Chapter 2
Intended Treatment of Fractions and Fraction Operations in Mathematics Curricula from Japan, Korea, and Taiwan

Tad Watanabe, Jane-Jane Lo, and Ji-Won Son

Abstract In spite of extensive research efforts, teaching and learning fractions remain challenging throughout the world. Although students' mathematics learning is influenced by many factors, one important factor is the learning opportunities afforded by their textbooks. Therefore, we examined how textbooks from Japan, Korea, and Taiwan—three high-achieving countries prominent in comparative studies—introduced and developed fraction concepts and fraction arithmetic. We used the content analysis method (National Research Council, On evaluating curricular effectiveness: Judging the quality of K-12 mathematics evaluations, 2004) to analyze the problems presented in the textbooks. Our analysis revealed that there were many similarities among the textbooks from these three countries, including the overall flow of the topics related to fraction concepts and fraction arithmetic. However, significant differences included how various fraction subconstructs were integrated in the textbooks and how fraction multiplication and division were discussed. These similarities and differences among high-achieving countries suggest fruitful directions for future research in the area of fraction teaching and learning.

Keywords Curriculum analysis • Textbook analysis • Fractions • Elementary school education • Number concepts and operations

T. Watanabe (✉)
Department of Mathematics, Kennesaw State University, Kennesaw, GA, USA
e-mail: twatanab@kennesaw.edu

J.-J. Lo
Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA
e-mail: jane-jane.lo@wmich.edu

J.-W. Son
Department of Learning and Instruction, University at Buffalo—The State University of New York, Buffalo, NY, USA
e-mail: jiwonson@buffalo.edu

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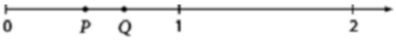
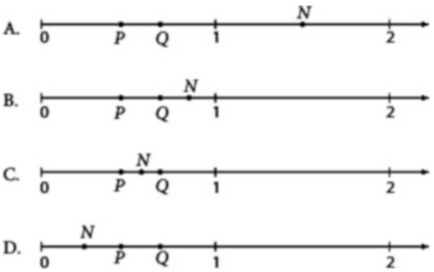
Introduction

The teaching and learning of fractions have long attracted the attention of mathematics education researchers (National Research Council [NRC], 2004). However, in spite of almost a half-century of research, these tasks continue to challenge mathematics teachers and students throughout the world. It is generally agreed that developing a deep understanding of fractions is critical for students' success in more advanced mathematics. The National Mathematics Advisory Panel (2008) in the United States, for example, listed fractions as one of the foundational topics for algebra.

While these challenges are widespread, cross-national comparison studies suggest that both teachers and students from East Asian countries seem to possess a deeper understanding of fractions than their counterparts in the United States (Mullis, Martin, & Foy, 2008; Son & Senk, 2010). For example, Mullis, Martin, and Foy (2008) noted that students from Hong Kong, Japan, Korea, Singapore, and Taiwan generally outperformed students from the United States. Table 2.1 shows some of the released Grade 8 mathematics problems from TIMSS 2011, for which significantly higher percentages of students from Japan, Korea, and Taiwan answered correctly compared to students from the United States. Comparing Chinese and US elementary school teachers, Ma (1999) noted that Chinese teachers possessed the deep understanding of elementary school mathematics, including division of fractions, necessary to teach it effectively. Similarly, Lo and Luo (2012) showed that Taiwanese prospective elementary school teachers understand division of fractions more deeply than their US counterparts.

Although a variety of factors influence student achievement, these performance differences might be attributed to variations in mathematical curricula (Reys, Reys, & Chávez, 2004). Textbooks are generally considered the bridge between the intended curriculum and the implemented curriculum. As Kilpatrick, Swafford, and Findell (2001) pointed out, "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36). Moreover, while the Chinese teachers in Ma's (1999) study gained their deep understanding of elementary mathematics, at least partly from studying their textbooks, Ball (1996) questioned whether US textbooks are written with teachers' learning in mind. Thus, examining the content of textbooks as a possible contributing factor to achievement gaps seems fruitful, and a growing number of cross-national researches analyzing the content of textbooks have been conducted in recent years. Some of those studies have examined the treatment of specific ideas related to fractions. For example, Charalambous, Delaney, Hsu, and Mesa (2010) examined the treatment of addition and subtraction of fractions in textbooks from Cyprus, Ireland, and Taiwan. Li, Chen, and An (2009) examined how selected textbooks from China, Japan, and the United States discussed division of fractions. Son and Senk (2010) also investigated the treatment of multiplication and division of fractions in textbooks from Korea and the United States.

Table 2.1 Student performance on selected TIMSS 2011 Grade 8 problems, by country (Mullis, Martin, Foy, & Arora, 2012)

Item number/problem statement	Student percent correct				
	JPN (%)	KOR (%)	TAI (%)	US (%)	Int'l Avg. (%)
M032064: Ann and Jenny divide 560 zeds between them. If Jenny gets $\frac{3}{8}$ of the money, how many zeds will Ann get?	45	67	60	25	27
M032094: $\frac{4}{100} + \frac{3}{1000}$	77	89	85	63	62
M032662:  P and Q represent two fractions on the number line above. $P \times Q = N$. Which of these shows the location of N on the number line?  	43	44	53	22	23
M052228: Which shows a correct method for finding $\frac{1}{3} - \frac{1}{4}$? A. $\frac{1-1}{4-3}$ B. $\frac{1}{4-3}$ C. $\frac{3-4}{3\times4}$ D. $\frac{4-3}{3\times4}$	65	86	82	29	37

The purpose of the current study is to add to the growing knowledge base on the content of textbooks from high-achieving East Asian countries. In particular, we hope to deepen our knowledge of how textbooks from Japan, Korea, and Taiwan introduce and develop the mathematically challenging idea of fractions.

Theoretical Perspectives

Textbook Analysis

Textbook analysis—in particular, cross-national textbook analysis—is a relatively new field of inquiry. Some of the existing research has investigated the overall structures of textbooks, often focusing on what mathematics is taught at what grade level (e.g., Schmidt, McKnight, Valverde, Houang, & Wiley, 1997), while other studies examined the treatment of a particular mathematical topic (e.g., Cai, Lo, & Watanabe, 2002; Son & Senk, 2010) or mathematical process (e.g., Fan & Zhu, 2007; Mayer, Sims, & Tajika, 1995). Charalambous et al. (2010) referred to the former approach as horizontal analysis and to the latter as vertical analysis, while Li et al. (2009) called the former type "macroanalysis" and the latter "microanalysis."

Although cross-national horizontal, or macro, analyses of textbooks give us a general sense of what topics are discussed in what grade level across different educational systems, they do not reveal much about the actual learning opportunities offered by different textbooks. On the other hand, because vertical, or micro, analyses of textbooks focus on a single mathematical topic, they can reveal different approaches taken by different textbooks. However, such an analysis does not reveal what influences other topics might have on the treatment of a particular topic. Furthermore, because mathematics consists of many interrelated "topics," it may be difficult to identify the boundaries of a single topic. For example, if we were to examine the treatment of a division algorithm, would we need to examine how division is introduced? What about the treatment of a multiplication algorithm or algorithms? Thus, some researchers chose to examine textbooks by integrating both horizontal (or macro) and vertical (or micro) analysis (e.g., Charalambous et al., 2010; Li et al., 2009).

Selecting which textbooks to include in a cross-national study is also an important consideration. Some researchers selected textbooks based on the characteristics of the education systems. For example, both Boonlerts and Inprasitha (2013) and Charalambous et al. (2010) selected their textbooks from countries with centralized education systems and national curriculum standards. Other studies consider the achievements of the targeted students, either explicitly or implicitly. Those studies will often include textbooks from high-achieving Asian countries and other countries of interest to the researchers. For example, Li et al. (2009) examined textbooks from China, Japan, and the United States, while Boonlerts and Inprasitha (2013) examined textbooks from Japan, Singapore, and Thailand. In many cases, the authors' familiarity with the textbook's language appears to play a role in the selection of textbooks.

Fractions

Teaching and learning fractions have been recognized as problematic for quite some time. Research has revealed a variety of misconceptions children possess about fractions. For example, some students do not appear to understand fractions as numbers or quantities, as the following excerpt from Simon (2002) shows:

> In a fourth-grade class, I asked the students to use a blue rubber band on their geoboards to make a square of a designated size, and then to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary, cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show halves of the square. Further, they were able to justify that assertion.
>
> I then asked the question, "Is Joe's half larger; is Mary's half larger, or are they the same size?" Approximately a third of the class chose each option. In the subsequent discussion, students defended their answers. However, few students changed their answers as a result of the arguments offered.
>
> (Simon, 2002, p. 992)

Another common misconception occurs when students misapply their understanding of whole numbers to fractions. Thus, some students conclude that $\frac{1}{3}$ is greater than $\frac{1}{2}$ because 3 is greater than 2. Larson (1980) noted that many students had difficulty locating fractions on number lines, and Greer (1987) reported on difficulties students had in selecting the appropriate operation to solve word problems. The fact that fractions comprise a multifaceted construct has been identified as contributing to these complexities (Lamon, 2007). Kieren (1976) articulated that fractions consist of five subconstructs—part-whole, measure, quotient, operator, and ratio. Table 2.2 provides a simple summary of these five subconstructs.

The goal of fraction instruction is to help students "recognize nuances in meaning; to associate each meaning with appropriate situations and operations; and, in general, to develop insight, comfort, and flexibility in dealing with the rational numbers" (Lamon, 2007, p. 636). Unfortunately, fraction instruction in the United States rarely extends beyond the part-whole meaning of fractions, despite the consensus that focusing solely on the part-whole meaning of fractions is limiting (e.g., Lamon, 2007).

Table 2.2 Interpretations of $\frac{3}{4}$ according to the five subconstructs (Lamon, 2007)

Part-whole	3 parts out of 4 equal parts of a unit
Measure	3 pieces of $\frac{1}{4}$-units, for example, the distance of $3\frac{1}{4}$-units on a number line
Operator	$\frac{3}{4}$ of something; $\frac{3}{4}$ is a rule that tells how to operate on a unit
Quotient	3 divided by 4
Ratio	3 of A are compared to 4 of B in a multiplicative sense

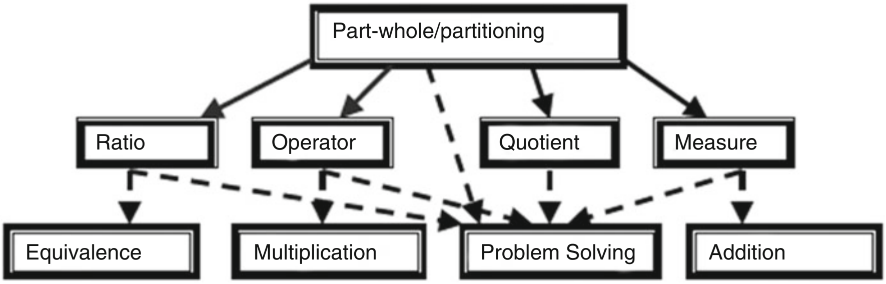


Fig. 2.1 Five subconstructs of fractions and their relationships (Behr et al., 1983)

Behr, Lesh, Post, and Silver (1983) further developed Kieren's (1976) ideas and proposed a theoretical model linking the different interpretations of fractions to the basic operations of fractions, as shown in Fig. 2.1.

According to Behr et al., the part-whole subconstruct of rational numbers is fundamental for developing understanding of the four subordinate constructs of fractions. Moreover, the operator and measure subconstructs are helpful for developing understanding of the multiplication and addition of fractions, respectively. However, there are many unanswered questions about how to incorporate these subconstructs in a mathematics curriculum. For example, is it better for students to be exposed to all five subconstructs early, or is it better to focus on one (beyond part-whole)? If it is better to focus on one, which? Do students need to understand all five subconstructs before algebra? These are some of the outstanding questions that demand mathematics education researchers' attention (Lamon, 2007).

Mack (1990, 1995) examined how educators might be able to take advantage of children's informal understanding of fractions in the formal study of fractions. Her studies suggest that instruction starting with partitioning of a whole might be effective. Pothier and Sawada (1983, 1989) also show that there is a pattern in young children's development of partitioning strategies and their justifications for equality of parts. Armstrong and Larson (1995) asked students in middle grades (Grades 4, 6, and 8) to compare areas of rectangles and triangles embedded in other geometric figures. They found that more students used justifications based on part-whole relationships or partitioning as they became more familiar with fractions. These studies suggest the foundational nature of partitioning activities in the early instruction of fractions.

Steffe, Olive, Tzur, and their colleagues have embarked upon an ambitious multipart study to articulate children's construction of fraction understanding (e.g., Olive, 1999; Steffe, 2002; Tzur, 1999, 2004). Their studies showed that children's whole number concepts did not interfere with their conceptualization of fractions (Olive, 1999; Steffe, 2002; Tzur, 1999, 2004). In fact, the types of units and operations children construct in their whole number sequences can support their development of fraction schemes. However, these studies suggest that teaching which supports students' development of fraction understanding requires a more coherent approach, not only toward fractions but also toward other related ideas, such as multiplication and division of whole numbers. For example, multiplication must go far beyond repeated addition: it must be understood as a way to quantify something when it is composed of several copies of identical size. Such an understanding of multiplication can help students view fraction $\frac{m}{n}$ as m times $\frac{1}{n}$ instead of "m out of n," which does not necessarily signify a quantity. Thompson and Saldanha (2003) noted that "we rarely observe textbooks or teachers discussing the difference between thinking of $\frac{3}{5}$ as 'three out of five' and thinking of it as '3 one fifths'" (p. 107).

Because fractions themselves are multifaceted constructs, students, and often teachers, may have difficulty with fraction arithmetic. As a result, a large number of studies have been conducted to examine students' understanding of fractions, including some cross-national examinations of textbooks (e.g., Charalambous et al., 2010; Li et al., 2009; Son & Senk, 2010). Fraction division in particular has attracted the attention of many researchers as it is recognized as one of the most challenging mathematics topics in the middle grades. Too often, fraction instruction focuses on the invert-and-multiply algorithm of division. However, a major difficulty for students is knowing when division is the appropriate calculation (e.g., Greer, 1987; Siegler & Lortie-Forgues, 2015), in part due to the difficulty of interpreting fraction division. While partitive division—that is, the divisor being the number of equal groups and the quotient being the group size—is more common with whole number division (e.g., Fischbein, Deri, Nello & Marino, 1985), it is easier to interpret fraction division with quotitive division than with partitive division. Some researchers (e.g., Zambat, 2015) recommend students first learn fraction division in quotitive situations, leading to the common denominator algorithm instead of the invert-and-multiply algorithm. On the other hand, many of the Chinese teachers Ma (1999) interviewed were able to give both partitive and quotitive problem situations for fraction division problems. This may suggest important differences in approaches to fraction multiplication and division in East Asia.

In the United States, the Common Core State Standards for Mathematics (CCSSM, Common Core State Standard Initiatives, 2010) suggests a progression of fraction arithmetic. The CCSSM approaches addition and subtraction of fractions by utilizing the idea of non-unit fractions as collections of unit fractions, which seems to be consistent with the idea of Steffe and his colleagues. With multiplication, the CCSSM first focuses on multiplication of fractions by whole numbers in Grade 4, and then multiplication of fractions by fractions in Grade 5. The CCSSM approaches division of fractions by first exploring division of unit fractions by whole numbers and whole numbers by unit fractions in Grade 5, and then fractions divided by fractions in Grade 6, leading to the invert-and-multiply algorithm.

Research Questions

In spite of the multitude of research studies and recommendations described above, fraction teaching and learning remain challenging, particularly in the United States. Our study focuses on textbooks from three high-achieving East Asian countries: Japan, Korea, and Taiwan. By examining their textbooks, we hope to gain some insights into how we might support both teachers and students as they tackle this mathematically challenging topic. Specifically, we examine the following questions:

1. What are the similarities and differences in the intended learning progressions of fraction concept development among the three high-achieving Asian curricula? In particular, what are the similarities and differences with respect to (1) the sequence of topics and (2) the integration of fraction subconstructs?
2. What are the similarities and differences in the treatment of the four arithmetic operations with fractions among the three high-achieving Asian curricula? In particular, what are the similarities and differences with respect to (1) the types of problem situations utilized in discussing each operation, (2) the intended computational algorithms, if any, and (3) the use of visual representations (set, line, or area)?

Because our study focuses on the single topic of fractions, it principally involves a *micro* analysis of the textbooks. However, because the topic is broader than addition/subtraction of fractions (Charalambous et al., 2010) or division of fractions (Li et al., 2009), our study also shares some characteristics of *macro* analysis. The scope of the analysis is still limited to topics directly related to fractions.

Methodology

NRC (2004) argues that content analysis should be about a specific standard and comparison curricula should be selected judiciously. For a cross-national study, there is no common standard on which to focus. Instead, we chose to use a specific mathematical topic, fractions, and examine how the selected textbooks introduce and develop the ideas of fractions and fraction operations. We focused on Japan, Korea, and Taiwan for three reasons. First, they are high-achieving countries in various international achievement studies. Second, their educational systems are similar—centralized with national curriculum standards published by the respective Ministries of Education. Finally, all three curricula complete the discussion of fractions within elementary school (i.e., Grades 1–6). The background of the research team members, who are natives of the three countries, was also a factor.

The textbooks selected (see Table 2.3) were in alignment with the national curriculum standards at the time of the study—the 2008 standards for Japan,

Table 2.3 Textbooks analyzed in this study

Country	Textbook series
Japan	Fujii, T. & Iitaka, S. (2011). *Atarashii Sansuu*. Tokyo: Tokyo Shoseki Co. Ltd.
Korea	Korean Ministry of Education and Human Resources Development. (2014). *Mathematics* (Grades 3–4). Seoul: DaeHan Printing and Publishing Co., Ltd. Korean Ministry of Education and Human Resources Development. (2015). *Mathematics* (Grades 5–6). Seoul: DaeHan Printing and Publishing Co., Ltd.
Taiwan	Kang Hsuan Educational Publishing Group. (2012). *Kang Hsuan elementary school mathematics textbooks*. (4A) Tainan, Taiwan: Author. Kang Hsuan Educational Publishing Group. (2013). *Kang Hsuan elementary school mathematics textbooks*. (3A, 4B, 5A, 6A) Tainan, Taiwan: Author. Kang Hsuan Educational Publishing Group. (2014). *Kang Hsuan elementary school mathematics textbooks*. (3B, 5B, 6B) Tainan, Taiwan: Author.

2013 for Korea, and 2008 (for Grades 1–3) and 2003 (for Grades 4–6) for Taiwan. The Japanese textbook series is commercially published by Tokyo Shoseki, one of six textbook series approved by the Ministry of Education. It is the most widely used elementary mathematics textbook (Naigaikyouiku, 2010). The Korean textbook series examined is the only elementary mathematics textbook series in Korea and is written by the Ministry of Education. The Taiwanese textbook series is one of the four commercially published textbooks in Taiwan. It is one of the two most widely used textbook series (S. Law, personal communication, July 13, 2015). The textbooks were analyzed in their original languages by researchers who are native speakers—the first author analyzed the Japanese textbooks, the second author the Taiwanese textbooks, and the third author the Korean textbooks. Because some aspects of our analysis—for example, word problem contexts and fraction subconstructs—are not visually verifiable, we used the English translation of the Japanese series (Fujii & Iitaka, 2012) to calibrate our analysis. The researchers independently analyzed segments of the translated Japanese textbook on those aspects, and then compared analyses. Whenever a discrepancy in the analyses occurred, the particular instance was discussed until a consensus was reached.

The study reported in this chapter is a content analysis of three Asian textbook series. The analysis took place in three stages. First, we identified the sequence and the grade placement of the major topics related to fractions in each textbook. Then, we analyzed each textbook's treatment of addition and subtraction, focusing on problem types, the use of diagrams, and the target algorithms, if any. For problem types, we first determined the frequencies of word problems, calculation exercises, and others. For word problems, we examined the addition/subtraction problem situations using the Cognitively Guided Instruction framework, which categorizes addition and subtraction word problems based on the four problem situations—join, separate, part-part-whole, and compare—and the unknown quantity in the situation (Carpenter, Fennema, & Romberg, 1992). Table 2.4 summarizes the 11 addition and subtraction word problem types according to this framework. Finally, we examined the treatment of multiplication and division in these textbooks, again focusing on problem types, the use of diagrams, and the target algorithms, if any.

Table 2.4 Problem types based on the CGI framework (Carpenter et al., 1992)

Problem type	Unknown factors		
Join (add to)	**(Result Unknown)** Connie had 5 marbles. Juan gave her 8 more marbles. How many marbles does Connie have altogether?	**(Change Unknown)** Connie had 5 marbles. How many marbles does she need to have 13 marbles altogether?	**(Start Unknown)** Connie had some marbles. Juan gave her 5 more. Now she has 13 marbles. How many marbles did Connie have to start with?
Separate (take from)	**(Result Unknown)** Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left?	**(Change Unknown)** Connie had 13 marbles. She gave some to Juan. Now she has 5 marbles left. How many marbles did Connie give to Juan?	**(Start Unknown)** Connie had some marbles. She gave 5 to Juan. Now she has 8 marbles left. How many marbles did Connie have to start with?
Part-Part-Whole (put together/ take apart)	**(Whole Unknown)** Connie has 5 red marbles and 8 blue marbles. How many marbles does she have altogether?		**(Part Unknown)** Connie has 13 marbles: 5 are red, and the rest are blue. How many blue marbles does Connie have?
Compare	**(Difference Unknown)** Connie has 13 marbles. Juan has 5 marbles. How many more marbles does Connie have than Juan?	**(Larger Unknown)** Juan has 5 marbles. Connie has 8 more than Juan. How many marbles does Connie have?	**(Smaller Unknown)** Connie has 13 marbles. She has 5 more marbles than Juan. How many marbles does Juan have?

Findings

Overall, the treatment of fractions in the three textbook series is more similar than different. However, there are some significant differences in the way some fraction topics are discussed in these textbooks. In the following sections, we will share the findings in accordance with the two research questions.

Intended Learning Progression

Table 2.5 summarizes the grade placements of the major fraction topics in the textbooks from each country. Clearly, some topics, like addition and subtraction of fractions, are discussed in multiple grade levels. However, by simply examining the grade level in which each topic is introduced, we found that all three textbooks introduce these topics in an identical order. Likewise, all three textbook series emphasize the idea that a non-unit fraction is made up of unit fractions. For

Table 2.5 Grade placements of major fraction topics in the textbooks from Japan, Korea, and Taiwan

	Japan	Korea	Taiwan
Fractions as equal shares	2	3	3
Fraction as number	3/4	3/4	3/4
Comparison	3/4/5	3/4/5	3/4
Addition/subtraction	3/4/5	3/4/5	3/4/5
Equivalent fractions	4/5	4/5	4
Fractions as quotients	5	4	4
Multiplication	5/6	5	4/5
Division	5/6	5/6	6

example, Fig. 2.2(a) shows how the Japanese textbook uses this idea to deal with fractions greater than 1 (Problem 2). Question (2) in Fig. 2.2(b) shows the Taiwanese textbook asking students "How many $\frac{1}{10}$ pieces are needed to make up a $\frac{7}{10}$ piece?" As we will see later, all three textbook series make use of this way of looking at fractions as they discuss addition and subtraction of fractions.

A few differences do occur in the overall flow of the curricula. First, in the textbooks from Japan and Korea, the idea of equivalent fractions is first introduced in Grade 4, but the formulas to create equivalent fractions, i.e., $\frac{a}{b} = \frac{a \times k}{b \times k}$ and $\frac{a}{b} = \frac{a \div k}{b \div k}$ (a, b, and k are nonzero whole numbers), are not discussed until Grade 5. However, the Taiwanese textbook develops this formula in Grade 4. Another difference is the grade placement of the quotient meaning of fractions; that is, $\frac{a}{b} = a \div b$ (a and b are whole numbers, $b \neq 0$). Both the Korean and the Taiwanese textbooks introduce this idea in Grade 4, but the Japanese textbook introduces it in Grade 5.

Whereas the treatments of addition and subtraction in all three textbooks are very similar, the treatments of multiplication and division illustrate significant differences among the three textbooks. (We will discuss the similarities and the differences of the actual treatments in greater detail later.) The Taiwanese textbook first introduces fraction multiplication in Grade 4. Both the Japanese and the Korean textbooks introduce multiplication and division of fractions in Grade 5, while the Taiwanese textbook does not introduce division of fractions until it completes the discussion of multiplication of fractions. Although multiplication and division are both introduced in Grade 5 in the Japanese and the Korean textbooks, the Korean textbook completes the discussion of multiplication in Grade 5 while the Japanese textbook extends the discussion of both operations into Grade 6.

Integration of Fraction Subconstructs

Table 2.6 summarizes which fraction subconstructs are discussed in the three Asian textbooks at different grade levels. Once again, the integration of various subconstructs among the three textbook series is more similar than different. All three textbook series integrate the five subconstructs into their discussions of fractions. Additionally, the part-whole and measure subconstructs clearly play a

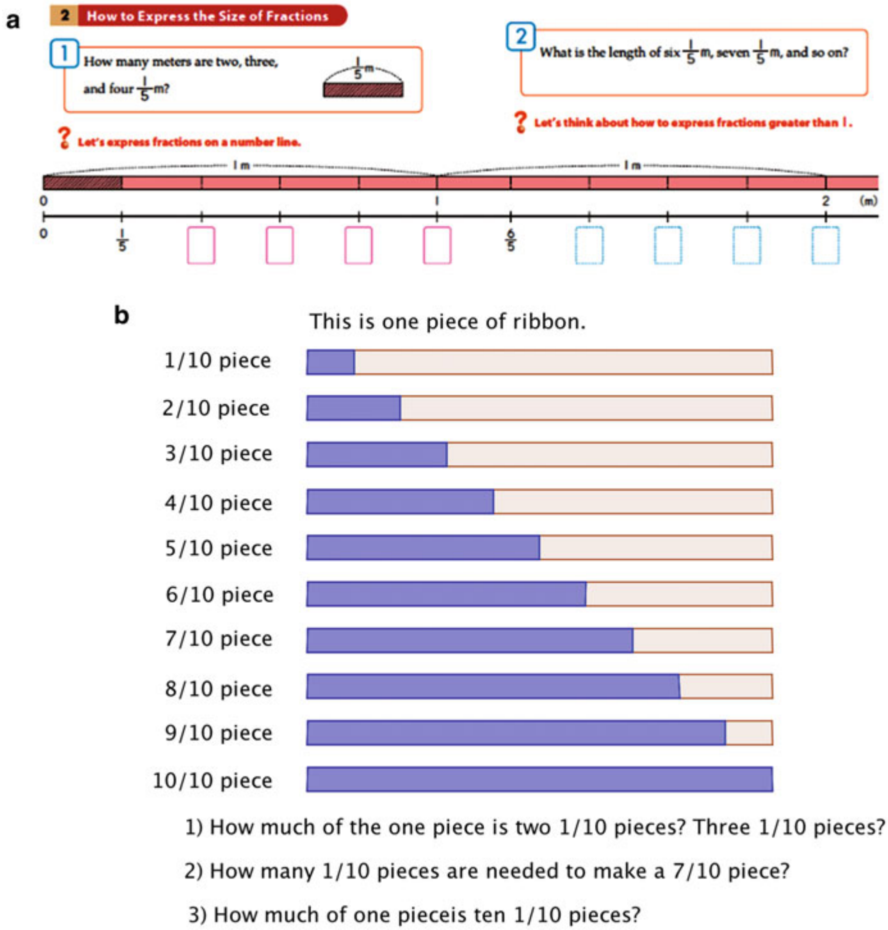


Fig. 2.2 (**a**) The idea that non-unit fractions are made up of unit fractions is emphasized in these Japanese Grade 3 problems (Fujii & Iitaka, 2012, Grade 3, pp. B48–B49). (Although the analysis was conducted using the original textbook in Japanese, we use images from the English translation so that we will not have to provide the translation separately.) (**b**) Problems from Taiwanese Grade 3 textbook (translated from Kang Hsuan Educational Publishing Group, 2014, Grade 3B, p. 35)

central role in the initial instruction on fractions in all three series. The quotient subconstruct is introduced, as expected, with the quotient meaning of fractions—during Grade 4 in Korea and Taiwan, and during Grade 5 in Japan. In all three series, the ratio subconstruct was introduced last.

Although the measure subconstruct seems to play a central role in all three textbook series in early grades, the operator subconstruct begins to play a more important role in the Korean and the Taiwanese textbooks than in the Japanese textbook when they discuss multiplication by fractions, that is, when the multiplier becomes a fraction. For example, Fig. 2.3 shows a problem from the Taiwanese

Table 2.6 Fraction subconstructs appearing in the three Asian textbooks

Grade	Japan	Korea	Taiwan
2	**Part-whole**		
3	Part-whole **Measure**	**Part-whole** **Measure** **Operator**	**Part-whole** **Measure**
4	Part-whole Measure	Part-whole Measure **Quotient**	Part-whole Measure **Quotient**
5	Part-whole Measure **Quotient**	Part-whole Measure Quotient Operator	Part-whole Measure Quotient **Operator**
6	Part-whole Measure **Operator** **Ratio**	Part-whole Measure Operator **Ratio**	Part-whole Measure Quotient **Ratio**

Note: Bold-faced letters indicate the first time the particular subconstruct is discussed in the textbook series

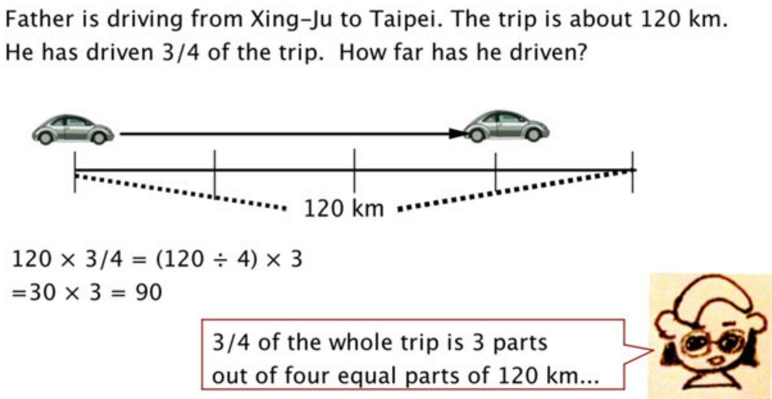


Fig. 2.3 A multiplication problem from the Taiwanese textbook (translated from Kang Hsuan Educational Publishing Group, 2014, Grade 5B, p. 7)

series. Note that in this problem $\frac{3}{4}$ is the multiplier, and it is given as an operator fraction.

In contrast, Fig. 2.4 shows an introductory word problem found in the Japanese Grade 6 unit on multiplication of fractions. In this case, $\frac{2}{3}$ is the multiplier and represents a measured quantity, not an operator fraction.

The Korean textbook actually incorporates the operator subconstruct much earlier than either the Japanese or the Taiwanese series. Figure 2.5 shows a Grade 3 problem from the Korean series. Although this problem can be interpreted as a

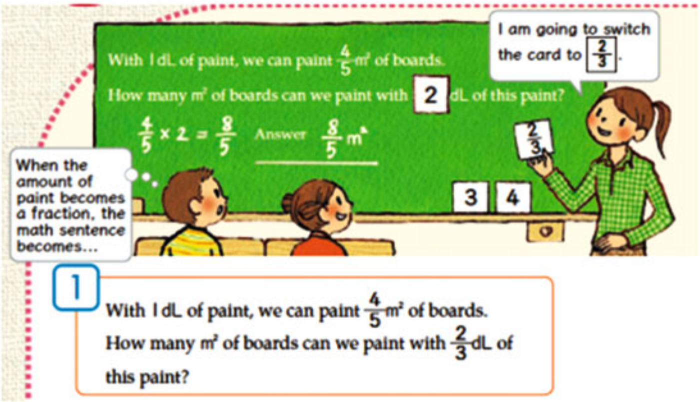


Fig. 2.4 An introductory problem in the unit on multiplication of fractions (Fujii & Iitaka, 2012, Grade 6, p. A 23)

Fig. 2.5 Translated from Korean Ministry of Education and Human Resources Development, 2014, Grade 3-B student book, p. 109

multiplication problem it appears in the introductory unit, in which the focus is helping students understand the meaning of fractions.

Find out how many are in $\frac{3}{4}$ of the set if 8 apples are a whole set.

Addition and Subtraction

The ways in which addition and subtraction are introduced and developed in all three series are quite similar. Addition and subtraction of fractions are first introduced in word problems. Table 2.7 summarizes the addition and subtraction situations used in the three textbook series. Although the Taiwanese series includes all but two of the possible addition and subtraction situations, all three series generally use simpler situations.

Figure 2.6 shows an introductory problem, of a part-part-whole whole unknown type, from the English translation of the Japanese series.

As noted earlier, all three textbook series emphasize the idea of a non-unit fraction being made up of unit fractions. In discussions regarding how to add or subtract fractions with like denominators, they all make use of this unitary perspective. Thus, $\frac{7}{10} - \frac{3}{10}$ can be thought as taking away 3 $\frac{1}{10}$-units from 7 $\frac{1}{10}$-units.

Table 2.7 Word problem situations found in the three Asian textbook series

Join result unknown *JKT*	Join change unknown *T*	Join start unknown
Separate result unknown *JKT*	Separate change unknown *T*	Separate start unknown *T*
Part-part-whole whole unknown *JKT*	Part-part-whole part unknown *T*	
Compare difference unknown *JKT*	Compare smaller unknown	Compare larger unknown *T*

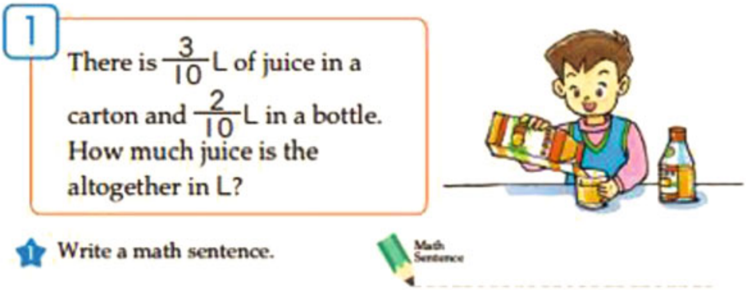


Fig. 2.6 An introductory problem addition of fractions from the Japanese series (Fujii & Iitaka, 2012, Grade 3, p. B51)

Therefore, the difference is (7–3) $\frac{1}{10}$-units, or 4 $\frac{1}{10}$-units, i.e., $\frac{4}{10}$. Figure 2.7 shows this approach as it appears in the Taiwanese series.

Although all three textbook series use word problems to introduce the addition and subtraction of fractions, they all seem to focus on helping students develop computational mastery once the reasoning behind the calculation is established. As a result, about $\frac{3}{4}$ of the problems found in the units on addition and subtraction are purely calculation exercises.

The three Asian textbook series incorporate a variety of visual representations to support students' reasoning with addition and subtraction. Figure 2.8(a) shows how the Korean textbook uses an area model to represent $1\frac{1}{5} + 2\frac{2}{5}$, while Fig. 2.8 (b) shows a linear model found in the Taiwanese series.

Multiplication and Division

Unlike addition and subtraction, more significant differences exist among the three East Asian textbook series in how they present multiplication and division of fractions. Overall, the Korean and the Taiwanese series' treatments of multiplication and division are similar, while the Japanese series incorporates some unique approaches. One common aspect among the three series is that they all discuss multiplication and division by whole numbers separately from multiplication and

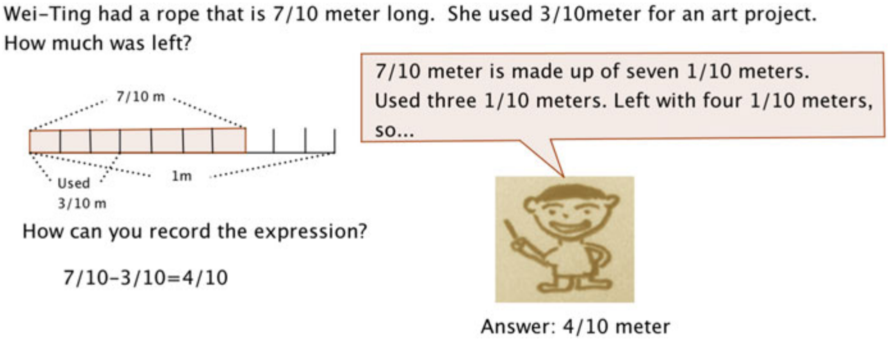


Fig. 2.7 This example from the Taiwanese series shows the typical approach, found in all three Asian series, to thinking about subtraction of fractions with like denominators (translated from Kang Hsuan Educational Publishing Group, 2014, Grade 3B, p. 20)

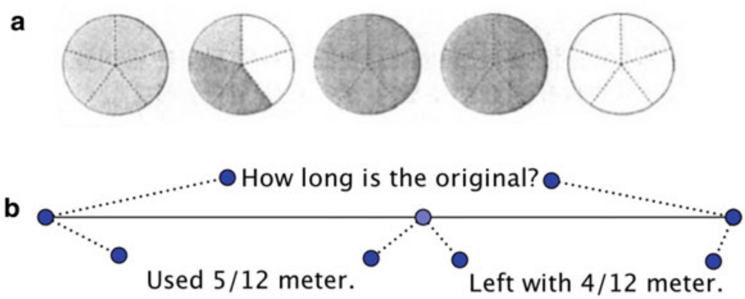


Fig. 2.8 (**a**) An area model from the Korean textbook (translated from Korean Ministry of Education and Human Resources Development, 2014, Grade 4B, p. 83). (**b**) A linear model showing $\frac{5}{12} + \frac{4}{12}$ in the Taiwanese series (translated from Kang Hsuan Educational Publishing Group, 2014, Grade 3B, p. 42)

division by fractions. Thus, the discussion of multiplication of fractions begins with situations where there are whole-number groups of fractional quantities (e.g., $3 \times 2/5$)—occurring in Grade 4 for the Taiwanese series and in Grade 5 for the Japanese and the Korean series. Furthermore, all three series continue to use the idea that non-unit fractions are made up of unit fractions to help students make sense of the process of multiplying fractions by whole numbers. Figure 2.9 shows an example from the Taiwanese series showing $\frac{2}{10}$ multiplied by 5.[1]

[1]In all three Asian textbook series, a multiplication equation is written in the form (multiplicand) $\times$ (multiplier) = (product), or (group size) $\times$ (number of groups) = (product). In this chapter, we adopt the convention that seems to be more common in English-speaking countries, (multiplier) $\times$ (multiplicand) = (product). However, we keep the Asian notation in figures or quotes taken directly from the textbooks.

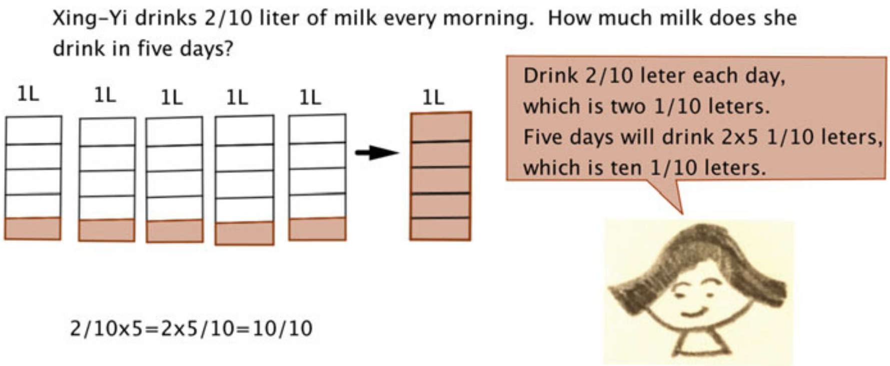


Fig. 2.9 An example of conceptualizing fraction multiplication through unit fraction (translated from Kang Hsuan Educational Publishing Group, 2012, Grade 4A, p. 102)

After this initial discussion of multiplication of fractions by whole numbers, the approach of the Japanese series diverges from both the Korean and the Taiwanese series. First, as noted earlier, both the Korean and the Taiwanese series wait to discuss division of fractions until they complete their discussions of multiplication of fractions, including multiplication by fractions. However, the Japanese series discusses dividing fractions by whole numbers in Grade 5, before discussing multiplication by fractions. Figure 2.10 shows the initial problem from the Japanese series that discusses division of a fraction by a whole number.

In addition to the difference in the overall sequencing of multiplication and division, there are differences in the sequences of topics related to multiplication of fractions among the three Asian textbook series. Table 2.8 summarizes the sequence of topics related to multiplication.

Sequence similarities exist between the Korean and the Taiwanese textbook series. However, unit fractions seem to play a foundational role in the Taiwanese series. Thus, as the series discusses multiplication of whole numbers by fractions, multiplication of fractions by whole numbers, and multiplication of two fractions, it starts with unit fractions. However, regarding multiplying whole numbers by fractions and fractions by whole numbers, the Korean series treats unit fractions as a special case of proper fractions. Thus, W × UF and UF × W appear in the exercise sets after the textbook discusses W × PF and PF × W, respectively. In the case of multiplying two fractions, however, both the Korean and the Taiwanese series start with the UF × UF situation.

The Japanese series also considers unit fractions as a special case of proper fractions. Thus, W × UF is found in the exercise set after it discusses W × PF, like in the Korean series. However, the Japanese series keeps the same perspective when it discusses multiplication of two fractions. Moreover, in their discussion of multiplying by fractions, the textbook authors seem to consider whole numbers as a special case of fractions. Thus, the series begins the discussion of fraction multipliers with a situation that involves multiplication of two proper fractions, e.g., $\frac{2}{3} \times \frac{4}{5}$, instead of P × W like the Korean series or UF × W like the Taiwanese series.

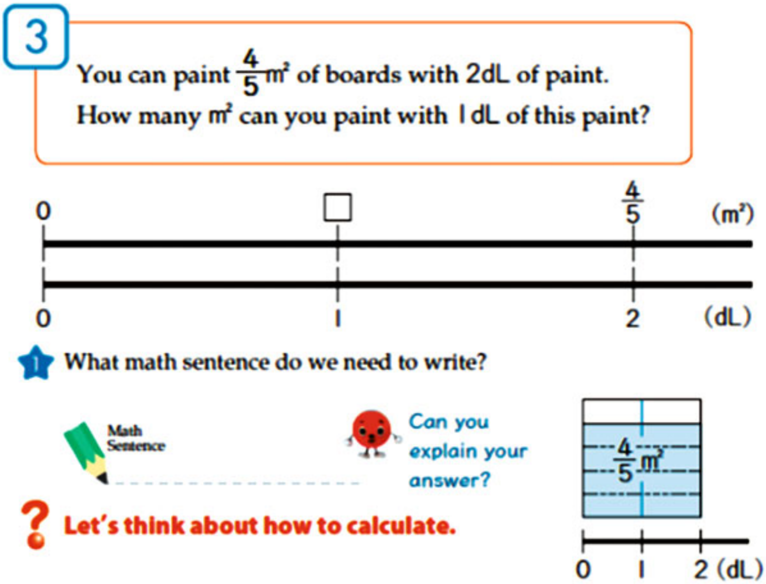


Fig. 2.10 The Japanese textbook discusses dividing fractions by whole numbers before discussing multiplication by fractions (Fujii & Iitaka, 2012, Grade 5, p. B91)

Table 2.8 Sequence of multiplication-related topics in the three Asian textbook series

Japan	Korea	Taiwan
Whole number multiplier		
W × PF[a]	W × PF[a]	W × UF
	W × M	W × PF
		W × M
Fraction multiplier		
P × P	P × W[b]	UF × W
P × W	M × W	P × W
P × M[c]	UF × UF	M × W
	P × P	UF × UF
	M × M	P × P
		P × M & M × P
		M × M

W: whole numbers; UF: unit fractions; PF: proper fractions; M: mixed fractions

[a]W × UF is included in the exercise set after this topic is discussed

[b]U × W is included in the exercise set after this topic is discussed

[c]M × M is included in the exercise set after this topic is discussed

Additional variations in the treatment of division of fractions occur among the three textbook series. Both the Japanese and the Korean textbooks discuss dividing fractions by whole numbers in Grade 5, but the Taiwanese textbook does not discuss this as a separate topic. In fact, the Taiwanese series only has one problem that considers dividing a fraction by a whole number, and it appears near the end of the discussion of division of fractions. Although the Japanese and the Korean series discuss division of fractions by whole numbers as a separate topic in Grade 5, there are some significant differences between these two series. In the Japanese series, division of fractions by whole numbers is treated immediately after multiplication of fractions by whole numbers and before the discussion of multiplication by fractions, a Grade 6 topic. In contrast, the Korean series discusses division of fractions by whole numbers after the completion of the discussion of multiplication by fractions. A major goal in the Japanese series is to develop the algorithm $\frac{a}{b} \div n = \frac{a}{b \times n}$. On the other hand, the Korean series tries to lay the foundation for the invert and multiply algorithm by helping students understand that division by a whole number is the same as multiplying by the unit fraction which is the reciprocal of the divisor.

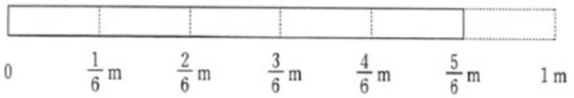
All three series discuss division of fractions by fractions in Grade 6. While the Japanese series begins with a word problem that is solved by $\frac{2}{5} \div \frac{3}{4}$, both the Korean and the Taiwanese textbooks start with word problems that involve dividing a fraction by a unit fraction with a common denominator: $\frac{5}{6} \div \frac{1}{6}$ for the Korean series and $\frac{8}{9} \div \frac{1}{9}$ for the Taiwanese. While the word problem for the Japanese series is a partitive division problem, both the Korean and the Taiwanese series use quotitive division problems. These two series follow up the initial problems with division problems where the numerator of the dividend is not divisible by the numerator of the divisor, which the Taiwanese series calls "fraction division with remainder." For example, in the Korean series, students are asked to find how many $\frac{2}{6}$ *m* are in $\frac{5}{6}$ *m*. The textbook provides a bar diagram showing $\frac{5}{6}$ *m* and then asks how many 2 *m* are in 5 *m*, accompanied by a bar diagram showing 5 *m* (see Fig. 2.11). Then, by comparing these two situations, the series develops the common denominator algorithm for division of fractions.

In both series, the invert-and-multiply algorithm is discussed only after the common denominator algorithm is established. For example, in the Taiwanese series, students are given a quotitive word problem that can be solved by $\frac{13}{12} \div \frac{5}{12}$. The textbook then illustrates the calculation process to show how the quotient may be found by multiplying the dividend by the reciprocal of the divisor:

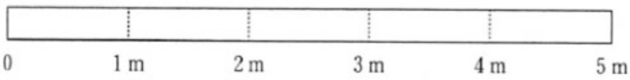
$$\frac{7}{8} \div \frac{3}{5} = \frac{7 \times 5}{8 \times 5} \div \frac{3 \times 8}{5 \times 8} = (7 \times 5) \div (8 \times 3) = \frac{7}{8} \times \frac{5}{3} = \frac{35}{24}.$$

Similarly, the Korean textbook series addresses how the common denominator method can be connected to the invert-and-multiply method with the problem $\frac{3}{4} \div \frac{2}{5}$, as follows:

Activity 1. Figure out how to calculate $\frac{5}{6} \div \frac{2}{6}$.



- Cut $\frac{5}{6}$ into $\frac{2}{6}$.
- When $\frac{5}{6}$ is divided by $\frac{2}{6}$, there are 2 pieces of $\frac{2}{6}$ and half of $\frac{2}{6}$.



- Cut 5 m into 2 m.
- When 5 m is divided by 2m, there are 2 pieces of 2m and half of 2m.
- Is the quotient of $5 \div 2$ the same as that of $\frac{5}{6} \div \frac{2}{6}$?
- Why do you think so?
- Construct an expression to calculate $\frac{5}{6} \div \frac{2}{6}$.

$$\frac{5}{6} \div \frac{2}{6} = [\ ] \div [\ ]$$

Fig. 2.11 A fraction division problem from the Korean textbook that requires students to use the common denominator algorithm for dividing fractions by comparing $\frac{5}{6} \div \frac{2}{6}$ and $5 \div 2$ (translated from Korean Ministry of Education and Human Resources Development, 2015, Grade 6–1, p. 2)

$$\frac{3}{4} \div \frac{2}{5} = \frac{3 \times 5}{4 \times 5} \div \frac{2 \times 4}{5 \times 4} = (3 \times 5) \div (2 \times 4) = \frac{3 \times 5}{2 \times 4}.$$

Because $\frac{3 \times 5}{2 \times 4} = \frac{3}{4} \times \frac{5}{2}$, $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2}$.

Note that these textbooks implicitly apply the commutative property at different steps. As stated above, while both the Korean and the Taiwanese textbook series introduce division by fractions using quotitive word problems, the Japanese series introduces division by fractions with a partitive word problem. Figure 2.12 shows the opening problem in the unit of division by fractions.

Not only is the problem situation partitive, the calculation involves dividing by a fraction less than 1, which has been shown to be challenging (Greer, 1987). Thus, the Japanese series' initial emphasis is helping students understand why this problem can be solved by $\frac{2}{5} \div \frac{3}{4}$. The textbook includes explanations by two hypothetical students. One student uses the generalized equation [Amount painted] $\div$ [Amounts of paint used (dL)] $=$ [Area we can paint with 1 dL], derived by thinking about whole-number divisors. The other student uses the double-number line representation to argue that $\frac{2}{5}$ is obtained by multiplying the missing quantity by $\frac{3}{4}$. Then, by using the relationship between multiplication and division,

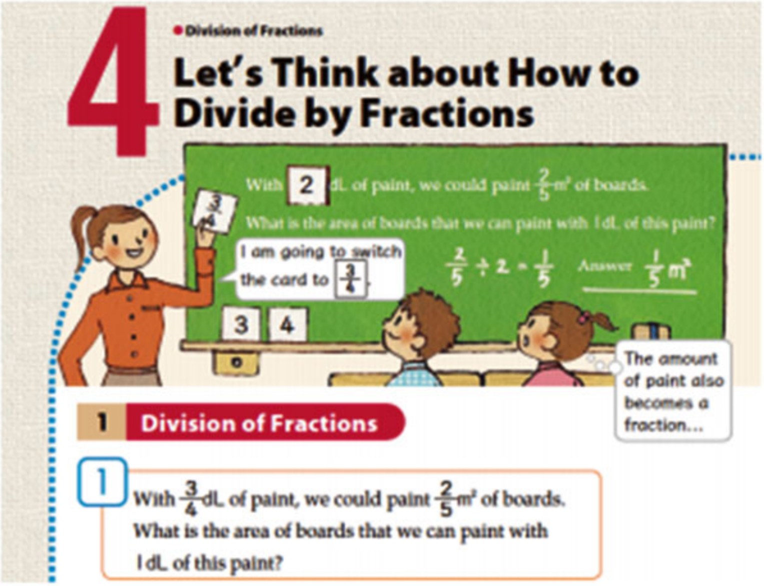


Fig. 2.12 The Japanese series introduces division by fractions using a partitive division situation (Fujii & Iitaka, 2012, Grade 6, p. A34)

the student justifies that the operation to find the missing quantity is $\frac{2}{5} \div \frac{3}{4}$ (see Fig. 2.13).

The use of double-number line diagrams to represent the relationships in a given problem situation is another unique feature of the Japanese textbook series. The diagram is used in all introductory problems as the authors discuss multiplying fractions by whole numbers, dividing fractions by whole numbers, multiplying fractions by fractions, and dividing fractions by fractions. In each instance, the double-number line diagram is used to justify the calculation needed to find the missing quantity. The Japanese series uses a different diagram to consider ways of actually carrying out the calculation. While the Korean and the Taiwanese textbooks use different diagrams to support students' reasoning with multiplication and division of fractions—area diagrams for multiplication and bar diagrams for division—the Japanese series uses a diagram that combines the area model of fractions with a number line (see Fig. 2.14).

Discussion and Implications

The findings discussed above clearly show that there are many similarities among the three Asian textbook series' initial treatment of fractions. In particular, all three series make heavy use of the measure subconstruct and the idea that non-unit fractions are collections of unit fractions. The three series approach addition,

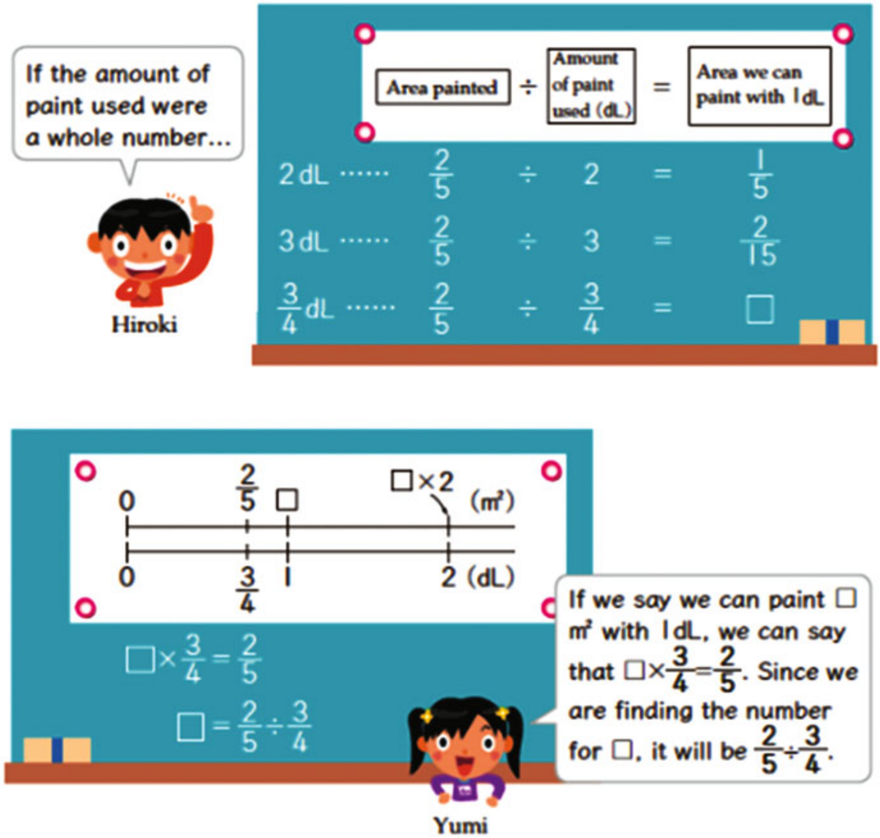


Fig. 2.13 Two ways the Japanese series justifies that the opening problem can be solved by $\frac{2}{5} \div \frac{3}{4}$ (Fujii & Iitaka, 2012, Grade 6, p. A35)

subtraction, and multiplication of fractions by whole numbers, i.e., whole-number groups of fractional quantities, using these tools. Their approach is consistent with Behr et al.'s (1983) hypothesis that the measure subconstruct supports students' development of addition and subtraction with fractions. As stated earlier, Thompson and Saldanha (2003) noted that thinking about non-unit fractions as collections of unit fractions is rare in US textbooks. However, this approach is emphasized in the CCSSM, and our findings support the CCSSM authors' claim that they have used high-achieving Asian curriculum materials as benchmarks.

In regard to addition/subtraction word problem situations, the three Asian curricula generally include simpler situations. It is as though the authors of these curricula attempt to develop the understanding that the operation necessary to answer a problem is determined by the situation and the missing quantity, not by the type of numbers. Once that understanding is achieved, they can then focus on helping students develop ways of calculating sums and differences in a meaningful manner.

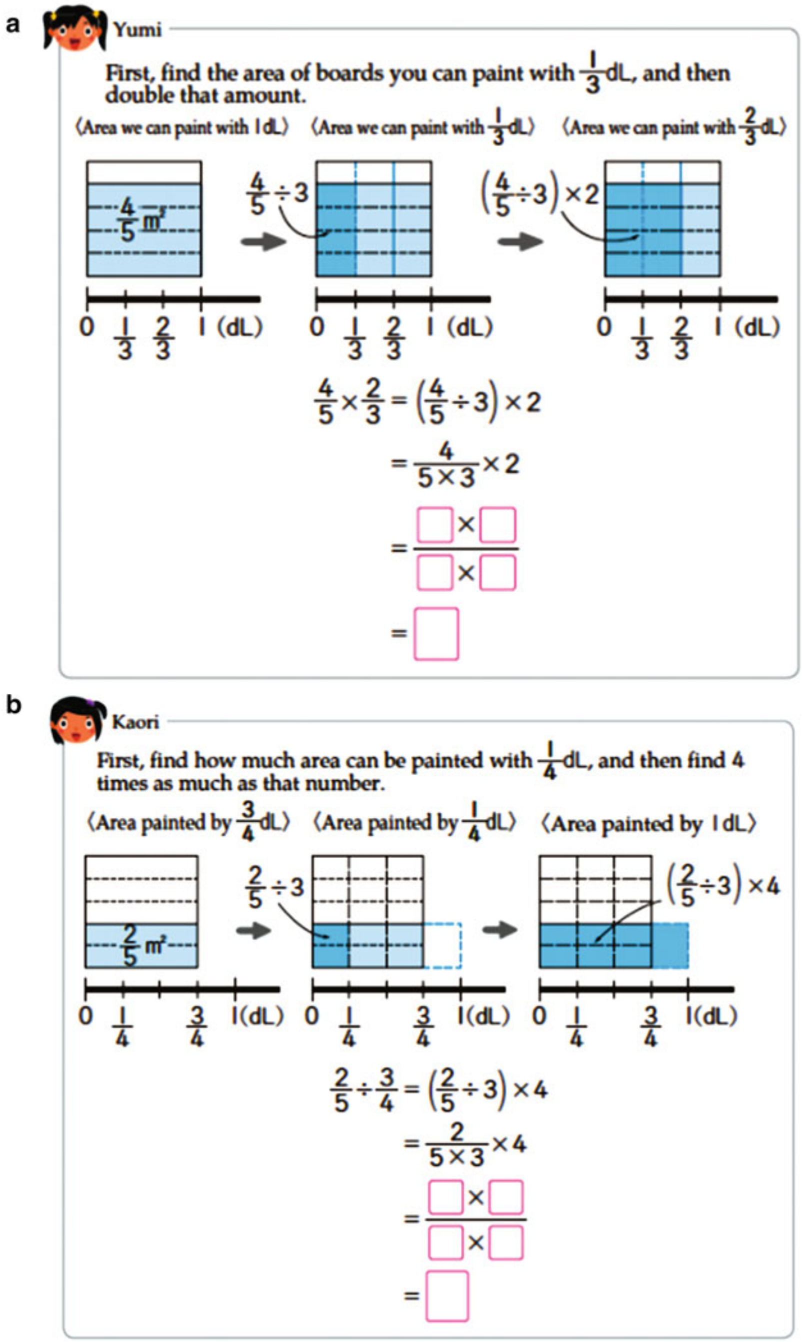


Fig. 2.14 The Japanese series uses a combination of the area model and the number line to illustrate the process of multiplying two fractions (**a**) and dividing a fraction by another fraction (**b**) (Fujii & Iitaka, 2012, Grade 6, p. A 25 & p. A36)

Although the three Asian textbook series' approaches to fractions support some aspects of the model proposed by Behr et al. (1983), they raise questions about other aspects. For example, according to Behr et al., the ratio subconstruct is helpful for developing the idea of equivalence. However, none of the three series incorporate the ratio subconstruct before they discuss equivalent fractions. Of course, this does not mean that the ratio subconstruct is not useful for developing the idea of equivalence. The three Asian textbook series simply show that there are other approaches to helping students develop the idea of equivalent fractions. The Asian models also suggest that the operator subconstruct is helpful for supporting students' development of multiplication. Indeed, both the Korean and the Taiwanese series utilize the operator construct to discuss multiplication by fractions by considering multiplication as an operation to find the fractional amount of the given quantity. However, the Japanese series approaches multiplication by fractions differently. While one justification for multiplication as the appropriate operation uses the idea of multiplicative comparison, the fractions in the problem situations are measured quantities. Further examination of the role the operator subconstruct may play in supporting students' development of multiplication is needed.

Lamon (2007) noted that "Is it better to teach one rational number subconstruct or all five?" and "If one, which should it be?" are two of the remaining researchable questions. As noted already, the three textbook series in the current study do not discuss the ratio subconstruct until the end of the fraction instruction in elementary schools. However, the Korean series seems to introduce the remaining four subconstructs intentionally early, while the Japanese series takes the most deliberate approach. Moreover, although the operator subconstruct plays a key role in the discussion of multiplication by fractions in both the Korean and the Taiwanese series, it is not quite clear what advantages the Korean textbook affords by introducing the subconstruct sooner than the Taiwanese series does. Thus, the current study offers mixed answers to these questions.

Because of the similar approaches taken by these three textbook series to the initial instruction of fractions, the differences in the way multiplication and division are treated are rather surprising. Overall, the approaches in the Korean and the Taiwanese series appear to be similar to US textbook series that are aligned with the CCSSM (Son, Lo, & Watanabe, 2015). However, the Japanese approach is intriguing for a couple of reasons. First, as noted in our findings, the measure subconstruct is a major emphasis of the early fraction instruction in all three Asian textbook series. However, in the Korean and the Taiwanese series, the measure subconstruct does not play a significant role in later instruction. On the other hand, in the Japanese approach, the idea of non-unit fractions being composed of unit fractions plays an important role in explaining the process of multiplication and division (see Fig. 2.14). In the CCSSM, 5.NF.4.a states that "Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.[2]" In the Japanese series, the quotient $q \div b$ is explicitly

[2]In order to match the verbal description, this expression should really be written as $a \times (q \div b)$.

interpreted as the amount corresponding to the unit fraction $\frac{1}{b}$. Moreover, because students must be able to divide fractions by whole numbers if q is a fraction, the Japanese series discusses division of fractions by whole numbers prior to multiplication by fractions.

Another interesting aspect of the Japanese approach is its consistency in problem context and visual representations across multiplication and division. As noted above, the Japanese series uses the same problem context to introduce multiplying and dividing fractions by whole numbers and multiplying and dividing by fractions. The series also uses the same representations—(1) double-number line diagrams to represent the relationships among the quantities in the problem situations, and (2) the combined area model and number line to illustrate the process of calculation. Although they discuss multiple ways to find the results of calculations, one approach involves the same reasoning process—first finding the amount corresponding to the unit fraction of the multiplier or the divisor, and then multiplying the result. These consistencies seem to emphasize the connection between multiplication and division operations, an important mathematical implication of the invert-and-multiply algorithm.

Limitations and Future Research

Because of the connoisseurial nature of textbook analyses, the NRC (2004) recommends that such a study make explicit the identity of those who conduct the analysis. The three researchers who conducted this study are natives of the three Asian countries whose textbooks were examined. As a result, they are fluent in the respective languages. They all received their doctorates in mathematics education from US institutions: Florida State University for the first two authors, and Michigan State University for the third author. Each has experience in content analysis of textbooks (e.g., Cai, Lo, & Watanabe, 2002; Son & Senk, 2010; Watanabe, 2003). The first two authors have also examined teaching and learning of fractions with children (e.g., Lo & Watanabe, 1996). Although none of the researchers are professional mathematicians, the first two authors hold master's degrees in mathematics. Thus, the researchers are well qualified to engage in this study. One limitation of the study, though, is that there is only one native speaker of each of the three Asian languages.

Another limitation of the study is that, for Japan and Taiwan, we examined only one of each country's existing elementary mathematics textbook series. Although each series is the most widely used series in its home country, there are other series. While past studies seem to suggest that textbooks from Japan are very similar (e.g., Li et al., 2009), nevertheless the differences in the way multiplication and division are treated in the three series make us wonder if there are some within-nation differences. Since each country has national standards, the grade placement of a particular topic should be the same in different textbooks. However, how topics within a grade level are ordered and developed can vary. Fujii (personal communication, 2010) noted that if Japanese mathematics education research has not reached a consensus on the teaching and learning of a particular mathematical idea, the ways different Japanese textbook series treat the topic can be different.

In this study, building on the existing studies, we intentionally expanded the scope of our analysis to the treatment of fractions from its introduction to its conclusion at the end of elementary school. We did so in part because we felt the way a particular idea is discussed is influenced by earlier discussion on related topics. Our findings show clear benefits of this expansion. For example, we see that the Korean textbook lays the foundation for multiplication by fractions by introducing the operator subconstruct of fractions in the introductory stage. We also see how the Japanese series utilizes a consistent approach to discuss both multiplication and division by fractions. However, our findings also suggest that it may be important to analyze how other related topics are treated in these textbooks. For example, how are decimal numbers introduced and developed? What are similarities and differences in the ways multiplication and division of decimal numbers are discussed? What about ratios and proportions? Are the ways ratios and proportions are discussed influenced by the ways fractions are treated in the textbooks? Further textbook analyses are definitely needed.

As we noted earlier, the difference in the way multiplication and division of fractions are treated in the three textbook series was a surprise for us. It will be interesting to see how the different emphases these textbook series place may impact students' understanding of fraction multiplication and division in particular. For example, how do Japanese students use visual representations in determining the appropriate operation for a given problem? Would they use a double-number line diagram, as emphasized by the textbook series?

Finally, it should be once again noted that textbooks are only an approximation of the implemented curriculum. They may reflect the image of the ideal implemented curriculum envisioned by the authors. However, it is obvious that teachers may use the same textbook and teach the same lesson very differently. For example, each of the three Asian textbook series includes a number of worked-out examples. However, how these examples are treated in actual classrooms can vary drastically. Some teachers may simply explain an example and assign students the exercise set that follows it. Other teachers may have the students actually tackle the problem on their own and use the worked-out solution only as one of the anticipated solutions by students. Clearly, those classrooms would be experiencing different implemented curricula. Thus, we need to be cautious how we interpret the results from this and other textbook content analyses.

References

Armstrong, B., & Larson, C. (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education, 26*, 2–19.

Ball, D. L. (1996). Connecting to mathematics as a part of teaching to learn. In D. Schifter (Ed.), *What's happening in math class? Reconstructing professional identities* (Vol. 2, pp. 26–45). New York: Teachers College Press.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). New York: Academic Press.

Boonlerts, S., & Inprasitha, M. (2013). The textbook analysis on multiplication: The case of Japan, Singapore and Thailand. *Creative Education, 4*, 259–262.

Cai, J., Lo, J., & Watanabe, T. (2002). Intended treatment of arithmetic average in US and Asian school mathematics textbooks. *School Science and Mathematics, 102*, 391–404.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (1992). Toward a unified discipline of scientific inquiry. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 1–11). NJ: Lawrence Erlbaum Associates.

Charalambous, Y. C., Delaney, S., Hsu, H., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *Mathematical Thinking and Learning, 12*, 117–151.

Common Core State Standard Initiatives (CCSSI) (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math Standards.pdf.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics, 66*, 61–75.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3–17.

Fujii, T., & Iitaka, S. (2011). *Atarashii sansuu*. Tokyo: Tokyo Shoseki Co. Ltd..

Fujii, T., & Iitaka, S. (2012). *Mathematics international*. Tokyo: Tokyo Shoseki Co. Ltd..

Greer, B. (1987). Non-conservation of multiplication and division involving decimals. *Journal for Research in Mathematics Education, 18*, 37–45.

Kang Hsuan Educational Publishing Group. (2012). *Kang Hsuan elementary school mathematics textbooks*. (4A) Tainan, Taiwan: Author.

Kang Hsuan Educational Publishing Group. (2013). *Kang Hsuan elementary school mathematics textbooks*. (3A, 4B, 5A, 6A) Tainan, Taiwan: Author.

Kang Hsuan Educational Publishing Group. (2014). *Kang Hsuan elementary school mathematics textbooks*. (3B, 5B, 6B) Tainan, Taiwan: Author.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144). Columbus, OH: ERIC/SMEAC.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Korean Ministry of Education and Human Resources Development (2014). *Mathematics*. (Grades 3–4) Seoul: DaeHan Printing and Publishing Co., Ltd.

Korean Ministry of Education and Human Resources Development (2015). *Mathematics*. (Grade 5–6) Seoul: DaeHan Printing and Publishing Co., Ltd.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics* *teaching and learning, National Council of Teachers of Mathematics* (pp. 629–668). Charlotte, NC: Information Age Publishing.

Larson, C. N. (1980). Locating proper fractions on number lines: Effect of length and equivalence. *School Science and Mathematics, 80*, 423–428.

Li, Y., Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: The case of fraction division. *ZDM Mathematics Education, 41*, 809–826.

Lo, J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education, 15*, 481–500.

Lo, J., & Watanabe, T. (1996). Developing ratio and proportion schemes: A story of a fifth grader. *Journal for Research in Mathematics Education, 28*, 216–236.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422–441.

Mayer, R. E., Sims, V., & Tajika, H. (1995). A comparison of how textbooks teach mathematical problem solving in Japan and the United States. *American Educational Research Journal, 32*, 443–460.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 International Mathematics Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades*. Chestnut Hill, MA: Boston College.

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 international results in mathematics*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Naigaikyouiku. (2010). *2011 nendo shougakko kyoukasho saitaku joukyou: Monkashou matome (Elementary school textbook market share for the 2011 school year: Summary by the Ministry of Education)*. Tokyo, Japan: Jijitsushinsha.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education.

National Research Council. (2004). On evaluating curricular effectiveness: Judging the quality of K-12 mathematics evaluations. Committee for a Review of the Evaluation Data on the Effectiveness of NSF-Supported and Commercially Generated Mathematics Curriculum Materials. In J. Confrey & V. Stohl (Eds.), *Mathematical Sciences Education Board, Center for Education, Division of Behavioral and Social Sciences and Education*. Washington, DC: The National Academies Press.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*, 279–314.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*, 307–317.

Pothier, Y., & Sawada, D. (1989). Children's interpretation of equality in early fraction activities. *Focus on Learning Problems in Mathematics, 11*(3), 27–38.

Reys, B. J., Reys, R. E., & Chávez, O. (2004). Why mathematics textbooks matter. *Educational Leadership, 61*(5), 61–66.

Schmidt, W. H., McKnight, C. C., Valverde, G., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims: A cross-national investigation of curricular intentions in school mathematics*. Dordrecht, The Netherlands: Kluwer.

Siegler, R. S., & Lortie-Forgues, H. (2015). Conceptual knowledge of fraction arithmetic. *Journal of Educational Psychology, 107*(3), 909–918.

Simon, M. A. (2002). Focusing on key developmental understandings in mathematics. In D. S. Mewborn et al. (Eds.), *Proceedings of the Twenty-Fourth Annual Meeting of the North*

American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), Volume 2 (pp. 991–998). Athens, Georgia: PME-NA.
Son, J., Lo, J., & Watanabe, T. (2015). Intended treatments of fractions, fraction addition and subtraction in mathematics curriculum from Japan, Korea, Taiwan, and US. In T. G. Bartell et al. (Eds.), *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 96–103)*. East Lansing, MI: Michigan State University.
Son, J., & Senk, S. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics, 74*(2), 117–142.
Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior, 20*, 267–307.
Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards in School Mathematics* (pp. 95–113). Reston, VA: National Council of Teachers of Mathematics.
Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30* (4), 390–416.
Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *The Journal of Mathematical Behavior, 23*, 93–114.
Watanabe, T. (2003). Teaching multiplication: An analysis of elementary school mathematics teachers' manuals from Japan and the United States. *The Elementary School Journal, 104*, 111–125.
Zambat, I. O. (2015). An alternative route to teaching fraction division: Abstraction of common denominator algorithm. *International Electronic Journal of Elementary Education, 7*(3), 399–422.