Undergraduate Lecture Notes in Physics

Alberto Vecchiato

Variational Approach to Gravity Field Theories

From Newton to Einstein and Beyond

Springer

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Variational Approach to Gravity Field Theories

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To Gabriella

Preface

The variational approach is probably the most powerful technique and unifying concept of theoretical physics, but in the same way it is not common to explain it to its full extent to undergraduate students, which later results in a splitting between the method and the languages of theorists and those of “the rest of the world,” at least for what concerns physicists. I think it is important to try to fill in this gap by introducing them as early as possible, thus keeping a common understanding in this discipline.

With the notable exception of Lagrangian and Hamiltonian mechanics, there is a consolidated teaching tradition in undergraduate physics courses in which there is little room for a complete exposition of variational techniques. This is indeed for good reason. Classical physics can be explicated in an extremely efficient way without resorting to these relatively advanced methods, which on the contrary can appear too abstract in such an elementary context.

However, there is a price to pay with this approach. At some point the student comes to more advanced subjects such as quantum or relativistic physics, where such techniques are extremely useful, if not necessary. The risk, then, is that these are perceived as completely detached from the familiar background, and accepted without a real understanding. In this way the nontheoretical physicist will quickly forget this “anomalous event.” This problem might be avoided if, after the regular exposition of classical physics, the same concepts were revised from the point of view of the variational approach.

This book undertakes the problem from the point of view of the gravity field theories, trying to introduce the variational approach by stressing its continuity from the classical to the relativistic realm. Such a job can be accomplished only by treating in parallel the evolution of the dynamics along the same theoretical path.

Despite its great power, however, the variational approach is ultimately a technique, whereas the essential physical meaning of theories lies on their fundamental principles. It is for this reason that the exposition tries to highlight as clearly as possible the connection between theories and principles at the very basic mathematical level.

The book is organized in four parts, which follow the basic ideas underlined above.

The first one tries to give a gentle introduction to variational principles using Newtonian dynamics and gravity as a case study. In the next chapters, the link between classical physics and Euclidean geometry is analyzed from the point of view of the founding principles. This part concludes with the discovery of the internal inconsistency between electrodynamics and these principles in their classical formulation.

The second part starts from the failure of classical physics with respect to the principle of relativity to arrive at an alternative formulation of such a principle. Building upon it to get to special relativity in its Minkowskian formulation, this part ends by observing the infeasibility of a special relativistic theory of gravitation.

By analyzing this problem in more detail, the third part deals with general relativity, some of its applications, and how and why this theory could be extended or modified.

In this book, I tried to gauge the exposition with a particular emphasis on the physical concepts, which sometimes required momentarily delaying some mathematical details. For this reason the appendices contained in the fourth part are not to be considered just as supplementary material. Rather, they carry essential information needed for the complete understanding of the text, and should be tackled in parallel to the respective chapters, where appropriate references can be found.

Finally, this book contains 42 exercises. It is important to solve all of them entirely, because in some cases they include other mathematical details referred to in the main text. Furthermore, this number might not be a mere coincidence, if it has to be the "Answer to the Ultimate Question of Life, the Universe, and Everything," as somebody argued.

I wish to thank my family for their support during the writing of this book, and I am deeply grateful to my wife. Without her continuous encouragement, steady support, and unshakeable patience this book would have never been completed.

Torino, Italy Alberto Vecchiato
November 2016

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Part I
Introduction and Classical Physics

"All models are wrong, but some are useful." George Box, 1978

Chapter 1
A Short Introduction to Field Theories and Variational Approach

In this chapter the reader is introduced to the concepts and language of the Lagrangian formalism of field theories. Only knowledge of the Newtonian theory of gravity, as usually explained in standard undergraduate courses is assumed, although some familiarity with classical mechanics is an asset. The goal is twofold:

1. One is to explain in the simplest and most natural way how one can resort to the Lagrangian formalism, giving some motivations for which this approach can look useful and appealing.
2. The second is to provide a general and ready-for-use picture of the "game" we are going to play in the following chapters, a sort of "case study" to guide us when more complex theories are treated.

In doing so, we do not claim theoretical completeness or full rigorousness. Indeed, in order to make the reasoning as clear as possible, although without hampering its logical completeness, when possible we skip some mathematical details and examples that are reported in the appendices.

1.1 What Is a Field Theory: A Naive View

The word *field* in physics is used to identify a physical quantity that takes a value at each point of space and time. A typical example of field is the temperature: one can define a quantity $T(\mathbf{x}, t)$ that represents the temperature at each point $\mathbf{x}$ and each instant of time t (Fig. 1.1a). Because the temperature can be defined by a scalar, such as a number, in this case we have a *scalar field*, but also other kind of quantities, such as vectors or tensors (see Appendix B) can be identified as fields. In such cases we have, quite obviously, *vector* and *tensor* fields such as, e.g., for one of velocity (Fig. 1.1b).

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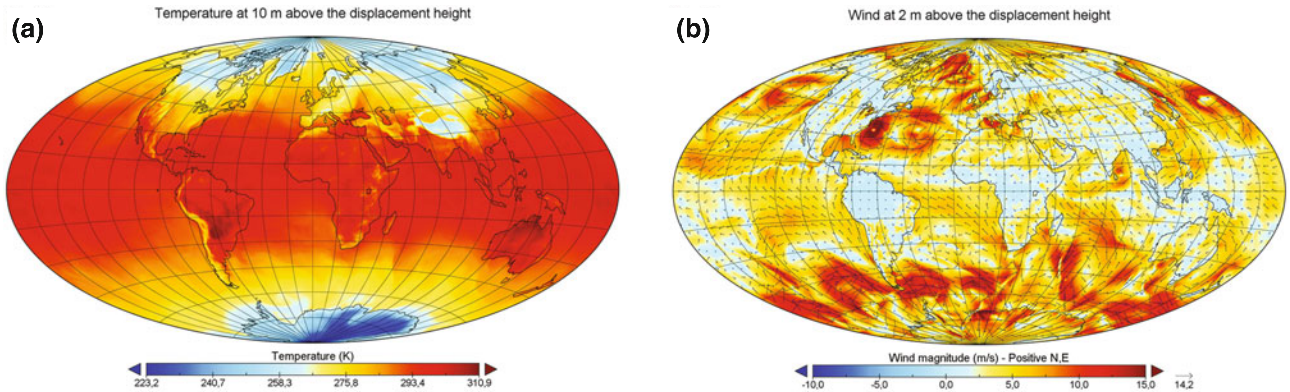

Fig. 1.1 Example of scalar field (left panel) and scalar/vector field (right panel). (a) Map of the surface Earth temperature at 10 m above the displacement height on October 29, 2012. (b) Map of the surface Earth winds at 2 m above the displacement height on October 29, 2012. The vector field shows both the magnitude and the direction, while the color map (scalar field) gives only the magnitude. Data from MERRA database (https://gmao.gsfc.nasa.gov/reanalysis/MERRA/). Maps were generated using Panoply Data Viewer software; Schmunk, Robert B. (2016). Panoply Data Viewer (version 4.6.2) [Software]. Available from http://www.giss.nasa.gov/tools/panoply/

One might now be tempted to give a precise definition of what a field theory is in physics, and then to use this definition to understand whether a physical model can be classified as a field theory. As always happens, however, historically a classification comes after the events that made it useful, and actually many theories were formulated before the concept of field theory was conceived. So it is better to proceed the other way round, i.e., to show the presently accepted classification and to deduce from them a general rule of thumb that was used to give such a classification.

The Newtonian theory of gravity and the Maxwell's electromagnetic theory are examples of the so-called *classical field theories*. These two theories describe, respectively, how the gravitational and the electromagnetic fields interact with massive and electrically charged particles. As a further specification, we can classify the former as *nonrelativistic* and the latter as *relativistic*, using these words to make it explicit that, contrary to the electromagnetic theory, Newtonian gravity is not covariant under the Lorentz transformations introduced by the special theory of relativity.[1] Needless to say, general relativity is a relativistic field theory, and it is also classical, where this word is used in opposition to *quantum* field theories to characterize those theories that do and do not incorporate quantum fields, like such as quantum electrodynamics (QED) or quantum chromodynamics (QCD). The treatment of these two theories is beyond the scope of this book, but following our line of reasoning above (that is, just in order to "extract" from practical examples which are the characteristics of a field theory) it is useful to recall that QED is the quantum and relativistic counterpart of classical electromagnetism: i.e., it deals with the interaction of the (quantized) electromagnetic field with electric charges. QCD instead is another quantum and relativistic field theory, but it describes so-called strong interaction, a kind of field that is felt by particles such as quarks that are provided with a physical property called *color*.

From all of these examples, one can thus infer that a field theory can be defined as a theory which describes the interaction of a physical field with some kind of matter. Naively speaking, such theories are characterized by two basic ingredients that can be summarized as:

1. There is "something" (called the *field source*) producing a physical field that evolves (in space and time) according to some mathematical laws, which are specific of the theory under consideration.
2. This physical field tells particles how to move according to the laws of dynamics.

In mathematical language, these two ingredients correspond to the *field equations* and to the *equations of motion*, respectively.

The Newtonian theory of gravity can thus be regarded as an example of field theory in which the field source is a property of particles called gravitational mass. A distribution of masses over a certain region of space having density ρ produces a gravitational field whose potential $\Phi\left(\mathbf{x}\right)$ satisfies the Poisson equation

$$\nabla^2\Phi\left(\mathbf{x}\right) = 4\pi G\rho\left(\mathbf{x}\right) \tag{1.1.1}$$

[1] These concepts are developed more rigorously in the following chapters.

that is, the field equation of this theory of gravitation. Once the Poisson equation for the gravitational potential is solved, the field Φ allows us to determine the motion of a particle with (inertial) mass m under the influence of gravity by means of the laws of dynamics because the gravitational potential energy $V = m\Phi$ obeys Newton's second law

$$-\nabla V = \mathbf{F} = m\mathbf{a}. \tag{1.1.2}$$

1.2 Equations of Motion

1.2.1 Lagrangian Formalism

The explicit determination of the motion of a particle with Eq. (1.1.2) requires the solution of a system of three differential equations that can be easily expressed in Cartesian coordinates as

$$\begin{aligned} m\ddot{x} &= F_x \equiv -\frac{\partial V}{\partial x} \\ m\ddot{y} &= F_y \equiv -\frac{\partial V}{\partial y} \\ m\ddot{z} &= F_z \equiv -\frac{\partial V}{\partial z} \end{aligned} \tag{1.2.1}$$

where, following a widely accepted convention, for any coordinate $\ddot{x}_i$

$$a_i \equiv \ddot{x}_i \equiv \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2},$$

however, it is often much more convenient to write the equations of motion with respect to a different coordinate system. (E.g., it is well known that the single-body problem for the gravitational force can be solved much more easily in spherical coordinates because of the spherical symmetry of the force field governing this case.) It is therefore mandatory to understand how these equations can be written in another coordinate system.

The equations of motion in a generalized coordinate system

A change of coordinates from Cartesian to a generic one q_1, q_2, q_3 is defined in a subset $S \subset \mathbb{R}^3$ by three functions

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3) \tag{1.2.2}$$

for which the determinant of the Jacobian matrix is different from zero for any point $\mathsf{P} \in S$, i.e.,

$$\det \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{pmatrix} \neq 0.$$

If P is pointed by the vector $\mathbf{x}\,(x, y, z)$, then the vectors

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{x}}{\partial q_i} = \frac{\partial x}{\partial q_i}\mathbf{i} + \frac{\partial y}{\partial q_i}\mathbf{j} + \frac{\partial z}{\partial q_i}\mathbf{k}, \qquad i = 1, 2, 3$$

are a basis associated with the coordinates q_i for the subspace S in P. The three equations for the coordinates q_i can then be obtained by projecting Eq. (1.1.2) on the three coordinate curves with $\mathbf{e}_i$

$$(m\mathbf{a} - \mathbf{F}) \cdot \mathbf{e}_i = 0. \tag{1.2.3}$$

It is immediate to notice that, when the basis is that of the Cartesian coordinates, this operation corresponds exactly to writing Eqs. (1.2.1).

The Euler–Lagrange equations

Considering that $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$, Eq. (1.2.3) can be rewritten as

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\partial \mathbf{x}}{\partial q_i} - \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_i} = 0$$

that is,

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v} \cdot \frac{\partial \mathbf{x}}{\partial q_i}\right) - m\mathbf{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{x}}{\partial q_i} - \mathbf{F} \cdot \frac{\partial \mathbf{x}}{\partial q_i} = 0. \tag{1.2.4}$$

However, from Eq. (1.2.2),

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \sum_{i=1}^{3} \frac{\partial \mathbf{x}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} \equiv \sum_{i=1}^{3} \frac{\partial \mathbf{x}}{\partial q_i}\dot{q}_i \tag{1.2.5}$$

(where the $\dot{q}_i$ are called *generalized velocities*) so that taking the derivative of this expression with respect to $\dot{q}_i$ one immediately has

$$\frac{\partial \mathbf{v}}{\partial \dot{q}_i} = \frac{\partial \mathbf{x}}{\partial q_i}, \tag{1.2.6}$$

and, because of the commutativity of the differential operators,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathbf{x}}{\partial q_i} = \frac{\partial \mathbf{v}}{\partial q_i}. \tag{1.2.7}$$

Substituting Eqs. (1.2.6) and (1.2.7) into Eq. (1.2.4) one gets

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(m\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \dot{q}_i}\right) - m\mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial q_i} - \mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial q_i} &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial q_i} &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - \mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial q_i} &= 0 \qquad (1.2.8)
\end{aligned}$$

where $T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$ is the kinetic energy of the particle.

Moreover, because we are considering the case of a conservative force it is $V = V\left(\mathbf{x}\left(q_1, q_2, q_3\right), t\right)$, and

$$\mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial q_i} = -\frac{\partial V}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{1.2.9}$$

$$\frac{\partial V}{\partial \dot{q}_i} = 0. \tag{1.2.10}$$

Therefore, if one defines a function $L = T - V$ called *Lagrangian*, Eq. (1.2.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{1.2.11}$$

which is called *Euler–Lagrange* equation or simply *Lagrange* equation. In other words, we have a simple "recipe" to obtain the equations of motion in a generic coordinate system q_1, q_2, q_3, just find the expression of the Lagrangian in this system and make the calculations indicated by the Euler–Lagrange equations above.

It can be proven explicitly (see Sect. 4.2) that the Euler–Lagrange equations are invariant for generic coordinate transformations; i.e., if one has a coordinate transformation $q_i = q_i\left(\tilde{q}_j\right)$, where $\det\left(\frac{\partial q_i}{\partial \tilde{q}_j}\right) \neq 0$, then the resulting equations of motion are still Euler–Lagrange equations with respect to $\tilde{\mathbf{q}}$, whose Lagrangian $\tilde{L}$ is simply obtained by L by coordinate substitution. It is worth noticing, however, that this is self-evident inasmuch as we obtained the Euler–Lagrange equations from Eqs. (1.2.2) and (1.2.3), in which the coordinate transformation was completely generic.

What we have deduced here for a single particle and for a conservative force can be extended to any number of particles and for systems subjected to both conservative and nonconservative forces[2] but this generalization is out of scope here, and it can be found in any textbook of classical mechanics such as Gantmacher (1970) or Arnol'd (1973).

The effort we have put in deriving the Euler–Lagrange equations was originally motivated by the need to find the equations of motion of a particle (or of a system of particles) in a generic coordinate system. In classical mechanics this problem is further developed to more general cases such as, as just pointed out, those of nonconservative forces, but also to systems with specific constraints, and the Lagrangian formulation (or reformulation) of Newton's mechanics is also motivated by more general reasons. For example, the complete freedom on the choice of the set of coordinates can be used to select the one that best fits the "structure" of the problem. This often makes it easier to find a solution, and although these equations may appear more "abstract" at first sight, this also makes the discovery of conserved quantities easier, which is usually tightly connected with the physical understanding of the problem. Finally, this formulation showed itself to be more adapted for describing complex systems and to be extended to other branches of physics, such as quantum mechanics.

Whatever the motivations, at this point we have understood that the Euler–Lagrange equations are equivalent to the equations of motion, i.e., *to Newton's second law of dynamics*. We show in the next section that there is another equivalent way to deduce the same equations of motion.

1.2.2 The Variational Approach

As explained more explicitly in Chap. 2, what we call "Newton's second law of dynamics" is actually a *principle*, i.e., a statement that we hold true without demonstration[3] (like axioms in mathematics) and that we use to deduce other laws (in the same way as we deduce theorems from the axioms). One can therefore wonder if there is another principle equivalent to this one, or what would happen if this principle would be proven to be not valid anymore. The answer to the first question is yes, and this principle is at the basis of the so-called *variational approach*, whose investigation also helps to answer the second question, i.e., to understand how Newton's second *principle* of dynamics can be superseded and extended.[4]

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[2]In this latter case the right-handside of the Euler–Lagrange equations is not zero anymore, but rather $Q_i = \mathbf{F}_{\rm nc} \cdot \partial \mathbf{x}/\partial q_i$ where $\mathbf{F}_{\rm nc}$ represent the nonconservative forces, and L still contains the potential energies of the conservative forces.

[3]More correctly, our "demonstration" is the experimental verifications, and the principle can be considered valid within the limits defined by the experimental accuracies.

[4]This issue is discussed with more detail in Chap. 3.

The action as a functional

Let us consider a system described by a Lagrangian $L\left(q\left(t\right), \dot{q}\left(t\right), t\right)$, i.e., a system with one degree of freedom. Following the conventions of classical mechanics, the independent variable is t, which we assume to be in the interval $t_0 \leq t \leq t_1$, and the *function* $q\left(t\right)$ represents the motion of a single particle. Therefore (see Appendix A)

$$S\left[q\right] = \int_{t_0}^{t_1} L\left(q\left(t\right), \dot{q}\left(t\right), t\right)\, \mathrm{d}t \tag{1.2.12}$$

is a *functional* of the system. This functional is called *Hamiltonian action* or, more briefly, just *action* and, from the definition of functionals, it is a quantity whose value depends on the choice of the specific motion followed by the particle in going from $q\left(t_0\right)$ to $q\left(t_1\right)$.

The principle of least action

From what is shown in Sects. A.2 and A.3 it is straightforward to understand that, under the condition that the arbitrary variations $\delta q\left(t\right)$ are zero at t_0 and t_1 (i.e., that, quite reasonably, we are dealing with motions having fixed endpoints) a specific motion $q\left(t\right)$ in the given interval of t identifies extremal values of $S\left[q\right]$ if and only if it satisfies the Euler–Lagrange equations (A.3.6) which, in this case, read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.2.13}$$

In other words, by imposing the condition $\delta S = 0$ (i.e., selecting those motions $q\left(t\right)$ between $q\left(t_0\right)$ and $q\left(t_1\right)$ that are extremal for the functional $S\left[q\right]$) is equivalent to requiring that these motions are solutions of the Euler–Lagrange equation. What we have just said for a system with one degree of freedom can be easily generalized to a system with any number n of degrees of freedom, namely to a Lagrangian $L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right)$, where $\mathbf{q} = \left(q_1\left(t\right), \ldots, q_n\left(t\right)\right)$: the variation of the Action S is (see Eq. (A.2.4))

$$\delta S = \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_o}^{t_1} - \int_{t_0}^{t_1} \left[\sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\delta q_i\right]\mathrm{d}t,$$

and, in particular, for variations fixed at the ends ($\delta q_i\left(t_0\right) = \delta q_i\left(t_1\right) = 0\ \forall\, i$)

$$\delta S = -\int_{t_0}^{t_1} \left[\sum_{i=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\delta q_i\right]\mathrm{d}t, \tag{1.2.14}$$

so that we can conclude that a motion with fixed ends in an n-dimensional space is extremal for the action, and therefore makes $\delta S = 0$ if and only if it obeys the system of Euler–Lagrange equations (1.2.11).

Because, from the conclusion of the previous section, this system is equivalent to Newton's equations of motion, also the condition $\delta S = 0$ is equivalent to hold the "$\mathbf{F} = m\mathbf{a}$ law" true. This means that $\delta S = 0$ and $\mathbf{F} = m\mathbf{a}$ are "fundamental at the same level," hence, in the same way as the second principle of dynamics, we can state the *principle of least action* which says that the motion of a system described by a set of coordinates $\mathbf{q}(t)$ between two fixed points $\mathbf{q}_0 = \mathbf{q}(t_0)$ and $\mathbf{q}_1 = \mathbf{q}(t_1)$ is the one that makes $\delta S = 0$.

Geodesic motions: The "geometrization" of the equations of motion

Before concluding this short review of the different approaches to the problem of finding the equations of motion, it is worth spending a few lines on a slightly different principle known as *Maupertuis' principle*. This is named after the French mathematician Pierre-Louis Moreau de Maupertuis who, in 1744, formulated a variational principle similar to that of least action but dealing with *trajectories* instead of motions, which are intended as "motions with no reference to time," i.e., just geometrical curves connecting two points. In modern language this means that the Lagrangian of such a system does not depend explicitly on time, i.e., $L = L(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q})$, and it is a well known theorem of classical mechanics that in these cases the total energy $E = T + V$ is a constant of motion. Moreover, it can be shown that for such systems, given the trajectories γ allowed between two fixed points $\mathbf{q}_0$ and $\mathbf{q}_1$,[5] the choice of a specific E identifies both a single trajectory and its generalized velocities $\dot{\mathbf{q}}$, and for such reason the functional can be defined

$$A[\gamma; E] = \int_\gamma \mathbf{p} \cdot d\mathbf{q} \tag{1.2.15}$$

called *Maupertuis' action*, or *reduced action*, where $\mathbf{p} = \partial L / \partial \dot{\mathbf{q}}$ is the *conjugate momentum* relative to the generalized coordinates $\mathbf{q}$.[6]

Now we need to know two more statements, whose demonstration can be found in any text of classical mechanics (see, e.g., Arnol'd (1973)) but that is skipped because it is out of scope here. The first one is that in the above conditions the Maupertuis principle, which states that the true motions of these systems minimize the reduced action, is equivalent to the least action principle; the second is that

$$\mathbf{p} \cdot \dot{\mathbf{q}} = E + L. \tag{1.2.16}$$

[5]The allowed trajectories are those for which the total energy is conserved, and $T = E - V > 0$.
[6]Such name is clearly justified because, from $L = T - V$ it results in $\mathbf{p} = m\mathbf{v}$.

Because of the equivalence between the two principles, we thus have that the true trajectories are those for which A is stationary; i.e., $\delta A = 0$.

Given this premise, we can notice that from Eqs. (1.2.15) and (1.2.16) it follows immediately that the reduced action can be rewritten as[7]

$$A = \int_\gamma \mathbf{p} \cdot \dot{\mathbf{q}}\, \mathrm{d}t = \int_\gamma \sqrt{2T}\sqrt{2T}\, \mathrm{d}t. \tag{1.2.17}$$

However, because of Eq. (1.2.5), the kinetic energy is

$$T = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m \sum_{i,j=1}^{3} \frac{\partial \mathbf{x}}{\partial q_i} \cdot \frac{\partial \mathbf{x}}{\partial q_j} \dot{q}_i \dot{q}_j \equiv \frac{1}{2} \sum_{i,j=1}^{3} a_{ij} \dot{q}_i \dot{q}_j \tag{1.2.18}$$

where $a_{ij} = m \frac{\partial \mathbf{x}}{\partial q_i} \cdot \frac{\partial \mathbf{x}}{\partial q_j}$ are the elements of the so-called *kinetic matrix*, and therefore

$$\sqrt{2T}\mathrm{d}t = \sqrt{\sum_{i,j=1}^{3} a_{ij} \mathrm{d}q_i \mathrm{d}q_j}. \tag{1.2.19}$$

The right-hand side of the above equation can be read as a sort of "generalization" of the Euclidean norm formula $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ for infinitesimal separations $\mathrm{d}s$. We can in fact reduce ourselves to this case by taking $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. Equation (1.2.19) can thus be interpreted as a sort of infinitesimal *distance* "weighted" by the components of the kinetic energy of the particle, that is, by its "dynamic content" because T depends on E and V.[8] Equation (1.2.17) then becomes

$$A = \int_\gamma \sqrt{2\,(E - V)} \sqrt{\sum_{i,j=1}^{3} a_{ij} \mathrm{d}q_i \mathrm{d}q_j} = \int_\gamma \mathrm{d}s,$$

where, following the same line of reasoning as above, in the last step we included the factor $\sqrt{2\,(E - V)}$ into the generalized distance by putting

$$\mathrm{d}s^2 = 2\,(E - V) \sum_{i,j=1}^{3} a_{ij} \mathrm{d}q_i \mathrm{d}q_j. \tag{1.2.20}$$

We have then shown that the action can be interpreted as a kind of length of the trajectory, measured with a particular "weighting factor" given by the dynamics of the problem. At the same time we could also state that this length is the one that

[7] Remember that $E + L = T + V + T - V = 2T$.

[8] This can be seen, perhaps more intuitively, also by writing $\sqrt{2T}\mathrm{d}t = \sqrt{m}v\, \mathrm{d}t = \sqrt{m}\mathrm{d}l$, where $\mathrm{d}l$ is the distance covered by the point with mass m in the time interval $\mathrm{d}t$. The above formula, however, allows us to write this distance with an object similar to what is later identified as a *metric tensor*.

would be measured over a strange kind of geometric surface, “curved” in such a way to give the correct result.[9] The Maupertuis principle can thus be interpreted as stating that the trajectories followed by the particles are those of *minimum distance* (geodesics) between two points if we measure the distances with Eq. (1.2.20). This is a first glimpse of what could be called a process of “geometrization of the dynamics,” which is at the basis of general relativity.

1.3 Field Equations and Variational Approach

In the previous section we deduced the equations of motion from a variational principle acting on a Lagrangian for the motions of the particles. In this section we show that also the Poisson equation, that is, the field equation of the Newtonian theory of gravity, can be deduced from (and therefore proved equivalent to) a specific variational principle acting on a Lagrangian for the gravitational potential. In doing so we neglect, in some cases, the full rigor in favor of a more intuitive reasoning that makes use of analogies, and the problem of guessing an appropriate Lagrangian is considered in the light of more fundamental principles in Chap. 4. The reader interested in an advanced introduction to this topic can refer to Doughty (1990).

1.3.1 The Euler–Lagrange Equations for the Fields

We can start by highlighting that the variational principle of the previous section is just an application of the variational calculus briefly explicated in Appendix A. Indeed, the Euler–Lagrange equations for the equations of motion are derived as a particular case of Eq. (A.2.4), i.e., by imposing the null variation (for motions fixed at the ends) of the Lagrangian-type functional of Eq. (A.2.1)

$$F\left[u\right] = \int_a^b L\left(u\left(x\right), u'\left(x\right), x\right)\,\mathrm{d}x$$

where L depends on u and on its derivative u'. This means that the same variational principle will produce Euler–Lagrange equations regardless of the actual physical meaning of the function $u\left(x\right)$, the only requirement being that u is continuously differentiable over the whole range of integration $[a, b]$.

In this regard the potential of the Poisson equation $\Phi\left(\mathbf{x}\right)$ is not much different from $u\left(x\right)$, so it is reasonable to wonder if there can exist an appropriate Lagrangian of Φ (and of its derivative) from which the field equation can be deduced by applying the same variational principle.

[9]The concept of curved space is discussed more rigorously in Appendix D.

Lagrangian and Lagrangian densities

The main difference between Φ and u to be considered is about the independent variables. Although $u\left(x\right)$ is a function of a single independent parameter (which in classical mechanics is the time t, because $u\left(x\right)$ is interpreted as a time-dependent generalized coordinate $q\left(t\right)$) $\Phi\left(\mathbf{x}\right)$ is one of a 3-dimensional vector. Because of this different dependence, it would be natural to rewrite the functional as

$$F\left[\Phi\right]=\int_{\Omega_3}\mathcal{L}\left(\Phi\left(\mathbf{x}\right),\nabla\Phi\left(\mathbf{x}\right),\mathbf{x}\right)\,\mathrm{d}^3\mathbf{x},$$

where Ω_3 represents the volume of the functional domain. In the above formula we have used a different symbol for the Lagrangian because the physical context of our problem makes it necessary to adopt a slightly different convention to write the functional. As is clear from the expression $L=T-V$ and from the definition of Eq. (1.2.12), the action has the dimensions of [energy]·[time]. These dimensions have to be preserved in order to give a consistent physical meaning to the variational principle we are assuming, therefore we want the functional of the field to represent an action as well. If we require that the Lagrangian of the field(s) is an energy as well as that of the motions, then we need to add a further integration with respect to time, so that the action of the field becomes

$$S\left[\Phi\right]=\int_{t_0}^{t_1}\int_{\Omega_3}\mathcal{L}\left(\Phi\left(\mathbf{x}\right),\nabla\Phi\left(\mathbf{x}\right),\mathbf{x}\right)\,\mathrm{d}^3\mathbf{x}\mathrm{d}t \tag{1.3.1}$$

where the integral

$$L=\int_{\Omega_3}\mathcal{L}\left(\Phi\left(\mathbf{x}\right),\nabla\Phi\left(\mathbf{x}\right),\mathbf{x}\right)\,\mathrm{d}^3\mathbf{x}$$

defines the *Lagrangian* of the field, and inasmuch as it is obtained by integration over a spatial domain, $\mathcal{L}\left(\Phi\left(\mathbf{x}\right),\nabla\Phi\left(\mathbf{x}\right),\mathbf{x}\right)$ is called the *Lagrangian density* of the field.

Euler–Lagrange equations for the fields

In the specific case of the Newtonian theory of gravitation the field Φ is a function of $\mathbf{x}$ only, but in general one can have a field $\phi\left(\mathbf{x},t\right)$ which is explicitly dependent on time as well. It is thus clear that the full action should be written as

$$S[\phi] = \int_{t_0}^{t_1}\int_{\Omega_3} \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial t}, \nabla\phi, \mathbf{x}, t\right) \mathrm{d}^3\mathbf{x}\mathrm{d}t$$
$$\equiv \int_{t_0}^{t_1}\int_{\Omega_3} \mathcal{L}\left(\phi, \dot{\boldsymbol{\phi}}, \mathbf{x}, t\right) \mathrm{d}^3\mathbf{x}\mathrm{d}t \tag{1.3.2}$$

where, to ease the notation and to stress the correspondence with Eq. (A.2.1), we have put $\dot{\boldsymbol{\phi}} \equiv \left(\frac{\partial\phi}{\partial t}, \nabla\phi\right)$.This does not make any difference for the Newtonian gravitation, as show later, but it is necessary if one wants to derive the complete Euler–Lagrange equations for the fields. It is quite straightforward now to proceed as in Appendix A. First the variation of S is

$$\delta S[\phi] = \int_{t_0}^{t_1}\int_{\Omega_3} \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\phi}}}\delta\dot{\boldsymbol{\phi}}\right) \mathrm{d}^3\mathbf{x}\mathrm{d}t; \tag{1.3.3}$$

then one has to notice that $\dot{\boldsymbol{\phi}}$ is a set of four functions, each of which thus contributes to its variation $\delta\dot{\boldsymbol{\phi}}$ and to the respective functional derivative $\partial\mathcal{L}/\partial\dot{\boldsymbol{\phi}}$, so that[10]

$$\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{\phi}}}\delta\dot{\boldsymbol{\phi}} = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta(\partial_t\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\delta(\partial_x\phi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_y\phi\right)}\delta\left(\partial_y\phi\right) + \frac{\partial\mathcal{L}}{\partial(\partial_z\phi)}\delta(\partial_z\phi)$$
$$\equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta(\partial_t\phi) + \frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\cdot\delta(\nabla\phi)\,.$$

Now the same reasoning as that of Eq. (A.2.3) can be applied to each of these functions in order to deduce that

$$\delta(\partial_t\phi) = \partial_t(\delta\phi)$$
$$\delta(\nabla\phi) = \nabla(\delta\phi)$$

which allow us to write Eq. (1.3.3) as

$$\delta S[\phi] = \int_{t_0}^{t_1}\int_{\Omega_3}\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\partial_t(\delta\phi) + \frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\cdot\nabla(\delta\phi)\right)\mathrm{d}^3\mathbf{x}\mathrm{d}t, \tag{1.3.4}$$

and after the usual integration by parts we obtain

$$\delta S[\phi] = \left[\int_{\Omega_3}\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\delta\phi\,\mathrm{d}^3\mathbf{x}\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}\mathrm{d}t\oint_{\Omega_2}\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\delta\phi\cdot\mathrm{d}\boldsymbol{\Omega}_2 -$$
$$\int_{t_0}^{t_1}\int_{\Omega_3}\left[\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi}\right]\delta\phi\,\mathrm{d}^3\mathbf{x}\mathrm{d}t \tag{1.3.5}$$

[10]In the formula we replaced $\partial\phi/\partial x_\alpha$ with the notation $\partial_\alpha\phi$, where $\alpha = t, x, y, z$ both for easier reading, and to highlight the fact that each partial derivative is actually a *function*.

where we used the divergence theorem to convert the integral over the volume Ω_3 into the integral over its boundary surface Ω_2. Finally, by imposing $\delta S = 0$ for null variations at the (spatial and temporal) boundaries, we have the Euler–Lagrange equation for the field $\phi(\mathbf{x}, t)$

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} \right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{1.3.6}$$

What we have just written is analogous to Eq. (1.2.13) for the fields, i.e., in the case of one field, but in general one could have a physical system described by many different fields ϕ_i and therefore, exactly happens for a system of particles with Eq. (1.2.11), we will have

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial (\partial_t \phi_i)} \right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \tag{1.3.7}$$

By analogy with Eq. (A.2.5), the quantity

$$\frac{\delta S}{\delta \phi_i} \equiv \partial_t \left(\frac{\partial \mathcal{L}}{\partial (\partial_t \phi_i)} \right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial (\nabla \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} \tag{1.3.8}$$

is called the *variational (or functional) derivative of the action* with respect to the fields.

1.3.2 Poisson Equations from Variational Principles

In order to be able to derive the Poisson equations from the variational principle for the fields that we have stated in the previous subsection, we need to find an appropriate Lagrangian for the Newtonian gravitational potential. To this end we use the Lagrangian for the equations of motion as a kind of guiding reference. Similarly to the latter, which describe the evolution of the motion of particles in time, we can consider the field equations as equations describing the evolution of the "motion" of the field in space (and in time, if $\phi = \phi(\mathbf{x}, t)$), i.e., as a sort of "equations of motion" of the field.

Lagrangian of a free field

The Lagrangian of a particle is a function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ of the generalized coordinates and velocities and of the time which in general writes (Eq. (1.2.18))

$$L = T - V = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} - V,$$

where V is the potential energy of the force interacting with it and (see Eq. (1.2.5)) $\mathbf{v} = \mathbf{v}(\dot{\mathbf{q}})$. Comparing it with the expression of the Lagrangian density $\mathcal{L}\left(\phi, \dot{\phi}, \mathbf{x}, t\right)$, we can reckon the quantity $\dot{\phi} \equiv (\partial_t \phi, \nabla\phi)$ as the equivalent of the velocity for the field. In particular, because for Newtonian gravitation the field $\Phi(\mathbf{x})$ does not depend explicitly on the time, we are seeking for a Lagrangian[11] of the kind $\mathcal{L}(\Phi, \nabla\Phi, \mathbf{x}, t)$ where the role of the velocities is played by just $\nabla\Phi$. It is then reasonable to assume by analogy that

$$\mathcal{L} = a\left(\nabla\Phi \cdot \nabla\Phi\right) + V_{\text{int}}(\Phi) \tag{1.3.9}$$

where a is a constant and the interaction potential energy $V_{\text{int}}(\Phi)$ can be regarded as a sort of "self-interaction" of the field with itself. Let us consider for the moment just the case of no self-interaction, namely that of the so-called *free field*, and let us put[12]

$$\mathcal{L} = \frac{1}{2}k\nabla\Phi \cdot \nabla\Phi \tag{1.3.10}$$

where $k \neq 0$ is a constant with the appropriate dimensions. From Eq. (1.3.6) we have then

$$\nabla \cdot \left(\frac{\partial\mathcal{L}}{\partial(\nabla\Phi)}\right) = k\nabla^2\Phi = 0 \ \Rightarrow\ \nabla^2\Phi = 0 \tag{1.3.11}$$

which means that our choice brought us to the correct statement that free fields obey the Laplace equation.[13]

Lagrangian for the Newtonian gravity field

The simplest choice for the interaction part of the Lagrangian is to take it linear in Φ.[14] We have also to consider that $\mathcal{L}$ must have dimensions compatible to the action functional we want to obtain, i.e., those of an energy density in space, and that obviously these dimensions must be the same for the "kinetic" and "interacting" parts *separately*. The dimensions of the gravitational field are those of an energy per unit mass, therefore it is then natural to multiply the latter by ρ. Conversely, the "kinetic" part is proportional to $\nabla\Phi \cdot \nabla\Phi$, which dimensionally corresponds to $[\text{energy}]^2 \cdot [\text{mass}]^{-2} \cdot [\text{length}]^{-2}$, so the constant a should be

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[11] From now on, as it is customary in field theory, for the sake of brevity we refer to Lagrangian density simply as "Lagrangian" when speaking of field equations.

[12] This is consistent with the analogous case of a Lagrangian of a free particle, i.e., a particle subject to no interacting force $L = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}$. The reason for putting a as a constant is made clear in the next chapter. Finally, the "self interaction" character of $V_{\text{int}}(\Phi)$ comes from the fact that, although the interacting entities are the matter and the field, as we show in a moment, the free term in this case is the field, which is also by definition the source of interaction.

[13] Once again, this is consistent with the parallel case of a free particle whose Lagrangian, as is immediate to see, brings to the equation $\ddot{\mathbf{x}} = 0$.

[14] We may think of it as a sort of first-order expansion in Φ.

[energy]$^{-1}$·[mass]^{2}·[length]$^{-1}$, which is equivalently [mass]·[length]$^{-3}$·[time]^{2}. The fact that these agree with the inverse of the gravitational constant G suggests that we are on the right path, and if we compare Eqs. (1.3.10) and (1.3.11) with Eq. (1.1.1), it is easy to find

$$k = \frac{1}{4\pi G},$$

so that

$$\mathcal{L} = \frac{1}{8\pi G}\nabla\Phi \cdot \nabla\Phi + \rho\Phi. \tag{1.3.12}$$

Using this Lagrangian density we finally have

$$\begin{aligned} 0 = \partial_t\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\Phi\right)}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\left(\nabla\Phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi} &= \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\left(\nabla\Phi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi} \\ &= \frac{1}{4\pi G}\nabla^2\Phi - \rho \end{aligned}$$

that is, exactly the Poisson equation for the gravitational field.

Summarizing our procedure, first we have shown that a functional corresponding to an action of a field should be represented in the form of Eq. (1.3.2), and afterward that a stationarity requirement for this action and for field variation vanishing on the boundaries of its domain implies, as for the principle of least action for particle motions, that the field evolves according to the Euler–Lagrange equation (1.3.6). Exploiting the analogies with the case of the Lagrangian of the mass particles we have then found a Lagrangian density of the gravitational field with which these equations can produce the Poisson equation exactly. This shows that the variational principles can be applied to fields as well and that, as in the case of the motion of particles, they can be used to formulate laws which are fundamental at the same level with respect to their differential counterparts.

1.4 Exercises

Exercise 1.1 Show that the Lagrangian of the two-body problem under the influence of a central force can be decomposed in two independent parts, which are functions of the coordinates of the barycenter and of the relative positions and velocities, respectively.

Solution 1.1 If $\mathbf{x}_1$ and $\mathbf{x}_2$ are the position vectors of the two bodies, having mass m_1 and m_2, respectively, the central force is characterized by a potential $V = V(r)$, where $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$. Its Lagrangian therefore is

$$L = T - V = \frac{1}{2}m_1\mathbf{v}_1 \cdot \mathbf{v}_1 + \frac{1}{2}m_2\mathbf{v}_2 \cdot \mathbf{v}_2 - V(r). \tag{1.4.1}$$

By definition, the position vector of the barycenter of the system is

$$\mathbf{x}_B = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2},$$

which obviously means that

$$\mathbf{v}_B = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}.$$

A simple calculation gives

$$(m_1 + m_2)\, \mathbf{v}_B \cdot \mathbf{v}_B = \frac{m_1^2 \mathbf{v}_1 \cdot \mathbf{v}_1 + 2 m_1 m_2 \mathbf{v}_1 \cdot \mathbf{v}_2 + m_2^2 \mathbf{v}_2 \cdot \mathbf{v}_2}{m_1 + m_2},$$

that is,

$$(m_1 + m_2)\, \mathbf{v}_B \cdot \mathbf{v}_B - 2 \frac{m_1 m_2}{m_1 + m_2} \mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{m_1^2 \mathbf{v}_1 \cdot \mathbf{v}_1 + m_2^2 \mathbf{v}_2 \cdot \mathbf{v}_2}{m_1 + m_2}.$$

Thus, by adding $[m_1 m_2/(m_1 + m_2)]\, (\mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2)$ to both sides we have

$$(m_1 + m_2)\, \mathbf{v}_B \cdot \mathbf{v}_B + \frac{m_1 m_2}{m_1 + m_2} (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) = m_1 \mathbf{v}_1 \cdot \mathbf{v}_1 + m_2 \mathbf{v}_2 \cdot \mathbf{v}_2. \tag{1.4.2}$$

Because $\mathbf{v}_r \equiv \mathbf{v}_2 - \mathbf{v}_1$ is by definition the relative velocity of the two bodies, by comparing Eqs. (1.4.2) and (1.4.1) it is easy to see that the above Lagrangian is also

$$L = \frac{1}{2} m \mathbf{v}_B \cdot \mathbf{v}_B + \frac{1}{2} \mu \mathbf{v}_r \cdot \mathbf{v}_r - V(r),$$

where $m = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$ is the *reduced mass* of the system. However, this Lagrangian is the sum of the two independent Lagrangians

$$L_B = \frac{1}{2} m \mathbf{v}_B \cdot \mathbf{v}_B$$
$$L_r = \frac{1}{2} \mu \mathbf{v}_r \cdot \mathbf{v}_r - V(r),$$

or in other words instead of having a single Lagrangian depending on the coordinates and velocities of the two bodies $\mathbf{x}_{1,2}$ and $\mathbf{v}_{1,2}$, we have disentangled the problem in two separate problems. The first one depends only on the velocity of the barycenter of the system and tells us that this point moves as a free particle. The second one depends only on the relative coordinates and velocity of the two bodies.

Exercise 1.2 Show by using the Euler–Lagrange equations that the motion of a test particle m under the influence of the gravitational potential $\Phi = -GM/r$ is planar.

Solution 1.2 The potential under consideration is that of a central force, so we can use the results of the previous exercise. Moreover, conventionally a system of two bodies, in which M is the central mass and m is a test particle, means that $m \ll M$, therefore the above case can be specialized with the further assumption that the total mass is in practice that of M and $\mu \simeq m$. We thus know that the barycenter of the system coincides in practice with the position of M, whose motion is therefore that of a free particle. Moreover, to ease the notation we can put $\mathbf{v}_\mathrm{r} \equiv \mathbf{v}$.

The motion of the test particle instead, comes from the Lagrangian $L_\mathrm{r} = T_\mathrm{r} - V(r)$, which in this case reads

$$\begin{aligned} L &= \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} + G \frac{mM}{r} \\ &= m \left[\frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\phi}^2 \right) + \frac{GM}{r} \right] \end{aligned}$$

in polar coordinates, and because for our purposes it is sufficient to consider the case of $m = \text{const}$, the Euler–Lagrange equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m \left[\frac{\mathrm{d}\dot{r}}{\mathrm{d}t} - r \left(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2 \right) + \frac{GM}{r^2} \right] = 0 \tag{1.4.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(r^2 \dot{\theta} \right) - r^2 \sin \theta \cos \theta \, \dot{\phi}^2 \right] = 0 \tag{1.4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = m \frac{\mathrm{d}}{\mathrm{d}t} \left(r^2 \sin^2 \theta \, \dot{\phi} \right) = 0. \tag{1.4.5}$$

We can immediately exclude the trivial solution $m = 0$, which implies that the particle does not interact with the gravitational field and has no inertial mass.[15] Then we can notice that the change of variable $\tilde{\theta} = \pi - \theta$, which implies that $\sin \tilde{\theta} = \sin \theta$, $\cos \tilde{\theta} = -\cos \theta$ and $\dot{\tilde{\theta}} = \dot{\theta}$, leaves these equations invariant. We can deduce from this that the motion is planar.

Let us suppose, in fact, that at a given instant, that we can choose $t = 0$ with no loss of generality; it is $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$. The planar solution passing from this point is $\theta(t) = \pi/2$ and $\dot{\theta}(t) = 0$ for any t, whereas a solution admitting a generic motion $\theta(t)$ different from the previous one would necessarily not be planar because it would require that $\ddot{\theta}(0) \neq 0$. However, if such nonplanar $\theta(t)$ existed, the function $\tilde{\theta}(t)$ would also be a solution and the system would then admit two different solutions from the same initial conditions, which is not possible.

The only possible solution for these initial conditions is therefore the planar one and the motion on the plane $\theta = \pi/2$ is stable. Moreover, this result can be extended to any plane passing from the origin, because this can lead back to the case of the equatorial plane with a simple rotation.

[15]The stage, this in practice means that there is nothing at all.

Exercise 1.3 From the solution of the previous exercise, show that there exists a class of bounded orbits and that it is periodical.

Solution 1.3 We have just seen that this does not imply any loss of generality, therefore we can put ourselves in the case of $\theta = \pi/2$ and $\dot{\theta} = 0$, so that the equations of motion reduce to

$$\frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = r\dot{\phi}^2 - \frac{GM}{r^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\phi}\right) = 0.$$

The second one immediately means that $r^2\dot{\phi} = h = \mathrm{const}$, which is the well-known Kepler's second law stating the constancy of the angular momentum, so that the first one can now be written

$$\ddot{r} = \frac{h^2}{r^3} - \frac{GM}{r^2}. \tag{1.4.6}$$

We neglect the almost trivial case of $h = 0$, whose solution is an accelerated radial motion toward the origin. In the case of $h \neq 0$, instead, because $r \geq 0$ we have that $\dot{\phi}$ is either always positive or negative, which means that the function $\phi(t)$ is a one-to-one relation and therefore can be inverted, using ϕ instead of t as a parameter for r.

In this sense we can then write

$$\begin{aligned}\ddot{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) &= \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) \\ &= \frac{h^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{r'}{r^2}\right),\end{aligned} \tag{1.4.7}$$

where we used the symbol "′" to indicate the derivation with respect to ϕ.

Now, from Eqs. (1.4.6) and (1.4.7), and following the standard procedure to solve the radial equation, we use the substitution $r = 1/u$, so that $r' = -u'/u^2$; therefore it is

$$-h^2u^2\frac{\mathrm{d}u'}{\mathrm{d}\phi} = h^2u^3 - GMu^2,$$

or

$$u'' + u = \frac{GM}{h^2}, \tag{1.4.8}$$

a standard second-order differential equation with constant coefficients whose solution is

$$u(\phi) = C\cos(\phi - \phi_0) + \frac{GM}{h^2}.$$

This function is periodical, with a period 2π, and thus the same is true for $r(\phi)$.

Incidentally, one could easily show that in the case of $C \geq 0$ the above equation represents an ellipse with eccentricity $e = Ch^2/GM$ and semi-major axis

$$a = \frac{GM/h^2}{\left(GM/h^2\right)^2 - C^2}.$$

In this way, in fact, it results in

$$u(\phi) = \frac{GM}{h^2}\left(1 + e\cos\left(\phi - \phi_0\right)\right)$$

and

$$\frac{h^2}{GM} = a\left(1 - e^2\right), \tag{1.4.9}$$

so that

$$u(\phi) = \frac{1 + e\cos\left(\phi - \phi_0\right)}{a\left(1 - e^2\right)}, \tag{1.4.10}$$

or

$$r(\phi) = \frac{a\left(1 - e^2\right)}{1 + e\cos\left(\phi - \phi_0\right)}. \tag{1.4.11}$$

Chapter 2
The Geometrical Character of Physics Theories

One of the powerful features of the variational approach is its generality. It is in fact based on a simple principle, stating that physical laws share the common characteristics of minimizing the action of a system, and as we have seen this can be applied to mechanics (i.e., the dynamics of particles) as well as to the dynamics of fields. This obviously cannot be regarded as the silver bullet of the physics problems, inasmuch as no one can anticipate which is the right expression of the Lagrangian for a given topic, and therefore that of the action to be minimized. The solution of such a task, however, is far from being purely arbitrary, and often many of the characteristics of a correct action can be deduced by some basic principles characterizing the physics theory of reference. Moreover, it is somewhat surprising to realize, as we do in the remainder of the book, that this technique can be applied to many different theories: Newtonian or relativistic, classical or quantum.[1] It is therefore extremely useful to understand which are these principles and how the differences among them can lead to completely different theories.

This is the goal of the next chapter but, even before that, it is worth anticipating that these principles are closely related to the first reason we advanced to justify the introduction of the variational approach, namely the possibility of deducing the equations of motion in any coordinate system. They in fact are the expression of a "geometrical connection" that stands at the very basis of our way of formulating physics theories.

[1]Given the scope of this book, the exposition is limited to classical physics. However, as long as the mechanism of the so-called *quantization* is understood, the same techniques can be applied to quantum models.

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2.1 A Scientific Theory as a Model

It might seem starting from afar, but inasmuch as this book is dealing with physics theories it is reasonable to ask ourselves if we actually realize what a physics theory is. As in the previous chapter, we choose to refrain from giving an abstract definition and instead start by describing what can probably be considered the first historical example of a scientific theory, when not even of a real theory of mathematical physics: Euclidean geometry.

Euclidean geometry as a “physics” theory

In today’s view it is difficult to see Euclidean geometry in its physical “nuance,” but in the past, at least until Newton, the feeling was completely different. To this aim it is sufficient to remember that Galileo considered the world “written in geometrical characters,” or that many demonstrations of Newton’s *Principia* are based on some theorems of Euclidean geometry which, albeit now considered exotic, at that time were part of the common knowledge of scientists as it presently for calculus. But this is even more meaningful if we consider Newton’s position precisely with regard to calculus. It is well known that he was one of the inventors of this powerful mathematical technique, which a modern physicist would consider essential to explain the theories of the English scientist. It thus can be surprising to realize that Newton himself never used calculus in the *Principia*, that rather his exposition is entirely based on Euclidean geometry, and that even the structure of the work closely reproduces the axiomatic and deductive one of Euclid’s *Elements*.

If we go even farther in the past, to the times of Hellenistic science, we can see more examples of such a connections because the physics and astronomy works of the time used Euclidean geometry to such an extent that it cannot be clearly stated, in terms of our mindset, whether they are mathematical or physical works (Russo 2004). The way this theory is now presented favors its misconception as a highly abstract mathematical work, with no connection to “reality”, but this vision might change if we realize that in its original version Euclid’s *Elements* contained not only definitions and theorems, but also what we could call *problems,* showing how some geometrical figures could be drawn, and in a certain sense could be considered as applications of the theory to practical tasks. This is not so strange if we think that geometrical methods are needed for many practical purposes and were used by other ancient civilizations way before Euclidean systematization. Does this mean that these cultures were using a scientific theory? The answer is no, because there are fundamental differences between their knowledge and how we specify a scientific theory today, and these are the same differences they show with respect to Euclid’s exposition.

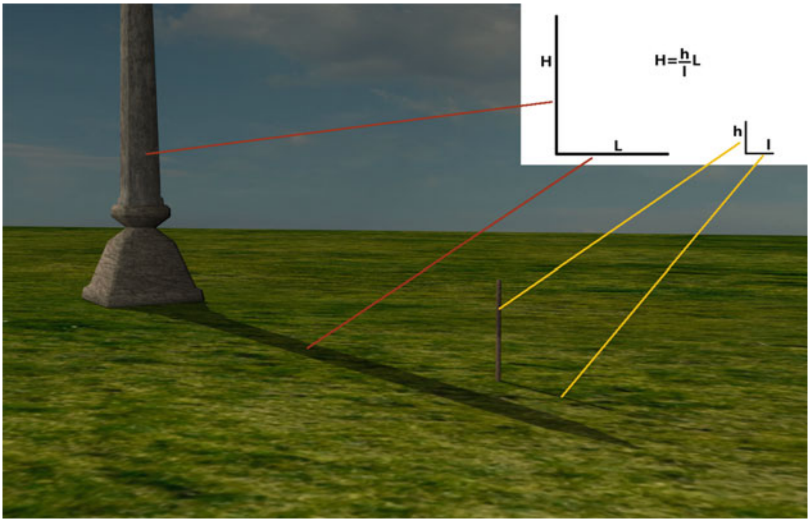

Fig. 2.1 The "rules of correspondence" between Euclidean geometry and reality in action.

Model versus reality in scientific theories

The latter starts with some definitions, concerning the concepts of point, line, surface, and so on, and with five assumptions or axioms,[2] that is, some statements which cannot be demonstrated by other, more elementary facts and thus taken as true *by hypothesis*.

Whereas the role of the latter is clear as the "game rules" of a theory, those of the definitions are usually less emphasized. Nonetheless they too play a fundamental role in a scientific theory, precisely that of establishing the "rules of correspondence" between the *real world* and *its representation given in the scientific model*.[3] For example, when we appeal to the triangles' similarity theorems to deduce the height of an obelisk from that of a smaller stick and the lengths of the two shadows (Fig. 2.1) we are implicitly doing the following reasoning: because the only property of interest in this case is a length, in any essential sense we can represent the two real objects and their shadows as segments of a triangle in our *model of reality* as far as the

[2] And also five common notions, which can be regarded as other axioms.

[3] The term "rules of correspondence" was first used in this sense by the philosopher Rudolf Carnap (1891–1970) in his concept of a scientific theory as an axiomatic formal system. In the case under discussion, strictly speaking, the definitions tell us which are the "characters of the game." Their correspondence with entities of the real world, however, is implicit in its use. For Euclid's *Elements* these regard the constructions that could be done with rulers and compasses, whereas in other works using these geometric theories as its fundamental tool this "mapping" could concern different objects.

solution of this problem is concerned. We are then allowed to say that the results of the theorems, which "live" in our model, correspond to a correct result in the real world.

We can also sketch a more elaborate picture. Let us imagine that initially a scientist starts with these "rules" (i.e., the "mapping" between the reality and our model, and the axioms) and wants to understand how accurately this model works. Then one could start off by deducing the consequences from the hypotheses, i.e., the theorems, and apply them back into reality using the rules of correspondence backwards. Sooner or later our imaginary scientist will run across the problem of the obelisk, which will represent a twofold step:

- If by direct measurement the *predictions* of our model are shown true, this constitutes an *experimental verification* that our model is a correct representation of that part of the reality covered by the rules of correspondence.
- On the other hand, the theorem will have given us access to a *new application/technology* that was previously unknown.

It is worth stressing that the last finding was made possible by the simplifications imposed in our model by the rules of correspondence. It would probably have been much harder, when not infeasible, to get the same results if one had to take into account, e.g., the color of the obelisk, its material, the day it was built, or also simply its actual shape. In other words, *simplification* and *schematization* play important roles in the definition of a scientific theory by filtering out of the model all the characteristics deemed in essential to its goals.

Although oversimplified, this schema well represents what in practice happens every day in science. It can have a more evident connection with mathematics in works such as those of Archimedes or of Hipparchus, where the rules of correspondence of the geometric entities refer to physical or astronomical objects, but the same principles apply, for example, to "softer" sciences such as biology, although more loosely.

2.2 Geometry and Physics: Tools for Modeling the Reality

Just the fact that Euclidean geometry can be regarded as the first historical example of a scientific theory, and that it is at the base of many ancient examples of physics theories, would be enough to show the argued connection between geometry and physics. But actually that is stronger than this.

Reference system and physical space

The concept of *reference system* is ubiquitous in physics, as well as that of geometrical objects including vectors, tensors, and the like that are directly connected with the former and constitute the building blocks of the equations of physics. In the framework depicted in the previous section, our rules of correspondence are set in such a way that the reference system represents the *model of physical space*, and therefore the geometrical objects "living" in it have to be matched with appropriate physical objects.

Choosing the right geometry

However, as we have stressed above, one geometry is defined by its postulates, examples of which are represented by the five axioms of Euclidean geometry, and in principle we have no limitations in choosing them. The choice, indeed, is sometimes believed to be governed by their alleged "self-evident truth," however:

1. The mere fact that we require them to correspond to some kind of truth implies that we are making a comparison with our perception of reality in the physical world, which means that geometry takes its origin from the need of modeling it.
2. Such so-called "self-evidence" can be just apparent, as in the famous case of Euclid's fifth postulate.

The history of this axiom represents an enlightening example of the difficulty of identifying the "right" set of postulates. First of all, a fundamental characteristic of an axiom is that it must be independent of the others, namely that it cannot be derived from other assumptions. It is well known that for at least 10 centuries many brilliant scientists unsuccessfully tried to demonstrate that the fifth postulate was not independent of the previous four, and these attempts continued until in the nineteenth century Beltrami (1868) proved that it was indeed the case.

Such proof came together with the discovery that other equally self-consistent geometries could be generated by taking another version of this postulate, which raises another strictly related problem: if these geometries are equivalent to each other from the point of view of their internal consistency, why should we prefer one or another as the correct model of the physical world? One might be tempted to say that Euclidean geometry is self-evident because every day it shows its adherence to reality. This, however, would just show once again the tight connection between geometry and physics, and moreover that the way we built it up was experience-driven.

We return to this concept in the next chapter, but now we want to remark that adherence to reality is what we ask of a physical theory, not geometry as an abstract mathematical construction. Thus physics and geometry are mutually interrelated because the goal of physics is that of finding the "governing principles" of the physical world, whereas geometry provides a way to translate them in a mathematical language through its rules of correspondence.[4]

[4]It has to be stressed that this connection must not be interpreted as a one-to-one identification between physics and geometry. For example, a particle in classical physics can correspond to a geometric point, but in a gravity theory this model includes a property called mass, and in electrodynamics we have an electric charge.

It can thus happen that the selection of specific physics principles implies the selection of a specific geometry which therefore represents the “best fit” for the mathematical description of that theory.

This process is based on a requirement called the “principle of covariance,” which we begin to explore in detail in the next chapter.

2.3 When Should a Scientific Theory Be Changed?

The possibility of having different physics theories even at the geometric level further enhances the need to understand the criteria that can help the selection of the best model. We can summarize them in the following list.

1. Comparison with experimental results
2. Compatibility issues between different theories
3. Unsolved “philosophical” or self-consistency issues

Actually one should not be deceived by this rigid classification. All these events can appear intermingled, which indeed is what commonly happens in the real world.

The first point of the list should be the most obvious for us inasmuch as it has already been mentioned in Sect. 2.1. A scientific theory can make predictions that can be seen as: (a) deductions of a chain of consequences starting from its postulates (theorems); (b) the translation of such theorems in the physical world by means of its rules of correspondence. These predictions are in principle subject to direct verification that can support or disprove the theory.

A typical example of the second point is the general relativity versus quantum physics issue, and it is also well known.

Finally, an easy example for the third case is Newtonian gravity theory, with its implication of an “action at distance” (see Sect. 4.1) which was “philosophically” difficult to accept even for Newton himself and moreover had a further problem of compatibility with classical electrodynamics, which instead required an interaction propagating at finite speed. Other less known examples are special-relativistic theories of gravity and their predictions for particles traveling at the speed of light.[5] Notably, it is possible to have theories accepting the existence of particles moving at the speed of light and getting deflected by a mass, but only if they are massive particles, whereas special relativity requires that only massless particles can travel at the speed of light, but they are not deflected by special-relativistic gravity.

In this book we find and explore in more detail examples of each of these cases.

[5] There are many such theories, and their predictions for these particles can vary considerably.

Chapter 3
Fundamental Principles of Classical Physics

In Chap. 1 we have shown how the classical Newtonian dynamics and Newtonian gravity can be formulated as a field theory based on the variational principle of least action. In this chapter we examine this formulation from a more fundamental standpoint. This approach makes it clearer which are the postulates at the basis of these theories in the "physical-theory-as-a-model" sense highlighted in the previous chapter. At the same time this is useful to better understand how the original theory can be extended (a "procedure" that is extensively used for other field theories) and also to start showing the "weak points" of Newtonian gravitation, which eventually brings us to special and general relativity.

3.1 Principle of Covariance

The introduction of the Lagrangian formalism of Sect. 1.2.1 was originated by the need of finding the expressions of the equations of motion (1.1.2) "projected" in a completely general coordinate system. Although this was necessary for the practical issue of solving those equations, actually one could argue that the physical meaning expressed by these equations should not depend on the choice of a specific coordinate system. The essential idea at the basis of this statement is that the coordinates we use to describe a physical system are just tools we need to address a practical problem, but they do not exist in reality, or in any case they have no fundamental role in the formulation of the laws of physics (and more generally of the laws of nature). This is an aspect of what is called the principle of covariance.

How can this be translated in a mathematical way? In a certain sense we already know the answer as we are used to expressing the laws of physics in vectorial form. It is in fact intuitive to realize that a vector does not change for any change of the coordinates, therefore, provided that our laws are expressed as vectorial expressions, we can be safe about their invariance or, to say better, covariance with respect to this kind of transformation. This answer, however, is only partial, and for a more

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complete and satisfactory view it is necessary to make a clearer statement on what can be meant with change of coordinates.

In a very rough sense, coordinates can be defined as mere “numbered labels” we can use to mark the “ingredients” of our physical laws that, with the help of appropriate “recipes,” can be conveniently used in some cases such as, e.g., to describe quantitatively our measurements, so a change of coordinates might in general be intended as a change of the numbers we associate with such ingredients, or more restrictively a change of the rules used to associate the elements of a physical model with numbers. It is then reasonable to assume that the very nature of such ingredients cannot depend on the label we decide to attach to them. Moreover, given the somewhat arbitrary nature of these assignments, it is also reasonable to require that our physical models can be formulated in a coordinate-independent way in order to avoid the need to resort to countless different and (apparently) unrelated equations describing the same physical phenomenology. Therefore what kind of association rules are we speaking about with the expression “change of coordinates”?

Actually, in the previous chapter and at the beginning of this section, we used, more precisely, the expression “change of *coordinate system*”. This has to do with a specific kind of rules we use to label the events of our space once we have set a reference system. Examples of such coordinate systems are the Cartesian or polar coordinates, but also more exotic ones such as the generic coordinates of Lagrangian mechanics. Although the covariance of physical laws expressed in vectorial formalism with respect to such kind of coordinate changes should be known to anyone used to this formalism and its coordinate representation, the reader can refer to Appendix B for a more formal definition of a coordinate system and for a mathematical exposition of such covariance.

But this is not the whole story, because the coordinates (i.e., the numbers on the “labels”) can obviously change also because of a change of the reference system itself, and although the covariance requirement in the case of coordinate systems carries little or no physical meaning, we show that asking that physical laws be covariant for changes of reference system has much more to do with physics, such that the choice of a particular kind of covariance instead of another means the selection of one physical model instead of another, as anticipated in the previous chapter. To this aim, we need to step back to a more elementary view of the vectorial formalism and of the geometry on which it is based.

3.1.1 Euclidean Space

Actually the mathematics we used to formulate our physics models thus far is based on Euclidean space and time. This implicitly means that our basic hypothesis is that physical laws and theories can be correctly modeled in this framework, therefore it makes sense to ask ourselves which are the basic properties of these mathematical objects. From a physical point of view the concept of Euclidean space takes its origin from its capacity to reproduce the three-dimensional space that we perceive with our senses giving us a way to describe it quantitatively, i.e., allowing a quantitative description of the *measurements* we can make. The two most fundamental measurements we can conceive in such space are those of lengths and of angles and, although the original formulation of the Euclidean geometry was in the completely different framework of the "rulers and compasses" approach of the Greek and Hellenistic period, which we might call *analogical geometry*,[1] for historical reasons it evolved to its present formulation based on numerical computation and coordinate representation that conversely is known as *analytical geometry*.

Euclidean space as a metric space

It is in this latter sense that we "build" the Euclidean space by taking the set $\mathbb{R}^3$ and, with the assumption that each of the three numbers of its elements has the meaning of a coordinate in a specific coordinate system, by providing it with a *metric*, namely with a "recipe" to define the distance between any pair of elements of this set, which is the translation or, in the sense given in Chap. 2, the *correspondence rule* of the analogical measurement in our "numbers and coordinates" framework. If we consider the Cartesian coordinate system then the distance d between the two elements (points) $\mathsf{P}_1 = \{x_1, y_1, z_1\}$ and $\mathsf{P}_2 = \{x_2, y_2, z_2\}$ is given by the formula

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

which has its obvious origin in the Pythagorean theorem. Taking the difference $\Delta\mathbf{x} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\} \equiv \{\Delta x, \Delta y, \Delta z\}$ as a "prototype" of a vector[2] then the above formula can be expressed as the result of the scalar product

$$\Delta s^2 \equiv d^2 = \Delta\mathbf{x} \cdot \Delta\mathbf{x} \equiv \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{3.1.1}$$

where, in general, the scalar product of the two vectors $\Delta\mathbf{x}_1$ and $\Delta\mathbf{x}_2$ is

$$\Delta\mathbf{x}_1 \cdot \Delta\mathbf{x}_2 \equiv \Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1 \Delta z_2.$$

The last equation also allows us to determine the angle α between the two vectors $\Delta\mathbf{x}_1$ and $\Delta\mathbf{x}_2$ as

$$\cos\alpha = \frac{\Delta\mathbf{x}_1 \cdot \Delta\mathbf{x}_2}{\sqrt{\Delta\mathbf{x}_1 \cdot \Delta\mathbf{x}_1}\sqrt{\Delta\mathbf{x}_2 \cdot \Delta\mathbf{x}_2}} \tag{3.1.2}$$

[1] In the sense that Euclidean geometry can be seen as a method to solve practical problems and to make computations with the help of rulers and compasses, as cited in Russo (2004) and as was evident in the original edition of Euclid's *Elements*.

[2] For the sake of generality we use coordinate differences instead of just coordinates; in fact the latter can be considered a particular case of the former, that is, the difference $\mathsf{P} - \mathsf{O}$ between the coordinates of the point P and those of the origin O of the reference system. Moreover, this allows us to naturally to the indexed components notation we use extensively throughout the book.

which, in the same way as the distance definition above, translates the compasses, operations in the language of analytical geometry giving an appropriate rule of correspondence.

Homogeneity and isotropy of space

The explicit reference to the correspondence between length and angle measurements in analytical and analogical geometry makes it natural to understand the origin of the covariance that is "embedded" in a vectorial formulation of an equation. Indeed, our everyday experience tells us that the length measurements we can make with a ruler do not change if we shift or *translate* the object in space nor will it change if we *rotate* it, and the same is true for angle measurements. Conversely, these measurements remain the same also for any observer in another position and with another orientation, that is to say from another reference system translated and/or rotated with respect to our own. We say that the Euclidean model of space is *homogeneous* and *isotropic*, respectively. These concepts are naturally enclosed in analogical geometry because there is a direct (actually analogical) connection between our senses and this model of space via its measurement instruments. These properties will therefore be transferred to our analytical formulation as long as we use the adequate "models" of rulers and compasses. Equations (3.1.1) and (3.1.2) give us such models, showing that they do not depend on coordinates but just on coordinate *differences*. Moreover, it is straightforward to realize that length and angles are left unchanged by a change of sign of the vectors, i.e., transforming $\Delta\mathbf{x}$ to $-\Delta\mathbf{x}$. Mathematically we just need to substitute the signed-reversed vectors into the two equations, whereas from a geometrical point of view we can easily understand why by noticing that the change of sign can be seen as reversing the tails and tips of vectors, which obviously has no influence in the determination of a distance or of an angle. This kind of operation is related to another transformation of reference systems called *parity reversal*.[3] We show more of this transformation in Exercise 3.1, however we do not enter into too much detail because it is beyond the scope of this book.

Covariance group of Euclidean space

It is now time to summarize quickly what has been shown up to now. We have stated that:

1. A coordinate-invariant or coordinate-covariant formulation of physical laws is convenient because it provides a concise and unique way to write them, but it also takes its origin from the use of Euclidean geometry, which is the mathematical

[3] Parity reversal can be obtained by the change of sign of just one component, but in three dimensions; this is equivalent to changing the sign of all the components. This is not true in two dimensions, where changing the sign of the two components is equivalent to a 180° rotation, instead.

model, in the sense stated in Chap. 2, of the three-dimensional space we perceive with our senses.
2. Such model is endowed "by construction" with length and angle measurements, which are directly connected with the working tools of analogical geometry, namely rulers and compasses.
3. Its analytical geometry version is based on vectors and coordinate representation, and its "correspondence rules" provide the formulae for lengths and angles in this framework.
4. The same everyday experience that contributed to the birth of the analogical version of Euclidean geometry suggests that lengths and angles must remain unchanged when their corresponding segments are moved, rotated, or reversed.

We have already found the equivalent mathematical operation of the segment reversal, which is the change of sign of its related vector, whereas translations and rotations can be written in the analytical geometry framework in terms of vectors' addition and matrix multiplication, or formally

$$\mathbf{x} \rightarrow \bar{\mathbf{x}} = -\mathbf{x} \tag{3.1.3}$$

$$\mathbf{x} \rightarrow \bar{\mathbf{x}} = \mathbf{x} + \mathbf{a} \tag{3.1.4}$$

$$\mathbf{x} \rightarrow \bar{\mathbf{x}} = R\mathbf{x}, \tag{3.1.5}$$

where $\mathbf{a}$ is a constant vector, and R is a rotation matrix, characterized by the properties $R^{\mathrm{T}} R = \mathbb{I}$ and $\det R = 1$. If our rules of correspondence are valid, then these transformations of reference systems should leave the lengths and angles measurements unchanged; this is shown in Exercise 3.3, which makes use of the mathematical framework detailed in Appendix B. Exercise 3.4 shows that the above transformations are also the only ones with the property of leaving these quantities unchanged.

It is instructive to notice that, although the former statement was a result of a simple translation of practical and intuitive observations in the language of a mathematical model, the latter was, on the contrary, an example of how a scientific model works in the sense intended by the "obelisk and stick example" of the previous chapter. Once the model and its rules have been established, following the consequences one can find other results (i.e., theorems) that were not obvious at first, and which possibly would have never been discovered otherwise. As long as the model is a correct representation of the "real world," we can expect that these conclusions are correct in it as well.

Another important fact is that this is one of the simplest examples of how the concept of covariance works to select the correct theories and/or models. By requiring that measurements in Euclidean space are preserved we have identified a specific set of transformations between admitted reference systems. Different requests driven by observations would have identified a different set of transformations; conversely, having a different set of transformations would be a sign of breakdown of our model. Special relativity would be an example in this sense.

Because these transformations form a group for the geometrical model of our three-dimensional world (see Exercise 3.5) they are called the *covariance group* for the Euclidean space.

3.1.2 Euclidean Space and Time

Homogeneity of time

Euclidean space is only one of the ingredients generally needed in a physical theory. Another fundamental one is time. In classical physics this can be conceived as a one-dimensional Euclidean space, i.e., as the set of real numbers $\mathbb{R}$ with the trivial metric $|\Delta t|$. A fundamental characteristics of Euclidean space and time is that they behave as two separate metric spaces. Mathematically, one could imagine a full four-dimensional Euclidean space, but in this case, as happens in the usual 3D space, in general a transformation affecting one dimension would involve the others as well. This is not what we want, because our everyday experience tells us that this is not what happens, but once again it has to be clear that the separate Euclidicity of space and time are just *experience-driven model assumptions*.

Because Euclidean space is homogeneous and isotropic, it is natural to expect that similar properties should hold for Euclidean time. This is partially true, as the length of a time interval does not depend on the initial or final time of the interval, but just on its differences; i.e., time is homogeneous. The duration of time does not change for a parity reversal transformation as well, but it is evident that, because this space has just one dimension, there cannot be anything like isotropy. Having only one "direction" of time, there is no way to imagine a rotation here.

Covariance group

The full covariance group of the Euclidean space and time is then composed of the above Eqs. (3.1.3), (3.1.4) and (3.1.5) augmented by the following time transformations,

$$
\begin{aligned}
t &\rightarrow \bar{t} = -t \\
t &\rightarrow \bar{t} = t + t_0,
\end{aligned}
$$

where t_0 is a constant. Because they have the property of keeping the lengths of space and time intervals invariant, they are also referred to as Euclidean isometry transformations.

3.1.3 Covariance Revisited

In the above subsections the origin of the covariance principle has been established as a natural requirement stemming from the very first mathematical model of our "world": Euclidean geometry and, by extension of the same procedure to the temporal realm, Euclidean space and time. In doing so we have largely restricted the scope of this principle. We had actually started with the requirement that *all the laws of physics* be independent of the specific set of coordinates used to write them. What we have shown so far, instead, is that Euclidean space (and time), as an experience-driven mathematical model of the "real" space and time, is built in a way to model and embed the properties of length and angle invariance with respect to a specific set of (coordinate and) reference system transformations.

The expression for lengths and angles can be regarded as particular examples of physical (or geometrical) laws, but they certainly do not represent the whole world, therefore we have to understand why covariance is required for all the laws of physics and the possible mechanism that brings it to this end. But what *is* a physical law, indeed?

What is a physical law?

This question may look trivial until one realizes that it usually gets answers based on long verbal explanations which are completely useless in this context, therefore leaving the subsequent reasoning on sloppy grounds. Actually, because we are trying to set covariance in a more formal and self-consistent context, we need a definition written (or which can be uniquely translated into) in a usable mathematical way. In this sense one can simply state that *a physical law is an equation*, i.e., a relation that in its (almost) bare bones reads

$$\text{"one thing"} = \text{"another thing"} \tag{3.1.6}$$

and where the "things" we use are the mathematical representations of the objects of our world and of their mutual relations. Like the question from which it originated, this answer may seem easy or ill-defined because it is too naive. However, a starting point is always necessary, and putting it into this quite general and elementary way allows us to notice that the equal symbol implies the logic requirement that the two "things" must be of the same type, which means that an even simpler way to write the equation is actually[4]

$$\text{"something"} = \text{"0"}, \tag{3.1.7}$$

[4]This, incidentally, derives from the third *common notion* of Euclidean geometry ("If equals are subtracted from equals, then the remainders are equal") which evidently is not restricted to this model only.

where "0" means that we obviously have to intend the right-hand side not just as the zero number but rather the null element of the object type on the left-hand side.

Therefore how should the covariance requirement be interpreted for this symbolic writing? At present we are compelled to stay on what we have discussed thus far, which can be listed in these two statements:

1. Any equation has to be written in the mathematical language of Euclidean geometry, which is based[5] on a coordinate representation.
2. The equations should be written in a coordinate-independent way.

"Constructivist" approach to physics laws

Considered from this point of view, at present we are constrained to a very restrictive interpretation of the "something" to be included in the currently available covariant "Euclidean physics laws" as a *scalar*. Indeed, even if intuitively we might consider (as we have already done above) vectors as possible objects to be inserted into a physical law, strictly speaking at this point our equations (in the sense of Eqs. (3.1.6) and (3.1.7)) are including just the scalar quantities of (Euclidean) lengths and angles, although defined with the help of vector quantities as well.[6]

We therefore can start understanding the sense of having the somewhat abstract representations of Eqs. (3.1.6) and (3.1.7). When the "something" in there can be considered a scalar as a whole, then *by definition* its value does not change for any Euclidean transformation, and therefore the equation does not depend on its coordinate representation.[7]

Having clarified in this most simple case the meaning of covariance for a prototypical physics law, we still have the problem of having a very limited "space" in which our equations can live. It is then interesting to notice that we can easily extend our application domain by using the same quantities to define other ones which are scalar by construction in the same sense of the lengths and angles.

The very first of these quantities is, e.g.,

$$\Delta A = |\Delta \mathbf{x}_1 \times \Delta \mathbf{x}_2| = |\Delta \mathbf{x}_1|\,|\Delta \mathbf{x}_2| \sin\alpha, \tag{3.1.8}$$

which has a well-known interpretation as the area included in the parallelogram having $\Delta\mathbf{x}_1$ and $\Delta\mathbf{x}_2$ as sides, forming an angle α in between, and which can be trivially understood to be a scalar as a product of three scalars. Similar reasoning easily reveals that also the quantity

[5] Because of its current analytical formulation.

[6] This has to be intended in the sense that the used vectors are combined in such a way that the resulting quantity is a scalar.

[7] Furthermore, because the scalar is fully characterized by its numerical value, it follows that in this case the independence of the equation from its coordinate representation, namely its covariance, means an *exact invariance*.

$$\Delta V = |\Delta \mathbf{x}_1 \cdot (\Delta \mathbf{x}_2 \times \Delta \mathbf{x}_3)| \,, \tag{3.1.9}$$

which can be interpreted as the volume delimited by the three vectors of the expression, is a scalar in the sense of representing, like lengths and angles, a quantity that is invariant under the transformations of the Euclidean covariance group. Additionally, in the language of the Euclidean tensor formalism of Appendix B, these latter quantities can be expressed as *infinitesimal* areas and volumes which, by integration, provide convenient formulae for any kind of figures and shapes, therefore extending in a general way the domain of investigation over the geometry of every surface and volume of the whole 3D Euclidean space.

It is now worth noticing that scalars have been introduced as a mathematical representation of a measurement. As we have seen in the above sections, this comes naturally from an experience-driven requirement endowed in Euclidean geometry. Under the same "constructivist" approach that we used to extend our "Euclidean physical model" to areas and volumes, it therefore makes perfect sense to ask ourselves if one can conceive other physical entities that could be reasonably and usefully interpreted as measurements. The answer is surely positive, and it includes quantities such as mass, temperature, energy, and the like. At present we do not know how they may be represented mathematically, but experience tells us that there will probably be a perfectly reasonable way to do this.[8] Making the identification "measurements = scalars", in the sense originally imagined for lengths and angles, naturally pushes us to require that any other quantity that we want to conceive as a measurement has the same properties of the "prototype scalars", and therefore to give the following.

Definition 3.1 A (Euclidean) *scalar* is a quantity identified by a number that remains unchanged under spatial translations and rotations.

We have therefore taken an important step that brings us from just two very particular objects to a potentially very rich set of objects belonging to the same class, in the sense that they inherit from the original ones their defining properties. It might not be useless now to highlight the difference between *numbers* and *scalars*. It is common practice, in fact, to consider scalars just as numbers, but their definition and the way we arrived at it makes it clear that this statement is wrong. It is important

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[8]It is clear that such a way of enriching physical objects is somewhat unnatural. The natural process goes from mathematical models of partial and apparently distinct fields of applications that, however, share a common mathematical background and are therefore eligible to a successive interpretation within a unified viewpoint. This usually becomes convenient after having reached a certain degree of maturity and complexity in the development of the various subjects. Proceeding by consistent, although apparently arbitrary, enlargement of our physics realm within an already unified framework and common principles is thus clearly overkill at first, when dealing with just one simple model. Its sense, however, is connected to the goal of this book of giving a self-consistent view of the procedure used in theoretical physics which otherwise might appear arbitrary when shown detached from its origins. For example, the definition of a scalar as a quantity that is invariant for transformations belonging to the Euclidean (and later the Lorentz) covariance group can certainly look arbitrary if introduced just as a definition, however, when presented as an extension of some initial well-understandable models it appears absolutely reasonable.

to understand this point because missing it usually leads to successive dangerous misunderstandings.

One of the reasons at the origin of such confusion lies in the fact that within a certain mathematical context scalars and numbers are actually synonymous, but it is evident from the definition that in geometry and in physics they are not! Actually a scalar is a number with some additional invariance properties. For example, it is clear that in this sense the length is a scalar but, e.g., the x component of a vector is a number but *not* a scalar. The fundamental distinction comes from the introduction of these invariance properties that are inextricably connected with the concept of a *reference system* through the identification of a scalar as a measurement. In other words, *a scalar cannot exist without a reference system* because it is an object that "lives" in a geometric/metric space, whereas a number is an object which can perfectly exist in a simpler space such as $\mathbb{R}$ or $\mathbb{N}$, not considering this difference is equivalent to confusing $\mathbb{R}^3$ with the 3D Euclidean space by neglecting the latter's additional metric structure.

It is also worth noticing that in this definition it is not important how the quantity is obtained. Lengths can be computed as the result of a scalar product, but, e.g., this is certainly not the case of temperatures or masses. Nonetheless, because both have the same transformation properties, they are both scalars. Moreover, with a little effort of imagination we can further broaden the list of "things" eligible to be the ingredients of some physical model. Objects such as temperature or a density field, in fact, can be easily regarded as "recipes" that produce a scalar at each point of a given domain of space and time. We can then therefore give the following further definition.

Definition 3.2 A (Euclidean) *scalar field* is a function of time and position $\phi(t, \mathbf{x})$ that remains unchanged under spatial translations and rotations acting simultaneously on $\mathbf{x}$ and on the functional form of the field.

Once again, the wording "remains unchanged" means that if we keep the same numerical value, so we can write more formally that, if we indicate with $\bar{\phi}(t, \bar{\mathbf{x}})$ the result of such transformations, then it has to be

$$\bar{\phi}(t, \mathbf{x} + \mathbf{a}) = \phi(t, \mathbf{x}) \quad \text{and} \quad \bar{\phi}(t, R\mathbf{x}) = \phi(t, \mathbf{x}) \,.$$

It is evident that the same meaning of covariance given for the equations involving scalars is valid for scalar fields as well, inasmuch as now (the numerical values of) such objects, by definition, do not depend on the coordinates we use.

3.1.4 Rotational Covariance

In Sect. 3.1.3, with the aim of making clearer the logical origin of the covariance principle required for the laws of physics, we identified such laws as equations relating "things" of the same kind. We then started literally to "build up" physics by exploiting the foregoing exposition of the properties of Euclidean geometry, which was taken as our prototype of a physics model, which sounds reasonable because this geometry provides the common mathematical background to formulate any other model, at least in classical physics.

We have seen above that the homogeneity and isotropy of Euclidean space is responsible for the invariance of the distance and angular measurements, denoted as scalars, and that these two properties "translate" in mathematical language by the translation and rotation transformations, respectively, which can be regarded in turn as the prototype mathematical formulation of the covariance principle.

Using these specific scalars as the very first "things" allowed to populate our laws of physics, and by identifying them as the result of a measurement process, we naturally extended this class by including other kinds of objects that can reasonably take the same measurement meaning. Although such extended elements can represent totally different concepts including temperature, energy, or even their corresponding fields, the requirement of inheriting the same properties of their prototypes within the context of Euclidean geometry allows them to incorporate automatically the same invariance prerequisites and therefore to satisfy by construction our definition of covariance, transmitting it to other fields of physics. Because new models have always been conceived with such a procedure, although unconsciously, this highlights why this principle can be regarded as general. This also shows how this principle represents a very reasonable requirement both because it is experience-driven, but also because it is convenient from a practical point of view, because having different representations of the same phenomena according to the reference system used is certainly possible, but probably not practical.

But if the scalar quantities of distances and lengths remain invariant, what happens to the vectors with which they are defined? In the context of the constructivist approach we are following, this question makes sense, as one can certainly imagine that the "things" of Eqs. (3.1.6) and (3.1.7) can also be vectors.

In the following we refer to the formalism and the results introduced in Appendix B, where infinitesimal displacements d$\mathbf{x}$ are used instead of $\Delta\mathbf{x}$, and the Einstein summation convention is adopted. Here we learn that it is generally possible to write a vector as

$$\mathrm{d}\mathbf{x} = \mathrm{d}x^i \mathbf{e}_i \tag{3.1.10}$$

where $\mathrm{d}x^i$ are the components of the vector with respect to the basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$, i.e., the projections of the vector onto each reference direction at the point of application.

Our intuitive model of a vector as a "directional segment" or a displacement suggests that a vector "as a whole" does not change under translations or rotations of the reference system. The mathematical formalism introduced in Appendix B puts these concepts in a more rigorous way. Here we've seen, in fact, that under a rotation transformation from coordinates x^i to $\bar{x}^j$ represented by a rotation matrix $R^{\bar{j}}{}_i$, vector components transform according to

$$\mathrm{d}\bar{x}^j = R^{\bar{j}}{}_i \mathrm{d}x^i = \frac{\partial \bar{x}^j}{\partial x^i}\mathrm{d}x^i, \quad i, j = 1, 2, 3 \tag{3.1.11}$$

and the basis vectors transform[9] according to the inverse transformation[10] $R^i{}_{\bar{j}} \equiv \left(R^{-1}\right)^{\bar{j}}{}_i$

$$\bar{\mathbf{e}}_j = R^i{}_{\bar{j}}\mathbf{e}_i = \frac{\partial x^i}{\partial \bar{x}^j}\mathbf{e}_i \tag{3.1.12}$$

which means that, as in Eq. (B.2.11),

$$\begin{aligned} \mathrm{d}\bar{\mathbf{x}} &= \left(\mathrm{d}x^{\bar{j}}\right)^{\mathrm{T}} \mathbf{e}_{\bar{j}} = \left(R^{\bar{j}}{}_i \mathrm{d}x^i\right)^{\mathrm{T}} R^i{}_{\bar{j}}\mathbf{e}_i \\ &= \left(\mathrm{d}x^i\right)^{\mathrm{T}} R_i{}^{\bar{j}} R^i{}_{\bar{j}}\mathbf{e}_i = \left(\mathrm{d}x^i\right)^{\mathrm{T}} R^{-1} R\, \mathbf{e}_i = \left(\mathrm{d}x^i\right)^{\mathrm{T}} \mathbf{e}_i = \mathrm{d}\mathbf{x}. \end{aligned}$$

In other words, the vector itself ("as a whole") does not change because the variation of the components corresponds to an inverse variation, or co-variation of the basis vectors, which clarifies the covariance attribute of this principle.[11]

Having clarified this point, we are now ready to understand what happens to our prototypical physical equations, Eqs. (3.1.6) and (3.1.7) when the "things" involved are vectors, instead of scalars. We start from the rotations, recalling that, as matrix products, they are linear and homogeneous transformations, from which it immediately follows that an equation written as $\mathbf{v} = 0$, where $\mathbf{v}$ is a vector, is automatically covariant. These two properties, in fact, imply that if $\mathbf{v}$ vanishes in a reference system, then it will vanish in any other reference system related to the first by a rotation.[12] Because by definition a vector vanishes if and only if all of its components are zero, then a vectorial equation implies that three scalar equations have to be satisfied simultaneously, and the change of reference system will not affect this condition because all the transformed equations will still hold separately. This result, as for the scalars, holds until we can put the vectorial equation in the form $\mathbf{v} = 0$, i.e., until we can consider the left-hand side a vector as a whole, but this condition is not difficult to meet inasmuch as it is automatically satisfied whenever we combine vectors with vectors, which therefore is the only requirement that has to be satisfied in order to guarantee this *rotational covariance*.

This also immediately explains why physical equations are necessarily covariant, by construction, for arbitrary changes of coordinate systems as well. We have seen that the essential reason of the rotational covariance of equations expressed in terms of vectors is that rotations are linear and homogeneous transformations for vector components, a fact guaranteed by the possibility of expressing any rotation with a

[9] Active and passive rotations.

[10] Recall that, because rotations are orthogonal transformations, $R^{-1} = R^{\mathrm{T}}$.

[11] For the sake of precision, the two transformation laws identify two different type of objects, *contravariant* and *covariant vectors*, respectively. As detailed in the mathematical appendices, in Euclidean geometry this distinction is just formal and we can safely use one or the other indifferently. However, because they are different in the more general cases of the Minkowskian and Riemaniann geometry used in special and general relativity, underlying explicitly the difference also in this case eases the transition of the next chapters.

[12] Or, more generally, by any orthogonal transformation, inasmuch as these are linear and homogeneous as well.

3×3 matrix. The same is true for arbitrary transformations of coordinate systems, as we know from Appendix B. In Sect. B.2 in fact we have shown that any change of coordinates, linear or not, induces a transformation on the vector components represented by its Jacobian matrix (B.2.5), or equivalently given by Eq. (B.2.6) which, not surprisingly, is identical to that of Eq. (3.1.11). Like rotations, this transformation is precisely linear and homogeneous in the derivatives $\partial \bar{x}^i / \partial x^j$, which allows one to make the same deductions on the covariance for such kinds of transformations.[13]

Extending the vectors class

As the scalar prototypes were length and angle *measurements*, to be later extended to any other type of measurement, we have learned that the vector prototype is a *spatial displacement* $\Delta \mathbf{x}$ (or temporal, if we include the 1D Euclidean time) that is to say a set of three quantities (vector components) defined with respect to a reference system which, when translated or rotated, transforms according to Eqs. (3.1.4) and (3.1.5).[14] It is therefore reasonable to ask ourselves whether an extension process similar to that of the previous section can also be used in this case to include in our mathematical models other physical meaningful quantities having the same characteristics.

Once again the answer is positive because any "directional" quantity is eligible for this identification even if, as for the scalars, they can represent something very different from the spatial displacement that is our prototype vector. So the identification "directional quantities = vectors" naturally pushes us to give the following.

Definition 3.3 A (Euclidean) *vector* is a set of three quantities that remain unchanged under spatial translations and transform under rotations as the components of a displacement vector.

As for the scalars, therefore, vectors are objects that cannot exist out of the context of a reference system, without which no transformations can be defined. This explains why they cannot be conceived of as just a set of three numbers. A set of three coordinates also cannot be considered a vector; in fact, even if coordinates by definition imply the existence of a reference system these will obviously change under spatial translations.

On the other hand, the above definition and a simple observation provide us an easy way to define new vectorial quantities. The observation is just that if we multiply a vector by a scalar quantity, or if we differentiate it by a scalar parameter, this does

[13]One more time, this also highlights the difference between coordinates and vector components. In this case, in fact, the coordinates change according to the generic functions of Eq. (B.2.2), which are completely general and can be highly nonlinear. These same laws, however, induce the transformation of Eq. (B.2.6) for the vector components, which is in any case linear regardless of the complications of its original coordinate change.

[14]And according to Eq. (3.1.3) with respect to a parity reversal transformation.

not change the transformation properties of the derived quantity, which thus is a new vector. In this way we can state on solid mathematical grounds that quantities ubiquitous in physics, such as the velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$, the acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$, the momentum $\mathbf{p} = m\mathbf{u}$, and the force $\mathbf{F} = m\mathbf{a}$ are well-defined vectors.

The list of vectorial objects can be further enriched without difficulty if, exactly as we did for scalar fields, we introduce the concept of *vector field* giving this

Definition 3.4 A (Euclidean) *vector field* is a set of three functions of time and position $\mathbf{v}(t, \mathbf{x})$ that remains unchanged under spatial translations and transforms under rotations according to

$$\bar{\mathbf{v}}(t, \bar{\mathbf{x}}) = R\mathbf{v}(t, R\mathbf{x}) .$$

We can easily understand that such kinds of objects correspond to the idea of, e.g., a velocity flux in a fluid, but also of fields such as gravitational, electric of magnetic fields, and, once again, such objects inherit the same covariance properties of the "parent."

Euclidean tensors of higher rank

In Appendix B Euclidean space is further enriched by the definition of other kinds of objects, called (Euclidean) tensor and tensor fields, which can be used in our equations. Mathematically they are "multi-indexed quantities" obtained from the so-called *direct product* of two or more vectors. The number of indexes defines the *rank* of the tensor, thus in this sense scalars and vectors are rank 0 and rank 1 tensors, respectively. From a physical point of view, instead, the introduction of such mathematical entities stems naturally when, e.g., it is necessary to deal with extended objects such as for the definition of the stress tensor given in Sect. B.6.

The way in which these objects are defined implies that a general tensor $T^{k_1\ldots k_n}{}_{k_{n+1}\ldots k_m}$ or tensor field $T^{k_1\ldots k_n}{}_{k_{n+1}\ldots k_{n+m}}(t, \mathbf{x})$ of rank $N = n + m$ transforms under rotations according to the formulae[15]

$$T^{\bar{k}_1\ldots\bar{k}_n}{}_{\bar{k}_{n+1}\ldots\bar{k}_{n+m}} = R^{\bar{k}_1}{}_{k_1}\ldots R^{\bar{k}_n}{}_{k_n} R^{k}{}_{\bar{k}_{n+1}}\ldots R^{k_{n+m}}{}_{\bar{k}_{n+m}} T^{k_1\ldots k_n}{}_{k_{n+1}\ldots k_{n+m}}$$
$$T^{\bar{k}_1\ldots\bar{k}_n}{}_{\bar{k}_{n+1}\ldots\bar{k}_{n+m}}(t, \mathbf{x}) = R^{\bar{k}_1}{}_{k_1}\ldots R^{\bar{k}_n}{}_{k_n} R^{k}{}_{\bar{k}_{n+1}}\ldots R^{k_{n+m}}{}_{\bar{k}_{n+m}} T^{k_1\ldots k_n}{}_{k_{n+1}\ldots k_{n+m}}(t, \mathbf{x})$$

which means that each index of the tensor transforms under rotations as that of a simple vector or, conversely, that any rotation matrix (consistently with the contraction rules) is applied to one single index of the tensor. Such equations are linear and homogeneous exactly in the same way as those for vectors, and therefore the same reasoning that brought us to understand that any equation written as a combination

[15] It can be easily shown that the same transformation rules are valid for any orthogonal matrix, i.e., for rotations and parity reversal, but the latter, which is less important in the nonquantum context of this book, is skipped from now on.

of vectors is automatically covariant can be generalized to tensors and tensor fields. In other words by writing an equation in such a way that every term is a tensor of the same rank is enough to guarantee that such an equation is form-invariant under rotational (or more generally orthogonal) transformations.

It is often said that the principle of covariance is a trivial consequence of having the equations in a vectorial form, and that it has little or no physical meaning at all. The first part of this statement is in a certain sense true, as we have seen for the rotational covariance, however, the second one could not be more wrong, as we have just seen. This point is sometimes stated in a different way by saying that "otherwise physics laws would make no sense," which is somewhat more adequate but certainly not so obvious as it might seem at first sight. The reason is clearer in the following sections, and in Sect. 5.4 a more comprehensive explanation is suggested.

3.2 Principle of (Galilean) Relativity

The second principle lying at the foundations of classical physics is the principle of relativity which, despite its name, is not exclusive to special or general relativity. In modern physics it dates back to about three centuries before, when it was introduced for the first time by Galileo Galilei in the "Dialogo sopra i due massimi sistemi del mondo" (Dialogue Concerning the Two Chief World Systems). By translating its original concept in a more modern language, it states that it is not possible for an observer to deduce by any experiment[16] its state of uniform motion relative to another observer. This, obviously, unless one can directly see it!

Although the importance of this principle is never underestimated, nowadays one might be tempted to regard it as intuitive. This is definitely a deceptive impression that comes from a centuries-long tradition of classical physics, and it could be surprising to check what a person not educated in the field would find intuitive instead. This statement, indeed, marks the transition to modern physics from the Aristotelian view to an alternative model.[17] The former was based on the idea of a universe with a motionless Earth at the center, and the relativity of motion had to be invoked in support of the latter, which rather required an Earth spinning and revolving around the Sun. The confrontation between these two models of the solar system, in fact, was based on two different and opposing dynamics.

Dynamical origin of the principle of relativity

According to the Aristotelian one it was always possible to deduce the motion with respect to the Earth (and therefore with respect to the universe) because the

[16]For obvious reasons, however, the exact meaning of that "any" varies in time. In practice the Italian scientist could only conceive of mechanical experiments.

[17]Namely the other "Chief World System."

natural state of a body is that of rest. It is easy to realize that this understanding is the most obvious with respect to several observations that everyone can commonly experience. For example, one could easily come to the conclusion about the state of rest being natural by observing that in order to keep a steady pace it is necessary to exert a continuous push, without which the motion sooner or later comes to an end. So if the Earth is moving, one should at the very least introduce a nonstopping force responsible for this. But more than this, everyone can notice that when moving with respect to the Earth we "feel" such movement, possibly from the wind which pushes us backwards, so if the Earth is moving itself why aren't we feeling anything nor have we been left behind? Such were the objections based on common sense that could be (and were!) raised against the idea of Earth's motion and therefore against the Copernican model, and there is little or no doubt that anyone starting today from scratch would probably come to the same conclusions as Aristotle.

It is now natural to understand why, if the Sun-centered model had to survive the competition, it needed a completely different kind of dynamics that could explain the apparently compelling evidence against it. The key question was the natural state of rest, and it required an outstanding exercise of abstraction, with the support of a remarkable amount of experimental evidence, to come to the conclusion, nowadays known as principle of inertia, that the natural state of a body was rather that of uniform motion, and that the above observation that anything that moves will eventually stop is actually due to the presence of the friction. However, if such a conclusion is correct, why should a specific motion be preferred over another one? The obvious deduction was that no one should be, a fact that was shown by Galileo with his famous example of the ship, and which puts into evidence how the principle of inertia can be regarded as the dynamical origin of the principle of relativity.

3.2.1 Principle of Relativity as a "Kinematic" Covariance Principle

It is quite easy to convince oneself that, from the point of view of a mathematical model, the observers mentioned in the principle of relativity can be identified with different *reference systems*. Indeed, the observer can perform an experiment only by doing some measurements, and we have already identified the mathematical counterpart of a measurement as a scalar "living" in a reference system. Therefore the principle of relativity can be reformulated by saying that the result of any experiment must be the same in any set of reference systems moving uniformly, i.e., with constant velocity, with respect to each other. Because any measurement has to be formulated mathematically in terms of the laws of physics, this statement is equivalent to saying that the laws of physics have to be independent from the transformation laws between two reference systems in uniform relative motion or, in other words, that the laws of physics must have the same form in reference systems related by a uniform relative velocity.

This way of formulating the principle of relativity clearly resembles that of the covariance principle, with just a different transformation of coordinates involved. Precisely, it extends the covariance requirement to a kinematic transformation between reference systems[18] dependent on their relative velocity.

It is worth noticing that, up to now, only two implicit requirements have been imposed on how this transformation has to be written, namely:

1. It has to depend only on the *relative* velocity $\mathbf{u}$ between the two reference systems.
2. Because any inertial reference system must be equivalent, the set of transformations, as for the Euclidean ones, must form a group.

We therefore consider the so-called *Galilean boost* transformations

$$\mathrm{d}\bar{t} = \mathrm{d}t \tag{3.2.1}$$

$$\mathrm{d}\bar{\mathbf{x}} = \mathrm{d}\mathbf{x} - \mathbf{u}\mathrm{d}t \tag{3.2.2}$$

without making any attempt to justify them save for the obvious reason that they agree with common sense. In doing so we follow the consequences of adopting these specific transformations and trying to pinpoint their untold implications.

First of all condition 1 is certainly satisfied, and it can be easily verified that the same is true for condition 2 (see Exercise 3.6). Another consequence, which is implicit in Eq. (3.2.1), is that any pair of events that appear to be simultaneous in an inertial reference system are simultaneous in any other inertial reference system. It is also easy to see from the above definitions that, if we name $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ the velocity of a body in a reference system, and $\bar{\mathbf{v}} = \mathrm{d}\bar{\mathbf{x}}/\mathrm{d}\bar{t}$ the velocity measured in the transformed reference system, then it is

$$\bar{\mathbf{v}} = \mathbf{v} - \mathbf{u}, \tag{3.2.3}$$

which is once again the velocity addition formula. It is worth noticing that this necessarily derives from the absolute simultaneity of Eq. (3.2.1), which means that this equation or the velocity addition formula are equivalent ways to characterize the Galilean boost transformations.

Galilean boost and simultaneity

In the previous section measurements were defined as scalar quantities characterized by specific invariance properties with respect to the transformation of the

[18] In this sense, therefore, the rotational covariance can be regarded as a purely geometrical one.

Euclidean covariance group. That of measurement, however, is a concept which is needed in *any* reference system, regardless of the transformations which are involved. It is thus reasonable to require the same kind of invariance properties with respect to the Galilean boost as well.

In this respect we can immediately see from Eq. (3.2.2) that the length $\mathrm{d}s = |\mathrm{d}\mathbf{x}|$ is not invariant with respect to such a transformation unless we require that $\mathrm{d}t = 0$, i.e., that the two events with respect to which this quantity is defined are simultaneous. This makes perfect sense as soon as we realize that this is what actually happens when, e.g., we use a rod: surely one would never think to define the length of a moving body by taking the differences between the coordinate of the tail at a certain time and that of its tip after it has moved! This type of invariant quantity is called the *Galilean-invariant*.[19] In general, therefore, one can obtain Galilean-invariants from their Euclidean counterparts simply by providing that they depend on differences of coordinates taken at the same time.

Using the same kind of reasoning for vectors and higher rank tensors, which are required to enjoy similar invariance (or covariance) properties, we are naturally led to define a *Galilean vector* (or higher rank tensor) as a set of three quantities that, in addition to the former, remain invariant for Galilean boosts as well. This puts some additional restrictions on the eligible candidates. As an example, Eq. (3.2.3) gives by definition the transformation law for the velocities, from which it can be immediately realized that these quantities are not Galilean vectors. On the other hand, if we derive one more time the above equation with respect to t, and remembering that $\mathrm{d}\bar{t} = \mathrm{d}t$, it is

$$\bar{\mathbf{a}} \equiv \frac{\mathrm{d}\bar{\mathbf{v}}}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} - \mathbf{u}) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \mathbf{a} \tag{3.2.4}$$

because the relative velocity between the two reference systems $\mathbf{v}$ is constant. In other words, the acceleration *is* invariant, and thus a Galilean vector, which is precisely the reason why the equations of motion of Newtonian physics involve this quantity and not others.

[19]It is worth noticing that the length defined in this way is still a scalar in the Euclidean sense, but it cannot be considered in the same way with respect to this larger set of transformations. Indeed, the concept of scalar is tightly associated with the possibility of defining a metric in a vector space. The Galilean boost, on the other hand, is a kind of transformation mixing up space and time coordinates, but we know that Euclidean space and time are two different metric spaces, endowed with two separate metrics. Because no unified “space plus time” metric can be defined, the term Galilean-invariant is preferred over Galilean scalar. This, however, is an example of the “philosophical issues” mentioned in Sect. 2.3. We started with the identification of a measurement with a scalar, but now we are extending them to objects that do not live in a metric space. At the moment it can seem an innocent stretching required by much needed practical purposes, but we show later that it is instead the first sign of a lack of self-consistency in the theory.

3.3 Equivalence Principle

The third and last principle of classical physics is the so-called equivalence principle. As is well known, it states the equivalence between the *inertial mass* and the *gravitational charge*. Let us quickly revisit it.

We used the term gravitational "charge" in order to stress the concept that gravity is a force or, to say it better, an *interaction*, that is generated by a specific source which we call gravitational charge by analogy with all the other interactions.[20] Thus we say that two bodies can have a gravitational interaction when they are both provided with gravitational charges m_1 and m_2, and the result of this interaction is a mutual attractive force

$$\mathbf{F} = -G\frac{m_1 m_2}{r_{12}^2}\mathbf{e}_r, \tag{3.3.1}$$

where G is the universal gravitational constant, r_{12} is the distance between the two bodies, and $\mathbf{e}_r$ is the unit vector of the segment connecting them.

However we cheated a bit when we used the expression "mutual attractive force," as forces are always acted upon something by something else. In other words the term "mutual" has to be intended in such a way that the first body exerts a force $\mathbf{F}_{12}$ on the second, and the second exerts a force $\mathbf{F}_{21}$ on the first. This might seem a specious distinction, because in the original formulation by Newton there was no distinction between m_1 and m_2 in these two different cases, thus by definition $\mathbf{F}_{12} = -\mathbf{F}_{21}$, which is the natural meaning of our original statement. Nonetheless, as first pointed out by Bondi (1957), strictly speaking the role of the two bodies is quite different in the two cases. In the former the source of the interaction is the first body, and the other one contributes to the force by reacting passively to such a source, whereas the opposite is true in the latter case.

Therefore, it makes sense to distinguish between two different gravitational properties of a body: an *active* gravitational charge m^{A} which is the property of a body to *generate* a gravitational force, and a *passive* gravitational charge m^{P}, which on the other hand is the property of the same body to *react* to a gravitational force generated by some other source. In formulae

$$\mathbf{F}_{12} = -G\frac{m_1^{\mathrm{A}} m_2^{\mathrm{P}}}{r_{12}^2}\mathbf{e}_{12}$$

$$\mathbf{F}_{21} = -G\frac{m_1^{\mathrm{P}} m_2^{\mathrm{A}}}{r_{12}^2}\mathbf{e}_{21}.$$

This distinction starts to fade out when one realizes that the momentum conservation or, which is equivalent, Newton's third law of dynamics,[21] requires that

[20] Equivalently, we say that the electromagnetic interaction is generated by the electric charge q.

[21] As is well known, one can assume the third law of dynamics, i.e., $\mathbf{F}_{12} = -\mathbf{F}_{21}$ and the additivity of the forces to deduce the conservation of the total momentum of a system, or the other way round wherein this conservation law can be assumed to deduce the "action = reaction" law.

$\mathbf{F}_{12} = -\mathbf{F}_{21}$, which implies $m_1^{\rm A} m_2^{\rm P} = m_1^{\rm P} m_2^{\rm A}$ for any pair of masses; i.e.,

$$\frac{m_1^{\rm A}}{m_1^{\rm P}} = \frac{m_2^{\rm A}}{m_2^{\rm P}} \quad \Rightarrow \quad \frac{m^{\rm A}}{m^{\rm P}} = k.$$

In other terms, the ratio between the active and and passive gravitational charges has to be a constant k for any body, and if not verified this would violate the basic assumptions of Newtonian dynamics, either in the form of a conservation law or of Newton's third law. It is therefore clear that the specific value of k can be easily set to 1 by an appropriate choice of measurement units; in fact one has

$$\mathbf{F}_{12} = -kG\frac{m_1^{\rm A} m_2^{\rm A}}{r_{12}^2}\mathbf{e}_{12},$$

therefore a new gravitational constant $\bar{G} = kG$ can be defined that incorporates, by a convenient change of the unit of measures, the distinction between active and passive gravitational charge. Therefore in the following we always assume $m^{\rm A} = m^{\rm P} = m_{\rm G}$ as the *gravitational charge*.

For the next part of our reasoning we consider the expression of the gravitational force as in Eqs. (1.1.1) and (1.1.2). In this case the force is originated by a gravitational field $\Phi\,(t, \mathbf{x})$ related to its source, i.e., the mass (charge) density $\rho_{\rm G}$, by means of the Poisson equation, namely[22]

$$\nabla^2\bar{\Phi} = k\rho_{\rm G},$$

where for the moment we used a generic constant k, instead of the normal $4\pi G$, indicating its associated potential with $\bar{\Phi}$. A body with gravitational charge $m_{\rm G}$ would then feel a force[23]

$$\mathbf{F} = -m_{\rm G}\boldsymbol{\nabla}\bar{\Phi}. \tag{3.3.2}$$

The motion of such a body is governed by Newton's second law of dynamics

$$\mathbf{F} = m_{\rm I}\mathbf{a} \tag{3.3.3}$$

whose meaning is that, upon the action of a force, this body would be accelerated to a quantity $\mathbf{a}$ which is inversely proportional to another property of the body $m_{\rm I}$ called the *inertial mass*, and combining Eqs. (3.3.2) and (3.3.3), we obtain

$$\mathbf{a} = -\frac{m_{\rm G}}{m_{\rm I}}\boldsymbol{\nabla}\bar{\Phi}.$$

[22] As the source of the gravitational field, from the above viewpoint ρ would play the role of active mass.

[23] And likewise, this body would play the role of the passive mass.

Being connected with two completely different physical aspects, namely the gravitational interaction and the way a body reacts to a force, m_G and m_I in principle could be totally uncorrelated, and therefore their ratio could vary for any reason. Thus, if this were the case, different bodies would accelerate in a different way under the influence of the same gravity field.

For example, although both are intuitively perceived to be roughly proportional to the "amount of matter" of a body (whatever this expression can mean) the coefficient could be different for the gravitational charge and the inertial mass. This would imply that different quantities of the same matter would fall with different acceleration, something that was actually believed, once again, in Aristotelian physics. Or maybe that could be the case of bodies with different composition, such as lead and aluminum, so that the same quantity of a different kind of matter would fall in different ways, which was another common belief before Galileo.

It was the Italian scientist who, for the first time in the modern age and within a consistent scientific framework, stated that all bodies fall with the same acceleration which depends, in modern language, only on the strength of the gravity field. For what we have said above, this implies, as for the case of active and passive gravitational charges, that the ratio m_G/m_I is constant for *any* body, so we can put

$$\frac{m_G}{m_I} = G \tag{3.3.4}$$

and therefore, with the same "trick" of choosing appropriate measurement units in order to set this ratio to 1, we can assume $m_G = m_I \equiv m$ and incorporate G into $\bar{\Phi}$. This justifies the common use of the name *gravitational mass* for m_G and transforms Eq. (3.3.2) into

$$\mathbf{F} = -mG\nabla\bar{\Phi} = -m\nabla\Phi$$

where $\Phi \equiv G\bar{\Phi}$ so that the constant in the Poisson equation can be set to the usual value $4\pi G$. The experimental evidence in support of the independence of the free fall with respect to any characteristic of the falling body, and therefore of the equivalence between the inertial and gravitational masses, is so strong that this has been assumed as a principle.

3.3.1 Equivalence Principle as a "Dynamic" Covariance Principle

Apparently the equivalence principle deals with a relation between two physical quantities that have nothing to do with coordinates or inertial motion, and therefore it can look alien to the two previous principles. Nonetheless, as in the case of the principle of relativity, it can be interpreted in terms of covariance of some physics laws with respect to some transformations between reference systems.

Let us first imagine a spaceship at rest on the Earth's surface. An astronaut inside will feel an acceleration keeping him or her attached to the floor caused by the gravitational pull of the planet.[24] At a certain point the spaceship (which is completely opaque from inside) turns the engines on and lifts up, reaching at some time a region far from any gravity sources and therefore with a negligible gravity field. The engines, however, are still on and their push is tuned to give the spaceship an acceleration equal to that of the Earth's gravity. From the point of view of the astronaut, therefore, nothing has changed with respect to the initial conditions, and he or she will not be able to distinguish between the gravitational pull of the Earth and the non-gravitational acceleration that is now responsible for keeping everything attached to the floor. In other words, a reference system at rest with respect to another one and "immersed" in a gravity field Φ is indistinguishable from a reference system having $\Phi = 0$ but uniformly accelerated with respect to the former with an acceleration having the same magnitude.

But is it really so? How can we say that this acceleration has exactly the same effect of the gravitational one? This point is worth a deeper investigation.

Let us thus imagine that the spaceship is now coming back to the Earth and at a certain point the engines are turned off. The astronaut inside will feel no acceleration, and therefore no force whatsoever, and everything will start to move freely around. The spaceship is now an *inertial reference system*.[25] While they are approaching their destination, however, the spaceship will start to fall toward our planet because of its gravitational pull, but nothing will change inside. How is it possible? The reason is that, because of the equivalence principle, everything inside the ship[26] will fall exactly with the same acceleration and therefore the objects will have no relative acceleration with respect to each other. In this way, everything will appear to "float" with respect to anything else exactly as before. The astronaut can move, throw objects around, do any mechanical experiment, but he or she will not be able to distinguish this condition from the previous inertial one. In other words, because of the equivalence principle, a reference system freely falling under the influence of a gravity field is indistinguishable from an inertial reference system. This is the same reason why we could state above the equivalence between the gravitational pull and the engines' push. They both accelerate everything in the spaceship in the same way because of the equivalence principle.

Indeed there is one possibility to detect a difference between the freely falling condition and a "true" inertial reference system, because in the former case the gravitational pull originates a nonuniform acceleration throughout the region covered by the reference system. For example, because the gravitational force is always directed toward the Earth's center, the accelerations of two points at the same distance

[24]Or better, there is a force pulling everything toward the center of the planet, but the acceleration with respect to the reference system attached to the surface is zero because this force is canceled by the opposite reaction of the ground.

[25]To be more precise, because this reference system is *realized* by the positions of "real" things like the objects inside the ship, its walls, and every single particle of this system, one should speak of a reference *frame* instead.

[26]And the ship itself, actually.

from it will not be parallel. Conversely, two points aligned with the center of the planet but at different distances will rigorously be affected by accelerations with slightly different magnitudes. These two conditions can be neglected provided that the dimensions of the reference system are small enough with respect to the typical scale at which the gravity field varies. Equivalently, one can say that, given a desired measurement accuracy, it is always possible to identify a region that is small enough to make the two conditions indistinguishable from the point of view of any experiment.

We can thus understand why the equivalence principle may be reformulated by saying that the laws of freely-falling test bodies have to be the same, in a small enough region, for an inertial reference system with a gravitational field and a uniformly accelerated reference system. In such form the equivalence principle already starts to recall a kind of "dynamic covariance" including the transformations to uniformly accelerated reference systems. It has to be stressed, however, that strictly speaking we are asking such covariance for the laws of freelyfalling test bodies only. We show in the next chapters that the extension of the covariance requirement to more general cases will bring us two different versions of the original principle, lying at the foundations of general relativity. This justifies the renaming of the former as the weak equivalence principle (WEP).[27]

Finally we have to stress the special role assumed by gravity among all the other interactions because of this principle.

3.4 Exercises

Exercises 3.1 Show that the parity reversal transformation of Eq. (3.1.3) in three dimensions can be represented by the matrix—$\mathbb{I}$, and therefore that it is an orthogonal transformation P with $\det P = -1$.

Solution 3.1 Parity reversal is defined as that transformation which changes the sign of each coordinate, namely

$$
\begin{aligned}
x &\rightarrow -x \\
y &\rightarrow -y \\
z &\rightarrow -z
\end{aligned}
$$

[27]It is shown in Chap. 6 how the basic hypothesis leading to the principle of Galilean relativity can be changed to obtain different transformation laws between inertial reference systems, which are at the basis of special relativity. Afterward Einstein was led, in its the search for a gravity theory, by the attempt of to extend the scope of the principle of relativity to reference systems in a more general relative motion, i.e., accelerating with respect to each other. Because acceleration is the natural extension of velocity, the relativistic extension to the Galilean relativity was called the "principle of special covariance" or relativity, which is at the origin of the theory's name, whereas that extended to the acceleration was called the "principle of general covariance" or relativity.

and it is therefore trivial to see that for any $\mathbf{x} = (x, y, z)^{\mathrm{T}}$ the operation $-\mathbb{I}\mathbf{x}$ realizes this transformation exactly. It is also trivial to show that $(-\mathbb{I})^{\mathrm{T}}(-\mathbb{I}) = \mathbb{I}^{\mathrm{T}}\mathbb{I} = \mathbb{I}$, which is the definition of an orthogonal transformation matrix, and that $\det(-\mathbb{I}) = -\det\mathbb{I} = -1$.

Exercises 3.2 Show that any orthogonal matrix O with $\det O = -1$ can be obtained as the product of an orthogonal matrix R with $\det R = 1$ and the parity matrix $P = -\mathbb{I}$.

Solution 3.2 By definition, a square matrix O is orthogonal if $O^{\mathrm{T}}O = OO^{\mathrm{T}} = \mathbb{I}$, which means that an orthogonal matrix is characterized by the property $O^{\mathrm{T}} = O^{-1}$. But because in general $\det(AB) = \det A \cdot \det B$ and $\det\left(A^{\mathrm{T}}\right) = \det A$ then

$$\det\left(O^{\mathrm{T}}O\right) = \begin{cases} (\det O)^2 \\ \det\mathbb{I} = 1, \end{cases}$$

which gives immediately that for any orthogonal matrix $\det O = \pm 1$.

If we then put $O = MN$, with M and N two generic matrices, it is necessarily

$$(MN)^{\mathrm{T}}(MN) = \mathbb{I}, \tag{3.4.1}$$

and the condition $\det O = -1$ implies

$$\det M \cdot \det N = -1.$$

It is trivial to understand that both these conditions are satisfied if $M \equiv R$ is an orthogonal matrix with $\det R = 1$ and $N \equiv -\mathbb{I}$.

Exercises 3.3 Show that the transformations

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{x} + \mathbf{a} \\ \bar{\mathbf{x}} &= O\mathbf{x}, \qquad O \mid O^{\mathrm{T}}O = \mathbb{I}, \end{aligned}$$

i.e., translations and transformations represented by orthogonal matrices O, leave any scalar product $\Delta\mathbf{x}_1 \cdot \Delta\mathbf{x}_2$, and therefore lengths and angles of Eqs. (3.1.1) and (3.1.2), unchanged.

Solution 3.3 The proof is trivial inasmuch as

1. translations leave *any* vector unchanged, being

$$\begin{aligned} \Delta\mathbf{x} &\equiv (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= \left(x_2 + a_x - x_1 - a_x, y_2 + a_y - y_1 - a_y, z_2 + a_z - z_1 - a_z\right) \\ &\equiv \Delta\bar{\mathbf{x}}; \end{aligned}$$

2. using matrix formalism the scalar product can be written

$$\Delta \mathbf{x}_1 \cdot \Delta \mathbf{x}_2 = (\Delta \mathbf{x}_1)^{\mathrm{T}} (\Delta \mathbf{x}_2) ,$$

therefore, if $O^{\mathrm{T}} O = \mathbb{I}$,

$$\begin{aligned} \Delta \bar{\mathbf{x}}_1 \cdot \Delta \bar{\mathbf{x}}_2 &= (O \Delta \mathbf{x}_1)^{\mathrm{T}} (O \Delta \mathbf{x}_2) \\ &= (\Delta \mathbf{x}_1)^{\mathrm{T}} O^{\mathrm{T}} O (\Delta \mathbf{x}_2) \\ &= (\Delta \mathbf{x}_1)^{\mathrm{T}} (\Delta \mathbf{x}_2) . \end{aligned}$$

Exercises 3.4 Show that the transformations of Exercise 3.3 are the only ones that preserve lengths and angles.

Solution 3.4 Let us start by seeking the most general form of coordinate transformation from Cartesian ones that leave (Euclidean) lengths invariant. The inverse reasoning can obviously be applied to the inverse transformation, and by composing these two we can apply this statement to the most general case.

Any coordinate transformation between the two reference systems $\bar{S}$ and S can be written as a set of three invertible functions $\bar{x}^i = \bar{x}^i \left(x^j\right)$, where $i, j = 1, 2, 3$, and if we impose the length invariance condition $\mathrm{d}\bar{s}^2 = \mathrm{d}s^2$ from Eq. (B.5.2) with $g_{ij} = \delta_{ij}$, it has to be

$$\delta_{ij} \mathrm{d}\bar{x}^i \mathrm{d}\bar{x}^j = \delta_{kl} \mathrm{d}x^k \mathrm{d}x^l. \tag{3.4.2}$$

As in Eq. (B.2.3), differentiation of the coordinate transformation functions gives

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^k} \mathrm{d}x^k, \tag{3.4.3}$$

and substituting it into Eq. (3.4.2) one has

$$\delta_{ij} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} \mathrm{d}x^k \mathrm{d}x^l = \delta_{kl} \mathrm{d}x^k \mathrm{d}x^l,$$

but this condition holds for any $\mathrm{d}x^k$ and $\mathrm{d}x^l$, and separately for any $\mathrm{d}x^k \mathrm{d}x^l$, thus

$$\delta_{ij} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} = \delta_{kl}. \tag{3.4.4}$$

Differentiating with respect to x^m the above equation becomes

$$\delta_{ij} \left(\frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^m} \frac{\partial \bar{x}^j}{\partial x^l} + \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial^2 \bar{x}^j}{\partial x^m \partial x^l} \right) = 0,$$

the first and second permutations of the three indexes *kml* give the two additional equations

$$\delta_{ij}\left(\frac{\partial^2\bar{x}^i}{\partial x^m\partial x^l}\frac{\partial\bar{x}^j}{\partial x^k}+\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial^2\bar{x}^j}{\partial x^l\partial x^k}\right)=0$$

and

$$\delta_{ij}\left(\frac{\partial^2\bar{x}^i}{\partial x^l\partial x^k}\frac{\partial\bar{x}^j}{\partial x^m}+\frac{\partial\bar{x}^i}{\partial x^l}\frac{\partial^2\bar{x}^j}{\partial x^k\partial x^m}\right)=0,$$

which can be combined to obtain

$$\delta_{ij}\left(\frac{\partial^2\bar{x}^i}{\partial x^k\partial x^m}\frac{\partial\bar{x}^j}{\partial x^l}+\frac{\partial\bar{x}^i}{\partial x^k}\frac{\partial^2\bar{x}^j}{\partial x^m\partial x^l}\right)-\delta_{ij}\left(\frac{\partial^2\bar{x}^i}{\partial x^m\partial x^l}\frac{\partial\bar{x}^j}{\partial x^k}+\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial^2\bar{x}^j}{\partial x^l\partial x^k}\right)$$
$$+\delta_{ij}\left(\frac{\partial^2\bar{x}^i}{\partial x^l\partial x^k}\frac{\partial\bar{x}^j}{\partial x^m}+\frac{\partial\bar{x}^i}{\partial x^l}\frac{\partial^2\bar{x}^j}{\partial x^k\partial x^m}\right)=0.$$

Because $\delta_{ij}=\delta_{ji}$ the latter can be rewritten as

$$\delta_{ij}\frac{\partial^2\bar{x}^i}{\partial x^k\partial x^m}\frac{\partial\bar{x}^j}{\partial x^l}+\delta_{ij}\frac{\partial\bar{x}^i}{\partial x^k}\frac{\partial^2\bar{x}^j}{\partial x^m\partial x^l}-\delta_{ji}\frac{\partial^2\bar{x}^j}{\partial x^m\partial x^l}\frac{\partial\bar{x}^i}{\partial x^k}-$$
$$\delta_{ij}\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial^2\bar{x}^j}{\partial x^l\partial x^k}+\delta_{ji}\frac{\partial^2\bar{x}^j}{\partial x^l\partial x^k}\frac{\partial\bar{x}^i}{\partial x^m}+\delta_{ji}\frac{\partial\bar{x}^j}{\partial x^l}\frac{\partial^2\bar{x}^i}{\partial x^k\partial x^m}=0,$$

which gives

$$2\delta_{ij}\frac{\partial^2\bar{x}^i}{\partial x^k\partial x^m}\frac{\partial\bar{x}^j}{\partial x^l}=0. \tag{3.4.5}$$

The above equation can be conveniently represented in an alternative way. First it can be observed that it is always possible to put

$$\frac{\partial\bar{x}^i}{\partial x^j}\equiv M^{\bar{i}}{}_j \tag{3.4.6}$$

i.e., to consider the partial derivatives of the coordinate transformation equation (3.4.3) as the components of a matrix[28] M. Multiplying both sides by δ^{hl}, Eq. (3.4.5) therefore becomes

$$0=\delta^{hl}\delta_{\bar{i}\bar{j}}\frac{\partial M^{\bar{i}}{}_k}{\partial x^m}M^{\bar{j}}{}_l=\frac{\partial M^{\bar{i}}{}_k}{\partial x^m}M_{\bar{i}}{}^l\equiv\frac{\partial M}{\partial x^m}M^{\mathrm{T}},$$

but inasmuch as the transformations $\bar{x}^i=\bar{x}^i\left(x^j\right)$ must admit an inverse by hypothesis, then the inverse matrix M^{-1} exists, and thus

[28]The bars over the indexes, like those over the coordinates in the derivatives, help us to distinguish between the components of $\bar{S}$ from those of S. In this case this means that, as it should be, the rows scan through the components of $\bar{S}$ and the columns refer to S.

$$0 = \frac{\partial M}{\partial x^m} M^{\mathrm{T}} \left(M^{\mathrm{T}}\right)^{-1} = \frac{\partial M}{\partial x^m}. \tag{3.4.7}$$

The latter represents a set of three simple differential equations (one for each component $\bar{x}^i$) whose solutions are

$$\bar{x}^i = M^{\bar{\imath}}{}_k x^k + \bar{a}^i, \tag{3.4.8}$$

where $\left\{\bar{a}^i\right\}$ is a set of three arbitrary constants. This proves the first statement, namely that the most general transformation formula is a linear equation. Let us now use the definition (3.4.6) to substitute the transformation matrix $M^{\bar{\imath}}{}_k$ into Eq. (3.4.4), thus obtaining

$$\delta_{\bar{\imath}\bar{\jmath}} M^{\bar{\imath}}{}_k M^{\bar{\jmath}}{}_l = \delta_{kl}.$$

Once again we can multiply both sides by δ^{hk} so that

$$\delta^{hk} \delta_{\bar{\imath}\bar{\jmath}} M^{\bar{\imath}}{}_k M^{\bar{\jmath}}{}_l = \delta^{hk} \delta_{kl} = \delta^h{}_l,$$

but because it is also

$$\delta^{hk} \delta_{\bar{\imath}\bar{\jmath}} M^{\bar{\imath}}{}_k = \delta^{hk} M_{\bar{\jmath}k} = M_{\bar{\jmath}}{}^h$$

we have

$$M_{\bar{\jmath}}{}^h M^{\bar{\jmath}}{}_l = \delta^h{}_l.$$

Finally, remembering that if $M = M^{\bar{\jmath}}{}_l$ then by definition $M^{\mathrm{T}} \equiv M_{\bar{\jmath}}{}^l$ and that we can interpret the Kronecker symbol in matrix form as $\delta^h{}_l \equiv \mathbb{I}$, we obtain

$$M^{\mathrm{T}} M = \mathbb{I}, \tag{3.4.9}$$

which means that M is an orthogonal matrix O.

Exercises 3.5 Show that the transformations of Eqs. (3.1.3), (3.1.4), and (3.1.5), together with an appropriate composition law, form a group.

Solution 3.5 Let us first recall that by definition $(G, \circ)$, namely a set of objects G with a binary operation "$\circ$" called the "product" that takes any two objects of G giving a new one, is a *group* if it satisfies the following four properties.

1. *Closure*: Given any two elements g_1 and g_2 of G, then $g_1 \circ g_2 \in G$.
2. *Identity element*: There exists an *identity element* i of G, such that $g \circ i = i \circ g = g$ for any $g \in G$.
3. *Inverse element*: For each $g \in G$ there exists a unique element of the group g^{-1} such that $g \circ g^{-1} = i$.
4. *Associativity*: For any three elements of the group $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

It is easy to see that the set having as elements the parity transformation matrix P and the identity matrix $\mathbb{I}$ constitutes a group with respect to the matrix product operation. The closure property comes immediately from the fact that any combination (i.e., matrix multiplication) between these two elements gives again either P or $\mathbb{I}$. Then by definition $\mathbb{I}$ is the identity element, and each one is its self-inverse. Finally, the associativity follows trivially from the matrix multiplication properties.

Translations, instead, constitute a group with respect to the vector sum operation, namely the sum of the vector components, where the identity element is the null vector $\mathbf{0} = (0, 0, 0)$, the inverse element is $-\mathbf{a}$, and the associativity is again a property of the algebraic operation.

Similar considerations hold for the rotation matrices complemented by the matrix product. The closure follows from the properties of the matrix products and of the determinants. For any matrix A and B, $(AB)^{\mathrm{T}} = B^T A^T$, thus if R and Q are two rotation matrices

$$(RQ)^T\,(RQ) = Q^{\mathrm{T}} R^{\mathrm{T}} R Q = Q^{\mathrm{T}} \mathbb{I} Q = \mathbb{I}.$$

Moreover, det $(AB) = \det A \cdot \det B$, hence if R and Q are orthogonal matrices det $(RQ) = \det R \cdot \det Q = 1$. These two properties mean that RQ is also a rotation matrix. Obviously the identity element is $\mathbb{I}$ and the inverse element is the rotation matrix around the same axis with an opposite angle, and once again the matrix product is associative (whose proof can be found in any algebra textbook).

Exercises 3.6 Show that the Galilean boost transformations of Eqs. (3.2.1) and (3.2.2) constitutes a group.

Solution 3.6 These transformations can be represented by the set of velocities $\mathbf{u}$, and the natural candidate for the composition law is the usual vectorial sum. In this way, if $\mathbf{u}_{(1)}$ and $\mathbf{u}_{(2)}$ are two members of the set, the composition result is $\mathbf{u}_{(1)} \circ \mathbf{u}_{(2)} = \mathbf{u}_{(1)} + \mathbf{u}_{(2)} \equiv \mathbf{u}$, which is again a velocity, i.e., a valid member of the set. This means that the set is closed with respect to the vectorial sum of velocities. Under Galilean boost transformations the time does not change, therefore this statement is equivalent to saying that the two transformations are composed in the following way:

$$\begin{aligned} \mathrm{d}t_{(1)} &= \mathrm{d}t \\ \mathrm{d}\mathbf{x}_{(1)} &= \mathrm{d}\mathbf{x} - \mathbf{u}_{(1)}\mathrm{d}t \end{aligned}$$

and then

$$\begin{aligned} \mathrm{d}t_{(2)} &= \mathrm{d}t_{(1)} = \mathrm{d}t \\ \mathrm{d}\mathbf{x}_{(2)} &= \mathrm{d}\mathbf{x}_{(1)} - \mathbf{u}_{(2)}\mathrm{d}t_{(1)} \\ &= \mathrm{d}\mathbf{x} - \left(\mathbf{u}_{(1)} + \mathbf{u}_{(2)}\right)\mathrm{d}t. \end{aligned}$$

The identity element is clearly $\mathbf{u} = 0$ and the inverse is $(\mathbf{u})^{-1} = -\mathbf{u}$. Finally, vector addition is associative, which means that the Galilean boost transformations constitute a group with respect to such a composition law.

Chapter 4
Classical Physics, Fundamental Principles, and Lagrangian Approach

In this chapter it is shown how some of the basic equations of classical physics behave with respect to the requirements of the fundamental principles we have discussed earlier. In particular the examples have been chosen with the aim of naturally ferrying our reasoning toward the realm of non-Newtonian physics. These exercises therefore want to show:

1. How, contrary to what is commonly expected, in some cases even the Euclidean (i.e., “geometrical”) covariance can hardly be regarded as self-evident
2. How the covariance, and therefore the fundamental principles from which it derives, can be used as a “minimum requirement” to guess some properties of the physical laws’ equations, but also as a way to explore extensions or different versions of an existing theory
3. That classical electromagnetism is incompatible with the principle of relativity in its Galilean form, whose consequences will lead us to special relativity.

It has to be stressed that in this chapter, when necessary, x^k are intended as vector components $\mathrm{d}x^k$, and not just as coordinates.

4.1 Equations of Motion and Newtonian Gravitational Force

Newton’s second law of dynamics

The very first case to start with is Newton’s law of dynamics $\mathbf{F} = m\mathbf{a}$. This is a very simple one because the force on the left-hand side of the equation is a vector in the Euclidean and Galilean sense by definition. In other words, we require that for the transformations of the Galilean covariance group (i.e., translations, rotations, and Galilean boosts) and for any transformation of the coordinate system it is $\bar{\mathbf{F}} = \mathbf{F}$.

A. Vecchiato, *Variational Approach to Gravity Field Theories*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2_4

This is unavoidable, as this equation is in practice the translation in mathematical terms of the principle of relativity. We also posit that there exists a property of the matter, called inertial mass, whose measurable value does not depend on the reference system. For what we have said above, this identifies m as a Euclidean scalar and, more than that, also as a Galilean-invariant. The equation is therefore covariant in the Euclidean sense if the acceleration $\mathbf{a}$ can be considered a Euclidean vector, because in this case the quantity $\mathbf{F} - m\mathbf{a}$ would be a Euclidean vector in its turn. This is immediate to show by the definition of acceleration as

$$\mathbf{a} \equiv \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2},$$

where $\mathbf{x}$ is a Euclidean vector and t is the Euclidean time. From Eq. (B.2.6) we have therefore

$$\mathrm{d}\bar{t} = \mathrm{d}t \text{ and } \bar{x}^k = \frac{\partial \bar{x}^k}{\partial x^j} x^j,$$

implying

$$\bar{a}^k \equiv \frac{\mathrm{d}^2\bar{x}^k}{\mathrm{d}\bar{t}^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \bar{x}^k}{\partial x^j} x^j = \frac{\partial \bar{x}^k}{\partial x^j}\frac{\mathrm{d}^2 x^j}{\mathrm{d}t^2} = \frac{\partial \bar{x}^k}{\partial x^j} a^j,$$

which means that the components of the acceleration transform as the components of a Euclidean vector by also considering Eq. (B.2.12).

Equation (3.2.4) also shows that $\bar{\mathbf{a}} = \mathbf{a}$ for Galilean boost transformations, which means that the acceleration is a Galilean vector, and this ensures the form invariance of Newton's second law of dynamics for all the transformations of the Galilean covariance group.

Newtonian gravity and action at a distance

The case of Newtonian gravity is more interesting because of the consequences which can be drawn by requiring that this law has to be covariant with respect to the transformations of the Galilean group.[1] Once again, we assume that the gravitational mass and the universal gravitational constant G are Euclidean scalars and Galilean-invariant, so that the expression of the force

$$\mathbf{F} = -m\nabla\Phi\,(t, \mathbf{x})$$

is form-invariant only if $\nabla\Phi$ is a vector both in the Euclidean and Galilean sense. However in Sect. B.3 it was shown that the operator ∇ is a vector,[2] therefore this

[1] With the term *Galilean group* we mean the full set of transformations including the Euclidean covariance group and the Galilean boost.

[2] More correctly the gradient operator is a one-form, as pointed out in the appendix, but in Euclidean geometry vectors and one-forms do coincide.

quantity is a Euclidean vector if Φ is a scalar field. This is true for the gravitational field because

$$\Phi\left(r\right) = -G\frac{M}{r},$$

where r is the distance of the test body m from the gravity source M and it is a Euclidean scalar (actually the very first we have introduced). In the Galilean sense we know that the distance r is a Galilean-invariant if it is treated as the length $\mathrm{d}s$, i.e., if we consider the position of the source and the test body at the same time. But in this case also all the components of the gradient are Galilean-invariant in fact, considering, e.g., that in Cartesian coordinates $r = \left[\sum_{k=1}^{3}\left(\Delta x^k\right)^2\right]^{1/2}$, for any component x^k and for a general function $f\left(r\right)$ it is

$$\frac{\partial f}{\partial x^k} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x^k} = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\Delta x^k}{r}$$

and

$$\frac{\partial \bar{f}}{\partial \bar{x}^k} = \frac{\mathrm{d}\bar{f}}{\mathrm{d}\bar{r}}\frac{\partial \bar{r}}{\partial \bar{x}^k} = \frac{\mathrm{d}\bar{f}}{\mathrm{d}\bar{r}}\frac{\Delta \bar{x}^k}{\bar{r}},$$

but because r is a Galilean-invariant, then under the same conditions $r = \bar{r}$, $\Delta x^k = \Delta \bar{x}^k$, and $f\left(r\right) = \bar{f}\left(\bar{r}\right)$, so

$$\frac{\partial f}{\partial x^k} = \frac{\partial \bar{f}}{\partial \bar{x}^k}.$$

We have then shown that the Newtonian gravity force is covariant with respect to the whole Galilean group under the conditions that the distances are taken at the same time, say t'. This, however, has an interesting consequence because it implies that the force applied to the test body must act at the same time or, if we combine the equations of motion with the gravity force,

$$\mathbf{a}\left(t'\right) = -\nabla\Phi\left(r\left(t'\right)\right),$$

which means that t' is also the time at which the body feels the acceleration due to the gravitational interaction. In other words, requiring the Galilean covariance of Newtonian gravity is equivalent to asking that the gravitational interaction propagates instantaneously, i.e., the so-called action at distance which is a well-known characteristic of this interaction.

4.2 Euler–Lagrange Equations

A less obvious example is that of the Euler–Lagrange equations (A.3.6). Actually we already know that these equations are form invariant for any change of coordinates, because in Chap. 1 we "built" them from Newton's second law with the specific purpose of having an alternative and coordinate-independent way to write the equations of motion. From what we have learned in this chapter, however, this should imply that the quantity

$$E_k \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^k} - \frac{\partial L}{\partial x^k}$$

should transform as the components of a Euclidean vector, or more precisely as those of a *co vector*, i.e., like the unit vectors of Eq. (3.1.12)[3]:

$$\bar{E}_k = \frac{\partial x^h}{\partial \bar{x}^k} E_h,$$

which is not evident at all. It is thus instructive to show this property.

First of all, let us recall that in Sect. 1.2.1 the Lagrangian was defined as a function $L = T - V$, where T and V were the kinetic and potential energies, respectively. This definition implies that L is a function of the coordinates $x^k(t)$ and of the velocities $\dot{x}^k(t)$, however, we have seen that the explicit dependence on the time t is also admitted.[4]

As stated in Eq. (1.2.2), we are considering a generic change of coordinates

$$x^k = x^k\left(\bar{x}^h\right) \tag{4.2.1}$$

with $\det J \neq 0$, or in other words, one that admits an inverse transformation

$$\bar{x}^k = \bar{x}^k\left(x^h\right),$$

and we have assumed that the value of the Lagrangian with respect to the new coordinates does not change; i.e.,

$$\bar{L}\left(t, \bar{x}^k, \dot{\bar{x}}^k\right) = L\left(t, x^h, \dot{x}^h\right), \tag{4.2.2}$$

namely that the Lagrangian is a (Euclidean) *scalar field*.

[3] We're using here the formalism from of Appendix B.

[4] When introducing the Lagrangian we used the symbols q_i and $\dot{q}_i$ for the coordinates, which were called *generalized coordinates* and *velocities*. Obviously using $\bar{x}^k$ and $\dot{\bar{x}}^k$ in their place does not change anything when we consider that in that chapter we were not making any distinction between covariant and contravariant vectors, as is always possible in Euclidean geometry.

Equation (4.2.1) immediately implies that

$$\dot{x}^k = \frac{\partial x^k}{\partial \bar{x}^h}\dot{\bar{x}}^h = \dot{x}^k\left(\bar{x}^h, \dot{\bar{x}}^h\right) \tag{4.2.3}$$

because $\dot{x}^k$ are vector components, and

$$\frac{\partial \dot{x}^k}{\partial \dot{\bar{x}}^h} = \frac{\partial x^k}{\partial \bar{x}^h}. \tag{4.2.4}$$

By substituting Eqs. (4.2.1) and (4.2.3) into (4.2.2) one then obtains the condition

$$\bar{L}\left(t, \bar{x}^k, \dot{\bar{x}}^k\right) = L\left(t, x^h\left(\bar{x}^k\right), \dot{x}^h\left(\bar{x}^k, \dot{\bar{x}}^k\right)\right).$$

Using this formula we can now obtain the transformation law for the first term of the Euler–Lagrange equation, which results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{\bar{x}}^k} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^h}\frac{\partial \dot{x}^h}{\partial \dot{\bar{x}}^k}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^h}\frac{\partial x^h}{\partial \bar{x}^k}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^h}\right)\frac{\partial x^h}{\partial \bar{x}^k} + \frac{\partial L}{\partial \dot{x}^h}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x^h}{\partial \bar{x}^k}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^h}\right)\frac{\partial x^h}{\partial \bar{x}^k} + \frac{\partial L}{\partial \dot{x}^h}\frac{\partial^2 x^h}{\partial \bar{x}^k \partial \bar{x}^j}\dot{\bar{x}}^j,\end{aligned} \tag{4.2.5}$$

where in the first line we used Eq. (4.2.4).

The second term is instead

$$\begin{aligned}\frac{\partial \bar{L}}{\partial \bar{x}^k} &= \frac{\partial L}{\partial x^h}\frac{\partial x^h}{\partial \bar{x}^k} + \frac{\partial L}{\partial \dot{x}^h}\frac{\partial \dot{x}^h}{\partial \bar{x}^k}\\ &= \frac{\partial L}{\partial x^h}\frac{\partial x^h}{\partial \bar{x}^k} + \frac{\partial L}{\partial \dot{x}^h}\frac{\partial^2 x^h}{\partial \bar{x}^k \partial \bar{x}^j}\dot{\bar{x}}^j,\end{aligned} \tag{4.2.6}$$

where the final expression can be easily derived from Eq. (4.2.3):

$$\frac{\partial \dot{x}^h}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^h}{\partial \bar{x}^j}\dot{\bar{x}}^j\right) = \frac{\partial^2 x^h}{\partial \bar{x}^k \partial \bar{x}^j}\dot{\bar{x}}^j.$$

By subtracting Eqs. (4.2.5) and (4.2.6) we finally have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{\bar{x}}^k} - \frac{\partial \bar{L}}{\partial \bar{x}^k} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^h}\right)\frac{\partial x^h}{\partial \bar{x}^k} - \frac{\partial L}{\partial x^h}\frac{\partial x^h}{\partial \bar{x}^k}\\ &= \frac{\partial x^h}{\partial \bar{x}^k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^h} - \frac{\partial L}{\partial x^h}\right)\end{aligned}$$

which is exactly what we wanted.

4.3 Lagrangian of a Free Particle and Variational Principles

In Chap. 1 we introduced the Lagrangian as a function $L = T - V$ and we showed that the resulting Euler–Lagrange equations were a form-invariant representation of the equations of motion. In the above section we have seen, on the other hand, that this is due to the fact that the equations themselves can be interpreted as components of a covariant vector regardless of the specific expression of the Lagrangian. Indeed the only requirement is that L, as a function of t, x^i and $\dot{x}^i$, is a scalar. It is then reasonable to ask ourselves if the actual form of the Lagrangian can be deduced from some basic principles or, to the contrary, that nothing can be said and we can only be driven by experiments.

The simplest case is that of a free particle. Starting from the standard expression of the Lagrangian this is the "force-free" case with $V = 0$, which means that $L = T = mv^2/2$. But what if we could start just from the "L is a scalar" condition?

Well, in this case we could appeal to the properties of the Euclidean space that are at the basis of the definition of a scalar quantity, i.e., to the homogeneity and isotropy of space. Moreover, because in general $L\left(t, x^i, \dot{x}^i\right)$ can also depend explicitly on time, we should also consider the hypothesis of homogeneity of time.

The homogeneity of space and time implies that, if we want L to be a scalar, the Lagrangian cannot change for translations in space or time. A possible way to meet this requirement, as for the vector lengths, would be to write L as function of coordinate *differences*, but because in this case it can depend on the coordinates of the free particle the only possibility is just to admit that there is no dependence on the coordinates at all, thus $L = L\left(\dot{x}^i\right)$.

The isotropy of space implies that the Lagrangian cannot change for rotations in space and parity reversal, therefore this function can depend only on the speed[5] $|v| = +\sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}$, however, it is easier to use the square of this quantity, i.e., $L = L\left(v^2\right)$.

We can finally appeal to another basic principle, that of relativity, which states the invariance of the equations of motion for transformations between reference systems in uniform relative motion. For what we have seen in the previous chapter, this is equivalent to saying that L has to be not just a Euclidean scalar, but also a Galilean-invariant.

Before this invariance requirement can be properly exploited, a known property of the Euler–Lagrange equations has to be recalled, namely that for any $f\left(t, x^i\right)$ the two Lagrangians $L_1\left(t, x^i, \dot{x}^i\right)$ and

$$L_2\left(t, x^i, \dot{x}^i\right) = kL_1\left(t, x^i, \dot{x}^i\right) + L_0\left(t, x^i, \dot{x}^i\right),$$

where k is a constant and $L_0\left(t, x^i, \dot{x}^i\right) = \frac{\mathrm{d}f}{\mathrm{d}t}$ give the same equation of motion. This can be easily understood by observing that the solution of the Euler–Lagrange equations does not change for a multiplication by a constant, and that

[5] This is because otherwise it would depend on the direction of the velocity, and thus it would change for rotations.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{x}^i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{x}^i}\frac{\mathrm{d}f}{\mathrm{d}t}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{x}^i}\left(\frac{\partial f}{\partial x^i}\dot{x}^i + \frac{\partial f}{\partial t}\right)\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\\ &= \frac{\partial L_0}{\partial x^i},\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_0}{\partial \dot{x}^i} - \frac{\partial L_0}{\partial x^i} = 0.$$

Going back to the principle of relativity, if the relative velocity between the two systems is $\mathbf{u}$ the Lagrangian becomes $L\left((\mathbf{v}+\mathbf{u})^2\right) = L\left(v^2 + 2\mathbf{v}\cdot\mathbf{u} + u^2\right)$. It is thus easy to see that, if $L\left(v^2\right) = kv^2$ (k is not the same as above) the transformed Lagrangian, because $\mathbf{u}$ is constant, can be written as

$$\begin{aligned}L\left((\mathbf{v}+\mathbf{u})^2\right) &= kv^2 + k\left(2\mathbf{v}\cdot\mathbf{u} + u^2\right)\\ &= kv^2 + k\frac{\mathrm{d}}{\mathrm{d}t}\left(2\mathbf{x}\cdot\mathbf{u} + tu^2\right)\\ &= L\left(v^2\right) + \frac{\mathrm{d}f}{\mathrm{d}t},\end{aligned}$$

therefore if L is linear in v^2 then the Galilean velocity transformation can be reduced to a total time derivative and the two Lagrangians bring about the same equations of motion. Conversely, if L is not linear in v^2 we can show that it is possible to find at least one case for which the transformed Lagrangian cannot be expressed as the original one plus a total time derivative. To this aim let us suppose that L is indeed not linear in v^2 and $u \ll v$. In this case we can always expand at first order $L\left((\mathbf{v}+\mathbf{u})^2\right)$ obtaining

$$L\left((\mathbf{v}+\mathbf{u})^2\right) = L\left(v^2\right) + 2\mathbf{v}\cdot\mathbf{u}\frac{\partial L}{\partial v^2},$$

from which it can be seen that the second term cannot be considered a total time derivative unless

$$\frac{\partial L}{\partial v^2} = \text{const} \tag{4.3.1}$$

because $\mathbf{u}$ is constant and $\mathbf{v}$ is already a total time derivative, but this condition is equivalent to asking that L is linear in v^2, contradicting our initial hypothesis.

We know that the Galilean principle of relativity states that free particles move with constant velocity. This is a trivial deduction from the free particle Lagrangian, but the properties we have just derived are already enough to imply this statement, in fact the Euler–Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\left(v^2\right)}{\partial \dot{x}^i}-\frac{\partial L\left(v^2\right)}{\partial x^i}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\left(v^2\right)}{\partial \dot{x}^i}=0, \tag{4.3.2}$$

but

$$\frac{\partial L\left(v^2\right)}{\partial \dot{x}^i}=\frac{\partial L\left(v^2\right)}{\partial v^2}\frac{\partial v^2}{\partial \dot{x}^i}=2\dot{x}^i\frac{\partial L\left(v^2\right)}{\partial v^2},$$

therefore Eq. (4.3.2) becomes

$$\ddot{x}^i\frac{\partial L\left(v^2\right)}{\partial v^2}+\dot{x}^i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\left(v^2\right)}{\partial v^2}=0,$$

which from Eq. (4.3.1) clearly requires that $\ddot{x}^i=0$ separately for any component, i.e., $\mathbf{v}=$ const.

This is a first example of how the actual form of a Lagrangian can be almost entirely deduced by starting just from fundamental principles. This method can be used also for the field equations, as we show in the next section.

As a final word of notice, the same and totally general "L must be a scalar" condition we used is perfectly valid also in the case of the principle of least action. In fact we should require that the action S of Eq. 1.2.12 is a scalar (and a Galilean-invariant) for any function $x^i(t)$, but this is true if and only if L is such because dt *is* a scalar and an integral can be safely considered as a continuous sum. This statement appears again in the next section, but it is also important to stress it in perspective, to better understand what happens in the relativistic case which is shown in Chap. 6.

4.4 Field Theories and Variational Approach: A "Not-so-Naive" View

In this section we explore in a more detailed way how the interactions of a dynamical system are treated in the Lagrangian approach, with specific reference to the case of fields, using as a driving example the Newtonian gravitational force.

Interactions in Lagrangian formalism

We have already established that the classical Lagrangian of a free particle is proportional to v^2, i.e., $L_{\text{free}}=kv^2$, with $k=m/2$ in Newtonian mechanics, and we know that the equations of motion can be obtained from the Euler–Lagrange equations with an appropriate Lagrangian $L = L_{\text{free}} - V_{\text{int}}$, where V_{int} represents the potential energy of the interaction of the no-longer-free particle. But how should such interactions be represented? One could stand on a force-based approach and state that such potential energy can be obtained from its relation $-\nabla V = \mathbf{F}$ for a conservative force. In this case we are following the typical scenario of particle mechanics, where the interaction of each particle of a system with an other can be described by a function $L_{\text{int}} = -V_{\text{int}}(\mathbf{r}_1(t), \dot{\mathbf{r}}_1(t), \ldots, \mathbf{r}_n(t), \dot{\mathbf{r}}_n(t))$ of the coordinates and velocities of the n particles. In order to represent an actual interaction, L_{int} cannot be a linear function of each $\mathbf{r}_i, \dot{\mathbf{r}}_i$; otherwise it would be possible to rearrange the total Lagrangian as $L = \sum_i L_i$ where

$$L_i = (L_{\text{free}} - V_{\text{int}})_i (\mathbf{r}_i(t), \dot{\mathbf{r}}_i(t))$$

and the equation of motion of each particle would be independent of the other, which is equivalent to saying that the system can be divided into several isolated (i.e., noninteracting) subsystems so that V does not represent an actual interaction. Finally, as in the previous section, in order to ensure the covariance of the equations of motion the action defined with such a Lagrangian must be a scalar,[6] which once again means that it therefore has to be L itself, because

$$S[\mathbf{r}_i] = \int_{t_0}^{t_1} L(\mathbf{r}_1(t), \dot{\mathbf{r}}_1(t), \ldots, \mathbf{r}_n(t), \dot{\mathbf{r}}_n(t), t)\, \mathrm{d}t,$$

and $\mathrm{d}t$ is a scalar as well. Thus, e.g., for two particles the total Lagrangian will be $L_{\text{tot}} = L_{\text{free}}^{(1)} + L_{\text{free}}^{(2)} + L_{\text{int}}$, where $L_{\text{int}} \equiv -V_{\text{int}}$, and the action will be a functional of the two trajectories $S[\mathbf{r}_1, \mathbf{r}_2]$.

Interactions in field theories

The same scenario, however, can be seen from another point of view. To this aim, we start from the example of the gravitational interaction, for which

$$V_{\text{int}} = V(\mathbf{r}_1(t), \mathbf{r}_2(t)) = -G\frac{m_1 m_2}{r}$$

where $r(t) = |\mathbf{r}_2(t) - \mathbf{r}_1(t)|$. If we consider the gravitational field generated by m_2, $\Phi = -Gm_2/r$, the equivalent formula

[6] Here by "scalar" we mean a Euclidean scalar and a Galilean-invariant, which is needed to guarantee the covariance with respect to the principle of Galilean relativity. In a general sense we can use this word to identify "numbers invariant with respect to a specific covariance group," as we show in the next chapters. The variational approach will always carry this requirement, but varying the basic principle, and therefore the selection of a different covariance group, will originate different theories.

$$V_{\mathrm{int}} = m_1 \Phi$$

can be considered as the expression for the interaction between the particle m_1 and the gravitational field, but in this case the total Lagrangian cannot contain the free term of the second particle anymore. Rather, because the interaction is between the first (and now the only) particle with the *field*, we have to substitute $L^{(2)}_{\mathrm{free}}$ with the Lagrangian of the *free field*.

For the particles, the Lagrangian depends on $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$, which means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ play the role of *dependent* variables, so that the Action becomes a *functional* of $\mathbf{r}$, and the real *independent* variable is t. A field $\phi(\mathbf{x}, t)$, instead, is a quantity that varies both in time and in space, so the counterpart of the particle trajectory $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ as dependent variables are the field itself and its total first derivative with respect to the independent variables t and $\mathbf{x}$, $\dot{\boldsymbol{\phi}} \equiv (\partial_t \phi, \nabla \phi)$ where, to ease the notation, we wrote $\partial_t \phi \equiv \partial \phi / \partial t$.

Lagrangian density and free fields

The introduction of $\mathbf{x}$ as independent variable and of the spatial partial derivatives of the field means that the Lagrangian for the fields has to depend on an integral over the space of a function, called the *Lagrangian density*, of the field and of its derivatives; i.e., in general

$$L = \int_{\Omega_3} \mathcal{L}(\phi, \partial_t \phi, \nabla \phi, \mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}.$$

We have already worked out in Sect. 1.3.2 the problem of finding the Lagrangian density that eventually brings us to the Poisson equation, namely to the field equation of Newtonian gravity. In this section we can look back at it from the vantage point of our now conscious understanding that it actually derives from an application of specific fundamental principles, which will therefore constitute the common driver for the development of this approach.

In particular, the same requirement made on L in the case of the particle dynamics still holds for the Lagrangian of the field, which implies that the Lagrangian density $\mathcal{L}$ has to be a scalar *field*. As in the previous section, this is an obvious consequence of the fact that

$$L = \int_{\Omega_3} \mathcal{L}(\phi, \partial_t \phi, \nabla \phi, \mathbf{x}, t)\, \mathrm{d}^3\mathbf{x}$$

and that the unit volume $\mathrm{d}^3\mathbf{x}$ is a scalar and a Galilean-invariant.

Moreover, considerations analogous to those used for free particles, once again, help us understand why we chose the Lagrangian (density) of the free field as

$$\mathcal{L} = k\,(\nabla \phi \cdot \nabla \phi)\,, \tag{4.4.1}$$

and the fact that we are taking the gravity field Φ as a working example gives, as shown in Sect. 1.3.2, $k = (8\pi G)^{-1}$, therefore

$$L^{(\Phi)}_{\text{free}} = \frac{1}{8\pi G} \int_{\Omega_3} \nabla\Phi \cdot \nabla\Phi \, \mathrm{d}^3\mathbf{x}. \tag{4.4.2}$$

Gravitational interaction and the total Action for Newtonian gravity

This, however, is not yet the end of the story. The interaction term up to now reads $L_{\text{int}} = -m_1\Phi$, whose corresponding action is, dropping the subscript index of the particle

$$S_{\text{int}} = -\int_{t_0}^{t_1} m\Phi \, \mathrm{d}t, \tag{4.4.3}$$

so the action obtained from the total Lagrangian $L_{\text{tot}} = L^{(\text{p})}_{\text{free}} + L^{(\Phi)}_{\text{free}} + L_{\text{int}}$ with the last term written as above would just give us the equation of motion of the particle because S_{int}, being expressed as an integral over time only, cannot include the field as a dependent variable, which would require a Lagrangian density instead.

In the case of the two particles we intended the expression of the total Lagrangian as a *mutual interaction* between them. This was made explicit by the fact that the action was a functional $S[\mathbf{r}_1, \mathbf{r}_2]$ of both particles’ trajectories. Likewise we should interpret the new Lagrangian as a mutual interaction between the particle and the field, which requires the complete expression to be written as a functional of both the particle’s trajectory and the field, $S[\mathbf{r}, \Phi(\mathbf{r})]$, which thus has to be varied with respect to the field as well. Thus, as just stated, the expression of $S_{\text{int}}[\mathbf{r}]$ in Eq. (4.4.3) should be modified to include a Lagrangian density of the field. In this way the total action will also become a functional $S_{\text{tot}}[\mathbf{r}, \Phi(\mathbf{r})]$ which, when varied with respect to the particle’s trajectory will give the equation of motion of the particle, whereas when varied with respect to the field, will return the field equations.

Writing S_{int} in terms of a Lagrangian density can be easily achieved by exploiting the well-known property of the 3D Dirac delta function

$$f(\mathbf{r}) = \int_{\Omega_3} f(\mathbf{x})\, \delta^3(\mathbf{x} - \mathbf{r})\, \mathrm{d}^3x$$

which allows us to write Eq. (4.4.3) as

$$S_{\text{int}} = -\int_{t_0}^{t_1} m \int_{\Omega_3} \Phi(\mathbf{x})\, \delta^3(\mathbf{x} - \mathbf{r})\, \mathrm{d}^3x\mathrm{d}t$$

or also, if we consider a continuous distribution of matter with density $\rho(\mathbf{x})$, as

$$S_{\text{int}} = -\int_{t_0}^{t_1} \int_{\Omega_3} \rho \Phi(\mathbf{x}) \, \delta^3(\mathbf{x} - \mathbf{r}) \, \mathrm{d}^3x \mathrm{d}t.$$

The total action for a particle with mass m in a gravitational field Φ therefore becomes

$$\begin{aligned} S_{\text{tot}}[\mathbf{r}, \Phi] &= S_{\text{free}}^{(\text{p})} + S_{\text{free}}^{(\Phi)} + S_{\text{int}} \\ &= \frac{1}{2} m \int_{t_0}^{t_1} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \, \mathrm{d}t + \frac{1}{8\pi G} \int_{\Omega_3} \nabla \Phi \cdot \nabla \Phi \, \mathrm{d}^3\mathbf{x} \\ &- \int_{t_0}^{t_1} m \int_{\Omega_3} \Phi(\mathbf{x}) \, \delta^3(\mathbf{x} - \mathbf{r}) \, \mathrm{d}^3x \mathrm{d}t. \end{aligned} \tag{4.4.4}$$

This example shows another characteristic of the total action, which is important to stress here for its future implications. As is evident from its derivation, the interaction term acts as a "bridge" between fields and matter, telling us the way the two can mutually interact. First of all, however, the meaning of the mass m we have taken above must be clarified in order to avoid possible misunderstandings. Actually it should be interpreted as the *inertial mass* of the test particle in the free term, whereas it has to be intended as the *gravitational mass* of the same body in the interaction part. As mentioned in Sect. 3.3 the equivalence principle tells us that, in this case, there is no difference between the two, provided that we use the appropriate unit of measures, but things would be different in the case of other interactions, e.g., the electromagnetic one where the interaction depends on electromagnetic density charges and currents rather than gravitational masses (i.e., gravitational density charges).

Another point to be stressed is that such (gravitational) charge density in the interaction part plays a different role according to how the action is varied. We have already said that by varying the particle's trajectory one gets its equations of motion under the influence of the interaction field which is assumed to be known a priori. On the other hand, by varying the field we obtain the field equations that tell how the field is generated and evolves from its sources. It is thus clear that in the first case the charge is that of a body interacting with an external field,[7] and in the second case the charge is that of the field source which, in a certain sense, is "interacting with its own field." This is why, in Sect. 1.3.2, when we first showed how the Poisson equation could be derived from a variational principle, we claimed that the interaction term could be regarded as a sort of "self-interaction" of the field with itself.

[7] In the sense that the field is generated by another source.

4.4.1 Variational Approach and Field Equations: General Review

In general, therefore, the total action is written as the sum of three terms, namely those of the free particle and of the free field, and the one describing the interaction between the particle and the field:

$$S_{\text{tot}}\left[\mathbf{r},\phi\right]=S_{\text{free}}^{(\text{p})}\left[\mathbf{r}\right]+S_{\text{free}}^{(\phi)}\left[\phi\right]+S_{\text{int}}\left[\mathbf{r},\phi\right], \tag{4.4.5}$$

thus it is clear that a variation of such action with respect to the field will affect only its components $S_{\text{free}}^{(\phi)}$ and S_{int} and the application of the variational principle for the field proceeds exactly as shown in Sect. 1.3.1. Thus, by imposing that the field equations are those for which $\delta S=0$ for null variations of the fields on the spatial and temporal boundaries one gets the previously shown Euler–Lagrange equation for the fields

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi\right)}\right)+\nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\left(\nabla\phi\right)}\right)-\frac{\partial\mathcal{L}}{\partial\phi}=0.$$

In general the advantages of working in the variational framework and in terms of fields are much more evident in relativistic and quantum physics rather than in the classical theories. Similarly to what happens for particle dynamics, this description of the field equations is coordinate independent, and the covariance requirements are made on a Euclidean and Galilean basis. Thus the limitations imposed by the necessity of operating on the two separate metric spaces of time and space often overcome the potential benefits of a field-based theory.

For example, a field treatment of the interactions in principle would allow us to build a physics theory that is local both in time and in space,[8] i.e., where the interactions do not propagate instantaneously among distant points, as shown, e.g., in Sect. 4.1. To this aim, however, the interacting field should also depend on $\partial_t\phi$. Those fields for which

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi\right)}=0$$

are in fact called *non propagating fields* because their configuration is determined everywhere at the same time by the generating sources, and therefore they are equivalent to using an action at a distance force.

Equation (4.4.4) shows immediately that this is indeed the case of Newtonian gravity, as one had to expect because its non locality is determined by the Galilean covariance[9] which, on the other hand, is not compatible with a propagating field.

Nonetheless, it is still useful to introduce such techniques in the classical framework for at least three reasons.

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[8]Roughly speaking, a particle reacts to the value that the field has locally at the point of interaction.

[9]Just by formulating the same theory with another language cannot change its founding principles.

First of all it will make the transition to relativistic physics much easier because one can become familiar with the essential characteristics of this new language by seeing it in action in an already known framework.[10]

Second, the most important part in the formulation of a new physics theory often lies in the changing of some basic principles. Even if the Lagrangian formulation of classical gravity might add little or nothing with respect to its usual exposition, the formulation of relativistic physics in classical language is at least not practical, when not infeasible. The former therefore is the only possibility to have both theories described with a common language, which will allow us to greatly improve our understanding of their differences and similarities solely in terms of their basic principles.

Finally, the Lagrangian formulation of field theories provides a natural and easy way to find extensions and/or modifications to already known theories. This is a common way to explore new possibilities in theoretical physics, and in the interest of a pedagogical introduction to such technique we can show an example of this procedure again in the familiar case of Newtonian gravity, which is the subject of the next section.

4.4.2 Newtonian Lagrangians: Poisson Equation and Its Extension

It is now worth recalling that we previously added a term $V_{\text{int}}(\Phi) = \rho\Phi$ just because this was the expression of the potential energy for Newtonian gravity, which was supposed to be already known. On the other hand, in Sect. 1.3.2 we started from scratch, supposing that the interaction term was not known, and that the expression was selected simply by asking it to be linear in Φ, and where the constant of proportionality ρ was chosen from dimensional considerations.[11] After noticing that this choice is compatible with the basic requirement on $\mathcal{L}$ because $\rho\Phi$ is a scalar field, one should consider also that this is not the only expression we can use from the point of view of the covariance requirement. For example, writing V_{int} with the more general expression

$$V_{\text{int}}(\Phi) = (\rho + C)\,\Phi,$$

with C constant, would not only save the Euclidean and Galilean covariance of the resulting action, but also the linearity in Φ. The constancy of C has to be preserved from the "point of view" of the field, thus we can still admit a dependence from t by

[10]Moreover, it has to be stressed that this formalism is ubiquitous in theoretical physics, therefore knowing it in advance can facilitate the task of understanding new theories expressed in an otherwise unknown language.

[11]We also recall that we were not considering the total Lagrangian because we were neglecting the free particle term. This explains why in that section we stated that $V_{\text{int}}(\Phi)$ could be regarded as a sort of "self interaction" of the field with itself.

taking $C = C(t)$. This new Lagrangian gives a modified version of the field equation of Newtonian gravity that we now explore.

Cosmological constant in Newtonian gravity

The constant term C has the dimensions of [mass]·[length]$^{-3}$, but for conventional reasons we write it as

$$C(t) = -\frac{\Lambda(t)}{4\pi G},$$

where $\Lambda(t)$ has the dimensions of [time]$^{-2}$. If we now use the resulting Lagrangian density

$$\mathcal{L} = \frac{1}{8\pi G}\nabla\Phi\cdot\nabla\Phi + \left[\rho - \frac{1}{4\pi G}\Lambda(t)\right]\Phi$$

in the Euler–Lagrange equation for fields (1.3.6), we easily obtain the new field equation

$$\nabla^2\Phi + \Lambda = 4\pi G\rho. \tag{4.4.6}$$

The effect of the Λ term that adds to the standard form of the Poisson equation can be quickly understood when this equation is integrated in the case of a spherically symmetric source $m = \int_{\Omega_3}\rho\,\mathrm{d}^3\mathbf{x}$ to obtain

$$\Phi(r) = -\frac{Gm}{r} - \frac{\Lambda r^2}{6} + \text{const}$$

outside the sphere, so that the gravitational force felt by a mass M is

$$\mathbf{F} = -M\frac{\partial\Phi}{\partial r}\mathbf{e}_r = \left(-G\frac{mM}{r^2} + \frac{1}{3}\Lambda M r\right)\mathbf{e}_r.$$

Two important things can be noticed about this force, namely:

1. It has two components, one of which is the normal Newtonian gravity, and the other one dependent on Λ which acts in the opposite direction if $\Lambda > 0$, or in the same if $\Lambda < 0$.
2. The magnitude of the second component does not depend on the mass of the source, and it increases proportionally to r.

From experimental evidence it is clear that, if it exists, the effects of this component must not be observable within the boundaries of the solar system at our accuracy, but because of its "accumulating" property something might be seen at larger scale.

Observations such as those of the flat rotation curves of the galaxies, the "impossible stability" of the galaxy clusters, or the accelerated expansion of the universe, seem to be the smoking guns of a breakdown of Newtonian gravity at some scale length, but it is in the cosmological context that the idea of having some kind of "Λ term" seems to be more appealing, and actually it is well known that such kind of mechanism has been invoked several times, in this case with a positive value of Λ or within a more complex scenario, to explain apparent deviations from Newtonian gravity at cosmological scales. This is why this term is called *cosmological*.

Variational approach as a tool for theoretical physics

What we have just seen here is a first pedagogical example of a technique often used in the variational approach. In addition to the above consideration, in fact, it is evident from this example that this framework provides a very easy and therefore powerful way to conceive new theories that preserve some fundamental physical principles, and to highlight their relations with the existing ones. As with all tools,, whether this reveals itself as an advantage or a danger depends solely on the users. It is interesting to notice, however, that the first historical appearance of a cosmological constant Λ happened in the context of general relativity to support a stationary model of the universe, and what we have seen above shows that this characteristic could have been easily requested in plain Newtonian gravity as well. The fact that it did not appear before probably depended on a combination of two factors: the lack of experimental data requiring an explanation and of the mathematical tools that would have facilitated its derivation.

4.5 Classical Electromagnetism

Studying the covariance of this theory is not immediate, because it involves quantities whose behavior under transformations between reference systems cannot be known in advance. The laws that are checked are the expression of the Lorentz electromagnetic force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{4.5.1}$$

and the four Maxwell equations, namely

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{4.5.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.5.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.5.4}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{4.5.5}$$

As we did in Newtonian gravity, we need to assume that quantities such as the electric charge q and the constants ε_0 and μ_0 involved in this interaction are both Euclidean scalars and Galilean-invariants. The invariance of q (and of the volume V) means that the charge density $\rho = q/V$ also has the same character. This immediately shows that we cannot appeal to the presence of the speed of light $c = 1/\sqrt{\varepsilon_0\mu_0}$ in Eq. (4.5.5) to claim the non-Galilean covariance of Maxwell equations, because the factor in this equation is actually $\varepsilon_0\mu_0$, which is Galilean-invariant and only incidentally (at least from the point of view of classical electromagnetism) has the meaning of a velocity.

Regarding the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and the current density $\mathbf{j}$, in general we cannot know how they transform in the various cases, however, because they are introduced in the mathematical framework of Euclidean geometry, it is practically impossible ("constructivist" approach) not to assume that they behave as do Euclidean vectors.

This is enough to ensure the Euclidean covariance of all these laws, but their Galilean covariance is quite a different story because we cannot know a priori the transformation laws of $\mathbf{E}$ and $\mathbf{B}$ among inertial frames. Actually, one might be tempted to deduce, from the presence of the velocity $\mathbf{v}$, a quantity that is not Galilean covariant and that the electromagnetic force is not form-invariant for such transformations either, however, not knowing anything about the behavior of $\mathbf{B}$, it would certainly be possible that, e.g., $\mathbf{v} \times \mathbf{B}$ is a Galilean covariant quantity.

What can be done instead is to require that Eq. (4.5.1) is form-invariant for Galilean transformations, and to deduce from this the transformation laws of the electric and magnetic fields, which are then applied to verify their consistency with the Maxwell equations. In doing this we follow closely the procedure shown in Preti et al. (2009).

We start therefore by assuming that, under Galilean transformations

$$\bar{t} = t \tag{4.5.6}$$

$$\bar{\mathbf{x}} = \mathbf{x} - \mathbf{u}t, \tag{4.5.7}$$

where $\mathbf{u}$ is the relative velocity between the two systems, it is

$$\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}} = \mathbf{E} + \mathbf{v} \times \mathbf{B}.$$

Because of Eq. (3.2.3), it is $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{u}$, therefore the above assumption implies that

$$\bar{\mathbf{E}} + \mathbf{v} \times \left(\bar{\mathbf{B}} - \mathbf{B}\right) = \mathbf{E} + \mathbf{u} \times \bar{\mathbf{B}}, \tag{4.5.8}$$

from which one can deduce the transformation laws of the fields being

$$\bar{\mathbf{E}}\left(\bar{\mathbf{x}}, \bar{t}\right) = \mathbf{E}\left(\mathbf{x}, t\right) + \mathbf{u} \times \mathbf{B}\left(\mathbf{x}, t\right) \tag{4.5.9}$$

$$\bar{\mathbf{B}}\left(\bar{\mathbf{x}}, \bar{t}\right) = \mathbf{B}\left(\mathbf{x}, t\right), \tag{4.5.10}$$

which are the only transformations able to satisfy Eq. (4.5.8) without using additional requirements on $\bar{\mathbf{v}}$ or $\mathbf{v}$.[12]

As regards the density current $\mathbf{j}$, it comes immediately from the definition $\mathbf{j} = \rho\mathbf{v}$ and from Eq. (3.2.3) that[13]

$$\bar{\mathbf{j}} = \bar{\rho}\bar{\mathbf{v}} = \rho\left(\mathbf{v} - \mathbf{u}\right) = \mathbf{j} - \rho\mathbf{u}.$$

In order to determine the form of the transformed Maxwell equations, we still have to understand how the differential operators change under Galilean transformations (4.5.6) and (4.5.7). According to these equations, an arbitrary function $f\left(\bar{\mathbf{x}}, \bar{t}\right)$ can be considered a function $f\left(\bar{\mathbf{x}}\left(\mathbf{x}, t\right), \bar{t}\left(t\right)\right)$ of $\mathbf{x}$ and t in which, at the same time,

$$\frac{\partial \bar{\mathbf{x}}}{\partial t} = \frac{\partial \bar{\mathbf{x}}}{\partial \bar{t}} = -\mathbf{u}.$$

It is, therefore,

$$\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^i} = \frac{\partial f}{\partial \bar{x}^i} \quad \Rightarrow \quad \nabla = \bar{\nabla} \tag{4.5.11}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial t} + \frac{\partial f}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = \frac{\partial f}{\partial \bar{t}} - v_0^i\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial \bar{t}} - \mathbf{u}\cdot\bar{\nabla} f \quad \Rightarrow \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} - \mathbf{u}\cdot\bar{\nabla}. \tag{4.5.12}$$

Armed with this knowledge, we can start from the simplest case, i.e., that of Eq. (4.5.4). It is immediate to see, from Eqs. (4.5.10) and (4.5.11), that

$$\nabla\cdot\mathbf{B} = \bar{\nabla}\cdot\bar{\mathbf{B}},$$

and therefore the magnetic Gauss law is Galilean-invariant. The same is true for Eq. (4.5.3); in fact

$$\begin{aligned}\nabla\times\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \bar{\nabla}\times\left(\bar{\mathbf{E}} - \mathbf{u}\times\bar{\mathbf{B}}\right) + \left(\frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}} - \left(\mathbf{u}\cdot\bar{\nabla}\right)\bar{\mathbf{B}}\right)\\ &= \bar{\nabla}\times\bar{\mathbf{E}} + \frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}} - \left[\bar{\nabla}\times\left(\mathbf{u}\times\bar{\mathbf{B}}\right) + \left(\mathbf{u}\cdot\bar{\nabla}\right)\bar{\mathbf{B}}\right]\end{aligned}$$

[12]As would happen, e.g., for the transformations $\bar{\mathbf{E}} = \mathbf{E}$ and $\bar{\mathbf{v}}\times\bar{\mathbf{B}} = \mathbf{v}\times\mathbf{B}$. The transformations cannot depend on the velocities of the bodies with respect to any reference system, but just on the relative velocity of the two reference systems.

[13]We recall that it is assumed that the charge density is a Galilean-invariant, thus $\bar{\rho} = \rho$.

$$
\begin{aligned}
&= \bar{\nabla} \times \bar{\mathbf{E}} + \frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}} - \left[\mathbf{u}\left(\bar{\nabla} \cdot \bar{\mathbf{B}}\right) - \left(\mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{B}} + \left(\mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{B}}\right] \\
&= \bar{\nabla} \times \bar{\mathbf{E}} + \frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}},
\end{aligned}
$$

where we used the property of the cross-product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ and $\bar{\nabla} \cdot \bar{\mathbf{B}} = 0$. Therefore Faraday's law is also Galilean-invariant.

The other two laws, however, are not form-invariant. Actually Eq. (4.5.2) becomes

$$
\begin{aligned}
\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} &= \bar{\nabla} \cdot \left(\bar{\mathbf{E}} - \mathbf{u} \times \bar{\mathbf{B}}\right) - \frac{\bar{\rho}}{\varepsilon_0} \\
&= \bar{\nabla} \cdot \bar{\mathbf{E}} - \frac{\bar{\rho}}{\varepsilon_0} - \bar{\nabla} \cdot \left(\mathbf{u} \times \bar{\mathbf{B}}\right) \\
&= \bar{\nabla} \cdot \bar{\mathbf{E}} - \frac{\bar{\rho}}{\varepsilon_0} + \mathbf{u} \cdot \left(\bar{\nabla} \times \bar{\mathbf{B}}\right)
\end{aligned}
$$

which is not form-invariant because in general $\mathbf{u} \cdot \left(\bar{\nabla} \times \bar{\mathbf{B}}\right) \neq 0$.[14] Finally, the transformation law of Eq. (4.5.5) is

$$
\begin{aligned}
\nabla \times \mathbf{B} - \mu_0 \mathbf{j} - \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} &= \bar{\nabla} \times \bar{\mathbf{B}} - \mu_0\left(\bar{\mathbf{j}} + \bar{\rho}\mathbf{u}\right) - \varepsilon_0 \mu_0 \left(\frac{\partial}{\partial \bar{t}} - \mathbf{u} \cdot \bar{\nabla}\right)\left(\bar{\mathbf{E}} - \mathbf{u} \times \bar{\mathbf{B}}\right) \\
&= \bar{\nabla} \times \bar{\mathbf{B}} - \mu_0 \bar{\mathbf{j}} - \varepsilon_0 \mu_0 \frac{\partial \bar{\mathbf{E}}}{\partial \bar{t}} - \mu_0 \bar{\rho}\mathbf{u} + \varepsilon_0 \mu_0 \left(\mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{E}} + \\
&\quad \varepsilon_0 \mu_0 \left(\frac{\partial}{\partial \bar{t}} - \mathbf{u} \cdot \bar{\nabla}\right)\left(\mathbf{u} \times \bar{\mathbf{B}}\right) \\
&= \bar{\nabla} \times \bar{\mathbf{B}} - \mu_0 \bar{\mathbf{j}} - \varepsilon_0 \mu_0 \frac{\partial \bar{\mathbf{E}}}{\partial \bar{t}} + \\
&\quad \varepsilon_0 \mu_0 \left\{\left(\mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{E}} + \mathbf{u} \times \left[\left(\frac{\partial}{\partial \bar{t}} - \mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{B}}\right]\right\} - \mu_0 \bar{\rho}\mathbf{u}
\end{aligned}
$$

which is therefore form-invariant only if

$$
\bar{\rho}\mathbf{u} = \varepsilon_0 \left\{\left(\mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{E}} + \mathbf{u} \times \left[\left(\frac{\partial}{\partial \bar{t}} - \mathbf{u} \cdot \bar{\nabla}\right)\bar{\mathbf{B}}\right]\right\},
$$

a condition that, once again, does not hold true in general.

We can thus conclude that the equations of classical electromagnetism are not Galilean-invariant because the two main groups of laws, namely the expression of the Lorentz force and the Maxwell equations, cannot be invariant at the same time: if we require the Galilean covariance of the former, then the transformation laws of the electric and magnetic fields deduced in this way make the latter non invariant. We explore the consequence of this fact in the next chapters.

[14] For the last passage we recall that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$.

Part II
Special Relativity

"Without experimentalists, theorists wouldn't have anything to explain"

Chapter 5
Special Relativity Setting Up

The non-Galilean covariance of the Maxwell equations has led us to a puzzling scenario with three possible alternatives:

1. One can simply state that the principle of relativity is valid for Newton's dynamics, but not for the electromagnetism.
2. Or maybe the Galilean principle of relativity holds for both the laws of Newton's dynamics and those of electromagnetism, but the latter are wrong.
3. Finally, it might be that one principle of relativity holds either for dynamics and electromagnetism, but not in its Galilean form, which implies that the Newtonian dynamics is wrong.[1]

5.1 Principle of Relativity Revisited

Historically, the deduction of the Lorentz transformations came after the first experimental evidence that eventually led to the present form of the principle of relativity and to the formulation of special relativity. It is, however, instructive and well suited for the purposes of this book to divert from the more common practice of illustrating first the Michelson and Morley experiment, which nonetheless is briefly recalled in the next section. We want instead to start from the third point of the above list. More precisely, we want to understand what is the most general form of the transformation laws that can be deduced by the principle of relativity in its present form, deriving afterward the consequences implied by such transformations.

[1] We recall that Newton's laws of dynamics are Galilean-invariant, and that in practice they are the mathematical translation of the Galilean principle of relativity, as shown in Sect. 3.2.

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5.1.1 Generalized "PoR-Aware" Transformations

Indeed, it is worth being recalled that the Galilean boost transformations do *not* derive by necessity from this principle which, in Sect. 3.2.1, we have reformulated as a "kinematic" covariance principle by stating that:

Claim The laws of physics have to be independent of the transformation laws between two reference systems in uniform relative motion.

The fact that the transformation laws between these reference systems take the form of the Galilean boost transformations of Eqs. (3.2.1) and (3.2.2), however, is just another a priori assumption, based on their reasonable and/or evident look with respect to our common experience, but it might well be the case that they are not the only ones admitted by such a claim.

Strictly speaking, one should be aware that the above statement lies "on top" of the Euclidean covariance requirements; i.e., any transformation law aiming at "implementing" the principle of relativity is admissible only if it does not break the Euclidean covariance. This is what actually happened with the Galilean transformations, that could be accepted only after we "cured" the potential problem of the Galilean-invariant. To this aim we had to agree that the length measurements are acceptable only under the (quite natural) requirement of simultaneity of the events defining the measured segment.

Following this line or reasoning, we can remember that Euclidean covariance was defined with respect to translations and rotations, which were used to put in mathematical language the so-called homogeneity and isotropy of the Euclidean space (and the homogeneity of Euclidean time). What we are thus seeking is the most general transformation between two reference systems, depending on their relative velocity, which preserves the homogeneity of the Euclidean space and time and the isotropy of the Euclidean space.[2]

The former hypothesis implies that the transformations between the coordinates of two reference systems S and $\bar{S}$, as explained below, have to be linear in t and $\mathbf{x}$; i.e.,

$$\bar{x}^\alpha = a^\alpha_\beta x^\beta \tag{5.1.1}$$

where the coefficients a^α_β do not depend on the spatial or temporal coordinates. In the above formula we have adopted once again the summation convention of the previous chapters, but to keep the formulae as compact as possible we have extended it by considering time and space coordinates together. In order to make it evident we used Greek indexes instead of Latin, so that $\alpha, \beta = 0, 1, 2, 3$, and x^0 means t. This convention is adopted consistently throughout this book, thus Latin indexes indicate spatial coordinates only, and the Greek ones include the temporal coordinate too.

[2] Although this way of deriving the Lorentz transformations is less common than the usual method based on the exchange of light signals, it is not new at all. It was shown for the first time by Waldemar von Ignatowsky in 1910 and later rediscovered and used many times in papers and textbooks. See, e.g., Liberati et al. (2002) and references therein.

However in this section, inasmuch as we are using Cartesian coordinates, to ease the notation we momentarily revert to the usual notation t, x, y, and z.

The linearity of Eqs. (5.1.1) required by the homogeneity hypothesis can be shown by remembering the meaning of the latter, which is that the distance between two events cannot depend on their coordinates. For the sake of simplicity, let us consider two events (t, x, y, z) and $(t, x + \mathrm{d}x, y, z)$ separated by a quantity $\mathrm{d}x^1 \equiv \mathrm{d}x$. This means that in the barred reference system their distance will be $\mathrm{d}\bar{s} \equiv |\mathrm{d}\bar{x}| = \left|a_1^1 \mathrm{d}x\right| = \left|a_1^1\right| \mathrm{d}s$, which implies that it cannot be $a_1^1 = a_1^1\,(x^\alpha)$ because in this case the distance in $\bar{S}$ would change for a rigid translation of the two events in the unbarred reference system. It is easy to convince oneself that this reasoning applies to any coefficient a_β^α.

We can appeal to homogeneity of time also to make the barred and unbarred reference systems' clocks set to zero when their origins coincide, while isotropy of space can be used to claim that there is no loss of generality if we rotate the two reference systems with the $\bar{x}$- and x-axes parallel to the direction of $\mathbf{v}$, such that we can consider $\mathbf{v} \equiv v^1 \equiv v_x \equiv v$.[3]

The above assumption on the coincidence of the two origins for $\bar{t} = t = 0$ implies also that the $\bar{x}$- and x-axes are not only parallel but coincident as well, which allows simplifying the writing of the transformation. This in fact means that, for any t and x, $y = z = 0 \;\Rightarrow\; \bar{y} = \bar{z} = 0$, which requires that[4]

$$a_0^2 = a_1^2 = a_0^3 = a_1^3 = 0.$$

Moreover, having the two x-axes (barred and unbarred) coincident and the two ys and zs pairwise parallel requires that the planes $\bar{x} - \bar{y}$ and $x - y$ are coincident too, as well as $\bar{x} - \bar{z}$ and $x - z$, i.e., that $z = 0$ (plane $x - y$) implies $\bar{z} = 0$, and similarly that $y = 0$ implies $\bar{y} = 0$, so that

$$a_3^2 = a_2^3 = 0.$$

The transformations of the y and z coordinates have now been reduced to

$$\bar{y} = a_2^2 y \quad \text{and} \quad \bar{z} = a_3^3 z,$$

but we can now appeal to the principle of relativity to show that $a_2^2 = a_3^3 = 1$. Let us in fact have two events in S with coordinates (t, x, y, z) and $(t, x, y + \mathrm{d}y_*, z)$. (We can imagine these two events as the endings of a rod parallel to the y-axis and at rest in the unbarred reference system.) Their distance in this reference system is $\mathrm{d}s_* = +\sqrt{(\mathrm{d}y_*)^2} = |\mathrm{d}y_*|$. In $\bar{S}$ the distance between the same events will then be

[3] Exercise 5.1 shows such transformations for a generic velocity.

[4] This is because such condition can be held true only if

$$\bar{y} = a_2^2 y + a_3^2 z \quad \text{and} \quad \bar{z} = a_2^3 y + a_3^3 z.$$

$\mathrm{d}\bar{s}_* = +\sqrt{(\mathrm{d}\bar{y}_*)^2} = \left|a_2^2 \mathrm{d}y_*\right| = \left|a_2^2\right| \mathrm{d}s_*$. If we reverse the situation and consider two events $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ and $(\bar{t}, \bar{x}, \bar{y} + \mathrm{d}\bar{y}_+, \bar{z})$ in $\bar{S}$ with $\mathrm{d}\bar{y}_+ = \mathrm{d}y_*$ (this is equivalent to saying that the rod is now at rest with respect to the barred reference system), then it is obviously $\mathrm{d}\bar{s}_+ = |\mathrm{d}\bar{y}_+| = |\mathrm{d}y_*| = \mathrm{d}s_*$. (We are considering the same rod at rest in two different reference systems.) Now $y = \bar{y}/a_2^2$, so in S the distance $\mathrm{d}\bar{s}_+$ will correspond to

$$\mathrm{d}s_+ = +\sqrt{(\mathrm{d}y_+)^2} = \left|\frac{\mathrm{d}\bar{y}_+}{a_2^2}\right| = \left|\frac{\mathrm{d}y_*}{a_2^2}\right| = \frac{\mathrm{d}s_*}{\left|a_2^2\right|} = \frac{\mathrm{d}\bar{s}_*}{\left|a_2^2\right|^2}.$$

But $\mathrm{d}s_+$ is the distance measured in S of a separation $\mathrm{d}\bar{y}_+$, whereas $\mathrm{d}\bar{s}_*$ is the distance measured in $\bar{S}$ of a separation $\mathrm{d}y_*$, which is the same as $\mathrm{d}\bar{y}_+$, thus the principle of relativity requires that $\mathrm{d}s_+ = \mathrm{d}\bar{s}_*$, which implies

$$\left|a_2^2\right|^2 = 1 \quad \Rightarrow \quad a_2^2 = \pm 1.$$

We can exclude the value -1 because when $v = 0$ the two reference systems must coincide,[5] so that $\bar{y} = y$, therefore we finally obtain that $a_2^2 = 1$. The same reasoning can be applied to the z coordinate, so that $a_3^3 = 1$, and we have

$$\bar{y} = y \tag{5.1.2}$$
$$\bar{z} = z. \tag{5.1.3}$$

We are now left with the two equations

$$\bar{t} = a_0^0 t + a_1^0 x + a_2^0 y + a_3^0 z$$
$$\bar{x} = a_0^1 t + a_1^1 x + a_2^1 y + a_3^1 z,$$

but we can immediately see that

$$a_2^0 = 0$$

because otherwise the times $\bar{t}_-$ and $\bar{t}_+$ of two clocks placed at coordinates $(t, x, -y, z)$ and (t, x, y, z), respectively, would be different, which cannot be because of the homogeneity of space. For the same reason it has to be

$$a_3^0 = 0.$$

Finally, $\bar{x}$ cannot depend on y and z; otherwise it would be

$$\bar{x} = a_0^1 t + a_1^1 x + a_2^1 \bar{y} + a_3^1 \bar{z}$$

[5] Actually, allowing $a_2^2 = -1$ would be equivalent to admitting a parity transformation on the y-axis between S and $\bar{S}$.

because of Eqs. (5.1.2) and (5.1.3), meaning that the coordinates in $\bar{S}$ are not independent, thus

$$a_2^1 = a_3^1 = 0.$$

Having been left with just four not null coefficients, we can ease the notation by renaming them $\alpha \equiv a_0^0$, $\beta \equiv a_1^0$, $\gamma \equiv a_1^1$, and $\delta \equiv a_0^1$, so that the transformations will read

$$\bar{t} = \alpha(v)\, t + \beta(v)\, x \tag{5.1.4}$$

$$\bar{x} = \delta(v)\, t + \gamma(v)\, x, \tag{5.1.5}$$

where we show that, because the principle of relativity states the equivalence of reference systems in uniform motion, all the coefficients must depend at most on the constant relative velocity v between S and $\bar{S}$.

By differentiation Eq. (5.1.5) becomes

$$\mathrm{d}\bar{x} = \delta(v)\, \mathrm{d}t + \gamma(v)\, \mathrm{d}x,$$

and this relation must hold for any $\mathrm{d}\bar{x}$; thus we can put $\mathrm{d}\bar{x} = 0$ in order to find

$$\delta(v) = -\gamma(v)\frac{\mathrm{d}x}{\mathrm{d}t} = -v\gamma(v), \tag{5.1.6}$$

which means that

$$\bar{x} = \gamma(v)(x - vt). \tag{5.1.7}$$

Therefore we can see that the relation of Eq. (5.1.6) is simply a consequence of the principle of relativity which requires that being at rest in $\bar{S}$ (which implies $\mathrm{d}\bar{x} = 0$) must be equivalent to being in (uniform) motion with velocity v (which in this case means along the x axis) in S.

Using the same procedure of Doughty (1990), which is reported for the reader's convenience in Appendix C.1, it can be shown that $\alpha(v) = \gamma(v) = \left(1 - kv^2\right)^{-1/2}$ and $\beta(v) = -k\gamma(v)$, where k is a constant independent of the reference system with the dimensions of the inverse of a velocity squared, i.e.,

$$\bar{t} = \frac{t - kvx}{\sqrt{1 - kv^2}} \tag{5.1.8}$$

$$\bar{x} = \frac{x - vt}{\sqrt{1 - kv^2}}, \tag{5.1.9}$$

and with the obvious constraint that $1 - kv^2 > 0$. Moreover the combination of two boosts with velocities v_1 and v_2 (both along the x-axis) is equivalent to a single boost with velocity

$$v = \frac{v_1 + v_2}{1 + kv_1v_2} \tag{5.1.10}$$

which represents the *generalized velocity addition law* for the x-component. This is a direct consequence of the closure property of the boost transformation shown in Appendix C.1.

5.1.2 Galilean and Lorentz Transformations

The explicit form of the transformations (5.1.8) and (5.1.9) depends on the actual value of the constant k, which can be negative, zero, or positive.

$k < 0$ and chronology violation

The case of $k < 0$ makes the transformation laws always possible, because $1 - kv^2 > 0$ for any v, but has to be rejected. Consider in fact two events $e_1 = (t_1, x_1, y, z)$ and $e_2 = (t_2, x_2, y, z)$ in S for which $t_2 - t_1 \equiv \mathrm{d}t > 0$. This can be interpreted by saying that e_1 comes before e_2 in S, and in principle this ordering of the events with respect to the time (i.e., the *chronology*) should be preserved, which means that for the two transformed events $\bar{e}_1 = \left(\bar{t}_1, \bar{x}_1, \bar{y}, \bar{z}\right)$ and $\bar{e}_2 = \left(\bar{t}_2, \bar{x}_2, \bar{y}, \bar{z}\right)$ it should be $\bar{t}_2 - \bar{t}_1 \equiv \mathrm{d}\bar{t} > 0$ as well. It is clear, however, that in this case, for any possible choice of $\mathrm{d}t$ and $\mathrm{d}x$, one can always find a velocity v for which $\mathrm{d}\bar{t} < 0$. It is in fact

$$\mathrm{d}\bar{t} = \frac{\mathrm{d}t - kv\mathrm{d}x}{\sqrt{1 - kv^2}} < 0 \quad \Rightarrow \quad \begin{cases} v > \frac{\mathrm{d}t}{k\mathrm{d}x} > 0 & \text{for } \mathrm{d}x < 0 \\ v < \frac{\mathrm{d}t}{k\mathrm{d}x} < 0 & \text{for } \mathrm{d}x > 0 \end{cases}$$

where the fact that $v > 0$ for $\mathrm{d}x < 0$ and vice versa comes from $k < 0$ (and $\mathrm{d}t > 0$). In practice, for any pair of events in S one can always find another inertial reference system $\bar{S}$ (moving with a sufficiently large velocity in absolute value) for which $\mathrm{d}\bar{t} < 0$, thus inverting the chronology of the events. It has to be said that what in general is forbidden is the so-called *causality violations*, a condition which is less restrictive than the chronology violation.[6] The idea at the basis of the chronology preservation is that an event can be interpreted as the cause of another one if the former comes before the latter, therefore preserving the chronology is a way to guarantee the cause–effect succession. However, asking to preserve the ordering of the events is not the right way to guarantee such causality. There is instead a general agreement on the fact that an event cannot be "before and after itself at the same time," or in other words that it cannot be possible to have a path composed of chronology-violating events which connects an event with itself. Such kinds of paths, in the language of relativistic physics, are called closed timelike curves, or CTC. Thus what is really forbidden is the existence of CTC, which is the most general accepted way to preserve physics

[6] It is clear, however, that at least some of these requirements about chronology and/or causality are a priori assumptions or, in other works, additional postulates.

from causality violations (Hawking 1992).[7] It is clear that this condition is weaker than the chronology preservation, so if a transformation does not admit chronology violation, then it does not admit CTC as well.

$k = 0$: Galilean transformations

If $k = 0$ the equations reduce to Galilean transformations. It is easy to deduce the well-known characteristics of these transformations from the point of view of the constraints used above. The condition $1 - kv^2 > 0$ in fact is always respected, meaning that no restriction can be put on the possible values of v, which can thus range from 0 to ∞. Moreover, $k = 0$ also implies that $\bar{t} = t$, namely that the observers associated with any reference system will measure the same time.[8] This guarantees that chronology is always preserved because $\mathrm{d}\bar{t} = \mathrm{d}t$, and therefore that no causality or chronology violation is possible.

$k > 0$: Lorentz transformations

For $k > 0$ it is useful to remember that this constant has the dimensions of the inverse of a velocity squared, therefore we can put $k = 1/c^2$, where now c is another universal constant having the dimensions of a velocity. Moreover, we can take $c > 0$ with no loss of generality, and the condition $1 - kv^2 > 0$ implies that $-c < v < c$. In other words c plays the role of an upper limit for the admitted relative velocities between two reference systems, whose transformations now read

$$\bar{t} = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \tag{5.1.11}$$

$$\bar{x} = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \tag{5.1.12}$$

which are the well-known *Lorentz transformations*. With regard to the chronology issue, let us now check if it is possible that, as for the case $k < 0$, two events e_1 and e_2 separated by a time interval $\mathrm{d}t > 0$ in S can have $\mathrm{d}\bar{t} < 0$ in $\bar{S}$. Following the same procedure as above, by imposing $\mathrm{d}\bar{t} < 0$ we have

$$\mathrm{d}t - \frac{v}{c^2}\mathrm{d}x < 0$$

which means

[7] Although it is not always accepted and actually the existence or not of CTC is still a matter of debate.

[8] Or in other words that time is absolute.

$$\begin{cases} dx > \frac{c^2}{v}dt & \text{if } v > 0 \\ dx < \frac{c^2}{v}dt & \text{if } v < 0, \end{cases}$$

but the condition $-c < v < c$ implies that

$$\begin{cases} dx > \frac{c^2}{v}dt > c dt & \text{if } v > 0 \\ dx < \frac{c^2}{v}dt < -c dt & \text{if } v < 0. \end{cases}$$

This shows that chronology is preserved for all the events whose spatial separation is $-c dt \leqslant dx \leqslant c dt$. The region defined by this condition is called the *light cone* of the reference system S. Events falling outside the light cone are said to be *causally disconnected*.

We have therefore shown that the requirements of the principle of relativity leave us with only two kinds of admissible transformations: the Galilean and the Lorentz ones. The former puts no restriction on the admitted relative velocities, which implies the "absoluteness of time," or in other words that any two events simultaneous in a reference system will be equally simultaneous ($dt = 0 = d\bar{t}$) in any other reference system in uniform relative motion. The latter has a characteristic velocity c that has to be the same in any reference system.

5.2 Experimental Footing of Special Relativity

The Michelson–Morley experiment was carried out in 1887 with the specific goal of detecting the motion of the Earth through the ether, which was considered at the time the realization of an inertial reference system. Indeed, there was solid experimental evidence that the Earth was not an inertial reference system. For example, the motion of a Foucault pendulum makes immediately evident the spin of our planet, and the measurement of stellar parallaxes highlights its orbital motion.

Apparently, however, the outcome of this experiment was in contrast with the above results because the speed of a light beam appeared the same in any direction. It is beyond the scope of this book to describe the details of the Michelson–Morley experiment, which moreover can be easily found in many classical textbooks (see e.g., Resnick 1968 or Jackson 1962), rather we just want to highlight its main concept.

A light ray (Fig. 5.1) travels from an emitter to a half-silvered mirror in front of it, where it is split in two orthogonal rays. The two beams reach two other mirrors and are reflected back to the splitter, where they recombine again into a single ray moving along a direction orthogonal to the original one until they reach a detector. The key concept is that, according to the principle of Galilean relativity, whereas the speed of light with respect to the aether is c, its velocity vector should sum with the one of the emitter, i.e., with that of the reference system of the experiment, so when the beam gets split the two parts have to move with different velocities with respect to the observer. In general, thus, the two paths are covered at different times,

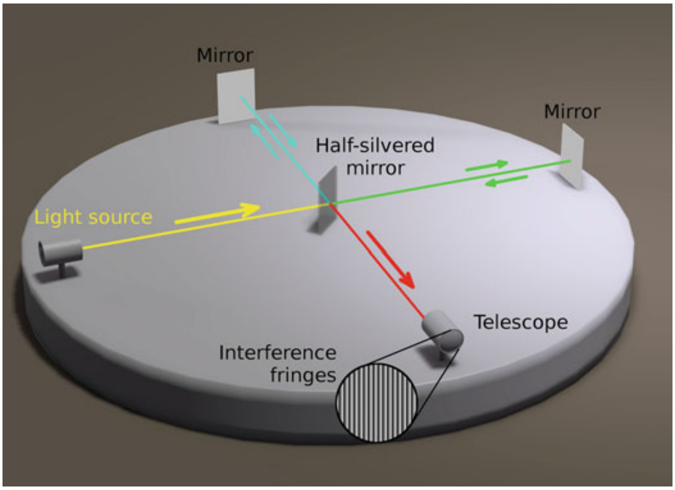

Fig. 5.1 Schematic representation of the light paths in the Michelson–Morley experiment

and in recombining the two rays should have different phases thus interfering with each other and producing characteristic fringes in the detector. Whatever the fringes at a certain moment, however, the other key point is that they should change when the experimental device is rotated, because the speed of light sums in a different way from that of the reference system, and therefore the two light rays travel with different velocities with respect to those of the previous configuration.

As is well known, the experiment "failed", in the sense that no change in the fringes was detected. The alternative transformations admitted by the principle of relativity offers an explanation for this outcome. In such transformations, in fact, the constant $k = 1/c^2$ does not depend on the reference system, and at the same time it sets a speed limit c that cannot be exceeded by any signal. Consequently, a signal moving with speed c in a reference system will move with the same speed in any reference system. This is shown explicitly in Exercise 5.3. The previous reasoning cannot tell anything about the actual value of c, but if we assume that such value is the speed of light then this would automatically explain the result of the Michelson–Morley experiment: the fringes do not change because the rotation of the apparatus, and the subsequent changed composition of the motion of the light rays with the velocity of the emitter, cannot change the speed of the two beams, which remains always c, whatever their direction of propagation.[9]

Obviously, this is not the only way to explain the results of the experiment, and indeed many attempts in this sense were proposed. Some attacked the problem by trying to modify the electromagnetic theory, i.e., following the second hypothesis of

[9]Indeed, the order of exposition is normally inverted, and the Lorentz transformations (or better their direct consequences in terms of length contraction and time dilation) are deduced in order to take into account the results of this experiment.

the list at the beginning of the chapter. Other attempts tried to modify the behavior of the aether and of its interaction with material bodies, which is equivalent to change the way an inertial reference system can be represented by this hypothetical substance. However, as shown in some detail in Resnick (1968), the only theory that passed all the experimental tests was the one based on the modified version of the principle of relativity, i.e., the special relativity theory.

We can therefore compare the advent of special relativity and the abandonment of classical dynamics and of the concept of the aether with our introductory comments of Chap. 2 about the need for a change of a scientific model. From the above considerations, in fact, it is clear that this transition was not just a matter of preference for a theory that offered a simpler and more elegant explanation of a single experiment, but rather it stood on quite vast experimental grounds and on unsolved self-consistency issues raised by classical physics, such as the tightly connected ones of the absoluteness of time and of the action-at-distance problem.

5.3 Basic Principles of Special Relativity and Relativistic Dynamics

According to the two previous sections the solution of the puzzling question raised by the non-Galilean covariance of the Maxwell equations seen in Sect. 4.5 is the third option of our list: *one principle of relativity holds either for dynamics and electromagnetism, but not in its Galilean form*.

Principle of relativity

More precisely, the principle of relativity is composed of two independent parts, namely:

1. The laws of physics have to be independent of the transformation laws between two reference systems in uniform relative motion.
2. The transformation laws admit an upper limit c to the possible speeds, which is the same for all the reference systems and coincides with the speed of light.

If we substitute the second part with the requirement of no upper limit on the possible speeds we get the principle in its Galilean form. From our derivation it is clear that other alternative but completely equivalent ways of formulating the principle of relativity are those stating the explicit transformation laws as the Galilean or the Lorentz ones.

It is also worth stressing once again that the validity of the Lorentz transformations, namely of the Einstein principle of relativity, imply that Newtonian dynamics is wrong. The latter, in fact, is practically a translation in mathematical language of the Galilean principle of relativity which required that these laws had to be covariant with respect to the Galilean transformations. For the same reason, then, the new laws of dynamics should be covariant with respect to the Lorentz transformations, but which exactly are these new laws, and how can we derive them?

One possibility is to exploit the results of the Michelson–Morley experiment, placing it in connection with the principle of relativity. Following its first statement, the "laws of physics" we are dealing with are not only those of mechanics which were implicit in Galileo's formulation, but also those of electromagnetism. And if we recall the claim of the Italian scientist about the impossibility, by any experiment, of measuring the absolute velocity of a reference system, we can immediately see that the Michelson–Morley experiment is exactly supporting this statement by including the electromagnetic and/or optics experiments.

Lorentz covariance of the wave equation

Because this test shows that the limit speed of the Lorentz transformations is that of light, it is natural to start by requiring the invariance of such speed in the appropriate equation for the electromagnetic field. The idea is to obtain in this way the transformation laws for the electric and magnetic fields, and use them within the Lorentz force to deduce the appropriate transformation laws of the force, and therefore of the laws of dynamics.

The law telling us that the electromagnetic field can be described by a wave moving with speed $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the wave equation, which can be derived from the Maxwell equations and in a space with no charges or currents writes as

$$\nabla^2 \boldsymbol{\Phi} = \varepsilon_0\mu_0 \frac{\partial^2 \boldsymbol{\Phi}}{\partial t^2}, \tag{5.3.1}$$

where $\boldsymbol{\Phi}$ is either $\mathbf{E}$ or $\mathbf{B}$. Requiring the Lorentz-covariance of this equation, namely that in a Lorentz-transformed reference system $\bar{S}$ the equation is

$$\bar{\nabla}^2 \bar{\boldsymbol{\Phi}} = \varepsilon_0\mu_0 \frac{\partial^2 \bar{\boldsymbol{\Phi}}}{\partial \bar{t}^2}, \tag{5.3.2}$$

means that the transformed field $\bar{\boldsymbol{\Phi}}$ is traveling with the same speed, which is exactly what the Michelson–Morley experiment showed. However this approach does not lead to any useful result; in fact, as shown in Exercise 5.7, the wave equation is automatically covariant under Lorentz transformations for any field (scalar, vector, or tensor) provided that the speed of the wave is c. This might seem weird at first sight, but it is easy to convince ourselves that this is what we should expect. The wave equation, in fact, just describes a kind of motion of a field regardless of the kind of field. If one could extract the transformation laws of the field from this equation, this would imply that all the fields should transform according to the same law, which is clearly contradictory.[10] Thus, the wave equation has to be always form-invariant under Lorentz transformation if it is correct from a relativistic point of view and as long as we ask that the wave is moving with speed c.[11]

Lorentz covariance of classical electromagnetism

Rather, we have to reason in the opposite way. What should suggest that classical electromagnetism can be compatible with Einstein's principle of relativity is the fact that, for both **E** and **B**, one can work out from the Maxwell equations a wave equation moving with speed c. Therefore, if the goal is to discover the relativistic laws of dynamics one should reverse the order of what was done in Sect. 4.5. First, the transformation laws for the fields can be deduced by imposing the Lorentz covariance of the Maxwell equations. Successively, the assumption that the Lorentz formula $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is a relativistic correct formula describing the electromagnetic force exerted by the fields **E** and **B** on a charge q moving with velocity **v** will lead us to the transformation laws for the forces and therefore to the new laws of dynamics.

Following this line of reasoning, it is possible to show (see Sect. C.2) that the electric and magnetic fields **E** and **B** leave the Maxwell equations form-invariant if they Lorentz transform as[12]

$$\bar{\mathbf{E}}_{\parallel} = \mathbf{E}_{\parallel} \tag{5.3.3}$$

$$\bar{\mathbf{E}}_{\perp} = \gamma\left[\mathbf{E}_{\perp} + (\mathbf{u} \times \mathbf{B})_{\perp}\right] \tag{5.3.4}$$

$$\bar{\mathbf{B}}_{\parallel} = \mathbf{B}_{\parallel} \tag{5.3.5}$$

$$\bar{\mathbf{B}}_{\perp} = \gamma\left[\mathbf{B}_{\perp} - \frac{1}{c^2}\,(\mathbf{u} \times \mathbf{E})_{\perp}\right]. \tag{5.3.6}$$

Now, if we use these transformations and Eq. (5.5.8) for the velocities, it is (Exercise 5.8)

$$\left(\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_{\parallel} = \frac{\mathbf{E}_{\parallel} + (\mathbf{v} \times \mathbf{B})_{\parallel} - \left(\mathbf{u}/c^2\right)(\mathbf{v} \cdot \mathbf{E})}{\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)} \tag{5.3.7}$$

$$\left(\bar{\mathbf{E}} + \bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_{\perp} = \frac{\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B})_{\perp}}{\gamma\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)} \tag{5.3.8}$$

[10]Because the field equations for **E** and **B** are different, e.g., this would imply that the Maxwell equations cannot be invariant under Lorentz transformations, which is exactly the opposite of what we are trying to show.

[11]Actually it can be shown (Spavieri 1985) that any wave equation is form-invariant under a general coordinate transformation.

[12]Here we have used with Eqs. (C.2.13) and (C.2.14) the obvious fact that for any vector **a** it is $(\mathbf{u} \times \mathbf{a}) = (\mathbf{u} \times \mathbf{a})_{\perp}$.

The transformation law of the force can thus be derived from these two by requiring that the transformed $\bar{\mathbf{F}}$ is again $q\left(\bar{\mathbf{E}}+\bar{\mathbf{v}}\times\bar{\mathbf{B}}\right)$ and under the above assumption that the electric charge is Lorentz-invariant (i.e., it is a scalar).

Clearly it is

$$\mathbf{F}_{\parallel}=q\left[\mathbf{E}_{\parallel}+(\mathbf{v}\times\mathbf{B})_{\parallel}\right], \tag{5.3.9}$$

and

$$q\,(\mathbf{v}\cdot\mathbf{E})=q\,[\mathbf{v}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})]=\mathbf{v}\cdot\mathbf{F}, \tag{5.3.10}$$

therefore, substituting Eqs. (5.3.9) and (5.3.10) into (5.3.7) it is

$$\bar{\mathbf{F}}_{\parallel}=\frac{\mathbf{F}_{\parallel}-\left(\mathbf{u}/c^{2}\right)\mathbf{v}\cdot\mathbf{F}}{\left(1-\mathbf{v}\cdot\mathbf{u}/c^{2}\right)}. \tag{5.3.11}$$

For the perpendicular component, instead, it is immediately

$$\bar{\mathbf{F}}_{\perp}=\frac{\mathbf{F}_{\perp}}{\gamma\left(1-\mathbf{v}\cdot\mathbf{u}/c^{2}\right)}. \tag{5.3.12}$$

In summary we have shown that:

- Given the appropriate transformations for charge and current densities, Eqs. (5.3.3) to (5.3.6) are the transformations of the electric and magnetic fields between two reference systems in uniform relative motion needed to leave the Maxwell equations form-invariant.
- These transformations and that for the velocity have been used to find the transformation law of the Lorentz force.
- The transformation we have just found, however, must be valid for any force, and these formulae show how in general the forces have to Lorentz-transform in order to be consistent with electromagnetism.

Relativistic laws of dynamics

With the above general transformation laws for the forces, we are now in position to understand how this reflects on the laws of dynamics. In principle, in fact, it should just be a matter of comparing them with the right-hand side of the classical equation $\mathbf{F}=m\mathbf{a}$, provided that the transformation laws of the accelerations are a problem of kinematics which can be solved independently (see Exercise 5.4) starting from the Lorentz transformations themselves.

We have already pointed out that Newton's equation of dynamics $\mathbf{F}=m\mathbf{a}$ is a direct consequence of the Galilean principle of relativity, and therefore that they cannot hold anymore as long as the latter is substituted by its relativistic counterpart. This means that in special relativity this equation is not covariant, so that under

Lorentz transformations it cannot be written in the form $\bar{\mathbf{F}} = m\bar{\mathbf{a}}$. However, it could be argued that one fundamental assumption of classical dynamics is that the inertial mass of a particle is a Galilean-invariant (i.e., a *scalar*) and so by allowing a relativistic law of the dynamics in the form $\bar{\mathbf{F}} = \bar{m}\bar{\mathbf{a}}$, i.e., dropping the hypothesis on the invariance of m, one could recover a relativistic consistent formulation of the dynamics. Unfortunately, this is not the case, as we show in a moment.

For our purposes it is sufficient to consider the simpler case of a particle at rest in S, so that $\mathbf{v} = 0$.[13] Under this condition, in fact, Eqs. (5.3.11) and (5.3.12) giving the components of the forces parallel and perpendicular to $\mathbf{u}$ become

$$\bar{\mathbf{F}}_\parallel = \mathbf{F}_\parallel \tag{5.3.13}$$

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \tag{5.3.14}$$

whereas those of the accelerations, using Eq. (5.5.10), are

$$\bar{\mathbf{a}}_\parallel = \frac{1}{\gamma^3}\mathbf{a}_\parallel \tag{5.3.15}$$

$$\bar{\mathbf{a}}_\perp = \frac{1}{\gamma^2}\mathbf{a}_\perp. \tag{5.3.16}$$

Under the hypothesis that the correct law is $\bar{\mathbf{F}} = \bar{m}\bar{\mathbf{a}}$, it is possible to derive the transformation law for the inertial mass from Eqs. (5.3.14) and (5.3.16). In fact, by indicating with m_0 the inertial mass of a particle at rest, so that $m = m_0$ in S, we have

$$\bar{m}\frac{1}{\gamma^2}\mathbf{a}_\perp = \frac{1}{\gamma}m_0\mathbf{a}_\perp,$$

that leads to the well-known conclusion

$$\bar{m} = m_0\gamma = \frac{m_0}{\sqrt{1 - u^2/c^2}}, \tag{5.3.17}$$

where the rest-mass m_0 can be regarded as a scalar in a broader sense which now includes the Lorentz transformations. However, if we use Eqs. (5.3.13) and (5.3.15), it is instead

$$\bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel = m_0\mathbf{a}_\parallel$$

which is clearly in contradiction to the previous result unless we make a distinction between a mass "felt" in the direction parallel to the force and a different one for the orthogonal component. Indeed, this concept was used in the past, but it has been

[13] And therefore $\bar{\mathbf{v}} = -\mathbf{u}$. Indeed, it is clear that a particle at rest in S will have a velocity $-\mathbf{u}$ in a frame $\bar{S}$ moving with velocity $\mathbf{u}$ with respect to S, but the reader can verify it explicitly by placing $\mathbf{v} = 0$ in Eq. (5.5.7).

abandoned inasmuch as its adoption requires that the mass cannot be considered a scalar anymore.

This is sufficient to show that $\mathbf{F} = m\mathbf{a}$ cannot be a relativistic consistent formulation of the law of dynamics even in the case of admitting a mass that varies with its velocity. Such a conclusion should not be surprising when it is recalled that Newton's law of dynamics, and therefore the Galilean principle of relativity, can be alternatively formulated as a principle of conservation of the momentum $\mathbf{p} = m\mathbf{v}$, and that we can also write it as $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$. This is, in fact, equivalent to $\mathbf{F} = m\mathbf{a}$ simply because m does not change with respect to t; however, by admitting that m can vary with the velocity of the particle, it is clear that the mass cannot in general be constant with respect to t. Thus, our comparison should start from the momentum-based formulation which, by considering that $\bar{\mathbf{v}}_\parallel = \bar{\mathbf{v}} = -\mathbf{u}$ and $\mathbf{v} = 0$, would write

$$\bar{\mathbf{F}}_\parallel = \frac{\mathrm{d}\bar{\mathbf{p}}_\parallel}{\mathrm{d}\bar{t}} = \bar{m}\bar{\mathbf{a}}_\parallel + \frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{t}}\bar{\mathbf{v}}_\parallel = \bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel - \frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{t}}\mathbf{u} = \mathbf{F}_\parallel = \frac{\mathrm{d}\mathbf{p}_\parallel}{\mathrm{d}t} = m_0\mathbf{a}_\parallel \quad (5.3.18)$$

$$\bar{\mathbf{F}}_\perp = \frac{\mathrm{d}\bar{\mathbf{p}}_\perp}{\mathrm{d}\bar{t}} = \bar{m}\bar{\mathbf{a}}_\perp + \frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{t}}\bar{\mathbf{v}}_\perp = \bar{m}\frac{1}{\gamma^2}\mathbf{a}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp = \frac{\mathrm{d}\mathbf{p}_\perp}{\mathrm{d}t} = \frac{1}{\gamma}m_0\mathbf{a}_\perp. \quad (5.3.19)$$

Actually, the equation involving the perpendicular component does not change, whereas that of the parallel component surely does, which explains the previous contradictory result.[14]

We can now regroup and make some considerations.

First it should be noticed that, although we have been able to use the Lorentz covariance of classical electromagnetism in order to deduce from it the Lorentz-covariant laws of dynamics, this was not an easy task, and it required long and sometimes cumbersome calculations. This might be even more surprising by comparing this derivation with the almost trivial proof of the Galilean covariance of the third Newton law of dynamics and the complicated path needed to get its modified relativistic expression.

[14]For the sake of completeness, it is worth while to add that the equation involving the parallel component of the force now gives the same result as of the perpendicular one; in fact it can be shown that in general, for a body of mass m moving with velocity $\mathbf{v}$, it is $\mathrm{d}m/\mathrm{d}t = \mathbf{F}\cdot\mathbf{v}/c^2$ (Exercise 6.5). In $\bar{S}$ we have therefore

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{t}} = -\frac{1}{c^2}\bar{\mathbf{F}}\cdot\mathbf{u},$$

so that

$$\bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel - \frac{\mathrm{d}\bar{m}}{\mathrm{d}\bar{t}}\mathbf{u} = \bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel + \frac{\mathbf{u}}{c^2}\bar{\mathbf{F}}\cdot\mathbf{u} = \bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel + \frac{u^2}{c^2}\frac{\bar{\mathbf{F}}\cdot\mathbf{u}}{u^2}\mathbf{u} \equiv \bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel + \frac{u^2}{c^2}\bar{\mathbf{F}}_\parallel,$$

but because in this case $\bar{\mathbf{F}}_\parallel = \mathbf{F}_\parallel = m_0\mathbf{a}_\parallel$, Eq. (5.3.18) becomes

$$\bar{m}\frac{1}{\gamma^3}\mathbf{a}_\parallel = m_0\mathbf{a}_\parallel\left(1 - \frac{u^2}{c^2}\right) = m_0\frac{1}{\gamma^2}\mathbf{a}_\parallel,$$

which again implies Eq. (5.3.17).

Moreover, in doing this the expressions we have obtained have lost one useful property of Galilean and Newtonian physics: the covariance that was automatically granted when the physical laws were expressed in vectorial formalism. In particular, this notation is absolutely unable to "produce" physical laws that are manifestly Lorentz covariant, but how much does this matter? And why? It is now time to remember our past considerations about the connection between physics and geometry, and our suggestion that the changing of the basic physical assumption might result in a change of the geometry which can fit best our needs.

5.4 Lorentz Transformations from a Geometrical Point of View

5.4.1 The Physical Meaning of Covariance

For what we have seen in Chap. 3, a covariance requirement is just a more formal way to ask that the mathematical model of a physics theory does not change with respect to some set of transformations between reference systems. This in practice ensures that any observer (represented by a specific reference system) would be able to deduce the same theory by observing the same physical phenomena, which is certainly a reasonable and very basic requisite for any useful description of the physical world.

Moreover, the set of transformations associated with each covariance requirement represents the formal mathematical formulation of some kind of assumptions we hold true about such a physical world. Therefore the Euclidean covariance, which is the covariance with respect to rotations and translations in space and to translations in time, comes from the hypothesis that the space is homogeneous and isotropic, and that the time is homogeneous,[15] whereas Galilean or special relativistic covariance derives from the assumption of the equivalence among reference systems moving with constant relative speed.

As a matter of fact, it seems that an important difference between these two covariances has to be stressed.

In practice Euclidean covariance involves purely geometrical transformations, and we have seen that the mathematical objects that we use to make our physical models are literally "built" on top of these homogeneity and isotropy hypotheses. It is for this reason that this kind of covariance is often regarded as obvious, forgetting that it is just a by-product of the experience-driven way by which Euclidean geometry was

[15]It is worth stressing here that we are intending "space" and "time" as *geometrical* rather than *physical* objects. In other words, this is not the homogeneity and isotropy assumption of cosmology. The fact that we can require that the geometry we use to represent the space is isotropic does not imply that the content of such space has to be distributed isotropically.

created: as long as we use Euclidean objects to write our physical models, it cannot be a surprise that the Euclidean covariance can be easily ensured.[16]

The covariance expressed by the principle of relativity, instead, is another story. Its basis, apparently, does not have a geometrical origin, but rather what we have called a "kinematic" character. Indeed (see Sect. 3.2.1) we managed to facilitate the identification of the Galilean-invariants, i.e., of those, among the Euclidean objects, that satisfy this principle in its Galilean form but, as stressed, this process cannot be brought to a complete and consistent geometrical formulation because it required a criterion of simultaneity of measurements involving both space and time. The concept of a measurement as a scalar[17] means that its definition rests on the availability of a metric space, but in Newtonian physics space and time are two separate metric spaces, so a Galilean scalar cannot be described as a measurement in such a strict geometric sense. Apparently the situation is even worse for the principle of relativity in its Einsteinian formulation, as the relativity of simultaneity makes it impossible to formulate an easy "recipe" to identify the relativistic invariants among our Euclidean objects. Sometimes, however, worse is for the better.

5.4.2 Lorentz-Invariant Quantities and Measurements

In Sect. 3.1.1 we observed that the identification of measurements with scalars can be referred to the definition of the distance between two points in Euclidean geometry as a prototypical model for scalars. Its fundamental property is that of defining a quantity, a function of coordinate differences, invariant for translations and rotations, which is the mathematical translation of the homogeneity and isotropy requirements. Armed with this previous experience, it makes perfect sense then to ask ourselves if it is possible to construct a similar quantity that is invariant for Lorentz transformations.

As can be easily seen, and as shown in Exercise 5.5, the quantity

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} \tag{5.4.1}$$

is invariant under Lorentz transformations, so we can imagine using it as a prototypical definition of "Lorentz scalar" which, similarly to what happens in Euclidean space, defines a sort of "distance" in special relativity.[18] It is evident, also by comparing it with Eq. B.5.2, that one can write it as

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[16]This, however, cannot be considered a good reason to diminish the importance of such a requirement, because it does not stand on the way it is assured, but rather on the assumption on which it is based. Even if a problem can be easily solved, this does not mean that it is less important or less fundamental than another more difficult, or less intuitive, one.

[17]Which, as we have seen, is at the very basis of a mathematical model of physics.

[18]The use of quotation marks in this case is justified by the fact that, unlike the Euclidean case, we can have $\mathrm{d}s^2 = 0$ even if the coordinate differences are not zero, and we can even have cases in which $\mathrm{d}s^2 < 0$. The problem of defining the measurements in special and general relativity, in fact, is connected with a correct definition of an observer who makes the measurements.

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta, \quad \alpha, \beta = 0, \ldots, 4, \tag{5.4.2}$$

where $g_{\alpha\beta} = \eta_{\alpha\beta}$ and[19]

$$\eta_{00} = -1$$
$$\eta_{ii} = 1$$
$$\eta_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta$$

and

$$dx^0 = c dt.$$

With this idea in mind, it is natural to identify ds as a distance between two "points", or events, $\boldsymbol{x} \equiv \{x^\alpha\} \equiv \{ct, x^1, x^2, x^3\}$ and $\boldsymbol{x} + d\boldsymbol{x} = \{x^\alpha + dx^\alpha\}$ of a four-dimensional "space" that includes both the usual time and 3D space and thus is called *spacetime*. But which is the geometry of this spacetime, if any?

Among its strange characteristics we can see that, unlike the Euclidean case, we can have $ds^2 = 0$ even if the coordinate differences are not zero, and we can even have cases in which $ds^2 < 0$, which is obviously due to the fact that the time coordinate is included with a negative metric coefficient. This brings us to an interesting point.

Difference between Lorentz-invariant and measurement

When we "built up" Euclidean geometry in Chap. 3 we based our reasoning on stressing the fact that the distance measurements could be taken as the prototypical scalars of such a geometrical set-up, and there was a tight connection between their invariance under the transformations of the Euclidean geometry (the Euclidean isometry group) and their association with a measurement.

Now it is difficult to affirm that the above Lorentz scalar can represent a measurement, at least in the sense we are used to conceive it. On the other hand, if we try to find a way to define a spatial length or a temporal interval (i.e., the two elementary measurements necessary for any physical theory) we soon face the problem represented by the length contraction and the time dilation (see Exercises 5.9 and 5.10). These are necessary consequences of the change of the principle of relativity from the Galilean to the Einsteinian form, and their meaning is that the results of these basic measurements are not invariant for a transformation between two reference systems anymore. More precisely, they are still invariant for Euclidean transformations, but not for those between two systems moving with constant relative velocity.

This fact was not unknown before special relativity,[20] but it introduces at the most basic level the concept of a measurement as a quantity that depends on the observer

[19] We are adopting the convention that Latin indexes run over the range $[1-3]$ and Greek indexes run over the range $[0, \ldots, 4]$.

[20] Just think about the idea of a speed as a scalar.

who makes it. Indeed, it completely breaks the identification of a "scalar" as both an invariant quantity and a measurement, but not because we can find a transformation for which some Euclidean scalar is not invariant. This was already true in Newtonian physics and we had to introduce the concept of Galilean-invariant to keep alive the idea of a scalar with these two properties. Rather, the disruption happens because now there is no single measurement that can keep its invariance properties for all the transformations.

In practice we cannot abandon the identification of a measurement as a Euclidean scalar, but since this is not an invariant quantity for any observer anymore, a consistent model cannot exist unless it is explicitly referred to a specific observer, i.e., to a reference system in which we can separate space and time and therefore give to the measurements the familiar representation in terms of the ordinary scalars.[21]

On the other hand, the requirement that the laws of physics do not depend on certain classes of transformations, which is always present in the basic principles as we have seen in the previous chapters, implies that the identification of invariant quantities can help to find a convenient form to express in a covariant way the laws of physics, as we show in the next chapter.

5.4.3 Spacetime, Four-Dimensional Hyperbolic Geometry, and Manifest Lorentz Covariance

Lorentz transformations as rotations in hyperbolic spacetime

We can now go back to the previous question of that is the geometry of the spacetime. We split the answer this and in the next subsections. Here it is worth concentrating on the fact that the Lorentz transformations, which leave Eq. (5.4.1) invariant, as shown in Sect. 5.1.1 are linear and homogeneous, like the rotations of Euclidean space. This means that they can be put in matrix form, and limiting ourselves to transformations involving just the t and x coordinates, we can write[22]

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[21]This theory is essential also in general relativity. The interested reader can refer to more advanced texts, such as de Felice and Bini (2010).

[22]We are neglecting the other two spatial coordinates, which anyway would simply modify the expression of the matrix Λ below as

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \Lambda\left(\frac{v}{c}\right)\begin{pmatrix} ct \\ x \end{pmatrix},$$

where, by putting $\beta = v/c$ and remembering that $\gamma = \left(1-\beta^2\right)^{-1/2}$,

$$\Lambda\left(\frac{v}{c}\right) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}.$$

It is immediate to verify that $\gamma^2 - (\beta\gamma)^2 = 1$, so from the properties of the hyperbolic functions[23] we can write

$$\Lambda = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix},$$

with $\gamma = \cosh\chi$, $\beta\gamma = \sinh\chi$, and $\beta = \tanh\chi$, therefore the spacetime can be seen as a four-dimensional space, also called *Minkowski space*, where the geometry of the time and of each spatial axis is hyperbolic, and the Lorentz transformations can be interpreted as rotations of an angle χ, called *rapidity*, in such a hyperbolic space.[24]

By combining two successive boosts with rapidity χ_1 and χ_2 we get, using the addition formulae of the hyperbolic functions,

$$\begin{aligned} \begin{pmatrix} ct_2 \\ x_2 \end{pmatrix} &= \Lambda_2\begin{pmatrix} ct_1 \\ x_1 \end{pmatrix} = \Lambda_2\Lambda_1\begin{pmatrix} ct \\ x \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\chi_1+\chi_2) & -\sinh(\chi_1+\chi_2) \\ -\sinh(\chi_1+\chi_2) & \cosh(\chi_1+\chi_2) \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix} \end{aligned}$$

which is another boost with rapidity $\chi = \chi_1 + \chi_2$. We had reached the conclusion that a combination of two boosts is a boost already at the end of Sect. 5.1.1, but now we can also say that it does not depend on the order of the two original transformations and that such a boost can be found simply by adding the rapidities, at least in the case where the two velocities are parallel. Thus, this particular set of Lorentz transformations is *closed* with respect to the multiplication of their matrix elements. It is also easy to see that it is *associative* as well, that it admits the *identity element* (which is simply $\mathbb{I} = \Lambda(\chi = 0) = \Lambda(v = 0)$), and the *inverse element* ($\Lambda(-\chi)$). In other words, these (parallel) Lorentz transformations form a *group* with respect to this matrix multiplication.

These considerations show us that our previous claim about the fundamental difference between the Euclidean and Lorentz covariances is not very well justified anymore. Indeed, Lorentz transformations can be interpreted as purely geometrical exactly as the rotations, with the only difference that the latter operates on a Euclidean space, whereas the former work on a hyperbolic one. It is not a surprise, then, that the

[23]In particular from the relation $\cosh^2\chi - \sinh^2\chi = 1$.

[24]This geometry is also named *pseudo-Euclidean* for reasons that are explained in the next subsection and in Sect. 5.4.4.

principle of relativity can be interpreted as another principle of covariance similarly to what was done for the Euclidean space and its rotations.

The Lorentz and the Poincaré groups

In the most general case things are not so simple, but it can be shown that:

1. The combination, in the sense of matrix multiplication, of two boosts along different directions can always be expressed as the multiplication of a boost and a spatial rotation where, however, the rotation matrix has to be "embedded" in a 4×4 matrix
$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_{3\mathrm{D}} & \\ 0 & & & \end{pmatrix},$$
and $R_{3\mathrm{D}}$ is its usual three-dimensional representation. In general the combination of any pair of boosts and rotations can always be expressed as a single rotation, or a single boost, or a rotation and a boost.
2. The four-dimensional identity matrix is the identity element of these transformations.
3. There exist the inverse elements for both rotations and boosts.
4. Any three transformations of this kind are associative.

Thus the set of all the rotations, boosts, and their combinations form a group with respect to the matrix multiplication that quite obviously is called *Lorentz group*.

If we also consider the set of transformations obtained by combining the elements of the Lorentz group with the translations, we get another group called the *Poincaré group*.

Four-vectors and Lorentz covariance

Finally, the geometrical interpretation of the Lorentz transformations naturally raises another question. In fact, as in the Euclidean case we found a way to define several geometrical objects in addition to the scalar ones, such as vectors and tensors of arbitrary rank, one can reasonably argue that, in addition to the Lorentz scalars, it is possible to conceive other geometrical objects "living" in this four-dimensional spacetime. Indeed, this is certainly feasible with a procedure similar to that used in the Euclidean case, where we used the spatial displacements as a model to define the three-dimensional Euclidean vectors. This was done in three steps:

1. These quantities can produce a scalar by means of an appropriate scalar product.
2. This scalar product is left invariant by any transformation belonging to the Euclidean isometry group.
3. A three-dimensional Euclidean vector is thus defined as any set of three quantities having the same transformation properties as the displacements, i.e., that does not change (as a whole) for any Euclidean transformation.

We can therefore compare this procedure with what we have in this four-dimensional case:

1. The set of these "four-dimensional displacements" d$\boldsymbol{x}$ has been used to define a scalar product characterized by $\eta_{\alpha\beta}$.
2. Moreover, we can easily notice that the Lorentz scalar ds^2 is invariant not only for boosts, but also for any transformation of the Poincaré (isometry) group; indeed, under translations both temporal and spatial displacements remain invariant, whereas spatial rotations do not affect the temporal part and leave invariant the spatial part of the scalar product, which is Euclidean.
3. Similarly to their three-dimensional counterparts (see the following subsection) the objects represented by such sets of four components do not change for the above transformations. We define these sets and any set of four components that transform in the same way as *four-vectors.*[25]

5.4.4 Minkowski Geometry

The above-depicted procedure results in another geometry, which is able to express a four-dimensional counterpart with respect to any Euclidean geometric object. The development of such a geometry, however, can be seen from several different perspectives. A more formal one is that based on differential manifolds, resting on the same framework presented in Appendix B. Indeed, it is easy to notice that the essence of this approach does not depend on the specific characteristics of the three-dimensional Euclidean space.

Manifold approach to special relativistic geometry

One can surely establish a homeomorphism, i.e., a coordinate system, between spacetime and $\mathbb{R}^4$ (not to mention the possibility of conceiving even higher-dimensional spaces). Then, as long as the manifold is differentiable or smooth, the same procedure used for the Euclidean space can be followed to define four-dimensional vectors, one-forms, and tensors of higher rank.

[25] And following the same reasoning we can obviously define four-tensors of any rank.

Naturally this also includes the possibility of defining basis vectors and one-forms $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\alpha$ linked by the usual relation $\langle \boldsymbol{e}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta$, which induce a representation of four-vectors and one-forms in terms of components

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}x^\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{\xi} = \xi_\alpha \boldsymbol{e}^\alpha$$

where those of the four-vectors have the usual meaning of infinitesimal displacements, but in this case in spacetime.

This procedure implies that the metric tensor in Cartesian coordinates corresponding to δ_{ab} of the Euclidean geometry is instead

$$\boldsymbol{\eta} = \eta_{\alpha\beta} \boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta$$

whose components are exactly the $\eta_{\alpha\beta}$ introduced in Eq. (5.4.2), and that the scalar product is

$$\mathrm{d}s^2 = \boldsymbol{\eta}\,(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) = \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta,$$

as required. As it should be for a metric tensor, its inverse exists, i.e., a $(2, 0)$ tensor whose components are $\eta^{\alpha\beta}$ related to $\boldsymbol{\eta}$ by the formula $\eta_{\alpha\gamma}\eta^{\gamma\beta} = \mathbb{I}$. These two tensors, as in Euclidean geometry can be used to "raise" and "lower" the indexes by contraction. In formulae

$$x_\alpha = \eta_{\alpha\beta} x^\beta \quad x^\alpha = \eta^{\alpha\beta} x_\beta.$$

This observation gives the opportunity to stress a fundamental difference between four-vectors and Euclidean vectors, whichever the dimension of the space. Indeed, in the same way that a Lorentz scalar is not just a four-dimensional Euclidean scalar, four-vectors cannot be considered just four-dimensional Euclidean vectors, because Minkowski space is a vector space with its own metric, different from the Euclidean one, defined by a $(0, 2)$ tensor whose components are $\eta_{\alpha\beta}$.

Usually in Euclidean geometry the scalar product is identified by an operation between two vectors, which as we have seen in Appendix B.4 is not completely correct inasmuch as it is rather an operation involving a vector and a one-form, or two vectors and the metric tensor. In this case, such kind of "abuse of notation" which is common in Euclidean geometry is not too dangerous because the covariant and contravariant Cartesian components are exactly the same. In Minkowski geometry, however, $x^a = x_a$, $a = 1, 2, 3$, and $x_0 = \eta_{0\alpha} x^\alpha = -x^0$, thus when dealing with relativistic spacetime it is important to distinguish between covariant and contravariant components, and between vectors and one-forms. In order to underline the analogy with the Euclidean formulae, in the next chapter we still adopt the convention of denoting the scalar product between two vectorial quantities $\boldsymbol{v}$ and $\boldsymbol{w}$ as $\boldsymbol{v} \cdot \boldsymbol{w}$, with the important understanding that, because we are using Minkowskian four-vectors,

$$\boldsymbol{v} \cdot \boldsymbol{w} \equiv \boldsymbol{\eta}\,(\boldsymbol{v}, \boldsymbol{w}) = \eta_{\alpha\beta} v^\alpha w^\beta. \tag{5.4.3}$$

Minkowski geometry is pseudo-Euclidean

Another fundamental difference, strictly related to the above one, is that the Minkowski metric is not positive definite as is the Euclidean one. This means, as already mentioned in Sect. 5.4.2, that the result of a scalar product is not necessarily positive. In particular, the norm of a four-vector can be positive ($\boldsymbol{\eta}\,(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) > 0$), zero ($\boldsymbol{\eta}\,(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) = 0$), or negative ($\boldsymbol{\eta}\,(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) < 0$) which leads to the well-known classification of four-vectors as *spacelike*, *null*, and *timelike*, accordingly. For this reason Minkowski geometry is called *pseudo-Euclidean*, the first word referring to the indefiniteness of its scalar product, and the second one to the fact that, as we show more dearly in Appendix D, it is intrinsically flat like the Euclidean one.

Zero-component lemma

It is now worth mentioning a useful theorem of Minkowski geometry, denoted as the *zero-component lemma* in Rindler (2006). It states that if a four-vector has a particular one of its four components zero in all inertial frames, then the entire vector must vanish.

This can be immediately shown as in the cited reference. Suppose that, say, the component x^1 of the vector $\boldsymbol{x} = \{x^\alpha\}$, $\alpha = 0, 1, 2, 3$, is always zero. This means that, e.g., this component will be zero also in another inertial reference frame $\bar{S}$ obtained by a Lorentz boost along that axis, i.e.,

$$\bar{x}^1 = \gamma \left(\frac{x^0 v}{c} - x^1 \right) = \gamma \frac{x^0 v}{c} = 0,$$

which implies $x^0 = 0$, and because the transformation is arbitrary, it means that x^0 must also vanish in all inertial frames. But then another Lorentz transformation in the x^2 direction gives

$$\bar{x}^0 = \gamma \left(-x^0 + \frac{x^2 v}{c} \right) = \gamma \frac{x^2 v}{c} = 0,$$

which similarly implies that $x^2 = 0$ as well. The same reasoning, eventually, can be applied to x^3.

The importance of this theorem, which is used in the next chapter, lays in its formulation in spacetime. Actually it is easy to see that the same theorem holds true for three vectors in the three-dimensional Euclidean space, but its consequences are much less useful in this context.

Finally, it is important to stress that these four-dimensional quantities play in the Minkowski spacetime the same role of the vectorial quantities in the Euclidean space, in particular for what concerns the covariance properties of physical laws. In

Chap. 3 in fact we showed that, in order to guarantee the rotational covariance of an equation, it was sufficient to write it in vectorial terms ($\mathbf{v} = 0$) because this was a direct consequence of the way Euclidean vectors (and tensors) had been defined. This implies that the same is true for four-vectors (and tensors) in Minkowski space with respect to the transformations of the Lorentz group. In order to guarantee the covariance of the equations with respect to Lorentz transformations, and therefore to ensure their compatibility with respect to both the principle of covariance (here intended in its purely Euclidean form) and to that of relativity, it is sufficient to write them in four-vectorial terms, such as $\boldsymbol{v} = 0$.

For the same reason, we show in the next chapter that the same "Lagrangian procedure" used in Sects. 4.2 and 4.4.2 to find the equations of motion and the field equations in the Euclidean case can be used for special relativity within its natural Minkowskian framework.

5.5 Exercises

Exercise 5.1 Find a general expression of the Lorentz transformations for space and time coordinates.

Solution 5.1 The Lorentz transformations between the reference system S, with coordinates (t, x, y, z) and $\bar{S}$, with coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$, moving with velocity u with respect to S along the x-axis and whose axes are parallel to those of S, are

$$\begin{aligned}\bar{t} &= \gamma\left(t - x\frac{u}{c^2}\right)\\ \bar{x} &= \gamma\,(x - ut)\\ \bar{y} &= y\\ \bar{z} &= z\end{aligned}$$

where $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$. The spatial part can be interpreted by saying that the components orthogonal to $\mathbf{u} \equiv (u_x, 0, 0)$ remain unchanged, whereas in the formula for t we can easily notice that xu is just the equivalent of $\mathbf{x} \cdot \mathbf{u}$ for this specific velocity. These transformations in vectorial form must be invariant for a generic relative velocity $\mathbf{u}$ so we can put in general

$$\bar{t} = \gamma\left(t - \frac{\mathbf{x}\cdot\mathbf{u}}{c^2}\right) \tag{5.5.1}$$

$$\bar{\mathbf{x}} = \mathbf{x}_\perp + \gamma\left(\mathbf{x}_\parallel - \mathbf{u}t\right) \tag{5.5.2}$$

where $\mathbf{x}_\parallel$, and $\mathbf{x}_\perp$ identify the decomposition of $\mathbf{x}$ along the direction of $\mathbf{u}$ and perpendicular to it respectively.[26] Using the fact that

[26]Obviously the inverse transformations can be easily found by considering that S is moving with velocity $-\mathbf{u}$ with respect to $\bar{S}$, so that

$$\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_\parallel$$
$$\mathbf{x}_\parallel = \frac{\mathbf{x}\cdot\mathbf{u}}{u}\frac{\mathbf{u}}{u}$$

where $u = \sqrt{\mathbf{u}\cdot\mathbf{u}}$, the transformations can be cast as

$$\bar{t} = \gamma\left(t - \frac{\mathbf{x}\cdot\mathbf{u}}{c^2}\right) \tag{5.5.3}$$
$$\bar{\mathbf{x}} = \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\right) + \gamma\left(\frac{\mathbf{x}\cdot\mathbf{u}}{u^2} - t\right)\mathbf{u}$$
$$= \mathbf{x} + \left[\frac{(\gamma-1)(\mathbf{x}\cdot\mathbf{u})}{u^2} - \gamma t\right]\mathbf{u}. \tag{5.5.4}$$

In component notation we have

$$\bar{t} = \gamma\left(t - \frac{\delta_{mn}x^m u^n}{c^2}\right) \tag{5.5.5}$$
$$\bar{x}^i = x^i + \left[\frac{(\gamma-1)\,\delta_{mn}x^m u^n}{u^2} - \gamma t\right]u^i, \tag{5.5.6}$$

where $u^2 = \delta_{kl}u^k u^l$.

Exercise 5.2 Find a general expression of the velocity transformations compatible with the Lorentz transformations.

Solution 5.2 From Eqs. (5.5.3) and (5.5.4) the velocity in the reference system $\bar{S}$ of a body moving with a velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ in S is

$$\bar{\mathbf{v}} = \frac{\mathrm{d}\bar{\mathbf{x}}}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}\mathbf{x} + \left[(\gamma-1)(\mathrm{d}\mathbf{x}\cdot\mathbf{u})/u^2 - \gamma \mathrm{d}t\right]\mathbf{u}}{\gamma\left(\mathrm{d}t - \mathrm{d}\mathbf{x}\cdot\mathbf{u}/c^2\right)}$$
$$= \frac{\mathbf{v} + \left[(\gamma-1)(\mathbf{v}\cdot\mathbf{u})/u^2 - \gamma\right]\mathbf{u}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)}$$
$$= \frac{\mathbf{v} + \left[(\gamma-1)\left(\mathbf{v}\cdot\hat{\mathbf{u}}\right)/u - \gamma\right]\mathbf{u}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)}, \tag{5.5.7}$$

where we used the relation $\mathbf{u} \equiv u\hat{\mathbf{u}}$.

(Footnote 26 continued)

$$t = \gamma\left(\bar{t} + \frac{\bar{\mathbf{x}}\cdot\mathbf{u}}{c^2}\right)$$
$$\mathbf{x} = \bar{\mathbf{x}}_\perp + \gamma\left(\bar{\mathbf{x}}_\parallel + \mathbf{u}t\right)$$

where $\bar{\mathbf{x}}_\parallel$ and $\bar{\mathbf{x}}_\perp$ decompose $\bar{\mathbf{x}}$ with respect to $\mathbf{u}$ in the same way as $\mathbf{x}$ in the direct transformations. And all the following formulae have a corresponding inverse that can be obtained by simply changing the sign of the velocity.

In the same way, but starting from Eqs. (5.5.1) and (5.5.2), we can write

$$\begin{aligned}\bar{\mathbf{v}} &= \left[\frac{\mathrm{d}\mathbf{x}_\perp}{\mathrm{d}t} + \gamma\left(\frac{\mathrm{d}\mathbf{x}_\parallel}{\mathrm{d}t} - \mathbf{u}\right)\right]\left[\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)\right]^{-1} \\ &= \left[\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)\right]\left[\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)\right]^{-1}\end{aligned} \tag{5.5.8}$$

where, as for the position vector $\mathbf{x}$, $\mathbf{v}_\perp \equiv \mathrm{d}\mathbf{x}_\perp/\mathrm{d}t$ and $\mathbf{v}_\parallel \equiv \mathrm{d}\mathbf{x}_\parallel/\mathrm{d}t$ identify the components of the velocity $\mathbf{v}$ orthogonal and parallel to $\mathbf{u}$.

Exercise 5.3 Show that a body moving with speed c in a reference system S will move with the same speed in any other reference system having velocity $\mathbf{u}$ with respect to S.

Solution 5.3 First we can find an expression for the transformed speed $\bar{v} = \sqrt{\bar{\mathbf{v}}\cdot\bar{\mathbf{v}}}$. To this aim it is easier to start from Eq. (5.5.8). In fact, because by definition $\mathbf{v}_\perp\cdot\mathbf{v}_\parallel = \mathbf{v}_\perp\cdot\mathbf{u} = 0$,

$$\begin{aligned}\bar{v} &= \frac{\sqrt{\left[\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)\right]\cdot\left[\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)\right]}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)} \\ &= \frac{\sqrt{\mathbf{v}_\perp\cdot\mathbf{v}_\perp + \gamma^2\left(\mathbf{v}_\parallel - \mathbf{u}\right)\cdot\left(\mathbf{v}_\parallel - \mathbf{u}\right)}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)}\end{aligned}$$

and if we indicate with α the angle between $\mathbf{u}$ and $\mathbf{v}$ the above formula becomes

$$\begin{aligned}\bar{v} &= \frac{\sqrt{v^2\sin^2\alpha + \gamma^2\left(v^2\cos^2\alpha - 2\mathbf{u}\cdot\mathbf{v}_\parallel + u^2\right)}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)} \\ &= \frac{\gamma\sqrt{\left(v^2/\gamma^2\right)\sin^2\alpha + v^2\cos^2\alpha - 2\mathbf{u}\cdot\mathbf{v} + u^2}}{\gamma\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)} \\ &= \frac{\sqrt{v^2\left(1 - u^2/c^2\right)\sin^2\alpha + v^2\cos^2\alpha - 2\mathbf{u}\cdot\mathbf{v} + u^2}}{\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)}\end{aligned} \tag{5.5.9}$$

where we have used the definition of $\gamma = \sqrt{1 - u^2/c^2}$ and the fact that $\mathbf{u}\cdot\mathbf{v}_\parallel = \mathbf{u}\cdot\mathbf{v}$. In vectorial form

$$\begin{aligned}\bar{v} &= \frac{\sqrt{\mathbf{v}\cdot\mathbf{v} - \left|\mathbf{u}\times\mathbf{v}\right|^2/c^2 - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u}}}{\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)} \\ &= \frac{\sqrt{(\mathbf{v} - \mathbf{u})\cdot(\mathbf{v} - \mathbf{u}) - \left|\mathbf{u}\times\mathbf{v}\right|^2/c^2}}{\left(1 - \mathbf{v}\cdot\mathbf{u}/c^2\right)},\end{aligned}$$

or, by remembering that $|\mathbf{u} \times \mathbf{v}|^2 = (uv)^2 - (\mathbf{u} \cdot \mathbf{v})^2$

$$\begin{aligned}
\bar{v} &= \frac{\sqrt{v^2\left(\hat{\mathbf{v}} \cdot \hat{\mathbf{v}}\right) - (uv/c)^2 + v^2\left(\mathbf{u} \cdot \hat{\mathbf{v}}\right)^2/c^2 - 2v\mathbf{u} \cdot \hat{\mathbf{v}} + u^2\left(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}\right)}}{\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)} \\
&= v\frac{\sqrt{1 - (u/c)^2 + \left(\mathbf{u} \cdot \hat{\mathbf{v}}\right)^2/c^2 - 2\mathbf{u} \cdot \hat{\mathbf{v}}/v + (u/v)^2}}{\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)}
\end{aligned}$$

which for $v = c$ becomes

$$\begin{aligned}
\bar{v} &= c\frac{\sqrt{1 - (u/c)^2 + \left(\mathbf{u} \cdot \hat{\mathbf{v}}\right)^2/c^2 - 2\mathbf{u} \cdot \hat{\mathbf{v}}/c + (u/c)^2}}{\left(1 - \hat{\mathbf{v}} \cdot \mathbf{u}/c\right)} \\
&= c\frac{\sqrt{1 - 2\mathbf{u} \cdot \hat{\mathbf{v}}/c + \left(\mathbf{u} \cdot \hat{\mathbf{v}}\right)^2/c^2}}{\left(1 - \hat{\mathbf{v}} \cdot \mathbf{u}/c\right)} \\
&= c\frac{\sqrt{\left(1 - \mathbf{u} \cdot \hat{\mathbf{v}}/c\right)^2}}{\left(1 - \hat{\mathbf{v}} \cdot \mathbf{u}/c\right)} = c.
\end{aligned}$$

Exercise 5.4 Find a general expression for the Lorentz-transformed acceleration.

Solution 5.4 The calculation is a bit long, but straightforward. Starting from Eq. (5.5.8) we have

$$\begin{aligned}
\bar{\mathbf{a}} \equiv \frac{d\bar{\mathbf{v}}}{d\bar{t}} &= d\left[\frac{\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)}{\gamma\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)}\right]\gamma^{-1}\left(dt - d\mathbf{x} \cdot \mathbf{u}/c^2\right)^{-1} \\
&= \frac{\left(d\mathbf{v}_\perp + \gamma d\mathbf{v}_\parallel\right)\gamma\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right) + \left[\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)\right]\gamma d\mathbf{v} \cdot \mathbf{u}/c^2}{\gamma^3\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3 dt} \\
&= \frac{\left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right)\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right) + \left[\mathbf{v}_\perp + \gamma\left(\mathbf{v}_\parallel - \mathbf{u}\right)\right]\mathbf{a} \cdot \mathbf{u}/c^2}{\gamma^2\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3} \\
&= \frac{\left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right) - \left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right)\mathbf{v} \cdot \mathbf{u}/c^2 + \left(\mathbf{v}_\perp + \gamma\mathbf{v}_\parallel\right)\mathbf{a} \cdot \mathbf{u}/c^2 - \gamma\mathbf{a} \cdot \mathbf{u}/c^2}{\gamma^2\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3} \\
&= \frac{\left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right) - \left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right)\mathbf{v} \cdot \mathbf{u}/c^2 + \left(\mathbf{v}_\perp + \gamma\mathbf{v}_\parallel\right)\mathbf{a} \cdot \mathbf{u}/c^2 - \gamma\mathbf{a}_\parallel u^2/c^2}{\gamma^2\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3} \\
&= \frac{\left(\mathbf{a}_\perp + \mathbf{a}_\parallel/\gamma\right) - \left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right)\mathbf{v} \cdot \mathbf{u}/c^2 + \left(\mathbf{v}_\perp + \gamma\mathbf{v}_\parallel\right)\mathbf{a} \cdot \mathbf{u}/c^2}{\gamma^2\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3} \\
&= \frac{\left(\gamma\mathbf{a}_\perp + \mathbf{a}_\parallel\right) + \gamma\left[\left(\mathbf{v}_\perp + \gamma\mathbf{v}_\parallel\right)\mathbf{a} \cdot \mathbf{u}/c^2 - \left(\mathbf{a}_\perp + \gamma\mathbf{a}_\parallel\right)\mathbf{v} \cdot \mathbf{u}/c^2\right]}{\gamma^3\left(1 - \mathbf{v} \cdot \mathbf{u}/c^2\right)^3}. \qquad (5.5.10)
\end{aligned}$$

Exercise 5.5 Show that the quantity $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$ is Lorentz-invariant.

Solution 5.5 From Eqs. (5.5.3) and (5.5.4)

$$\mathrm{d}\bar{t} = \gamma\left(\mathrm{d}t - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{c^2}\right)$$
$$\mathrm{d}\bar{\mathbf{x}} = \left(\mathrm{d}\mathbf{x} - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\right) + \gamma\left(\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2} - \mathrm{d}t\right)\mathbf{u}$$

therefore

$$\begin{aligned}
-c^2\left(\mathrm{d}\bar{t}\right)^2 + \mathrm{d}\bar{\mathbf{x}}\cdot\mathrm{d}\bar{\mathbf{x}} &= -c^2\gamma^2\left(\mathrm{d}t - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{c^2}\right)^2 + \left[\left(\mathrm{d}\mathbf{x} - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\right) + \gamma\left(\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u} - \mathbf{u}\mathrm{d}t\right)\right]^2 \\
&= -c^2\gamma^2\left(\mathrm{d}t\right)^2 + 2c^2\gamma^2\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{c^2}\mathrm{d}t - c^2\gamma^2\frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{c^4} \\
&+ \left(\mathrm{d}\mathbf{x} - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\right)^2 + 2\gamma\left(\mathrm{d}\mathbf{x} - \frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\right)\cdot\left(\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u} - \mathbf{u}\mathrm{d}t\right) + \gamma^2\left(\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u} - \mathbf{u}\mathrm{d}t\right)^2 \\
&= -c^2\gamma^2\left(\mathrm{d}t\right)^2 + 2\gamma^2\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)\mathrm{d}t - \frac{\gamma^2}{c^2}\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2 \\
&+ \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - 2\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathrm{d}\mathbf{x}\cdot\mathbf{u} + \frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{u^4}\mathbf{u}\cdot\mathbf{u} \\
&+ \gamma^2\frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{u^4}\mathbf{u}\cdot\mathbf{u} - 2\gamma^2\frac{\mathrm{d}\mathbf{x}\cdot\mathbf{u}}{u^2}\mathbf{u}\cdot\mathbf{u}\mathrm{d}t + \mathbf{u}\cdot\mathbf{u}\mathrm{d}t \\
&= -c^2\gamma^2\left(1 - \frac{u^2}{c^2}\right)\left(\mathrm{d}t\right)^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} \\
&+ \gamma^2\left(1 - \frac{u^2}{c^2}\right)\frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{u^4}\mathbf{u}\cdot\mathbf{u} - 2\frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{u^4}\mathbf{u}\cdot\mathbf{u} + \frac{\left(\mathrm{d}\mathbf{x}\cdot\mathbf{u}\right)^2}{u^4}\mathbf{u}\cdot\mathbf{u} \\
&= -c^2\left(\mathrm{d}t\right)^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}.
\end{aligned}$$

Exercise 5.6 Find the expressions for the Lorentz-transformed gradient operator and for the time derivative.

Solution 5.6 In Cartesian coordinates the gradient operator ∇ reads $\left\{\frac{\partial}{\partial x^i}\right\}$, $i = 1, 2, 3$, and in general, considering $\bar{x}^i$ as $\bar{x}^i\left(x^j, t\right)$,

$$\nabla \equiv \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^i} + \frac{\partial}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial x^i}, \quad i = 1, 2, 3$$

From Eqs. (5.5.5) and (5.5.6) it is therefore

$$\begin{aligned}
\frac{\partial}{\partial x^i} &= \left[\delta_i^j + \left(\frac{\gamma - 1}{u^2}\delta_{mn}\frac{\partial x^m}{\partial x^i}u^n\right)u^j\right]\frac{\partial}{\partial \bar{x}^j} - \frac{\gamma}{c^2}\delta_{mn}\frac{\partial x^m}{\partial x^i}u^n\frac{\partial}{\partial \bar{t}} \\
&= \left[\delta_i^j + \left(\frac{\gamma - 1}{u^2}\delta_{mn}\delta_i^m u^n\right)u^j\right]\frac{\partial}{\partial \bar{x}^j} - \frac{\gamma}{c^2}\delta_{mn}\delta_i^m u^n\frac{\partial}{\partial \bar{t}} \\
&= \left(\delta_i^j + \frac{\gamma - 1}{u^2}u_i u^j\right)\frac{\partial}{\partial \bar{x}^j} - \frac{\gamma}{c^2}u_i\frac{\partial}{\partial \bar{t}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \bar{x}^i} + \frac{\gamma - 1}{u^2} u_i \left(u^j \frac{\partial}{\partial \bar{x}^j} \right) - \frac{\gamma}{c^2} u_i \frac{\partial}{\partial \bar{t}} \\
&\equiv \bar{\boldsymbol{\nabla}} + \frac{\gamma - 1}{u^2} \mathbf{u} \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right) - \frac{\gamma}{c^2} \mathbf{u} \frac{\partial}{\partial \bar{t}}. \qquad (5.5.11)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial}{\partial \bar{x}^i} \frac{\partial \bar{x}^i}{\partial t} + \frac{\partial}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t} \\
&= -\gamma \left(u^i \frac{\partial}{\partial \bar{x}^i} - \frac{\partial}{\partial \bar{t}} \right) \\
&\equiv -\gamma \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} - \frac{\partial}{\partial \bar{t}} \right) \qquad (5.5.12)
\end{aligned}
$$

Exercise 5.7 Show that any wave moving with speed c in a reference system S will move with the same speed in any other reference system having velocity $\mathbf{u}$ with respect to S.

Solution 5.7 What is asked in this exercise might be immediately taken as proven from the result of Exercise 5.3, as it should make no difference whether what is moving is a solid body or something else, such as a wave. However, the real meaning of this one depends on the equations involved. In particular, because the principle of relativity hrequires that the speed of a wave, if it is c, has to be independent from the reference system, this means that the wave equation (5.3.1) has to be covariant with respect to the Lorentz transformations, which is not evident at all.

The wave equation of a field $\boldsymbol{\Phi}$ traveling with speed c is

$$
\nabla^2 \boldsymbol{\Phi} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Phi}}{\partial t^2},
$$

or, in component notation,

$$
\left(\delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) \boldsymbol{\Phi} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\Phi}}{\partial t^2}.
$$

From the definition of ∇^2 and from Eq. (5.5.11) it is

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} &= \left(\bar{\boldsymbol{\nabla}} + \frac{\gamma - 1}{u^2} \mathbf{u}\, \mathbf{u} \cdot \bar{\boldsymbol{\nabla}} - \frac{\gamma}{c^2} \mathbf{u} \frac{\partial}{\partial \bar{t}} \right)^2 - \frac{1}{c^2} \gamma^2 \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} - \frac{\partial}{\partial \bar{t}} \right)^2 \\
&= \bar{\nabla}^2 + 2 \frac{\gamma - 1}{u^2} \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right) \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right) - 2 \frac{\gamma}{c^2} \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right) \frac{\partial}{\partial \bar{t}} \\
&\quad + \frac{(\gamma - 1)^2}{u^4} (\mathbf{u} \cdot \mathbf{u}) \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right)^2 - 2 \frac{\gamma}{c^2} \frac{\gamma - 1}{u^2} (\mathbf{u} \cdot \mathbf{u}) \left(\mathbf{u} \cdot \bar{\boldsymbol{\nabla}} \right) \frac{\partial}{\partial \bar{t}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma^2}{c^4}(\mathbf{u}\cdot\mathbf{u})\frac{\partial^2}{\partial\bar{t}^2}-\frac{\gamma^2}{c^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)^2+2\frac{\gamma^2}{c^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)\frac{\partial}{\partial\bar{t}}-\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial\bar{t}^2}\\
&=\bar{\nabla}^2-\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial\bar{t}^2}+\frac{\gamma^2}{c^4}u^2\frac{\partial^2}{\partial\bar{t}^2}\\
&+2\frac{\gamma-1}{u^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)^2+\frac{(\gamma-1)^2}{u^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)^2-\frac{\gamma^2}{c^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)^2\\
&-2\frac{\gamma}{c^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)\frac{\partial}{\partial\bar{t}}-2\frac{\gamma}{c^2}(\gamma-1)\left(\mathbf{u}\cdot\bar{\nabla}\right)\frac{\partial}{\partial\bar{t}}+2\frac{\gamma^2}{c^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)\frac{\partial}{\partial\bar{t}}\\
&=\bar{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial\bar{t}^2}-\frac{1}{u^2}\left(\mathbf{u}\cdot\bar{\nabla}\right)^2+\frac{\gamma^2}{u^2}\left(1-\frac{u^2}{c^2}\right)\left(\mathbf{u}\cdot\bar{\nabla}\right)^2\\
&=\bar{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial\bar{t}^2}.
\end{aligned}
$$

Therefore the wave equation is Lorentz-covariant because it is the d'Alembert operator $\Box^2 \equiv \nabla\cdot\nabla - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ which is itself form-invariant with respect to Lorentz transformations regardless of its argument.

The symbol $\Box^2$ is used here for the d'Alembert operator, instead of the more common $\Box$. This allows a more consistent notation with respect to the three-dimensional operators and a more natural transition to the four-dimensional geometry described in Sects. 5.4.3 and 5.4.4.

In this way, in fact, we can write the four-dimensional "gradient" as $\Box \equiv \left(c^{-1}\partial/\partial t, \nabla\right) = \partial_\alpha$, and because this exercise has shown that the d'Alembert operator is invariant for Lorentz transformations, consistently with ∇^2 it is also

$$
\Box^2 = \Box\cdot\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta,
$$

in the sense that ∂_α can be considered the components of a one-form in the Minkowsky geometry.

Exercise 5.8 Prove Eqs. (5.3.7) and (5.3.8) which give the transformation laws of the components of the Lorentz force, respectively, parallel and perpendicular to the direction of the velocity $\mathbf{u}$ between two reference systems S and $\bar{S}$ in uniform relative motion.

Solution 5.8 The parallel and perpendicular components of $\mathbf{F}/q = \mathbf{E} + (\mathbf{v}\times\mathbf{B})$ are by definition $\mathbf{E}_\parallel + (\mathbf{v}\times\mathbf{B})_\parallel$ and $\mathbf{E}_\perp + (\mathbf{v}\times\mathbf{B})_\perp$, respectively, but

$$
\begin{aligned}
\mathbf{v}\times\mathbf{B} &= \left(\mathbf{v}_\parallel + \mathbf{v}_\perp\right)\times\left(\mathbf{B}_\parallel + \mathbf{B}_\perp\right)\\
&= \left(\mathbf{v}_\parallel\times\mathbf{B}_\parallel\right) + \left(\mathbf{v}_\parallel\times\mathbf{B}_\perp\right) + \left(\mathbf{v}_\perp\times\mathbf{B}_\parallel\right) + (\mathbf{v}_\perp\times\mathbf{B}_\perp)\\
&= \left(\mathbf{v}_\parallel\times\mathbf{B}_\perp\right) + \left(\mathbf{v}_\perp\times\mathbf{B}_\parallel\right) + (\mathbf{v}_\perp\times\mathbf{B}_\perp)\,.
\end{aligned}
$$

The first two terms of this formula are orthogonal to $\mathbf{u}$, whereas the last one, being the cross-product of two vectors perpendicular to $\mathbf{u}$, is parallel to the relative velocity, therefore

$$(\mathbf{v} \times \mathbf{B})_{\parallel} = (\mathbf{v}_{\perp} \times \mathbf{B}_{\perp}) \tag{5.5.13}$$

$$(\mathbf{v} \times \mathbf{B})_{\perp} = \left(\mathbf{v}_{\parallel} \times \mathbf{B}_{\perp}\right) + \left(\mathbf{v}_{\perp} \times \mathbf{B}_{\parallel}\right). \tag{5.5.14}$$

Obviously all these relations do not depend on the reference system, and are also valid for the corresponding quantities in $\bar{S}$.

Equation (5.5.8) can also be rewritten as

$$\bar{\mathbf{v}}_{\parallel} = K\left(\mathbf{v}_{\parallel} - \mathbf{u}\right) \tag{5.5.15}$$

$$\bar{\mathbf{v}}_{\perp} = K_{\gamma}\mathbf{v}_{\perp} \tag{5.5.16}$$

where

$$K = \left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right)^{-1} \tag{5.5.17}$$

$$K_{\gamma} = \left[\gamma\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right)\right]^{-1}. \tag{5.5.18}$$

Hence from $\left(\bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_{\parallel} = \left(\bar{\mathbf{v}}_{\perp} \times \bar{\mathbf{B}}_{\perp}\right)$ and Eq. (5.3.6) it is

$$\begin{aligned}
\bar{\mathbf{E}}_{\parallel} + \left(\bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_{\parallel} &= \mathbf{E}_{\parallel} + K_{\gamma}\gamma\mathbf{v}_{\perp} \times \left(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{u} \times \mathbf{E}_{\perp}\right) \\
&= \mathbf{E}_{\parallel} + K\left(\mathbf{v}_{\perp} \times \mathbf{B}_{\perp}\right) - \frac{K}{c^2}\mathbf{v}_{\perp} \times \left(\mathbf{u} \times \mathbf{E}_{\perp}\right) \\
&= K\left\{\frac{\mathbf{E}_{\parallel}}{K} + (\mathbf{v} \times \mathbf{B})_{\parallel} - \frac{1}{c^2}\left[\mathbf{u}\left(\mathbf{v}_{\perp} \cdot \mathbf{E}_{\perp}\right) - \mathbf{E}_{\perp}\left(\mathbf{v}_{\perp} \cdot \mathbf{u}\right)\right]\right\} \\
&= K\left[\mathbf{E}_{\parallel}\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right) + (\mathbf{v} \times \mathbf{B})_{\parallel} - \frac{\mathbf{u}}{c^2}\left(\mathbf{v}_{\perp} \cdot \mathbf{E}_{\perp}\right)\right] \\
&= K\left[\mathbf{E}_{\parallel} - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\mathbf{E}_{\parallel} + (\mathbf{v} \times \mathbf{B})_{\parallel} - \frac{\mathbf{u}}{c^2}\left(\mathbf{v}_{\perp} \cdot \mathbf{E}_{\perp}\right)\right] \\
&= K\left[\mathbf{E}_{\parallel} + (\mathbf{v} \times \mathbf{B})_{\parallel} - \frac{\mathbf{u}}{c^2}\left(\mathbf{v}_{\parallel} \cdot \mathbf{E}_{\parallel}\right) - \frac{\mathbf{u}}{c^2}\left(\mathbf{v}_{\perp} \cdot \mathbf{E}_{\perp}\right)\right],
\end{aligned} \tag{5.5.19}$$

where the last equality holds because by definition $\mathbf{u} \| \mathbf{E}_{\parallel}$. However, it is straightforward to see that

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{E} &= \left(\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}\right) \cdot \left(\mathbf{E}_{\parallel} + \mathbf{E}_{\perp}\right) \\
&= \mathbf{v}_{\parallel} \cdot \mathbf{E}_{\parallel} + \mathbf{v}_{\perp} \cdot \mathbf{E}_{\perp},
\end{aligned}$$

and therefore Eq. (5.5.19) becomes

$$\bar{\mathbf{E}}_{\parallel} + \left(\bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_{\parallel} = K\left[\mathbf{E}_{\parallel} + (\mathbf{v} \times \mathbf{B})_{\parallel} - \frac{\mathbf{u}}{c^2}(\mathbf{v} \cdot \mathbf{E})\right]$$

which is exactly Eq. (5.3.7).

The second relation requires the computation of $\bar{\mathbf{E}}_\perp + \left(\bar{\mathbf{v}} \times \bar{\mathbf{B}}\right)_\perp$ which from Eq. (5.5.14) reads

$$\bar{\mathbf{E}}_\perp + \left(\bar{\mathbf{v}}_\parallel \times \bar{\mathbf{B}}_\perp\right) + \left(\bar{\mathbf{v}}_\perp \times \bar{\mathbf{B}}_\parallel\right).$$

Remembering that, for any vector $\mathbf{a}$, $(\mathbf{u} \times \mathbf{a})_\perp = \mathbf{u} \times \mathbf{a}_\perp$, and using Eqs. (5.3.4) to (5.3.6), and Eqs. (5.5.15) to (5.5.18), we have

$$\bar{\mathbf{E}}_\perp = K_\gamma \left[\frac{\gamma^2}{K}\left(\mathbf{E}_\perp + \mathbf{u} \times \mathbf{B}_\perp\right)\right] \tag{5.5.20}$$

$$\bar{\mathbf{v}}_\parallel \times \bar{\mathbf{B}}_\perp = K_\gamma \left[\gamma^2\left(\mathbf{v}_\parallel - \mathbf{u}\right) \times \left(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{u} \times \mathbf{E}_\perp\right)\right] \tag{5.5.21}$$

$$\bar{\mathbf{v}}_\perp \times \bar{\mathbf{B}}_\parallel = K_\gamma \left(\mathbf{v}_\perp \times \mathbf{B}_\parallel\right). \tag{5.5.22}$$

Now, because K_γ is already the denominator of Eq. (5.3.8), we need to perform the calculations only on the remaining factors of these equations. In particular the first one is

$$\frac{\gamma^2}{K}\left(\mathbf{E}_\perp + \mathbf{u} \times \mathbf{B}_\perp\right) = \gamma^2\left(\mathbf{E}_\perp + \mathbf{u} \times \mathbf{B}_\perp\right)\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right), \tag{5.5.23}$$

and the second can be written as

$$\gamma^2\left[\left(\mathbf{v}_\parallel \times \mathbf{B}_\perp\right) - \left(\mathbf{u} \times \mathbf{B}_\perp\right) - \frac{1}{c^2}\mathbf{v}_\parallel \times \left(\mathbf{u} \times \mathbf{E}_\perp\right) + \frac{1}{c^2}\mathbf{u} \times \left(\mathbf{u} \times \mathbf{E}_\perp\right)\right],$$

and after using the well-known relation $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$ and the obvious property $\mathbf{v}_\parallel \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u}$,

$$\gamma^2\left[\left(\mathbf{v}_\parallel \times \mathbf{B}_\perp\right) - \left(\mathbf{u} \times \mathbf{B}_\perp\right) + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\mathbf{E}_\perp - \frac{u^2}{c^2}\mathbf{E}_\perp\right]. \tag{5.5.24}$$

Summing Eqs. (5.5.23) and (5.5.24) results in

$$\begin{aligned}
\gamma^2\left[\left(\mathbf{E}_\perp + \mathbf{u} \times \mathbf{B}_\perp\right)\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\right) + \left(\mathbf{v}_\parallel \times \mathbf{B}_\perp\right) - \left(\mathbf{u} \times \mathbf{B}_\perp\right) + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\mathbf{E}_\perp - \frac{u^2}{c^2}\mathbf{E}_\perp\right] &= \\
\gamma^2\left[\mathbf{E}_\perp - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}\mathbf{u} \times \mathbf{B}_\perp + \mathbf{v}_\parallel \times \mathbf{B}_\perp - \frac{u^2}{c^2}\mathbf{E}_\perp\right] &= \\
\gamma^2\left(\mathbf{E}_\perp + \mathbf{v}_\parallel \times \mathbf{B}_\perp\right)\left(1 - \frac{u^2}{c^2}\right) &= \\
\left(\mathbf{E}_\perp + \mathbf{v}_\parallel \times \mathbf{B}_\perp\right)&
\end{aligned}$$

where the last step results from considering that

$$(\mathbf{v}\cdot\mathbf{u})\,(\mathbf{u}\times\mathbf{B}_\perp) = \left(\mathbf{v}_\parallel\cdot\mathbf{u}\right)(\mathbf{u}\times\mathbf{B}_\perp)$$
$$= u^2\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\right)\left(v_\parallel\hat{\mathbf{u}}\times\mathbf{B}_\perp\right)$$
$$= u^2\left(\mathbf{v}_\parallel\times\mathbf{B}_\perp\right).$$

In this way it is easy to realize that summing Eqs. (5.5.20), through (5.5.22) we get

$$\bar{\mathbf{E}}_\perp + \left(\bar{\mathbf{v}}\times\bar{\mathbf{B}}\right)_\perp = K_\gamma\left(\mathbf{E}_\perp + \mathbf{v}_\parallel\times\mathbf{B}_\perp + \mathbf{v}_\perp\times\mathbf{B}_\parallel\right)$$
$$= \frac{\mathbf{E}_\perp + (\mathbf{v}\times\mathbf{B})_\perp}{\gamma\left(1-\mathbf{v}\cdot\mathbf{u}/c^2\right)}$$

as required.

Exercise 5.9 Show that Euclidean time differences are not invariant quantities anymore with respect to Lorentz transformations (time dilation).

Solution 5.9 Let us consider two inertial reference systems S and $\bar{S}$, the latter moving with constant relative velocity $\mathbf{u}$ with respect to the former. Two signals are emitted at different times t_1 and t_2 in S, so that an observer is this reference system would record a time interval between them equal to $\varDelta T = t_2 - t_1$.

The same time interval in $\bar{S}$, from Eq. (5.5.1) is

$$\varDelta\bar{T} = \bar{t}_2 - \bar{t}_1 = \gamma\left[(t_2 - t_1) - \frac{(\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{u}}{c^2}\right],$$

but the signal is emitted from the same point, which in S has the same spatial coordinates $\mathbf{x}_1 = \mathbf{x}_2$, thus

$$\varDelta\bar{T} = \gamma\,(t_2 - t_1) = \gamma\varDelta T. \tag{5.5.25}$$

In other words, the same intervals of time appear longer when measured in a moving reference system. For example, two identical clocks, one at rest with respect to an observer and another one moving, would be seen ticking at different rates. Actually, the one at rest will tick faster and the moving one will tick slower, a phenomenon usually denoted *time dilation*.

Exercise 5.10 Show that distances are no invariant quantities with respect to Lorentz transformations (length contraction).

Solution 5.10 We consider two reference systems S and $\bar{S}$ as in the previous exercise, and two points P_1 and P_2, whose positions in a reference system S are given by the two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. The spatial distance between these two points (which can be considered at rest in S) is simply $d = |\mathbf{x}_2 - \mathbf{x}_1|$. The same distance, as seen in another reference system $\bar{S}$ moving with velocity $\mathbf{u}$ with respect to S, will obviously be $\bar{d} = |\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1|$; hence from Eq. (5.5.2)

$$\bar{d} = \left|(\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 1}) + \gamma\left[\left(\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right) - \mathbf{u}\,(t_2 - t_1)\right]\right|$$
$$= \sqrt{|\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 1}|^2 + \gamma^2\left|\left(\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right) - \mathbf{u}\,(t_2 - t_1)\right|^2}. \tag{5.5.26}$$

As in the Galilean case, the distance is equivalent to the length of a "rod" stretching from P_1 to P_2, which has to be taken at equal instants of time in $\bar{S}$. We have therefore to express the time difference $(t_2 - t_1)$ in S as a function of that in $\bar{S}$. As already shown above, from Eq. (5.5.1) this is immediately

$$\bar{t}_2 - \bar{t}_1 = \gamma\left[(t_2 - t_1) - \frac{(\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{u}}{c^2}\right],$$

that is, by observing that only the parallel component of d "survives" in the scalar product with $\mathbf{u}$,

$$t_2 - t_1 = \gamma^{-1}\left(\bar{t}_2 - \bar{t}_1\right) + \frac{(\mathbf{x}_2 - \mathbf{x}_1)\cdot\mathbf{u}}{c^2}$$
$$= \gamma^{-1}\left(\bar{t}_2 - \bar{t}_1\right) + \frac{d_\parallel u}{c^2},$$

Substituting this result in Eq. (5.5.26) one has

$$\bar{d} = \sqrt{|\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 1}|^2 + \gamma^2\left|\left(\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right) - \mathbf{u}\left[\gamma^{-1}\left(\bar{t}_2 - \bar{t}_1\right) + \frac{d_\parallel u}{c^2}\right]\right|^2}$$
$$= \sqrt{|\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 1}|^2 + \gamma^2\left|\left(\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right) - \mathbf{u}\left(\frac{d_\parallel u}{c^2}\right)\right|^2}. \tag{5.5.27}$$

In this expression one can easily recognize in $d_\perp \equiv |\mathbf{x}_{\perp 2} - \mathbf{x}_{\perp 1}|$ the projection of the distance between the two points along the direction perpendicular to $\mathbf{u}$ and in $d_\parallel \equiv \left|\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right|$ the projection along the parallel direction, so that in S it is $d = \sqrt{(d_\perp)^2 + \left(d_\parallel\right)^2}$. Moreover, it is also

$$\left(\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}\right) - \mathbf{u}\left(\frac{d_\parallel u}{c^2}\right) = d_\parallel - d_\parallel\frac{u^2}{c^2}$$
$$= d_\parallel\gamma^{-2}$$

because the two vectors $\mathbf{x}_{\parallel 2} - \mathbf{x}_{\parallel 1}$ and $\mathbf{u}$ by hypothesis are parallel to each other thus only the norms are needed in the formula. By comparing this quantity with Eq. (5.5.27) it is therefore easy to deduce that the total length in $\bar{S}$ is shorter than that measured in S because $\bar{d}_\perp = d_\perp$, whereas

$$\bar{d}_\parallel = \gamma d_\parallel\gamma^{-2} = \gamma^{-1}d_\parallel, \tag{5.5.28}$$

with $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$.

Because in S the imaginary rod connecting the two points is at rest, and in $\bar{S}$ is moving with velocity $-\mathbf{u}$, it is clear that moving objects appear shortened along the direction parallel to their velocity (with respect to their length at rest) which explains why it is customary to refer to this consequence as *length contraction*.

Exercise 5.11 Find a general expression for Lorentz boost in matrix form.

Solution 5.11 The easiest approach is to transform Eqs. (5.5.5) and (5.5.6) into matrix form. The first one in fact can be immediately written as (here we stay with the Euclidean convention of denoting the components with subscripts in order to avoid confusion with the powers)

$$\bar{t} = \gamma \begin{pmatrix} 1 & -u_1/c^2 & -u_2/c^2 & -u_3/c^2 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and the second one is

$$\bar{x}_i = \begin{pmatrix} -\gamma u_i & \delta_{ij} x_j + (\gamma-1)\, u_1 u_i/u^2 & \delta_{ij} x_j + (\gamma-1)\, u_2 u_i/u^2 & \delta_{ij} x_j + (\gamma-1)\, u_3 u_i/u^2 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

These two equations can be easily recognized as the result of the following matrix multiplication

$$\begin{pmatrix} \bar{t} \\ \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \Lambda \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$\Lambda = \begin{pmatrix} \gamma & -\frac{u_1}{c^2} & -\frac{u_2}{c^2} & -\frac{u_3}{c^2} \\ -\gamma u_1 & x_1 + (\gamma-1)\,\frac{(u_1)^2}{u^2} & (\gamma-1)\,\frac{u_2 u_1}{u^2} & (\gamma-1)\,\frac{u_3 u_1}{u^2} \\ -\gamma u_2 & (\gamma-1)\,\frac{u_1 u_2}{u^2} & x_2 + (\gamma-1)\,\frac{(u_2)^2}{u^2} & (\gamma-1)\,\frac{u_3 u_2}{u^2} \\ -\gamma u_3 & (\gamma-1)\,\frac{u_1 u_3}{u^2} & (\gamma-1)\,\frac{u_2 u_3}{u^2} & x_3 + (\gamma-1)\,\frac{(u_3)^2}{u^2} \end{pmatrix}.$$

Such a matrix can be written in a more synthetic way if we instead give the transformation of $\boldsymbol{x}^\alpha$, with $x^0 = ct$. In this case the matrix components can be expressed as

$$\begin{aligned} \Lambda_{00} &= \gamma \\ \Lambda_{0i} = \Lambda_{i0} &= -\gamma \beta^i \\ \Lambda_{ij} &= \delta_{ij} + (\gamma-1)\,\frac{\beta^i \beta^j}{\beta^2}, \end{aligned}$$

where $\beta^2 = \delta_{ij}\beta^i\beta^j$ and $\beta^i = u^i/c$.

Chapter 6
Special Relativity in Minkowskian Spacetime

In the last chapter we introduced the idea that, using the same approach followed in Chap. 3, one can define four-dimensional objects which are the counterparts of the 3D Euclidean scalars, vectors, and tensors in the Minkowski spacetime. Because, by construction, they are covariant for any transformation of the Poincaré group, they will provide the most convenient geometrical framework to show the manifest Lorentz-covariance of the laws of physics, and therefore their compatibility with the principle of covariance and that of (Einsteinian) relativity.[1]

If our goal is to provide a relativistically consistent version of any physics theory, it is therefore clear that we have to identify the Minkowskian replacements for the needed Euclidean quantities. In particular, as we have already stressed in the previous chapter, the transition from Galilean to special relativity first of all comes together with the redefinition of the kinematic quantities, including position vectors, velocities, accelerations, and the like and with a modification of the principle of relativity which implies a different formulation of the dynamics, i.e., of quantities such as momentum, energy, forces, and so on.

A detailed exposition of these topics is beyond the needs of this book and can be found in many sources (see, e.g., Weinberg 1972; Rindler 2006) but it is useful to show an overview of some aspects of Minkowskian kinematics and dynamics before reverting to the Lagrangian methods foreshadowed in the conclusions of the last chapter. The twofold aim of this short summary is:

1. To give another example of a practical realization of a mathematical model, intended as an axiomatic and self-consistent experience-driven theory with well-defined rules of correspondence.
2. To provide indications and justifications for the transition to the Minkowskian geometry of the Lagrangian approach, which are shown at the end of this chapter

[1] For example, from what has been previously shown, a relativistic compatible dynamics in Euclidean geometry is difficult to conceive and to understand.

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6.1 Kinematics

Proper time and coordinate time

In the Euclidean case we started to "build" scalar and vectorial quantities by using the distance between two points and its components as their prototypes. In the previous chapter we introduced the possibility of using the invariant quantity of Eq. (5.4.1) and its components $\mathrm{d}x^\alpha$ as the prototypical four-dimensional scalars and vectors. We also stressed the difficulty of identifying the former with the result of a measurement, however, if $\mathrm{d}t \neq 0$ and remembering that $\mathrm{d}x^i/\mathrm{d}t = v^i$, we have

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right) = -\frac{c^2\mathrm{d}t^2}{\gamma^2(v)},$$

thus one can define a quantity $\mathrm{d}\tau$ such as $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$ which has the dimensions of a time and is related to $\mathrm{d}t$ by

$$\mathrm{d}\tau = \gamma^{-1}\mathrm{d}t. \tag{6.1.1}$$

This relation is nothing else than a rewriting of the Lorentz transformation for the time (Eq. (5.1.11)) in differential form and considering $\mathrm{d}x^i = 0$. In other words, an observer at rest in a reference system moving with relative velocity v with respect to another inertial reference system would measure time intervals $\mathrm{d}\tau$. For this reason this quantity is called *proper time*, whereas $\mathrm{d}t$, which can be considered as the tick rate of a clock in motion with velocity v as seen by this observer, and which is different for different observers, is called *coordinate time*.

Four-velocity

Proper time is useful also because it is a Lorentz-invariant quantity that, similarly to what happens in the Euclidean case, can be used to define new four-vectors from their original prototype easily, i.e., the four-dimensional Lorentz-transformed displacement $\mathrm{d}\boldsymbol{x}$. In Sect. 3.1.4 we noticed that differentiating a 3D vector with respect to a scalar quantity gives another vector, and in this way one can define velocities and accelerations as vectorial quantities.

Similarly, differentiating a four-vector with respect to a Lorentz-invariant quantity (a "Minkowskian scalar") results in another four-vector. In this way it is easy to understand that

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} \equiv \left\{\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\right\} \equiv \left\{\frac{\mathrm{d}ct}{\mathrm{d}\tau}, \frac{\mathrm{d}x^a}{\mathrm{d}\tau}\right\} \tag{6.1.2}$$

is a four-vector tangent to the *worldline* $x^\alpha(\tau)$, namely the set of events x^αs constituting a curve in the spacetime which, in this definition, is parameterized by the proper time τ. This is the four-dimensional analogy to the three-dimensional velocity (which is the vector $\mathrm{d}x^a/\mathrm{d}t$ tangent to the curve $x^a(t)$ parameterized by the time t) and therefore is conveniently called *four-velocity*. From Eq. (6.1.1) it immediately results in $\boldsymbol{v} = \gamma(v)\{c, \mathbf{v}\}$, where $\mathbf{v}$ is the (three-)velocity of a point like body moving along the worldline, and from Eq. (5.4.3) we find that

$$\boldsymbol{v} \cdot \boldsymbol{v} = \eta_{\alpha\beta} v^\alpha v^\beta = \gamma^2(v)\left(v^2 - c^2\right) = -c^2. \tag{6.1.3}$$

It is worth stressing that this is a completely general result because the (Minkowskian) scalar product of four-vectors is a Lorentz-invariant. This in fact can be seen even more easily if we consider the point of view of an observer at rest with respect to $\boldsymbol{v}$. In this case $\mathbf{v} = 0$, so the four-velocity becomes $\bar{\boldsymbol{v}} = \{c, \mathbf{0}\}$ and obviously $\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} = -c^2$.

It also has to be noticed, however, that for particles moving at the speed of light ($v = c$) such as photons, the proper time cannot be defined. Such particles move on null worldlines, for which it is always $-c\mathrm{d}t = \mathrm{d}\mathbf{x}$,[2] therefore the four-dimensional separation of any two events on these worldlines, or equivalently $\mathrm{d}\tau$, is always zero.[3] For this reason it is not possible to define a four-velocity for photons, although their three-velocity makes perfect sense, which is equivalent to saying that, as is known, it is not possible to have a reference frame in which photons are at rest. It is thus clear that from now on when we use four-velocities or proper time of a particle we implicitly deal with those moving at speed $v < c$.

Let us now take two crossing worldlines with four-velocities $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively, at their crossing point. In general it is

$$\boldsymbol{v} = \gamma(v)\{c, \mathbf{v}\} \quad \boldsymbol{w} = \gamma(w)\{c, \mathbf{w}\},$$

where $\mathbf{v}$ and $\mathbf{w}$ are the instantaneous velocities of the two worldlines with respect to the starting reference system. From Eq. (5.4.3) it is

$$\boldsymbol{v} \cdot \boldsymbol{w} = \eta_{\alpha\beta} v^\alpha w^\beta = \gamma(v)\gamma(w)\left(-c^2 + \mathbf{v} \cdot \mathbf{w}\right). \tag{6.1.4}$$

Now let us put ourselves in the rest frame $\bar{S}$ of $\boldsymbol{w}$. In this case the two four-velocities will be $\bar{\boldsymbol{v}} = \gamma(\bar{v})\{c, \bar{\mathbf{v}}\}$ and $\bar{\boldsymbol{w}} = \{c, \mathbf{0}\}$, where $\bar{\mathbf{v}}$ is the relative velocity of $\boldsymbol{v}$ with respect to $\boldsymbol{w}$,[4] thus

$$\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{w}} = -\gamma(\bar{v})\, c^2. \tag{6.1.5}$$

By exploiting once again the Lorentz invariance of the scalar product we have $\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{w}} = \boldsymbol{v} \cdot \boldsymbol{w}$, and from Eqs. (6.1.4) and (6.1.5), after some simple algebra, one can

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[2]Which is exactly why we can tell that the particle is moving at the speed of light, because $v = |\mathrm{d}\mathbf{x}/\mathrm{d}t| = c$.

[3]It is evident that in this case Eq. (6.1.1) does not hold even if $\mathrm{d}t$ can still be defined.

[4]Or, which is the same, the velocity $\mathbf{v}$ as measured in the rest frame $\bar{S}$ of $\boldsymbol{w}$.

find the expression of the relative speed $\bar{v}$ as function of $\mathbf{v}$ and $\mathbf{w}$

$$\bar{v}^2 = c^2 \left[1 - \frac{\left(c^2 - v^2\right)\left(c^2 - w^2\right)}{\left(-c^2 + \mathbf{v} \cdot \mathbf{w}\right)^2} \right]. \tag{6.1.6}$$

This formula immediately shows that if any of the two speeds is c, then the relative speed is also c, as one should expect. Moreover it is instructive, and a standard test in relativistic physics, to check that the latter reduces to its classical form when $v, w \ll c$ or, which is the same, in the limit of $c \to \infty$. We show it for parallel velocities ($\mathbf{v} \parallel \mathbf{w}$), for which it is well known that classically $|\bar{v}| = |v - w|$. In this case some straightforward calculations give

$$\begin{aligned} |\bar{v}| &= c \sqrt{1 - \frac{\left(c^2 - v^2\right)\left(c^2 - w^2\right)}{\left(-c^2 + vw\right)^2}} \\ &= c \sqrt{\frac{\left(v^2 - 2vw + w^2\right)}{c^2 \left(1 - vw/c^2\right)^2}} \\ &= \frac{|v - w|}{1 - vw/c^2}, \end{aligned}$$

which clearly reduces to the classical case for $c \to \infty$.

Four-acceleration

In a similar fashion we can also define the four-acceleration

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}\tau^2}, \tag{6.1.7}$$

but its relation with the three-dimensional acceleration is more complicated than that between four- and three-velocities. Using the same procedure it is

$$\begin{aligned} \boldsymbol{a} &= \gamma(v) \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \gamma(v) \frac{\mathrm{d}}{\mathrm{d}t} \left[\gamma(v) \{c, \mathbf{v}\}\right] \\ &= \gamma(v) \left\{ c \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \mathbf{v} \frac{\mathrm{d}\gamma}{\mathrm{d}t} + \mathbf{a}\gamma(v) \right\} \end{aligned} \tag{6.1.8}$$

where, because we can write $\gamma(v) = \left(1 - v^2/c^2\right)^{-1/2} = \left(1 - \mathbf{v} \cdot \mathbf{v}/c^2\right)^{-1/2}$,

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma^3 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2}, \tag{6.1.9}$$

and it is easy to deduce that in the frame comoving with the particle, the instantaneous rest frame of the particle, $\boldsymbol{a} = \{0, \mathbf{a}\}$ because $\mathbf{v} = 0$, therefore in this reference frame the norms of $\boldsymbol{a}$ and $\mathbf{a}$ are the same. The same reference frame, as we have seen above, is characterized by the fact that its *proper* and *coordinate* times coincide, i.e., that $\tau = t$ when $\mathbf{v} = 0$, which explains why $\mathbf{a}$, i.e., the three-acceleration of a particle in its instantaneous rest frame, is called *proper acceleration*.

In Sect. 5.3, in our process of showing the infeasibility of a relativistic formulation of the $\mathbf{F} = m\mathbf{a}$ law of dynamics, we found that the proper acceleration $\mathbf{a}$ transforms as in Eqs. (5.3.15) and (5.3.16).[5] We can now obtain the same result in a easier way by combining Eqs. (6.1.8) and (6.1.9). In general it is

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{a} &= \gamma^2\left(-c^2\gamma^6\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^4} + v^2\gamma^6\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^4} + 2\gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v}\cdot\mathbf{a} + a^2\gamma^2\right)\\ &= \gamma^2\left[-\gamma^6\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2}\left(1-\frac{v^2}{c^2}\right) + 2\gamma^4\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2} + a^2\gamma^2\right]\\ &= \gamma^6\frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2} + \gamma^4 a^2. \end{aligned} \tag{6.1.10}$$

In Eqs. (5.3.15) and (5.3.16) $\mathbf{a}_\perp$ and $\mathbf{a}_\parallel$ were the accelerations in the rest frame S, where the velocity of the body was $\mathbf{v} = 0$, respectively, perpendicular and parallel to $\mathbf{u}$, which was the relative velocity between S and another frame $\bar{S}$. In the latter thus the same particle has a non zero velocity $\bar{\mathbf{v}} = -\mathbf{u}$ and its transformed accelerations are indicated with $\bar{\mathbf{a}}_\perp$ and $\bar{\mathbf{a}}_\parallel$, respectively. In the above formula, therefore, the acceleration on the right-hand side corresponds to $\bar{\mathbf{a}}$ (and $\mathbf{v}$ to $\bar{\mathbf{v}}$). If we adopt the same notation, and if we exploit the Lorentz invariance of $\boldsymbol{a}\cdot\boldsymbol{a}$ as for the four-velocities, we obtain

$$a^2 \equiv \mathbf{a}\cdot\mathbf{a} = \gamma^6\frac{(\bar{\mathbf{v}}\cdot\bar{\mathbf{a}})^2}{c^2} + \gamma^4\bar{a}^2, \tag{6.1.11}$$

which naturally splits into

$$\begin{aligned} a_\perp^2 &= \gamma^4\bar{a}_\perp^2\\ a_\parallel^2 &= \gamma^6\bar{a}_\parallel^2 \end{aligned}$$

when we decompose $\mathbf{a}$ and $\bar{\mathbf{a}}$ in their components parallel and perpendicular to $\mathbf{u}$.[6]

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[5] We remind the reader that these equations had been derived in the case of $\mathbf{v} = 0$.

[6] The second equation can be obtained by the relation $(\mathbf{a}\times\mathbf{b})^2 = a^2b^2 - (\mathbf{a}\cdot\mathbf{b})^2$, which allows us to rewrite Eq. (6.1.11) as

$$\mathbf{a}\cdot\mathbf{a} = \gamma^6\frac{\bar{v}^2\bar{a}^2 - (\bar{\mathbf{v}}\times\bar{\mathbf{a}})^2}{c^2} + \gamma^4\bar{a}^2.$$

In fact, for $\mathbf{a}_\parallel$ this becomes

$$a_\parallel^2 = \gamma^6\frac{\bar{v}^2\bar{a}_\parallel^2}{c^2} + \gamma^4\bar{a}_\parallel^2 = \gamma^6\bar{a}_\parallel^2\left(\frac{1}{\gamma^2}+\frac{\bar{v}^2}{c^2}\right) = \gamma^6\bar{a}_\parallel^2\left(1-\frac{\bar{v}^2}{c^2}+\frac{\bar{v}^2}{c^2}\right) = \gamma^6\bar{a}_\parallel^2.$$

6.2 Dynamics

In the previous section we defined the four-dimensional relativistic equivalents of the most basic entities used in kinematics: velocities and accelerations. These have been used to show some examples of how, thanks to their Lorentz invariance, it is possible to recover easily the transformation properties analyzed in the previous chapter.

Actually this is just the first step, the next one being the formulation of a dynamics compatible with the Einsteinian principle of relativity. As we have already seen, this is not the case of Newtonian dynamics, and we know that, by construction, this requirement is automatically satisfied if we write our equations with the objects of Minkowskian geometry. This is the relativistic counterpart of what happens by postulating the principle of covariance (rotational and translational) which is automatically met if we model the physical world (in the sense of Chap. 2 of representing it) with Euclidean geometry.

However, relativistic dynamics cannot be deduced from Minkowskian geometry (i.e., by the principle of relativity in its Einsteinian form) exactly as Newtonian dynamics cannot be deduced by Euclidean geometry, whose implicit covariance requirements constitute just part of its basic hypothesis, but rather it is built from some additional assumption. We can summarize the situation in this way:

1. The (rotational and translational) covariance requirements selects Euclidean geometry as the preferred geometrical model of the sensible world.
2. The principle of relativity in its Galilean form sets another (kinematic) covariance constraint, namely with respect to the Galilean boost transformations.
3. Newtonian dynamics can be developed from its three hypotheses, i.e., the principle of inertia, Newton's second law, and the action–reaction principle.

These three statements are strictly related; in fact the assumption that space and time can be modeled as two separate Euclidean vector spaces makes possible the concept of *absolute time*,[7] which is compatible with the principle of relativity only in its Galilean form. In its turn this version of the principle of relativity implicitly selects a specific form of the velocity transformations, namely the Galilean boost transformations, which are compatible with Newton's second law of dynamics. For these reasons, a modification of any of these hypotheses has consequences on all the others.

We modified the principle of relativity, which is now incompatible with the idea of absolute time. This called for a different kind of geometry that does not consider space and time separately, and which is no longer Euclidean, and at the same time this requires a modification of Newton's principles, but how?

One reasonable choice would be to start by postulating the validity of a four-dimensional version of Newton's second law and thus defining a four-force $\boldsymbol{f}$ as the product of the four-acceleration of the particle by its rest-mass. From now on, unless

[7] And therefore of an absolute concept of simultaneity among events.

explicitly stated, this is the only kind of mass we use, because the velocity-dependent one used in the last chapter is not a Lorentz-invariant quantity, and dropping the subscript "0" the rest-mass is simply indicated with m. Therefore the tentative formulation of a relativistic law of dynamics might be $\boldsymbol{f} = m\boldsymbol{a}$, which is clearly a four-vector because it is obtained from the product of a Lorentz-scalar and a four-vector. However, it is well known that Newton's principles can also be laid on some conservation principles (see, e.g., Doughty 1990). In classical dynamics the latter are completely equivalent to Newton's laws, in the sense that we can use one set of principles to deduce the other as well as proceed the other way round.

In relativistic dynamics it is more convenient to start from the conservation principles because: (a) these principles have a broader range of applicability because they are also used in quantum mechanics; (b) whereas in Newtonian dynamics the concept of "force" is quite pervasive, this is no longer true in relativistic dynamics even if one tries to use its four-dimensional version; in particular, although in the former it is expected that any interaction can be described by an appropriate force, it is not possible to formulate a special relativistic theory of gravity[8]; (c) the concept of four-force based on the kinematic quantity $\boldsymbol{a}$ automatically excludes from dynamics photons and any other particles moving at the speed of light because, as we have noted above, for such particles four-velocities and four-accelerations cannot be defined.

Four-momentum and mass-energy equivalence

One specific conservation law is that of momentum, therefore, with this guiding idea in mind, we start by a tentative definition of the *four-momentum* as $\boldsymbol{p} \equiv m\boldsymbol{v}$, which is again a four-vector as is the previously defined four-force. From the definition of four-velocity, we can also write $\boldsymbol{p} = m\gamma(v)\{c, \mathbf{v}\}$, which means that the spatial part of this four-vector can be regarded as the relativistic (three-) momentum $\mathbf{p} = m\gamma(v)\,\mathbf{v}$ of a body with rest-mass m, which differs from the classical one by the factor $\gamma(v)$. The temporal part $m\gamma(v)\,c$, as is obvious, also has the dimensions of a momentum, so that $m\gamma(v)\,c^2$ has the dimensions of an energy, and if we expand this quantity in series of $\epsilon \equiv v/c$ when $v \ll c$, it is

$$m\gamma(v)\,c^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\epsilon^4\right),$$

which is formed by the constant term mc^2 plus a velocity-dependent part whose most significant term is the classical kinematic energy of a body with mass m.

It thus makes sense to make the two posits that (a) the quantity $E = m\gamma(v)\,c^2$ represents the total energy of the particle with rest-mass m, and (b) the kinetic energy of such a particle is $T = E - mc^2 = mc^2(\gamma - 1)$. Clearly, this is just a reasonable

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[8] Or at least no one in agreement with the experimental data.

assumption,[9] and there is no compelling argument in favor of this definition, therefore the best we can do is to reason on this definition within our model and try to extrapolate its consequences in terms of its rules of correspondence.

First of all, one can notice that in classical physics the total energy is the sum of the kinetic and the potential energy, thus one may wonder why this should be called "total" in a absence of a potential term. Actually a potential energy is associated with an interaction or a force which, however, in the context of a field theory can be accounted for by the energy of the field itself, as we have already seen in Sect. 4.4. This will turn out to be the more convenient way to satisfy the conservation of energy in relativistic field theories.

Photons-aware definition of four-momentum

Second, these definitions imply that a particle possesses some kind of energy due to its rest-mass m even when at rest. This was one of the results highlighted by Einstein in his paper of 1905, which was later experimentally verified also by showing that mass and energy can be converted to each other.

Coming back to the four-momentum, from the above definitions this can be equivalently written as

$$\boldsymbol{p} = \left\{ \frac{E}{c}, \mathbf{p} \right\}. \tag{6.2.1}$$

This expression for the four-momentum can be made more general than our original definition $\boldsymbol{p} = m\boldsymbol{v}$. The latter is clearly valid only in the case of particles with $v < c$ whereas the above one can be used also for those moving at the speed of light, as we show in a moment.

Indeed, from Eq. (6.1.3), we have

$$\boldsymbol{p} \cdot \boldsymbol{p} = -m^2 c^2$$

for "normal" particles, i.e., for those moving at speed $v < c$ to which a proper time and a four-velocity can be assigned. Moreover, if we consider how the γ factor enters in the expressions for the four-momentum and of the total energy, we can easily associate it also with the rest-mass and define one "relativistic inertial mass" $m_\mathrm{r} \equiv m\gamma(v)$. In this sense, thus, one could say that the four-momentum and the total energy can be defined also by means of m_r, c, and $\mathbf{v}$ with no influence on their character of Minkowskian invariance. However,

$$\lim_{v \to c} m_\mathrm{r} = \lim_{v \to c} \frac{m}{\sqrt{1 - v^2/c^2}} = \infty, \tag{6.2.2}$$

[9] For example, the definition of kinetic energy satisfies the basic requirement of coinciding with its classical definition when $v \ll c$.

therefore a particle with $m \neq 0$, denoted a *massive particle*, would need an infinite energy and would have infinite four-momentum to reach the speed of light, which is another way to explain why massive particles must move at $v < c$. Conversely, the only way to allow for particles moving at $v = c$ and admit that they have sensible energy and four-momentum, is to require that in their case $m = 0$. In other words, only massless particles can move at the speed of light.

Mathematically this is still not sufficient to solve the problem of the motion at $v = c$; in fact $m/\sqrt{1 - v^2/c^2}$ is simply undefined for $m = 0$ and $v = c$, therefore in principle one could think that there cannot exist such things as massless particles and nothing can move at the speed of light, or alternatively that they can exist but they do not have any sensible energy and momentum.

This is one of those cases, cited in Sect. 2.3 in which the experimental evidence and the presence of self-consistency issues demonstrate the need of changing the rules of correspondence in order to formulate a consistent and evidence-based theory. If we accept the experimental results and the explanation in terms of quantum physics, in fact, we have to admit that light can also be interpreted (that is to say, we can formulate a specific rule of correspondence) as a collection of particles moving at the speed of light which do have a well-defined and measurable energy and a momentum.[10] This means that: (a) if they have to enter in the framework of special relativity, they must be treated as massless particles; and (b) the special relativity theory cannot be considered complete if it is not able to care for the motion and the dynamics of massless particles.

Now the solution to this problem is at the same time obvious and stunning. Inasmuch as we cannot always associate an energy and a four-momentum with a given mass, but we can rather do the opposite, the obvious conclusion is that we can use the energy itself to define the four-momentum and to compute its equivalent mass (either the relativistic inertial one or the rest-mass) when it is possible. thus it turns out that saying that mass and energy are equivalent, or that energy is a form of mass, is not completely correct. Actually energy is a more fundamental quantity than the mass, and mass is rather a form of energy, but obviously not the only one.

Understanding this point is of utmost importance because its consequences largely overcome that of the redefinition of the four-momentum at a more fundamental level as Eq. (6.2.1)

$$\boldsymbol{p} = \frac{E}{c} \{1, \mathbf{v}\} .$$

Rather, this is the key point that introduces in a seamless and logical way the practical necessity of concepts such as

[10]To convince ourselves of this it is sufficient to notice that we can assign an energy $E = h\nu$ to a photon with frequency ν and to remember that the fact that light can exert a pressure is shown in many cases and at different levels, from small didactic experiments to objects the size of solar sails.

1. Gauge theories
2. The stress-energy tensor as the source of gravity and, the most important of all
3. The modification of the equivalence principle from its weak form

Conservation of four-momentum and four-force

We can now turn back to the problem of finding the laws of relativistic dynamics in a Poincaré covariant form by first recalling that the other axiom of relativistic dynamics, driven by its classical parallel, is that of the conservation of the four-momentum in an isolated system. As with the classical, three-dimensional one, it is assumed that the four-momentum is an additive quantity, which means that if we have a set of n particles, each with its own $\boldsymbol{p}_i, i = 1, \ldots, n$, then the total four-momentum of the system is $\boldsymbol{p} = \sum_i \boldsymbol{p}_i$, and for massive particles it is constant over the (proper) time of the observer if such a system is isolated:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = 0. \tag{6.2.3}$$

The meaning of such a hypothesis can be better understood if we consider the spatial and temporal parts of the above equation separately. The two in fact are the relativistic counterparts of the momentum and energy conservation laws of classical dynamics, respectively, and once again these two reduce to their classical form for $v \ll c$. Moreover, because this law is obtained by differentiating a four-vector with respect to a four-scalar, the quantity on the left-hand side is again a four-vector, so the zero-component lemma holds, by which we can see that any of the conservation laws, energy or momentum, implies the other and therefore its whole four-dimensional version. In the case of massless particles, we can just see this as the two separate (but now connected in a covariant way) laws of the conservation of energy and three-momentum, namely

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0, \text{ and } \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0. \tag{6.2.4}$$

The need of distinguishing between the cases of massive and massless particles, moreover, implies that any equation of motion has to be paired with a condition on the four-momentum which is

$$\boldsymbol{p} \cdot \boldsymbol{p} \equiv \eta_{\alpha\beta} p^\alpha p^\beta = \begin{cases} -m^2c^2 & \text{for } m \neq 0 \text{ particles} \\ 0 & \text{for } m = 0 \text{ particles.} \end{cases} \tag{6.2.5}$$

These two different formulations can be seen as an anticipation of the need for different gauges in physics theories, which translate in the choice of a specific constraint for the nonsingular Lagrangians defined in a four-dimensional spacetime, as we show in the next section.

It is then natural to conceive a relativistic extension of Newton's second law of dynamics which, stemming from its $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$ form, uses the four-dimensional counterpart of the forces. The latter, with little fantasy, are called *four-forces*. Similarly to what is postulated in classical dynamics, the four-force $\boldsymbol{f}$ acting on a non isolated system is posited to equal the variation of the total four-momentum of such a system that, for massive particles, reads

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau}. \tag{6.2.6}$$

In the four-momentum the three-momentum/spatial components show together with an energy/temporal component, and all four are linked by a constraint given by the invariant Minkowskian norm of this four-vector. In a similar fashion it is evident that the four-force components include a (relativistic) three-force (or a three-impulse) and its power constrained by the norm of such a four-vector. Once again we have to use this constraint to give a relativistic acceptable treatment of the dynamics of massless and massive particles in the same theory, but the equivalent formulation of dynamics in the framework of a field theory provides an easier way to deal with the two different cases of massive and massless particles so, even if we analyze in more detail the case of electromagnetism in Sect. C.3 as an explicit example of four-force, here we limit ourselves to explore a bit the consequences of this definition interpreting it as a modification of classical dynamics.

In order to show it, and because we want to compare the results obtained in Sect. 5.3 with the following ones, we momentarily revert to the previous notation by writing the rest-mass with m_0 and putting $m = m_0\gamma(v)$. With this in mind we can write

$$\boldsymbol{f} = \gamma(v)\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{E}{c}, \mathbf{p}\right\} = \gamma(v)\left\{\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, \mathbf{F}\right\}, \tag{6.2.7}$$

where we have defined such a three-force as $\mathbf{F} = \mathrm{d}(m\mathbf{v})/\mathrm{d}t$. This definition is justified by the fact that it becomes the Newtonian force in the usual limit of $v \ll c$; in fact

$$\begin{aligned}\mathbf{F} &= \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v} + m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\\ &= m_0\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{v} + m_0\gamma\mathbf{a},\end{aligned}$$

and from Eq. (6.1.9)

$$\mathbf{F} = m_0\gamma^3\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} + m_0\gamma\mathbf{a}, \tag{6.2.8}$$

from which clearly

$$\lim_{v\to 0}\mathbf{F} = \lim_{c\to\infty}\mathbf{F} = m_0\mathbf{a}.$$

Finally, we want to stress here a further interesting observation. Equation (6.2.8) shows that the relativistic expression of the three-force necessarily contains terms that are explicitly dependent on the velocity. In other words, the three-force can be a part of a Lorentz-invariant quantity (a four-vector) only if it depends on the velocity, which explains why a velocity-independent force cannot be "relativistic", i.e., Lorentz-invariant. More than this, we recall that the Lorentz electromagnetic force is velocity-dependent, and this is why such a force can be compatible with special relativity, although a generic dependence on the velocity is not sufficient to guarantee the relativistic covariance of a force.

6.3 Lagrangian Formulation

We have already seen that in classical physics the equations of dynamics for both the particles and the fields, in this case with particular reference to the gravitational interaction, can be entirely deduced by requiring the validity of a variational principle that imposes the condition $\delta S = 0$ to an appropriately defined functional $S[\mathbf{r}, \phi]$ of the particle motion $\mathbf{r}$ and of the field ϕ called Action. The equations of motion of the particle are obtained by imposing the $\delta S = 0$ condition for null variations at the extremal points of the particle's trajectories, whereas those of the fields can be derived when the same condition is enforced for null variations of the field at the spatial and temporal boundaries of the domain of integration. The Action functional is in general defined from a Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}, t)$ as

$$S[\mathbf{r}] = \int_{t_0}^{t_1} L(\mathbf{r}, \dot{\mathbf{r}}, t)\, \mathrm{d}t$$

in the case of particle dynamics, and from a Lagrangian density $\mathcal{L}(\phi, \partial_t \phi, \nabla\phi, t)$ as

$$S[\phi] = \int_{t_0}^{t_1} \int_{\Omega_3} \mathcal{L}(\phi, \partial_t \phi, \nabla\phi, t)\, \mathrm{d}^3\mathbf{x}\mathrm{d}t$$

in the case of the fields, and the resulting equations of motion are the Euler–Lagrange equations (1.2.13) and (1.3.6), respectively. In order to ensure their covariance with respect to the transformations of the Euclidean isometry group and of those of Galilean relativity, S has to be a Euclidean scalar and a Galilean-invariant, which implies, because $\mathrm{d}t$ and $\mathrm{d}^3\mathbf{x}$ already satisfy these requirements, the condition that L must be a scalar and $\mathcal{L}$ a scalar field.

Variational approach and relativistic prerequisites

If we want to transfer the same mathematical machinery in the context of special relativity, clearly we need to require the covariance of the action with respect to the transformations of the Poincaré group. The first thing to notice is about the integrating variables and their respective domains which, in the classical case, are t for the particle dynamics and $\mathbf{x}$, t for the fields. Although in Euclidean geometry they are both scalars, this is not the case in Minkowski spacetime and, as we realized at the beginning of this chapter, although it would still be possible to continue using them, this might not be the most convenient choice.

Dynamics of particles

In the case of particles' dynamics t, as independent variable, plays the role of a parameter for the trajectories and therefore as a derivation variable, so that $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} \equiv \dot{\mathbf{r}} = \mathrm{d}\mathbf{r}/\mathrm{d}t$. Therefore the most natural replacement would be the proper time τ of the reference system, for which we can put $\boldsymbol{r} = \boldsymbol{r}(\tau)$ and $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}\tau$. In this case then, similarly to the classical case, the expression for the action will be in general

$$S[\boldsymbol{r}] = \int_{\tau_0}^{\tau_1} L(\boldsymbol{r}, \boldsymbol{v}, \tau)\, \mathrm{d}\tau,$$

with the requirement on L to be a Lorentz scalar to guarantee the covariance of the equations of motion with respect to relativistic transformations (and therefore to satisfy the principle of relativity in its Einsteinian form). The action defined in this way coincides with the general case treated in Sect. A.2, therefore

$$\delta S[\boldsymbol{r}] = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^{\alpha}} - \frac{\partial L}{\partial x^{\alpha}} = 0, \tag{6.3.1}$$

for variations δx^{α} null at the endpoints ($\delta x^{\alpha}(\tau_0) = \delta x^{\alpha}(\tau_1) = 0\ \forall\, \alpha = 0, \ldots, 3$) where $v^{\alpha} \equiv \dot{x}^{\alpha} \equiv \mathrm{d}x^{\alpha}/\mathrm{d}\tau$.

We stress again that we can take the proper time as the parameter only because we have assumed to be confined to the case of particles' dynamics. This could not be admitted for massless particles, for which proper time cannot be defined, but this problem is addressed in the next section.

Dynamics of fields

A similar operation can be done for the Lagrangian density, but in this case we can exploit the Lorentz-invariance of the infinitesimal four-volume element $\mathrm{d}^4\boldsymbol{x} = c\mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ to put an analogous constraint for the expression of the action, which now becomes

$$S[\phi] = \int_{\Omega_4} \mathcal{L}(\phi, \partial_{\alpha}\phi, \boldsymbol{x})\, \mathrm{d}^4\boldsymbol{x} \tag{6.3.2}$$

where, similarly to the case of the particles, $\mathcal{L}$ must be a Lorentz scalar field and $\partial_{\alpha}\phi \equiv \partial\phi/\partial x^{\alpha}$.

The equations of motion for the field are then obtained following the same calculations shown in Sect. 1.3.1, with the difference that we do not need any distinction between space and time coordinates as in Eq. (1.3.2). Eventually the condition of null variation of the action at the spacetime boundaries brings to the relativistic equivalent of Eq. (1.3.6), i.e.,

$$\delta S\left[\phi\right] = 0 \quad \Leftrightarrow \quad \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha \phi\right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{6.3.3}$$

where, once again, the Euler–Lagrange equations can be defined as the variational derivative of the action

$$\frac{\delta S}{\delta \phi} \equiv \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha \phi\right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi}.$$

Contrary to the case discussed above, we did not use the proper time, but just the four-dimensional volume element, so the above equation is completely general.

6.3.1 Free Particles

In Sects. 4.3 and 4.4 we used the fundamental principles of Euclidean covariance and of Galilean relativity to deduce almost entirely the actual form of the Lagrangian for a free particle. The same reasoning can be applied to the Lagrangian of special relativity by considering the principle of relativity in its Einsteinian form, instead of the Galilean one.

First of all, one can appeal to the required invariance for translations in spacetime to claim that the Lagrangian can depend on the four-velocity only, and to the need that it is a scalar with respect to “hyperbolic rotations” in the Minkowskian spacetime (namely with respect to Lorentz transformations) to make the second claim that $L^{(\mathrm{p})}_{\mathrm{free}} = L\left(\boldsymbol{v} \cdot \boldsymbol{v}\right)$.

Lagrangian of free massive particles

Finally, we can demand that, in the case of massive particles, its spatial part reduces to the classical form for $v \ll c$ or, in formulae, that

$$L\left(\boldsymbol{v} \cdot \boldsymbol{v}\right)\, \mathrm{d}\tau \simeq \frac{1}{2} m v^2\, \mathrm{d}t \quad \text{for } v \ll c.$$

Taking into account that $\mathrm{d}\tau = \gamma^{-1}\left(v\right)\, \mathrm{d}t$, the above condition becomes

$$L\left(\boldsymbol{v} \cdot \boldsymbol{v}\right) \gamma^{-1}\left(v\right)\, \mathrm{d}t = \frac{1}{2} m v^2\, \mathrm{d}t \quad \text{for } v \ll c.$$

This requirement can be easily met if we put $L = m\boldsymbol{v} \cdot \boldsymbol{v} = -mc^2$; in fact in this case $L\left(\boldsymbol{v} \cdot \boldsymbol{v}\right)\gamma^{-1}\left(v\right)\,\mathrm{d}t = -mc^2\gamma^{-1}\left(v\right)\,\mathrm{d}t$, and it is easy to see that, in the slow-speed limit,

$$L\,\mathrm{d}\tau = -mc^2\sqrt{1-\frac{v^2}{c^2}}\,\mathrm{d}t \simeq \left(-mc^2 + \frac{1}{2}mv^2\right)\mathrm{d}t.$$

This is the desired approximation except for a constant term $-mc^2$ which, however, does not contribute to the equations of motion. As it happened when we defined the total relativistic energy, above in this chapter, we are assuming that the same multiplication by the rest-mass m which is needed for the spatial part also brings the temporal part to the correct expression. The fact that this is the simplest way to achieve the desired result does not allow us to neglect the experimental verification.

The action of a free particle in special relativity therefore results in

$$S^{(\mathrm{p})}_{\mathrm{free}}\left[\boldsymbol{r}\right] = -mc^2\int_{\tau_0}^{\tau_1}\mathrm{d}\tau. \tag{6.3.4}$$

Now, in order to recover the equation of motion of the particle using the Euler-Lagrange equations, we need a Lagrangian expressed as function of the four-velocity. For this reason we can again exploit the relation $-c^2 = \boldsymbol{v} \cdot \boldsymbol{v}$, and remembering that $\boldsymbol{v} \cdot \boldsymbol{v} = \eta_{\alpha\beta}v^\alpha v^\beta$ it is

$$\frac{\partial L_{\mathrm{free}}}{\partial \dot{x}^\alpha} \equiv \frac{\partial L_{\mathrm{free}}}{\partial v^\alpha} = 2m\eta_{\alpha\beta}v^\beta = 2p_\alpha$$

for any $\alpha = 0, \ldots, 3$, and therefore from Eq. (6.3.1) the equations of motion can be written

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = 0$$

as assumed in Eq. (6.2.3). As we have seen, the above equation implies the energy and momentum conservation, and in particular the latter means that a free particle moves on straight lines.

This derivation follows very closely its non relativistic counterpart, but has a subtle issue that can be sensed when it is recalled that the problem of the free relativistic motion is completely solved only when the constancy of the four-momentum is paired with the appropriate condition which, in this case, is $\boldsymbol{p} \cdot \boldsymbol{p} = -m^2c^2$. Actually one could obtain the former simply by starting from a Lagrangian

$$S^{\mathrm{p}}_{\mathrm{free}}\left[\boldsymbol{r}\right] = m\int_{\tau_0}^{\tau_1}\boldsymbol{v} \cdot \boldsymbol{v}\,\mathrm{d}\tau,$$

but it is only from Eq. (6.3.4) that the condition for massive particles can be assumed, which explains why $L = m\boldsymbol{v} \cdot \boldsymbol{v}$ cannot be assumed as the correct Lagrangian for the free relativistic particle. Before settling this issue in a more rigorous way, it is worth noticing that the same Lagrangian can be interpreted from another point of view. From Eqs. (6.1.3) and (6.3.4) the action can also be written as

$$S^{\mathrm{p}}_{\mathrm{free}}\left[\boldsymbol{r}\right] = -mc\int_{\tau_0}^{\tau_1} c\,\mathrm{d}\tau = -mc\int_{s_0/c}^{s_1/c}\sqrt{-\mathrm{d}s^2}, \tag{6.3.5}$$

with $\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$,which unlike the classical mechanics view highlights the fact that the motion of a particle is the one that minimizes the length of the four-dimensional trajectory between two points.

Lagrangian for free massless particles

As pointed out in the previous section, using proper time and four-velocity confines ourselves to the case of massive particles, and neither Eq. (6.3.4) or Eq. (6.3.5) are valid actions in the case of massless particles.[11] We therefore need to find a more general action that can allow us to include such cases.

6.3.2 General Free Particles' Dynamics and Local Gauge Freedom

Before facing this task, we want to stress a fundamental difference between the classical and relativistic action for the free particles. Whereas in the former the time and space variables are the coordinates of two completely independent metric spaces, in the latter all these variables are the coordinates of one single metric space. This allows us to use the time t in classical physics as an independent parameter for the trajectories of the particles. The equation of motion of classical physics is then three equations giving the evolution of each spatial coordinate parameterized by an independent parameter t. In order to replicate this scenario in relativistic physics, the proper time used to parameterize the worldlines of massive particles and allows us to define their four-velocities should be considered an independent parameter, but we know this is not possible because it is defined as a relation between space and time coordinates.

A straight correspondence of the two situations would require that the equations of motion, in the form of Eq. (6.2.3) or of Eq. (6.3.1), have to be a set of four independent differential equations, but this is not possible. Indeed, as we have also emphasized in the above derivation of the Lagrangian for free particles, they are related to each other by a specific relation that characterizes the nature of the worldline as timelike or null (or spacelike). This relation leaves us with just three independent equations

[11] The latter both because $m = 0$ and because for massless particles $\mathrm{d}s^2 = 0$.

plus one that is not independent anymore, but acts as a "constraint" linking the four together.

Local gauge invariance and Euler–Lagrange equations

Moreover, it is always possible to choose a specific parameterization that fixes the actual form of the equations of motion and that of the constraint. A practical example is given by Eqs. (6.2.3) and (6.2.4). These two sets describe exactly the same situation and are equivalent to each other, but they use the proper and coordinate time, respectively, as parameters. The arbitrariness of this choice is referred to as *local gauge invariance* of the system, and the act of choosing the form of the constraint, and therefore the parameterization to be used for the equations of motion, is called *gauge fixing*. In the first form (i.e., with τ as parameter) the equations of motion are manifestly covariant, whereas in the second they are more easily interpreted in the reference frame of an external observer. For this reason the latter is often named the *laboratory gauge*.

The existence of a local gauge invariance depends on that of one or more degrees of freedom (also called *local gauge freedom*) in the parameterization, and the latter is strictly connected to the number of independent equations of motion of the physical system.

As shown in Exercise 6.4, in the Lagrangian formalism this can be easily tested by checking the Hessian matrix $H = \left(\partial^2 L/\partial\dot{q}^\alpha\partial\dot{q}^\beta\right)$ of the Lagrangian. A generic Lagrangian $L\left(\mathbf{q}, \dot{\mathbf{q}}, \lambda\right)$, where the dot indicates the differentiation with respect to a suitable parameter λ of the worldlines, is singular when H is rank deficient, i.e., when $\det(H) = 0$. In this case, however, only a subset of coordinates, whose number coincides with the rank of H, can "produce" independent equations of motion in the form $\ddot{q}^\alpha = f^\alpha\left(\mathbf{q}, \dot{\mathbf{q}}, \lambda\right)$, and the system admits an infinite number of solutions. A unique solution can be obtained only by adding a number of (independent) equations equal to the number of "missing variables", and the solution will depend on which equation has been adopted. This is why the coordinates associated with the independent equations of motion are called *true dynamical coordinates*, and the remaining are the *constraints* of the system.

This is exactly the situation described above, therefore the Lagrangian exhibits a local gauge freedom if it is singular, and the operation of fixing the gauge corresponds to impose a specific constraint.

In our case it is evident that the Lagrangian of Eq. (6.3.4) is not singular, which is exactly what one would expect because the relation among the spacetime coordinates was neglected. What we are trying to solve, then, is not a "free" Lagrangian problem, but rather a *constrained* Lagrangian problem. It is well known that these can be conveniently solved with the technique of the Lagrange multipliers and in fact a completely general Lagrangian, valid for both massive and massless particles, can be written in this way.

Einbein Lagrangian for free massive a massless particles

Before proceeding, we briefly recall how the technique of the Lagrangian multipliers works to find the extremal points of a function $f(\mathbf{q})$ constrained by another function $g(\mathbf{q})$ through the condition $g(\mathbf{q}) = c$, where c is a constant (a more detailed explanation of this technique can be found in several textbooks, e.g., Arfken and Weber (2012)). In practice the problem is solved by defining a third function, called Lagrangian[12] as

$$L = f(\mathbf{q}) - l\,(g(\mathbf{q}) - c) \tag{6.3.6}$$

by the introduction of a constant l called the *Lagrange multiplier*, and by imposing the condition $\boldsymbol{\nabla} L = 0$, where the gradient has to be intended as the full set of derivatives with respect to all the variables q_i *and* l. The first set will provide the essential condition $\partial_i f = l \partial_i g$, and the derivative with respect to l will give the constraint. The method can be easily generalized in the case of many constraints adding more independent multipliers.

Our case differs in some aspects with respect to the simplest case summarized above. The most important one is that we do not have a function of some independent variables, but of a set of coordinates depending on a generic independent parameter λ. In this case the Lagrange multiplier is not to be regarded as a simple constant, but rather as a *constant function* $e\,(\lambda)$, called *einbein*. The second difference is that we are trying to find the extremal points of a *functional* instead of those of a function, so the "gradient" we are taking is the functional derivative, or the functional variation which, as we know, is equivalent to the Euler–Lagrange equations of the Lagrangian.

We have now to find a suitable Lagrangian for our problem. The solution is not unique, thus for pedagogical reasons we first show one more similar to the expression of Eq. (6.3.6). The role of function f is clearly $\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$, where we have used $\dot{x}^\alpha$ instead of $\boldsymbol{v}$ as in the previous case to demonstrate that we are indeed considering the tangent four-vector to the particle's trajectory, which always exists, but the differentiation parameter is a generic λ rather than the proper time, which can be defined only for massive particles. Moreover, inasmuch as we cannot always assure the condition $m \neq 0$, we have dropped the usual factor m before the scalar product. The condition function can be taken as that of Eq. (6.2.5); in fact we know that the four-momentum is another quantity which can be always defined, whereas the explicit expression of the condition simply depends on the value of m, which is the parameter we want to use to distinguish the two cases of massive and massless particles.

Thus in general the Lagrangian can be written

$$L\,(\dot{x}^\alpha\,(\lambda)\,, e\,(\lambda)) = \eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - e\,(\lambda)\left(\eta_{\alpha\beta}p^\alpha p^\beta + m^2c^2\right),$$

[12]Not to be confused with the Lagrangian of the variational approach.

and the Euler–Lagrange equations will result in

$$2\eta_{\alpha\beta}\ddot{x}^{\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial}{\partial\dot{x}^{\alpha}}e(\lambda)\left(\eta_{\alpha\beta}p^{\alpha}p^{\beta}+m^2c^2\right) = 0$$
$$\eta_{\alpha\beta}p^{\alpha}p^{\beta} = -m^2c^2,$$

the second one coming from the derivation with respect to $e(\lambda)$, i.e.,[13]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial\dot{e}} - \frac{\partial L}{\partial e} = 0.$$

The explicit expression of both the Lagrangian and the Euler–Lagrange equations is adjusted according to the specific case under investigation by imposing a condition on the einbein.

In general, in fact, the dimensions of the two terms forming the Lagrangian have to match, thus for massive particles $e(\lambda)$ has to be a constant function with the dimensions of m^{-2}, and indeed we can safely take $e(\lambda) = m^{-2}$. This implies that we are fixing the gauge of our problem by taking $\lambda = \tau$, thus we can write $p^{\alpha} = m\dot{x}^{\alpha}$. We have therefore

$$2\eta_{\alpha\beta}\ddot{x}^{\alpha} - e(\lambda)\,m^2\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial}{\partial\dot{x}^{\alpha}}\left(\eta_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}+c^2\right) = 4\eta_{\alpha\beta}\ddot{x}^{\alpha} = 0,$$

and after having canceled out all the unnecessary constant factors the above equations become

$$\ddot{x}^{\alpha} = 0$$
$$\boldsymbol{p}\cdot\boldsymbol{p} = -m^2c^2,$$

where the first of them is clearly equivalent to Eq. (6.2.3) in the case of free particles.

For massless particles, instead, we have $m = 0$, so $\eta_{\alpha\beta}p^{\alpha}p^{\beta} = 0$, and we can just consider $e(\lambda)$ a constant function able to cancel out the factor (proportional to E^2/c^2) coming from the scalar product of the moments, thus obtaining the obvious equations

$$\ddot{x}^{\alpha} = 0$$
$$\boldsymbol{p}\cdot\boldsymbol{p} = 0.$$

[13] It is thus clear that the introduction of the einbein has introduced one more "coordinate" in the Lagrangian, which, however, is not a true dynamical coordinate because $\partial L/\partial\dot{e} = 0$. From this point of view, therefore, the function of the einbein is to show the local gauge freedom of the problem by writing a singular Lagrangian and making it possible to fix a specific gauge.

The same results can be obtained if we use the equivalent form of the Lagrangian

$$L\left(\dot{x}^{\alpha}(\lambda), e(\lambda)\right) = \frac{\eta_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}{e(\lambda)} - e(\lambda)\, m^2 c^2 \tag{6.3.7}$$

whose connection with the standard form of Eq. (6.3.6) is less clear, but from which the results are easier to work out. As in the previous case, the einbein has exactly the same function of as the Lagrange multiplier. The equations of motion for such a Lagrangian are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\dot{x}^{\alpha}(\lambda)}{e(\lambda)}\right) = 0$$
$$\eta_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = -e^2 m^2 c^2$$

which alternatively reduce to those of the photons when $e(\lambda) = \text{const}$, whereas they become those of massive particles for $e(\lambda) = m^{-1}$, so that, once again, choosing a specific expression for $e(\lambda)$ is equivalent to gauge fixing.

6.3.3 Particle Dynamics Under the Influence of a Scalar Field

In Sect. 4.4 we learned that the total action for a field theory is in general formed by three parts: a free part for the particle(s), one for the free field(s), and another one describing the interaction between them. Limiting ourselves to the derivation of the equations of motion, however, only first and the third parts are needed, because the equations of motion can be found by varying the action with respect to the motion $\boldsymbol{r}$ of the particle, to which the free field action does not contribute.

Potential energy in the case of a Lorentz-invariant scalar field

In this case the interaction part would obviously play the role that the potential energy plays in the classical particle Lagrangian $L = T - V$, and the first obvious case to explore is that of a potential energy as a scalar field, namely that of an interaction mediated by a scalar field. The difference with respect to the classical case lies on the fact that the background geometry is now Minkowskian, thus we are calling for a *Lorentz* scalar field ϕ multiplied by a convenient *Lorentz* scalar κ.[14]

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[14]We recall that L has to be a Lorentz scalar to guarantee the covariance of the resulting equations with respect to any transformation of the Poincaré group.

In the case of massive particles, the total action (except for the now unnecessary free field contribution) can therefore be written as

$$S[\boldsymbol{r}, \phi] = -\int_{\tau_1}^{\tau_2} mc^2 \, \mathrm{d}\tau - \int_{\tau_1}^{\tau_2} \kappa\phi \, \mathrm{d}\tau, \tag{6.3.8}$$

where it is clear that it has to be $\kappa \neq 0$ in order to have the particle m and the field ϕ interacting with each other. This explains why this quantity is said to characterize the *coupling* of the field with matter. More precisely, neglecting possible numerical factors introduced for convenience, we can think of κ as the product of a "charge" by a *coupling constant*. The former will be a property of the matter characterizing the interaction mediated by the field, such as the mass for gravity or the electric charge for electromagnetic interaction, whereas the actual value of the latter is a sort of "measure" of the interaction strength.

The relativistic equations of motion are then obtained from Eq. (6.3.1) with $L = -\left(mc^2 + \kappa\phi\right)$. This form of the Lagrangian is chosen to highlight a particularity of the special relativistic case. At first sight, in fact, it might seem that $\partial L/\partial \dot{x}^\alpha = 0$, but we have already seen that this is not the case in the previous section because $-c^2 = \eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$. This also means that

$$-\frac{1}{c^2}\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$$

and that the previous Lagrangian could be equivalently written as

$$L = -\left(mc^2 + k\phi\right)\sqrt{-\frac{1}{c^2}\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}.$$

This apparently arbitrary modification is justified by remembering that we are applying the principle of least action, according to which we require that the variation of the Action functional takes zero value for null variations of the particle's trajectory at its ending points. In this case we are dealing with a *four-dimensional trajectory*, which means that, as pointed out in the last subsection, because of the gauge freedom of the problem the differential $\mathrm{d}\tau$ is inextricably connected with the trajectory itself and it has to be considered in the computation of the Action variation. This is therefore a way to take into account this issue in the context of the Euler–Lagrange equations[15] which is equivalent to, e.g., the derivation shown in Padmanabhan (2010) by the direct computation of δS.

[15]One could equivalently say that $-c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ and therefore $\mathrm{d}\tau = \sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta/c^2}\,\mathrm{d}\tau$ so that $\mathrm{d}\tau$ is actually a quantity related to the four-velocity of the particle.

Equation (6.3.1) then results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^\alpha}-\frac{\partial L}{\partial x^\alpha} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{c^2}\left(mc^2+\kappa\phi\right)\eta_{\alpha\beta}\dot{x}^\beta\left(-\frac{1}{c^2}\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right)^{-1/2}\right]+\kappa\frac{\partial\phi}{\partial x^\alpha}\\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{c^2}\left(mc^2+\kappa\phi\right)\dot{x}_\alpha\right]+\kappa\frac{\partial\phi}{\partial x^\alpha}\\ &= \frac{1}{c^2}\left(mc^2+\kappa\phi\right)\frac{\mathrm{d}\dot{x}_\alpha}{\mathrm{d}\tau}+\frac{1}{c^2}\kappa\dot{x}_\alpha\frac{\mathrm{d}\phi}{\mathrm{d}\tau}+\kappa\frac{\partial\phi}{\partial x^\alpha}=0,\end{aligned}$$

and because

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=\frac{\partial\phi}{\partial x^\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}=\frac{\partial\phi}{\partial x^\alpha}\dot{x}^\alpha,$$

renaming appropriately the dummy indices we can write

$$\frac{\mathrm{d}\dot{x}_\gamma}{\mathrm{d}\tau}=-\frac{\kappa\dot{x}_\gamma\dot{x}^\beta}{\left(mc^2+\kappa\phi\right)}\frac{\partial\phi}{\partial x^\beta}-\frac{\kappa c^2}{\left(mc^2+\kappa\phi\right)}\frac{\partial\phi}{\partial x^\gamma}$$

and finally, by raising the covariant indices with $\eta^{\alpha\gamma}$,

$$\frac{\mathrm{d}\dot{x}^\alpha}{\mathrm{d}\tau}=-\frac{\kappa\dot{x}^\alpha\dot{x}^\beta}{\left(mc^2+\kappa\phi\right)}\frac{\partial\phi}{\partial x^\beta}-\frac{\kappa c^2\eta^{\alpha\gamma}}{\left(mc^2+\kappa\phi\right)}\frac{\partial\phi}{\partial x^\gamma}. \tag{6.3.9}$$

This somewhat abstract result is used in the next chapter, where we examine the possibility of writing a theory of gravity in the context of special relativity, but the presence of the term $-\left[\kappa\dot{x}^\alpha\dot{x}^\beta\left(mc^2+\kappa\phi\right)\partial\phi/\partial x^\beta\right]$ in the four-dimensional "force" of the right-hand side of the equation is already evident. The second term is similar to the classical gradient of the potential, however, the first one depends on the velocity of the particle, thus reaffirming what we already discovered in Sect. 6.2, i.e., the need for velocity-dependent terms in a force to be a Lorentz-invariant.

6.3.4 Field Equations

In the previous section we briefly explored the relativistic implementation of an action describing the dynamics of particles under the influence of a scalar field. A complete action also requires another component, namely the action for the free field, and we know that it is possible to derive the field equations by varying the resulting formula with respect to the field and assuming variations null at the boundaries. The details of such treatment, however, depend on the specific interaction one has to model. Just for example, the above choice of a scalar field was motivated by its simplicity and because, given its classical counterpart, it is the most obvious option if the goal is to write a relativistic theory of gravity, but in general one might naturally imagine fields in the form of higher rank tensors.

In concluding this chapter, therefore, we are completing the picture for a scalar field started above, but limiting ourselves to a sketch of its basic features, and postponing a more detailed treatment to the next one, where the same problem is faced for the specific case of a gravity theory.

By comparing the classical case and the special relativistic Lagrangian for free particles, it is natural to assume

$$\mathcal{L}\left(\partial_{\alpha}\phi\right) = -\frac{1}{2}k\eta_{\alpha\beta}\partial^{\alpha}\phi\partial^{\beta}\phi = -\frac{1}{2}k\partial_{\alpha}\phi\partial^{\alpha}\phi \tag{6.3.10}$$

as the Lagrangian for free fields, with k a constant introduced for future uses, and

$$S_{\text{free}}^{(\phi)}\left[\phi\right] = -\frac{1}{2}k\int_{\Omega_4}\partial_{\alpha}\phi\partial^{\alpha}\phi\,\mathrm{d}^4x \tag{6.3.11}$$

for the corresponding action. It should be noted that the relativistic Lagrangian depends also on $\partial_t\phi$ which, recalling what was said in Sect. 4.4.1, classifies ϕ as a *propagating field*. The meaning of such a definition is that the relativistic field propagates at finite speed and does not admit instantaneous interactions. Indeed, the Euler–Lagrangian equation (6.3.3) gives therefore

$$\partial_{\alpha}\partial^{\alpha}\phi = 0 \tag{6.3.12}$$

for the equation of motion of ϕ, which can be equivalently written as $\Box^2\phi = 0$ and is the special relativistic counterpart of the Laplace equation, just with the d'Alembert operator substituting the Laplacian. This operator is the same as the wave equations in vacuum, which makes it easy to understand why the field propagates with finite speed c.

It is also immediate to see that in the Newtonian limit of $v \ll c$ (or of $c \to \infty$) Eq. (6.3.12) reduces to $\nabla^2\phi = 0$.

The field equations can be obtained in a similar fashion, following a procedure similar to that of Sect. 4.4 for the definition of the interaction term of the action, and remembering that now we are working in a Minkowskian spacetime, so that

$$S = S_{\text{free}}^{(\phi)}\left[\phi\right] + S_{\text{int}}\left[\mathbf{r},\phi\right] = K\int_{\Omega_4}\partial_{\alpha}\phi\partial^{\alpha}\phi\,\mathrm{d}^4x + \int_{\Omega_4}\kappa\phi\,\mathrm{d}^4x, \tag{6.3.13}$$

where the values of the constants depend on the specific problem under consideration.

6.4 Exercises

Exercise 6.1 Show that the four-velocity and the four-acceleration are orthogonal, namely that $\boldsymbol{v} \cdot \boldsymbol{a} = 0$.

Solution 6.1 The proof is immediate by deriving Eq. (6.1.3), in fact,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{v} \cdot \boldsymbol{v}) = -\frac{\mathrm{d}\left(c^2\right)}{\mathrm{d}\tau} = 0,$$

but it is also

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(\boldsymbol{v} \cdot \boldsymbol{v}) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\eta_{\alpha\beta} v^\alpha v^\beta\right) \\ &= 2\eta_{\alpha\beta} v^\alpha \frac{\mathrm{d}v^\beta}{\mathrm{d}\tau} = 2\eta_{\alpha\beta} v^\alpha a^\beta \\ &= 2\boldsymbol{v} \cdot \boldsymbol{a}\end{aligned}$$

from the four-acceleration definition of Eq. (6.1.7).

Exercise 6.2 Find the relativistic formula for the aberration of light.

Solution 6.2 The term "aberration of light" denotes a phenomenon caused by the relative motion between two observers, according to which they shall disagree on the direction of the light coming from a source observed in their respective reference systems. Actually, this is not limited to light propagation, rather it can be generally experienced for any signal observed by these observers. Indeed, an intuitive explanation of the aberration is usually given in terms of the different direction of the falling rain as seen by these two observers: the larger the horizontal speed, the more tilted the rain will be seen.

This is why, qualitatively, such phenomena can be predicted also in the context of classical physics.[16] In this case, in fact, we consider for the sake of brevity the special case of a light ray moving on the xy-plane of S and the reference system $\bar{S}$ of a body with the axes parallel to those of S and moving with speed u along the x axis.

This means that the incoming direction of a photon will have components $v_x = c\cos\alpha$ and $v_y = c\sin\alpha$ in S, because its speed is c. In $\bar{S}$ we can follow the same reasoning, but the incoming direction is given by the transformed components $\bar{v}_x$ and $\bar{v}_y$, which will be related to the corresponding angle $\bar{\alpha}$ by the same formulae, namely $\bar{v}_x = c\cos\bar{\alpha}$ and $\bar{v}_y = c\sin\bar{\alpha}$. Hence, we just need to know the transformation laws of the velocity components.

[16]The aberration of a light was detected in the seventeenth century in the form of the so-called *stellar aberration*. The first explanation was given by Bradley in 1727, who derived the non relativistic equation of the light aberration.

In the classical/Galilean case it is $\bar{v}_x = v_x - u$ and $\bar{v}_y = v_y$, therefore

$$\cos\bar{\alpha} = \frac{\bar{v}_x}{c} = \frac{v_x - u}{c} = \cos\alpha - \frac{u}{c}$$
$$\sin\bar{\alpha} = \frac{\bar{v}_y}{c} = \sin\alpha,$$

or[17]

$$\tan\bar{\alpha} = \frac{\sin\alpha}{\cos\alpha - u/c}. \tag{6.4.1}$$

In special relativity, instead, by specializing Eq. (5.5.8) to our case, it is

$$\bar{v}_x = \frac{v_x - u}{1 - v_x u/c^2}$$
$$\bar{v}_y = \frac{v_y\sqrt{1-(u/c)^2}}{\left(1 - v_x u/c^2\right)},$$

which gives

$$\cos\bar{\alpha} = \frac{\cos\alpha - u/c}{1 - (u/c)\cos\alpha}$$
$$\sin\bar{\alpha} = \frac{\sin\alpha\sqrt{1-(u/c)^2}}{1 - (u/c)\cos\alpha},$$

that is,

$$\tan\bar{\alpha} = \frac{\sin\alpha\sqrt{1-(u/c)^2}}{\cos\alpha - u/c}. \tag{6.4.2}$$

It is immediate to verify that for $u \ll c$, the relativistic formula reduces to the classical one inasmuch as

$$\sqrt{1-(u/c)^2} = 1 - \frac{1}{2}\left(\frac{u}{c}\right)^2 + \mathcal{O}\left(u^4/c^4\right),$$

we can neglect $(u/c)^2 \ll 1$, but u/c can be retained because it is of lower order.

Exercise 6.3 Show that, for a body of mass m moving with velocity $\mathbf{v}$, it is $\mathrm{d}m/\mathrm{d}t = \mathbf{F}\cdot\mathbf{v}/c^2$.

[17] In considering the tangent, however, we are are introducing a sign ambiguity.

Solution 6.3 From Eq. (6.2.8), remembering that $m = m_0\gamma$,

$$\mathbf{F} \cdot \mathbf{v} = \left(\gamma^2 \frac{v^2}{c^2} + 1\right) m\mathbf{a} \cdot \mathbf{v}, \tag{6.4.3}$$

but from Eq. (6.1.9) it is

$$\mathbf{v} \cdot \mathbf{a} = \frac{c^2}{\gamma^3} \frac{\mathrm{d}\gamma}{\mathrm{d}t},$$

and because

$$\frac{\mathrm{d}m}{\mathrm{d}t} = m_0 \frac{\mathrm{d}\gamma}{\mathrm{d}t},$$

we have immediately

$$m\mathbf{v} \cdot \mathbf{a} = \frac{c^2}{\gamma^2} \frac{\mathrm{d}m}{\mathrm{d}t}.$$

Equation (6.4.3) therefore becomes

$$\mathbf{F} \cdot \mathbf{v} = \left(v^2 + \frac{c^2}{\gamma^2}\right) \frac{\mathrm{d}m}{\mathrm{d}t} = c^2 \frac{\mathrm{d}m}{\mathrm{d}t},$$

from which the result can be immediately derived.

Exercise 6.4 Show that the Euler–Lagrange equations of a generic Lagrangian admit a unique solution only if the Hessiam matrix $\left(\partial^2 L/\partial \dot{q}_i \partial \dot{q}_j\right)$ is non singular.

Solution 6.4 Although in this chapter we are working in the context of special relativity, this is a completely generic result that holds for any Lagrangian $L\left(\mathbf{q}, \dot{\mathbf{q}}, \lambda\right)$ which is function of a set of generalized coordinates $\mathbf{q} = \{q_i\left(\lambda\right)\}$, $i = 1, \ldots, n$, parameterized with respect to a parameter λ and with generalized velocities $\dot{\mathbf{q}} = d\mathbf{q}/\mathrm{d}\lambda$. For this reason we refrain from making any distinction between vectors and one-forms, and for this exercise the non relativistic notation of putting the indexes as subscript is adopted. The Euler–Lagrange equations of such a system can thus be written as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

By expanding the total derivative with respect to λ results in

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \frac{\mathrm{d}\dot{q}_j}{\mathrm{d}\lambda} + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \frac{\mathrm{d}q_j}{\mathrm{d}\lambda} + \frac{\partial^2 L}{\partial \dot{q}_i \partial \lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda} = \frac{\partial L}{\partial q_i},$$

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial \lambda}. \tag{6.4.4}$$

This alternative way to represent the generic solution of a system of Euler–Lagrange equations can be put in matrix form $H\mathbf{x} = \mathbf{b}$ where

$$H = \left\{ \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\}$$

is the *Hessian matrix*, $\mathbf{x} = \left\{ \ddot{q}_j \right\}^{\mathrm{T}}$ and

$$\mathbf{b} = \frac{\partial L}{\partial \mathbf{q}} - \frac{\partial^2 L}{\partial \dot{\mathbf{q}} \partial \mathbf{q}} \dot{\mathbf{q}}^{\mathrm{T}} - \frac{\partial^2 L}{\partial \dot{\mathbf{q}} \partial \lambda}.$$

As is well known from linear algebra theory, this system admits a unique solution if and only if the Hessian matrix is non singular, i.e., if $\det(H) \neq 0$. L is called a *non singular* or *singular Lagrangian* whether the above condition holds or not, respectively.[18]

[18] As a further note, if the system is not only singular, but also $\partial^2 L / \partial \dot{q}_i \partial \dot{q}_j = 0$ for all i, j then it transforms to

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial \lambda}$$

that in its turn can be solved only if

$$\det \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \right) \neq 0.$$

If $\partial L / \partial \dot{q}_j = 0$ for a specific j, then the coordinate q_j is said to be *non propagating* and the Lagrangian is *non dynamic* for that coordinate. A Lagrangian is simply non dynamic as a whole if the condition holds for all the q_j.

Chapter 7
Gravity and Special Relativity

We know that, contrary to electromagnetism, the Newtonian theory of gravity is compatible with Newtonian dynamics, and because we have just realized that the latter has to be replaced by special relativity, it is easy to understand that the former has to be superseded by another theory of gravity which is compatible with the new dynamics.

Armed with the encouraging results of the previous chapter, we can now make the first attempt at developing a relativistic theory of gravity using the framework of the variational approach. Although we already know that this attempt cannot be successful, it is very useful from a pedagogical point of view inasmuch as it makes clearer the reasons that eventually lead to the actual formulation of general relativity.

7.1 Gravity as a Lorentz-Invariant Scalar Field

By comparison with its classical form, it seems reasonable to follow the historical attempts of Nordstrom (1912, 1913) and make the tentative hypothesis that, similarly to the Newtonian case, the gravitational interaction is mediated by a Lorentz scalar field.[1] Moreover, we initially limit our discussion to the case of massive particles. This allows a more gentle transition to the complete picture and a clear are exposition of the motivations for this transition.

In this case the dynamics of the particles interacting with such are scalar field are described by Eq. (6.3.9), with its specific coupling constant μ:

$$\frac{\mathrm{d}\dot{x}^{\alpha}}{\mathrm{d}\tau}=-\frac{\mu\dot{x}^{\alpha}\dot{x}^{\beta}}{\left(mc^{2}+\mu\phi\right)}\frac{\partial\phi}{\partial x^{\beta}}-\frac{\mu c^{2}\eta^{\alpha\gamma}}{\left(mc^{2}+\mu\phi\right)}\frac{\partial\phi}{\partial x^{\gamma}} \tag{7.1.1}$$

[1]For a comprehensive historical review of this subject in English the interested reader can refer to Norton (2007).

A. Vecchiato, *Variational Approach to Gravity Field Theories*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2_7

which has been renamed to stress the role of this constant in a gravitational theory. Indeed, simply by recalling the role of the free and of the interacting parts of the action with the classical case, it is easy to understand that m has to be interpreted as as the *inertial mass* m_I, and μ, being coupled with the field, can be related to the *gravitational mass* m_G of the same particle.

This equation, keeping m and μ separated, considers the inertial and gravitational masses as two different quantities, whereas they are claimed equal by the equivalence principle in its weak form, as described in Sect. 3.3. Such a principle has been tested up to an accuracy of $\sim 10^{-13}$ (Schlamminger et al. 2008), therefore it is appropriate to analyze how this statement can be incorporated in the theory and what can be deduced from it.

7.1.1 Equivalence Principle and "Geometrization" of Gravity

The quantity $\mu\phi$ has the dimensions of an energy, therefore, to integrate the equivalence principle into our theory, we can just take $\mu = k m_\mathrm{G} = Gm$, and at the same time $\Phi = G\phi$, with G constant, has the same dimension of the Newtonian potential. The particle's equations of motion thus become

$$\begin{aligned} \frac{\mathrm{d}\dot{x}^\alpha}{\mathrm{d}\tau} &= -\frac{Gm\dot{x}^\alpha\dot{x}^\beta}{\left(mc^2 + Gm\phi\right)}\frac{\partial\phi}{\partial x^\beta} - \frac{Gmc^2\eta^{\alpha\gamma}}{\left(mc^2 + Gm\phi\right)}\frac{\partial\phi}{\partial x^\gamma} \\ &= -\frac{\dot{x}^\alpha\dot{x}^\beta}{\left(1 + \Phi/c^2\right)}\frac{\partial}{\partial x^\beta}\left(\frac{\Phi}{c^2}\right) - \frac{\eta^{\alpha\gamma}}{\left(1 + \Phi/c^2\right)}\frac{\partial}{\partial x^\gamma}\left(\frac{\Phi}{c^2}\right), \end{aligned} \tag{7.1.2}$$

which shows that as a consequence of the equivalence principle the motion of the particle will not depend on its mass.

Another consequence of this principle can be seen from the Action, which now reads

$$\begin{aligned} S\left[\boldsymbol{r}, \Phi\right] &= -\int_{\tau 0_1}^{\tau_1} \left(mc^2 + \mu\phi\right)\, \mathrm{d}\tau \\ &= -mc^2 \int_{\tau_0}^{\tau_1} \left(1 + \frac{G\phi}{c^2}\right) \mathrm{d}\tau \\ &= -mc^2 \int_{\tau_0}^{\tau_1} \left(1 + \frac{\Phi}{c^2}\right) \mathrm{d}\tau \end{aligned} \tag{7.1.3}$$

We already met a similar situation in Sect. 1.2.2 when, with Eq. (1.2.20), we observed that the reduced action could be interpreted also as the distance between two points "weighted" by the potential energy of the interaction. At that time we remarked that this could also be interpreted as if distances were not measured in the ordinary Euclidean space but in a curved one. Moreover, the change of a physical principle such as that of relativity brought us from the classical to the relativistic dynamics, and to the transition from Euclidean space and time to Minkowskian spacetime.

In this case it is even easier to understand the geometrical connection, which can be given an interesting twofold interpretation. With Eq. (6.3.5) we used the property $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$ of the Minkowskian length element to highlight the fact that the Action could be interpreted as a distance in the four-dimensional Minkowskian spacetime, therefore the same reasoning can be generalized by noticing that the factor $\left(1 + \Phi/c^2\right)$ can be included in the definition of four-dimensional lengths, so that in presence of a gravity field the distances are measured as

$$\mathrm{d}s_{\mathrm{G}}^2 = \left(1 + \frac{\Phi}{c^2}\right)^2 \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta, \tag{7.1.4}$$

i.e., in a spacetime where the metric tensor is not $\eta_{\alpha\beta}$ anymore, but rather $g_{\alpha\beta} = \eta_{\alpha\beta}\left(1 + \Phi/c^2\right)^2$. Moreover, because $\Phi = \Phi(\boldsymbol{x})$, such metric tensor will not be constant, but a function of space and time.[2]

Gravitational time dilation

On the other hand, in this spacetime geometry it would be $\mathrm{d}s_{\mathrm{G}}^2$, rather than the Minkowskian one, the invariant line element. This implies that the Minkowskian definition of proper time $-c^2\mathrm{d}\tau_{\mathrm{M}}^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ cannot be valid anymore, or that this does not represent the time measured by an observer at rest with respect to a reference system under the influence of a gravity field. In other words, an immediate consequence of this geometrization of gravity is that gravity influences the flow of time.

Inasmuch as a reasonable definition of proper time has to depend on the invariant line element, with a gravity field Φ this quantity will be

$$-c^2\mathrm{d}\tau_{\mathrm{G}}^2 = \mathrm{d}s_{\mathrm{G}}^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{7.1.5}$$

In the so-called weak field limit, namely when $\Phi \ll c^2$, we can write

$$-c^2\mathrm{d}\tau_{\mathrm{G}}^2 \simeq \left(1 + \frac{2\Phi}{c^2}\right)\eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta,$$

which implies that the proper time measured by an observer with fixed coordinates ($\mathrm{d}x^i = 0$) feeling the influence of a gravitational potential Φ is related to that measured by an observer at infinity $\mathrm{d}t$ by the equation

[2]The "$\eta_{\alpha\beta}$ to $g_{\alpha\beta}$" operation is one of the characteristics of the transition from special to general relativity. As we have seen, this derives directly from the equivalence principle which on its turn is connected to the way matter couples with gravity. This explains why it also enters into action in the so-called *minimal coupling* mentioned in the next chapters.

$$d\tau_G \simeq \sqrt{1 + \frac{2\Phi}{c^2}}\, dt,$$

and because outside of a body of mass M we have $\Phi = -GM/r$, the ticking of the observer's proper time with respect to t slows down when Φ gets stronger, i.e., when the observer is closer (i.e., r is smaller) or M is larger, which justifies the naming of *gravitational time dilation* given to this effect. Clearly at infinity the two times will coincide.

Another important observation coming from Eq. (7.1.5) is that we can also interpret the action of Eq. (7.1.3) as

$$S[\boldsymbol{r}, \Phi] = -mc^2 \int_{\tau_0}^{\tau_1} d\tau_G$$

i.e., that of a free particle moving in a spacetime with a different geometry. For what we have seen, this is a consequence of assuming the equivalence principle.

Field equations

The application to the gravitational case of the Lagrangian formulation described in Sect. 6.3, because of the equivalence principle, required the interpretation of the coupling constant κ in terms of masses by means of the assumption $\kappa \equiv \mu = km_G = Gm$. Limiting once again our reasoning to massive particles, the same assumption should thus be done in the derivation of the field equations, where this time κ is related to the mass density ρ.

Therefore the relativistic counterpart of Eq. (4.4.4), neglecting the free-particle component, can be written

$$S[\Phi] = -\frac{1}{2}k \int_{\Omega_4} \partial_\alpha \Phi \partial^\alpha \Phi \, d^4x - \int_{\Omega_4} \rho \delta^4 (\boldsymbol{x} - \boldsymbol{r})\, \Phi(\boldsymbol{x})\, d^4x, \tag{7.1.6}$$

which immediately gives the field equations

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = -k\, \square^2\, \Phi + \rho = 0,$$

that is,

$$\square^2\, \Phi = \frac{1}{k}\rho. \tag{7.1.7}$$

It is immediate to verify that in the Newtonian limit ($c \to \infty$) this equation reduces to the Poisson equation for $k = (4\pi G)^{-1}$, and recalling the observation made in the previous chapter, the d'Alembert operator is the same as the wave equations of electrodynamics, which means that this theory predicts the existence of gravitational waves that in vacuum propagate at the speed of light.

Once again, as we noted in the previous chapter, the field equations imply that the gravity field propagates with a finite speed, which is an important result because it gives an answer to the action-at-distance problem. Moreover we have seen that this theory already gives an interesting prediction such as that of the gravitational time dilation, and its field equations have the correct Newtonian limit. All these points would encourage us to proceed with the generalization of the treatment with the inclusion of massless particles, however, already at this stage some problems of this approach can be seen, as we show in the following.

7.1.2 Particle Orbit and Perihelion Shift

It is known that Newtonian dynamics cannot account for a $43''$/century excess of perihelion shift observed in the orbit of Mercury. This is a general feature of the dynamics of massive bodies, according to which in the two-body problem the orbit of the test particle is not an ellipse, and actually it is not closed anymore, but it can rather be considered as the superposition of a Newtonian ellipse with an additional apsidal precession, i.e., a slow rotation of the apsidal line around the center of mass.

We want to see if the scalar theory of gravity we have sketched above is able to deduce this feature, which cannot be taken into account in Newtonian gravity.

We have to start from the Lagrangian of the above action for the dynamics of particles, namely $L = \mathrm{d}\tau_{\mathrm{G}}$, which as in the previous chapter can be recast as a function of the four-velocities, i.e.,

$$L = \frac{\sqrt{-\mathrm{d}s_{\mathrm{G}}^2}}{c} = \frac{1}{c}\sqrt{-\left(1+\frac{\Phi}{c^2}\right)^2 \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta} = \frac{1}{c}\sqrt{-\left(1+\frac{\Phi}{c^2}\right)^2 \eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\, \mathrm{d}\tau_{\mathrm{G}}$$

with the condition, coming from Eq. (7.1.5),

$$-c^2 = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \left(1+\frac{\Phi}{c^2}\right)^2 \eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta. \tag{7.1.8}$$

In polar coordinates the Lagrangian reads[3]

$$L = \frac{1}{c}\sqrt{\left(1+\frac{\Phi}{c^2}\right)^2 \left(c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2\right)}\, \mathrm{d}\tau_{\mathrm{G}}.$$

[3]We are using φ as the longitude of the polar coordinates to avoid misunderstandings with the scalar field ϕ.

Now the same reasoning of Exercise 1.3 can be followed for the above equations, showing that the motion on the spatial coordinates has to be planar, so that we can study the case of $\theta(t) = \pi/2$, $\dot{\theta}(t) = 0$ with no loss of generality. We can therefore study the case of

$$L = \frac{1}{c}\sqrt{\left(1+\frac{\Phi}{c^2}\right)^2 \left(c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\varphi}^2\right)}\,\mathrm{d}\tau_{\mathrm{G}} \tag{7.1.9}$$

from which, remembering the condition of Eq. (7.1.8), it is immediately

$$\begin{aligned}\frac{\partial L}{\partial \dot{t}} &= \frac{1}{2c}\left[\left(1+\frac{\Phi}{c^2}\right)^2 \left(c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\varphi}^2\right)\right]^{-1/2} 2c^2\left(1+\frac{\Phi}{c^2}\right)^2 \dot{t} \\ &= \left(1+\frac{\Phi}{c^2}\right)^2 \dot{t},\end{aligned}$$

and similarly

$$\frac{\partial L}{\partial \dot{\varphi}} = -\frac{1}{c^2}\left(1+\frac{\Phi}{c^2}\right)^2 r^2\dot{\varphi}.$$

Because L does not depend on t and φ, it is

$$\frac{\partial L}{\partial \dot{t}} = \text{const}, \qquad \frac{\partial L}{\partial \dot{\varphi}} = \text{const},$$

therefore we can obtain the first two equations of motion by defining two arbitrary constants k and h

$$\left(1+\frac{\Phi}{c^2}\right)^2 \dot{t} = k$$
$$\left(1+\frac{\Phi}{c^2}\right)^2 r^2\dot{\varphi} = h.$$

The second one, in particular, resembles the Kepler's second law $r^2\dot{\varphi} = h$, in which the angular momentum h is rescaled by a factor depending on the gravitational field.

A direct application of the Euler–Lagrange equation for r is unnecessarily complicated; the equation for the radial coordinate, instead, can be more conveniently derived by exploiting the condition of Eq. (7.1.8), which in our case reads

$$\left(1+\frac{\Phi}{c^2}\right)^2 \left(c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\varphi}^2\right) = c^2. \tag{7.1.10}$$

As usual the target is to find this equation using φ as a parameter for r, and to make the substitution $r = 1/u$ so that, by denoting with "′" the derivative with respect to φ, one has

$$\dot{r} = \frac{\mathrm{d}u^{-1}}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau_\mathrm{G}} = -\frac{u'}{u^2}\dot{\varphi}.$$

In summary

$$\left(1+\frac{\Phi}{c^2}\right)^2 c^2\dot{t}^2 = \left(1+\frac{\Phi}{c^2}\right)^{-2} c^2k^2$$
$$\left(1+\frac{\Phi}{c^2}\right)^2 r^2\dot{\varphi}^2 = \left(1+\frac{\Phi}{c^2}\right)^{-2} h^2u^2$$
$$\left(1+\frac{\Phi}{c^2}\right)^2 \dot{r}^2 = \left(1+\frac{\Phi}{c^2}\right)^{-2} h^2\left(u'\right)^2$$

which for the above-mentioned condition gives

$$c^2k^2 - h^2\left(u'\right)^2 - h^2u^2 = c^2\left(1+\frac{\Phi}{c^2}\right)^2.$$

Using the potential of a single body of mass M, $\Phi = -GM/r$, this equation can be written as

$$\left(u'\right)^2 + u^2 = \frac{c^2\left(k^2-1\right)}{h^2} + \frac{2GMu}{h^2} - \frac{GM}{h^2}\left(\frac{GMu^2}{c^2}\right),$$

from which one can easily obtain the equation of motion (parameterized by φ) by taking the derivative with respect to φ and dividing the result by $2u'$, as in Exercise 1.5

$$u'' + u = \frac{GM}{h^2} - \frac{GM}{h^2}\left(\frac{GMu}{c^2}\right). \tag{7.1.11}$$

A straightforward comparison with Eq. (1.4.8) shows that this case differs from the Newtonian one for the additional term $-u\,(GM/hc)^2$, which is small compared to the classical term GM/h^2. We can resort to a perturbative approach to find the solution of the above equation. In particular it can be assumed that $u = u_0 + \epsilon_1 u_1$, where

$$u_0 = \frac{1 - e\cos\varphi}{a\left(1-e^2\right)}$$

is the well-known solution for the Newtonian ellipses (with initial condition $\varphi_0 = 0$) for which

$$u_0'' + u_0 = \frac{GM}{h^2}$$

and $\epsilon_1 u_1$ is a small perturbation. In this case, as shown in Exercise 7.1, the complete solution can be written as

$$u(\varphi) \simeq \frac{1 - e\cos(\varphi + \epsilon\varphi/2)}{a(1 - e^2)} \tag{7.1.12}$$

where $\epsilon = GM/\left(c^2 a\left(1 - e^2\right)\right)$ and after taking into account that $\epsilon \ll 1$.

This formula shows that the actual orbit in this theory is not closed because of the additional term $\epsilon e\varphi/2$; in fact the pericenter is characterized by the condition

$$\varphi\left(1 + \frac{1}{2}\epsilon\right) = 2\pi n,$$

where n is an integer, which means that these angles can be expressed as function of n as

$$\varphi(n) = \frac{2\pi n}{1 + \epsilon/2}.$$

Because $\epsilon > 0$, the above formula means that the angle between two consecutive pericenters is than the Newtonian value 2π, which is qualitatively in disagreement with the known value, actually larger than 2π. More precisely, the shift "defect" between two pericenters is

$$\begin{aligned}\Delta\varphi &= \varphi(n+1) - \varphi(n) - 2\pi \\ &= \frac{2\pi}{1 + \epsilon/2} - 2\pi = -\frac{\pi\epsilon}{1 + \epsilon/2}.\end{aligned}$$

In summary, despite the encouraging features mentioned in the previous section the relativistic theory of gravity based on a scalar field is a blatant failure with respect to the particle's motion. This, however, is only one of the problems of this approach, but others that are explained in the next section help us to settle the equivalence principle in a consistent relativistic framework.

7.2 Relativistic Gravity Sources: The Stress-Energy Tensor and the Equivalence Principle

The tentative expression for the particles' action of Eq. (7.1.6) actually has a serious issue that was intentionally neglected. We know in fact that the Lagrangian has to be a scalar, i.e., an invariant quantity, so if Φ is a scalar field the same should be for the property that couples with it. However, ρ is not an invariant in special relativity, so such action is not viable. One could think to fix this problem simply by using the rest-mass density ρ_0 instead of ρ (after all, in the free particle action we use the rest-mass) but in this case another difficulty advises against such a solution.

The stress-energy tensor trace as gravitational source

Already in Sect. 6.2, indeed, we have stressed that a complete special relativistic dynamics has to be able to treat the dynamics of massless particles. This problem was solved there by observing that energy is a more fundamental quantity than the mass, and resorting to a definition of four-momentum that was proportional to E, instead. Energy, however, once again is not an invariant quantity, but rather a component of a four-vector; moreover, by analogy with mass and mass density, we would need something related to the *energy density* rather than just the energy.

Now, if a four-dimensional scalar is needed but there is none that fits our needs, the only possibility is to build one up by a scalar product of two four-vectors, or by contracting a rank two tensor. Because we want something that contains the energy, the four-momentum is a natural candidate, so we could try with $\eta_{\alpha\beta}p^{\alpha}p^{\beta}$, but in order to avoid the square of the energy a better choice could be $\eta_{\alpha\beta}p^{\alpha}v^{\beta}$. Then we recall that we are seeking a *spatial* density, which means that we have to introduce the usual Kronecker delta, but in four-dimensions, and to integrate over the proper time or one equivalent parameter λ

$$\int \eta_{\alpha\beta}p^{\alpha}\delta^{4}\left(\boldsymbol{x}-\boldsymbol{r}\left(\lambda\right)\right)v^{\beta}\mathrm{d}\lambda, \tag{7.2.1}$$

in which we can easily recognize the stress-energy tensor of Eq. (C.4.1). Indeed, from Eq. (C.4.1), this expression is nothing else than the *trace* of the stress-energy tensor $T \equiv T^{\alpha}{}_{\alpha}$, so that the correct expression for the action of Eq. (7.1.6) becomes

$$S\left[\Phi\right]=-\frac{1}{2}k\int_{\Omega_4}\partial_{\alpha}\Phi\partial^{\alpha}\Phi\ \mathrm{d}^4x-\int_{\Omega_4}T\Phi\left(\boldsymbol{x}\right)\ \mathrm{d}^4x,$$

and its corresponding field equation results in

$$\Box^2\,\Phi=\frac{1}{k}T. \tag{7.2.2}$$

This modification implies an important change of perspective.

First of all, it establishes a mechanism, which is also exploited in general relativity, according to which the source of gravity is not just the mass, but any quantity that can provide a valid stress-energy tensor.

Einstein equivalence principle

Second, this changes the interpretation of the equivalence principle. In order to understand this fact, let us recall that in its weak form it simply posits the equivalence of the inertial and gravitational masses, and in Sect. 3.3.1 we used this assumption to put it in a form which states that the laws of freely falling test bodies are the same in an inertial reference systems under the influence of a gravitational field and in a uniformly accelerated reference system. However, this equivalence, from the way we derived it, strictly depends on the mass of the objects, thus it does not apply to massless objects. This explains why it is said that it holds only for classical mechanics (which applies only to bodies with a non zero inertial mass) and gravitational experiments (non zero gravitational mass).

In classical physics, which ultimately does not contemplate the existence of massless objects, this is not a problem. The concept of light as a massless particle was completely unnecessary to classical physicists, and indeed Soldner in 1801 (Jaki, 1978) could conceive the idea of light bending on the basis of the equivalence principle applied to a light particle, (whatever its value, if it was a particle by definition it had to have a non zero mass and the calculation of the light deflection required only the knowledge of the mass of the attracting body and of the light speed) but when the light was interpreted as a wave this possibility was immediately abandoned, and nobody else thought to apply the equivalence principle to the light until the advent of special relativity.

The possibility of a dynamics (i.e., mechanics) of massless particles, however, obviously raises immediately the issue. As long as only special relativity is concerned, gravity is not involved, and thus neither is the equivalence principle, but what happens when we try to include gravitation in a relativistic context? A natural question arises: how does light "fall" in gravity fields?

Actually, as we have seen in Eq. (7.1.3), the effect of the equivalence principle exhibits in the factorization of the inertial/gravitational mass and in the consequent modification of the line element of special relativity, which is multiplied by a factor containing the gravitational potential.

It has been shown that in the case of massless particles the Action (Eq. (6.3.7)) can be obtained by giving an appropriate parameterization to the null trajectory and introducing a constraint einbein in the Lagrangian whose free particle term contains the part $\eta_{\alpha\beta} e^{-1}(\lambda)\, \dot{x}^{\alpha} \dot{x}^{\beta}$. The trace of the stress-energy tensor in Eq. (7.2.1) contains exactly the same term because, with the same assumptions on the einbein, one can write $p^{\alpha} = e^{-1}(\lambda)\, v^{\alpha}$ thus ending with a Lagrangian

$$L = -\frac{1}{2} e^{-1}(\lambda) \left(1 + \frac{\Phi}{c^2}\right)^2 \eta_{\alpha\beta} v^{\alpha} v^{\beta} + e^{-1}(\lambda)\, m^2 c^2$$

that clearly adopts the same "trick" used for massive particles. In other words we can apply the equivalence principle to massless particles as well, and from the perspective of classical physics, we would say that this principle states the equivalence of the "inertial systems+gravity" and of the uniformly accelerated reference systems also for nongravitational experiments (the propagation of a light beam is nongravitational physics from this point of view). Such an extended version is usually called *einstein* *equivalence principle*.[4] Using this version one can deduce, for example, that the frequency of light has to be red- or blue-shifted, respectively, if the photons “climb” or “fall into” the well of a gravitational potential. But if this principle is applied to the motion of photons by postulating the equivalence of the two types of reference systems, the light bending of Soldner can be easily deduced, as Einstein did already in 1911.

Unfortunately, the null condition $\eta_{\alpha\beta}v^{\alpha}v^{\beta} = 0$ immediately implies that the trace of the stress-energy tensor of Eq. (7.2.1) is zero too, which means that light (massless particles) does not couple with gravity. Not only does this tell that light cannot be a gravity source because of Eq. (7.2.2), but also that the motion of massless particles is unaffected by the gravitational field. Scalar gravity does not admit light deflection, in contrast with the Einstein equivalence principle (which used in another way tells us the opposite) and above all in contrast with the observational evidence as already happened with the problem of the orbital motion.

It is also worth noting that if we consider light as an electromagnetic field, and as explained in Sect. C.4 we compute its stress-energy tensor through the definition valid for the fields, which in this case is the four-potential A^{α}, we consistently come to the identical conclusion that $T = 0$.

Strong equivalence principle

The possibility of defining a stress-energy tensor for the fields gives us the opportunity of making a final observation. As mentioned above, in fact, the mass-energy equivalence forces the admission as a gravity source of any collection of material particles or of fields having $T \neq 0$. The interesting point is that the stress-energy tensor of the gravitational field has a non vanishing trace, thus so we come to the compelling conclusion that gravity couples with itself, and therefore a consistent scalar theory of gravity must be nonlinear with a Lagrangian such as

$$L = -\frac{1}{2}k\partial_{\alpha}\Phi\partial^{\alpha}\Phi + (T_{\mathrm{M}} + T_{\Phi})\,\Phi,$$

where T_{Φ} is the trace of the field stress-energy tensor and T_{M} that of the other components interacting with Φ, and

$$\Box^2\Phi = \frac{1}{k}\,(T_{\mathrm{M}} + T_{\Phi})$$

[4]Actually in the literature there are several definitions of the different versions of this principle, each with its own peculiar interpretation. This is especially valid for the Einstein equivalence principle, which is also called the semi-strong equivalence principle.

therefore it can be said that gravity "self-gravitates". Once again, it can be shown that the corresponding action appeals to the equivalence principle, and therefore we have a third version of this principle which applies to self-gravitating objects as well, called *strong equivalence principle*.

7.3 Gravity Potential as a Nonscalar Relativistic Field

Gravity theory and four-vector potentials

The failure of a model of gravity based on a scalar field cannot necessarily be the end of the story for special relativistic gravitation. For example, we know from Sect. C.3.2 that it is possible to set up electrodynamics in the framework of special relativity using the four-potential A^α, thus gravity as well might be generated by another four-vector potential, rather than a scalar one. This could also seem more reasonable inasmuch as we have just seen that even in scalar gravity we had to resort to T as the gravity source, but only because we needed to couple the field with another scalar, whereas its actual origin was the four-momentum p^α.

With a rank 1 tensor gravity potential Φ^α, instead, it would be natural to write the interaction part as

$$\mathcal{L} \propto p_\alpha \Phi^\alpha,$$

where the proportionality means that we will obviously introduce some appropriate coupling constant. This Lagrangian is based on that for the electromagnetic four-potential A^α with the important difference that it would have to embed the equivalence principle.

Unfortunately this approach cannot work, and for a very simple reason. We know that one fundamental property of electrodynamics is that there exist two different kinds of charges, and that this interaction is repulsive among like charges, and attractive among opposite charges. On the other hand gravity, as far as we know, has only one type of charge, and it is an attractive interaction. Maybe we cannot a priori exclude the existence of two types of charges in gravity as well, but we know for sure that any model must reproduce the irrevocable fact that like gravitational charges attract each other.

This is exactly the point which makes the hypothesis of a four-vector gravitational potential untenable; in fact it can be shown that the repulsion of like charges is not a specific characteristic of electrodynamics, but it is a general property of a four-vector potential, or better of an odd-ranked tensor field. Likewise, the attraction between like charges is a general property of the even-ranked tensor fields, which encompasses the case of a scalar gravity into this common picture. This interesting result had been noted by many authors in the past, but apparently without a rigorous proof, which eventually was given by Jagannathan and Singh (1986). This paper is quite advanced, and the interested reader can find a more pedagogical exposition in Padmanabhan (2010).

Gravity theory for a rank-2 tensor field

In the same reference a somewhat detailed discussion tackles the next, quite obvious, question. If a rank 1 tensor has to be excluded, what about a rank 2 tensor instead? In this case the above considerations ensure that the resulting interaction is attractive, as one should expect, and the interaction Lagrangian could be $\mathcal{L} \propto p_\alpha u_\beta g^{\alpha\beta}$, where we have denoted the potential with $g^{\alpha\beta}$, a symmetric rank 2 tensor. Eventually this attempt fails for several reasons. For example, it does not succeed in producing a consistent stress-energy tensor, namely a symmetric and divergenceless rank 2 tensor. In particular the condition $\partial_\alpha T^{\alpha\beta} = 0$ holds only as an approximation, which in turns implies that also the whole theory can hold as a first approximation. However, it has to be stressed that the best approximation that can be obtained from such approach is susceptible to a geometrical interpretation such as we have seen with its scalar "sister". This is once again a consequence of the equivalence principle, which we further explore in the next section in preparation for general relativity.

7.4 Equivalence Principle and Special Relativity: Accelerated Frames

We started this chapter by stressing the necessity of replacing the Newtonian theory of gravity with another one compatible with the dynamics of special relativity, however, we have seen that all the attempts to include a gravitational field of any sort (scalar, vectorial, tensorial) in a relativistic context eventually have to fail. This basically is because even in the most favorable situations, it does not seem possible to recover some known experimental results, namely the excess of perihelion shift of planetary orbits and the deflection of light.

Actually, in our quest for a relativistic theory of gravity, we faced another interesting point. One of the essential aspects of Newtonian gravitation is the equivalence principle, stating the equivalence between the gravitational and inertial masses. Although one could surely conceive a violation of this principle as a starting point for a gravity theory, the experimental verification of this posit is so accurate that it is much more reasonable to hold it as true unless there exist more compelling reasons demanding its rejection. This is why, while attempting to formulate the theories of this chapter, at a certain point it has always been introduced.

At the same time, however, this brought to what could be called a "geometric formulation" of gravity, in the sense that, because of the equivalence principle, the interaction term of the action can be absorbed by the free particle term, and therefore the equations of motion can be thought of as the motion of a "free particle" in a spacetime with a generic metric tensor $g_{\alpha\beta}$ whose coefficients depend on the potential.[5] The particle can be considered free in the sense that its motion depends only on the gravitational source, or more in detail:

1. It does not depend on its (inertial) mass.
2. I depends only on some kind of geometry because the potential is equivalent to a change of metric, or in less rigorous words on the way "spacetime distances" are defined.

It has to be stressed that this "geometrical view" strictly depends on the application of the equivalence principle, which makes it possible to consider the source of the gravity field as the only property needed to characterize the motion of the test body, and therefore it is specific only to the gravitational field. The motion in the presence of an electromagnetic field, for example, depends on the ratio of the mass and charge of the moving body and cannot be seen as an intrinsic characteristic of the spacetime or of the electromagnetic source, independent of the body itself.

It is useful then to remember the alternative interpretation of the equivalence principle as a "dynamic" covariance principle that was given in Sect. 3.3.1, where it was stressed that, as a consequence, the laws of freely falling test bodies in an inertial reference system under the influence of a gravitational field have to be, at least locally, indistinguishable from those of a uniformly accelerated reference system.

For this reason, in preparation for the general relativistic formulation be given in the next chapter, it is convenient to understand what happens to the Minkowski metric of special relativity when we write it in the coordinates of a reference system that is uniformly accelerating with respect to an inertial one.

To make the calculations simpler, we consider the case in which the acceleration is only in the x direction. In this case one can put $\mathbf{v}_\perp = \mathbf{a}_\perp = 0$ and $\mathbf{v}_\parallel = v_x, \mathbf{a}_\parallel = a_x$ in Eq. (5.5.10), which gives immediately

$$\bar{a}_x = \frac{\left(1 - u^2/c^2\right)^{3/2}}{\left(1 - u v_x/c^2\right)^3} a_x .$$

Now let us assume that we have a particle at rest in the reference system $\bar{S}$, so that instant by instant $\bar{v}_x = 0$, $u = v_x$ and $\bar{S}$ is comoving with this body, which in this system feels a constant acceleration $\bar{a}_x \equiv g$. Then the above formula gives

$$g = \left(1 - \frac{v_x^2}{c^2}\right)^{-3/2} a_x .$$

[5] Actually the same kind of reasoning could be followed in the classical case by noting that the Lagrangian $L = T - V$ in the case of the gravitational potential does not depend on the mass of the moving body.

Because $a_x = \mathrm{d}u_x/\mathrm{d}t$, and taking the initial condition such as $u_x(0) = 0$, we have

$$\int_0^{u_x} \left(1 - \frac{v_x^2}{c^2}\right)^{-3/2} \mathrm{d}u_x = \int_0^t g\,\mathrm{d}t,$$

so

$$\frac{v_x}{\sqrt{1 - v_x^2/c^2}} = gt, \tag{7.4.1}$$

and inverting the formula

$$v_x = \frac{gt}{\sqrt{1 + (gt/c)^2}}. \tag{7.4.2}$$

A second integration with initial condition $x(0) = c^2/g$ results[6] in

$$x = \frac{c^2}{g}\sqrt{1 + \left(\frac{gt}{c}\right)^2}. \tag{7.4.3}$$

At this point we can derive the relation between the proper time τ of the particle and the coordinate time of S from Eq. (6.1.1) (with the convenient initial condition $t(0) = 0$)

$$\tau = \int \sqrt{1 - \frac{v_x^2}{c^2}}\,\mathrm{d}t = \int \frac{\mathrm{d}t}{\sqrt{1 + (gt/c)^2}} = \frac{c}{g}\sinh^{-1}\left(\frac{gt}{c}\right),$$

where the second equality has been obtained by combining Eqs. (7.4.1) and (7.4.2). The last formula can be easily inverted to provide the expression of the coordinate time of S parameterized by the proper time of the accelerated particle

$$t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) \tag{7.4.4}$$

which can be used in Eq. (7.4.3) to get that of the x coordinate as well

$$x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) \tag{7.4.5}$$

which is consistent with the previous initial condition. In fact this formula gives that $x(\tau = 0) = c^2/g$, but $t(\tau = 0) = 0$, which reproduces the initial condition of Eq. (7.4.3), as it should be.

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[6] This and the next apparently strange initial conditions are taken just for a matter of convenience. In this case in fact we obtain the simplest expressions as a function of the proper time, and it also helps us also in the exercises.

Up to now we have dealt with the coordinates of S, and indeed the two last formulae are the equations of motion of the accelerated particle in S when we use its proper time as parameter. Now we need to find a coordinate system adapted to the moving body, i.e., the system of coordinates that an observer comoving with the particle would "see" instead of t, x, y, z. It is worth noting that the Lorentz transformations we have used at the beginning to establish the relation between the two accelerations in S and $\bar{S}$, respectively, namely a and $\bar{a}$, are valid only instantaneously, or in other words they can just give the transformation laws between two systems S and, say, $\bar{S}'(\tau)$, which is moving with a constant velocity equal to the value of v_x at a given proper time τ.

As shown in Padmanabhan (2010), an observer moving along a generic trajectory $x = f(\tau)$, $t = h(\tau)$ parameterized by his or her proper time τ, would assign to a generic event in spacetime a set of coordinates t', x', y', z' where $y' = y$, $z' = z$, and

$$x - ct = f\left(t' - \frac{x'}{c}\right) - ch\left(t' - \frac{\bar{x}}{c}\right)$$
$$x + ct = f\left(t' + \frac{x'}{c}\right) + ch\left(t' + \frac{x'}{c}\right).$$

By applying these formulae to Eqs. (7.4.4) and (7.4.5) we find

$$\begin{aligned} x - ct &= \frac{c^2}{g}\cosh\left[\frac{g}{c}\left(t' - \frac{x'}{c}\right)\right] - \frac{c^2}{g}\sinh\left[\frac{g}{c}\left(t' - \frac{x'}{c}\right)\right] \\ &= \frac{c^2}{g}\mathrm{e}^{-g\left(t'-x'/c\right)/c} \end{aligned} \tag{7.4.6}$$

and

$$\begin{aligned} x + ct &= \frac{c^2}{g}\sinh\left[\frac{g}{c}\left(t' + \frac{x'}{c}\right)\right] + \frac{c^2}{g}\cosh\left[\frac{g}{c}\left(t' + \frac{x'}{c}\right)\right] \\ &= \frac{c^2}{g}\mathrm{e}^{g\left(t'+x'/c\right)/c} \end{aligned} \tag{7.4.7}$$

so one can take their differentials to obtain

$$\mathrm{d}(x - ct)\,\mathrm{d}(x + ct) = -c^2\mathrm{e}^{2gx'/c^2}\,\mathrm{d}\left(t' - \frac{x'}{c}\right)\mathrm{d}\left(t' + \frac{x'}{c}\right),$$

but in general $\mathrm{d}(a - b)\,\mathrm{d}(a + b) = \mathrm{d}a^2 - \mathrm{d}b^2$ and therefore the transformation of the line element reads

$$-c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = c^2\mathrm{e}^{2gx'/c^2}\left(-c^2\mathrm{d}t'^2 + \mathrm{d}x'^2\right) + \mathrm{d}y'^2 + \mathrm{d}z'^2.$$

It would be desirable to separate the spatial and temporal part of the right-hand side term. This can be done by taking a new coordinate system $\bar{t}$, $\bar{x}$, $\bar{y}$, $\bar{z}$ where

$$e^{gx'/c^2} = 1 + \frac{g\bar{x}}{c^2} \tag{7.4.8}$$

and the others remain unchanged.[7] It is easy to see that the above transformation gives $d\bar{x} = e^{gx'/c^2}dx'$, thus obtaining

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2\left(1 + \frac{g\bar{x}}{c^2}\right)^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$$

which tells us that the metric "seen" by the accelerated observer in not Minkowskian anymore.

Moreover, by applying the transformation of Eqs. (7.4.8) to (7.4.6) and (7.4.7) one obtains that the correspondent relations with the barred coordinates are

$$x - ct = \left(\frac{c^2}{g} + \bar{x}\right)e^{-g\bar{t}/c}$$
$$x + ct = \left(\frac{c^2}{g} + \bar{x}\right)e^{g\bar{t}/c}.$$

Summing the above

$$\begin{aligned} x &= \frac{1}{2}\left(\frac{c^2}{g} + x\right)\left[\cosh\left(\frac{g\bar{t}}{c}\right) - \sinh\left(\frac{g\bar{t}}{c}\right) + \cosh\left(\frac{g\bar{t}}{c}\right) + \sinh\left(\frac{g\bar{t}}{c}\right)\right] \\ &= \left(\frac{c^2}{g} + x\right)\cosh\left(\frac{g\bar{t}}{c}\right) \end{aligned} \tag{7.4.9}$$

and subtracting them

$$ct = \left(\frac{c^2}{g} + x\right)\sinh\left(\frac{g\bar{t}}{c}\right) \tag{7.4.10}$$

which are the transformations between the coordinates of S and $\bar{S}$. As expected, the above equations at the lowest order write

[7] It is worth stressing that here we are not changing the reference system, in the sense that we are still in $\bar{S}$ defined as the reference system comoving with the particle. Rather, this is just a change of one coordinate labeling the axis of $\bar{S}$, and precisely one where $\bar{x}$ is shifted by the quantity c^2/g at $x' = 0$.

$$
\begin{aligned}
ct &\simeq \left(\frac{c^2}{g}+\bar{x}\right)\frac{g\bar{t}}{c} = c\bar{t}+\bar{x}\frac{g\bar{t}}{c} \simeq c\bar{t} \\
x &\simeq \left(\frac{c^2}{g}+\bar{x}\right)\left(1+\frac{1}{2}\frac{g^2\bar{t}^2}{c^2}\right) = \frac{c^2}{g}+\bar{x}+\frac{1}{2}g\bar{t}^2+\frac{1}{2}\bar{x}\left(\frac{g\bar{t}}{c}\right)^2 \\
&\simeq \frac{c^2}{g}+\bar{x}+\frac{1}{2}g\bar{t}^2
\end{aligned}
$$

which coincide to the Newtonian limit of an accelerated motion and show that the transformed coordinates, except for the irrelevant shift factor c^2/g of the barred system, are exactly

$$
\bar{x} = x - \frac{1}{2}gt^2,
$$

consistent with our assumption of an observer moving with uniform acceleration.

This analysis shows that an accelerated observer naturally induces a transformation of coordinates which makes the spacetime metric curved. At the same time it is implicit that, whatever the transformation, by applying its inverse to the coordinate system of the accelerated observer we can transform the curved spacetime into a globally flat spacetime, i.e., into that of an inertial observer.

This kind of transformation was explored because the equivalence principle implies that an accelerated reference system can mimic a gravitational field. There is, however, an important limitation to the extent of such equivalence. An accelerated observer, by definition, can be associated only with a uniform acceleration. It is exactly this characteristic that makes it possible to transform between inertial and accelerated observers in a global sense. On the other hand, a gravitational field does not induce strictly uniform accelerations in the motion of test bodies. At the very least, tidal effects are caused precisely by the nonuniformity of the acceleration field. This suggests that it is not possible in general to have a "gravitational spacetime" whose geometry can be globally reduced to the Minkowskian one. This fact represents the bridge between special and general relativity, as we discover in the next chapter.

7.5 Exercises

Exercise 7.1 Solve Eq. (7.1.11) with perturbative methods, showing that a general solution can be written as $u = u_0 + \epsilon u_1 + \mathcal{O}\left(\epsilon_1^2\right)$, where

$$
u_0 = \frac{1 - e\cos\varphi}{a\left(1-e^2\right)}
$$

is the solution of the classical case and

$$u_1 = -\frac{1}{2}\frac{GM}{h^2}e\varphi\sin\varphi = -\frac{1}{2}\frac{e\varphi\sin\varphi}{a\left(1-e^2\right)}.$$

Solution 7.1 Substituting the tentative solution in Eq. (7.1.11) yields

$$u_0'' + u_0 + \epsilon\left(u_1'' + u_1\right) = \frac{GM}{h^2} - \frac{GM}{h^2}\left(\frac{GM}{c^2}\right)(u_0 + \epsilon u_1),$$

but because $GM/c^2 \ll 1$, one can say that

$$\frac{GM}{h^2}\left(\frac{GM}{c^2}\right) \ll \frac{GM}{h^2}.$$

Moreover, h is the angular momentum of the orbit, which is proportional to $\mathbf{r}\times\mathbf{v}$, and this allows us to assume that $GM \ll h^2$ for common orbits, namely for all the bodies not simply falling on another body.

We can thus take

$$\epsilon = \frac{GM}{h^2}\left(\frac{GM}{c^2}\right) = \frac{GM}{c^2 a\left(1-e^2\right)},$$

where the second step comes from Eq. (1.4.9), so that the above formula can be rewritten as

$$u_0'' + u_0 + \epsilon\left(u_1'' + u_1\right) = \frac{GM}{h^2} - \epsilon u_0 + \mathcal{O}(\epsilon).$$

In order to solve this equation one has to find separate solutions for each equal power of ϵ; i.e.,

$$u_0'' + u_0 = \frac{GM}{h^2}$$

and

$$u_1'' + u_1 = -u_0.$$

However, the first one is just the Newtonian equation and we know from Chap. 1 that its solution is

$$u_0 = \frac{GM}{h^2}(1 + e\cos\varphi)$$

which allows us to recast the second differential equation as

$$u_1'' + u_1 = -\frac{GM}{h^2}(1 + e\cos\varphi). \tag{7.5.1}$$

A solution of this equation can be searched in the family of functions

$$u_1 = A + B\varphi \sin\varphi;$$

in fact

$$u_1'' = 2B\cos\varphi - B\varphi\sin\varphi$$

and

$$u_1'' + u_1 = A\left(1 + 2\frac{B}{A}\cos\varphi\right),$$

which coincides with Eq. (7.5.1) for

$$\begin{aligned} A &= -\frac{GM}{h^2} \\ B &= \frac{1}{2}Ae = -\frac{1}{2}\frac{GM}{h^2}e. \end{aligned}$$

The complete solution thus reads

$$\begin{aligned} u &= u_0 + \epsilon u_1 \\ &= \frac{GM}{h^2}\left[1 + e\cos\varphi - \frac{GM}{h^2}\left(\frac{GM}{c^2}\right)\left(1 + \frac{1}{2}e\varphi\cos\varphi\right)\right] \\ &= \frac{GM}{h^2}\left[1 + e\cos\varphi - \epsilon\left(1 + \frac{1}{2}e\varphi\cos\varphi\right)\right] \end{aligned}$$

however, the term ϵ can be neglected in this expression, not only because it is of higher order with respect to $1 + e\cos\varphi$, but also because $\epsilon e\varphi\cos\varphi$ is proportional to φ, which increases indefinitely and therefore gives rise to contributions that build up over time, hence

$$u \simeq \frac{GM}{h^2}\left(1 + e\cos\varphi - \frac{1}{2}\epsilon e\varphi\cos\varphi\right).$$

Finally, it can be easily verified that the last two terms between parentheses are the first-order Taylor expansion around 0 of

$$e\cos\left[\varphi\left(1 + \frac{1}{2}\epsilon\right)\right],$$

which gives

$$u \simeq \frac{GM}{h^2}\left\{1 + e\cos\left[\varphi\left(1 + \frac{1}{2}\epsilon\right)\right]\right\},$$

namely the solution of Eq. (7.1.12) if we consider the Newtonian relation (1.4.9).

Exercise 7.2 A rocket is accelerating with a constant proper acceleration $g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$. Compute when, with respect to an observer in the rocket, it reaches a speed $v = c/2$.

Solution 7.2 This exercise is a simple application of the results of Sect. 7.4 about the motion of an accelerated reference frame. Indeed, the rocket can be considered the reference system $\bar{S}$ accelerated with respect to S. We need to compute the proper time τ in $\bar{S}$ at which the velocity of the rocket in S, $v = \mathrm{d}x/\mathrm{d}t$ reaches half of the speed of light.

Equations (7.4.4) and (7.4.5) give the coordinates t and x in S as function of τ, and their differentiation gives

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = c\tanh\left(\frac{g\tau}{c}\right).$$

Because $-1 < \tanh(x) < 1$, this formula shows that the speed of the rocket in S cannot exceed the speed of light, which will be asymptotically reached after an infinite time in $\bar{S}$. Its inversion, then, yields

$$\tau = \frac{c}{g}\tanh^{-1}\left(\frac{v}{c}\right),$$

and for $v = c/2$

$$\tau \simeq 0.55\frac{c}{g} \simeq 194.67\,\mathrm{days}.$$

Part III
General Relativity and Beyond

"Be careful what you wish for. You might just get it."

Chapter 8
Lagrangian Formulation of General Relativity

In Chap. 6 it has been shown that the formulation of relativistic dynamics naturally fits into the Minkowskian geometry, which is covariant with respect to the Lorentz transformations. Then we also showed that special relativity, as any theory of dynamics should do, can handle any interaction, such as the electromagnetic one, with the exception of gravity inasmuch as it leads to incorrect or inconsistent predictions. Finally, in the last section of the last chapter, we show that the difference between the electromagnetic and gravitational fields comes from the equivalence principle, and when, as required by such principle, one tries to "mimic" the presence of a constant gravitational field with a constant acceleration this leads to the same equations of motion.

These last two facts imply that gravitation, cannot be reduced to a field of some kind on a Minkowskian background geometry obeying the equivalence principle and, on the other side, that it neither can be explained by translating the equivalence principle in the form of an "acceleration-induced" curved spacetime.

This seems to hint that a relativistic theory of gravity has to be found in another way, different from those used for field theories, such as electromagnetism, for which equivalence principle does not apply. A way that, once again, has a geometrical background.

8.1 Gravity and Geometry: Curved Spacetime and General Covariance

A clue on the possible missing link for a relativistically correct description of gravity comes by realizing how the electromagnetic force/field, or in general any force is treated in our context. In both Newtonian and relativistic dynamics, we first state that there exists a class of privileged observers (namely the *inertial observers*) according

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2_8

to which we can describe the motion of test particles in the absence of forces. Then, by definition, the difference between the inertial and the actual motion with respect to an inertial reference system is ascribed to the presence of a force. This procedure succeeds because in general it is not possible to cancel out the effects of a force by a suitable transformation of reference systems. In other words, the inertial motion is a sort of *global* "background motion" that can always be used as a reference.

Gravity is different, however, because the equivalence principle by definition makes it impossible to find such background motion. By incorporating this principle in special relativity we have seen that it inevitably leads to a change of the metric tensor from the Minkowskian $\eta_{\alpha\beta}$ to a more general form $g_{\alpha\beta}$ which depends on the gravitational field.

Now the next step is to observe that:

1. This procedure alone, as shown in the previous chapter is not able to give a description of gravity which is completely consistent and/or which takes into account of the known experimental evidence.
2. The metric change is not the most general one.

This suggests that the correct way to incorporate gravity in a relativistic way might be a different, more general, change of the metric tensor.

As we have already seen between Newtonian and relativistic dynamics, the change of metric tensor describing the invariant line element can be equivalent to a change of geometry. In our case, requiring that when gravity comes into play the invariant line element is of the kind

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{8.1.1}$$

means that this field can be described in a geometrical fashion, but such geometry is that of an intrinsically curved spacetime, whereas that of special relativity is intrinsically flat. Leaving a more rigorous explanation of this statement in Appendix D.2, we can give it a first, intuitive understanding in connection with gravity by using the above-mentioned background motion of special relativity.

Special relativity and flat geometry

In Sect. 5.4.4 we concluded that Minkowski geometry is flat. As explained in more detail in Appendix D, a more formal way to identify a flat space (time) is by verifying that its Riemann curvature tensor is identically zero, which is true in both the Euclidean and Minkowski geometries. In appendix we have also stressed that this property is directly related to Euclid's fifth postulate, namely with the property that parallel straight lines never intersect. Therefore, because both these spaces are intrinsically flat, the difference between Euclidean and Minkowski geometry (not considering the dimensions of the two spaces) all lies in the signature of the metric tensor, i.e., on in the existence of coefficients of opposite signs for the temporal and spatial coefficients. For this reason the latter is also referred to as *pseudo-Euclidean* geometry.[1] In short, one can say that the word "pseudo-" has to do with the norm of the vectors, which is always greater than or equal to zero in Euclidean geometry, whereas it can have any sign in pseudo-Euclidean ones, and the word "Euclidean" refers to the "parallels never intersect" postulate, and we identify this as the fundamental character of the "flatness", so that by definition a geometry is flat if and only if it is Euclidean or pseudo-Euclidean.

The background motion of special relativity is that of free test particles, and it can be considered the materialization of an inertial reference system.[2] From Eq. (6.3.5) we also know that this motion minimizes the length of the four-dimensional trajectory between two events, which agrees with our idea of "straight line," therefore two particles moving freely according to the geometry of special relativity will never meet each other if their four-velocities are parallel at any instant. This gives a physical meaning to our identification of flat space which can help us understand why a geometrical description of gravity cannot stand in terms of a flat geometry.

Gravity and curved geometry

Saying that gravity can be modeled in a geometrical way, in fact, means that the motion of test particles under the influence of a gravity field is that of free particles moving in an appropriate geometry. But these trajectories, namely the geodesics mentioned in Sect. 1.2.2, are the equivalent of straight lines in such geometry. Let us suppose now we have two test particles in a gravity field. They are positioned in different locations at the same distance from the gravitational source, and initially at rest with respect to this source, so that their trajectories are not only "straight lines" but also "parallel" because they have the same initial velocity. Nevertheless, as long as they move toward the source, the will also get closer to each other, and because, in this ideal condition, we can safely suppose to have a point source, the two paths have to meet each other at the source's location. We know that a fundamental property of parallel straight lines in a Euclidean space is that their distance remains constant, thus the geometry equivalent to a gravity field cannot be flat.[3]

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[1] In a more formal way we can say also that in Euclidean geometry the metric is positive definite, whereas in pseudo-Euclidean it is indefinite or, equivalently, the signature of an N-dimensional Euclidean metric is $(N, 0)$, and for the pseudo-Euclidean geometry it is (N_+, N_-), where $N_+ + N_- = N$.

[2] The same mechanism is used, e.g., in astrometry. We can consider an ideal celestial reference system as a grid "attached" to the sky according to which stars and other objects can be given appropriate coordinates. On the other hand, when these coordinates are assigned to celestial objects we can consider their set as a representation, a "materialization" of the reference system.

[3] This way of characterizing a curved in contrast to a flat space(time) is shown in a more mathematically rigorous way in Appendix D.6, with the equation of the geodesic deviation.

General covariance and physical theories

As a final consideration, we have to notice that assuming that the form of the invariant line element is that of Eq. (8.1.1), similarly to what happened with classical physics and special relativity, it has to introduce implicitly a requirement on the covariance group of the theory. This is worth a more in-depth consideration for it has always been a central point of controversy since the very beginning of the formulation of general relativity. We try to get a synthetic grasp on it starting from our discussion about the "covariant interpretation" of the relativity and the equivalence principles of Chap. 3, but the reader can find a lot of material about this long-standing and somewhat complex debate in the literature, e.g., in Misner et al. (1973) or in the dedicated historical review of Norton (1993) and references therein.

In classical physics we identified the existence of a Euclidean line element invariant $\mathrm{d}s^2 = \delta_{ab}\mathrm{d}x^a\mathrm{d}x^b$ (plus a separate Euclidean time invariant) with a principle of covariance, dubbed as "geometric" inasmuch as it relied on the form-invariance of the equations of physics with respect to the transformations of the Euclidean isometry group, namely rotations and translations (and parity transformations). That a valid equation had to be covariant with respect to such transformations derived "by construction" from the fact that daily experience naturally leads us to base the geometrical model of reality on the Euclidean geometry used in classical physics. In no way is this sufficient, because physics needs further fundamental principles for its models, such as the principle of relativity for the dynamics and the principle of equivalence for the gravity theory.[4] Nonetheless, the covariance requirement helps to identify the Euclidean one as the most convenient geometry to use in physics, because we know that once an equation is written in terms of its "objects" it is automatically (or "manifestly") covariant in the classical sense. It is worth while stressing, moreover, that writing an equation is such a fashion can be done in a coordinate independent way, which means that we do not need to write it explicitly in terms of its components, but rather just using symbols representing the whole object (be it a scalar, a vector or a higher rank tensor) in a specific reference system.

A similar picture is displayed eventually in special relativity, where having the $\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$ invariant meant that the Einstein principle of relativity held, instead of the Galilean one, and that the equations of physics had to be covariant with respect to the transformations of the Poincaré group, i.e., rotations, translations, and boosts, which is a "kinematic" covariance in the language of Chap. 3. In geometrical terms, this translated in the identification of the Minkowski one as our preferred geometry. This does not prevent us from continuing to use Euclidean geometry, but physics "likes" to be written with Minkowskian four-dimensional geometrical objects in the same sense as of above, because in this way we can immediately check the Lorentz covariance of an equation, and we can do it in a coordinate independent way.

---

[4] Which nevertheless is in itself not sufficient to "produce" the Newtonian theory of Gravity.

From a physical point of view the principle of relativity, in any form we take it, states the equivalence of all the inertial reference systems, which are related to each other by transformations involving a constant velocity. It is probably natural then to wonder if this can be extended to more general transformations, especially when one realizes that the equivalence principle states, with all the due constraints, the equivalence between gravity and an accelerated reference system.

This is what at some point drove Einstein on the quest for his theory of gravity, believing that as in the previous cases the covariance requirement could help him to determine the correct form of the equations. Actually, one gets almost immediately to a crossroad while exploring this path. The first and simplest extension beyond a constant velocity is a constant and uniform acceleration, which we already realized above to be not enough, because this can transform out gravity only locally, both in a spatial and temporal sense. We are then compelled to uncover a more general transformation, and the most general one can only be an acceleration that is neither uniform nor constant. In other words, what we are seeking out is a kind of completely general covariance, to replace the more special covariance of the Einstein theory of dynamics. This is why, in the end, Einstein turned out to name general relativity his gravity theory, after which it was natural to refer to his dynamics as the *special* relativity. At this point, however, things take an unexpected twist.

Both the "geometric" and "kinematic" covariance led us to a specific kind of preferred geometry to use for the equations of physics, which is the Euclidean and Minkowski one respectively, therefore what kind of preferred geometry is required in the case of "dynamic", or general, covariance? We have seen that apparently the most natural choice is to have an invariant line element in the form $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$, but this just leads to two requirements, namely that:

1. As shown in Appendix D.2 the geometry can be described by a *symmetric metric tensor* $g_{\alpha\beta}$.
2. The general covariance holds for any infinitesimal transformation, which is another direct consequence of the equivalence principle when it states that the possibility of transforming out gravity is valid only locally.

Gravity can thus be described in a geometrical way, but apart from the constraints of possessing a metric and, equivalently to the second statement of above, of reducing to the Minkowskian one when no gravity is involved, there is no "special" or "preferred" geometry at all (Table 8.1) in the same sense intended for classical or special relativistic physics.[5] In addition to this, from the fact that we know in advance that the metric tensor has to depend on the gravity field, it is clear that such geometry is dynamic, in the sense that in general there is no a priori preferred coordinate system to deal with and that it evolves as a true dynamical variable, with the sole constraint of leaving invariant throughout spacetime the quantities linked to the geometry of the spacetime itself. This leads us to the almost inevitable choice of using the differential

[5] Actually, this is not completely true for general relativity, whose geometry is pseudo-Riemannian and torsionless, as explained in Appendix D. Moreover in Chap. 10 it is mentioned how even the route of avoiding the symmetry condition of the metric tensor was attempted.

Table 8.1 Covariance of classical physics, special, and general relativity in comparison

	Covariance	Transformations (covariance group)	Preferred geometry (dimensions)
Classical physics	"Geometric"	Rotations, translations, parity Rotations, translations, parity, Galilean boosts	Euclidean (3+1)
Special relativity	"Kinematic"	Rotations, translations, parity, Lorentz boosts	Minkowski (4)
General relativity	"Dynamical"	Any infinitesimal transformation	Riemann (4)

geometry, and its related absolute differential calculus as the suitable mathematical tool for the formulation of a relativistic theory of gravity.

As customary in this book, we give the mathematical details of this topic in Appendix D, but there is one final "coup de théâtre" in this interesting story. The original idea was that of using the general covariance requirement as a direction to guess the correct form of the equations in the presence of a gravity field, and we already adopted a similar strategy when we used the principle of relativity to switch from the classical to the special relativistic form of the equations of dynamics. However, as first pointed out by Kretschmann (1917), it is always possible to reformulate any equation in a way that is generally covariant, which is reasonable if one thinks that differential geometry provides a mathematical machinery that is intrinsically able to give a coordinate-free description of any geometry (and thus also of flat geometries). The difference with our earlier statement about using Euclidean geometry for special relativistic equations is striking: instead of saying that the "true" equations can also be written using the "non preferred" geometry, we are rather pointing out the opposite, i.e., that any arbitrary "false" equation can be rendered generally covariant! This was shown for the first time by Cartan (1923) regarding Newton's theory of gravity, and the interested reader can refer to Chap. 12 of Misner et al. (1973) for a more modern exposition of the same problem.

With this possibility in hand, it seems that the general covariance requirement loses all of its predictive power and, contrary to what Einstein argued, it is in no way the fundamental axiom of general relativity such as that of relativity was for the theory of special relativity. What remains of this statement is that this language plays for general relativity the same role that Minkowski geometry has for special relativity or the Euclidean one for classical physics: the equations of general relativity take the simplest form when written in a generally covariant way. In other words, the general covariance requirement simply selects the most convenient geometrical stage to write a manifestly covariant version of Einstein's gravity theory.

In addition to this, when developed in a general covariant formalism general relativity is the theory having the simplest form[6] and Einstein argued that this criterion of simplicity alone gave an heuristic predictive power to the general covariance requirement. It has to be said, that such a simplicity requirement is more a philosophical

[6]Newton's theory, for example, looks a lot more complicated.

approach which can be seen as an "heuristic aide". Surely it does not have the same predictive force of a physical principle, and it cannot be used as a full replacement of a well-defined axiomatic principle.

Requiring that the equations of classical physics have to be covariant in a Euclidean sense just "selects" the Euclidean geometry and its "3D tensorial objects" as the best fitted to write such equations, and it establishes an "exclusion principle" according to which if an equation is not covariant then it is not a valid one. However, this does not tell anything else about the correct formulation of Newtonian gravity. In the same way, the general covariance requirement "selects" a pseudo-Riemaniann geometry and its "spacetime tensorial objects" as the stage and scenery tools for relativistic physics, with an analogue exclusion principle, but in no way can it determine the correct form of the field equations of a relativistic gravity theory.

The simplicity requirement is an attractive criterion, but in the end it is the experimental evidence that rules. Indeed, nothing prevents us from writing different and more complex theories of gravity both in classical and relativistic physics. The MOND theory of Milgrom (2014, 2015) and the "cosmological extension" of Newtonian gravity of Sect. 4.2.2 are examples for the former, and we show briefly in Chap. 10 similar examples (even coming from Einstein himself!) for relativistic physics.

I hope that at this point the reader will remain convinced that, as anticipated, the debate around the relation between general relativity and general covariance is quite complex and it is not settled yet. Actually there are authors questioning, and with some reason, even the role of the equivalence principle in the formulation of general relativity (see, e.g., Synge (1960)) but the discussion is well beyond the scope of this book.

8.2 The Geodesic Equations

If, by the equivalence principle, the motion of a test particle in a gravity field can be modeled as that of a free particle in a curved geometry characterized by the metric $g_{\alpha\beta}$, i.e., by the line invariant of Eq. 8.1.1, we can find its equations of motion by means of the usual variational technique, considering an action

$$S^{(p)}_{\text{free}} = \int_{s_0}^{s_1} \sqrt{-\mathrm{d}s^2}, \qquad (8.2.1)$$

which is formally the same as Eq. 6.3.5 used for special relativity. Such formal analogy conceals the decisive fact that the preferred geometry is not the Minkowski one anymore, but a general pseudo-Riemaniann one using more "general" spacetime tensorial objects. Scalars have to be invariant with respect to any transformation and thus the actual form of the (invariant) line element is determined by $g_{\alpha\beta}$ instead of $\eta_{\alpha\beta}$. Only the former, in fact, ensures the essential condition of the variational approach that the action is a scalar in the appropriate geometry.

Another fundamental difference with respect to our usual procedure of "assembling" the action is that we do not need any interaction part as long as only gravity is concerned inasmuch as the latter is automatically absorbed in the free part because of the equivalence principle. The action needs to be complemented with an interaction part when it is necessary to describe the motion in the presence of gravity and any additional field, such as the electromagnetic one, which will introduce an extra "force" term in the geodesic equations.

8.2.1 General Covariant Geodesic Equations

The above form of the action cannot be further elaborated unless an appropriate parameterization of the trajectories is provided. In principle any monotonic function able to map the points of such a trajectory on unique values of the parameter represents a valid alternative, therefore first let us assume that some parameter λ exists with these properties. In this case we can write Eq. (8.2.1) as

$$S^{(\mathrm{p})}_{\mathrm{free}} = \int_{\lambda_1}^{\lambda_2} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda, \tag{8.2.2}$$

and we know that the equations of motion can be obtained by imposing the condition $\delta S^{(\mathrm{p})}_{\mathrm{free}} = 0$ for variations of the path null at the extremal points. As usual, this is equivalent to write the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\rho} - \frac{\partial L}{\partial x^\rho} = 0,$$

where in this case $\dot{x}^\rho \equiv \mathrm{d}x^\rho/\mathrm{d}\lambda$ and $L = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$.

A straightforward calculation gives

$$\frac{\partial L}{\partial x^\rho} = -\frac{1}{2L}\partial_\rho g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}$$

$$\frac{\partial L}{\partial \dot{x}^\rho} = -\frac{1}{2L}g_{\mu\nu}\left(\delta^\mu_\rho\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\delta^\nu_\rho\right) = -\frac{1}{L}g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\rho} &= \frac{1}{L^2}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} - \frac{1}{L}\frac{\mathrm{d}g_{\mu\rho}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} - \frac{1}{L}g_{\mu\rho}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}\\ &= -\frac{1}{L}g_{\mu\rho}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} - \frac{1}{L}\partial_\nu g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} + \frac{1}{L^2}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda},\end{aligned}$$

where in the last equation we applied the chain rule to $\mathrm{d}g_{\mu\rho}/\mathrm{d}\lambda$. The Euler-Lagrange equations then become

$$g_{\mu\rho}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\frac{1}{2}\left(2\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda},$$

and because μ and ν are dummy indexes

$$2\partial_\nu g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=\partial_\nu g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}+\partial_\mu g_{\nu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda},$$

so that, by using the symmetry of the metric tensor, we have

$$g_{\mu\rho}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2}+\frac{1}{2}\left(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda}g_{\mu\rho}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}.$$

Multiplying by $g^{\alpha\rho}$ and remembering that $g^{\alpha\rho}g_{\mu\rho}=\delta^\alpha_\mu$ it finally results in

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}+\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}=\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}, \tag{8.2.3}$$

where

$$\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\rho}\left(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu}\right) \tag{8.2.4}$$

are the connection coefficients of Eq. D.4.12. We stress that the connection coefficients derived in this way, from a Lagrangian that is compatible with the equivalence principle, are not the most general ones, but rather the *metric connection*. This explains what mentioned in Appendix D.4, i.e., that general relativity assumes a torsion-free manifold. Remembering the expression of Eq. (D.4.6) for the covariant derivative and because L is a function of λ, this formula can be written as

$$\dot{x}^\mu\nabla_\mu\dot{x}^\alpha=f\left(\lambda\right)\dot{x}^\alpha \tag{8.2.5}$$

where we put $f\left(\lambda\right)\equiv L^{-1}\mathrm{d}L/\mathrm{d}\lambda$.

Affine parameters and geodesic equations

Equation (8.2.3) is the result for a completely generic parameterization of the trajectory, however it is clear that the proper length of the geodesic s can by definition be a good parameter, and the same is true for its proper time τ, at least for timelike geodesics, defined by the known relation $\mathrm{d}s=-c\mathrm{d}\tau$. Proceeding with these parameters as in the case of λ, however, we get a different result. In the case of s it is

$$-\mathrm{d}s^2 = -g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\mathrm{d}s^2,$$

which means that $g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = -1$, whereas similarly, in the case of τ it results in $g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -c^2$. The constancy of the line element for these parameters means that it is

$$\frac{\mathrm{d}L}{\mathrm{d}s} = \frac{\mathrm{d}L}{\mathrm{d}\tau} = 0, \tag{8.2.6}$$

and by indicating with a dot and a double dot, respectively, the first and second derivative with respect to these parameters, Eq. (8.2.3) becomes

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{8.2.7}$$

Any parameter with the property of giving the geodesic equation in this form is called an *affine parameter*, and it is easy to see that if, say, s is an affine parameter, any parameter related to s by a linear relation $\lambda = as + b$ is an affine parameter as well. This can be proven by taking Eq. (8.2.7) and finding how its expression changes after a generic reparameterization $\lambda = g(s)$.

It is

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}g}{\mathrm{d}s},$$

$$\begin{aligned}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}g}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}g}{\mathrm{d}s} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}^2g}{\mathrm{d}s^2} \\ &= \frac{\mathrm{d}g}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}g}{\mathrm{d}s} + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}^2g}{\mathrm{d}s^2} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}^2g}{\mathrm{d}s^2},\end{aligned} \tag{8.2.8}$$

$$\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \Gamma^\alpha_{\mu\nu}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda},$$

and therefore Eq. (8.2.7) becomes

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}^2g}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$$

which gives

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = -\frac{\mathrm{d}^2g}{\mathrm{d}s^2}\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^{-2}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}. \tag{8.2.9}$$

The above equation shows that the reparameterization has recast the geodesic equation in a form similar to that of Eq. (8.2.5) (the factor at the right-hand side can always be regarded as a function of λ) and, more importantly, this expression reduces to the same form of Eq. (8.2.7) if

$$\frac{d^2 g}{ds^2}\left(\frac{dg}{ds}\right)^{-2} = 0,$$

which naturally implies that g has to be a linear function of the affine parameter s so that, as mentioned above, λ can be an affine parameter only if g is linear in s. Exercise 8.1 shows that this uncertainty can be avoided if we derive the geodesic equations by direct variation of the action.

Indeed, there is good reason to use affine parameters for the geodesics. We in fact started from an action $\sqrt{-ds^2} = \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}$ which is einvariant for reparametrization, in the sense that

$$\sqrt{-ds^2} = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}}ds = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}d\lambda.$$

This is physically meaningful, since one should not expect that the geodesic equation should change for such transformations. On the contrary, the geodesic equations derived from the Euler–Lagrange equations associated with this action, as we have seen, are not invariant for reparameterization unless we use affine parameters.

Another reason is that using an affine parameter makes explicit the concept that the geodesic is the generalization in curved spaces and in any kind of coordinates of the definition of a straight line in flat spaces as the paths along which vectors are kept parallel to themselves inasmuch in this case (see Appendix D.5)

$$\dot{x}^\mu \nabla_\mu \dot{x}^\alpha = 0. \tag{8.2.10}$$

The possibility of using different affine parameters in the parameterization of the geodesic ensures that we can treat properly the gauge freedom connected with massive and massless particles first encountered in Sect. 6.3.1. As in special relativity, in fact, it is not possible to define the proper time for massless particles, but in this case it is always possible to parameterize the null geodesic with another affine parameter that preserves its coordinate-free expression.

Reduction to special relativity

Finally, it is trivial to notice that the geodesic equations automatically comply with the requirement cited in the previous section, i.e., that the laws of relativistic gravity reduce to that of special relativity when no gravity is involved. This is immediate from the definition of the action, because without gravity we recover the Minkowskian one, and it can be easily seen that, as one had to expect, this property is correctly transferred to the geodesic equations.

This is evident from both Eqs. (8.2.7) and (8.2.10) simply by considering the fact that when $g_{\alpha\beta} = \eta_{\alpha\beta}$ the connection coefficients of Eq. (8.2.4) are zero, and the covariant derivative coincides with the partial derivative.

8.2.2 Classical Limit of the Geodesic Equations

An obvious requirement for any equation of relativistic gravity is that, in the slow motion and weak field approximation, they reduce to their classical counterparts. In general this helps to give an interpretation in classical terms of the geometric quantities involved in the equations. We show how it works in the case of the geodesic equations.

Mathematical definition of the classical limit

The two assumptions of slow motion and weak field mathematically translate in the conditions

$$v \ll c \;\Rightarrow\; \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau} \text{ for } i = 1, 2, 3 \tag{8.2.11}$$

and

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \text{ where } h_{\alpha\beta} \ll 1. \tag{8.2.12}$$

Calculation of the approximate geodesic equations

The first one means that, if we choose to parameterize geodesics with the proper time,[7] Eq. (8.2.7) become

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 = 0$$

because $\dot{x}^0\dot{x}^0 \gg \dot{x}^0\dot{x}^i \gg \dot{x}^i\dot{x}^j$ and thus we can neglect the terms with $\mu, \nu = 1, 2, 3$.

Inasmuch the geodesic equations refer to the case of a gravity theory, the classical equation we want to recover is $\boldsymbol{a} = -\nabla\Phi$, where $\Phi(\boldsymbol{x}, t)$ is the gravity field.[8] In the simplest case the goal is the formula $\boldsymbol{F}/m = -GM/r^2$ for the force exerted on the test particle by a fixed mass M, which implies a static potential $\Phi(\boldsymbol{x}) = -GM/r$.

[7] This is always possible in this case because proper time cannot be defined only in the case of photons or zero-rest mass particles, which, however, does not comply with the slow motion requirement.

[8] Remember that the weak equivalence principle holds in Newtonian gravity as well, therefore the equation derives directly from $m\boldsymbol{a} = -\nabla V$, where $V = m\Phi$ and m is the constant mass of the test particle.

We already know from Chap. 7 that the equivalence principle requires that $h_{\alpha\beta}$ is a function of Φ, therefore in the static case the metric cannot depend on the time, thus

$$\begin{aligned}\Gamma^{\alpha}_{00} &= \frac{1}{2}g^{\alpha\rho}\left(\partial_0 g_{0\rho} + \partial_0 g_{\rho 0} - \partial_\rho g_{00}\right)\\ &= -\frac{1}{2}g^{\alpha\rho}\partial_\rho g_{00} = -\frac{1}{2}g^{\alpha i}\partial_i g_{00}\end{aligned}$$

with $i = 1, 2, 3$. Considering Eq. (8.2.12), $\partial_i g_{00} = \partial_i h_{00}$ and keeping only the terms at the first order in $h_{\alpha\beta}$ it is

$$\Gamma^{\alpha}_{00} \simeq -\frac{1}{2}\eta^{\alpha i}\partial_i h_{00}, \tag{8.2.13}$$

therefore it is easy to see that $\Gamma^0_{00} \simeq 0$ because $\eta^{0i} = 0$. The first equation thus results in

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\tau^2} \equiv c\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0, \tag{8.2.14}$$

which implies

$$\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = c\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{const}, \tag{8.2.15}$$

namely that proper and coordinate time coincide, as it should be in the classical limit.

It is also interesting to realize that, because by definition $p^0 = m\dot{x}^0 = E/c$ where E is the energy of the test particle, this is just a statement asserting the conservation of energy.

The spatial components are instead (from $\eta^{\alpha i} = \eta^{ii}$ and then remembering that raising and lowering the spatial indexes with the Minkowski metric implies $\partial^i = \partial_i$)

$$\Gamma^{i}_{00} \simeq -\frac{1}{2}\partial^i h_{00} = -\frac{1}{2}\partial_i h_{00}, \tag{8.2.16}$$

so that for the spatial components it is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} \simeq \frac{1}{2}\partial_i h_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 = \frac{1}{2}c^2\partial_i h_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2.$$

It is now useful to not that Eq. (8.2.15) implies that $t = a\tau + b$, and therefore in this case t is an affine parameter like τ, which means that under the current conditions the geodesic equations are invariant for a $\tau \to t$ reparameterization. For this reason we know that the previous equation does not change if we use the parameter t instead of τ, and it results[9]

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[9]This can be seen also by considering the result of a general reparameterization given by Eq. (8.2.8) with τ, and t replacing s and λ respectively, which gives

$$\frac{d^2x^i}{dt^2} \simeq c^2\frac{1}{2}\partial_i h_{00}\left(\frac{dt}{dt}\right)^2 = c^2\frac{1}{2}\partial_i h_{00}.$$

Comparison with Newtonian equations of motion

Comparing the above equation with $\boldsymbol{a} = -\nabla\Phi$ it is immediate to see that this reduces to the classical one by putting

$$h_{00} = -\frac{2\Phi}{c^2}, \tag{8.2.17}$$

or $h_{00} = 2GM/c^2r$, in this case.

As a final consideration which becomes useful in the next section, Eq. (8.2.17) allows us to give a physical interpretation also to the Newtonian order of the connection coefficients. Equation (8.2.16) in fact shows that

$$\Gamma^i_{00} \simeq \frac{\partial_i\Phi}{c^2}, \tag{8.2.18}$$

suggesting an interesting parallelism between the gravitational force $\boldsymbol{F} = -m\nabla\Phi$ and the temporal components of the connection coefficients.

8.3 Field Equations of General Relativity

The Newtonian limit of the relativistic geodesic equations reveals how the metric can be identified as "a gravitational field in a geometrical disguise." This fact allows us to make some guesses on the action of a relativistic theory of gravity.

The gravitational field in the relativistic action

As already recalled in Sect. 6.3, in the Euclidean case we can define the action in the presence of a generic scalar field ϕ as a functional $S[\mathbf{r}, \phi]$, therefore it appears reasonable to assume that when switching to relativistic gravity the functional will become $S\left[x^\alpha, g_{\alpha\beta}\right]$. This is also consistent with the observation that the gravitational

(Footnote 9 continued)

$$\frac{d^2x^\alpha}{d\tau^2} = \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{dx^\alpha}{dt}\frac{d^2t}{d\tau^2} = \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{d\tau}\right)^2$$

where the rightmost term derives from Eq. (8.2.14).

field has to be represented by an even rank tensor. Although the parallelism between the classical gravitational field and the metric tensor was shown only for the g_{00} component, proposing an action that depends only on specific components is not reasonable because the resulting action cannot be built as a Riemannian scalar in this way. The variational principle tells that the equations of motion of the particle come from the condition $\delta S = 0$ for null variations at the extremal points of the particle's trajectories, whereas the field equations from the same condition for null variations of the field at the spatial and temporal boundaries of the domain of integration. It is then natural to assume that the same holds true now: the geodesic equations result from the variation of S with respect to x^α, and the field equations will be obtained by varying the action with respect to the metric. The field is therefore not represented by a scalar quantity anymore, but rather has a tensorial nature, as already tried in a special relativistic context.

Action for the field equations

The problem of the geodesic equations was already solved in the previous section where we saw that the action resembles that of a free particle, as it is missing the interaction term. This is due to the equivalence principle which, more precisely, allows us to incorporate the interaction term in the free particle one. The field equations, however, are a completely different story, in the sense that, as we tried to highlight at the beginning of this chapter, the Equivalence principle does not allow us to establish a priori a specific form of the action for the gravitational field.

This cannot be surprising, in the sense that it is not a novel situation for a gravity theory. Historically, Newton did not write its formula for the gravity force out of some principle. Rather, it was obtained to explain the astronomical observations of the time, so that Newton's gravity theory had no specific "founding principle" but an experimental justification. This is obviously true for any theory but it implies that, if we want to consider this gravity theory as an axiomatic formal system, the force formula, and therefore its equivalent field equation, in a certain sense are the starting axioms of the theory themselves.

The same holds true for a relativistic theory of gravity. In the context of the variational approach its action has to comply with some a priori constraints (it must be a scalar) and we can appeal to some reasonable assumptions to guess its final form, but in the end this will leave us with many possibilities, out of which one has to choose.

The most important of these reasonable assumptions is that we can expect that the Lagrangian density used to define the action has to comply with some characteristics that we have already met in the case of Newtonian gravity and of special relativity. if we refer to Eq. (6.3.2) as the model from which the action has to be figured out, then $\mathcal{L}$ has to be a (Riemannian) scalar, but this implies that the equivalent of d^4x has to be a scalar as well.

In special relativity d^4x is an invariant, because it does not change under a transformation belonging to the Poincaré group. However, as shown in Exercise 8.2, under a generic transformation $\bar{x}^\alpha = \bar{x}^\alpha\left(x^\beta\right)$ the invariant volume element is instead $\sqrt{-g}\mathrm{d}^4x$, where $g \equiv \det\left(g_{\alpha\beta}\right)$ is the determinant of the metric tensor.[10] We have thus to expect an action having the form

$$S\left[g_{\alpha\beta}\right] = \int_{\Omega_4} \sqrt{-g}\mathcal{L}\left(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \boldsymbol{x}\right) \mathrm{d}^4\boldsymbol{x}.$$

The equivalence principle tells us that the gravity field can be described in a geometrical way, but says nothing about the sources of such a field. This means that, even if the interaction term of S can be incorporated in the free term in the case of the equations of motion as long as only gravity is involved, this cannot be true for the field equations. It will then be possible to split the Lagrangian density in two parts, equivalent to the "free field" and "interaction" to which we are accustomed. The first one, alone, will produce the equivalent of the Laplace equation, i.e., the equations of motion of the gravity field in vacuum, whereas together they will give the equivalent of the Poisson equation. Given the geometrical formulation of the relativistic theory of gravity, it is customary to name these two parts *geometry* and *matter* Lagrangian, respectively. The action will then be something like

$$S = \int_{\Omega_4} \sqrt{-g}\left(\mathcal{L}_\mathrm{G} + \mathcal{L}_\mathrm{M}\right) \mathrm{d}^4\boldsymbol{x}. \tag{8.3.1}$$

Geometric Lagrangian density

It is possible to get some hints on the actual form of $\mathcal{L}_\mathrm{G}$ from the assumption that this Lagrangian alone has to give the equivalent of the Laplace equation. Its counterpart in the classical case in fact, as shown in Sect. 4.4, is the Lagrangian density of the free field, which is quadratic in the (spatial) derivatives of the field. From the above-mentioned correspondence between the metric coefficients and the field we can therefore reasonably ask that $\mathcal{L}_\mathrm{G}$ be at least quadratic in the spacetime derivatives of the metric or, because of Eq. (8.2.4), quadratic with respect to the connection coefficients. In Appendix D we have seen that the latter, however, are not in general a tensorial quantity.

Equations (D.6.4)–(D.6.12) show that the most obvious combinations of the connection coefficients that can produce tensorial quantities are the Riemann and the Ricci tensor, and the Ricci scalar. Moreover we stress once again that the covariance requirement implies that $\mathcal{L}_\mathrm{G}$ has to be a *scalar* in a Riemannian geometry, so either we can try to build a scalar out of the Riemann and the Ricci tensors or, obviously, we

[10] In the sense that we take the determinant of the matrix formed by the components of the metric tensor.

can directly try with a generic function of the Ricci scalar R. The simplest possibility is clearly to put $\mathcal{L}_{\mathrm{G}} = \sqrt{-g}R$ and thus

$$S = \int_{\Omega_4} \sqrt{-g}\,(R + \mathcal{L}_{\mathrm{M}})\ \mathrm{d}^4\boldsymbol{x}. \tag{8.3.2}$$

Matter Lagrangian density

The matter Lagrangian density $\mathcal{L}_{\mathrm{M}}$, as not before, should contain the sources of the gravity field, or better of the metric. For what has been said in Chap. 7, one should expect that this will be related with the stress-energy tensor of the system, but we postpone further comments on this problem to the following section.

8.3.1 Derivation of the Field Equations

The action defined in Eq. (8.3.2), with an additional constant factor $1/2\kappa$ multiplying the Ricci scalar, is called *Einstein–Hilbert action*, where the second name is that of the mathematician David Hilbert, who first introduced this action and worked out the field equations from variational principles. Obviously the constant is just a matter of conventions, and is later adjusted to an appropriate value.

Variation of the Geometry Lagrangian

In this case, instead of using the Euler–Lagrange equations, it is easier to operate directly on the action. Let us first find the variation of the geometric part of the action

$$S_{\mathrm{G}} = \frac{1}{2\kappa}\int_{\Omega_4} R\sqrt{-g}\,\mathrm{d}^4x \tag{8.3.3}$$

with respect to the metric. By making explicit calculations of this variation, one has

$$\begin{aligned}\delta S_{\mathrm{G}} &= \frac{1}{2\kappa}\int_{\Omega_4}\delta\left(\sqrt{-g}R\right)\ \mathrm{d}^4x = \frac{1}{2\kappa}\int_{\Omega_4}\left(R\delta\sqrt{-g} + \sqrt{-g}\delta R\right)\ \mathrm{d}^4x\\ &= \frac{1}{2\kappa}\int_{\Omega_4}\left[R\delta\sqrt{-g} + \sqrt{-g}\delta\left(g^{\alpha\beta}R_{\alpha\beta}\right)\right]\ \mathrm{d}^4x\\ &= \frac{1}{2\kappa}\int_{\Omega_4}\left(R\delta\sqrt{-g} + \sqrt{-g}R_{\alpha\beta}\delta g^{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right)\ \mathrm{d}^4x\end{aligned} \tag{8.3.4}$$

where in the third step we made use of the definition of the Ricci scalar of Eq. (D.6.12).

The first part of the integral argument involves the variation $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$, which is proved in Exercise 8.3. It is more convenient to compute the variation with respect to contravariant components of the metric, inasmuch as the second part is already expressed in this way. For the last part we have, from Exercise 8.4,

$$\int_{\Omega_4} \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta}\, \mathrm{d}^4 x = \int_{\Omega_4} \sqrt{-g} \nabla_\mu V^\mu\, \mathrm{d}^4 x,$$

where V^μ is a four-vector, but because $\sqrt{-g}\nabla_\mu V^\mu$ is the total derivative of V^μ, then from the Gauss theorem we have

$$\int_{\Omega_4} \sqrt{-g} \nabla_\mu V^\mu\, \mathrm{d}^4 x = \int_{\partial\Omega_4} \mathrm{d}V^\mu = 0$$

for variations that are zero at the boundaries of the domain of integration, as required by the variational principle.

Equation (8.3.4) then becomes

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \delta g^{\alpha\beta}\, \mathrm{d}^4 x, \tag{8.3.5}$$

and the requirement that $\delta S_{\mathrm{G}} = 0$ for any $\delta g^{\alpha\beta}$ null at boundaries implies as usual that the argument of the integral has to be zero; i.e.,

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0. \tag{8.3.6}$$

These are called the *Einstein field equations in vacuum* because this result was obtained neglecting the matter terms. Remembering that $g^{\alpha\mu} g_{\alpha\nu} = \delta^\mu_\nu$, which implies $g^{\alpha\beta} g_{\alpha\beta} = g^\alpha_\alpha = 4$, it is immediate to see that by contracting the above equation we have

$$0 = g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = R - 2R = -R, \tag{8.3.7}$$

so in vacuum it is $R = 0$, and Eq. (8.3.6) is equivalent to

$$R_{\alpha\beta} = 0. \tag{8.3.8}$$

Variation of the Matter Lagrangian

Regarding the matter Lagrangian density, one has to remember that, by hypothesis, we have assumed that the total Lagrangian density is a functional of the metric and of its derivatives $\mathcal{L}\left(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \boldsymbol{x}\right)$, so in general we can say that this assumption is valid for $\mathcal{L}_{\mathrm{M}}$ separately. In this case we know from Appendix A that

$$\begin{aligned}\delta S_{\mathrm{M}} &= \int_{\Omega_4} \frac{\delta S_{\mathrm{M}}}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} \sqrt{-g}\, \mathrm{d}^4 x \\ &= \int_{\Omega_4} \delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\left(g_{\alpha\beta}, \partial_\mu g_{\alpha\beta}, \boldsymbol{x}\right)\right) \mathrm{d}^4 x \\ &= \int_{\Omega_4} \left[\partial_\mu \left(\frac{\partial\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\right)}{\partial\left(\partial_\mu g^{\alpha\beta}\right)}\right) - \frac{\partial\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\right)}{\partial g^{\alpha\beta}}\right] \delta g^{\alpha\beta}\, \mathrm{d}^4 x. \qquad (8.3.9)\end{aligned}$$

This is a general result of the functional calculus, and, e.g., we have applied the same reasoning also in Sect. 1.3.1, to derive Eq. (1.3.7).

We have said above that the matter Lagrangian has to be connected with the stress-energy tensor, thus for the moment let us define such a tensor with the relation

$$\frac{\delta S_{\mathrm{M}}}{\delta g^{\alpha\beta}} = \partial_\mu \left(\frac{\partial\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\right)}{\partial\left(\partial_\mu g^{\alpha\beta}\right)}\right) - \frac{\partial\left(\sqrt{-g}\mathcal{L}_{\mathrm{M}}\right)}{\partial g^{\alpha\beta}} \equiv -\frac{1}{2} T_{\alpha\beta}\sqrt{-g}. \qquad (8.3.10)$$

A first intuitive justification for this definition can be given by remembering that in the Hamiltonian formalism of classical physics the Hamiltonian function associated with a physical system $H \equiv \mathbf{p} \cdot \dot{\boldsymbol{q}} - L$ represents the energy of such system,[11] and because it is also $\boldsymbol{p} \equiv \partial L / \partial \dot{\boldsymbol{q}}$ this function can be written

$$E\,(t) \equiv H\,(t) = \dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L.$$

which has a clear "visual" analogy to the above formula when recalling the analogy between fields and coordinates when passing from discrete to continuous degrees of freedom.[12] Another justification is given in a moment, after the derivation of the field equations.

Einstein field equations, Bianchi identities, and gauge conditions

In this way, combining Eqs. (8.3.5), (8.3.9), (8.3.2), and (8.3.10) yields

$$\delta S = \frac{1}{2} \int_{\Omega_4} \sqrt{-g} \left[\frac{1}{\kappa}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right) - T_{\alpha\beta}\right] \delta g^{\alpha\beta}\, \mathrm{d}^4 x,$$

[11]This is immediate to understand in the case of a mechanical system, where from the definition of $L = T - V$ one has $H = T + V$, which is clearly the total energy of the system.

[12]Actually, the classical formula gives the *total* energy, while one should rather think to the energy *density* to have a more straight correspondence and similarity.

and the condition $\delta S = 0$ for any $\delta g^{\alpha\beta}$ implies

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta} \tag{8.3.11}$$

which is the complete *Einstein field equation* and where the tensor $G_{\alpha\beta}$ defined as $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is called the *Einstein tensor*. Such a tensor has an important property that follows directly from the Bianchi identities of Appendix D.6. As shown in Exercise 8.5, in fact, its covariant divergence is zero, namely

$$\nabla_\alpha G^{\alpha\beta} = 0. \tag{8.3.12}$$

This immediately implies that $\nabla_\alpha T^{\alpha\beta} = 0$, but being divergenceless is exactly what should be required for a proper stress-energy tensor, because it is the general covariant version of $\partial_\alpha T^{\alpha\beta} = 0$ that as we have seen in Sect. C.4 holds in special relativity. This also tells us that the conservation laws implied by the zero divergence of the stress-energy tensor are eventually a consequence of the Bianchi identities. Moreover, from the symmetry of $g_{\alpha\beta}$ it follows immediately that $T^{\alpha\beta} = T^{\beta\alpha}$, namely the second fundamental property of a suitable stress-energy tensor.[13] Finally, it can be verified that such a definition leads to the correct expressions of the stress-energy tensor for the known fields.

Equation (8.3.12) can be considered as a set of four conditions on the metric tensor $g_{\alpha\beta}$. We have already met a similar situation in Sect. 6.3.1 when discussing the local gauge invariance of the equations of motion for relativistic particles induced by the special relativistic metric condition. The symmetry of the metric tensor would leave 10 independent components, but the four above conditions play the role of constraints on these coefficients, which implies that the physical model we are dealing with has just six dynamical variables.

These four constraints are thus another consequence of the Bianchi identities, and yield an equal number of degrees of freedom in the choice of coordinates. In other words, one can always find a *gauge transformation*, namely a coordinate transformation (four equations) that changes the "potentials" ($g_{\alpha\beta}$) but has no effect on the "fields" (the curvature tensor). This property, as in electrodynamics, is often exploited to put an arbitrary constraint, called a *gauge condition*, that makes the solution of the field equations easier, as show see in the next chapter.

This Einstein field equation can also be put in an alternative form by contracting its two sides, i.e., taking its trace

$$G^{\alpha}{}_{\alpha} = g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = \kappa g^{\alpha\beta} T_{\alpha\beta},$$

[13] It is worth stressing that this property derives directly from the definition above, contrary to that of the stress-energy pseudotensor of Eq. C.4.9 which requires the additional sum with a rank 3 tensor antisymmetric in the last two indexes.

in fact, remembering that $g^{\alpha\beta}\left(R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R\right)=-R$ from Eq. (8.3.7) and $T^{\alpha}{}_{\alpha}\equiv T$, we have

$$-R=\kappa T$$

so that the einstein field equation can alternatively be written as

$$R_{\alpha\beta}=\kappa\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right). \tag{8.3.13}$$

We are now left with one pending task, namely to find the value of κ.

8.3.2 Classical Limit of Field Equations

As anticipated above, if our working hypotheses are correct, under the weak field and slow limit approximation presented in Sect. 8.2.2, it should be possible to recover the Newtonian relations, namely the Laplace and Poisson equations from the Einstein field equations in vacuum and in the presence of a matter source, respectively. This procedure also provides the appropriate value for the constant κ.

Poisson equation from Einstein field equations

Because we want to recover the Poisson equation we can start from a simple stress-energy tensor involving only the matter density ρ, i.e., the dust case described in Sect. C.4.1 which has the form $T_{00}=\rho c^2$ and its other components are zero. This choice can be further justified by observing that the general stress-energy tensor for a perfect fluid is $T_{\alpha\beta}=\left(\rho+p/c^2\right)u_\alpha u_\beta+pg_{\alpha\beta}$, but in the weak field and slow motion limit it is $p/c^2\ll\rho$ and $g_{\alpha\beta}\sim\eta_{\alpha\beta}$, thus $T_{\alpha\beta}\simeq\rho u_\alpha u_\beta$. Once again, in the slow motion limit $u^i\ll-u^0\simeq c$, so $u_0=g_{0\alpha}u^\alpha=\left(\eta_{0\alpha}+h_{0\alpha}\right)u^\alpha\simeq\eta_{0\alpha}u^\alpha\simeq\eta_{00}u^0\simeq c$ and $u_i\ll u_0$ which implies that the only significant term is $T_{00}=\rho c^2$. This immediately implies that $T=-\rho c^2$.[14] Moreover, it is easier to proceed from Eq. (8.3.13) because we only have to compute the components of the Ricci tensor. In this case the time–time component of the right-hand side of the field equation becomes

$$\kappa\left(T_{00}-\frac{1}{2}g_{00}T\right)=\kappa\left[\rho c^2-\frac{1}{2}\left(\eta_{00}+h_{00}\right)\rho c^2\right].$$

[14]This would have been true in any case, in fact $T=g^{\alpha\beta}T_{\alpha\beta}\simeq\rho g^{\alpha\beta}u_\alpha u_\beta$ because of the slow motion hypothesis. But $g^{\alpha\beta}u_\alpha u_\beta=-c^2$, thus $T\simeq-\rho c^2$.

Neglecting the h_{00} term the above expression becomes

$$\kappa\left(T_{00}-\frac{1}{2}\eta_{00}T\right)\simeq\frac{1}{2}\kappa\rho c^2. \tag{8.3.14}$$

The same component of the left-hand side is instead

$$R_{00}=R^{\alpha}{}_{0\alpha0}=\partial_\alpha\Gamma^{\alpha}{}_{00}-\partial_0\Gamma^{\alpha}{}_{0\alpha}+\Gamma^{\alpha}{}_{\rho0}\Gamma^{\rho}{}_{00}-\Gamma^{\alpha}{}_{\rho0}\Gamma^{\rho}{}_{0\alpha}.$$

However, from Eq. (8.2.4) and the weak field assumption it is

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu}&=\frac{1}{2}\left(\eta^{\alpha\rho}+h^{\alpha\rho}\right)\left(\partial_\mu h_{\nu\rho}+\partial_\nu h_{\rho\mu}-\partial_\rho h_{\mu\nu}\right)\\&\simeq\frac{1}{2}\eta^{\alpha\rho}\left(\partial_\mu h_{\nu\rho}+\partial_\nu h_{\rho\mu}-\partial_\rho h_{\mu\nu}\right),\end{aligned}$$

therefore any connection coefficient is $\sim\partial h$. The products of two different Γs, therefore, are $\sim(\partial h)^2$ thus they can all be neglected in this approximation. which reduces to

$$R_{00}\simeq\partial_\alpha\Gamma^{\alpha}{}_{00}-\partial_0\Gamma^{\alpha}{}_{0\alpha},$$

but we also have to consider that, as observed in Sect. 8.2.2, the requirement of a static potential in the classical limit implies a metric independent of time, therefore $\partial_0\Gamma^{\alpha}{}_{0\alpha}=0$. This condition applies also to the temporal component of the first term, thus leaving us only with the spatial components

$$R_{00}\simeq\partial_i\Gamma^{i}{}_{00}=-\frac{1}{2}\partial_i\left(\eta^{ij}\partial_j h_{00}\right),$$

where the second equality comes from Eq. (8.2.13). Because $\eta^{ij}=\delta^{ij}$ we finally have

$$R_{00}\simeq-\frac{1}{2}\partial_i^2 h_{00}\equiv\nabla^2 h_{00} \tag{8.3.15}$$

and by combining Eqs. (8.3.14) and (8.3.15), and using the approximation of Eq. (8.2.17) we have

$$\nabla^2\Phi\simeq\frac{1}{2}\kappa\rho c^4$$

which reduces to the Poisson equation $\nabla^2\Phi=4\pi G\rho$ if one takes

$$\kappa=8\pi Gc^{-4}. \tag{8.3.16}$$

Moreover, in vacuum the stress-energy tensor is identically zero, therefore from Eqs. (8.3.15) and (8.3.13) it is immediate to see that the Newtonian limit of the Einstein field equations in vacuum is exactly the Laplace equation.

The complete form of the einstein field equations is therefore

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}.$$

8.4 An Alternative Formulation: The Palatini Approach

The derivation of the Einstein field equations of Sect. 8.3.1 makes use of the variation of the Ricci tensor with respect to the metric $g^{\alpha\beta}$. Equation (8.3.4) in fact can be written

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa} \int_{\Omega_4} \left[\sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \delta g^{\alpha\beta} + \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta} \right] \mathrm{d}^4 x \qquad (8.4.1)$$

and to proceed further we have to find the variation $\delta R_{\alpha\beta}$ with respect to $g^{\alpha\beta}$. This variation vanished at the boundaries because we assumed the metric connection of Eq. (D.6.12). Indeed, the chain of reasoning starts from the assumption that the spacetime can be represented as a torsion-free manifold. This allows us to derive the metric connection, which in its turn implies that $\nabla_\mu g_{\alpha\beta} = \nabla_\mu g^{\alpha\beta} = 0$, as we have seen in Sect. D.4. Finally, for this reason we have that $g^{\alpha\beta} \delta R_{\alpha\beta}$ is the total derivative of a four-vector and thus vanishes on the boundary of the domain of integration.

In principle, however, this assumption might be relaxed, with the consequence that the metric connection does not hold. In this case the Ricci tensor is defined by Eq. (D.6.10) just in terms of the connection coefficients, and the action results in a functional of *two independent fields*, namely the metric and the connection coefficients. Having assumed the existence of two independent fields implies that there are two field equations. The first one is the usual equation for the metric, and the second one will be that for the connection coefficients, obtained by varying the action with respect to $\Gamma^\mu{}_{\alpha\beta}$. This method of deriving of the field equations is called the *Palatini approach*, after Attilio Palatini, an Italian mathematician who first introduced this method in 1919, which was then put in its present form by Einstein in 1925 (Ferraris et al. 1982).

The field equations for the metric are trivial, in fact in this case the Ricci tensor depends on the connection coefficients and on its partial derivatives, but because we are considering the metric and $\Gamma^\mu{}_{\alpha\beta}$ as two independent fields, this time we are not authorized to assume that $R_{\alpha\beta}$ depends on the metric. Therefore its variation with respect to $g^{\alpha\beta}$ is identically zero and Eq. (8.4.1) directly becomes

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \delta g^{\alpha\beta} \, \mathrm{d}^4 x,$$

which immediately gives the Einstein field equations in vacuum

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0. \tag{8.4.2}$$

Regarding the connection coefficients, we have from Eqs. (8.4.1) and (8.4.2),

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta} \, \mathrm{d}^4 x.$$

Here we can use Eq. (8.5.8) from Exercise 8.4, which was derived regardless of the relation between the metric and the connection. With this substitution the above equation becomes

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} g^{\alpha\beta} \left(\nabla_\mu \delta \Gamma^\mu{}_{\alpha\beta} - \nabla_\beta \delta \Gamma^\mu{}_{\alpha\mu} \right) \mathrm{d}^4 x,$$

which gives the variation with respect to $\Gamma^\mu{}_{\alpha\beta}$. In the standard derivation we could exploit the fact that the covariant derivative of the metric vanishes, but this property strictly depends on the validity of the metric connection, as shown in Appendix D.4, which cannot be assumed here if the two fields are taken as independent of each other. We can write instead

$$\begin{aligned}\delta S_{\mathrm{G}} =& \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \nabla_\beta \left(g^{\alpha\mu} \delta \Gamma^\beta{}_{\alpha\mu} - g^{\alpha\beta} \delta \Gamma^\mu{}_{\alpha\mu} \right) \mathrm{d}^4 x \\ &- \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left[\nabla_\beta \left(g^{\alpha\mu} \right) \delta \Gamma^\beta{}_{\alpha\mu} - \nabla_\beta \left(g^{\alpha\beta} \right) \delta \Gamma^\mu{}_{\alpha\mu} \right] \mathrm{d}^4 x.\end{aligned}$$

Now, as in the standard approach, the first integral vanishes if consistently with the variational hypothesis we make the requirement of null variations of the connection coefficients on the boundary; thus so we are left with the second integral, which can be rearranged as

$$\begin{aligned}\delta S_{\mathrm{G}} =& \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left[\nabla_\beta \left(g^{\alpha\beta} \right) \delta \Gamma^\mu{}_{\alpha\mu} - \nabla_\beta \left(g^{\alpha\mu} \right) \delta \Gamma^\beta{}_{\alpha\mu} \right] \mathrm{d}^4 x \\ =& \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left[\nabla_\nu \left(g^{\alpha\nu} \right) \delta^\mu_\beta \delta \Gamma^\beta{}_{\alpha\mu} - \nabla_\beta \left(g^{\alpha\mu} \right) \delta \Gamma^\beta{}_{\alpha\mu} \right] \mathrm{d}^4 x \\ =& \frac{1}{2\kappa} \int_{\Omega_4} \sqrt{-g} \left[\nabla_\nu \left(g^{\alpha\nu} \right) \delta^\mu_\beta - \nabla_\beta \left(g^{\alpha\mu} \right) \right] \delta \Gamma^\beta{}_{\alpha\mu} \, \mathrm{d}^4 x. \end{aligned} \tag{8.4.3}$$

It has to be observed that $\delta \Gamma^\beta{}_{\alpha\mu}$ is a tensor that is symmetric in the two lower indexes because we are assuming a torsionless geometry. On the other hand, we know that $\nabla_\nu \left(g^{\alpha\nu} \right) \delta^\mu_\beta - \nabla_\beta \left(g^{\alpha\mu} \right)$ is a tensor as well, but we do not know anything about its symmetry properties. Actually we know that $\nabla_\beta \left(g^{\alpha\mu} \right)$ is symmetric in the same indexes α and μ, and it is always possible (see Exercise 8.6) to break down $\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta$ into

$$\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta = \frac{1}{2}\left[\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta + \nabla_\nu\left(g^{\mu\nu}\right)\delta^\alpha_\beta\right] + \frac{1}{2}\left[\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta - \nabla_\nu\left(g^{\mu\nu}\right)\delta^\alpha_\beta\right],$$

where the first part is symmetric and the second antisymmetric in α and μ. Therefore Eq. (8.4.3) can be written as

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa}\int_{\Omega_4}\sqrt{-g}\left[S_\beta{}^{\alpha\mu} + A_\beta{}^{\alpha\mu}\right]\delta\Gamma^\beta{}_{\alpha\mu}\,\mathrm{d}^4x,$$

where

$$S_\beta{}^{\alpha\mu} = \frac{1}{2}\left[\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta + \nabla_\nu\left(g^{\mu\nu}\right)\delta^\alpha_\beta\right] - \nabla_\beta\left(g^{\alpha\mu}\right)$$

is symmetric in α and μ, and

$$A_\beta{}^{\alpha\mu} = \frac{1}{2}\left[\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta - \nabla_\nu\left(g^{\mu\nu}\right)\delta^\alpha_\beta\right]$$

is antisymmetric. Because, as observed above, $\delta\Gamma^\beta{}_{\alpha\mu}$ is symmetric on the same indexes, then (see Exercise 8.7) $A_\beta{}^{\alpha\mu}\delta\Gamma^\beta{}_{\alpha\mu} = 0$ and the variation of the action with respect to the connection coefficients becomes

$$\delta S_{\mathrm{G}} = \frac{1}{2\kappa}\int_{\Omega_4}\sqrt{-g}\left(S_\beta{}^{\alpha\mu}\right)\delta\Gamma^\beta{}_{\alpha\mu}\,\mathrm{d}^4x,$$

which means that the requirement $\delta S_{\mathrm{G}} = 0$ for variations of the connection coefficients vanishing on the boundary gives

$$\frac{1}{2}\left[\nabla_\nu\left(g^{\alpha\nu}\right)\delta^\mu_\beta + \nabla_\nu\left(g^{\mu\nu}\right)\delta^\alpha_\beta\right] - \nabla_\beta\left(g^{\alpha\mu}\right) = 0.$$

The last equation implies that the covariant derivative of $g^{\alpha\beta}$ vanishes, and therefore that it is also $\nabla_\mu g_{\alpha\beta} = 0$, but from Eq. (D.4.8)

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma^\nu{}_{\alpha\mu}g_{\nu\beta} - \Gamma^\nu{}_{\mu\beta}g_{\alpha\nu},$$

so

$$\partial_\mu g_{\alpha\beta} = \Gamma^\nu{}_{\alpha\mu}g_{\nu\beta} + \Gamma^\nu{}_{\mu\beta}g_{\alpha\nu}.$$

This equation is identical to Eq. (D.4.9), except for two negligible swappings for $\mu\beta$ and $\alpha\nu$ in the second term, therefore the same permutations done in Appendix D.4 leads to the equivalent of Eqs. (D.4.10) and (D.4.11), and eventually to

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\beta\alpha} \right).$$

Therefore in the end we have shown that the "field equations" for the connection exactly imply the metric connection. The Palatini approach, in the case of general relativity, is completely equivalent to the standard approach. It is worth noting that this strictly depends on the assumption of the validity of the Einstein–Hilbert action, which cannot be necessarily true in general. As we show in Sect. 10.3.1, in fact, for more general actions the two approaches can bring out different types of connections.

8.5 Exercises

Exercise 8.1 Derive the geodesic equation by direct variation of the action defined in Eq. (8.2.1).

Solution 8.1 One can use the relation $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ to deduce

$$\begin{aligned}
\delta S^{(\mathrm{p})}_{\mathrm{free}} &= -\int_{s_1}^{s_2} \frac{1}{2\sqrt{-\mathrm{d}s^2}} \delta \left(g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \right) \\
&= -\int_{s_1}^{s_2} \frac{1}{2\sqrt{-\mathrm{d}s^2}} \left(\partial_\gamma g_{\alpha\beta} \delta x^\gamma \, \mathrm{d}x^\alpha \mathrm{d}x^\beta + 2 g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}\delta x^\beta \right) \\
&= -\int_{s_1}^{s_2} \left(\frac{1}{2} \partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \delta x^\gamma + g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}\delta x^\beta}{\mathrm{d}s} \right) \mathrm{d}s.
\end{aligned}$$

As in the normal procedure of Appendix A.2, the second term of the above integral can be integrated by parts, thus obtaining

$$\begin{aligned}
\delta S^{(\mathrm{p})}_{\mathrm{free}} &= \int_{s_1}^{s_2} \left[\frac{1}{2} \partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \delta x^\gamma - \frac{\mathrm{d}}{\mathrm{d}s} \left(g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \right) \delta x^\beta \right] \mathrm{d}s + \left[g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \delta x^\beta \right]_{s_1}^{s_2} \\
&= \int_{s_1}^{s_2} \left[\frac{1}{2} \partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} - \frac{\mathrm{d}}{\mathrm{d}s} \left(g_{\alpha\gamma} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \right) \right] \delta x^\gamma \mathrm{d}s + \left[g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \delta x^\beta \right]_{s_1}^{s_2}
\end{aligned}$$

and after the usual requirement that $\delta S^{(\mathrm{p})}_{\mathrm{free}} = 0$ for null variations at the extremal points one gets

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(g_{\alpha\gamma} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \right) - \frac{1}{2} \partial_\gamma g_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0.$$

By expanding the derivative of the first term and switching the dummy indexes α and β in the second term the above equation can be written as

$$
\begin{aligned}
0 &= g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \frac{\mathrm{d}g_{\alpha\gamma}}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} - \frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} \\
&= g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \partial_\beta g_{\alpha\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} - \frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} \\
&= g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \frac{1}{2}\partial_\beta g_{\alpha\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} + \frac{1}{2}\partial_\alpha g_{\beta\gamma}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} - \frac{1}{2}\partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} \\
&= g_{\alpha\gamma}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\beta\gamma} - \partial_\gamma g_{\alpha\beta}\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}.
\end{aligned}
$$

Finally, contraction of this expression with $g^{\mu\gamma}$ yields

$$
\begin{aligned}
0 &= \delta^\mu_\alpha\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} \\
&= \frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s},
\end{aligned}
$$

where in the last step $\partial_\beta g_{\alpha\gamma} = \partial_\beta g_{\gamma\alpha}$ from the symmetry of the metric tensor.

Exercise 8.2 Show that $\sqrt{-\bar{g}}\mathrm{d}^4\bar{x} = \sqrt{-g}\mathrm{d}^4x$, where $g = \det\left(g_{\alpha\beta}\right)$.

Solution 8.2 It is well known that under a generic coordinate transformation $\bar{x}^\alpha = \bar{x}^\alpha\left(x^\beta\right)$ the volume element changes according to the formula $\mathrm{d}^4\bar{x} = \det(J)\,\mathrm{d}^4x$, where $\det(J)$ is the determinant of the Jacobian matrix $J = \left(\partial\bar{x}^\alpha/\partial x^\beta\right)$. Under the same transformation the metric tensor, by definition, transforms as

$$
\bar{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial\bar{x}^\alpha}\frac{\partial x^\nu}{\partial\bar{x}^\beta}g_{\mu\nu}.
$$

The last equation can be interpreted as a matrix product, and because, by the rules of determinants, if $A = BC$ then $\det(A) = \det(B)\det(C)$, it is

$$
\bar{g} = \frac{g}{[\det(J)]^2}. \tag{8.5.1}
$$

With our conventions on the metric it is $g < 0$, which means that $\det(J) = \sqrt{-g}/\sqrt{-\bar{g}}$, and therefore

$$
\mathrm{d}^4\bar{x} = \det(J)\,\mathrm{d}^4x = \frac{\sqrt{-g}}{\sqrt{-\bar{g}}}\mathrm{d}^4x,
$$

which gives the required expression. The sign before the determinant of the metric can also be understood in another way, which exploits the fact that, if this is a covariant expression, this has to be valid in any coordinate system. In this sense there is no loss of generality in starting from the Minkowskian metric, so that in the above computation we put $g_{\alpha\beta} = \eta_{\alpha\beta}$. In this case then $g = \det\left(\eta_{\alpha\beta}\right) = -1$, and from Eq. (8.5.1) we have

$$\det(J) = \sqrt{-1/\bar{g}} = 1/\sqrt{-\bar{g}}.$$

This means that in the case of the volume element in the Minkowski case we have $\sqrt{-\bar{g}}\mathrm{d}^4\bar{x} = \mathrm{d}^4 x$, but once again, this is a covariant expression and the same has to be true for any transformation of coordinates and any transformed metric coefficient, say $\tilde{g}^{\alpha\beta}$, so that $\sqrt{-\bar{g}}\mathrm{d}^4\bar{x} = \sqrt{-\tilde{g}}\mathrm{d}^4\tilde{x}$.

Exercise 8.3 Compute δg and then show that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}.$$

Solution 8.3 For this quantity we need to recall two fundamental formulas for matrices. The first one is the expression of the element i, j of A^{-1}

$$\left(A^{-1}\right)_{ij} = \frac{(-1)^{i+j}\, M_{ji}}{\det A}. \tag{8.5.2}$$

The second is the classical expansion used in the calculation of the determinant of a matrix A

$$\det A = a_{ij}\, (-1)^{i+j}\, M^i_j \tag{8.5.3}$$

where a_{ij} are the elements of A, and M_{ij} is the corresponding minor,[15] i.e., the determinant of the sub matrix obtained by dropping the ith row and jth column.

If we apply these formulas to the case of $A = g_{\alpha\beta}$, and therefore $A^{-1} = g^{\alpha\beta}$ and $\det A = g$, Eq. (8.5.2) becomes

$$g^{\alpha\beta} = \frac{(-1)^{\alpha+\beta}\, M_{\beta\alpha}}{g} \quad \Rightarrow \quad (-1)^{\alpha+\beta}\, M_{\beta\alpha} = g g^{\alpha\beta},$$

and because $g_{\alpha\beta} = g_{\beta\alpha}$, also

$$(-1)^{\alpha+\beta}\, M_{\beta\alpha} = (-1)^{\alpha+\beta}\, M_{\alpha\beta}.$$

Equation (8.5.3) instead corresponds to[16]

$$g = g_{\alpha\beta}\, (-1)^{\alpha+\beta}\, M^\alpha_\beta,$$

[15] In the formula it is written as M^i_j to use the index summation convention.

[16] Once again, we wrote the minor in such a way that the Einstein summation convention can be applied, but raising and lowering indices does not change $M_{\alpha\beta}$.

and therefore

$$\begin{aligned}\delta g &= \delta g_{\alpha\beta}\,(-1)^{\alpha+\beta}\,M^{\alpha}_{\beta}\\ &= g g^{\alpha\beta}\delta g_{\alpha\beta}.\end{aligned} \tag{8.5.4}$$

On the other hand, applying the same formulae to $A = g^{\alpha\beta}$, we have[17] $\det(A) = 1/g$, thus

$$(-1)^{\alpha+\beta}\,M_{\alpha\beta} = g_{\alpha\beta}/g$$

and

$$1/g = g^{\alpha\beta}\,(-1)^{\alpha+\beta}\,M^{\beta}_{\alpha},$$

therefore

$$-\frac{\delta g}{g^2} = \frac{g_{\alpha\beta}\delta g^{\alpha\beta}}{g},$$

and finally

$$\delta g = -g g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{8.5.5}$$

Equations (8.5.4) and (8.5.5) imply that

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}} g\, g^{\alpha\beta}\,\delta g_{\alpha\beta}\\ &= \frac{1}{2}\sqrt{-g}\,g^{\alpha\beta}\,\delta g_{\alpha\beta}\\ &= -\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}.\end{aligned} \tag{8.5.6}$$

Exercise 8.4 Compute $\delta R_{\alpha\beta}$ and use the result to show that

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_{\beta} V^{\beta},$$

where V^{β} is a four-vector.

Solution 8.4 Let us first take the variation of the Riemann curvature tensor of Eq. (D.6.4). Remembering that the partial derivative and the variation commute we have:

$$\begin{aligned}\delta R^{\mu}{}_{\alpha\nu\beta} = \partial_{\nu}\delta\Gamma^{\mu}{}_{\alpha\beta} - \partial_{\beta}\delta\Gamma^{\mu}{}_{\alpha\nu}\\ + \delta\Gamma^{\mu}{}_{\rho\nu}\Gamma^{\rho}{}_{\alpha\beta} - \delta\Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\alpha\nu}\\ + \Gamma^{\mu}{}_{\rho\nu}\delta\Gamma^{\rho}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\rho\beta}\delta\Gamma^{\rho}{}_{\alpha\nu}.\end{aligned}$$

[17] Because $\det\left(A^{-1}\right) = 1/\det(A)$.

Because the difference of two connection coefficients behaves as a tensor, it is possible to take the covariant derivative of $\delta\Gamma^{\mu}{}_{\alpha\beta}$, whose result is another tensor

$$\nabla_{\nu}\delta\Gamma^{\mu}{}_{\alpha\beta} = \partial_{\nu}\delta\Gamma^{\mu}{}_{\alpha\beta} + \Gamma^{\mu}{}_{\rho\nu}\delta\Gamma^{\rho}{}_{\alpha\beta} - \delta\Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \delta\Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\alpha\nu}.$$

Similarly

$$\nabla_{\beta}\delta\Gamma^{\mu}{}_{\alpha\nu} = \partial_{\beta}\delta\Gamma^{\mu}{}_{\alpha\nu} + \Gamma^{\mu}{}_{\rho\beta}\delta\Gamma^{\rho}{}_{\alpha\nu} - \delta\Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\beta\nu} - \delta\Gamma^{\mu}{}_{\rho\nu}\Gamma^{\rho}{}_{\alpha\beta},$$

but remembering that the connection coefficients are symmetric in the covariant indexes, it results in

$$\begin{aligned}\nabla_{\nu}\delta\Gamma^{\mu}{}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\mu}{}_{\alpha\nu} &= \partial_{\nu}\delta\Gamma^{\mu}{}_{\alpha\beta} - \partial_{\beta}\delta\Gamma^{\mu}{}_{\alpha\nu} \\ &\quad + \Gamma^{\mu}{}_{\rho\nu}\delta\Gamma^{\rho}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\rho\beta}\delta\Gamma^{\rho}{}_{\alpha\nu} \\ &\quad + \delta\Gamma^{\mu}{}_{\rho\nu}\Gamma^{\rho}{}_{\alpha\beta} - \delta\Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\alpha\nu} \\ &= \delta R^{\mu}{}_{\alpha\nu\beta}.\end{aligned} \tag{8.5.7}$$

By contracting the variation of the Riemann curvature tensor, one obtains that of the Ricci tensor as

$$\delta R_{\alpha\beta} = \delta R^{\mu}{}_{\alpha\mu\beta} = \nabla_{\mu}\delta\Gamma^{\mu}{}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\mu}{}_{\alpha\mu}, \tag{8.5.8}$$

and therefore, by remembering from Eq. (D.4.13) that the covariant derivative of the metric is zero and relabeling dummy indexes to factor in the metric tensor,

$$\begin{aligned}g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta}\left(\nabla_{\mu}\delta\Gamma^{\mu}{}_{\alpha\beta} - \nabla_{\beta}\delta\Gamma^{\mu}{}_{\alpha\mu}\right) \\ &= \nabla_{\beta}\left(g^{\alpha\mu}\delta\Gamma^{\beta}{}_{\alpha\mu} - g^{\alpha\beta}\delta\Gamma^{\mu}{}_{\alpha\mu}\right).\end{aligned} \tag{8.5.9}$$

The right-hand side is thus the covariant derivative of a (rank 1) tensor because differences of connection coefficients such as $\delta\Gamma^{\mu}{}_{\alpha\beta}$ are tensors.

Exercise 8.5 Show that the Einstein tensor is divergenceless, i.e., that

$$\nabla_{\alpha}G^{\alpha\beta} = 0.$$

Solution 8.5 First one has to contract the Bianchi identity (D.6.9) with the product of three metric tensors:

$$g^{\mu\epsilon}g_{\alpha}{}^{\gamma}g^{\beta\delta}\left(\nabla_{\epsilon}R^{\alpha}{}_{\beta\gamma\delta} + \nabla_{\delta}R^{\alpha}{}_{\epsilon\beta\gamma} + \nabla_{\gamma}R^{\alpha}{}_{\beta\delta\epsilon}\right) = 0.$$

Because $\nabla_{\gamma}g^{\alpha\beta} = 0$ the metric tensor can raise and lower indexes of the arguments of covariant derivatives, which gives

$$\nabla_\epsilon \left(g^{\mu\epsilon} g_\alpha{}^\gamma g^{\beta\delta} R^\alpha{}_{\beta\gamma\delta}\right) +$$
$$\nabla_\delta \left(g^{\mu\epsilon} g_\alpha{}^\gamma g^{\beta\delta} R^\alpha{}_{\epsilon\beta\gamma}\right) +$$
$$\nabla_\gamma \left(g^{\mu\epsilon} g_\alpha{}^\gamma g^{\beta\delta} R^\alpha{}_{\beta\delta\epsilon}\right) = 0,$$

and therefore

$$\nabla_\epsilon \left(g^{\mu\epsilon} g_\alpha{}^\gamma g^{\beta\delta} R^\alpha{}_{\beta\gamma\delta}\right) +$$
$$\nabla_\delta \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\epsilon\beta\gamma}\right) +$$
$$\nabla_\gamma \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\delta\epsilon}\right) = 0. \tag{8.5.10}$$

From the definitions of the Ricci tensor and scalar, the first term of this equation becomes

$$\begin{aligned}\nabla_\epsilon \left(g^{\mu\epsilon} g_\alpha{}^\gamma g^{\beta\delta} R^\alpha{}_{\beta\gamma\delta}\right) &= \nabla_\epsilon \left(g^{\mu\epsilon} g^{\beta\delta} R^\alpha{}_{\beta\alpha\delta}\right)\\ &= \nabla_\epsilon \left(g^{\mu\epsilon} g^{\beta\delta} R_{\beta\delta}\right)\\ &= \nabla_\epsilon \left(g^{\mu\epsilon} R\right) = \nabla_\epsilon \left(g^{\epsilon\mu} R\right)\end{aligned}$$

where in the last step we used the symmetry of the metric.

We can now use the symmetry properties of the Riemann tensor to do some "index gymnastics" on the second and third terms. Equations (D.6.7) and (D.6.5) give

$$\nabla_\delta \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\epsilon\beta\gamma}\right) = -\nabla_\delta \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\gamma\beta\alpha\epsilon}\right),$$

and similarly, but now using the antisymmetry on the second pair of indexes,

$$\nabla_\gamma \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\delta\epsilon}\right) = -\nabla_\gamma \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\delta\epsilon\beta\alpha}\right).$$

These two expressions can be made functions of the Ricci tensor by contracting with $g^{\alpha\gamma}$ and $g^{\beta\delta}$, respectively, and the remaining metric tensors are used to get the raised-indexes version. The first one becomes

$$\begin{aligned}-\nabla_\delta \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\gamma\beta\alpha\epsilon}\right) &= -\nabla_\delta \left(g^{\mu\epsilon} g^{\beta\delta} R_{\beta\epsilon}\right)\\ &= -\nabla_\delta R^{\delta\mu},\end{aligned}$$

and the second term gives

$$\begin{aligned}-\nabla_\gamma \left(g^{\mu\epsilon} g^{\alpha\gamma} g^{\beta\delta} R_{\delta\epsilon\beta\alpha}\right) &= -\nabla_\gamma \left(g^{\mu\epsilon} g^{\alpha\gamma} R_{\epsilon\alpha}\right)\\ &= -\nabla_\gamma \left(R^{\mu\gamma}\right) = -\nabla_\gamma \left(R^{\gamma\mu}\right),\end{aligned}$$

where this time the symmetry of $R^{\alpha\beta}$ has been exploited in the last equality.

Back-substituting all these results in Eq. (8.5.10) and arranging the dummy indexes appropriately one finally obtains

$$\begin{aligned}0 &= \nabla_\epsilon\left(g^{\mu\epsilon}R\right) - \nabla_\delta R^{\mu\delta} - \nabla_\gamma\left(R^{\mu\gamma}\right)\\ &= \nabla_\alpha\left(g^{\alpha\beta}R\right) - 2\nabla_\alpha R^{\alpha\beta}\\ &= \nabla_\alpha\left(g^{\alpha\beta}R - 2R^{\alpha\beta}\right)\\ &= -\frac{1}{2}\nabla_\alpha G^{\alpha\beta}.\end{aligned}$$

Exercise 8.6 Show that any second rank tensor can be broken down in the sum of a symmetric and an antisymmetric tensor.

Solution 8.6 If T_{ij} is a generic rank (0, 2) tensor,[18] then by construction

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

is a (0, 2) symmetric tensor, and

$$A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

is antisymmetric. Moreover,

$$T_{ij} = S_{ij} + A_{ij}.$$

Finally, it is easy to see that a similar decomposition holds for any second rank tensor, regardless of the distribution of covariant and contravariant components.

Exercise 8.7 Prove that the inner product of a symmetric by an antisymmetric second rank tensor is identically zero.

Solution 8.7 Let S_{ij} be a symmetric tensor and A_{ij} and antisymmetric one. Because $S^{ij} = S^{ji}$ and $A_{ij} = -A_{ji}$ it is

$$S^{ij}A_{ij} = -S^{ji}A_{ji},$$

but i and j are dummy indexes, so we can swap them in the left-hand side of this equation; therefore it is also

$$S^{ij}A_{ij} = -S^{ij}A_{ij},$$

which implies that $2S^{ij}A_{ij} = 0$, and thus

$$S^{ij}A_{ij} = 0.$$

[18] In this exercise and in the following one the results are completely general, so we are not considering the usual distinction between Latin and Greek indexes.

Chapter 9
Applications

The aim of this chapter is to show how general relativity can be applied to some selected significant physical problems. The nature of this book prevents a detailed treatment of these and other important applications, which, however, can be found in many other places (see, e.g., Weinberg 1972 or Misner et al. 1973). Indeed, the rationale of these choices is not to give an extensive and complete overview of the applications of the Einsteinian theory of gravitation, but rather to provide a basis for the successive investigation of possible alternative theories, which is the next and final chapter of this work.

In this sense, the Schwarzschild solution is the starting point for a first description of the gravitational behavior of massive bodies. The so-called linearized gravity is shown because it is essential in treatment of the gravitational radiation, namely the last of the fundamental predictions of general relativity that have been missing an experimental confirmation until very recently. The post-Newtonian limit of general relativity, instead, is widely used to compare different gravity theories in the weak-field and slow motion limit. Finally, not only was general Relativity the first theory which produced a consistent and testable cosmological model, but also at such scales we can find experimental data, such as those on the accelerated expansion of the universe or on the estimation of the Hubble constant, which are providing clues of some unsolved problems in the theory.

9.1 The Schwarzschild Solution of Field Equations

This solution of the Einstein field equations, derived by Karl Schwarzschild in 1916, aims at representing the relativistic gravitational field out of a massive static and spherically symmetric body. Implicitly, this assumes that externally the spacetime is empty.

These assumptions help to constrain the expressions of the metric coefficients $g_{\alpha\beta}$ before having to use the field equations explicitly, thus making the task much easier. From the spherical symmetry assumption, it is reasonable to use spherical coordinates

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r, θ, φ to map the spatial part of the metric. Another natural assumption is that far from the body its gravitational influence goes to zero. Mathematically, this translates as the posit that at infinity the spacetime is flat, and therefore that asymptotically the line element is that of special relativity $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, or

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2$$

in spherical coordinates. As is rigorously shown, e.g., in D'Inverno (1992) or in Wald (1984), the spherical symmetry in spacetime can be expressed with the condition that the angular part of the line element is a 2-sphere, i.e., that in these coordinates

$$\mathrm{d}s^2 = r^2\left(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\varphi^2\right), \tag{9.1.1}$$

which means that $g_{\theta\theta} = r^2$ and $g_{\varphi\varphi} = r^2\sin\theta$, and that the off-diagonal terms $g_{\theta\varphi}$ and $g_{\varphi\theta}$ are identically zero.[1]

Alternatively, one can deduce from the definition of spherical symmetry as the form invariance of the line element for spatial rotations that (Exercise 9.1) the most general admissible form of the line element is

$$\mathrm{d}s^2 = -\bar{f}(r,t)\,\mathrm{d}t^2 + \bar{g}(r,t)\,\mathrm{d}t\,\mathrm{d}r + \bar{h}(r,t)\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

The assumption that the metric is static, then, also requires that its coefficients cannot depend explicitly on t, and that it does not change for a time reversal transformation $t \to -\bar{t}$, which implies $g_{0\alpha} = g_{\alpha 0} = 0$ or, in our case, $\bar{g}(r,t) = 0$. We are then left with a line element in the form

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{9.1.2}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ and we have extracted a minus sign out of the g_{tt} metric coefficients for an easier comparison with the Minkowski line element. We can in fact already use the assumption on the asymptotic flatness of the metric to infer that at infinity it has to be

$$\lim_{r\to\infty} f(r)\cdot h(r) = c^2. \tag{9.1.3}$$

To obtain the explicit expressions of these functions, the Einstein field equations for the metric in Eq. (9.1.2) have to be solved. We are seeking the solution in the supposedly empty space out of a finite spatial region that represents our isolated

---

[1]It has to be not that in general the spherical symmetry condition just requires that the radius of curvature is a general function of a radial coordinate ρ and of the time t, so that $\mathrm{d}s^2 = f(\rho,t)\left(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\varphi^2\right)$, but we can drop the time dependence because of the assumption of a static solution. Moreover, the choice of the radial coordinate is somewhat arbitrary, and having put $f(\rho) = r^2$ we have simply selected the coordinate that makes the hypersurfaces of constant radius have an area of $4\pi r^2$.

body, so here the stress-energy tensor of the source is $T_{\alpha\beta} = 0$ and we need to solve the field equations in vacuum; i.e., $R_{\alpha\beta} = 0$.

It can be seen both by long but straightforward calculations, and by symmetry considerations that $R_{\alpha\beta} = 0$ identically for any $\alpha, \beta = 0, \ldots, 3$ such that $\alpha \neq \beta$, and that the only non zero equations are thus the diagonal ones, for which it is

$$\begin{aligned}
R_{tt} &= \frac{1}{2h}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} - \frac{1}{4h}\frac{\mathrm{d}f}{\mathrm{d}r}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right) + \frac{1}{rh}\frac{\mathrm{d}f}{\mathrm{d}r} \\
R_{rr} &= -\frac{1}{2f}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{4f}\frac{\mathrm{d}f}{\mathrm{d}r}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right) + \frac{1}{rh}\frac{\mathrm{d}h}{\mathrm{d}r} \\
R_{\theta\theta} &= 1 - \frac{1}{h} - \frac{r}{2h}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right) \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta.
\end{aligned}$$

So the off-diagonal Einstein field equations in vacuum; $R_{\alpha\beta} = 0$ are automatically satisfied, and the same equations for each of the above diagonal terms are real equations and not mere identities. Our task is now to manipulate the resulting equations to obtain further information on the unknown functions $f(r)$ and $h(r)$.

Multiplying the first equation by (h/f) and adding to the second one it results in

$$\frac{h}{f}R_{tt} + R_{rr} = 0,$$

so that

$$\begin{aligned}
&\frac{1}{2f}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} - \frac{1}{4f}\frac{\mathrm{d}f}{\mathrm{d}r}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right) + \frac{1}{rf}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{1}{2f}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} \\
&\quad + \frac{1}{4f}\frac{\mathrm{d}f}{\mathrm{d}r}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}r}\right) + \frac{1}{rh}\frac{\mathrm{d}h}{\mathrm{d}r} = 0
\end{aligned}$$

$$\frac{1}{rf}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{1}{rh}\frac{\mathrm{d}h}{\mathrm{d}r} = 0$$

and, because $r \neq 0$, we can multiply the above by rfh to obtain

$$h\frac{\mathrm{d}f}{\mathrm{d}r} + f\frac{\mathrm{d}h}{\mathrm{d}r} = 0.$$

This is obviously equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}r}(fh) = 0,$$

which implies that $f(r) \cdot h(r) = \text{const}$ and thus, from Eq. (9.1.3), it is

$$f(r) \cdot h(r) = c^2$$

for any r. These functions can now be found by substituting $h = c^2/f$ in the equation for $R_{\theta\theta}$, which gives

$$\begin{aligned} c^2 &= f + \frac{rf}{2}\left(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{f}{c^2}\frac{c^2}{f^2}\frac{\mathrm{d}f}{\mathrm{d}r}\right) \\ &= f + r\frac{\mathrm{d}f}{\mathrm{d}r}, \end{aligned}$$

and therefore

$$\frac{\mathrm{d}(rf)}{\mathrm{d}r} = c^2.$$

This equation can be easily integrated and its solution is

$$rf(r) = c^2(r + K),$$

where K is a constant. The two functions are then

$$\begin{aligned} f(r) &= c^2\left(1 + \frac{K}{r}\right) \\ h(r) &= \left(1 + \frac{K}{r}\right)^{-1}. \end{aligned}$$

The only unknown term is now K, which, however, can be easily determined by remembering that in the weak-field approximation

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right),$$

where $\Phi = -GM/r$ is the Newtonian potential, so that $K = -2GM/c^2$ and the line element of the Schwarzschild solution finally becomes

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{9.1.4}$$

which is interpreted as the metric generated by a body of mass M outside the body itself.

Schwarzschild metric in isotropic coordinates

We obtained this expression for the Schwarzschild metric by imposing a spherically symmetry condition that makes the choice of using spherical coordinates natural. This condition, upon fixing the metric of the 2-sphere as in Eq. (9.1.1), says that θ and φ are the usual colatitude and longitude coordinates and that r is such that the area of the 2-sphere is $4\pi r^2$. With future applications in mind, it is useful to have a set of coordinates t, ρ, θ, φ with which the spatial part of the line element is Euclidean

$$\mathrm{d}s^2 = -c^2 F(\rho)\,\mathrm{d}t^2 + H(\rho)\left(\mathrm{d}\rho^2 + \mathrm{d}\Omega^2\right).$$

These will not preserve the original meaning of θ and φ, but the Euclidean form of its spatial coordinates makes them useful for comparison with the classical case. As shown in Exercise 9.2, the transformation

$$r = \rho\left(1 + \frac{C}{\rho}\right)^2, \tag{9.1.5}$$

where $C = GM/\left(2c^2\right)$, puts the line element in the desired form

$$\mathrm{d}s^2 = -c^2\left(\frac{1 - C/\rho}{1 + C/\rho}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{C}{\rho}\right)^4 \left(\mathrm{d}\rho^2 + \mathrm{d}\Omega^2\right). \tag{9.1.6}$$

By inverting Eq. (9.1.5) one has

$$\rho = \frac{1}{2}\left[(r - 2C) \pm \sqrt{r(r - 4C)}\right],$$

which shows that isotropic coordinates cannot be defined for $r < 4C = 2GM/c^2$. This is not a problem in the weak-field limit, because is the context in which they are normally used, since the so-called *Schwarzschild radius* $r_s = 2GM/c^2$ in this case is smaller than the physical dimension of the body.

9.1.1 Pericenter Advance

The dynamics of massive bodies, and in particular the advance of the orbital pericentre, can be found in the same way as shown in Sect. 7.1.2. One starts from the Lagrangian

$$
\begin{aligned}
L &= g_{\alpha\beta} v^\alpha v^\beta \\
&= -c^2\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2+\left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\varphi}^2, \quad (9.1.7)
\end{aligned}
$$

where the dots indicate the derivation with respect to the proper time τ. Once again, the motion is planar and therefore we can reduce our analysis to the case of $\theta(t) = \pi/2$, $\dot{\theta}(t) = 0$ with no loss of generality, which gives

$$
L = -c^2\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2+\left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2+r^2\dot{\varphi}^2. \quad (9.1.8)
$$

In this Lagrangian the coordinates t and φ are cyclic,[2] therefore their moments $p_\alpha = \partial L/\partial \dot{x}^\alpha$ are conserved because the respective Euler–Lagrange equations write

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{t}} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\varphi}} &= 0
\end{aligned}
$$

which give the first two equations of motion as conservation laws:

$$
\left(1-\frac{2GM}{c^2 r}\right)\dot{t} = k \quad (9.1.9)
$$

$$
r^2\dot{\varphi} = h. \quad (9.1.10)
$$

As already done in the previous chapter, instead of using the Euler–Lagrange equation for r, it is much easier to exploit the relation $g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -c^2\mathrm{d}\tau^2$, valid for massive particles, which translates into a constraint for the Lagrangian:

$$
-c^2 = g_{\alpha\beta}v^\alpha v^\beta = -c^2\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2+\left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2+r^2\dot{\varphi}^2. \quad (9.1.11)
$$

By substituting Eqs. (9.1.9) and (9.1.10) into (9.1.11) and considering that since Eq. (9.1.10) implies that $\varphi(\tau)$ is monotonic we can use, as in the Newtonian case, φ as a parameter for r instead of τ. Therefore $\dot{r} = r'\dot{\varphi}$, where we put $r' \equiv \mathrm{d}r/\mathrm{d}\varphi$, and after some straightforward manipulation one can obtain

$$
r'^2 + r^2\left(1+\frac{c^2 r^2}{h^2}\right)\left(1-\frac{2GM}{c^2 r}\right) - \frac{c^2 k^2 r^4}{h^2} = 0.
$$

The usual "trick" of making the substitution $u = 1/r$ can also be adopted in this case, so that the above equation becomes

[2] That is, the Lagrangian does not depend on t and φ.

$$u'^2 + u^2 = \frac{c^2\left(k^2-1\right)}{h^2} + \frac{2GM}{h^2}u + \frac{2GM}{c^2}u^3,$$

of which we take the derivative with respect to φ, dividing the result by $2u'$, thus obtaining

$$u'' + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2. \tag{9.1.12}$$

Once again, as happened in the case of Sect. 7.1.2, this equation is almost identical to Eq. (1.4.8), which we solved in the case of Newtonian orbits, with just the addition of the term $3GMu^2/c^2$.

As shown in Exercise 9.3, and similarly to the case of the previous chapter, this differential equation can be solved by means of perturbative methods giving a solution

$$u \simeq \frac{GM}{h^2}\left\{1 + e\cos\left[\varphi\left(1-\epsilon\right)\right]\right\},$$

where

$$\epsilon = \frac{3G^2M^2}{c^2h^2}.$$

The perihelion and the aphelion are the extremal points of u, namely those for which

$$u' \simeq -\frac{GM}{h^2}\left\{e\left(1-\epsilon\right)\sin\left[\varphi\left(1-\epsilon\right)\right]\right\} = 0,$$

hence in these points the angles φ satisfy the relation

$$\varphi\left(1-\epsilon\right) = n\pi.$$

If, by convenience, we take $n = 0$ (i.e., $\varphi = 0$) as a perihelion, then the next one will be after a complete period of the function, i.e., at

$$\varphi = \frac{2\pi}{1-\epsilon} \simeq 2\pi\left(1+\epsilon\right).$$

This implies that the next perihelion does not happen after 2π, but rather when the body has advanced of a further angle

$$\Delta\varphi \simeq 2\pi\epsilon = \frac{6\pi G^2M^2}{c^2h^2}.$$

At first order one can take the Newtonian relation of Exercise 1.5

$$\frac{h^2}{GM} = a\left(1-e^2\right)$$

which gives

$$\Delta\varphi \simeq \frac{6\pi GM}{c^2 a\left(1-e^2\right)}$$

as the *advance* (because $\Delta\varphi > 0$) of the perihelion per orbit. This can be put as a function of the orbital period again using a Newtonian relation, namely Kepler's third law

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM},$$

thus obtaining

$$\Delta\varphi = \frac{24\pi^3 a^2}{c^2 T^2\left(1-e^2\right)}.$$

For Mercury this evaluates at $\Delta\varphi \simeq 5\cdot 10^{-7}$ rad per orbit, which gives the well-known value of about 42.9 arcseconds per century.

9.1.2 Light Deflection

The equations of motion of the photon start from a Lagrangian having the same form as Eq. (9.1.7), which means that Eqs. (9.1.9) and (9.1.10) are still valid. However, proper time cannot be defined for photons, therefore the differentiation uses another affine parameter, say λ. Moreover, they move on null geodesics, which implies that the previous relation $L = -c^2$ has to be substituted by $L = 0$. In practice Eq. (9.1.11) becomes

$$-c^2\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0$$

and, with the same calculation of the previous section,

$$u'^2 + u^2 = \frac{c^2 k^2}{h^2} + \frac{2GM}{c^2}u^3$$

and

$$u'' + u = \frac{3GM}{c^2}u^2. \qquad (9.1.13)$$

Once again, we can resort to perturbative methods to compute the solution in the case of interest. Indeed, unless one has to deal with the motion in the vicinity of compact objects, for any orbit it is always $r \gg 3GM/c^2$ and therefore the right-hand

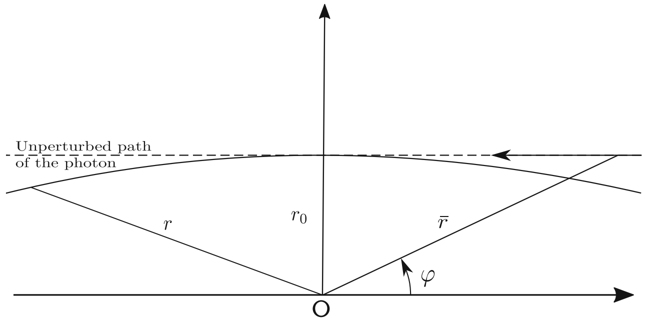

Fig. 9.1 Path of the unperturbed light ray. The equation $\bar{u} = a \sin\varphi$ implies that $r_0 = \bar{r}\sin\varphi$ is constant, and therefore the non relativistic path is a straight line

side of the equation is very small.[3] We can then assume that the complete solution can be written as

$$u = \bar{u} + \tilde{u}, \tag{9.1.14}$$

where $\tilde{u} \ll \bar{u}$ and $\bar{u}$ is the solution in the limit for which the right-hand side can be neglected $\bar{u}'' + \bar{u} = 0$, i.e., that of the simple harmonic oscillator

$$\bar{u} = a \sin\varphi + b \cos\varphi$$

with a and b constants. This can be easily recognized to represent a straight line in r if we take a particular solution of this family, e.g., $\bar{u} = a \sin\varphi$, where $a = 1/r_0$.[4] The resulting equation $r_0 = r \sin\varphi$ in fact, as shown in Fig. 9.1, means that for $\phi = \pi/2$ the (coordinate) distance from the centre of the body is $r = r_0$, and that for any angle φ the distance is such that the product is kept constantly equal to r_0, going to infinity for $\varphi = 0$ and for $\varphi = \pi$.

By substituting Eq. (9.1.14) into (9.1.13), and considering that $\bar{u}'' + \bar{u} = 0$, it results in

$$\bar{u}'' + \bar{u} + \tilde{u}'' + \tilde{u} = \tilde{u}'' + \tilde{u} = \frac{3GM}{c^2}\left(\bar{u} + \tilde{u}\right)^2 \simeq \frac{3GM}{c^2}\bar{u}^2 = \frac{3GM}{c^2 r_0^2}\sin^2\varphi,$$

where in the right-hand side we used the hypothesis $\tilde{u} \ll \bar{u}$ to neglect all the terms in $\tilde{u}$. This equation can also be written

$$\tilde{u}'' + \tilde{u} = \frac{3GM}{2c^2 r_0^2}\left(1 - \cos 2\varphi\right),$$

[3] For example, $3GM/c^2 \simeq 4.5$ km in the case of the sun. Because we are dealing with an exterior solution, only the orbits outside the physical dimensions of the body can be treated by this metric, so these 4.5 km have to be compared with the radius of the sun $R \simeq 7 \cdot 10^5$ km.

[4] It can be easily understood also by noting that Eq. (9.1.13) becomes $u'' + u = 0$ for $M = 0$.

whose general solution is

$$\tilde{u} = \frac{3GM}{2c^2 r_0^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + C_1 \sin\varphi + C_2 \cos\varphi.$$

Inasmuch as the constants of integration C_1 and C_2 are arbitrary, and they only fix the starting conditions for $\tilde{u}$ and $\tilde{u}'$, we can put them both equal to zero and get

$$\tilde{u} = \frac{3GM}{2c^2 r_0^2}\left(1 + \frac{1}{3}\cos 2\varphi\right), \tag{9.1.15}$$

so that we can write the complete solution as

$$u = \frac{1}{r_0}\sin\varphi + \frac{3GM}{2c^2 r_0^2}\left(1 + \frac{1}{3}\cos 2\varphi\right),$$

and taking into account that $\lim_{r\to\infty} u = 0$, when r goes to infinity it is

$$\frac{1}{r_0}\sin\varphi = -\frac{3GM}{2c^2 r_0^2}\left(1 + \frac{1}{3}\cos 2\varphi\right).$$

At these points, because the solution is a perturbation of the "straight line solution", $\varphi \simeq 0$ or $\varphi \simeq \pi$, therefore $\cos 2\varphi \simeq 1$, thus

$$\sin\varphi \simeq -\frac{2GM}{c^2 r_0} \ll 1,$$

which implies that $\sin\varphi \simeq \varphi$. We thus have that when a null geodesic (i.e. a photon) is "on one side" (corresponding, e.g., to $\varphi \simeq 0$) then $\varphi \simeq -2GM/\left(c^2 r_0\right)$, whereas "on the other side" ($\varphi \simeq \pi$) it means that $\varphi \simeq \pi + 2GM/\left(c^2 r_0\right)$, which gives the conclusion that the null geodesic undergoes a deflection angle from $-\infty$ to $+\infty$ equal to

$$\delta\varphi = \frac{4GM}{c^2 r_0}. \tag{9.1.16}$$

This quantity, as can be easily verified, corresponds to the well-known value of 1.75 arcseconds for light rays grazing the solar limb, i.e., for $M = M_\odot$ and $r_0 = R_\odot$.

9.2 Linearized Gravity and Gravitational Waves

General relativity, as we have seen above, predicts a deviation from Keplerian motion that since its very onset provided an experimental support to the theory. Some years later, in 1919, came observation of the light deflection (Dyson et al. 1920) which gave a second experimental confirmation to the new gravitational theory. In the succeeding years other phenomena, such as the gravitational redshift,[5] the gravitational time dilation, or the Shapiro time delay, always confirmed the prediction of Einstein's gravity theory.

Yet one of the most important predictions, despite indirect confirmations coming from the determination of the orbit of massive stellar binaries, continued to be missing a direct observation until recently. Finally in 2015, exactly 100 years after the formulation of the theory, the LIGO observatory detected the signal of a gravitational wave coming from the merging of two stellar-sized black holes (Abbott et al. 2016).

Weak-field limit and linearized gravity

As has been considered in the previous chapter, geometrization of gravity implies that there is a direct connection between the gravity field, the Newtonian gravitational potential, the components of the metric, and the coefficients of the connection. In particular we know that in the absence of gravity the laws of physics are those of special relativity, and therefore in the weak-field limit the metric can be written as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{9.2.1}$$

with $h_{\alpha\beta} \ll 1$, which is nothing else than requiring that gravity is described as a perturbation of the flat metric of special relativity. The so-called *linearized gravity* is a method of working out solutions of the field equations in the weak-field limit. In the next section we give a short summary of another method called post-Newtonian approximation.

In this approach we want to solve the Einstein equations in vacuum under the weak field hypothesis. As show see, this will lead us to a linearized field equation resembling a wave equation, whose solution therefore predicts the existence of wavelike perturbations of the metric (or the gravitational field) propagating at a finite speed in spacetime, namely the *gravitational waves*.

The Einstein field equations in vacuum are simply

$$R_{\alpha\beta} = 0,$$

which are obtained by setting $T_{\alpha\beta} = 0$ on Eq. (8.3.13). The alternative form is particularly convenient in avoiding the complications of the extra calculations caused by the Ricci scalar in the Einstein tensor $G_{\alpha\beta}$.[6]

---

[5] Which however is not strictly related to general relativity because it is a general consequence of the equivalence Principle.

[6] For the same reason this method is preferred here over the derivation of the linearized field equations from the variation of the linearized Einstein–Hilbert action.

Linearized field equation

By using the weak-field metric of Eq. (9.2.1) in the connection coefficients, the derivatives of the flat part vanish because of their constancy, whereas the products of $h_{\alpha\beta}$ by $\partial_\gamma h_{\alpha\beta}$ can be neglected because they are at second order in h, so it results in

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\beta h_{\delta\alpha} + \partial_\alpha h_{\delta\beta} - \partial_\delta h_{\alpha\beta}\right), \tag{9.2.2}$$

which, incidentally, implies that in this case indexes can be raised and lowered with the Minkowski metric at the required order.

Similarly, in the Ricci tensor the products of the connection coefficients are $\sim h^2$; therefore

$$\begin{aligned} R_{\alpha\beta} &= \partial_\gamma \Gamma^\gamma{}_{\alpha\beta} - \partial_\beta \Gamma^\gamma{}_{\alpha\gamma} \\ &= \frac{1}{2}\left(\partial_\gamma\partial_\alpha h^\gamma{}_\beta - \partial_\gamma\partial^\gamma h_{\alpha\beta} - \partial_\alpha\partial_\beta h^\gamma{}_\gamma + \partial_\beta\partial_\delta h_\alpha{}^\delta\right), \end{aligned} \tag{9.2.3}$$

which after some manipulation can be reduced to the form[7]

$$R_{\alpha\beta} = \frac{1}{2}\left(-\Box^2 h_{\alpha\beta} + \partial_\alpha\partial_\gamma\eta^{\gamma\delta}h_{\delta\beta} - \frac{1}{2}\partial_\alpha\partial_\beta\eta^{\gamma\delta}h_{\gamma\delta} + \partial_\beta\partial_\gamma\eta^{\gamma\delta}h_{\delta\alpha} - \frac{1}{2}\partial_\beta\partial_\alpha\eta^{\gamma\delta}h_{\gamma\delta}\right).$$

On defining the covector

$$k_\alpha \equiv \partial_\gamma\eta^{\gamma\delta}h_{\delta\alpha} - \frac{1}{2}\partial_\alpha\eta^{\gamma\delta}h_{\gamma\delta}$$

the above equation becomes

$$R_{\alpha\beta} = -\frac{1}{2}\left(\Box^2 h_{\alpha\beta} + \partial_\alpha k_\beta + \partial_\beta k_\alpha\right), \tag{9.2.4}$$

which, by that we are seeking the Einstein equations in vacuum $R_{\alpha\beta} = 0$, yields the linearized field equations

$$\Box^2 h_{\alpha\beta} - \partial_\alpha k_\beta - \partial_\beta k_\alpha = 0. \tag{9.2.5}$$

[7] Remembering that the d'Alembert operator is defined as $\Box^2 \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$.

Gauge freedom

As pointed out in the previous chapter, the gauge freedom of the Einstein field equations implies that we can try to determine a specific gauge condition, namely a coordinate transformation, that makes the solution easier. However, the weak field limit means that the set of admissible transformations has to preserve the metric in the form $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, which restricts the possible choices. In practice, this can be ensured by a "small" coordinate transformation, i.e., some $\bar{x}^\alpha = x^\alpha + \epsilon^\alpha\left(x^\beta\right)$ that admits an inverse and having $\epsilon^\alpha\left(x^\beta\right) \sim h_{\alpha\beta}$. Indeed, under this hypothesis it is $x^\alpha = \bar{x}^\alpha - \epsilon^\alpha\left(x^\beta\right) = \bar{x}^\alpha - \epsilon^\alpha\left(\bar{x}^\beta\right) + \mathcal{O}\left(\epsilon^2\right)$, therefore the infinitesimal displacement becomes

$$\mathrm{d}x^\alpha = \mathrm{d}\bar{x}^\alpha + \left(\partial_\beta \epsilon^\alpha\right) \mathrm{d}\bar{x}^\beta.$$

Substituting this relation in the line element $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ shows that the weak-field metric is transformed into $\bar{g}_{\alpha\beta} = \eta_{\alpha\beta} + \bar{h}_{\alpha\beta}$ where, putting $\epsilon_\alpha = \eta_{\alpha\beta}\epsilon^\beta$,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \epsilon_\beta - \partial_\beta \epsilon_\alpha, \tag{9.2.6}$$

and because $\epsilon^\alpha\left(x^\beta\right) \sim h_{\alpha\beta}$ we can conclude that $\bar{h}_{\alpha\beta}$ still complies with the weak-field condition $\bar{h}_{\alpha\beta} \ll \eta_{\alpha\beta}$, as required. The physical meaning of this transformation, as for the gauge transformations of the four-potential of electromagnetism, is that if $h_{\alpha\beta}$ is a solution of the field equations, then $\bar{h}_{\alpha\beta}$ is a solution as well and vice versa.

By comparison with Eq. (9.2.5), we are naturally led to choose $k_\alpha = 0$, i.e.,

$$\partial_\gamma \eta^{\gamma\delta} h_{\delta\alpha} - \frac{1}{2}\partial_\alpha \eta^{\gamma\delta} h_{\gamma\delta} = 0 \tag{9.2.7}$$

as our gauge condition, which is called a Lorenz gauge by analogy with Electromagnetism since it immediately implies that the linearized field equation (9.2.5) becomes

$$\Box^2\, h_{\alpha\beta} = 0, \tag{9.2.8}$$

in which the reader can easily identify a wave equation.[8] In particular, as it was stressed in the case of special relativity, the d'Alembert operator requires that the perturbation of the gravity field move in vacuum at the speed of light.

[8]In electrodynamics the gauge transformation for the four-potential reads $\bar{A}^\alpha = A^\alpha - \partial^\alpha \chi$, which is the counterpart of Eq. (9.2.6). This can be more easily understood if we define the so-called *transverse-traceless perturbation*

$$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h,$$

where $h = \eta^{\gamma\delta} h_{\gamma\delta}$. Similarly, the Lorenz gauge $\partial_\alpha \bar{A}^\alpha = 0$ gives the field equations in the form $\Box^2 \bar{A}^\alpha = 0$. This resembles that of Eq. (9.2.7), which in terms of the transverse-traceless perturbation reads $\partial_\alpha \bar{h}^{\alpha\beta} = 0$.

Solution of the field equations: gravitational waves

A detailed discussion of the solutions of Eq. (9.2.8) is beyond the scope of this book. We thus limit our exposition to a short summary of its main characteristics.

The easiest thing to be stressed is that the plane-wave equation with

$$h_{\alpha\beta} = A_{\alpha\beta} \exp\left(i k_\gamma x^\gamma\right) \tag{9.2.9}$$

is a solution of the wave equation (9.2.8) if k^γ is a null four-vector. This can be shown simply by substituting and doing the calculations, which give

$$\Box^2 h_{\alpha\beta} = \eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\alpha\beta} = \eta^{\gamma\delta} k_\gamma k_\delta h_{\alpha\beta} = k_\gamma k^\gamma h_{\alpha\beta},$$

but this yields a solution of the wave equation $\Box^2 h_{\alpha\beta} = 0$ only if $k_\gamma k^\gamma = 0$.

Similarly, by substituting the solution in the gauge condition (9.2.7), we have the further requirement that

$$k_\alpha A^\alpha{}_\beta = \frac{1}{2} k_\beta A^\alpha{}_\alpha. \tag{9.2.10}$$

It follows directly from the properties of $h_{\alpha\beta}$ that the amplitude matrix $A_{\alpha\beta}$ is by definition a symmetric rank 2 tensor. The number of independent components, however, is constrained by the four conditions of Eq. (9.2.10), which means that this is reduced to six. Moreover, any gauge transformation that fulfills the Lorenz condition produces another equivalent solution, which gives four more constraints and further reduces the number of independent components of $A_{\alpha\beta}$ to two. Such components are interpreted as two possible polarizations of the gravitational waves.

9.3 The Post-Newtonian Limit of General Relativity

In Sect. 8.2.2 it was shown how the einstein field equations and the geodesic equations of general relativity reduce to their Newtonian counterparts under appropriate conditions. In the first section of this chapter, instead, we showed that it is possible to derive an exact solution of the Einstein field equations in the simple case of a spherically symmetric mass. Unfortunately, this is rather an exception, and typically in the general cases of interest it is not possible to find an exact solution. Therefore, it is often necessary to resort to approximate ones, instead.

This task is made easier by general methods that allow us to find approximate solutions to any degree of approximation expressed as functions of specific smallness parameters. We have seen the linearized gravity approach, which is useful in the treatment of gravitational waves. Another of those methods was born to work in the realm of the (almost) "everyday world" such as the solar system dynamics or the study of stellar interiors, namely the N-body problem and the hydrodynamics of systems where the weak-field hypothesis is supplemented by the so-called slow motion and low energy conditions, whose meaning we now clarify.

Post-Newtonian smallness parameters

In the previous section we introduced the weak-field limit, namely a condition in which a weak gravity field allows us to write the metric as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $h_{\alpha\beta} \ll 1$. Indeed, from Sect. 8.2.2 we know that[9] $h_{00} = -2U/c^2$; i.e., this condition translates into one for the Newtonian potential U which, in the solar system, is always $\lesssim 10^{-5}$.

If we want to apply this technique to the N-body problem it is reasonable to make the further hypothesis that the bodies' motion are virialized, i.e., that their kinetic and potential energies are of the same order, which immediately translates into the condition $v^2 \sim U$. This is the *low-velocity limit* in the sense that the previous condition is equivalent to

$$\frac{v^2}{c^2} \sim \frac{U}{c^2} \ll 1.$$

This constraint enters in the left-hand side of the field equations because it also contains the derivatives of the metric coefficients, among which there is at least

$$\frac{\partial h_{00}}{\partial t} = \frac{v^i}{c}\frac{\partial h_{00}}{\partial x^i}.$$

This relation holds for any α, β in $h_{\alpha\beta}$, therefore in the calculations one has to take into account that

$$\frac{\partial h_{00}}{\partial t} \sim \frac{v}{c}\frac{\partial h_{00}}{\partial x^i}.$$

The metric is only one of the constituents of the field equations, the other being the stress-energy tensor $T_{\alpha\beta}$, and because the computation of the left-hand side will necessarily lead to a series in terms of the above smallness parameters we then expect that this quantity will be expanded as well and that it can be eventually written as a series of $\overset{(k)}{T}_{\alpha\beta}$, where the (k) above denotes the order of expansion in terms of the appropriate smallness parameters. This leads us to introduce the so-called *low-energy limit*. The stress-energy tensor, in fact, is in general expressed as a function of the matter density ρ (always appearing as ρc^2) of the pressure p, the velocity, and of the specific internal energy Π of the system. The latter contribution, which was never introduced before, is significant only for hydrodynamics. For the N-body problem particles have no internal structure, or if they have one this is just considered

[9] Because in this context it is customary to use the symbol Φ for another potential, we use U for the Newtonian gravitational potential.

a "geometric property" in the sense that it just describes how an ensemble of non-interacting test particles are configured to give the shape of the body and therefore to produce the explicit expression of U. In general, therefore, the stress-energy tensor for a perfect fluid gets the following expression

$$T^{\alpha\beta} = \left[\left(\rho + \rho\frac{\Pi}{c^2}\right) + \frac{p}{c^2}\right] u^\alpha u^\beta + p g^{\alpha\beta}. \tag{9.3.1}$$

If we denote with $\epsilon \sim v/c$ the general order of magnitude, the weak-field and small-velocity conditions give therefore

$$\epsilon^2 \sim \frac{v^2}{c^2} \sim \frac{U}{c^2} \sim \frac{p}{\rho c^2} \sim \frac{\Pi}{c^2}$$

and

$$\epsilon \sim \frac{1}{c}\frac{\partial/\partial t}{\partial/\partial x^i}.$$

Newtonian and Post-Newtonian limits

It now has to be stressed once again that the Newtonian limit of the field equation of general relativity is the Poisson equation, which can be recovered by taking for the metric $h_{00} = -2U/c^2$ and $h_{0i} = h_{ij} = 0$. From this observation it is clear that there is no physically meaningful approximation of general relativity at the order $\mathcal{O}(\epsilon)$ for the metric. The stress-energy tensor in this case is $T_{00} = \rho c^2$, with $T_{0i} = T_{ij} = 0$ because:

1. The internal energy density Π of the body does not play any role in the Poisson equation.
2. Under the hypothesis that $p \ll \rho c^2$ the fluid is pressureless at the Newtonian order.
3. In the further hypothesis of slow-motion limit, i.e., for $v \ll c$, the stress-energy becomes that of the dust.

Because from Eq. (8.3.16) $\kappa \sim \epsilon^4$, the order of the right-hand side of the field equation is at least always $\kappa T_{00} \sim \epsilon^2$, $\kappa T_{0i} \sim \epsilon^3$ and $\kappa T_{ij} \sim \epsilon^4$.

Moreover, one has reasonably to expect that in proceeding with further approximations the next order for h_{00} should contain the second power of the gravitational potential, which implies that it has to be at the ϵ^4 order. This reflects a general characteristic of these expansions which was first pointed out by Einstein, Infeld, and Hoffmann (1938), namely that metric components having an even number of temporal indexes contain only even powers of c, whereas those with an odd number contain only odd powers. Typically the latter derive from terms such as Uv.

This is all is needed to define the next order of approximation, which quite obviously is referred to as the *post-Newtonian limit*. Regarding the metric, it is thus characterized by expanding h_{00} up to $\mathcal{O}\left(\epsilon^4\right)$, h_{0j} up to $\mathcal{O}\left(\epsilon^3\right)$ and h_{ij} up to $\mathcal{O}\left(\epsilon^2\right)$, and the expansion of the stress-energy tensor takes its origin from the above three points, namely:

1. As in the Newtonian case, one can consider or not the contribution of Π in the case of hydrodynamics or of N-body dynamics, respectively.
2. Because p is at the next order with respect to ρc^2 it cannot be neglected in the post-Newtonian expansion of $T_{\alpha\beta}$ anymore.
3. Identical considerations hold for the velocity of the particles, therefore the spatial components of u_α cannot be neglected as in the Newtonian case.

Procedure for the Post-Newtonian expansion

In the following we sketch a general procedure that can be used to obtain the post-Newtonian limit of the field equations. More details about it can be found in the original paper by Chandrasekhar (1965), but the interested reader can also refer to the different approach based on the Landau–Lifshitz formulation of general relativity which is thoroughly explicated in Poisson and Will (2014).

The procedure can be summarized in the following steps.

1. Compute the Post-Newtonian expansion of the stress-energy tensor and of its trace.
2. Compute the Ricci tensor in terms of the perturbations of the metric coefficients $h_{\alpha\beta}$ in a convenient gauge.
3. Solve the resulting perturbed field equations to obtain the expressions for the Post-Newtonian metric.
4. Use the resulting metric coefficients to compute the explicit post-Newtonian expansion of the connection coefficients.

The latter, finally can be used to find the post-Newtonian expressions of the equations of motion.

Stress-energy tensor

As shown in Exercise 9.4, the four-velocity components expanded to the post-Newtonian order read

$$
\begin{aligned}
u_0 &= c\left[-1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right) + \mathcal{O}\left(\epsilon^4\right)\right] \\
u_i &= v_i + \mathcal{O}\left(\epsilon^3\right).
\end{aligned}
$$

The four-velocity is needed to compute the stress-energy tensor components

$$
\begin{aligned}
T_{00} &= \left[\left(\rho + \rho\frac{\Pi}{c^2}\right) + \frac{p}{c^2}\right](u_0)^2 + p g_{00} \\
&= \left[\left(\rho + \rho\frac{\Pi}{c^2}\right) + \frac{p}{c^2}\right] c^2 \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right)\right]^2 + p\left(-1 + h_{00}\right) \\
&= \rho c^2 \left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi\right)\right] + \mathcal{O}\left(\epsilon^2\right),
\end{aligned}
\tag{9.3.2}
$$

where in substituting h_{00} one can consider only its Newtonian expansion because of the κ factor. Similarly one gets

$$T_{0i} = -\rho c v_i + \mathcal{O}(\epsilon), \tag{9.3.3}$$

$$T_{ij} = \rho v_i v_j - \delta_{ij} p, \tag{9.3.4}$$

Raising the indexes of the four-velocities by considering the Newtonian order of $g_{\alpha\beta}$ gives the contravariant components of the stress-energy tensor just from Eq. (9.3.1), and therefore its trace from $T = T^{\alpha}{}_{\alpha} = -T_{00} + T_{11} + T_{22} + T_{33}$.

Field equations

Using the alternative form of Eq. (8.3.13) for the field equations implies that the only missing ingredient now is the Ricci tensor. From the above considerations we know that the h_{ij} components are $\mathcal{O}\left(\epsilon^2\right)$ at the post-Newtonian order, and because the equivalence principle, as derived for h_{00}, implies that this term is $-2U/c^2$, then it is not necessary to solve the field equations for the space–space components because we already know that

$$g_{ij} = 1 - \frac{2U}{c^2}.$$

Only the R_{00} and R_{0i} components of the Ricci tensor are then needed. The calculations are quite long, but straightforward, and for the sake of brevity here we remember only the guiding principles, leaving their details to the original sources.[10]

First of all, we know from the definition of the Ricci tensor of Eq. (D.6.10) that it contains two types of terms: those depending on the second derivatives of the metric and those that are the product of two connection coefficients. It turns out that at the post-Newtonian order the latter are all negligible except for $\Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{00}$.

[10] The cited work of Chandrasekhar, however, uses a different convention for the signature of the metric tensor.

Post-Newtonian coordinates and harmonic gauge

Second, it has to be remembered from the previous chapter that in solving the field equations we always have the freedom of choosing a coordinate transformation, namely to fix the gauge of our specific problem. In this case the second derivatives of R_{00} gives an expression that can be considerably more tractable if one adopts the gauge condition

$$\frac{\partial h^i{}_0}{\partial x^i} - \frac{1}{2}\frac{\partial h^i{}_i}{\partial x^0} = 0, \tag{9.3.5}$$

called the *temporal gauge*.

Metric tensor and connection coefficients

The field equations obtained in this way can be solved by using some additional functions, which are called potentials because they are defined by means of differential equations resembling those for familiar potentials. For example, the first one is Φ, where

$$\nabla^2 \Phi = -4\pi G \rho \phi$$

and $\phi \equiv v^2 + U + \Pi/2 + 3p/2\rho$. This allows us to get h_{00}, which reads

$$h_{00} = -2\frac{U}{c^2} + \frac{1}{c^4}\left(2U^2 - 4\Phi\right) + \mathcal{O}\left(\epsilon^6\right).$$

The off-diagonal terms, instead, require the definition of the so-called superpotential χ and of the vector potential V_i given by

$$\nabla^2 \chi = -2U$$
$$\nabla^2 V_i = -4\pi G \rho v_i,$$

which give

$$h_{0i} = \frac{1}{c^3}\left(4V_i - \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x^i}\right) + \mathcal{O}\left(\epsilon^5\right).$$

It can be verified by explicit substitution that this solution fulfills the gauge condition of Eq. (9.3.5), which finally leads to the post-Newtonian metric coefficients

$$g_{00} = -1 - 2\frac{U}{c^2} + \frac{1}{c^4}\left(2U^2 - 4\Phi\right) + \mathcal{O}\left(\epsilon^6\right) \tag{9.3.6}$$

$$g_{0i} = \frac{1}{c^3}\left(4V_i - \frac{1}{2}\frac{\partial^2\chi}{\partial t\partial x^i}\right) + \mathcal{O}\left(\epsilon^5\right) \tag{9.3.7}$$

$$g_{ij} = \left(1 - 2\frac{U}{c^2}\right)\delta_{ij} + \mathcal{O}\left(\epsilon^4\right). \tag{9.3.8}$$

Once again, long but straightforward calculations made by simply substituting these metric coefficients and their derivatives into the definition of Eq. (8.2.4) lead to the post-Newtonian expansion of the connection coefficients, which can be used in the geodesic equations to obtain the post-Newtonian equations of motion.

9.4 Cosmology

Attempts at exploring the possibility of describing the evolution of the whole universe under the prescriptions of Newtonian physics (gravity and dynamics) were probably done by Newton himself, and it is well known that these are undermined by a serious difficulty that makes any Newtonian cosmology untenable.

The non convergence problem of Newtonian cosmology

On its bare bones, as has been thoroughly explained by Norton (1999), the problem stems from two facts, namely:

1. That the Newtonian gravitational force is inversely proportional to the square of the distance, propagating with an infinite speed.
2. That it is assumed that the universe is infinite and with a uniform matter distribution, which implies that the amount of matter increases with the square of the distance.

For any given distance and any given direction, we have the same amount of matter exerting the same amount of force on the two opposite sides. Moreover, the force exerted by the matter at a given distance is constant, i.e., independent of the distance itself.

At first one can be tempted to conclude that this is a strong argument in favor of a static universe, however, it is easy to recognize that such a universe is totally unstable: any slight movement at any point would inevitably destroy the equilibrium, therefore the apparent accordance of this model with the static appearance of the universe at large scales is not a help, but rather a problem because it requires that, because of some “miracle” the universe does not evolve at all, although we know that, at least at the scales of our solar system, evolution happens.

Moreover, if one tries to find the total gravitational pull by integrating over the infinite space, because we observed above, the force exerted by a shell of constant radius does not depend on the distance and is constant, this integral can not converge. This means that, although it might seem that on the basis of the above symmetry considerations the total force is zero and the universe is static, the same symmetry can also tell us that the total force is different from zero, or better, and that it is not possible to compute the total gravitational force acting on a given body. [11]

A finite and dynamic universe, on the other hand, was denied by Newton on the basis of the large-scale static appearance of the celestial sphere. It is likely that, as also happened with Einstein 250 years later, this statement was influenced by philosophical preferences as well, which prevented any tenable cosmological model.

9.4.1 The Cosmological Principle and the Friedmann–Lemaître–Robertson–Walker Metric

The fact that Newtonian gravity fails to provide a reliable cosmological model does not mean that it had no influence on the development of these theories. The transition from Aristotelian physics to present-day understanding, in fact, can be seen as a gradual shift from a "we are the center of the universe" to a "there is no special place in the universe" philosophical tenet. It is to this process, started in the modern era with the Copernican revolution and fully embedded in classical dynamics, that we can ascribe the origin of the so-called cosmological principle, which states that, at large scales,[12] the Universe is homogeneous and emphisotropic. In practice this translates to the fact that the universe looks the same in all directions (isotropy) and that it should look isotropic from any place (homogeneity). This is the starting hypothesis (we could say the "axiom") essential for modern cosmological models based on general Relativity or relativistic theories of Gravity.

In principle, any relativistic cosmological model, in attempting to take into account the evolution of the universe driven by gravitational interaction, should solve the Einstein field equations given some kind of hypothesis on the expression of the stress-energy tensor. As already said in the previous section, this problem cannot be reasonably tackled in general, and it is the cosmological principle that makes the problem of relativistic cosmology feasible, by providing another exact solution of the Einstein field equations which is apparently representative of the geometry of the universe at large scales.

Indeed, the cosmological principle alone is able to constrain the line element of a metric theory of gravity in almost all of its parts without any reference to a

[11] Such argument was considered since the early advent of Newtonian gravity by the theologian and scholar an Richard Bentley and exposed in mathematical detail by Hugo Seeliger at the end of the nineteenth century.

[12] With this expression one can reasonably mean scales of the order of 100 Mpc.

specific gravity theory,[13] and without specifying the form of the stress-energy tensor in advance. Actually, as is shown in many sources (see, e.g., Robertson and Noonan 1968 or Weinberg 1972) the homogeneity and isotropy of the universe necessarily implies that the line element takes the form

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + \mathrm{d}\Omega^2\right), \tag{9.4.1}$$

where k is a constant that can take the values -1, 0 or $+1$ and $a(t)$ is an unknown function of the time t. This is known as the *Friedmann–Lemaître–Robertson–Walker* (FLRW) metric after the names of the people who introduced and studied it in the second and third decades of the twentieth century. Now, although the derivation of the metric is beyond the scope of this book, it is worth stressing the meaning of its characteristic quantities.

First of all, the spatial part of the metric bears its origin from that of a generally curved 3D space

$$\mathrm{d}s^2 = \frac{\mathrm{d}\bar{r}}{1-K\bar{r}^2} + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\varphi^2 \tag{9.4.2}$$

where K is the Gaussian curvature of such a space. It is easy to not that when $K = 0$ this reduces to the usual metric of a flat (Euclidean) 3D space. On the other hand, when $K > 0$ the curvature is positive and the metric is the three-dimensional version of that of a spherical surface,[14] whereas for $K < 0$ the curvature is negative and we have the 3D equivalent of an hyperbolic surface.

Cosmic time

One can always imagine that the Gaussian curvature can vary with time, but it cannot depend on the spatial coordinates because of the homogeneity and isotropy assumption. The cosmological principle in fact demands that at any instant of time the space appears isotropic everywhere (i.e., from any point of the universe) but it says nothing about how it has to evolve. It is therefore admissible a universe whose overall geometry changes with time, which has a very important consequence. We can imagine such a universe as a sequence of "spatial geometries" specific for each instant of time that thus will have the meaning of marking each "configuration" which will be valid for the whole universe. In principle, then, any observer from his or her location could measure some global geometrical properties of the universe that

[13] At least as long as it is a metric theory of gravity.

[14] The geometry on the surface of a sphere can be studied as a function of its two-dimensional coordinates only. This means that one can define the *intrinsic* geometric properties of a manifold using measures of angles, lengths, and areas completely defined within the manifold itself, i.e., without considering it embedded in a space with more dimensions. The first generic and comprehensive systematization of such technique was done by Carl Friedrich Gauss in the first half of the nineteenth century.

will determine unambiguously a specific time which, because of the homogeneity and isotropy, would be the same in all the three-dimensional locations. This means that any observer, in determining his or her proper time with respect to the whole universe, would obtain the same value given a specific "geometrical configuration", which explains why t is referred to as *cosmic time*. In a more rigorous way, one can say that the homogeneity and isotropy conditions allow us to define for any t a global hypersurface of simultaneity orthogonal to the temporal coordinate axis.

Comoving coordinates

From Eq. (9.4.2), the geometry of the universe at a given instant of time is determined by K which has to depend on t if the overall geometric change mentioned above must be allowed and, in general, the same has to be for the radial coordinate $\bar{r}$. However it is always possible to redefine this coordinate and incorporate all the temporal variation in a single parameter a, i.e., by means of the coordinate transformation

$$\bar{r}(t) = a(t)\, r, \tag{9.4.3}$$

where a therefore represents a time-dependent a scale factor. It is customary to attribute the dimension of a length to such scale factor, thus leaving r dimensionless.

In other words, any point of the universe will have the same spatial coordinates r, θ, φ at any time, and any evolution will be taken into account by the parameter a, as if the spatial coordinates were "moving with the evolving universe". For this reason these are called *comoving coordinates*. The line element of Eq. (9.4.2) then becomes

$$\mathrm{d}s^2 = a(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right)$$

where $k = K(t)\, a^2(t)$ can be made constant as for r, so that the $K \lesseqgtr 0$ conditions on the Gaussian curvature can be easily led back to the previously mentioned $k = -1, 0, +1$ conditions, by a mere rescaling.

The Hubble constant and the expansion of the universe

The proper distance d between two points is particularly simple to define when we put the spatial origin of the coordinates at one of these points, which is always admissible in the FLRW metric because of the homogeneity of space.[15] In this case the two points can be connected by a radial curve whose extremes are, say, $r = 0$ and $r = r_{\mathrm{p}}$, and the distance can be written as

[15] The homogeneity of space implies that we can consider any point as the origin of the spatial coordinates with no influence on the distance measurements.

$$d(t) = \int_0^{r_{\mathrm{p}}} \sqrt{g_{rr}}\,\mathrm{d}r = a(t) \int_0^{r_{\mathrm{p}}} \frac{1}{1-kr^2}\,\mathrm{d}r. \tag{9.4.4}$$

If we indicate with a dot the derivative of this distance with respect to t, i.e., the relative radial velocity between the two points, we get

$$v_r(t) \equiv \dot{d}(t) = \dot{a}(t) \int_0^{r_{\mathrm{p}}} \frac{1}{1-kr^2}\,\mathrm{d}r, \tag{9.4.5}$$

and therefore, dividing Eq. (9.4.5) by Eq. (9.4.4),

$$v_r(t) = \frac{\dot{a}(t)}{a(t)} d(t). \tag{9.4.6}$$

This relation means that on each "three-dimensional slice" the relative velocity between two points is proportional to their distance by a factor $H(t) = \dot{a}(t)/a(t)$ called *Hubble parameter*. The value of this parameter at the present time, conventionally written as $t = 0$ is called the *Hubble constant* and is indicated with H_0, therefore one has the well-known Hubble law

$$v = H_0 d. \tag{9.4.7}$$

If, as noticed by Hubble in 1929, the velocity of recession from us of any "point" (represented by distant galaxies) is increasing with their distance, this means that $\dot{a} > 0$ (from Eq. (9.4.3) it is necessarily $a > 0$), which means that the scale factor a is increasing with time and thus the universe is "expanding".

The cosmological redshift

Another effect related to the changing of the scale factor of the universe is the frequency shift experienced by light rays incoming from distant sources. This can be shown by starting again from the FLRW metric of Eq. (9.4.1) and considering that the light propagates on the null geodesic for which $\mathrm{d}s = 0$. In our case this implies that

$$c^2\mathrm{d}t^2 = a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + \mathrm{d}\Omega^2\right).$$

Moreover, if we consider a purely radial path from the source to the observer (which again we can put at the origin with $r = 0$ with no loss of generality) it is $\mathrm{d}\theta = \mathrm{d}\varphi = 0$ and we have

$$c\mathrm{d}t = \pm a(t)\frac{\mathrm{d}r}{\sqrt{1-kr^2}},$$

where we adopt the convention that the minus sign represents an incoming photon, so that its radial coordinate decreases while the time increases. We can then separate the variables and integrate the previous equation from the two events of emission and reception, respectively indicated as t_e, r_e, and t_r, r_r, thus

$$c\int_{t_e}^{t_r}\frac{dt}{a(t)} = -\int_{r_e}^{r_r}\frac{dr}{\sqrt{1-kr^2}} = -\int_{r_e}^{0}\frac{dr}{\sqrt{1-kr^2}}, \tag{9.4.8}$$

the last part due to the fact that we put the observer at the origin, so that $r_r = 0$.

Suppose now that the wavelength of the light emitted from the source is λ_e, so that one complete oscillation at the source will happen after a time $\Delta t_e = \lambda_e/c$. Because the spatial coordinates are comoving, and we are assuming that neither the source nor and the observer is in motion, they will not change. This complete oscillation then will characterize two events, at the source and at the reception, respectively, with coordinates $(t_e + \Delta t_e, r_e)$ and $(t_r + \Delta t_r, 0)$, for which the same relation as above will hold, i.e.,

$$c\int_{t_e+\Delta t_e}^{t_r+\Delta t_r}\frac{dt}{a(t)} = -\int_{r_e}^{0}\frac{dr}{\sqrt{1-kr^2}}. \tag{9.4.9}$$

Because

$$\int_{t_e+\Delta t_e}^{t_r+\Delta t_r}\frac{dt}{a(t)} = \int_{t_r}^{t_r+\Delta t_r}\frac{dt}{a(t)} + \int_{t_e}^{t_r}\frac{dt}{a(t)} + \int_{t_e+\Delta t_e}^{t_e}\frac{dt}{a(t)}, \tag{9.4.10}$$

by combining Eqs. (9.4.8), (9.4.9), and (9.4.10) we get

$$\int_{t_r}^{t_r+\Delta t_r}\frac{dt}{a(t)} = \int_{t_e}^{t_e+\Delta t_e}\frac{dt}{a(t)}.$$

It is reasonable to assume that $a(t)$ can be considered constant during both time intervals Δt_r and Δt_e, so that $a(t) \simeq a(t_e)$ for $t_e < t < t_e + \Delta t_e$ and $a(t) \simeq a(t_r)$ for $t_r < t < t_r + \Delta t_r$, therefore the previous equation reads $\Delta t_r/a(t_r) \simeq \Delta t_e/a(t_e)$, or equivalently

$$\frac{\Delta t_r}{\Delta t_e} \simeq \frac{a(t_r)}{a(t_e)}.$$

The above equation tells us that:

- If the scale factor does not change between the emission and reception times, then the time interval for a complete oscillation will be the same at the source and at the observer.
- If the scale factor changes in such a way that $a(t_r) > a(t_e)$, i.e., that $\dot{a}(t) > 0$ if we suppose a monotonic change, then the period for a complete oscillation at the reception will be longer than that at the emission, and therefore the observer will see an increase of the wavelength and a decrease of the frequency of the light

with respect to that at the source, i.e., a *redshift*; the monotonic $\dot{a}(t) > 0$ means that the overall geometry of the universe is changing in such a way that the proper distances are steadily increasing, which justifies the denomination of "expanding universe" attributed to such a condition.
- If the scale factor changes in such a way that $a(t_\mathrm{r}) < a(t_\mathrm{e})$, i.e., that the universe is contracting, than the observer will see a decrease of the wavelength (a *blueshift*) of the light with respect to that at the source.

In formulae, we can remember that it was assumed $\Delta t_\mathrm{e} = \lambda_\mathrm{e}/c = 1/\nu_\mathrm{e}$, where ν_e is the frequency of the light at emission so that, by definition, it is also $\Delta t_\mathrm{r} = \lambda_\mathrm{r}/c = 1/\nu_\mathrm{r}$, therefore

$$\frac{\lambda_\mathrm{r}}{\lambda_\mathrm{e}} = \frac{\nu_\mathrm{e}}{\nu_\mathrm{r}} \simeq \frac{a(t_\mathrm{r})}{a(t_\mathrm{e})}.$$

It is common practice to define a quantity called *redshift* as

$$z = \frac{\nu_\mathrm{e} - \nu_\mathrm{r}}{\nu_\mathrm{r}} = \frac{\lambda_\mathrm{r} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}},$$

therefore the last equation can be written as

$$z = \frac{\lambda_\mathrm{r}}{\lambda_\mathrm{e}} - 1 = \frac{\nu_\mathrm{e}}{\nu_\mathrm{r}} - 1 = \frac{a(t_\mathrm{r})}{a(t_\mathrm{e})} - 1. \tag{9.4.11}$$

As is often remembered, such a frequency shift resembles a kind of Doppler effect, however, one has to consider that the "relative velocity" at its basis is due to the change of the geometry of the universe through the temporal evolution of its scale factor. Indeed, this frequency shift holds among points of the "coordinate grid" of the spacetime, which by definition of comoving coordinates are fixed, and therefore there is no such thing as a variation in time of the coordinates themselves. This effect is thus related to some properties of the universe as a whole rather than of some peculiar motion of its points, and it is for this reason that it takes the name of *cosmological* redshift or blueshift.

9.4.2 Friedmann Equations

It is noteworthy to realize that everything we have shown thus far depends only on the cosmological principle, i.e., on the assumption of the spatial homogeneity and isotropy of the universe. We have not used the Einstein field equations yet, so these results are independent of the specific model of universe we decide to adopt. It is even independent of the specific gravity theory itself, provided that it can be expressed as a metric theory and the FLRW metric represent a valid solution for the field equation of the theory under consideration.[16] This has to remind us that, up to now, we do not know if this metric can be a solution for the field equations of general relativity as well, which is what we to do in this section.

First of all, we have to establish the "composition of our universe", or more rigorously the form of the stress-energy tensor to use in the field equations. A reasonable choice is to consider the cosmic energy and matter as a perfect fluid which, as we know from Chap. 8, can be written as

$$T_{\alpha\beta} = \left(\rho c^2 + p\right) u_\alpha u_\beta + p g_{\alpha\beta}, \tag{9.4.12}$$

where ρ and p represent the fluid density and pressure, respectively, and u^α is the four-velocity of the "particles" of the fluid.

We can now use Eqs. (9.4.1) and (9.4.12) with the Einstein field equations. It is convenient to take the latter in the form of Eq. (8.3.13), which requires us to compute the components of the Ricci tensor through Eqs. (D.6.10) and (8.2.4). The explicit computation is quite long, but straightforward, therefore here we just give the final results. In particular it is $R_{\alpha\beta} = 0$ for $\alpha \neq \beta$, whereas the only non-zero components are the diagonal ones, which read

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{9.4.13}$$

$$R_{11} = \left(a\ddot{a} + 2\dot{a}^2 + 2kc^2\right) / \left[c^2\left(1 - kr^2\right)\right] \tag{9.4.14}$$

$$R_{22} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2kc^2\right)/c^2 \tag{9.4.15}$$

$$R_{33} = r^2 \sin^2\theta \left(a\ddot{a} + 2\dot{a}^2 + 2kc^2\right)/c^2 \tag{9.4.16}$$

Regarding the right-hand side of the field equations, it has to be remembered that, as for the redshift case above, because of the cosmological principle we are neglecting any relative motion between particles unless it is caused by the gravitational evolution of the universe as a whole, therefore in the comoving coordinates of the metric it has to be $u^\alpha = \delta^\alpha_0$. The trace of the stress-energy tensor then results in

$$T = -\rho c^2 + 3p \tag{9.4.17}$$

and

$$T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta} = \left(\rho c^2 + p\right) c^2 \delta^0_\alpha \delta^0_\beta + \frac{1}{2}\left(\rho c^2 - p\right) g_{\alpha\beta}, \tag{9.4.18}$$

[16]This is true even beyond the expectations. Indeed, the fact that the assumptions of the homogeneity and isotropy of space(-time) fit perfectly in a Euclidean geometry, has the even more noticeable consequence that everything we derived above, and everything be shown in the following can be devised in the framework of Newton's theory of gravity, as first proved by Milne (1934) and McCrea and Milne (1934). In particular we obtain very similar results by assuming the constancy of the speed of light, hence showing that the problems of Newtonian cosmology rather came from the additional assumption of an infinite and static universe. An in-depth and brilliant exposition of these considerations can be found in Bondi (1961).

which once again is zero for $\alpha \neq \beta$. In the $\alpha = \beta = 0$ case we then have

$$-3\frac{\ddot{a}}{a} = \frac{8\pi G}{c^4}\left(\rho c^2 + p - \frac{1}{2}\rho c^2 + \frac{1}{2}p\right)c^2$$

which simplifies to

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right);$$

for $\alpha = \beta = 1$ it is

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2}{c^2\left(1 - kr^2\right)} = \frac{4\pi G}{c^4}\left(\rho c^2 - p\right)\frac{a^2}{1 - kr^2},$$

that is,

$$a\ddot{a} + 2\dot{a}^2 + 2kc^2 = \frac{4\pi G}{c^2}\left(\rho c^2 - p\right)a^2, \tag{9.4.19}$$

and for the two remaining cases we always obtain the same equation.[17]

In summary, we have shown that the FLRW metric is a correct solution of the Einstein field equations if its scale factor $a\,(t)$ satisfies the two Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right) \tag{9.4.20}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho \tag{9.4.21}$$

where we divided Eq. (9.4.19) by a^2, using then Eq. (9.4.20) to eliminate $\ddot{a}$ from Eq. (9.4.19) and write the last formula.

In a simplified rendition, Friedmann equations just tell us that the universe, in the cosmological sense represented by the scale factor, behaves like a cannonball: the second equation says that it can be expanding or contracting ($\dot{a}$ can be either greater or less than zero); the first one states that, because gravity is attractive, i.e., because ρ and p are always positive, the expansion is always slowing down.

A cannonball have three possibilities: falling back down to the ground, escaping from Earth with asymptotic velocity, or escaping with more than asymptotic velocity, respectively, when its initial velocity is below, equal to or above the escape velocity of the Earth, which depends on the mass of our planet. Following this parallelism, and with not much surprise, one can expect that, according to its initial conditions, the universe could expand and contract or to expand forever, either in an asymptotic or non asymptotic way, according to two initial conditions, namely expansion velocity and the stress-energy density. This is what we to discover in the next section.

[17] Basically, this is due to the homogeneity and isotropy hypothesis.

9.4.3 Standard Cosmological Models

The solution of the Friedmann equations determines the temporal evolution of the scale factor, and therefore that of the cosmological geometry of the universe,[18] however, we have still been left with one last degree of freedom on the value of k. As said in Sect. 9.4.1 we can have the three cases $k = -1, 0, +1$ corresponding to a negative, zero, or positive value of the Gaussian curvature K, respectively. For what has been said above, these can be named hyperbolic, flat, and spherical spatial geometries.

Critical density

The value of k is linked with the stress-energy density of the universe; in fact remembering that $\dot{a}/a$ is the Hubble parameter $H(t)$, one can write Eq. (9.4.21) as

$$\frac{kc^2}{a^2H^2} + 1 = \frac{8\pi G}{3H^2}\rho \equiv \Omega,$$

that is,

$$\Omega - 1 = \frac{kc^2}{a^2H^2},$$

and therefore, because $(c/aH)^2 > 0$, the value of k is completely determined by Ω, so that:

1. If $\Omega < 1$ then $k < 0$ and the geometry of the universe is hyperbolic.
2. If $\Omega = 1$ the $k = 0$ and we are in the flat case.
3. If $\Omega > 1$ then $k > 0$ and we have the spherical case.

The sign of k is thus related to the value of ρ; in fact we have the first, second, or third case when ρ is, respectively less then, equal to or greater than

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G},$$

which has the dimensions of a density and is therefore named *critical density*. Conveniently, Ω is referred to as the *density parameter.* Current experimental data e.g., the results from the BOOMERanG experiment (de Bernardis et al. 2000) which is cited in a little more detail in Sect. 10.1.2, are in support of a $\Omega \simeq 1$, flat Universe.

[18] We can interpret the word "cosmological" as "global". The scale factor specifies the geometry of the overall universe "as a whole", once the type of geometry, i.e., the value of k, is set.

Equation of state

Given the goal of this book, we do not go through the details of the derivation of the explicit solutions of the Friedmann cosmological equations, but rather we illustrate the general picture and the results that are most important for the comparisons we present in the next chapter. To ease the notation, in the following and until the end of this chapter we adopt the widespread convention of setting $c = 1$.

First, it is needed to know the behavior of the components of the universe in terms of an *equation of state*, i.e., of a relationship between density and pressure which, for a perfect fluid, generally reads

$$p = w\rho \tag{9.4.22}$$

where w is a constant. The two main cases usually considered are those of *dust*, with $w = 0$, and of *radiation*, corresponding to $w = 1/3$.[19] The meaning of the former is that of a pressureless fluid made of point particles moving at $v \ll c$, which in the cosmological context is a reasonable model for typical stars and galaxies and it is therefore thought to represent the present state of the universe. The latter instead applies to pure electromagnetic radiation or to high-speed particles, which easily explains the value of w in this case when we substitute in Eq. (9.4.17) the value of $T = 0$ characteristic of the electromagnetic field. According to present knowledge, this equation of state applies to the early stage of the universe, when it was much smaller and dominated by radiation.

Normal matter is expected to have positive density, which means that $p \geq 0$. Using Eq. (9.4.20) we can thus draw our first deduction, namely that the universe, both in the dust and radiation cases, has $\ddot{a} < 0$.

Universe models: radiation

The solutions in the three different cases for the radiation models are:

$$a(t) = \begin{cases} F_{\mathrm{r}}\sqrt{\left(1 + \frac{t}{F_{\mathrm{r}}}\right)^2 - 1} & \text{for } k = -1 \\ \sqrt{\sqrt{2}F_{\mathrm{r}}t} & \text{for } k = 0 \\ F_{\mathrm{r}}\sqrt{1 - \left(1 - \frac{t}{F_{\mathrm{r}}}\right)^2} & \text{for } k = +1 \end{cases} \tag{9.4.23}$$

[19] As pointed out in Appendix C.4.1, in cosmology dust (also called matter) and radiation are both particular cases of a perfect fluid. Here we can see that the characterization given by the equation of state coincides with the definition of dust used there.

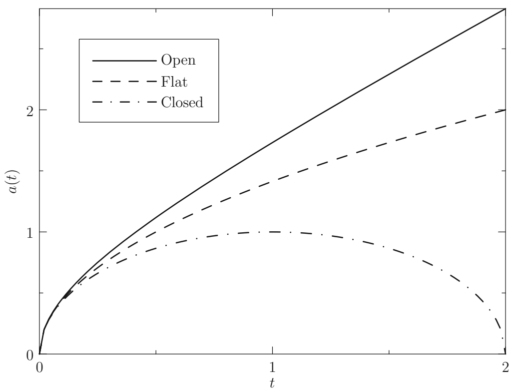

Fig. 9.2 The evolution of the scale factor $a(t)$ in the open, flat, and closed cosmological models

where $F_{\mathrm{r}} = a^2\sqrt{8\pi G\rho/3}$ is constant because, in the radiation case, it is $\rho \propto a^{-4}$ (Exercise 9.5) and therefore $a^2\sqrt{\rho} = \text{cost}$. These are plotted[20] in Fig. 9.2, from which it is clear why the first and the last case are named "open" and "closed" universes: in the former the scale factor increases monotonically with time, whereas in the latter it reaches a maximum and then it goes back to zero.

In a more mathematical way, we can use Eq. (9.4.21) to say that in general[21]

$$\begin{aligned}\dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - k \\ &= \frac{8\pi G}{3}\frac{\rho a^4}{a^2} - k \\ &= \frac{F_{\mathrm{r}}^2}{a^2} - k,\end{aligned} \tag{9.4.24}$$

and because F_{r} is a constant, it is

$$\lim_{a\to\infty} \dot{a}^2 = -k,$$

which, by restoring for a moment the c factor, gives

$$\lim_{a\to\infty} \dot{a}^2 = -kc^2.$$

This implies that when the density is less than ρ_{crit} and therefore in the case of $k = -1$, not only the scale factor increases monotonically with time, but that as a goes to infinity (i.e. as t goes to infinity) the expansion rate reaches the asymptotic value of c. On the other hand in the "flat" case (i.e., for $k = 0$), this limit is zero,

[20] In convenient units of measure where $F_{\mathrm{r}} = 1$.

[21] Remember that here we set the unit of measures in such a way that $c = 1$.

and therefore the universe continues to expand but its expansion rate goes to zero as t goes to infinity.

Using this relation in the $k = +1$ case requires that we take into account that it is always $\dot{a}^2 \geq 0$, which means that a cannot go to infinity anymore, but rather there will be a maximum value $a_{\max}$ for which $\dot{a}^2 = 0$. Therefore

$$a_{\max}^2 = F_r^2 = a_{\max}^4 \frac{8\pi G}{3}\rho,$$

and

$$a_{\max} = \sqrt{\frac{3}{8\pi G\rho}}. \tag{9.4.25}$$

After that value, because from Eq. (9.4.20) it is always $\ddot{a} < 0$, the expansion rate will became negative and the universe will began to "close on itself", ideally reaching the value of $a = 0$.

Universe models: dust

Actually this short analysis cannot be applied to our universe because we know that at the present time it is not dominated by radiation, and therefore it makes little sense to extrapolate its evolution to an infinite time. Nonetheless this was pedagogical because similar considerations can be applied to the dust case solution, whose interpretation is less intuitive.

In this case in fact the dependence on t can be expressed only in an indirect way. The solution is usually expressed using an auxiliary variable ψ called the *development angle*, and it reads

$$\begin{cases} a(\psi) = \dfrac{F_d}{2}(\cosh\psi - 1) \\ t(\psi) = \dfrac{F_d}{2}(\sinh\psi - \psi) \end{cases} \tag{9.4.26}$$

for $k = -1$ and taking $F_d = 8\pi G\rho a^3/3 = \text{const}$ inasmuch as in the case of dust $\rho \propto a^{-3}$,

$$a(t) = \left(\frac{9}{4}F_d t^2\right)^{1/3} \tag{9.4.27}$$

for $k = 0$, and

$$\begin{cases} a(\psi) = \dfrac{F_d}{2}(1 - \cos\psi) \\ t(\psi) = \dfrac{F_d}{2}(\psi - \sin\psi) \end{cases} \tag{9.4.28}$$

for $k = +1$. The plots of these solutions would look similar to the previous ones, however, it is easier to not that the above reasoning can also be applied in this case, with minor changes in the formulae coming from the fact that in the dust models we have to exploit the definition of F_d, which is proportional to a^3 instead of a^4. Thus, e.g., Eq. (9.4.24) becomes

$$\dot{a}^2 = \frac{F_d}{a} - k,$$

but in the cases of $k = -1$ and $k = 0$ we can draw exactly the same conclusions because the limit for $a \to \infty$ is the same, whereas for the closed universe of $k = +1$ it results in

$$a_{\max} = F_d = a_{\max}^3 \frac{8\pi G}{3} \rho,$$

which again gives Eq. (9.4.25).

Putting things together: the "ingredients" of the universe

If the cosmological models can be seen as a "recipe" for the making of a universe, then from the above considerations it is easy to understand that its origin, its ultimate fate and its history as a whole depend on the "ingredients" of the recipe. Previously we have mentioned that everything stands on the value of the density parameter Ω. Its value, however, can be written as the sum $\Omega = \sum_i \Omega_i$, in which each kind of matter-energy i gives a contribution to the overall density according its ratio with respect to the critical density. As already stressed above, it is reasonable to assume that at present times the universe is matter-(dust-)dominated, so that $\Omega_r \ll \Omega_m$. However dynamical and experimental considerations (Larson et al. 2011; Komatsu et al. 2011; Planck Collaboration et al. 2016a) lead us to accept that $\Omega_m \simeq 0.3$, and detectable (baryonic) matter can explain about $\Omega_b = 0.04$. This seems to suggest the presence of non visible matter, called for this reason *dark matter*, whose nature is not known at the moment. This scenario is made even less clear from the previously cited estimation of $\Omega \simeq 1$ calling for a further unknown ingredient of our recipe, which is considered in the next chapter.

9.5 Exercises

Exercise 9.1 Show that the most general form of the line element of a spherically symmetric spacetime is

$$ds^2 = -f(r,t)\, dt^2 + g(r,t)\, dt\, dr + h(r,t)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Solution 9.1 The basic requirement for a spatial spherical symmetry is that there exists a point, (i.e., the origin of the reference system) for which the metric does not change for spatial rotations. This implies the existence of a series of 2-spheres with line element

$$a(\bar{r}, t)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

In fact the coordinates θ and ϕ cannot appear in other elements of the metric if the line element has to remain invariant for these changes. If the latter contained mixed terms $g_{\theta\phi}$ then the metric would not be invariant for rotations, whereas if θ or ϕ appeared in other coefficients of the metric they would not be invariant for a rotation or a reversal of the corresponding angular coordinate, such as a transformation from ϕ to $-\phi$.

Moreover, the coefficients $a(\bar{r}, t)$ in this case have a precise geometric meaning, namely that the area of each of these 2-spheres is $A = 4\pi a(\bar{r}, t)$. We can thus define a radial coordinate r such that $a(\bar{r}, t) = r^2$, or $r = \sqrt{A/4\pi}$; in other words, the radial coordinate r will be the one that will make the area of a 2-sphere at constant t (and constant r) coincident with that of an ordinary three-dimensional 2-sphere.

This provides an additional natural constraint to the metric coefficients; in fact the invariance under rotations requires that the basis vectors for θ and ϕ have to be orthogonal with respect to that of r. Dropping this condition would identify a privileged direction in space, contrary to the hypothesis of spherical symmetry. In formulae this condition translates to $g_{r\theta} = (\mathbf{e}_r \cdot \mathbf{e}_\theta) = 0 = \left(\mathbf{e}_r \cdot \mathbf{e}_\phi\right) = g_{r\phi}$. The same condition holds for the time coordinate, because the condition of spherical symmetry must apply for the whole spacetime and this means that the 2-spheres must be orthogonal also to the temporal direction, therefore $g_{t\theta} = g_{t\phi} = 0$.

In summary, we have shown that with for a spherically symmetric spacetime the only admissible off-diagonal element is g_{tr}, that the diagonal t and r metric coefficients cannot depend on θ and ϕ, and that the angular part of $g_{\alpha\beta}$ can be recast as $r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$ by defining a radial coordinate with a convenient geometrical meaning. Finally, the requirement that it reduces to the Minkowski form in flat spacetimes determines the signature of the metric, which eventually takes the form claimed at the beginning.

Exercise 9.2 Show that the coordinate transformation

$$r = \rho\left(1 + \frac{GM}{2c^2\rho}\right)^2$$

puts the Schwarzschild metric of Eq. (9.1.4) in the isotropic form expressed by Eq. (9.1.6).

Solution 9.2 By direct substitution of the above definition it is

$$\left(1 - \frac{2GM}{c^2 r}\right) = \left[1 - \frac{2GM}{c^2\rho}\left(1 + \frac{GM}{2c^2\rho}\right)^{-2}\right]$$

$$= \left(1+\frac{GM}{2c^2\rho}\right)^{-2}\left[\left(1+\frac{GM}{2c^2\rho}\right)^2-\frac{2GM}{c^2\rho}\right]$$
$$= \left(1+\frac{GM}{2c^2\rho}\right)^{-2}\left(1-\frac{GM}{2c^2\rho}\right)^2, \tag{9.5.1}$$

and differentiation of the same formula gives

$$\begin{aligned}\mathrm{d}r^2 &= \left[\mathrm{d}\rho\left(1+\frac{GM}{2c^2\rho}\right)^2+2\rho\left(1+\frac{GM}{2c^2\rho}\right)\left(-\frac{GM}{2c^2\rho^2}\right)\mathrm{d}\rho\right]^2\\ &= \left(1+\frac{GM}{2c^2\rho}\right)^2\left[\left(1+\frac{GM}{2c^2\rho}\right)+2\rho\left(-\frac{GM}{2c^2\rho^2}\right)\right]^2\mathrm{d}\rho^2\\ &= \left(1+\frac{GM}{2c^2\rho}\right)^2\left(1+\frac{GM}{2c^2\rho}-\frac{GM}{c^2\rho}\right)^2\mathrm{d}\rho^2\\ &= \left(1+\frac{GM}{2c^2\rho}\right)^2\left(1-\frac{GM}{2c^2\rho}\right)^2\mathrm{d}\rho^2.\end{aligned} \tag{9.5.2}$$

Eqs. (9.5.1) and (9.5.2) imply

$$\begin{aligned}\left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 &= \left(1+\frac{GM}{2c^2\rho}\right)^2\left(1-\frac{GM}{2c^2\rho}\right)^{-2}\left(1+\frac{GM}{2c^2\rho}\right)^2\left(1-\frac{GM}{2c^2\rho}\right)^2\mathrm{d}\rho^2\\ &= \left(1+\frac{GM}{2c^2\rho}\right)^4\mathrm{d}\rho^2,\end{aligned} \tag{9.5.3}$$

and another direct calculation yields

$$r^2\mathrm{d}\Omega^2 = \left(1+\frac{GM}{2c^2\rho}\right)^4\left(\rho^2\mathrm{d}\theta^2+\rho^2\sin^2\theta\,\mathrm{d}\phi^2\right). \tag{9.5.4}$$

Finally, by substituting Eqs. (9.5.1), (9.5.3), and (9.5.4) in

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2GM}{c^2r}\right)\mathrm{d}t^2+\left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2,$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2$, gives

$$\begin{aligned}\mathrm{d}s^2 = &-c^2\left(1+\frac{GM}{2c^2\rho}\right)^{-2}\left(1-\frac{GM}{2c^2\rho}\right)^2\mathrm{d}t^2\\ &+\left(1+\frac{GM}{2c^2\rho}\right)^4\left(\mathrm{d}\rho^2+\rho^2\mathrm{d}\theta^2+\rho^2\sin^2\theta\,\mathrm{d}\phi^2\right).\end{aligned}$$

Exercise 9.3 Solve Eq. 9.1.12 with perturbation methods.

Solution 9.3 This problem closely resembles the one already investigated in Exercise 7.1, and in fact Eq. (9.1.12) is similar to the analogous equation of the Newtonian problem, with just the additional $3GMu^2/c^2$ term, which is, however, proportional to a coefficient that is small for ordinary masses. One can thus resort to perturbative methods and search for a solution

$$u = u_0 + \epsilon_1 u_1 + \mathcal{O}\left(\epsilon_1^2\right), \tag{9.5.5}$$

where $\epsilon_1 \ll 1$. Let us rewrite Eq. (9.1.12) as

$$u'' + u = \frac{GM}{h^2} + \epsilon_1 u^2,$$

where $\epsilon_1 = 3GM/c^2$. Substitution of Eq. (9.5.5) in this expression yields

$$\begin{aligned} u_0'' + u_0 + \epsilon_1 \left(u_1'' + u_1\right) &= \frac{GM}{h^2} + \epsilon_1 \left(u_0 + \epsilon_1 u_1\right)^2 \\ &= \frac{GM}{h^2} + \epsilon_1 \left(u_0^2 + 2\epsilon_1 u_0 u_1 + \epsilon_1^2 u_1^2\right) \\ &= \frac{GM}{h^2} + \epsilon_1 u_0^2 + \mathcal{O}\left(\epsilon_1^2\right). \end{aligned}$$

The solution of this equation can be found by equating each equal power of ϵ_1, so it has to be

$$\begin{aligned} u_0'' + u_0 &= \frac{GM}{h^2} \\ u_1'' + u_1 &= u_0^2, \end{aligned}$$

but the first one is just the usual Newtonian equation of Exercise 1.5, which gives[22]

$$u_0 = \frac{GM}{h^2}\left(1 + e\cos\varphi\right), \tag{9.5.6}$$

so that

$$u_1'' + u_1 = \left(\frac{GM}{h^2}\right)^2 \left(1 + 2e\cos\varphi + e^2\cos^2\varphi\right). \tag{9.5.7}$$

It is clear that a solution of this equation has to contain an appropriate combination of trigonometric functions. A convenient choice can be

$$u_1 = A + B\varphi\sin\varphi + C\cos^2\varphi,$$

[22]We can consider with no loss of generality the case with $\varphi_0 = 0$.

in fact

$$u_1'' = 2B\cos\varphi - B\varphi\sin\varphi - 2C\cos^2\varphi + 2C\sin^2\varphi$$

and

$$\begin{aligned} u_1'' + u_1 &= A + 2B\cos\varphi + C\left(2\sin^2\varphi - \cos^2\varphi\right) \\ &= A + 2B\cos\varphi + C\left(2 - 3\cos^2\varphi\right) \\ &= (A+2C)\left(1 + \frac{2B}{A+2C}\cos\varphi + \frac{-3C}{A+2C}\cos^2\varphi\right). \end{aligned} \tag{9.5.8}$$

By comparing Eqs. (9.5.7) and (9.5.8) it is easy to deduce that these coincide for

$$\begin{aligned} A + 2C &= \left(\frac{GM}{h^2}\right)^2 \\ \frac{B}{A+2C} &= e \\ \frac{-3C}{A+2C} &= e^2, \end{aligned}$$

which gives

$$\begin{aligned} A &= \left(\frac{GM}{h^2}\right)^2\left(1 + \frac{2}{3}e^2\right) \\ B &= e\left(\frac{GM}{h^2}\right)^2 \\ C &= -\frac{1}{3}e^2\left(\frac{GM}{h^2}\right)^2 \end{aligned}$$

and

$$u_1 = \left(\frac{GM}{h^2}\right)^2\left(1 + \frac{2}{3}e^2\right) + e\left(\frac{GM}{h^2}\right)^2\varphi\sin\varphi - \frac{1}{3}e^2\left(\frac{GM}{h^2}\right)^2\cos^2\varphi. \tag{9.5.9}$$

Substituting Eqs. (9.5.6) and (9.5.9) into (9.5.5) we thus have

$$\begin{aligned} u &= u_0 + \epsilon_1 u_1 \\ &= \frac{GM}{h^2}\left[1 + e\cos\varphi + \epsilon\left(1 + \frac{2}{3}e^2 + e\varphi\sin\varphi - \frac{1}{3}e^2\left(\frac{GM}{h^2}\right)^2\cos^2\varphi\right)\right] \end{aligned}$$

where $\epsilon = 3\left[GM/(ch)\right]^2$. Among all the terms multiplied by ϵ one can retain only the third one because it is proportional to φ, which means that it increases indefinitely and therefore it can become much larger than the others, which are always small, and its effects can accumulate over time, thus obtaining

$$u \simeq \frac{GM}{h^2}\left(1 + e\cos\varphi + \epsilon e\varphi\sin\varphi\right).$$

The factor in parentheses contains the term $\cos\varphi + \epsilon\varphi\sin\varphi$, which is the Taylor expansion at first order of $\cos\left[\varphi\left(1-\epsilon\right)\right]$ around ϵ, therefore

$$u \simeq \frac{GM}{h^2}\left\{1 + e\cos\left[\varphi\left(1-\epsilon\right)\right]\right\}$$

as reported in the main text.

Exercise 9.4 Compute the post-Newtonian expansion of the four-velocity components.

Solution 9.4 The four-velocity is defined as $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$, where τ is the proper time of the particle, and because of the normalization condition it is also $g_{\alpha\beta}u^\alpha u^\beta = -c^2$, thus

$$\left(u^0\right)^2\left(g_{00} + 2g_{0i}\frac{u^i}{u^0} + g_{ij}\frac{u^i}{u^0}\frac{u^j}{u^0}\right) = -c^2.$$

However, because $x^0 = ct$, one has

$$\frac{u^i}{u^0} = \frac{1}{c}\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{v^i}{c}, \tag{9.5.10}$$

hence

$$u^0 = c\left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i}{c}\frac{v^j}{c}\right)^{-1/2}.$$

We do not know yet the post-Newtonian expansion of the metric coefficients, but from the considerations discussed in Sect. 9.3 it is understood that at this order $g_{00} = -1 - 2U/c^2 + \mathcal{O}\left(\epsilon^4\right)$, $g_{0i} = \mathcal{O}\left(\epsilon^3\right)$ and $g_{ij} = \delta_{ij} + \mathcal{O}\left(\epsilon^2\right)$, where $-2U/c^2$ is the Newtonian order and the remaining terms are the missing post-Newtonian parts, therefore

$$\begin{aligned} u^0 &= c\left(1 + \frac{2U}{c^2} - \frac{v^2}{c^2} + \mathcal{O}\left(\epsilon^4\right)\right)^{-1/2} \\ &= c\left(1 - \frac{U}{c^2} + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\epsilon^4\right)\right). \end{aligned}$$

The spatial components can then be found from Eq. (9.5.10),

$$u^i = \frac{v^i}{c}u^0 = \left(1 - \frac{U}{c^2} + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\epsilon^4\right)\right)v^i.$$

The covariant components are therefore

$$\begin{aligned} u_0 &= g_{0\alpha}u^{\alpha} = g_{00}u^0 + \mathcal{O}\left(\epsilon^3\right) u^i \\ &= \left(-1 - \frac{2U}{c^2}\right) u^0 + \mathcal{O}\left(\epsilon^4\right) \\ &= c\left(-1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{U}{c^2} + \mathcal{O}\left(\epsilon^4\right)\right) \end{aligned}$$

and

$$\begin{aligned} u_i &= g_{i\alpha}u^i = \left(1 + \mathcal{O}\left(\epsilon^2\right)\right)\left(1 + \mathcal{O}\left(\epsilon^2\right)\right) v^i \\ &= v_i + \mathcal{O}\left(\epsilon^3\right), \end{aligned}$$

where in the last step we used the fact that at first order $v_i = v^i$.

Exercise 9.5 Prove the relations $\rho \propto a^{-4}$ and $\rho \propto a^{-3}$ for radiation and dust, respectively.

Solution 9.5 The equation of state (9.4.22) gives a relationship between density and pressure of a perfect fluid, for which dust (matter) and radiation constitute two special cases. We know that ρ and p are not independent quantities also from the divergence-less condition of the stress-energy tensor. We used this equation in Apprndix C.4.1 to derive the continuity equation and Euler's equation of fluid dynamics in a special relativistic context, where such condition read $\partial_\alpha T^{\alpha\beta}$. We know, however, that in general relativity this becomes

$$\nabla_\alpha T^{\alpha\beta} = 0$$

which from Eq. (D.4.8) can be written

$$\partial_\alpha T^{\alpha\beta} + \Gamma^{\alpha}{}_{\alpha\mu} T^{\mu\beta} + \Gamma^{\beta}{}_{\alpha\mu} T^{\alpha\mu} = 0.$$

The connection coefficients can be found by direct computation from the FLRW metric, and by using this result with Eq. (9.4.12) it can be shown after some long but straightforward calculations that the spatial components of the divergence conditions are automatically satisfied (this is actually a consequence of the spatial isotropy) and the time components give

$$\nabla_\alpha T^{\alpha 0} = \dot{\rho} + 3\left(\rho + p\right)\frac{\dot{a}}{a} = 0. \tag{9.5.11}$$

But the time component, as shown in Appendix C.4.1, is just the continuity equation which now reads

$$\frac{\dot{\rho}}{\rho} = -3\left(1 + w\right)\frac{\dot{a}}{a}. \tag{9.5.12}$$

From the constancy of w it follows that in general

$$\rho \propto a^{-3(1+w)}. \tag{9.5.13}$$

This gives immediately the required relations, given that $w = 1/3$ for radiation and $w = 0$ for the dust.

Chapter 10
Beyond General Relativity

General relativity is almost universally considered one in the most successful theories of the history of science. Since its very beginning, the elegance of its basic tenets and of its formulation constituted a fundamental reason for admiration and consideration. Moreover, it is capable of making many new predictions. These became more and more testable with time, especially during the last 50–60 years, and current results put Einstein's theory of gravity on extremely solid experimental grounds.

Nonetheless, general relativity is like any scientific theory: it is a now useful model that, allowing a sufficient for to the scientific investigation, sooner or later will be replaced by a better one. In Sect. 2.3 we identified three "signals" for the need of a theory switchover, and Einstein's theory, despite its undeniable strength, is (or at least it could be) "positive" to all of them.

Signal number 1 is the comparison with experimental results. Although general relativity fits impressively well the results of all the current experiments at small scales, both in the weak and in the strong field regime, at larger scales the deviations from the expected results are equally impressive. In an extremely short way, we can simplify the status by saying that:

1. The dynamic of massive and massless objects starting from the galactic scales is completely disregarded unless a matter–energy density in the form of undetected particles (dark matter) 5 times larger than the currently detectable one is introduced.
2. The dynamic of the universe, i.e., at cosmological scales, is as well completely disregarded unless a matter–energy density under the form of an unknown field 3 times larger than the sum of dark and known matter (dark energy) is introduced.

This does not mean that general relativity is necessarily wrong, but as the first signal of the inadequacy of Newton's gravitational theory was a defect in the prediction of the dynamics of the solar system requiring the introduction of an undetected internal planet, the same might be true in this case.

Signal number 2 is the presence of compatibility issues between different theories. In this regard it is well known that general relativity and quantum physics are not

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compatible, in the sense that quantum physics cannot be put in the form of a classical field theory, whereas general relativity is not quantizable.[1]

Signal number 3 is the presence of "philosophical" or self-consistency issues. General relativity admits singularities in some cosmological models (Big Bang) as well as for the Black Holes models. Also in this case, one is not forced to interpret this issue as a breakdown of the theory as long as they do not represent measurable quantities, but in general singularities, such as ke infinities, are not welcome in a physical theory. Another kind of philosophical issue is linked to the importance that Einstein ascribed to Mach's principle and the origin of inertia. As is more completely explained later, general relativity failed to incorporate this principle fully. Again, judging this a serious issue or not can be a taste, and it may even be true that the principle itself is meaningless, yet it is a matter of fact that this was one of Einstein's goals and it is missing from general relativity.

This extremely rough sketch of the problems of general relativity explains why this final chapter is therefore devoted to the presentation of some alternatives to general relativity. It would be impossible to give a complete overview of all the possibilities, both because of their huge number and the often advanced level of many of these topics. This exposition therefore is limited to some of the most known or easiest examples, with a level of detail following from the pedagogic slant of the book. Indeed, given that each of these topics would require a book on its own, the clarification of the basic concepts and the physical motivations is given priority over the mathematical details. More often than not, these "details" are essential to understand more than superficially the specialized literature and to work out the necessary calculations, thus an extensive use of references is done in order to provide the reader with sufficient indications to analyze the specific subject in more detail.

Finally, the subject is further complicated by the fact that sometimes even the agreement about what can be defined as "general relativity" is missing in the scientific community. It is therefore necessary to make a choice in this regard, which in this case favored what was deemed to be a pedagogically attractive exposition.

10.1 General Relativity Beyond General Relativity

In Sect. 4.4.2 we introduced a cosmological term in the Newtonian theory of gravity with a slight modification of its standard Lagrangian density. An analogous procedure can be easily done for general relativity. Actually it is such a natural option in the context of a variational approach that one version was first suggested by Einstein himself in 1917, one year after the presentation of his gravity theory. For this reason, usually, this theory is considered within the same framework of general relativity. On the other hand, it can be surely considered an alternative formulation of a relativistic theory of gravity, for the Newtonian limit of these field equations is different from the Poisson equation.

[1]Obviously, at the same time this might be a problem of quantum physics as well.

Indeed, the idea that general relativity with a cosmological term is still general relativity can be done with some reason, because it affects the gravitational behavior of the bodies only at cosmological scales. At the time of its conception general relativity had to compete with Newton's theory of gravity, whose natural realm was the solar system, and not the cosmological arena (see Sect. 9.4) which instead was a sort of unexplored territory. Classical gravity is neutral in this respect because it had never been able to "set the standard", therefore to speak, so if General relativity has to be considered as the replacement of the universal theory of gravity, then both its versions, with and without the cosmological term, can be put on an equal footing.

Moreover, general relativity was formulated at the dawn of the era in which observational data could be used to discriminate between different cosmological models. As we have seen, it was the first theory that was able to produce a consistent cosmological model, and because of this it nurtured the development of cosmology as an independent and more rigorous field of application.

Similar arguments can be invoked for models that have assumed the existence of different sources of gravity in addition to those usually established. The only constraint imposed by general relativity is that of $\nabla_\alpha T^{\alpha\beta} = 0$ on the stress-energy tensor and as long as the matter Lagrangian density satisfies this requirement, modifications to the field equations fall back to the general relativistic model.

10.1.1 The Cosmological Constant: Einstein Versus General Relativity

Defined by Einstein himself as "the greatest blunder" of his life as a scientist his version of the field equations with an additional, cosmological term is instead a quite natural alternative to the original formulation of general relativity.

In the Newtonian case we introduced a cosmological scalar field by modifying the interaction term $\rho\Phi$ to $(\rho + C)\,\Phi$, with C a constant or admitting at most a dependency on t. Its introduction could be allowed because that of being linear was the sole constraint imposed on such interaction term, and it was observed that it brought in an additional term to the force, which was repulsive when its sign was negative and attractive in the opposite case. Moreover, contrary to the standard inverse-square-distance law, the magnitude of the new component of the force increased with the distance. This immediately suggests that an analogous procedure could be followed in general relativity.

Einstein field equations with a cosmological constant

In Sect. 8.3 the geometry Lagrangian density was defined as $\mathcal{L}_\mathrm{G} \propto R$. As already observed there, this choice was just a matter of convenience, inasmuch as it was the simplest possibility to comply with the requirement for $\mathcal{L}_\mathrm{G}$. If the Lagrangian has to be a Riemannian scalar, however, putting $\mathcal{L}_{\mathrm{G}} = \sqrt{-g}\,(R/(2\kappa) + C)$,where $\kappa = 8\pi G/c^4$ and C is a constant, equally does not infringe upon such a requirement.[2] For conventional reasons we put $C = -2\Lambda$, i.e.,

$$\mathcal{L}_{\mathrm{G}} = \sqrt{-g}\left(\frac{1}{2\kappa}R - 2\Lambda\right), \tag{10.1.1}$$

so that following exactly the same procedure of Sect. 8.3.1 the field equations become

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + g_{\alpha\beta}\Lambda = \frac{8\pi G}{c^4}T_{\alpha\beta}, \tag{10.1.2}$$

or alternatively, by contracting the equation with $g^{\alpha\beta}$ (Exercise 10.1)

$$R_{\alpha\beta} - g_{\alpha\beta}\Lambda = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) \tag{10.1.3}$$

From the point of view of the fundamental principles this does not constitute such a great problem, as the left-hand side is still symmetric and continues to agree with the null divergence condition $\nabla_\alpha\left(G^{\alpha\beta} + g^{\alpha\beta}\Lambda\right) = 0$, provided that the metric connection holds. It is immediate to see that the Newtonian limit of the above equation is Eq. (4.4.6),[3] which, however, is again not a serious problem, as we stressed above. The most notable difference, instead, is that the field equations in vacuum are not $R_{\alpha\beta} = 0$, but rather

$$R_{\alpha\beta} = g_{\alpha\beta}\Lambda \tag{10.1.4}$$

which implies that in vacuum we do not return to a flat Minkowskian spacetime as in "standard" general relativity.

By computing the curvature K out of the Ricci tensor we have

$$K = \frac{1}{3}\Lambda,$$

thus a positive value of Λ gives, in vacuum, a positive curvature spacetime, called a *de Sitter space*, which is the Riemannian counterpart of the repulsive force we met in the Newtonian extension. The opposite sign, corresponding to a negative curvature spacetime, is called *anti-de Sitter space*. Intuitively, then, one could build a cosmological static universe model by considering Eqs. (10.1.2) with a positive value of Λ chosen to balance exactly the attractive gravitational pull of the matter density at our time. Another way of seeing the same thing is making the apparently

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[2]As Rindler (2006) points out, this addition plays the same role of an additive constant to an indefinite integral.

[3]Except for a factor c^2 multiplying Λ, which is due to dimensional reasons. In Sect. 4.4.2 in fact the cosmological constant had the dimensions of $[\text{time}]^{-2}$, whereas the relativistic one has to be chosen with dimensions of $[\text{length}]^{-2}$.

innocent switch of the cosmological term from the left-to the right-hand side of the field equations and interpreting the quantity $g_{\alpha\beta}\Lambda$ as proportional to a kind of stress-energy tensor $T^{(\Lambda)}_{\alpha\beta}$ of the vacuum, which is produced by a "vacuum perfect fluid" with equation of state $p_\Lambda = -\rho_\Lambda c^2$ (Exercise 10.2) whereas the original and "normal" matter part is characterized by its own density $\rho_{\rm m}$ and pressure $p_{\rm m}$.

Conditions for a static model of the universe

For a universe with positive curvature and sufficiently large to be considered matter-dominated, matter contributions come only from dust, which implies $p_{\rm m} = 0$. In this case, as shown in Exercise 10.3, the static Universe can be obtained by requiring

$$\frac{8\pi G}{3}\left(\rho_{\rm m} + \rho_\Lambda\right) = \frac{kc^2}{a^2} \tag{10.1.5}$$

and

$$2\rho_\Lambda = \rho_{\rm m}. \tag{10.1.6}$$

At the beginning of the past century, before Hubble discovered the expansion of the universe, the philosophical preference for an everlasting and static universe was still dominant, which is the reason why Einstein suggested this version of his field equations. Such preference, indeed, seemed to be supported by the astronomical observations, so before general relativity there was no reasonable motivation to expect or desire an expanding universe. In this regard therefore Einstein's choice of introducing a cosmological constant was by no means any "big blunder".

However, as shown in Exercise 10.4 the static universe represents an unstable equilibrium point so that any "movement" could have produced a slight deviation from the exact equilibrium conditions of Eqs. (10.1.5) and (10.1.6), thus inevitably disrupting the static nature of the Universe. This is similar to what happened with Bentley's objections to Newtonian cosmology, with the notable difference that in that case we needed an infinite universe to support the static requirement, whereas here the latter can survive on its own in a finite universe.[4] Normally any model relying on the existence of an unstable equilibrium point should be looked at with great suspicion, to say the least, so the error was rather not to realize that this model looked quite untenable. If so, Einstein could have followed the cosmological implications of its original equations and, preceding Friedmann, Lemaitre, Robertson, and Walker, he could have repeated exploit he made with the light deflection, predicting the expansion of the universe some 10 years before its discovery.

[4] The fact that the critical point is the hypothesis of a static universe as in the Newtonian models mentioned in Sect. 9.4.2, shows once again that the fundamentals of cosmology have little or nothing to do with general relativity.

10.1.2 Exotic Matter Lagrangians

Apparently, however, the fate of the cosmological constant was quite different from such an untimely end. After Einstein "unleashed" it, in a perfect Pandora's box-style, the "thing" refused to give up so simply.

The reprise of the cosmological constant

It is well known (see, e.g., Jackson 2015) that since its very beginning the interpretation of the expanding universe in the framework of a FLRW cosmological model was disturbed by recurrent difficulties in reconciling the age of the universe from the estimation of the Hubble constant H_0 with those of other known objects.[5] The so-called Hubble "constant" is no constant whatsoever, but rather it is the value of the Hubble parameter $H(t)$ at the present time, whose alleged constancy is justified only by the fact that at human time scales it can appear such because we are looking at a tiny part of the Hubble diagram or, equivalently, of the plot of Fig. 9.4.1. Estimating the age of the universe by a simple linear extrapolation from H_0 is therefore a gross approximation, but taking into account this error rather makes things worse because it is easy to see that in this way the age of the universe can only be further reduced because of the attractive nature of gravity, which is responsible for the $\ddot{a} < 0$ condition. In this context a source of repulsive force such as the cosmological constant can be (and it was!) invoked as a way to solve the age problem.

It is evident, in fact, that as the introduction of an attractive mechanism reduces the age of the universe, a repulsive one has the opposite effect because it is possible to have $\ddot{a} > 0$. If in Sect. 9.4.2 we compared the universes of the FLRW cosmologies to a cannonball, those with a repulsive mechanism can be associated with a rocket, because we got access to a sort of "engine" that can be used for fine tuning the evolution of $H(t)$ and $a(t)$.

But "be careful what you wish for, you might just get it." Indeed, a cosmological constant component is a new degree of freedom in our models that can make things more complicated, e.g., opening the possibility of universe models with *accelerating* expansion. As noted above this could be demostrated only by following the shape of $a(t)$ up to cosmological distances, i.e., to high redshift, and this is exactly what happened at the end of the last century (Riess et al. 1998; Perlmutter et al. 1999).

[5]For example, Hubble's initial estimation of $H_0 = 500$ km s^{-1} Mpc^{-1} was not even compatible with the known geological age of the Earth. Later estimations steadily lowered the value of H_0, shifting the incompatibility problem to the astronomical realm.

Although recent measurements have cast some doubts on its evidence, which is presently widely accepted by the scientific community, it is indisputable that such a scenario could not be managed by a classical cosmological model and thus calls for a new interpretation.[6]

Before proceeding it is necessary to recall the "apparent innocent switching" of the cosmological term we made in the last section. Formally, everything on the left-hand side of the field equations is "geometry", and on the right-hand side there is the "matter". In this sense the switch implicitly changes the nature of the "thing" at the origin of the associated behavior.

In the cosmological context it is easier to deal with this term when it is on the matter side. In this case, in fact, similarly to those for matter and radiation a vacuum density ρ_Λ can be associated with another density fraction Ω_Λ which adds to that of the other components so that the condition determining the overall geometry of the universe becomes

$$\Omega \equiv \Omega_{\mathrm{m}} + \Omega_\Lambda \lesseqgtr 1.$$

As in the previous case, a flat universe still corresponds to the case $\Omega = 1$ in the sense that it implies $k = 0$ in the FLRW metric, however, now this situation can be reached in three different ways, namely with either Ω_{m} or Ω_Λ equal to zero and the other equal to one, or with a combination of two non zero density fractions.

Together with the above-cited measurements, the totally different experiments involving the so-called cosmic microwave background (CMB) radiation helps to put more stringent constraints on the composition of the universe. It is beyond the scope of this book to deal with this subject, and here it is enough to recall that the presence of a uniform microwave radiation background is among the classical predictions of the Big Bang-like cosmological theories. At the same time, the presence of current structures in the universe tells us that at a certain level such radiation must deviate from a perfect uniformity. The details of such deviations are related with several characteristics of the cosmological models, and in particular one of these is the density parameter Ω of Sect. 9.4.3. The BOOMERanG experiment cited therein (de Bernardis et al. 2000) used exactly this technique to constrain the universe geometry around a flat model.

In summary, although these data tell us that the observations are compatible with a flat ($\Omega \simeq 1$) geometry, together they fix the "recipe" for the ingredients of the universe to $\Omega_{\mathrm{m}} = 0.3$ and $\Omega_\Lambda = 0.7$. It is worth stressing, however, that the densities vary with time because they depend on the varying scale factor a. However, on cosmological time scales these densities are ruled by completely different dependencies (Exercises 9.9 and 10.2), thereforein general these values can be totally dissimilar, depending on the cosmological epoch. It thus looks strange that, apparently with no

[6]Recent measurements (Riess et al. 2016) demonstrated an even higher acceleration, with a possible disagreement in the determination of H_0 with a different determination from Planck Collaboration et al. (2016b) whose calibration origin, however, cannot be excluded yet. At the same time Nielsen et al. (2016), thanks to a larger database of supernovae, suggested that the evidence is much weaker than previously supposed and it is even still compatible with a constant expansion rate.

reason, we are living in an epoch where $\Omega_{\mathrm{m}} \sim \Omega_\Lambda$. This is called the *coincidence problem.*

The actual nature of the cosmological term

This is not the end of the story yet. If it seems established that some "cosmological-constant-like" thing has to be used in cosmology, determining its nature is a totally different thing, and the source of a good share of headaches.

Let us suppose that, as anticipated in the previous section, the origin of Λ Alies on the hypothesis that the vacuum is a perfect fluid with an equation of state with $w = -1$. This immediately implies a constant ρ_Λ, which is good, otherwise the effects of a non homogeneous density could (at least in principle) be detected through astronomical observations. Qualitatively, quantum physics could provide a mechanism that could justify such an assumption, because according to its predictions a vacuum must have a minimum energy level with a constant density. It is well known, however (see e.g., Weinberg 1989), that although the value of ρ_Λ expected from astronomical considerations is extremely low, the one derived from quantum physics estimation is larger than this by a factor 10^{120}. Such an utter disagreement between the two expectations, known as the *cosmological constant problem*, makes this possibility infeasible.

Another option is to assume that there exists some kind of a field of unclear origin able to give birth to a repulsive force. It has to be stressed that such field could well have a non gravitational origin.[7] In this case it simply gives origin to an additional component of the action, as was done in Sect. 7.3, but without the implications of the equivalence principle. The other difference form special relativity is that, as usual, the invariant volume element is $\sqrt{-g}\mathrm{d}^4x$.

In the case of a scalar field ϕ, historically one of the first considered options (Ratra and Peebles 1988), it has been called *quintessence*[8] (Caldwell et al. 1998), and recalling Eq. (6.3.13) its action would be

$$S_{\mathrm{Q}} = \int \left(-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right)\sqrt{-g}\mathrm{d}^4x.$$

With the further simplifying assumption that the field is spatially homogeneous (i.e. that $\partial_i\phi = 0$) one can use the FLRW metric and the corresponding connection coefficients to show that the "equation of motion" of ϕ results

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{10.1.7}$$

[7]For example, an electromagnetic field can surely influence the dynamics of the bodies.

[8]The name is justified by the fact that a field is representative of a still unknown interaction of some kind, like a "fifth force", whose source could then well be called "quintessence".

Solving this equation demonstrates that, when H is sufficiently large, $\nabla_\alpha \phi \ll V(\phi)$, and recalling that its stress-energy tensor is given by the variational derivative of the action S_G with respect to the metric, namely

$$T_{\alpha\beta} = \frac{\delta S_G}{\delta g^{\alpha\beta}} = \nabla_\alpha \phi \nabla_\beta \phi + g_{\alpha\beta} \left(\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - V(\phi) \right),$$

the above approximation gives

$$T_{\alpha\beta} \simeq -g_{\alpha\beta} V(\phi),$$

so when $V(\phi)$ has a region almost constant in ϕ, the stress-energy tensor resembles that of a cosmological constant.

This is the simplest option for implementing what in general is defined as *dark energy*. In the more general case the model has the only requirement of giving an equation of state with a negative w, or more precisely $w < -1/3$, which is sufficient to guarantee the creation of the needed repulsive force.[9]

Antimatter as dark energy?

In a bare-bones summary of the above discussion, we can reasonably state that something might have to be added to the Einstein field equations in order to cope with the cosmological data, but if that something is of the "right-hand sided nature", i.e., something which contributes to the stress-energy tensor in the form of a new kind of source and/or interaction, then we have little or no idea about its actual nature.

The only thing that we know is that it has to introduce a repulsive interaction. A lot of work has been done to constrain the the appearance of its generating field, but this tells us nothing about the origin of such a field, whose characteristics, admittedly, are quite strange. We do not know precisely the equation of state of its sources, and the only reasonable candidate we have thus far, e.g., the energy of the quantum vacuum, gives a number in blatant disagreement with our expectations by an exceptional 120 order of magnitude factor.

There exist tentative proposals to identify such an elusive entity with antimatter on the basis of its supposed repulsive gravitational behavior. As is well known, antimatter has was been proposed in the context of quantum mechanics by Dirac in 1928, and experimentally discovered[10] by Anderson in 1932. Its elementary particles' constituents are characterized by an opposite electric charge with respect to normal

[9] As a side note, it is worth noting that the need for a repulsive interaction in cosmology seems not be limited to the "recent" acceleration of the expansion. At the other end of the history of the universe, another kind of "repulsive mechanism" in form of a field called *inflation* (Guth 1981) has been invoked to explain, among other things, the apparent uniformity of our Universe on causally disconnected regions.

[10] With the detection of the first antimatter particles, namely the positrons.

matter, and by a similar characterization in regard to the charges of other fundamental interactions.[11] In particular, there is a long-standing debate about their gravitational charge and interaction. The original position was that of rejecting the possibility of gravitational repulsion for antimatter (Morrison 1958; Schiff 1958, 1959; Good 1961) and stating that gravity is always attractive. Identifying the two statements as a single one, however, is misleading and it can be admitted only if it is implicitly assumed that there exist only one type of gravitational charge.

In order to better understand this last statement, let us recall Sect. 7.3, where it was shown that for even-rank fields the corresponding interaction is attractive between like charges, whereas for odd-rank fields the interaction between like charge is *repulsive*. The electromagnetic potential is a rank-one field, and like charges repel each other, but there exist two types of charges, and opposite ones interact attractively. Gravity is a rank-two field (or a rank-0 in non relativistic models) so like charges attract, but if another type of charge would exist, two particles of the opposite type would attract exactly like two "regular" ones, whereas two different charges could surely repel each other. Therefore from this point of view nothing prevents the existence of repulsive gravity.

Actually, the equivalence principle seems to suggest that such things as two types of gravitational charges cannot exist, but

1. This principle can be tested only for ordinary matter.
2. Mathematically, this reasoning makes sense because the equations of motion would change in the case of antimatter.

On the other hand, other studies have challenged the validity of the previous criticism (Nieto and Goldman 1991) and supported the existence of a gravitational repulsion (Chardin and Rax 1992; Chardin 1993, 1997). In particular Villata 2011) has shown that the equivalence principle is compatible with a repulsive antimatter when the latter is supposed to be CPT-transformed matter. This model requires that antimatter "travels backward in time", in the sense that normal matter "sees" an inverted temporal evolution for antimatter and vice versa, an hypothsis that had also been proposed by Feynman (1948, 1949). The debate is still unsettled (see, e.g., Cabbolet 2014 for an alternative explanation of gravitational repulsion) and it could probably come to an end only when the AEgIS (Scampoli and Storey 2014), ALPHA (Alpha Collaboration et al. 2013) or GBAR (Perez and Sacquin 2012) experiments at CERN will be able to observe experimentally the behavior of atoms of antihydrogen in free-fall.

Currently there exist two main models based on antimatter with gravitational repulsion which that can be cited for a certain level of development. One is that of Villata (2011, 2012, 2013, 2015) according to which:

1. Antimatter and matter have opposite gravitational charges and thus their gravitational interaction is repulsive (Villata 2011).
2. Matter and antimatter exist in similar quantities in the universe, an assumption that prevents the currently unexplained problem of the matter–antimatter asymmetry.

[11] For example, an opposite strange ness for the quarks.

3. The latter, located in the so called "cosmic voids", is responsible for the accelerated expansion, and the resulting cosmological model can reproduce current observational data with no need for dark energy and avoids other common issues such as that of the coincidence or the horizon problems, and does not request any initial singularity (Villata 2012, 2013).

The other one is that of Hajdukovic (2012, 2014) which shares with the previous one the claim about a gravitationally repulsive antimatter, but differs from it in that:

1. There is no significant amount of antimatter in the universe, however, it pervasively populates the quantum vacuum, where the existence of virtual pairs of particles–antiparticles is currently necessary in quantum physics models.
2. Thanks to the opposite charges of their components, these pairs behave as (virtual) gravitational dipoles and the vacuum energy density computed following this hypothesis is compatible with the cosmological expectations and solves the cosmological constant problem; as a consequence, a model in agreement with current cosmological observations can be conceived.
3. The model can be adjusted to explain also the inflationary phase of the universe, and leads to a cosmological model with no Big Bang and no initial singularity.

10.2 Beyond the Metric Tensor

The models described in the previous section can all be considered part of general relativity, in the sense that they do not infringe upon the basic assumption of this theory, which basically can be identified with the association between gravity and the metric tensor. We have seen, in fact, that the modification of the dynamics of the bodies sensitive to the gravitational interaction can always be attributed to the matter part of the Lagrangian. In other words, all of these "modifications" can be described either as an unknown non gravitational field or to an unknown gravity source. This implies that we can write $T_{\alpha\beta}$ in other ways which that although, while they can be strange and still unknown, do not change the essence of the field equations, namely the description of how gravity actually operates. Changing the geometry Lagrangian in a way that could not be "transferred on the right-hand side", instead, would have the completely different meaning of giving a distinct model of how gravity works.

Several theories of this kind have been conceived, but a large number of them have in common the presence of an additional "gravitationally coupled" field, i.e., a field expressed as a (pseudo-)Riemannian scalar, vector, or tensor (or better a rank 0, 1, or 2 and even larger order tensor) which an with the metric, instead of the stress-energy tensor. In this section, as in the next one, we mainly refer to Clifton et al. (2012) and to the references therein as a comprehensive and authoritative mathematical review of alternatives to general relativity, but when needed other sources will are be cited.

Here we have to refrain from the temptation of plainly giving a description of the different mathematical possibilities. All the previously exposed models started from some physical or experimental motivations, and in order to stress the physical content of theories it is important to take a similar starting point in this case too, although this means restricting our exposition to scalar-tensor theories only, which, however, are probably the most promising and the most studied theories among those included in this section. Mach's principle is quite important in this case, because it allows to show very clearly how scalar-tensor theories can emerge from its implementation.

10.2.1 Mach's Principle

Mach, Newtonian dynamics and the problem of absolute space

The equivalence principle, in its weak form, stresses the fact that the inertial and gravitational masses are exactly proportional, and Einstein incorporates such a principle in general relativity by identifying gravity and the spacetime geometry. Mach's principle instead arises from the need to explain what the inertial massis. Inertia is a property embedded in Newton's theory of dynamics when it asserts:

1. The existence of an absolute space in which bodies are moving
2. The possibility to establish an inertial reference system, with respect to which the bodies move in uniform rectilinear motion if no external forces are present

These two statements implicitly require that a test body in an otherwise empty space would move in rectilinear motion. In 1893 Mach argued that such a conclusion is nonsensical. This in fact asks for conceiving a motion with respect to an absolute space, whereas the principle of relativity at the foundations of Newton's dynamic itself is based on the fact that the motion of a body can make sense only when it is referred to another body.

This matter-of-principle problem can be paired with another, apparently obvious, coincidence. One can tell simply by observing a Foucault pendulum that the Earth is rotating with respect to an inertial frame. The same motion can be deduced independently, by direct observation of the motion of the stars[12] with respect to the Earth. These two independent motions identify exactly the same inertial reference system, in the sense that both the pendulum and the stars, infinitely separated from each other and with no apparent connection among them, are observed to be at rest with respect to this reference system. In other words they seem to support the existence of an a priori "absolute inertial space" with no direct connection with material bodies. Such absolute space, as we have already noted, does not have a good relationship with the principle of relativity. Therefore, if the pendulum looks "connected" with the absolute space and the stars exhibit the same connection, but absolute space is a disturbing concept, why can't we simply throw away such a concept and transitively

[12]Nowadays it would rather be more accurate to say "of the QSOs," but this is irrelevant to the reasoning.

suppose a direct connection between the two objects? The critical point in the definition of an inertial system, with respect to which the inertial properties are defined, is the need to include only relative motions in physical theories, on the basis that absolute space and absolute motion are not physically meaningful concepts.

Mach's Principle

This naturally to what is known as *Mach's Principle*, which posits that the inertial reference system, and therefore the inertial properties of the bodies, can be defined by some average motion among all the existing bodies. This generic assumption, though, is difficult to be "materialized", inasmuch as it implies that one would have to model some kind of simultaneous connection among all the bodies of the universe as a function of their relative motions. A task which that is difficult to conceive.

The subtlety of the problem raised by Mach does not prevent the development of the highly successful Newtonian framework, which, together with the difficulty of defining it mathematically, can explain why this issue is deemed important in principle but generally neglected in practice. Everything has its price, however, and what is confidently chased out through the door, usually sooner than later, sneaks back in through the window. Such a price is very well known, and it is called the "equivalence principle".

Mach's principle, equivalence principle, gravity and cosmology

Inertial mass can be interpreted as a measure of how much a body is "linked" to the absolute space, and the equivalence principle says that it is exactly proportional to the gravitational mass, thus the latter is linked in the same way to such space. But gravity, as an infinite-range interaction, is certainly connecting all the bodies, although not simultaneously, therefore why not suppose that the inertial properties of the bodies can be defined with respect to the relative gravitational motion of all the bodies among themselves? This tempting solution would allow us to reduce two principles (the existence of the absolute space and the equivalence principle) to one (something built on Mach's principle) and two entities that have to be "artificially" connected (the inertial and the gravitational masses) to one. Seemingly, this solution would possess other attractive properties. For example, we know that local bodies cannot have a great influence on the inertial properties. This is clear because what we call the "inertial properties" are indeed the simple, familiar, inertial mass, which in the Newtonian view is an intrinsic property of each body. This therefore requires that inertia, i.e., the "origin" of the inertial mass, as a first approximation should not depend on distance. The reasoning mentioned in Sect. 9.4 about the constancy of the integrated gravitational force has exactly this characteristic and can be applied to its motion as well. Consequently, connecting inertia with the relative motion among the gravitational sources would imply that it should not change from place to place. In the

Newtonian interpretation the inertial mass is a scalar, thus it behaves exactly in this way. This therefore would mimic one fundamental property of the absolute space. All these thoughts rely on the homogeneity of the space in the cosmological sense, therefore the characteristics of inertia would also be consistent with this assumption, which stands at the foundations of cosmology. Moreover, the same is true for isotropy, because if there existed preferred directions in the distribution of the gravitational motion they would be reflected in the same way in an anisotropy of the inertial properties as well. This would imply that the inertial mass could not be a scalar.

In summary, we can imagine that Mach's principle can be implemented by means of a gravity theory. Moreover, from our reasoning we can single out one fundamental property that such a theory must have, namely that the inertial mass, or its equivalent in such a theory, has to be fully determined by the (gravitational) mass distribution in the universe. A theory conceived in this way would be capable of defining a sort of "dynamical inertial reference system" without any need to resort to absolute space.

These considerations, although very appealing, are just interesting speculations, and once again they are difficult to conceive in precise mathematical terms, however, they lead us directly to Einstein's point of view.

10.2.2 Mach's Principle and General Relativity

In spite of these problems, at this point it is easy to understand why Einstein, whose theories pursued explicitly the suppression of the concept of absolute space, deemed of the utmost importance the incorporation of Mach's principle in his general relativity theory. Moreover, as we show in a moment, this provided him another strong reason for the introduction of the cosmological constant.

By assuming the equivalence principle there is no distinction between inertial and gravitational masses, which are substituted by the more general concept of stress-energy tensor as the sources of the gravity field, represented by geometric means with the metric tensor. Additionally, the field equations seem to go in the right direction because the metric field completely substitutes the concept of absolute space and of inertial reference field, and moreover it is dynamic, in the sense that it is determined by the distribution of mass-energy of the universe. Therefore, if one could prove that $g_{\alpha\beta}$ is fully determined by such a distribution, then it could be concluded that Einstein's implementation of the equivalence principle implies also Mach's principle, leaving us with just one assumption, as desired.

The problem of the boundary conditions at infinity

The metric $g_{\alpha\beta}$ comes from the solution of a set of differential equations, thus it can be fully determined only by imposing some initial conditions. Einstein (1917), however, found that a difficulty arose when these conditions had to be set at spatial infinity, namely that:

1. Either one has to assume the familiar condition that at infinity the metric reduces to $\eta_{\alpha\beta}$
2. Or one has to avoid the problem of initial conditions at infinity by introducing the cosmological term which had the consequence of producing an unbounded, but *finite* Universe

Einstein argued that the first solution was unsatisfactory because:

1. It implies the choice of a preferred reference system.
2. A test particle at infinity would still have an *inertial mass*, whereas to the contrary, because r "[...] there can be no inertia relatively to space, but only an inertia of masses relatively to one another [...]" one had to require that on grounds inertia of a test particle at infinity and away from any other gravity source has to vanish because the relative motion tends to zero.

It is not difficult to recognize Mach's principle in the second statement, which can also be viewed in another way. Because by the equivalence principle inertia is given by the metric tensor, then having an "irreducible" component $\eta_{\alpha\beta}$ at infinity implies that an "[...] inertia would indeed be influenced but not be conditioned by matter [...]" or in other words that there is a corresponding "irreducible" part of inertia at infinity, thus $g_{\alpha\beta}$ is *influenced* but not fully determined by $T_{\alpha\beta}$.

The Cosmological constant and Mach's principle

Thus we come to the surprising conclusion that Einstein arrived at the cosmological constant through the need of including Mach's principle into general relativity, and the static universe came only as a consequence of this attempt. Unfortunately, this effort shared the same fate of that constant, but not because of the discovery of the expansion of the universe. Einstein in fact thought that in the case of a positive value of the cosmological constant and empty space ($T_{\alpha\beta} = 0$) Eq. (10.1.4) admitted no solution, or the trivial solution $g_{\alpha\beta} = 0$. This would be perfect in Mach's sense, because it would consistently imply that in the absence of matter there is also no inertia. However we have already seen how this equation admits the non trivial solution of the de-Sitter space, which is named after the person who discovered such a result, thus proving that Einstein's claim was false.

Left only with the first of the above choices, we are forced to admit that inertia is influenced but not fully determined by the mass-energy distribution, and thus that Mach's principle can be only partially incorporated in general relativity.

10.2.3 Mach's Principle and Scalar-Tensor Theories

The failing of general relativity in respect to Mach's principle can also be put in a different way. Previously we mentioned that the connection between gravity and inertia, together with the assumption of the homogeneity of space, would imply that inertia cannot change from place to place (and in time) and therefore a constant inertial mass. This, however, is not completely true. Homogeneity of space, actually, is just an average property, so the alleged constancy can be required only on average, or better at large scales. On the contrary, if Mach's principle requires that inertia depends on the dynamical status of the mass-energy of the universe, we have to expect that, in principle, a change of such system in time or space, would bring to a change of the inertial properties, i.e., of the inertial mass, of the matter.

The connection between Mach's principle and scalar fields

We give more details on such variation below, but first let us assume that this hypothesis makes sense and try to deduce the consequences on the mathematical formulation of a gravity theory. As we already know, the link between the inertial and gravitational masses is established by the weak equivalence principle, and it is mathematically represented by their ratio, i.e., by (Eq. (3.3.4)) the i.e., gravitational constant $G = m_{\rm G}/m_{\rm I}$. Therefore, if the gravitational mass is a constant scalar but we have to admit a variation of $m_{\rm I}$, the only possibility is to assume that G is no longer a constant, but that it can vary in space and/or time. We can thus draw the first important conclusion that any theory aiming at incorporating Mach's principle should admit the possibility of a variation of G, whereas on the contrary requiring $G = \text{const}$ implies the denial of such principle. This is why, eventually, general relativity failed in this respect. Experiments trying to estimate the variation $\dot{G}/G$ are thus also testing the effects of Mach's principle. Moreover, the assumption that $G \equiv G\left(x^{\alpha}\right)$ defines the coupling parameter between matter and Gravity as a scalar field, rather than just as a constant. This suggests that Einstein's goal could be better served by introducing a scalar field as a replacement of G in field theories of gravity.

Before setting forth the mathematical exposition of such alternative theories, it is worth stressing two points.

Variable G and the equivalence principle

The first one is about some expectations about the magnitude of the changes of G, and therefore of the scalar field. Surely these changes have to be tiny, and this not only because of the experimental evidence, but also in view of the fact that large-scale motion variations, which contribute as an "integral" over a large space, have to be preponderant over small-scale ones, but at the same time they have to be tiny because the motion variations tend to cancel out on average. However tiny they have to be, these changes cannot be avoided in consideration of the fact that we know for sure that there is at least one case in which Mach's principle requires that inertia has to change, and in an all-but-tiny way, namely when a test particle is at infinity and inertia has to vanish. Therefore the expectation is that local variations of inertia are larger when large-scale dishomogeneities in the relative motion of gravitational sources can be singled out from the "average" background. In this sense the evidence for variations available in our everyday experience should be much tinier than those possibly present at the edge of a galaxy or of a galaxy cluster, which is qualitatively in agreement with very well-known observational results.

The second consideration is about the equivalence principle. Up to now we have always stated the "equivalence of two versions of the weak equivalence principle". The original version declares the equality of the inertial and gravitational masses, and we showed that this could be transformed in an assertion about the equivalence between gravity and accelerated reference systems. In Chap. 7 we pointed out that the existence of gravitational sources different from masses called for a broadening of the application range of this principle, and thus to its extension to Einstein's equivalence principle. The weak form, however, remained unchanged in its own validity range. Now we realized that the notion of a "static" inertial mass has to fade out when Mach's principle is called into action, so the version referring to the reference systems should in any case be considered more general.

Mathematical set up of scalar-tensor theories

In order to get a general understanding of the mathematical formulation of a scalar-tensor theory of gravity, let us recall the expression of the Lagrangian density of general relativity. We refer to the most general case of Eq. (10.1.1) which, together with the matter term, reads

$$\mathcal{L}_{\mathrm{GR}} = \sqrt{-g}\left(\frac{1}{2\kappa}R - 2\Lambda\right) + \mathcal{L}_{\mathrm{M}}\left(\Psi, g_{\alpha\beta}\right). \tag{10.2.1}$$

A scalar-tensor theory that aims at incorporating Mach's principle,[13] as we have seen, cannot retain the $G = \mathrm{const}$ assumption, and as a "general rule" it has to be substituted with a field ϕ wherever the former appears. This would suggest that in this case the corresponding Lagrangian density could be written

$$\mathcal{L}_{\mathrm{ST}} = \frac{c^4}{16\pi}\sqrt{-g}\left(\frac{1}{\phi}R - 2\Lambda\left(\phi\right)\right) + \mathcal{L}_{\mathrm{M}}\left(\Psi, \phi g_{\alpha\beta}\right),$$

[13] Which is thus far the only physical motivation we have given to introduce an additional scalar field.

but this expression is not completely correct, nor it is the most general one. First, and most important, it has to be remembered that we have added a new dynamical field, which will necessarily bring in a component in the geometric part referring to its "free-field action". As usual this component has to be written like the equivalent of the Euclidean quantity $\nabla\phi \cdot \nabla\phi/\left(8\pi G\right)$ in a four-dimensional pseudo-Riemannian space, i.e., $\left(\nabla_\mu\phi\nabla^\mu\phi\right)/\phi = \left(\partial_\mu\phi\partial^\mu\phi\right)/\phi$, where the equality holds because covariant and partial derivatives coincide in the case of scalar fields. Moreover, one could reasonably assume that the dependence on ϕ of the factors multiplying the components of the action might be different in a more general formulation, which brings us to the expression (Clifton et al. 2012)

$$\mathcal{L}_{\mathrm{ST}} = \frac{c^4}{16\pi}\sqrt{-g}\left(f\left(\phi\right)R - g\left(\phi\right)\partial_\mu\phi\partial^\mu\phi - 2\Lambda\left(\phi\right)\right) + \mathcal{L}_{\mathrm{M}}\left(\Psi, h\left(\phi\right)g_{\alpha\beta}\right),$$

where $f\,()$, $g\,()$, and $h\,()$ are generic functions. We can notice that in this formula the field, as anticipated, couples to the metric tensor, as G did in General Relativity. This justifies the name of scalar-tensor theories attributed to these theories, and the definition of non minimal coupling (of ϕ and $g_{\alpha\beta}$) because the two gravitational-related fields couple directly with each other. Although this is the most general expression for such kind of theories one can always transform the last one by putting $h\left(\phi\right)g_{\alpha\beta} \rightarrow g_{\alpha\beta}$. This kind of transformation is called *conformal transformation*, because it leaves unaltered the angles between two vectors and it is equivalent to choosing a reference system in which matter does not couple directly with the scalar field, but just with the metric. Such a reference system is called a *Jordan frame*.

This obviously will modify the definition of f, g, and Λ, and because they are arbitrary with respect to the other components of the Lagrangian but at the same time they have to be interrelated, with no loss of generality we can decide to put $f\left(\phi\right) \equiv \phi$ as long as we keep their mutual relations by also putting[14] $g\left(\phi\right) = \omega\left(\phi\right)/f\left(\phi\right) = \omega\left(\phi\right)/\phi$. In summary the most general scalar-tensor Lagrangian in the Jordan frame results in

$$\mathcal{L}_{\mathrm{ST}} = \frac{c^4}{16\pi}\sqrt{-g}\left(\phi R - \frac{\omega\left(\phi\right)}{\phi}\partial_\mu\phi\partial^\mu\phi - 2\Lambda\left(\phi\right)\right) + \mathcal{L}_{\mathrm{M}}\left(\Psi, g_{\alpha\beta}\right), \qquad (10.2.2)$$

which puts in to evidence the degree of freedom we have in choosing the coupling with the metric by means of the so-called coupling parameter $\omega\left(\phi\right)$.

It is important to observe that, in varying the resulting action to obtain the field equations of a scalar-tensor theory it is not sufficient to compute the variation with respect to $g^{\alpha\beta}$, but also with respect to ϕ because of its additional dependence on this field.

[14] It is agreed that, whatever for the transformation, one can keep the same symbol in the cosmological part, which therefore will constitute a definition per se.

An example: the Brans–Dicke theory

Historically, the first and simplest representative of this class of theories is the *Brans–Dicke theory* (Brans and Dicke, 1961), which was explicitly conceived with the goal missed by Einstein, i.e., that of implementing to a full extent Mach's principle in a gravity theory. Its action can be obtained from Eq. (10.2.2) by putting $\omega(\phi) \equiv \omega = \text{const}$ and $\Lambda = 0$. The condition of ω is in fact just the simplest assumption one can make on the coupling between the scalar field and the metric, whereas, if the cosmological constant was introduced for the same reason that brought in the field, it is reasonable to expect that ϕ can be sufficient to reach the goal.

Varying the action with respect to the metric, and defining the generally covariant d'Alambertian operator $\Box$ as

$$\Box^2 \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$$

gives the field equations[15]

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi}{c^4}\phi^{-1}T_{\alpha\beta} + \frac{1}{\phi}\left(\nabla_\alpha\partial_\beta\phi - \Box^2\phi\right) + \frac{\omega}{\phi^2}\left(\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) \tag{10.2.3}$$

whereas with respect to the scalar field one can obtain the simplest expression after eliminating the Ricci scalar by contraction with the metric tensor, thus resulting in the end

$$\Box^2\phi = \frac{8\pi}{(2\omega+3)\,c^4}T. \tag{10.2.4}$$

10.3 Going Beyond in Different Ways

Things quickly get very complicated when it comes to this point. The following are in fact definitely subjects for advanced theoretical studies and therefore are shown in an extremely concise way. The goal of this section is that of providing a glimpse of other possibilities, along with some of the motivations that are at the basis of these which ideas and that in many cases between the two have in common, methods, without harassing the reader with a mathematical explanation, except for the few cases in which this can give a better understanding of some selected issues.

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[15]The term $\nabla_\alpha\partial_\beta\phi$ would in general be written as $\nabla_\alpha\nabla_\beta\phi$, but in this case the first covariant derivative can be substituted with a scalar one because of the scalar nature of ϕ, and the second one remains a covariant derivative because $\partial_\beta\phi$ is a four-vector.

10.3.1 Beyond the Einstein–Hilbert Action

When in Sect. 8.3 the problem arose of defining the geometric part of the Lagrangian density for a relativistic theory of gravity, we opted for $\mathcal{L}_{\mathrm{G}} \propto R$. This choice was basically motivated by its simplicity, and indeed it is the simplest possible Lagrangian that is quadratic in spacetime derivatives. This condition has to be required in order to produce non trivial solutions for $g_{\alpha\beta}$, and at the same time to reduce to the Poisson equations in the Newtonian regime.

We can stay perfectly fine with this choice as long as there are no reasons to try more complicated versions of this action, but at the same time it must always be remembered that:

1. From a logical point of view, the fact that up to now all the accepted gravity theories have been second-order is not a compelling reason to suppose that the Lagrangian of the "correct" model of gravity has to be quadratic in spacetime derivatives.
2. This is true especially because the only physically compelling reason that allows us to reject a Lagrangian is the case in which this cannot reduce to the Poisson equation,[16] but from a mathematical point of view this condition does not necessarily imply a limit at the second-order derivatives.

In principle, therefore, field equations of higher order in spacetime derivatives and the corresponding higher-order gravity theories cannot be excluded a priori. On the other hand, there are at least two good reasons to admit the possibility of higher-order theories of gravity.

The first is already familiar to us: it is a cosmological motivation. As mentioned in several reviews, e.g., Capozziello and de Laurentis (2011), Clifton et al. (2012) and in references therein, they can be good for cosmological models where general relativity needs additional assumptions, i.e., both at the early and late ends of the universe history. Not only some of these theories seem able to fit the observational scenario, but also their approximations can be arranged to reproduce the aspect of simpler theories, such a general relativity with inflation.

The second is somewhat new in our discussion, and has to do with the possibility of formulating a quantum theory of gravity. We have occasionally mentioned the well-known problem of the incompatibility between quantum physics and general relativity (see, e.g., Feynman et al. 1995). Although general relativity and quantum theories are both in very good agreement with the experimental results[17] the accepted cosmological models include phases in which energies are in a regime that probably can be addressed only by a quantized theory of gravity. This fact alone would be enough to justify the research in this field.

[16] And even in this case, as we have just seen, the cases with and without the cosmological term are admitted.

[17] Obviously the former falls in to troubles where dark matter and dark Energy are needed, but in principle, as explained in this chapter, this could not be a problem of the theory, but just that of identifying some missing components.

Field equations of quantum gravity theories are higher-order, and their classical counterpart can be seen as a sort of more tractable "testing field" that can help to select a good candidate in the quantum realm.

There is a very natural way to introduce higher-order terms in field equations, i.e., introducing an action with a more general form than the Einstein–Hilbert one. In Sect. 8.3.1 we noticed that even R alone would generate these terms from the variation $\delta R_{\alpha\beta}$ of the Ricci tensor. These did not enter in the final result simply because they could be "confined" to the boundary of the integration domain, where these are zero by definition. However, it would be sufficient to admit a more general Lagrangian to make them unavoidable.

If we limit to the option of having only the Ricci scalar in the Lagrangian, then we can simply imagine admitting a geometric action where R is replaced by a generic function $f(R)$

$$S_{\mathrm{G}} = \frac{1}{\chi} \int \sqrt{-g} f(R) \, \mathrm{d}^4 x, \tag{10.3.1}$$

which explains the name of *$f(R)$-theories* (see, e.g., Capozziello et al. 2010, Capozziello and Faraoni 2011; Capozziello and de Laurentis 2011; Capozziello and Laurentis 2015 for detailed and comprehensive expositions of these theories) given to such a very popular branch of the extended theories of gravity. Here χ is a constant as κ in the Einstein–Hilbert action. We gave it a different symbol to stress that the two can be different.

Applying the usual variational procedure to the complete action with the matter term one gets the field equations

$$f'(R) R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} - \nabla_\alpha \nabla_\beta f'(R) + g_{\alpha\beta} \Box f'(R) = \frac{\chi}{2} T_{\alpha\beta}, \tag{10.3.2}$$

where $T_{\alpha\beta}$ as usual is the stress-energy tensor defined by the variational derivative of $\mathcal{L}_{\mathrm{M}}$ with respect to the metric and the prime indicates the derivative with respect to R. The above expression, as anticipated, is clearly fourth order.

The apparently innocent expression "usual variational procedure" actually hides some non trivial assumptions. First all this can be an approximate relation in neglecting the contribution of some boundary terms. Second, this is the result of the so-called an metric approach, which computes the variation only with respect to the metric.

We recall that there is another possibility, the so-called Palatini approach, in which no a priori relation between the metric and the connection coefficients is assumed, so that these two quantities can be assumed as independent fields.

We have seen in Sect. 8.4 that for general relativity these two methods are completely equivalent: the metric approach assumes the vanishing of the covariant derivative of the metric thus deriving the field equations, whereas the Palatini approach derives two field equations, one for $R_{\alpha\beta}$ and the other for the connection coefficients, and from the second it derives (and thus requires) the metric connection and the condition on the vanishing of the covariant derivative of the metric.

In general, for $f(R)$ theories, these two approaches are not equivalent and the Palatini approach can bring in to evidence different types of connections.

10.3.2 Beyond Spacetime

The quest for quantum theories of gravity has also followed another way, which basically rests on the idea that the correct geometrical model of our world has more dimensions than the usual four of the spacetime.

This apparently weird concept can be dated back to the attempts of Kaluza and Klein (Kaluza 1921; Klein 1926) on the unification of gravity and electromagnetism in a quantum-aware framework. These theories were built on a five-dimensional pseudo-Riemannian spacetime supplemented with an extra spatial dimension. In order to comply with the experimental evidence that supports the existence of just the ordinary $3+1$ space and time, they assumed that such extra dimension was, as it is usually said, *compactified*. We can roughly interpret this assumption by saying that this dimension is "small". In slightly more rigorous terms it can be said that the coordinate axis of the extra dimension does not stretch "linearly" in the range $(-\infty, +\infty)$, but on a much smaller range constituting the radius of a small circle, say L, which thus represents the "extension" and the "sphere of influence" of the effects of this extra dimension.

Save for that of our eyes, probably the clearest evidence of the $3+1$ structure of the spacetime lies in the characteristic $1/r^2$ behavior of the gravity force. If, in quantum-mechanical fashion, one imagines gravity as an interaction mediated by some elementary particles (called bosons) and that, roughly speaking, the intensity of such an interaction at a certain distance is proportional to the number of particles reaching the unit (hyper-)surface, then in three dimensions the force has to be $\propto r^{-2}$. This is immediately understood by noting that the density of the particles will decrease with the same law in three dimensions, because the surface surrounding a volume of radius r is proportional to r^2. In a general number of dimensions N, however, the same force will be proportional to $r^{-(N-1)}$ because the area of the hypersurface is r^{N-1}.

This reasoning obviously works exactly only in Euclidean spaces, and it has been chosen for the sake of clarity, but analogous considerations are perfectly valid in a pseudo-Riemannian geometry. Proceeding along this path it is also intuitive that if the extra dimension is small, in the sense explained above, the area of a hypersurface with a radius $r \gg L$ is still almost proportional to r^2, so that deviations from the r^{-2} law to be small. At the same time, the most evident requirement of a quantum theory is that the quantum effects start to be "visible" at small temporal and spatial scales. In our idealized geometric model indeed it happens that the deviations mentioned above become evident when the theory works at scales in which $r \sim L$, which explains why this compactification mechanism has been adopted for such models.

These ideas have been pursued in many models, and are used also in the so-called string theories that rely on a space with at least 10 dimensions (see, e.g., Zwiebach

(2009) for an accessible introduction to string theories). One difficulty arises from the fact that the compactification can be done in many different ways. First of all, the only requirement of the extra dimension is that its coordinate must have a finite range, instead of going from $-\infty$ to $+\infty$, and that the extremities of such range can be identified as a single point. We have associated such characteristic with a “circle” but this has not to be intended literally. An “ellipse”, a “square” or any other closed curve would have done the work eventually. Second, and more important, if the extra dimensions are more than one they can be compactified together, opening the possibility of having compact subspaces with several different topologies, like e.g. (in the case of two extra dimensions) that of a torus, of a sphere, or that of another more complex figure. The more the extra dimensions, the more the ways these can be compactified, which correspond to different theories.

Another way the extra-dimensional geometries can be accommodated without contradiction with the experimental evidence is in the so-called *Brane-world* scenario (Blumenhagen et al. 2013), which is strictly related to that of string theories. The basic idea of this paradigm is that the spacetime can have many more dimensions than what we normally perceive, and that such dimensions can also develop on larger scales with respect to the compactification scenario. However, the fields describing matter and non-gravitational interactions propagate only along 3+1 subspaces called branes. The branes, in turn, have their own dynamics, generally ruled by gravity, which is not strictly confined within a single brane.

10.4 A Mathematical Tool for Testing Gravity Theories: The Parametrized Post-Newtonian Formalism

After the very quick survey of the last sections on the many theoretical possibilities for developing a theory of gravity alternative to general relativity, and with the goal of showing an example of the mathematical tools developed to help in the testing of such alternatives, we conclude the chapter and the book with a final section having a more experimental flavour. Indeed, the reasons briefly touched at the beginning of the chapter, and the relative ease on at finding new theories of gravity, stimulated massive theoretical production over the years, of which those quickly touched upon above constitute only a tiny part. Moreover, although in many cases they can be ruled out by the usual criteria presented in Sect. 2.3, several are still viable (see, e.g., Will 1993; Hořava 2009; de Rham et al. 2011 for reviews and references). Moreover, approximately from the 1960s, the experimental testing of general Relativity started to build momentum, marking the beginning of what has been called a “golden era” for studies in gravitation.

This theoretical abundance and the simultaneous need to compare the experimental results in the theoretical arena inspired the development of a mathematical formalism useful to compare a wide range of theories in the weak-field limit called the parametrized Post-Newtonian formalism, or PPN formalism. There are other "tools" like this available in different contexts (Lightman and Lee 1973; Mattingly 2005) and the PPN one is valid only under some specific assumptions, however, it is probably the most known and widely used formalism of this kind, which is the reason why it has been included in this book.

Assumptions and validity of the PPN formalism

More precisely, as suggested by its name, this formalism can be considered a generalization of the post-Newtonian limit described in Sect. 9.3, thus sharing with it the same assumptions, namely the weak-field, slow-velocity, and low-energy regimes, and even before this its applicability to metric theories of gravity only. In this framework the post-Newtonian order of a generic metric theory depends on a set of ten parameters, coming from the five constraints listed below.[18] Setting these parameters to specific values reproduces the PN order of a particular theory, which in practice is characterized by such a set of values.

The first condition makes the PPN formalism the most suitable framework for the description of solar system-based tests. On the other hand any non metric theory of gravity must necessarily violate the equivalence principle, which therefore represents a natural filter between metric and non-metric theories of gravity. The reason for such a clear relation can be intuitively understood by remembering that it was exactly the assumption of the validity of the equivalence principle that led us to one of the essential statements of a metric theory of gravity, namely that of geometrization of gravity. This issue, however, is extensively discussed and explained in Di Casola et al. (2015).

This allows us to understand why the restriction to metric theories only is generally not deemed a serious limitation for the PPN formalism. We had already highlighted that the equivalence principle is extremely well tested, and in fact current experimental limits rule out all the known non metric theories of gravity.

10.4.1 The PPN Formalism in a Nutshell

In Sect. 9.3 we showed that the post-Newtonian limit of general relativity is that of Eqs. (9.3.6)–(9.3.8). The full derivation is too long to be given here, but it can be found in standard texts such as Will (1993) or Misner et al. (1973).[19]

The essential point is that under the five assumptions cited above the same metric coefficients can always be written as

[18] To be more precise, as pointed out by Misner et al. (1973), only four of them are strictly necessary. Giving up the sixth would increase the number of parameters, whereas retaining it implies excluding a few metric theories, which, however, are ruled out by experiments.

[19] These two texts use two different set of parameters to define the generalized PN limit.

$$g_{00} = -1 + 2U + h_{\mathrm{t}}\left(U^2, \Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \mathcal{A}\right) \tag{10.4.1}$$

$$g_{0i} = h_{\mathrm{m}}\left(V_i, W_i\right) \tag{10.4.2}$$

$$g_{ij} = 1 + h_{\mathrm{s}}\left(\delta_{ij} U\right) \tag{10.4.3}$$

where the h terms indicate linear functions of some post-Newtonian potentials for the temporal, mixed, and spatial components of the metric. The exact form of the potentials is not important in this context, and it can be found in the above standard texts. It should not be surprising to know, however, that their expressions are closely related to those derived for the post-Newtonian order of general relativity. This is also because the PPN formalism is generally used with reference to the dynamics of the solar system, which means that the starting stress-energy tensor is that of a perfect fluid, as in the previous case.

Constraints on the post-Newtonian corrections

What is important to stress here, instead, is that such potentials and the corresponding corrections can be found from the constraints mentioned above, which are now worth being listed:

1. Obviously, the corrections must be of the post-Newtonian order.
2. Because the metric coefficients are dimensionless, the same must be true for the hs.
3. The corrections h_{00}, h_{0i}, and h_{ij} have to behave as three-dimensional scalars, vectors, and rank 2 tensors, because the post-Newtonian coordinate system is quasi-Cartesian, and therefore we can treat space and time coordinates as in Euclidean geometry; this condition implies that the sought functions can depend only on differences of position vectors (i.e., $r \equiv |\boldsymbol{x}_2 - \boldsymbol{x}_1|$, r_i, $r_i r_j$).
4. Far from the gravity sources the metric should become that of special relativity, which means that $\lim_{r\to\infty} g_{\alpha\beta} = \eta_{\alpha\beta}$. This also means that not only $h_{\alpha\beta} \ll \eta_{\alpha\beta}$ because of the PN approximation, but also that $\lim_{r\to\infty} h_{\alpha\beta} = 0$.
5. The corrections cannot depend on the gradients (i.e., the spatial derivatives) of the gravity sources, namely by mass, pressure, and energy densities. Products of velocities and time derivatives are admitted. This last constraint simplifies the calculations and reduces the final number of PPN parameters, but as stressed above it is not strictly necessary.

The spatial PPN correction

Just to give an example of how the calculations can proceed, we show here the simplest case, i.e., the one of g_{ij}. We know from Sect. 9.3 that the corrections of the spatial components must be at the ϵ^2 level, that is equivalent to constraint 1.

Constraints numbers 3 and 4 allow us to select the only two functions that are rank 2 Euclidean tensors vanishing at infinity, namely $\delta_{ij}U$ and U_{ij}, where

$$U(\boldsymbol{x},t)=\int G\frac{\rho(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x$$

is the usual gravitational potential and

$$U_{ij}(\boldsymbol{x},t)=\int G\frac{\rho(\boldsymbol{x}',t)\,(x_i-x_i')\left(x_j-x_j'\right)}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}^3x.$$

Finally, in order to keep the correction dimensionless (constraint 2) we have to multiply it by the usual factor c^{-2}.

We could then write the correction as a linear combination of these two potentials

$$h_{\mathrm{s}}=2\gamma\delta_{ij}\frac{U}{c^2}+2\Gamma\frac{U_{ij}}{c^2},$$

where γ and Γ are two post-Newtonian parameters. It is worth stressing that in this context the expression "linear combination" has to be intended with reference to the potentials. In practice the two parameters do not necessarily have to assume numerical values, as show see in the next section. Moreover, it is easy to note, following Misner et al. (1973), that it is always possible to set $\Gamma=0$ by means of an infinitesimal coordinate transformation, so that

$$g_{ij}=1+2\gamma\delta_{ij}\frac{U}{c^2}, \tag{10.4.4}$$

as required as Eq. 10.4.3. The factor 2 was added just by a matter of convenience so that general relativity can be recovered by putting $\gamma=1$. From the above formula it is evident the physical interpretation that is usually given to this parameter as the "measure" of how much spatial curvature is produced by a given Newtonian potential, which is by a specific rest mass. Similar interpretations are given also to the other nine parameters, as shown in Table 10.1.

The γ parameter in general relativity

We mentioned that the PPN formalism was a convenient tool to have a common parametric description of the predictions of many theories. This can be seen immediately by looking at the light deflection phenomenon. Using the same procedure of Sect. 9.1.2, but with a Schwarzschild PPN metric, one can find that the amount of deflection for null geodesics from $-\infty$ to $+\infty$ is equal to (see, e.g., Will 1993)

Table 10.1 The 10 PPN parameters in the conventions of Will (1993, 2014) and their physical interpretations

PPN parameter	Physical interpretation
γ	Amount of spatial curvature produced by unit rest mass
β	Amount of non linearity in the superposition of the Newtonian gravitational potential
ξ	Existence of preferred-location effects
α_1 α_2 α_3	Existence of preferred-frame effects
ζ_1 ζ_2 ζ_3 ζ_4	Non conservation of total momentum

$$\delta\varphi = \frac{2\,(1+\gamma)\,GM}{c^2 r_0}, \tag{10.4.5}$$

which, as expected, reduces to the value predicted by general relativity of Eq. (9.1.16) if one puts $\gamma = 1$. On the other hand, using this formalism one can easily convert a measurement of light deflection into an estimation of the value of γ, which then can be used as a quick and comprehensive measure of the experimental constraints set by this phenomenon on all the gravity theories accessible with the PPN formalism.

10.4.2 The Weak-Field Approximation of the Brans–Dicke Theory and Its PPN parameters

As an example for a theory of gravity different from general relativity, we to sketch the derivation of the weak-field approximation of the Brans–Dicke theory following closely the procedure outlined in the original paper of Brans and Dicke (1961). From these expressions we are easily able to deduce the expression for the PPN γ parameter in this case.

The metric tensor is thus expressed as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, but in this case the weak-field has to refer also to ϕ, which is expanded to first order in mass densities so that $\phi = \phi_0 + \xi$, where ϕ_0 is a constant. Moreover, the case of a static sphere of mass M is considered for the source of $T_{\alpha\beta}$, so that the integrated stress-energy tensor gives $2Mc^2$.

In this approximation the field equation for ϕ becomes

$$\Box^2\phi \simeq \Box^2\xi \simeq \eta^{\alpha\beta}\partial_\alpha\partial_\beta\xi = \frac{8\pi}{(2\omega+3)\,c^4}T$$

which can be integrated giving a retarded time solution. Because we are considering a static stress-energy source the retarded time in the solution has no meaningful influence, and it results in

$$\phi = \phi_0 + \xi = \phi_0 + \frac{2M}{(2\omega+3)\,c^2}. \tag{10.4.6}$$

The solution for the metric tensor is instead

$$\begin{aligned} g_{00} &= -1 + \left(\frac{2M}{\phi_0 c^2 r}\right)\left(1 + \frac{1}{2\omega+3}\right) \\ g_{ii} &= 1 + \left(\frac{2M}{\phi_0 c^2 r}\right)\left(1 - \frac{1}{2\omega+3}\right)\delta_{ii} \\ g_{\alpha\beta} &= 0 \quad \text{for } \alpha \neq \beta. \end{aligned}$$

Because at the Newtonian order $g_{00} = -1 + 2GM/c^2r$, it is

$$G = \frac{1}{\phi_0}\left(1 + \frac{1}{2\omega+3}\right),$$

which is incorporating the Machian idea of a gravitational constant that can evolve in space and time. Obviously this definition sets a relation between the value of ϕ at the present time (ϕ_0) and ω. This yields

$$\begin{aligned} g_{ii} &= 1 + \left(\frac{2MG}{c^2 r}\right)\left(1 - \frac{1}{2\omega+3}\right) / \left(1 + \frac{1}{2\omega+3}\right)\delta_{ii} \\ &= 1 + \frac{2GM}{c^2 r}\left(\frac{1+\omega}{2+\omega}\right)\delta_{ii}. \end{aligned}$$

The above formula, by comparison with Eq. (10.4.4) finally gives

$$\gamma = \frac{1+\omega}{2+\omega}.$$

10.5 Exercises

Exercise 10.1 Derive the alternative forms of the complete and vacuum Einstein field equation, Eqs. (10.1.3) and (10.1.4), respectively, in the case with the cosmological constant.

Solution 10.1 The derivation easily follows from the contraction of Eq. (10.1.2), which gives, remembering that $g^{\alpha\beta}g_{\alpha\beta}=4$,

$$R=-\frac{8\pi G}{c^4}T+4\Lambda,$$

so the left-hand side of the field equations becomes

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\left(-\frac{8\pi G}{c^4}T+4\Lambda\right)+g_{\alpha\beta}\Lambda=R_{\alpha\beta}+\frac{4\pi G}{c^4}g_{\alpha\beta}T-g_{\alpha\beta}\Lambda.$$

The complete field equations then become immediately

$$R_{\alpha\beta}-g_{\alpha\beta}\Lambda=\frac{8\pi G}{c^4}\left(T_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}T\right),$$

and in vacuum $T=0$ and $T_{\alpha\beta}=0$ therefore

$$R_{\alpha\beta}-g_{\alpha\beta}\Lambda=0.$$

Exercise 10.2 Show how the cosmological constant can be interpreted as if the vacuum were a perfect fluid with equation of state $p_\Lambda=-\rho_\Lambda c^2$.

Solution 10.2 Writing Eq. (10.1.2) as

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=\frac{8\pi G}{c^4}\left(T_{\alpha\beta}-g_{\alpha\beta}\frac{\Lambda c^4}{8\pi G}\right)$$

makes it evident that the cosmological term can be interpreted as if $-g_{\alpha\beta}\Lambda c^4/8\pi G$ were some kind of "vacuum stress-energy tensor" $T^{(\Lambda)}_{\alpha\beta}$. It is important to stress that this is not just a matter of "changing the side" of the cosmological term, but it is assured from the fact that because $\nabla_\alpha g^{\alpha\beta}=0$ then $T^{(\Lambda)}_{\alpha\beta}$ is divergence-free; i.e., it complies with the fundamental requirement of a stress-energy tensor. Assuming now that vacuum behaves like a perfect fluid, for which it is in general $T_{\alpha\beta}=\left(\rho c^2+p\right)u_\alpha u_\beta+pg_{\alpha\beta}$, implies

$$T^{(\Lambda)}_{\alpha\beta}=-g_{\alpha\beta}\frac{\Lambda c^4}{8\pi G}=\left(\rho_\Lambda c^2+p_\Lambda\right)u_\alpha u_\beta+p_\Lambda g_{\alpha\beta}.$$

This is satisfied if

$$p_\Lambda=-\frac{\Lambda c^4}{8\pi G}$$

and

$$\rho_\Lambda=-\frac{p_\Lambda}{c^2}=\frac{\Lambda c^2}{8\pi G}. \tag{10.5.1}$$

Vacuum can thus be seen as a perfect fluid with equation of state $p = w\rho$ where $w = -1/c^2$, or $w = -1$ in geometric units.

Note also that Eq. (10.5.1) implies that, because Λ is assumed to be constant, so it has to be for ρ_Λ. This is consistent with Eq. (9.5.13) and the derivation of $w = -1$ inasmuch as it means that in the case of vacuum

$$\rho_\Lambda \propto a^0 = \text{constant}.$$

Exercise 10.3 Show how the einstein field Equations with a positive cosmological constant can lead to a static cosmological model.

Solution 10.3 First of all, one has to notice that the form of the FLRW metric does not depend on the form of the field equations, but just on the hypotheses of homogeneity and isotropy of the spacetime. We expect then that changing the Einstein field equations will simply cause a change in the equations describing the evolution of the universe, namely the Friedmann equations.[20]

This greatly simplifies the calculations, because nothing changes in Eqs. (9.4.14) to (9.4.18), which can be used as in the derivation of the equivalent to the Friedmann equations, and thus can be easily written simply by considering the additional term $g_{\alpha\beta}\Lambda$ in Eq. (10.1.3).

From Eqs. (9.4.13) and (9.4.18), and remembering that $g_{00} = -c^2$ the time–time equation is

$$-3\frac{\ddot{a}}{a} + c^2\Lambda = \frac{4\pi G}{c^2}\left(\rho c^2 + 3p\right),$$

that is,

$$\frac{\ddot{a}}{a} - \frac{1}{3}c^2\Lambda = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right). \tag{10.5.2}$$

The spatial components, as in the Friedmann equations, are all equal and with similar calculations give

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{1}{3}c^2\Lambda = \frac{8\pi G}{3}\rho. \tag{10.5.3}$$

In practice, the equations of the cosmological evolution differ from the original Friedmann equations only for a common term $-c^2\Lambda/3$ on the left-hand side.

[20] It is worth while remembering that the derivation of the cosmological equations is a sort of "backwards working" in the sense that it shows that the FLRW metric, i.e., the hypotheses of homogeneity and isotropy of spacetime, is a correct solution of the field equations only if the scale factor a of such a metric evolves according to the cosmological equations. Thus one can assume that the metric is a correct solution of the field equations and say that the latter transform in the laws telling how the scale factor evolves with respect to the cosmic time t.

If we require that the universe is made of a perfect fluid, then the continuity equation (9.5.11) holds, and multiplying both sides by a^3 it becomes

$$\dot{\rho} a^3 + 3\rho \dot{a} a^2 = -3p\dot{a}a^2,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) = -p\frac{\mathrm{d}a^3}{\mathrm{d}t}.$$

The static universe was the attempt made by Einstein to explain the present appearance of the Universe, which we know as matter-dominated and thus has $p = 0$. The last equation then implies $\rho a^3 = \text{const}$, a condition that can simply be interpreted as the constancy of the total mass of the universe $M = 4\pi\rho a^3/3$. For pressureless matter Eq. (10.5.2) rewrites

$$\frac{\ddot{a}}{a} - \frac{1}{3}c^2\Lambda = -\frac{4\pi G}{3}\rho \tag{10.5.4}$$

A static solution is simply $a(t) = a_0 = \text{const}$, therefore by substituting a_0 in the above equation, because in this case $\ddot{a} = \dot{a} = 0$, we have

$$\frac{1}{3}c^2\Lambda = \frac{4\pi G}{3}\rho, \tag{10.5.5}$$

but $\rho > 0$, thus we have recovered the deduction made in the main text, namely that a static solution requires $\Lambda > 0$. Equation (10.5.3) instead becomes

$$\frac{kc^2}{a_0^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}c^2\Lambda = c^2\Lambda, \tag{10.5.6}$$

and because we have just concluded that the cosmological constant must be positive, this equation implies $k > 0$. We know that the options for k are ± 1 and 0, respectively, associated with closed, open, and flat universe. This means that a static solution is admitted, provided that the universe is closed and $\Lambda > 0$.

We can now use the results of the previous exercise to derive further implications of the static universe assumption. It was shown therein that Λ could also be interpreted as a perfect fluid whose density was given by Eq. (10.5.1)

$$\Lambda = \frac{8\pi G}{c^2}\rho_\Lambda.$$

Substitution in Eq. (10.5.6) gives the first requirement cited in the text

$$\frac{kc^2}{a_0^2} = \frac{8\pi G}{3}\left(\rho_{\mathrm{m}} + \rho_\Lambda\right)$$

where we put $\rho \equiv \rho_{\mathrm{m}}$ to make it clear that in the original equation that was the density of the ordinary matter. Substitution in Eq. (10.5.5) gives instead

$$\frac{8\pi G}{3}\rho_{\Lambda} = \frac{4\pi G}{3}\rho_{\mathrm{m}},$$

which is the second requirement $2\rho_{\Lambda} = \rho_{\mathrm{m}}$.

Exercise 10.4 Show that the static universe is intrinsically unstable.

Solution 10.4 Let us imagine that the static solution a_0 suffers a small perturbation described by a small parameter $\epsilon \ll 1$, so that we can write

$$a\,(t) = a_0\,[1 + \epsilon\,(t)]\,.$$

Under such a hypothesis Eq. (10.5.4) becomes

$$\begin{aligned}\frac{\mathrm{d}^2\epsilon}{\mathrm{d}t^2} &= \left(\frac{1}{3}c^2\Lambda - \frac{4\pi G}{3}\rho\right)(1+\epsilon)\\ &= \left(\frac{1}{3}c^2\Lambda - \frac{GM}{a_0^3}\,(1+\epsilon)^{-3}\right)(1+\epsilon)\\ &= \frac{1}{3}c^2\Lambda\,(1+\epsilon) - \frac{GM}{a_0^3}\,(1+\epsilon)^{-2}\end{aligned}$$

where in the second term of the right-hand side we have used the relation $M = 4\pi\rho a^3/3$ found in the previous exercise.

A Taylor expansion gives $(1+\epsilon)^{-2} \simeq 1 - 2\epsilon$, so that

$$\frac{\mathrm{d}^2\epsilon}{\mathrm{d}t^2} \simeq \left(\frac{1}{3}c^2\Lambda - \frac{GM}{a_0^3}\right) + \left(\frac{1}{3}c^2\Lambda + \frac{2GM}{a_0^3}\right)\epsilon$$

but we know from Eq. (10.5.5) that a static universe implies

$$\frac{1}{3}c^2\Lambda - \frac{GM}{a_0^3} = 0,$$

therefore the above equation reduces to

$$\frac{\mathrm{d}^2\epsilon}{\mathrm{d}t^2} = C\epsilon, \tag{10.5.7}$$

where

$$C = \frac{1}{3}c^2\Lambda + \frac{2GM}{a_0^3} > 0$$

is a constant because of the constancy of each term on the right-hand side. It is well known that the most general solution of Eq. (10.5.7) is

$$\epsilon = C_1 \mathrm{e}^{\sqrt{C}t} + C_2 \mathrm{e}^{-\sqrt{C}t},$$

with C_1 and C_2 constants of integration, and the first term immediately shows that the perturbation is growing exponentially with time.

The static solution is therefore not stable because it represents an unstable equilibrium point, in the sense that any small perturbation from a_0 would necessarily evolve in a non limited way. As it was once said, “If you sneeze, the universe collapses.”

Appendix A
Functionals and Calculus of Variations

In the same way a function $f(x)$ is a map from a space of numbers ($\mathbb{R}$, $\mathbb{C}$, etc.) to another space of numbers, a *functional* $F[u]$ is a map from a space of functions U to a space of numbers, or in other words it is a law that associates a number with any function $u \in U$. An example of a functional is therefore the integral

$$F[u] = \int_a^b u(x)\,\mathrm{d}x \tag{A.0.1}$$

defined in the space of the integrable functions in the domain $[a, b]$; another can be the value of a function u or of its derivative u' at a given point $\bar{x}$: $F[u] = u(\bar{x})$, $F[u] = u'(\bar{x})$.[1] Following the same analogy, within the realm of the functionals *calculus of variations* (or variational calculus) is the counterpart of differential calculus for the functions.

The aim of this appendix is to provide a concise overview of this subject, which represents an essential tool for the Variational formalism. For a more detailed introduction to this topic the reader can refer to Arfken and Weber (2012), and a more advanced exposition can be found in Lovelock and Rund (1989).

A.1 Variation of a Functional

The variation δF of a functional $F[u]$ can be defined in analogy to that of the differential $\mathrm{d}f$ of a function $f(x)$. The latter is a first-order evaluation of the difference $\Delta f = f(x + \mathrm{d}x) - f(x)$; i.e., $\mathrm{d}f$ is the part of Δf "linear in $\mathrm{d}x$". Similarly one can consider a variation δu of a function u in the domain U of F as a function such that $u + \delta u \in U$, and the variation of the functional is the part of $\Delta F = F[u + \delta u] - F[u]$ that is "linear in δu".

[1] It is worthwhile emphasizing that in all these cases what is varying is the function u. The boundaries of the integral and the value of $\bar{x}$ are fixed.

A. Vecchiato, *Variational Approach to Gravity Field Theories*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2

In order to rigorously define this concept we introduce a family of functions

$$u^{\alpha}(x) \equiv u(x) + \alpha \delta u(x), \tag{A.1.1}$$

with $\alpha \in \mathbb{R}$, representing the possible variations of u. For a given pair u, δu, the functional $F[u^{\alpha}]$ is a function of α, and the variation of the functional is defined as

$$\delta F[u] = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} F\left[u^{\alpha}\right]\right|_{\alpha=0}. \tag{A.1.2}$$

For example, applying this definition to the functional of Eq. (A.0.1) we have

$$\begin{aligned}\delta F[u] &= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b (u(x) + \alpha\delta u(x))\, \mathrm{d}x\right|_{\alpha=0} \\ &= \int_a^b \delta u(x)\, \mathrm{d}x,\end{aligned}$$

whereas for the functional $F[u] = \int_a^b u^2(x)\, \mathrm{d}x$ the result is

$$\begin{aligned}\delta F[u] &= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b (u(x) + \alpha\delta u(x))^2\, \mathrm{d}x\right|_{\alpha=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_a^b \left\{u^2(x) + 2\alpha u(x)\,\delta u(x) + \left[\alpha^2(\delta u(x))^2\right]\right\}\, \mathrm{d}x\right|_{\alpha=0} \\ &= \left.\int_a^b \left[2u(x)\,\delta u(x) + 2\alpha\delta u(x)^2\right]\, \mathrm{d}x\right|_{\alpha=0} \\ &= \int_a^b 2u(x)\,\delta u(x)\, \mathrm{d}x.\end{aligned} \tag{A.1.3}$$

It is worth noting that the results of these two examples correspond exactly to those obtained by taking the linear part (in δu) of $\Delta F = F[u + \delta u] - F[u]$. It is in fact, quite trivially,

$$\begin{aligned}\Delta F &= \int_a^b (u(x) + \delta u(x))\, \mathrm{d}x - \int_a^b u(x)\, \mathrm{d}x \\ &= \int_a^b \delta u(x)\, \mathrm{d}x\end{aligned}$$

for the first case, and

$$\begin{aligned}\Delta F &= \int_a^b (u(x) + \delta u(x))^2 \, \mathrm{d}x - \int_a^b u^2(x) \, \mathrm{d}x \\ &= \int_a^b 2u(x)\,\delta u(x) \, \mathrm{d}x + \int_a^b (\delta u(x))^2 \, \mathrm{d}x\end{aligned}$$

for the second one which reduces to Eq. (A.1.3), because we have to drop the non linear parts in δu.

It is quite easy to understand that the definition of Eq. (A.1.2) still holds for a family of $u^\alpha(x)$ that is more general than the one of Eq. (A.1.1). In particular, if we consider those $u^\alpha(x)$ for which $u^0(x) = u(x)$, and if we define

$$\delta u(x) = \frac{\mathrm{d}}{\mathrm{d}\alpha} u^\alpha(x)\Big|_{\alpha=0}, \tag{A.1.4}$$

then it can be shown that the above definition of δF gives the same result for any choice of $u^\alpha(x)$. It is straightforward to notice that, consistently, for the family of Eq. (A.1.1) the condition $u^0(x) = u(x)$ holds true and the definition (A.1.4) is an identity.

A.2 Variation of a Lagrangian-Type Functional

Let us now consider a functional of particular interest in classical mechanics, i.e.,

$$F[u] = \int_a^b L\left(u(x), u'(x), x\right) \mathrm{d}x, \tag{A.2.1}$$

where L is a function of u and of its derivative u'.[2]

From Eq. (A.1.2) the variation of F is

$$\delta F[u] = \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_a^b L\left(u^\alpha(x), u'^\alpha(x), x\right) \mathrm{d}x\Big|_{\alpha=0},$$

but

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_a^b L\left(u^\alpha(x), u'^\alpha(x), x\right) \mathrm{d}x =$$

[2]Because the derivative of a function is completely determined by the function itself, the functional eventually depends just on u even in the cases where F contains derivatives of any order of u. For the variational principles of mechanics, however, only u and u' are important.

$$\int_a^b \left[\frac{\partial}{\partial u} L\left(u^\alpha(x), u'^\alpha(x), x\right) \frac{\mathrm{d}}{\mathrm{d}\alpha} u^\alpha(x) \right.$$
$$\left. + \frac{\partial}{\partial u'} L\left(u^\alpha(x), u'^\alpha(x), x\right) \frac{\mathrm{d}}{\mathrm{d}\alpha} u'^\alpha(x) \right] \mathrm{d}x.$$

Evaluating the above expression in $\alpha = 0$ we have

$$\delta F[u] = \int_a^b \left[\frac{\partial}{\partial u} L\left(u^0(x), u'^0(x), x\right) \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} u^\alpha(x)\right|_{\alpha=0} + \right.$$
$$\left. \frac{\partial}{\partial u'} L\left(u^0(x), u'^0(x), x\right) \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} u'^\alpha(x)\right|_{\alpha=0} \right] \mathrm{d}x$$

which, given the condition $u^0(x) = u(x)$ and Eq. (A.1.4) becomes

$$\delta F[u] = \int_a^b \left[\frac{\partial}{\partial u} L\left(u(x), u'^0(x), x\right) \delta u(x) + \right.$$
$$\left. \frac{\partial}{\partial u'} L\left(u(x), u'^0(x), x\right) \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} u'^\alpha(x)\right|_{\alpha=0} \right] \mathrm{d}x.$$

However $u^\alpha(x)$ can be considered as a function of the two variables α and x, thus

$$u'^0(x) = \left.\frac{\mathrm{d}}{\mathrm{d}x} u^\alpha(x)\right|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}x} u^0(x) = u'(x),$$

which means that the definition (A.1.4) also applies to u' and

$$\delta u'(x) = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} u'^\alpha(x)\right|_{\alpha=0},$$

therefore

$$\delta F[u] = \int_a^b \left(\frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial u'} \delta u' \right) \mathrm{d}x \tag{A.2.2}$$

where, for the sake of brevity, we have omitted to write the explicit dependence of $L\left(u(x), u'(x), x\right)$, $\delta u(x)$, and $\delta u'(x)$.[3] Furthermore, it is straightforward to notice that in general the variation δ and the derivative operator are commutative, i.e.,[4]

[3]Moreover, this concise writing makes it evident that the symbol of variation δ works like the differential operator, i.e., it goes under the integral and operates on each argument of L.

[4]Its verification is trivial by considering that in the most general case, by definition, the variation δu of a function u produces another function $\bar{u} = u + \delta u$; i.e., $\delta u = \bar{u} - u$. Applying the same definition it is

$$\delta u' = \bar{u}' - u' = \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\bar{u} - u) = \frac{\mathrm{d}}{\mathrm{d}x}\delta u.$$

$$\delta u' = \frac{\mathrm{d}}{\mathrm{d}x}\delta u, \tag{A.2.3}$$

so that

$$\delta F[u] = \int_a^b \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\frac{\mathrm{d}}{\mathrm{d}x}\delta u \right) \mathrm{d}x.$$

Integrating by parts it is

$$\int_a^b \frac{\partial L}{\partial u'}\frac{\mathrm{d}}{\mathrm{d}x}(\delta u)\,\mathrm{d}x = \left[\frac{\partial L}{\partial u'}\delta u\right]_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'}\delta u\,\mathrm{d}x,$$

so finally Eq. (A.2.2) becomes

$$\delta F[u] = \left[\frac{\partial L}{\partial u'}\delta u\right]_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}\right)\delta u\,\mathrm{d}x. \tag{A.2.4}$$

The quantity

$$\frac{\delta F}{\delta u} \equiv -\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u}\right) \tag{A.2.5}$$

is a special case of the so-called *functional derivative*, or *variational derivative*, which are the generalizations of the derivatives for functions, namely in spaces of functions, so that the variation of the functional F can in general be written as

$$\delta F[u] = \int_a^b \frac{\delta F}{\delta u}\delta u\,\mathrm{d}x + \text{boundary terms}. \tag{A.2.6}$$

A.3 The Euler–Lagrange Equations

In the same way differential calculus can be used to find the maxima and minima of the functions (i.e., those numbers of the domain for which a function gets its maximum or minimum values). Calculus of variations makes it possible to find the extremal points of the functionals which, in this case, are those functions of its domain providing the maximum or minimum values of the functional. This is indeed the problem that was as the origin of the calculus of variations.

Keeping the analogy with differential calculus, we define u as an extremal or stationary point for the functional F if $\delta F[u] = 0$ for any variation δu. More rigorously, given a family of curves u^α for which $u^0(x) = u(x)$, the extremal point u of F can be found by imposing the condition (see Eq. (A.1.2))

$$\delta F[u] \equiv \frac{\mathrm{d}}{\mathrm{d}\alpha}F\left[u^\alpha\right]\Big|_{\alpha=0} = 0. \tag{A.3.1}$$

A classical example is the problem of finding, among all the possible curves, those of minimum length connecting two fixed points A and B. For the sake of simplicity we restrict ourselves to the case of the functions in the Cartesian plane xy. In general the length of a curve between two extremes is the integral of its infinitesimal norm ds:

$$\ell(u) = \int_{\mathsf{A}}^{\mathsf{B}} \mathrm{d}s.$$

In the Cartesian plane $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$ and if $u(x)$ is a real, differentiable function between these two points with domain $a \le x \le b$, where $\mathsf{A} = (a, u(a))$ and $\mathsf{B} = (b, u(b))$, the above integral writes

$$\ell(u) = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_a^b \sqrt{1 + u'^2(x)}\,\mathrm{d}x.$$

It is immediate to not that this functional belongs to the more general class of functionals of Eq. (A.2.1), therefore the condition for u to be stationary, from Eqs. (A.2.4) and (A.3.1), is

$$\delta\ell[u] = \left[\frac{\partial\tilde{L}}{\partial u'}\delta u\right]_a^b - \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\tilde{L}}{\partial u'} - \frac{\partial\tilde{L}}{\partial u}\right)\delta u\,\mathrm{d}x = 0 \quad \forall\,\delta u, \tag{A.3.2}$$

where $\tilde{L}\left(u(x), u'(x), x\right) \equiv \tilde{L}\left(u'(x)\right) = \sqrt{1 + u'^2(x)}$. Moreover, we imposed the additional condition that the endpoints A and B of the curve are fixed, i.e., that $\delta u(a) = \delta u(b) = 0$ for any u, thus the above condition becomes

$$\delta\ell[u] = -\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\tilde{L}}{\partial u'} - \frac{\partial\tilde{L}}{\partial u}\right)\delta u\,\mathrm{d}x = 0 \tag{A.3.3}$$

and it is clear that if

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial\tilde{L}}{\partial u'} - \frac{\partial\tilde{L}}{\partial u} \tag{A.3.4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial u'}\sqrt{1 + u'^2(x)}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\frac{u'}{\sqrt{1 + u'^2(x)}}, \tag{A.3.5}$$

this condition can be satisfied for any δu. This means

$$\frac{u'}{\sqrt{1 + u'^2(x)}} = k$$

with k = costant or

$$u'^2 = \frac{k^2}{1-k^2}.$$

This equation tells the well-known fact that the curves of minimum length on the Cartesian plane are those with constant first derivative, i.e., straight lines.[5]

It is worth noting that the conditions of Eqs. (A.3.2) to (A.3.4) are not valid for just the curve length functional, which means that for any functional expressed as in Eq. (A.2.1), with the additional condition of fixed boundaries $\delta u\,(a) = \delta u\,(b) = 0$, *u is an extremal point if it satisfies the* Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} = 0. \tag{A.3.6}$$

On the other hand, if u does not satisfy the above equation it is not an extremal point of the functional or, in other words, u is stationary if and only if it is a solution of the Euler–Lagrange equations. The latter statement can be deduced as a particular case, with

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u'} - \frac{\partial L}{\partial u} = f\,(x)$$

and $\delta u\,(x) = g\,(x)$, of the following, more general, fundamental lemma of the calculus of variations, which states that

Theorem A.1 *If $f\,(x)$ is an n-times continuously differentiable function ($f \in \mathcal{C}^n$) on the interval* $[a, b]$ *and*

$$\int_a^b f\,(x)\,g\,(x)\;\mathrm{d}x = 0 \quad \forall\, g \in \mathcal{C}^n$$

on the same interval $[a, b]$ *with* $g\,(a) = g\,(b) = 0$, *then* $f\,(x) = 0\;\forall\, x \in [a, b]$.

Proof Let us assume that $\exists\, \bar{x} \in [a, b]$ for which $f\,(\bar{x}) \neq 0$. For the continuity of f, it will be possible to find a real number $\epsilon > 0$ such that $f\,(x)$ has the same sign of $f\,(\bar{x})\;\forall\, x$ in the interval $(\bar{x} - \epsilon, \bar{x} + \epsilon) \subset [a, b]$. We first consider the case $f\,(\bar{x}) > 0$, and the function

$$\bar{g}\,(x) = \begin{cases} \left[\epsilon^2 - (x - \bar{x})^2\right]^n & \text{for } x \in (\bar{x} - \epsilon, \bar{x} + \epsilon) \\ 0 & \text{for } x \notin (\bar{x} - \epsilon, \bar{x} + \epsilon) \end{cases}$$

which is $\mathcal{C}^n$ in $[a, b]$. Moreover it is $\bar{g}\,(x) > 0\;\forall\, x \in (\bar{x} - \epsilon, \bar{x} + \epsilon)$, which implies $f\,(x)\,\bar{g}\,(x) > 0$ in such interval and

[5] Another way to find the same result is to compute the derivative of Eq. (A.3.5) which gives $u''/\left(1 + u'^2\right)^{3/2} = 0$ that implies $u'' = 0$.

$$\int_a^b f(x)\,\bar{g}(x)\,\mathrm{d}x = \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon} f(x)\,\bar{g}(x)\,\mathrm{d}x > 0.$$

We have thus shown that assuming the existence of a point $\bar{x}$ in the domain of f for which $f(\bar{x}) > 0$ implies the contradiction of the starting hypothesis that the integral has to be zero. An identical reasoning can be followed if we consider instead the case of $f(\bar{x}) < 0$ simply by taking $\tilde{g}(x) = -\bar{g}(x)$. In this case in fact it will be $\tilde{g}(x) < 0\,\forall x \in (\bar{x}-\epsilon, \bar{x}+\epsilon)$, so that $f(x)\,\bar{g}(x) < 0$ everywhere in $(\bar{x}-\epsilon, \bar{x}+\epsilon)$, again contradicting the starting hypothesis. This finally shows that $f(x)$ cannot be greater than zero or less than zero anywhere in the interval, thus proving the theorem. □

This result is extremely important because it shows, for this kind of functional, that the stationary condition is equivalent to a specific differential equation. Such equivalence in classical mechanics, and in other branches of physics, is a fundamental "bridge" between the differential and the variational approaches.

Appendix B
Tensor Algebra in Euclidean Geometry

In this appendix we give a quick review of the tensor algebra in Euclidean geometry. It is assumed that the reader is familiar with the subject at the level needed for undergraduate courses in physics. As for the previous one, more detailed and advanced expositions of this topic can be found in Arfken and Weber (2012) and Lovelock and Rund (1989), whereas here we concentrate on the basic ideas and techniques needed to address the concepts and calculations of this book. Moreover, the subject is introduced using from the very beginning the more general modern way, which relies on the concept of differentiable manifolds. Although this is not strictly needed in Euclidean geometry, it allows us to become familiar with concepts used in differential geometry, which is the natural framework of general relativity. We make explicit reference to a three-dimensional space homeomorphic (see below) to $\mathbb{R}^3$, but the same reasoning can be extended to a Euclidean space of any dimension, i.e., to $\mathbb{R}^n$ with no effort. We show that this approach is equivalent to the concept of vector as spatial displacement between two points that has been adopted in Chap. 3 as long as such displacement is infinitesimal, which justifies a posteriori the choice made later in that chapter of leaving the "Δ" for the "d". A rigorous introduction to topological spaces and manifolds can be found, i.e., in de Felice and Clarke (1992) and a more complete one in Dodson and Poston (1991) or Isham (1999).

B.1 Vectors as Directional Derivatives

If we model our "abstract" 3D space $\mathcal{S}$ with the set of points $\mathsf{P} \in \mathbb{R}^3$, we are implicitly giving a natural topological structure to $\mathcal{S}$, in the sense that we can always map the points of this space into elements of $\mathbb{R}^3$, i.e., with their coordinates. The way of mapping the points with their coordinates is definitely not unique, and this translates

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2

to saying that each map is associated with a specific *coordinate system*. In topology this is called a (3D) *manifold*.[6]

In this way a transformation between two coordinate systems can be seen as the composition of two maps, say M_1 and M_2, or more precisely as $M_2 \circ M_1^{-1}$, which in practice is a set of functions from $\mathbb{R}^3$ to $\mathbb{R}^3$. If these functions are differentiable, then it is said that the manifold is *differentiable*.[7] We take as true this reasonable assumption for our 3D space in the following.

The definition of coordinate system stems naturally from the concept of a *curve* in a manifold, which is a subset of points $\mathcal{C} \subset \mathcal{S}$ that can be mapped as a differentiable function in $\mathbb{R}$. Formally, if we indicate with $p(\lambda)$, $\lambda \in \mathbb{R}$ the function that "produces" the points $\mathsf{P} \in \mathcal{C}$, we can write

$$\begin{aligned} p : \mathbb{R} &\longrightarrow \mathcal{S} \\ \lambda &\longmapsto p(\lambda) = \mathsf{P} \in \mathcal{C}. \end{aligned}$$

The function $p(\lambda)$ is called *the parameterization of* $\mathcal{C}$ *with parameter* λ, and the special curves parameterized by the coordinates x^i themselves are called *coordinate lines* which, in other words, are the points of $\mathbb{R}^3$ characterized by having any coordinate x^j constant for $j \neq i$.

Vectors and basis vectors

Given these definitions, a vector can be defined as a *directional derivative* of a scalar field defined on the manifold. In formulae, if we have a differentiable function $f : \mathcal{S} \to \mathbb{R}$, a vector is defined as an operator that, given a curve $\mathcal{C}$, maps f into a real number at each point $\mathsf{P} = p(\lambda) \in \mathcal{C}$. Such mapping is given by the ordinary derivative of $f(p(\lambda))$ computed at a specific point $\tilde{\mathsf{P}} = p\left(\tilde{\lambda}\right)$:

$$\mathbf{v}(f) = \left.\frac{\mathrm{d}f}{\mathrm{d}\lambda}\right|_{\tilde{\lambda}}. \tag{B.1.1}$$

Intuitively it is possible to identify this definition with the familiar tangent to a curve at P_0. Taking the specific case of the coordinate lines $p\left(x^i\right)$ it is not difficult to recognize in the above formula the partial derivative of f with respect to the corresponding coordinate x^i, and in this case we call the related vector

[6]More rigorously, in topology we would say that it is always possible to define a *chart* or a *coordinate system* in the 3D space $\mathcal{S}$, which is a homeomorphism (roughly speaking, a continuous map) between $\mathcal{S}$ and $\mathbb{R}^3$. It is possible to have several charts on the same space, and it is also possible that some charts can map only part of the whole space. An *atlas* of $\mathcal{S}$ is a set of N charts, each mapping a corresponding subspace $\mathcal{S}_i$, when $\cup_{i=1}^N \mathcal{S}_i = \mathcal{S}$.

[7]Moreover, if $M_2 \circ M_1^{-1} \in C^\infty$ the manifold is smooth.

$$\boldsymbol{\partial}_i (f) = \left.\frac{\partial f}{\partial x^i}\right|_{\tilde{x}^i} \tag{B.1.2}$$

a *basis vector* for the coordinate system $\{x^i\}$. Such naming can be easily understood by expanding the calculation of Eq. (B.1.1) for a specific set of coordinates. Actually, in this case the generic curve of that formula is represented by a set of three functions $\{x^i(\lambda)\}$, so that the directional derivative becomes[8]

$$\mathbf{v}(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \equiv \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = \boldsymbol{\partial}_i (f) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda},$$

where in the last formula we have adopted the so-called *Einstein summation convention*, i.e., that when two quantities with the same index, one raised and the other lowered, are multiplied, it has to be interpreted as a sum over the entire range of variation of the indexes of the product of the components.[9] The above equation does not depend on the scalar field, therefore we can drop f from Eqs. (B.1.1) and (B.1.2), and because the curve is completely generic any vector tangent to $\mathcal{S}$ at $\tilde{\mathsf{P}}$ can be decomposed as[10]

$$\mathbf{v} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\boldsymbol{\partial}_i \equiv v^i \boldsymbol{\partial}_i, \tag{B.1.3}$$

which justifies the above definition of $\boldsymbol{\partial}_i$ as basis vectors. More precisely, the latter are called *natural basis vectors* associated with the coordinates (x^i), whereas the *real numbers* v^i (we stress again that the functions $v^i(\lambda)$ are evaluated at a specific value $\lambda = \tilde{\lambda}$) are the *components* of the vector $\mathbf{v}$. Finally, the set of all vectors tangent to $\mathcal{S}$ at $\tilde{\mathsf{P}}$ is denoted $\mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S})$ and called the *tangent vector space to* $\mathcal{S}$ *at* $\tilde{\mathsf{P}}$.

Infinitesimal displacements as vectors

Without entering into the mathematical details that can be found in Wald (1984) if we consider two points connected by a curve $\mathcal{C}$, in the limit their separation along that curve goes to zero, we can consider the quantity

$$\mathrm{d}\mathbf{x} = \mathbf{v}\mathrm{d}\lambda = \mathrm{d}x^i \boldsymbol{\partial}_i \tag{B.1.4}$$

as the infinitesimal separation or *infinitesimal displacement* between these two points. Intuitively, it is as if, in this limit, we identify the arc between the two points with

[8]To make explicit that the operation is done at a specific point from now on we omit, coinciding to the traditional *point of application* of a vector. This allows a simpler notation, but the reader has to remember, that the actual result indeed depends on this implicit detail.

[9]In formulae, if e.g., x_i and y_i are sets of n numbers, $x^i y_i \equiv \sum_{i=1}^{n} x_i y_i$.

[10]This is no different from considering a function $f()$ independently of its arguments.

a “straight segment” as a first-order integration of Eq. (B.1.3). This quantity can be regarded as a vector, inasmuch as it is the product of a vector by a scalar, and the above formula provides its expression in terms of the coordinate system $\left(x^i\right)$. The basis vectors obviously are the same, and its components are simply the familiar infinitesimal coordinate differences $\mathrm{d}x^i$.

B.2 Vectors and Change of Coordinate System

From the previous section we can write a vector as $(\mathbf{e}_i \equiv \boldsymbol{\partial}_i)$[11]

$$\mathrm{d}\mathbf{x} = \mathrm{d}x^i \mathbf{e}_i , \tag{B.2.1}$$

and any change of coordinate system between $\left(x^i\right)$ and $\left(\bar{x}^i\right)$ is a set of transformation laws

$$\bar{x}^i = \bar{x}^i \left(x^j\right), \quad i, j = 1, 2, 3 \tag{B.2.2}$$

which must admit an inverse transformation $x^j = x^j \left(\bar{x}^i\right)$ at least in a finite region of the coordinate space $R \subset \mathbb{R}^3$.

Transformation of vector components

The corresponding transformation laws for the vector components can be easily found by considering that they have to transform as infinitesimal displacements, which can be obtained simply by differentiating Eq. (B.2.2):

$$\mathrm{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} \mathrm{d}x^j . \tag{B.2.3}$$

The last formula can be equivalently written, in the sense of linear algebra, as a matrix-vector product

$$\begin{pmatrix} \mathrm{d}\bar{x}^1 \\ \mathrm{d}\bar{x}^2 \\ \mathrm{d}\bar{x}^3 \end{pmatrix} = J \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix}, \tag{B.2.4}$$

where

$$J \equiv J^{\bar{i}}{}_j = \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^1}{\partial x^3} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \frac{\partial \bar{x}^2}{\partial x^3} \\ \frac{\partial \bar{x}^3}{\partial x^1} & \frac{\partial \bar{x}^3}{\partial x^2} & \frac{\partial \bar{x}^3}{\partial x^3} \end{pmatrix} \tag{B.2.5}$$

[11] Strictly speaking, the natural ones are not the only basis vectors that can be used to decompose a vector. In general these are denoted with $\mathbf{e}_i$, but here we are confusing the two because the notation $\boldsymbol{\partial}_i$ is more common in differential geometry, than in physics, and we want to emphasize the connection with the language of the main text. As regards their transformation properties, which we are discussing here, the two are equivalent.

is called the *Jacobian matrix* of the coordinates transformation. It is therefore clear that the transformation can admit an inverse only if $\det J \neq 0$ for any point $\mathsf{P} \in R$.[12] In summary, even if the transformation laws of Eq. (B.2.2) are completely general, and therefore not necessarily linear, the components v^i of a vector $\mathbf{v}$ will change according to

$$\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} v^j \tag{B.2.6}$$

which is a linear and homogeneous equation.

Transformation of basis vectors

It is natural now, after the vector components, to complete the picture and ask how the basis vectors transform for the same change of coordinates. In the previous section we introduced the basis vectors of a coordinate system x^i as the set of tangent vectors $\{\mathbf{e}_i\}$ along each coordinate line. For this reason these vectors are called the natural basis, and are also written as $\boldsymbol{\partial}_i \equiv \partial/\partial x^i$ because we showed that they operate on any scalar field $f\left(x^i\right)$ such as the partial derivative of f with respect to x^i. We can use the same definition to determine $\bar{\boldsymbol{\partial}}_i \equiv \partial/\partial \bar{x}^i$, namely the basis vectors of the barred coordinate system. In this case in fact we have to consider the inverse transformation $x^j = x^j\left(\bar{x}^i\right)$ to show that

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i},$$

and therefore, because this relation holds for any $f\left(x^j\left(\bar{x}^i\right)\right)$, it can in general be written

$$\bar{\mathbf{e}}_i \equiv \bar{\boldsymbol{\partial}}_i \equiv \frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial}{\partial x^j} \equiv \frac{\partial x^j}{\partial \bar{x}^i}\boldsymbol{\partial}_j \equiv \frac{\partial x^j}{\partial \bar{x}^i}\mathbf{e}_j. \tag{B.2.7}$$

As for the vector components, this equation, equivalently, reads

$$\begin{pmatrix} \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \\ \bar{\mathbf{e}}_3 \end{pmatrix} = \bar{J}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{B.2.8}$$

[12] In this statement we are using the homeomorphism between the manifold $\mathcal{S}$ and $\mathbb{R}^3$ to identify the subset of the latter with $\mathcal{R} \subset \mathcal{S}$, but strictly speaking we should say that $\mathsf{P} \in \mathcal{R}$.

where

$$\bar{J} \equiv J^j{}_{\bar{i}} = \begin{pmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} & \dfrac{\partial x^1}{\partial \bar{x}^2} & \dfrac{\partial x^1}{\partial \bar{x}^3} \\ \dfrac{\partial x^2}{\partial \bar{x}^1} & \dfrac{\partial x^2}{\partial \bar{x}^2} & \dfrac{\partial x^2}{\partial \bar{x}^3} \\ \dfrac{\partial x^3}{\partial \bar{x}^1} & \dfrac{\partial x^3}{\partial \bar{x}^2} & \dfrac{\partial x^3}{\partial \bar{x}^3} \end{pmatrix}.$$

The above equations give the transformation law of the basis vectors for a change of coordinates, and as happened for the vector components, it can also be considered an equivalent definition of a basis vector.

Invariance of the vector for changes of coordinates

By using Eqs. (B.2.3) and (B.2.7) we can therefore write

$$\mathrm{d}\bar{\mathbf{x}} = \mathrm{d}\bar{x}^i \bar{\mathbf{e}}_i = \frac{\partial \bar{x}^i}{\partial x^j} \mathrm{d}x^j \frac{\partial x^k}{\partial \bar{x}^i} \mathbf{e}_k = \mathrm{d}x^j \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^i} \mathbf{e}_k. \tag{B.2.9}$$

The last step is easier to understand using the matrix-vector linear algebra mentioned above. In this formalism, in fact, we can write Eq. (B.2.1) as the "row-by-column" product of the arrays

$$\mathrm{d}x^i \equiv \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} \text{ and } \mathbf{e}_i \equiv \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix},$$

more precisely

$$\mathrm{d}\mathbf{x} = \left(\mathrm{d}x^1 \ \mathrm{d}x^2 \ \mathrm{d}x^3\right) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \left(\mathrm{d}x^i\right)^{\mathrm{T}} (\mathbf{e}_i),$$

where $\left(\mathrm{d}x^i\right)^{\mathrm{T}}$ is the transpose of the column array $\mathrm{d}x^i$. The same obviously holds for the transformed vector which, by means of Eqs. (B.2.4) and (B.2.8), reads

$$\begin{aligned} \left(\mathrm{d}\bar{x}^i\right)^{\mathrm{T}} (\bar{\mathbf{e}}_i) &= \left(J \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix}\right)^{\mathrm{T}} \bar{J} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \left(\mathrm{d}x^j\right)^{\mathrm{T}} J^{\mathrm{T}} \bar{J} \, (\mathbf{e}_k) \\ &= \left(\mathrm{d}x^j\right)^{\mathrm{T}} J_j{}^{\bar{i}} J^k{}_{\bar{i}} \, (\mathbf{e}_k) \end{aligned} \tag{B.2.10}$$

which is exactly Eq. (B.2.9) and where we mixed the Einstein summation convention with the other usual convention of exchanging the index order to indicate the transpose of a matrix. Using the chain rule, and noting that we are combining a transformation with its inverse to obtain a function $x^k\left(\bar{x}^i\left(x^j\right)\right)$, it is easy to see that $J_j{}^{\bar{\imath}} J^k{}_{\bar{\imath}} = \mathbb{I} \equiv \delta^k{}_j$. Equation (B.2.9) thus becomes

$$\mathrm{d}\bar{\mathbf{x}} = \mathrm{d}\bar{x}^i \bar{\mathbf{e}}_i = \mathrm{d}x^j \mathbf{e}_j = \mathrm{d}\mathbf{x}, \tag{B.2.11}$$

which is interpreted in the sense that the vector itself does not change for a coordinate transformation. This was the expected result, and here we have understood that we translated this intuitive requirement by developing a mathematical formalism in which the transformations of vector components and of the basis vectors cancel each other on the whole.

Rotation transformations

If R is a rotation matrix then it transforms the coordinates of a reference system according to

$$\bar{x}^j = R^{\bar{\jmath}}{}_i x^i,$$

thus it is not difficult to see that, because of the constancy of R, the components of the vectors transform exactly as in a pure change of coordinates

$$\mathrm{d}\bar{x}^j = R^{\bar{\jmath}}{}_i \mathrm{d}x^i \equiv \frac{\partial \bar{x}^j}{\partial x^i} \mathrm{d}x^i. \tag{B.2.12}$$

We now remark that we could arrive at Eq. B.2.11 because, $J_j{}^{\bar{\imath}} J^k{}_{\bar{\imath}} = \mathbb{I}$, or in pure matrix language $J^{\mathrm{T}} = J^{-1}$. This, however, is the definition of an orthogonal transformation, and it is exactly the same property of rotation matrices. We can thus conclude that also on the basis vectors rotations behave as a standard change of coordinates, i.e.,

$$\bar{\mathbf{e}}_j = R^i{}_{\bar{\jmath}} \mathbf{e}_i \equiv \frac{\partial x^i}{\partial \bar{x}^j} \mathbf{e}_i$$

and that they leave vectors unchanged.

It is worth while to remember that in Chap. 3 we set the displacement $\Delta\mathbf{x}$, with a later transition to its infinitesimal version $\mathrm{d}\mathbf{x}$, as the prototype of a vector, stating as a defining attribute the experience-driven property of being invariant not only by change of coordinates but also by rotations. Hence what we have shown up to now is the demonstration that the alternative and more formal definition stemming from the topological properties of the manifolds is consistent with such an intuitive representation.

Another essential ingredient of that "heuristic" model of space was a "recipe" to define the distances between any two points, i.e., the length of a vector, or the angle between two vectors, that used what we called metric and scalar product. In the following we thus provide the translation of this other concept in the context of the same formal approach.

B.3 One-Forms and Dual Space

As stressed in Chap. 3, measurements are defined as scalars, i.e., numerical quantities invariant for changes of reference system. In traditional Euclidean vector algebra a pair of vectors can produce scalars by means of a scalar product, which is a linear mapping of vectors into real numbers. In the more general context of this appendix, however, this mapping is realized with a different geometrical entity called *one-form*. As we show, there is no difference between the components of vectors and those of the one-forms in Euclidean geometry, but this is not true in general, so we introduce this distinction now for future convenience, even if it can look unnecessary.

As 3D vectors are indicated with bold Roman letters, one-forms are denoted by bold Greek letters according to the definition

$$\begin{aligned} \boldsymbol{\theta} : \mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S}) &\longrightarrow \quad \mathbb{R} \\ \mathbf{v} \quad &\longmapsto \langle \boldsymbol{\theta}, \mathbf{v} \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is a symbol that generalizes the concept of scalar product, having the meaning that when "filled" with a one-form and a vector it gives a scalar. Moreover, in order to be a linear mapping, this operation has to satisfy the following relations given any pair of vectors $\mathbf{v}$ and $\mathbf{w}$, and any pair of one-forms $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$:

$$\begin{aligned} \langle \boldsymbol{\theta}, a\mathbf{v} + b\mathbf{w} \rangle &= a \langle \boldsymbol{\theta}, \mathbf{v} \rangle + b \langle \boldsymbol{\theta}, \mathbf{w} \rangle \\ \langle a\boldsymbol{\theta} + b\boldsymbol{\omega}, \mathbf{v} \rangle &= a \langle \boldsymbol{\theta}, \mathbf{v} \rangle + b \langle \boldsymbol{\omega}, \mathbf{v} \rangle \end{aligned} \quad \forall\, a, b \in \mathbb{R} \tag{B.3.1}$$

which means that as one-forms are linear operators on vectors, vectors can be considered linear operators on one-forms.[13] As we consider vectors independently of the functions upon which they act, we can look at the one-forms independently of the vectors. In this way one-forms constitute another vector space, different from that of the vectors but "complementary" to it. Such space is denoted with $\mathcal{T}^*_{\tilde{\mathsf{P}}}(\mathcal{S})$ and is called the *dual space of* $\mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S})$. The two spaces are isomorphic, in the sense that there is a one-to-one correspondence between vectors and one-forms, and in particular the one-form corresponding to the natural basis $\boldsymbol{\partial}_i$ is the *dual basis* $\mathbf{d}x^i$ (or $\mathbf{e}^i$)[14] of $\boldsymbol{\partial}_i$ defined by the relation

$$\left\langle \mathbf{d}x^i, \boldsymbol{\partial}_j \right\rangle = \delta^i{}_j, \tag{B.3.2}$$

[13] This notation recalls the bra ($\langle \cdot |$)/ket ($| \cdot \rangle$) one of quantum physics, and indeed both are using the abstract concept of vector space on specific objects and with specific operations. In the latter case the Hilbert space is a space of functions having the structure of a vector space with respect to the integral operation, and the equivalent of the scalar product between two functions gives the probability amplitude of a given state.

[14] Even if it is a one-form, we are following the common convention of indicating the basis with a bold "e" regardless of the type of object. Moreover, the same consideration we did for $\boldsymbol{\partial}_i$ and $\mathbf{e}_i$ hold here, and the same relation $\left\langle \mathbf{e}^i, \mathbf{e}_j \right\rangle = \delta^i{}_j$ defines the dual basis of any basis $\mathbf{e}_i$ of $\mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S})$.

which can be used to represent any one-form in the corresponding coordinate system as

$$\boldsymbol{\theta} = \theta_i \mathbf{d}x^i$$

where, as for the vectors, θ_i are the *components* of the one-form $\boldsymbol{\theta}$ in such a system.

Using Eqs. (B.3.1) and (B.3.2) on the vector d$\mathbf{x}$ of Eq. (B.2.1) one immediately has

$$\left\langle \mathbf{d}x^i, \mathrm{d}\mathbf{x} \right\rangle = \mathrm{d}x^j \left\langle \mathbf{d}x^i, \mathbf{e}_j \right\rangle = \mathrm{d}x^i, \tag{B.3.3}$$

whereas in the same way it is

$$\langle \boldsymbol{\theta}, \mathbf{v} \rangle = \theta_i v^j \left\langle \mathbf{d}x^i, \mathbf{e}_j \right\rangle = \theta_i v^i. \tag{B.3.4}$$

The most common example of a one-form is the *gradient*, which is denoted with the symbol $\boldsymbol{\nabla}$. The use of the same name and symbol of the common gradient of calculus is not casual; in fact, for a given function f, $\boldsymbol{\nabla} f$ is defined as that one-form which gives the variation of f when applied to the infinitesimal displacement d$\mathbf{x}$. We know that in calculus the variation of f is

$$\mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i, \tag{B.3.5}$$

thus the definition implies that $\langle \boldsymbol{\nabla} f, \mathrm{d}\mathbf{x} \rangle = \mathrm{d}f$. Using the dual basis the gradient can be written as $\boldsymbol{\nabla} f = \boldsymbol{\nabla}_i f \, \mathbf{d}x^i$, therefore

$$\mathrm{d}f = \langle \boldsymbol{\nabla} f, \mathrm{d}\mathbf{x} \rangle = \boldsymbol{\nabla}_i f \, \mathrm{d}x^j \left\langle \mathbf{d}x^i, \mathbf{e}_j \right\rangle = \boldsymbol{\nabla}_i f \, \mathrm{d}x^i, \tag{B.3.6}$$

and by comparison with Eq. (B.3.5) we have that the components of the gradient one-form in $\left(x^i\right)$ are just the partial derivatives with respect to x^i.

It could be objected that in calculus the gradient is rather a vector, and not a one-form. However, this is just the result of applying this formalism in Euclidean space. Indeed, in this context we can define a basis vector in two independent ways (Foster and Nightingale 2006), namely as *directional derivative* with respect to a *coordinate curve* or as a gradient of the level surface of the same coordinate. These are exactly the same vectors if the coordinate systems are orthogonal and the basis vectors are normalized, and therefore the isomorphism between one-forms and vectors allows us to identify these two different entities, however, two important points make us distinguish between the two.

First of all, the transformation laws for one-forms are different from those of vectors. In particular it is easy to see that its components transform according to

$$\bar{\theta}_i = \frac{\partial x^j}{\partial \bar{x}^i} \theta_j \tag{B.3.7}$$

and the dual basis with

$$\bar{\mathbf{e}}^i = \frac{\partial \bar{x}^i}{\partial x^j} \mathbf{e}^j . \tag{B.3.8}$$

In other words the components of a one-form transforms as the basis vectors, and the dual basis transform as the components of a vector. The components of a one-form are often called *covariant* components, and those of a vector *contravariant*.

Second, a single coordinate x^i can be considered a scalar field on the space $\mathcal{S}$; in fact, it is an application that associates a number with each point of the space. Therefore we can take $f = x^i$ in Eq. (B.3.6), and remembering that $\nabla f = \nabla_i f \, \mathbf{d}x^i$, where $\nabla_i f = \partial f / \partial x^i$, we have

$$\nabla x^i = \mathbf{d}x^i ,$$

which means that the dual basis of the natural basis is the set of gradients of the coordinates.

Once again, this distinction is not very important in Euclidean geometry, but it become essential in general relativity.

B.4 Tensors

In the previous sections we identified two complementary (dual) classes of objects: vectors and one-forms. Eventually, they have been characterized by their transformation properties with respect to coordinate changes, however, the introduction of one-forms as a way to implement the scalar product revealed another, equivalent way to define them. The latter, in fact, can be seen as a sort of "mathematical machinery" that associates a vector with a number. It is easy to see that this picture is perfectly symmetric in the sense that, conversely, a vector can also be interpreted as a tool to associate one-forms with scalars or, formally

$$\begin{aligned} \mathbf{v} : \mathcal{T}^*_{\tilde{\mathsf{P}}} (\mathcal{S}) &\longrightarrow \quad \mathbb{R} \\ \boldsymbol{\theta} \quad &\longmapsto \langle \boldsymbol{\theta}, \mathbf{v} \rangle . \end{aligned}$$

Definition of tensor

From this point of view, a natural generalization of this approach is having another bit of machinery that takes a number m of one-forms and another number n of vectors and gives a scalar. Such machinery is called a *tensor of rank* (m, n), and is denoted with the abstract symbol $\mathbf{t}^{(m)}_{(n)}$ or with $\mathbf{t}^{(m)}_{(n)} (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ when its arguments will be made explicit. It is easy to understand that this quantity operates on a space which combines m independent vector spaces of one-forms and n of vectors, called *the outer product* of these spaces. Moreover, if tensors have to be the generalization of vectors and one-forms, we have to require their linearity with respect to each argument separately. In other words, if a one-form is a (0, 1) tensor linear in its vector arguments, a (0, 2) tensor will be bilinear in its two vector arguments. Conversely a (2, 0) tensor will be bilinear in its two one-forms arguments, and so on. This definition can be formally written as

$$\begin{aligned} \mathbf{t}^{(m)}_{(n)} : \mathcal{T}^*_{\tilde{\mathsf{P}}}(\mathcal{S}) \otimes \ldots \otimes \mathcal{T}^*_{\tilde{\mathsf{P}}}(\mathcal{S}) \otimes \mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S}) \otimes \ldots \otimes \mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S}) &\longrightarrow \mathbb{R} \\ \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n &\longmapsto \mathbf{t}^{(m)}_{(n)}(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n), \end{aligned}$$

where the symbol "⊗" stands for the outer product.

Tensors decomposition

Tensors can be represented by means of a basis, just like vectors and one-forms. Indeed the basic idea stems from what we know about the latter. For example, from the above definitions it is $\mathbf{v}\left(\mathbf{e}^i\right) \equiv \left\langle \mathbf{e}^i, \mathbf{v}\right\rangle = v^j \left\langle \mathbf{e}^i, \mathbf{e}_j\right\rangle = v^i$, and similarly $\boldsymbol{\theta}\left(\mathbf{e}^i\right) = \theta_i$, therefore for consistency a (0, 2) tensor is written $\mathbf{t}^{(0)}_{(2)} = t_{ij}\mathbf{e}^{ij}$, and its components must be

$$t_{ij} = \mathbf{t}^{(0)}_{(2)}\left(\mathbf{e}_i, \mathbf{e}_j\right),$$

which, following the above schema, implies that for the basis it is

$$\mathbf{e}^{ij}\left(\mathbf{e}_k, \mathbf{e}_l\right) = \delta^i{}_k \delta^j{}_l.$$

But in general $\delta^i{}_j = \mathbf{e}^i\left(\mathbf{e}_j\right) \equiv \left\langle \mathbf{e}^i, \mathbf{e}_j\right\rangle$ so the "basis tensors" are just the outer products of the basis of each single basis, namely

$$\mathbf{t}^{(0)}_{(2)} = t_{ij}\mathbf{e}^{ij} = t_{ij}\mathbf{e}^i \otimes \mathbf{e}^j.$$

Tensors' transformation law

This means that, by construction, such a tensor can be decomposed as the outer product of two one-forms, namely $\mathbf{t}^{(0)}_{(2)} = \boldsymbol{\theta} \otimes \boldsymbol{\omega}$, and its operation on the two vectors $\mathbf{v}$ and $\mathbf{w}$ can be written

$$\mathbf{t}^{(0)}_{(2)}\left(\mathbf{v}_1, \mathbf{v}_2\right) = \langle \boldsymbol{\theta}, \mathbf{v}\rangle \langle \boldsymbol{\omega}, \mathbf{w}\rangle = t_{ij} v_1^k v_2^l \left\langle \mathbf{e}^i, \mathbf{e}_k\right\rangle \left\langle \mathbf{e}^j, \mathbf{e}_l\right\rangle = t_{ij} v_1^i v_2^j, \tag{B.4.1}$$

that is, in general any index behaves independently as a vector or a one-form, in the sense that, if we imagine to fix all the indexes but u_i in the components of a (m, n) tensor $t^{u_1 \ldots u_m}{}_{d_1 \ldots d_n}$, then $t^{u_1 \ldots u_i \ldots u_m}{}_{d_1 \ldots d_n}$ will behave as a one-form. This gives immediately the transformation law for the components of an (m, n) tensor which, from Eqs. (B.2.6) and (B.3.7), reads

$$\bar{t}^{u_1 \ldots u_m}_{d_1 \ldots d_n} = \frac{\partial x^{u_1}}{\partial \bar{x}^{u'_1}} \cdots \frac{\partial x^{u_m}}{\partial \bar{x}^{u'_m}} \frac{\partial \bar{x}^{d'_1}}{\partial x^{d_1}} \cdots \frac{\partial \bar{x}^{d'_n}}{\partial x^{d_n}} t^{u'_1 \ldots u'_m}{}_{d'_1 \ldots d'_n} \tag{B.4.2}$$

which, as for the vectors and one-forms, can be intended as an alternative definition of tensors.

These considerations imply another general property of a tensor, which can also be intended as a machinery mapping tensors to other tensors. If, for example, the (0, 2) tensor of Eq. (B.4.1) operates just on one vector **v**, then we obtain

$$\mathbf{t}^{(0)}_{(2)}(\mathbf{v}, \cdot) = t_{ij} v^i \mathbf{e}^j \equiv \mathbf{t}^{(0)}_{(1)}(\cdot) = \boldsymbol{\theta}$$

i.e., a one-form, which is consistent with the fact that we left unfilled one “slot” of the tensor. In general, therefore, if $M > m$ and $N > n$, an (M, N) the tensors maps (m, n) tensors to $(M - m, N - n)$ tensors. This operation is called *contraction* of the two tensors and it implies that if one has an (M, N) indexed “quantity” which, by operating on a (m, n) tensor, produces an $(M - m, N - n)$ tensor, that the former is a tensor as well.

The decomposition of Eq. (B.4.1) makes it easy to understand that, in general $\mathbf{t}^{(0)}_{(2)}(\mathbf{v}_1, \mathbf{v}_2) \neq \mathbf{t}^{(0)}_{(2)}(\mathbf{v}_2, \mathbf{v}_1)$; i.e., $t_{ij} \neq t_{ji}$. When this is happens, the tensor is called *symmetric*, whereas if on the contrary it is $t_{ij} = -t_{ji}$, one has an *antisymmetric* tensor. This reasoning can be easily generalized to tensors of any rank for which the symmetry (or anti symmetry) property can hold separately for each pair of indexes.

B.5 Metric

Up to this point manifolds have been described just as collections of points, and vectors, one-forms, and tensors have been defined starting from the concepts of tangent space and directional derivatives. All of these elements are connected with the topological properties of the differentiable manifold, but they do not provide any definition of *distance* between two points. The closest approximation we have thus far is the connection between the infinitesimal displacement and the dual basis of Eq. (B.3.3), which, however, does not provide a way to define a distance between two points on the manifold. Moreover, the scalar product defined by the linear mapping of Eq. (B.3.4) is not the desired tool either, because it is just a way to map pairs of vectors and one-forms into real numbers, but we did not give these numbers the meaning of distance.

To this aim, we thus need to add further structure to the topological space and transform it into a metric space. The right tool is the so-called *metric tensor* $\mathbf{g} = g_{ij}\mathbf{e}^i \otimes \mathbf{e}^j$, which is defined as a (0, 2) tensor that

1. Is *symmetric*, i.e., for which $g_{ij} = g_{ji}$, or equivalently $\mathbf{g}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{g}(\mathbf{v}_2, \mathbf{v}_1)$ for any $\mathbf{v}_1, \mathbf{v}_2$
2. Is *non degenerate*, which means that if at any point of the manifold it is $\mathbf{g}(\mathbf{u}, \mathbf{v}) = 0$ for any vector $\mathbf{v}$, then $\mathbf{u} = 0$

From this definition and Eqs. (B.3.2) and (B.4.1) one can immediately obtain the components of the metric tensor as

$$\mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right) = g_{ij}.$$

Moreover, it is evident that such a definition has the right properties to implement the scalar product as it is normally intended. Indeed, the scalar (dot) product is symmetric and non degenerate. In addition to this, by applying this tensor to the infinitesimal displacement vector d$\mathbf{x}$ one has

$$\mathbf{g}(\mathrm{d}\mathbf{x}, \mathrm{d}\mathbf{x}) = g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{B.5.1}$$

which can be compared with the usual expression for the dot product

$$\mathrm{d}s^2 = \left(\mathrm{d}x^i\mathbf{e}_i\right) \cdot \left(\mathrm{d}x^j\mathbf{e}_j\right) = \left(\mathbf{e}_i \cdot \mathbf{e}_j\right)\mathrm{d}x^i\mathrm{d}x^j. \tag{B.5.2}$$

This means that Eq. (B.5.1) gives the Euclidean norm if we put $g_{ij} = \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = \left(\mathbf{e}_j \cdot \mathbf{e}_i\right) = g_{ji}$, which also correctly reduces to its standard expression in Cartesian coordinates inasmuch as in this case it is $\left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = \delta_{ij}$. This justifies the definition $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ as the (squared) norm of d$\mathbf{x}$.

In the previous section the possibility of leaving "unfilled slots" in the arguments of a (m, n) tensor was observed, thus producing tensors of lower rank, which implies that $\mathbf{g}(\mathbf{v}, \cdot) = g_{ij}v^j$ is a one-form, say $\boldsymbol{\theta}$. Moreover, this is a very specific one-form, and precisely the one that, applied to $\mathbf{v}$ gives its norm, i.e.,

$$\langle \boldsymbol{\theta}, \mathbf{v} \rangle = v^2.$$

This establishes a correspondence between the elements of the tangent space $\mathcal{T}_{\tilde{\mathsf{P}}}(\mathcal{S})$ to those of the dual space $\mathcal{T}^*_{\tilde{\mathsf{P}}}(\mathcal{S})$ such that

$$g_{ij}v^j = v_i, \tag{B.5.3}$$

where v_i is the one-form whose scalar product with $\mathbf{v}$ gives its norm $v^2 = v_i v^i$. This operation is called *index lowering*.

It is clear that the components of a rank 2 tensor can be represented in matrix form, thus it makes perfect sense to take its determinant. In the case of the metric tensor we will denote it with $g \equiv \det\left(g_{ij}\right)$. The non degeneration property also implies that $g \neq 0$, because by definition the system of equations $g_{ij}u^i = 0$ admits only the solution $\mathbf{u} = 0$, and therefore it is not rank deficient. For this reason there always exists the inverse of the matrix g_{ij}, say g^{ij}, such that

$$g_{ik}g^{kj} = \delta^j{}_i, \tag{B.5.4}$$

however both g_{ij} and $\delta^j{}_i$ are tensors, which implies that g^{ij}, or better $\mathbf{g}^{-1} = g^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, has to be a tensor as well.[15]

From Eqs. (B.5.3) and (B.5.4) it is therefore

$$g^{ij}v_j = g^{ij}g_{jk}v^k = \delta^i{}_k v^k = v^i,$$

which are the components of a vector. This shows that the correspondence between the tangent and dual spaces is a one-to-one relation, or an isomorphism, and demonstrates that the inverse of the metric tensor can be associated with the *index raising* operation, the inverse of the lowering produced by g_{ij}.

These two basic operations, obviously, can be performed on indexes of tensors of any rank, so that, e.g, formulae like

$$g_{ik}t^{ljk} = t_i{}^{jk}$$

are regular tensorial expressions.

Finally, all the above exposition shows as origin of the familiar Euclidean geometry when the metric $g_{ij} = \delta_{ij}$ is assumed in Cartesian coordinates, but the approach followed here allows a seamless extension to more general cases of non positive-definite metrics and to curved spaces, as shown in Appendix D.

B.6 Stress Tensor

An example of a tensor in Euclidean space is the so-called *stress tensor*, which we also are mentioning for future applications. Indeed, its four-dimensional counterpart (the *stress-energy* or *energy-momentum* tensor) plays a central role in the development of relativistic physics. It is thus with this goal in mind (and with no claim of completeness and full rigor for which the interested reader can refer to specialized books on the relevant subjects including those on fluid dynamics or theory of elasticity) that we are going to show its link with dynamics and some of its properties which are generalized in special and general relativity.

B.6.1 Definition and Properties

We start by considering a small cubic volume of matter $\mathrm{d}V$ subject to internal forces. In general (Fig. B.1) we can imagine the forces acting pairwise on each couple of parallel planes, so that we can "attach" a specific force per unit area $\mathbf{f}_i$ to each pair

[15] More precisely, a (2, 0) tensor.

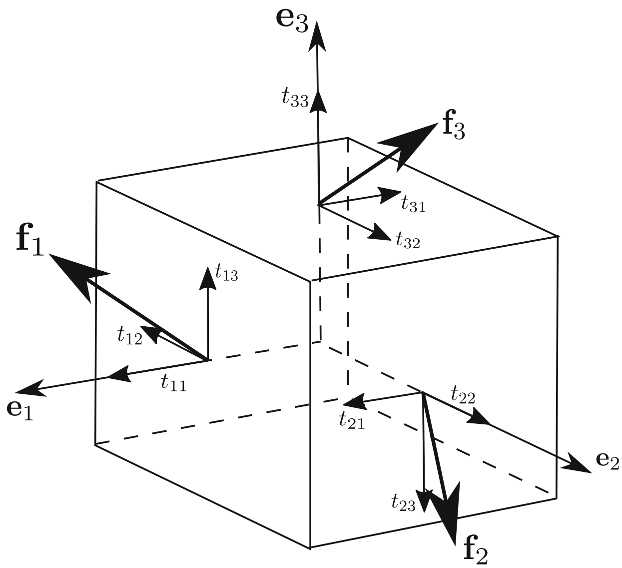

Fig. B.1 The internal forces of a small volume of matter can be decomposed into three forces per unit area acting on each pair of parallel surfaces. In its turn, each of these forces is a vector that can be decomposed into its three components in the given reference system, for a total of nine components of a rank 2 tensor

of faces of the cube, where the value of the index $i = 1, 2, 3$ refers to the x, y, and z faces, i.e., those orthogonal to the $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ axes, respectively. These forces are oriented in a completely general way so each one, on its turn, can be decomposed in its x-, y-, and z-components as

$$\mathbf{f}_i = t_{ij}\hat{\mathbf{e}}_j.$$

The above formula immediately tells us that by definition these nine quantities t_{ij} return a vector $\mathbf{f}_j$ when they operate on a vector and thus, for the considerations of the previous section, they form the components of a rank 2 tensor[16] called *a stress tensor*, and it is easy to understand that, if we put it into matrix form,

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \equiv \mathsf{T},$$

its diagonal terms t_{ii} represent the forces per unit area perpendicular to the ith face, i.e., the *pressure* on that face. The off-diagonal components t_{ij} instead are named *shear* because, being parallel to the ith face, they induce a shear force. The surface

[16]Because we are dealing with Euclidean objects in Cartesian coordinates we are not allowed to make no distinction between vectors and one-forms, or between covariant and contravariant components.

S delimiting a generic finite volume V can be thought as a set of infinitesimal planar surfaces defined by their area and orientation as $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\mathrm{d}S$, where $\hat{\mathbf{n}} = \{n^i\}$ is their normal unit vector. The quite intuitive *Cauchy stress principle* claims that the total force on such a surface is the sum of the forces per unit area orthogonal to each $\mathrm{d}\mathbf{S}$, which are clearly $\mathbf{f}_{\hat{\mathbf{n}}} = t_{ij}n^j$. Integrating over the whole surface we have

$$\mathbf{F} = \int_S t_{ij}n^j \,\mathrm{d}S. \tag{B.6.1}$$

Properties of the stress tensor

We now apply the principles of conservation of momentum and of angular momentum to show two fundamental properties of the stress tensor. These two principles apply to the cases of an isolated body, subject to internal forces only, or to a body subject to both internal and external forces that are in equilibrium, but we are interested in the properties of the stress tensor per se, thus for our purposes these two cases are totally equivalent. The latter in fact can always beled back to the former by "extending the body" to all the forces or, in other words, by considering the stress tensor of a "body" composed of the sources generating all the forces.

The momentum conservation of an isolated body implies that the total force is zero, therefore

$$\int_S t_{ij}n^j \,\mathrm{d}S = 0. \tag{B.6.2}$$

We can safely suppose that the volume we are dealing with is sufficiently "well-behaved" (i.e., that such volume is closed and bound) to make the Gauss theorem applicable, in which case we have[17]

$$\int_S t_{ij}n^j \,\mathrm{d}S \equiv \int_S \mathbf{T}\cdot\mathbf{n}\,\mathrm{d}S = \int_V \nabla\cdot\mathbf{T}\,\mathrm{d}V \equiv \int_V \frac{\partial t_{ij}}{\partial x^j}\,\mathrm{d}V$$

so

$$\int_V \frac{\partial t_{ij}}{\partial x^j}\,\mathrm{d}V = 0$$

and because this equation holds for arbitrary volumes we can say that, in general, the momentum conservation implies that the stress tensor of an isolated region of space has the property

$$\frac{\partial t_{ij}}{\partial x^j} = 0. \tag{B.6.3}$$

[17]Usually the Gauss theorem is refers to vector fields, but it can be applied to tensors of any rank because the additional free indexes do not affect the result of the integration.

Let us take the angular momentum of the body with respect to an arbitrary point O. From the definition of the three-dimensional *Levi-Civita symbol*[18] ε_{ijk} and Eq. (B.6.2) it is[19]

$$\int_S \varepsilon_{ijk} r^j t^{km} n_m \, \mathrm{d}S = 0.$$

Using once again the Gauss theorem the above equation becomes

$$\int_V \frac{\partial}{\partial x^m} \left(\varepsilon_{ijk} r^j t^{km} \right) \mathrm{d}V = 0,$$

therefore, expanding the derivative,

$$0 = \int_V \varepsilon_{ijk} \left(\frac{\partial r^j}{\partial x^m} t^{km} + r^j \frac{\partial t^{km}}{\partial x^m} \right) \mathrm{d}V = \int_V \varepsilon_{ijk} \delta^j_m t^{km} \, \mathrm{d}V = \int_V \varepsilon_{ijk} t^{kj} \, \mathrm{d}V$$

in which we used Eq. (B.6.3) and the fact that $\partial r^j / \partial x^m = \delta^j_m$. As for the moment conservation, the arbitrary volume implies that

$$\varepsilon_{ijk} t^{kj} = 0.$$

This is a set of three independent equations for $i = 1, 2, 3$, and in each of these, from the definition of the Levi–Civita symbol, the only "surviving" terms are the even or odd permutations of $\{1, 2, 3\}$. Thus, e.g., for the $i = 1$ equation

$$\varepsilon_{1jk} t^{kj} = 0 \quad \Rightarrow \quad \varepsilon_{123} t^{32} + \varepsilon_{132} t^{23} = 0,$$

and because $\varepsilon_{123} = -\varepsilon_{132}$ it has to be $t^{23} = t^{32}$. Identical considerations can be done with $i = 2$ ($t^{13} = t^{31}$) and for $i = 3$ ($t^{12} = t^{21}$) so that in general it can be stated that the conservation of angular momentum implies that the stress tensor has to be symmetric; i.e.,

$$t_{ij} = t_{ji}. \tag{B.6.4}$$

[18] This quantity is defined as $\varepsilon_{ijk} = (-1)^p \varepsilon_{123}$ where each index can be 1, 2 or 3, and p is the number of permutations between i, j, k and 1, 2, 3. This explains its alternative denomination of *permutation symbol*. If p is even then $\varepsilon_{ijk} = 1$, otherwise it is -1. If any of the index is repeated, then $\varepsilon_{ijk} = 0$.

[19] We are using the components formalism with which the product vector can be written $\mathbf{a} \times \mathbf{b} \equiv \varepsilon_{ijk} a^j b^k$, so that the angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{f}_{\hat{\mathbf{n}}}$ of $\mathbf{f}_{\hat{\mathbf{n}}} = t_{ij} n^j \hat{\mathbf{e}}_i$ with respect to the position vector $\mathbf{r} = r_i \hat{\mathbf{e}}_i$ with respect to O is $l^i = \varepsilon_{ijk} r^j f^k = \varepsilon_{ijk} r^j t^{km} n_m$. The positions of the indexes take into account that in Euclidean geometry and Cartesian coordinates there is no difference between covariant and contravariant components, and we take the liberty of using the summation convention only for the components.

B.6.2 Stress Tensor as Momentum Flux

In loose terms the “flux of something” can be intended as the flow of something across a unit surface per unit time, where with “something” we mean any physical property that varies in space and time. Inasmuch as the physical property can be a tensorial quantity of any rank, the flux can be described by a tensor of any rank as well. Mathematically it can be defined by its surface integral formula, as in this example;

$$F_S = \int_S \mathbf{f}_p \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{f}_p \cdot \mathbf{n}\, \mathrm{d}S. \tag{B.6.5}$$

Here the *flux* of the property p is represented by a vector $\mathbf{f}_p$, so that the net “transfer” of the property across the surface element $\mathrm{d}S$ per unit time is $\mathbf{f}_p \cdot \mathbf{n}\, \mathrm{d}S$ and the total transfer per unit time across the surface S is F_S.

The seemingly obscure meaning of such definitions can be easily clarified with a well-known example. Suppose we have a certain quantity of matter $\mathrm{d}m$ with density ρ in a small volume $\mathrm{d}V$, such as $\mathrm{d}m = \rho \mathrm{d}V$. If such a mass is moving with velocity $\mathbf{v}$, then one can think that it swarms through the volume $\mathrm{d}V = \mathrm{d}\mathbf{S} \cdot \mathbf{v}\mathrm{d}t$ during the time interval $\mathrm{d}t$, or alternatively that there exists an instantaneous mass current $j_m = \mathrm{d}m/\mathrm{d}t$ flowing at velocity $\mathbf{v}$ through the surface $\mathrm{d}\mathbf{S}$; in formulae

$$j_m = \frac{\mathrm{d}m}{\mathrm{d}t} = \rho \mathbf{v} \cdot \mathbf{n}\mathrm{d}S. \tag{B.6.6}$$

The total amount of mass transfer per unit time across the surface S is then

$$J_m = \int_S \rho \mathbf{v} \cdot \mathbf{n}\mathrm{d}S,$$

thus, by comparison with Eq. (B.6.5), the flux of mass density ρ is $\mathbf{f}_\rho = \rho \mathbf{v}$.

Now we can just do the same reasoning with the momentum multiplying Eq. (B.6.6) by $\mathbf{v}$:

$$\mathbf{j_p} = \frac{\mathrm{d}m}{\mathrm{d}t}\mathbf{v} = (\rho \mathbf{v})\, \mathbf{v} \cdot \mathbf{n}\mathrm{d}S, \tag{B.6.7}$$

so that $\mathbf{j_p}$ can be interpreted as there is an instantaneous momentum density current transferring $(\rho \mathbf{v})\, \mathbf{v} \cdot \mathbf{n}\mathrm{d}S$ momentum per unit time across the surface $\mathrm{d}S$. The total momentum transfer per unit time across the surface S will then be

$$\mathbf{J_p} = \int_S (\rho \mathbf{v})\, \mathbf{v} \cdot \mathbf{n}\mathrm{d}S.$$

But $\mathbf{J_p}$ is the variation of the total momentum per unit time through the surface S, thus it is equivalent to the total force $\mathbf{F}$ on the surface of Eq. (B.6.1) and thus[20]

$$\int_S \mathsf{T} \cdot \mathbf{n} \, \mathrm{d}S = \int_S (\rho \mathbf{v}) \, \mathbf{v} \cdot \mathbf{n} \mathrm{d}S$$

or, in component notation,

$$\int_S t_{ij} n^j \, \mathrm{d}S = \int_S (\rho v_i) \, v_j \cdot n^j \, \mathrm{d}S$$

which means that the stress tensor can also be interpreted as the flux $(f_\pi)_{ij}$ of momentum density $\pi_i = \rho v_i$; i.e.,[21]

$$t_{ij} = (f_\pi)_{ij} = \rho v_i v_j = \pi_i v_j \tag{B.6.8}$$

that in abstract notation can be written

$$\mathsf{T} = \mathbf{f}_\pi = \rho \mathbf{v} \otimes \mathbf{v}.$$

In the special case of a single particle of mass m, we could write $\rho = m\delta^3 (\mathbf{x} - \mathbf{r}(t))$ and $\pi_i = m\delta^3 (\mathbf{x} - \mathbf{r}(t)) \, v_i = p_i \delta^3 (\mathbf{x} - \mathbf{r}(t))$, where $\mathbf{r}(t)$ is the particle's trajectory and $\delta^3(x)$ is the three-dimensional Kronecker delta, so that the stress tensor can be written as

$$t_{ij} = m\delta^3 (\mathbf{x} - \mathbf{r}(t)) \, v_i v_j = p_i \delta^3 (\mathbf{x} - \mathbf{r}(t)) \, v_j.$$

This alternative expression of Eq. (B.6.8) for the stress tensor shows immediately the symmetry property of Eq. (B.6.4).

[20] It is worth noting that, because we are assuming that $\mathbf{v}$ is constant, $\mathbf{j_p} = \mathrm{d}\mathbf{p}/\mathrm{d}t$ and thus it is nothing else than the force on the surface $\mathrm{d}S$.

[21] Note that this time the flux is a rank 2 tensor.

Appendix C
Special Relativity

C.1 Derivation of Eqs. (5.1.8) and (5.1.9)

Using the requirement of homogeneity of space and time and that of isotropy of space, we have shown that the transformations between two reference systems S and $\bar{S}$ moving with relative velocity v along the x-axis must be linear functions in t, x, y, and z having the form of Eqs. (5.1.4) and (5.1.7), namely

$$\begin{aligned}\bar{t} &= \alpha(v)\, t + \beta(v)\, x \\ \bar{x} &= \gamma(v)\,(x - vt) \\ \bar{y} &= y \\ \bar{z} &= z,\end{aligned}$$

where α, β, and γ are functions of v only.

For the next steps, it can first be noticed that we can always define two new functions of v, $\lambda(v)$ and $\mu(v)$, such that $\alpha(v) = \gamma(v)\,\lambda(v)$ and $\beta(v) = -\gamma(v)\,\mu(v)$, thus getting

$$\bar{t} = \gamma(v)\,[\lambda(v)\, t - \mu(v)\, x] \tag{C.1.1}$$

from Eqs. (5.1.4). If we now consider a change of sign of, e.g., the x coordinate, then the transformations will depend on a different velocity, namely $\tilde{v}$, and the space separations between two events in $\bar{S}$ will be reversed as well; i.e.,

$$\begin{aligned}\mathrm{d}\bar{t}\,(t, x, v) &= \mathrm{d}\bar{t}\,(t, -x, \tilde{v}) \\ \mathrm{d}\bar{x}\,(t, x, v) &= -\mathrm{d}\bar{x}\,(t, -x, \tilde{v})\,.\end{aligned}$$

By applying this condition to Eqs. (C.1.1) and (5.1.7) we obtain

$$\begin{aligned}\gamma(v)\,[\lambda(v)\,\mathrm{d}t - \mu(v)\,\mathrm{d}x] &= \gamma(\tilde{v})\left[\lambda(\tilde{v})\,\mathrm{d}t + \mu(\tilde{v})\,\mathrm{d}x\right] \\ \gamma(v)\,(\mathrm{d}x - v\mathrm{d}t) &= -\gamma(\tilde{v})\,(-\mathrm{d}x - \tilde{v}\mathrm{d}t)\end{aligned}$$

A. Vecchiato, *Variational Approach to Gravity Field Theories*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2

for any dt and dx. From the first of these two equations it has to be

$$\gamma(v)\,\lambda(v) = \gamma(\tilde{v})\,\lambda(\tilde{v}) \quad \text{and} \quad \gamma(v)\,\mu(v) = -\gamma(\tilde{v})\,\mu(\tilde{v})\,,$$

and from the second we have the conditions

$$\gamma(v) = \gamma(\tilde{v}) \quad \text{and} \quad v\gamma(v) = -\tilde{v}\gamma(\tilde{v})\,. \tag{C.1.2}$$

The second set of conditions immediately implies the intuitive requirement that $\tilde{v} = -v$,[22] and thus also

$$\gamma(v) = \gamma(-v)\,, \tag{C.1.3}$$

from which it can be easily deduced that

$$\lambda(v) = \lambda(-v) \quad \text{and} \quad \mu(v) = -\mu(-v)\,. \tag{C.1.4}$$

We now appeal to the requirement that in general the set of transformations must form a group with the set of reference systems to proceed further. The first of such requirements is that these transformations admit an identity, i.e., that there exists a value of v for which $\bar{S} \equiv S$ or, which is the same, $\bar{t} = t$ and $\bar{x} = x$. Moreover, it is clear that such value must be $v = 0$ because by definition in this case the barred and unbarred reference systems must be the same. This means that

$$\gamma(0) = \lambda(0) = 1 \quad \text{and} \quad \mu(0) = 0.$$

Another requirement is that there exists an inverse transformation, i.e., a velocity $\hat{v}$ that "brings" the transformed reference system $\bar{S}$ back into S by means of Eqs. (C.1.1) and (5.1.7), this time used to go from the barred to the unbarred system, namely

$$t = \gamma\left(\hat{v}\right)\left[\lambda\left(\hat{v}\right)\bar{t} - \mu\left(\hat{v}\right)\bar{x}\right] \tag{C.1.5}$$
$$x = \gamma\left(\hat{v}\right)\left(\bar{x} - \hat{v}\bar{t}\right)\,. \tag{C.1.6}$$

From Eqs. (C.1.1) and (5.1.7) we have instead

$$t = \frac{\bar{t} + \mu(v)\,\bar{x}}{\gamma(v)\left[\lambda(v) - v\mu(v)\right]} \tag{C.1.7}$$
$$x = \frac{v\bar{t} + \lambda(v)\,\bar{x}}{\gamma(v)\left[\lambda(v) - v\mu(v)\right]}, \tag{C.1.8}$$

which require that $\gamma(v) \neq 0$ and $\lambda(v) \neq v\mu(v)$. By comparing Eqs. (C.1.5) and (C.1.6) with Eqs. (C.1.7) and (C.1.8), and remembering that they must hold for any

[22] In other words if we reverse the orientation of the x-axis, the transformations hold equivalent to the previous by reversing the velocity.

$\bar{t}$ and $\bar{x}$, the following relations can be found

$$\hat{v} = -\frac{v}{\lambda(v)} \tag{C.1.9}$$

$$\mu(\hat{v}) = -\frac{\mu(v)}{\lambda(v)} \tag{C.1.10}$$

$$\lambda(\hat{v}) = \frac{1}{\lambda(v)} \tag{C.1.11}$$

$$\gamma(\hat{v}) = \frac{\lambda(v)}{\gamma(v)\left[\lambda(v) - v\mu(v)\right]}, \tag{C.1.12}$$

with the additional requirement that $\lambda(v) \neq 0$. From Eqs. (C.1.9), (C.1.11), and (C.1.4) it is

$$\lambda\left(\frac{v}{\lambda(v)}\right) = \frac{1}{\lambda(v)},$$

which is clearly satisfied by $\lambda(v) = 1$. Moreover, Lévy-Leblond (1976) showed that the latter is also the only solution of the above functional equation. This means that $\hat{v} = -v$ or, in other words, that the inverse transformation which brings $\bar{S}$ back to S is the original one but having the opposite velocity. From Eqs. (C.1.3) and (C.1.12) we have then

$$\gamma^2(v) = \frac{1}{1 - v\mu(v)},$$

and from the condition $\gamma(0) = 1$ it has to be

$$\gamma(v) = \frac{1}{\sqrt{1 - v\mu(v)}}, \tag{C.1.13}$$

so that the transformations laws become

$$\bar{t} = \frac{t - \mu(v)x}{\sqrt{1 - v\mu(v)}} \tag{C.1.14}$$

$$\bar{x} = \frac{x - vt}{\sqrt{1 - v\mu(v)}} \tag{C.1.15}$$

with the condition that $v\mu(v) < 1$.

The third propriety of the groups that we can use is that of the closure. In practice we require that if one applies a transformation like that of Eqs. (C.1.14) and (C.1.15) with parameter $\tilde{v}$, followed by another one with parameter $\hat{v}$, the net result is equivalent to another transformation of the same kind with a parameter v to be found with an appropriate velocity composition law.

The first transformation is then

$$\tilde{t} = \frac{t - \mu(\tilde{v})\, x}{\sqrt{1 - \tilde{v}\mu(\tilde{v})}}$$
$$\tilde{x} = \frac{x - \tilde{v}t}{\sqrt{1 - \tilde{v}\mu(\tilde{v})}},$$

and the application of the second one leads us to

$$\hat{t} = \frac{\tilde{t} - \mu\left(\hat{v}\right)\tilde{x}}{\sqrt{1 - \hat{v}\mu\left(\hat{v}\right)}}$$
$$\hat{x} = \frac{\tilde{x} - \hat{v}\tilde{t}}{\sqrt{1 - \hat{v}\mu\left(\hat{v}\right)}}$$

which means that

$$\hat{t} = \frac{\left[1 + \mu\left(\hat{v}\right)\tilde{v}\right] t - \left[\mu(\tilde{v}) + \mu\left(\hat{v}\right)\right] x}{\sqrt{\left[1 - \hat{v}\mu\left(\hat{v}\right)\right]\left[1 - \tilde{v}\mu(\tilde{v})\right]}} \tag{C.1.16}$$
$$\hat{x} = \frac{\left[1 + \mu(\tilde{v})\,\hat{v}\right] x - \left(\tilde{v} + \hat{v}\right) t}{\sqrt{\left[1 - \hat{v}\mu\left(\hat{v}\right)\right]\left[1 - \tilde{v}\mu(\tilde{v})\right]}}. \tag{C.1.17}$$

By requiring that $\bar{t} = \hat{t}$ and $\bar{x} = \hat{x}$ for any t and x, one has immediately

$$\frac{\left[1 + \mu\left(\hat{v}\right)\tilde{v}\right]}{\sqrt{\left[1 - \hat{v}\mu\left(\hat{v}\right)\right]\left[1 - \tilde{v}\mu(\tilde{v})\right]}} = \frac{1}{\sqrt{1 - v\mu(v)}}$$
$$\frac{\left[1 + \mu(\tilde{v})\,\hat{v}\right]}{\sqrt{\left[1 - \hat{v}\mu\left(\hat{v}\right)\right]\left[1 - \tilde{v}\mu(\tilde{v})\right]}} = \frac{1}{\sqrt{1 - v\mu(v)}}$$

which implies

$$1 + \mu\left(\hat{v}\right)\tilde{v} = 1 + \mu(\tilde{v})\,\hat{v},$$

that is,

$$\frac{\mu\left(\hat{v}\right)}{\hat{v}} = \frac{\mu(\tilde{v})}{\tilde{v}}. \tag{C.1.18}$$

Because this condition does not depend on the choice of the velocity, it is in general $\mu(v)/v = k$, for any v, or

$$\mu(v) = kv, \tag{C.1.19}$$

where k is a universal constant. The last relation can be substituted in Eqs. (C.1.14) and (C.1.15) giving

$$\bar{t} = \frac{t - kvx}{\sqrt{1 - kv^2}} \tag{C.1.20}$$

$$\bar{x} = \frac{x - vt}{\sqrt{1 - kv^2}} \tag{C.1.21}$$

which are exactly Eqs. (5.1.8) and (5.1.9).

The velocity composition law can be easily found by using again Eqs. (C.1.17) and (C.1.15), which give[23]

$$\frac{v}{\sqrt{1 - v\mu(v)}} = \frac{\tilde{v} + \hat{v}}{\sqrt{\left[1 - \hat{v}\mu(\hat{v})\right]\left[1 - \tilde{v}\mu(\tilde{v})\right]}}.$$

Because of Eq. (C.1.19) this equation becomes

$$\frac{v}{\sqrt{1 - kv^2}} = \frac{\tilde{v} + \hat{v}}{\sqrt{\left(1 - k\hat{v}^2\right)\left(1 - k\tilde{v}^2\right)}}, \tag{C.1.22}$$

which can be squared giving us, after some simple algebra,

$$v^2 = \left(\frac{\tilde{v} + \hat{v}}{1 + k\hat{v}\tilde{v}}\right)^2,$$

that is,

$$v = \frac{\tilde{v} + \hat{v}}{1 + k\hat{v}\tilde{v}}, \tag{C.1.23}$$

where in the latter formula the positive sign has to be chosen because otherwise, in the case of, e.g., $\hat{v} = 0$, it would be $v = -\tilde{v}$, which is in contradiction with Eq. (C.1.22) that we started from, which instead gives $v = \tilde{v}$ for the same case. This result is intuitive as well, inasmuch as the $\hat{v} = 0$ assumption is equivalent to having just one transformation.

C.2 Lorentz Covariance of Classical Electromagnetism

In Sect. 4.5 we showed that the equations of classical electromagnetism does not satisfy the principle of relativity in its Galilean form, i.e., that it is not possible to leave the Lorentz electromagnetic force and the Maxwell equations form-invariant under

[23] The same result can be obtained with Eqs. (C.1.16) and (C.1.14).

Galilean transformations at the same time. However, we discovered that the only other possibility to formulate the principle of relativity is the one which involves the Lorentz transformations between two reference systems in uniform relative motion, and the additional requirement of the existence of a characteristic constant c, having the dimensions of a velocity, which is the same for all the inertial reference systems. It is therefore natural to of of the possibility that the equations of classical electromagnetism can be form-invariant with respect to these transformations, i.e., that these laws satisfy the principle of relativity in such a form.

Although there is a more direct way to show it, for pedagogical reasons we show in this section how this statement can be proved in the framework of the classical Euclidean framework. This is shown again in Sect. C.3 using the four-vector formalism, which allows us to make an interesting parallelism between the principle of covariance in 3D Euclidean space and the principle of relativity in 4D Minkowskian spacetime.

As in Sect. 4.5, the problem is to find how both the Lorentz electromagnetic force

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{C.2.1}$$

and the Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{C.2.2}$$

transform under Lorentz transformation, but contrary to the classical case now we have to start from the latter instead of the former. Actually, by proceeding as previously done one would impose the invariance of Eq. (C.2.1) ending up with some transformation laws. In the classical case these laws provided the transformation laws for **E** and **B**, but this worked only because we knew that, because **F** does not change under Galilean transformations, change in the right-hand side of the formula could only be ascribed to the transformation laws of the fields. In this case, however, we do not know how forces transform in special relativity, therefore we would not be able to separate the final result in a part relative to the forces and another relative to the fields.

On the other hand, from what has been shown in Chap. 5 we can assume on an experimental basis that the Maxwell equations are form-invariant under Lorentz transformations. We can also assume that ε_0 and μ_0 are Lorentz scalars, i.e., their values remain unchanged for transformations of the Poincaré group and also that $\varepsilon_0\mu_0 = c^{-2}$. In a reference system $\bar{S}$ moving with velocity **u** with respect to S, Maxwell equations must read

$$\begin{aligned}
\bar{\nabla} \cdot \bar{\mathbf{E}} &= \frac{\bar{\rho}}{\varepsilon_0} \\
\bar{\nabla} \times \bar{\mathbf{E}} &= -\frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}} \\
\bar{\nabla} \cdot \bar{\mathbf{B}} &= 0 \\
\bar{\nabla} \times \bar{\mathbf{B}} &= \mu_0 \bar{\mathbf{j}} + \frac{1}{c^2} \frac{\partial \bar{\mathbf{E}}}{\partial \bar{t}}.
\end{aligned} \tag{C.2.3}$$

In these equations we know from Exercise 5.11 the transformation formulae for the gradient and the time derivative operators, but we do not know those of the fields, which actually are those we are looking for, as well as those of the charge and current density. One possible approach can be that of using the field equations in vacuum and deriving only the transformation for the fields, as shown in Kennedy (2012), but it is also possible to play with the general case because the equations for ρ and $\mathbf{j}$ can be obtained without any reference to the fields. This is what we do in the next subsection. These formulae are then used in the subsequent one to obtain those of $\mathbf{E}$ and $\mathbf{B}$.

The final step, i.e., showing the form-invariance of the Lorentz electromagnetic force, cannot be obtained without the assumption that forces do not transform as in Newtonian physics. The procedure then starts from calculating how the right-hand side of the equation, namely of $q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, is Lorentz-transformed, then it requires that the result represents a Lorentz force having the same form to obtain the equations for the forces. In other words, it is not a "proof" of the covariance of the Electromagnetism in special relativity, but rather a "discovery" of the transformation laws of the forces in this theory, which is exactly what we need because we know that the Einstein version of the relativity principle requires changing the laws of dynamics. This is why this last step has not been included in this section, but rather in Chap. 5, dealing with the relativistic dynamics treated with the tools of Newtonian physics.

C.2.1 Transformation of Charge and Current Density

The first task is to derive the transformation laws for the charge and current densities. These are defined as $\rho = q/V$, namely the charge per unit volume and $\mathbf{j} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of the charges. The fact that charges are given in discrete quantities carried by isolated particles, rather than being a continuous distribution, raises the problem of the meaning of the charge velocities. If, e.g, we have a charge Q formed by two single charges q_1 and q_2, each can have its own velocity, different in both speed and direction, and although this does not make any difficulty for the definition of ρ, it is natural to wonder which is the velocity defining the corresponding current density. This doubt can be addressed in two equivalent ways.

First of all, given a sufficiently large number of charges, the problem can be treated in a statistical way as in fluid dynamics. Second, it is always possible to reduce to the case of a single charge and to derive the transformations in this circumstance. The vectorial nature of these quantities then, allows them to be later added in this sense and to define the desired density. For this reason we assume in the following to be in the case of a single charge, or in the equivalent eventuality of a group of charges all having the same velocity.

The problem therefore can be defined like this: given a reference system S in which the charges with density ρ move with velocity $\mathbf{v}$, and are thus associated with a density current $\mathbf{j}$, we want to derive the transformation laws giving the corresponding $\bar{\rho}$ and $\bar{\mathbf{j}}$ in a second reference system $\bar{S}$ moving with velocity $\mathbf{u}$ with respect to S and assuming the Lorentz invariance of the charge q.

For our purposes it is easier to tackle this problem using a third reference system in which the charges are at rest ($\mathbf{v} = 0$) and as long as three reference systems are involved, in order to avoid possible misunderstandings we temporarily use the following heavier but unambiguous notation.

1. The reference system with respect to which the charges are at rest is denoted with S_0, and S and $\bar{S}$ are identified with S_1 and S_2, respectively.
2. The same number subscript denote the corresponding quantities of interest in each reference system, e.g., ρ_0, ρ_1, and ρ_2.
3. The symbol $\mathbf{v}_i$ is used for the velocities of the charges with respect to the reference system S_i, and $\mathbf{u}_{(ij)}$ indicates the velocity of the reference system S_i with respect to S_j.

The original notation with S and $\bar{S}$ is recovered in the end, when the final transformation formulae no longer make use of S_0.

In S_0 therefore it is $\mathbf{v}_0 = 0$, and $\mathbf{j}_0 = \rho_0 \mathbf{v}_0 = 0$.

S_1 moves with velocity $\mathbf{u}_{(10)}$ with respect to S_0, and thus the charges move with opposite velocity in S_1; in formulae $\mathbf{v}_1 = -\mathbf{u}_{(10)}$. In this system the charge density can be easily expressed with respect to that of S_0, i.e., ρ_0, in fact $\rho_1 = q/V_1$, where V_1 is the volume, as seen in S_1, containing the charge q. But given the Lorentz factor $\gamma_{(10)}$ of S_1 with respect to S_0 it is $V_1 = V_0/\gamma_{(10)}$. This can be easily understood in the simple case of two reference systems with parallel x-axes and $\mathbf{u}_{(10)}$ parallel to x. Under these conditions the volume of a parallelepiped with sides l_x, l_y, l_z is $V_1 = l_{x1} l_{y1} l_{z1}$, but from Eq. (5.5.28) $l_{x1} = l_{x0}/\gamma_{(10)}$, therefore, because of the assumption of the invariance of q,

$$\rho_1 = \frac{q}{l_{x1} l_{y1} l_{z1}} = \gamma_{(10)} \frac{q}{l_{x0} l_{y0} l_{z0}} = \gamma_{(10)} \rho_0. \tag{C.2.4}$$

In general we can always consider an appropriately rotated the reference system, reverting to the previous case.

The current density thus, in terms of the above quantities, can be written as

$$\mathbf{j}_1 = \rho_1 \mathbf{v}_1 = -\gamma_{(10)} \rho_0 \mathbf{u}_{(10)}. \tag{C.2.5}$$

We can obviously apply the same reasoning to S_2, thus getting

$$\rho_2 = \gamma_{(20)}\rho_0 \tag{C.2.6}$$

and

$$\mathbf{j}_2 = -\gamma_{(20)}\rho_0\mathbf{u}_{(20)}, \tag{C.2.7}$$

so that

$$\rho_2 = \frac{\gamma_{(20)}}{\gamma_{(10)}}\rho_1. \tag{C.2.8}$$

The problem with this formula is that it still relates the charge densities in S_1 and S_2 with the help of the third reference system. However, using Eq. (5.5.8) the velocity $\mathbf{v}_2 = -\mathbf{u}_{(20)}$ of the charges in S_2 can be also expressed as function of $\mathbf{v}_1$ and of the relative velocity $\mathbf{u}_{(21)}$ of S_2 with respect to S_1:

$$\mathbf{v}_2 = \left[\mathbf{v}_{1\perp} + \gamma_{(21)}\left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right)\right]\left[\gamma_{(21)}\left(1 - \frac{\mathbf{v}_1 \cdot \mathbf{u}_{(21)}}{c^2}\right)\right]^{-1}, \tag{C.2.9}$$

where as usual we separate the components of $\mathbf{v}_1$ perpendicular ($\perp$) and parallel ($\parallel$) to $\mathbf{u}_{(21)}$ and

$$\gamma_{(21)} = \left(1 - \frac{u_{(21)}^2}{c^2}\right)^{-1/2}.$$

We thus have

$$\begin{aligned}
u_{(20)}^2 &= \mathbf{u}_{(20)} \cdot \mathbf{u}_{(20)} = \mathbf{v}_2 \cdot \mathbf{v}_2 \\
&= \frac{\left[\mathbf{v}_{1\perp} + \gamma_{(21)}\left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right)\right] \cdot \left[\mathbf{v}_{1\perp} + \gamma_{(21)}\left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right)\right]}{\gamma_{(21)}^2\left(1 - \mathbf{v}_1 \cdot \mathbf{u}_{(21)}/c^2\right)^2} \\
&= \frac{\mathbf{v}_{1\perp} \cdot \mathbf{v}_{1\perp} + \gamma_{(21)}^2\left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right) \cdot \left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right)}{\gamma_{(21)}^2\left(1 - \mathbf{v}_1 \cdot \mathbf{u}_{(21)}/c^2\right)^2},
\end{aligned}$$

and therefore, putting $K = \left[\gamma_{(21)}\left(1 - \mathbf{v}_1 \cdot \mathbf{u}_{(21)}/c^2\right)\right]^{-2}$,

$$\begin{aligned}
1 - \frac{u_{(20)}^2}{c^2} &= 1 - \frac{\mathbf{v}_{1\perp} \cdot \mathbf{v}_{1\perp} + \gamma_{(21)}^2\left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right) \cdot \left(\mathbf{v}_{1\parallel} - \mathbf{u}_{(21)}\right)}{c^2\gamma_{(21)}^2\left(1 - \mathbf{v}_1 \cdot \mathbf{u}_{(21)}/c^2\right)^2} \\
&= K\left[\gamma_{(21)}^2\left(1 + \frac{\left(\mathbf{v}_1 \cdot \mathbf{u}_{(21)}\right)^2}{c^4} - \frac{\mathbf{v}_{1\parallel} \cdot \mathbf{v}_{1\parallel}}{c^2} - \frac{\mathbf{u}_{(21)} \cdot \mathbf{u}_{(21)}}{c^2}\right) - \frac{\mathbf{v}_{1\perp} \cdot \mathbf{v}_{1\perp}}{c^2}\right] \\
&= K\left[\gamma_{(21)}^2\left(1 + \frac{v_{1\parallel}^2 u_{(21)}^2}{c^4} - \frac{v_{1\parallel}^2}{c^2} - \frac{u_{(21)}^2}{c^2}\right) - \frac{v_{1\perp}^2}{c^2}\right]
\end{aligned}$$

$$= K\left[\gamma_{(21)}^2\left(1-\frac{u_{(21)}^2}{c^2}\right)\left(1-\frac{v_{1\parallel}^2}{c^2}\right)-\frac{v_{1\perp}^2}{c^2}\right]$$
$$= \frac{1-\left(v_{1\parallel}^2+v_{1\perp}^2\right)/c^2}{\gamma_{(21)}^2\left(1-\mathbf{v}_1\cdot\mathbf{u}_{(21)}/c^2\right)^2} = \frac{1-v_1^2/c^2}{\gamma_{(21)}^2\left(1-\mathbf{v}_1\cdot\mathbf{u}_{(21)}/c^2\right)^2}.$$

This gives, remembering that $\mathbf{v}_1 = -\mathbf{u}_{(10)}$,

$$\gamma_{(20)} = \frac{1}{\sqrt{1-u_{(20)}^2/c^2}} = \frac{\gamma_{(21)}\left(1-\mathbf{v}_1\cdot\mathbf{u}_{(21)}/c^2\right)}{\sqrt{1-v_1^2/c^2}}$$
$$= \gamma_{(21)}\gamma_{(10)}\left(1-\mathbf{v}_1\cdot\mathbf{u}_{(21)}/c^2\right).$$

Substituting this relation in Eq. (C.2.8), and because $\mathbf{j}_1 = \rho_1\mathbf{v}_1$, we have

$$\rho_2 = \gamma_{(21)}\left(1-\frac{\mathbf{v}_1\cdot\mathbf{u}_{(21)}}{c^2}\right)\rho_1$$
$$= \gamma_{(21)}\left(\rho_1-\frac{\mathbf{u}_{(21)}\cdot\mathbf{j}_1}{c^2}\right)$$

or, reverting to the original notation,

$$\bar{\rho} = \gamma\left(\rho-\frac{\mathbf{j}\cdot\mathbf{u}}{c^2}\right). \tag{C.2.10}$$

The current density transformation now can be easily recovered from Eq. (C.2.9) (again in the initial notation where $\mathbf{v}_2 \equiv \bar{\mathbf{v}}$, $\mathbf{v}_1 \equiv \mathbf{v}$, $\mathbf{u}_{(21)} \equiv \mathbf{u}$ is the relative velocity of S_2 with respect to S_1 and $\gamma_{(21)} \equiv \gamma$ is the corresponding Lorentz factor) and (C.2.10) remembering that by definition $\mathbf{j} = \rho\mathbf{v}$:

$$\bar{\mathbf{j}} = \bar{\rho}\bar{\mathbf{v}} = \gamma\left(\rho-\frac{\mathbf{u}\cdot\mathbf{j}}{c^2}\right)\frac{\mathbf{v}_\perp+\gamma\left(\mathbf{v}_\parallel-\mathbf{u}\right)}{\gamma\left(1-\mathbf{v}\cdot\mathbf{u}/c^2\right)}$$
$$= \rho\frac{\left(1-\mathbf{v}\cdot\mathbf{u}/c^2\right)\left[\mathbf{v}_\perp+\gamma\left(\mathbf{v}_\parallel-\mathbf{u}\right)\right]}{\left(1-\mathbf{v}\cdot\mathbf{u}/c^2\right)}$$
$$= \mathbf{j}_\perp+\gamma\left(\mathbf{j}_\parallel-\mathbf{u}\rho\right). \tag{C.2.11}$$

C.2.2 Transformation of Electric and Magnetic Fields

As anticipated, we assume the invariance of the constants ε_0 and μ_0. In order to simplify the calculations, however, we assume $\mathbf{u} = (u, 0, 0)$, i.e., a relative velocity along the x-axis. In this case such an assumption can be made with no loss of generality, because the general formulae can be immediately recovered as a function of the parallel and orthogonal parts of the fields, as we show later.

In this case it is easy to see that the formula for the gradient computed in Exercise 5.11 becomes simply

$$\bar{\nabla} = \left(\gamma\frac{\partial}{\partial x} + \gamma\frac{u}{c^2}\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

We can now use the first of Eqs. (C.2.3) and (C.2.10) to find

$$\bar{\nabla} \cdot \bar{\mathbf{E}} = \gamma\frac{\partial \bar{E}_x}{\partial x} + \gamma\frac{u}{c^2}\frac{\partial \bar{E}_x}{\partial t} + \frac{\partial \bar{E}_y}{\partial y} + \frac{\partial \bar{E}_z}{\partial z} = \gamma\frac{\rho}{\varepsilon_0} - \gamma\frac{u}{\varepsilon_0 c^2} j_x$$

which, remembering that we assumed $\varepsilon_0\mu_0 = c^{-2}$, can be rearranged as

$$\gamma\frac{\partial \bar{E}_x}{\partial x} + \frac{\partial \bar{E}_y}{\partial y} + \frac{\partial \bar{E}_z}{\partial z} = \gamma\frac{\rho}{\varepsilon_0} - \gamma u\left(\mu_0 j_x + \frac{1}{c^2}\frac{\partial \bar{E}_x}{\partial t}\right).$$

It is easy to recognize in the left-hand side of this equation a resemblance to $\gamma\nabla\cdot\mathbf{E}$, which could be paired with $\gamma\rho/\varepsilon_0$, as long as it is

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma E_y, \quad \bar{E}_z = \gamma E_z,$$

but it is also clear that this transformation would leave an undesired term $-u\left(\mu_0 j_x + c^{-2}\partial_t E_x\right)$. The latter, however, is just $-u$ times the x component of the right-hand side of the fourth Maxwell equation, which in its turn should be paired to $-u$ times the x component of $\nabla \times \mathbf{B}$ to give the desired form-invariant transformed equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

This result can therefore be obtained by requiring

$$\begin{aligned}
\bar{E}_x &= E_x \\
\frac{\partial \bar{E}_y}{\partial y} + \frac{\partial \bar{E}_z}{\partial z} &= \gamma\left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} - u\left(\nabla \times \mathbf{B}\right)_x\right] \\
&= \gamma\left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} - u\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\right] \\
&= \frac{\partial}{\partial y}\left[\gamma\left(E_y - uB_z\right)\right] + \frac{\partial}{\partial z}\left[\gamma\left(E_z + uB_y\right)\right],
\end{aligned}$$

which implies

$$\begin{aligned}\bar{E}_x &= E_x \\ \bar{E}_y &= \gamma\left(E_y - uB_z\right) \\ \bar{E}_z &= \gamma\left(E_z + uB_y\right).\end{aligned} \tag{C.2.12}$$

The general expression can be easily retrieved if we consider that $u \equiv u_x$, which leads immediately to the conclusion that $-uB_z\mathbf{e}_y + uB_y\mathbf{e}_z$ is just the explicit expression of $(\mathbf{u} \times \mathbf{B})$ in this particular case. Moreover, because for the same reason we can consider that the components of $\mathbf{E}$ are the specific expressions of

$$E_x\mathbf{e}_x \equiv \mathbf{E}_\parallel \quad \text{and} \quad E_y\mathbf{e}_y + E_z\mathbf{e}_z \equiv \mathbf{E}_\perp,$$

the previous transformations can be rewritten as

$$\begin{aligned}\bar{\mathbf{E}}_\parallel &= \mathbf{E}_\parallel \\ \bar{\mathbf{E}}_\perp &= \gamma\left[\mathbf{E}_\perp + (\mathbf{u} \times \mathbf{B})\right].\end{aligned} \tag{C.2.13}$$

The transformation laws for $\mathbf{B}$ can be found using the third of Eq. (C.2.3), i.e., the magnetic Gauss law

$$\bar{\nabla} \cdot \bar{\mathbf{B}} = \gamma\frac{\partial \bar{B}_x}{\partial x} + \gamma\frac{u}{c^2}\frac{\partial \bar{B}_x}{\partial t} + \frac{\partial \bar{B}_y}{\partial y} + \frac{\partial \bar{B}_z}{\partial z} = 0$$

that is,

$$\gamma\frac{\partial \bar{B}_x}{\partial x} + \frac{\partial \bar{B}_y}{\partial y} + \frac{\partial \bar{B}_z}{\partial z} = -\gamma\frac{u}{c^2}\frac{\partial \bar{B}_x}{\partial t}.$$

Given the previous procedure for $\mathbf{E}$, it is obvious that similar considerations of the magnetic Gauss' law and the Faraday law $\nabla \times \mathbf{E} = -\partial_t\mathbf{B}$ allows us to make the following requirements

$$\begin{aligned}\bar{B}_x &= B_x \\ \frac{\partial \bar{B}_y}{\partial y} + \frac{\partial \bar{B}_z}{\partial z} &= \gamma\left[\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} + \frac{u}{c^2}(\nabla \times \mathbf{E})_x\right],\end{aligned}$$

which eventually leads to the transformations for $\mathbf{B}$:

$$\begin{aligned}\bar{\mathbf{B}}_\parallel &= \mathbf{B}_\parallel \\ \bar{\mathbf{B}}_\perp &= \gamma\left[\mathbf{B}_\perp - \frac{1}{c^2}(\mathbf{u} \times \mathbf{E})\right].\end{aligned} \tag{C.2.14}$$

Up to now we have shown that Gauss law and the magnetic Gauss' law are covariant with respect to the Lorentz transformations if $\mathbf{E}$ and $\mathbf{B}$ transform according to Eqs. (C.2.13) and (C.2.14). Finally, a straightforward calculation after substituting the latter in the Faraday and Ampere laws is sufficient to show that these transformations also leave the two remaining Maxwell equations form-invariant.

C.3 Electrodynamics in Minkowski Geometry

With the above calculations we have derived the transformation laws of the fundamental quantities of electrodynamics, namely the charge and current densities, and the electric and magnetic fields. This was done strictly within the boundaries of the three-dimensional formalism of classical physics. In Chap. 6, however, it has been shown that Minkowski geometry is the preferred formalism to demonstrate the covariance of the equations of kinematics and dynamics with respect to the transformations of the Poincaré group of special relativity. The same is true for electrodynamics.

C.3.1 Maxwell Equations and Lorentz Electromagnetic Force

Four-dimensional current and continuity equation

First of all, by comparing the transformation laws of Eqs. (C.2.10) and (C.2.11) with those of Eqs. (5.5.1) and (5.5.2) it is immediate to not that the electric charge density and the current density transform exactly as time and space coordinates. Because the latter, by means of their infinitesimal displacements, identified our first prototypical four-vector, this means that ρ and $\mathbf{j}$ also constitute the (contravariant) components of a four-vector, and remembering that in order to have components with homogeneous dimensions we put $x^0 = ct$ we can write such a current vector as

$$\boldsymbol{j} = c\rho \boldsymbol{e}_0 + j_x \boldsymbol{e}_x + j_y \boldsymbol{e}_y + j_z \boldsymbol{e}_z,$$

which in component notation is denoted by j^α. Moreover, inasmuch as we know from Chap. 6 that the four-velocity can be written as $\boldsymbol{v} = \gamma(v)\{c, \mathbf{v}\}$, from Eqs. (C.2.4) and (C.2.5) (or from Eqs. (C.2.6) and (C.2.7) equivalently) the current vector can also be expressed as

$$\boldsymbol{j} = \rho_0 \boldsymbol{v},$$

or in component notation $j^\alpha = \rho_0 v^\alpha$ which is the analogous to the definition of the four-momentum $\boldsymbol{p}$ where the charge density at rest ρ_0 plays the role of the rest-mass m.[24]

In Exercise 5.13 it was stressed that the four-dimensional gradient operator $\partial_\alpha = \left(c^{-1}\partial/\partial t, \boldsymbol{\nabla}\right)$ is a one-form, therefore it makes perfect sense to write the covariant expression

$$\partial_\alpha j^\alpha = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0,$$

where the last equality follows because of the continuity equation. The above definitions therefore makes it possible to write such an equation in covariant and compact form simply by $\partial_\alpha j^\alpha = 0$.

Four-potential and Maxwell equations

The next step is to find a four-dimensional version of the Maxwell equations. In this case it is more natural to resort to the version of these equations involving the potentials, instead of that based on the fields, therefore we need to briefly recall this alternative formulation. As shown in many classical textbooks (see, e.g., Jackson 1962) the four first-order equations (C.2.2) are equivalent to the two second-order differential equations

$$\begin{aligned} \nabla^2 \phi + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \cdot \mathbf{A}\right) &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) &= -\mu_0 \mathbf{j} \end{aligned} \tag{C.3.1}$$

where ϕ and $\mathbf{A}$ are the electric and the vector potentials, respectively, and the electric and magnetic fields are defined in terms of these two by

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{C.3.2}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{C.3.3}$$

These definitions are arbitrary with respect to a transformation of ϕ and $\mathbf{A}$ depending on a generic scalar function f. In particular, $\mathbf{E}$ and $\mathbf{B}$ are left unchanged, thus exhibiting a so-called *gauge invariance*, for any function $f(\mathbf{x}, t)$ and by the two

[24] It is worth observing that this definition does not depend on the "type" of density (and of current) we use, and actually could be easily extended to any kind of "rest density times four-velocity", e.g., the mass density, or "density, current density" pair, such as the energy-momentum density. The four-vector character of this entity in fact descends as a consequence of the fact that its transformation laws were derived using the time and space transformations, which as we know constitute our prototype four-vector.

transformations $\phi' = \phi - \partial_t f$ and $\mathbf{A}' = \mathbf{A} + \nabla f$, called *gauge transformations*. The choice of a specific function determines a relation between the resulting ϕ and $\mathbf{A}$ that, eventually, changes the form of the second-order equations (C.3.1) even if their physical meaning remains the same because of the arbitrariness of the definition of the fields.

If, starting from a general ϕ and $\mathbf{A}$, f is chosen such that the resulting transformed potentials satisfy the condition

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t},$$

and because by construction

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2},$$

then

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \qquad \text{(C.3.4)}$$

which means that the gauge freedom on the potentials always allows them to be chosen to obey the above Lorenz condition.[25] Substituting this condition in Eqs. (C.3.1), it is easy to see that they change to

$$\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\mathbf{j} \end{aligned} \qquad \text{(C.3.5)}$$

which therefore, together with Eqs. (C.3.2) and (C.3.3), represent another set of second-order differential equations equivalent to the original four Maxwell equations.

Because $\varepsilon_0\mu_0 = c^{-2}$, the first of these two equations can be written as

$$\nabla^2\left(\frac{\phi}{c}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\phi}{c}\right) = -\mu_0 c\rho,$$

and remembering the definitions of ∂_α and j^α, by putting $A^\alpha = (\phi/c, \mathbf{A})$ Eqs. (C.3.5) can be rewritten in terms of four-dimensional Minkowski objects as

$$\Box^2 A^\alpha = -\mu_0 j^\alpha, \qquad \text{(C.3.6)}$$

where $\Box^{2X} = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alambertian operator. The quantity A^α is called a *four-potential*, and its writing as a four-vector is justified a posteriori by the form of Eq. (C.3.6). Indeed, if ∂_α is a one-form, then $\Box^2$ is a Lorentz scalar, such as μ_0

[25]Named after the Danish physicist Ludwig Lorenz, who first introduced it in 1867.

by hypothesis; moreover we have just shown that j^α are the components of a four-vector, thus A^α must be a four-vector as well. At the same time, the Lorenz gauge condition becomes $\partial_\alpha A^\alpha = 0$.

Lorentz force and Electromagnetic field tensor

Attempting to put the formula for the Lorentz force in covariant form leads us to the definition of a new rank 2 tensor that allows incorporating the electric and magnetic fields into an object belonging to the Minkowski geometry.

Recalling the definition of four-force of Eq. (6.2.6) and its decomposition in terms of three-dimensional entities of Eq. (6.2.7) gives

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = \gamma \left(\frac{1}{c}\frac{\mathrm{d}W}{\mathrm{d}t}, \mathbf{F} \right),$$

where we used W for the energy instead of E to avoid confusing it with the electric field. Inasmuch as in general

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \mathbf{F} \cdot \mathbf{v},$$

substituting in the above formulae the equation of the Lorentz electromagnetic force gives

$$\boldsymbol{f} = \gamma q \left(\frac{1}{c}\mathbf{E} \cdot \mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B} \right).$$

In components this equation can be written

$$\begin{aligned}
\frac{\mathrm{d}p^0}{\mathrm{d}\tau} &= q \left(\frac{E_x}{c}\gamma v_x + \frac{E_y}{c}\gamma v_y + \frac{E_z}{c}\gamma v_z \right) \\
\frac{\mathrm{d}p^1}{\mathrm{d}\tau} &= q \left(\frac{E_x}{c}\gamma c + \gamma v_y B_z - \gamma v_z B_y \right) \\
\frac{\mathrm{d}p^2}{\mathrm{d}\tau} &= q \left(\frac{E_y}{c}\gamma c - \gamma v_x B_z + \gamma v_z B_x \right) \\
\frac{\mathrm{d}p^3}{\mathrm{d}\tau} &= q \left(\frac{E_z}{c}\gamma c + \gamma v_x B_y - \gamma v_y B_x \right)
\end{aligned}$$

which can be put in matrix form

$$\begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix} = q \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}.$$

Remembering the relation $\boldsymbol{v} = \gamma \{c, \mathbf{v}\}$ from Sect. 6.1 the rightmost column of this formula is just the four-vector v^α, and the left-hand side of the equation is the four-force which is another valid four-vector and q is a Lorentz-invariant by hypothesis. The matrix therefore must represent the components of a mixed rank 2 tensor called the *electromagnetic field tensor* that can be defined as

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{C.3.7}$$

in order to put the above equation in its covariant form

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = q F^\alpha{}_\beta v^\beta. \tag{C.3.8}$$

The spatial components of this equation represent the Lorentz electromagnetic force, and the temporal one is nothing else than the amount of energy per unit time spent by the force, namely its power.

Electromagnetic field tensor and Maxwell equations

A straightforward calculation shows that

$$\partial^\alpha A^\beta - \partial^\beta A^\alpha = F^{\alpha\beta},$$

therefore the antisymmetric tensor $\partial^\alpha A^\beta - \partial^\beta A^\alpha$ on the left-hand side of the above equation can be taken as another definition of the electromagnetic field tensor. Moreover, such a tensor allows us also to recast in a four-dimensional covariant form the Maxwell equations for the fields.

For example, it is immediate to see that $\partial_\beta F^{0\beta} = c^{-1}\nabla \cdot \mathbf{E}$, thus we can recover the Gauss law with

$$\partial_\beta F^{0\beta} = \frac{1}{c}\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0 c} = \frac{c\rho}{\varepsilon_0 c^2} = \mu_0 j^0,$$

whereas if we take ith "row" we get

$$\partial_\beta F^{i\beta} = -\frac{1}{c^2}\frac{\partial E_i}{\partial t} + (\nabla \times \mathbf{B})_i = \mu_0 j_i = \mu_0 j^i,$$

which is the ith component of Ampere's law. This shows that the first and last Maxwell equations can be condensed in the covariant expression

$$\partial_\beta F^{\alpha\beta} = \mu_0 j^\alpha. \tag{C.3.9}$$

Similarly, for any combination of different spatial indexes i, j, k the magnetic Gauss law can be recast as

$$\partial^i F^{jk} + \partial^j F^{ki} + \partial^k F^{ij} = \pm\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = \pm\nabla\cdot\mathbf{B} = 0,$$

while the similar expression with the temporal component gives

$$\partial^0 F^{jk} + \partial^j F^{k0} + \partial^k F^{0j} = \pm\frac{1}{c}\left((\nabla\times\mathbf{E})_i + \frac{\partial \mathbf{B}_i}{\partial t}\right) = 0$$

which is the ith component of Faraday's law, thus in the end the second and third Maxwell equations can be written as

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \tag{C.3.10}$$

with α, β, and γ any combination of three different spacetime indexes.

C.3.2 *Variational Approach to Electrodynamics: Particles and Fields*

In the last section we showed how the basic equations of electrodynamics, namely the Maxwell equations and the Lorentz force, can be naturally recast in a Lorentz covariant way stemming from the fact that the scalar and vector potentials of the Euclidean formulation form a four-vector potential $A^\alpha = (\phi/c, \mathbf{A})$ and that the same holds for the charge and current densities, which can be interpreted as the components of the current four-vector $j^\alpha = (c\rho, \mathbf{j})$.

For the sake of completeness, and in order to give a more exhaustive justification to the statements of Sect. 7.3, it is sketched how these equations can be derived using the same variational procedure adopted for interactions modeled by scalar fields.

We have as usual to give an expression for the three components of the Lagrangian, namely the free particle, free fields, and the interacting part. Contrary to what we did in the main chapters, however, they are already written in their final form, with all the constants set to the correct value, to shorten the discussion.

The total action then results in

$$S = \frac{1}{2}\int m\eta_{\alpha\beta}v^\alpha v^\beta\,\mathrm{d}\tau - \frac{1}{2\mu_0}\int \partial_\alpha A_\beta \partial^\alpha A^\beta\,\mathrm{d}^4x + \int \eta_{\alpha\beta} j^\alpha A^\beta\,\mathrm{d}^4x,$$

where the three terms are, respectively, the action for the free particle, in which we do not need to consider the case of massless particles and m is the *inertial* mass, the action for free fields whose Lagrangian density is proportional to $\eta_{\alpha\mu}\eta_{\beta\nu}\partial^\mu A^\nu \partial^\alpha A^\beta$, and finally, the interaction term, which as usual is proportional to the product of the source and the potential. The latter is clearly written in this way to get a scalar out of the two four-vectors j^α and A^α.

Dynamics of charged particles

The equations of motion for charged particles are thus

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^\gamma} - \frac{\partial L}{\partial x^\gamma} = 0,$$

where as always only the first and last term of the action contribute, and the latter can be integrated over a three-dimensional space considering that $j^\alpha = \rho_0 v^\alpha$ and $\rho_0 = q\delta^3\left(\boldsymbol{x} - \boldsymbol{r}\left(\tau\right)\right)$, so that

$$\int \eta_{\alpha\beta} j^\alpha A^\beta \, \mathrm{d}^4 x = q \int \eta_{\alpha\beta} v^\alpha A^\beta \, \mathrm{d}\tau.$$

The above Euler–Lagrange equations therefore can be recast as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{2} m \eta_{\alpha\beta}\left(\delta^\alpha_\gamma v^\beta + v^\alpha \delta^\beta_\gamma\right) + q\eta_{\alpha\beta}\delta^\alpha_\gamma A^\beta\right] - q v^\alpha \frac{\partial A_\alpha}{\partial x^\gamma} = 0,$$

but $\eta_{\alpha\beta}\delta^\alpha_\gamma v^\beta = \delta_{\beta\gamma} v^\beta = v_\gamma = \eta_{\alpha\beta} v^\alpha \delta^\beta_\gamma$, and similarly $\eta_{\alpha\beta}\delta^\alpha_\gamma A^\beta = A_\gamma$, therefore

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m v_\gamma + q A_\gamma\right) - q v^\alpha \frac{\partial A_\alpha}{\partial x^\gamma} = 0.$$

From the definition of four-momentum $p^\gamma = m v^\gamma$ it is

$$\begin{aligned}\frac{\mathrm{d}p_\gamma}{\mathrm{d}\tau} &= q\left(-\frac{\mathrm{d}A_\gamma}{\mathrm{d}\tau} + v^\alpha \partial_\gamma A_\alpha\right)\\ &= q\left(-\partial_\alpha A_\gamma \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} + v^\alpha \partial_\gamma A^\alpha\right)\\ &= q\left(-v^\alpha \partial_\alpha A_\gamma + v^\alpha \partial_\gamma A_\gamma\right)\\ &= -q F_{\alpha\gamma} v^\alpha = q F_{\gamma\alpha} v^\alpha,\end{aligned}$$

and finally the Lorentz force in covariant form of Eq. (C.3.8) can be obtained by contracting both sides with $\eta^{\beta\gamma}$ and relabeling dummy indexes.

Field dynamics for vector potential

Similarly, the electromagnetic field equations can be obtained by substituting ϕ with A^α in the Euler–Lagrange equations for the scalar fields of Eq. (6.3.3)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A^\nu\right)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0,$$

with

$$\mathcal{L} = -\frac{1}{2\mu_0} \eta_{\alpha\mu} \eta_{\beta\nu} \partial^\mu A^\nu \partial^\alpha A^\beta + \eta_{\alpha\beta} j^\alpha A^\beta.$$

Straightforward calculations along the lines of the above ones give

$$-\frac{1}{\mu_0} \partial^\mu \partial_\mu A_\nu = j_\nu$$

which once again can be recast in the Lorentz-covariant form of the Maxwell equations of Eq. (C.3.6) by contracting both sides with $\eta^{\alpha\nu}$.

C.4 Stress-Energy Tensor

In Appendix B.6 we presented the stress a tensor as an example of tensor in Euclidean space, anticipating that it was the "ancestor" of a four-dimensional quantity fundamental for the development of relativistic physics. Here we introduce this quantity, namely the stress-energy tensor or the energy-momentum tensor, beginning with the mere necessity of having a consistent four-dimensional extension of its Euclidean predecessor and deducing its general form upon this requirement. Then we apply this definition to some specific cases used in the text and more generally in relativistic physics. Finally, we generalize this concept by showing how its definition naturally applies to different entities, such as the fields, demonstrating its connection with the variational approach.

C.4.1 The Stress-Energy Tensor for Matter

When the Euclidean definition of the stress tensor was introduced, it was highlighted that the diagonal components of this quantity can be interpreted as the pressure on the three faces of the volume element and the off-diagonals represent the shear of the forces.

In the relativistic framework, however, such a Euclidean quantity cannot identify a tensorial object, which prevents its use in special or general relativity. It is then normal to wonder if, as happens with other vectorial quantities such as velocities, momentum, and the like there is a way to translate this concept into a four-dimensional geometry.

Stress-energy tensor as four-momentum flux

The fact that Chaps. 7 and 8 show that this tensor represents the actual source of gravity in relativistic theories makes such a transition even more compelling. It is therefore relieving to know that the answer to the above question is positive. Actually the most natural way to achieve this goal is to exploit the other interpretation of the Euclidean tensor as a momentum flux of a swarm of particles and make an equivalent relation based on the four momentum, instead. The mathematical rendition of this expression lies in Eq. (B.6.8), and it is based on the definition of momentum density $\pi_i = \rho v_i = m\delta^3(\mathbf{x} - \mathbf{r}(t))\, v_i = p_i\delta^3(\mathbf{x} - \mathbf{r}(t))$. By analogy, one can imagine using the covariant definition

$$t^{\alpha\beta} = p^\alpha \delta^4(\boldsymbol{x} - \boldsymbol{r}(\lambda))\, v^\beta,$$

where λ is an appropriate quantity used to parameterize the particles' trajectories and $v^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$. However, it is easy to realize that such a definition could not give the classical three-dimensional quantity in the non relativistic limit, because we have introduced the four-dimensional Kronecker delta, which means a kind of additional "temporal density" that we need to avoid. This leads to the definition of the tensor

$$T^{\alpha\beta} = \int p^\alpha \delta^4(\boldsymbol{x} - \boldsymbol{r}(\lambda))\, v^\beta \mathrm{d}\lambda \tag{C.4.1}$$

which is called the *stress-energy tensor*. This definition can be specialized for massive particles, for which one can use the proper time τ as parameter and $p^\alpha = mv^\alpha$ so that

$$T^{\alpha\beta} = mc \int \delta^4(\boldsymbol{x} - \boldsymbol{r}(\lambda))\, v^\alpha v^\beta \mathrm{d}\tau$$

or for massless particles with $p^\alpha = (E/c)\, v^\alpha$. A completely general formula, instead, can be obtained using the einbein introduced in Sect. 6.3.1

$$T^{\alpha\beta} = c \int \mathrm{e}^{-1}(\lambda)\, \delta^4(\boldsymbol{x} - \boldsymbol{r}(\lambda))\, v^\alpha v^\beta \mathrm{d}\lambda. \tag{C.4.2}$$

This name evidently comes from the fact that the four-momentum is a four-vector composed of a temporal part having the meaning of energy and a spatial part giving the momentum of the particle. This means that $\pi^\alpha \equiv T^{\alpha 0}$ is a four-vector that can rightly be called *four-momentum density*. Its temporal component T^{00} is the *energy density* and the spatial components T^{i0} are the *momentum density*. As in the Euclidean case multiplication by v^α (i.e., their outer product, in the tensorial language of the previous appendix) produces a rank 2 tensor which is the *flux* of the four-momentum density, and splitting again the temporal and spatial parts we can easily observe that the spatial–spatial components T^{ij} keep the original Euclidean meaning of (spatial) *momentum density current*, i.e., the *stress tensor*. On the other hand the T^{0j} components can be interpreted as the *energy density current*.

Properties of the stress-energy tensor

In Appendix B.6.1 it was shown that its definition implies that the stress tensor is symmetric, and that the momentum conservation implies that in an isolated region $\partial t_{ij}/\partial x^j = 0$. Now the definition of Eq. (C.4.2) immediately recasts the symmetry property also in the four-dimensional case, so that

$$T^{\alpha\beta} = T^{\beta\alpha}. \tag{C.4.3}$$

Moreover, as for the the Euclidean quantity, the conservation of four-momentum implies that

$$\partial_\alpha T^{\alpha\beta} = 0. \tag{C.4.4}$$

This derives from the fact that the Gauss theorem is also valid in the Minkowski space (this can be intuitively understood because its proof in the Euclidean space does not depend on its dimensions or metric, but see, e.g., Misner et al. (1973) for a detailed proof) namely

$$\int_S T^{\alpha\beta} n_\beta \, \mathrm{d}S = \int_V \partial_\alpha T^{\alpha\beta} \mathrm{d}V$$

where $\mathrm{d}V \equiv \mathrm{d}^4x$ and $\mathrm{d}\boldsymbol{S} \equiv \boldsymbol{n}\mathrm{d}S = \boldsymbol{n}\mathrm{d}^3x$ is the three-dimensional oriented boundary of V. Now, for what we have just observed, the left-hand side of this formula is exactly the four-momentum, because it can be reconstructed from its density by integrating $T^{\alpha\beta}$ over this surface

$$p^\alpha = \int_S T^{\alpha\beta} n_\beta \, \mathrm{d}S.$$

To be more precise, the above definition is valid in general, but as in the Euclidean case we are considering an isolated system, and therefore this integral is the total four-momentum of the system. Once again, then it has to be $\boldsymbol{p} = 0$ constantly for the conservation of four-momentum, which leads to Eq. (C.4.4) by means of the four-dimensional Gauss theorem because the condition

$$\int_V \partial_\alpha T^{\alpha\beta} \mathrm{d}V = 0$$

must hold for any four-dimensional volume.

Stress-energy tensor for dust and perfect fluids

In order to have a more intuitive understanding of the physical meaning of the vanishing divergence condition, we can examine the two simplest and most used cases of the stress-energy tensor: those of the so-called *dust* and of the *perfect fluid*.

These two media are defined from the properties they exhibit in a special reference system where the particle is instantaneously at rest. In this system the former refers to a set of non interacting particles that also form a *pressureless* medium, and the latter indicates another set of non-interacting particles that have two properties in a reference frame:

1. They do not conduct heat.
2. They have zero viscosity.

Dust therefore has the simplest possible form of stress-energy tensor; in fact this means that in the instantaneously rest frame the only property of the particles is their density so $T^{00} = \rho c^2$. They cannot move otherwise the medium would have a pressure, appearing as the diagonal spatial components T^{ii}. At the same time, because in this reference system particles are at rest by definition, $v^\alpha = (c, 0, 0, 0)$, therefore we can write

$$T^{\alpha\beta} = \rho v^\alpha v^\beta, \tag{C.4.5}$$

but $T^{\alpha\beta}$ is a tensor, so this expression has to be covariant, i.e., valid in any reference system, thus the above formula is also the general expression of the stress-energy tensor for the dust.

As regards the perfect fluid, it has to be observed that the no-heat-conduction constraint can be rephrased with the absence of energy transmission due to heat, or in other words as a particular form of energy flux, which as we know is connected with the T^{0i} components. In principle an energy flux might also come from the particles' motion, but in our reference system particles do not move, therefore this condition means $T^{0i} = 0$. The zero spatial velocity also implies $T^{i0} = 0$, but the same conclusion could be drawn from the symmetry of $T^{\alpha\beta}$. The zero-viscosity property instead is related to the condition $T^{ij} = 0$ for $i \neq j$, because viscosity is intended as the equivalent of friction for the fluids, i.e., a force parallel to the faces of the infinitesimal volume. This, however, does not exclude the existence of pressure, which would mean $T^{ii} = 0$. Putting everything together and making the same considerations on v^α used for the dust we can write the general expression of the stress-energy tensor for perfect fluids as

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) v^\alpha v^\beta + p\eta^{\alpha\beta}. \tag{C.4.6}$$

From this equation it is evident that dust can be considered a particular case of perfect fluid for which $p = 0$, consistent with our previous definitions. This point of view is usually adopted in cosmology, where the dust is also denoted “matter”.

We can now return to the vanishing divergence condition, breaking it in its temporal and spatial components

$$\partial_\alpha T^{\alpha 0} = 0$$
$$\partial_\alpha T^{\alpha j} = 0.$$

If we consider the small-velocity and weak-energy case, i.e., Newtonian conditions, it can be assumed that $v^i \ll c$, and $p \ll \rho c^2$. The components of $T^{\alpha\beta}$ for a perfect fluid read

$$T^{00} = \left(\rho + \frac{p}{c^2}\right) v^0 v^0 - p = \rho c^2$$
$$T^{0j} = \left(\rho + \frac{p}{c^2}\right) v^0 v^j = \rho c v^j + \frac{p}{c^2} c v^j \simeq \rho c v^j$$
$$T^{ij} = \left(\rho + \frac{p}{c^2}\right) v^i v^j + p\delta^{ij} \simeq \rho v^i v^j + p\delta^{ij}.$$

Substituting them into the first equation gives

$$\frac{1}{c}\frac{\partial\left(\rho c^2\right)}{\partial t} + \frac{\partial\left(\rho c v^j\right)}{\partial x^j} = 0$$

which becomes the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0.$$

The second equation instead is

$$\frac{1}{c}\frac{\partial\left(\rho c v^j\right)}{\partial t} + \frac{\partial\left(\rho v^i v^j + p\delta^{ij}\right)}{\partial x^j} = 0.$$

A simple calculation gives

$$\frac{1}{c}\frac{\partial\left(\rho c v^j\right)}{\partial t} + \frac{\partial\left(\rho v^i v^j + p\delta^{ij}\right)}{\partial x^j} = \frac{\partial \rho}{\partial t} v^j + \rho \frac{\partial v^j}{\partial t} + \frac{\partial\left(\rho v^i v^j\right)}{\partial x^j} + \frac{\partial p}{\partial x^i},$$

and because from the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial\left(\rho v^j\right)}{\partial x^j}$$

the above equation can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla p$$

which can be easily recognized as Euler's equation of fluid dynamics in case of incompressible fluids.

C.4.2 The Stress-Energy Tensor for the Fields

In the previous section the density of the energy-momentum four-vector gives the origin of a rank 2 tensor with a structure

$$\begin{pmatrix} \text{Energy density} & \text{Energy current density} \\ \text{Momentum density} & \text{Momentum current density} \end{pmatrix}$$

which justifies the alternative name of *energy-momentum tensor* also given to $T^{\alpha\beta}$. This simple observation allows us to draw an interesting conclusion. Indeed, a swarm of particles is not the only physical entity to which energy and momentum can be attributed. Fields as well can carry these properties, and the most obvious example is that of the electromagnetic field.

It is well known (see, e.g., Jackson (1962) or any classical textbook) that the energy density of the electromagnetic field can be written as

$$\mathcal{E} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right),$$

and the flux of energy, i.e., the momentum density, is given by the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

These two quantities together form a perfectly valid four-momentum density, so it should be possible to associate a stress-energy tensor to the electromagnetic field.

A heuristic way to derive it is observing that E^2 and B^2 can be built from the electromagnetic field tensor $F^{\mu\nu}$. From Eq. (C.3.7) it is

$$F^{0\nu} F_{\nu 0} = \frac{E^2}{c^2}$$

and

$$F^{\mu\nu}F_{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right),$$

so

$$\frac{1}{\mu_0}\left(F^{0\nu}F_{\nu 0} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) = \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right) = \frac{1}{2\mu_0}\left(\epsilon_0\mu_0 E^2 + B^2\right) = \mathcal{E}.$$

Now the problem with this expression is that it is just the sum of something resembling a component of a (1, 1) tensor ($F^{0\nu}F_{\nu 0}$) and a scalar, whereas we need something that could be identified with T^{00}, i.e., the component of a rank 2 tensor. These two problems can be fixed by raising the 0 index in $F_{\nu 0}$ and the second term by multiplying by η^{00}, but because $F_\nu{}^0 = -F_{\nu 0}$ and $\eta^{00} = -1$, we have to introduce an additional -1 factor to keep the correct sign of $\mathcal{E}$. Finally, by considering that $F^{\mu\nu}$ is antisymmetric, we can use $F^0{}_\nu = -F_\nu{}^0$ to obtain

$$T^{00} = \frac{1}{\mu_0}\left(F^{0\nu}F^0{}_\nu - \frac{1}{4}\eta^{00}F^{\mu\nu}F_{\mu\nu}\right). \tag{C.4.7}$$

Now we can make an "educated guess" and suppose that the same relation is valid for all the components, so that

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(F^{\alpha\mu}F^\beta{}_\mu - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right), \tag{C.4.8}$$

but how much is this posit reliable?

We can start by noting that, according to this definition, not only T^{00} coincides with the energy density, but also T^{0j} coincides with the Poynting vector, in fact

$$\begin{aligned} T^{0j} &= \frac{1}{\mu_0}\left(F^{0\mu}F^j{}_\mu - \frac{1}{4}\eta^{0j}F^{\mu\nu}F_{\mu\nu}\right) \\ &= \frac{1}{\mu_0}F^{0\mu}F^j{}_\mu = \frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right)_j . \end{aligned}$$

In other words, the T^{0j} components have the expected meaning of energy current density.

We also know that a fundamental property of a stress-energy tensor is to be symmetric, namely Eq. (C.4.3). This is a trivial consequence of the definition, moreover one could also see by direct calculation that T^{j0} is again equal to the Poynting vector, which can equivalently be interpreted as the momentum density.

The second fundamental property of $T^{\alpha\beta}$ is that of having zero divergence, namely Eq. (C.4.4). This can be straightforwardly obtained in the source-free case using the Maxwell equations (C.3.9) and (C.3.10). When the sources are not zero the divergence of $T^{\alpha\beta}$ is not zero, but rather it is equal to the Lorentz force, as should be expected. In other words, as for the continuity equation, the divergence of the electromagnetic stress-energy tensor satisfies the laws of conservation of electromagnetic energy and momentum density.

This reasoning can be extended to any field, obviously. Intuitively, this can be easily understood by realizing that in practice the field is nothing else than a "local replacement" for particles. The gravity force between two particles with mass m and M is proportional to their product mM, which means that m feels a force $\mathbf{F}$ caused by M and M feels a force $-\mathbf{F}$ from m. We can attribute energy and momentum to any particle, but when fields come into play the general picture is that M produces a field Φ and m feels a force $\mathbf{F}$ proportional to $\nabla\Phi$. As we have seen, this approach allows a local description of the interaction, i.e., a way to avoid the action-at-distance problem, but if Φ has to replace M this has to be at all events, so it must be possible to attribute the particle's energy and momentum to the field. This is a completely general reasoning, that can be applied to any kind of interacting field.

C.4.3 The Stress-Energy Tensor in the Variational Approach

A further interesting observation is that the stress-energy tensor (and its three-dimensional predecessor, the stress tensor) was initially introduced in direct connection with the motion of particles, which later evolved into a more abstract definition. Indeed, the former approach demonstrated an ideal "recipe" to create a stress-energy tensor. Actually, if a rank 2 tensor can be built out of anything having an energy-momentum four-vector (i.e., a four-momentum) by taking this four-vector and "appending its current aside", and if such tensor is also symmetric and has a vanishing divergence, then it can be rightly called the stress-energy (or the energy-momentum) tensor for "that anything".

In other words, it does not matter that we are dealing with matter or fields; as long as the object has energy and momentum, it might be eligible to produce a correct energy-momentum tensor. Working out the problem of the stress-energy for the dust, for the perfect fluids, or for the electromagnetic field, made us understand that what really matters is not the specific formula that the energy and the momentum are expressed with, but just the "recipe" in which these quantities are transformed into a tensor and the two properties of the resulting object. This naturally leads to another, even more advanced generalization.

Energy and momentum can also be given in terms of a Lagrangian. For example, it is well known that in classical physics, for a conservative force, the definition of a Lagrangian for a system of particles implies that

$$E = \mathbf{p}\cdot\dot{\mathbf{q}} - L$$
$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}},$$

thus, if it does not matter how these two quantities are given, one can apply the usual method and build a stress-energy tensor for a system of particles out of its Lagrangian. There is a complication in the fact that what we really need are densities of energy and momentum, but this does not invalidate the basic idea.

Moreover, the same reasoning applies in the case of fields, because the variational approach in this case can be seen as a generalization to an infinite number of degrees of freedom of that for the particles. This transition can be realized in practice by appropriate replacements of the original quantities with equivalent counterparts defined in terms of fields. For example, in the case of a scalar field ϕ, $\dot{\mathbf{q}}$ is replaced by $\partial_\alpha \phi$ and $\mathbf{p}$ by $\pi^\alpha = \partial \mathcal{L} / \partial \left(\partial_\alpha \phi \right)$. This means that the energy of ϕ can be written as

$$\mathcal{E} = \pi^0 \left(\partial_0 \phi \right) - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi \right)} \partial_0 \phi - \delta^0_0 \mathcal{L}.$$

Actually it is worth mentioning that for the fields this procedure is even easier, inasmuch as here we are already dealing with densities.

Obviously enough, when this procedure is applied to a Lagrangian (density) for matter the result is called a *matter* stress-energy tensor, whereas in the case of fields we denote it a *field* stress-energy tensor.

The success obtained with the electromagnetic field can justify our perseverance along the heuristic way, which would lead us to interpret the above formula as the $T^0{}_0$ component of a stress-energy tensor expressed in Lagrangian terms. In the same way, therefore, one could be tempted to assume that[26]

$$\begin{aligned} T^\alpha{}_\beta &= - \left(\pi^\alpha \partial_\beta \phi - \delta^\alpha_\beta \mathcal{L} \right) \\ &= - \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha \phi \right)} \partial_\beta \phi - \delta^\alpha_\beta \mathcal{L} \right) \end{aligned} \tag{C.4.9}$$

This definition immediately implies the vanishing divergence condition; in fact considering that $\mathcal{L} = \mathcal{L} \left(\phi \left(x^\alpha \right), \partial_\alpha \phi \left(x^\alpha \right) \right)$, it results in

$$\begin{aligned} \partial_\alpha T^\alpha{}_\beta &= - \left[\partial_\alpha \pi^\alpha \partial_\beta \phi + \pi^\alpha \partial_\alpha \partial_\beta \phi - \frac{\partial \mathcal{L}}{\partial \phi} \partial_\beta \phi - \frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha \phi \right)} \partial_\beta \partial_\alpha \phi \right] \\ &= - \left[\partial_\alpha \pi^\alpha \partial_\beta \phi + \pi^\alpha \partial_\alpha \partial_\beta \phi - \frac{\partial \mathcal{L}}{\partial \phi} \partial_\beta \phi - \pi^\alpha \partial_\alpha \partial_\beta \phi \right] \\ &= - \left(\partial_\alpha \frac{\partial \mathcal{L}}{\partial \left(\partial_\alpha \phi \right)} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \partial_\beta \phi = 0 \end{aligned}$$

because of Eq. (6.3.3).

The second condition however, i.e., the symmetry of $T^{\alpha\beta}$, is not met. As shown, e.g., in Landau and Lifshitz (1975), the problem can be fixed by redefining the stress-

[26]More precisely, this is the so-called canonical stress-energy pseudo-tensor. The negative sign is taken to have a positive energy and to obtain the functional derivative of the action in the following formula.

energy tensor as

$$\hat{T}^{\alpha\beta} = T^{\alpha\beta} + \partial_\gamma S^{\alpha\beta\gamma},$$

where $S^{\alpha\beta\gamma}$ is an antisymmetric tensor in β and γ called the Belinfante tensor. In this case in fact

$$\begin{aligned}\hat{T}^{\alpha\beta} &= T^{\alpha\beta} + \partial_\gamma S^{\alpha\beta\gamma} = T^{\alpha\beta} + \partial_\gamma S^{\alpha\beta\gamma} \\ &= \hat{T}^{\beta\alpha},\end{aligned}$$

moreover

$$\begin{aligned}\partial_\alpha \hat{T}^{\alpha\beta} &= \partial_\alpha T^{\alpha\beta} + \partial_\alpha \partial_\gamma S^{\alpha\beta\gamma} \\ &= 0.\end{aligned}$$

This redefinition does not change anything in terms of its expression as a function of the Lagrangian density; in fact it can be shown that the Belinfante tensor contributes only with surface terms to the variation of the action, which therefore will vanish as required by the variational principle.

Finally, because the action is expressed with respect to the Lagrangians, it can be expected that this definition of the stress-energy tensor can be equivalently expressed in terms of the action. This last issue, however, is left to Chap. 8, inasmuch as it applies more naturally to the context of general relativity.

Appendix D
Elements of Differential Geometry

Differential geometry refers to the study of the geometrical properties of differentiable manifolds, a concept that we introduced in Appendix B.1 in the context of the Euclidean geometry. The latter can be considered a particular case of this general approach, in which the manifold is Euclidean: i.e., it is *flat* and is endowed with a *positive definite metric*. But in giving this characterization the careful reader is aware that, although the metric was already defined in a sufficiently rigorous way, nothing has been said about the idea of flat and/or curved spaces yet. Throughout this appendix we suspend the convention that Latin indexes go from 1 to 3 and Greek ones from 0 to 3 because the results discussed hereafter are valid in any number of dimensions of any type. Therefore, only Latin indexes are used unless needed to avoid misunderstandings, and they can refer to any type of coordinate for a space of any dimension. Similarly, we are indicating tensorial quantities in components-free notation just in bold typeface, thus dropping the convention that used upright bold for Euclidean quantities and italic bold for spacetime.

D.1 Flat Versus Curved Spaces

We have indeed an intuitive mental depiction of them in terms of surfaces embedded in a three-dimensional space, but nothing more than this. Moreover, this intuitive understanding is limited by the fact that it relies on the existence of a space with more dimensions in which the curved object is "immersed". In this way the properties of the latter then can be determined from theorems of the former space, which is Euclidean. For this reason this is denoted the *extrinsic* approach to curved geometry, and it is clear that it cannot go beyond the studies of curves and surfaces. There is another approach, however, that stands on the *intrinsic* properties of the space, or those that can be defined without any reference to an embedding space. The advantage of this formulation is the possibility of extending it to spaces of any dimensions. Moreover, its independence from a larger Euclidean space means that it can be regarded as a

A. Vecchiato, *Variational Approach to Gravity Field Theories*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-51211-2

geometry on its own, with the same self-consistency of the former but with different characteristics.

The idea of building a geometry that is completely consistent, but at the same time different from the Euclidean one, is more ancient than one can usually believe. The first known treatise of this kind is the *Sphaerica* of Menelaus of Alexandria (who lived approximately between the first and second century AD), in which methods of intrinsic geometry akin to those of Euclid's *Elements* for the plane are used to develop a spherical trigonometry with applications to astronomy.

A more general approach was developed by Gauss in his *Disquisitiones generales circa superficies curvas* (General investigations on curved surfaces) of 1828, where these methods introduced of concepts such as the curvature of a surface, and which can be considered the first systematic anticipation of differential geometry. It is generally believed that Gauss had already grasped the idea of non-Euclidean geometry, but preferred not to publish anything in order to avoid embarking himself upon long debates with the scientific community.

Contrary to the German scientist, about in the same years two other mathematicians, Nikolaj Lobačevskij and János Bolyai, published works in which a non-Euclidean geometry with constant negative curvature was presented. Nevertheless, a strong discussion about their inconsistency continued until, as already mentioned in Chap. 2, Eugenio Beltrami proved that any inconsistency in any non-Euclidean geometry with constant curvature would have implied a parallel inconsistency in the Euclidean one, thus showing that Euclidean and non-Euclidean geometries were equivalent from the point of view of their self consistency.

In the meantime, Bernhard Riemann had developed a complete geometric formalism for differential geometry, extending previous works to the more general case of manifolds with non-constant curvature.

D.2 Introduction to Curvature

As is well known, the concept of curved in contrast to flat space bears its origin in the fifth postulate of Euclidean geometry. It can also be shown that the parallels' postulate is equivalent to the Pythagorean theorem, in the sense that one could take the latter as axiom and then deduce the former. Similarly, the other well-known theorem of Euclidean geometry that the sum of the internal angles of a triangle is two right angles is also equivalent to Euclid's fifth postulate.

We denote Euclid's geometry and the corresponding model of space as *flat* in contrast to the geometry of curved surfaces, intended in the exterior sense mentioned above. For example, the sum of the internal angles of a spherical triangle (see Fig. D.1) is greater than two right angles and at the same time all the great circles[27] containing one of the vertices of the triangle intersect the great circle connecting the other two

[27] Great circles on a sphere are the equivalent of the straight lines on a plane in the sense that both are the shortest lines connecting any pair of points on the respective surfaces.

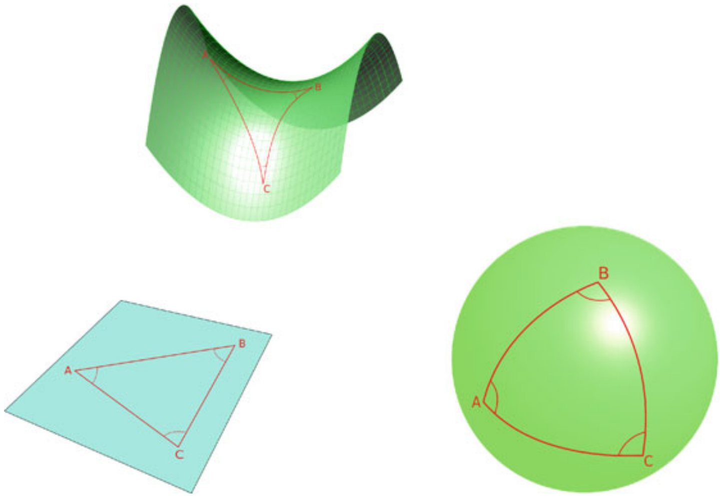

Fig. D.1 Hyperbolic, plane, and spherical triangles

points. On a hyperbolic paraboloid, instead, the sum of the angles of a triangle is less than two right angles and there exist an infinite number of "straight" lines through one of the vertices that never intersect the one connecting the two opposite points. Similarly, different versions of the Pythagorean theorem hold on these two surfaces.[28]

If we admit the possibility of using these properties from the point of view of the intrinsic geometry it is clear that this reasoning can be extended to our three-dimensional space and to spaces with any number of dimensions. Thus, if it would be possible to notice a departure from the Pythagorean theorem, or an excess or defect with respect to π in the sum of the angles of a triangle, one could rightly admit that the correct geometry is different from the Euclidean one.

Moreover, in the above examples, the amount of the departure from the law of the sum of the angles is the same for equal triangles traced on any part of the surface. This is one of the practical ways in which the concept of constant curvature mentioned in the previous section can be meant, which implies that, conversely, a non constant deviation can be interpreted as an effect of a geometry with a variable curvature.

In Appendix B Euclidean geometry was introduced in the relatively unfamiliar framework of the differential manifolds. After the definition of vectors, one-forms, and tensors of higher rank, the topological manifold was enriched with the introduction of a special rank two tensor, the *metric tensor*, which defines the dis-

[28]On a sphere the equivalent is $\cos C = \cos A \ \cos B$, which reduces to its plane version for small triangles. In this case, in fact, $\cos x \simeq 1 + x^2/2$ and the above formula gives

$$1 + \frac{1}{2}c^2 = \left(1 + \frac{1}{2}a^2\right)\left(1 + \frac{1}{2}b^2\right) = 1 + \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{4}a^2b^2$$

which reduces to the Pythagorean theorem at $\mathcal{O}\left(a^2\right) \sim \mathcal{O}\left(b^2\right) \sim \mathcal{O}\left(c^2\right)$.

tance between any two points (or the norm of a vector, which is the same). In this way the topological manifold becomes a *metric space*.

By means of this object the infinitesimal distance between two points with coordinate separations $\mathrm{d}x^i$ was written as

$$\mathrm{d}s = \left(g_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)^{1/2},$$

and finally Euclidean geometry was recovered when, in the case of Cartesian coordinates, $g_{ij} = \delta_{ij}$, because in this way the distance becomes the familiar Pythagorean theorem in two or three dimensions.

This can, with reason, appear as an unnecessarily complicated way to derive the Euclidean geometry, however:

1. This technique allows us effortlessly to use Euclidean geometry in different coordinate systems.
2. Because we identified in the departure from this theorem as one of the distinctive signs of a curved geometry, the metric tensor might be the right tool to build up a rigorous and general mathematical description of such geometries.

Actually, it has to be stressed that these two points are intimately different from each other, even if both can lead to a $g_{ij} \neq \delta_{ij}$ metric tensor. The first one gives a different metric tensor because of a change of coordinate system, but it does not alter the intrinsic geometrical properties of the space. For example, in spherical coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2,$$

so that

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix},$$

but the space is still Euclidean. The derivation is straightforward from the coordinate transformation and the properties of tensors; in fact,

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta, \end{aligned}$$

but from Eq. (B.4.2), putting $x^i = x, y, z$ for $i = 1, 2, 3$, respectively, it has to be, e.g.,

$$\begin{aligned} g_{rr} &= \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r}\delta_{ij} = \frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial r}\delta_{ii} = 1 \\ g_{r\theta} &= \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial \theta}\delta_{ij} = \frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial \theta}\delta_{ii} = 0, \end{aligned}$$

and the other components follow immediately by similar calculations.

The second point, instead, requires that the metric tensor can express a more complex meaning. It is not clear yet what precisely this meaning is we are seeking, however, it is evident that, if one could obtain any metric tensor simply by a coordinate transformation from δ_{ij}, then any infinitesimal distance could lead back to the Pythagorean theorem with its inverse. In this case, as for the spherical coordinates above, the meaning of g_{ij} would be simply that of expressing a flat geometry in a different coordinate system, and no "space" for curved geometries would exist in the metric tensor.

Fortunately, it easy to understand that this is not the case. Indeed, any coordinate transformation in a three-dimensional space is defined by three independent functions linking the two sets of points of $\mathbb{R}^3$, but the metric tensor has six independent components, i.e., six independent functions of the coordinates, therefore in general it is not possible to transform back any metric tensor in δ_{ij}, and the additional components of g_{ij} contain the information that allow us to define mathematically a curved space. It is interesting to not that this reasoning can be extended to a metric space of a general number of dimensions N. In such space the number of independent functions of a coordinate change is N, and the number of independent components of g_{ij} (we continue to use Latin indexes but in this case $1 \leq i, j \leq N$) is $N\,(N+1)\,/2$, which shows that the case of $N = 1$ is the only one in which it can be guaranteed that any pair of metric tensors can be linked by an appropriate change of coordinates. This is in agreement with our intuitive understanding that we can continuously transform, or superimpose, any line to any other line, but this is no longer true for two-dimensional surfaces as cartographers know very well!

Even if we cannot always find a coordinate transformation that can give the metric in its flat Cartesian form, it is worth understanding to which extent this goal can be pursued. This provides the first indication of the mathematical meaning of curvature, which indeed can be intended in a "negative" sense as the deviation from flatness.

Our intuition comes to our aid regarding how this can be obtained by realizing that a smooth manifold at any point P can be approximated by its tangent plane, and the deviations from this approximation can be represented by the coefficients of a Taylor expansion around such a point. The expansion can be considered from both sides of the metric tensor and of the coordinate transformation, in the sense that we can write

$$\begin{aligned} x^i\left(\bar{x}^j_{\mathsf{P}}\right) &= \bar{x}^i_{\mathsf{P}} + \left.\frac{\partial x^i}{\partial \bar{x}^k}\right|_{\mathsf{P}} \left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right) + \frac{1}{2}\left.\frac{\partial^2 x^i}{\partial \bar{x}^k \partial \bar{x}^l}\right|_{\mathsf{P}} \left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right)\left(\bar{x}^l - \bar{x}^l_{\mathsf{P}}\right) \\ &\quad + \frac{1}{6}\left.\frac{\partial^3 x^i}{\partial \bar{x}^k \partial \bar{x}^l \partial \bar{x}^m}\right|_{\mathsf{P}} \left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right)\left(\bar{x}^l - \bar{x}^l_{\mathsf{P}}\right)\left(\bar{x}^m - \bar{x}^m_{\mathsf{P}}\right) + \mathcal{O}\left(\left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right)^4\right), \end{aligned}$$

and similarly

$$\bar{g}_{ij}\left(\bar{x}^k\right) = \bar{g}_{ij}\left(\bar{x}^k_{\mathsf{P}}\right) + \left.\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right|_{\mathsf{P}} \left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right) + \frac{1}{2}\left.\frac{\partial^2 \bar{g}_{ij}}{\partial \bar{x}^k \partial \bar{x}^l}\right|_{\mathsf{P}} \left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right)\left(\bar{x}^l - \bar{x}^l_{\mathsf{P}}\right) + \mathcal{O}\left(\left(\bar{x}^k - \bar{x}^k_{\mathsf{P}}\right)^3\right),$$

where, in this case, we want that

$$\begin{aligned} \bar{g}_{ij}\left(\bar{x}^k_{\mathsf{P}}\right) &= \delta_{ij} \\ \left.\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right|_{\mathsf{P}} &= 0 \\ \left.\frac{\partial^2 \bar{g}_{ij}}{\partial \bar{x}^k \partial \bar{x}^l}\right|_{\mathsf{P}} &= 0 \\ &\dots \end{aligned} \tag{D.2.1}$$

for as many orders of the series as possible. We can arrange the degrees of freedom made available by the first series to accommodate the conditions of Eqs. (D.2.1). In view of a more general understanding of this issue, we can consider the case of an N-dimensional space.

The first condition requires us to supply $N(N+1)/2$ independent quantities to set the zero-order symmetric metric tensor to the δ_{ij} values. We can use the first-order coefficient of the coordinate transformation series $\partial x^i/\partial \bar{x}^k$, whose N^2 independent values are more than sufficient because $N^2 - N(N+1)/2 = N(N-1)/2$. Therefore, we can always have $\bar{g}_{ij}\left(\bar{x}^k_{\mathsf{P}}\right) = \delta_{ij}$. It is also worth stressing that, e.g., for $N = 2$ it is $N(N-1)/2 = 1$, whereas for $N = 3$ it is $N(N-1)/2 = 3$. In both cases these numbers coincide with the number of possible independent rotations in these spaces, which are the transformations that can leave δ_{ij} invariant.

The second condition requires cancelling $N^2(N+1)/2$ quantities to make all the first derivatives of the metric at P vanish, and like the previous one we have to use the second-order coefficients $\partial^2 x^i/\partial \bar{x}^k \partial \bar{x}^l$. These have three indexes and they are symmetric in two of them, precisely as $\partial \bar{g}_{ij}/\partial \bar{x}^k$, therefore the two have the same number of independent quantities and it is always possible to have the transformed metric tensor identical to δ_{ij} at the first order.

The number of independent quantities needed for the second-order approximation, instead, requires $N^4(N+1)^2/4$ because the derivation is symmetric in the two lower indexes as is the metric tensor. In this case, however, we have only the coefficients of the third derivatives, which are symmetric for any pair of the lower indexes and therefore are $N^2(N+2)(N+1)/6$. This number is less than the required one, therefore the diagonal form is not possible, in general, at the second order. The number of nonzero second-order derivatives of the metric is $N^2\left(N^2-1\right)/12$, which are thus the quantities characterizing the curvature of the space. For $N = 2, 3$, and 4 these numbers are 1, 6, and 20, respectively. The last case characterizes the Riemann tensor of the spacetime.

In comparing the general metric tensor to a Euclidean one, we have implicitly assumed that this quantity is *positive definite*, i.e., that $g_{ij}v^i v^j \geq 0$, vanishing only if

$\mathbf{v} = 0$. The resulting geometry is called *Riemannian.* In special and general relativity, however, the norm of the line element can be also negative, and $g_{ij}v^i v^j = 0$ does not necessarily imply that the argument is zero. We have in this case a *pseudo-Riemanninan* geometry.

It is worth noting that the validity of the above expansion of the metric tensor at a given point P does not depend on the value of the line element. The only difference with respect to the previous results is that the diagonal elements of the zero order at P are not all positive, but rather they can be positive or negative as well. For example, in General Relativity the zero order will be $\eta_{\alpha\beta}$, whereas in a general N-dimensional space it will result in a series of ± 1 constituting the *signature* of the metric, which is the trace of the zero-order part. Actually there is a certain arbitrariness in assigning the signature. Continuing with the General Relativity example, this book adopts the so-called *spacelike convention*, for which the signature is $+2$, whereas other books use the *timelike convention*, with signature -2.[29] The resulting formulae can differ for some signs, but this does not affect in any sense the overall validity and significance of the theory. The signature is also written as (N_+, N_-), where $N_+ + N_- = N$ and N_+ and N_- are the number of positive and negative numbers, respectively. In a Riemannian geometry $N_- = 0$, and in a Pseudo-Riemannian one $N_- \neq 0$.

Tensorial quantities from Euclidean to curved spaces

From the above discussion it can now be understood how the approach described in Appendix B is general enough to provide a common framework applicable to different geometries, from flat Euclidean ones to curved pseudo-Riemannian. The "trick" lies in the transformation properties of tensorial quantities of Eq. (B.4.2) that in the end can be taken as their definition. These ultimately ensure the covariance of tensors of any rank with respect to coordinate and reference systems transformations. When these are taken to be orthogonal 3D transformations, namely rotations, one gets Euclidean geometry, whereas in the case of Lorentz transformations (i.e., hyperbolic rotations) we obtain the pseudo-Euclidean geometry of special relativity.

D.3 Calculus in Curved Spacetime

A complete mathematical toolbox for physics theories cannot be limited to tensorial algebra. Calculus is also an essential mathematical tool that, like algebra, has to be adapted to the objects of generically curved N-dimensional spaces.

The main problem is that derivatives of order higher than the first are generally not tensors, but let us recap the situation. Vectors are defined as the directional derivatives

[29] Spacelike and timelike conventions are denoted in this way because in the former the spatial coordinates are taken with the "normal" plus sign, while the latter reserves the familiar positive sign for the (coordinate) time intervals.

of a scalar field, which is a first-order derivative giving the tangent to a curve. On the other hand, gradients, whose components are first-order derivatives with respect to the coordinates of the space, are one-forms, therefore they obey the transformation law

$$\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial}{\partial x^k}.$$

But if, e.g., we take the derivative of the gradient $\partial/\partial x^i$, its transformation law is

$$\begin{aligned}\frac{\partial^2}{\partial \bar{x}^i \partial \bar{x}^j} &= \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial}{\partial x^k} + \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial^2}{\partial x^k \partial \bar{x}^j} \\ &= \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial}{\partial x^k} + \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial^2}{\partial x^k \partial x^l},\end{aligned}$$

where in the second step we used again the transformation law of $\partial/\partial \bar{x}^j$. This formula clearly shows that a second differentiation does not give the components of a (0, 2) tensor because of the first term.

The origin of this problem can be better understood if we go back to the definition of vectors as directional derivatives. In this case it is easy to understand that the basis vectors are in their turn the tangents to the curves of the coordinate lines, thus their orientation may not be constant in general, in the sense that they are functions of the coordinates as well. This happens, e.g., when one uses curved coordinate lines, like the spherical ones. Furthermore, it is unavoidable when the space is intrinsically curved. The immediate consequence is that just differentiating the components of a vector is not sufficient to obtain a tensorial quantity because the differentiation with respect to a given coordinate x^i, in order to give a vector, must be the tangent vector when it is transported from the original point x^i to $x^i + \mathrm{d}x^i$. This is clearly not possible when one takes the derivative of the components only, because the resulting quantities are projected on the basis at the point of differentiation or, in other words, because this operation does not take into account that the basis vectors are in general variable as well.

Nonetheless, derivation is a fundamental operation in physics, thus it is necessary to find a generally covariant way to define this operator.

D.4 Covariant Derivative and Metric Connection

In Euclidean geometry the derivative of a vector (or better, of a vector field) with respect to its Cartesian components is just a matter of deriving its components

$$\partial_j \mathbf{v} \equiv \frac{\partial \mathbf{v}}{\partial x^j} = \frac{\partial v^i}{\partial x^j} \mathbf{e}_i$$

because its basis vectors are constant over the entire space, and identical considerations can be done in Minkowskian spacetime. However, as pointed out in the previous section, when vectors are expressed in a different coordinate system one cannot in general assume that any basis vector constitutes a constant field, therefore the derivative should read

$$\nabla_j \mathbf{v} \equiv \frac{\partial v^i}{\partial x^j}\mathbf{e}_i + v^i \frac{\partial \mathbf{e}_i}{\partial x^j}. \tag{D.4.1}$$

This expression can be made more useful if we decompose $\partial \mathbf{e}_i / \partial x^j$ in terms of basis vectors by defining

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^k{}_{ij}\mathbf{e}_k, \tag{D.4.2}$$

so that by substituting Eq. (D.4.2) into Eq. (D.4.1) and renaming the dummy indexes it results in

$$\nabla_j \mathbf{v} = \left(\frac{\partial v^i}{\partial x^j} + v^k \Gamma^i{}_{kj}\right)\mathbf{e}_i \tag{D.4.3}$$

where $\Gamma^i{}_{kj}$ can be interpreted as the ith component of the derivative of the kth basis vector with respect to x^j. The last formula is called the *covariant derivative* of $\mathbf{v}$ with respect to x^j.

Connection coefficients

Equation (D.4.2) defines the components of the so-called *Christoffel symbol of the second kind*, which in modern textbooks are more frequently denoted *connection coefficients* for reasons clarified below. Despite that they appear as indexed quantities these coefficients are not the components of a tensor, which should be expected inasmuch as they are combined with another non tensorial quantity, namely $\partial v^i / \partial x^j$. However it is convenient to not that $\partial \mathbf{e}_i / \partial x^j$, intended as a derivative of a basis vector with respect to a fixed parameter x^j, is indeed a vector. This has to be meant in the sense that the components of connection coefficients running on the upper index behave as do those of a contravariant vector.

We can thus exploit this property to deduce the transformation properties of $\Gamma^k{}_{ij}$, because this observation implies that it makes perfect sense to multiply both sides of Eq. (D.4.2) by $\mathbf{e}^k$, thus obtaining

$$\left\langle \mathbf{e}^k, \frac{\partial \mathbf{e}_i}{\partial x^j}\right\rangle = \Gamma^k{}_{ij}\left\langle \mathbf{e}^k, \mathbf{e}_k\right\rangle = \Gamma^k{}_{ij}. \tag{D.4.4}$$

The same relation will hold in another coordinate system, which means

$$\bar{\Gamma}^k_{ij} = \left\langle \bar{\mathbf{e}}^k, \frac{\partial \bar{\mathbf{e}}_i}{\partial \bar{x}^j} \right\rangle.$$

Using the transformation properties of Eqs. (B.2.7) and (B.3.8) the right-hand side becomes

$$\begin{aligned}
\left\langle \frac{\partial \bar{x}^k}{\partial x^c}\mathbf{e}^c, \frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^a}{\partial \bar{x}^i}\mathbf{e}_a\right)\right\rangle &= \left\langle \frac{\partial \bar{x}^k}{\partial x^c}\mathbf{e}^c, \left(\frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j}\mathbf{e}_a + \frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial \mathbf{e}_a}{\partial \bar{x}^j}\right)\right\rangle \\
&= \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j}\left\langle \mathbf{e}^c, \mathbf{e}_a\right\rangle + \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial x^a}{\partial \bar{x}^i}\left\langle \mathbf{e}^c, \frac{\partial \mathbf{e}_a}{\partial \bar{x}^j}\right\rangle \\
&= \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j}\delta^c_a + \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial x^a}{\partial \bar{x}^i}\left\langle \mathbf{e}^c, \frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial \mathbf{e}_a}{\partial x^b}\right\rangle \\
&= \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial^2 x^c}{\partial \bar{x}^i \partial \bar{x}^j} + \frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial x^a}{\partial \bar{x}^i}\frac{\partial x^b}{\partial \bar{x}^j}\Gamma^c{}_{ab},
\end{aligned} \tag{D.4.5}$$

and from these relations it can be shown that the components of the covariant derivative

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + v^k \Gamma^i{}_{kj} \tag{D.4.6}$$

are those of a (1, 1) tensor. More formally, one can say that the covariant derivative of a vector is the outer product of the differential operator $\boldsymbol{\nabla} \equiv \nabla_j \mathbf{e}^j$ and the vector $\mathbf{v} = v^i \mathbf{e}_i$, i.e., $\boldsymbol{\nabla}\mathbf{v} = \nabla_j \mathbf{e}^j \otimes v^i \mathbf{e}_i = \nabla_j v^i \mathbf{e}^j \otimes \mathbf{e}_i$.

Covariant derivatives for generic tensors

Up to now the covariant derivative has been defined only on vectors, so we still need to explore its effect on tensors of different rank. The easiest case is that of a scalar field, i.e., a rank 0 tensor. Because this quantity does not depend on basis vectors, it is natural to assume that $\Gamma^k{}_{ij} = 0$ and

$$\nabla_i \phi = \partial_i \phi$$

for any scalar field ϕ. This also allows us to derive the behavio of the covariant derivative operator on a one-form; in fact, for any vector field v^i and one-form θ_i it must be

$$\nabla_j \left(\theta_i v^i\right) = \partial_j \left(\theta_i v^i\right).$$

Now we only have to require the reasonable condition that ∇_i obeys the Leibniz rule for the products,[30] which gives

[30]This assumption is perfectly reasonable because the covariant derivative is intended as the extension to curved spaces of the partial derivatives. On the other side, it is indeed an assumption because its effect on (0, 1) tensors is not known.

$$\left(\nabla_j \theta_i\right) v^i + \theta_i \left(\partial_j v^i + v^k \Gamma^i{}_{kj}\right) = \left(\partial_j \theta_i\right) v^i + \theta_i \left(\partial_j v^i\right)$$

and therefore

$$\left(\nabla_j \theta_i\right) v^i = \left(\partial_j \theta_i\right) v^i - \theta_i v^k \Gamma^i{}_{kj}.$$

Finally, by relabeling the dummy indexes in the last equation so that v^i can be factored out,[31] we obtain

$$\nabla_j \theta_i = \partial_j \theta_i - \theta_k \Gamma^k{}_{ij}. \tag{D.4.7}$$

The same technique of building a scalar from a tensor of any rank and an appropriate number of arbitrary vectors and one-forms can be used to find how the covariant derivative works on higher-rank tensors. The calculations are pretty long but straightforward, thus we can just give the final results for a (m, n) tensor $t^{i_1,\dots,i_m}{}_{j_1,\dots,j_n}$

$$\begin{aligned}\nabla_k t^{i_1,\dots,i_m}{}_{j_1,\dots,j_n} = {} & \partial_k t^{i_1,\dots,i_m}{}_{j_1,\dots,j_n} + \Gamma^{i_1}{}_{lk} t^{l,i_2,\dots,i_m}{}_{j_1,\dots,j_n} + \cdots + \Gamma^{i_m}{}_{lk} t^{i_1,\dots,i_{m-1},l}{}_{j_1,\dots,j_n} \\ & - \Gamma^l{}_{j_1 k} t^{i_1,\dots,i_m}{}_{l,j_2,\dots,j_n} - \cdots - \Gamma^l{}_{j_n k} t^{i_1,\dots,i_{m-1},l}{}_{j_1,\dots,j_{n-1},l}.\end{aligned} \tag{D.4.8}$$

In practice, the general rule is that a connection coefficient with a plus sign has to be introduced for any contravariant index and one with a minus sign for any covariant index, which can be summarized with the mnemonic rule "up means plus, down means minus."

Connection coefficients, curved manifolds, and metric

Loosely speaking, the connection coefficients can be interpreted as a way to compare vectors on different points of the manifold, providing the needed information about how the basis vectors change along the path connecting such points. We return to this idea in more detail in the next section, but first it has to be stressed that, although we started from the case of curvilinear coordinates, all the above considerations do not depend on this assumption. Actually, Eq. (D.4.1) simply assumes that the basis vectors are functions of the coordinates of the manifold, an hypothesis which can be seamlessly transferred to genuinely curved spaces. Previously it has been shown that the intrinsic curvature of a manifold is directly related to the metric tensor, therefore it is natural to wonder if there is a relation between the latter and the connection coefficients.

Before proceeding on this way it should be observed that in defining the connection coefficients we did not pay attention to the order of the lower indexes. Actually, nothing could prevent us from putting $\partial_j \mathbf{e}_i = \Gamma^k{}_{ji} \mathbf{e}_k$, and moreover Eq. (D.4.4) shows that it can surely happen that $\Gamma^k{}_{ij} \neq \Gamma^k{}_{ji}$. In order to distinguish the two cases in which the indexes are or are not symmetric the fact that in general the difference of two connection coefficients is a rank 3 tensor is usually exploited. Thus

[31] In the sense that the relation $\left(\nabla_j \theta_i\right) v^i = \left(\partial_j \theta_i - \theta_k \Gamma^k{}_{ij}\right) v^i$ is valid for any v^i.

the *torsion tensor* defined as

$$T^k{}_{ij} \equiv \Gamma^k{}_{ij} - \Gamma^k{}_{ji}$$

is zero only if the connection coefficients are symmetric in the two lower indexes. This is assumed as true in general relativity, therefore in the following it is taken as $T^k{}_{ij} = 0$.

Under this hypothesis the relation with the metric tensor can be found by taking the dot product of Eq. (D.4.2) and remembering that $\mathbf{g}\left(\mathbf{e}_i, \mathbf{e}_j\right) = g_{ij} = \left(\mathbf{e}_i \cdot \mathbf{e}_j\right)$. Therefore

$$\frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}_l = \Gamma^k{}_{ij} \mathbf{e}_k \cdot \mathbf{e}_l = \Gamma^k{}_{ij} g_{kl},$$

but $\left(\partial_j \mathbf{e}_i\right) \cdot \mathbf{e}_l = \partial_j \left(\mathbf{e}_i \cdot \mathbf{e}_l\right) - \left(\partial_j \mathbf{e}_l\right) \cdot \mathbf{e}_i$, thus the previous equation can be rearranged to obtain

$$\partial_j g_{il} = \Gamma^k{}_{ij} g_{kl} + \left(\partial_j \mathbf{e}_l\right) \cdot \mathbf{e}_i = \Gamma^k{}_{ij} g_{kl} + \Gamma^k{}_{lj} g_{ki}. \tag{D.4.9}$$

The same calculations can be made on different combinations of the indexes i, j, and l, thus from $\left(\partial_l \mathbf{e}_j\right) \cdot \mathbf{e}_i$ one gets

$$\partial_l g_{ji} = \Gamma^k{}_{jl} g_{ki} + \Gamma^k{}_{il} g_{kj}, \tag{D.4.10}$$

and from $(\partial_i \mathbf{e}_l) \cdot \mathbf{e}_j$

$$\partial_i g_{lj} = \Gamma^k{}_{li} g_{kj} + \Gamma^k{}_{ji} g_{kl}. \tag{D.4.11}$$

From these equations, and using the symmetry of the connection coefficients, it is

$$\begin{aligned} \partial_j g_{il} + \partial_l g_{ji} - \partial_i g_{lj} &= \Gamma^k{}_{ij} g_{kl} + \Gamma^k{}_{lj} g_{ki} + \Gamma^k{}_{jl} g_{ki} + \Gamma^k{}_{il} g_{kj} - \Gamma^k{}_{li} g_{kj} - \Gamma^k{}_{ji} g_{kl} \\ &= 2\Gamma^k{}_{jl} g_{ki}, \end{aligned}$$

and finally, from $g_{ki} g^{mi} = \delta^m_k$, and relabeling the indexes in order to match with Eq. (D.4.4)

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ji}\right). \tag{D.4.12}$$

This formula is called the *metric connection*, and it is valid only for torsion-free manifolds.

Finally, an important result can be shown by applying Eq. (D.4.8) to the metric tensor, which gives

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^l{}_{ik} g_{lj} - \Gamma^l{}_{kj} g_{il},$$

but substituting in this $\partial_k g_{ij}$ from Eq. (D.4.10) we have

$$\nabla_k g_{ij} = 0, \tag{D.4.13}$$

where in obtaining this result we did not use any hypothesis on the symmetry of the connection coefficients, which means that this result holds in any manifold, regardless of its torsion properties. This result also means that the metric can be used to raise and lower indexes "inside" the covariant derivative, in the sense that $g_{ki}\nabla_j v^k = \nabla_j v_i$. This is because the covariant derivative retains the properties of the standard derivative with respect to the product, therefore

$$\nabla_j v_i = \nabla_j \left(g_{ki} v^k\right) = \nabla_j \left(g_{ki}\right) v^k + g_{ki}\nabla_j v^k = g_{ki}\nabla_j v^k.$$

D.5 Parallel Transport and Geodesic Equations

In Sect. D.2 the first cited criterion to characterize flat and curved spaces was that of the parallels' postulate: a space is flat if such apostulate holds, and is curved otherwise. However, the concept of parallel lines is intuitive and even well defined in Euclidean geometry, but here it needs to be extended, as we show.

Parallel transport in Cartesian coordinates

In Euclidean geometry a straight line is parallel to another one if their mutual distance is constant. It is possible to translate this definition in the language of analytical geometry by saying that, in Cartesian coordinates, two straight lines are parallel if at any point their directional derivatives, namely the tangent vectors, have the same constant components. It is worth stressing that in Appendix B.1 vectors were defined as directional derivatives of *scalar fields* along a curve, whereas here we need to look at *variations* of vectors, a long the whole line, which implies a directional derivative of the vector along the curve.

Mathematically, if there exist a vector field $\mathbf{v}\left(x^j\right) = v^i\left(x^j\right)\mathbf{e}_i$ and a curve $\mathcal{C} = \left\{\mathsf{P} = x^i\left(\lambda\right) \mid \lambda \in \mathbb{R}\right\}$, then the derivative of $\mathbf{v}$ along the curve is just

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\lambda} = \frac{\mathrm{d}v^i}{\mathrm{d}\lambda}\mathbf{e}_i = \frac{\partial v^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\mathbf{e}_i. \tag{D.5.1}$$

Using Cartesian coordinates in this case is fundamental because their basis vectors are constant vector fields over the manifold, thus in the differentiation only the components can vary. A straight line, therefore, can be defined by requiring that

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\lambda} = 0,$$

for any vector field $\mathbf{v}$; i.e.,

$$\frac{\mathrm{d}v^i}{\mathrm{d}\lambda} = 0 \quad \forall\, i. \tag{D.5.2}$$

An often used intuitive depiction of this object is that along a straight line the directional derivative "transports" vectors "parallel to themselves", which explains the name of "parallel transport" given to this condition.

Absolute derivative and parallel transport in curved spaces

In curvilinear coordinates one should generalize Eq. (D.5.1) by considering that $\mathbf{e}_i = \mathbf{e}_i\left(x^j\right)$ as well. We have already played this game in the previous section, with the definition of covariant derivative, but before going on with our reasoning it is worth stressing that in a flat space this operation could be just a matter of convenience. In principle, in order to find the directional derivative of a vector in this case, one could transform the curve in Cartesian coordinates, do the ordinary directional derivative, and then transform back to the original coordinate system. In a genuinely curved space, instead, this is strictly impossible because we have seen that here nothing like global Cartesian coordinates exists. Therefore in curved spaces this generalization of the directional derivative is rather a necessity.

This operation is called the *absolute derivative*, and quite naturally it writes

$$\begin{aligned}\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}\lambda} &\equiv \frac{\mathrm{d}v^i}{\mathrm{d}\lambda}\mathbf{e}_i + v^i\frac{\mathrm{d}\mathbf{e}_i}{\mathrm{d}\lambda}\\ &= \frac{\partial v^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\mathbf{e}_i + v^i\frac{\partial \mathbf{e}_i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\\ &= \left(\frac{\partial v^i}{\partial x^j} + v^k\Gamma^i{}_{kj}\right)\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\mathbf{e}_i\\ &= \left(\nabla_j v^i\right)\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\mathbf{e}_i,\end{aligned} \tag{D.5.3}$$

so that the absolute derivative can be considered the generalization of the directional derivative in the same way the covariant derivative is the generalization of the partial derivative. By analogy with the previous case, then, it is also natural to conceive the generalization of a straight line and parallel transport in curved spaces by imposing the condition

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}\lambda} = 0,$$

for any vector field $\mathbf{v}$, or equivalently

$$\frac{\mathrm{D}v^i}{\mathrm{D}\lambda} \equiv \left(\nabla_j v^i\right)\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = 0 \quad \forall\, i. \tag{D.5.4}$$

Moreover, the derivation done in Sect. 8.2 from a variational principle showed that this definition is consistent with the other fundamental characteristic of straight lines of being paths of extremal length connecting two points.

Geodesic equation

Parallel transport, in fact, can be defined for any vector $\mathbf{v}$, so it can be applied also to the particular case of the vector tangent to the curve itself. In this case, from Eq. (B.1.3), it is $v^i = \mathrm{d}x^i/\mathrm{d}\lambda$, so the parallel transport condition of Eq. (D.5.4) becomes

$$\left(\nabla_j \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right)\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{kj}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = 0, \tag{D.5.5}$$

which is called the *geodesic equation*. Strictly speaking, this form of the geodesic equation implies that λ is an affine parameter for the geodesic, as explained in Sect. 8.2.

It is worth noting that in flat spaces and Cartesian coordinates this condition reads[32]

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = 0,$$

which is exactly the usual property of straight lines in ordinary analytical geometry.

Obviously, even if the absolute derivative has been introduced as an operation on vectors, like the covariant derivative it can be applied to any tensor, for which

$$\frac{\mathrm{D}\mathbf{t}^{(m)}_{(n)}}{\mathrm{D}\lambda} = \left(\nabla_k \mathbf{t}^{(m)}_{(n)}\right)\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\mathbf{e}^{j_1}\otimes\cdots\otimes\mathbf{e}^{j_n}\otimes\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_m}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}\lambda}\left(t^{i_1,\ldots,i_m}{}_{j_1,\ldots,j_n}\right) = \left(\nabla_k t^{i_1,\ldots,i_m}{}_{j_1,\ldots,j_n}\right)\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}.$$

D.6 Again on Curvature of Manifolds

Having acquired the parallel transport in our toolbox, we are now ready to complete the picture and finally give a rigorous mathematical criterion to determine the intrinsic geometric properties of a space. We can approach the problem (almost) from the point of view of the parallels' postulate.

Actually we agreed on the fact that if two parallel straight lines never intersect then Euclid's fifth axiom implies that their distance always remains unchanged. We are now in the position of translating this statement in the language of differential geometry, and to explore the behavior of two parallel "straight lines" in a generic geometry. We expect that the intrinsic flatness of a space would reveal itself when the

[32] This easily comes from the fact that $\Gamma^i{}_{kj} = 0$ in this case, or also simply from Eq. (D.5.2).

distance of these lines does not change, whereas the opposite has rather to characterize a curved one.

We know from the last section that the equivalent of a straight line in the generalized context of a differential manifold is that of a geodesic of Eq. (D.5.5), thus we should take two parallel geodesics passing through two nearby points A and B. However, finding the condition that makes two geodesics parallel to each other would require us to parallel transport the tangent vector on A to B; moreover the result depends on the actual path connecting the two points. For this reason, our task would be much simpler if, in investigating the evolution of the geodesics' separation, we would drop the requirement of parallelism. We know in fact that, as two parallel straight lines keep a constant distance in Euclidean geometry, for two generic straight lines such distance is just a linear function. Our goal, therefore, is to find the equation describing how such separation changes along two geodesic curves; we expect that this equation will involve some quantities that will characterize the intrinsic curvature of the manifold, and that when their value is that of a flat space, the equation will reduce to a linear dependence.

The distance between these points is small enough to allow us to use the same parameter for both, which is thus denoted $x^i_{\mathsf{A}}(\lambda)$ and $x^i_{\mathsf{B}}(\lambda)$, and each of them satisfies a geodesic equation, namely

$$\frac{\mathrm{d}^2 x^i_{\mathsf{A,B}}}{\mathrm{d}\lambda^2} + \Gamma^i{}_{kj} \frac{\mathrm{d}x^k_{\mathsf{A,B}}}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j_{\mathsf{A,B}}}{\mathrm{d}\lambda} = 0,$$

where the subscript A, B means that the equation can be considered separately for the set of points $x^i_{\mathsf{A}}(\lambda)$ or $x^i_{\mathsf{B}}(\lambda)$, and the connection coefficients obviously have to be computed at the points belonging to the corresponding geodesic.

If we denote with $\epsilon^i(\lambda)$ the separation of the two curves then by definition

$$x^i_{\mathsf{B}}(\lambda) = x^i_{\mathsf{A}}(\lambda) + \epsilon^i(\lambda)\,, \tag{D.6.1}$$

and under the hypothesis of infinitesimal distance, by Taylor expanding the connection coefficients around x^i_{A},

$$\Gamma^i{}_{kj}\left(x^i_{\mathsf{B}}\right) = \Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right) + \partial_l \Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\epsilon^l, \tag{D.6.2}$$

where for the sake of brevity we omitted to write explicitly the dependence on λ.

Substituting Eqs. (D.6.1) and (D.6.2) in the geodesic equation for x^i_{B} and retaining only the terms at the first order in ϵ^i we have

$$\begin{aligned}
&\frac{\mathrm{d}^2 x^i_{\mathsf{A}}}{\mathrm{d}\lambda^2} + \frac{\mathrm{d}^2 \epsilon^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\frac{\mathrm{d}x^k_{\mathsf{A}}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j_{\mathsf{A}}}{\mathrm{d}\lambda} + \Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\frac{\mathrm{d}x^k_{\mathsf{A}}}{\mathrm{d}\lambda}\frac{\mathrm{d}\epsilon^j}{\mathrm{d}\lambda} \\
&+\Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\frac{\mathrm{d}\epsilon^k}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j_{\mathsf{A}}}{\mathrm{d}\lambda} + \Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\frac{\mathrm{d}x^k_{\mathsf{A}}}{\mathrm{d}\lambda}\frac{\mathrm{d}\epsilon^j}{\mathrm{d}\lambda} + \partial_l\Gamma^i{}_{kj}\left(x^i_{\mathsf{A}}\right)\epsilon^l\frac{\mathrm{d}x^k_{\mathsf{A}}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j_{\mathsf{A}}}{\mathrm{d}\lambda} = 0.
\end{aligned}$$

This result can be simplified by remembering that x^i_{A} satisfies the geodesic equation, and thus

$$\frac{d^2\epsilon^i}{d\lambda^2} + \Gamma^i{}_{kj}\frac{dx^k}{d\lambda}\frac{d\epsilon^j}{d\lambda} + \Gamma^i{}_{kj}\frac{d\epsilon^k}{d\lambda}\frac{dx^j}{d\lambda} + \partial_l\Gamma^i{}_{kj}\epsilon^l\frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda} = 0,$$

where, again to shorten the writing, we made the dependence on x^i_{A} of the connection coefficients implicit. Because in general

$$\frac{d}{d\lambda}\left(\frac{d\epsilon^i}{d\lambda} + \Gamma^i{}_{kj}\epsilon^k\frac{dx^j}{d\lambda}\right) = \frac{d^2\epsilon^i}{d\lambda^2} + \partial_l\Gamma^i{}_{kj}\frac{dx^l}{d\lambda}\epsilon^k\frac{dx^j}{d\lambda} + \Gamma^i{}_{kj}\frac{d\epsilon^k}{d\lambda}\frac{dx^j}{d\lambda} + \Gamma^i{}_{kj}\epsilon^k\frac{d^2x^j}{d\lambda^2}$$

the previous equation can be recast as

$$\frac{d}{d\lambda}\left(\frac{d\epsilon^i}{d\lambda} + \Gamma^i{}_{kj}\epsilon^k\frac{dx^i_{\mathsf{A}}}{d\lambda}\right) - \partial_l\Gamma^i{}_{kj}\frac{dx^l_{\mathsf{A}}}{d\lambda}\epsilon^k\frac{dx^j_{\mathsf{A}}}{d\lambda} - \Gamma^i{}_{kj}\epsilon^k\frac{d^2x^j_{\mathsf{A}}}{d\lambda^2} + \Gamma^i{}_{kj}\frac{dx^k_{\mathsf{A}}}{d\lambda}\frac{d\epsilon^j}{d\lambda} + \partial_l\Gamma^i{}_{kj}\epsilon^l\frac{dx^k_{\mathsf{A}}}{d\lambda}\frac{dx^j_{\mathsf{A}}}{d\lambda} = 0.$$

The geodesic equation, again, implies that

$$\frac{d^2x^j_{\mathsf{A}}}{d\lambda^2} = -\Gamma^j{}_{lm}\frac{dx^l_{\mathsf{A}}}{d\lambda}\frac{dx^m_{\mathsf{A}}}{d\lambda},$$

and it is easy to see from Eq. (D.5.3) that the absolute derivative of ϵ^i can also be written as

$$\frac{D\epsilon^i}{D\lambda} = \frac{d\epsilon^i}{d\lambda} + \Gamma^i{}_{kj}\epsilon^k\frac{dx^i_{\mathsf{A}}}{d\lambda}.$$

Using these two relations, the previous equation can be recast in the so-called *equation of the geodesic deviation*

$$\frac{D^2\epsilon^i}{D\lambda^2} + R^i{}_{jkl}\epsilon^j\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0, \quad \text{(D.6.3)}$$

in which we dropped the last reference to the specific geodesic x^i_{A} because this refers to any pair of geodesics with infinitesimal separation ϵ^i, and where the *Riemann curvature tensor*, or *Riemann tensor* has been defined as

$$R^i{}_{jkl} \equiv \partial_k\Gamma^i{}_{jl} - \partial_l\Gamma^i{}_{kj} + \Gamma^i{}_{km}\Gamma^m{}_{lj} - \Gamma^i{}_{ml}\Gamma^m{}_{kj}. \quad \text{(D.6.4)}$$

This tensor is precisely the quantity that mathematically characterizes the curvature of a space that we were seeking. We can understand which are the appropriate conditions by recalling that in a flat space the absolute derivative reduces to the usual total derivative, and that all the connection coefficients are zero. In a flat space, therefore, we have

$$R^i{}_{jkl} = 0$$

and

$$\frac{D^2\epsilon^i}{D\lambda^2} = \frac{d^2\epsilon^i}{d\lambda^2} = 0.$$

The last equation implies that in flat spaces the separation between the two geodesics obeys the linear equation in λ $\epsilon^i = a^i\lambda + b^i$, where a^i and b^i are constants of integration, consistent with our expectations. Conversely, if the Riemann tensor does not vanish, the separation cannot be linear, therefore the space cannot be flat. In other words, we have found that the space is flat when $R^i{}_{jkl}$ vanishes, although it is curved otherwise, which justifies the name given to this quantity.

Another interesting observation comes from the early characterization of curved spaces made in Appendix D.2. There we observed that in a genuinely curved space we can find a coordinate transformation that makes the metric tensor equal to that of a flat space up to the first order in the derivatives of g_{ij}, but at least a minimum number of second-order derivatives have to survive eventually. We commented on this fact saying that the quantity characterizing the curvature had to depend on the second derivatives of the metric tensor. Because the connection coefficients, in the case of a metric connection, are functions of the first derivatives of the metric, and the Riemann tensor is a function of the first derivative of the coefficients, then $R^i{}_{jkl}$ is just a function of the second derivatives of the metric, in agreement with our forecast.

Moreover, we observed that the number of independent second derivatives was $N^2\left(N^2-1\right)/12$, where N was the number of dimensions of the space. In this case it might seem that the number of independent coefficients of the fourth-rank Riemann tensor is simply N^4, which does not match our expectations, however, a more careful investigation shows that there are a certain number of symmetries that reduce this number to the expected value.

To this aim, first of all we need to transform the tensor in its completely covariant form $R_{ijkl} = g_{im}R^m{}_{jkl}$. Moreover, in doing this it is convenient to show explicitly the dependencies on the second-order derivatives of the metric which, after straightforward but long calculations, give

$$R_{ijkl} = \frac{1}{2}\left(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}\right) + g_{mn}\left(\Gamma^m{}_{jk}\Gamma^n{}_{il} - \Gamma^m{}_{jl}\Gamma^n{}_{ik}\right).$$

We know that it is always possible to find a change of coordinates in which all the first derivatives of the metric vanish, and therefore $\Gamma^i{}_{jk} = 0$. In such a reference system it is

$$R_{ijkl} = \frac{1}{2}\left(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik}\right).$$

The symmetry properties of the tensor must be independent on the reference system, thus we are free to choose the one that better fits our purposes.

The first property can be found by swapping the first two indexes. In this way the first term becomes the third one and vice versa, but with opposite signs, and the same happens to the second and fourth terms, so that

$$R_{ijkl} = -R_{jikl}. \tag{D.6.5}$$

Similar considerations can be done when swapping k and l, which gives

$$R_{ijkl} = -R_{ijlk}, \tag{D.6.6}$$

whereas exchanging the pairs ij with kl keeps the tensor unchanged; i.e.,

$$R_{ijkl} = R_{klij}. \tag{D.6.7}$$

Finally, from the symmetry of the metric tensor and the commutative property of partial derivatives, one can easily deduce that

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{D.6.8}$$

Each symmetry property decreases the number of independent coefficients of the Riemann tensor. For example, the first two anti symmetries mean that in an N-dimensional space there are only $M = N(N-1)/2$ independent ways to select independent pairs of indexes. At the same time, the third symmetry on the pair exchanging tells us that the number of such independent pairs is $M(M+1)/2$. Finally, the latter symmetry implies that the fourth relation is always true unless the four indexes have all distinct values, which puts another constraint on their combinations.

Eventually, by putting together all these constraints we are left with only $N^2\left(N^2-1\right)/12$ independent coefficients, which is the same number that was deduced in Appendix D.2.

This analysis has shown that the Riemann curvature tensor satisfies all the properties that might be required to determine unambiguously the curvature of a Riemannian (or pseudo-Riemannian) space. Moreover, in the same reference system where the connection coefficients vanish, it is

$$\nabla_m R^i{}_{jkl} = \partial_m \partial_k \Gamma^i{}_{jl} - \partial_m \partial_l \Gamma^i{}_{jk},$$

from which it is not difficult to show that $R^i{}_{jkl}$ also fulfills a differential property called *Bianchi identity* which reads[33]

[33]It is interesting to not that this relation is the analogue of Eq. (C.3.10) in electrodynamics. From this point of view, indeed, the Einstein field equations play the same role of the Maxwell

$$\nabla_m R^i{}_{jkl} + \nabla_l R^i{}_{mjk} + \nabla_k R^i{}_{jlm} = 0, \tag{D.6.9}$$

and because this is a tensorial equation, it has to be valid in any reference system. This relation is quite important for general relativity inasmuch as it is connected with the vanishing of the divergence of the stress-energy tensor, and therefore with the conservation of the four-momentum and of the angular momentum, as shown in Chap. 8.

From the Riemann tensor it is possible to build two more tensorial quantities called the *Ricci (curvature) tensor* and the *Ricci (curvature) scalar*, obtained by successive contraction on the pair of indexes. The first one, the Ricci tensor, is defined as

$$R_{ij} \equiv R^k{}_{ikj} = \partial_k \Gamma^k{}_{ij} - \partial_j \Gamma^k{}_{ik} + \Gamma^k{}_{lk}\Gamma^l{}_{ij} - \Gamma^k{}_{lj}\Gamma^l{}_{ik}. \tag{D.6.10}$$

This definition can be easily justified by considering that it is also $R_{ij} = g^{kl} R_{kilj}$, but we know from the algebraic properties of the Riemann tensor shown above that $R_{ijkl} = -R_{jikl}$, which implies that contracting the first two indexes gives $R^k{}_{kij} = g^{kl} R_{klij} = 0$. The same considerations hold for the second pair of indexes for the antisymmetry in k and l. Finally, contracting on the first and last indexes gives

$$R^k{}_{ijk} = -R^k{}_{ikj}.$$

This is also an acceptable definition of the Ricci tensor. The actual choice is a matter of convention which, like that on the signature of the metric, determines a change of sign.

The Ricci tensor is symmetric, a property that can be proved with some simple calculations. By contracting Eq. (D.6.8) with g^{ik} it is

$$\begin{aligned} 0 &= g^{ik}\left(R_{ijkl} + R_{iklj} + R_{iljk}\right) \\ &= R^k{}_{jkl} + g^{ik} R_{iklj} + R^k{}_{ljk}, \end{aligned}$$

and from the definition of the Ricci tensor and the above-mentioned properties $g^{kl} R_{klij} = 0$ and $R^k{}_{ijk} = -R^k{}_{ikj}$, one immediately obtains

$$R_{jl} - R_{lj} = 0. \tag{D.6.11}$$

Finally, the Ricci scalar is simply obtained by contracting the Ricci tensor in the only possible way, namely

$$R \equiv R^i{}_i = g^{ij} R_{ij}. \tag{D.6.12}$$

These two tensors, as explained in Chap. 8, define the so-called Einstein tensor $G_{ij} = R_{ij} - g_{ij} R/2$, whose properties are investigated there.

(Footnote 33 continued)
equations (C.3.9). Moreover, as is show in Chap. 8, the Bianchi identities imply the existence of a gauge freedom in the field equations, which are the gravitational counterpart of the gauge freedom in the four-potential of the electrodynamics.

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