bar{g}(\xi)} \min_{v \in U(\xi)} \xi f(z, v). \tag{5.27}
$$

Moreover, if the function g is differentiable at z, then ̃ [\(5.27\)](#page-7-0) *reduces to*

t Constraints to Mixed Constraints ...
\n
$$
\tilde{g}
$$
 is differentiable at z, then (5.27) reduces to
\n
$$
0 = L_f \tilde{g}(z, \bar{u}(\bar{t})) = \min_{u \in U(z)} L_f \tilde{g}(z, u).
$$
\n(5.28)

Remark 5.1 Note that [\(5.28\)](#page-8-0) significantly differs from [\(5.8\)](#page-3-3) on several aspects: in [\(5.28\)](#page-8-0), $U(z)$ replaces U, where *z* is such that $\tilde{g}(z) = 0$; moreover, in (5.28), if g_i effectively depends on *u* for *i* ∈ $\iint_{\mathcal{E}} \tilde{g}(z, u)$ is not generally differs from (5.8) on several aspects: in (5.28), *U*(*z*) replaces *U*, where *z* is such that $\tilde{g}(z) = 0$; moreover, in (5.28), if g_i effective

5.3.3 The Barrier Equation (Mixed Case)

The next necessary conditions are essential to construct the integral curves running along the barrier.

Theorem 5.2 (Minimum-like Principle (Mixed Case) [\[7](#page-16-8)]) *Under the assumptions* The next necessary conditions are essential to construct the integral curves running
along the barrier.
Theorem 5.2 (Minimum-like Principle (Mixed Case) [7]) *Under the assumptions*
of Proposition [5.6,](#page-7-1) consider an integ *ing necessary conditions. ̇***eorem 5.2** (Minimum-like Principle (Mixed Case) [7]) *Under the assumptions Proposition 5.6, consider an integral curve* $x^{\bar{u}}$ *on* $[\partial A]_ \cap$ \cap $C(\text{int}(A))$ *and assume t the control function* \bar{u} *is piecewis multipliers* \overline{a} *i* \overline{b} *i f proposition* 5.6, *consider an integral c* that the control function \overline{u} is piecewise coing necessary conditions.
There exists a nonzero absolutely continuality *pli* \geq that the control function \bar{u} is piecewise continuous. Then \bar{u} and $x^{\bar{u}}$ satisfy the follow-

 ū ū ū

\n
$$
\text{ssary conditions.}
$$
\n

\n\n $\text{exists a nonzero absolutely continuous adjoint } \lambda^{\bar{u}} \text{ and piecewise continuous}$ \n

\n\n $\text{arg } \mu_i^{\bar{u}} \geq 0, \, i = 1, \ldots, p, \text{ such that}$ \n

\n\n $\dot{\lambda}^{\bar{u}}(t) = -\left(\frac{\partial f}{\partial x}(x^{\bar{u}}(t), \bar{u}(t)) \right)^T \lambda^{\bar{u}}(t) - \sum_{i=1}^p \mu_i^{\bar{u}}(t) \frac{\partial g_i}{\partial x}(x^{\bar{u}}(t), \bar{u}(t))$ \n

\n\n "complementary slackness condition"\n

\n\n $\mu_i^{\bar{u}}(t)g_i(x^{\bar{u}}(t), \bar{u}(t)) = 0, \quad i = 1, \ldots, p$ \n

\n\n (5.30)\n

with the "complementary slackness condition" .ū

$$
\text{try slackness condition} \tag{5.30}
$$
\n
$$
\mu_i^{\bar{u}}(t)g_i(x^{\bar{u}}(t), \bar{u}(t)) = 0, \quad i = 1, \dots, p \tag{5.30}
$$
\n
$$
\lambda^{\bar{u}}(\bar{t})^T \in \text{arg max} \, \xi \, f(\bar{z}, \bar{u}(\bar{t})) \tag{5.31}
$$

and final conditions

$$
t)g_i(x^{\bar{u}}(t), \bar{u}(t)) = 0, \quad i = 1, \dots, p
$$
\n
$$
\lambda^{\bar{u}}(\bar{t})^T \in \arg\max_{\xi \in \partial \bar{g}(z)} \xi f(z, \bar{u}(\bar{t}))
$$
\n(5.31)

and final conditions
 $\lambda^{\bar{u}}(\bar{t})^T \in \arg \max_{\xi \in \partial \bar{g}(z)} \xi f(z, \bar{u}(\bar{t}))$ (5.31)

where $z = x^{\bar{u}}(\bar{t})$ with \bar{t} such that $z \in G_0$, i.e., $\min_{u \in U} \max_{i=1,...,p} g_i(z, u) = 0$, $\partial \tilde{g}(z)$

being the generalized gradient such that $z \in G_0$, *i.*
 H(*x*^{*ū*}(*t*)*, tn*^{*i*}(*t*)*, u*) = (
 H(λ ^{*ū*}(*t*)*, x*^{*ū*}(*t*)*, u*) = (*e.*, $\min_{u \in U} \max_{i=1}$
 by (5.23), at z.
 nian
 $\lambda^{\bar{u}}(t)$ ^T $f(x^{\bar{u}}(t), u)$ *ū being the generalized gradient of* \tilde{g} *, defined by (5.23), at z.*

Moreover, at almost every t, the Hamiltonian

$$
H(\lambda^{\bar{u}}(t), x^{\bar{u}}(t), u) = (\lambda^{\bar{u}}(t))^{T} f(x^{\bar{u}}(t), u)
$$

$$
H(\lambda^{\bar{u}}(t), x^{\bar{u}}(t), u) = (\lambda^{\bar{u}}(t))^{T} f(x^{\bar{u}}(t), u)
$$

is minimized over the set $U(x^{\bar{u}}(t))$ and equals zero

$$
\min_{u \in U(x^{\bar{u}}(t))} \lambda^{\bar{u}}(t)^{T} f(x^{\bar{u}}(t), u) = \min_{u \in U} \left[(\lambda^{\bar{u}}(t))^{T} f(x^{\bar{u}}(t), u) + \sum_{i=1}^{p} \mu_{i}^{\bar{u}}(t) g_{i}(x^{\bar{u}}(t), u) \right]
$$

$$
= \lambda^{\bar{u}}(t)^{T} f(x^{\bar{u}}(t), \bar{u}(t)) = 0
$$
(5.32)

Remark 5.2 If *^g̃* is differentiable at the point *^z*, condition [\(5.31\)](#page-8-1) indeed reduces to its smooth counterpart, i.e., $\lambda^{\bar{u}}(\bar{t})^T = D\tilde{g}(z)$ (*̄t*) its smooth counterpart, i.e., $\lambda^{\bar{u}}(\bar{t})^T = D\tilde{g}(z)$

Remark 5.3 The assumption that $x^{(\bar{u},\bar{x})} \in [\partial \mathcal{A}]$ \cap cl(int(\mathcal{A})) means that we possibly miss isolated trajectories which are in $\mathcal{A} \setminus cl(int(\mathcal{A}))$. The existence and computation of such trajectories, if they exist, are open questions.

5.4 Examples

5.4.1 Double Integrator, Mixed Constraint

Let us go back to the double integrator introduced in Sect. [5.2.3,](#page-3-4) the pure state constraint *x*₁ \leq 1 being now replaced by the mixed constraint *x*₁ \leq *u ẋ* double integrator introduced in Sect. 5.2.3, the pure state con-

ow replaced by the mixed constraint $x_1 \le u$
 $x_1 = x_2$, $\dot{x}_2 = u$, $|u| \le 1$, $x_1 - u \le 0$ (5.33)

$$
\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \le 1, \quad x_1 - u \le 0 \tag{5.33}
$$

We will show that this apparently innocuous change dramatically modifies the admissible set and its barrier since, in the mixed case, the latter does not intersect G_0 any-
more (compare Figs. 5.1 and 5.2) more (compare Figs. [5.1](#page-4-1) and [5.2\)](#page-11-0). will show that this apparently innocuous change dramatically modifies the admis-
le set and its barrier since, in the mixed case, the latter does not intersect G_0 any-
re (compare Figs. 5.1 and 5.2).
We readily get \til We will show that this apparently innocuous change dramatically modifies the admissible set and its barrier since, in the mixed case, the latter does not intersect G_0 any-
more (compare Figs. 5.1 and 5.2).
We readily g

that, at this point, $U(z) = \{1\}$ is reduced to a single element. The minimal Hamiltonian is given by We readily get $\tilde{g}(x) = x_1 - 1$ and $G_0 = \{(x_1, x_2) : x_1 = 1\}$. The ultimate tangen-Ind $G_0 = \{(x_1, x_2) : x_1 : x_2 \}$
 $\delta \tilde{g}(z) f(z, u) = z_2 = 0, \text{ or}$

Educed to a single element
 $x_1x_2 + \lambda_2 u = 0, \text{ a.e. } t.$

$$
\min_{u \in U(x)} \lambda_1 x_2 + \lambda_2 u = 0, \quad \text{a.e. } t.
$$

Thus:

$$
\min_{u \in U(x)} \lambda_1 x_2 + \lambda_2 u = 0, \quad \text{a.e. } t.
$$

if $\lambda_2(t) < 0$, $\bar{u}(t) = 1$ if $x_1 \in]\infty, 1]$
if $\lambda_2(t) > 0$, $\bar{u}(t) = \begin{cases} x_1 \text{ if } x_1 \in [-1, 1] \\ -1 \text{ if } x_1 \in]-\infty, -1[\end{cases}$
if $\lambda_2(t) = 0$, $\bar{u}(t) = \text{arbitrary}$.

The costate equations are given by

$$
\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda
$$

with $\lambda(\bar{t}) = D\tilde{g}(z) = (1,0)^T$. From here we deduce that $\lambda_2(t) = -t + \bar{t}$ for all $t \in$ $\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda$
with $\lambda(\bar{t}) = D\tilde{g}(z) = (1,0)^T$. From here we deduce that $\lambda_2(t) = -t + \bar{t}$ for all $t \in (-\infty, \bar{t}]$, and thus $\lambda_2(t) > 0$ for all $t \in (-\infty, \bar{t}]$. Integrating backwards from the point z $\lambda = \begin{pmatrix} 0 & \lambda \\ -1 & 0 \end{pmatrix}$
 z = (1,0)^T. From here we deduce that $\lambda_2(t) = -t + \overline{t}$ for all $t \in (-\infty, \overline{t}]$, and thus $\lambda_2(t) > 0$ for all $t \in (-\infty, \overline{t}]$. Integrating backwards from the point $z = (1, 0)$, we find t cannot be part of the barrier.

However, let us show that the barrier indeed exists and that it remains in *G*− for all time. From Pure State and Input Constraints to Mixed Constraints ...

However, let us show that the barrier indeed exists and that it remains in G_{-} for

time.

When the control $\hat{u}(t) = x_1(t)$ is applied to [\(5.33\)](#page-9-0), the anal

However, let us show that the barrier
all time.
When the control $\hat{u}(t) = x_1(t)$ is appl
at $t = 0$ from $x_0 \triangleq (x_{1,0}, x_{2,0})$ is given by

$$
\hat{u}(t) = x_1(t) \text{ is applied to (5.33), the anal:}
$$
\n
$$
x_{1,0}^{(\hat{u},x_0)}(t) = \frac{x_{1,0} + x_{2,0}}{2} e^t + \frac{x_{1,0} - x_{2,0}}{2} e^{-t}
$$
\n
$$
x_2^{(\hat{u},x_0)}(t) = \frac{x_{1,0} + x_{2,0}}{2} e^t - \frac{x_{1,0} - x_{2,0}}{2} e^{-t}.
$$

It is thus immediately seen that, with this control, the origin is a saddle point, the line $x_1 + x_2 = 0$ being the associated stable manifold, and $x_1 - x_2 = 0$ the unstable one. Solution that the line segment \mathcal{L} △ ² ² ² ² ² ² ² *x***₁ +** *x***₂ = 0***,* **being the associated stable manifold, and** *x***₁ −** *x***₂ = 0***,* **the unstable as

We now prove that the line segment** \mathcal{L} It is thus immedia

line $x_1 + x_2 = 0$ be

one.

We now prove to a subset of $[\partial \mathcal{A}]_-.$

Clearly \mathcal{L} is no

Clearly, $\mathcal L$ is positively invariant and every integral curve starting on it asymptotione.
We now prove that the line segment $\mathcal{L} \triangleq \{(x_1, x_2) : x_1$ -
a subset of $[\partial \mathcal{A}]_$.
Clearly, \mathcal{L} is positively invariant and every integral curv
cally approaches the origin. Moreover, $g(x^{(\hat{u},x_0)}(t), \hat{u}(t))$ $+x_2$
 e sta
 $\frac{(\hat{u},x_0)}{1}$ 1 (*t*) λ *u*^{*i*} λ *ui* λ *ui* λ such that $-1 \le x_1^{(\hat{u},x_0)}(t) < 1$. Let $h(x) \triangleq x_1 + x_2$ and denote $x_i(t) \triangleq x_i^{(\hat{u})}$ for simplicity's sake. If at a suitable time t_1 , the state satisfies $x(t_1)$ $h(x(t_1)) = 0$, using any other admissible control $v > \hat{$ External the line segment $\mathcal{L} \triangleq \{(x_1, x_2) : x_1 + x_2\}$
 \rightarrow
 (*u*^{*i*}) $\hat{u}(t)$
(*u*^{*i*},*x*₀) $-1 \le x_1 < 1$ is

on it asymptoti-
 $\hat{u}(t) = 0$ for all t
 $\hat{u}^{(\hat{u},x_0)}(t), i = 1, 2,$
 $\hat{i} \in \mathcal{L}$, i.e. with for simplicity's sake. If at a suitable time t_1 , the state satisfies $x(t_1) \in \mathcal{L}$, i.e. with $h(x(t_1)) = 0$ using any other admissible control $y > \hat{y}(t_1) = x_1(t_1)$ with $|y| < 1$ we *Clearly, <i>£* is positively invariant and every integral curve starting on it asymptotically approaches the origin. Moreover, $g(x^{(\hat{u},x_0)}(t), \hat{u}(t)) = x_1^{(\hat{u},x_0)}(t) - \hat{u}(t) = 0$ for all *t* such that $-1 \le x_1^{(\hat{u},x_0$ get

$$
Dh(x(t_1)) \cdot f(x(t_1), v) = x_2(t_1) + v > -x_1(t_1) + x_1(t_1) = 0.
$$

Therefore, any other admissible control results in the state entering the set $B \triangleq$ *h*(*x*(*t*₁)) = *x*₁ (*x*₁) + *x*₂ (*t*₁) + *y* > -*x*₁(*t*₁) + *x*₁(*t*₁) = 0.
get
Bh(*x*(*t*₁)),*f*(*x*(*t*₁),*v*) = *x*₂(*t*₁) + *v* > -*x*₁(*t*₁) + *x*₁(*t*₁) = 0.
H(*x*) = *x*₁ *Dh*(*x*(*t*₁)),*f*(*x*(*t*₁),*v*) = *x*₂(*t*₁) + *v* > −*x*₁(*t*₁) + *x*₁(*t*₁) = 0.

Therefore, any other admissible control results in the state entering the set *B* [≙]

{*h*(*x*) = *x*₁ + *x*₂ > *x*₁ ≤ 1, which implies that all trajectories starting from *B* cross the constraint *x*₁ = 1 and hence are not admissible, i.e., $B \subset A^C$. Moreover, starting from any point in the complement, i.e., such that $x_1 + x_2$ Therefore, any other admissible control results in the state entering the set $B \triangleq \{h(x) = x_1 + x_2 > 0\}$. Moreover, in B , all trajectories are such that h is non-decreasing for all admissible control $v: L_f h(x, v) = x_2 + v > -x$ complement, i.e., such that $x_1 + x_2 \le 0$, denoted by C in Fig. [5.2,](#page-11-0) it is straightforward
to verify that \hat{u} ensures that the corresponding integral curve remains in G for all time decreasing for all admissible control v: $L_f h(x, v) = x_2 + v > -x_1 + x_1 = 0$ as long as $x_1 \le 1$, which implies that all trajectories starting from *B* cross the constraint $x_1 = 1$ and hence are not admissible, i.e., $B \subset A^C$. reover, starting from any point in the
by C in Fig. 5.2, it is straightforward
tegral curve remains in G for all time
 $[\partial \mathcal{A}]_-.$
 $\frac{1}{2} > 1$, by the integral curve starting
ith the control $\bar{v}(t) = -1$ for all $t \in$ backwards from the point $(x_1, x_2) = (-1, 1)$, with the control $\bar{u}(t) \equiv -1$ for all time which proves the assertion that \mathcal{L} is a subset of $[\partial \mathcal{A}]$.
We now prove that the corresponding integral curve remains in G f

We now prove that the barrier extends, for *x*to verify that \hat{u} ensures that the corresponding integral curve remains in *G* for all time
which proves the assertion that \mathcal{L} is a subset of $[\partial \mathcal{A}]_-.$
We now prove that the barrier extends, for $x_2 > 1$, by ich proves the assertion that *L* is a subset of $[\partial A]_$.
We now prove that the barrier extends, for $x_2 > 1$, by the integral curve starting
kwards from the point $(x_1, x_2) = (-1, 1)$, with the control $\bar{u}(t) \equiv -1$ for all Which proves the assertion that Σ is a stocked to $[\partial \mathcal{A}]_+$.
We now prove that the barrier extends, for $x_2 > 1$, by the integral curve starting
backwards from the point $(x_1, x_2) = (-1, 1)$, with the control $\bar{u}(t) \$

backwards from the point $(x_1, x_2) = (-1, 1)$, with the control $\bar{u}(t) \equiv -1$
 $]-\infty, \bar{t}$.

By Theorem 5.2, assuming that \bar{u} is piecewise continuous, any traje

ning along the barrier, generated by \bar{u} , satisfies beherated by *u*, satisfies Eqs. (5.25), (5.36) and (5) state $\lambda^{\bar{u}}$ and piecewise continuous multipliers $\mu^{\bar{u}}$. Extra distribution of the point $(x_1, x_2) = (-1, 1, 1)$, which is continuous, any trajectory run-
By Theorem 5.2, assuming that \bar{u} is piecewise continuous, any trajectory run-
g along the barrier, generated by \bar{u} , By Theorem 5.2, assuming that \bar{u} is piecewise continuous,
ning along the barrier, generated by \bar{u} , satisfies Eqs. (5.29), (5.3
absolutely continuous costate $\lambda^{\bar{u}}$ and piecewise continuous mult
Consider the

set $U(\xi)$ at that point is equal to $[-1, 1]$. By (5.32) we must have *ū T f*^{λ *ū*} and piecewise continuo
L, denoted by ξ , of coordinuo
to [-1, 1]. By (5.32) we must
 $T_f(\xi, u) = \min_{u \in [-1,1]} \lambda_1^{\bar{u}}(t) + \lambda_2^{\bar{u}}$ *ū*

$$
\min_{u \in U(\xi)} \lambda^{\bar{u}}(t)^{T} f(\xi, u) = \min_{u \in [-1, 1]} \lambda_{1}^{\bar{u}}(t) + \lambda_{2}^{\bar{u}}(t)u = 0
$$

and, by continuity of the Hamiltonian on \mathcal{L} , since we had $\bar{u} = x_1$, considering the $\min_{u \in U(\xi)} \lambda^{\bar{u}}(t)^{T} f(\xi, u) = \min_{u \in [-1, 1]} \lambda_{1}^{\bar{u}}(t) + \lambda_{2}^{\bar{u}}(t)u = 0$
and, by continuity of the Hamiltonian on *L*, since we had $\bar{u} = x_{1}$, considering the
limit of the Hamiltonian for $x \to \xi, x \in \mathcal{L}$, we de

Fig. 5.2 Figure showing some of the sets referred to in Sect. [5.4.1,](#page-9-1) along with a curve obtained by the barrier **Fig. 5.2** Figure showing some of the sets referred to in Sect. 5.4.1, along with a curve obtained by backward integration from the point (−1, 1) which we have shown to be the backward extension of the barrier
is orthogo

backward integration from the point $(-1, 1)$ which we have shown to be the backward extension of
the barrier
is orthogonal to the vector $(1, -1)^T$, i.e., $\lambda(\overline{t}) = k(1, 1)^T$, with k a positive constant,
and the minimizing since *x*9 **onal to the vector** $(1, -1)^T$, i.e., $λ(τ) = k(1, 1)^T$, with *k* a positive constant,

a minimizing \bar{u} is thus $\bar{u}(t) = -\text{sign}(\lambda_2(t)) = -1$. Therefore, in $[∂A]_ ∖ Σ$,
 $\frac{1}{1} < -1$, the constraint $x_1 - u$ is nowhe the costate equation reads and the minimizing \bar{u} is thus $\bar{u}(t) = -\text{sign}(\lambda_2(t)) = -1$. Therefore, in $[\partial \mathcal{A}]_{-} \setminus \mathcal{L}$, $sign(\lambda_2(t)) = -1.$
 nowhere active an
 λ , $\lambda(\overline{t}) = k(1, 1)$

$$
\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda, \quad \lambda(\bar{t}) = k(1, 1)
$$

from which we deduce that $\lambda_1(t) \equiv k$ and $\lambda_2(t) = -k(t - \overline{t}) + k$, $t \in (-\infty, \overline{t}]$ and $\overline{n}(t) = -\sin(1-t)$ = -1 . Note that this solution indeed satisfies the piecewise con- $\lambda = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \lambda$, $\lambda(\overline{t}) = k(1, 1)$

from which we deduce that $\lambda_1(t) \equiv k$ and $\lambda_2(t) = -k(t - \overline{t}) + k$, $t \in (-\infty, \overline{t}]$ and $\overline{u}(t) = -\text{sign}(\lambda_2(t)) \equiv -1$. Note that this solution indeed satisfies the piecewis wards as in Fig. [5.2.](#page-11-0) We have also included a few of the vectograms along the extension of the barrier in order to emphasize that this is indeed an "extremal" trajectory $\bar{u}(t) = -\sin(\lambda_2(t)) \equiv -1$. Note that this solution indeed satisfies the piecewise continuous assumption of \bar{u} in Theorem 5.2. The barrier is thus further extended backwards as in Fig. 5.2. We have also included a few o which we have shown to be a subset of A^C .

5.4.2 Constrained Spring I

Consider the following constrained
 $\begin{pmatrix} \dot{x}_1 \\ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \end{pmatrix} \begin{pmatrix} x_1 \\ \end{pmatrix}$

Consider the following constrained mass–spring–damper model:

1 *ẋ*) (0 1 −2 −2) (*^x*¹ *x*) (0) *^u,* [|]*u*[|] [≤] ¹*, ^x*2 [−] *^u* [≤] ⁰

where x_1 is the mass's displacement. The spring stiffness is here equal to 2 for a mass equal to 1 and the friction coefficient is equal to 2 *u* is the force applied to mass equal to 1 and the friction coefficient is equal to 2. μ is the force applied to the mass.

We identify $g(x, u) = x_2 - u$, $U = [-1, 1]$ and $\tilde{g}(x) = x_2 - 1$. We also identify the following sets: $G = \{x \in \mathbb{R}^2 : x_2 \le 1\}$, $G_0 = \{x \in G : x_2 = 1\}$ and $U(x) = \{u \in$
 $U : x_1 \le u \le 1\}$ Note that if $z = \frac{2}{v}$, $z_2 = C_0$, i.e. $z_3 = 1$, then $U(z)$ is the singleton mass equal to 1 and the friction coefficient is equal to 2. *u* is the force applied to the mass.

We identify $g(x, u) = x_2 - u$, $U = [-1, 1]$ and $\tilde{g}(x) = x_2 - 1$. We also identify the following sets: $G = \{x \in \mathbb{R}^2 : x_2 \le 1$ $U(z) = \{1\}.$ We identify $g(x, u) = x_2 - u$, $U = [-1, 1]$ and $\tilde{g}(x) = x_2 - 1$. We also identify the
lowing sets: $G = \{x \in \mathbb{R}^2 : x_2 \le 1\}$, $G_0 = \{x \in G : x_2 = 1\}$ and $U(x) = \{u \in$
 $\therefore x_2 \le u \le 1\}$. Note that if $z \triangleq (z_1, z_2) \in G_0$, i.e. *U* : $x_2 \le u \le 1$. Note that if $z \triangleq (z_1, z_2) \in G_0$, i.e. $z_2 = 1$, then $U(z)$ is the singleton $U(z) = \{1\}$.
We have $\partial \tilde{g}(z) = \{(0, 1)\} = D\tilde{g}(z)$ (\tilde{g} being indeed differentiable everywhere) and

the ultimate tangentiality condition reads

$$
\min_{u \in U(z)} D\tilde{g}(z) f(z, u) = 0
$$

which gives

$$
\min_{u \in U(z)} -2z_1 - 2z_2 + u = -2z_1 - 2 + 1 = 0
$$

Thus $z = (-\frac{1}{2}, 1)$.
The final costa

 \overline{c} ich gives
 $\min_{u \in U(z)} -2z_1 - 2z_2 + u = -2z_1 - 2 + 1 = 0$

us $z = (-\frac{1}{2}, 1)$.

The final costate $\lambda(\bar{t})$, according to [\(5.31\)](#page-8-1), which here reduces to [\(5.28\)](#page-8-0), is given
 $\lambda^T(\bar{t}) - D\bar{\theta}(z) = (0, 1)$ $\min_{u \in U(z)}$
Thus $z = (-\frac{1}{2}, 1)$.
The final costate $\lambda(\overline{t})$, i
by $\lambda^T(\overline{t}) = D\tilde{g}(z) = (0, 1)$.
The Hamiltonian bein If $z = (-\frac{1}{2}, 1)$.

The final costate $\lambda(\bar{t})$, according to (5.31), which here reduces to (5.28), is given $\lambda^T(\bar{t}) = D\tilde{g}(z) = (0, 1)$.

The Hamiltonian being here $H(x, \lambda, u) = \lambda_1 x_2 + \lambda_2(-2x_1 - 2x_2 + u)$, condition

32 *o* (5.31), which here reduces to (5.28), is given

ere $H(x, \lambda, u) = \lambda_1 x_2 + \lambda_2(-2x_1 - 2x_2 + u)$, condition
 $\lambda_1 x_2 + \lambda_2(-2x_1 - 2x_2 + u) = 0$ (5.34)

[\(5.32\)](#page-8-5) reads The Hamiltonian being here $H(x, \lambda, u) = \lambda_1 x_2 + \lambda_2(-2x_1 - 2x_2 + u)$, condition

$$
\min_{x_2 \le u \le 1} \lambda_1 x_2 + \lambda_2 (-2x_1 - 2x_2 + u) = 0
$$
\n(5.34)

which gives the control \bar{u} associated with the barrier

$$
x_2 \le u \le 1
$$
\nwhich gives the control \bar{u} associated with the barrier

\n
$$
\text{if } \lambda_2(t) < 0, \quad \bar{u}(t) = 1
$$
\n
$$
\text{if } \lambda_2(t) > 0, \quad \bar{u}(t) = \begin{cases} x_2 & \text{if } x_2 \in]-1, 1] \\ -1 & \text{if } x_2 \in]-\infty, -1] \end{cases}
$$
\nif $\lambda_2(t) = 0, \quad \bar{u}(t) = \text{arbitrary}$

\nWe note from condition (5.29) that if the constraint is active (i.e., $g(x, u) = 0$), the state differential equation is given by

costate differential equation is given by *ū* $\overline{ }$ *ū ū*

$$
\vec{a} \cdot \vec{a} = -\frac{\partial f}{\partial x} \hat{i} - \mu \vec{a} \frac{\partial g}{\partial x} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \lambda^{\bar{u}} - \mu^{\bar{u}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$
\n(5.35)

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and is otherwise (when $g(x, u) < 0$) given by

W. Esterhuizen and J. Lévine
\n
$$
\lambda^{\bar{u}} = -\frac{\partial f}{\partial x}^T \lambda^{\bar{u}} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \lambda^{\bar{u}}.
$$
\n(5.36)

Recall that $\lambda_2(\bar{t}) > 0$ and $x_2(\bar{t}) = z_2 = 1 > 0$. Therefore, because λ and x are continuous, $\bar{u}(t) = x_2(t)$ over an interval before \bar{t} . We can show that $\bar{u}(t) \neq 1$ over this $\lambda^{\bar{u}} = -\frac{\partial f}{\partial x}$ $\lambda^{\bar{u}} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \lambda^{\bar{u}}$. (5.36)

Recall that $\lambda_2(\bar{t}) > 0$ and $x_2(\bar{t}) = z_2 = 1 > 0$. Therefore, because λ and x are continuous, $\bar{u}(t) = x_2(t)$ over an interval before $\bar{t$ Recall that $\lambda_2(\bar{t}) > 0$ and $x_2(\bar{t}) = z_2 = 1 > 0$. Therefore, because λ and x are continuous, $\bar{u}(t) = x_2(t)$ over an interval before \bar{t} . We can show that $\bar{u}(t) \neq 1$ over this interval: if $x_2 = 1$ and $u =$ $\stackrel{2}{\in}$ $\left[\bar{t} - \eta, \bar{t}\right], \ \eta > 0$ *t* the contraint $\lambda_2(t) \ge 0$ and $\lambda_2(t) = z_2 - T \ge 0$. Therefore, seedals λ and λ are contrinuous, $\bar{u}(t) = x_2(t)$ over an interval before \bar{t} . We can show that $\bar{u}(t) \ne 1$ over this interval: if $x_2 = 1$ and $2 + 1 = 0$, or $x_1 = -\frac{1}{2}$, meaning that
Thus $\dot{x}_1 = 0$ for all $t \in]\bar{t} - \eta, \bar{t}], \quad \eta > t \in]\bar{t} - \eta, \bar{t}].$
Therefore, only the constraint g [\(5.32\)](#page-8-5), we obtain μ over this interval

(5.32), we obtain μ over this interval
 $\frac{\partial H}{\partial u} + \mu \frac{\partial g}{\partial u} = \lambda_2 - \mu$

thus $\lambda_2 = \mu$ and the adjoint equation [\(5.35\)](#page-12-0) reads

$$
l_1, l_2, l_3, l_4 > 0
$$
, which collated
varinert real

$$
\frac{\partial H}{\partial u} + \mu \frac{\partial g}{\partial u} = \lambda_2 - \mu = 0
$$

$$
\frac{\partial H}{\partial u} + \mu \frac{\partial g}{\partial u} = \lambda_2 - \mu = 0
$$

as $\lambda_2 = \mu$ and the adjoint equation (5.35) reads

$$
\dot{\lambda} = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \lambda, \quad \forall t \in]\bar{t} - \eta, \bar{t}]
$$
(5.37)
Let us next analyze the switching condition of \bar{u} , or more precisely the change of

Let us ne
signum of λ
we want to $\lambda = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix} \lambda$, $\forall t \in]\overline{t} - \eta, \overline{t}]$ (5.37)

2. We know that, in an interval $|\overline{t} - \eta, \overline{t}|$ with $\eta > 0$, we have $\lambda_2 > 0$ and

characterize *n* such that $\lambda_2(t) < 0$ for $t < \overline{t} - n$ and $\lambda_2(\overline$ Let us next analyze the switching condition of \bar{u} , or more precisely the change of

signum of λ_2 . We know that, in an interval $[\bar{t} - \eta, \bar{t}]$ with $\eta > 0$, we have $\lambda_2 > 0$ and

we want to characterize η su that λ_2 cannot vanish over a nonempty open interval since then, according to [\(5.36\)](#page-13-0) Let us next analyze the switching condition of \bar{u} , or more precisely the change of signum of λ_2 . We know that, in an interval $|\bar{t} - \eta, \bar{t}|$ with $\eta > 0$, we have $\lambda_2 > 0$ and we want to characterize η such $\lambda_2(\bar{t} - \eta) > 0$, which is equivalent to $\lambda_1(\bar{t} - \eta) < 0$. Thus, expressing (5.34) at time *z*₂ *v t t*₂*t t t*₂*t t t*₂*t t t*₂*t t c*_{*t*} *t f*_{*n*} *t c*_{*n*} *t c*_{*n*} *t c*_{*n*} *t c*_{*n*} *t c*_{*n*} *cn t*₂*t c*_{*n*} *t c*_{*n*} *c*_{*n*} *cn c*_{*n*} we want to entracted by such that $\lambda_2(t) \le 0$ for $t \le t - \eta$ and $\lambda_2(t - \eta) = 0$. Note
that λ_2 cannot vanish over a nonempty open interval since then, according to (5.36)
or (5.37), we would also get $\lambda_1 = 0$ which is vanish. Thus, since λ_2 is locally increasing in a neighborhood of $\overline{t} - \eta$, we must have $\lambda_2(\overline{t} - \eta) > 0$, which is equivalent to $\lambda_1(\overline{t} - \eta) < 0$. Thus, expressing (5.34) at time $\overline{t} - \eta$, we get $x_2(\overline{$

crosses for a second time the x_1 axis and it can be checked that, at this time, λ_2 also vanishes. Therefore, the last section of the barrier is made of the trajectory generated $\bar{t} - \eta$, we get $x_2(\bar{t} - \eta) = 0$ and $\bar{u}(t) = 1$ for $t < \bar{t} - \eta$.
As long as λ_2 remains different from zero we keep crosses for a second time the x_1 axis and it can be ch vanishes. Therefore, the last section

Remark 5.4 Note that Assumption (A7) does not hold true at the final point *z* since there are two active constraints for only one control. However, since this condition is violated only at this point, we may conclude by continuity that condition [\(5.31\)](#page-8-1) still holds.

5.4.3 Constrained Spring II

Consider the same mass–spring–damper system with the same constants as in the previous example, but with a richer constraint

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad |u| \le 1, \quad x_2(x_2 - u) \le 0 \tag{5.38}
$$

We identify $\tilde{g}(x) = x_2^2 - |x_2|$, and $G_0 = \{x : x_2 = 0 \text{ or } x_2 = \pm 1\}$. \tilde{g} is differentiable for $x_2 \ne 0$ and from (5.31) and (5.32) we identify, in same manner as in the pre-

tiable for $x_2 \neq 0$ and from [\(5.31\)](#page-8-1) and [\(5.32\)](#page-8-5) we identify, in same manner as in the pre-
vious example, two points of ultimate tangentiality, namely $z = (-1, 1)$ along with vious example, two points of ultimate tangentiality, namely $z = (-\frac{1}{2}, 1)$ along with .
1 We identify $\tilde{g}(x) = x_2^2 - |x_2|$, and $G_0 = \{x : x_2 = 0 \text{ or } x_2 = \pm 1\}$. \tilde{g} is differen-
able for $x_2 \neq 0$ and from (5.31) and (5.32) we identify, in same manner as in the pre-
ious example, two points of ultimate the x_1 axis, which is also in G_0 , to the discussion below.
From the minimisation of the Hamiltonian, which is

From the minimisation of the Hamiltonian, which is the same as in the previous example except that $U(x)$ now corresponds to $u \ge x_2$ if $x_2 \ge 0$ and $u \le x_2$ if $x_2 \le 0$, we find the control \bar{u} $\lambda(\bar{t}) = (0, 1)$, and $z =$
the x_1 axis, which is
From the minimis
example except that
we find the control \bar{u} also in G_0 , to the discussion below.

sation of the Hamiltonian, which is the sation of the Hamiltonian, which is the sation of the Hamiltonian, which is the sation of $U(x)$ now corresponds to $u \ge x_2$ if $x_2 \in]0, 1]$

if
$$
\lambda_2(t) < 0
$$
 $\bar{u}(t) = \begin{cases} 1 & \text{if } x_2 \in]0, 1] \\ x_2 & \text{if } x_2 \in]-1, 0[\end{cases}$
if $\lambda_2(t) > 0$ $\bar{u}(t) = \begin{cases} x_2 & \text{if } x_2 \in]0, 1] \\ -1 & \text{if } x_2 \in]-1, 0[\end{cases}$
if $\lambda_2(t) = 0$ $\bar{u}(t) = \text{arbitrary}$

If we now integrate backwards from the points $\left(-\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ with the con-
 $\frac{1}{2}(t)$ we obtain the barrier as in Fig. 5.4. It turns out that $\frac{1}{2}(t) = x_1(t)$ all along as in Fig. 5.4. It turns out that \bar{u} If $x_2(t) > 0$ $u(t) = \begin{cases} -1 \text{ if } x_2 \in]-1,0[$

if $\lambda_2(t) = 0$ $\bar{u}(t) =$ arbitrary

If we now integrate backwards from the points $\left(-\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ with the control $\bar{u}(t)$ we obtain the barrier as in if $\lambda_2(t) = 0$ $\bar{u}(t) =$ arbitrary

If we now integrate backwards from the points $\left(-\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ with the trol $\bar{u}(t)$ we obtain the barrier as in Fig. 5.4. It turns out that $\bar{u}(t) = x_2(t)$ all **College** *g* $\frac{u}{u}$. If we
trol $\bar{u}(t)$
both cur
yields λ *x* now integrate backwards from the points $\left(-\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$ with the conve obtain the barrier as in Fig. 5.4. It turns out that $\bar{u}(t) = x_2(t)$ all along ves: the reader may easily check that, the If we now integrate backwards from the points $(-\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$ with the control $\bar{u}(t)$ we obtain the barrier as in Fig. 5.4. It turns out that $\bar{u}(t) = x_2(t)$ all along both curves: the reader may easily trol $\bar{u}(t)$ we obtain the barrier as in Fig. 5.4. It turns out that $\bar{u}(t) = x_2(t)$ all along
both curves: the reader may easily check that, the necessary condition $\frac{\partial H}{\partial u} + \mu \frac{\partial g}{\partial u} = 0$
yields $\lambda_2 - \mu x_2 = 0$ a both curves: the reader may easily check that, the yields $\lambda_2 - \mu x_2 = 0$ and, since $\frac{\partial g}{\partial x} = (0, 2x_2 - u)^T$ as (5.37) when $\bar{u} = x_2$, and conclude that $\lambda_2(t)$ is j which implies that $\bar{u} = x_2$, and $\lambda_2(t)$ must

Fig. 5.4 Admissible set of the constrained spring from Sect. [5.4.3](#page-13-2)

EXECUTE: x_1
 Let us now turn to the x_1 **axis, where** $\tilde{g} = x_2^2 - |x_2|$ **is non-differentiable. For any n the** *x***₁ axis, where** $\tilde{g} = x_2^2 - |x_2|$ **is non-differentiable. For any n the** *x***₁ axis, we have U(x)** *z* on the *x*₁ axis, we have $U(z) = [-1, 1]$ and $\partial \tilde{g}(z) = \bar{co}((0, -1)^{T}, (0, 1)^{T}) = \{0\} \times [-1, 1]$ and we must have dmissible set of the constrained spring from Sect. 5.4.3

now turn to the x_1 axis, where $\tilde{g} = x_2^2 - |x_2|$ is non-differentiable
 \tilde{g}_1 axis, we have $U(z) = [-1, 1]$ and $\partial \tilde{g}(z) = \tilde{\textbf{co}}((0, -1)^T, (0, 1)^T)$

d we Let us now turn to the z on the x_1 axis, we have $[-1, 1]$ and we must have *i*₁ axis, where $\tilde{g} = x_2^2 - |x_2|$ is non-differentiable. For any
 $J(z) = [-1, 1]$ and $\partial \tilde{g}(z) = \bar{co}((0, -1)^T, (0, 1)^T) = \{0\} \times$
 $\xi f(\tilde{z}, u) = 0 = \min_{u \in [-1, 1]} \max_{\xi_2 \in [-1, 1]} \xi_2(-2x_1 + u)$ (5.39) $\ddot{}$

$$
\lim_{u \in [-1,1]} \max_{\xi \in \partial \tilde{g}(\xi)} \xi f(\tilde{z}, u) = 0 = \min_{u \in [-1,1]} \max_{\xi_2 \in [-1,1]} \xi_2(-2x_1 + u) \tag{5.39}
$$

[-1, 1] and we must have
 $\min_{u \in [-1,1]} \max_{\xi \in \partial \bar{g}(\bar{z})} \xi f(\bar{z}, u) = 0 = \min_{u \in [-1,1]} \max_{\xi_2 \in [-1,1]} \xi_2(-2x_1 + u)$ [\(5.39\)](#page-15-1)

For each $-\frac{1}{2} \le z_1 \le \frac{1}{2}$ Eq. (5.39) has a solution given by $\xi = (0, \text{sign}(-2z_1 + u))$ from

which min max $\xi f(\tilde{z}, u) = 0 = \min_{u \in [-1, 1]}\max_{\xi_2 \in [-1, 1]}\xi_2(-2x_1 + u)$ (5.39)
For each $-\frac{1}{2} \le z_1 \le \frac{1}{2}$ Eq. (5.39) has a solution given by $\xi = (0, \text{sign}(-2z_1 + u))$ from
which we deduce that $\bar{u} = 2z_1$. However, one can d curves of [\(5.38\)](#page-14-1) with endpoints in the set $[-\frac{1}{2}]$
correspond to admissible curves (integrated ba $, \overline{a}$ curves of (5.38) with endpoints in the set $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ with the control $u = x_2$ all correspond to admissible curves (integrated backwards) and therefore do not belong which we deduce that $\bar{u} = 2z_1$. However, one can directures of (5.38) with endpoints in the set $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ correspond to admissible curves (integrated backwards) to the barrier, but that they make the co $\dot{t} = (0, \text{sign}(-2z_1 + u))$ from
ctly verify that the integral
with the control $u = x_2$ all
and therefore do not belong
 $(t), \bar{u}(t))$ equal to 0 for $\bar{u} =$
hat our conditions are only *x* $\ddot{}$ *,* $, \overline{,}$ x_2 for all $\bar{x} \in [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ and for all *t*. This attests that our conditions are only necessary and far from being sufficient.

Remark 5.5 Note that, as in Sect. [5.4.2,](#page-12-2) Assumption (A7) does not hold true at the final points $z \in G_0$ since there are two active constraints for only one control. Again, we conclude by a continuity argument that condition (5.31) still holds.

5.5 Conclusion

In this paper, we have demonstrated on elementary examples of systems subject to pure or mixed constraints, the effectiveness of the results obtained in $[6, 7]$ $[6, 7]$ $[6, 7]$ $[6, 7]$, which allowed us to give a complete construction of their barriers and admissible sets. We also pointed out some significant differences in these constructions. In particular, we have shown, in the mixed constrained case, that the barrier does not need to intersect the boundary G_0 of the constraint set; that, according to the feedback nature

of the control, due to the state dependence of the control set, the equilibria and their stability could be modified to be repelled from G_0 ; that the *nonsmooth* version of the *necessary* ultimate tangentiality condition though useful is far from being sufficient: necessary ultimate tangentiality condition, though useful, is far from being sufficient; and that Assumption (A7) is, even in simple examples, not everywhere satisfied. Higher dimensional examples are presently under investigation and will be published elsewhere.

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