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Vladik Kreinovich *Editor*

Uncertainty Modeling

Dedicated to Professor Boris Kovalerchuk on his Anniversary

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Preface

This volume is devoted to the 65th birthday of Dr. Boris Kovalerchuk. Dr. Kovalerchuk's results cover many research areas. Many of these areas are reflected in this volume.

In this preface, I would like to emphasize his contributions to research areas which are the closest to my own research: data processing under uncertainty, especially *fuzzy* data processing, when uncertainty comes from the imprecision of expert opinions.

Fuzzy research area: successes and challenges. Fuzzy techniques have many successful practical applications, especially in intelligent control, where expert knowledge—originally formulated in terms of imprecise (fuzzy) words from natural language—is successfully translated into a computer-understandable form and then used in automated decision making.

However, there are still many applications problems (and even whole application areas) where, at present, we are not that successful in formalizing and using imprecise expert knowledge. To be able to use this knowledge, we must overcome several important challenges. In all these challenges, Dr. Kovalerchuk plays an important role as a research leader.

First challenge: need to select appropriate techniques. The first challenge is that, in contrast to (more traditional) probabilistic methods—which are based on solid foundations—many fuzzy techniques are, by nature, heuristic.

There are usually many ways to translate imprecise expert knowledge into precise terms, and the success of an application often depends on selecting the most adequate translation. To be able to select such a translation, we need to have a general description of all possible translations and ways to deal with them. This activity is known as *foundations* of fuzzy techniques.

This is a very complex area of research, an area that requires deep knowledge of mathematics, computer science, foundations and philosophy of science, and—since the ultimate goal is applications—a good understanding of many application areas.

Boris has all these skills, and he has used them successfully in his numerous seminal papers on fuzzy foundations. His papers appeared as chapters in the Springer book series “Studies in Fuzziness and Soft Computing” (see, e.g., [1, 2]; one of the first was his 1994 chapter [1] devoted to the difficult problem of optimization of an uncertain (fuzzy) objective function under uncertain (fuzzy) constraints.

Second challenge: need to combine fuzzy and probabilistic techniques. The second major challenge is related to the fact that, in addition to *subjective* expert knowledge, we also have measurement-based *objective* information about the corresponding system, information usually formulated in probabilistic terms. To solve the corresponding practical problems, we need to adequately combine fuzzy and probabilistic uncertainty. Here, we face two problems:

- a *foundational* problem—which is the best way to combine these two types of uncertainty?—and
- a *communication* problem, caused by the fact that the two communities are not very familiar with each other’s research and, as a result, have misunderstandings about the other research areas, misunderstandings that prevent successful collaboration.

Boris is one of the main research leaders in solving both these problems.

He has published several seminal papers on selecting the best way of combining these two types of knowledge; see, e.g., [3, 4]; I would like to specifically mention his 2012 Springer chapter [2].

He has also done a great job of describing probability ideas to fuzzy community and fuzzy ideas to probability researchers, in particular, by showing that—contrary to the widely spread misunderstanding—fuzzy-related techniques do not violate the main idea of probability, and moreover, many such fuzzy techniques can be meaningfully reformulated (and explained) in probabilistic terms.

In particular, he has shown that many real-life applications of fuzzy techniques can actually be reformulated in probabilistic terms—and that the combination of such reformulated terms and traditional probabilistic techniques can enhance the probabilistic approach. He has also shown that a seeming inconsistency between fuzzy methods (based on t-norms) and probabilistic approach can be resolved within a new formalism that Boris called Exact Complete Context Spaces (ECCS). His series of publications starting with his 1994 paper [5], in which he showed that exact complete context spaces link fuzzy logic and probability theory in a new rigorous way. Specifically, he has shown how the use of ECCS can explain numerous successes of fuzzy control in application; this was the main topic of his 1996 paper [6] that was welcomed by Lotfi Zadeh. This work had been expanded in his other publications published in the proceedings of the IEEE World Congresses on Computational Intelligence WCCI’2008–2012, International Conferences on Information Processing and Management of Uncertainty in Knowledge-Based Systems IPMU’2012–2014, World Congress of International Fuzzy Systems

Association IFSA/NAFIPS'2013, and in several seminal Springer book chapters published in 2012 and 2013; see also [7, 8, 9].

Third challenge: dynamic character of human reasoning. The third challenge is that, in contrast to the objective knowledge, which, once established, remains stable, subjective knowledge changes with time, it is dynamic: an expert may learn new things and/or realize that some of his/her previous opinions were imprecise or even incorrect. To make applications of expert knowledge more adequate, we need to take into account the dynamic nature of human reasoning. This is a very difficult task.

In solving this task, Boris was one of the pioneers. With Leonid Perlovsky and Gregory Wheeler, he established a formal mechanism for modeling such dynamic character, a mechanism that they called Dynamic Logic of Phenomena. This is an approach to solve real-world tasks via a dynamic process of synchronous adapting the task and the solution criteria when both are uncertain. Boris started this research under the grant from the US National Research Council (NRC) when he was working at the US Air Force Research laboratory in 2007–2008. His main results are overviewed in his seminal 2012 paper published in a prestigious Journal of Applied Non-Classical Logics [10].

Fourth challenge: dealing with (somewhat) inconsistent expert knowledge. The fourth challenge is that, due to imprecision of expert reasoning, some of the expert statements are, strictly speaking, contradictory to one another. It is desirable to be able to deal with such seemingly inconsistent knowledge. The logic of such inconsistent knowledge bases is known as *paraconsistent logic*. This a very active and a very difficult area of research, so difficult that at present, it has are very few applications to real-life situations, and most of these applications only deal with "crisp" (non-fuzzy) expert statements.

In his pioneer 2006–2010 joint research with Germano Resconi, Boris developed a theory of *irrational* (=inconsistent) *agents*, a theory that combined fuzzy logic, probability theory, and paraconsistent logic into a general techniques for handling both rational and irrational agents [11, 12, 13–20].

Fifth challenge: translating computer results into human-understandable form. The fifth major challenge is related to the fact that, in contrast to fuzzy control where often a decision needs to be made urgently and thus, has to be automated, in many other application areas—e.g., in many cases of medical diagnostics—there is no such hurry. So, it is desirable to first show the resulting computer-generated decision proposal to an expert, to make sure that the automated system properly took all the experts' knowledge into account. To be able to do that, we face a problem which is reverse to the above-mentioned translation problem underlying fuzzy techniques—a problem of how to better translate numerical results of the computer data processing into expert-understandable form. There are two ways we humans get the information:

- in terms of words, and
- in terms of pictures.

Thus, we need to translate the computer results both into words and into pictures. On both tasks, Boris did a pioneer work.

The question of translating computer results into words is handled in Boris's publications on *interpretability* of fuzzy operations. Not only he analyzed this problem theoretically, he also proposed and conducted empirical studies that established the scope of applicability of different "and"-operations (=t-norms) of fuzzy logic. This work was published in Fuzzy Sets and Systems—the main journal of our community—in Elsevier's Journal of General Systems [21], in proceedings of IEEE WCCI'2010–2012 [22], and in many other places (see, e.g., [23]).

In terms of visualization, Boris is a recognized expert in analytical and visual data mining, and in visual analytics. He has published two related books: *Data Mining in Finance* [24] and *Visual and Spatial Analysis* [25]. Most recently (2014) Boris published a series of four conference papers (jointly with his colleague Vladimir Grishin) on lossless visualization of multi-D data in 2-D; see, e.g., [26, 27].

This is an interesting new development, with a potential for a breakthrough in the critical area of big data research. Boris introduced new concepts of collocated paired coordinates and general line coordinates that dramatically expand the scope of lossless multi-D data visualizations [1, 27, 28].

Need for applications. Finally, once all these challenges are resolved, it is important to actively pursue new applications of the corresponding techniques. Boris has many application papers, ranging:

- from applications to medicine, including breast cancer diagnostics [29, 30];
- to finance [24]
- to geospatial analysis—in a series of SPIE publications during the last 10 years; see, e.g., [31–33], and in [34];
- to efficient applications of his new visualization techniques to World Hunger data analysis and the Challenger disaster.

Dr. Kovalerchuk is a world-renowned researcher. All this research activity has made Boris Kovalerchuk a world-renowned expert in systems and uncertainty modeling.

For example, in 2012, he was invited to present a 3-h tutorial on Fuzzy Logic, Probability, and Measurement for Computing with Words at the IEEE World Congress on Computational Intelligence WCCI'2012.

Service to the research community. In addition to doing research, Boris is also very active in the fuzzy research community. He regularly posts short tutorials and opinions on the relation between possibility and probability to the Berkeley

Initiative Soft Computing (BISC) mailing list, often at the explicit invitation of Dr. Zadeh himself.

He also makes an important contribution to conferences. He chaired two Computational Intelligence Conferences [35, 36]. In 2015, he serves as a technical co-chair of the North American Fuzzy Information Processing Society (NAFIPS) Conference to be held in Redmond, Washington (August 2015). At the IEEE Symposium on Computational Intelligence for Security and Defense Applications, CISDA (New York State, May 2015), he organized and mediated a panel of leading experts from multiple organizations including DARPA on Current Challenges of Computational Intelligence in Defense and Security.

Conclusion. Dr. Boris Kovalerchuk is an excellent well-recognized world-level researcher in the area of fuzzy techniques and uncertainty modeling in general, he is one of the leaders in this research area. We wish him happy birthday and many many more interesting research results!

El Paso, Texas, USA Vladik Kreinovich

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Contents

MapReduce: From Elementary Circuits to Cloud

Răzvan Andonie, Mihaela Maliţa, and Gheorghe M. Ştefan

Abstract We regard the MapReduce mechanism as a unifying principle in the domain of computer science. Going back to the roots of AI and circuits, we show that the MapReduce mechanism is consistent with the basic mechanisms acting at all the levels, from circuits to Hadoop. At the circuit level, the elementary circuit is the smallest and simplest MapReduce circuit—the elementary multiplexer. On the structural and informational chain, starting from circuits and up to Big Data processing, we have the same behavioral pattern: the MapReduce basic rule. For a unified parallel computing perspective, we propose a novel starting point: Kleene's partial recursive functions model. In this model, the composition rule is a true MapReduce mechanism. The functional forms, in the functional programming paradigm defined by Backus, are also MapReduce type actions. We propose an abstract model for parallel engines which embodies various forms of MapReduce. These engines are represented as a hierarchy of recursive MapReduce modules. Finally, we claim that the MapReduce paradigm is ubiquitous, at all computational levels.

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1 Introduction

MapReduce is a programming framework invented by engineers at Google [4] and used to simplify data processing across massive data sets. Beside Googles' implementation, a very popular MapReduce platform is Hadoop [16], used by companies like Yahoo!, Facebook, and New York Times. Other MapReduce implementations are Disco [5], MapReduce-MPI [6], and Phoenix [14].

In MapReduce, users specify a *map* function that processes a key/value pair to generate a set of intermediate key/value pairs, and a *reduce* function that merges all intermediate values associated with the same intermediate key. Computational processing can occur on data stored either in a file system (unstructured) or within a database (structured). MapReduce operates only at a higher level: the programmer thinks in terms of functions of key and value pairs, and the data flow is implicit. The two fundamental functions of a MapReduce query are[1]:

"Map" function: The master node takes the input, chops it up into smaller subproblems, and distributes those to worker nodes. A worker node may do this again in turn, leading to a multi-level tree structure. The worker node processes that smaller problem, and passes the answer back to its master node.

"Reduce" function: The master node then takes the answers to all the sub-problems and combines them in a way to get the output—the answer to the problem it was originally trying to solve.

Programs written in this functional style are automatically parallelized and executed on a large cluster of commodity machines (e.g., on a computer cloud). The run-time system takes care of the details of partitioning the input data, scheduling the program's execution across a set of machines, handling machine failures, and managing the required inter-machine communication [4]. This allows programmers to easily utilize the resources of a large distributed system.

Why is MapReduce so popular today? One simple reason would be that it comes from Google. Well, this is not enough, and we have to look closer at what the problems are with Big Data storage and analysis. Yes, we know, Big Data is here. It is difficult to measure the total volume of data stored electronically. We can estimate that this data is in the order of one disk drive for every person in the world [16]. The bad news is that we are struggling to access fast Big Data:

- While the storage capacity of hard drives has increased massively over the years, access speed has not kept up. On one typical drive from 1990 you could read all the data from the full drive in about five minutes. Presently, it takes more than two and a half hours to read all the data from a Tera byte drive. Access speed depends on the transfer rate, which corresponds to a disk's bandwidth.
- Seeking time improves more slowly than transfer rate (seeking time characterizes the latency of a disk operation, i.e., the delay introduced by the rotation of the disk to bring the required disk sector).

[1] http://www.mapreduce.org/what-is-mapreduce.php.

The problem is that we expect to keep data access time constant (if not to reduce), whereas the data volume is dramatically increasing. MapReduce solves this problem by collocating data with computer code, so data access is fast since it is local. This way, MapReduce tries to avoid the data access bottleneck. MapReduce is a good answer to the execution time concern, being a linearly scalable model. If you also double the size of the input data, a job will run twice as slow, whereas if you double the size of the cluster, a job will run as fast as the original one.

Even if MapReduce looks revolutionary and Google has been granted a patent, there are authors who consider that MapReduce is too similar to existing products.[2] What is perhaps missing in this debate is the connection between the principles of MapReduce and old results from functional and distributed computing. It may be surprising to see how old ideas, put in a new framework (Big Data in this case), can be rediscovered and made practical. Let us consider the following example.

Here are two of the most used functional forms written in Scheme/Lisp:

```
(define (myMap func list)
 (cond ((null? list) ())
       (#t (cons (func (car list))
                (myMap func (cdr list))))
 )
)

(define (myReduce binaryOp list)
 (cond ((null? (cdr list)) (car list))
       (#t (binaryOp (car list)
                    (myReduce binaryOp (cdr list))))
 )
)
```

The first form maps the function `func` over a list of elements we call `list`. It is a recursive definition which takes, by turn, each element of the list `list` as argument for the function `func`. The selection is done using the Lisp function `car`, which selects the first element from a list. The selected element is extracted from the list using another Lisp function, `cdr`, which returns what remains from `list` after removing its first element. The final result is provided recursively, step by step, using the Lisp function `cons`, which builds the list of results attaching in front of the shortening list of arguments the value of `(func (car list))`. The process ends when the Lisp conditional function, `cond`, evaluates `list` to the empty list, `()`. For example:

```
(myMap inc '(3 5 8)) -> (4 6 9)
```

The second form reduces the list `list` to an atom by repeated application of the binary function `binaryOp`. It is also a recursive application which stops when

[2] http://www.dbms2.com/2010/02/11/google-MapReduce-patent/.

`list` contains just one element. The result occurs, step by step, applying by turn the binary operation `binaryOp` to the first element, `(car(list))`, and `binaryOp` applied on the rest, `(cdr(list))`. For example:

```
(myReduce + '(3 5 8)) -> 16
```

In MapReduce, the combination of these two forms is considered the fundamental programming model for big and complex data processing. This observation gave us the motivation for our work.

Actually, we claim more: MapReduce is a key concept, from elementary digital circuits to the main tasks performed in the cloud, going through the (parallel) computing model proposed by Stephen Kleene, the functional languages envisaged by Alonzo Church (Lisp, the first to be considered), and the one-chip parallel architectures.

We do not aim to question the originality of Google's patent, or the practical value of MapReduce and its implementations. But going back to the roots of AI and circuits, we aim to show that the MapReduce mechanism is consistent with the basic mechanisms acting at all the levels, starting from circuits to Hadoop. On the structural and informational chain, starting from circuits to Big Data processing, we recognize the same MapReduce behavioral pattern. We discover a unifying principle which guarantees a natural structuring process, instead of an *ad hoc* way of scaling up data processing systems. David Patterson offered us a vivid image about what is meant by an *ad hoc* solution, when, expressing serious concerns about one-chip parallel computing, he wrote [10]:

> The semiconductor industry threw the equivalent of a Hail Mary pass when it switched from making microprocessors run faster to putting more of them on a chip – doing so without any clear notion of how such devices would in general be programmed.

We start by looking at the roots of the MapReduce mechanisms in Kleene's model of computation, and describing the main theoretical models related to MapReduce. We will refer both to the recursive function level and to the computer architecture level (circuits and cloud computing). We discover the MapReduce mechanism as a basic module in circuits. We present the abstract model of parallelism using five particular forms of Kleene's composition, and we define the corresponding generic parallel computational engines. These engines will be represented as a hierarchy of recursive MapReduce modules. At the highest level of abstraction, we analyze the MapReduce mechanism in the cloud computing context. In the Concluding Remarks, we synthesize the results of our study.

2 Theoretical Models

Computation starts with *mathematical models of computation*. For sequential computation, Turing's model worked very well because of its simplicity and an almost direct correspondence with the first actual embodiments. Indeed, the distance from Turing Machine to von Neumann's abstract model is short. For parallel computation, we propose a novel starting point: Stephan Kleene's partial recursive functions model [7]. This model is directly supported by circuits and it supports the functional programming style.

2.1 A Mathematical Model of Computation

Kleene's model consists of three basic functions (zero, increment, selection) and three rules (composition, primitive recursivity, minimalization). In [8, 13], we proved that primitive recursivity and minimalization are special forms of composition. Therefore, only the composition rule should be considered.

Composition captures directly the process of computing a number of $p + 1$ functions by:

$$f(x_1, \dots, x_n) = \boldsymbol{g}(h_1(x_1, \dots, x_n), \dots, h_p(x_1, \dots, x_n))$$

where, at the first level, we have functions $h_1, \dots, h_p$ and one p-variable reduction function $\boldsymbol{g}$. Function $\boldsymbol{g}$ (the reduction function) is sometimes implementable as a $(\log\ p)$-depth binary tree of $p - 1$ functions.

The circuit structure associated to the composition rule is presented in Fig. 1. At the first level, functions h_i $(i = 1, 2 \dots, p)$ are processed in parallel. The reduction function is computed at the second level. In the general case, the input variables are sent to all the p cells processing functions h_i, and the resulting vector $\{h_1(x_1, \dots, x_n), \dots, h_p(x_1, \dots, x_n)\}$ is reduced by $\boldsymbol{g}$ to a scalar.

The modules used in this representation can be implemented in various ways, from circuits to programmable structures. The first level implements a *synchronous parallelism*, while between the two levels there is a pipelined, *diachronic parallelism*. In other words, the first level represents the *map* step and the second level the *reduce* step.

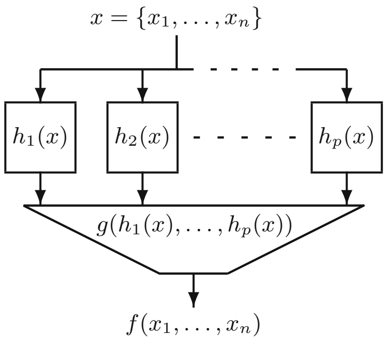

Fig. 1 The physical embodiment of the composition rule. The *map* step performs a synchronous parallel computation, while the *reduce* step is diachronic parallel with the *map* step

Thus, the mathematical model of computation proposed by Kleene in 1936 is an intrinsic model for parallel computing, and we consider it the starting point for defining the MapReduce parallel computation.

2.2 Circuits

Behind Kleene's model, we have the circuit level. Circuits can be used to implement any h_i or $\boldsymbol{g}$ function. Using Kleene's definition, the three basic functions (zero, increment, projection) can be reduced to one, the elementary selection:

$$s(i, x_1, x_0) = x_i$$

$i \in \{0, 1\}$, because any Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ can be recursively defined by the Boolean expression:

$$f(x_{n-1}, \ldots, x_0) = x_{n-1} \cdot G + x'_{n-1} \cdot H$$

where:

$$G = \boldsymbol{g}(x_{n-2}, \ldots, x_0) = f(1, x_{n-2}, \ldots, x_0)$$

$$H = h(x_{n-2}, \ldots, x_0) = f(0, x_{n-2}, \ldots, x_0)$$

Therefore, any circuit can be seen as a tree of basic circuits corresponding to the elementary `if-then-else` control structure. Indeed, if

$$out = \textbf{if } (sel = 1) \textbf{ then } in1 \textbf{ else } in0$$

is performed for the one bit variables *out, sel, in1, in0*, then the associated circuit is the ***elementary multiplexer*** expressed in Boolean form as:

$$O = S \cdot I_1 + S' \cdot I_0$$

For $I_0 = 1$ and $I_1 = 0$, the circuit performs the NOT function, i.e., $O = S'$. For $I_0 = 0$, the circuit performs the AND function, i.e., $O = S \cdot I_1$. Thus, we are back to Boolean algebra.

Figure 2a displays the elementary multiplexer. In the world of digital systems, this is the simplest and smallest form of MapReduce.

If we connect the output of this multiplexer to one of the selected inputs, we obtain the basic memory circuit. If $O = I_0$ (see Fig. 2b) and the clock (applied on the selection input) switches from 1 to 0 the value applied on the input I_1 is latched into the circuit, i.e., it is stored, maintained to the output of the circuit as long as the clock stays on 0. In a complementary configuration, if $O = I_1$, the input I_0 is latched when the clock switches from 0 to 1.

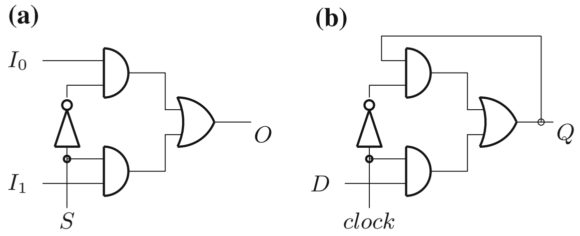

Fig. 2 **a** The elementary multiplexer: the first level of ANDs is the *map* step, while the OR circuit makes the reduction. **b** The elementary storage element: the loop closed over the elementary multiplexer provides the 1-bit clocked latch

All digital circuits, from the simplest to the most complex programmable systems, are build on the MapReduce elementary structure formally described as an *if-then-else*. Hence, the basic "brick" used in the circuit domain is a MapReduce circuit.

2.3 Programming Style

The lambda calculus was introduced by Church [2] as part of an investigation into the foundations of mathematics. Together with Kleene's partial recursive function model, it has significantly contributed to the development of the Lisp programming language. In our opinion, the most important contribution for the extension of functional programming to parallel systems was provided by Backus [1], much later: the *functional forms*.

Functional forms are based on functions which map objects into objects, where an object can be [1]:

- an atom, x; special atoms are: T (true), F (false), ϕ (empty sequence)
- a sequence of objects, $\langle x_1, \ldots, x_p \rangle$, where x_i are atoms or sequences
- an undefined object $\perp$

The set of functions contains:

- primitive functions which manage:
  - atoms, using functions defined on constant length sequences of atoms, returning constant length sequence of atoms
  - p-length sequences, where p is the number of computational cells
- functional forms for:
  - expanding to sequences the functions defined on atoms
  - defining new functions
- definitions—the programming tool used for developing applications.

A functional form is made of functions which are applied to objects. They are used to define complex functions starting from the set of primitive functions.

We are interested in the following functional forms:

- **Apply to all**: the same function is applied to all elements of the sequence.

$$\alpha f : x \equiv (x = \langle x_1, \ldots, x_p \rangle) \to \langle f : x_1, \ldots, f : x_p \rangle$$

- **Insert**: the function f has as argument a sequence of objects and returns an object. Its recursive form is:

$$/f : x \equiv ((x = \langle x_1, \ldots, x_p \rangle) \,\&\, (p \geq 2)) \to f : \langle x_1, /f : \langle x_2, \ldots, x_p \rangle \rangle$$

  The resulting action looks like a sequential process executed in $O(p)$ cycles, but it can be executed as a reduction function in $O(\log\ p)$ steps.
- **Construction**: the same argument is used by a sequence of functions.

$$[f_1, \ldots, f_n] : x \equiv \langle f_1 : x, \ldots, f_n : x \rangle$$

- **Threaded construction**: is a special case of construction, when $f_i = \boldsymbol{g}_i \circ i$

$$\theta[f_1, \ldots, f_p] : x \equiv (x = \langle x_1, \ldots, x_p \rangle) \to \langle \boldsymbol{g}_1 : x_1, \ldots, \boldsymbol{g}_p : x_p \rangle$$

  where: $\boldsymbol{g}_i : x_i$ represents an independent thread.
- **Composition**: a pile of functions is applied to a stream of objects. By definition:

$$(f_q \circ f_{q-1} \circ \ldots \circ f_1) : x \equiv f_q : (f_{q-1} : (f_{q-2} : (\ldots : (f_1 : x) \ldots)))$$

  This form is a pipelined computation if a ***stream*** of objects, $|x_n, \ldots, x_1|$, is inserted into a pipe of cells, starting with x_1. In this case, each two successive cells will perform:

$$f_i(f_{i-1} : (f_{i-2} : (\ldots : (f_1 : x_j) \ldots)))$$

$$f_{i+1}(f_i : (f_{i-1} : (\ldots : (f_1 : x_{j-1}) \ldots)))$$

The functional forms `apply to all`, `construction`, and `threaded construction` are obviously map-type functions. The function `composition` results by serially connecting map-type functions. The function `insert` is a reduction function. Consequently, Backus's formalism for functional form uses predominantly MapReduce mechanisms.

3 MapReduce Architectures

In this section, we generate abstract models for computing engines and recursive structures, especially for solving Big Data problems. We start our construction from

Kleene's parallel computation model and from Backus' functional forms. We also look at the MapReduce paradigm in the context of cloud computing.

3.1 Abstract Model for Parallel Engines

The computational model defined by Turing was followed in the mid 1940s by the von Neumann [15] and Harvard [3] architectures. Kleene's formalism is also an abstract computational model [7]. Inspired by Kleene's model, in a previous work [9], we have generated five particular composition forms which correspond to the main abstract parallel engines:

1. *Data-parallel*
   If we consider $h_i(x_1, \dots, x_p) = h(x_i)$ and $\boldsymbol{g}(y_1, \dots, y_p) = \{y_1, \dots, y_p\}$, then
   $$f(x_1, \dots, x_p) = \{h(x_1), \dots, h(x_p)\}$$
   where $x_i = \{x_{i1}, \dots, x_{im}\}$ are data sequences. This composition form corresponds to Backus' `apply to all` functional form.
2. *Reduction-parallel*
   If $h_i(x_i) = x_i$, then the general form becomes:
   $$f(x_1, \dots, x_p) = \boldsymbol{g}(x_1, \dots, x_p)$$
   which operates a reduction from vector(s) to scalar(s). This corresponds to Backus' `insert` functional form.
3. *Speculative-parallel*
   If the functionally different cells – h_i – receive the same input variable, while the reduction performs the identity function, $\boldsymbol{g}(y_1, \dots, y_p) = \{y_1, \dots, y_p\}$, then,
   $$f(x) = \{h_1(x), \dots, h_p(x)\}$$
   where: x is a sequence of data. This corresponds to Backus' `construction` functional form.
4. *Time-parallel*
   For the special case when $p = 1$, $f(x) = \boldsymbol{g}(h(x))$. Here we have no synchronous parallelism. Only the pipelined, diachronic parallelism is possible, if in each "cycle" we have a new input value. Many applications of type $f(x) = \boldsymbol{g}(h(x))$ result in the m-level *"pipe"* of functions:
   $$f(x) = f_m(f_{m-1}(\dots f_1(x)\dots))$$
   where x is an element in the stream of data. This corresponds to the `composition` functional form.

5. *Thread-parallel*
   If $h_i(x_1, \ldots, x_n) = h_i(x_i)$ and $\boldsymbol{g}(h_1, \ldots, h_p) = \{h_1, \ldots, h_p\}$, then the general composition form reduces to

   $$f(x_1, \ldots, x_p) = \{h_1(x_1), \ldots, h_p(x_p)\}$$

   where x_i is a data sequence. Each $h_i(x_i)$ represents a distinct and independent *thread*. This corresponds to the `threaded construction` functional form.

There is a synergic relation between the five abstract parallel engines resulting from Kleene's model and the functional forms proposed by Backus:

Kleene's parallelism	↔	**Backus' functional forms**
data-parallel	↔	`apply to all`
reduction-parallel	↔	`insert`
speculative-parallel	↔	`construction`
time-parallel	↔	`composition`
thread-parallel	↔	`threaded construction`

The MapReduce oriented functional forms find their correspondents in the particular forms of Kleene's MapReduce shaped rule.

3.2 Recursive Structures

The above MapReduce generic parallel engines can be integrated as a recursive hierarchy of MapReduce cells (see Fig. 3), where **cell** consists of **engine** & **memory**. At the lowest level[3] in the hierarchy, for example, we can have:

- **engine**: consists of the **MAP** array of many cells and the **REDUCTION** network loop coupled through CONTROL, with:
  - engine: an execution/processing unit of 8–64 bits
  - memory: a 2–8 KB of static RAM
  - CONTROL: a 32-bit sequential processor
- **memory**: is MEMORY, a 1–4 GB of dynamic RAM (sometimes expanded in a 1–4 TB hard disc)

The above described **cell**, let us call it $cell_1$, can be used to build recursively the next level, $cell_2$, with the same organization but based on $cell_1$ cells. And so on.

In the MAP module, the cellular structure has the simplest interconnections network: a linearly connected network. Each cell contains a local engine and a local memory. The REDUCTION module is a *log*-depth network. At the lowest level, the *log*-depth reduction network contains simple circuits (adders, comparators, logic

[3] The lowest level of the generic engine was implemented as BA1024 SoC for HDTV applications [12].

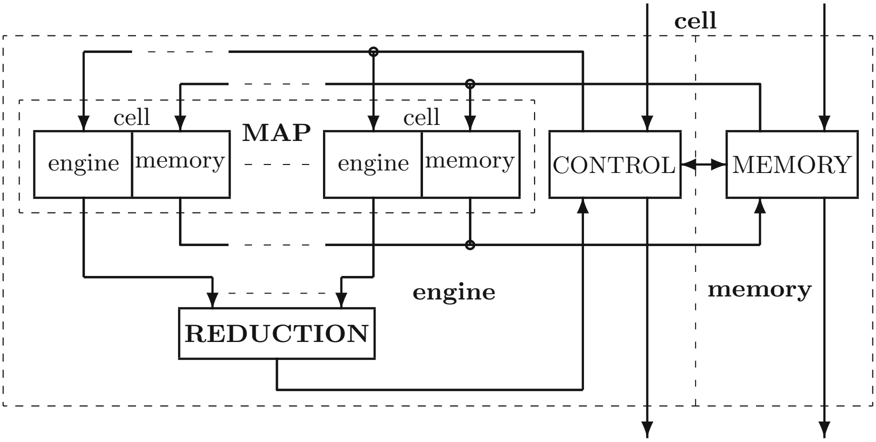

Fig. 3 The recursive definition of the hierarchy of generic MapReduce engines. At each level of this hierarchy, the MAP stage is a linear array of cells, while the REDUCTION stage is a *log*-depth network of circuits

units), while at the highest level, the network is used almost exclusively for communication.

The MapReduce communication pattern allows an easy coherence maintenance of the MEMORY content. The local memory in each $cell_i$ provides the premises for a very high global bandwidth between the storage support and the processing support.

The recursiveness of the structure allows the uniformity of the associated programming environment. Let it be of the form suggested in the first section, where a Lisp-like functional language was used to introduce the map and reduction operations.

3.3 MapReduce and the Cloud

MapReduce programs prove to be very useful for processing big data in parallel. This is performed by dividing the workload across a large number of processing nodes which are not allowed to share data arbitrarily. This feature is one of the explanation of the scalability of a MapReduce application: the communication overhead required to keep the data on the nodes synchronized at all times would prevent the system from performing efficiently on a large scale. In other words, all data elements in MapReduce are immutable, meaning that they cannot be updated. There are several benefits of MapReduce over conventional data processing techniques:

- The model is easy to use, even for programmers without experience with distributed systems, since it hides the implementation details.

- A large variety of problems are easily expressible as MapReduce computations. For example, MapReduce is used for the generation of data for Google's production web search service, for sorting, for data mining, for machine learning, and many other systems.
- MapReduce enables scaling of applications across large clusters of machines comprising thousands of nodes, with fault-tolerance built-in for ultra-fast performance.

The popularity of MapReduce is deeply connected to a very "hot" distributed computer architecture: *the cloud*. MapReduce usually runs on computer clouds and is highly scalable: a typical MapReduce computation processes many Tera bytes of data on thousands of machines. Actually, MapReduce has become the weapon of choice for data-intensive analyses in the cloud and in commodity clusters due to its excellent fault tolerance features, scalability and the ease of use.

In the simplest terms, cloud computing means storing and accessing data and programs over the Internet instead of your computer's hard drive. Cloud computing is the next stage in the Internet's evolution, providing the means through which everything from computing power to computing infrastructure, applications, business processes to personal collaboration can be delivered to you as a service wherever and whenever you need. Cloud computing groups together large numbers of commodity hardware servers and other resources to offer their combined capacity on an on-demand, pay-as-you-go basis. The users of a cloud have no idea where the servers are physically located and can start working with their applications.

Like with MapReduce, the primary concept behind cloud computing is not a new idea. John McCarthy, in the 1960s, imagined that processing amenities is going to be supplied to everyone just like a utility (see [11]). The cloud computing concept is motivated by latest data demands as the data stored on web is increasing drastically in recent times. Combined with MapReduce, the cloud architecture attempts to minimize (and make transparent) the communication bottlenecks between cells and between cells and the system memory.

4 Concluding Remarks

Almost everyone has heard of Google's MapReduce framework which executes on a computer cloud, but few have ever hacked around with the general idea of map and reduce. In our paper, we looked at the fundamental structure of the map and reduce operations from the perspective of recursive functions and computability. This took us back to the foundations of AI and digital circuits.

We can synthesize now the results of our study:

In MapReduce, all is composition We have identified the *MapReduce chain* with six meaningful stages:

- Circuits—for both, combinational and sequential circuits, the basic "brick" is the elementary multiplexer: one of the simplest and smallest MapReduce structure.

- Mathematical model—the composition rule in Kleene's model (the only independent one) is MapReduce.
- Programming style—the functional forms in Backus's approach are MapReduce-type forms.
- Abstract machine model—the five forms of parallelism (data-, reduction-, speculative-, time-, threaded-parallelism) are synergic with the functional forms of Backus and follow the MapReduce mechanism.
- Hierarchical generic parallel structures—represent a direct recursive embodiment of the MapReduce mechanism.
- Cloud computing—is an optimal distributed architecture for MapReduce operations.

MapReduce is the only true parallel paradigm According to our study, this is the only parallel mechanism which solves data synchronization overhead using explicit local buffer management. It is true, MapReduce uses an embarrassingly simple parallelization (a problem that is obviously decomposable into many identical, but separate subtasks is called embarrassingly parallel) and not all problems can be easily parallelized this way.

The programming language matches the structure The composition, embodied in the MapReduce mechanism, generates both the language and the physical structure. It is more "natural" than in the Turing triggered approach, where the language and the engine emerge in two distinct stages: first the engine and then the idea of language as a symbolic entity.

The match between language & structure is maximal at the lowest level in the recursive hierarchy: the one-chip parallel engine level For this reason the power and area performances are in the order of hundred GOPS/Watt[4] and tens GOPS/mm^2 [12].

We claim that the MapReduce chain works as a spinal column in computer science, providing the ultimate coherence for this apparently so heterogeneous domain. We have to observe that the standards for connecting the computer systems and the software needed to make cloud computing work are not fully defined at present time, leaving many companies to define their own cloud computing technologies. Our approach may be useful in this direction.

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[4] GOPS stands for Giga Operations Per Second.

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On the Helmholtz Principle for Data Mining

Alexander Balinsky, Helen Balinsky, and Steven Simske

Abstract Keyword and feature extraction is a fundamental problem in text data mining and document processing. A majority of document processing applications directly depend on the quality and speed of keyword extraction algorithms. In this article, an approach, introduced in [1], to rapid change detection in data streams and documents is developed and analysed. It is based on ideas from image processing and especially on the Helmholtz Principle from the Gestalt Theory of human perception. Applied to the problem of keywords extraction, it delivers fast and effective tools to identify meaningful keywords using parameter-free methods. We also define a level of meaningfulness of the keywords which can be used to modify the set of keywords depending on application needs.

1 Introduction

Automatic keyword and feature extraction is a fundamental problem in text data mining, where a majority of document processing applications directly depend on the quality and speed of keyword extraction algorithms. The applications ranging from automatic document classification to information visualization, from automatic filtering to security policy enforcement—all rely on automatically extracted keywords [2]. Keywords are used as basic documents representations and features to perform higher level of analysis. By analogy with low-level image processing, we can consider keywords extraction as low-level document processing.

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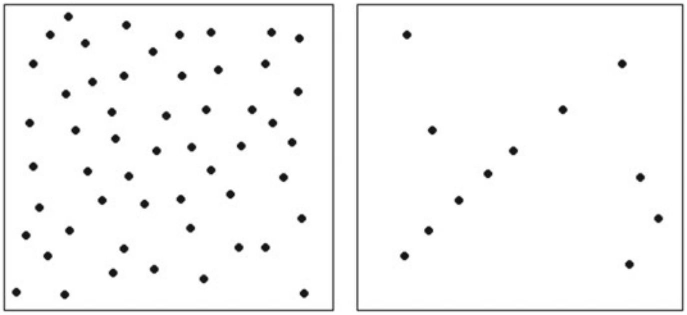


Fig. 1 The Helmholtz principle in human perception

The increasing number of people contributing to the Internet and enterprise intranets, either deliberately or incidentally, has created a huge set of documents that still do not have keywords assigned. Unfortunately, manual assignment of high quality keywords is expensive and time-consuming. This is why many algorithms for automatic keywords extraction have been recently proposed. Since there is no precise scientific definition of the meaning of a document, different algorithms produce different outputs.

The main purpose of this article is to develop novel data mining algorithms based on the Gestalt theory in Computer Vision and human perception. More precisely, we are going to develop Helmholtz principle for mining textual, unstructured or sequential data.

Let us first briefly explain the Helmholtz principle in human perception. According to a basic principle of perception due to Helmholtz [3], an observed geometric structure is perceptually meaningful if it has a very low probability to appear in noise. As a common sense statement, this means that "events that could not happen by chance are immediately perceived". For example, a group of seven aligned dots exists in both images in Fig. 1, but it can hardly be seen on the left-hand side image. Indeed, such a configuration is not exceptional in view of the total number of dots. In the right-hand image we immediately perceive the alignment as a large deviation from randomness that would be unlikely to happen by chance.

In the context of data mining, we shall define the Helmholtz principle as the statement that meaningful features and interesting events appear as large deviations from randomness. In the cases of textual, sequential or unstructured data we derive qualitative measure for such deviations.

Under *unstructured data* we understand data without an explicit *data model*, but with some internal geometrical structure. For example, sets of dots in Fig. 1 are not created by a precise data model, but still have important geometrical structures: nearest neighbours, alignments, concentrations in some regions, etc. A good example is textual data where there are natural structures like files, topics, paragraphs, documents etc. Sequential and temporal data also can be divided into natural blocks like days, months or blocks of several sequential events. In this article, we will assume that data comes packaged into objects, i.e. files, documents or containers. We can also have several layers of such structures; for example, in 20Newsgroups all words are packed into 20 containers (news groups), and each group is divided into individual news. We would like to detect some unusual behaviour in these data and automatically extract some meaningful events and features. To make our explanation more precise, we shall consider mostly textual data, but our analysis is also applicable to any data that generated by some basic set (words, dots, pair of words, measurements, etc.) and divided into some set of containers (documents, regions, etc.), or classified.

The current work introduces a new approach to the problem of automatic keywords extraction based on the following intuitive ideas:

- keywords should be responsible for topics in a data stream or corpus of documents, i.e. keywords should be defined not just by documents themselves, but also by the context of other documents in which they lie;
- topics are signalled by "unusual activity", i.e. a new topic emerges with some features rising sharply in their frequency.

For example, in a book on C++ programming language a sharp rise in the frequency of the words "file", "stream", "pointer", "fopen" and "fclose" could be indicative of the book chapter on "File I/O".

These intuitive ideas have been a source for almost all algorithms in Information Retrieval. One example is the familiar TF-IDF method for representing documents [4, 5]. Despite being one of the most successful and well-tested techniques in Information Retrieval, TF-IDF has its origin in heuristics and it does not have a convincing theoretical basis [5].

Rapid change detection is a very active and important area of research. A seminal paper by Jon Kleinberg [6] develops a formal approach for modelling "bursts" using an infinite-state automation. In [6] bursts appear naturally as state transitions.

The current work proposes to model the above mentioned unusual activity by analysis based on the Gestalt theory in Computer Vision (human perception). The idea of the importance of "sharp changes" is very natural in image processing, where edges are responsible for rapid changes and the information content of images. However, not all local sharp changes correspond to edges, as some can be generated by noise. To represent meaningful objects, rapid changes have to appear in some coherent way. In Computer Vision, the Gestalt Theory addresses how local variations combined together to create perceived objects and shapes.

As mention in [7], the Gestalt Theory is a single substantial scientific attempt to develop principles of visual reconstruction. Gestalt is a German word translatable as "whole", "form", "configuration" or "shape". The first rigorous approach to quantify basic principles of Computer Vision is presented in [7]. In the next section, we develop a similar analysis for the problem of automatic keywords extraction.

The paper is organized as follows. In Sect. 2 we present some results from [1] and further analyse the Helmholtz Principle in the context of document processing and derive qualitative measures of the meaningfulness of words. In Sect. 3 numerical results for State of the Union Addresses from 1790 till 2009 (data set from [8]) are presented and compared with results from [6, Sect. 4]. We also present some preliminary numerical results for the 20Newsgroups data set [9]. Conclusions and future work are discussed in the Sect. 4.

2 The Helmholtz Principle and Meaningful Events

We have defined Helmholtz principle as the statement that meaningful features and interesting events appear as large deviations from randomness. Let us now develop a more rigorous approach to this intuitive statement.

First of all, it is definitely not enough to say that interesting structures are those that have low probability. Let us illustrate it by the following example. Suppose, one unbiased coin is being tossed 100 times in succession, then *any* 100-sequence of heads (ones) and tails (zeros) can be generated with the same equal probability $(1/2)^{100}$. Whilst both sequences

$$s_1 = 10101\ 11010\ 01001\ \ldots\ 00111\ 01000\ 10010$$

$$s_2 = \underbrace{111111111\ldots111111}_{\text{50 times}}\ \underbrace{000000000\ldots000000}_{\text{50 times}}$$

are generated with the same probability, the second output is definitely not expected for an unbiased coin. Thus, low probability of an event does not really indicates its deviation from randomness.

To explain why the second output s_2 is unexpected we should explain what an expected output should be. To do this some global observations (random variables) on the generated sequences are to be considered. This is similar to statistical physics where some macro parameters are observed, but not a particular configuration. For example, let μ be a random variable defined as the difference between number of heads in the first and last 50 flips. It is no surprise that the expected value of this random variable (its mean) is equal to zero, which is with high level of accuracy true for s_1. However, for sequence s_2 with 50 heads followed by 50 tails this value is equal to 50 which is very different from the expected value of zero.

Another example can be given by the famous ‘Birthday Paradox’. Let us look at a class of 30 students and let us assume that their birthdays are independent and uniformly distributed over the 365 days of the year. We are interested in events that some students have their birthday on a same day. Then the natural random variables will be C_n, $1 \leq n \leq 30$, the number of n-tuples of students in the class having the same birthday. It is not difficult to see that the expectation of the number of pairs of students having the same birthday in a class of 30 is $E(C_2) \approx 1.192$. Similar, $E(C_3) \approx 0.03047$ and $E(C_4) \approx 5.6 \times 10^{-4}$. This means that ‘on the average’ we can expect to see 1.192 pairs of students with the same birthday in each class. So, if we have found that two students have the same birthday we should not really be surprised. But having tree or even four students with the same birthday would be unusual. If we look in a class with 10 students, then $E(C_2) \approx 0.1232$. This means that having two students with the same birthday in a class of 10 should be considered as unexpected event.

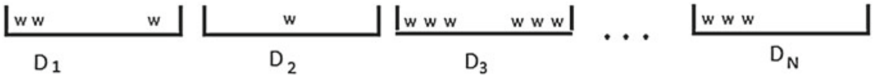
More generally, let Ω be a probability space of all possible outputs. Formally, an output $\omega \in \Omega$ is defined as unexpected with respect to some observation μ, if the value $\mu(\omega)$ is very far from expectation $E(\mu)$ of the random variable μ, i.e. the bigger the difference $|\mu(\omega) - E(\mu)|$ is, the more unexpected outcome ω is. From Markov's inequalities for random variables it can be shown that such outputs ω are indeed very unusual events.

The very important question in such setup is the question of how to select appropriate random variables for a given data. The answer can be given by standard mathematical and statistical physics approach. Any structure can be described by its symmetry group. For example, if we have completely unstructured data, then any permutation of the data is possible. But if we want to preserve a structure, then we can do only transformations that preserve the structure. For example, if we have set of documents, then we can not move words between documents, but can reshuffle words inside each documents. In such a case, the class of suitable random variables are functions which are invariant under the group of symmetry.

2.1 Counting Functions

Let us return to the text data mining. Since we defined keywords as words that correspond with a sharp rise in frequency, then our natural measurements should be counting functions of words in documents or parts of document. To simplify our description let us first derive the formulas for expected values in the simple and ideal situation of N documents or containers of the same length, where the length of a document is the number of words in the document.

Suppose we are given a set of N documents (or containers) $D_1, \ldots, D_N$ of the same length. Let w be some word (or some observation) that present inside one or more of these N documents. Assume that the word w appear K times in all N documents and let us collect all of them into one set $S_w = \{w_1, w_2, \ldots, w_K\}$.



Now we would like to answer the following question: *If the word w appears m times in some document, is this an expected or unexpected event*? For example, the word "*the*" usually has a high frequency, but this is not unexpected. From other hand, in a chapter on how to use definite and indefinite articles in any English grammar book, the word "*the*" usually has much higher frequency and should be detected as unexpected.

Let us denote by C_m a random variable that counts how many times an m-tuple of the elements of S_w appears in the same document. Now we would like to calculate the expected value of the random variable C_m under an assumption that elements from S_w are randomly and independently placed into N containers.

For m different indexes $i_1, i_2, \ldots, i_m$ between 1 and K, i.e. $1 \le i_1 < i_2 < \ldots < i_m \le K$, let us introduce a random variable $\chi_{i_1,i_2,\ldots,i_m}$:

$$\begin{cases} 1 & \text{if } w_{i_1}, \ldots, w_{i_m} \text{ are in the same document,} \\ 0 & \text{otherwise.} \end{cases}$$

Then by definition of the function C_m we can see that

$$C_m = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le K} \chi_{i_1,i_2,\ldots,i_m},$$

and that the expected value $E(C_m)$ is sum of expected values of all $\chi_{i_1,i_2,\ldots,i_m}$:

$$E(C_m) = \sum_{1 \le i_1 < i_2 < \ldots < i_m \le K} E(\chi_{i_1,i_2,\ldots,i_m}).$$

Since $\chi_{i_1,i_2,\ldots,i_m}$ has only values zero and one, the expected value $E(\chi_{i_1,i_2,\ldots,i_m})$ is equal to the probability that all $w_{i_1}, \ldots, w_{i_m}$ belong to the same document, i.e.

$$E(\chi_{i_1,i_2,\ldots,i_m}) = \frac{1}{N^{m-1}}.$$

From the identities above we can see that

$$E(C_m) = \binom{K}{m} \cdot \frac{1}{N^{m-1}}, \tag{1}$$

where $\binom{K}{m} = \frac{K!}{m!(K-m)!}$ is a binomial coefficient.

Now we are ready to answer the previous question:

If in some document the word w appears m times and $E(C_m) < 1$, *then this is an unexpected event.*

Suppose that the word w appear m or more times in each of several documents. *Is this an expected or or unexpected event*? To answer this question, let us introduce another random variable I_m that counts number of documents with m or more appearances of the word w. It should be stressed that despite some similarity, the random variables C_m and I_m are quite different. For example, C_m can be very large, but I_m is always less or equal N. To calculate the expected value $E(I_m)$ of I_m under an assumption that elements from S_w are randomly and independently placed into N containers let as introduce a random variable $I_{m,i}$, $1 \le i \le N$ with

$$I_{m,i} = \begin{cases} 1 & \text{if } D_i \text{ contains } w \text{ at least } m \text{ times,} \\ 0 & \text{otherwise.} \end{cases}$$

Then by definition

$$I_m = \sum_{i=1}^{N} I_{m,i}.$$

Since $I_{m,i}$ has only values zero and one, the expected value $E(I_{m,i})$ is equal to the probability that at least m elements of the set S_w belong to the document D_i, i.e.

$$E(I_{m,i}) = \sum_{j=m}^{K} \binom{K}{j} \left(\frac{1}{N}\right)^{j} \left(1 - \frac{1}{N}\right)^{K-j}.$$

From the last two identities we have

$$E(I_m) = N \times \sum_{j=m}^{K} \binom{K}{j} \left(\frac{1}{N}\right)^{j} \left(1 - \frac{1}{N}\right)^{K-j}. \tag{2}$$

We can rewrite (2) as

$$E(I_m) = N \times \mathcal{B}(m, K, p),$$

where $\mathcal{B}(m, K, p) := \sum_{j=m}^{K} \binom{K}{j} p^j (1-p)^{K-j}$ is the *tail of binomial distribution* and $p = 1/N$.

Now, if we have several documents with m or more appearances of the word w and $E(I_m) < 1$, then this is an unexpected event.

Following [7], we will define $E(C_m)$ from (1) as the *number of false alarms* of a m-tuple of the word w and will use notation $NFA_T(m, K, N)$ for the right hand side of (1). The NFA_T of an m-tuple of the word w is the expected number of times such an m-tuple could have arisen just by chance. Similar, we will define $E(I_m)$ from (2) as the number of false alarms of documents with m or more appearances of the word w, and us notation $NFA_D(m, K, N)$ for the right hand side of (2). The NFA_D of an the word w is the expected number of documents with m or more appearances of the word w that could have arisen just by chance.

2.2 Dictionary of Meaningful Words

Let us now describe how to create a dictionary of meaningful words for our set of documents. We will present algorithms for NFA_T. The similar construction is also applicable to NFA_D.

If we observe that the word w appears m times in the same document, then we define this word as *a meaningful word* if and only if its NFA_T is smaller than 1. In other words, if the event of appearing m times has already happened, but the expected number is less than *one*, we have a meaningful event. The set of all meaningful words in the corpus of documents $D_1, \ldots, D_N$ will be defined as a set of keywords.

Let us now summarize how to generate the set of keywords $KW(D_1, \ldots, D_N)$ of a corpus of N documents $D_1, \ldots, D_N$ of the same or approximately same length:

For all words w from $D_1, \ldots, D_N$

1. Count the number of times K the word w appears in $D_1, \ldots, D_N$.
2. For i from 1 to N
   (a) count the number of times m_i the word w appears in the document D_i;
   (b) if $m_i \geq 1$ and
   $$NFA_T(m_i, K, N) < 1, \tag{3}$$
   then add w to the set $KW(D_1, \ldots, D_N)$ and mark w as a meaningful word for D_i.

If the NFA_T is less than ϵ we say that w is *ϵ-meaningful*. We define a set of ϵ-keywords as a set of all words with $NFA_T < \epsilon, \epsilon < 1$. Smaller ϵ corresponds to more important words.

In real life examples we can not always have a corpus of N documents $D_1, \ldots, D_N$ of the same length. Let l_i denote the length of the document D_i. We have three strategies for creating a set of keywords in such a case:

- Subdivide the set $D_1, \ldots, D_N$ into several subsets of approximately equal size documents. Perform analysis above for each subset separately.
- "Scale" each document to common length l of the smallest document. More precisely, for any word w we calculate K as $K = \sum_{i=1}^{N}[m_i/l]$, where $[x]$ denotes an integer part of a number x and m_i counts the number of appearances of the word w in a document D_i. For each document D_i we calculate the NFA_T with this K and the new $m_i \leftarrow [m_i/l]$. All words with $NFA_T < 1$ comprise a set of keywords.
- We can "glue" all documents $D_1, \ldots, D_N$ into one big document and perform analysis for one document as will be described below.

In a case of one document or data stream we can divide it into the sequence of disjoint and equal size blocks and perform analysis like for the documents of equal size. Since such a subdivision can cut topics and is not shift invariant, the better way is to work with a "moving window". More precisely, suppose we are given a document D of the size L and B is a block size. We define N as $[L/B]$. For any word w from D and any windows of consecutive B words let m count number of w in this windows and K count number of w in D. If $NFA_T < 1$, then we add w to a set of keywords and say that w is meaningful in these windows. In the case of one big document that has been subdivided into sub-documents or sections, the size of such parts are natural selection for the size of windows.

If we want to create a set of ϵ-keywords for one document or for documents of different size we should replace the inequality $NFA_T < 1$ by an inequality $NFA_T < \epsilon$.

2.3 Estimating of the Number of False Alarms

In real examples calculating $NFA_T(m, K, N)$ and $NFA_D(m, K, N)$ can be tricky and is not a trivial task. Numbers m, K and N can be very large and NFA_T or NFA_D can be exponentially large or small. Even relatively small changes in m can results in big fluctuations of NFA_T and NFA_D. The correct approach is to work with

$$-\frac{1}{K}\log NFA_T(m, K, N) \tag{4}$$

and

$$-\frac{1}{K}\log NFA_D(m, K, N) \tag{5}$$

In this case the meaningful events can be characterized by $-\frac{1}{K}\log NFA_T(m, K, N) > 0$ or $-\frac{1}{K}\log NFA_D(m, K, N) > 0$.

There are several explanations why we should work with (4) and (5). The first is pure mathematical: there is a unified format for estimations of (4) and (5) (see [7] for precise statements). For large m, K and N there are several famous estimations for large deviations and asymptotic behavior of (5): law of large numbers, large deviation technique and Central Limit Theorem. In [7, Chap. 4, Proposition 4] all such asymptotic estimates are presented in uniform format.

The second explanations why we should work with (4) and (5) can be given by statistical physics of random systems: these quantities represent 'energy per particle' or energy per word in our context. Like in physics where we can compare energy per particle for different systems of different size, there is meaning in comparison of (4) and (5) for different words and documents.

Calculating of (4) usually is not a problem, since NFA_T is a pure product. For (5) there is also possibility of using Monte Carlo method by simulating Bernoulli process with $p = 1/N$, but such calculations are slow for large N and K.

2.4 On TF-IDF

The TF-IDF weight (term frequency—inverse document frequency) is a weight very often used in information retrieval and text mining. If we are given a collection of documents $D_1, \ldots, D_N$ and a word w appears in L documents $D_{i_1}, \ldots, D_{i_L}$ from the collection, then

$$IDF(w) = \log\left(\frac{N}{L}\right).$$

The TF-IDF weight is just 'redistribution' if IDF among $D_{i_1}, \ldots, D_{i_L}$ according to *term frequency* of w inside of $D_{i_1}, \ldots, D_{i_L}$.

The TF-IDF weight demonstrates remarkable performance in many applications, but the IDF part is still remain a mystery. Let us now look at IDF from number of false alarms point of view.

Consider all documents $D_{i_1}, \ldots, D_{i_L}$ containing the word w and combine all of them into one document (*the document about* w) $\widetilde{D} = D_{i_1} + \cdots + D_{i_L}$. For example, if $w =$ 'cow', then $\widetilde{D}$ is all about 'cow'. We now have a *new collection* of documents (containers): $\widetilde{D}, D_{j_1}, \ldots, D_{j_{N-L}}$, where $D_{j_1}, \ldots, D_{j_{N-L}}$ are documents of the original collection $D_1, \ldots, D_N$ that do not contains the word w. In general, $\widetilde{D}, D_{j_1}, \ldots, D_{j_{N-L}}$ are of different sizes. For this new collection $\widetilde{D}, D_{j_1}, \ldots, D_{j_{N-L}}$ the word w appear only in $\widetilde{D}$, so we should calculate number of false alarms or 'energy' ((4) or (5)) per each appearance of w only for $\widetilde{D}$.

Using adaptive window size or 'moving window', (4) and (5) become

$$-\frac{1}{K}\log\left(\binom{K}{K}\frac{1}{\widetilde{N}}\right),$$

i.e.

$$\frac{K-1}{K}\cdot\log\widetilde{N}, \quad \text{where} \quad \widetilde{N} = \frac{\sum_{i=1}^{N}|D_i|}{|\widetilde{D}|}. \tag{6}$$

If all documents $D_1, \ldots, D_N$ are of the same size, then (6) becomes

$$\frac{K-1}{K}\cdot IDF(w),$$

and for large K is almost equal to $IDF(w)$. But for the case of documents of different lengths (which is more realistic) our calculation suggest that more appropriate should be *adaptive IDF*:

$$AIDF(w) := \frac{K-1}{K}\cdot\log\frac{\sum_{i=1}^{N}|D_i|}{|\widetilde{D}|}, \tag{7}$$

where K is term count of the word w in all documents, $|\widetilde{D}|$ is the total length of documents containing w and $\sum_{i=1}^{N}|D_i|$ is the total length of all documents in the collection.

3 Experimental Results

In this section we present some numerical results for State of the Union Addresses from 1790 till 2009 (data set from [8]) and for the famous 20 Newsgroups data set [9].

The performance of the proposed algorithm was studied on a relatively large corpus of documents. To illustrate the results, following [6], we selected the set of all U.S. Presidential State of the Union Addresses, 1790–2009 [8]. This is a very rich data set that can be viewed as a corpus of documents, as a data stream with natural timestamps, or as one big document with many sections.

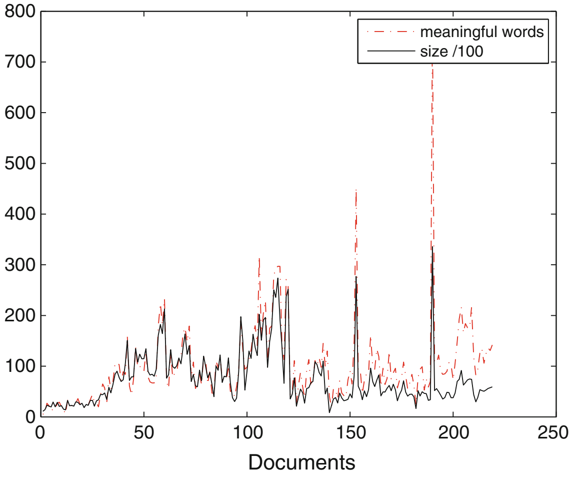


Fig. 2 Document lengths in hundreds of words is shown by the *solid line*; the document average length is equal to 7602.4 and the sample deviation is 5499.7

It is important to emphasize that we do not perform any essential pre-processing of documents, such as stop word filtering, lemmatization, part of speech analysis, and others. We simply down-case all words and remove all punctuation characters.

For the first experiment, the data is analyzed as a collection of $N = 219$ individual addresses. The number of words in these documents vary dramatically, as shown in Fig. 2 by the solid line.

As expected, the extraction of meaningful or ϵ-meaningful words using formula (3) from the corpus of different length documents performs well for the near-average length documents. The manual examination of the extracted keywords reveals that

- all stop words have disappeared;
- meaningful words relate to/define the corresponding document topic very well;
- the ten most meaningful words with the smallest NFA follow historical events in union addresses.

For example, five of the most meaningful words extracted from the speeches of the current and former presidents are

Obama, 2009: lending, know, why, plan, restart;
Bush, 2008: iraq, empower, alqaeda, terrorists, extremists;
Clinton, 1993: jobs, deficit, investment, plan, care.

However, the results for the document outliers are not satisfactory. Only a few meaningful words or none are extracted for the small documents. Almost all words are extracted as meaningful for the very large documents. In documents with size more than 19K words even the classical stop word “the” was identified as meaningful.

To address the problem of the variable document length different strategies were applied to the set of all Union Addresses: *moving window*, *scaling to average* and *adapting window size* described in Sect. 2. The results are dramatically improved for outliers in all cases. The best results from our point of view are achieved using an *adaptive window size* for each document, i.e. we calculate (3) for each document with the same K and m_i but with $N = L/|D_i|$ with L being the total size of all documents and $|D_i|$ is the size of the document D_i. The numbers of meaningful words ($\epsilon = 1$) extracted for the corresponding documents are shown by the dashed line in Fig. 2. The remarkable agreement with document sizes is observed.

Our results are consistent with the existing classical algorithm [6]. For example, using a moving window approach, the most meaningful words extracted for The Great Depression period from 1929 till 1933 are: “loan”, “stabilize”, “reorganization”, “banks”, “relief” and “democracy”, whilst the most important words extracted by [6] are “relief”, “depression”, “recovery”, “banks” and “democracy”.

Let us now look at the famous Zipf’s law for natural languages. Zipf’s law states that given some corpus of documents, the frequency of any word is inversely proportional to some power γ of its rank in the frequency table, i.e. frequency(rank) $\approx$ const/rank$^{\gamma}$. Zipf’s law is mostly easily observed by plotting the data on a *log-log* graph, with the axes being log(rank order) and log(frequency). The data conform to Zipf’s law to extend the plot is linear. Usually Zipf’s law is valid for the upper portion of the log-log curve and not valid for the tail.

For all words in the Presidential State of the Union Address we plot rank of a word and the total number of the word’s occurrences in log-log coordinates, as shown in Fig. 3.

Let us look into Zipf’s law for only the meaningful words of this corpus ($\epsilon = 1$). We plot rank of a meaningful word and the total number of the wold’s occurrences in log-log coordinates, as shown in Fig. 4. We still can observe the Zipf’s law, the curve become smoother and the power γ becomes smaller.

If we increase level of meaningfulness (i.e. decrease the ϵ), then the curve becomes even more smoother and conform to Zipf’s law with smaller and smaller γ. This is very much in a line with what we should expect from good feature extraction and dimension reduction: to decrease number of features and to decorrelate data.

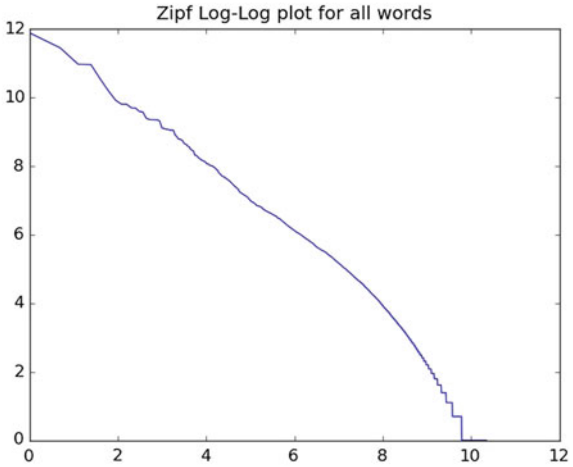


Fig. 3 The total number of words in the Presidential State of the Union Addresses as a function of their rank in log-log coordinates

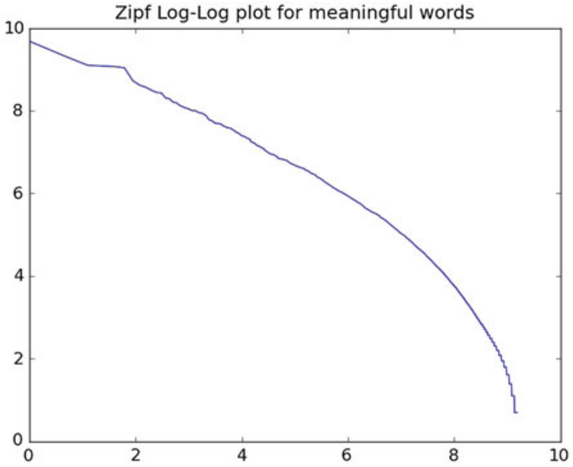


Fig. 4 The total number of meaningful words as a function of their rank in log-log coordinates

For two sets S_1 and S_2 let us use as a measure of their similarity the number of common elements divided by the number of elements in their union: $W(S_1, S_2) = |S_1 \bigcap S_2|/|S_1 \bigcup S_2|$. After extracting meaningful words we can look into similarity of the Union Addresses by calculating similarity W for their sets of keywords. Then, for example, Barack Obama, 2009 speech is mostly similar to George H.W. Bush, 1992 speech with the similarity $W \approx 0.132$ and the following meaningful words in common:

set([‘everyone’, ‘tax’, ‘tonight’, ‘i’m’, ‘down’, ‘taxpayer’, ‘reform’, ‘health’, ‘you’, ‘tell’, ‘economy’, ‘jobs’, ‘get’, ‘plan’, ‘put’, ‘wont’, ‘short-term’, ‘long-term’, ‘times’, ‘chamber’, ‘asked’, ‘know’]).

George W. Bush, 2008 speech is mostly similar to his 2006 speech (which is very reasonable) with the similarity $W \approx 0.16$ and the following meaningful words in common:

set([‘terrorists’, ‘lebanon’, ‘al-qaeda’, ‘fellow’, ‘tonight’, ‘americans’, ‘technology’, ‘enemies’, ‘terrorist’, ‘palestinian’, ‘fight’, ‘iraqi’, ‘iraq’, ‘terror’, ‘we’, ‘iran’, ‘america’, ‘attacks’, ‘iraqis’, ‘coalition’, ‘fighting’, ‘compete’]).

From all the Presidential State of the Union Address most similar are William J. Clinton 1997 speech and 1998 speech. Their similarity is $W \approx 0.220339$ and the following meaningful words in common:

set([‘help’, ‘family’, ‘century’, ‘move’, ‘community’, ‘tonight’, ‘schools’, ‘finish’, ‘college’, ‘welfare’, ‘go’, ‘families’, ‘education’, ‘children’, ‘lifetime’, ‘row’, ‘chemical’, ‘21st’, ‘thank’, ‘workers’, ‘off’, ‘environment’, ‘start’, ‘lets’, ‘nato’, ‘build’, ‘internet’, ‘parents’, ‘you’, ‘bipartisan’, ‘pass’, ‘across’, ‘do’, ‘we’, ‘global’, ‘jobs’, ‘students’, ‘thousand’, ‘scientists’, ‘job’, ‘leadership’, ‘every’, ‘know’, ‘child’, ‘communities’, ‘dont’, ‘america’, ‘lady’, ‘cancer’, ‘worlds’, ‘school’, ‘join’, ‘vice’, ‘challenge’, ‘proud’, ‘ask’, ‘together’, ‘keep’, ‘balanced’, ‘chamber’, ‘teachers’, ‘lose’, ‘americans’, ‘medical’, ‘first’]).

3.1 20 Newsgroups

In this subsection of the article some numerical results for the famous 20 Newsgroup data set [9] will be presented.

This data set consists of 20000 messages taken from 20 newsgroups. Each group contains one thousand Usenet articles. Approximately 4% of the articles are cross-posted. Our only preprocessing was removing words with length ≤ 2. For defining meaningful words we use NFA_T and consider each group as separate container. In Fig. 5, group lengths (total number of words) in tens of words is shown by blue line and number of different words in each group is shown by green line. The highest peek in group lengths correspond to the group 'talk.politics.mideast', and the highest peek in number of different words correspond to the group 'comp.os.ms-windows.misc'.

After creating meaningful words for each group based on NFA_T with $\epsilon = 1$ and removing non-meaningful words from each group, the news group lengths (total number of meaningful words) in tens of words is shown my blue line in Fig. 6. The number of different meaningful words in each group is shown by green line on the same Fig. 6.

Let us now look into the Zipf's law for 20 Newsgroups. We plot rank of a word and total number of the word's occurrences in log-log coordinates, as shown in Fig. 7, and we also plot rank of a meaningful word and total number of the word's occurrences in log-log coordinates, as shown in Fig. 8. As we can see, meaningful word also follow Zipf's law very close.

Similar to the State of the Union Addresses, let us calculate similarity of groups by calculating W for corresponding sets of meaningful words. We will index the groups by integer $i = 0, \ldots, 19$ and denote ith group by Gr[i], for example, Gr[3] = 'comp.sys.ibm.pc.hardware', as shown on the Table 1. The similarity matrix W is 20×20-matrix and is too big to reproduce in the article. So, we show in the Table 1 most similar and most non-similar groups for each group, together with corresponding measure of similarity W. For example, the group 'comp.windows.x'

Fig. 5 Group lengths (total number of words) in tens of words and the number of different words in each group

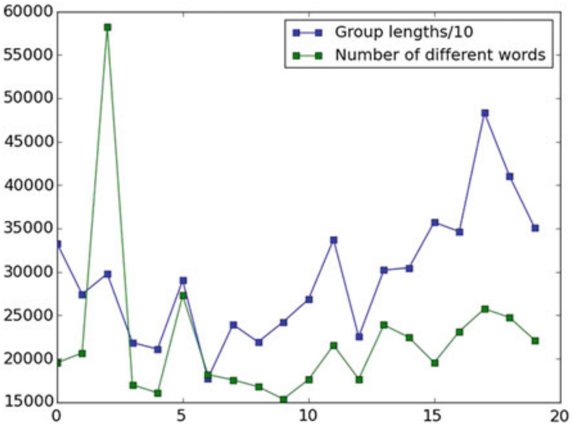


Fig. 6 News group lengths (total number of meaningful words) in tens of words and the number of different meaningful words in each group

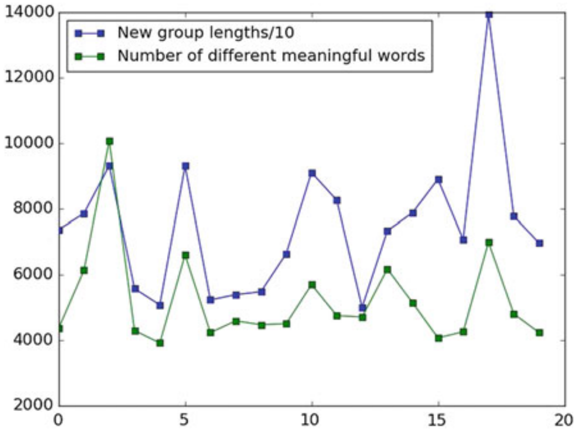
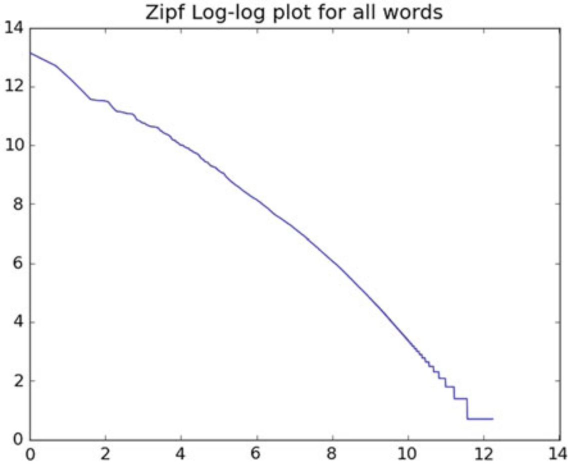


Fig. 7 The total number of words in the 20Newsgroups dataset as a function of their rank in log-log coordinates



(index = 5) is most similar to the group Gr[1] = 'comp.graphics' with similarity = 0.038, and most non-similar with the group Gr[19] = 'talk.religion.misc' with similarity = 0.0012. As we can see, our feature extraction approach produce very natural measure of similarity for the 20 Newsgroups.

Let us now investigate how sets of meaningful words change with number of articles inside groups. Let us create so called mini-20Newsgroups by selecting randomly 10% of articles in each group. In the mini-20Newsgroups there are 100 articles in each group. We have used for our numerical experiments the mini-20Newsgroups from [9]. After performing meaningful words extraction from the mini-20Newsgroup with NFA_T and $\epsilon = 1$, let us plot together number of meaningful words in each group of original 20Newsgroups, number of meaningful words in each group of mini-20Newsgroup and number of common meaningful words for these two data set. The results are shown in Fig. 9.

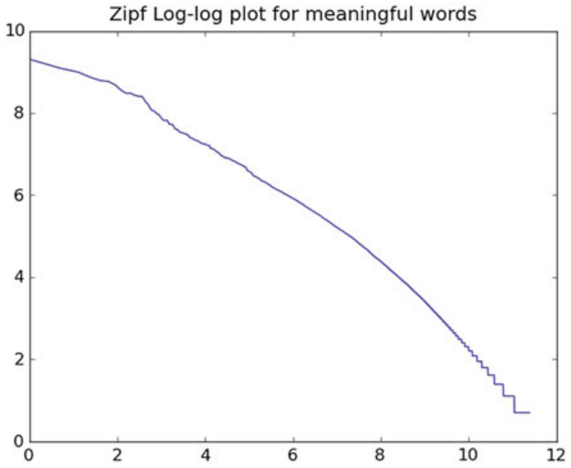


Fig. 8 The total number of meaningful words in the 20Newsgroups dataset as a function of their rank in log-log coordinates

Table 1 An example of group similarities

Index	News groups	Highest similarity	Lowest similarity
0	alt.atheism	Gr[19], 0.12	Gr[2], 0.0022
1	comp.graphics	Gr[5], 0.038	Gr[15], 0.0023
2	comp.os.ms-windows.misc	Gr[3], 0.0197	Gr[15], 0.0023
3	comp.sys.ibm.pc.hardware	Gr[4], 0.041	Gr[17], 0.0024
4	comp.sys.mac.hardware	Gr[3], 0.041	Gr[17], 0.0023
5	comp.windows.x	Gr[l], 0.038	Gr[19], 0.0012
6	misc.forsale	Gr[12], 0.03	Gr[0], 0.0024
7	rec.autos	Gr[8], 0.035	Gr[15], 0.0025
8	rec.motorcycles	Gr[7], 0.035	Gr[2], 0.0033
9	rec.sport.baseball	Gr[10], 0.036	Gr[19], 0.0043
10	rec.sport.hockey	Gr[9], 0.036	Gr[15], 0.0028
11	sci.crypt	Gr[16], 0.016	Gr[2], 0.0025
12	sci.electronics	Gr[6], 0.030	Gr[17], 0.0028
13	sci.med	Gr[12], 0.012	Gr[2], 0.0035
14	sci.space	Gr[12], 0.016	Gr[2], 0.0045
15	soc.religion.christian	Gr[19], 0.044	Gr[2], 0.0014
16	talk.politics.guns	Gr[18], 0.042	Gr[2], 0.0021
17	talk.politics.mideast	Gr[18], 0.022	Gr[2], 0.0018
18	talk.politics.misc	Gr[19], 0.043	Gr[5], 0.0017
19	talk.religion.misc	Gr[0], 0.120	Gr[5], 0.0012

As we can see, large part of meaningful words survive when we increase number of articles by ten times, i.e. when we go from mini to full 20Newsgroup data: red and green lines are remarkable coherent.

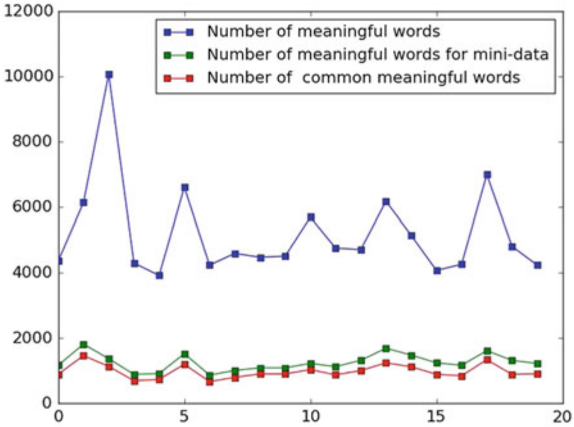


Fig. 9 The number of meaningful words in each news group of the original 20Newsgroups, the number of meaningful words in each group of the mini-20 Newsgroups and the number of common meaningful words for these two data sets

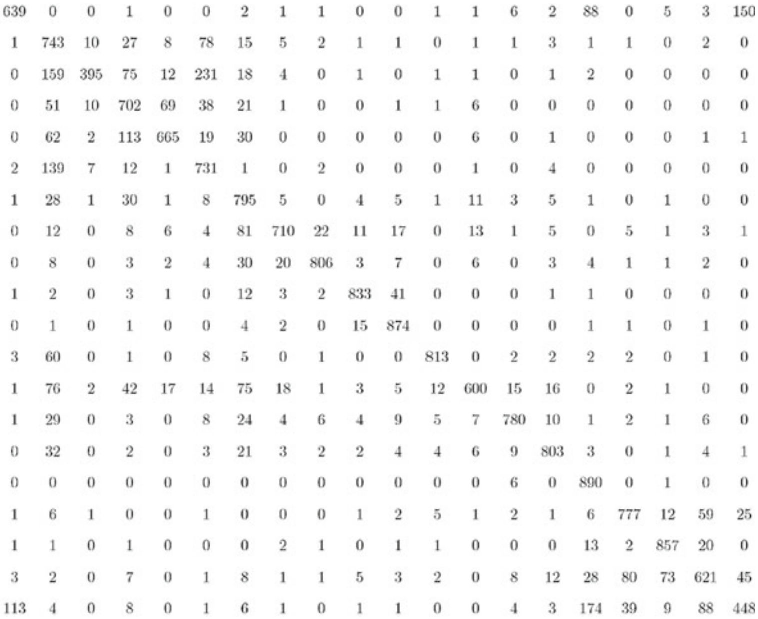
Let us now check how all these meaningful words perform in classification tasks. We would like to have a classifier for finding appropriate newsgroup for a new message. Using 10% of news as training set we have created 20 sets of meaningful words, $MW[i], i = 0, \ldots, 19$. Let us introduce the simplest possible classifier C from messages to the set of 20 Newsgroups. For a message M let us denote by $set(M)$ the set of all different words in M. Then $C(M)$ is a group with largest number of words in $set(M) \bigcap MW[i]$. If there are several groups with the same largest number of words in $set(M) \bigcap MW[i]$, then we select as $C(M)$ a group with smallest index. In the case when all intersections $set(M) \bigcap MW[i]$ are empty, we will mark a message M as “unclassifiable”.

The results of applying this classifier to the remaining 90% of 20Newsgroups can be represented by the classification confusion matrix CCM Fig. 10. CCM is a 2020 integer value matrix with CCM(i,j) is the number of messages from ith group classified into jth group. For ideal classifier CCM is a diagonal matrix. For calculating this matrix we used 18000 messages from 20Newsgroups excluding the training set.

REMARK: In each row of the CCM, the sum of its elements is equal to 900, which is the number of messages in each group. The exception is the row corresponding to the group “soc.religion.christian”, where the sum is equal to 897, because 3 messages from this group remained unclassified, their intersection with the set of meaningful words in each group was empty.

It is also useful to check the classifier performance on the training set itself to validate our approach for selecting meaningful words. The classification confusion matrix for the training set only is shown in Fig. 11.

We now calculate the precision, recall and accuracy of our classifier for each of 20 groups.



639	0	0	1	0	0	2	1	1	0	0	1	1	6	2	88	0	5	3	150
1	743	10	27	8	78	15	5	2	1	1	0	1	1	3	1	1	0	2	0
0	159	395	75	12	231	18	4	0	1	0	1	1	0	1	2	0	0	0	0
0	51	10	702	69	38	21	1	0	0	1	1	6	0	0	0	0	0	0	0
0	62	2	113	665	19	30	0	0	0	0	0	6	0	1	0	0	0	1	1
2	139	7	12	1	731	1	0	2	0	0	0	1	0	4	0	0	0	0	0
1	28	1	30	1	8	795	5	0	4	5	1	11	3	5	1	0	1	0	0
0	12	0	8	6	4	81	710	22	11	17	0	13	1	5	0	5	1	3	1
0	8	0	3	2	4	30	20	806	3	7	0	6	0	3	4	1	1	2	0
1	2	0	3	1	0	12	3	2	833	41	0	0	0	1	1	0	0	0	0
0	1	0	1	0	0	4	2	0	15	874	0	0	0	0	1	1	0	1	0
3	60	0	1	0	8	5	0	1	0	0	813	0	2	2	2	2	0	1	0
1	76	2	42	17	14	75	18	1	3	5	12	600	15	16	0	2	1	0	0
1	29	0	3	0	8	24	4	6	4	9	5	7	780	10	1	2	1	6	0
0	32	0	2	0	3	21	3	2	2	4	4	6	9	803	3	0	1	4	1
0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	890	0	1	0	0
1	6	1	0	0	1	0	0	0	1	2	5	1	2	1	6	777	12	59	25
1	1	0	1	0	0	0	2	1	0	1	1	0	0	0	13	2	857	20	0
3	2	0	7	0	1	8	1	1	5	3	2	0	8	12	28	80	73	621	45
113	4	0	8	0	1	6	1	0	1	1	0	0	4	3	174	39	9	88	448

Fig. 10 Classification confusion matrix, where $CCM(i, j)$ is the number of messages from the ith group classified into the jth group

Precision for ith group is defines as

$$P(i) = \frac{CMM(i,i)}{\sum_j CMM(i,j)},$$

Recall for ith group is defines as

$$R(i) = \frac{CMM(i,i)}{\sum_j CMMj, i)},$$

and

Accuracy for ith group is defines as harmonic mean of precision and recall

$$A(i) = \frac{2P(i)R(i)}{P(i)+R(i)}.$$

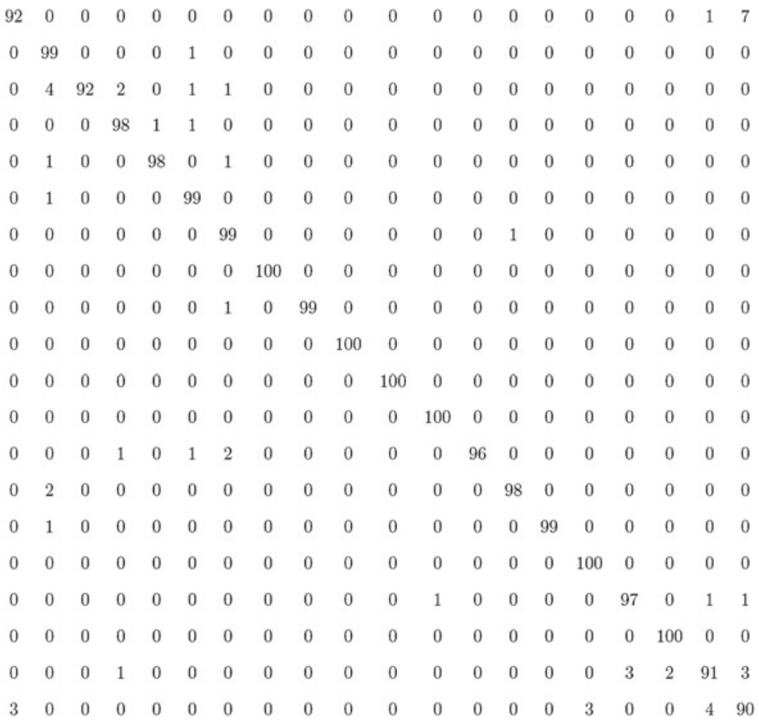
The results of calculation of precision, recall and accuracy of our classifier for each of 20 groups is shown in the Table 2.

As we can see from the Table 2, this simple classifier performs impressively well for the most of news groups, thus illustrating the success of NFA_T for selecting meaningful features. The smallest accuracy of around 57% has been observed for the group "talk.religion.misc". From the classification confusion matrix *CCM* (Fig. 10), we can see that many articles from "talk.religion.misc" have been classified as belonging to "alt.atheism", "soc.religion.christian" or "talk.politics.misc" groups.

4 Conclusion and Future Work

In this article, the problem of automatic keyword and feature extraction in unstructured data is investigated using image processing ideas and especially the Helmholtz principle. We define a new measure of keywords meaningfulness with good performance on different types of documents. We expect that our approach may not only establish fruitful connections between the fields of Computer Vision, Image Processing and Information Retrieval, but may also assist with the deeper understanding of existing algorithms like TF-IDF.

In TF-IDF it is preferable to create a stop word list, and remove the stop word before computing the vector representation [2]. In our approach, the stop words are removed automatically. It would be very interesting to study the vector model for text



92	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	7
0	99	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	4	92	2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	98	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	98	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	99	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	99	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	99	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0	0	0	0	0
0	0	0	1	0	1	2	0	0	0	0	0	96	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0	98	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	99	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	97	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	100	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	3	2	91	3
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	4	90

Fig. 11 Classification confusion matrix for the training set with data presented as in Fig. 10

Table 2 Precision, recall and accuracy

News groups	Precision	Recall	Accuracy
alt.atheism	0.71	0.8331	0.7667
comp.graphics	0.8256	0.5251	0.6419
comp.os.ms-windows.misc	0.4389	0.9229	0.5949
comp.sys.ibm.pc.hardware	0.78	0.6756	0.7241
comp.sys.mac.hardware	0.7389	0.8504	0.7907
comp.windows.x	0.8122	0.6362	0.7135
misc.forsale	0.8833	0.6925	0.7764
rec.autos	0.7889	0.9103	0.8452
rec.motorcycles	0.8956	0.9516	0.9227
rec.sport.baseball	0.9256	0.9423	0.9339
rec.sport.hockey	0.9711	0.9001	0.9343
sci.crypt	0.9033	0.961	0.9312
sci.electronics	0.6667	0.9091	0.7692
sci.med	0.8667	0.9319	0.8981
sci.space	0.8922	0.9209	0.9063
soc.religion.christian	0.9922	0.7325	0.8428
talk.politics.guns	0.8633	0.852	0.8576
talk, politics, mideast	0.9522	0.8899	0.92
talk.politics.misc	0.69	0.7657	0.7259
talk.religion.misc	0.4977	0.6677	0.5703

mining based with $-\log(NFA)$ as a weighting function. Even the simplest classifier based on meaningful events performs well.

One of the main objectives in [7] is to develop parameter free edge detections based on maximal meaningfulness. Similarly, algorithms in data mining should have as few parameters as possible—ideally none. Developing a similar approach to the keyword and feature extraction, i.e. defining the maximal time or space interval for a word to stay meaningful, is an exciting and important problem. It would also be interesting to understand the relationship between the NFA and [6].

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A Method of Introducing Weights into OWA Operators and Other Symmetric Functions

Gleb Beliakov

Abstract This paper proposes a new way of introducing weights into OWA functions which are popular in fuzzy systems modelling. The proposed method is based on replicating the inputs of OWA the desired number of times (which reflect the importances of the inputs), and then using a pruned *n*-ary tree construction to calculate the weighted OWA. It is shown that this tree-based construction preserves many useful properties of the OWA, and in fact produces the discrete Choquet integral. A computationally efficient algorithm is provided. The tree-based construction is universal in its applicability to arbitrary symmetric idempotent *n*-ary functions such as OWA, and transparent in its handling the weighting vectors. It will be a valuable tool for decision making systems in the presence of uncertainty and for weighted compensative logic.

1 Introduction

One of the fundamental aspects in dealing with various forms of uncertainty is operations with uncertainty degrees, for example in the *if-then-else* rule statements. Theories that extend classical logic require extensions of the logical operations, such as conjunction, disjunction, negation and implication. In fuzzy logic in particular, membership degrees from the unit interval are combined by using aggregation functions [4, 14], which in the most general form are monotone increasing functions f with the boundary conditions $f(0,\dots,0) = 0$ and $f(1,\dots,1) = 1$, so they ensure matching the classical logical operations in the limiting cases.

The first class of prominent aggregation functions are the triangular norms and their duals, the triangular conorms [17]. The product was the first t-norm to appear alongside with the minimum in L. Zadeh's original paper on fuzzy sets [30]. When trying to mimic human decision making empirically, however, it became clear that

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it is a combination of the t-norm and t-conorm, which itself is neither conjunctive nor disjunctive operation, that fits the data best [19, 31]. These functions expressed compensation properties, so that an increase in one input could be compensated by a decrease in another.

Another prominent class of aggregation functions that are in use since the early expert systems MYCIN and PROSPECTOR are the uninorms [4, 7, 26]. These are associative mixed operations that behave as conjunction on one part of their domain and as a disjunction on another. These functions, when scaled to the interval $[-1, 1]$, are useful bi-polar operations, where the information can be interpreted as "positive" or "negative", supporting or negating the conclusion of a logical statement or a hypothesis.

The class of averaging functions is the richest in terms of the number of interesting families. Averaging functions, whose prototypical examples are the arithmetic mean and the median, allow compensation between low values of some inputs and high values of the others. Such functions are also important for building decision models in weighted compensative logic [8], where the concept of Generalized Conjunction/Disjunction (GCD) play a role [10, 12].

Along with many classical means [5], the class of averaging functions contains such constructions as ordered weighted averages (OWA) [24] and fuzzy integrals [15]. The OWA functions in particular became very popular in fuzzy systems community [13, 28, 29]. These are symmetric functions which associate the weights with the magnitude of the inputs rather than with their sources.

The inability of OWA functions to associate weights with the specific arguments in order to model the concepts of importance and reliability of information sources was a stumbling block for their usage in some areas. However the concept of weighted OWA (WOWA) [20] resolved this issue. In WOWA functions, two vectors of weights are used. One vector has the weights associated with the arguments, thus modelling importance of the inputs, whereas the second vector associates weights with the magnitude of the inputs. It was later shown that WOWA are a special class of the discrete Choquet integral [18].

A different way of introducing weights into OWA was presented in [25]. In this paper, the OWA function was applied to the modified arguments, and the modifying function was found by using fuzzy modelling techniques. One such function was a linear combination of the argument and the orness (see Definition 3) of the respective OWA function. This method did not produce idempotent functions except in a few special cases.

In this contribution we present a different approach to introducing weights into OWA, based on n-ary tree construction and recursive application of the base OWA function. Such an approach based in binary trees was recently introduced by Dujmovic [9] in order to incorporate weights into bivariate symmetric means, as well as to extend bivariate means to n variables. More detailed analysis of the properties of this method is in [2, 11].

This paper is structured as follows. After presenting preliminaries in Sect. 2, we describe the construction of WOWA functions by Torra [20] in Sect. 3. In Sect. 4 we describe the method of introducing weights into arbitrary bivariate averaging functions from [9]. Our main contribution is in Sect. 5, where we introduce the n-ary tree construction, outline some of its theoretical properties, and present an efficient computational algorithm. The conclusions are presented in Sect. 6.

2 Preliminaries

Consider now the following definitions adopted from [4, 14]. Let $\mathbb{I} = [0, 1]$, although other intervals can be accommodated easily.

Definition 1 A function $f : \mathbb{I}^n \to \mathbb{R}$ is **monotone** (increasing) if $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{I}^n, \mathbf{x} \leq \mathbf{y}$ then $f(\mathbf{x}) \leq f(\mathbf{y})$, with the vector inequality understood componentwise.

Definition 2 A function $f : \mathbb{I}^n \to \mathbb{I}$ is **idempotent** if for every input $\mathbf{x} = (t, t, ..., t), t \in \mathbb{I}$, the output is $f(\mathbf{x}) = t$.

Definition 3 A function $f : \mathbb{I}^n \to \mathbb{I}$ is a **mean** (or is averaging) if for every $\mathbf{x}$ it is bounded by $\min(\mathbf{x}) \leq f(\mathbf{x}) \leq \max(\mathbf{x})$.

Averaging functions are idempotent, and monotone increasing idempotent functions are averaging. We consider weighting vectors $\mathbf{w}$ such that $w_i \geq 0$ and $\sum w_i = 1$ of appropriate dimensions.

Definition 4 A function $f : \mathbb{I}^n \to \mathbb{I}$ is **shift-invariant** (stable for translations) if $f(\mathbf{x} + a\mathbf{1}) = f(\mathbf{x}) + a$ whenever $\mathbf{x}, \mathbf{x} + a\mathbf{1} \in \mathbb{I}^n$. A function $f : \mathbb{I}^n \to \mathbb{I}$ is **homogeneous** (of degree 1) if $f(a\mathbf{x}) = af(\mathbf{x})$ whenever $\mathbf{x}, a\mathbf{x} \in \mathbb{I}^n$.

Definition 5 For a given generating function $g : \mathbb{I} \to [-\infty, \infty]$, and a weighting vector $\mathbf{w}$, the **weighted quasi-arithmetic mean** (QAM) is the function

$$M_{\mathbf{w},g}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right). \tag{1}$$

Definition 6 Let $\varphi : \mathbb{I} \to \mathbb{I}$ *be a bijection. The* $\boldsymbol{\varphi}$**-transform** of a function $f : \mathbb{I}^n \to \mathbb{I}$ is the function $f_\varphi(\mathbf{x}) = \varphi^{-1}\left(f\left(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n)\right)\right)$.

The weighted QAM is a φ-transform of the weighted arithmetic mean with $\varphi = g$.

Definition 7 For a given weighting vector w, $w_i \geq 0, \sum w_i = 1$, the **OWA function** is given by

$$OWA_w(x) = \sum_{i=1}^{n} w_i x_{(i)}, \tag{2}$$

where $x_{(i)}$ denotes the i-th largest value of x.

The main properties of OWA are summarised below.

- As with all averaging aggregation functions, OWA are increasing (strictly increasing if all the weights are positive) and idempotent;
- OWA functions are continuous, symmetric, homogeneous and shift-invariant;
- OWA functions are piecewise linear, and the linear pieces are joined together where two or more arguments are equal in value.
- The OWA functions are special cases of the Choquet integral with respect to symmetric fuzzy measures.
- The special case of OWA include the arithmetic mean, the median and the minimum and maximum operators among others.
- The dual of an OWA with respect to the standard negation is the OWA with the weights reversed.

The orness measure allows one to qualify an OWA function as OR-like or AND-like based on whether it behaves more disjunctively or more conjunctively than the arithmetic mean. The expression for the orness measure is given by the following simple formula

$$orness(OWA_w) = \sum_{i=1}^{n} w_i \frac{n-i}{n-1} = OWA_w\left(1, \frac{n-2}{n-1}, \dots, \frac{1}{n-1}, 0\right). \tag{3}$$

The OWA functions are OR-like if $orness(OWA_w) \geq \frac{1}{2}$ and AND-like if $orness(OWA_w) \leq \frac{1}{2}$. If the weighting vector is decreasing, i.e., $w_i \geq w_j$ whenever $i < j$, OWA is OR-like and is in fact a convex function. The respective (symmetric) fuzzy measure in this case is sub-modular [3]. The OWA functions with increasing weights are AND-like, concave functions which correspond to the Choquet integral with respect to a super-modular fuzzy measure. OWA with decreasing weighting vectors can be used to define norms [3, 23].

Similarly to quasi-arithmetic means, OWA functions have been generalized with the help of generating functions $g : \mathbb{I} \to [-\infty, \infty]$ as follows.

Definition 8 Let $g : \mathbb{I} \to [-\infty, \infty]$ be a continuous strictly monotone function and let w be a weighting vector. The function

$$GenOWA_{w,g}(x) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_{(i)})\right) \tag{4}$$

is called a **generalized OWA**. As for OWA, $x_{(i)}$ denotes the i-th largest value of x.

The generalized OWA is a φ-transform of the OWA function with $\varphi = g$. One special case is the Ordered Weighted Geometric (OWG) function studied in [16, 22]. It is defined by

Definition 9 For a given weighting vector w, the **OWG function** is

$$OWG_w(x) = \prod_{i=1}^{n} x_{(i)}^{w_i}. \tag{5}$$

Similarly to the weighted geometric mean, OWG is a special case of (4) with the generating function $g(t) = \log(t)$. Another special case is the Ordered Weighted Harmonic (OWH) function, where $g(t) = 1/t$.

A large family of generalized OWA functions is based on power functions, similar to weighted power means [27]. Let g_r denote the family of power functions

$$g_r(t) = \begin{cases} t^r, & \text{if } r \neq 0, \\ \log(t), & \text{if } r = 0. \end{cases}$$

Definition 10 For a given weighting vector w, and a value $r \in \mathbb{R}$, the function

$$GenOWA_{w,[r]}(x) = \left(\sum_{i=1}^{n} w_i x_{(i)}^r \right)^{1/r}, \tag{6}$$

if $r \neq 0$, and $GenOWA_{w,[r]}(x) = OWG_w(x)$ if $r = 0$, is called a **power-based generalized OWA**.

3 Weighted OWA

The weights in weighted means and in OWA functions represent different things. In weighted means w_i reflects the importance of the i-th input, whereas in OWA w_i reflects the importance of the i-th largest input. In [20] Torra proposed a generalization of both weighted means and OWA, called WOWA. This aggregation function has two sets of weights w, p. Vector p plays the same role as the weighting vector in weighted means, and w plays the role of the weighting vector in OWA functions.

Consider the following motivation. A robot needs to combine information coming from n different sensors, which provide distances to the obstacles. The reliability of the sensors is known (i.e., we have weights p). However, independent of their reliability, the distances to the nearest obstacles are more important, so irrespective of the reliability of each sensor, their inputs are also weighted according to their numerical value, hence we have another weighting vector w. Thus both factors, the size of the inputs and the reliability of the inputs, need to be taken into account. WOWA provides exactly this type of aggregation function.

WOWA function becomes the weighted arithmetic mean if $w_i = \frac{1}{n}, i = 1, \ldots, n$, and becomes the usual OWA if $p_i = \frac{1}{n}, i = 1, \ldots, n$.

Definition 11 Let w, p be two weighting vectors, $w_i, p_i \geq 0$, $\sum w_i = \sum p_i = 1$. The following function is called **Weighted OWA** function

$$WOWA_{w,p}(x) = \sum_{i=1}^{n} u_i x_{(i)},$$

where $x_{(i)}$ is the i-th largest component of x, and the weights u_i are defined as

$$u_i = g\left(\sum_{j \in H_i} p_j\right) - g\left(\sum_{j \in H_{i-1}} p_j\right),$$

where the set $H_i = \{j | x_j \geq x_i\}$ is the set of indices of i largest elements of x, and g is a monotone non-decreasing function with two properties:

1. $g(i/n) = \sum_{j \leq i} w_j$, $i = 0, \ldots, n$ (of course $g(0) = 0$);
2. g is linear if all w_i are equal.

Thus computation of WOWA involves a very similar procedure as that of OWA (i.e., sorting components of x and then computing their weighted sum), but the weights u_i are defined by using both vectors w, p, a special monotone function g, and depend on the components of x as well. One can see WOWA as an OWA function with the weights u.

Of course, the weights u also depend on the generating function g. This function can be chosen as a linear spline (i.e., a broken line interpolant), interpolating the points $(i/n, \sum_{j \leq i} w_j)$ (in which case it automatically becomes a linear function if these points are on a straight line), or as a monotone quadratic spline, as was suggested in [20, 21], see also [1] where Schumaker's quadratic spline algorithm was used, which automatically satisfies the straight line condition when needed.

It turns out that WOWA belongs to a more general class of Choquet integral based aggregation functions [18]. It is a piecewise linear function whose linear segments are defined on the simplicial partition of the unit cube $[0, 1]^n$: $\mathscr{S}_i = \{x \in [0, 1]^n | x_{p(j)} \geq x_{p(j+1)}\}$, where p is a permutation of the set $\{1, \ldots, n\}$. Note that there are exactly $n!$ possible permutations, the union of all $\mathscr{S}_i$ is $[0, 1]^n$, and the intersection of the interiors of $\mathscr{S}_i \cap \mathscr{S}_j = \emptyset, i \neq j$.

The next two sections introduce an alternative and generic construction to incorporate weights into any symmetric averaging function. In particular, it will work for OWA and will not have a somewhat unclear issue of selecting the function g in Torra's WOWA.

4 Binary Tree Construction

Consider now a method of incorporating weights into a symmetric bivariate idempotent function f, presented [9] and then extended in [2, 11]. To introduce the weights we use the approach from [6], where each argument x_i is replicated a suitable number of times. To be more precise, we consider an auxiliary vector of arguments

$\mathbf{X} = (x_1, \ldots, x_1, x_2, \ldots, x_2)$, so that x_1 is taken k_1 times and x_2 is taken k_2 times, so that $\frac{k_1}{2^L} \approx w_1$, $\frac{k_2}{2^L} \approx w_2$, and $k_1 + k_2 = 2^L$. Here w_i are the desired weights, and $L \geq 1$ is a specified number of levels of the binary tree shown in Fig. 1. One way of doing so is to take $k_1 = \lfloor w_1 2^L + \frac{1}{2} \rfloor$ and $k_2 = 2^L - k_1$. The vector $\mathbf{X}$ needs to be sorted in the increasing or decreasing order.

Next, let us build a binary tree presented in Fig. 1, where at each node a value is produced by aggregating the values of two children nodes with the given bivariate symmetric averaging function f (denoted by B on the plot and with weights equal to $\frac{1}{2}$). We start from the leaves of the tree which contain the elements of the vector $\mathbf{X}$. In this example we took $w_1 = \frac{5}{8}$ and $w_3 = \frac{3}{8}$. The value y at the root node will be the desired output of the n-variate weighted function.

A straightforward binary tree traversal algorithm for doing so, which starts from the vector $\mathbf{X}$, is as follows:

Aggregation by Levels (ABL) Algorithm

```
1. Compute $k_1 := \lfloor w_1 2^L + \frac{1}{2} \rfloor$, $k_2 := 2^L - k_1$, and create the array
   $X := (x_1, \ldots, x_1, x_2, \ldots, x_2)$ by taking $k_1$ copies of $x_1$ and $k_2$ copies of $x_2$;
2. $N := 2^L$;
3. Repeat $L$ times:
   (a) $N := N/2$;
   (b) For $i := 1 \ldots N$ do $X[i] := f(X[2i-1], X[2i])$;
4. return $X[1]$.
```

The algorithm is obviously terminating. The runtime of the ABL algorithm is $O(2^L)$, which can make its use prohibitive even for moderate L. Fortunately an efficient algorithm based on pruning the binary tree was presented in [2].

The pruning of the binary tree is done by using the idempotency of f, see Fig. 1, right. Indeed no invocation of f is necessary if both of its arguments are equal. Below

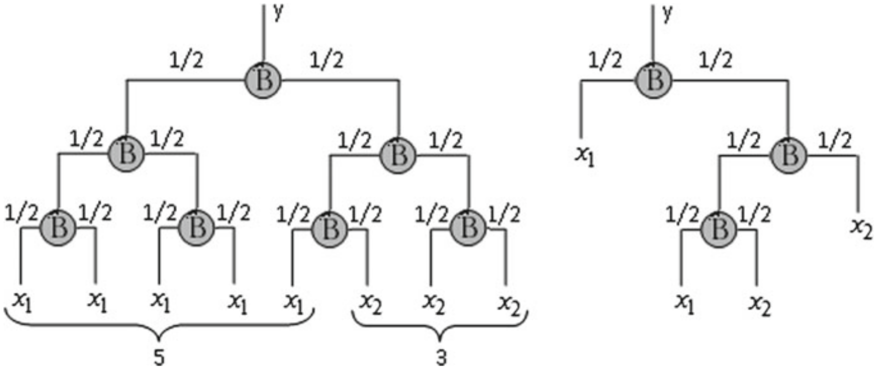

Fig. 1 Representation of a weighted arithmetic mean in a binary tree construction. The tree on the *right* is pruned by using idempotency

we present the pruned tree algorithm whose worst case complexity is $O(L)$, which makes it practically applicable for larger L.

The algorithm is recursive depth-first traversing of the binary tree. A branch is pruned if it is clear that all its leaves have exactly the same value, and by idempotency this is the value of the root node of that branch.

Pruned Tree Aggregation (PTA) Algorithm

```
function $node(m, N, K, x)$

1. If $N[K] \geq 2^m$ then do:
     (a) $N[K] := N[K] - 2^m$;
     (b) $y := x[K]$;
     (c) If $N[K] = 0$ then $K := K + 1$;
     (d) return $y$;
   else
2. return $f(node(m-1, N, K, x), node(m-1, N, K, x))$.

function $f\_n(w, x, L)$

1. create the array $N := (k_1, k_2)$ by using
   $k_1 := \lfloor w_1 2^L + \frac{1}{2} \rfloor$, and $k_2 := 2^L - k_1$;
2. $K := 1$;
3. return $node(L, N, K, x)$.
```

In this algorithm, the array N serves as a counter of how many copies of each of $x[K]$ remains. If there are more than 2^m copies, they belong to a branch that can be pruned, so the function *node* just returns $x[K]$ and never visits the nodes of that branch. If $N[K] = 1$ then the last remaining copy of $x[K]$ is returned and the value of K is incremented. Every time a branch whose leaves contain identical arguments is encountered (which is detected by the counter $N[K] \geq 2^m$), this branch is pruned.

As an example, consider the binary tree in Fig. 1. Here $L = 3$, $k_1 = 5$ and $k_2 = 3$. In the first call to function *node* instruction passes to step 2 where *node* is called recursively twice. In the first recursive call $N[1] = 5 \geq 2^2$ at step 1, hence x_1 is returned and $N[1]$ is set to 1. In the second call to *node* the instruction goes to step 2, where *node* is called recursively twice. In the first of those calls the recursion continues until $m = 0$, at which point x_1 is returned and K is incremented. In the subsequent call to *node* x_2 is returned and then $f(x_1, x_2)$ is computed and returned (the bottom level in the pruned tree in Fig. 1, right). At this point $N[2]$ becomes 2, and in the subsequent call to *node* step 1 is executed, x_2 is returned and subsequently aggregated in $f(f(x_1, x_2), x_2)$ (middle level of the tree in Fig. 1, right). That last output is aggregated with x_1 at the top level of the tree, and the recursive algorithm terminates, producing the output $y = f(x_1, f(f(x_1, x_2), x_2))$.

To see the complexity of this algorithm note that f is never executed (nor the corresponding node of the tree is visited) if its arguments are the same. There is exactly one node at each level of the tree where the child nodes contain distinct arguments, hence f is executed exactly L times. Also note that both N and K are input-output parameters, so that the two arguments of f at step 2 are different as N and K change from one invocation of the function *node* to another, however the order of execution of the calls to *node* does not matter as the lists of formal parameters are identical.

The ABL and PTL algorithms produce identical outputs but differ in computational complexity. For this reason it may be convenient to formulate (or prove) the results in terms of the complete tree processed by algorithm ABL.

Several useful properties of the binary tree construction were presented in [2]. In particular, the weighted function f_w inherits many properties of the base aggregator f, such as idempotency, monotonicity, continuity, convexity (concavity), homogeneity and shift-invariance, due to preservation of these properties in function composition. Furthermore, when the weights are given in a finite binary representation (as is always the case in machine arithmetic), the sequence of the outputs of the ABL (and hence PTA) algorithm with increasing $L = 2, 3, \ldots$ converges to a weighted mean with the specified weights, and in fact L needs not exceed the number of bits in the mantissa of the weights w_i to match these weights exactly. Finally, when f is a quasi-arithmetic mean, f_w is a weighted quasi-arithmetic mean with the same generator.

Another contribution made in [2, 11] is the extension of the symmetric bivariate means to weighted n-variate means by using essentially the same approach, i.e., by replicating the n inputs a suitable number of times and constructing a binary tree with the desired numbed of levels L. The ABL and PTA algorithms in fact remain the same, safe the definition of the array of multiplicities N which is now n-dimensional.

The big advantage of the binary tree construction is its universality and transparency. It is applicable to any bivariate idempotent function f without modification, and the role of the weights as the respective multiplicities of the arguments as argued in [6] is very clear. The availability of a fast and uncomplicated algorithm for computing the output makes this method immediately applicable.

However the binary tree construction is not suitable for introducing weights into OWA functions, as here we already start with an n-variate function. The approach presented in the next section is an adaptation of the binary tree approach to n-variate OWA functions.

5 Importance Weights in OWA Functions

Our goal here is to incorporate a vector p of non-negative weights (which add to one) into a symmetric n-variate function, by replicating the arguments a suitable number of times. As in the binary tree construction we build an n-ary tree with L levels, as shown in Fig. 2. As the base symmetric aggregator f we take an OWA function OWA_w with specified weights w (although the origins of f are not important for the algorithm).

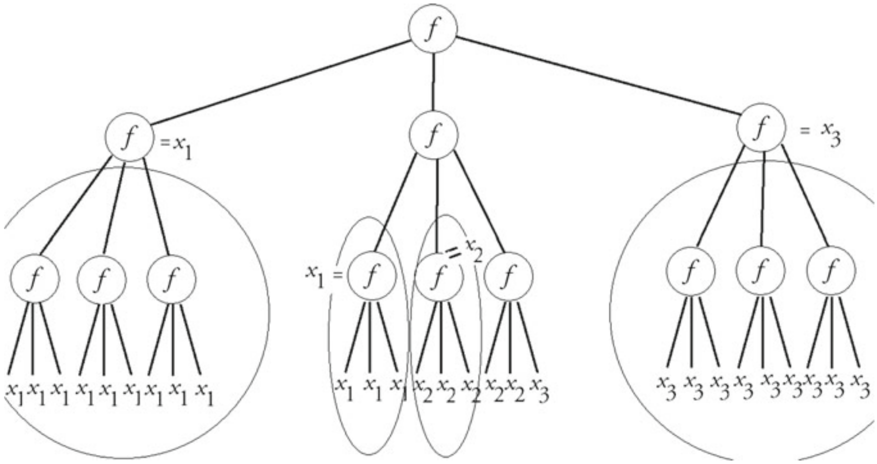
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Fig. 2 Representation of a weighted tri-variate function f in a ternary tree construction. The weights are chosen as $\mathbf{p} = (\frac{12}{27}, \frac{5}{27}, \frac{10}{27})$ and $L = 3$. The *circled branches* are pruned by the algorithm

Now, let us create an auxiliary vector $\mathbf{X} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_n, \ldots, x_n)$, so that x_1 is taken k_1 times, x_2 is taken k_2 times, and so on, and $\frac{k_1}{n^L} \approx p_1$, $\frac{k_2}{n^L} \approx p_2$, …, and $\sum k_i = n^L$, where $L \geq 1$ is a specified number of levels of the tree shown in Fig. 2. One way of doing so is to take $k_i = \lfloor p_i n^L + \frac{1}{n} \rfloor$, $i = 1, \ldots, n-1$ and $k_n = n^L - k_1 - k_2 - \cdots - k_{n-1}$.

Pruned n-Tree Aggregation (PnTA) Algorithm

```
function $node(n, m, N, K, x)$

1. If $N[K] \geq n^m$ then do:
      (a) $N[K] := N[K] - n^m$;
      (b) $y := x[K]$;
      (c) If $N[K] = 0$ then $K := K + 1$;
      (d) return $y$;
   else
2. for $i := 1, \ldots, n$ do
      $z[i] := node(n, m-1, N, K, x)$
3. return $f(z)$.

function $f\_n(n, x, w, p, L)$

1. create the array $N := (k_1, k_2, \ldots, k_n)$ by using
   $k_i := \lfloor p_i n^L + \frac{1}{n} \rfloor$, $i = 1, \ldots, n-1$, and $k_n := n^L - k_1 - \cdots - k_{n-1}$;
2. $K := 1$;
3. return $node(n, L, N, K, x)$.
```

The algorithm PnTA works in the same way as the PTA algorithm for binary trees. The vector of counters N helps determine whether there are more than n^m identical elements of the auxiliary array X, in which case they are the leaves of a branch of the tree with m levels. This branch is pruned. The function f is executed only when some of its arguments are distinct, and since the elements of X are ordered, there are at most $n-1$ such possibilities at each level of the tree, hence the complexity of the algorithm is $O((n-1)L)$.

Note that the complexity is linear in terms of L, as that of the PTA algorithm, which means that the dimension of the base aggregator f does not matter in this respect. Of course, nominally the n-ary tree is larger than the binary tree, but since we only track the multiplicities of the arguments, never creating the array $\mathbf{X}$ explicitly, memorywise the complexity of the PnTA algorithm is the same as that of PTA.

We also reiterate that the vector $\mathbf{X}$ needs to be sorted, which is equivalent to sorting the inputs $\mathbf{x}$ jointly with the multiplicities of the inputs N (i.e., using the components of $\mathbf{x}$ as the key).

Let us list some useful properties of the function f_p generated by the PnTA algorithm.

Theorem 1 (The Inheritance Theorem) *The weighted extension f_p of a function f by the PnTA algorithm preserves the intrinsic properties of the parent function f as follows:*

1. *f_p idempotent since f is idempotent;*
2. *if f is monotone increasing then f_p is monotone increasing;*
3. *if f is continuous then f_p is continuous;*
4. *if f is convex (resp. concave) then f_p is convex (resp. concave);*
5. *if f is homogeneous then f_p is homogeneous;*
6. *if f is shift-invariant then f_p is shift-invariant;*
7. *f_p has the same absorbing element as f (if any);*
8. *if f generates f_p then a φ-transform of f generates the corresponding φ-transform of f_p.*

Proof The proof easily follows from the properties of composition of the respective functions and idempotency of f. For the φ-transform notice that at each inner level of the tree the composition $\varphi^{-1} \circ \varphi = Id$, while φ is applied to the leaves of the tree and φ^{-1} is applied to the root. □

Now we focus on the OWA functions as the base aggregator f. Here we can show the following.

Theorem 2 *Let $f = OWA_w$. Then the algorithm PnTA generates the weighted function f_p which is the discrete Choquet integral (and is hence homogeneous and shift-invariant).*

Proof The Choquet integral is a piecewise linear continuous aggregation function where the linear pieces are joined together at the intersections of the canonical simplices $\mathscr{S}_i$, i.e., where two or more components of $\mathbf{x}$ are equal. Since OWA_w is continuous and piecewise linear, so is f_p, by the properties of function composition. Now, let us show that the function f_p is not differentiable only on the sets where some of the components of the input vector $\mathbf{x}$ are equal, which will imply that the result is the discrete Choquet integral. Indeed at each node in the n-ary tree there is a function (OWA) not differentiable when some of inputs are equal. At the L-th level of the tree these are the points where the components of $\mathbf{x}$ are equal.

At the level $L-1$, the arguments of the OWA function are equal if and only if the arguments of the child nodes are equal, because the smallest argument of the left child node is no smaller than the largest argument of the right node (we recall that $\mathbf{X}$ is sorted). Continuing this recursion, we end up with the root node where the resulting function is not differentiable if and only if some of the arguments of the nodes at the bottom level L are equal, which are exactly the components of $\mathbf{x}$. Hence f_p is a piecewise linear, continuous aggregation function, and the linear pieces are joined together at the intersections of $\mathscr{S}_i$, so f_p is the discrete Choquet integral. □

As the special cases of Choquet integral we have the following results.

Theorem 3 *Let $f = OWA_w$. Then the algorithm PnTA generates the weighted function f_p with the following properties:*

1. *for the weights $w_i = \frac{1}{n}$, f_p is the weighted arithmetic mean with the weights* $\mathbf{p}$*;*
2. *for the weights $p_i = \frac{1}{n}$, f_p is OWA_w;*
3. *when $f = OWA_w = \min$ (or $= \max$) and $p_i > 0$ for all i, f_p is also* min *(respectively,* max*);*
4. *when $f = OWA_w = median$ and n is odd, f_p is the weighted median;*
5. *if OWA_w generates f_p, then the dual OWA_w^d generates the dual f_p^d, and in particular an OWA with the reverse weights generates the respective weighted OWA with the reverse weights.*

Proof 1. In this case OWA_w is the arithmetic mean, and hence f_p is the weighted arithmetic mean with the respective weights.

2. In this case, each argument x_i is repeated exactly n^L times, hence the inputs to each node of the n-ary tree (except the root node) are all equal, and by idempotency the tree is pruned to just one level, and hence delivers the original OWA_w function.
3. The function at each node in the tree returns the minimum (maximum) of its arguments, hence the result does not depend of the weights if they are strictly positive. However, when the weights p_i could be 0, the result is not true, as the smallest (largest) component of $\mathbf{x}$ can be excluded from the calculation of the minimum (maximum). Note that f_p is not a weighted minimum or weighted maximum functions, as those are the instances of the Sugeno and not Choquet integral.
4. The weighted median can be written as the median of a vector with the components repeated the relevant number of times, i.e., $median(\mathbf{X})$ [4], p. 121. While in general the median of medians of subsets of inputs is not the median of the whole set of inputs, for an odd n at each level of the tree the median is the value of the central child node, which in turn is the value of its central child node, and so on until

the bottom level where we get the centralof **X**; see Fig. 2 for an illustration. This statement does not work for an even n if we consider the lower (respectively, upper) medians, because now we need to take the value of the right middle child node at every level, but the lower median of **X** sorted in the decreasing order corresponds to $X_{n^L/2+1}$, which happens to be the left child of its parent node. This is clearly seen in the case of $n = 2$ where the lower median of the bivariate function f is the minimum of its arguments, hence $f_p(\mathbf{x}) = \min(\mathbf{X})$ which clearly does not always coincide with the median of **X**. For example, in the binary tree in Fig. 1 $median(\mathbf{X}) = X_5 = x_1$ whereas $f_p(x_1, x_2) = X_8 = x_2$.

5. Follows from the preservation of φ-transform. □

Theorem 4 *Let $f = OWA_w$ and let the weighting vector be decreasing (increasing). Then the algorithm PnTA generates a Choquet integral with respect to a submodular (supermodular) fuzzy measure.*

Proof An OWA function with decreasing weights is convex (respectively concave for increasing weights), and hence is a special case of the Choquet integral with respect to a submodular (supermodular) fuzzy measure [3]. Since the convexity (concavity) is preserved in the n-ary tree construction as per Theorem 1, the resulting weighted function f_p is also convex (concave), and hence it is a Choquet integral with respect to a submodular (supermodular) fuzzy measure [3]. □

This result is useful when constructing weighted norms from OWA with decreasing weights, see [3, 23].

On the technical size we note that we do not need to sort the arguments in each OWA function in the n-ary tree, as the vector $\mathbf{x}$ is already sorted, hence only one sort operation for the inputs is required. Another note is that when the weights $\mathbf{p}$ are specified to m digits in base n, $L = m$ levels of the n-ary tree is sufficient to match these weighs exactly. For example if $\mathbf{p}$ are specified to 3 decimal places and $n = 10$, we only need to take $L = 3$. Therefore to match the weights to machine precision (e.g., 53 bits for data type `double`) n^L need not exceed the largest 64-bit integer, and hence the algorithm PnTA can be implemented with 64-bit data types. The source code in C++ is presented in Fig. 3.

Finally by using Definition 8 we can introduce weights into generalized OWA functions in the same was as for OWA functions, by using the n-ary tree construction. This can be done in two ways: a) by using $GenOWA_{\mathbf{w},g}$ function as f, or b) by using a φ-transform of a weighted OWA with $\varphi = g$, that is, by applying g and g^{-1} only to the leaves and to the root node of the tree, relying on the preservation of φ-transforms. The second method is computationally more efficient as functions g and g^{-1} need not be used in the middle of the tree, where they cancel each other.

This way we also obtain the special cases of weighted OWG and weighted power based generalized OWA functions.

```
 double OWA(int n, double x[],double w[])
 { /* no sorting is needed when used in the tree */
    double z=0;
    for(int i=0;i<n;i++) z+=x[i]*w[i];
    return z;
 }
 double node(int n,double x[],long int N[],long int C,int & k,
     double w[],double(*F)(int,double [],double[]),double* z)
 {
   /* recursive function in the n-ary tree processing
   Parameters: x input vector, N vector of multiplicities of x
   m current level of recursion counted from root node L to 0
   k input-output parameter, the index of x being processed */
   if(N[k]==0) k++;
   if(N[k]>= C) {  /* we use idempotency to prune the tree */
       N[k] -= C;
       if(N[k]<=0) return x[k++]; else return x[k];
   }
    C /= n;
   /* tree not pruned, process the children nodes */
   for(int i=0;i<n;i++) z[i]=node(n,x,N,C,k,w,F,z+n);
   return F(n,z,w);
}

double weightedf(double x[], double p[], double w[], int n,
         double(*F)(int, double[],double[]), int L)
/*
 Function F is the symmetric base aggregator.
 p[] = array of weights of inputs x[],
 w[] = array of weights for OWA, n = the dimension of x, p, w.
 the weights must add to one and be non-negative.
 L = number of binary tree levels. Run time = O[(n-1)L]  */
{
   long int t=0, C=1;
   int k=0;
   for(int i=0;i<L;i++) C*=n;  /* C=n^L */
   sortpairs(x, x+n, p);
   long int N[n];/* multiplicities of x based on the weights*/
   for(int i=0;i<n-1;i++)  { N[i]=p[i]*C+1./n;   t+=N[i]; }
   N[n-1]=C-t;
   double z[n*L]; /* working memory */
   return node(n,x,N,C,k,w,F,z);
}
/* example: calling the function */
int n=4, L=4;
double x[4]={0.2,0.2,0.4,0.8};
double w[4]={0.1,0.2,0.3,0.4};
double p[4]={0.3,0.2,0.1,0.4};
double y=weightedf(x,p,w,n,&OWA,L);
```

Fig. 3 A C++ implementation of the pruned n-ary tree algorithm PnTA. The function `sortpairs` (not shown) implements sorting of an array of pairs (x_i, p_i) in the order of decreasing x_i

6 Conclusions

The proposed method of introducing weights into n-ary symmetric functions has several advantages. Firstly, it is a generic method universally applicable to any symmetric idempotent function, in particular to OWA functions. Secondly, the handling of the weights is transparent and intuitive: the weights correspond to the multiplicities of the arguments. Thirdly, many important properties of the base symmetric aggregator are preserved in the n-ary tree construction, which is very useful as these properties need to be verified only for the base aggregator. Finally, the pruned n-ary tree algorithm delivers a numerically efficient way of calculating weighed averages, among them the weighted OWA. This algorithm has complexity linear in n and the number of levels of the tree L, and L is bounded by the desired accuracy of the weights. We believe the n-ary tree algorithm constitutes a very competitive alternative to the existing weighted OWA approaches.

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Uncertainty Management: Probability, Possibility, Entropy, and Other Paradigms

Bernadette Bouchon-Meunier

Abstract Uncertainty modeling is a domain explored by researchers for centuries and it is difficult to bring stones to this long history of thought. Boris Kovalerchuk has been concerned for many years with specificities, relations and complementarities of the main paradigms enabling the construction of automatic systems able to handle imperfect information in a real-world environment, mainly probability theory and fuzzy set theory. We point out the necessity to cope with several aspects of uncertainty apparent in complex systems, mainly due to the complexity of natural phenomena and, more recently, to the size and diversity of artifacts, in addition to the necessity to take observers of the phenomena into account.

Keywords Uncertainty · Fuzzy sets · Probability · Complexity · Entropy · Perception · Subjectivity · Imprecision · Incomplete information

1 Introduction

Uncertainty modeling is a domain explored by researchers for centuries and it is difficult to bring stones to this long history of thought. After the preeminence of probability theory until the 1960s, the necessity to built automatic systems able to handle imperfect information in a real-world environment drove to the emergence of new paradigms, among which fuzzy set and possibility theories have led the field, mainly because of their efficiency to cope with the complexity of real-world problems. These theories are surrounded by evidence theory, imprecise probabilities, interval methods, non-classical logics, to name but a few among the most important paradigms to represent uncertainty and imperfect information. Boris Kovalerchuk has been concerned for many years with specificities of several of these paradigms, their relations and their complementarities and he has keenly explored solutions

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to make them collaborate, in particular in the Computing With Words approach [2, 13, 14]. In this paper, we point out the necessity to cope with several aspects of uncertainty underlying complex systems, mainly due to the complexity of natural phenomena and, more recently, to the size and diversity of artifacts, in addition to the necessity to take observers of the phenomena into account. We can consider that the imperfection of available real-world information takes three main forms bringing uncertainty, the first one being a doubt on the outcome of an experiment or in the forecasting of a system state. The second one is incompleteness of information, which entails uncertainty on unknown aspects of the studied phenomenon. The third one is imprecision and vagueness resulting either from the observer, due to natural ability of humans to cope with imprecise concepts and values or to inaccuracies of observation tools; imprecision and vagueness also exist in nature and provide uncertainty on precise values or states.

In Sect. 2, we insert the question of uncertainty management in approaches to complexity, mainly proposed by E. Morin and J.-L. Le Moigne. In Sect. 3, we review concepts of entropy as methods to deal with uncertainty, showing that several approaches are available, according to the level of complexity of the observed system we take into account. In Sect. 4, we propose to differentiate several dimensions of uncertainty to show that they must work together and adapt to fit the requirements of any given real-world problem.

2 Complexity Intelligence

In a restricted view, complexity intelligence could be regarded as an approach to deal with the present complexity of available data, due to the large amount and the heterogeneity of information available in a digital environment in the so-called big data paradigm. This aspect of complexity intelligence corresponds to the use of artificial intelligence approaches and, in particular, the efficiency of computational intelligence methods, to deal with large amounts of data, to explore them, to extract relevant information, to discover patterns, to detect exceptional elements, outliers or weak signals, in order to support decisions and to predict risk. This aspect is certainly a major one in the modern environment, but it does not cover the whole concept of complexity intelligence advocated by E. Morin [16] and J.-L. Le Moigne [15].

They consider that, in its general form, complexity intelligence refers to the impossibility to separate the observed world from the observer, and the necessity to take into account perception and context in the analysis of complex phenomena. Natural intelligence is then involved in the handling of complexity inherent in all systems, in the construction of models and the communication of knowledge. According to E. Morin, consciousness of complexity is necessary to understand a phenomenon and to generate science. Progresses made by humans on scientific certainties induce progresses on their uncertainties and recognition of their ignorance associated with their knowledge. E. Morin claims that principles of incompleteness and uncertainty are at the heart of complexity intelligence, even though it is based on a kind of multidimensional thinking tending to take into account all aspects of the observed phenomenon.

In [15], the authors point out the necessity for Science to be conscious of its own complexity, impossible to avoid because of relations between humans and the observed world, be it natural or artifactual. They recommend that science of complexity produces concepts and theories supporting intelligibility and being in the realm of the possible, rather than the necessary. They claim that, before being conscious of the need to approach complexity through intelligence, perception and understandability, science was based on four "pillars of certainty" tending to eliminate complexity and to introduce simplicity in order to grasp complex phenomena. E. Morin [15] considers that the first pillar of certainty is the absolute order, considering that a strict ordering governs everything in the universe and disorder can only be the consequence of a lack of knowledge. The second pillar of certainty is the separability principle, associated with the capability to split any problem into simple elements, forgetting the whole and the existing relations between parts of the system. The third pillar of certainty is the reduction principle, reducing knowledge to measurable quantities. The fourth pillar of certainty is inductive/deductive logic, denying concepts like creation, abduction and hypothesis-based reasoning.

E. Morin insists on the need to take into account qualitative, as well as quantitative effects and to accept contradictions in observations. Uncertainty is unavoidable, as is complexity inherent in the real world, and it takes various forms, such as "logical uncertainty", "empirical uncertainty", or "cognitive uncertainty" referring to human mental categories. E. Morin and Le Moigne's vision is embedded in system science, the foundations of which have been laid by Ludwig von Bertalanffy in the fifties, at the same time as cybernetics was emerging, disseminating the concept of entropy. It is worth noting that their work is related to the so-called second-order cybernetics, invented by H. von Foerster [1, 17] and regarded as the cybernetics of observing systems, as opposed to the cybernetics of observed systems. The observer is clearly involved in the analysis of complexity and the management of uncertainty, involving perception and taking into account context and environment of the system.

Such an approach leads to search for solutions to deal with complexity through the representation of incomplete information, the preservation of possible situations even in the case where they seem incompatible and the capacity to revise and update information in non-classical logics. Furthermore, it appears that the world complexity cannot be grasped by means of the only utilization of probabilities to cope with so diverse uncertainties. We can think of L.A. Zadeh's search for the involvement of perception in the automated management of information, and in particular his perception-based theory of probabilistic reasoning [20] or a computational theory of perception [19] adding to classic probability theory or predicate logic the capacity to compute and reason with perception-based information.

More generally, a fuzzy set-based knowledge representation following L.A. Zadeh's seminal work and the associated possibility theory provide solutions to escape the four above-mentioned pillars of certainty. Among fuzzy relations, fuzzy orders, similarity and indistinguishability relations provide methods to avoid absolute orders. Partial membership, fuzzy partitions, overlapping classes enable to weaken the separability principle, according to human capabilities of considering unsharp categories and to accept contradictions to some extent. Fuzzy sets, providing a numerical/symbolic interface, are source of expressiveness and linguistic summarization of numerical data can be achieved, as an alternative proposal to reduction principle. Finally, approximate reasoning, general modus ponens, fuzzy inductive reasoning, fuzzy abduction and case-based reasoning, are solutions to avoid the rigidity of deduction in classical logic, when suitable.

3 Entropy

After means of representing uncertainty reviewed in Sect. 2, we can consider means of evaluating uncertainty, which are mainly studied in information theory. We can establish a parallel between these two paradigms and their history.

The previous proposals to deal with complexity go beyond traditional knowledge representation methods, by means of the involvement of the observer and his perceptions in the representation of uncertainty in complex systems. They can be compared to novel information theory that also appeared in the 1960s, after the first concept of entropy proposed by C. Shannon in coding theory and N. Wiener at the origin of cybernetics, in 1948. This concept was dedicated to communication and information transmission and was considered as syntactical, with no semantic value. This is the reason why a more general theory of information was proposed by J. Kampé de Fériet and B. Forte [10] as a theoretical approach to the concept of information, based on axioms [8] not necessarily dependent on probabilities and taking an observer into account. We see again, as we did in E. Morin's approach to uncertain information representation, that observers take part in the evaluation of uncertain knowledge, with a role assigned to a "headquarter" [11] in the case where there exist several observers. The objective of the authors is to accept subjectivity in information.

Another extension of the original concept of entropy has been simultaneously proposed in [3, 9] to take into account qualitative aspects of information related to its utility for the fulfillment of a goal, considering again the context of the observed phenomena. A fresh look at the concept of entropy was taken by De Luca and Termini [7] in their definition of a non-probabilistic entropy regarded as a measure of a quantity of information not necessarily related to random experiments, provided by a different kind of uncertainty based on a fuzzy set-based knowledge representation.

Such works were the presages of a long list of extensions of the concept of entropy [12] supposed to go along with various uncertain knowledge representation, such as intuitionistic entropy [6], or entropy in a mathematical theory of evidence [18]. The reasons why the introduced quantities are called entropy, as well as their properties, are diverse [5], but in any case they are assumed to evaluate a degree of imperfection taking a part in some uncertainty inherent in an observed system, whatever the chosen means of representing uncertainty are.

4 Dimensions of Uncertainty

Introduced and developed in the 17th century by B. Pascal, P. de Fermat and J. Bernoulli, probability has been the main mathematical concept to represent uncertainty during three centuries. Subjective probabilities were a first attempt to soften this concept in order to take into account the observer, to some extent, and they were proposed by F. Ramsey and B. de Finetti in the 1930s. Another alternative to classic probabilities was introduced by A. Dempster in 1967 to encompass a higher form of uncertainty, regarded as uncertainty on degrees of uncertainty, by means of upper and lower probabilities, then developed by G. Shafer in 1976 as evidence theory. All these works were addressing the problem of *doubt and uncertainty about the outcome* of an experiment or the state of a phenomenon [2], which can be regarded as the first level of uncertainty in complex systems. Examples of such intrinsic uncertainty are "Peter may attend the meeting today" or "Mr. X will probably be elected by the assembly".

A second level of uncertainty corresponds to a *doubt resulting from an imprecision, an approximation, or from incomplete knowledge*. For instance, "I will be at the railway station around noon" is based on imprecise information and entails an uncertainty on my exact arrival time and the fact that I can catch a train leaving at noon. This level of uncertainty was first tackled by L.A. Zadeh in 1965 in his seminal paper on fuzzy sets, and also by Ramon Moore in 1966 in his book on interval analysis addressing a narrower aspect of imprecise information. It is only in the framework of fuzzy set theory that the induced uncertainty itself is represented by means of possibilities, introduced by L.A. Zadeh in 1978.

A third level of uncertainty can be identified, corresponding to a *doubt due to a subjective appreciation or a judgment* expressed by an individual. For instance, "I don't believe that Peter is in Paris" or "I am not sure that Mr. X will be elected by the assembly". Such appreciation can autonomously rate the doubt on a fact or be regarded as some meta-uncertainty added to an uncertainty of the first or second level. In addition to possibilistic logic developed in the environment of possibility theory, extensive work has been published on non-classical logics such as modal logics since the 1960s, nonmonotonic reasoning to cope with evolving information and its revision since 1980, Truth Maintenance Systems introduced by J. Doyle in 1979 to enable the revision of beliefs, autoepistemic logics introduced by Robert Moore in 1983 to take into account partial information with semantical considerations, probabilistic logic introduced by N.J. Nilsson in 1986.

This list is not exhaustive, as the development of non-classical logics has been intense, but our purpose is to point out the existence of solutions to cope with uncertainty, answering E. Morin and J.-L. Le Moigne's concerns and accepting uncertain information instead of eliminating it, considering the need to revise beliefs and to take context and observer into account in the representation of complex situations.

This typology of levels of uncertainty can be completed by a view of uncertain information representation methods along three dimensions according to the nature of the addressed uncertainty [2], with the aim of choosing an appropriate method when facing a real-world problem.

The first dimension refers to the distinction between *numerical and symbolic information*: probabilities or masses of assignment in evidence theory are examples of methods to deal with numerical information ("Mr. X is expected to be elected with 62% of the votes"), while modal logics manage symbolic uncertain information ("I believe that Mr. X will be elected"). Fuzzy set-based methods and the associated possibility theory are intermediate between these kinds of information, coping with numerical data by means of a symbolic representation, while equipping knowledge representation with capabilities of interpretability and expressiveness ("Mr. X is expected to be elected with a large majority of the votes").

The second dimension of uncertain information corresponds to the distinction between *intrinsic and extrinsic uncertainty*, the first one being attached to the real world phenomenon, the second one being due to the process of observation itself. Interval analysis can be used to handle extrinsic uncertainty deriving for instance from measurement errors ("within 10% of the measured value"), while fuzzy classes can also address various cases of intrinsic uncertainty, such as the contours of regions in a digital image or categories with blurred boundaries like "young" and "old".

The third dimension of uncertain information in a complex world indicates a gradual degree of *subjectivity*, from an objective measurement to a subjective evaluation [4], from physical properties of an object to its perceived properties, for instance from the numerical representation of colors from red ("wavelength between 620–700 nm", "RGB: 255, 36, 0") in a digital image to the perception of a dominant color by a human agent ("red"), and furthermore to a subjective appreciation, such as a feeling or an emotion ("passion", "energy").

It is worth noting that these dimensions of uncertain information do not generally appear independently and representation methods can be regarded in this three dimensional space. We give a few examples of elements in this space.

The most classic uncertainty representation regards numerical intrinsic uncertainty, dealt with by means of objective or subjective probabilities, according to the importance attached to subjectivity.

Examples of objective symbolic intrinsic uncertainty can be associated with imprecise descriptions of variables managed through fuzzy sets in the case of objective and measured data. For instance [2], in the case of descriptions of spots on a mammography, qualified as "round" by a medical doctor, an uncertainty comes from the imprecise description, since "round" is only a linguistic approximation of the mathematical characterization "circular", and the membership function is obtained in a machine learning-based process from measurements of various criteria such as convexity or elongation, performed during the image processing.

Objective numerical extrinsic uncertainty is present in sensor imprecision or in estimations. Interval analysis, fuzzy sets, or confidence intervals can be used to deal with it.

A subjective intrinsic symbolic uncertainty can be identified in felt probabilities (such as "highly probable"), which can be represented by fuzzy values, whose membership functions can be obtained by means of a psychometric approach.

Subjective extrinsic numerical uncertainty is identified in web-based information quality scoring, and it can be managed by means of possibility theory or evidence theory, for instance.

A symbolic extrinsic source of uncertainty is related to the difficulty to characterize a given complex phenomenon, solved by means of linguistic expressions. An example of objective such uncertainty appears when it is impossible to obtain precise values from an observer, for instance evaluating a distance ("far from the house"), this information being easily represented through fuzzy sets. The case of a subjective symbolic extrinsic uncertainty is observed when the observer expresses a doubt on the validity of data ("I believe these news"), and modal logic is one of the candidates to cope with it.

In real-world applications, the selection of an uncertainty modeling technique relies on the nature of uncertainty present in the problem to solve or the phenomenon to observe. It is also oriented by the necessity to obtain an expressible result and/or a numerical coefficient or mark expressed by users. Let us remark that, in some situations, it may be interesting to preserve the uncertainty associated with the outcomes of a system and to enable the user to use his/her expertise to make a final decision. In other words, it may be important to know that two different events may occur and to be prepared to both of them, which can be achieved by means of possibility theory or fuzzy logic, rather than to look for a deterministic decision. In addition, it is often interesting to use several uncertainty modeling techniques simultaneously in a given environment, for instance probabilities and fuzzy sets, which is possible in the so-called Soft Computing paradigm.

5 Conclusion

We have presented a prism to study uncertainty in an operational environment, with the aim of helping users to appropriately choose a knowledge representation method. We have situated this study in the analysis of complex systems, in which uncertainty is fundamentally inherent. We have shown three levels and three dimensions of uncertainty and we claim that it is impossible to reduce one form of uncertainty to another one to deal with complex systems.

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Relationships Between Fuzziness, Partial Truth, and Probability in the Case of Repetitive Events

Jozo Dujmović

Abstract Frequency-based probabilistic models are suitable for the quantitative characterization and analysis of repetitive events. Quantitative models based on the concept of fuzzy set can be applied both in the case of repetitive events and in cases where frequency-based probabilistic models are not appropriate. In the case of repetitive events there is a possibility of comparison of relationships between probabilistic and fuzziness-based interpretations of the same physical or perceptual reality. In this paper we analyze these relationships using three characteristic examples and show that in the case of repetitive events, fuzziness, probability and partial truth are three coexisting compatible interpretations of the same reality, i.e. practically equivalent concepts.

1 Introduction

Relationships between fuzziness and probability have a long history of controversial opinions and hot discussions. The latest contribution to this area can be found in [6, 8]. In spring 2014, BISC community (Berkeley Initiative in Soft Computing) was a forum for exchanging a variety of opinions about fuzziness, probability, possibility and partial truth, as formalisms for dealing with uncertainty. This paper includes material that was initially used as author's contribution to that discussion.

Repetitive events are events that have predecessors and successors. Any form of life creates repetitive events. Some sequences of events have the first and the last event in the sequence, but the nature of sequence is still repetitive. Even the first event in the sequence can frequently be interpreted as a repetition of the first event in some similar previous sequence that consisted of similar events. Since the repetitive events are defined using predecessors and successors, singular events should be defined as events without predecessor and successor. Taking into account that predecessors and successors only need to be sufficiently similar to an analyzed event, it seems

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to be extremely difficult to find truly singular events. Even if such events can be identified, it is very easy to show that they are a negligible minority compared to repetitive events. Therefore, whenever in this paper we speak about probability and probabilistic models we always assume frequency-based probability and frequency-based probabilistic models derived from observation of repetitive events.

Most of human experiences are repetitive. For example, human work in any profession is essentially a repetitive experience that consists of many days spent working with similar people, offering similar services, and solving similar problems. Regular use of machines and tools (e.g. cars and computers), by both individuals and corporate users creates repetitive events where some experiences are more positive (machines provided satisfactory service) and some are less positive (performance of machines was not sufficiently satisfactory). Human physical and other properties (e.g. weight, height) are also repetitive and can be analyzed using data about selected populations of human subjects.

It is useful to note that some events are only seemingly singular, i.e. they are interpreted with intention to be classified as singular. A favorite recent example is the possible election of a female president in a country that had a long sequence of male presidents. Of course, one could argue that this is a singular event, but it would be much easier to see that the election of female president follows the same rules as the election of all previous male presidents, and that such a "singular event" already occurred many times in many countries that had a sequence of male presidents followed by a female president. Interpreting events as repetitive is much easier than interpreting them as singular.

Generally, repetitive experiences are neither identical nor certain—they always vary in a specific range and create human perceptions of satisfaction, suitability, quality, value, etc. In such cases some properties are objectively measurable (e.g. the height of a car driver), but much more frequently we have to deal with human percepts that are not measurable. Of course, human perceptions can be modeled using formal models, and such models create the areas of computing with words [9], perceptual computing [7], and aggregation logic [1, 5].

Uncertainty is a human property that primarily reflects a lack of information and human inability to accurately predict future events. Formalisms for dealing with uncertainty are developed with general intention to reduce uncertainty and help in precision of meaning and better describing perceptions in a natural language.

The frequency-based probabilistic approach to reducing uncertainty consists of observing and recording frequencies of past events, and then using them to predict the likelihood of similar future events (e.g. to determine, with a given degree of confidence, intervals where events are likely to occur). Fuzzy approach is based on reducing uncertainty by using fuzzy set membership functions where high degree of membership reflects a high certainty that an object satisfies conditions that define members of a specific fuzzy set. Logic approach to reducing uncertainty consists of computing a degree of truth of assertion that an object has specific properties [2]. If the degree of truth is high, then the certainty of assertion is also high, contributing to precision and reducing the uncertainty. In this paper, our goal is to use characteristic examples to show that in the case of repetitive events fuzzy approach, probabilistic approach, and soft computing logic approach are similar and very frequently equivalent.

2 The Case of Tall Drivers

Let us consider a set of n licensed car drivers (in the USA $n > 200 \times 10^6$). Let h denote the height of a car driver (written in each driver license), and let us sort car drivers according to their height:

$$h_{\min} = h_1 < h_2 < \cdots < h_n = h_{\max}.$$

The heights are real numbers, and for simplicity we assume that all heights are different. The function $P : [h_{\min}, h_{\max}] \to [0, 1]$ denotes the fraction of remaining $n - 1$ drivers that are shorter than the i-th driver:

$$P(h_i) = (i - 1)/(n - 1), \quad i = 1, \ldots, n$$

$$P(h_1) = 0, \quad P(h_n) = 1$$

A typical shape of this function (for large n) is shown in Fig. 1.

Let us now consider a fuzzy set of tall drivers. The membership function of such a fuzzy set $\mu_{\text{tall}}(H)$ could be defined in many different ways using various arbitrarily selected intervals of height H. Since the distribution in Fig. 1 represents the objective reality, it is very reasonable to claim that only the tallest of all n drivers has the distinct privilege to be the "full member" of the tall driver fuzzy set, i.e., $\mu_{\text{tall}}(h_n) = 1$. Once we decided the status of the tallest driver, we must do the same with the shortest driver: we declare the shortest driver to be the one and only full member of the fuzzy set of short drivers, i.e. $\mu_{\text{short}}(h_1) = 1$. Furthermore, $\mu_{\text{tall}}(H) + \mu_{\text{short}}(H) = 1$.

Another self-evident decision is related to the driver whose membership satisfies $\mu_{\text{tall}}(H) = \mu_{\text{short}}(H) = 1/2$. The most natural way to select such a driver is by using the median of all heights $H = h_{med}$; assuming that n is odd, the median driver has the height $h_{med} = \mu^{-1}_{\text{short}}(1/2) = \mu^{-1}_{\text{tall}}(1/2)$, i.e., approximately 50% of all drivers are

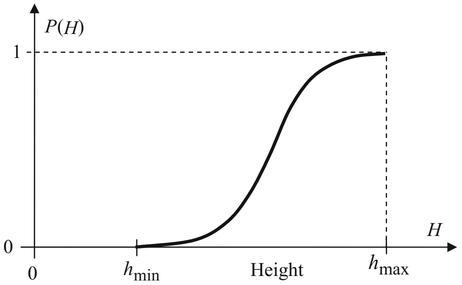


Fig. 1 A characteristic shape of the $P(H)$ function

taller than the driver *med* and 50% of all drivers are shorter than the driver *med*. If this reasoning is acceptable, then the most realistic membership functions are the following:

$$\mu_{\text{tall}}(h_i) = (i-1)/(n-1)$$

$$\mu_{\text{short}}(h_i) = (n-i)/(n-1)$$

Consequently, $\mu_{\text{tall}}(H)$ is exactly the function shown in Fig. 1.

Let us now investigate an arbitrary driver D whose height is h_m, and let us find the degree of truth of the statement "D is a tall driver." Again, it seems impossible to find reasons why this degree of truth should differ from $\mu_{\text{tall}}(h_m) = (m-1)/(n-1)$, which is the fraction of drivers who are shorter than D. In other words, the degree of truth for "D is a tall driver" is the same as the degree of membership in the fuzzy set of tall drivers, and the probability that a randomly selected driver is shorter then D.

The height of a driver is obviously a repetitive event that occurs with measurable frequency. The number of drivers having the height $h \in [a, b]$ is $n[P(b) - P(a)]$. If we measure frequencies of drivers $f_1, \ldots, f_k$ in k equidistant subintervals of $[h_{\min}, h_{\max}]$, and normalize the frequencies, $p_i = f_i/(f_1 + \ldots + f_k)$, $i = 1, \ldots, k$, then, for large k and n we can consider that h is a continuous random variable, and we get the probability density function of driver height $p(H)$, $0 < h_{\min} \le H \le h_{\max}$ and the (cumulative) probability distribution function $P(H) = \int_{h_{\min}}^{H} p(t)\,dt$. Unsurprisingly, for all practical purposes related to modeling human reasoning, the function shown in Fig. 1 is also the probability distribution function for the height of a driver, i.e. $\Pr[h \le H] = P(H)$ and $\Pr[a < h \le b] = P(b) - P(a)$. The mean height of drivers is the following:

$$\bar{h} = n^{-1}(h_1 + \ldots + h_n) = \int_{h_{\min}}^{h_{\max}} t p(t)\,dt =$$

$$\int_0^{h_{\max}} [1 - P(t)]\,dt = h_{\max} - \int_{h_{\min}}^{h_{\max}} P(t)\,dt.$$

Therefore, in the repetitive case of a measurable physical property (height) of a large number of drivers, we have that the fuzzy set membership function coincides with the probability distribution function. In addition, the degree of fuzzy set membership of an object can be naturally interpreted as the degree of truth of a statement claiming that the object is a full member of the set.

3 The Case of Job Satisfaction

Job satisfaction is not an objective physical property. However, it is a well defined perceptual variable. Each worker has a clear percept of the degree of job satisfaction, and such percept can easily be quantified. For example, consider the statement "I am asked to indicate, on a scale from 0 to 1, the degree to which I like my job. My answer is: 0.7." This is a value statement (it reports the result of an intuitive evaluation). Obviously, this statement indicates that the degree of truth of the statement "I have a perfect job" is 0.7. Following are two interesting questions related to this statement:

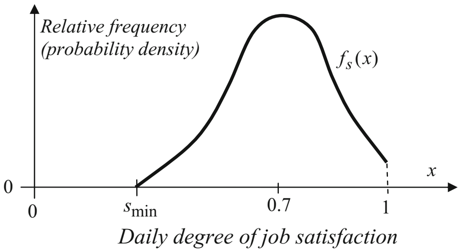
(a) Is this statement a consequence of the repetitive nature of job satisfaction?
(b) Is this statement yielding equivalent fuzzy, probabilistic, and truth value interpretations?

Job satisfaction is a compound perception. It is affected by many factors, including monetary compensation, fringe benefits, opportunities for professional growth, social recognition, degree of job-related stress, relationships with coworkers and managers, etc. There is no doubt that each job creates repetitive experiences every day spent at work. Job satisfaction can be interpreted as a quantifiable perception that can be created at the end of each working day. The degree to which a worker likes a job can be interpreted as an overall perception of job satisfaction obtained by averaging all repetitive daily job satisfaction degrees.

A typical normalized relative frequency distribution of a daily job satisfaction perception s is shown in Fig. 2. So, $\int_{s_{\min}}^{1} f_s(x)\,dx = 1$ and the corresponding probability distribution function of job satisfaction is $F_s(x) = \Pr[s \le x] = \int_{s_{\min}}^{x} f_s(t)\,dt$. The average daily job satisfaction is $S = \overline{s} = \int_{s_{\min}}^{1} x f_s(x)\,dx = 1 - \int_{s_{\min}}^{1} F_s(x)\,dx$ and it can also be interpreted as the degree of truth of the statement "I have a perfect job". So, S is a cumulative perception of job satisfaction that in the above example was reported to be $S = 0.7$. This is also a degree of membership of the current job in the fuzzy set of "worker's ideal jobs," as well as a degree of likelihood that a randomly selected day at work will completely satisfy worker's expectations. On the other hand, the probability distribution function $F_s(x)$ shows the probability $\Pr[s \le x]$, and it can be interpreted as the membership function of the daily job satisfaction perception x in a fuzzy set of "high job satisfaction days."

Note that all these interpretations hold only for one specific individual. In the next iteration we can analyze a group of n related individuals (e.g. medical doctors, or corn farmers, or software engineers) with respect to their perceptions of the average job satisfaction $S_1, \ldots, S_n$. Here we can directly apply the same reasoning used for the fuzzy set of tall drivers. For example, we can sort the average job satisfaction of all medical doctors, $S_{\min} = S_1 < \cdots < S_n = S_{\max}$, and then the

Fig. 2 A typical shape of the daily job satisfaction distribution for a specific worker



membership in the fuzzy set of job-satisfied medical doctors is the same as the probability distribution function of the average MD job satisfaction: $P_{MD}(S_i) = \Pr[S \leq S_i] = (i-1)/(n-1) = \mu_{MD}(S_i)$. The mean value of this probability distribution $\overline{S} = s_{\max} - \int_{S_{\min}}^{S_{\max}} P_{MD}(x)\,dx$ can be used as the degree of truth of the statement "medical doctors are completely satisfied with their jobs."

Therefore, this example demonstrates that job satisfaction experiences are repetitive both at the level of individual worker, and at he level of a professional group. In both cases, probabilistic, fuzzy and truth value interpretations are equivalent: the job satisfaction probability distribution is the same as the corresponding membership function in the fuzzy set of satisfied workers, and the mean degree of satisfaction can be interpreted as the degree of truth of the statement claiming a complete satisfaction.

4 The Case of Computer Selection

The percept of value is one of the most frequent human percepts. It is not an objectively measurable property of evaluated object. Our final example investigates computer evaluation and selection, i.e. a decision process of predicting the value of a computer system for a specific stakeholder. The stakeholder is a company (or an individual) who buys a computer system in order to attain specific goals. The computer selection process consists of computing the overall suitability of each competitive computer system and selecting the computer that has the maximum suitability.

Each computer has many quantitative and qualitative parameters and there is no simple probabilistic model for computing the probability of satisfaction of specific stakeholder. Similar difficulties are encountered in defining a precise overall membership function of each competitive computer in the fuzzy set of computers that completely satisfy the stakeholder. What can be done, however, is to use the soft computing evaluation logic methodology and compute the degree of truth of the statement claiming that a given computer completely satisfies justifiable requirements of the stakeholder. That degree of truth is a function of degrees of truth of statements that significant computer attributes satisfy stakeholder's requirements. The computation of the overall degree of truth can be based on a propositional calculus presented in [2, 3]; that calculus is the main component of the Logic Scoring of Preference (LSP) evaluation method [4].

If the evaluation is based on the LSP criterion model, then the first step consists of identifying a set of attributes $a_1, a_2, \ldots, a_n$ that affect the capability of computer system to attain stakeholder's goals. For complex computer systems typical value of n is from 80 to 120. For each attribute it is necessary to define an attribute criterion that specifies the attribute degree of suitability. For example, if a_i denotes the memory capacity, then a simple form of attribute criterion for computing the memory capacity suitability degree x_i might be the following:

$$x_i = g_i(a_i) = \max\left[0, \min\left(1, \frac{a_i - M_{\min}}{M_{\max} - M_{\min}}\right)\right], \quad a_i > 0, \quad 0 \leq x_i \leq 1$$

If $a_i \le M_{\min}$, then $x_i = 0$ and if $a_i \ge M_{\max}$ then $x_i = 1$. Between $M_{\min}$ and $M_{\max}$, the degree of suitability is approximated as a linear function of a_i. Therefore, the available memory must be greater than the threshold value $M_{\min}$; similarly, $M_{\max}$ denotes the maximum necessary memory (buying more would not improve performance but would decrease affordability). The selections of both the attributes and the parameters of their criteria are based on experiences with previous (and/or similar) computer systems, i.e. on repetitive events of satisfaction/dissatisfaction with attributes of previously owned computers running stakeholder's workload. The suitability degree x_i is interpreted as the degree of truth of assertion that the available memory perfectly (completely) satisfies stakeholder's requirements. It can also be interpreted as the degree of membership in the fuzzy set of fully satisfied memory capacity requirements.

After the specification of all attribute criteria and attribute evaluation we have n attribute suitability degrees $x_1, x_2, \ldots, x_n$ that are continuous logic variables affecting the overall degree of suitability x. The aggregation of $x_1, x_2, \ldots, x_n$ is a logic process because these values must satisfy various logic conditions: some groups must be simultaneously satisfied, some attributes can replace each other, some are mandatory, some are optional, etc. Similarly to attribute criteria, these conditions are also derived from experiences generated using previous stakeholder's computer systems. The result of logic aggregation is the overall suitability

$$x = L(x_1, \ldots, x_n) = L(g_1(a_1), \ldots, g_n(a_n)), \quad 0 \le x \le 1.$$

A more detailed description of the logic aggregation process based on the Logic Scoring of Preference aggregators can be found in [2, 4], and discussion of relationships of mathematical models of aggregation and observable properties of human reasoning can be found in [1].

In agreement with the interpretation of attribute suitability degrees, the overall suitability degree x is interpreted as the degree of truth of assertion that the evaluated computer system perfectly (i.e. completely) satisfies all stakeholder's requirements. Similarly to previous examples, it can also be interpreted as the degree of membership of the analyzed computer in the fuzzy set of perfectly suitable computers. In addition, x can also be used as a predictor of the likelihood (or probability) that at any time in the future the analyzed computer will perfectly satisfy stakeholder's needs. In other words, if we have the equivalence between the overall degree of truth, fuzzy membership, and probability, then any one of them can be used to determine or approximate the other two. Compared to previous examples, the case of computer selection has the least visible repetitive properties. Regardless a generally modest number of previously owned computers, the experiences that are necessary to specify evaluation criteria (the selection, structure and parameters of attribute criteria, and the structure and parameters of the suitability aggregation models) can also be obtained from similar users having similar computers and running similar workload, as well as from domain expert knowledge and professional literature. In other words, wherever decisions are made using professional experiences of stakeholders, evaluators, and domain experts, such knowledge can only come from previous repetitive experiences accumulated during education, training, practical work and/or solving similar problems.

5 Discussion and Conclusions

Presented examples show that in the case of repetitive events we can identify an observable object and select, collect, and analyze one or more of its measurable attributes. Each selected attribute can be an objective physical property or a perceptual variable. In all cases we can assume the existence of a number of measurable observations of each selected attribute. In such cases the attribute probability distribution function, the fuzzy set membership function, and the degree of partial truth of the statement claiming the full satisfaction of membership requirements are *equivalent interpretations* of the same physical reality. The prerequisite for the equivalency is the repetitive character of the analyzed variable (e.g. the height of driver) and an appropriate and justifiable interpretation of the concepts of fuzziness, probability, and partial truth.

Generally, for any observable repetitive attribute a that has the probability distribution function $\Pr[a \le X] = P_a(X), a_{\min} \le a \le a_{\max}$ where $P_a(X) = 0$ for $X \le a_{\min}$ and $P_a(X) = 1$ for $X \ge a_{\max}$ we can define a fuzzy set of "objects that have a large value of attribute a" (i.e. objects whose value of a is close to $a_{\max}$). The membership function in such a fuzzy set for an object having $a = x$ is $\mu_{\text{large } a}(x) = P_a(x)$. The degree of truth of the statement "the object having $a = x$ is the largest" is $\mu_{\text{large } a}(x)$.

It is important to note that a very large number of cases fit in this model. In particular, all professional evaluation problems fall in this category. Indeed, all evaluation criteria are based on evaluator's previous experiences and reflect repetitive character of the evaluation process. It is not possible to differentiate good and bad values of attributes unless the evaluator has knowledge derived from repetitively performing similar evaluations. If a justifiable evaluation model generates for object Ω a resulting degree of suitability $S \in [0, 1]$, then S can be equivalently interpreted in three ways:

(1) S is a degree of truth of the statement "Ω is perfectly satisfying all requirements."
(2) S is the degree of membership of Ω in a fuzzy set of perfect objects.
(3) S is a degree of likelihood (probability) that Ω will deliver a perfect performance.

Let us emphasize that in evaluation problems S is a quantitative estimate (or predictor) of the human percept of the overall (compound) value/quality/suitability. Thus, there is no objective measurable value that could be compared with S. On the other hand, everybody understands that there are stakeholders and their goals and requirements, and that these goals and requirements can regularly be only partially satisfied. The likelihood of full satisfaction, whatever label is used for it (probability, or fuzzy membership, or partial truth), is going to be predicted using the only indicator we have, the overall suitability S. In this situation the concepts of probability, fuzzy membership, and partial truth become equivalent interpretations of a clearly observable but objectively nonmeasurable human percept. Of course, the equivalence of a probability distribution function and a fuzzy set membership function is *not* a general mathematical property based on fundamental assumptions or axiomatic origins of these concepts. However, it is a rather frequent consequence of *interpretations and use* of those concepts in the context of observable human reasoning.

The repetitive nature of human experiences is frequently underestimated or neglected. In many discussions that compare fuzzy and probabilistic approach some authors use the concept of singular events, as unpredictable events that happen without predecessors and consequently their probability cannot be determined using frequency-based probability theory. However, it is extremely difficult, if not impossible, to find events without predecessors, and those that might (partially) qualify for that status usually deserve significantly less attention than the huge majority that has predecessors and can be characterized as repetitive. Since the observable repetitiveness in finite populations (clearly visible in the case of tall drivers) yields probability distribution function that can also serve as a fuzzy membership function, as well as a degree of truth of a related assertion, it seems that the situations where the concepts of probability, fuzzy membership, and partial truth are very close or fully equivalent significantly dominate the situations where that is not the case.

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Using Extended Tree Kernel to Recognize Metalanguage in Text

Boris A. Galitsky

Abstract The problem of classifying text with respect to metalanguage and language-object patters is formulated and its application areas are proposed. We extend parse tree kernels from the level of individual sentences towards the level of paragraphs to classify texts at a high level of abstraction. The method targets the text classification tasks where keyword statistics is insufficient for text classification tasks. We build a set of extended trees for a paragraph of text from the individual parse trees for sentences. Conventional parse trees are extended across sentences based on anaphora and rhetoric structure relations between the phrases in different sentences. Tree kernel learning is applied to extended trees to take advantage of additional discourse-related information. We evaluate our approach in the security-related domain of the design documents. These are the documents which contain a formal well-structured presentation on how a system is built. Design documents need to be differentiated from product requirements, architectural, general design notes, templates, research results and other types of documents, which can share the same keywords. We also evaluate classification in the literature domain, classifying text in Kafka's novel "The Trial" as metalanguage versus novel's description in scholarly studies as a mixture of metalanguage and language-object.

Keywords Tree kernel · Formalizing discourse · Language-object and metalanguage · Document analysis

1 Introduction

In the majority of text classification problems, keywords statistics is sufficient to determine a class. Keywords are sufficient information to determine a topic of a text or document, such as software vs hardware, or pop rock vs punk. However, there are classification problems where distinct classes share the same keywords, and document phrasing, style and other kinds of text structure information needs

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to be taken into account. To perform text classification in such domain, discourse information such as anaphora and rhetoric structure needs to be taken into account.

We are interested in classifying a text belonging to metalanguage or language-object. If a text tells us how to do things, or how something has been done, we classify this text as a language-object. If a text is saying how to write a document which explains how to do things, we classify it as metalanguage. Metalanguage is a language or symbolic system used to discuss, describe, or analyze another language or symbolic system. In theorem proving, metalanguage is a language in which proofs are manipulated and tactics are programmed, as opposed to the logic itself (the object-language). In logic, it is a language in which the truth of statements in another language is being discussed.

Obviously, using just keyword information would be insufficient to differentiate between texts in metalanguage and language-object. Use of parse trees [11] would give us specific phrases in use by texts in metalanguage, but still it will not be sufficient for systematic exploration of metalanguage-related linguistic features. It is hard to identify these features unless one can analyze the discourse structure, including anaphora, rhetoric relations, and interaction scenarios by means of communicative language [28]. Furthermore, to systematically learn these discourse features associated with metalanguage, we need a unified approach to classify graph structures at the level of paragraphs [5, 8, 14].

The design of syntactic features for automated learning of syntactic structures for classification is still an art nowadays. One of the approaches to systematically treat these syntactic features is the set kernels built over syntactic parse trees. Convolution tree kernel [2] defines a feature space consisting of all subtree types of parse trees and counts the number of common subtrees as the syntactic similarity between two parse trees. Tree kernels have found applications in a number of NLP tasks, including syntactic parsing re-ranking, relation extraction, named entity recognition [4] and Semantic Role Labeling [23, 32], relation extraction, pronoun resolution [18], question classification and machine translation [29, 30].

The kernel's ability to generate large feature sets is useful to assure we have enough linguistic features to differentiate between the classes, to quickly model new and not-well-understood linguistic phenomena in learning machines. However, it is often possible to manually design features for linear kernels that produce high accuracy and fast computation time, whereas the complexity of tree kernels may prevent their application in real scenarios. Support Vector Machines (SVM [31]) can work directly with kernels by replacing the dot product with a particular kernel function. This useful property of kernel methods, that implicitly calculates the dot product in a high-dimensional space over the original representations of objects such as sentences, has made kernel methods an effective solution to modeling sentence-level structures in natural language processing (NLP).

An approach to build a kernel based on more than a single parse tree has been proposed, however for a different purpose than treating multi-sentence portions of text. To perform classification based on additional discourse features, we form a single tree from a tree forest for a sequence of sentences in a paragraph of text. Currently, kernel methods tackle individual sentences. For example, in question answering, when a query is a single sentence and an answer is a single sentence, these methods work fairly well. However, in learning settings where texts include multiple sentences, we need to represent structures which include paragraph-level information such as discourse.

A number of NLP tasks such as classification require computing semantic features over paragraphs of text containing multiple sentences. Doing it at the level of individual sentences and then summing up the score for sentences will not always work. In the complex classification tasks where classes are defined in an abstract way, the difference between them may lay at the paragraph level and not at the level of individual sentences. In the case where classes are defined not via topics but instead via writing style, discourse structure signals become essential. Moreover, some information about entities can be distributed across sentences, and classification approach needs to be independent of this distribution [12–14]. We will demonstrate the contribution of paragraph-level approach versus the sentence level in our evaluation.

1.1 Design Documents Versus Design Meta-Documents

We define design document as a document which contains a thorough and well-structured description of how to build a particular engineering system. In this respect a design document according to our model follows the reproducibility criteria of a patent or research publication; however format is different from them. What we exclude is a document which contains meta-level information relatively to the design of engineering system, such as *how to write design docs* manuals, standards design docs should adhere to, tutorials on how to improve design documents, and others.

We need to differentiate design documents from the classes of documents which can be viewed as ones containing meta-language, whereas the genuine design document consists of the language-object. Below we enumerate such classed of *meta-documents*:

(1) design requirements, project requirement document, requirement analysis, operational requirements
(2) construction documentation, project planning, technical services review
(3) design guidelines, design guides, tutorials
(4) design templates (template for technical design document)
(5) research papers on system design
(6) general design-related notes
(7) educational materials on system design
(8) the description of the company, which owns design documents
(9) resume of a design professional
(10) specifications for civil engineering
(11) functional specifications
(12) ‘best design practices’ description
(13) project proposals

Naturally, design documents are different from similar kinds of documents on the same topic in terms of style and phrasing. To extract these features, rhetoric relations are essential.

Notice that meta-documents can contain object-level text, such as design examples. Object-level documents (genuine design docs) can contain some author reflections on the system design process (which are written in metalanguage). Hence the boundary between classes does not strictly separate metalanguage and language-object. We use statistical language learning to optimize such boundary, having supplied it with a rich set of linguistic features up to the discourse structures. In the design document domain, we will differentiate between texts in mostly meta-language and the ones mostly in language-object.

1.2 Novel in Metalanguage Versus Novel in Language-Object

A mixture of object-language and metalanguage descriptions can be found in literature. Describing the nature, a historical event, an encounter between people, an author uses a language-object. Describing thoughts, beliefs, desires and knowledge of characters about the nature, events and interactions between people, an author uses a metalanguage. The entities/relations of such metalanguage range over the expressions (phrases) of the language-object. In other words, the physical world is usually described in language-object, and the mental world (theory of mind, the world of thoughts) typically combines both levels.

One of the purest examples of use of metalanguage in literature is Franz Kafka's novel "The Trial". According to our model, the whole plot is described in metalanguage, and object-level representation is absent. This is unlike a typical work of literature, where both levels are employed. In "The Trial" a reader learns the main character Joseph K. is being prosecuted, his thoughts are described, meeting with various people related to the trial are presented. However, no information is available about a reason for the trial, the charge, the circumstances of the deed. The novel is a pure example of the presence of meta-theory and absence of object-level theory, from the standpoint of logic. The reader is expected to form the object–level theory herself to avoid ambiguity in interpretation of the novel.

Exploration of "The Trial" would help to understand the linguistic properties of metalanguage and language-object. For example, it is easy to differentiate between a mental and a physical words, just relying on keywords. However, to distinguish meta-language from language-object in text, one need to consider different discourse structures, which we will automatically learn from text.

The following paragraph of text can be viewed as a fragment of an algorithm for how to solve an abstract problem of acquittal. Since it suggests a domain-independent approach (it does not matter what an accused did), it can be considered as a meta-algorithm.

'There are three possibilities: absolute acquittal, apparent acquittal and deferment. Absolute acquittal is the best, but there is nothing I could do to get that sort of outcome.

I don’t think there’s anyone at all who could do anything to get an absolute acquittal. Probably the only thing that could do that is if the accused is innocent. As you are innocent it could actually be possible and you could depend on your innocence alone. In that case you will not need me or any other kind of help.’

In some sense this algorithm follows along the lines of a ‘vanilla’ interpreter in Prolog, a typical example of a meta-program:
achieve_acquittal (true).
achieve_acquittal ((A,B)):- achieve_acquittal (A), achieve_acquittal (B).
achieve_acquittal (A):- clause(A, B), achieve_acquittal (B).
where the novel enumerates various *clauses*, but never ground terms expressing the details of a hypothetical crime (no instances of A or B). *clause(A, B)* is expression of the format A :- B, where A is a term being defined (a clause head) and B is a sequence of defining terms (a body of this clause). This interpreter shows multiple possibilities a term can be proved, similarly to multiple possibilities of acquittal spelled out by Kafka.

We hypothesize that a text expressing such a meta-program, Kafka’s text, should have specific sequences of rhetoric relation, infrequent in other texts. We will attempt to find distinct discourse patterns associated with metalanguage and differentiate it with other texts.

In the literature domain, we will attempt to draw a boundary between the pure metalanguage (peculiar works of literature) and a mixed level text (a typical work of literature).

2 Extending Tree Kernel Towards Discourse

To Why can sentence-level tree kernels be insufficient for classification? Important phrases can be distributed through different sentences. So we want to combine/merge parse trees to make sure we cover the phrase of interest.

For the following text:

This document describes the design of back end processor. Its requirements are enumerated below.

From the first sentence, it looks like we got the design document. To process the second sentence, we need to disambiguate the preposition ‘its’. As a result, we conclude from the second sentence that it is a requirements document (not a design document).

2.1 Leveraging Structural Information for Classification

How can a sentence structural information be indicative of the class?

The idea of measuring similarity between the question-answer pairs for question answering instead of the question-answer similarity turned out to be fruitful

[23]. The classifier for correct vs incorrect answers processes two pairs at a time, $<q_1, a_1>$ and $<q_2, a_2>$, and compare q_1 with q_2 and a_1 with a_2, producing a combined similarity score. Such a comparison allows to determine whether an unknown question/answer pair contains a correct answer or not by assessing its distance from another question/answer pair with a known label. In particular, an unlabeled pair $<q_2, a_2>$ will be processed so that rather than "guessing" correctness based on words or structures shared by q_2 and a_2, both q_2 and a_2 will be compared to their correspondent components q_1 and a_1 of the labeled pair $<q_2, \quad a_2>$ on the grounds of such words or structures. Since this approach targets a domain-independent classification of answer, only the structural cohesiveness between a question and answer is leveraged, not 'meanings' of an answers.

We take this idea further and consider an arbitrary sequence of sentences instead of question-sentence and answer-sentence pair for text classification. Our positive training paragraphs are "plausible" sequences of sentences for our class, and our negative training paragraphs are "implausible" sequences, irrespectively of the domain-specific keywords in these sentences.

In our opinion, for candidate answer selection task, such structural information is important but insufficient. At the same time, for the text classification tasks just structure analysis can suffice for proper classification.

Given a positive sequence and its parse trees linked by RST relations:

'A hardware system contains classes such as GUI for user interface, IO for importing and exporting data between the emulator and environment, and Emulator for the actual process control. Furthermore, a class Modules is required which contains all instances of modules in use by emulation process.' (Fig. 1).

And a negative sequence and its linked parse trees:

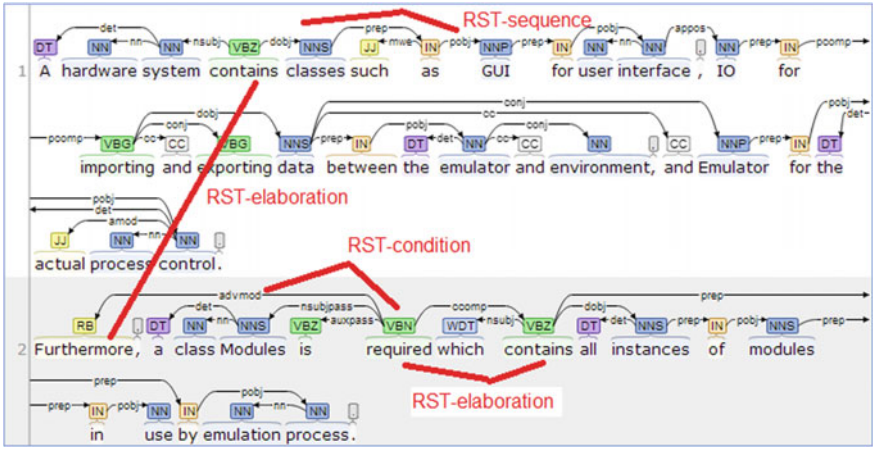

RST-sequence
1 A hardware system contains classes such as GUI for user interface , IO for
importing and exporting data between the emulator and environment, and Emulator for the
RST-elaboration
actual process control.
RST-condition
2 Furthermore, a class Modules is required which contains all instances of modules
in use by emulation process.
RST-elaboration

Fig. 1 A sequence of parse trees and RST relations for a positive example

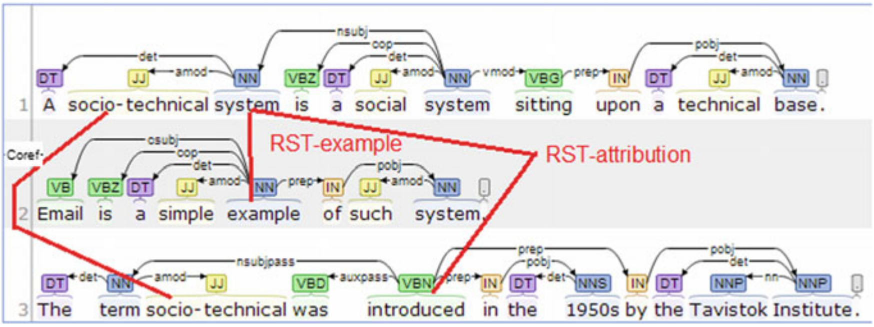

1 A socio-technical system is a social system sitting upon a technical base.
RST-example
RST-attribution
Coref
2 Email is a simple example of such system.
3 The term socio-technical was introduced in the 1950s by the Tavistok Institute.

Fig. 2 A sequence of parse trees and RST relations for a negative example

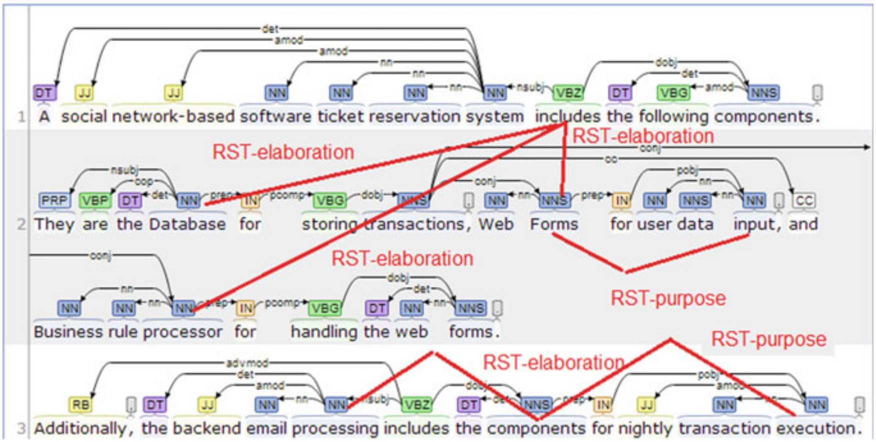

1 A social network-based software ticket reservation system includes the following components.
RST-elaboration
RST-elaboration
2 They are the Database for storing transactions, Web Forms for user data input, and
RST-elaboration
RST-purpose
Business rule processor for handling the web forms.
RST-purpose
RST-elaboration
3 Additionally, the backend email processing includes the components for nightly transaction execution.

Fig. 3 A sequence of parse trees and RST relations for a text to be classified

‘A socio-technical system is a social system sitting upon a technical base. Email is a simple example of such system. The term socio-technical was introduced in the 1950s by the Tavistok Institute.’ (Fig. 2).

We want to classify the paragraph

‘A social network-based software ticket reservation system includes the following components. They are the Database for storing transactions, Web Forms for user data input, and Business rule processor for handling the web forms. Additionally, the backend email processing includes the components for nightly transaction execution.’ (Fig. 3).

One can see that this paragraph follows the rhetoric structure of the top (positive) training set element, although it shares more common keywords with the bottom (negative) element. Hence we classify it as a design document text, since it describes the system rather than introduces a terms (as the negative element does).

To illustrate the similar point in the question answering domain, we use a simple query example. If q_1 is '*What is plutocracy?*' and the candidate answers are a_1 = '*Plutocracy may be defined as a state where …*' versus a_0 = '*Plutocracy affects the wills of people …*', comparison with the correct pair formed by q_2 = '*What is a source control software?*' and a_2 = '*A source control software can be defined as a…*' will induce the kernel method to prefer a_1 to a_0. One can see that a_1 has a similar wording and structure to a_2, hence $< q_1, a_1 >$ will get a higher score than $< q_1, a_0 >$ using the kernel method. In contrast, the opposite case would occur using a similarity score matching q_1 with a_1 as compared with matching q_1 with $a_{0,}$ since both a_1 and a_0 contain keywords *plutocracy* from $q_{1.}$ This explains why even a bag-of-words kernel adjusting its weights on question/answer pairs has a better chance to produce good results than a bag-of-words question/answer similarity.

2.2 Anaphora and Rhetoric Relations for Sentiments Topicality Classification

I sentiment analysis, we classify sentences and documents with respect to sentiments they contain. Let us consider the sentiment classes for the following sentences. We are interested in both polarity and topicality classes:

'I would not let my dog stay in this hotel' (Fig. 4)
'Our dog would have never stayed in this hotel'
'Filthy dirty hotel, like a dog house'
'They would not let my dog stay in this hotel'
'The hotel management did not let me in when I was with my dog'
'We were not allowed to stay there with our dog'

What one observes is that polarity = negative in both cases, whereas topics are totally different. Topic1 = '*hotel is dirty*', and

topic2 = '*dogs are not allowed*'.

This is rather difficult task for keyword-based text classification problems because both classes share the same keywords.

Notice these classes are different from the topic3 = '*hotels intended for dogs*', polarity = '*neutral*'

If you have never been to a dog hotel, now is the time.

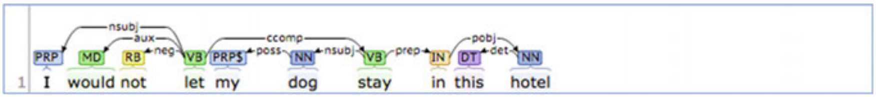
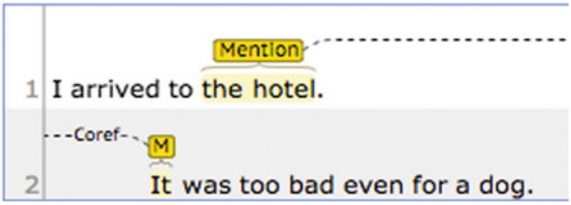


Fig. 4 A parse tree for an individual sentence



1 I arrived to the hotel.

2 It was too bad even for a dog.

Fig. 5 A fragment of text with a coreference relation

It is even harder to perform classification, when information about ‘staying in a hotel and having a dog’ is spread through different sentences. An easier case is anaphora:

‘I arrived to the hotel. It was too bad even for a dog’ (Fig. 5).

The hardest case is when rhetoric structure is needed to link information about a hotel and a dog:

‘I was traveling for business. My partner invited me to stay at his place, however it looked like a house for dogs.’

‘I was traveling with my dog for business. I was not going to stay at a hotel but at my partner’s place, however he turned out to be allergic to dogs. Sadly, the hotel did not let us in.’ (Fig. 6).

In the above cases, the parts of the parse trees (sub-trees) essential to determine the meanings occur in different sentences, so needs to be connected. Anaphora is a natural way to do that, but is not always sufficient. Hence we need rhetoric relations to link ‘*travel, dog owner, hotel’* and permission relationships.

2.3 Anaphora and Rhetoric Relations for Classification Tasks

We introduce a domain where a pair-wise comparison of sentences is insufficient to properly learn certain semantic features of texts. This is due to the variability of ways information can be communicated in multiple sentences, and variations in possible discourse structures of text which needs to be taken into account.

We consider an example of text classification problem, where short portions of text belong to two classes:

- Tax liability of a landlord renting office to a business.
- Tax liability of a business owner renting an office from landlord.

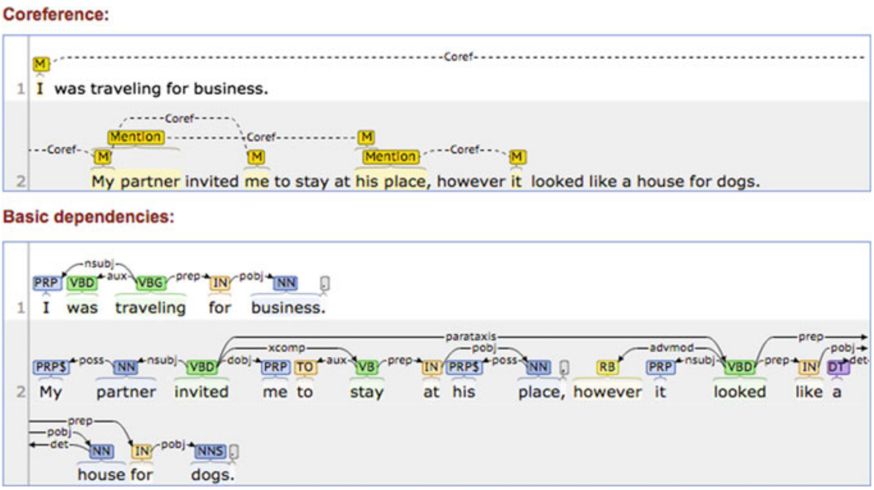

Coreference:
M
Coref
1 I was traveling for business.
Coref
Mention
Coref
M
M
M
Mention
Coref
M
2 My partner invited me to stay at his place, however it looked like a house for dogs.
Basic dependencies:
PRP VBD VBG IN NN
nsubj aux prep pobj
1 I was traveling for business.
PRP$ NN VBD PRP TO VB IN PRP$ NN RB PRP VBD IN DT
poss nsubj dobj xcomp aux prep pobj poss parataxis advmod nsubj prep prep pobj det
2 My partner invited me to stay at his place, however it looked like a
NN IN NNS
prep pobj det pobj
house for dogs.

Fig. 6 The parse tree and respective coreference arcs for two sentences

'I rent an office space. This office is for my business. I can deduct office rental expense from my business profit to calculate net income.

To run my business, I have to rent an office. The net business profit is calculated as follows. Rental expense needs to be subtracted from revenue.

To store goods for my retail business I rent some space. When I calculate the net income, I take revenue and subtract business expenses such as office rent.

I rent out a first floor unit of my house to a travel business. I need to add the rental income to my profit. However, when I repair my house, I can deduct the repair expense from my rental income.

I receive rental income from my office. I have to claim it as a profit in my tax forms. I need to add my rental income to my profits, but subtract rental expenses such as repair from it.

I advertised my property as a business rental. Advertisement and repair expenses can be subtracted from the rental income. Remaining rental income needs to be added to my profit and be reported as taxable profit.'

Note that keyword-based analysis does not help to separate the first three paragraph and the second three paragraphs. They all share the same keywords *rental/office/income/profit/add/subtract*. Phrase-based analysis does not help, since both sets of paragraphs share similar phrases.

Secondly, pair-wise sentence comparison does not solve the problem either.

Anaphora resolution is helpful but insufficient. All these sentences include ‘I’ and its mention, but other links between words or phrases in different sentences need to be used.

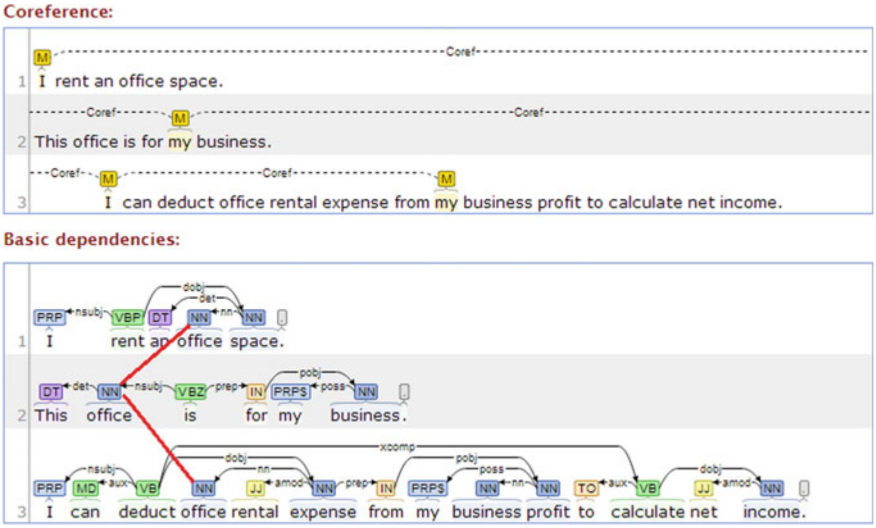
Rhetoric structures need to come into play to provide additional links between sentences. The structure to distinguish between

‘renting for yourself and deducting from total income’ and

‘renting to someone and adding to income’ embraces multiple sentences. The second clause about *‘adding/subtracting incomes’* is linked by means of the rhetoric relation of *elaboration* with the first clause for *landlord/tenant*. This rhetoric relation may link discourse units within a sentence, between consecutive sentences and even between first and third sentence in a paragraph. Other rhetoric relations can play similar role for forming essential links for text classification.

Which representations for these paragraphs of text would produce such common sub-structure between the structures of these paragraphs? We believe that extended trees, which include the first, second, and third sentence for each paragraph together can serve as a structure to differentiate the two above classes.

The dependency parse trees for the first text in our set and its coreferences are shown in Fig. 7. There are multiple ways the nodes from parse trees of different sentences can be connected: we choose the rhetoric relation of elaboration which links the same entity office and helps us to form the structure *rent-office-space—for-my-business—deduct-rental-expense* which is the base for our classification. We used Stanford Core NLP, coreferences resolution [19] and its visualization to form Figs. 1 and 2.



Coreference:

1 I rent an office space.

2 This office is for my business.

3 I can deduct office rental expense from my business profit to calculate net income.

Basic dependencies:

1 I rent an office space.

2 This office is for my business.

3 I can deduct office rental expense from my business profit to calculate net income.

Fig. 7 Coreferences and the set of dependency trees for the first text

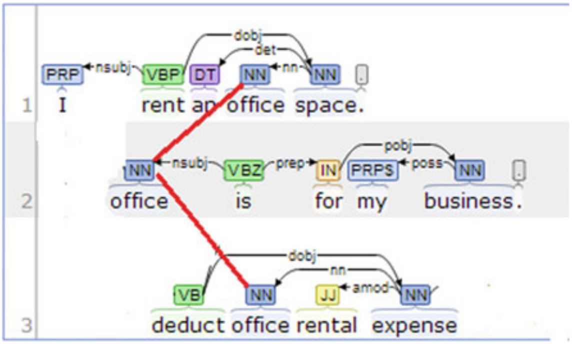

dobj
det
PRP nsubj VBP DT NN nn NN .
1 I rent an office space.
pobj
NN nsubj VBZ prep IN PRP$ poss NN .
2 office is for my business.
dobj
nn
VB NN JJ amod NN
3 deduct office rental expense

Fig. 8 Extended tree which includes 3 sentences

Figure 8 shows the resultant extended tree with the root ‘I’ from the first sentence. It includes the whole first sentence, a verb phrase from the second sentence and a verb phrase from the third sentence according to rhetoric relation of elaboration. Notice that this extended tree can be intuitively viewed as representing the ‘main idea’ of this text compared to other texts in our set. All extended trees need to be formed for a text and then compared with that of the other texts, since we don’t know in advance which extended tree is essential. From the standpoint of tree kernel learning, extended trees are learned the same way as regular parse trees.

3 Building Extended Trees

For every arc which connects two parse trees, we derive the extension of these trees, extending branches according to the arc (Fig. 9).

In this approach, for a given parse tree, we will obtain a set of its extension, so the elements of kernel will be computed for many extensions, instead of just a single tree. The problem here is that we need to find common sub-trees for a much higher number of trees than the number of sentences in text, however by subsumption (sub-tree relation) the number of common sub-trees will be substantially reduced.

If we have two parse trees P_1 and P_2 for two sentences in a paragraph, and a relation $R_{12} : P_{1i} \rightarrow P_{2j}$ between the nodes P_{1i} and P_{2j}, we form the pair of extended trees $P_1 * P_2$:

$\ldots, P_{1i-2}, P_{1i-1}, P_{1i}, P_{2j}, P_{2j+1}, P_{2j+2}, \ldots$

$\ldots, P_{2j-2}, P_{2j-1}, P_{2j}, P_{1i}, P_{1i+1}, P_{2i+2}, \ldots,$

which would form the feature set for tree kernel learning in addition to the original trees P_1 and P_2. Notice that the original order of nodes of parse trees are retained under operation ‘*’.

The algorithm for building an extended tree for a set of parse trees T is presented below:

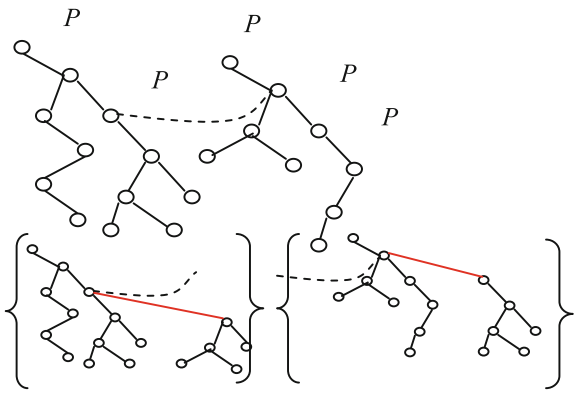


Fig. 9 An arc which connects two parse trees for two sentences in a text (on the *top*) and the derived set of extended trees (on the *bottom*)

```
Input:
    (1) Set of parse trees $T$.
    (2) Set of relations $R$, which includes relations $R_{ijk}$ between the nodes of
$T_i$ and $T_j$ : $T_i \in T$, $T_j \in T$, $R_{ijk} \in R$. We use index $k$ to range over multiple
relations between the nodes of parse tree for a pair of sentences.
Output: the exhaustive set of extended trees $E$.
Set $E = \emptyset$;
For each tree $i = 1 : |T|$
    For each relation $R_{ijk}$, $k = 1 : |\mathrm{R}|$
    Obtain $T_j$
    Form the pair of extended trees $T_i * T_j$;
    Verify that each of the extended trees do not have a super-tree in $E$
    If verified, add to $E$;
Return $E$.
```

Notice that the resultant trees are not the proper parse trees for a sentence, but nevertheless form an adequate feature space for tree kernel learning.

To obtain the inter-sentence links, we employed coreferences from Stanford NLP [19, 24]. Rhetoric relation extractor based on our rule-based approach to finding relations between elementary discourse units [7, 9, 14]. We combined manual rules with automatically learned rules derived from the available discourse corpus by means of syntactic generalization.

Rhetorical Structure Theory (RST [20]) is one of the most popular approach to model extra-sentence as well as intra-sentence discourse. RST represents texts by labeled hierarchical structures, called Discourse Trees (DTs). The leaves of a DT correspond to contiguous Elementary Discourse Units (EDUs). Adjacent EDUs are connected by rhetorical relations (e.g., Elaboration, Contrast), forming larger discourse units (represented by internal nodes), which in turn are also subject to this relation linking. Discourse units linked by a rhetorical relation are further distinguished based on their relative importance in the text: nucleus being the central part, whereas satellite being the peripheral one. Discourse analysis in RST involves two subtasks: discourse segmentation is the task of identifying the EDUs, and discourse parsing is the task of linking the discourse units into a labeled tree.

3.1 Kernel Methods for Parse Trees

Kernel methods are a large class of learning algorithms based on inner product vector spaces. Support

Vector machines (SVMs) are mostly well-known algorithms. The main idea behind SVMs is to learn a hyperplane,

$$H(\vec{x}) = \vec{w} \cdot \vec{x} + b = 0, \tag{1}$$

where $\vec{x}$ is the representation of a classifying object o as a feature vector, while $\vec{w} \in \Re^n$ (indicating that $\vec{w}$ belongs to a vector space of n dimensions built on real numbers) and $b \in \Re$ are parameters learned from training examples by applying the Structural Risk Minimization principle [31]. Object o is mapped into $\vec{x}$ via a feature function

$$\phi : \mathcal{O} \to \Re^n,$$

where $\mathcal{O}$ is the set of objects; o is categorized in the target class only if

$$H(\vec{x}) \geq 0.$$

The decision hyperplane can be rewritten as:

$$H(\vec{x}) = \left(\sum_{i=1,\ldots,l} y_i \alpha_i \vec{x}_i \right) \cdot \vec{x} + b = \sum_{i=1,\ldots,l} y_i \alpha_i \vec{x}_i \cdot \vec{x} + b = \sum_{i=1,\ldots,l} y_i \alpha_i \phi(o)_i \cdot \phi(o) + b, \tag{2}$$

where y_i is equal to 1 for positive examples and to -1 for negative examples

$$\alpha_i \in \Re \text{ (with } \alpha_i \geq 0, o_i \, \forall i \in \{1, \ldots, l\})$$

are the training instances and

$$K(o_i, o) = \langle \phi(o_i) \cdot \phi(o) \rangle$$

is the kernel function associated with the mapping $\phi\cdot$.

Convolution kernels as a measure of similarity between trees compute the common sub-trees between two trees T_1 and $T_{2.}$ Convolution kernel does not have to compute the whole space of tree fragments. Let the set $\mathcal{T} = \{t_1, t_2, ..., t_{|\mathcal{T}|}\}$ be the set of subtrees of an extended parse tree, and $\chi_i(n)$ be an indicator function which is equal to 1 if the subtree t_i is rooted at a node n, and is equal to 0 otherwise. A tree kernel function over trees T_1 and T_2 is

$$TK(T_1, T_2) = \sum_{n_1 \in N_{T_1}} \sum_{n_1 \in N_{T_2}} \Delta(n_1, n_2), \tag{3}$$

where N_{T1} and N_{T2} are the sets of T_1‘s and T_2‘s nodes, respectively and

$$\Delta(n_1, n_2) = \sum_{i=1}^{|\mathcal{T}|} \chi_i(n_1)\chi_i(n_2). \tag{4}$$

(4) calculates the number of common fragments with the roots in n_1 and n_2 nodes.

3.2 Learning System Architecture

The architecture of learning system is shown in Fig. 10. Once Stanford NLP performs parsing and identifies coreferences, VerbNet components obtains verb signatures, and Stanford NLP also builds anaphora relations, we proceed to finding the same-entity and sub-entity links. After that, we perform segmentation into elementary discourse units and find RST relations, using our own templates (also, third party RST parsers can be incorporated).

As a result, we obtain parse thicket for a text as a sequence of parse trees with additional discourse level arcs linking parse trees for different sentences. For a parse

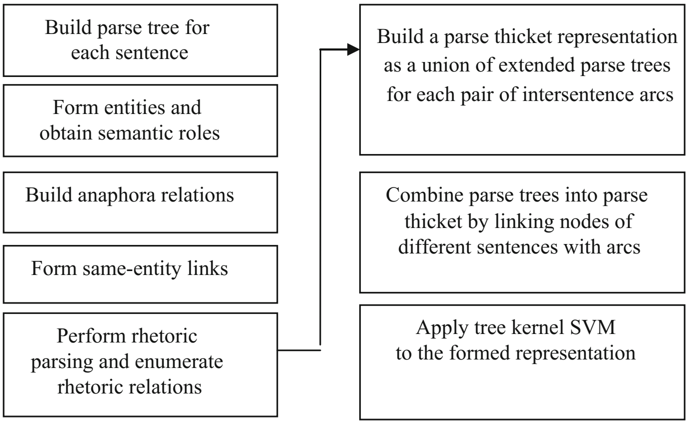

Build parse tree for each sentence
Form entities and obtain semantic roles
Build anaphora relations
Form same-entity links
Perform rhetoric parsing and enumerate rhetoric relations
Build a parse thicket representation as a union of extended parse trees for each pair of intersentence arcs
Combine parse trees into parse thicket by linking nodes of different sentences with arcs
Apply tree kernel SVM to the formed representation

Fig. 10 The architecture of learning system

thicket, we form the set of all phrases and then derive the set of extended phrases according to inter-sentence arcs. Once we build the totality of extended phrases, we form a representation for SVM tree kernel learning, which inputs the set of extended parse trees.

4 Metalanguage and Works of Literature

4.1 Frantz Kafka's Novel "The Trial"

For the example use of metalanguage we consider Frantz Kafka's (1883–1924) novel "The Trial". This novel, written in 1915, has been puzzling many generations of literature critics. Kafka's novels belong to the modernist trends in the literature, which uses various philosophical concepts and new ways of depicting reality. In his works, Franz Kafka went largely beyond other modernists, since his novel structure and language feature very sophisticated artistic expression. The language of Kafka, apparently devoid of any revelations and interpretations, contains an inexhaustible material for linguistic theories. Through scientific analysis of the novel "The Trial", one can see that the writer's use of language is a pointer to an understanding of the underlying aspects of his work. Novel "The Trial" consists of sixteen chapters and an appendix titled "Dream". During his life Franz Kafka could not complete this work. Max Brod, Kafka's closest friend and executor helped publish this novel and gave it the title.

Undoubtedly, many unfinished manuscripts of Kafka makes treating his work with even greater interest, since the absence of ties and open ending always contribute to the construction of the set of inconsistent theories.

"The Trial" does not represent a complex, multi-passage structure, at the first glance. The story presented by Kafka is fairly simple. Joseph K. is the alter ego of the writer. From the first lines of the novel he became involved in a lawsuit. The reader does not get any information about the reason and meaning of this trial, instead the reader is given the common description of its flow and the suspects' encounters with a series of law enforcement officials. It turns out that Joseph K. himself does not know why he ended up in this situation. There are no clues and hints as to why the hero suddenly became defendants and for what crime he may be judged.

4.2 Use of Metalanguage in the Novel

Incomprehensibility of what is happening in the novel makes us scrutinize the peculiarities of language used by Franz Kafka to describe the events in "The Trial".

To comprehend the meaning of the novel as a whole, we need to ascend to the certain level of abstraction. To systematically treat the plot, we consider the text of the novel as written in meta-language, and the remaining part about the reason and the subject of the trial as hypothetically represented in the language-object. Hence what seems to be most interesting and informative to the reader is the theory-object (which is absent), and what is available is the meta-theory (what is explicitly described in the novel) [10].

A metalanguage includes the relations over the expressions of language-object, such as the features of the trial flow. On the contrary, a traditional literature style relies on language-object only, which includes the relations between subjects such as objects of physical and mental world. In most works of literature, metalanguage might be present but its role is secondary and its purpose is to present some generalizations of characters and observations of a writer. However, in “The Trial” the language-object level is totally absent.

Kafka describes all events in metalanguage only, relatively to the subject of the trial. It is easy to verify this statement, having taken an arbitrary paragraph. It describes some discussion of a detail of the trial with some individual involved in law enforcement, such as the following: Object-level details are absent here.

‘If you want an apparent acquittal I’ll write down an assertion of your innocence on a piece of paper. The text for an assertion of this sort was passed down to me from my father and it’s quite unassailable. I take this assertion round to the judges I know. So I’ll start off with the one I’m currently painting, and put the assertion to him when he comes for his sitting this evening. I’ll lay the assertion in front of him, explain that you’re innocent and give him my personal guarantee of it. And that’s not just a superficial guarantee, it’s a real one and it’s binding.’

We attempt to represent the novel “The Trial” at the meta-level only. One can observe that the key character meets with other characters to discuss the details of the trial. All these characters are aware of what is happening, the plot is fully focused on the trial procedures, but the knowledge of the process itself (the object-level) is absent. Franz Kafka naturally used his peculiar writing style and way of narration working on a novel. Many of Kafka’s novels are full with mystery and deep analysis of characters’ mental states, and the author often uses meta-language as a means of expression. What are the reasons for it?

In the novel the use of metalanguage symbolizes the impossibility to come up with a term for the whole trial. The author is so appalled by what he is describing that he is unable to name it (this name would belong to language-object). Possibly, the described trial is not correlated with the socio-historical context of the 1910s. Metalanguage describes all the specific information about Joseph K., for example, the reader learns from the first pages where Joseph K. lives and what he receives for breakfast and when. But we cannot say for sure that Kafka describes the physical reality, not a human condition between sleeping and waking. In other words, the writer enhances the effect of involvement of the reader in the phantasmagoric and absurd world with the blurred boundaries between dream and reality.

For Franz Kafka’s representation of sleep and awakening, it is an inexhaustible source for the exploration of the world and its characters. It was after a restless sleep (troubled dreams) that Gregor Samsa has turned into an insect in the novel “The Metamorphosis” of 1912. On the contrary, a partial detachment from the real world and the abundance of household items in “The Trial” gives the author an opportunity to create a completely different reader perception, subject to human comprehension to a even smaller degree.

4.3 Use of Metalanguage and the Author’s Attachment to the Novel

Kafka’s scholars consider his biography for better comprehension of his novel. It is important for “The Trial” since the details of Kafka’s life are similar to Joseph K.’s. They both occupy an office position (Joseph K. was a bank’s vice president). Kafka is familiar with bureaucratic structures which includes even cathedrals. All venues in the novel change as if they are theater decorations, dwellings are instantly turned into a courtroom. Other characters also belong to the same common bureaucratic/mechanical system, as its indispensable parts. Many characters in the novel are not bright personalities at all, for example, the officers who arrested Joseph K. The writer aims to convincingly capture the whole bureaucratic world, relying on the expressiveness of metalanguage instead of satirical motifs. Apparently, the elements of satire in “The Trial” were deliberately rejected by the author to make the novel sound as a parable.

Creating a novel in which the metalanguage is the only means of descriptions of what is happening, Kafka continues the tradition of historical writings and legends. “The Trial” is a novel where ancient cultural traditions are deeply intertwined with the subconscious mind of the author. In the story of Josef K., described in the metalanguage, there is a lot of social terms: the court, the law, the judge, the arrest, the process of accusation. These terms could have been used from ancient times to the modern era of social development. For example, the process of Joseph K. may well be a reference to the Last Judgment and his sentence denotes the divine punishment. In the novel “The Trial” much can be seen from the perspective of theology and spirituality.

For sure, Franz Kafka believed in an idea, not in a reality, being a modernist. His idea came from his own subconscious, which can be described by metalanguage means only [1, 25] to represent processes in which only the metalanguage matters. All moral reflection of the author, his spiritual studies and innermost thoughts reflect his special method of expression.

To some extent, the novel can be attributed to the detective genre. The events occur in the environment Joseph K. is familiar with, and all the characters are described by the author in a fairly routine manner. Yet “The Trial” does not belong to the classical detective because its logical constructions cannot be assessed as a truth. This is because the facts are represented in metalanguage and the object-level facts are missing, so a reader cannot appreciate a solution to a problem, an inherent part of a detective novel. Initiation of “The Trial” occurs in the object level outside of text and cannot be used to accuse the main character.

Kafka's use of metalanguage has a significant impact on the perception of the whole process as something absurd, not subject to any expected rules. All the actions of Joseph K. cannot be viewed from the standpoint of a legal system and are deliberately fairly generic in nature. The writer presents the Court as an omnipotent organization, but in fact this court makes no investigation of the main character, at least from what we know from the novel. The investigation itself cannot exist in this novel, given the style of the description of the whole process to the metalanguage.

The lack of investigation and of disclosure of secrets do not prevent Kafka from maintaining the suspense until the last pages of the novel. One gets the impression that the final sentence of Joseph K. in all circumstances of the case may not be an acquittal. The very first cause of any judicial process lies in the fact of charges being brought. Metalanguage only emphasizes the bureaucratic proceedings, which is the foundation and skeleton of the novel.

Endless absurdity of the described process reminds us about the famous novel by Charles Dickens “Bleak House”. Completed in 1853, “Bleak House” was forerunner of a new narrative paradigm. It is important to note that the plot of a Dickens novel in many respects anticipates the history of “The Trial”. All events of “Bleak House” take place during an endless litigation process in which the Court of Chancery determines the fate of the characters of the novel.

Dickens conducted to a certain extent an artistic experiment, outlining the events in a very detached way, but not depriving them of a secret meaning. Kafka and Dickens are united in their desire to express in new linguistic forms the meaninglessness and injustice of what is happening in the courts. Both writers were looking for the truth, which is mired in unpleasant courtrooms and stuffy offices. Franz Kafka, perfectly familiar with the works of Dickens, partially enriched “The Trial” by the narrative technique of “Bleak House.” But the metalanguage of Kafka does not express deep personal experiences of the character. Instead, it is limited to the unattached vision of distant events.

Commenting and clarification of events by the author is absent in “The Trial”. Interestingly, the nature of the novel is close to the authors' perception of what is happening in the society. However, the writer expresses his attitude about the society implicitly, in the metalanguage. The absence of any background and causes of the trial makes the reader focus primarily on the fate of Josef K. Perhaps Kafka strongly felt connection with his main character and considered supplementing the text with his subjective emotions unnecessary. In addition to the above, Kafka's metalanguage hides his deep personal alienation from the system of justice. By the very use of metalanguage the author emphasizes that any litigation is a priori senseless. Also, any man is doomed to become a victim of the system, where bureaucratic principles dominate.

“The Trial” is both illogical and ordered through the use of metalanguage. Joseph K. was not ready for the trial, but at the same time it is clear that without the involvement of a trial this character would be no longer interesting for the reader. Kafka carefully hides mentions of the past of the main character, and also does not give us any opinion about him by the other novel characters. It becomes difficult for the reader to determine whether Josef K. is guilty or not guilty. Metalanguage does not provide any clues about previously occurred events in the novel. The metalanguage of “The Trial” prevents its unequivocal interpretation and leaves the ending of the novel open and ambiguous.

The expectation of the novel outcome is rather strong. The metalanguage expressive means reinforce the impression of the reader via the seemingly meaningless wandering of the main character through the courtrooms. Here the author attempts to create a hostile atmosphere, where the usual course of time is replaced by eternal awaiting of a sentencing. The conflict is manifested not in the struggle of the main character for his honor and dignity, but in the endless discussion of details. Having ruled out by the use of the metalanguage, any information about the reasons for the charges of Josef K., the writer depicts the stages of the judicial process in detail. Even the passage of time in the novel is completely subordinated to the trial. Any decision depends on the unknown higher powers, and the activity of the main character cannot affect the situation.

Metalanguage of “The Trial” is not only a way to describe events, but it also gives this novel an artistic value, independent of the historical and philosophical concepts of the era. Undoubtedly, Franz Kafka does not give any answer in the novel, instead, this novel only raises questions. Writer’s method manifests itself in the complete absence of any reasons and plot ties, so that the boundaries of the reader’s imagination are unlimited. Thanks to the metalanguage, the writer creates an artistic world where thoughts and speculations become the only possible form of the perception of the novel. Through the analysis of “The Trial”, it becomes clear that any conclusions on the product may not be final. Metalanguage creates a paradoxical situation where the lack of basic information makes it possible for any interpretations of the text. By applying a metalanguage in the novel, Franz Kafka included it in the list of works of literature appealing for new generations of readers.

5 Evaluation

5.1 Baseline Classification Approaches

*TF*IDF Nearest Neighbor approach* finds a document in the training set which is the closest to the given one being recognized. Nearest Neighbor feature is implemented via the search in inverse index of Lucene [3] where the search result score is computed based on TF*IDF model [27]. The query is formed from each sentence of the documents being classified as a disjunctive query including all words except stop-words. The resultant classes along with their TF*IDF scores are weighted and aggregated on the basis of a majority vote algorithm such as [22].

A Naive Bayes classifier is a simple probabilistic classifier based on applying Bayes’ theorem (from Bayesian statistics) with strong (naive) independence assumptions. This classifier assumes that the presence (or absence) of a particular feature of a class is unrelated to the presence (or absence) of any other feature. For example, a fruit may be considered to be an apple if it is red, round, and about 4 inches in diameter. Even if these features depend on each other or upon the existence of the other features, a naive Bayes classifier considers all of these properties to independently contribute to the probability that this fruit is an apple. Depending on the precise nature of the probability model, naive Bayes classifiers can be trained very efficiently in a supervised learning setting. In many practical applications, parameter estimation for naive Bayes models uses the method of maximum likelihood. In this study, we use a Naïve Bayes classifier from WEKA package based on [16].

5.2 Forming Training Datasets

For design documents, we built a web mining utility which searched for public design documents on the web in a number of engineering and science domains. We use the following keywords to add to a query for *design document*: *material, technical, software, pharmaceutical, bio, biotech, civil engineering, construction, microprocessor, C++, python, java, hardware, processor, architectural, creative, web.* As a result we formed a set of 1200 documents, it turned out we had 90 % of non-design engineering documents of the classes we want to exclude (meta-documents) and 10% of genuine design documents.

For the literature domain, we collected 200 paragraphs from Kafka's novel "The Trial" describing interaction with people related to the court, as a training set of meta-documents. As a set of object-level documents we manually selected 100 paragraphs of text in the same domain (scholarly articles about "The Trial"). A good example of such language-object documents is a Wikipedia article on the novel, which is a language-object/mixed paragraph (it described the actions of a person):

'K. visits the lawyer several times. The lawyer tells him incessantly how dire his situation is and tells many stories of other hopeless clients and of his behind-the-scenes efforts on behalf of these clients, and brags about his many connections. The brief is never complete. K.'s work at the bank deteriorates as he is consumed with worry about his case.'

On the contrary, a paragraph in metalanguage (an abstract note on the procedure of *apparent acquittal*) looks like this paragraph which we already sited earlier:

'If you want an apparent acquittal I'll write down an assertion of your innocence on a piece of paper. The text for an assertion of this sort was passed down to me from my father and it's quite unassailable. I take this assertion round to the judges I know. So I'll start off with the one I'm currently painting, and put the assertion to him when he comes for his sitting this evening. I'll lay the assertion in front of him, explain that you're innocent and give him my personal guarantee of it. And that's not just a superficial guarantee, it's a real one and it's binding.'

We split the data into five subsets for training/evaluation portions [17]. For the design documents, evaluation results were assessed by quality assurance personnel. For the literature domain, the evaluation was done by the author.

Table 1 Classifying text into metalanguage and language-object

	Design document %				Literature %			
Method	Precision	Recall	F-measure	Standard deviation	Precision	Recall	F-measure	Standard deviation
Nearest neighbor classifier—tf*idf based	55.8	61.4	58.47	2.1	51.7	54.0	52.82	3.4
Naive Bayesian classifier (WEKA)	57.4	59.2	58.29	3.2	52.4	50.4	51.38	4.2
Manual keyword-based rule selection	93.1	97.5	95.25	1.3	88.1	90.3	89.19	1.3
Manual parse-tree based rules	95.3	97.8	96.53	1.2	N/A	N/A	N/A	1.2
Tree kernel—regular parse trees	73.4	77.6	75.44	2.8	65.7	67.3	66.49	3.8
Tree kernel SVM—extended trees for anaphora	77.0	79.3	78.13	3.1	67.1	67.8	67.45	4.4
Tree kernel SVM—extended trees for RST	78.3	81.5	79.87	2.6	69.3	72.4	70.82	3.7
Tree kernel SVM—extended trees for both anaphora and RST	82.1	85.2	83.62	2.7	70.8	73.0	71.88	4.2

5.3 Evaluation Results

We report the standard deviation of the recognition accuracy expressed as F-measure over five folds achieved by different methods. Table 1 shows evaluation results for the both domains, Design document and Literature. Each row shows the results of a particular classification method.

Keyword statistic-based methods, including Nearest-Neighbor classification and Naïve Bayes, produced rather poor results. Conversely, a manual rule-based system produces a very high accuracy result, especially when manually formed rules go beyond the keywords/phrases and take into account part-of-speech information.

In addition to automated learning, we relied on manual rules for classes. An increase in accuracy by a few percent is achieved in design documents by using manually collected cases for of expressions indicating a use of metalanguage. Also, the rules for writing styles associated with meta-documents have been compiled. These rules also included regular expressions relying on specific document formatting, including a table of content and structure of sections. In the literature domain, that was not possible. Manual rule performance is shown by grayed rows.

Performance of the tree kernel based methods improves as the sources of linguistic properties become richer. For both domains, there is a few percent improvement by using RST relations compared with baseline tree kernel SVM which relies on parse trees only. For the literature domain, the role of anaphora was rather low.

6 Conclusions

In our previous papers we showed how employing a richer set of linguistic information such as syntactic relations between words assists relevance tasks [6, 12, 13]. To take advantage of semantic discourse information, we introduced parse thicket representation and proposed the way to compute similarity between texts based on generalization of parse thickets [8]. We built the framework for generalizing PTs as sets of phrases on one hand, and generalizing PTs as graphs via maximal common subgraphs, on the other hand [14].

In this study we focused on how discourse information can help with a fairly abstract text classification tasks by means of statistical learning. We selected the domain where the only difference between classes lays in phrasing and discourse structures and demonstrated that both are learnable. We compared two sets of linguistic features:

- The baseline, parse trees for individual sentences,
- Parse trees and discourse information,

and demonstrated that the enriched set of features indeed improves the classification accuracy, having the learning framework fixed. We demonstrated that the

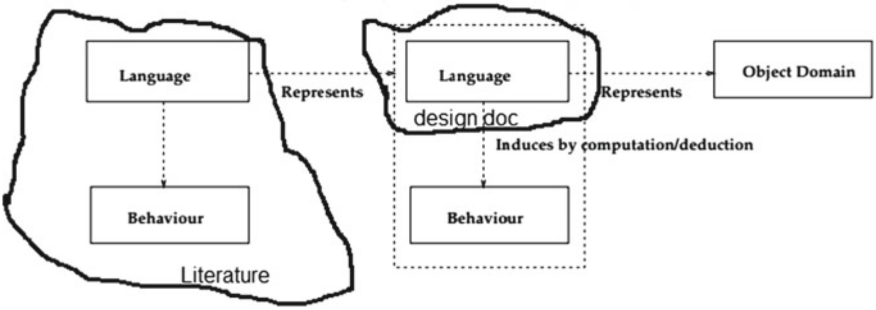


Fig. 11 Meta-reasoning chart: mutual relationships between major classes of our interest

baseline text classification approaches perform rather poorly in the chosen classification domain. Also, kernel-based learning was unable to reach the performance of manually structure-based rules, and we hypothesize that a vast amount of discourse information is not employed in the proposed learning framework.

Meta-reasoning addresses a question of how to give a system its own representation to manipulate. Meta-reasoning needs both levels for both languages and domain behavior. We depict out two main classes of interest in Fig. 11.

[26] outlines a general approach to meta-reasoning in the sense of providing a basis for selecting and justifying computational actions. Addressing the problem of resource-bounded rationality, the authors provide a means for analyzing and generating optimal computational strategies. Because reasoning about a computation without doing it necessarily involves uncertainty as to its outcome, probability and decision theory were selected as main tools.

A system needs to implement metalanguage to impress peers of being human-like and intelligent, needs to be capable of thinking about one's own thinking. Traditionally within cognitive science and artificial intelligence, thinking or reasoning has been cast as a decision cycle within an action-perception loop [21]. An intelligent agent perceives some stimuli from the environment and behaves rationally to achieve its goals by selecting some action from its set of competencies. The result of these actions at the ground level is subsequently perceived at the object level and the cycle continues. Meta-reasoning is the process of reasoning about this reasoning cycle. It consists of both the meta-level control of computational activities and the introspective monitoring of reasoning. In this study we focused on linguistic issues of text which describes such cognitive architecture. It turns out that there is a correlation between a cognitive architecture and a discourse structure used to express it in text. Relying on this correlation, it is possible to automatically classify texts with respect to metalanguage they contain.

In our previous studies we considered the following sources of relations between words in sentences: coreferences, taxonomic relations such as sub-entity, partial case, predicate for subject etc., rhetoric structure relations and speech acts [8]. We demonstrated that a number of NLP tasks including search relevance can be improved if search results are subject to confirmation by parse thicket generalization, when answers occur in multiple sentences. In this study we employed coreferences and rhetoric relation only to identify correlation with the occurrence of metalanguage in text.

Traditionally, machine learning of linguistic structures is limited to keyword forms and frequencies. At the same time, most theories of discourse are not computational, they model a particular set of relations between consecutive states. In this work we attempted to achieve the best of both worlds: learn complete parse tree information augmented with an adjustment of discourse theory allowing computational treatment.

In this paper, we used extended parse trees instead of regular ones, leveraging available discourse information, for text classification. This work describes one of the first applications of tree kernel to industrial scale NLP tasks. The advantage of this approach is that the manual thorough analysis of text can be avoided for complex text classification tasks where the classes are rather abstract. The feasibility of suggested approach to classification lays in the robustness of statistical learning algorithms to unrelated and unreliable features inherent in NLP.

The experimental environment, multi-sentence queries and the evaluation framework is available at https://github.com/bgalitsky/relevance-based-on-parse-trees.

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Relationships Between Probability and Possibility Theories

Boris Kovalerchuk

Abstract The goal of a new area of Computing with Words (CWW) is solving computationally tasks formulated in a natural language (NL). The extreme uncertainty of NL is the major challenge to meet this ambitious goal requiring computational approaches to handle NL uncertainties. Attempts to solve various CWW tasks lead to the methodological questions about rigorous and heuristic formulations and solutions of the CWW tasks. These attempts immediately reincarnated the long-time discussion about different methodologies to model uncertainty, namely: Probability Theory, Multi-valued logic, Fuzzy Sets, Fuzzy Logic, and the Possibility theory. The main forum of the recent discussion was an on-line Berkeley Initiative on Soft Computing group in 2014. Zadeh claims that some computing with words tasks are in the province of the fuzzy logic and possibility theory, and probabilistic solutions are not appropriate for these tasks. In this work we propose a useful constructive probabilistic approach for CWW based on sets of specialized K-Measure (Kolmogorov's measure) probability spaces that differs from the random sets. This work clarifies the relationships between probability and possibility theories in the context of CWW.

1 Introduction

Computing with Words (CWW) is a new area [1, 2] that intends to solve computationally the tasks formulated in a natural language (NL). Examples of these tasks are: "What is the sum of (about 5) + (about 10)?" and "What is the possibility or probability that Mary is middle-aged, if she is 43 and is married for about 15 years?"

In these problems we are given *information I* in the form of membership functions (MFs) such as $\mu_{about-5}$, $\mu_{about-15-years}$, μ_{young}, $\mu_{middle-aged}$, and the respective probability density function (pdfs), e.g., age pdf $\mathrm{p}_{\mathrm{age}}$. Similarly for other tasks information I can include a membership function μ_{tall} (Height(X)), where X is a person (e.g., John) and a *probability density function* of height p_{H} (Height(John)) in

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the height interval U = [u_1, u_n], e.g., u_1 = 160 cm and u_n = 200 cm. The probability space for P_H contains a set of elementary events e_i = "Height of John is u_i": e_1 = "Height of John is u_1", e_2 = "Height of John is u_2",…, e_n = "Height of John is u_n".

Attempts to solve various CWW tasks lead to the methodological questions about rigorous and heuristic formulations and solutions of the CWW tasks which have a tremendous level of uncertainty in both formulations and solutions [3, 4].

These attempts immediately reincarnated the long-time discussion about different methodologies to model uncertainty [5], namely: probability theory (PrT), multivalued logic (MVL), Fuzzy Sets (FS), Fuzzy Logic (FL), and the Possibility theory. The main forum of the recent discussion was an on-line Berkeley Initiative on Soft Computing (BISC) group in 2014.

Zadeh claims that some computing with words tasks are in the province of the fuzzy logic, and probabilistic solutions are not appropriate for these tasks. "Representation of grade of membership of probability has no validity" [6].

He uses this clam for producing a much wider claim of fundamental limitation of the probability theory for computing with words. "Existing logical systems—other than fuzzy logic—cannot reason and compute with information which is described in natural language [6].

In this paper, we show that this and similar claims may be true if we only consider simple probabilistic models and techniques, like the ones used in routine engineering applications. However, modern probability theory contains more sophisticated concepts and ideas that go way beyond these techniques. We show that, by using such more sophisticated concepts and ideas, we can come up with a probabilistic interpretation of fuzzy sets.

To be more precise, such interpretations have been proposed in the past, e.g., the interpretation using random sets [7–10]. Unfortunately, the random sets interpretation is quite complicated, requires mathematical sophistication from users, and often a lot of data that is not always realistic in many practical applications to provide an efficient computational tool. In contrast, our interpretation that uses a set of specialized Kolmogorov-type probability measures (K-measures, for short), is intuitively understandable and computationally efficient.

In this discussion Zadeh did not specify his term "no validity". This led to the difficulty for an independent observer to test his opinion. To move from an opinion level to a scientific level we need the independently verifiable criteria to test validity of an approach that we propose below for the probabilistic solutions.

Any solution of the task to be a *valid probabilistic solution* must meet criteria of the Axiomatic Probability Theory (APT):

(1) The K-measure spaces (probability spaces) must be created.
(2) All computations must be in accordance with the APT.

In the course of BISC discussion and as result of it we proposed probabilistic formulations and solutions that met criteria (1) and (2) for several CWW test tasks listed below:

(1) "Tall John" (Zadeh's task) [11],

(2) “Hiring Vera, Jane, and Mary” that is based on their age and height (Zadeh’s task), [12],
(3) “Degree of circularity of the shape” [11],
(4) “Probably Very is middle-aged” [11, 12],
(5) “Sum of two fuzzy sets” (Zadeh’s task) [4].

The proposed K-measure based solutions are at least of the same usefulness as a solution with the fuzzy logic min/max operation. In [4] we proposed several alternative explications of task 5 and their valid solutions within APT.

This work focuses on three problems:

P1: Establishing relations between concepts of possibility and probability;
P2: Establishing relations between grades of membership of a fuzzy set and probabilities;
P3: Outlining future rigorous possibility theory.

The paper is organized as follows: problem P1 is considered in Sect. 2, problem P2 is considered in Sect. 3, problem P3 is considered in Sects. 4 and 5 provides a conclusion.

Within problem P1 we discuss the following topics:

- Possibilistic Semantics,
- Relationship between possibility and probability in natural language (NL),
- Is NL possibility easier than NL probability for people to use?
- Relation between NL words “possibility” and “probability” and the formal math probability concept,
- NL possibility versus modal and non-modal probability,
- Can the sum of possibilities be greater than 1?
- Context of probability spaces for modeling possibility.

Within problem P2 we discuss:

- Are fuzzy sets and possibility distributions equivalent?
- Can we model degrees of membership probabilistically by Kolmogorov’s measures spaces (K-spaces)?
- What is a more general concept fuzzy set or probability?
- Is interpreting grade of membership as probability meaningful?
- Relationship among unsharpness, randomness and axiomatic probability?

Within problem P3 we discuss:

- Possibility as upper probability,
- Should evaluation structure be limited by [0, 1] or lattice?
- Exact numbers vs. intervals to evaluate uncertainty.

2 Relations Between Concepts of Possibility and Probability

2.1 Possibilistic Semantics

The literature on the Possibility Theory is quite extensive [8, 9, 13–22]. These studies mostly concentrate on conceptual developments. The important aspects of empirical justification of these developments via psychological and linguistic experiments are still in a very earlier stage. As a result it is difficult to judge whether the current possibility concepts are descriptive or prescriptive. A descriptive theory should reproduce the actual use of the concept of possibility in NL. A prescriptive theory should set up rules for how we should reason about possibilities. Both theories require identification of the semantics of the concept of possibility.

Below we outline semantics of possibility in the natural language based on review in [17]:

- An object/event A that did not yet occur (have not yet been seen) can be a possible event, $\text{Pos}(A) \geq 0$.
- The object/event A is possible (not prohibited) if $0 < \text{Pos}(A) \leq 1$.
- The possibility of the observed object A is a unitary possibility, $\text{Pos}(A) = 1$.
- If to characterize the object/event A somebody selects the statement "A is possible" over a statement "A is probable" then the chances for A to occur are lower than if the second statement would be selected. This is consistent with the common English phrases "X is possible, but not probable.", i.e., $\text{Pos}(X) \geq \text{Pr}(X)$.
- Types of possible events:

  Type 1: *Merely possible events* are events that may not occur (earthquake) while occurred in the past, Pos(A) is unknown.
  Type 2: *Eventual events* are events that are guaranteed to occur sooner or later (rain), $\text{Pos}(A) = 1$

In the Sect. 4.2 we discuss possible deviations from this semantics for some complex situations.

Other aspects of possibilistic semantics are discussed in [14] with four meanings of the word "*possibility*" identified:

- *Feasibility*, *ease* of achievement, the solution of a problem, satisfying some constraints with linguistic expressions such as "it is possible to solve this problem".
- *Plausibility*, the propensity of events to occur with linguistic expressions such as "it is possible that the train arrives on time".
- *Epistemic*, logical consistency with available information, a proposition is possible, meaning that it does not *contradict* this information. It is an all-or-nothing version of plausibility.
- *Deontic*, possible means *allowed,* permitted by the law.

The expressions below illustrate the semantic difference between NL concepts of possibility and probability where it seems that the expressions with word "probable" are equivalent, but the expressions with word "possible" are not equivalent:

It is *not probable* that "not A" vs. It is probable that A

It is *not possible* that "not A" vs. It is possible that A.

2.2 Is Easiness an Argument for Possibility Theory?

Below we analyze arguments for and against the possibility theory based on [6]. [12] argued for the possibility theory by stating: "*Humans find it difficult to estimate probabilities. Humans find it easy to estimate possibilities*." To support this claim he presents "Vera's task" that we reconstruct below from his sketchy description.

Vera is middle-aged and middle-aged can be defined as a probability distribution or a possibility distribution. Age is discrete. It is assumed that neither probability nor possibility distributions are given and a person must use his/her judgment to identify them by answering questions for specific ages, e.g.,

(Q1) What is your estimate of the *probability* that Vera is 43?

(Q2) What is your estimate of the *possibility* that Vera is 43?

Zadeh stated that if he would be asked Q1 he "could not come up with an answer. Nor would *anyone* else be able to come up with an answer based on one's perception of middle-aged."

Accordingly answering Q2 must be easier than answering Q1. In fact he equates *possibility* with a *grade of membership*. "Possibility is *numerically equal to grade of membership*. 0.7 is the degree to which 43 fits my perception of middle-aged, that is, it is the grade of membership of 43 in middle-aged." Thus, he is answering question Q3:

(Q3) What is your estimate of the *grade of membership* that middle-aged Vera is 43?

This question is a result of *interpretation* of possibility as a grade of membership. In other words, Zadeh's statement is not about easiness of possibility versus probability for humans, but about *ability or inability to interpret grades of membership as probabilities or possibilities*. We had shown the ability to interpret grades of membership as probabilities in [12] for Zadeh's examples "Mary is middle-aged" and "John is tall". It is done by reinterpreting probabilistically Zadeh's own solution [23] using a set of K-Measure spaces outlined in this paper in Sect. 3.2. For the current example "Vera's age" the reinterpretation is the same. Thus to test easiness of possibility versus probability for humans other arguments are needed.

2.3 Words Possibility and Probability Versus Formal Math Probability

In the BISC discussion Zadeh made several statements:

- "Probabilistic information is *not derivable* from possibilistic information and vice versa" [6].
- "A basic reason is that count and boundary are distinct, *underivable* concepts." [6].
- "0.7 is the possibility that Vera is 43 and has *no relation to the probability* of middle-aged" [6].
- "The possibility that Hans can eat n eggs for breakfast is *independent* of the probability that Hans eats n eggs for breakfast, with the understanding that if the possibility is zero then so is the probability" [6].

Below we show deficiencies of the first three claims and explain independence in the last claim in a way that is consistent with probabilistic interpretation of the concept of possibility. Zadeh bases these statements on claims that, in large measure, the probability theory is "*count*-oriented", fuzzy logic is "fuzzy-*boundary*-oriented", and the possibility theory is "*boundary*-oriented".

Making these statements, Zadeh references Chang's claim [24] that possibility relates to: "Can it happen?" and probability relates to: "How often?" [25] commented on these statements, noting that Zadeh has accepted for his claims the *frequency interpretation* of probability, while there exist five or more, interpretations of probability [26] and in the same way likely multiple interpretations of possibility can be offered depending on some features such as the number of alternatives.

Below we list four well-known different meanings of probability and different people are not equal in using these meanings:

(a) probability as it is known in a *natural language* (NL) without any relation to any type of probability theory,
(b) probability as it is known in the *frequency-based* probability theory,
(c) probability as it is known in the *subjective* probability theory [27],
(d) probability as it is known in the formal *axiomatic* mathematical probability theory (Kolmogorov's theory).

We show deficiencies of Zadeh's statement for meanings (b)–(d) in this section and for meaning (a) later in this paper. It is sufficient to give just a single example of relations for each (b)–(d) that would interpret the possibility that Vera is 43 as a respective probability (b)–(d) of middle-aged.

Consider two events e_1 = {Vera is middle-aged} and e_2 = {Vera is not middle-aged}. In the formal mathematical theory of probability, events are objects of any nature. Therefore, we can interpret each of these two events simply as a sequence of words or symbols/labels, e.g., we can assign the number 0.7 to e_1 and the number 0.3 to e_2. In this very formal way we fully satisfy the requirement of (d) for 0.7 and 0.3 to be called probabilities in the Kolmogorov's axiomatic probability theory [28].

In addition meanings (b) and (c) allow getting 0.7 and 0.3 meaningfully by asking people questions similar to questions like "What is your subjective belief about event e?". An interpretation (b) is demonstrated in depth by Jozo Dujmovic in this volume [29].

To counter Zadeh's claim about the example with n eggs it is sufficient to define two probabilities:

p(Hans *eats* n eggs for breakfast) and p(Hans *can eat* n eggs for breakfast).

The last probability is, in fact, the possibility that Hans eats n eggs for breakfast as the Webster dictionary defines the word "possibility" and we discuss in the next section in details. The idea of that probability (probability of a modal statement with the word "can") was suggested by Cheeseman a long time ago [30]. Thus, we simply have two different probabilities, because we have different sets of elementary events {eat, not eat} and {can eat, cannot eat}. Nobody claims and expects that two probabilities from two different probability spaces must be equal. In the same way, we have two different probabilities, when we evaluate probabilities of rain with two sets of elementary events {daytime rain, no daytime rain} and {nighttime rain, no nighttime rain}.

2.4 Possibility and Probability in Natural Language

The Webster dictionary provides meanings of the words "possibility" and "probability" in the Natural Language (NL). Table 1 contrasts Webster NL definitions of these words. It shows that the major difference is in "will happen", "is happened" associated with word "probability" and "might happen" associated with word "possibility". In other words, we have different levels of chance to happen.

The word "might" expresses a lower chance to happen. Thus even in the pure NL setting there is a strong relation between probability and possibility.

Respectively questions Q1–Q2 can be reformulated into *equivalent* NL forms:

(Q1.1) What is your estimate of the chance that Vera *is* 43?

(Q2.1) What is your estimate of the chance that Vera *might be* 43?

Note that question (Q3): "What is your estimate of the *grade of membership* that middle-aged Vera is 43?" contains term "*grade of membership*" which is a part of

Table 1 Webster NL definitions of words possibility and probability

Possibility	Probability
a *chance* that something *might exist, happen, or be true*	the *chance* that something *will happen*
something that might be done or might happen	something that has a *chance of happening*
abilities or qualities that could make someone or something better in the future	a measure of *how often* a particular event will happen if something (such as tossing a coin) is done repeatedly

the *professional language* not the basic NL, i.e., it requires interpretation. In contrast Q1.1 and Q2.1 do not contain any professional words and phrases. Note that the Webster interpretation of possibility is fully consistent with Cheeseman's modal probability [30] with the modal word "can" that we discussed above for the modal event "n people can be in the car", P(n people can be in the car).

2.5 Experiment: Is Possibility Easier for People than Probability in NL?

We conducted an experiment asking students to answer the question Q1.1 – Q2.1 in the form shown below, i.e., to answer questions about probability and possibility as it follows from the Webster dictionary:

Please provide your personal opinion for the questions below in the following situation. It is known that Vera is middle-aged.

1. What is the chance that she **is 43**?

   Circle your answer: 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 X
   Circle X if it is difficult for you to assign any number.

2. What the chance that she **might be 43**?

   Circle your answer: 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 X
   Circle X if it is difficult for you to assign any number.

In these questions we deliberately avoided both words "probability" and "possibility", but used the word "chance" in combinations with words "is" and "might be" to provide a meanings of these concepts which is derived from the Webster dictionary. It is important to distinguish wording from semantics of the concepts of "probability" and "possibility". Questions must preserve semantics of these concepts, but can be worded differently. The word "chance" is more common in NL and is clearer to the most of the people. By asking questions with the word "chance" we preserve the semantics of concepts of "probability" and "possibility" and made the task for the respondents easier.

A total of 27 sophomore Computer Science university students answered these questions. Practically all of them are English native speakers and none of them took fuzzy logic or probability theory classes before. Answering time was not limited, but all answers were produced in a few minutes. Two strong students took an initiative and offered formulas and charts to compute the answers and generated answers based on them. Table 2 shows all answers of this experiment and Table 3 shows the statistical analysis of these answers.

This experiment leads to the following conclusion:

1. Most of the students (80%) do not have any problem to answer both questions Q1.1 (probability) and Q2.1 (possibility).

Table 2 Answers of respondents

Person	Q1.1	Q2.1
1	0.1	0.3
2	0.1	1
3	0.1	1
1	0.2	0.3
2	0.2	1
3	0.3	0.5
4	0.3	0.6
5	0.3	0.9
6	0.3	1
7	0.3	1
8	0.5	0.8
9	0.5	1
10	0.7	0.8
11	0.7	0.9
12	0.8	1
13	0.8	0.6
14	1	0
15	1	0.4
16	0.4	0.1
17	1	1
18	0.4	0.4
19	0.3	0.3
20	0.1	X
21	X	0.3
22	X	1
23	X	0.3
24	X	1
25	X	X
26	X	X
27	X	X

2. Answers produced quickly (in less than 5 min).
3. Answers are consistent with expected higher value for Q2.1 than for Q1.1. The average answer for Q2.1 is 0.67 and the average answer for Q1.1 is 0.45.
4. Out of 27 students three students found difficult to answer both questions (11.1% of all respondents).
5. Out of 27 students four students found difficult to answer only question Q1.1 (14.8% of all respondents).
6. Out of 27 students one student found difficult to answer only question Q2.1 (3.7% of all respondents).

Table 3 Statistical analysis of answers

Characteristic	Q1.1	Q2.1
Average answer (all numeric answers)	0.45	0.67
Std. dev. (all numeric answers)	0.3	0.34
Number of all numeric answers	20	24
Average answer for cases with answer Q2.1 > answer Q1.1	0.36	0.81
Std. dev. for cases with answer Q2.1 > answer Q1.1	0.23	0.26
Number of answers where answer Q2.1 > answer Q1.1	12	12
Average answer for cases with answer Q1.1 > answer Q2.1	0.63	0.49
Std. dev. for cases with answer Q1.1 > answer Q2.1	0.37	0.36
Number of answers where answer Q1.1 > answer Q2.1	4	4
Average answer for cases with answer Q1.1 = answer Q2.1	0.57	0.57
Std. dev. for cases with answer Q1.1 = answer Q2.1	0.38	0.38
Number of answers where answer Q1.1 = answer Q2.1	3	3
Number of respondents who refused to answer both questions	3	3
Number of respondents who refused to answer Q1.1 only	4	0
Number of respondents who refused to answer Q2.1 only	0	1
Total number of respondents who refused to answer.	7	4

Thus only conclusion 5 can serve as a partial support for Zadeh's claim that answering Q2.1 is impossible. However, it is applicable only to less than 15 % of participants. The other 85% of people have the same easiness/difficulty to answer both Q1.1 and Q2.1.

These results seem to indicate that the probabilistic approach to CWW is at least as good as the possibilistic approach for most of the people who participated in this experiment. The expansion of this experiment to involve more respondents is desirable to check the presented results. We strongly believe that experimental work is necessity in fuzzy logic to guide the production of meaningful formal methods and the scientific justification of existing formal methods, which is largely neglected.

2.6 Is Sum of Possibilities Greater Than 1?

Consider a probability space with two events e_{m1} and e_{m2}

$$e_{m1} = \text{“A } \textit{may} \text{ happen”}, e_{m2} = \text{“A } \textit{may not} \text{ happen”},$$

Thus, for A = “rain” we will have events

$$e_{m1} = \text{“rain } \textit{may} \text{ happen”}, e_{m2} = \text{“rain } \textit{may not} \text{ happen”},$$

Respectively for A = “no rain” we will have events

$$e_{m1} = \text{“no rain } \textit{may} \text{ happen”}, e_{m2} = \text{“no rain } \textit{may not} \text{ happen”},$$

Note, that “dry weather” is not equal to “no rain” (negation of rain), because snow is another part of negation of rain in addition to dry weather.

Next consider another probability space with two events without “may” and “may not”,

$$e_1 = \text{“A”}, e_2 = \text{“notA”}$$

Denote probabilities in spaces with “may” and “may not” as P_m and without them as P_e, respectively. In both cases in accordance with the definition of probability spaces sums are equal to 1:

$$P_m(e_{m1}) + P_m(e_{m2}) = 1, P_e(e_1) + P_e(e_2) = 1,$$

e.g., $P_m(e_{m1}) = 0.4, P_e(e_{m2}) = 0.6$, and $P_e(e_1) = 0.2, P_e(e_2) = 0.8$.

Note that in example $P_m(e_{m1}) + P_e(e_2) = 0.4 + 0.8 = 1.2 > 1$. In fact, these probabilities are from different probability spaces and are not supposed to be summed up.

The confusion takes place in situations like presented below. Let

$$e_{m1} = \text{“rain } \textit{may} \text{ happen” and } e_2 = \text{“no rain”}$$

instead of

$$e_{m2} = \text{“rain } \textit{may not} \text{ happen”}.$$

Probability of e_2 = “no rain” is probability of a *physical event* (absence of a physical event). In contrast probability of e_{m1} = “rain *may not* happen” is probability of a *mental event*. It depends not only on physical chances that rain will happen, but also on the *mental interpretation* of the word “might” that is quite subjective.

This important difference between events e_2 and e_{m2} can be easily missed. It actually happens with a resulting claim that the possibility theory must be completely different from the probability theory based on such mixing events from different probability spaces and getting the sum of possibilities greater than 1.

2.7 Context of Probability Spaces

Consider a sum for possibilities of events A and not A: Pos(A) + Pos(not A) and a sum of probabilities Pr(A) + Pr(not A). For the last sum it is assumed in the Probability Theory that we have the *same context* for Pr(A) and Pr(not A) when it is defined that Pr(A) + Pr(not A) = 1. Removing the requirement of the same context can make the last sum greater than 1. For instance, let Pr(A) = 0.6 for A = “rain”, which is computed using data for the last spring month, but Pr(not A) = 0.8 is computed using data for the last summer month. Thus, Pr(A)+Pr(not A) = 0.6 + 0.8 = 1.4 > 1.

The same *context shift* can happen for computing Pos(A) + Pos(not A). Moreover, for mental events captured by a possibility measure checking that the context is the same and not shifted is *extremely difficult*. Asking “What is the possibility of the rain?”, “What is possibility that rain will not happen?”, “What is the chance that rain may happen?” and “What is the chance that no rain may happen?” without controlling the context can easily produce the sum that will be greater or less than 1. Sums of possibility values from such “context-free” questions cannot serve as a justification for rejecting the probability theory and for introducing a new possibility theory with the sum greater than 1. The experiments should specifically control that *no context shift* happen.

2.8 Natural Language Possibility Versus Modal and Non-modal Probability

Joslyn [17] provided an example of the difference between probability and possibility in the ordinary natural language for a six-sided die. Below we reformulate it to show how this difference can be interpreted as a difference between *modal probability* [30] and *non-modal probability*. A six-sided die has six *possible* outcomes (outcomes that *can occur*). It is applied to both balanced and imbalanced dies. In other words, we can say, each face is *completely possible***,** possible with *possibility 1*, Pos(s) = 1 or *can occur for sure*. In contrast, different faces of the imbalanced die *occur* with different probabilities $P(s_i)$.

Thus, for all sides s_i we have $Pos(s_i) = 1$, but probability $P(s_i) < 1$. Respectively $\Sigma_{i=1:6}Pos(s_i) = 6$ and $\Sigma_{i=1:6}P(s_i) = 1$. At the first glance, it is a strong argument that possibility does not satisfy Kolmogorov’s axioms of probability, and respectively possibility is not probability even from the formal mathematical viewpoint, not only as a NL concept.

In fact, Pos(s_i) satisfies Kolmogorov's axioms, but considered not in a single probability space with 6 elementary events (die sides), but in 6 probability spaces with pairs of elementary events: {s_1, not s_1}, {s_2, not s_2}, ..., {s_6, not s_6} each.

Consider {s_1, not s_1} with Pos(s_1) as an answer for the question: "What is the probability that side s_1 *may occur/happen*?" and Pos(not s_1) as an answer for the question: "What is the probability that side s_1 *may not occur/happen*?". The common sense answers are Pos(s_1) = 1, and Pos(not s_1) = 0 that are fully consistent with the Kolmogorov's axioms. This approach is a core of our approach with the *set of probability spaces* and probability distributions [31–33].

Consider another version of the same situation of mixing of probabilities and possibilities from different spaces as an incorrect way to justify superadditivity for both of them. It is stated in [34]: "…probabilistic relationship between p(A) and p(not A) is fully determined. By contrast, P(A) and P(not A) are weakly dependent in real life situations like medical diagnosis. For instance, given a particular piece of *evidence*, A can be *fully possible* and not A can be *somewhat possible* at the same time. Therefore, a "superadditivity" inequality stands: Pos(A) + Pos(not A) $\geq$ 1."

Let us analyze this medical diagnosis example for a pair of events

$$\{\mathrm{A, notA}\} = \{e_1, e_2\} =$$
$$= \{\text{malignant tumor occurred for the patient, malignant tumor did not occur for the patient}\}$$

Then according to the probability theory we must have $P_e(e_1) + P_e(e_2) = 1$ and it seems reasonable if these probabilities will be based on frequencies of the A and not A under the given evidence. Here in {e_1, e_2} the physical entity (malignant tumor) is negated to get e_2 from e_1, and no modality is involved.

Next consider another pair of events (modal events with the words "might" and "might not"):

{e_{m1}, e_{m2}} = {malignant tumor might occur for the patient,malignant tumor might *not* occur for the patient}

Having $P_m(e_{m1}) + P_m(e_{m2}) = 1$ is also seems reasonable for these events if these probabilities will be based on frequencies of subjective judgments of experts about e_{m1} and e_{m2} under given evidence. The experts will estimate the same physical entity "malignant tumor" for two different (opposite) modalities "can" and "cannot". In other words here we *negate modality* ("can") not a *physical entity* (malignant tumor) as was the case with {e_1, e_2}. The same property is expected for the events:

{e_{m3}, e_{m4}} = {benign tumor might occur for the patient,benign tumor might not occur for the patient}

The pairs {e_{m1}, e_{m2}} and {e_{m3}, e_{m4}} differ from another pair that mix them:

{e_{m1}, e_{m3}} = {malignant tumor might occur for the patient, *no* malignant tumor (benign tumor) *might* occur for the patient}

For this pair, $P_m(e_{m1}) + P_m(e_{m3}) = 1$ seems less reasonable. Here in e_{m3} the *physical entity* (malignant tumor) is *negated*, but *modality* ("might") is the same. In essence, e_{m1} and e_{m3} are from different modal probability spaces, and should not be added, but operated with as it is done in the probability theory with probabilities from different probability spaces.

This example shows that probability spaces for modal entities must be built in a *specific way* that we identified as N1 below. Let A be a modal statement. It has two components: a physical entity E and a modal expression M, denote such A as A = (E, M). Respectively, there are three different not A:

(N1): not A = (E, not M)
(N2): not A = (not E, M)
(N3): not A = (not E, not M)

As we have seen above, it seems more reasonable to expect additivity for N1 than for N2 and N3. The next example illustrates this for N1 and N2. Consider A = (E, M) where E = "sunrise", not E = "sunset", M = "can", and not A = (E, not M) for N1, and not A = (not E, M) for N2.

For N1 the questions are:

"What is the probability that John *can* watch the sunrise tomorrow?" and

"What is the probability that John *cannot* watch the sunrise tomorrow?

For N2 these questions are:

"What is the probability that John can watch the *sunrise* tomorrow?" and

"What is the probability that John can watch the *sunset* tomorrow?

These questions in the possibilistic form can be:

"What is the possibility that John will watch the *sunrise* tomorrow?" and

"What is the possibility that John will watch the *sunset* tomorrow?"

Both probabilities/possibilities P(A) and P(not A) can reach 1 for N2, therefore building spaces with N1 and N3 should be avoided. The situation is the same as in the probability theory itself—not every set of events can be used as a set of meaningful elementary events.

Consider the next example, based on the following joke: "Can misfortune make a man a millionaire? Yes if he is a billionaire. In this example the first two pairs of questions are in N1, and the last pair is in N2:

What is the probability that misfortune *can* make a man a millionaire if he is a billionaire?

What is the probability that misfortune *cannot* make a man a millionaire if he is a billionaire?

What is the possibility that misfortune *will* make a man a millionaire if he is a billionaire?

What is the possibility that misfortune *will not* make a man a millionaire if he is a billionaire?

What is the possibility that misfortune will make a man a millionaire if he is a billionaire?

What is the possibility that negligence will make a man a millionaire if he is a billionaire?

Consider another example on possibility of n people in the car [30]. Let n = 10 and A is "10 people are in the car". What is the possibility of "10 people in the car"? Let Pos(A) = 0.8 then "What is the possibility of not A?", i.e., possibility of "not 10 people in the car". We need to define a probability space {A, notA}. Note that "not 10 people in the car" includes 9 people. Thus

$$\mathrm{Pos(notA)} \geq \mathrm{Pos(9)},$$

Next it is logical to assume that Pos(9) $>$ Pos(10). Thus, Pos(not A) $\geq$ 0.8 with

$$\mathrm{Pos(A)} + \mathrm{Pos(notA)} > 1.$$

As we noted above such "superadditivity" often is considered as an argument for the separate possibility theory with superadditivity.

In fact this cannot be such argument because there is a space with normal additivity (P(A) + p(notA) = 1). In that space we have a different A. It is not an event that 10 people are in the car, but a modal statement, where A = "10 people *can be* in the car", and respectively not A = "10 people *cannot be* in the car", e.g., with P(A) = 0.8 and P(notA) = 0.2.

Thus in general superadditivity situations N2 and N3 can be modeled with a Set (pair) of probability (K-spaces) Spaces (SKS) of N1 type without any superadditivity. This is in line with our approach [32] discussed above.

Note that superadditivity can really make sense in the probability theory and be justified, but very differently, and not as an argument for the separate possibility theory. See a chapter by Resconi and Kovalerchuk in this volume.

3 Relations Between Grades of Membership of Fuzzy Set and Probabilities

3.1 *What is More General Concept Fuzzy Set, Probability or Possibility?*

Joslyn stated [17] that neither is fuzzy theory specially related to possibility theory, nor is possibility theory specifically related to fuzzy sets. From his viewpoint "both probability distributions and possibility distributions are *special cases of fuzzy sets*". It is based on the fact an arbitrary fuzzy set can be specialized to be probability or possibility by imposing additional properties. For probability this property is that the sum (integral) or all values must be equal to 1. For possibility this property is that the maximum must be equal to 1.

The specialization of an arbitrary fuzzy set to probability or possibility can be done as follows. Let $m(x)$ be a membership function of a fuzzy set on the interval [a, b]. We can normalize values of $m(x)$ by dividing them by the value S which is the sum (integral) of m(x) values in this interval and get probabilities, $p(x) = m(x)/S$. Alternatively, we can divide $m(x)$ by the value M which is the maximum of m(x) values in the interval [a, b], $\text{pos}(x) = m(x)/M$.

However, these transformations do not establish the actual relation between theories: Fuzzy Set Theory (FST), Fuzzy Logic (FL) and Probability Theory (PrT) where the *main role play operations* with membership functions m and probabilities p. The concept of the fuzzy set is only a part of the fuzzy set theory, we should not base the comparison of the theories based only one concept, but should analyze other critical concepts such as operations.

Probability theory operations of intersection and union ($\cap$ and $\cup$) are *contextual*, $\text{p}(\text{x} \cap \text{y}) = \text{f}(\text{p}(\text{x}), \text{p}(\text{y}|\text{x})) = \text{p}(\text{x})\text{p}(\text{y}|\text{x})$, where conditional probability p(y|x) expresses contextual dependence between x and y. In contrast, operations in FST and FL, i.e., t-norms and t-conforms are "*context-free*", $m(\text{x}\ \&\ \text{y}) = f(\text{m}(\text{x}), \text{m}(\text{y}))$, that is no conditional properties, dependencies are captured. In fact FL, FST present rather a *special case* of the Probability Theory because all t-norms are a *subset* of copulas that represent n-D probability distributions [4, 35].

Joslyn [17] also made an interesting comment that the same researchers, who object to confusion of membership grades with probability values, are not troubled with confusion of membership grades with possibility values. In fact both confusions have been resolved by:

(In1) interpreting a **fuzzy set** as a **set of probability distributions** (**SPD**) not a single probability distribution [4, 31, 32].

(In2) interpreting **possibility** as **modal probability** as proposed by Cheeseman [30].

In the next section we illustrate the first interpretation and show how fuzzy sets are interpreted/modeled by SPD. Note that the interpretation of membership functions as a *single probability distribution* is too narrow and often is really confusing. The interpretation In2 already has been discussed in the previous sections.

3.2 What Is the Relation of a Membership Function to Probability?

Piegat [25, 36] stated that a membership function of the crisp set is not pdf and not probability distribution, noting that it only informs that a set element belongs to the considered set.

While it is true that in general a membership function (MF) is not a pdf or a probability distribution it is not a necessary to compare them in a typical way, i.e., by one-to-one mapping. This relation can be expressed in different ways.

The relation can be one-to-many similarly to the situation when we compare a linear function and a piecewise linear function. While these functions differ, a set of linear functions accurately represents a piecewise linear function. In the case of MF a set of pdfs represents any MF of a crisp set or a fuzzy set [32]. This technique uses a set of probability Spaces that are *Kolmogorov's measure (K-Measure) spaces* (S*KM*) [4].

Let $f(\mathrm{x}) = 1$ be a MF of the interval [a, b]. Consider a pair of events

$$\{\mathrm{e}_1(\mathrm{x}), \mathrm{e}_2(\mathrm{x})\} = \{\text{x belongs to [a, b], x does not belong to [a, b]}\}$$

with pdf $\mathrm{P_x(e(x))} = 1$, if x belongs to [a, b] else $\mathrm{P_x(e(x))} = 0$. Thus, $\mathrm{P_x(e_1(x))} = f(\mathrm{x}) = 1$, $\mathrm{P_x(e_2(x))} = 0$ and $\mathrm{P_x(e_1(x))} + \mathrm{P_x(e_2(x))} = 1$. This is a simplest probability space that satisfies all Kolmogorov's axioms of probability. A set of these pdfs $\mathbf{p} = \{\mathrm{P_x}\}$ for all x from [a, b] is equivalent to MF of the crisp set.

Note that here we use the term probability density function (pdf) in a general sense as any function with sum(integral) of its values equal to one on its domain. It can be a traditional pdf defined on a continuous variable or on a set of any nature continuous or discrete that will be called a set of elementary events. As we will see below typically these sets will be pairs of NL sentences or phrases.

For a fuzzy set with a membership function $f(\mathrm{x})$ we have probabilities

$$\mathrm{P_x(e_1(x))} = f(\mathrm{x}), \mathrm{P_x(e_2(x))} = 1 - f(\mathrm{x})$$

that satisfy all Kolmogorov's axioms of probability, where again a set of these pdfs $\mathbf{p} = \{\mathrm{P}_x\}$ for all x from [a, b] is equivalent to MF of the fuzzy set $\langle [\mathrm{a, b}], f\rangle$.
Figure 1 shows these probability spaces for each x that is human's height h.

In other words MF is a *cross-section* of a set of pdfs **p**. More details are in [32]. The advantage of MF over a set of pdfs is in *compactness* of the representation for many MFs such as triangular MFs. Each of these MFs "compresses" an infinite number of pdfs. A probability space S(180) is shown as a pair of circles on a vertical line at point h = 180. As we can see from this figure, just two membership functions serve as a compact representation of many simple probability spaces. This is a *fundamental representational advantage* of Zadeh's fuzzy linguistic variables [37, 38] versus multiple small probability spaces.

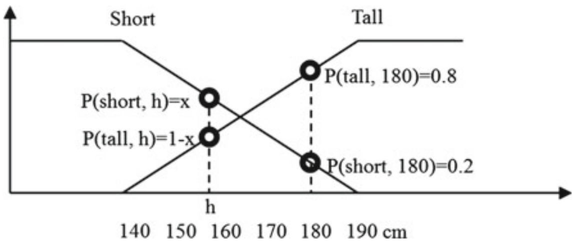

Fig. 1 Sets of probability spaces S(h) for elementary events {short(h), tall(h)} for each height h

It is important to note that having two MFs $m_1(x)$ and $m_2(x)$ for two fuzzy sets the operations with them must follow the rules of probabilistic operations AND and OR with sets of respective pdfs $\mathbf{p}_1$ and $\mathbf{p}_2$ on two sets of probability spaces to be *contextual*.

An example of these operations is presented in [4] using convolution-based approach. In this context it is very important to analyze the relation of t-norms and t-conorms used in fuzzy logic with sets of probability distributions. One of such analyses is presented in [4, 35] in this volume.

The common in fuzzy logic "context-free" heuristic min and max operations and other t-norms and t-conorms for AND and OR operations on fuzzy sets are not correct probabilistic operations in general. These context-free operations can only serve as approximations of contextual probabilistic operations for AND and OR operations. For the tasks where the context is critical context-free operations will produce incorrect results.

Thus, membership functions and sets of pdfs are mutually beneficial by combining *fast* development of pdfs (coming from MFs) and rigor of *contextual operations* (coming from probability).

In this sense, fuzzy membership functions and probabilities are *complimentary not contradictory*. It would be incorrect to derive from advantages of MFs outlines above the conclusion that probabilistic interpretation only brings a complication to a simple fuzzy logic process for solving practical problems. The probabilistic interpretation of MFs removes a big chunk of heuristics from the solution and brings a rigorous mechanism to incorporate context into AND and OR operations with fuzzy sets.

3.3 Are Fuzzy Sets and Possibility Distributions Equivalent?

Zadeh [22] defined a possibility distribution as a fuzzy set claiming that possibilistic concepts are inherently more appropriate for fuzzy theory than probabilistic concepts.

This equivalence was rejected in earlier studies [17] which reference works with a *wider view* on the concept of possibility that includes:

- its generalization to the *lattice* [13],
- involvement of *qualitative relations* [16],
- the semantics of *betting* [19],
- *measurement* theory [21],
- *abstract* algebra [20].

Other alternatives include:

- *modal probabilities* [30] and the later work by
- *elaborated lattice* approach [39].

Also a general logic of uncertainty based on the lattice can be traced to Gains [18, 40].

3.4 Is Interpreting Grade of Membership as Probability Meaningful?

Zadeh [6] objects to equating a grade of membership and a truth value to probability, but equates a grade of membership and a truth value as obvious. He states that "equating grade of membership to probability, is *not meaningful*", and that "consensus-based definitions may be formulated *without* the use of probabilities".

To support this statement he provided an example of a *grade of membership* 0.7 that Vera is middle-aged if she is 43. Below we explicate and analyze his statement for this example that is reproduced as follows.

Grade of Membership that Vera is middle-aged being 43:

$$\text{GradeMembership}(\text{Vera is middle-aged}|\text{Age} = 43) = 0.7, \tag{1}$$

Truth value for Age = 43 to be middle-aged,

$$\text{TruthValue}(\text{Age} = 43 \text{ is middle-aged}) = 0.7. \tag{2}$$

Possibility that Vera is middle-aged for the Age = 43,

$$\text{Poss}(\text{Age} = 43|\ \text{Vera is middle-aged}) = 0.7 \tag{3}$$

Probability that Vera is middle-aged for the Age = 43,

$$\text{Prob}(\text{Vera is middle-aged}|\ \text{Age} = 43) = 0.7 \tag{4}$$

In other words, Zadeh equates (1), (2) and (3) and objects to equating all of them to (4) as *not meaningful* and *not needed.*

Consider a crisp voting model that equates the grade of membership to probability *meaningfully.* In this model, the statement "Vera is middle-aged" is classified as true or false by each respondent. Respectively the grade of membership is defined as the *average* of votes. Jozo Dujmovic presented this approach in details in his chapter in [41] and in this volume [29].

In a *flexible voting model* the voter indicates a grade of membership on the scale from 0 to 1 as a subjective/declarative probability of each voter. In the case of multiple voters the consensus-based grade of membership is defined as the *average* of subjective/declarative probabilities of voters. Thus, a meaningful probabilistic interpretation of a grade of membership exists for long time and is quite simple.

Next we comment on Zadeh's statement that "consensus-based definitions may be formulated *without* the use of probabilities". If the intent of the whole theory is to get results like in (1) then it can be obtained without using the word "probability". In fact both fuzzy logic and probability theory are interested to get more complex answers than (1). This leads to the need to *justify operations* on combination such as t-norms and t-conorms used in fuzzy logic as normative or descriptive. These operations differ from probability operations. Moreover there are multiples of such operations for both conjunction and disjunction in contrast with the probability theory, where the conjunction (intersection) and disjunction (union) are unique. Thus each t-norm and t-conorm must be justified against each other and probabilistic operations. Note that experimental studies [42, 43] with humans do not justify min t-norm versus product t-norm, which is a probabilistic operation for independent events. In general the justification of operations continues to be debated [3, 44] while extensive experimental studies with humans still need to be conducted in the fuzzy logic justification research.

3.5 Unsharpness of Border And Randomness Versus Axiomatic Probability

Chang [24] differenced fuzzy logic and probability theory stating that (1) *fuzzy sets theory* deals with *unsharpness*, while the *probability theory* deals with *randomness* and (2) the problems with unsharpness *cannot be converted* into "randomness problems". This statement was made to justify that there are problems which cannot be solved by using the probability theory, but which can be solved by using the fuzzy sets and fuzzy logic. In this way, he attempted to counter my arguments that probability theory has a way to solve Zadeh's CWW test tasks by using the formal axioms of probability theory (called by Tschantz [45] axioms of *K-measure space*).

Tschantz [45] objected to Chang's statement: "As Boris has said many times, he is not attempting to convert reasoning about vague words into reasoning about randomness. He is merely attempting to *reuse the formal axioms of Kolmogorov*. While these axioms were first found useful in modeling randomness, the fact that they are useful for modeling randomness does not mean that they cannot also be useful for modeling vagueness. To put it another way, someone can use the same model M to model both X or Y even if X and Y are not the same thing. Thus, I do not buy CL Chang's argument that Boris is incorrect. If he [24] shows that vagueness has some feature that makes it behave in such a way that is so different from randomness that no model can be good at modeling both, then I would buy his argument. However, I would need to know *what that feature is*, what "good" means in this context, and how he proves that no model can be good at both. It seems far easier to point to some contradiction or unintuitive result in Boris's model."

So far no contradiction or unintuitive results have been presented by opponents to reject the approach based on the Space of K-Measures.

4 What Is the Future Rigorous Possibility Theory?

4.1 Is Possibility an Upper Estimate of Probability for Complex Events?

Below we analyze abilities to interpret possibility Pos as an upper estimate of probability Pr using the current min-max possibility theory where "&" operator is modeled as min and "or" operator is modeled as max.

Section 2.4 had shown how questions with words "possibility" and "probability" can be reworded to be in a comparable form using their Webster definitions. Thus Webster leads to the conclusion that possibility must be an upper estimate of probability in NL for every expression X,

$$\text{Pos(X)} \geq \text{Pr(X)}$$

Respectively when possibility of A&B is defined as

$$\text{Pos(A\&B)} = \text{min(Pos(A), Pos(B))}$$

it gives an *upper estimate* of Pr(A&B),

$$\text{Pos(A\&B)} = \text{min(Pos(A), Pos(B))} \geq \text{Pr(A\&B)}$$

However in contrast when Pos(A or B) is defined as

$$\text{Pos(A or B)} = \text{max(Pos(A), Pos(B))}$$

it gives a *lower estimate* of Pr(A or B) = P(A) + P(B) - P(A&B)

$$\text{Pos(A or B)} = \text{max(Pos(A), Pos(B))} \leq \text{Pr(AorB)}$$

Thus, the NL property Pos(X) ≥ Pr(X) does not hold when X = (A or B) in the min-max possibility theory advocated by Zadeh and others.

Let us compute the possibility of another composite expression X = A&(B or C) using the min-max possibility theory:

$$\text{Pos(A\&(B or C)} = \text{min(Pos(A), Pos(B or C)))} =$$
$$\text{min (Pos(A), max (Pos(B), Pos(C))).}$$

The value min (Pos(A), Pos(B or C))) is supposed to be an upper estimate for Pr(A&(B or C)). However, min(Pos(A), max(Pos(B), Pos(C))) is not an upper estimate for Pr(A&(B or C)) because max (Pos(B), Pos(C)) is a lower estimate for Pr(B or C). The infusion of this lower estimate to min (Pos(A), Pos(B or C))) can destroy an upper estimate.

Below we show it with a numeric example. Consider sets A, B and C such that A has a significant overlap with (B or C) with probabilities

$$\Pr(\mathrm{A}) = 0.4, \Pr(\mathrm{B}) = 0.3, \Pr(\mathrm{C}) = 0.32, \Pr(\mathrm{B}\text{ or }\mathrm{C}) = 0.5, \Pr(\mathrm{A}\&(\mathrm{B}\text{ or }\mathrm{C})) = 0.35$$

From these numbers we see that

$$\Pr(\mathrm{A}) = 0.4 < \Pr(\mathrm{B}\text{ or }\mathrm{C}) = 0.5$$

$$\Pr(\mathrm{B}\text{ or }\mathrm{C}) = 0.5 > \max(\Pr(\mathrm{B}), \Pr(\mathrm{C})) = \max(0.3, 0.32)) = 0.32$$

$$\begin{aligned}&\min(\Pr(\mathrm{A}), \Pr(\mathrm{B}\text{ or }\mathrm{C}))) = \min(0.4, 0.5) = 0.4 =\\ &\mathrm{Pos}(\mathrm{A}\&(\mathrm{B}\text{ or }\mathrm{C})) > \Pr(\mathrm{A}\&(\mathrm{B}\text{ or }\mathrm{C})) = 0.35.\\ &\mathrm{Pos}(\mathrm{A}\&(\mathrm{B}\text{ or }\mathrm{C})) = \min(\Pr(\mathrm{A}), \max(\Pr(\mathrm{B}), \Pr(\mathrm{C}))) =\\ &\qquad \min(0.4, 0.32) = 0.32 < \Pr(\mathrm{A}\&(\mathrm{B}\text{ or }\mathrm{C})) = 0.35\end{aligned}$$

Thus, upper estimate 0.4 of probability 0.35 is converted to the lower estimate 0.32 of this probability 0.35.

This confirms that for composite expressions that involve "or" operator, we cannot ensure that possibility of that expression is an upper estimate of its probability in the min-max possibility theory. Thus, a future possibility theory should differ from the current min-max possibility theory to be able to hold the property that Pos(X) ≥ Pr(X) for all composite expressions X.

4.2 *Should Evaluation Structure Be Limited by [0, 1] or Lattice?*

In Sect. 3.3 we referenced several studies that generalized the possibility theory to a lattice (partial order of evaluations) from the full order in [0.1]. However, to the best of our knowledge, none of the previous studies went beyond a *lattice structure*. The lattice assumption means that the *absolute positive* and *negative possibility* exists with max value commonly assigned to be equal to 1 and min value to be equal to 0. It does not mean that x with Pos(x) = 1 always happens, but such x is one of the real alternatives, e.g., x is physically possible. Similarly Pos(x) = 0 can mean that x physically cannot happen.

There are NL concepts where this assumption can be too restrictive. Consider the concept of happiness and a chance that somebody might by absolutely happy. Can somebody be absolutely happy (unhappy) to the level that he/she cannot be happier in the future? The lattice assumption tells that if x is absolutely happy at time t, Happy(x) = 1, then x cannot be happier later. If x tells that he/she was absolutely happy yesterday, and even happier today then under the lattice model this person will be inconsistent or irrational. This person "exceeded" the absolute maximum. Another example is "possibility of absolute understanding". What is a chance that somebody might understand something absolutely today and cannot increase understanding later? In other words, we question the concept of absolute maximum for possibilities such as absolute happiness, Pos(Happy(x)) = 1 and absolute understanding, Pos(Understand(x)) = 1. The number of such "unlimited" NL concepts is quite large.

How to build a model to accommodate such behavior? We can simply remove limit 1 (or any constant given in advance) and allow the unlimited values of possibility of happiness, in this example and, in a general measure of possibility.

Other difficulties for constructing an evaluation medium are that often a set of possible alternatives is uncertain, and assigning the possibility values to a known alternative is highly uncertain and difficult. For instance, the NL expression "unlimited possibilities" often means that a set of alternatives is huge, and not fully known, being uncertain to a large extent. Often it is easier to say that A is more possible than B, and both alternatives are not fully possible, than to give numeric Pos(A) and Pos(B). We simply may have no stable landmark for this. The NL expression "unlimited possibilities" can mean that for a possible A, there can exist a B, which is more *possible* than A. The next difficulty is that adding more possible alternatives to the consideration changes the possibility values. Next the fixed value of max of possibility, e.g., $\max_{x \in X} \mathrm{Pos}(x) = 1$, is not coming from the NL, but is just *imposed* by the mathematical formalisms. We advocate the need for developing the formalism with "unlimited" possibilities.

4.3 From an Exact Number to an Interval

Another common viewpoint is that possibility expresses a *less constraining form of uncertainty* than probability [46], and even that the probability theory brings in the excess of constraints [17] requiring an *exact number* instead of an interval of belief and plausibility [Bel, Pl] as in the evidence measure in the belief theory.

Again while it is true that probability is a single number, not an interval, the issue is not in the formal expansion to the interval, but in the justification of operations to produce the intervals and operations with intervals. We need well justified *empirical procedures* to get them. Without this generalization, the intervals and other alternative formalisms have little value. Somebody must bring the "life meaning" to it. Respectively the claims of successful applications need to be closely scrutinized to answer a simple question: "Is the application successful because of the method, or in spite of it?" In other words: "Does this method capture the properties of the application better than the alternative methods, in our case the pure probabilistic methods?" It short, deficiency of one method should not be substituted by deficiency of another method.

5 Conclusion

This work clarified the relationships between the probability and the possibility theories as tools for solving Computing with Words tasks that require dealing with extreme uncertainty of the Natural Language. Multiple arguments have been provided to show deficiencies of Zadeh's statement that CWW tasks are in the province of the fuzzy logic and possibility theory, and probabilistic solutions are not appropriate for these tasks. We proposed a useful constructive probabilistic approach for CWW based on sets of specialized K-Measure (Kolmogorov's measure) probability spaces. Next we clarified the relationships between probability and possibility theories on this basis, outlined a future rigorous possibility theory and open problems for its development.

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Modeling Extremal Events Is Not Easy: Why the Extreme Value Theorem Cannot Be As General As the Central Limit Theorem

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Abstract In many real-life situations, a random quantity is a joint result of several independent factors, i.e., a *sum* of many independent random variables. The description of such sums is facilitated by the Central Limit Theorem, according to which, under reasonable conditions, the distribution of such a sum tends to normal. In several other situations, a random quantity is a *maximum* of several independent random variables. For such situations, there is also a limit theorem—the Extreme Value Theorem. However, the Extreme Value Theorem is only valid under the assumption that all the components are identically distributed—while no such assumption is needed for the Central Limit Theorem. Since in practice, the component distributions may be different, a natural question is: can we generalize the Extreme Value Theorem to a similar general case of possible different component distributions? In this paper, we use simple symmetries to prove that such a generalization is not possible. In other words, the task of modeling extremal events is provably more difficult than the task of modeling of joint effects of many factors.

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1 Sums and Maxima of Independent Factors: Formulation of the Problem

Why normal distributions are ubiquitous: sums of many independent factors. In many practical situations, we have a joint effects of many independent small factors. This happens, e.g., in measurements, when:

- after eliminating all major sources of possible measurement error,
- we end up with a measurement error which results from a joint effect of multiple difficult-to-eliminate independent sources of measurement uncertainty.

In this case, the measurement error—i.e., the difference $\Delta X = \widetilde{X} - X$ between the measurement result $\widetilde{X}$ and the actual value X of the corresponding quantity—can be represented as a sum $\Delta X = \sum\limits_{i=1}^{n} X_i$ of a large number n of small independent random variables X_i.

The description of the resulting probability distribution for this sum ΔX is facilitated by the well-known *Central Limit Theorem*, according to which, under some reasonable conditions, the distribution of such a sum tends to Gaussian as n tends to infinity; see, e.g., [13]. This limit result means that for large n, the distribution of the sum ΔX is close to Gaussian.

This is indeed the case for most measuring instruments: experimental analysis shows that for about 60% of them, the measurement error is normally distributed [10, 11]. The Central Limit Theorem also explains why normal distribution is ubiquitous in many other areas as well: the familiar bell-shaped curve indeed describes many phenomena, from distribution of people by height or by weight to distribution of molecules by velocity.

Extremal events: maxima of many independent factors. In many other practical situations, we are interested in describing the *maxima* of several independent factors. For example, in structural engineering, to estimate the structure's stability under catastrophic events such as hurricanes and earthquakes, it is important to estimate the probability that this structure collapses—i.e., that in one of its components, the tension exceeds the stability threshold. The condition that one of the tension values X_i exceeds the threshold x is equivalent to $X \geq x$, where $X \stackrel{\text{def}}{=} \max\limits_i X_i$ is the maximum of several independent components. Thus, to study such extremal events, it is important to know the probability distribution of such maxima.

Similar arguments show the need to study similar maxima in earth sciences, in finances, in hydrology, and in many other areas where rare disastrous events are possible; see, e.g., [1–7, 9, 12].

Limit theorems for extreme events: what is known. Similarly to the Central Limit Theorem that describes the limit of sums, there are the limit theorems that describe the limits of maxima of many independent random variables. The most well-known of these limit theorems is the *Extreme Value Theorem* (also known as the *Fisher-Tippett-Gnedenko Theorem*), according to which, if we have a sequence of independent identically distributed random variables X_i and the distributions of their maxima $M_n \stackrel{\text{def}}{=} \max(X_1, \ldots, X_n)$ has a limit, then this limit has one of the following forms [5]:

- *Weibull law*, with cumulative distribution function (cdf) $F(x) = \exp\left(-\left|\dfrac{x-b}{a}\right|^{\alpha}\right)$ for $x \le b$ and $F(x) = 1$ for $x \ge b$;
- *Gumbel law* $F(x) = \exp\left(-\exp\left(\dfrac{b-x}{a}\right)\right)$; and
- *Fréchet law* $F(x) = \exp\left(-\left(\dfrac{x-b}{a}\right)^{-\alpha}\right)$ for $x > b$ and $F(x) = 0$ for $x \le b$.

Formulation of the problem: what is available for the central limit theorems but missing for extreme value theorems. In many formulations of the Central Limit Theorem, it is not necessary to require that all components random variables X_i are identically distributed. These theorems are applicable to many situations in which different variables X_i have different distributions.

In contrast, the Extreme Value Theorem is only known for the case when all the component random variables X_i are identically distributed. In practical applications, the distributions of the corresponding random variables X_i—e.g., variables describing stability of different part of the construction—are, in general, different. A natural question arises: can we generalize the Extreme Value Theorem so that it can be applied to the case when we have different distributions X_i?

What we prove in this paper. In this paper, we prove that such a generalization is not possible. In this sense, the task of modeling extremal events is provably harder than the task of modeling a joint effect of several factors.

2 Analysis of the Problem

What do we mean by the desired generalization? Both in case of the Central Limit Theorem and in the case of the Extreme Value Theorem, we have a finite-parametric family of limit distributions such that, under certain reasonable conditions, the distribution of the corresponding sum or maxima tends to one of the distributions from this class.

From this viewpoint, when we say that we are looking for a generalization of the Extreme Value Theorem—which would be similar to the current (generalized) versions of the Central Limit Theorem—we mean that:

- we are looking for a finite-dimensional family $\mathcal{F}$ of probability distributions,
- such that, that, under certain reasonable conditions, the distribution of the corresponding maxima tends to one of the distributions from the class $\mathcal{F}$.

Let us describe which properties of the class $\mathcal{F}$ follow from this desired description.

First desired property: the class $\mathcal{F}$ should be invariant under shifts and scalings. We are interested in the distributions of physical quantities X_i. Of course, in all data processing, we deal with the numerical values of the corresponding physical quantities. The numerical value of a quantity depends on the choice of a measuring unit and on the choice of a starting point. For example, we can measure time in years or days or seconds, we can start measuring time with year 0 or with year 1950 (as is sometimes done in astronomical computations), etc.

When we change a starting point for measuring X by a new starting point which is smaller by b units, then the numerical value X changes to $X' = X + b$. This shift changes the numerical expression for the cdf: instead of the original probability $F(x) = \text{Prob}(X \le x)$ that $X \le x$, for $X' = X + b$, we have

$$F'(x) = \text{Prob}(X' \le x) = \text{Prob}(X + b \le x) = \text{Prob}(X \le x - b) = F(x - b).$$

When we change a measuring unit to a new one which is $a > 0$ times smaller, then the numerical value X changes to $X' = a \cdot X$. For example, if instead of meters, we use centimeters, a unit which is $a = 100$ times smaller, then all numerical values of length are multiplied by 100: e.g., $X = 2$ m becomes $X' = 200$ cm. This scaling changes the numerical expression for the cdf: instead of the original probability $F(x) = \text{Prob}(X \le x)$ that $X \le x$, for $X' = a \cdot X$, we have

$$F'(x) = \text{Prob}(X' \le x) = \text{Prob}(a \cdot X \le x) = \text{Prob}\left(X \le \frac{x}{a}\right) = F\left(\frac{x}{a}\right).$$

In general, if we change both the starting point and the measuring unit, we get a new cdf

$$F'(x) = F\left(\frac{x - b}{a}\right).$$

If we perform this transformation, then all the values X_i are replaced by the new values $X_i' = a \cdot X_i + b$. For the maximum, we similarly have

$$M_n' = \max(X_1', \ldots, X_n') = \max(a \cdot X_1 + b, \ldots, a \cdot X_n + b) =$$

$$a \cdot \max(X_1, \ldots, X_n) + b = a \cdot M_n + b.$$

Thus, if in the original units, we had a limit distribution $F(x)$, in the new units, we will have a limit distribution $F'(x) = F\left(\dfrac{x - b}{a}\right)$.

The desired limit theorem should not depend on the choice of the starting point or on the choice of a measuring unit. Thus, it is reasonable to require that if the class $\mathcal{F}$ of limit distributions contains a cdf $F(x)$, then it should also contain a re-scaled and shifted distribution $F'(x) = F\left(\dfrac{x - b}{a}\right)$.

Second desired property: the class $\mathcal{F}$ should be closed under multiplication of cdfs. Let us assume that $F(x)$ and $F'(x)$ are two cdfs from the desired class $\mathcal{F}$. By definition of the class $\mathcal{F}$, all distributions from this class are limits of distributions of the maxima. In particular:

- the cdf $F(x)$ is the limit of the distributions $F_n(x)$ of $M_n = \max(X_1, \ldots, X_n)$ for some sequence of independent random variables X_i, and
- the cdf $F'(x)$ is the limit of the distributions $F'_n(x)$ of $M'_n = \max(X'_1, \ldots, X'_n)$ for some sequence of independent random variables X'_i.

Then, for a combined sequence $X''_i \stackrel{\text{def}}{=} X_1, X'_1, \ldots, X_n, X'_n, \ldots$, the corresponding maxima will have the form

$$M''_{2n} = \max(X_1, X'_1, \ldots, X_n, X'_n) =$$

$$\max(\max(X_1, \ldots, X_n), \max(X'_1, \ldots, X'_n)) = \max(M_n, M'_n).$$

The distribution of M_n is close to $F(x)$, the distribution of M'_n is close to $F'(x)$. The cdf $F''_{2n}(x)$

for the maximum M''_{2n} can be thus described as follows:

$$F''_{2n}(x) = \text{Prob}(M''_n \le x) = \text{Prob}(\max(M_n, M'_n) \le x) =$$

$$\text{Prob}(M_n \le x \,\&\, M'_n \le x).$$

Since the variables X_i and X'_i are independent, their maxima M_n and M'_n are also independent, so

$$F''_{2n}(x) = \text{Prob}(M_n \le x \,\&\, M_n \le x) = \text{Prob}(M_n \le x) \cdot \text{Prob}(M'_n \le x),$$

i.e., $F''_{2n}(x) = F_n(x) \cdot F'_n(x)$. In the limit, the distribution of M_n tends to $F(x)$ and the distribution of M'_n tends to $F'(x)$, so the distribution for the new sequence tends to the product $F(x) \cdot F'(x)$.

Thus, with every two cdfs $F(x)$ and $F'(x)$, the class $\mathcal{F}$ should also contain their product $F(x) \cdot F'(x)$.

The class $\mathcal{F}$ should be finite-dimensional. The previous property is easier to describe if we consider logarithms $\ln(F(x))$: the logarithm of product of the cdfs is the sum of their logarithms, so the class $\mathcal{L}$ of such logarithms should be closed under addition.

One can easily check that if the original class $\mathcal{F}$ is closed under shifts and scalings, then the class $\mathcal{L}$ of logarithms of functions $F(x) \in \mathcal{F}$ should also be similarly closed.

Each such class can be naturally extended to a linear space. We can show that this space should also be closed under shift and scaling.

The fact that the original set is finite-dimensional (= finite-parametric) implies that this space should also be finite-dimensional, i.e., all its functions should have the form

$$\ell(x) = C_1 \cdot e_1(x) + \cdots + C_m \cdot e_m(x),$$

where m is the dimension of this space, $e_i(x)$ are given functions, and C_i are arbitrary real values.

Now, we are ready to formulate and prove our main result.

3 Definitions and the Main Result

Definition We say that a finite-dimensional linear space $\mathcal{L}$ of differentiable functions is *shift-* and *scale-invariant* if with every function $\ell(x)$ and for every two real numbers $a > 0$ nd b, this class also contains the function

$$\ell'(x) = \ell\left(\frac{x-b}{a}\right).$$

Proposition *For every shift- and scale-invariant finite-dimensional linear space $\mathcal{L}$ of differentiable functions, all its elements are polynomials.*

Comment. All the proofs are given in the following section.

Corollary *If $F(x)$ is a cdf, then its logarithm* $\ln(F(x))$ *cannot be an element of a shift- and scale-invariant finite-dimensional linear space.*

Discussion. This result shows that a finite-dimensional limit class $\mathcal{F}$ is not possible. Thus, the Extreme Value Theorem indeed cannot be extended to the general case when variables X_i are not necessarily identically distributed.

4 Proofs

Proof of the Proposition. The main ideas of this proof can be found in [8].

1°. The fact that the linear space $\mathcal{L}$ is shift-invariant means, in particular, that for every basis function $e_i(x)$ and for every real value b, the shifted function $e_i(x+b)$ also belongs to this linear space. Since all the function from the linear space are linear combinations of the basis functions $e_1(x), \ldots, e_m(x)$, this means that for every b, there exist values $C_{i,j}(b)$ for which

$$e_i(x+b) = C_{i,1}(b) \cdot e_1(x) + \cdots + C_{i,m}(b) \cdot e_m(x). \tag{1}$$

For each b, we can take m different values $x_1, \ldots, x_m$, and get m resulting equalities:

$$e_i(x_1+b)=C_{i,1}(b)\cdot e_1(x_1)+\cdots+C_{i,m}(b)\cdot e_m(x_1);$$

$$\ldots$$

$$e_i(x_j+b)=C_{i,1}(b)\cdot e_1(x_j)+\cdots+C_{i,m}(b)\cdot e_m(x_j);\tag{2}$$

$$\ldots$$

$$e_i(x_m+b)=C_{i,1}(b)\cdot e_1(x_m)+\cdots+C_{i,m}(b)\cdot e_m(x_m).$$

We thus get a system of m linear equations for m unknowns $C_{i,1}(b)$, …, $C_{i,m}(b)$. By using Cramer's rule, we can describe the values $C_{i,j}(b)$ as ratios of polynomials in terms of the coefficients $e_j(x_k)$ and the right-hand sides $e_i(x_j+b)$. Since the functions $e_i(x)$ are differentiable, we can conclude that the dependence $C_{i,j}(b)$ on b is differentiable as well.

2°. We can now combine the Eq. (1) corresponding to different functions $e_i(x)$. As a result, we get the following system of m equations:

$$e_1(x+b)=C_{1,1}(b)\cdot e_1(x)+\cdots+C_{1,m}(b)\cdot e_m(x);$$

$$\ldots$$

$$e_i(x+b)=C_{i,1}(b)\cdot e_1(x)+\cdots+C_{i,m}(b)\cdot e_m(x);\tag{3}$$

$$\ldots$$

$$e_m(x+b)=C_{m,1}(b)\cdot e_1(x)+\cdots+C_{m,m}(b)\cdot e_m(x).$$

Differentiating both sides of these equations by b and taking $b=0$, we get the following system of differential equations:

$$\frac{de_1(x)}{dx}=c_{1,1}\cdot e_1(x)+\cdots+c_{1,m}\cdot e_m(x);$$

$$\ldots$$

$$\frac{de_i(x)}{dx}=c_{i,1}\cdot e_1(x)+\cdots+c_{i,m}\cdot e_m(x);\tag{4}$$

$$\ldots$$

$$\frac{de_m(x)}{dx}=c_{m,1}\cdot e_1(x)+\cdots+c_{m,m}\cdot e_m(x),$$

where we denoted $c_{i,j}\stackrel{\text{def}}{=}\dfrac{dC_{i,j}(b)}{db}_{|b=0}$.

We have a system (4) of linear differential equations with constant coefficients. It is known that a general solution to this system is a linear combination of expressions of the type $x^k \cdot \exp(\lambda \cdot x)$, where:

- the value λ is a (possible complex) eigenvalue of the matrix $c_{i,j}$, and
- the value k is a natural number; this number should be smaller than the multiplicity of the corresponding eigenvalue.

Thus, each function $e_i(x)$ is such a linear combination.

3°. Let us now use scale-invariance. The fact that the linear space $\mathcal{L}$ is scale-invariant means, in particular, that for every basis function $e_i(x)$ and for every real value a, the shifted function $e_i(a \cdot x)$ also belongs to this linear space. Since all the function from the linear space are linear combinations of the basis functions $e_1(x), \ldots, e_m(x)$, this means that for every a, there exist values $A_{i,j}(a)$ for which

$$e_i(a \cdot x) = A_{i,1}(a) \cdot e_1(x) + \cdots + A_{i,m}(a) \cdot e_m(x). \tag{5}$$

For each a, we can take m different values $x_1, \ldots, x_m$, and get m resulting equalities:

$$e_i(a \cdot x_1) = A_{i,1}(a) \cdot e_1(x_1) + \cdots + A_{i,m}(a) \cdot e_m(x_1);$$

$$\ldots$$

$$e_i(a \cdot x_j) = A_{i,1}(a) \cdot e_1(x_j) + \cdots + A_{i,m}(a) \cdot e_m(x_j); \tag{6}$$

$$\ldots$$

$$e_i(a \cdot x_m) = A_{i,1}(a) \cdot e_1(x_m) + \cdots + A_{i,m}(a) \cdot e_m(x_m).$$

We thus get a system of m linear equations for m unknowns $A_{i,1}(a)$, ..., $A_{i,m}(a)$. By using Cramer's rule, we can describe the values $A_{i,j}(a)$ as ratios of polynomials in terms of the coefficients $e_j(x_k)$ and the right-hand sides $e_i(a \cdot x_j)$. Since the functions $e_i(x)$ are differentiable, we can conclude that the dependence $A_{i,j}(a)$ on a is differentiable as well.

4°. We can now combine the Eq. (5) corresponding to different functions $e_i(x)$. As a result, we get the following system of m equations:

$$e_1(a \cdot x) = A_{1,1}(a) \cdot e_1(x) + \cdots + A_{1,m}(a) \cdot e_m(x);$$

$$\ldots$$

$$e_i(a \cdot x) = A_{i,1}(a) \cdot e_1(x) + \cdots + A_{i,m}(a) \cdot e_m(x); \tag{7}$$

$$\ldots$$

$$e_m(a \cdot x) = C_{m,1}(a) \cdot e_1(x) + \cdots + A_{m,m}(a) \cdot e_m(x).$$

Differentiating both sides of these equations by a and taking $a = 1$, we get the following system of differential equations:

$$x \cdot \frac{de_1(x)}{dx} = a_{1,1} \cdot e_1(x) + \cdots + a_{1,m} \cdot e_m(x);$$

$$\ldots$$

$$x \cdot \frac{de_i(x)}{dx} = a_{i,1} \cdot e_1(x) + \cdots + a_{i,m} \cdot e_m(x); \tag{8}$$

$$\ldots$$

$$x \cdot \frac{de_m(x)}{dx} = a_{m,1} \cdot e_1(x) + \cdots + a_{m,m} \cdot e_m(x),$$

where we denoted $a_{i,j} \stackrel{\text{def}}{=} \frac{dA_{i,j}(a)}{da}_{|a=1}$.

5°. To solve this new system of equations, we can introduce a new variable $t \stackrel{\text{def}}{=} \ln(x)$, for which $\frac{dx}{x} = dt$. Here, $x = \exp(t)$, so for the new functions $E_i(t) \stackrel{\text{def}}{=} e_i(\exp(t))$, the system (8) takes the following form:

$$\frac{dE_1(t)}{dt} = a_{1,1} \cdot E_1(t) + \cdots + a_{1,m} \cdot E_m(t);$$

$$\ldots$$

$$\frac{dE_i(t)}{dt} = a_{i,1} \cdot E_1(t) + \cdots + a_{i,m} \cdot E_m(t); \tag{9}$$

$$\ldots$$

$$\frac{dE_m(x)}{dx} = a_{m,1} \cdot E_1(t) + \cdots + a_{m,m} \cdot E_m(t).$$

This is a system of linear differential equations with constant coefficients; so, each function $E_i(t)$ is linear combination of the expressions of the type $t^k \cdot \exp(\lambda \cdot t)$, Thus, for $e_i(x) = E_i(\ln(x))$, we conclude that each function $e_i(x)$ is a linear combination of the expressions

$$(\ln(x))^k \cdot \exp(\lambda \cdot \ln(x)) = (\ln(x))^k \cdot x^\lambda.$$

6°. We have proven that:

- on the one hand, each function $e_i(x)$ is a linear combination of the expressions $x^k \cdot \exp(\lambda \cdot x)$, where k is a natural number;
- on the other hand, each function $e_i(x)$ is a linear combination of the expressions $(\ln(x))^k \cdot x^\lambda$.

One can easily see that the need to be represented in the second form excludes the possibility of $\lambda \neq 0$. Thus, each function $e_i(x)$ is a linear combination of the expressions of the type x^k with natural k—i.e., a polynomial. Every function from the linear space $\mathcal{L}$ is a linear combination of the basis functions $e_i(x)$ and is, thus, also a polynomial.

The proposition is proven.

Proof of the Corollary. Let us prove this result by contradiction. Let us assume that for some cdf $F(x)$, its logarithm $\ln(F(x))$ belongs to a shift-and scale-invariant linear space $\mathcal{L}$. Due to Proposition, this implies that this logarithm is a polynomial $P(x)$: $\ln(F(x)) = P(x)$ and thus, $F(x) = \exp(P(x))$.

When $x \to -\infty$, we have $F(x) \to 0$, so we should have

$$P(x) = \ln(F(x)) \to -\infty.$$

For the corresponding polynomial $P(x) = a_0 \cdot x^k + a_1 \cdot x^{k-1} + \cdots$, this means that:

- either k is even and $a_0 < 0$,
- or k is odd and $a_0 > 0$.

When $x \to +\infty$, then:

- in the first case, we have $P(x) \to -\infty$, while
- in the second case, we have $P(x) \to +\infty$.

However, we should have $F(x) \to 1$ and thus, $P(x) = \ln(F(x)) \to \ln(1) = 0$.

This contradiction shows that our assumption was wrong, and logarithms for cdfs cannot belong to shift-and scale-invariant linear spaces.

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Information Quality and Uncertainty

Marie-Jeanne Lesot and Adrien Revault d'Allonnes

Abstract The quality of a piece of information depends, among others, on the certainty that can be attached to it, which relates to the degree of confidence that can be put in it. This paper discusses various components to be considered when assessing this certainty level. It shows that they cover a wide range of different types of uncertainty and provide a highly relevant application domain for theoretical questioning about uncertainty modelling. It also describes several frameworks that have been considered for this task.

Keywords Information scoring · Information processing · Uncertainty type · Competence · Reliability · Plausibility · Credibility · Linguistic uncertainty

1 Introduction

Information quality (see e.g. [1]) and its implementation in the domain of information evaluation (see e.g. [2]) aim at providing guidance and help to users in the drowning quantity of information they are nowadays overwhelmed with, in particular due to the dramatic increase of Web usage, e.g. through blogs and social networks, such as Facebook and Twitter. One specificity of these new media is that everyone can participate in the information spread and be a source of information, making the question of a relevance measure of the available information crucial. As a consequence, it is necessary to dispose of tools for automatically assessing their quality: there is an acute need for automatic methods to identify the "best", e.g. understood as the most useful, pieces of information.

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Numerous criteria and properties have been proposed and considered to that aim [1, 2]. This paper[1] focuses on the certainty dimension, numerically evaluated as a degree of certainty that can be attached to any piece of information. In a schematic view, it exploits the argument according to which a certain piece of information is worthier than a doubtful one. Insofar, it is related to the task that aims at assessing the trust that can be put in a piece of information. It can be underlined that such a degree of trust can mean either evaluating the reality of the fact the piece of information reports [3–5] or the extent to which the rater is convinced, based on the process with which he forms an opinion about this piece of information [6–8].

Even if uncertainty is only one of its components, information quality appears as a highly relevant application framework for the theoretical domain of uncertainty modelling. Indeed, it turns out to be a very challenging one, raising critical requirements that lead to question existing models and possibly to develop new ones. As discussed in this paper, information processing involves several types of uncertainty that must be distinguished, appropriately modelled and possibly combined: information-related uncertainty covers a wide spread spanning over several dimensions. As detailed in the following, one can mention distinctions between objective and subjective uncertainty, as well as between general versus contextual uncertainty.

This paper first discusses various kinds of uncertainty that can be attached to a piece of information in Sect. 2, organising them according to their cause, i.e. the characteristic of the considered piece of information that triggers them. Section 3 discusses the two axes objective-subjective and general-contextual. Section 4 briefly describes some theoretical frameworks that have been proposed to model uncertainty for information evaluation.

2 Sources of Uncertainty in the Information Processing Framework

This section discusses 5 sources of uncertainty that can be considered in the framework of information processing, structuring them according to their cause: it distinguishes the uncertainties respectively triggered by the content of a piece of information, its source, its context, its formulation and its automatic extraction.

In order to illustrate these types, it considers the following fictitious piece of information together with two basic meta-data, namely author and publication date:

> On February 15th 2015, the International Olympic Committee declared
> "In 2048, the Summer Olympic Games will probably take place in November"

[1] This paper is based on part of the panel which has been organised by Prof. Kovalerchuk at IPMU2012 on the general topic "Uncertainty Modelling".

2.1 Content-Related Uncertainty: What is Said?

The degree of uncertainty attached to a piece of information obviously depends on its content, i.e. the answer to the question "what does it say?": for the running example, it for instance relates to the assertion that can be schematically written as "2048 Summer Olympic Games dates = November".

More precisely, the piece of knowledge provided by the considered information can trigger a surprise effect that influences its uncertainty level: an unexpected fact can, at least at first, appear as more uncertain than a known one. The surprise effect can be measured with respect to two types of background, leading to distinguish between the notions of plausibility and credibility.

Knowledge context: plausibility Surprise can be defined as compared to the personal background of the information rater, i.e. as the compatibility of the considered piece of information with his/her knowledge, which is defined as *plausibility* [8].

For instance for the running example, the asserted date may appear highly atypical, in particular to people living in the North hemisphere, who usually do not associate November with summer. As a consequence, they may receive the information with more caution and consider it as more uncertain than people living in the South hemisphere. Along the same lines, for someone with knowledge about the history of the Olympic Games, for instance knowing that the situation where the summer games take place in November already occurred (in 1956, for the Melbourne Games), the fact may appear as less uncertain.

Plausibility can be considered as the first component in the conviction establishing process [8], that determines an a priori confidence level attached to a considered piece of information.

Other information context: credibility Surprise can also be defined with respect to other available pieces of information, e.g. other assertions provided in the same period regarding the location and dates of the Olympic Games: in this case, the considered piece of information is compared to other statements, building the *credibility* component [3, 8].

More precisely, the assessment of credibility relies on the identification of corroboration or invalidation of the considered piece of information, defining another type of background for the evaluation of its attached uncertainty. This dimension both depends on the content of the information and the context of its assertion, it is more detailed in the section discussing the latter (Sect. 2.3).

2.2 Source-Related Uncertainty: Who Says it?

The uncertainty attached to an assertion also depends on its source, i.e. the answer to the question "who says it?": for the running example, it for instance relates to the fact that the International Olympic Committee provides it, who can be considered as a qualified source. The question is then to define the characteristics that make a source "qualified", this section discusses some of them, a more complete discussion can be found in [9] for instance.

It must be underlined that, altogether, the qualification of a source is contextual: it may not be the same for all pieces of information and may for instance depend on their topics, i.e. on their contents. However, some of its components remain topic-independent and general.

Source trustworthiness: reliability The reliability of the source corresponds to an a priori assessment of its quality, independently of the considered piece of information: it indicates whether, in general, the assertions it provides can be trusted or should be considered with caution.

In the seminal model for information evaluation [3], reliability plays a major role: this model represents the information score as a bi-gram defined as the concatenation of two symbols measured on two discrete graded scales associated to linguistic labels. The first one is called reliability, although it may depend on several distinct dimensions [10]: its explicit and direct presence in the final score underlines its crucial role.

This subjective dimension, that may take different values for different raters, is difficult to define formally and thus to measure. It can be related to the concept of source reputation although the latter may be as difficult to model and quantify. In the case of Twitter sources, it has for instance been proposed to establish it from measurable quantities such as the number of followers or the source social status [11].

Reliability can also be assessed by comparing previous source assertions with the ground truth when the occurrence of events has make it possible to establish whether the source was right or wrong [12, 13]. This approach highlights the fact that reliability is a dynamic concept whose measure should evolve with time. It also relates this dimension to validity [4], according to which if the source produces a piece of information, then the latter is true.[2]

Another component of reliability can be derived from the formulation used by the source: the number of citations that are contained in its publications allows to evaluate the extent to which it cites its own sources [9, 14, 15]. Now, offering a possibility to track back the origin of the provided information contributes to its reliability. Another indication can be derived from the amount of grammatical and spelling errors [9, 14, 15]: it is argued that a grammatically mistake-free text advocates for analysis capacity and critical way of thinking, which are desirable qualities for reliable sources. Although these quantities are related to the question "how is it said", discussed in Sect. 2.4, they capture an uncertainty originated from the source, allowing to infer some of the source characteristics, whereas the components described in Sect. 2.4 measure uncertainty originated from the expression itself.

Source expertise level: competence A distinct component of source-related uncertainty comes from its competence, that measures the extent to which it is entitled to provide the information, i.e. whether it is legitimated to give it [7, 9].

[2] Conversely, a source is said to be *complete* if, when a piece of information is true, the source provides it [4]. This useful characterisation, related both to the source omniscience and "sharing communication type", is however less central for the assessment of the information uncertainty.

In the considered example, it can for instance be considered that the IOC is much more competent than a taxi driver would be, leading to a lower uncertainty regarding the date of the 2048 Olympic Games than would occur if the latter provided the information.

Competence relates to the source expertise and appears to be a topic-dependent component: the IOC would be significantly less competent to provide information about the World Football Champions' cup; likewise, the taxi driver would be a legitimate source about efficient routes or traffic jams for instance, leading to less uncertain pieces of information regarding these topics.

It is worth noticing that two types of competence can be distinguished, an essential one and a more accidental one, that respectively apply to experts and witnesses [9]. Indeed, an essential competence can be established from the source fields of study and possibly diplomas, or from official roles: they provide a theoretical expertise and indeed entitle a source to make assertions about a given topic. On the other hand, a geographical or temporal proximity to an event provides an empirical competence, granting witnesses a local expertise level.

Source intention The assessment of the certainty degree attached to a piece of information, or the degree of trust put in it, can also depend on source characteristics even more difficult to establish, related to its *intention*: indeed, a source may for instance pursue an aim of desinformation, with the intention to lure the information rater. The certainty degree should obviously be reduced in such a communication paradigm, if it can be recognised as such.

This dimension is related to a *sincerity* feature, which captures the tendency of the source to tell the truth or not (see also [4]): it can be considered that sincerity is a general characteristic of the source, describing its global tendency, whereas its intention is more contextual and varies for each piece of information. Sincerity can be considered as being related to the source reliability, as they both depend on the truth of the source assertions. The notion of sincerity may be seen as integrating a judgment component, that takes into account the source intention when interpreting the reason why it is wrong.

Source implication Another source characteristic is captured by its *commitment degree*, i.e. the extent to which it is involved in the propagation of the information it produces. Commitment depends on what the source may loose if it produces erroneous information, and, insofar, can be seen as related to its reputation.

It has for instance been proposed, in the case of Twitter sources, to measure the commitment degree as a function of the energy they put in their accounts [9, 15], in turn quantified by the richness of their profile, e.g. the number of filled fields, the presence of a picture or the number of publications.

The source commitment also influences the uncertainty that can be attached to its assertions, under the interpretation that a highly committed source should be less prone to produce erroneous content and may be trusted.

Successive sources: hearsay A specific case for the evaluation of the source of an information occurs when the piece of information is not directly obtained, i.e. when it results from a series of successive sources, following a scheme of the form "S_1 says that S_2 says that … S_n says that F" where F is the fact and S_i, $i = 1, \ldots, n$

the sources. Dedicated models have been proposed to process such cases, see e.g. [16, 17].

Indeed, for such pieces of information, all previous source-uncertainty related components are measured not with respect to F (except for S_n) but, for S_i, with respect to "S_{i+1} says that… S_n says that F": competence then for instance measures whether S_i is entitled to report the assertions of S_{i+1}.

2.3 Context-Related Uncertainty: When is it Said?

Another meta-data that influences the certainty attached to a considered piece of information relates to the context of its assertion, understood as the global answer to the question "when is it said?". Different components can be considered, a purely temporal one as well as a more global one that depends on other available assertions.

Temporal context The date associated to an assertion contributes to the certainty level that can be attached to it, both in comparison with the date of the reported event and with the current date.

Indeed, the gap between the reported event and the assertion influences the uncertainty: information provided too much in advance may be considered with caution, decreasing their certainty level. For instance for the running example, if the assertion is about the Olympic Games in 2084, it may be interpreted as less certain.

On the other hand, a comparison with the current date can influence the importance that should be granted to a considered piece of information: when faced with an information stream, it can be useful to take into account older, and possibly out-of-date, pieces of information to a lesser degree than the more recent ones. It has for instance been proposed to associate each assertion with a currentness score [5], so as to weight down the pieces of information according to their possible obsolescence. It can be underlined that such a model makes the evaluation sensitive to the information order, possibly leading to different results if a piece of information I_1 is published before I_2 or reciprocally. Such a behaviour can be considered as a realistic approach to model the uncertainty evolution when faced with an information stream.

It can be noted that beside these relative date comparisons, with respect to the information content and the current date, an absolute effect can be considered: some dates do bear meaning and influence the evaluation of their content. This component depends on a cultural dimension that makes difficult its general implementation. For instance, one can consider that information produced on April 1st is less certain than others; announcements contained in election campaigns may also require a specific processing.

Other assertion context: credibility The evaluation of the uncertainty attached to a piece of information classically includes a cross-checking step, aiming at identifying complementary information backing up or undermining it: confirmations and invalidations respectively increase and decrease its certainty level. The *credibility* dimension can be understood as a degree of confirmation resulting from comparison of the piece of information to be rated with the available information [3, 5, 7, 18].

In the seminal model [3], the second symbol of the bigram measures this confirmation degree, as indicated by the description it is accompanied by. It can be underlined that its linguistic labels mainly describe the information certainty, across the scale *improbable, doubtful, possibly true, probably true, confirmed by other sources*, showing the relation with this underlying essential component.

The principle of credibility evaluation [5, 7] consists in aggregating several assertions, said to be homologous, that refer to the same content. It thus depends on the choice of a similarity measure that measures the degree of confirmation by assessing the extent to which an homologous piece of information corroborates the information to be rated (see e.g. [5] for a discussion on such eligible measures and their components).

The aggregation step can take into account various dimensions, among which the previous degree of confirmation, the individual uncertainty attached to the homologous information [5, 7, 18], but also the relations between the sources [5]: one can consider a refined notion of confirmation and invalidation, weighting them according to affinity or hostility relations between sources. Indeed, a confirmation provided by sources known to be in affinity relation should have a lower influence than a confirmation by independent, not to say hostile, sources: friendly sources are expected to be in agreement and to produce somehow redundant information.

As the temporal component, the credibility dimension makes uncertainty evaluation sensitive to the order of the pieces of information in a stream, taking into account more subtle relations than their publication dates only. This dynamical behaviour, source of many a theory of argumentation, considers that two confirmations followed by an invalidation may lead to a different level of uncertainty than a confirmation followed by a contradiction and another confirmation might [18].

2.4 Formulation-Related Uncertainty: How is it Said?

The words used in a piece of information play a major role on the attached uncertainty level, both because of the imprecision they convey and the uncertainty they intrinsically convey. The additional role of linguistic quality, that influences the assessment of the source reliability, has been discussed in Sect. 2.2.

Natural language is often imprecise (see e.g. [19]), allowing for fuzziness of the conveyed message, which can lead to uncertainty: if, for instance, the IOC asserts that the 2048 Games will take place “around the end of the year”, some uncertainty is attached to the fact that the games will take place in November. In this case, uncertainty arises from the approximate compatibility between the rated piece of information and the query (e.g. regarding the Games date): only a partial answer is available. Such imprecision also plays a role in the identification of homologous information involved in the cross-checking step of credibility assessment discussed in Sect. 2.3.

Beside imprecision, the used words also convey uncertainty: they give indication regarding the source own level of uncertainty and influence the overall evaluation of the uncertainty [20]. In the case of the considered example for instance, the linguistic expression contains the adverb "probably" whose presence increases the uncertainty of the final evaluation.

Linguistic works (see e.g. [21, 22]) propose classification of uncertainty bearing terms, making it possible to assess the global expressed uncertainty. Such terms include adjectives (such as certain, likely or improbable), modal verbs (e.g. may, might, could, should), adverbs (such as certainly, possibly or undeniably) or complex idiomatic structures. Modifiers such as "very" can be used to reinforce or weaken the previous linguistic tags.

2.5 Automatic Processing-Related Uncertainty: How is it Extracted?

A fifth level of uncertainty comes from the fact that the available pieces of information are automatically processed, which can introduce errors in the content identification and thus for many of the components mentioned in the previous sections.

Indeed, the evaluation of the uncertainty attached to a piece of information according to the previously cited dimensions for instance include the use of tools for named entity detection, event and relationship identification and date extraction [22]. They also require to solve difficult linguistic tasks, as negation handling and anaphora resolution, that still are challenges for automatic text processing systems. These uncertainties can be measured automatically, for instance through performance rates of the corresponding methods, i.e. using recognition rate, recall or precision.

Among the examples of the encountered difficulties, one can for instance mention possible errors in the text topic identification, possibly leading to erroneous assessment of the source competence (see Sect. 2.2). Similarly, the identification of the date in the processed document may result in mistakes in the evaluation of the temporal content (see Sect. 2.3). The most impacted dimension is probably credibility (Sect. 2.3), that relies on the extraction of homologous pieces of information, and therefore both on all the documents processing and the computation of their similarities. It can be noticed that this task is sometimes performed semi-automatically, in order to guarantee its quality, crucial for the whole system [5].

3 Uncertainty Types for Information

Form a formal point of view, the various uncertainty types discussed in the previous section can be classified according to two axes, opposing objective versus subjective uncertainties as well as general versus contextual ones.

It can be underlined that the considered uncertainties also differ in their very nature: for instance, some express structural doubts about the phenomena, as content plausibility or recognition rate for instance, whereas the linguistically triggered uncertainty on the other hand captures an imprecision level.

Objective versus subjective uncertainty A first axis discriminating the listed uncertainty types refers to the position of the rater and his/her implication in the evaluation: some of them actually do not depend on the rater and constitutes objective dimensions, whereas others are subjective.

Indeed, the evaluation of the uncertainty triggerend by the automatic processing step for instance is objective and can be automatically measured. Similarly, the evaluation of the degree of confirmation between two pieces of information, i.e. the credibility dimension, does not depend on the rater and is identical for all users.

On the other hand, the plausibility dimension is subjective: it is measured by comparison to the rater's background knowledge and therefore varies from one rater to another. Likewise, most source evaluation criteria can be considered as subjective: for instance, not all users may agree on the competence fields of a given source, nor on its intention.

General versus contextual uncertainty Another discriminating axis refers to the dependence of the dimension to the rated piece of information: some criteria are evaluated generally, a priori, i.e. independently of any information, whereas others characterise the considered one.

As an example, the source reliability does not depend on the rated piece of information and similarly applies to all the source assertions. The category of general criteria also involve the evaluation of the uncertainty triggered by automatic processing step, which is measured globally, for all types of information. Similarly, the measure of the formulation-related uncertainty relies on a linguistic modelling of uncertainty expression: the latter is built generally, not for a specific piece of information.

On the other hand, the source competence for instance is topic-dependent and thus varies from one piece of information to the other. In that sense, it is considered to be contextual. Obviously, the content credibility, as well as the temporal dimension, are contextual too.

4 Formal Frameworks for Information Scoring

As discussed in the previous sections, the uncertainty to be considered in the domain of information quality covers different types. As a consequence, distinct formal frameworks have been considered to represent it or some of its components. A central issue is to dispose of aggregation operators to combine the individual uncertainty scores obtained for each considered component. It can be observed that some propositions focus on this aggregation issue, in a multi-criteria aggregation approach, using for instance Choquet integrals [9].

This section briefly discusses the main existing uncertainty modelling frameworks applied to the case of information evaluation, distinguishing them depending on whether they model symbolic, ordered or numerical uncertainties.

Symbolic framework Symbolic approaches in the domain of information evaluation include logical representation, in particular in the framework of modal logics

[4, 23–25], that allow to perform logical inferences to characterise the sources and the pieces of information. However, they usually do not model the attached uncertainty.

The first formal framework for information evaluation considering uncertainty has been proposed in the seminal model [3]: it represents the information score as a bi-gram defined as the concatenation of two symbols measured on two discrete graded scales associated to linguistic labels: according to the descriptions they are accompanied by, the first one captures the source reliability and the second one the information credibility. However it has been shown [6, 10, 26] that this symbolic approach raises some difficulties, among others regarding the manipulation and comparison of the obtained scores.

Ordered framework: extended multivalued logic In order to ease the manipulation of uncertainty scores, it has been proposed to exploit an extended multivalued logic framework [8, 27] to model the process of trust building: trust can be defined on a single discrete graded scale, clarified with linguistic labels, improving the legibility of a unique degree with a semantic interpretation. Moreover, this framework is equipped with formal tools to combine the truth degrees through logical operations that generalise conjunction, disjunction or implication, as well as arithmetical ones [28].

The extension [8, 27] of classical multivalued logic consists in introducing an additional degree that allows to distinguish between facts that are 'neither true nor false', i.e. that have a neutral truth value, and facts whose truth values cannot be evaluated: it makes it possible to distinguish between ignorance and neutral knowledge, which is for instance required to distinguish between a source whose reliability is unknown from a source with intermediate reliability.

Numerical frameworks: probability, possibility and evidence Probability theory is one of the most frequent framework used to model uncertainties. In the case of information evaluation, it can for instance naturally be used to quantify the uncertainty related to the extraction process, e.g. to measure error recognition rates of the applied automatic tools. However, many components of information evaluation uncertainty cannot be considered as having a probabilistic nature. Moreover, they need to distinguish between ignorance and uniform distribution, as sketched above, which cannot be implemented in the probabilistic framework. Furthermore, probabilities impose strong axiomatic constraints, restricting the choice of aggregation operators. Finally, probability theory often requires to set a priori distributions, which may be a difficult task in the case of information evaluation.

Possibility theory [29] allows to represent the ignorance case separately from the neutral one and offers a wide variety of aggregation operators allowing to model many different behaviours for the combination of the considered uncertainty dimensions. It has for instance be applied to assess the uncertainty that can be attached to an event e, to answer the question "did e take place?", based on a set of pieces of information, enriching the binary answer yes/no with a confidence level [5].

The theory of belief functions [30] generalises the probability and the possibility theories, offering a very rich expression power. It has been applied to information evaluation in particular to the issues of reported information [16, 26] and source reliability measures [25, 31].

5 Conclusion

This chapter considered the issue of uncertainty in the domain of information evaluation, discussing the various types of uncertainty that can be attached to a piece of information, describing either the event it reports or its intrinsically attached trust. Many components can be distinguished, whose combination builds to a complex notion for which several theoretical frameworks have been considered, so as to capture its diverse facets.

Among other topics related to uncertainty in the context of information evaluation, dynamics and validation offer challenging issues opening the way to research directions. The need for modelling the temporal evolution of uncertainty comes from the availability of information streams, beyond the individual pieces of information, as briefly mentioned previously. It also comes from the possible evolution of the general components of the source characteristics: if, for instance, the reliability of a source proves to change over time, it may require to re-evaluate the uncertainty attached to previously assessed pieces of information this source had provided, and, consequently, also to the information they are analogous to.

The issue of validation aims at assessing the quality of the proposed uncertainty models, both regarding the considered components and the chosen formal framework. Now its difficulty comes from the lack of data allowing to perform empirical studies: in the case of real data, it is difficult to dispose of expected scores to which the computed ones can be compared. The use of artificial data raises the challenge of their realistic generation controlling their relevance.

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Applying Anomalous Cluster Approach to Spatial Clustering

Susana Nascimento and Boris Mirkin

Abstract The concept of anomalous clustering applies to finding individual clusters on a digital geography map supplied with a single feature such as brightness or temperature. An algorithm derived within the individual anomalous cluster framework extends the so-called region growing algorithms. Yet our approach differs in that the algorithm parameter values are not expert-driven but rather derived from the anomalous clustering model. This novel framework successfully applies to the issue of automatically delineating coastal upwelling from Sea Surface Temperature (SST) maps, a natural phenomenon seasonally occurring in coastal waters.

1 Introduction

In our previous work [1, 2], we automated the process of delineation of upwelling regions and boundaries using a fuzzy clustering method supplemented with the anomalous cluster initialization process [3]. Yet that method operates over temperature data only, without any relation to the spatial arrangement of the pixels involved. Therefore, we decided to modify the anomalous cluster framework in such a way that it applies to the pixels spatially located on geographical maps. We apply the view that an upwelling region grows step-by-step by involving nearest cold water pixels. The process is controlled by a function expressing the similarity of pixel temperatures to those already in the region. In a self-tuning version of the algorithm the homogeneity

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threshold is locally derived from the approximation criterion over a window around the pixels under consideration. This window serves as a boundary regularize.

The paper is organized in three main sections. Section 2 describes a version of the anomalous cluster model and method relevant to the task. Section 3 describes a modification of this method applicable to clustering pixels according to the sea surface temperature maps. Section 4 gives a glimpse on application of the method to real temperature map data. Conclusion outlines the contents and sketches future work.

2 Anomalous Cluster Model and Alternating Method to Optimize It

Consider a set of objects P characterized by just one feature x so that, for every $p \in P$, $x(p)$ is a real number representing the value of the feature at p. A subset $C \subseteq P$ will be characterized by a binary vector $z = (z_p)$ such that $z_p = 1$ if $p \in C$ and $z_p = 0$ otherwise. We find such a C that approximates the feature x as closely as possible. To adjust the quantitative expression of C with z to the measurement scale of x, vector z should be supplied with an adjustable scaling coefficient λ. Also, a preliminary transformation of the x scale can be assumed by shifting the zero point of x into a point of interest, e.g. the mean value of x. Therefore, following Mirkin [3, 4], an approximation model can be stated as

$$y_p = \lambda z_p + e_p \tag{1}$$

where y_p are the preprocessed feature values, $z = (z_p)$ is the unknown cluster membership vector and λ is the scaling factor value, also referred to as the cluster intensity value [3, 4]. The items e_p represent errors of the model; they emerge because the vector λz_p may have only two different values, 0 and λ, whereas rescaled feature y may have different values. Anyway, the model requires that the errors should be made as small as possible.

Consider the least squares criterion $\Phi^2 = \sum_{p \in P} e_p^2 = \sum_{p \in P} (y_p - \lambda z_p)^2$ for fitting the model (1).

This is a more or less conventional statistics criterion. Yet in the clustering context, Φ^2 bears a somewhat unconventional meaning. Indeed, any $z_p = 0$ contributes y_p^2 to Φ^2 independently of the λ value. Therefore, to minimize the criterion, $z_p = 0$ should correspond to those objects at which pre-processed feature values $y(p)$ are zero or near zero. In contrast, those objects $p \in P$ at which maximum or almost maximum absolute values of the feature hold, should be assigned with $z_p = 1$. Moreover, these must be either maximum positive values or minimum negative values but not both. Indeed, as it is well known, the optimum λ at any given C must be the average of $y(p)$ over all $p \in C$, $\lambda(C) = \sum_{p \in C} y_p / |C|$. Substituting this value into criterion Φ^2, one can easily derive the following decomposition

$$\Phi^2 = \sum_{p \in P} y_p^2 - \lambda^2(C)|C| \tag{2}$$

or, equivalently,

$$\sum_{p \in P} y_p^2 = |C|\lambda^2(C) + \Phi^2. \tag{3}$$

The latter expression is a Pythagorean decomposition of the data scatter (on the left) in explained and unexplained parts. The smaller the unexplained part, Φ^2, the greater the explained part,

$$g^2(C) = |C|\lambda^2(C). \tag{4}$$

Equation (4) gives an equivalent reformulation to the least-squares criterion: an optimal C must maximize it. This cannot be achieved by mixing in C objects with both high positive and high negative y values because this would make the average $\lambda(C)$ smaller.

Therefore, an optimal C must correspond to either highest positive values of y or lowest negative values of y. Assume, for convenience, the latter case and consider a local search algorithm for finding a suboptimal C by adding objects one by one starting from a singleton $C = \{p\}$. What singleton? Of course that one corresponding to the lowest negative value of y, to make the value of criterion (4) as high as possible.

In publications [3, 4] only local search algorithms were considered. In these algorithms, entities are added (or removed) one by one to warrant a maximum possible local increment of the criterion until that becomes negative. Here, we develop a method which is more suitable for temperature map data. In this new method iterations are carried on in a manner similar to that of the well known clustering k-means method which is described in every text on data mining or clustering (see, for example [3]). Specifically, given a central value $c = \lambda(C)$, we add to cluster C all the relevant objects at once, after which the central value is recomputed and another iteration is applied. The computations converge to a stable solution that cannot be improved with further iterations.

To arrive at this "batch" clustering method, let us derive a different expression for the criterion Φ^2 by "opening" parentheses in it. Specifically, since $z_p^2 = z_p$ because z_p accepts only 0 and 1 values, we may have

$$\Phi^2(C, \lambda) = \sum_{p \in P}(y_p - \lambda z_p)^2 = \sum_{p \in P} y_p^2 - 2\lambda \sum_{p \in P}(y_p - \lambda/2)z_p \tag{5}$$

As the data scatter $\sum_{p \in P} y_p^2$ is constant, minimizing (5) is equivalent to maximizing the scoring function

$$f(C, \lambda) = \sum_{p \in P} \lambda(y_p - \lambda/2)z_p = \sum_{p \in C} \lambda(y_p - \lambda/2). \tag{6}$$

This can be rewritten as

$$f(C, \lambda, \pi) = \sum_{p \in C} \left(\lambda y_p - \pi\right) \tag{7}$$

where π is a parameter that is, optimally, equals $\pi = \lambda^2/2$, and yet can be considered a user-defined threshold in criterion (7). This criterion may be considered as depending on two variables, that are to be determined: λ and C. Therefore, the method of alternating optimization can be applied to maximize it. Each iteration of this method would consist of two steps. First, given λ, find all $p \in P$ such that $\lambda y_p > \pi$ and put them all as C. Of course, these y-values must be negative since $\lambda < 0$ in our setting. Second, given a C, find λ as the within-C average of y_p. Since both steps are optimal with respect to the corresponding variable, this method increases the value of g^2 at each step and, therefore, must converge because there are a finite number of different subsets C. In the follow-up the alternating anomalous clustering algorithm will be referred to as AA-clustering.

3 Adapting AA-clustering to the Issue of Delineating Upwelling Areas on Sea Surface Temperature Maps

Consider the set of pixels of a Sea Surface Temperature (SST) map of an ocean part as the set P, the feature x being the surface temperature. Such maps are used in many applications of which we consider the problem of automatic delineation of coastal upwelling. This is a phenomenon that occurs when the combined effect of wind stress over the coastal oceanic waters and the Coriolis force cause these surface waters to move away from the coast. Therefore, deep, cold and nutrient-rich waters move to the surface to compensate for the mass deficiency due to this surface water circulation. As such, it has important implications in ocean dynamics and for the understanding of climate models. The identification and continuing monitoring of upwelling is an important part of oceanography.

Unfortunately the current state is far from satisfactory. Although a number of approaches for segmentation of upwelling have been proposed, they suffer from too complex computational processes needed to get more or less satisfactory results (see, for instance, [5–9]).

Therefore, we decided to apply the self-tuning AA-clustering to pixels of the temperature map starting from the coldest pixel which, in fact, corresponds to the nature of upwelling.

Let $P = R \times L$ be a map under consideration where R is the set of rows and L, the set of columns, so that a pixel p can be presented as $p = (i, j)$ where $i \in L$ is its row coordinate and $j \in L$ its column coordinate. Then a corresponding sea surface temperature map can be denoted as $x = (x(i, j))$, for all $i \in R$ and $j \in L$. First of all, let us center the temperature, that is, subtract the average temperature

$t^* = mean(x)$ of the temperature map x from the temperature values at all pixels in $R \times L$. Let the centered values be denoted as $t(i, j)$, $(i, j) \in R \times L$. The algorithm finds a cluster $C \subseteq R \times L$ in the format of a binary map $Z(R, L)$ with elements z_{ij} defined as $z_{ij} = 1$ if $(i, j) \in C$ and $z_{ij} = 0$, otherwise. Since the pixels are not just arbitrary objects but rather elements of a spatial grid, we need to introduce this property into the AA clustering approach.

For this purpose, let us consider each pixel $p = (i, j)$ as an element of a square window of a pre-specified size $W(i, j)$ centered at p. Based on preliminary experimentation we define the window size as 7×7 (pixels). The usage of a window system appears to be useful not only as a device for maintaining continuity of the cluster C being built, but also that its boundary is of more or less smooth shape. We refer to the pixels in window $W(i, j)$ as the neighborhood of $p = (i, j)$.

The algorithm starts by selecting a seed pixel, $o = (i_o, j_o)$, as a pixel with the lowest temperature value. The cluster C is initialized as the seed $o = (i_o, j_o)$ together with pixels within the window $W(i_o, j_o)$ satisfying the similarity condition

$$c \times t(i, j) \geq \pi, \tag{8}$$

where c is the reference temperature taken at the start as the temperature of the seed pixel o, and π, a similarity threshold, as described in the previous section. For convenience, let us refer to pixels in cluster C as labeled.

Once cluster C is initialized, its boundary set F is defined as the set of such unlabeled pixels, that their neighborhood intersects the cluster. Therefore,

$$F = \left\{(i', j') \notin C | W(i', j') \cap C \neq \oslash\right\} \tag{9}$$

Then the algorithm proceeds iteratively expanding the cluster C step by step by dilating its boundary F until it is empty. For each boundary pixel (i', j') in F we define the boundary expansion region as the subset of pixels (i, j) of C that intersect the exploring window centered at pixel (i', j'), that is, $(i, j) \in W(i', j') \cap C$, and we define c^* as the average temperature of those pixels.

The homogeneity criterion of the algorithm is defined by the following similarity condition (10). This condition involves the reference temperature $c^* = mean\left(T\left(W(i', j') \cap C\right)\right)$, the mean temperature of the window pixels within the expanding region:

$$c^* \times t(i', j') \geq \pi \tag{10}$$

Therefore, in the following text we take the self-tuned value for the similarity threshold as half the squared average temperature over the cluster C.

A more or less smooth shape of the growing region is warranted by the averaging nature of the similarity criterion and by involving windows around all pixels under consideration in the frontline.

This method, called Seed Expanding Cluster (SEC) [25], is a specific case of Seeded Region Growing (SRG) approach introduced by Adams and Bischof [10] for region based image segmentation (see also [11–14]). The SRG approach tries to find a region which is homogeneous according to a certain *feature of interest* such as intensity, color, or texture. The algorithm follows a strategy based on the growth of a region, starting from one or several 'seeds' and by adding to them similar neighboring pixels. The growth is controlled by using a homogeneity criterion, so that the adding decision is generally taken based only on the contrast between the evaluated pixel and the region. However, it is not always easy to decide when this difference is small (or large) enough to make a reasonable decision.

The Seeded Region Growing image segmentation approach has been widely used in various medical image applications like magnetic resonance image analysis and unsupervised image retrieval in clinical databases [15–18]. The approach has been also successfully applied in color image segmentation with applications in medical imaging, content-based image retrieval, and video [14, 19, 20], as well as in remote sensing image analysis [21, 22].

Main challenging issues that arise with SRG methods are:

(i) selection of the initial seed(s) in practical computations to find a good segmentation;
(ii) choosing the homogeneity criterion and specifying its threshold;
(iii) efficiently ordering pixels for testing whether they should be added to the region.

Most approaches of SRG involve homogeneity criteria in the format of the difference in the feature of interest between its value at the pixel to be labeled and the mean value at the region of interest [10, 11, 13, 14]. A weak point of these algorithms is the definition of the non-homogeneity threshold at which the pixels under consideration are considered as failing the homogeneity test and, therefore, cannot be added to the region. Such a definition is either expert driven or supervised in most of the currently available algorithms [11, 14].

Many SRG algorithms grow the regions using a sequential list which is sorted according to the dissimilarity of unlabeled pixels to the growth region [10, 14, 15]. The disadvantage is that the segmentation results are very much sensitive to this order.

As can be readily seen, our approach avoids these issues altogether. It utilizes a mathematically derived, though somewhat unusual, homogeneity criterion, in the format of a product rather than the conventional difference between the pixel and the mean of the region of interest. To this end, we first subtract the average temperature value from all the temperature values. This process is implemented by using the concept of window of a pre-specified size around the pixels under consideration: only those within the window are involved in the comparison processes. This provides for both the spatial homogeneity and smoothness of the growing region. Indeed, only borderline pixels are subject to joining in, because the windows around remote pixels just do not overlap the growing region. Therefore, there is no need in specifying the order of testing for labeling among pixels: all those borderline pixels can be considered and decided upon simultaneously. The process starts from a cluster consisting of just one pixel, the coldest one, according to the approximation clustering criterion. The preprocessed temperature of this pixel is negative with a relatively large absolute value. Our region growing process initializes with a fragment of the coldest pixels, which is rather robust. Moreover, the simultaneous borderline labeling considerably speeds up the SRG procedure.

In our experiments, we used also the similarity threshold π derived according to Otsu's thresholding method [23]. This method fine-tunes the similarity threshold by finding the maximum inter-class variance that splits between warm and cold waters, and is considered one of the most popular threshold method in the literature [24].

4 Experimental Testing

The newly developed method and its Otsu's competitor have been applied to a group of 28 images of size 500 × 500 showing the upwelling phenomenon in Portugal coastal waters; a detailed description can be found in [25]. The selected images cover different upwelling situations. Specifically:

(i) SST images with a well characterized upwelling situation in terms of fairly sharp boundaries between cold and warm surface waters measured by relatively contrasting thermal gradients and continuity along the coast (two topmost images);
(ii) SST images showing distinct upwelling situations related to thermal transition zones offshore from the North toward the South and with smooth transition zones between upwelling regions;
(iii) noisy SST images with clouds, so that information for defining the upwelling front is lacking (fourth-line image).

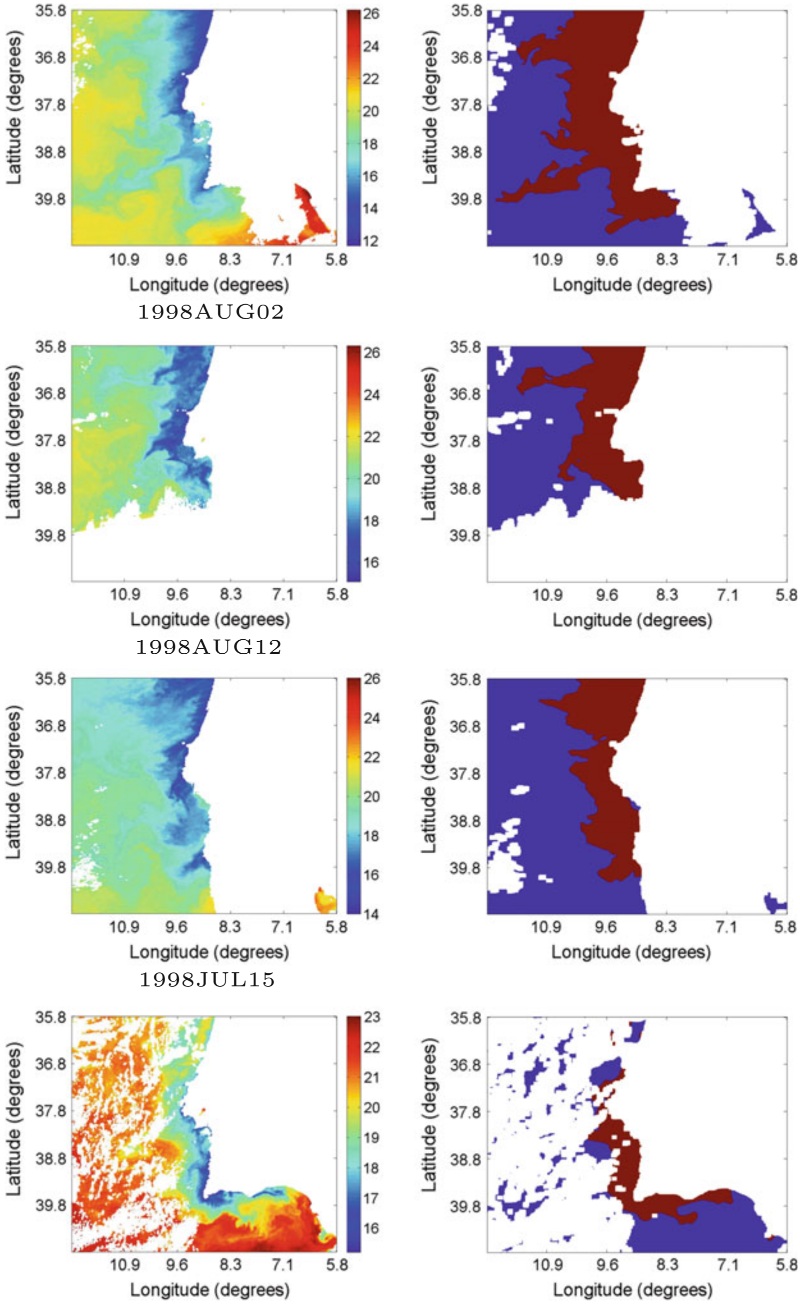
Figure 1 (left column) illustrate these types of situations. These images have been manually annotated by expert oceanographers regarding the upwelling regions (binary ground truth maps), which are shown in the right column of Fig. 1.

Here we report of experiments on the SEC method at which the value of parameter π has been determined by either as the optimal $\lambda^2/2$ (SelfT-SEC) or by using the Otsu method (Otsu-SEC) applied to ground truth maps.

To compare the performance of seed region growing algorithms, we use the popular precision and recall characteristics, as well as their harmonic mean, the F-measure. Precision corresponds to the probability that the detection is valid, and recall to the probability that the ground truth is detected.

Overall, the segmentations are rather good, with 82% of F-scores ranging between 0.7 and 0.98. On analyzing segmentations obtained by the self-tuning threshold version of the algorithm we obtained good results in 75% of the cases. The majority of the lower value scores occur for the images with weak gradients. Figure 2 (left column) illustrates the segmentation results obtained by the self-tuning SEC algorithm for three SST images presented in Fig. 1.

By comparing the relative performances of the two unsupervised thresholding versions of SEC algorithm (Otsu-SEC and SelfT-SEC), we came up with the following conclusions. The Otsu-SEC wins in 53.6% of the cases whereas the self-tuning version wins in 46.4% of images.



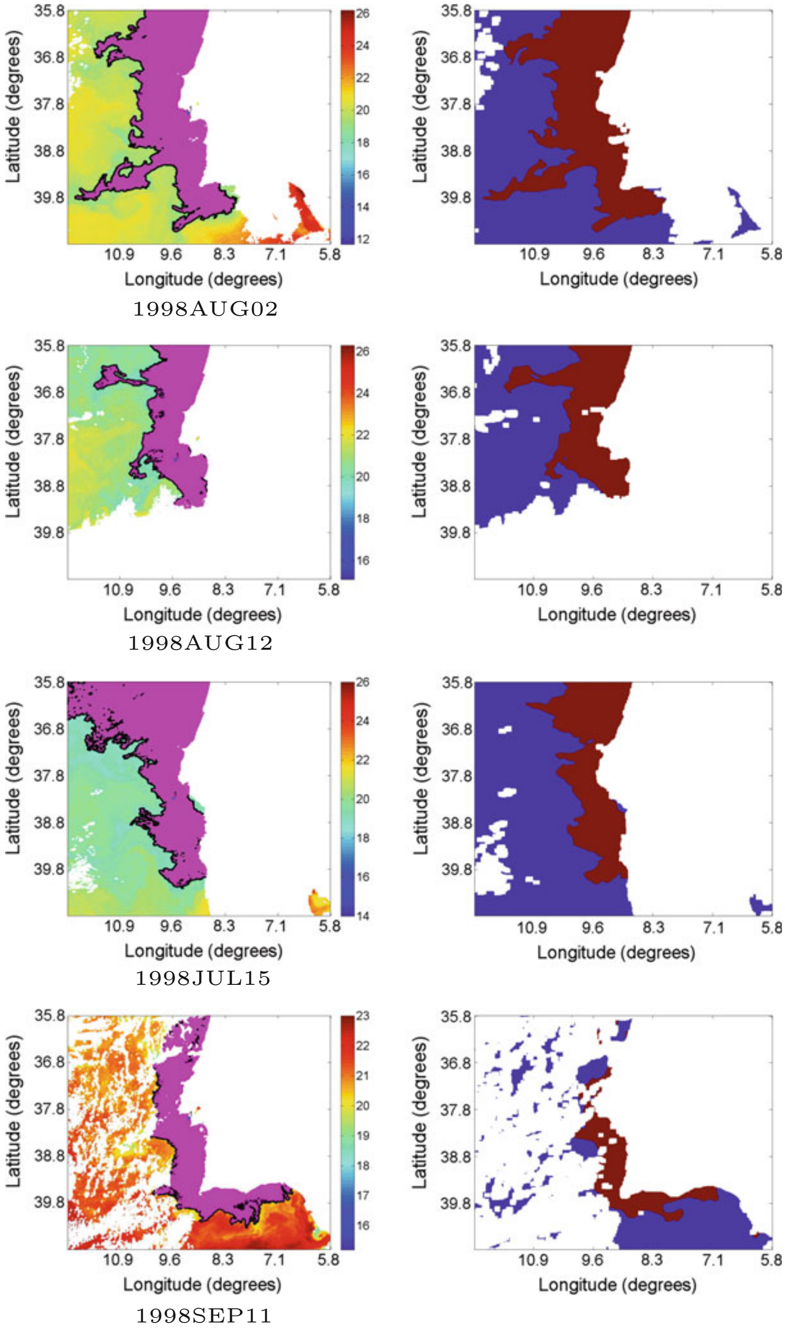
1998AUG02

1998AUG12

1998JUL15

1998SEP11

Fig. 1 Four SST images of Portugal showing different upwelling situations (*left column*); corresponding binary ground-truth maps (*right column*)



1998AUG02

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Fig. 2 Upwelling areas found by the self-tuning version of SEC algorithm on SST images of Portugal and her coastal waters (*left column*) versus the binary ground-truth maps (*right column*)

The two versions of the algorithm are implemented in MatLab R2013a. The experiments have been run on a computer with a 2.67 GHz Intel(R) core(TM) i5 processor and 6 Gbytes of RAM. The operating system is Windows 8.1 Pro, 64-bit. The elapsed time of segmentation of an SST image with the Otsu's thresholding version takes 25 s, whereas the self-tuning version takes 22 s for the task.

5 Conclusion

We have proposed a new method for image segmentation combining ideas of AA clustering and Seed Region Growing. This algorithm involves a novel homogeneity criterion (10), no order dependence of the pixel testing, and a version with self-tuning threshold derived from the approximation criterion.

The Otsu's version of the algorithm leads to high F-measure values at segmenting SST images showing different upwelling situations. The self-tuning version of the algorithm succeeds at all images presenting contrasting gradients between the coastal cold waters and the warming offshore waters of the upwelling region, and at some images with weak gradients for upwelling.

Further research should be directed toward both extending the SEC algorithm to situations with many clusters and applying it to other image segmentation problems.

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Why Is Linear Quantile Regression Empirically Successful: A Possible Explanation

Hung T. Nguyen, Vladik Kreinovich, Olga Kosheleva, and Songsak Sriboonchitta

Abstract Many quantities describing the physical world are related to each other. As a result, often, when we know the values of certain quantities $x_1, \ldots, x_n$, we can reasonably well predict the value of some other quantity y. In many application, in addition to the resulting estimate for y, it is also desirable to predict how accurate is this approximate estimate, i.e., what is the probability distribution of different possible values y. It turns out that in many cases, the quantiles of this distribution linearly depend on the values $x_1, \ldots, x_n$. In this paper, we provide a possible theoretical explanation for this somewhat surprising empirical success of such linear quantile regression.

1 Formulation of the Problem

What is regression: a brief reminder. Many things in the real world are related to each other. As a result, if we know the values of some quantities $x_1, \ldots, x_n$, then we can often reasonable well estimate the value of some other quantity y.

In some cases, the dependence of y on $x_1, \ldots, x_n$ is known. In many other situations, we do not know this dependence, so we need to find this dependence based on

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the empirical data. The desired dependence of y on $x_1, \ldots, x_n$ is known as a *regression function* $y \approx f(x_1, \ldots, x_n)$, and the methodology of determining the regression function from the empirical data is known as *regression analysis*.

In many practical situations, the dependence of y on $x_1, \ldots, x_n$ is well described by a linear function

$$y \approx \beta_0 + \sum_{i=1}^{n} \beta_i \cdot x_i. \tag{1}$$

Such linear dependence is known as *linear regression*.

What is quantile regression. Traditionally, the emphasis of regression analysis has been on finding the actual dependence $y \approx f(x_1, \ldots, x_n)$. However, finding this dependence is not enough. As we have mentioned earlier, the value $f(x_1, \ldots, x_n)$ is only an *approximation* to y. It is good to know this approximation, but it is also important to know *how accurate* is this approximation. In other words, we want to know not only the estimate of y for given x_i, we also want to know how the conditional probability distribution of y depends on the inputs $x_1, \ldots, x_n$.

One of the empirically efficient ways for finding this dependence is the method of *quantile regression*. One of the possible ways to describe the conditional probability distribution $P(y \mid x_1, \ldots, x_n)$ is to describe, for each probability p, the p-th quantile y_p of this distribution, i.e., that value for which the conditional probability $\text{Prob}(y \le y_p \mid x_1, \ldots, x_n)$ is equal to p. In particular:

- for $p = 0.5$, we get the median,
- for $q = 0.25$ and $q = 0.75$, we get the quartiles, etc.

One of the most empirically successful methods of describing the dependence of the conditional probability distribution on x_i is a method of *quantile regression*, when, for each p, we find a regression function $y_p = f_p(x_1, \ldots, x_n)$ that describes the dependence of the corresponding quantile y_p on the inputs x_i.

In particular, somewhat surprisingly, in many practical situations, this dependence turns out to be linear:

$$y_p \approx \beta_{0,p} + \sum_{i=1}^{n} \beta_{i,p} \cdot x_i \tag{2}$$

for appropriate coefficients $\beta_{i,p}$; see, e.g., [2–6].

Why is linear quantile regression empirically successful? Why is this linear quantile regression empirically successful in many practical applications? In this paper, we provide a possible explanation for this empirical success.

The structure of this paper is as follows. First, in Sect. 2, we provide fundamental reasons why linear regression is often empirically successful. In Sect. 3, we expand this result to the case of *interval uncertainty*, when instead of predicting the exact value of the quantity y, we predict the interval of its possible values. Finally, in Sect. 4, we show how this result can be expanded from interval to probabilistic uncertainty—thus explaining the empirical success of linear quantile regression.

2 Why Linear Regression Is Often Empirically Successful: A General Explanation

What we do in this section. Our goal is to explain why linear *quantile* regression is empirically successful. To explain this empirical phenomenon, let us first provide a possible explanation of why linear regression *in general* is empirically successful.

Empirical fact. Linear regression is empirically successful in many real-life situations, often in situations when the known empirical dependence is non-linear.

In this section, we will provide a possible explanation for this empirical fact.

Basis for our explanation: possibility of different starting points for measuring the corresponding quantities. We are interested in the dependence between the *quantities* x_i and y. To describe this dependence between *quantities*, we describe the dependence between the *numerical values* of these quantities.

The difference between the quantity itself and its numerical value may be perceived as subtle but, as will show, this difference is important—and it provides the basis for our explanation. The reason why there is a difference in the first place is that the numerical value of a quantity depends on the starting point for measuring this quantity. If we change this starting point to the one which is a units earlier, then all the numerical values of this quantity change from the previous value x to the new value $x + a$.

For example, we can start measuring temperature with the absolute zero (as in the Kelvin scale) or with the temperature at which the ice melts (as in the Celsius scale), the corresponding numerical values differ by $a \approx 273°$. Similarly, we can start measuring time with the birth year of Jesus Christ or, as the French Revolution decreed, with the year of the French Revolution.

It may be not so clear, but when we gauge many economic and financial quantities, there is also some arbitrariness in the selection of the starting point. For example, at first glance, unemployment is a well-defined quantity, with a clear starting point of 0%. However, economists who seriously study unemployment argue that starting it from 0 is somewhat misleading, since this may lead to an unrealistic expectation of having 0 unemployment. There is a natural minimal unemployment level of approximately 3%, and a more natural way of measuring unemployment is:

- not by its absolute value,
- but by the amount by which the current unemployment level exceeds its natural minimum.

Similarly, a person's (or a family's) income seems, at first glance, like a well-defined quantity with a natural starting point of 0. However, this does not take into account that 0 is not a possible number, a person needs to eat, to get clothed. Thus, a more reasonable way to gauge the income is:

- not by the absolute amount,
- but by how much the actual income exceeds the bare minimum needed for the person's survival.

The changes in the starting points should not affect the actual form of the dependence. In general, we can have different starting points for measuring each of the input quantities x_i. As a result, for each i, instead of the original numerical values x_i, we can have new values $x_i' = x_i + a_i$, for some constants a_i.

The change in the starting point:

- *changes* the numerical value, but
- *does not change* the actual quantity.

Thus, it is reasonable to require that the exact form of the dependence between x_i and y should not change if we simply change the starting points for all the inputs.

Of course, even for the simplest dependence $y = x_1$, if we change the starting point for x_1, then the numerical value of y will change as well, by the same shift—and thus, while the numerical value of y changes, the quantity y does not change—because the change in the starting point for x_1 simply implies that we correspondingly change the starting point for y.

In general, it is therefore reasonable to require that for each combination of shifts $a_1, \ldots, a_n$:

- once we shift the inputs to $x_i' = x_i + a_i$ and apply the function f to these shifted values,
- the resulting value $y' = f(x_1', \ldots, x_n')$ should simply be obtained from the original pre-shifted value $y = f(x_1, \ldots, x_n)$ by an appropriate shift:

$$y' = y + s(a_1, \ldots, a_n).$$

Thus, we arrive at the following definition.

Definition *We say that a function $f(x_1, \ldots, x_n)$ is* shift-invariant *if for every tuple $(a_1, \ldots, a_n)$ there exists a value $s(a_1, \ldots, a_n)$ such that for all tuples $(x_1, \ldots, x_n)$, we have*

$$f(x_1 + a_1, \ldots, x_n + a_n) = f(x_1, \ldots, x_n) + s(a_1, \ldots, a_n). \tag{3}$$

The desired dependence should be continuous. The values x_i are usually only approximately known—they usually come from measurements, and measurement are always approximate. The actual values x_i^{act} of these quantities are, in general, slightly different from the measurement results x_i that we use to predict y. It is therefore reasonable to require that when we apply the regression function $f(x_1, \ldots, x_n)$ to the (approximate) measurement results, then the predicted value $f(x_1, \ldots, x_n)$ should be close to the prediction $f(x_1^{\text{act}}, \ldots, x_n^{\text{act}})$ based on the actual values x_i^{act}.

In other words, if the inputs to the function $f(x_1, \ldots, x_n)$ change slightly, the output should also change slightly. In precise terms, this means that the function $f(x_1, \ldots, x_n)$ should be *continuous.*

Now that we have argued that the regression function be shift-invariant and continuous, we can explain why linear regression is empirically successful.

Proposition *Every shift-invariant continuous function $f(x_1, \ldots, x_n)$ is linear, i.e., has the form*

$$f(x_1, \ldots, x_n) = \beta_0 + \sum_{i=1}^{n} \beta_i \cdot x_i \tag{4}$$

for appropriate coefficients β_i.

Proof Substituting the values $x_i = 0$ into the equality (3), we conclude that

$$f(a_1, \ldots, a_n) = f(0, \ldots, 0) + s(a_1, \ldots, a_n) \tag{5}$$

for all possible tuples $(a_1, \ldots, a_n)$. In particular, this is true for the tuples $(x_1, \ldots, x_n)$ and $(x_1 + a_1, \ldots, x_n + a_n)$, i.e., we have:

$$f(x_1, \ldots, x_n) = f(0, \ldots, 0) + s(x_1, \ldots, x_n) \tag{6}$$

and

$$f(x_1 + a_1, \ldots, x_n + a_n) = f(0, \ldots, 0) + s(x_1 + a_1, \ldots, x_n + a_n). \tag{7}$$

Substituting the expressions (6) and (7) into the equality (3) and cancelling the common term $f(0, \ldots, 0)$ in both sides of the resulting equality, we conclude that

$$s(x_1 + a_1, \ldots, x_n + a_n) = s(x_1, \ldots, x_n) + s(a_1, \ldots, a_n) \tag{8}$$

Such functions are known as *additive.*

From the equality (5), we conclude that

$$s(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) - f(0, \ldots, 0). \tag{9}$$

Since the function $f(a_1, \ldots, a_n)$ is continuous, we can conclude that the function $s(a_1, \ldots, a_n)$ is continuous as well. So, the function $s(x_1, \ldots, x_n)$ is continuous and additive.

It is known (see, e.g., [1]) that every continuous additive function is a homogeneous linear function, i.e., has the form

$$s(x_1, \ldots, x_n) = \sum_{i=1}^{n} \beta_i \cdot x_i \tag{10}$$

for some real numbers β_i. Thus, from the formula (5), we can conclude that

$$f(x_1, \ldots, x_n) = \beta_0 + s(x_1, \ldots, x_n) = \beta_0 + \sum_{i=1}^{n} \beta_i \cdot x_i, \tag{11}$$

where we denoted $\beta_0 \overset{\text{def}}{=} f(0, \ldots, 0)$.

The proposition is proven.

Comment. It is easy to see that, vice versa, every linear function (4) is continuous and shift-invariant: namely, for each such function, we have:

$$f(x_1 + a_1, \ldots, x_n + a_n) = \beta_0 + \sum_{i=1}^{n} \beta_i \cdot (x_i + a_i) =$$

$$\beta_0 + \sum_{i=1}^{n} \beta_i \cdot x_i + \sum_{i=1}^{n} \beta_i \cdot a_i = f(x_1, \ldots, x_n) + s(a_1, \ldots, a_n),$$

where we denoted $s(a_1, \ldots, a_n) \overset{\text{def}}{=} \sum\limits_{i=1} \beta_i \cdot a_i$.

3 Case of Interval Uncertainty

Description of the case. In the previous section, we have shown that when we try to predict a numerical value y, then it is often beneficial to use linear regression. As we have mentioned, predicting a single value y is often not enough:

- in addition to the approximate value y,
- it is also necessary to know how accurate is this approximate value, i.e., which values y are possible.

Because of this necessity, in this section, we consider a situation, in which, for each inputs $x_1, \ldots, x_n$:

- instead of predicting a *single* value y,
- we would like to predict the *interval* $\left[\underline{y}(x_1, \ldots, x_n), \overline{y}(x_1, \ldots, x_n)\right]$ of all the values of y which are possible for given inputs $x_1, \ldots, x_n$.

Why linear regression. In the case of interval uncertainty, instead of a *single* regression function $y = f(x_1, \ldots, x_n)$, we have *two* regression functions:

- a regression function $\underline{y} = \underline{f}(x_1, \ldots, x_n)$ that describes the lower endpoint of the desired interval, and
- a regression function $\overline{y} = \overline{f}(x_1, \ldots, x_n)$ that describes the upper endpoint of the desired interval.

It is reasonable to require that each of these two functions is

- continuous, and
- does not change if we change the starting points for measuring the inputs—i.e., is *shift-invariant* (in the sense of the above Definition).

Thus, due to our proposition, each of these functions is linear, i.e., we have

$$\underline{f}(x_1, \ldots, x_n) = \underline{\beta}_0 + \sum_{i=1}^{n} \underline{\beta}_i \cdot x_i \tag{12}$$

and

$$\overline{f}(x_1, \ldots, x_n) = \overline{\beta}_0 + \sum_{i=1}^{n} \overline{\beta}_i \cdot x_i. \tag{13}$$

for appropriate values $\underline{\beta}_i$ and $\overline{\beta}_i$.

4 Case of Probabilistic Uncertainty

Description of the case. We consider the situation in which for each combination of inputs $x_1, \ldots, x_n$, in addition to the set of possible values of y, we also know the probability of different possible values of y. In other words, for each tuple of inputs $x_1, \ldots, x_n$, we know the corresponding (conditional) probability distribution on the set of all possible values y.

What is the relation between a probability distribution and the set of possible values? From the previous section, we know how to describe regression in the case of interval uncertainty. We would like to extend this description to the case of probabilistic uncertainty. To be able to do that, let us recall the usual relation between the probability distribution and the set of possible values.

This relation can be best illustrated on the example of the most frequently used probability distribution—the normal (Gaussian) distribution, with the probability density

$$\rho(y) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right). \tag{14}$$

The ubiquity of this distribution comes from the Central Limit Theorem, according to which the probability distribution caused by the joint effect of many small independent random factors is close to Gaussian; see, e.g., [7].

From the purely mathematical viewpoint, a normally distributed random variable can attain any real value. Indeed, the corresponding probability density is always positive, and thus, there is always a non-zero probability that we will have a value far away from the mean μ.

However, values which are too far from the mean have such a low probability that from the practical viewpoint, they are usually considered to be impossible. It is well know that:

- with probability 95%, the normally distributed random variable y is inside the two-sigma interval $[\mu - 2\sigma, \mu + 2\sigma]$;

- with probability 99.9%, y is inside the three-sigma interval $[\mu - 3\sigma, \mu + 3\sigma]$, and
- with probability 1–10^{-8}, y is inside the six-sigma interval $[\mu - 6\sigma, \mu + 6\sigma]$.

In general, a usual way to transform a probability distribution into an interval of practically possible values is to use a *confidence interval*, i.e., an interval for which the probability to be outside this interval is equal to some pre-defined small value p_0. A usual way to select such an interval is to select the bounds $\underline{y}$ and $\overline{y}$ for which:

- the probability to have y smaller than $\underline{y}$ is equal to $\dfrac{p_0}{2}$, and
- the probability to have y larger than $\overline{y}$ is equal to $\dfrac{p_0}{2}$.

One can easily see that:

- the lower endpoint $\underline{y}$ of this confidence interval is the quantile $y_{p_0/2}$, and
- the upper endpoint $\underline{y}$ of this confidence interval is the quantile $y_{1-p_0/2}$.

Depending on the problem, we can have different probabilities p_0, so we can have all possible quantiles.

Conclusion: why linear quantile regression is empirically successful. For each combination of inputs $x_1, \ldots, x_n$, based on the related (conditional) probability distribution of y, we can form the interval of practically possible values, in which both endpoints are quantiles y_p corresponding to some values p.

In the previous section, we have shown that reasonable requirements imply that each of these endpoints is a linear function of the inputs. Thus, we conclude that for each p, we have

$$y_p \approx \beta_{0,p} + \sum_{i=1}^{n} \beta_{i,p} \cdot x_i, \tag{2}$$

for appropriate values $\beta_{i,p}$.

This is exactly the formula for linear quantile regression. Thus, we have provided the desired first-principles for linear quantile regression formulas. The existence of such a justification can explain why linear quantile regression is empirically successful.

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A Rapid Soft Computing Approach to Dimensionality Reduction in Model Construction

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Abstract A rapid soft computing method for dimensionality reduction of data sets is presented. Traditional approaches usually base on factor or principal component analysis. Our method applies fuzzy cluster analysis and approximate reasoning instead, and thus it is also viable to nonparametric and nonlinear models. Comparisons are drawn between the methods with two empiric data sets.

Keywords Dimension reduction · Factor analysis · Principal component analysis · Fuzzy cluster analysis · Fuzzy reasoning

1 Introduction

In model construction large observation or input variable sets arouse various problems and thus we usually attempt to simplify our examinations by applying data compression or dimensionality reduction. Then we may operate with fewer observations or variables. In statistical multivariate analysis this means that in the former case we may compress our data matrices by applying cluster analysis (CA), whereas the number of variables is reduced by combining similar original variables for variable groups with such methods as the principal component analysis (PCA) and factor analysis (FA) [1].

Today soft computing (SC) systems have also proven to be useful in statistical modeling and model construction in general, and its CA and regression models provide good examples of these [2–4]. On the other hand, we still face certain challenges when applying SC techniques to dimensionality reduction.

One open problem is how to reduce the dimensionality in our data matrix if the traditional PCA and FA approaches are insufficient. Typical limitations of PCA and FA are that they are only appropriate for linear models, their data sets should be sufficiently large and their variables are expected to be normally distributed [1, 5].

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In many practical cases, however, actual relationships are nonlinear and/or sample sizes are small. In such situations, SC techniques have often been very successful in data processing. It is therefore reasonable to believe that SC techniques will be also efficient in reducing dimensionality.

Below we apply SC models to dimensionality reduction and we aim at a simple and an easily understandable method that is also robust and good in practice. This approach thus provides us with a "quick-and-dirty" method in the practice of model construction. We also attempt to draw an analogy between our approach and the PCA and FA. Some SC approaches to dimensionality reduction are available already, but they seem to be fuzzified versions of PCA or FA [6–18].

On the other hand, some papers have applied fuzzy similarity measures to this problem area, but they do not seem to correspond with the theoretical background or the goodness criteria of PCA and FA [13, 14]. Our approach, in turn, applies fuzzy similarity measures but we also yield and assess our outcomes according to PCA and FA and their goodness criteria. We also maintain Lotfi Zadeh's original idea on fuzzy systems, viz. instead of only using fuzzy mathematics or set theory, we also apply fuzzy reasoning in an understandable manner [19–21]. Thanks for the good available fuzzy methods in CA and approximate reasoning, that are well-known in the fuzzy community already, we adopt a general, a meta-level, approach, and thus detailed calculations are precluded.

Section 2 presents basic ideas on PCA and FA. Section 3 introduces our method. Section 4 provides two real-world examples. Section 5 concludes our examination.

2 Dimensionality Reduction in Statistics

In the explorative studies in human sciences [22] we aim at reducing the number of the original variables by grouping first the similar variables and then specifying such new variables that constitute these variable groups. These new variables are often referred to as sum variables because they are usually the (possibly weighted) sums the original variables. In this manner we may understand better the nature of our data and can perform simpler calculations. We may even attempt to find new "latent" variables behind the original variables in which case we can also label these new variables according to our interpretations, if necessary. For example, if we notice that a certain group of variables in our data matrix actually measures the same general feature, such as person's mathematical skills, from various standpoints, we may specify the sum variable "mathematical skills" that captures the meanings of the corresponding original variables. In the confirmatory studies, in turn, we may apply the available background theories to our dimensionality reduction.

In the traditional statistics we may specify our sum variables directly on our intuitive basis by calculating the sums of the selected original variables. We may also apply PCA or FA, in which case the obtained sum variables are referred to as principal components or factors, respectively. In fact, these methods operate with the standard scores of the original variables, i.e., for each variable its mean is subtracted from the observations, and then these differences are divided by its standard deviation, standard score = (observation − mean)/standard deviation. This transformation indicates us how the observations are distributed around the mean in the units of the standard deviation. Hence, when in PCA and FA we calculate the sum variable values for the original observations, viz. the principal component or factor scores, we also obtain the standard scores, and this may complicate the interpretation of our outcomes. Thus simpler sum variable specifications are also available, and these are discussed below.

We usually proceed as follows with the traditional PCA and FA [1]:

1. We assume that there are sufficiently high linear inter-correlations between the original variables, because the similarities between the variables are usually based on these correlations.
2. We also assume that the sample size is sufficiently large (e.g., at least five observations per variable), outliers are excluded, and the variables are measured at least at the level of good ordinal scales. In FA the variables should also be normally distributed and multicollinearity is not accepted.
3. We calculate the so-called principal component or factor loadings that are the correlations between the original variables and the principal components or factors.
4. We apply rotation to these components or factors in order to better interpret our loadings.
5. We select the appropriate principal components or factors, and these will be used in our sum variable specifications.

One distinction between the PCA and FA is that in the former case we always obtain the same unique components, whereas in FA the factors may vary according to the established number of factors.

In a sense, PCA and FA aim to find variable clusters according to the linear intercorrelations between the variables. The variable groups having intercorrelations will constitute variable clusters, and thus the obtained components or factors are the corresponding cluster centers. The principal component or factor loadings may now be regarded as being the “degrees of membership” of the variables to these clusters. Another approach is to consider that the principal components or factors span such vector spaces in which the loadings of the variables denote their coordinates.

On the other hand, we may also assume that in dimensionality reduction we aim to find clusters of variables according to their distances, and thus we can apply CA or multidimensional scaling. Hence, instead of the so-called Q-techniques of clustering, we are now applying the R-techniques [23]. In practice, when our data set is not too large, we may then operate with the transposed version of the original data matrix and then apply CA, and this approach is adopted below. We also apply fuzzy rule-based reasoning in our analyses. Thanks for the good fuzzy clustering techniques and usable approximate reasoning methods, the SC approach is more robust, applicable and user-friendly than the traditional methods.

Table 1 Example of a data matrix and its principal components

Case nr.	Variables and principal components					
	X_1	$X_2 \ldots$	X_n	C_1	$C_2 \ldots$	$C_{q \leq n}$
1	X_{11}	X_{12}	X_{1n}	C_{11}	C_{12}	C_{1q}
2	X_{21}	X_{22}	X_{2n}	C_{21}	C_{22}	C_{2q}
…	…			…		
m	X_{m1}	$X_{m2} \ldots$	X_{mn}	C_{m1}	C_{m2}	C_{mq}

3 Soft Computing and Dimensionality Reduction

Our SC approach below applies both fuzzy clustering and approximate reasoning directly to dimensionality reduction. In this manner we may also apply dimensionality reduction in a nonparametric manner to nonlinear data sets. Since we adopted a meta-level approach, i.e., only methods of general nature are considered and detailed mathematical analyses are precluded, we apply the prevailing fuzzy clustering methods and fuzzy rule-based models, and these have also proved to be useful in practice. However, for the sake of consistency, our goodness criteria for the outputs are those of PCA and FA. In this respect fuzzy mountain clustering [24, 25] and fuzzy c-means clustering methods [10, 11, 26–34] are analogous to PCA and FA, respectively.

For example, in mountain clustering and PCA we specify the unique outputs iteratively one at a time starting from the densest or largest group of observations or variables, whereas within the fuzzy c-means method and FA our outcomes vary according to the established number of clusters or factors. We focus on the mountain clustering method and PCA below, because these techniques may bring better understanding to our approach.

Given now the original data matrix with *m* cases or observations (rows) and *n* variables (columns), if we apply fuzzy clustering method to dimensionality reduction, we may proceed as follows (Table 1).

1. We focus on such groups of variables that are close to each other. In other words, the distances between these variables, X_i, are small. In practice, we may operate, for example, with their standard scores, ZX_i, and our task stems from the calculation of the norms, $||ZX_i - ZX_j||$ $(i \neq j)$. Alternative transformations may also be used, but in any case the original variables should be transformed into similar scales, because otherwise our variables have unequal weights.
2. Our method uses the transpose of the original data matrix (columns become rows), and then we apply fuzzy cluster analysis to the variables.
3. The obtained cluster centers of the variables, C_k, will be our "principal components" or "factors", and our loadings are now the correlations between the variables and these centers. We may also use linguistic values in this context.
4. We assess the goodness of our outcomes by applying such prevailing criteria as the communalities and eigenvalues of the variables.

When we specify the sum variables according to our loadings, we use functions that express sufficiently well the relationships between the variables and the cluster centers. Traditionally we may apply linear regression analysis in order to obtain such corresponding sum variables, $S_k \approx C_k$,

$$S_k = \sum_i w_{ik} \cdot ZX_i, \ (i = 1, 2, \ldots, n), \tag{1}$$

in which C_k is the dependent variable, the weights, w, are the regression coefficients and *ZX* are the standard scores of the original variables [1, 5, 35–37]. This method is used for calculating the principal component or factor scores.

However, (1) also includes the irrelevant variables in the sum variables, i.e., the variables with low loadings, and thus it may yield more or less misleading outcomes. Hence, in practice, we quite often simply calculate the sum of the relevant variables in each component, i.e., we only select the variables with the high loadings. In the case of the similar original scales we prefer the averages of these variables. This idea is widely used in PCA and FA in the human sciences. We may then justify our decisions by applying item analysis with Cronbach's alpha reliability coeffiecients, which are based on correlation coefficients, even though this method is not foolproof for this task [1, 5]. Another, sometimes more reliable method is based on the factor score covariance matrix [1]. The former usually minimizes and the latter maximizes the reliability coefficients.

Within our SC framework, we may also apply the fuzzy rule-based models, *F*, for all relevant variables in a component, i.e., the cluster center, C_k, is the dependent variable in the model

$$S_k = F_k(X_q, \ldots, X_r), \ q \geq 1, r \leq n, \tag{2}$$

Method (2) seems better in practice because it is also appropriate for nonlinear relationships.

Our SC approach may nevertheless arouse problems if variable clusters are unavailable or we have large observation sets. In the former case it may be difficult to find plausible cluster centers with any method, and the latter case may lead to quite heavy computations due to the large number of parameters.

Below we apply the idea on mountain clustering that is analogous to PCA [24, 25]. Hence, our method specifies the first cluster center according to the greatest or densest variable cluster. The second center is assigned to the second densest cluster with the restriction that it is not in the neighborhood of the first cluster. The next center, in turn, belongs to the third densest cluster, but it should also locate far from the previous clusters, and so forth. The number of cluster centers is determined by the user in the manner of PCA.

From the mathematical standpoint, the general idea for our clustering is that our first cluster center, C_1, is obtained when we minimize this type of penalty function,

$$\Sigma_i \mu_{C1}(ZX_i) \cdot ||ZX_i - C_1||, \ i = 1, 2, \ldots, n, \tag{3}$$

in which μ is an appropriate fuzzy triangular or bell-shaped membership function with its maximum value at C_1. This method finds the vector C_1 to be the center of the densest variable cluster.

The second cluster center, C_2, should represent the second densest cluster, and thus the neighborhood of C_1 should be excluded from our analysis. Hence, our minimizing penalty function should now also contain the exclusion function, *Ex*, that excludes the first cluster,

$$\Sigma_i \mathrm{Ex}_{C2}(\mathrm{ZX_i}) \cdot \mu_{C2}(\mathrm{ZX_i}) \cdot ||\mathrm{ZX_i} - C_2||, i = 1, 2, \ldots, n, \tag{4}$$

in which, for example, $\mathrm{Ex}_{C2}(\mathrm{ZX_i}) = (1 - \mu_{C1}(\mathrm{ZX_i}))^s(s > 1)$. This means that in the second round the variables close to $C_1$are irrelevant.

In the third round, the variables close to C_1 and C_2 are irrelevant, i.e.,

$$\mathrm{Ex}_{C3}(\mathrm{ZX_i}) = \min(\mathrm{Ex}_{C2}(\mathrm{ZX_i}), (1 - \mu_{C2}(\mathrm{ZX_i}))^s) \tag{5}$$

and the penalty function for C_3 is

$$\Sigma_i(\mathrm{Ex}_{C3}(\mathrm{ZX_i}) \cdot \mu_{C3}(\mathrm{ZX_i}) \cdot ||\mathrm{ZX_i} - C_3||, \mathrm{i} = 1, 2, \ldots, \mathrm{n}. \tag{6}$$

We will continue till all the values of $\mathrm{Ex}_{Ci}(\mathrm{ZX_i})$ are small, this meaning that we have examined all variable clusters.

These operations may be carried out conveniently with such methods as the genetic algorithms if custom-made models are preferred. We may also apply the original mountain clustering method directly, if the number of variables is sufficiently large. Below we will provide examples with the empiric data sets.

4 Real-World Examples

We examine below two real-world data sets with both PCA and our SC method. We aim to demonstrate that our method, that is simpler and more robust than PCA and FA, is also plausible for dimensionality reduction. We use Matlab™ version 2014b and IBM SPSS™ version 22 in our calculations.

4.1 The Iris Data

Fisher's Iris data is the widely-used benchmark data in cluster analysis. Fuzzy clustering methods have already proved their usability in this context of grouping the objects, but only some indirect methods have been suggested for dimensionality reduction of variables. This data set is challenging to us because it contains problematic clusters.

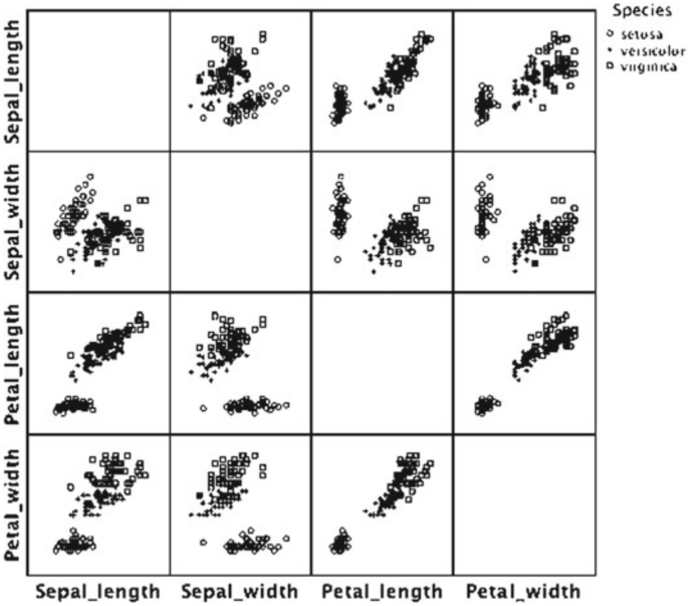


Fig. 1 Scatter plots of flowers in the Iris data

The Iris data contains 150 of these flowers and four feature variables that measure in millimeters their *Sepal lengths* (Sl), *Sepal widths* (Sw), *Petal lengths* (Pl) and *Petal widths* (Pw; Fig. 1). In the cluster analysis for these flowers we should find three clusters, and good fuzzy CA methods are able to perform this. Since sufficiently high inter-correlations prevail between the variables, we may also attempt to use PCA and our SC method for dimensionality reduction.

4.1.1 The PCA Approach

In PCA the values of the inter-correlations between our variables may first be analyzed with such rules of thumb as the Kaiser-Mayer-Olkin measure and Bartlett's test, and if the former yields values greater than 0.6 and the latter rejects its null hypothesis, our correlations seem to be sufficiently high [1, 5]. In our data the former yields 0.54 and the latter rejects the null hypothesis (at the level of significance <0.05). Hence, the former value is not fully satisfactory, but the latter fulfills the conditions.

On the other hand, the communalities of the variables are higher than 0.9, and hence we assume that PCA is justified with all our feature variables in this context. The communalities are the *rsquares* in those regression models in which the feature variable is the dependent variable and the principal components are the independent variables. Hence, the communalities indicate how well the selected principal components can explain or predict the variances of the variables.

PCA yields first at the extraction stage the initial component loadings (i.e., the correlations between the variables and the components) for the variables in each component by starting from the largest variable group. In the first component we thus obtain the highest absolute values of the loadings, the second component has the second highest values, and so forth. There is also the restriction that the components must be orthogonal, i.e., they have no intercorrelations.

We also calculate the sums of squares of these loadings in each component, and these sums are referred to as the eigenvalues. We are usually interested in those principal components that yield eigenvalues greater than or equal with unity. The sums of the eigenvalues of our components divided by the number of the original variables, in turn, reveals us how much our components explain of the total variance of our variables.

In order to better understand our outputs, rotation is also carried out, and it modifies our original principal component loadings. The rotation aims to yield either high or low loadings, and in addition to the orthogonal methods, we may now apply oblique methods. The latter methods allow intercorrelations between the principal components, and this situation is usual in the human sciences. However, unlike in orthogonal rotation, in oblique rotation the loadings are not the correlations between the components and variables, but rather the weights that show us the importance of the variables in each component. Typical examples of orthogonal and oblique methods are Varimax and Promax, respectively.

Summing up the foregoing measures, given a table of original or orthogonally rotated principal component loadings, the row-wise sums of squares of the loadings yield the communalities, whereas the corresponding column-wise values yield the eigenvalues.

We prefer the oblique "Procrustean" Promax method in rotation below, and these principal component loadings are presented in Table 2 and Fig. 2 (the loadings less than the absolute value of 0.3 are omitted below because they are irrelevant). We select two principal components, because they already explain approximately 96% of the total variance of the variables (i.e., the sum of these two eigenvalues/

Table 2 Rotated component matrix of Iris data

	Component	
	1	2
Sepal_length	1.000	
Petal_length	0.933	
Petal_width	0.929	
Sepal_width		1.000

Extraction Method: Principal Component Analysis, Rotation Method: Promax with Kaiser Normalization, Rotation converged in 3 iterations

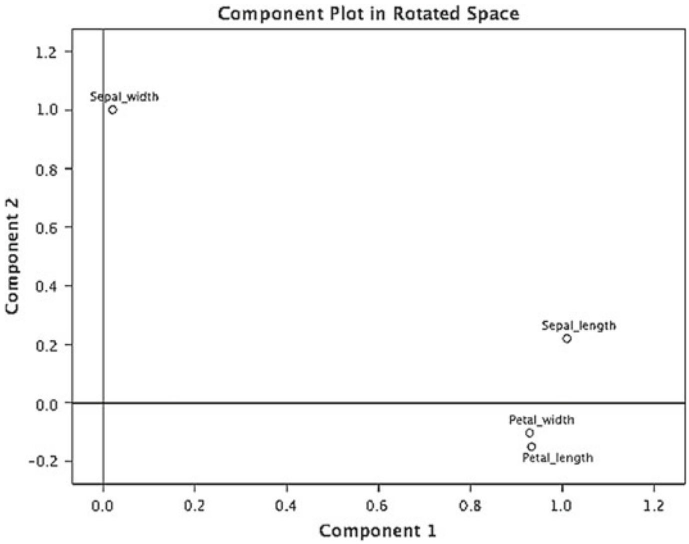


Fig. 2 Component plot in the rotated space with the Iris data

Table 3 The intercorrelations between the variables and the components in Iris data

	Component	
	1	2
Sepal_length	0.982	−0.455
Petal_length	0.962	−0.407
Petal_width	0.938	
Sepal_width	−0.307	0.995

Extraction Method: Principal Component Analysis, Rotation Method: Promax with Kaiser Normalization

4 × 100% = 96%) even though the eigenvalue of the second component was slightly less than unity.

We notice that, according to the first component in our rotated table, we may generate a sum variable that includes the variables *Sepal length*, *Petal length* and *Petal width*. The second component only includes one high loading, viz. for *Sepal width*. Hence, instead of the original variables, we may use two principal variables within the Iris data, if necessary. Table 3 presents the loadings that are also the correlations between the variables and the components because in oblique rotation the loadings in Table 2 are not correlations (as in the orthogonal rotation). The latter loadings are better comparable with our SC analyses below. We notice that these loadings are slightly more blurred with respect to sum variable specification. Both of these loading tables are nevertheless used in the conduct of inquiry.

According to Table 4, that presents the regression coefficients for the component scores, our first sum variable would now be

Table 4 Component score coefficient matrix for Iris data

	Component	
	1	2
Sepal_length	0.366	−0.200
Petal_length	0.005	−0.923
Petal_width	0.339	−0.140
Sepal_width	−0.338	−0.097

Extraction Method: Principal Component Analysis, Rotation Method: Promax with Kaiser Normalization

$$S_1 = 0.366 \cdot ZSl + 0.005 \cdot ZSw + 0.339 \cdot ZPl + 0.338 \cdot ZPw \tag{7}$$

if this prevailing method is applied to the standardized feature variables. Since *Sepal width* is irrelevant to S_1 and the rest of the loadings are quite similar, in practice we may use for the original variables their nonweighted sum instead,

$$S_1 = Sl + Pl + Pw, \tag{8}$$

or their average, if their standard scores are used. In item analysis Cronbach's alpha is greater than 0.9 for S_1, and this result also corresponds to this sum variable construction.

Hence, PCA provided us with one plausible sum variable, and this was due to the high linear intercorrelations between the feature variables.

4.1.2 The Soft Computing Approach

If we apply our SC method, we principally utilize the distances between the variables and, in the manner of the PCA, we operate with the standard scores of the original variables. Then, within the Iris data, we notice in the dendrogram in Fig. 3 that Sepal width is clearly distinct from the others and the rest of the feature variables seem to belong to same cluster. The multidimensional scaling analysis (SPSS Proxcal), that allocates the variables into a 2-D space according to their distances, also seems to support quite well this resolution (Fig. 4). Hence, it seems that we may specify one sum variable as above.

According to our cluster analysis approach, we will proceed as follows:

1. We specify two cluster centers, and these are our principal components, C_i. The correlations between the variables and principal components will be our component loadings.
2. The communalities are the *rsquares* of the fuzzy models F_i: $(C_1,C_2) \rightarrow ZX_i$, i = 1, 2, …, n, i.e., we consider how well our components explain or predict the variables.
3. The eigenvalues are the squared column-wise sums of the loadings as above.

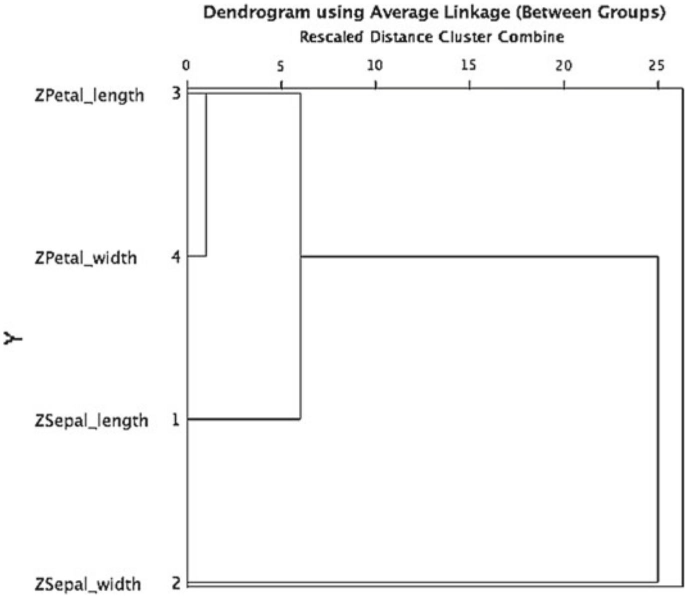


Fig. 3 Dendrogram based on the average linkage method and distances between four standard score variables in the Iris data

Fig. 4 The locations of the standardized variables according to the multidimensional scaling

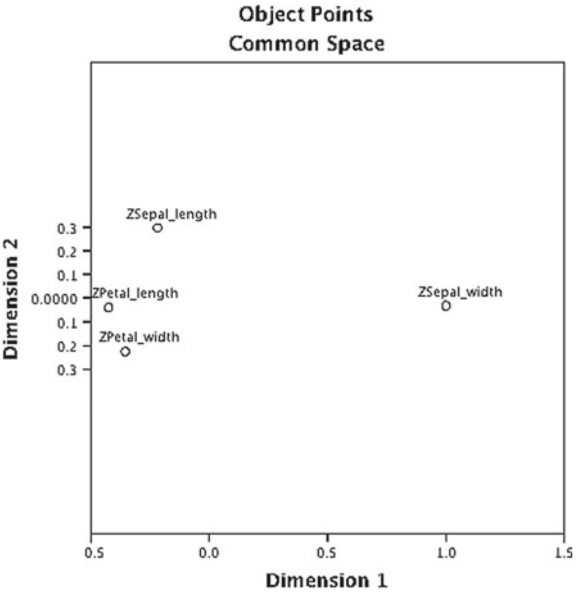


Table 5 Intercorrelations between the standardized variables and principal components in the Iris data

Variable	C_1	C_2	Communalities
ZSepal length	0.999		1.000
ZSepal width		1.000	1.000
ZPetal length	0.888		0.944
ZPetal width	0.838		0.873
Eigenvalues	2.49	1.000	

Table 6 Rmse values of the fuzzy models F_{ij}: $C_j \rightarrow ZX_i$

	C1	C2
ZX1	0.042	0.342
ZX2	0.282	0.001
ZX3	0.099	0.503
ZX4	0.134	0.522

4. The final sum variable, S_1, only constitutes the relevant variables of the first principal component, and its specification is similar to that of the PCA method. We may apply the fuzzy model $S_1 = F_1(ZX_1, ZX_3, ZX_4)$ by using C_1 as the dependent variable instead, if necessary.

Our mountain clustering method for the feature variables seems to yield two plausible cluster centers, and the corresponding principal components contain sufficiently high loadings. We used Matlab's Fuzzy Logic Toolbox and Takagi-Sugeno reasoning for these tasks [38]. Table 5 presents these correlations, or loadings (the absolute values less than 0.3 are omitted as above). This Table is analogous to Table 3 within the PCA. For the illustrative purposes, we also calculated the corresponding *rmse* values, and naturally they were consistent with our loadings (Table 6).

Table 5 also presents the communalities and eigenvalues, and fuzzy rule-based systems with seven rules and Takagi-Sugeno reasoning were used in an above-mentioned manner in this context. Our eigenvalues and communalities indicate that two components yield high loadings and all the variables are relevant in this context. These values also correspond to the PCA outcomes above. There is a slight negative correlation between C_1 and C_2, and in this respect we have an oblique resolution.

Figure 5 depicts the locations of our variables in the principal component space and it also corresponds quite well to the PCA approach. Figure 6 depicts the locations of the variables as well as the principal components based on the PCA and our method when the multidimensional scaling is applied. We notice that our outcomes are slightly dissimilar to those of the PCA. In fact, our components are closer to singular variables and thus we should possibly fine-tune our model. This procedure is nevertheless precluded here because we have adopted the meta-level approach and our outcomes are already sufficiently plausible.

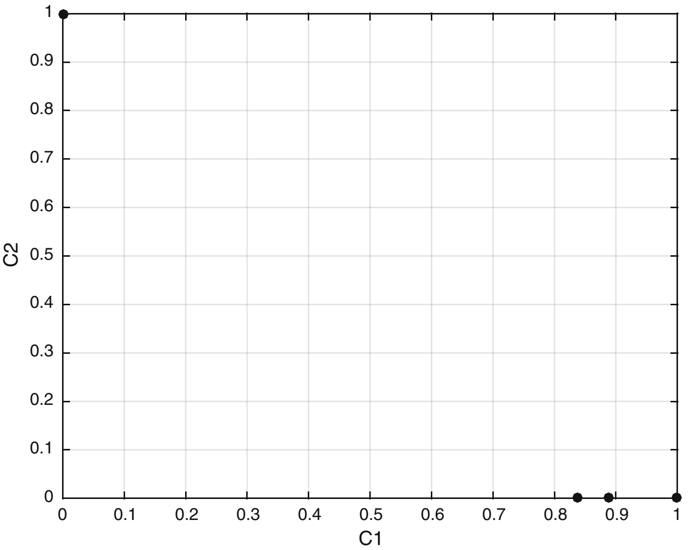
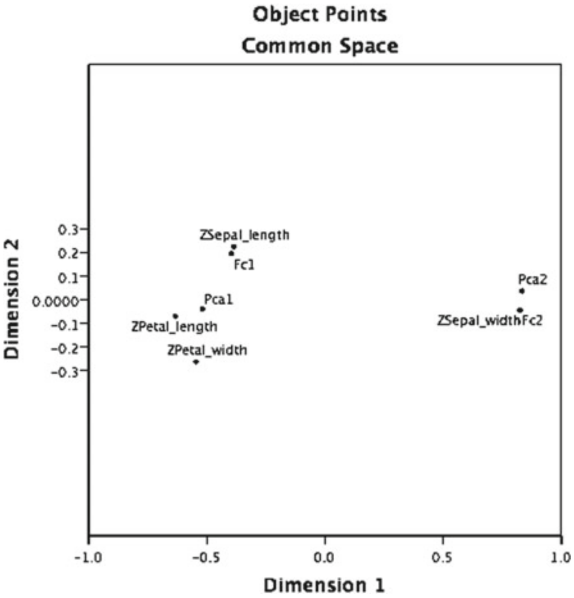


Fig. 5 The loadings of the standardized variables in the principal component space with Iris data

Fig. 6 The locations of the standardized variables as well as the PCA components (Pca) and SC components (Fc) according to the multidimensional scaling



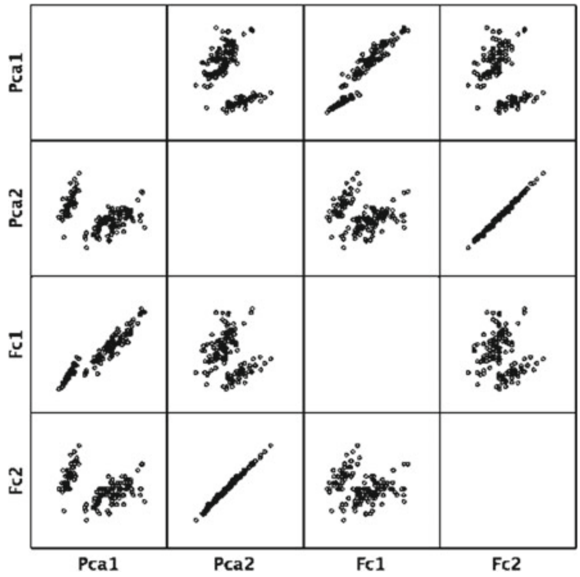


Fig. 7 Scatterplots of the PCA (Pca) and SC (Fc) components in the Iris data

If we will generate the linear sum variable, $S_1 \approx C_1$, it would be similar to that of (7). The corresponding fuzzy-model approach, in turn, will base on the rule-based system, F_1,

$$S_1 = F_1(\mathrm{Sl}, \mathrm{Pl}, \mathrm{Pw}) \tag{9}$$

when C_1 is used as the dependent variable in the model construction.

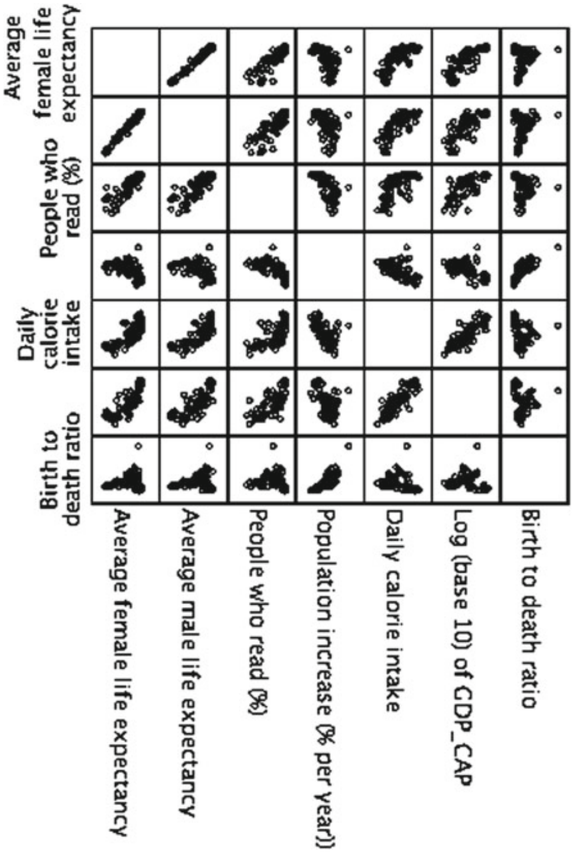
The intercorrelations between the PCA and our components are depicted in Fig. 7, and, as expected, they indicate high positive correlations.

Since fuzzy systems are now applied, we could also establish that the closer the variables are to the components, the higher their degrees of membership, and vice versa. We could even replace our loadings with these memberships, if necessary. However, then the comparison between the distinct component extractions would be more difficult than in the case of correlations.

4.2 The World95 Data

Our second example deals with the benchmark data collected within the international world survey from 109 countries in 1995 (World95 data), and this is included in the SPSS example data sets, inter alia [39]. We focus on seven variables, *Average female life expectancy*, *Average male life expectancy*, *People who read (%)*, *Population increase (% per year)*, *Daily calorie intake*, *Log (base 10) of GDP per capita* and *Birth to death ratio*. Figure 8 depicts the inter-correlations between our variables.

Fig. 8 Scatter plots of variables in the World95 data



4.2.1 The PCA Approach

We used again PCA with Promax oblique rotation. Both the Kaiser-Mayer-Olkin measure and Bartlett's test now fulfilled the conditions on the satisfactory inter-correlations. The communalities were at least 0.91 and thus all the original variables seemed relevant in our analysis. We selected three components, even though only two components had eigenvalues higher than unity, because our decision seemed to reflect better the variable groups. These components explained approximately 95% of the total variance of the variables. The first two components have a quite high correlation (.753), and thus oblique rotation is justified.

Table 7 and Fig. 9 indicate that three variables have high loadings in the first principal component (the loadings less than 0.3 are omitted). The second and the third components seem to include two variables with high loadings. In our outcome *Population increase* is not having a clear membership to any component. Since the foregoing table will not yield the correlations in oblique rotation, Table 8 presents the corresponding loading matrix based on the correlations between the variables and

Table 7 Rotated component matrix of Word95 data

	Component		
	1	2	3
Average female life expectancy	0.849		
Average male life expectancy	0.830		
People who read (%)	1.000		
Population increase (% per year)	−0.432		0.759
Daily calorie intake		0.951	
Log (base 10) of GDP/CAP		0.767	
Birth to death ratio			1.000

Extraction Method: Principal Component Analysis, Rotation Method: Promax with Kaiser Normalization

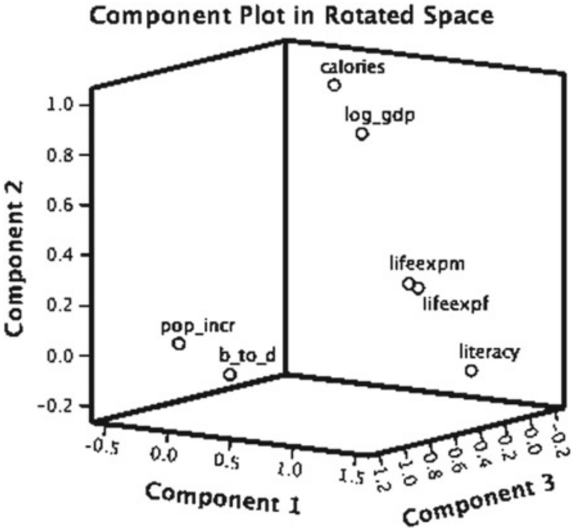


Fig. 9 Component plot in the rotated space with the World95 data

components. We notice that if we relied on this, the sum variable specification would be more problematic, but on the other hand, these values are better comparable to our SC-method outputs below.

Table 8 The intercorrelations between the variables and the components in the World95 data

	Component		
	1	2	3
Average female life expectancy	0.973	0.815	
Average male life expectancy	0.955	0.807	
People who read (%)	0.955	0.672	−0.396
Population increase (% per year)	−0.668	−0.569	0.892
Daily calorie intake	0.733	−0.968	−0.349
Log (base 10) of GDP/CAP	0.800	0.941	−0.373
Birth to death ratio			970

Extraction Method: Principal Component Analysis, Rotation Method: Promax with Kaiser Normalization

If we specify now three sum variables, we may proceed as above by merely calculating the sums of those variables that have high loadings in the principal components. For example,

$$\mathrm{S}_2 = \text{Daily calorie intake} + \text{Log (base 10) of GDP per CAP}, \tag{10}$$

or their averages, if the standard scores are used.

Hence, it seems plausible to specify sum variables among this data set even though now this task is more challenging than with the Iris data. Next we apply our SC method to this task.

4.2.2 The Soft Computing Approach

When our SC method is applied to three components, the correlation between the first two components is −0.565, and thus in this respect we also apply an "oblique" method. According to multidimensional scaling, three cluster centers also seem plausible, even though clear clusters are now unavailable (Fig. 10).

Our intercorrelations between the variables and the components seem somewhat distinct from the PCA outcomes with oblique loadings (Table 9, Figs. 11 and 12). Now the first component seems to comprise three variables common to both the PCA and SC approaches. In the SC model *ZLog (base 10) of GDP per CAP* is also having a high loading in the first component, and the same outcome is found in the correlation Table 8 above, and we must bear in mind that this table presents the correlations between the variables and the components in the manner of our SC method. The third component in PCA and the second component in the SC approach, in turn, provide quite similar outcomes.

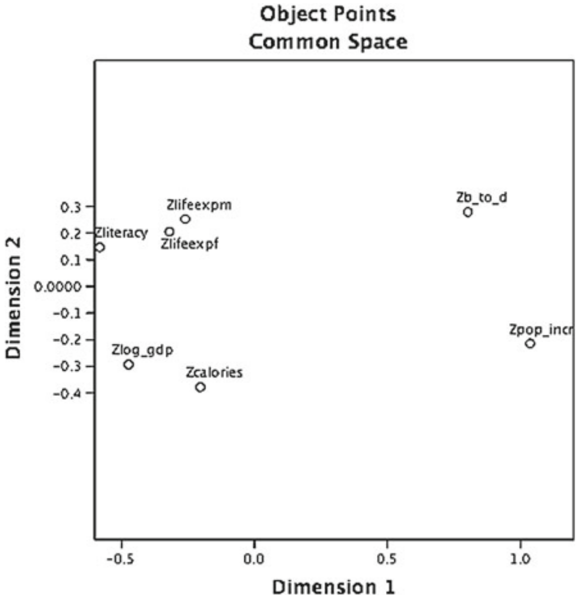


Fig. 10 The locations of the standardized variables according to the multidimensional scaling

Table 9 Intercorrelations between the standardized variables and principal components in the World95 data

Variable	C_1	C_2	C3	Communalities
ZAverage femal life expectancy	0.999			0.999
ZAverage male life expectancy	0.982			0.982
ZPeople who read (%)	0.771			0.795
ZPopulation increase (% per year)		0.999		1.000
ZDaily calorie intake			1.000	1.000
ZLog (base 10) of GDP per CAP	0.837			0.876
ZBirth to death ratio		0.816		0.939
Eigenvalues	3.261	1.665	1.000	

Hence, the variables *ZLog (base 10) of GDP per CAP* and *ZDaily calorie intake* seem to yield distinct outcomes, but even in this case the correlation Table 8 corresponds quite well to our results. Figure 13 depicts the scatter plots of our components.

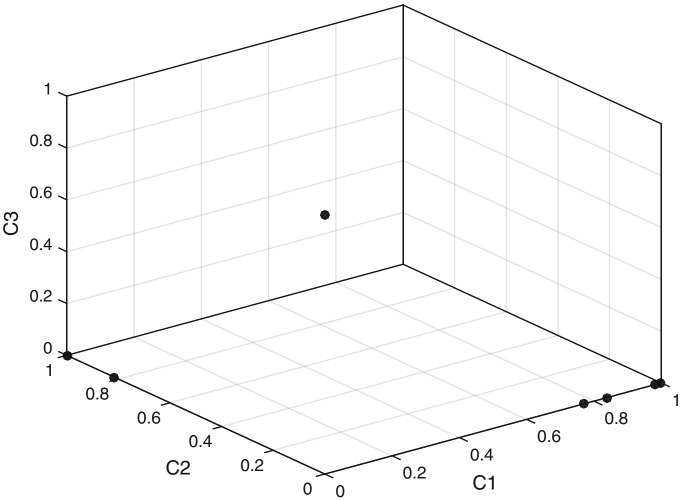
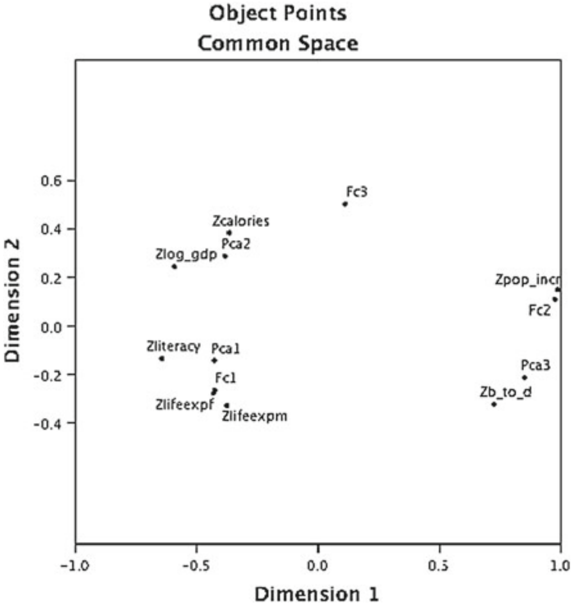


Fig. 11 The loadings of the standardized variables in the principal component space with World95 data

Fig. 12 The locations of the standardized variables as well as the PCA components (Pca) and SC components (Fc) according to the multidimensional scaling



As regards our communalities (that were calculated according to the fuzzy models) and eigenvalues, they seem to fulfill the given conditions.

Our sum variables, again, are the sums of the relevant variables in each component, for example,

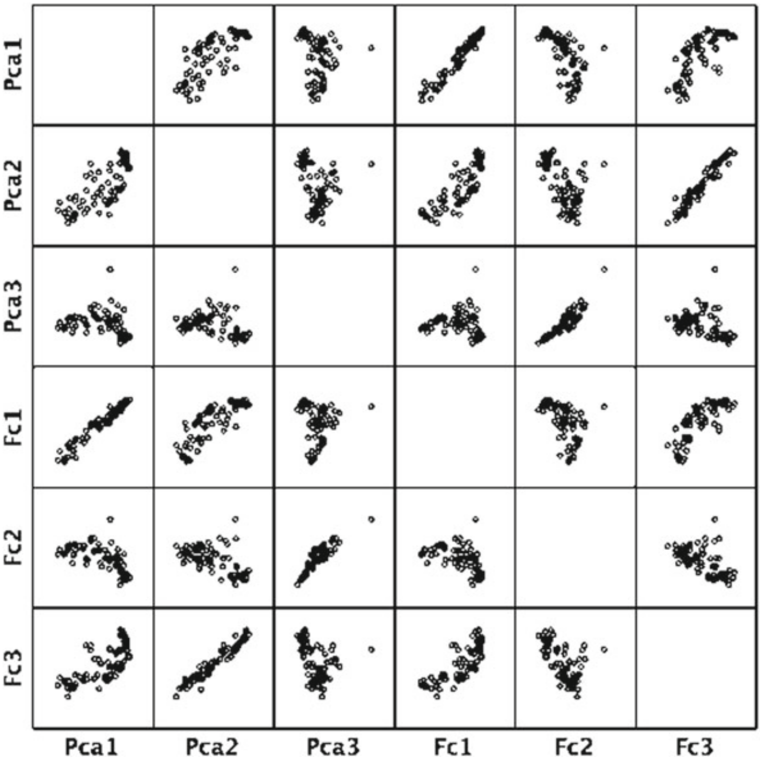


Fig. 13 Scatter plots of the PCA (Pca) and SC (Fc) components in World95 data

$$S_2 = \text{Population increase (\% per year)} + \text{Birth to death ratio}, \tag{11}$$

or, by applying the corresponding fuzzy model,

$$S_2 = F_2(\text{Population increase(\% per year), Birth to death ratio}), \tag{12}$$

with the dependent variable C_2 (Fig. 14).

Hence, in this context the SC method yields somewhat distinct outcomes, and this is due to our clustering approach and merely tentative calculations based on our general approach. In addition, the three-component approach seemed not fully justified in this context. On the other hand, in the human sciences the variables in the real world data often contain quite much noise and borderline cases.

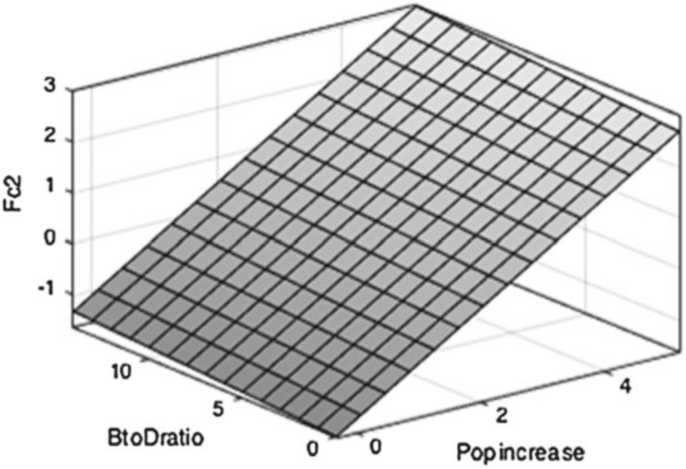


Fig. 14 A fuzzy model fitting to sum variable $S_2 = F$ (Population increase (% per year), Birth to death ratio)

5 Conclusions

We have considered above how the dimensionality reduction analogous to PCA may be carried out with fuzzy clustering method and fuzzy reasoning in an understandable manner, and two real-world data sets were also used as examples. Such prevailing traditional methods as PCA and FA are only appropriate to fairly limited usage because they presuppose linear correlations between the variables and normally distributed data sets, inter alia. Our SC approach, in turn, also seems usable to nonlinear and nonparametric data sets. The central idea in our approach is that we use fuzzy clustering method for finding the appropriate cluster centers to our variables, and these centers provide a basis for our sum variable construction.

In order to draw comparisons to the traditional approaches, our component loadings and goodness criteria based on various intercorrelations and *rsquares* between the variables and the principal components, and in this context we also applied fuzzy reasoning. However, in the long term we could replace the loadings with the degrees membership, as well as use even more fuzzy reasoning models. The loadings could also be linguistic values, if necessary. In this manner we could even better attain Lotfi Zadeh's recent idea on the fuzzy extended logic.

Since the SC community can already provide us with good model construction methods, we did not formulate any novel calculation technics but rather we focused at meta-level on constructing a tentative and an analogous system to PCA. Our contribution was to apply the fuzzy R-technique to data matrix and then construct fuzzy models for assessing the goodness of our outcomes with the loadings, communalities and eigenvalues. We also used fuzzy models for sum variable specifications because they are also appropriate to nonlinear cases.

We still have some open questions. First, due to the clustering approach, we encounter such their prevailing problems as the nonspherical clusters, selection of the correct metrics or the appropriate number of clusters. The lack of variable clusters or the great number of observations and original variables may also arouse problems.

Second, we still expect such standard goodness criteria for our dimensionality reduction within SC that may replace those of the PCA and FA. Examples of these are communalities and eigenvalues. Finally, if the degrees of membership are used for loadings, we still have various alternatives for specifying them.

One new frontier is to apply the fuzzy c-means clustering to this problem area, and this method would be analogous to FA. This is an interesting topic for the future studies and now it was mainly precluded due to the lack of space.

Despite the foregoing open problems, our SC approach seems nevertheless promising in practice as a "quick-and-dirty" method for the dimensionality reduction. However, further studies are still expected in this problem area.

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Physics of the Mind, Dynamic Logic, and Monotone Boolean functions

Leonid I. Perlovsky

Abstract The chapter discusses physics of the mind, a mathematical theory of higher cognition developed from the first principles, including concepts, emotions, instincts, the knowledge instinct, and aesthetic emotions leading to understanding of the emotions of the beautiful. The chapter briefly discusses neurobiological grounds as well as difficulties encountered by previous attempts at mathematical modeling of the mind encountered since the 1950s. The mathematical descriptions are complemented with detailed conceptual discussions so the content of the chapter can be understood without necessarily following mathematical details. Formulation of dynamic logic in terms of monotone Boolean functions outlines a possible future direction of research.

Keywords Physics of the mind · Concepts · Cognition · The instinct for knowledge · Aesthetic emotions · Beautiful · Cognitive science · Psychology · Dynamic logic · Monotone boolean functions

1 Mechanisms of the Mind

How the mind works has been a subject of discussions for millennia, from Ancient Greek philosophers to mathematicians, to modern cognitive scientists [1]. This chapter describes physics of the mind, a mathematical theory built from the first principles including dynamic logic, higher cognitive functions, and its further development using monotone Boolean functions. This technique could serve two purposes. First, it would lead to the development of smart computers and intelligent robots. Second, it would help to unify and clarify complex issues in philosophy, psychology, neurobiology, and cognitive science. This chapter is a step toward developing "physics of the mind," a theory of the mind concentrating on developing a mathematical model of the mind from a limited number of the first principles.

A broad range of opinions exists about the mathematical methods suitable for the description of the mind. Founders of artificial intelligence, including

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Newell and Minsky [64] and Marvin [62], thought that formal logic was sufficient and no specific mathematical techniques would be needed to describe the mind. An opposite view was advocated by [37, 67], suggesting that the mind cannot be understood within the current knowledge of physics; new unknown yet physical phenomena will have to be accounted for explaining the working of the mind; quantum computational processes are necessary for understanding the mind [27, 37, 67]. This chapter develops a point of view that the main difficulty toward a mathematical theory of the mind since the 1950s has been using logic, and the new mathematical theory of dynamic logic enables explaining the mind from the "the first principles" [68, 69, 71, 91, 95].

This chapter presents a mathematical theory of dynamic logic proposed to be intrinsic to operations of the mind, a suggestion that have been experimentally proven [6, 56]. It discusses difficulties encountered by previous attempts at mathematical modeling of the mind and how the new theory overcomes these difficulties. I show an example of solving a problem related to perception that was unsolvable in the past, argue that the theory is related to an important mechanism of "the knowledge instinct", KI, as well as to other cognitive functions, including interactions of language and cognition. I discuss neurobiological foundations, cognitive, psychological, and philosophical connections, experimental verifications, and further mathematical developments using monotone Boolean functions originally developed jointly with Prof. Boris Kovalerchuk [50–53, 109, 110].

2 Logic and the Mind

For a long time logic was considered the best way to deduce scientific truths. In the 1930s [23] proved that logic is inconsistent and cannot serve this foundational purpose, nevertheless artificial intelligence, mathematical and psychological models of the mind until today are logical, misleading intuitions of psychologists and mathematicians modeling the mind.

The beginning of this story is usually attributed to Aristotle, the inventor of logic. He was proud of this invention and emphasized, "nothing in this area existed before us" (Aristotle, IV BCE). However, Aristotle did not think that the mind works logically; he invented logic as a supreme way to argument already discovered truths, not as a theory of the mind. To explain the mind, Aristotle developed a theory of forms, the fundamental mechanism of the mind as a process, in which an illogical "form-as-potentiality" "meets matter" and becomes a logical "form-as-actuality". Today this process is called an interaction between top-down and bottom-up neural signals (BU, TD). A mathematical model of this process, dynamic logic is described later.

During centuries following Aristotle not all subtleties of his thoughts were understood. With the advent of science, the idea that intelligence is equivalent to logic was gaining grounds. In the 19th century mathematicians turned their attention to logic. George Boole thought that Aristotle did not complete a theory of the mind, and it should be improved by making logic more exact. The foundation of logic, since Aristotle (Aristotle, IV BCE, c), was the law of excluded middle (or excluded third): every statement is either true or false, any middle alternative is excluded.

Boole thought that the contradiction between exactness of the law of excluded third and vagueness of language should be corrected, and a new branch of mathematics, formal logic was born. Prominent mathematicians contributed to the development of formal logic including, in addition to Boole, Gottlob Frege, Georg Cantor, Bertrand Russell, David Hilbert, and Kurt Gödel. Logicians 'threw away' uncertainty of language and founded formal mathematical logic based on the law of excluded middle. Hilbert developed an approach named formalism which rejected the intuition as a part of scientific investigation and thought to define scientific objects formally in terms of axioms or rules. Hilbert was sure that his logical theory also described mechanisms of the mind, "The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds" (see [31]). In the 1900 he formulated famous Entscheidungsproblem: to define a set of logical rules sufficient to prove all past and future mathematical theorems. This entailed formalization of scientific creativity and the entire human thinking. This illustrates the difference between mathematics and physics, whereas mathematics concentrates on internal structure of the theory, physics concentrates on the fundamental laws of nature and their mathematical description.

Almost as soon as Hilbert formulated his formalization program the first hole appeared. In 1902 Russell exposed an inconsistency of formal procedures by introducing a set R as follows: *R is a set of all sets which are not members of themselves.* Is R a member of R? If it is not, then it should belong to R according to the definition, but if R is a member of R, this contradicts the definition [103]. Thus either way we get a contradiction. This became known as the Russell's paradox. Its jovial formulation is as follows: A barber shaves everybody who does not shave himself. Does the barber shave himself? Either answer to this question (yes or no) leads to a contradiction. This barber, like Russell's set can be logically defined, but cannot exist. For the next 25 years mathematicians where trying to develop a self-consistent mathematical logic, free from paradoxes of this type. But in 1931, Gödel proved that it is not possible, formal logic was inexorably inconsistent and self-contradictory [23].

Belief in logic has deep psychological roots related to functioning of the human mind. A major part of any perception and cognition, illogical Aristotelian process involving forms-as-potentialities is not accessible to consciousness. We are conscious about the 'final states' of these processes, crisp forms-as-actualities which are perceived by our minds as 'concepts' approximately obeying formal logic. For this reason prominent mathematicians believed in logic. Even after Gödelian proof, founders of artificial intelligence still insisted that logic is sufficient to explain how the mind works.

3 Cognition, Logic, and Complexity

Object perception involves signals from sensory organs and mental representations of objects. During perception, the mind associates subsets of signals corresponding to objects with object representations in the memory. This recognition activates brain signals leading to mental and behavioral responses.

Mathematical models of this *recognition* step in this seemingly simple association-recognition-understanding process has not been easy, a number of difficulties have been encountered over the last sixty years. These difficulties were summarized under the notion of combinatorial complexity, CC, [70]. CC refers to multiple combinations of various elements in a complex system; for example, recognition of an object usually requires concurrent recognition of the multiple elements of the scene that could be encountered in various combinations. CC is prohibitive because the number of combinations is very large: for example, the number of combinations of 100 elements (not too large a number) is 100^{100}, exceeding the number of all elementary particle events in the life of the Universe; no computer would ever be able to compute that many combinations.

It has been proven that CC is mathematically similar to the Gödelian incompleteness. Therefore using logic leads to CC. But logic is used by all popular mathematical approaches to developing "cognitive algorithms" and modeling the mind, even fuzzy logic and neural networks, approaches specifically designed to overcome limitations of logic, still use logic in their training or learning procedures, every training example is a separate logical statement (e.g. "this is food"). CC of engineering algorithms and mathematical approaches to theories of the mind is related to the fundamental inconsistency of logic.

4 Fundamental Mechanisms of the Mind

Concepts are representations of the world events in the mind. According to [40], they are the contents of pure reason. They model events or simulate them. For this reason [7] calls them "simulators", [35] prefers to use the word *model*.

Instincts are sensory-like neural mechanism measuring vital bodily parameters. When these parameters are outside safe bounds, the instinctual mechanism sends evaluative neural signals to decision-making parts of the brain [26].

Emotions refer to many different mechanisms in the mind and body. According to [26] emotional signals evaluate concepts for the purpose of instinct satisfaction. Recognition of objects or situations that can potentially satisfy vital needs of the organism (instincts) receive preferential attention. A number of neural and physiological realizations of this mechanism have been identified [22].

Aesthetic emotions [41] emphasized the role of emotions in learning: aesthetic emotions are related to learning; they are judgments about correspondence between an object-event and its concept, which today we relate to interactions between BU and TD neural signals. Grossberg and Levine [26] theory of instinctual drives and emotions has been developed along Kantian ideas by introducing KI [71, 74, 75, 77, 98]. KI is an instinct measuring similarities between object-events and their concepts. Satisfactions or dissatisfactions of KI are indicated by aesthetic emotions [92, 93] that drive improvement of concepts in correspondence with experience. KI and aesthetic emotions are the foundations of all human higher cognitive abilities [58, 72, 82, 92], including emotions of the beautiful.

Emotions evaluating satisfaction or dissatisfaction of the knowledge instinct are not directly related to bodily needs. Therefore, they are 'spiritual' or aesthetic emotions. I would like to emphasize that aesthetic emotions are not peculiar to perception of art; they are inseparable from every act of perception and cognition. In the next sections we describe a mathematical theory of conceptual-emotional recognition and understanding. As we discuss, in addition to concepts and emotions, it involves mechanisms of intuition, imagination, conscious, and unconscious. This process is intimately connected to an ability of the mind to think, to operate with symbols and signs. The mind involves a hierarchy of multiple layers of concept-models, from simple perceptual elements to concept-models of objects, to relationships among objects, to complex scenes, and up the hierarchy… toward the concept-models of the meaning of life and purpose of our existence. Hence the tremendous complexity of the mind, yet relatively few fundamental principles of the mind organization go a long way explaining this system. The mind is not a kludge [84].

5 Physics of the Mind

Physics of the mind is different from psychology in that it searches for the fundamental principles of the mind. Among these principles are mechanisms of dynamic logic, concept-models, emotions, the knowledge instinct, and aesthetic emotions discussed above. Few more fundamental principles will be gradually introduced and discussed. These fundamental mechanisms are organized in a multi-layer, hierarchical system [25, 71]. The mind is not a strict hierarchy; there are multiple feedback connections among several adjacent layers, this approximately-hierarchical organization is another fundamental principle. At each layer there are concept-models encapsulating the mind's knowledge; they generate TD signals interacting with BU signals; the interaction between TD and BU neural signals is a fundamental principle constituting the essence of thinking and cognition. These interactions are governed by the knowledge instinct, which drives concept-model learning, adaptation, and formation of new concept-models for better correspondence to experience.

I begin with describing a basic mechanism of interaction between TD and BU signals between two adjacent hierarchical layers. At each layer, output signals are concepts recognized in (or formed from) BU signals. BU signals are associated with (or recognized, or grouped into) concepts according to the models and the knowledge instinct at this layer. This general structure of the theory of the mind corresponds to our knowledge of neural structures in the brain; although it is a physical intuition of the general principle, not necessarily one-to-one mapping to actual neural connections. How actual brain neuronal connection "implement" models and the knowledge instinct is an area of active research. The knowledge instinct is modeled mathematically as maximization of a similarity measure between bottom-up and top-down signals. In the process of learning and understanding input signals, models are adapted for better representation of the input signals so that similarity between the models and signals increases. This increase in similarity satisfies the knowledge instinct and is felt as aesthetic emotions.

5.1 The Knowledge Instinct

At a particular hierarchical layer, we enumerate neurons by index $\mathrm{n} = 1, \ldots \mathrm{N}$. These neurons receive BU signals, $\mathbf{X}(\mathrm{n})$, from lower layers in the processing hierarchy. $\mathbf{X}(\mathrm{n})$ is a field of BU neuronal synapse activations, coming from neurons at a lower layer. Each neuron has a number of synapses; for generality, we describe each neuron activation as a set of numbers, $\mathbf{X}(\mathrm{n}) = \{\mathrm{X_d(n)}, \mathrm{d} = 1, \ldots \mathrm{D}\}$. TD, or priming signals to these neurons are sent by concept-models, $\mathbf{M}_\mathrm{m}(\mathbf{S}_\mathrm{m}, \mathrm{n})$; we enumerate models by index $\mathrm{m} = 1, \ldots \mathrm{H}$. Each model is characterized by its parameters, $\mathbf{S}_\mathrm{m}$; in the neuron structure of the brain they are encoded by strength of synaptic connections, mathematically, we describe them as a set of numbers, $\mathbf{S}_\mathrm{m} = \{\mathrm{S^a}_\mathrm{m}, \mathrm{a} = 1, \ldots \mathrm{A}\}$. Models *represent* signals in the following way. Say, signal $\mathbf{X}(\mathrm{n})$, is coming from sensory neurons activated by object *m*, characterized by parameters $\mathbf{S}_\mathrm{m}$. These parameters may include position, orientation, or lighting of an object *m*. Model $\mathbf{M}_\mathrm{m}(\mathbf{S}_\mathrm{m}, \mathrm{n})$ predicts a value $\mathbf{X}(\mathrm{n})$ of a signal at neuron *n*. For example, during visual perception, a neuron *n* in the visual cortex receives a signal $\mathbf{X}(\mathrm{n})$ from retina and a priming signal $\mathbf{M}_\mathrm{m}(\mathbf{S}_\mathrm{m}, \mathrm{n})$ from an object-concept-model *m*. A neuron *n* is activated if both BU signal from lower-layer-input and TD priming signal are strong. Various models compete for evidence in the BU signals, while adapting their parameters for better match as described below. This is a simplified description of perception. The most benign everyday visual perception uses many layers from retina to object perception. Perception of minute features, or everyday objects, or cognition of complex abstract concepts is due to the same mechanism described below. Perception and cognition involve models and learning. In perception, models correspond to objects; in cognition models correspond to relationships, situations, and more abstract entities.

Learning is an essential part of perception and cognition, and it is driven by the knowledge instinct. It increases a similarity measure between the sets of models and signals, L({**X**},{**M**}). The similarity measure is a function of model parameters and associations between the BU and TD signals. For concreteness I refer here to an object perception using a simplified terminology, as if perception of objects in retinal signals occurs in a single layer.

In constructing a mathematical description of the similarity measure, it is important to acknowledge two principles. First, the exact content of the visual field is unknown before perception occurred. Important information could be contained in any BU signal; therefore, the similarity measure is constructed so that it accounts for all input information, **X**(n),

$$\mathrm{L}(\{\mathbf{X}\},\{\mathbf{M}\}) = \prod_{n \in N} \mathrm{l}(\mathbf{X}(\mathrm{n})). \tag{1}$$

This expression contains a product of partial similarities, l (**X**(n)), over all BU signals; therefore it forces the mind to account for every signal (even if one term in the product is zero, the product is zero, the similarity is low and the knowledge instinct is not satisfied); this is a reflection of the first principle. Second, before perception occurs, the mind does not know which retinal neuron corresponds to which object. Therefore a partial similarity measure is constructed so that it treats each model as an alternative (a sum over models) for each input neuron signal. Its constituent elements are conditional partial similarities between signal **X**(n) and model $\mathbf{M}_{\mathrm{m}}$, l (**X**(n)|m). This measure is "conditional" on object m being present,[1] therefore, when combining these quantities into the overall similarity measure, L, they are multiplied by r(m), which represent the measure of object m actually being present. Combining these elements with the two principles noted above, a similarity measure is constructed as follows:

$$\mathrm{L}(\{\mathbf{X}\},\{\mathbf{M}\}) = \prod_{n \in N} \sum_{m \in M} \mathrm{r(m)l}(\mathbf{X}(\mathrm{n})|\mathrm{m}). \tag{2}$$

The structure of (2) follows standard principles of the probability theory: a summation is taken over alternatives, *m*, and various pieces of evidence, *n*, are multiplied. This expression is not necessarily a probability, but it has a probabilistic structure. The name "conditional partial similarity" for l (**X**(n)|m) (or simply l (n|m)) follows the probabilistic terminology. If learning is successful, l(n|m) becomes a conditional probability density function, a probabilistic measure that signal in neuron *n* originated from object *m*. Coefficients r(m), called priors in probability theory, contain preliminary biases or expectations, expected objects *m* have relatively high r(m) values; their true values are usually unknown and should be learned, like other parameters

[1] Mathematically, the condition that the object m is present with 100 % certainty, is expressed by normalization condition: $\int \mathrm{l(X|m)dX} = 1$. We should also mention another normalization condition: $\int \mathrm{l(X(n))dX(n)} = 1$, which expresses the fact that, if a signal is received, some object or objects are present with 100 % certainty.

$\mathbf{S}_{\mathrm{h}}$. If learning is successful, L approximates a total likelihood of observing signals $\{\mathbf{X}(\mathrm{n})\}$ coming from objects described by models $\{\mathbf{M}_{\mathrm{m}}\}$ and leads to near-optimal Bayesian decisions.

Note. In the probability theory, a product of probabilities usually assumes that evidence is independent. Expression (2) contains a product over n, but it does not assume independence among various signals $\mathbf{X}(\mathrm{n})$. There is a dependence among signals due to models: each model $\mathbf{M}_{\mathrm{m}}(\mathbf{S}_{\mathrm{m}}, \mathrm{n})$ predicts expected signal values in many neurons n.

During the learning process, concept-models are constantly modified. From time to time a system forms a new concept, while retaining an old one as well; alternatively, old concepts are sometimes merged or eliminated. Even so functional forms of models, $\mathbf{M}_{\mathrm{m}}(\mathbf{S}_{\mathrm{m}}, \mathrm{n})$, are fixed and learning-adaptation involves only model parameters, $\mathbf{S}_{\mathrm{m}}$, still structural variation of models can be achieved as we discuss below [73, 74]. Formation of new concepts and merging or elimination-forgetting of old ones require a modification of the similarity measure (2); the reason is that more models always result in a better fit between the models and data. This is a well known phenomenon, a similarity measure fit to the data is biased toward a larger value. To obtain an unbiased estimation the similarity (2) should be reduced by using a "penalty function," p(N,M) that grows with the number of models M, and this growth is steeper for a smaller amount of data N. For example, an asymptotically unbiased maximum likelihood estimation leads to multiplicative $\mathrm{p(N, M)} = \exp(-\mathrm{N}_{\mathrm{par}}/2)$, where $\mathrm{N}_{\mathrm{par}}$ is a total number of adaptive parameters in all models (using this penalty function is known as Akaike Information Criterion, see [74] for further discussion and references).

5.2 Dynamic Logic

The learning process consists in estimating model parameters $\mathbf{S}$ and associating signals with concepts by maximizing the similarity (2). Note, all possible combinations of signals and models are accounted for in expression (2). This can be seen by expanding a sum in (2), and multiplying all the terms; it would result in M^{N} items, a huge number. This is the number of combinations between all signals (N) and all models (M). Here is the source of CC of many popular algorithms. For example, a popular multiple hypothesis testing algorithm [106] attempts to maximize similarity L over model parameters and associations between signals and models, in two steps. First it takes one of the M^{N} items, that is one particular association between signals and models; and maximizes it over model parameters. Second, the largest item is selected (that is the best association for the best set of parameters). Such a program inevitably faces a wall of CC, the number of computations on the order of M^{N}.

Our theory solves this problem by using dynamic logic [68, 71, 74]. An important aspect of dynamic logic is matching vagueness or fuzziness of similarity measures to the uncertainty of models. Initially, parameter values are not known, and uncertainty of models is high; so is the fuzziness of the similarity measures. In the process of learning, models become more accurate and the similarity measure more crisp, the value of the similarity increases. This is the mechanism of dynamic logic in which vague models-representations evolve into crisp models-representations. This vague-to-crisp process is a mathematical description of the Aristotelian theory of the mind, in which illogical "forms-as-potentialities" "meet matter" and turn into logical "forms-as-actualities" [1].

Mathematically it is described as follows. First, assign any values to unknown parameters, $\{\mathbf{S}_{\mathrm{m}}\}$. Then, compute association variables f(m|n),

$$\mathrm{f(m|n)} = \mathrm{r(m)}\,\mathrm{l}\,(\mathbf{X}(\mathrm{n})|\mathrm{m}) / \sum_{m' \in M} \mathrm{r(m')l}(\mathbf{X}(\mathrm{n})|\mathrm{m'}). \tag{3}$$

Equation (3) looks like the Bayes formula for a posteriori probabilities; if l (n|m) in the result of learning become conditional likelihoods, f(m|n) become Bayesian probabilities for signal *n* originating from object *m*. The next step defines a joint dynamics of the association variables and model parameters,

$$\mathrm{df(m|n)/dt} = \mathrm{f(m|n)} \sum_{m' \in M} [\delta_{\mathrm{mm'}} - \mathrm{f(m'|n)}] \cdot [\partial \mathrm{lnl(n|m')} / \partial \mathbf{M}_{\mathrm{m'}}] \partial \mathbf{M}_{\mathrm{m'}} / \partial \mathbf{S}_{\mathrm{m'}} \cdot \mathrm{d}\mathbf{S}_{\mathrm{m'}}/\mathrm{dt}, \tag{4}$$

$$\mathrm{d}\mathbf{S}_{\mathrm{m}}/\mathrm{dt} = \sum_{n \in N} \mathrm{f(m|n)}[\partial \mathrm{lnl(n|m)} / \partial \mathbf{M}_{\mathrm{m}}] \partial \mathbf{M}_{\mathrm{m}} / \partial \mathbf{S}_{\mathrm{m}}, \tag{5}$$

here

$$\underset{\mathrm{mm'}}{\delta} \text{ is 1 if } \mathrm{m} = \mathrm{m'},\ \text{0 otherwise.} \tag{6}$$

The following theorem was proven [71].

Theorem *Equations (3) through (6) define a convergent dynamic system with stationary states defined by* $\max_{\{\mathrm{Sm}\}}\mathrm{L}$.

It follows that the stationary states of the system are the maximum similarity states satisfying the knowledge instinct. When partial similarities are specified as probability density functions (pdf), or likelihoods, the stationary values of parameters $\{\mathbf{S}_{\mathrm{m}}\}$ are asymptotically unbiased and efficient estimates of these parameters [15]. A computational complexity of the MF method is linear in N.

In plain English, this means that dynamic logic is a convergent process. It converges to the maximum of similarity, and therefore satisfies the knowledge instinct. Several aspects of this convergence are discussed in the next section. If likelihood is used as similarity, parameter values are estimated efficiently (that is, in most cases, parameters cannot be better learned using any other procedure). Moreover, as a part of the above theorem, it is proven that the similarity measure increases at each iteration (until the system receives new data). The psychological interpretation is that the knowledge instinct is satisfied at each step: a modeling field system with dynamic logic *enjoys* learning.

5.3 Example of Dynamic Logic Operations

Finding patterns below noise could be an exceedingly complex problem. If an exact pattern shape is not known and depends on unknown parameters, these parameters should be found by fitting the pattern model to the data. However, when location and orientation of patterns are not known, it is not clear which subset of the data points should be selected for fitting. A standard approach for solving this kind of problems, as discussed, multiple hypothesis testing [106] tries all combinations of subsets and models, it faces combinatorial complexity. In this example, we are looking for 'smile' and 'frown' patterns in noise shown in Fig. 1a without noise, and in Fig. 1b with noise, as actually measured. The image size in this example is 100 × 100 points (N = 10,000). The true number of patterns is 3 plus 1 for noise. The number of patterns is not known, therefore, at least 4 patterns +1 noise (= 5) should be fit to the data, to decide that 3 patterns fit best. This yields an incomputable combinatorial complexity $\mathrm{M}^{\mathrm{N}} = 5^{10,000} = 10^{3,000}$.

Nevertheless, this problem unsolvable due to CC becomes solvable using dynamic logic, as illustrated in Fig. 1: (c) illustrates initial vague dynamic logic state, corresponding to randomly selected parameter values; on the first iteration, (d), vagueness is somewhat reduced; (d) through (h) show improved models at various iteration stages (total of 22 iterations). Between iterations (d) and (e) the algorithm decided, that it needs three Gaussian models for the 'best' fit. There are several types of models: one uniform model describing noise (it is not shown) and a variable number of blob models and parabolic models, which number, location and curvature are estimated from the data. Until about stage (g) the algorithm used simple blob models, at (g) and beyond, the algorithm decided that it needs more complex parabolic models to describe the data. Iterations stopped at (h), when similarity stopped increasing.

5.4 The Mind Hierarchical Organization

The previous sub-sections described a single processing layer of a cognitive mind system. At each layer of a hierarchy there are BU signals from lower layers, models, similarity measures (2), aesthetic emotions, which psychologically measure satisfaction of KI, and mathematically are changes in similarity (2), and actions; actions include adaptation, behavior satisfying the knowledge instinct—maximization of similarity, equations (3) through (6). An input to each layer, the BU signals are a set of signals $\mathbf{X}(\mathrm{n})$, or in neural terminology, an input field of neuronal activations. The result of learning at a given layer are activated models, or concepts *m* recognized in the BU signals *n*; these models along with the corresponding instinctual signals and emotions may activate behavioral models and generate behavior at this layer.

The activated models initiate other actions. They serve as input signals to the next, higher processing layer, where more general concept-models are recognized

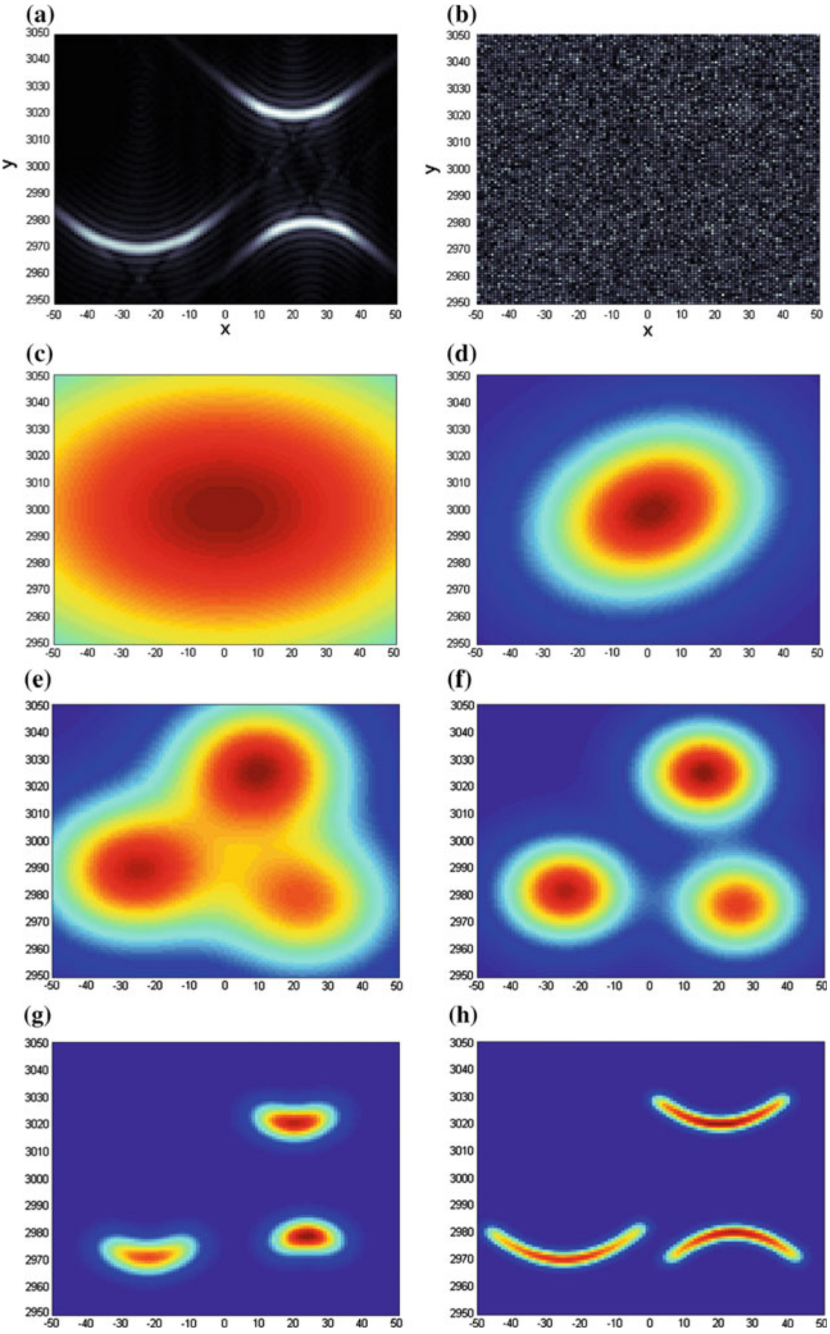

Fig. 1 (Continued)

◀**Fig. 1** Finding 'smile' and 'frown' patterns in noise, an example of dynamic logic operation: **a** true 'smile' and 'frown' patterns are shown without noise; **b** actual image available for recognition (signal is below noise, signal-to-noise ratio is between –2 dB and –0.7 dB); **c** an initial fuzzy blob-model, the fuzziness corresponds to uncertainty of knowledge; **d** through **h** show improved models at various iteration stages (total of 22 iterations). Between stages **d** and **e** the algorithm tried to fit the data with more than one model and decided, that it needs three blob-models to 'understand' the content of the data. There are several types of models: one uniform model describing noise (it is not shown) and a variable number of blob-models and parabolic models, which number, location and curvature are estimated from the data. Until about stage **g** the algorithm 'thought' in terms of simple blob models, at (**g**) and beyond, the algorithm decided that it needs more complex parabolic models to describe the data. Iterations stopped at (**h**), when similarity (2) stopped increasing. This example is discussed in more details in [93]

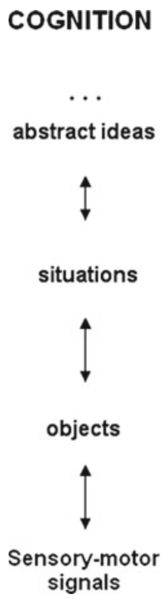

Fig. 2 The hierarchical cognitive system. At each layer of a hierarchy there are models, similarity measures, and actions (including adaptation, maximizing the knowledge instinct—similarity). High levels of partial similarity measures correspond to concepts recognized at a given layer. Concept activations are output signals at this layer and they become BU input signals to the next layer, propagating knowledge up the hierarchy

or created. Output signals from a given layer, serving as input, BU to the next layer, could be model activation signals, $\mathrm{a_m}$, defined as

$$\mathrm{a_m} = \sum_{n \in N} \mathrm{f(m|n)}. \tag{7}$$

In addition, output signals may include model parameters. The hierarchical cognitive system is illustrated in Fig. 2. Within the hierarchy of cognition, each concept-model finds its “mental” meaning and purpose at a higher layer (in addition to other purposes). For example, consider a concept-model “chair.” It has a “behavioral” purpose of initiating sitting behavior (if sitting is required by the body), this is the “bodily” purpose at the same hierarchical layer. In addition, it has a “purely mental” purpose at a higher layer in the hierarchy, a purpose of helping to recognize a more general concept, say of a "concert hall," which model contains rows of chairs.

A mathematical formulation of the hierarchical similarity is a product of similarity measures at every layer. We denote similarity (1) at the layer h as L(h). Then, the total similarity is

$$\mathrm{L} = \prod_{h \in H} \mathrm{L(h)}. \tag{8}$$

In this total hierarchical similarity, concepts formed at the level h become BU signals for the layer h+1, whereas concepts formed at the level h+1 become TD signals for the layer h. Whereas conditional similarities l (n|m) in Eq. (2), used in the example of Fig. 1 were designed for this example using specific functional shapes corresponding to image patterns, now a uniform conditional similarity is needed, suitable for every hierarchical level and capable to be learned for all current and future concepts. This new conditional similarity is based on the idea that higher-level concepts are formed from subsets of the BU signals. For simplicity of notations for now we leave out the index h. At every level in the hierarchy, each higher-level concept-model m is characterized by parameters-probabilities, $\mathbf{p}_\mathrm{m} = (\mathrm{p_{m1}}, .. \mathrm{p_{mi}}, ...)$, where $\mathrm{p_{mi}}$ is the probability of BU signal $\mathrm{x_{ni}}$ being part of the higher model m. The model-conditional similarity measures, l (n|m), are defined as follows,

$$\mathrm{l(n|m)} = \prod_{i=1} \mathrm{p_{mi}}^{\mathrm{x_{ni}}} (1 - \mathrm{p_{mi}})^{(1-\mathrm{x_{ni}})}. \tag{9}$$

Applying the dynamic logic learning equation we obtain a surprisingly simple learning-estimation equation for the model parameters, $\mathrm{p_{mi}}$,

$$\mathrm{p_{mi}} = \sum_{n \in N} \mathrm{f(m|n) x_{ni}} / \sum_{n' \in N} \mathrm{f(m|n')}. \tag{10}$$

The dynamic logic learning converges in few iterations [97]. Equations (8, 9, 10) complete the definition of the hierarchical system of the mind.

Models at higher layers in the hierarchy are defined as subsets of lower-level models and therefore more general than models at lower layers. For example, at the very bottom of the hierarchy, if we consider vision system, models correspond (roughly speaking) to retinal ganglion cells and perform similar functions; they detect simple features in the visual field; at higher layers, models correspond to functions performed at V1 and higher up in the visual cortex, that is detection of more complex features, such as contrast edges, their directions, elementary moves, etc. Visual hierarchical structure and models are studied in details ([24, 111]). At still higher cognitive layers, models correspond to objects, to relationships among objects, to situations, and relationships among situations, etc. [71, 97]. Still higher up are even more general models of complex cultural notions and relationships, like family, love, friendship, and abstract concepts, like law, rationality, etc. Contents of these models correspond to cultural wealth of knowledge, including writings of Shakespeare and Tolstoy.

Let me repeat that concept-models at every layer evolved in evolution with the purpose to unify lower-level concepts. This is the fundamental organizational principle of the hierarchy that helps us to understand contents and meanings of the concepts near the top of the hierarchy. These "top" concepts evolve with the purpose to unify the entire life experience. We perceive these concepts as the meaning of life. This clarifies Kantian conclusion that at the top of the hierarchy of the mind, there are concept-models of the meaning and purpose of our existence, unifying our knowledge, and the corresponding behavioral models aimed at achieving this meaning.

We discussed that satisfaction of the knowledge instinct is accompanied with aesthetic emotions. At lower levels of the hierarchy, say below objects, these aesthetic emotions usually are below the threshold of consciousness. Near the top of the hierarchy, where concepts address meanings important for life, aesthetic emotions.

6 Higher Cognition, Beautiful, and the Dual Hierarchy

We discussed that satisfaction of the knowledge instinct is accompanied by aesthetic emotions. At lower levels of the hierarchy, say below objects, these aesthetic emotions usually are below the threshold of consciousness. We are not getting emotionally excited when we understand an everyday object, say a refrigerator. Near the top of the hierarchy, where concepts address meanings important for life, aesthetic emotions acquire similarly important meanings. Aesthetic emotions felt when understanding the meaning and purpose of life are emotions of the beautiful [83]. The essential aspects of this theory of the beautiful follows the Kantian aesthetics (1790), the following mathematical theory helps understanding the details that have escaped Kantian analysis.

Can this mathematical analysis be used for exactly elucidating what is the meaning of life and emotions of the beautiful? The answer is unexpectedly negative: there cannot be a clear-cut prescription for what the meaning of life is and what the beautiful is. This negative answer is not entirely unexpected, it corresponds to our intuitions about these highest concepts and emotions. And yet, it calls for clarification of how it is possible that mathematical descriptions leads to denial of clear-cut definitions. To say it more exactly, the mathematical analysis clarifies the meanings of these highest concepts and emotions [92], as well as the corresponding meaning of "exactness."

Let us examine more closely exactness of perception of everyday objects. Look at an object in front of you, then close eyes and imagine this object with closed eyes. Imaginations are not as exact as perception of objects with opened eyes, it is impossible to recollect all the details with closed eyes. We know the mechanism of visual imagination: imagination, say of an object, is experiencing a neural projections of the object concept-model-representation onto the visual cortex. Vagueness of imaginations testifies to the vagueness of concept-representations. This fact has been proven in [6, 80]. Moreover this publication demonstrated that vague states and processes in the mind are also less accessible to consciousness.

If representations of the everyday objects are vague and not completely conscious, what does it say about representations near the top of the hierarchy. These "top" representations built on multiple layers of representations throughout the hierarchy must be much vaguer and much less conscious. But does not it contradict our ability to discuss in details the meaning of life and what is beautiful? To answer this question we have to analyze the difference between cognition and language, and the function of language in cognition.

6.1 Language and Cognition in Thinking

Do we think with language, or is language just a communication device used for expression of completed thoughts? What is a difference between language and cognition? Chomsky [12] suggested that these two abilities are separate and independent. Cognitive linguistics emphasizes a single mechanism for both [16]. Evolutionary linguistics considers the process of transferring language from one generation to the next one [11, 13, 34]. This process is a "bottleneck" that forms the language. Brighton, Smith, and Kirby [10] demonstrated emergence of compositional language due to this bottleneck. Still, none of these approaches resulted in a computational theory explaining how humans acquire language and cognition. Below I discuss a computational model overcoming previous difficulties and unifying language and cognition as two separate and closely integrated abilities. I identify their functions and discuss why human thinking ability requires both language and cognition. The fundamental difference between language and cognition is that language does not directly interacts with the world, language is learned from culturally evolved surrounding language. Cognition interacts with the world and is learned from experience in real world under the guidance of language.

We have discussed the difficulty of mathematical models of cognition, it is related to a need to consider combinations of sensor signals, objects, and events. The number of combinations is very large and even a limited number of signals or objects form a very large number of combinations, exceeding all interactions of all elementary particles in a lifetime of the Universe, a combinatorial complexity, CC. This difficulty in modeling the mind has been overcome by dynamic logic. Whereas classical logic considers static statements such as "this is a chair," dynamic logic models processes from vague to crisp representations. These processes do not need to consider combinations, an initial vague state of a "chair" matches any object in the field of view, and at the end of the process it matches the chair actually present, without CC.

Yet, there remains another difficulty, similar still even more complex. It is related to the fact that "events" and "situations" in the world do not exist "ready for cognition." There are many combinations of percepts and objects, a near infinity, events and situations important for understanding and learning have to be separated from those that are just random collections of meaningless percepts or random objects [97].

Events and situations recognized by non-human animals are very limited compared to human abilities to differentiate events in the world. Human cognitive abilities acquire their power due to language. Language is "easier" to learn than cognitive representations for the following reason, language representations: words, phrases exist in the surrounding language "ready-made," created during millennia of cultural evolution. Therefore language could be learned without much real-life experience; only interactions with language speakers are required. Every child learns language early in life before acquiring full cognitive understanding of events and their cognitive meanings. Thus language is learned early in life with only limited cognitive understanding of the world [78, 87, 91]. Cognitive representations of situations and abstract concepts are vague. Throughout the rest of life, language guides acquisition of cognitive representations from experience. Vague cognitive representations become more crisp and concrete. Thinking involves both language and cognition, and as we discuss later thinking about abstract ideas usually involves language more than cognition, not too different from thinking by children.

6.2 The Dual Hierarchy

A mathematical description of this interacting language-cognition system, connecting language, world and cognition, requires the dual hierarchy illustrated in Fig. 3. Cognitive hierarchy from sensor-motor percepts near "bottom," to objects "higher up," to situations, and to still more abstract cognitive representations have been illustrated in Fig. 2 and mathematically modeled in the previous sections. Language representations are organized in a parallel hierarchy from sounds, and words for objects and situations, to phrases, and to more abstract language representations. Our previous discussion is mathematically modeled as a dual hierarchy [78, 87, 91, 97] illustrated in Fig. 3.

Language is learned from the surrounding language, where it exists "ready-made" for cognition, and therefore after 5 or 7 language representations are crisp and conscious throughout the hierarchy. But cognitive representations, as discussed are vague and less conscious. For this reason abstract concepts are mostly known to us through language. This explains why concepts at the top of the hierarchy, such as meaning of life, and the corresponding emotions of the beautiful can be discussed in great details, still their exact contents are not known.

Hierarchical organization of cognition and related brain structures are reviewed in [4]. In particular, anterior-posterior axis corresponds to a gradient of abstract-concrete cortex functions. Hierarchical organization of language functions is also well established. However, hierarchical organization of language does not correspond to a particular spatial axis in the brain, it is distributed [100]. Therefore, the dual hierarchy in Fig. 3 is a functional hierarchy not organized along a spatial axis in the brain as in this figure. A fundamental aspect of acquiring language is interaction with the surrounding language as well as between BU and TD representations. Language learning is grounded in experience with the surrounding language. Learning

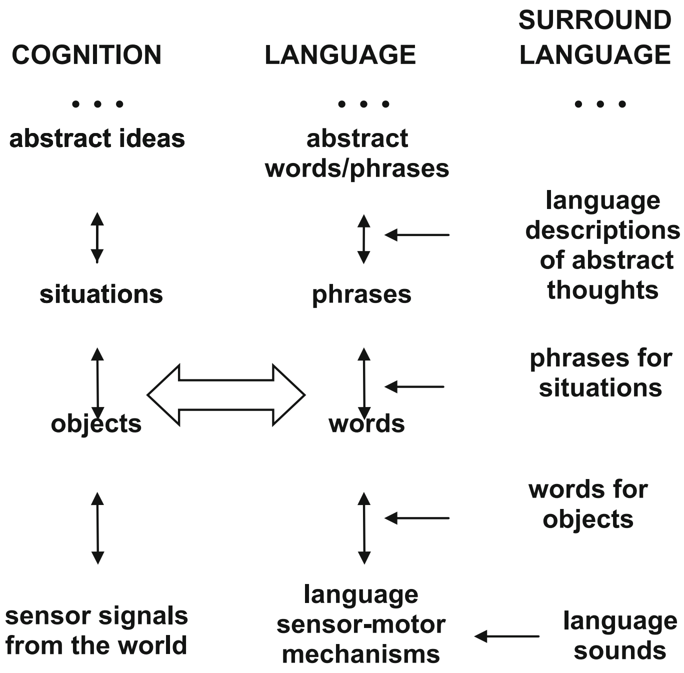

Fig. 3 The dual hierarchy. Language and cognition are organized into approximate dual hierarchy. Learning language is grounded in the surrounding language throughout the hierarchy. Cognitive hierarchy is grounded in experience only at the very *bottom*

of cognitive mental representations is directly grounded in experience only at the very bottom of the hierarchy; the rest of the hierarchy is located inside the brain. But in addition to interactions between BU and TD signals (indicated in Fig. 3 by vertical arrows), cognition interacts with language representations, and learning cognition is grounded in experience guided by language. In this interaction a lower layer representations are organized in more abstract and general concept-representations at a higher layer.

The dual model makes a number of experimentally testable predictions, and some of these have been confirmed [78, 87, 91, 95, 97]. It explains functions of language and cognition in thinking: cognitive representations model surrounding world, relations between objects, events, and abstract concepts. Language stores culturally accumulated knowledge about the world, yet language is not directly connected to objects, events, and situations in the world. Language guides acquisition of cognitive representations from random percepts and experiences, according to what is considered worth learning and understanding in culture. Events that are not described in language are likely not even noticed or perceived in cognition. (2) Whereas language is acquired early in life, acquiring cognition takes a lifetime. The reason is that language representations exist in surrounding language "ready-made," acquisition of language requires only interaction with language speakers, but does not require much life experience. Cognition on the opposite requires life experience. (3) This is the reason why abstract words excite only language regions of brain, whereas concrete words excite also cognitive regions [8]. The dual model predicts that abstract concepts are often understood as word descriptions, but not in terms of objects, events, and relations among them. (4) This model explains why language is acquired early in life, whereas cognition takes a lifetime. It also explains why children can acquire the entire hierarchy of language including abstract words without experience necessary for understanding them. (5) Since dynamic logic is the basic mechanism for learning language and cognitive representations, the dual model suggests that language representations become crisp after language is learned (5–7 years of age), however cognitive representations may remain vague for much longer; the vagueness is exactly the meaning of "continuing learning", this takes longer for more abstract and less used concepts. (6) The dual model gives mathematical description of the recursion mechanism [97]. Whereas Hauser et al. [30] postulate that recursion is a fundamental mechanism in cognition and language, the dual model suggests that recursion is not fundamental, hierarchy is a mechanism of recursion.

(7) Another mystery of human-cognition, not addressed by cognitive or language theories, is basic human irrationality. This has been widely discussed and experimentally demonstrated following discoveries of Tversky and Kahneman [108], leading to the 2002 Nobel Prize. According to the dual hierarchy model, the "irrationality" originates from the dichotomy between cognition and language. Language is crisp and conscious while cognition might be vague and ignored when making decisions. Yet, collective wisdom accumulated in language may not be properly adapted to one's personal circumstances, and therefore be irrational in a concrete situation. In the 12th c. Maimonides wrote that Adam was expelled from paradise because he refused original thinking using his own cognitive models, but ate from the tree of knowledge and acquired collective wisdom of language [57].

6.3 Emotional Prosody and Its Cognitive Function

The language instinct drives the development of the language part of the dual hierarchy [99]. The knowledge instinct drives the development of the cognitive hierarchy and the language guidance of the cognition, indicated by a wide horizontal arrow in Fig. 3. These neural connections have to be developed and maintained. This requires motivation, in other words, emotions. These aesthetic emotions corresponding to the KI must be in addition to utilitarian meanings of words, otherwise only practically useful words would be connected to their cognitive meanings. Also these emotions must "flow" from language to cognition and they must act fast, so that language is able to perform its cognitive function of guiding acquisition of cognitive representations, organizing experience according to cultural contents of language. These emotions therefore must be contained in language sounds, before cognitive contents are acquired [90].

This requirement of emotionality of language sounds is surprising and contradictory to assumed direction of evolution of language. Evolution of the language ability required rewiring of human brain in the direction of freeing vocalization from uncontrollable emotions [19, 79]. Still, the dual model requires that language sounds be emotional. This emotionality is necessary to motivate connecting sounds and cognitive meanings. Emotionality of human voice is most pronounced in songs [81, 85, 88, 90]. Emotions of everyday speech are low, unless affectivity is specifically intended. We may not notice emotions in everyday "non-affective" speech. Nevertheless, this emotionality is important for developing the cognitive part of the dual model. If language is highly emotional, speakers are passionate about what they say, however evolving new meanings might be slow, emotional ties of sounds to old meanings might be "too strong." If language is low-emotional, new words are easy to create, however motivation to develop the cognitive part of the dual model might be low, the real-world meaning of language sound might be lost. Cultural values might be lost as well. Indeed languages differ in how strong are emotional connections between sounds and meanings. This leads to cultural differences. Thus the dual model leads to Emotional Sapir-Whorf Hypothesis [76, 79, 86]. Strength of emotional connections between sound and meaning depends on language inflections. In particular, after English lost most of its inflections, it became a low emotional language, powerful for science and engineering. At the same time English is losing autonomous connections to cultural values that used to be partially inherent in language sounds. Fast change of cultural values during recent past in English-speaking cultures is usually attributed to progress in thinking, whereas effects of change in emotionality of language sounds have not been noticed.

6.4 Musical Emotions

At lower levels KI acts automatically: sensory-motor experiences are directly embodied. But at higher levels abstract knowledge is called abstract exactly because it does not exist pre-formed in the world, it is created through the interaction of the world and the mind. But cognitive dissonance (CD), a mechanism opposite to KI, might interfere at higher levels. CD is a discomfort caused by holding conflicting cognitions [14, 21, 29]. This discomfort is usually resolved by devaluing or discarding the conflicting cognition. This discarding often occurs below the level of consciousness; it is fast and momentary [36]. It is also known that the majority of new knowledge originates through the differentiation of previous knowledge, which is the mechanism for several broad empirical laws: Zipf's law, the power law, Pareto law emerge when new entities (or usage) evolve from pre-existing ones [65, 66, 105]. Therefore, almost all knowledge contradicts other knowledge to some extent and according to CD theory, any knowledge should be discarded before its usefulness becomes established.

As language began emerging, every word brought new knowledge. However, CD should have interfered with this process. Before the usefulness of new knowledge could be established, it should have been discarded along with language [89]. To overcome CD and enable the evolution of language and knowledge, a word should "grab attention" preconsciously. Emotion of prosody accomplishes this [91]. This same mechanism is alive and well today. Over millions of years many musical and prosodial emotions have evolved and have been culturally inherited by everyone. Yet the diversity of culture is "too much" for some people, leading to new knowledge often being discarded in school and later. Recall that Einstein never received a Nobel Prize for Theory of Relativity. The emotions of language prosody embody the meanings of language and connect sounds to cognitive meanings in everyday conversations as well as in science (experimental evidence is discussed below). While language evolved toward more semantic and less emotional sounds, emotionality of voice, inherited from our animal past, evolved toward stronger emotions, to songs and music. And today song lyrics may affect us stronger than the same text without music.

Did music evolve to connect abstract thoughts and KI? This hypothesis has been confirmed experimentally. In [3] children devalued a toy if they could not play with it. The desire 'to have' contradicts the inability 'to attain'; this creates CD, which is resolved by discarding the contradiction. This experiment repeated many times [14] was first described in Aesop's fable 2500 years ago: the fox unable to attain the grape devalues a contradictory cognition by deciding: "the grape is sour."

Does music help in overcoming CD? [60, 85, 86] have reproduced the above experiment with music in the background and observed that the toy is *not* devalued. Another experiment demonstrated that academic test performance may improve while listening to music. Perlovsky and Cabanac [94] demonstrated (1) that students allocate *less* time to more difficult and stressful tests (as expected from CD theory), and (2) with music in the background students *can* tolerate stress, allocate *more* time to stressful tests, and improve grades. These experiments confirmed the hypothesis that music helps in overcoming CD. It is likely that music emerged and evolved for a fundamental cognitive function: music makes the accumulation of knowledge and human evolution possible.

For thousands of years philosophers and psychologists wondered about the origin of dissonances and consonances. Masataka and Perlovsky [61] demonstrated that consonant music helps "everyday" decision-making in the presence of cognitive interfering evidence, whereas dissonant music increases interference effects. Is music limited to a few emotions, or does every musical phrase evoke a different shade of emotion? Researchers take opposite sides of this issue [17, 38, 39, 44, 107, 112].

How can multiplicity of emotions be explained and justified from a cognitive and evolutionary standpoint, and why has this ability emerged? The proposed hypothesis relating music to CD suggests the following explanation. CD produces a variety of emotional discomforts, different emotions for every combination of knowledge—in other words, a huge number of emotions. Most of these emotions are barely noticed because they lie below the level of consciousness, and in these unconscious states they produce disincentives for knowledge. Music helps to overcome these emotional discomforts by developing a huge number of conscious musical emotions. The mind being conscious of the multiplicity of emotions can bring into consciousness emotions of CD, and thus be prepared to tolerate them. We enjoy even sad and difficult musical emotions for their positive effect of overcoming difficult CD. Possibly this explains the mysterious enjoyment of sad music: it helps us to overcome CD of life's unavoidable disappointments, including the ultimate one, the knowledge of our finiteness in the material world.

Melody, harmony, and other musical devices produce complex, uniquely human musical emotions—they are related to knowledge and therefore are aesthetic emotions ([9, 96]). They expand KI toward a differentiated instinct sensitive not only to unifying knowledge and the world, but also to unifying multiplicity of contradictions among various aspects of knowledge. While CD split our psyche into differentiated knowledge, KI and music unify our psyche. Musical emotions embody abstract knowledge and unify our mental life, language, and body.

These are the reasons why music affects us so strongly. Music connects thinking and intuition to the world. Our spiritual life is embodied through music. Uniquely human refined musical emotions embody our abstract thoughts from the everyday to the most exalted experience. Our highest mental representations near the top of the mental hierarchy attempt to unify our entire life experience. As discussed we perceive them as the meaning of life; their cognitive representations are vague but their feelings are strong, we feel them as emotions of the beautiful [82, 83]. These representations cannot be matched to anything "objectively existing" in the world outside of our brain-mind. Their deep meanings have been created in cultural evolution. Every individual human being receives this cultural knowledge through language. However, this cultural wisdom is not received in an embodied form. It might remain as meaningless disembodied text in books. It is up to everyone's personal effort to create an embodied meaning of life from one's own life experience. Music helps us to embody the meaning of life. The beautiful and sublime, art and religious experience, emotions that embody the meaning of life, as well as the highest spiritual experiences are all embodied through music.

For thousands of years music has been an unexplainable mystery. Aristotle [2] listed the power of music among the great unsolved problems. Darwin wrote (1871) that musical ability "must be ranked amongst the most mysterious with which (man) is endowed." Nature published a series of essays on music [20]. The authors of these essays agreed that "none... has yet been able to answer the fundamental question: why does music have such power over us?" [5]. Today with the help of the physical theory of the mind we have an answer to this question.

7 Future Mathematical Development. Monotone Boolean Functions

This section uses mathematical formalization of dynamic logic for further development of the foundation of the mathematical theory of the mind formulated above; here we name the above theory modeling field theory, MFT. This section is based on previous joint development with Prof. Boris Kovalerchuk [50–53, 109].

Empirical data, E is any data that can be used to identify a model. In logical terms it is defined as a pair

$$\mathbf{E} = <\mathrm{A}, \Omega>,$$

where A is a set of objects, $\Omega = \{P_i\}$ is a signature, that is a set of predicates P_i of arity n_i, e.g., predicate $P_1(x,y)$ with $n_i = 2$ can mean that length of x is no less than the length of y, $l(x) \geq l(y)$.

Definition The pair $<A, \Omega>$ is called an ***empirical system*** in Krantz et al. [55].

Definition A pair $<A, \Omega>$ is knows in logic as **a*model* (of the system of axioms T)** [59].

Alfred Tarski proposed the name ‘model theory’ in 1954 [32]. A variety of other names are also used that include a *relational system* Krantz et al. [55], a *protocol of experiment* ([47, 104]). To distinguish this model from model in MFT we will also call this model a **logic model** or first-order model [33]. Application of algebraic methodology to System Software is presented in [102].

The next important concept in MFT is a concept of **a priori model** (of reality), M. In logic formalization it can be matched with a **system of axioms** T.

Definition A **system of axioms** T is a set of closed first order logic (FOL) formulas (sentences) in the signature Ω.

This means that every variable x_i is presented in the formula with the existential (∃) or the universal (∀) quantifier, e.g., $\forall\, x_i \exists x_j\ P_1(x_i, x_j)$.

The quick comparison of MFT and logic approaches in these two concepts reveals a fundamental difference between them. Logic is going from a *very formal* (syntactical) axiomatic system T to something more real called a model $\mathbf{A}_T = <A_T, \Omega>$ of that formal system T. MFT is going in the opposite way—from *very informal* reality to more formal models. As a result the concepts of a model are quite different in two theories. Empirical data in MFT is a model $\mathbf{E} = <A, \Omega>$ in logic, if we interpret empirical data as an empirical system E [55]. On the other hand MFT model is not a model in logic it leans more to a set of axioms about the class of the logic models.

This difference was well described in Hodges [32]: “To *model* a phenomenon is to construct a formal theory that describes and explains it. In a closely related sense, you *model* a system or structure that you plan to build, by writing a description of it. These are very different senses of ‘model’ from that in model theory: the ‘model’ of the phenomenon or the system is not a structure but a theory, often in a formal language”.

The next MFT concept is a **similarity measure** L(M, E) between empirical data E and a priory model M that is assigned individually to each specific problem and E:

$$\mathrm{L}: \quad \{(\mathrm{M}, \mathrm{E})\} \rightarrow \quad \mathrm{R},$$

where r is a set or real numbers.

The closest concept in the logic for this concept is a statement that m = <a, Ω> is a model of the system of the axioms t.

Definition Pair e = <a, Ω> is a model of the system of the axioms t if every formula from t is true on e.

Definition Boolean similarity measure b(t, e) is defined to be equal to 1, l(t, e) = 1, if m is a model of t, else l(t, e) = 0.

7.1 Concept of Uncertainty, Generality and Simplicity

7.1.1 Uncertainty, Generality and Simplicity Relations Between Models

Below we introduce concepts of uncertainty, generality and simplicity relations. These concepts can be applied to both logic and MFT models.

Notation An **uncertainty relation between models** is denoted as "$\geq_{\mathrm{Mu}}$", relation $M_i \geq_{\mathrm{Mu}} M_j$ is read: "Model M_i is equal in uncertainty or *more uncertain* than model M_j" or "Model M_j is no less certain than model M_j" This relation is a partial order.

Notation A **generality relation between models** is denoted as "$\geq_{\mathrm{Mg}}$" and relation $M_i \geq_{\mathrm{Mg}} M_j$ is read: "Model M_j is a *specialization* of the measure M_i" or "Model $M_i$is a *generalization* of the measure M_j". This relation is a partial order too.

Notation A **simplicity relation between model** is denoted as "$\geq_{\mathrm{Ms}}$" and relation $M_i \geq_{\mathrm{Ms}} M_j$ is read: "Model M_i is equal in simplicity of *simpler* than Model M_j". This relation also is a partial order.

For the MFT models that are represented as a system of axioms the generality relation can be defined as follows.

Definition $T_i \geq_{gen} T_j$ if and only if $T_i \subset T_j$, i.e., system of axioms $T_i$is equal to or more general than the system of axioms T_j if and only if T_i contains less axioms than $T_{j,} T_i \subset T_j$.

7.1.2 Uncertainty, Generality and Simplicity Relations Between Similarity Measures

Below we introduce concepts of uncertainty, generality and simplicity relations for similarity measures. These concepts can be applied to both logic and MFT similarity measures.

Notation An **uncertainty relation** between similarity measures is denoted as "$\geq_{\mathrm{Lu}}$" and relation $L_i \geq_{\mathrm{Lu}} L_j$ is read: "Measure L_i is equal to in uncertainty or more uncertain than measure L_j". This is a partial order relation.

Notation A **generality relation between similarity measures** is denoted as "$\geq_{Lg}$" and relation $L_i \geq_{Lg} L_j$ is read: "Measure L_j is a *specialization* of measure L_i and measure $L_i$is a *generalization* of the measure L_j" This relation also is a partial order.

Notation A **simplicity relation between similarity measures** is denoted as "$\geq_{Ls}$" and relation $L_i \geq_{Ls} L_j$ is read: "Measure L_j is equal in simplicity or *simpler* than measure L_i". This relation also is a partial order.

Definition Mapping F between a set of models {M} and a set of similarity measures {L}

$$F : \{M\} \rightarrow \{L\}$$

is called a **match mapping** if F preserves uncertainty, generality and simplicity relations between models and measures in the form of homomorphism from relational system $<\{M\}, \geq_{Mg}, \geq_{Mu}>$ to relational system $<\{L\}, \geq_{Lg}, \geq_{Lu}>$, i.e.,

$$\forall M_a, M_b(M_a \geq_{Mg} M_{b\Rightarrow}F(M_a) \geq_{Lg} F(M_b)),$$

$$\forall M_a, M_b(M_a \geq_{Mu} M_{b\Rightarrow}F(M_a) \geq_{Lu} F(M_b)).$$

A homomorphism in contrast with an isomorphism allows several models to be mapped to the same similarity measure L.

Specific match mappings may need additional properties such as simplicity relation between models, that is if two models that are equal in generality and uncertainty the preference is given to a simpler one.

7.2 Partial Order of Models

Two different models can be at the same level of uncertainty ($M_1 =_u M_2$), one model can be more uncertain than another one ($M_1 >_u M_2$), or these models can be incomparable for uncertainty. Symbol "$\geq_u$" is also can be viewed as a disjunction of relations $>_u$ and $=_u$.

We may define *model uncertainty* in such way that two different quadratic models M_1: $2x^2$ +3y and model M_2: 5x+4y^2 will have the same level of uncertainty $M_1 =_u M_2$. The *number of unknown coefficients* is a possible ways to do this. For M_1 and M_2 these numbers m_1 and m_2 are equal to zero. All coefficients are known and thus both models are certain.

Definition *NUC measure of polynomial model uncertainty* is defined as the Number of Unknown Coefficients (NUC) in the model.

The generality relation between models M_1 and M_2 can also be defined. For instance, it can be the *highest power n* of the polynomial model. Both models M_1 and M_2 are quadratic with $n_1 = n_2 = 2$ and, thus, have the same generality.

Definition *HP measure of polynomial model generality* is defined as the *Highest Power n* of the polynomial model.

Alternatively we may look deeper and notice that M_1 contains x^2 and M_2 contains y^2. We may define the *generality* of a polynomial model as its *highest polynomial variable,* which are x^2 for M_1 and y^2 for M_2. We cannot say that one of them is more general and can call them incomparable in generality.

Definition *HPV measure of polynomial model generality* is defined as the *Highest Power Variable (HPV)* of the polynomial model.

Consider model M_3: $5x + by^2$. Using NUC measure this model is *more uncertain* than model M_2: $5x+4y^2$, $M_3 >_{Mu} M_2$, because coefficient b in M_3 is not known, that is NUC for M_2 is $n_2 = 0$ and NUC for M_3 is $n_3 = 1$ and $n_3 > n_2$.

We can also consider M_3 as *more general* than M_2: $5x+4y^2$, $M_3 >_{Mg} M_2$, because M_2 is a specialization of M_3 with $b = 4$. Similarly model M_4: $ax+cx^2+by^2$ is more general and uncertain than models M_1: $2x^2 +3y$, M_2: $5x+4y^2$ and M_3: M_3: $5x + by^2$, because all coefficients in M_4 are uncertain, but none of the coefficients is uncertain in M_1, M_2 and M_3. In these examples uncertainty and generality are consistent and it is hard to distinguish them. In the next section we provide an example with clear difference between them. To formalize this idea we need to introduce some concepts that also will be described in the next section.

7.3 Examples

7.3.1 Uncertainty and Generality of Polynomial Models

In this section we discuss uncertainty and generality of polynomial models based on a parameterization idea. At first we consider an example of models with increasing levels of uncertainty:

Level 0: $3x + 4y + 5y^2$. All coefficients are known at level 0 (no uncertainty).
Level 1: $ax + 4y + 5y^2$. One coefficient is unknown at level 1.
Level 2: $ax + by + 7y^2$. Two coefficients are unknown at level 2.

We may notice that models $3x + 4y + 5y^2$, $ax + 4y + 5y^2$and $ax+by+7y^2$ form a chain from a more specific model (level 0) to a less specific model (level 2) as well as from a more certain model to a more uncertain model.

In contrast models $3x + 4y + 5y^2$, $ax + 9y + 5y^2$and $ax + by + 7y^2$ form an *increasing uncertainty chain* by UNC measure, but they *do not form an increasing generality*. We cannot get $3x + 4y + 5y^2$ by specializing $ax+9y+5y^2$and $ax + by + 7y^2$, because y coefficients 4 and 9 are different. Similarly we cannot get $ax + 9y + 5y^2$ by specializing $ax + by + 7y^2$, because y^2 coefficients 5 and 7 are different. This example illustrates the difference between uncertainty and generality relations.

Another example provides us five models at five uncertainty levels:

Uncertainty level n = 4: $M_4 = ax^2 + by + cx + d$
Uncertainty level n = 3: $M_3 = ax^2 + by + 7x + 10$
Uncertainty level n = 2: $M_2 = ax^2 + 3y + 7x + 10$
Uncertainty level n = 1: $M_1 = x^2 + 3y + 7x + 10$
Uncertainty level n = 0: $M_0 = 9x^2 + 3y + 7x + 10$

Models M_4, M_3, M_2, M_1, M_0 form a *uncertainty decreasing chain* with UNC uncertainty relation defined above:

$$M_4 >_{Mu} M_3 >_{Mu} M_2 >_{Mu} M_1 >_{Mu} M_0$$

They also form a *generality decreasing chain*

$$M_4 >_{Mg} M_3 >_{Mg} M_2 >_{Mg} M_1 >_{Mg} M_0$$

Here model M_3 can be obtained by specialization of parameters of model M_4 and so on, but we did not define the generality concept for them formally yet. Below it is done by using parameterization approach.

Each considered model has 4 parameters, $p_1, p_2, p_{3,}$ and p_4. For instance, for model $M_2 = ax^2 + 3y + 7\ x + 10$ parameter $p_1 = 1$ represents uncertainty of ax^2, where coefficient a is unknown. Similarly, $p_2 = p_3 = p_4 = 0$, because further coefficients 3, 7 and 10 are known. In this notation we can represent each model as a Boolean vector, $\mathbf{v}_i = (v_{i1}, v_{i2}, .., v_{ik}, \ldots, v_{in})$:

$$M4 : v_4 = 1111;\ M3 : v_3 = 1110;\ M2 : v_2 = 1100;\ M1 : v_1 = 1000;\ M0 : v_0 = 0000.$$

Definition Parametric model M_i is *no less general* than model M_j if

$$\mathbf{v}_i \geq \mathbf{v}_j, \text{ i.e., } \forall\, k\ v_{ik} \geq v_{jk}.$$

In accordance with this definition we have

$$1111 \geq 1110 \geq 1100 \geq 1000 \geq 0000$$

that is isomorphic to $M_4 >_{Mg} M_3 >_{Mg} M_2 >_{Mg} M_1 >_{Mg} M_0$.

7.4 Uncertainty and Generality of Kernel Models

In this section we discuss uncertainty and generality of parametric kernel models. Consider a model that consists of n Gaussian kernels. Each kernel K_i has two parameters, p_{i1} and p_{i2} that are mean and standard deviation respectively (or covariation matrix in a multidimensional case). We define the following levels of uncertainty:

Level 0: All 2n coefficients p_{i1} and p_{i2} are known at level 0 (no uncertainty).
Level 1: Only one coefficient is unknown at level 1.
Level 2: Two coefficients are unknown at level 2.
Level 2n: All 2n coefficients are unknown at level 2n.

Consider level n with all $p_{i1}$are known and all $p_{i2}$are not known. Assume also that the maximum possible value $p_{2\max}$ of all p_{i2} is known. This value is considered as a priory value of $p_{i2}$for all i in the initial a priory model M_0. Assume that a learning operator C(M_0, E) produced a model M_1 that shrinks the standard deviation max value $p_{2\max}$, to smaller numbers $p_{i2\max}(M_1)$ for i = 1.

The **similarity measure** S(M_1, E) can be defined by kernel overlap. If kernels do not overlap in the 2 standard deviations, $\pm 2p_{i2\max}(M_1)$ then $S(M_1, E) = 1$, else $S(M_1, E) < 1$.

Now we can apply learning operator C(M_j, E) and produce a chain of models, where each model M_{j+1} is more specific then model M_j with decreasing parameters p_{i2}.

$$M_n >_{Mg} M_{Mn-1....}M_{j+1} >_{Mg} M_{j....}M_2 >_{Mg} M_1 >_{Mg} M_0.$$

Each considered model has 2n parameters, p_{11}, p_{12}, p_{21}, p_{22},..., p_{n1}, p_{n2}. We already assumed that n parameters p_{i1} are known. Now we encode known parameters as 1 and unknown as 0, thus we have for model M_0 a 2n-dimensional Boolean vector

$$\mathbf{v}_0 = (v_{01}, v_{02}, .., v_{0k}, \ldots, v_{0n}) = (1010....10)$$

In this notation we can represent each model for n = 3

$$M_3 : v_2 = 111111; M_2 : v_2 = 011111; M_1 : v_1 = 011110; M_0 : v_0 = 101010.$$

and

$$111111 \geq 011111 \geq 011110 \geq 101010.$$

that is consistent with $M_3 >_{Mg} M_2 >_{Mg} M_1 >_{Mg} M_0$.

A more detailed uncertainty parameterization can be developed if Boolean vectors are substituted by k-valued vectors $\mathbf{u}_i = (u_{i1}, u_{i2}, .., u_{ik}, \ldots, u_{in})$ with

$$u_{ij} \in U = \{0, 1/(k-1), 2/(k-1), \ldots k-2/(k-1), 1\}.$$

Definition Parametric model M_i is *no less general* than model M_j if

$$\mathbf{u}_i \geq \mathbf{u}_j, \text{ i.e., } \forall\, k\; v_{im} \geq v_{jm}.$$

Above we encoded known parameters as 1 and unknown as 0. Now we can assign a level of parameter uncertainty u_{i2} by computing $p_{i2}(Mj)/p_{i2max}$ and assigning u_{i2} as a nearest number from {0, 1/(k-1), 2/(k-1), …k-2/(k-1), 1}, e.g., let $p_{i2}(Mj)/p_{i2max} = 0.75$, but the nearest k-value if 0.8, then $u_{i2} = 0.8$.

7.5 *Similarity Maximization*

Now we can define **a similarity maximization** problem in MFT using the definition of the similarity measure provided above.

Definition A similarity L_{fin} measure is called a **final similarity measure** if

$$\forall M, E, L_i \;\; L_i(M, E)_{\geq Lu} L_{fin}(M, E)$$

The goal of setting up the final similarity measure is to set up the level of certainty of model similarity of the data that we want to reach.

Definition The **static model optimization problem (SMOP)** is to find a model M_a such that

$$L_{fin}(M_a, E) = \text{Max}_{\,i\in I} L_{fin}(M_i, E) \tag{11}$$

subject to conditions (12) and (13):

$$\forall M_j \, L_{fin}(M_a, E) = L_{fin}(M_j, E) \Rightarrow M_a \geq_{Mu} M_{j,} \tag{12}$$

$$\forall M_j((L_{fin}(M_a, E) = L_{fin}(M_j, E) \;\&\; ((M_j \geq_{Mg} M_a)v((M_a \geq \;_{Mg} M_j))) \Rightarrow M_a \geq_{Mg} M_j)) \tag{13}$$

The goal of conditions (12) and (13) is prevent model overfitting. In addition conditions (12) and (13) can be beneficial computationally if further specification of the model requires more computations.

Condition (12) means that if $\mathrm{M_a}$ and $\mathrm{M_j}$ have the same similarity measure with E, $\mathrm{L_{fin}(M_a, E) = L_j(M_j, E)}$, then uncertainty of $\mathrm{M_a}$ should be no less than uncertainty of $\mathrm{M_j}$. Say if model $\mathrm{M_a}$ has three uncertain coefficients and model $\mathrm{M_j}$ has only one uncertainty coefficient then we prefer model $\mathrm{M_a}$.

Condition (13) means that if $\mathrm{M_a}$ and $\mathrm{M_j}$have the same similarity measure with E, $\mathrm{L_a(M_a, E) = L_j(M_j, E)}$, and $\mathrm{M_j}$and $\mathrm{M_a}$ are comparable relative to generality relation "$\geq_{\mathrm{Mg}}$" then $\mathrm{M_a}$ should be no less general than $\mathrm{M_j}$, $\mathrm{M_j \geq_{Mu} M_a}$. This means that model $\mathrm{M_j}$ can be obtained by specification of model $\mathrm{M_a}$. Say if models $\mathrm{M_a}$ and $\mathrm{M_j}$ have all the same coefficients, but coefficient c, and c can be any number from [1, 3] in $\mathrm{M_a}$ then we would prefer this model to a model $\mathrm{M_j}$ with a more specific $c = 2.1$.

Definition The dynamic logic model optimization (DLPO) problem is to find a model $\mathrm{M_a}$ such that

$$\mathrm{L_a(M_a, E) = Max_{i \in I} L_i(M_i, E)} \tag{14}$$

subject to conditions (14) and (15):

$$\forall \mathrm{M_j}\, \mathrm{L_a(M_a, E) = L_j(M_j, E) \Rightarrow M_a \geq_{Mu} M_{j,}} \tag{15}$$

$$\forall \mathrm{M_j((L_{fin}(M_a, E){=}\ L_j(M_j, E)\ \&\ ((M_j \geq_{Mg} M_a) v((M_a \geq_{Mg} M_j))) \Rightarrow M_a \geq_{Mg} M_j))}. \tag{16}$$

This is a non-standard optimization problem. In standard optimization problems the optimization criterion L is static, which is given at the beginning of the optimization process and is not changed in the course of the optimization and does not depend on the model M_i to be optimized. Only models M_i are changed in the standard (static) optimization process:

$$\mathrm{Max_{i \in I} L(M_i, E)}. \tag{17}$$

In the dynamic logic model optimization problem the criterion L is changing dynamically with models M_i. MFT shows that this is an effective way to cut down computational (combinatorial) complexity of finding an optimal model. Since the focus of MFT approach is in cutting **computational complexity** (CC) of model optimization a **dual optimization problem** can be formulated.

Definition Mapping {M}→{M} is called a **learning (adaptation) operator** C,

$$\mathrm{C(M_i, E) = M_{i+1}},$$

where E are data and $\mathrm{M_i \geq_{Mu} M_{i+1}}$, $\mathrm{M_i \geq_{Mg} M_{i+1}}$, this operation represents a *cognitive learning process* c of a new model $\mathrm{m_{i+1}}$ from a given model $\mathrm{m_i}$ and data e. in other words it is an adaptation of model $\mathrm{m_i}$ to data e that produce model $\mathrm{m_{i+1}}$.

Definition An optimization problem of finding a **shortest sequence** of matched pairs (M_i, L_i) of models M_i and optimization criteria (similarity measures) L_i that **solves** the optimization problem (4)–(6) for a given data E is called a **dual dynamic logic model optimization (DDLMO)** problem, that is finding a sequence of n matching pairs

$$(M_1, L_1), (M_2, L_2), \ldots, (M_n, L_n)$$

such that

$$L(M_n, E) = Max_{i \in I} L(M_i, E).$$

$$\forall M_i L_i = F(M_i), C(M_i, E) = M_{i+1}, M_i \geq_{Mu} M_{i+1}, M_i \geq_{Mg} M_{i+1}, M_n = M_a, L_n = L_a$$

This means finding a sequence of more specific and certain models for given data E, matching operator F and learning operator C that maximizes L(M_i, E).

7.6 Monotonicity, Monotone Boolean and K—valued Functions

We consider a Boolean function f: $\{0,1\}^n \rightarrow \{0,1\}$.

Definition A Boolean function f is a **monotone Boolean function** if:

$$\forall \mathbf{v}_i \geq \mathbf{v}_j \,\& \;\Rightarrow f(\mathbf{v}_j) \;\geq f(\mathbf{v}_i).$$

This means that

$$\forall (\mathbf{v}_i \geq \mathbf{v}_j \,\& \, f(\mathbf{v}_i) = 0) \Rightarrow f(\mathbf{v}_j) = 0$$

$$\forall (\mathbf{v}_i \geq \mathbf{v}_j \,\& \, f(\mathbf{v}_j) = 1) \Rightarrow f(\mathbf{v}_i) = 1.$$

Function f is a non-decreasing decreasing function.

Now we consider fixed E, M_i parameterized by $\mathbf{v}_i$ and explore interpretation of L(M_i, E) as f($\mathbf{v}_i$), i.e., L(M_i, E) = f($\mathbf{v}_i$). Let us assume for now that L(M_i, E) has only two values (unacceptable-0 and acceptable-1). Later on it can be generalized to a k-value case. If L(M_i, E) is monotone then

$$\forall \mathbf{v}_i \geq \mathbf{v}_j \;\Rightarrow L(M_i, E)) \geq L(M_j, E),$$

e.g., if $L(M_{3-1110}, E) = L(M_{2-1100}, E) = 0$ then

$$\forall\, (\mathbf{v}_i \geq \mathbf{v}_j \,\&\, L(M_{3-1110}, E) = 0) \Rightarrow L(M_{2-1100}, E) = 0 \tag{18}$$

$$\forall\, (\mathbf{v}_i \geq \mathbf{v}_j \,\&\, L(M_{2-1100}, E) = 1) \Rightarrow L(M_{3-1110}, E) = 1 \tag{19}$$

This means that if a model with more unknown parameters $\mathbf{v}_i$ failed then a model with less unknown parameters $\mathbf{v}_j$ will also fail. In other words, if at higher level of uncertainty the model in not acceptable $L(M_i, E) = 0$, then it can not be acceptable on the lower level of uncertainty, $L_j(M_j, E) = 0$. If we conclude that a quadratic polynomial model (M_2) is not acceptable, $L(M_2, E) = 0$, then a more specific quadratic model M_3 also cannot be acceptable, $L(M_3, E) = 0$. Thus, we do not need to test model M_3. This is an idea how monotonicity can help to decrease computational complexity. To be able to use this principle of monotonicity in a task we need to check that it takes place for that task.

In essence, we can use

$$\forall(\mathbf{v}_i \geq \mathbf{v}_j \,\&\, f(\mathbf{v}_i) = 0) \Rightarrow f(\mathbf{v}_j) = 0$$

for rejection models and we can use

$$\forall\, (\mathbf{v}_i \geq \mathbf{v}_j \,\&\, f(\mathbf{v}_j) = 1) \Rightarrow f(\mathbf{v}_i) = 1$$

for confirming models.

In the case of **model rejection test** for data E the main focus is not quick building a model but quick rejecting a model M (Popper's principle). In essence the test $L_3(M_3, E) = 0$ means that the whole class of the models M_3 with 3 unknown parameters fails. For testing M_3positively for data E we need to find 4 correct parameters. This may mean searching in a large 4-D parameter space $[-100, +100]^4$ for single vector, say $(p_1, p_2, p_3, p_4) = (9, 3, 7, 10)$, if each parameter varies in the interval $[-100, 100]$. For rejection we may need only, 4 training vectors (x,y,u) from data E and 3 test vectors. The first four vectors will allow us to build a quadratic surface in 3-D as a model. We would just need to test that three test vectors from E do not fit this quadratic surface.

Definition K-valued function f of n variable $f\colon U^n \rightarrow U$ is called a **monotone k-valued function** if

$$\mathbf{u}_i \geq \mathbf{u}_j \Rightarrow f(\mathbf{u}_i) \geq f(\mathbf{u}_j)$$

This function can be applied similarly in the case when we have more uncertainty levels between 0 and 1.

7.7 Search Process

7.7.1 Search Process in Monotone Functions Terms

In the optimization process we want to keep a track of model rejections and be able to guide dynamically what model will be tested next to minimize the number of tests. This is a part of dynamic logic in MFT that we formalize below using the theory of Monotone Boolean and k-valued Functions. Formulas (a) and (b) are key formulas to minimize tests, but we need the whole strategy how to minimize the number of tests and formalize it. One of the ways of formalization is to minimize **Shannon function** φ [28, 45].

$$\min_{A\in\mathbf{A}}\max_{f\in\mathbf{F}}\ \varphi(f, A),$$

where **A** is a set of algorithms, F is a set of monotone functions and φ(f,A) is a number of tests that algorithm A does to fully restore function f. Each test means computing a value f(**v**) for a particular vector **v**. In the theory of monotone Boolean functions it is a assumed that there is an **oracle** that is able to produce the value f(**v**), thus each test is equivalent to a request to the oracle [28, 45, 48, 49]. Minimization of Shannon function means that we search for the algorithm that needs smallest number of tests for its worst case (function f that needs maximum number of tests relative to other functions). This is a classic min-max criterion.

It was proven in [28, 45] that

$$\min_{A\in\mathbf{A}}\max_{f\in\mathbf{F}}\ \varphi(f, A) = \binom{n}{\lfloor n/2\rfloor} + \binom{n}{\lfloor (n/2\rfloor + 1},$$

$\lfloor x \rfloor$ is a floor of x (an integer that smaller than x and closest to x).

The proof of this theorem allows us to derive an algorithm based on the structure called Hansel chains. These chains designed by Hansel cover the whole n-dimensional binary cube $\{1,0\}^n$. The steps of the algorithm are presented in detail in [48]. The main idea of these steps is building Hansel chains, starting from testing his smallest chains, expanding each tested value using (a) and (b) formulas presented above and test values that are left not expanded on the same chains then move to larger chains until no chains left.

In mathematical terms the goal of the search is to find a **smallest lower unit v**, i.e., a Boolean vector such that f(**v**) = 1, and for every w < v f(w) = 0, and for every u>v |u| > |v|. A less challenging problem could be to find any lower unit of f.

The difference of the approach based on the Hansel chains from traditional one when individual parameters are added sequentially to the list of certain parameters is that the simple sequence does not optimize Shannon function, that is it may require more steps than Hansel chains. Mathematical results are also known for k-valued monotone functions [42, 43].

7.7.2 Search in Logic and Probabilistic Terms

The search problem in logic terms can be formulated as a satisfiability problem:

Find a system of axioms T_a such that

$$L_a(T_a, E) = 1$$

subject to the condition

$$\forall T_j L_a(T_a, E) = L_j(T_j, E) \Rightarrow T_j \geq_{Mu} T_{a,}$$

i.e., if T_a and $T_j$have the same similarity with E, $L_a(T_a, E) = L_j(T_j, E)$, then M_a should have a lower uncertainty than T_j, e.g., $T_j \geq_{Mu} M_{a.}$

If a similarity measure is defined as a **probabilistic measure** with values in [0, 1] then the probabilistic version of the task of finding system of axioms T for model A that maximizes a probabilistic similarity measure is:

$$\mathrm{Max}_{i \in I} L(M_i, E),$$

where $L = F(M_i)$ and I is a set of models. This task can be further developed by using Occam principle—to select the simplest model out of two equally appropriate.

7.8 Summary of Formalization

In this section formalization of the concept of the dynamic logic in the terms of the first order logic, logic model theory and theory of Monotone Boolean functions has been introduced. It concentrates on the main idea of dynamic logic of matching levels of uncertainty of the problem/model and levels of uncertainty of the evaluation criterion used dramatically minimize search for model identification. When a model becomes more certain then the evaluation criterion is also adjusted dynamically to match an adjusted model. This dynamic logic process of model construction is likely mimics a process of a natural evolution.

This section introduced the concepts of partial order on the models with respect to their uncertainty, generality and simplicity. These concepts are also introduced for similarity measures and examples provided for models and measures. Next these partial orders are represented using a set of Boolean parameters. The theory of monotone Boolean functions is used for guiding and visualizing search in the parameter space in the dynamic logic setting.

The proposed formalization creates a framework for developing specific applications, that will consist of a sets of models, matching similarity measures, processed for testing them and model learning processed for specific problems in pattern recognition, data mining, optimization, cognitive process modeling and decision making.

Further theoretical studies may reveal a dipper links with classical optimization search processes and significantly advance then by adding an extra layer of actual constructing optimization criteria not only using them.

In the area of logic further theoretical studies may also reveal a dipper links with classical logic problems such as decidability, completeness and consistency.

In the areas of machine learning further theoretical studies may also reveal a dipper links with analytical machine learning [63], inductive and probabilistic logic programming [18, 101], relational machine learning [54], where a priory models play a critical role in the learning process.

8 Conclusion

This chapter summarizes previous development of physics of the mind, a theory of cognition based on fundamental principles of the mind operation, and dynamic logic a mathematical foundation of the physics of the mind that enables to overcome combinatorial complexity, which has prevented previous developments of cognitive theories since the 1950s. Mathematical formulations of the fundamental principles of the mind have been presented including dynamic logic, the knowledge instinct, mechanisms of concepts, emotions, aesthetic emotions, emotions of the beautiful, language, the dual model of the interactions between language and cognition, including the fundamental role of prosodial emotions in this interaction. Physics of the mind predicted a number of psychological phenomena, many of which have been confirmed in experiments.

The chapter concludes with the development of mathematical formalization of dynamic logic using first order logic, logic model theory, and monotone Boolean functions that could be used for further mathematical development.

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Fuzzy Arithmetic Type 1 with Horizontal Membership Functions

Andrzej Piegat and Marek Landowski

Abstract The chapter shortly (because of the volume limitation) presents multidimensional fuzzy arithmetic based on relative-distance-measure (RDM) and horizontal membership functions which considerably facilitate calculations. This arithmetic will be denoted as MD-RDM-F one. It delivers full, multidimensional problem solutions that further enable determining, in an accurate and unique way, various representations of the solutions such as span (maximal uncertainty of the solution), cardinality distribution of possible solution values, center of gravity of the solution granule, etc. It also allows for taking into account relations and dependencies existing between variables, what is absolutely necessary e.g. in calculations with fuzzy probabilities that always should sum up to 1 or in equation system solving.

Keywords Fuzzy arithmetic · Fuzzy mathematics · Uncertainty theory · Granular computing · Soft computing

1 Introduction

Fuzzy arithmetic [10–12, 15, 17–19, 25–27, 34] is extension of interval arithmetic [20, 22–24, 28–32] from calculation on standard intervals to calculation on fuzzy intervals and fuzzy numbers. This arithmetic is connected with uncertainty theory [6], soft computing [14], granular computing [25], grey systems [21], etc. It is necessary for solving problems of Computing with Words [37, 38], for solving linear and nonlinear equations and fuzzy equation systems [1, 2, 5, 7, 8, 13] which occur in many real problems of economy, engineering, medicine, environmental protection, etc. [3, 4, 9, 16, 17, 36, 39]. Fuzzy arithmetic has been developed for many

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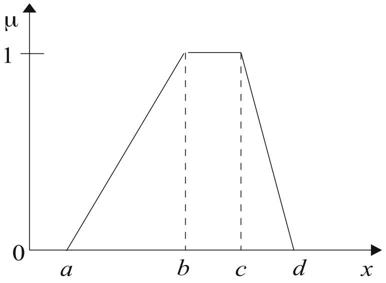


Fig. 1 Trapezoidal membership function or a fuzzy interval

years. There exist many methods of fuzzy arithmetic such as L-R arithmetic [10, 17, 18, 26], fuzzy arithmetic based on discretized fuzzy numbers and Zadeh's extension principle, fuzzy arithmetic based on decomposed fuzzy numbers (α-cuts and Moore interval arithmetic), standard fuzzy arithmetic based on decomposed fuzzy numbers, advanced fuzzy arithmetic based on transformation method or on extended transformation method [17], constrained fuzzy arithmetic [19] and other. Overview of many methods can be found in [12, 17, 25]. Simpler types of fuzzy arithmetic (FA) allow for analytical determining (by hand) of solutions. More complicated methods determine solutions numerically and require computer application. The fact that new types of FA have been developed all the time means that existing methods of FA are not ideal and can be improved. Such improvement proposal in form of MD-RDM-F arithmetic will be presented in this chapter. Concept of this arithmetic has been elaborated by A. Piegat. This arithmetic uses horizontal membership functions (horizontal MFs). They were already, introductory presented in [35, 36]. Figure 1 shows a trapezoidal MF (fuzzy interval).

A vertical model (1) of the fuzzy interval from Fig. 1 expresses the dependence $\mu = f(x)$ in the usual way known from literature of fuzzy sets.

$$\mu(x) = \begin{cases} (x-a)/(b-a) & \text{for } x \in [a, b] \\ 1 & \text{for } x \in (b, c] \\ (d-x)/(d-c) & \text{for } x \in (c, d] \end{cases} \tag{1}$$

Formula (1) expresses unique dependence of the "vertical" variable μ from the "horizontal" variable x. However, a question can be asked: can a "horizontal" model of MF $x = f(\mu)$ be determined? It seems impossible because such dependence would be not unique and thus it would not be function. However, let us consider horizontal cut of the MF on level μ, which further on will be called not α-cut but μ-cut, Fig. 2a.

Variable α_x, $\alpha_x \in [0, 1]$, called RDM-variable [20, 30, 31] determines relative distance of a point $x^* \in [x_L(\mu), x_R(\mu)]$ from the origin of the local coordinate-system positioned on the left side of MF-cut made on level μ, Fig. 2. Thus, this variable introduces Cartesian-coordinate-system inside of interval. The left border $x_L(\mu)$ and the right border $x_R(\mu)$ of MF is determined by formula (2).

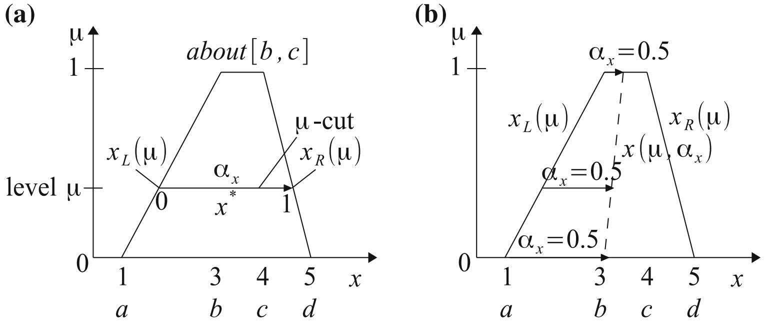
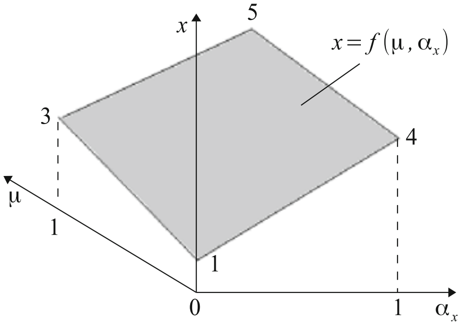


Fig. 2 Visualization of μ-cut (**a**) and of the horizontal approach to description of membership functions (**b**)

Fig. 3 Horizontal membership function $x = (1 + 2\mu) + (4 - 3\mu)\alpha_x$, $\alpha_x \in [0, 1]$, corresponding to the function shown in Fig. 2, as unique function in 3D-space



$$x_L = a + (b - a)\mu, \;\; x_R = d - (d - c)\mu \tag{2}$$

The left border $x_L(\mu)$ is transformed into the right border $x_R(\mu)$ by the RDM-variable α_x. Contour line $x(\mu, \alpha_x)$ of constant values of α_x is determined by formula (3).

$$x(\mu, \alpha_x) = x_L + (x_R - x_L)\alpha_x, \;\; \alpha_x \in [0, 1] \tag{3}$$

This line is set of points lying at equal relative distance α_x from the left border $x_L(\mu)$ of MF. Precise form of formula (3) given by formula (4) can be called horizontal membership function.

$$x = [a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x, \;\; \alpha_x \in [0, 1] \tag{4}$$

Formula (4) describes function $x = f(\mu, \alpha_x)$, which is function of two variables defined in 3D-space. As Fig. 3 shows, this function is unique.

Because the horizontal function $x = f(\mu, \alpha_x)$ defines not one value of variable x but a set of possible values of this variable corresponding to a given μ-cut level, it defines an information granule and will be denoted as x^{gr}. In this chapter the horizontal model (4) for the trapezoidal MF was shown. However, if $b = c$ then model (4) describes triangular MF and if $a = b$ and $c = d$ then it describes rectangular MF. Figure 2 presents MF with line borders $x_L(\mu)$ and $x_R(\mu)$. However, the borders can also be nonlinear, e.g. of Gauss-type. To derive formulas for horizontal MF in this case, nonlinear formulas for the left and right border of the MF should be determined and inserted in the general formula (2).

2 Basic Operations of MD-RDM-F Arithmetic

Let $x^{gr}(\mu, \alpha_x)$ be horizontal MF representing fuzzy interval X and $y^{gr}(\mu, \alpha_y)$ be horizontal MF representing fuzzy interval Y, formula (5) and (6).

$$X : x^{gr} = [a_x + (b_x - a_x)\mu] + [(d_x - a_x) - \mu(d_x - c_x + b_x - a_x)]\,\alpha_x,\ \ \mu, \alpha_x \in [0, 1] \tag{5}$$

$$Y : y^{gr} = \left[a_y + (b_y - a_y)\mu\right] + \left[(d_y - a_y) - \mu(d_y - c_y + b_y - a_y)\right]\alpha_y,\ \ \mu, \alpha_y \in [0, 1] \tag{6}$$

Addition $X + Y = Z$ of two independent intervals

$$X + Y = Z : x^{gr}(\mu, \alpha_x) + y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y),\ \ \mu, \alpha_x, \alpha_y \in [0, 1] \tag{7}$$

Example: for $x^{gr}(\mu, \alpha_x) = (1 + 2\mu) + (4 - 3\mu)\alpha_x$ representing trapezoidal MF $(1, 3, 4, 5)$ and $y^{gr}(\mu, \alpha_y) = (1 + \mu) + (3 - 2\mu)\alpha_y$ representing MF $(1, 2, 3, 4)$ the sum is given by (8).

$$z^{gr}(\mu, \alpha_x, \alpha_y) = x^{gr}(\mu, \alpha_x) + y^{gr}(\mu, \alpha_y) = (2 + 3\mu) + (4 - 3\mu)\alpha_x + (3 - 2\mu)\alpha_y,\\ \mu, \alpha_x, \alpha_y \in [0, 1] \tag{8}$$

If we are interested in the span $s(z^{gr})$ of the 4-D result granule $z^{gr}(\mu, \alpha_x, \alpha_y)$ then it can be determined with known methods of function examination, formula (9).

$$s(z^{gr}) = \left[\min_{\alpha_x, \alpha_y} z^{gr}\left(\mu, \alpha_x, \alpha_y\right), \max_{\alpha_x, \alpha_y} z^{gr}\left(\mu, \alpha_x, \alpha_y\right)\right] \tag{9}$$

In frame of the function examination its extremes should be found. The extremes can lie on borders of the function domain or inside of it, in zeroing points of derivatives in respect of RDM-variables. Because the addition result function is monotonic, its extrema lie on the domain borders. Examination of function (8) shows that its minimum corresponds to $\alpha_x = \alpha_y = 0$ and maximum to $\alpha_x = \alpha_y = 1$. Thus, the span of the result granule is finally expressed by formula (10).

$$s(z^{gr}) = [2 + 3\mu, 9 - 2\mu],\ \ \mu \in [0, 1] \tag{10}$$

One should noticed that span of the 4D-result $z^{gr}(\mu, \alpha_x, \alpha_y)$ is not the full result but only 2D-information about the maximal uncertainty of the result. On the basis of formula (9) determining the full, multidimensional addition result, apart from the span also other features of the result can be determined, such as cardinality distribution $card(z)$ of all possible result values, position of center of gravity (x_{CofG}, y_{CofG}), core position $core(z)$ etc. However, it is not discussed in this chapter because of its volume limitation.

Subtraction $X - Y = Z$ of two independent intervals

$$X - Y = Z : x^{gr}(\mu, \alpha_x) - y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y),\ \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (11)$$

x^{gr} and y^{gr} are determined by general formulas (5) and (6). If $x^{gr}(\mu, \alpha_x) = (1 + 2\mu) + (4 - 3\mu)\alpha_x$ and $y^{gr}(\mu, \alpha_y) = (1 + \mu) + (3 - 2\mu)\alpha_y$ then the subtraction result is given by (12).

$$z^{gr} = x^{gr} - y^{gr} = \mu + (4 - 3\mu)\alpha_x - (3 - 2\mu)\alpha_y,\ \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (12)$$

If we interested in span $s(z^{gr})$ of the 4D-result granule, then it can be determined from (13). Min(z^{gr}) corresponds in this formula to $\alpha_x = 0, \alpha_y = 1$ and max(z^{gr}) to $\alpha_x = 1, \alpha_y = 0$.

$$s(z^{gr}) = \left[\min_{\alpha_x,\alpha_y} z^{gr}, \max_{\alpha_x,\alpha_y} z^{gr}\right] = [-3 + 3\mu, 4 - 2\mu],\ \ \mu \in [0, 1] \quad (13)$$

Multiplication $XY = Z$ of two independent intervals

$$XY = Z : x^{gr}(\mu, \alpha_x)y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y),\ \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (14)$$

x^{gr} and y^{gr} are determined by general formulas (5) and (6). If $x^{gr}(\mu, \alpha_x) = (1 + 2\mu) + (4 - 3\mu)\alpha_x$ and $y^{gr}(\mu, \alpha_y) = (1 + \mu) + (3 - 2\mu)\alpha_y$ then the multiplication result z^{gr} is given by (15).

$$z^{gr} = x^{gr} \cdot y^{gr} = [(1 + 2\mu) + (4 - 3\mu)\alpha_x][(1 + \mu) + (3 - 2\mu)\alpha_y],\ \ \mu, \alpha_x, \alpha_y \in [0, 1] \quad (15)$$

If we are interested in span $s(z^{gr})$ of the result, then it is expressed by formula (16), where min(z^{gr}) corresponds to $\alpha_x = \alpha_y = 0$ and max(z^{gr}) to $\alpha_x = \alpha_y = 1$.

$$s(z^{gr}) = \left[\min_{\alpha_x,\alpha_y} z^{gr}, \max_{\alpha_x,\alpha_y} z^{gr}\right] = [(1 + 2\mu)(1 + \mu), (5 - \mu)(4 - \mu)] \quad (16)$$

Division X/Y of two independent intervals

$$X/Y = Z : x^{gr}(\mu, \alpha_x)/y^{gr}(\mu, \alpha_y) = z^{gr}(\mu, \alpha_x, \alpha_y),\ \ \mu, \alpha_x, \alpha_y \in [0, 1],\ \ 0 \notin Y \quad (17)$$

x^{gr} and y^{gr} are determined by general formulas (5) and (6). If $x^{gr}(\mu, \alpha_x) = (1 + 2\mu) + (4 - 3\mu)\alpha_x$ and $y^{gr}(\mu, \alpha_y) = (1 + \mu) + (3 - 2\mu)\alpha_y$ then the division result z^{gr} is given by (18).

$$\begin{aligned} z^{gr} &= x^{gr}/y^{gr} \\ &= [(1 + 2\mu) + (4 - 3\mu)\alpha_x]/[(1 + \mu) + (3 - 2\mu)\alpha_y], \;\; \mu, \alpha_x, \alpha_y \in [0, 1] \end{aligned} \tag{18}$$

If we are interested in span $s(z^{gr})$ of the multidimensional result then it is expressed by (19).

$$s(z^{gr}) = \left[\min_{\alpha_x,\alpha_y} z^{gr}, \max_{\alpha_x,\alpha_y} z^{gr}\right] = [(1 + 2\mu)/(4 - \mu), (5 - \mu)/(1 + \mu)] \tag{19}$$

3 Some Mathematical Properties of MD-RDM-F Arithmetic

Commutativity
MD-RDM-F arithmetic is commutative. For any fuzzy intervals X and Y Eqs. (20) and (21) are true.

$$X + Y = Y + X \tag{20}$$

$$XY + YX \tag{21}$$

Associativity
MD-RDM-F arithmetic is associative. For any fuzzy intervals X, Y, Z Eqs. (22) and (23) are true.

$$X + (Y + Z) = (X + Y) + Z \tag{22}$$

$$X(YZ) = (XY)Z \tag{23}$$

Neutral Elements of Addition and Multiplication
In MD-RDM-F arithmetic there exist additive and multiplicative neutral elements such as degenerate fuzzy interval 0 and 1 for any interval X, as shown in Eqs. (24) and (25).

$$X + 0 = 0 + X = X \tag{24}$$

$$X \cdot 1 = 1 \cdot X = X \tag{25}$$

Inverse Elements in MD-RDM-F Arithmetic
In MD-RDM-F arithmetic an element (fuzzy interval) $-X : -x^{gr} = -[a + (b - a)\mu] - [(d - a) - \mu(d - a + b - c)]\alpha_x, \alpha_x \in [0, 1]$ is an additive inverse element

of fuzzy interval $X : x^{gr} = [a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x$, $\alpha_x \in [0, 1]$. It is explained by formula (26).

$$\begin{aligned}&X - X : x^{gr} - x^{gr} = \{[a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x\}\\&-\{[a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x\} = 0,\\&\mu, \alpha_x \in [0, 1]\end{aligned} \tag{26}$$

If parameters of two fuzzy intervals are equal: $a_x = a_y, b_x = b_y, c_x = c_y, d_x = d_y$, then interval $-Y$ is only then the additive inverse interval of X when also the inner RDM-variables are equal, $\alpha_x = \alpha_y$, which means full coupling (correlation) of both uncertain values modeled by the intervals. A multiplicative inverse element of fuzzy interval $X{:}x^{gr}{=}[a + (b - a)\mu]{+}[(d - a){-}\mu(d - a){-}\mu(d - a + b - c)]$ α_x, $\alpha_x \in [0, 1]$, if $0 \notin X$, in MD-RDM-F arithmetic is $1/X : x^{gr} = 1/ = [a + (b - a)\mu] + [(d - a) - \mu(d - a) - \mu(d - a + b - c)\alpha_x$, $\alpha_x \in [0, 1]$. The above is explained by formula (27).

$$\begin{aligned}&X/X : x^{gr}/x^{gr} = \{[a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x\}\\&/\,\{[a + (b - a)\mu] + [(d - a) - \mu(d - a + b - c)]\alpha_x\} = 1,\\&\mu, \alpha_x \in [0, 1]\end{aligned} \tag{27}$$

It should be noted that if parameters of two fuzzy intervals X and Y are equal: $a_x = a_y$, $b_x = b_y$, $c_x = c_y$, $d_x = d_y$, then interval $1/Y$ is the multiplicative inverse interval of X only when also the inner RDM-variables are equal, $\alpha_x = \alpha_y$, which means full coupling (correlation) of both uncertain values modeled by the intervals. Full or partial couplings between uncertain variables occur in many real problems.

Sub-distributive Law

The sub-distributive law is given by (28).

$$X(Y + Z) = XY + XZ \tag{28}$$

In MD-RDM-F arithmetic this law holds. Proof: for any three fuzzy intervals described in terms of the RDM horizontal notation,

$$X : x^{gr} = [a_x + (b_x - a_x)\mu] + [(d_x - a_x) - \mu(d_x - a_x + b_x - c_x)]\alpha_x,\ \alpha_x \in [0, 1]$$

$$Y : y^{gr} = [a_y + (b_y - a_y)\mu] + [(d_y - a_y) - \mu(d_y - a_y + b_y - c_y)]\alpha_y,\ \alpha_y \in [0, 1]$$

$$Z : z^{gr} = [a_z + (b_z - a_z)\mu] + [(d_z - a_z) - \mu(d_z - a_z + b_z - c_z)]\alpha_z,\ \alpha_z \in [0, 1]$$

analysis of the sub-distributive law results in conclusions expressed by (29).

$$\begin{aligned}
&X(Y+Z): x^{gr}(y^{gr}+z^{gr}) = \\
&= \{[a_x+(b_x-a_x)\mu]+[(d_x-a_x)-\mu(d_x-c_x+b_x-a_x)]\alpha_x\} \\
&\cdot \{[a_y+(b_y-a_y)\mu]+[(d_y-a_y)-\mu(d_y-c_y+b_y-a_y)]\alpha_y \\
&+[a_z+(b_z-a_z)\mu]+[(d_z-a_z)-\mu(d_z-c_z+b_z-a_z)]\alpha_z\} \\
&= \{[a_x+(b_x-a_x)\mu]+[(d_x-a_x)-\mu(d_x-c_x+b_x-a_x)]\alpha_x\} \\
&\cdot \{[a_y+(b_y-a_y)\mu]+[(d_y-a_y)-\mu(d_y-c_y+b_y-a_y)]\alpha_y\} \\
&+\{[a_x+(b_x-a_x)\mu]+[(d_x-a_x)-\mu(d_x-c_x+b_x-a_x)]\alpha_x\} \\
&\cdot \{[a_z+(b_z-a_z)\mu]+[(d_z-a_z)-\mu(d_z-c_z+b_z-a_z)]\alpha_z\} \\
&= x^{gr}y^{gr}+x^{gr}z^{gr} : XY+XZ,\ \ \mu,\alpha_x,\alpha_y,\alpha_z \in [0,1]
\end{aligned} \tag{29}$$

Because in MD-RDM-F arithmetic the sub-distributive law holds transformations of equations are admissible because they do not change the result.

Cancellation Law for Addition
Cancellation law (30) for addition of fuzzy intervals holds both for the standard C-αC-F arithmetic and for MD-RDM-F one.

$$X+Z=Y+Z \Rightarrow X=Y \tag{30}$$

Cancellation Law for Multiplication
This law has form of (31).

$$XZ=YZ \Rightarrow X=Y \tag{31}$$

Let us assume, for simplicity, three triangle fuzzy numbers: $X=(1,2,3)$, $Y=(2,2.5,3)$ and $Z=(-1,0,1)$.

The fuzzy numbers are expressed in terms of horizontal MFs (32).

$$X=\{x : x=[a_x+(b_x-a_x)\mu]+(d_x-a_x)(1-\mu)\alpha_x,\ \ \mu,\alpha_x \in [0,1]\} \tag{32}$$

Fuzzy numbers $X=(1,2,3)$, $Y=(2,2.5,3)$, $Z=(-1,0,1)$ are expressed by (33).

$$\begin{aligned}
X&=\{x : x=(1+\mu)+2(1-\mu)\alpha_x,\ \ \mu,\alpha_x \in [0,1]\} \\
Y&=\{y : y=(2+0.5\mu)+(1-\mu)\alpha_y,\ \ \mu,\alpha_y \in [0,1]\} \\
Z&=\{z : z=(-1+\mu)+2(1-\mu)\alpha_z,\ \ \mu,\alpha_z \in [0,1]\}
\end{aligned} \tag{33}$$

Now, particular products can be calculated, (34).

$$\begin{aligned}
&XZ=\{xz : xz=[(1+\mu)+2(1-\mu)\alpha_x][(-1+\mu)+2(1-\mu)\alpha_z]\} \\
&YZ=\{yz : yz=[(2+0.5\mu)+(1-\mu)\alpha_y][(-1+\mu)+2(1-\mu)\alpha_z]\} \\
&\text{where } \mu,\alpha_x,\alpha_y,\alpha_z \in [0,1]
\end{aligned} \tag{34}$$

Analysis of products XZ and YZ given by (34) shows that these products are different because $Y \neq Z$. If necessary, in frame of MD-RDM-F arithmetic, on the basis of (34) spans of XZ and YZ can be calculated, formulas (35) and (36).

$$s(XZ) = \left[\min_{\alpha_x,\alpha_z} XZ, \max_{\alpha_x,\alpha_z} XZ\right] = [(3-\mu)(-1+\mu), (3-\mu)(1-\mu)] \quad (35)$$
$$\mu, \alpha_x, \alpha_z \in [0, 1]$$

Because XZ and YZ are monotonic functions, $\min(XZ)$ occurs on its domain border for $\alpha_x = 1, \alpha_y = 0$. $\text{Max}(XZ)$ occurs for $\alpha_x = 1, \alpha_z = 1$. $\text{Min}(YZ)$ occurs for $\alpha_y = 1$ and $\alpha_z = 0$, and $\max(YZ)$ for $\alpha_y = 1$ and $\alpha_z = 1$, $\alpha_x, \alpha_y, \alpha_z \in [0, 1]$.

$$s(YZ) = \left[\min_{\alpha_y,\alpha_z} YZ, \max_{\alpha_y,\alpha_z} YZ\right] = [(3-\mu)(-1+\mu), (3-\mu)(1-\mu)] \quad (36)$$
$$\mu, \alpha_y, \alpha_z \in [0, 1]$$

As formula (33) shows precise, multidimensional products XZ and YZ are different. Only their spans (35) and (36), in this special case, are identical. However, the spans are not precise multiplication results of the intervals, they only are information about

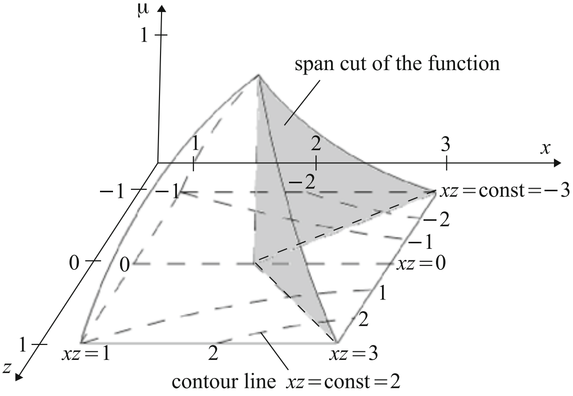


Fig. 4 3D-projection of the 4D result granule of multiplication $XZ = (1, 2, 3)(-1, 0, 1)$ of two fuzzy numbers from the full space $XZ \times \mu \times X \times Z$ on the space $\mu \times X \times Z$

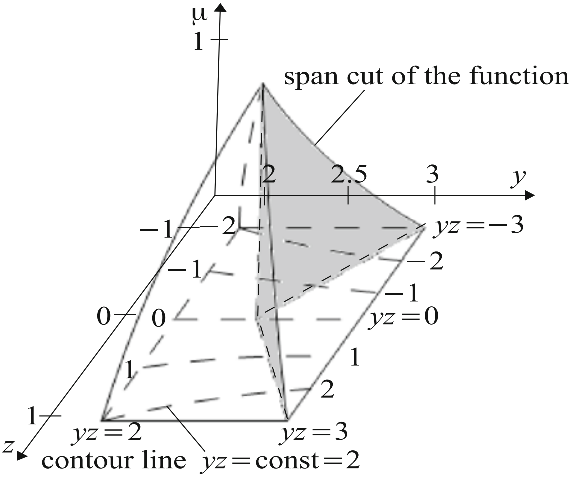


Fig. 5 3D-projection of the 4D result-granule of multiplication $YZ = (2, 2.5, 3)(-1, 0, 1)$ of two fuzzy numbers from the full space $YZ \times \mu \times Y \times Z$ on the space $\mu \times Y \times Z$

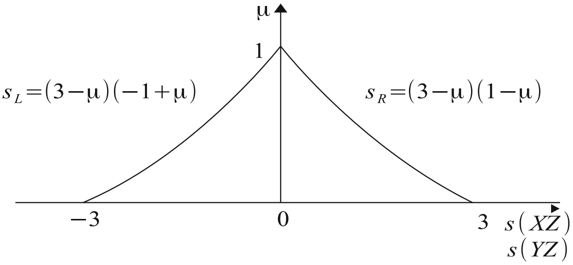


Fig. 6 Span function $\min s = f_L(\mu)$ and $\max s = f_R(\mu)$ identical for two different 4D multiplication results $XZ = (1, 2, 3)(-1, 0, 1)$ and $YZ = (2, 2.5, 3)$ $(-1, 0, 1)$

the results. To better understand this on Figs. 4 and 5 were presented 3D-projections of 4D-products XZ and YZ. Values of the fourth variable z were shown on this projections by contour lines of constant z-values. On these figures also cuts were shown which generate the span function shown in Fig. 6. The span functions $s(XZ)$ and $s(YZ)$ are identical in this case. They inform about the span width on particular μ-levels.

4 Paradox of Hukuhara Difference

Let us assume triangle fuzzy numbers $X = (4, 6, 9)$ and $Y = (0, 2, 3)$. The standard difference Z_s of these numbers has form (37).

$$Z_s = X - Y = (4, 6, 9) - (0, 2, 3) = (4 - 3, 6 - 2, 9 - 0) = (1, 4, 9) \tag{37}$$

However, equation $Z_s = X - Y$ is not equivalent to $X = Y + Z_s$ (38).

$$Y + Z_s = (0, 2, 3) + (1, 4, 9) = (1, 6, 12) \neq X = (4, 6, 9) \tag{38}$$

This was the reason of introducing by Hukuhara the second way of interval difference calculation Z_H that is achieved on the basis of equation $X = Y + Z_H$ [4, 33], later called Hukuhara difference (H-difference). Before Hukuhara, this difference had been developed by Kaucher. Hence, it also can be called Kaucher-difference.

Definition 1 Given $X, Y \in F$, the H-difference Z_H of X and Y is defined by $X \ominus Y \Leftrightarrow X = Y + Z_H$: if $X \ominus Y$ exists, it is unique and its μ-cuts are $[X - Y]_\mu$.

Example of H-difference Z_H of X and Y is given by (39).

$$\begin{aligned} X = Y + Z_H &= (4, 6, 9) = (0, 2, 3) + (z_{H1}, z_{H2}, z_{H3}) \\ &= (z_{H1}, 2 + z_{H2}, 3 + z_{H3}) : Z_H = (4, 4, 6) \end{aligned} \tag{39}$$

H-difference is frequently used in solving e.g. fuzzy differential equations [25, 33], but not only. Introducing H-difference in fuzzy arithmetic created a very strange and paradoxical situation. Now, there exist two different ways of the difference calculation: the standard way from equation $Z_s = X - Y$ and Hukuhara-way from equation $X = Y + Z_H$. However, to be on this trail, one could also add two next ways of the difference calculation: difference Z_3 calculated from equation $Y = X - Z_3$ and difference Z_4 calculated from equation $X - Y - Z_4 = (0, 0, 0)$! It is a paradoxical situation. The paradox of many various differences $Z = X - Y$ does not exist in MD-RDM-F arithmetic, where only one difference of fuzzy numbers and intervals exist, and equations $Z = X - Y$, $X = Y + Z$, $Y = X - Z$, $X - Y - Z = (0, 0, 0)$ are equivalent! What was said above will now be explained. Formulas (40) and (41) present horizontal models of fuzzy numbers $X = (4, 5, 6)$ and $Y = (0, 2, 3)$.

$$X = \{x : x = 4 + 2\mu + 5(1 - \mu)\alpha_x,\ \ \mu, \alpha_x \in [0, 1]\} \tag{40}$$

$$Y = \{y : y = 2\mu + 3(1 - \mu)\alpha_y,\ \ \mu, \alpha_y \in [0, 1]\} \tag{41}$$

Formula (42) shows their difference $Z_s = X - Y$ calculated in terms of MD-RDM-F arithmetic.

$$\begin{aligned} &Z_s = X - Y \\ &= \{z_s = x - y : x - y = [4 + 2\mu + 5(1 - \mu)\alpha_x] - [2\mu + 3(1 - \mu)\alpha_y], \\ &\mu, \alpha_x, \alpha_y \in [0, 1]\} \end{aligned} \tag{42}$$

Formula (43) shows H-difference Z_H calculated from equation $X = Y + Z_H$.

$$\begin{aligned} &Z_H : X = Y + Z_H \\ &= \{z_H : x = y + z_H = [4 + 2\mu + 5(1 - \mu)\alpha_x] = [2\mu + 3(1 - \mu)\alpha_y] + z_H : \\ &z_H = [4 + 2\mu + 5(1 - \mu)\alpha_x] - [2\mu + 3(1 - \mu)\alpha_y], \mu, \alpha_x, \alpha_y \in [0, 1]\} \end{aligned} \tag{43}$$

As (43) shows H-difference Z_H calculated from equation $X = Y + Z_H$ is identical as the difference Zs calculated in the standard way from equation $Z_s = X - Y$. Similar results are achieved if the difference Z is calculated from other possible formulas $Y = X - Z$ or $X - Y - Z = (0, 0, 0)$. It means that in MD-RDM-F arithmetic all possible equations forms $Z = X - Y$, $X = Y + Z$, $Y = X - Z$, $X - Y - Z = (0, 0, 0)$ are equivalent. If we are interested in span $s(Z)$ of the difference then it can be calculated from (44).

$$\begin{aligned} s(Z) &= \left[\min_{\alpha_x,\alpha_y} Z, \max_{\alpha_x,\alpha_y} Z\right] \\ &= \left[\min_{\alpha_x,\alpha_y}(4 + (1 - \mu)(5\alpha_x - 3\alpha_y)), \max_{\alpha_x,\alpha_y}(4 + (1 - \mu)(5\alpha_x - 3\alpha_y))\right] \end{aligned} \tag{44}$$

Because $Z = f(\mu, \alpha_x, \alpha_y)$ is monotonic function, its extrema occur on its domain borders. Examination of this function shows that its minimum occurs for $\alpha_x = 0$, $\alpha_y = 1$ and the maximum for $\alpha_x = 1$ and $\alpha_y = 0$. Thus, span $s(Z)$ is determined by formula (45).

$$s(Z) = [1 + 3\mu, 9 - 5\mu] \tag{45}$$

It should be once more reminded that span $s(Z)$ of the difference $Z = f(\mu, \alpha_x, \alpha_y)$ is not the difference itself but only a 2D-representation of it.

5 Application Example of Fuzzy RDM-Arithmetic with Horizontal Membership Functions

Experts have formulated following prognoses of economic growth in a country: "*Strong economic growth (SG) will take place with a medium (M) probability, moderate growth with a less than medium (LM) probability, stabilization of the growth (ST) will occur with a small probability (S) and recession (R) with a very small (VS) probability.*" Experts are also of the opinion that $M \geq LM \geq S \geq VS$ independently of how large are precise values of particular probabilities. Fuzzy definitions of understanding linguistic values of growth are given in Fig. 7 and of probability in Fig. 8.

The task consists in determining expected value of the economic growth x_{exp}^{gr} in multidimensional granular form and its 2D-representation in form of span $s(x_{exp}^{gr}) = f(\mu)$.

Horizontal MFs of particular linguistic values of economic growth are given by (46) and of probability by (47), α_i and β_j are RDM-variables.

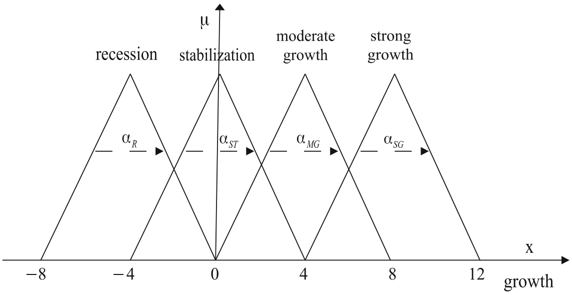


Fig. 7 Membership functions of linguistic values of the economic growth

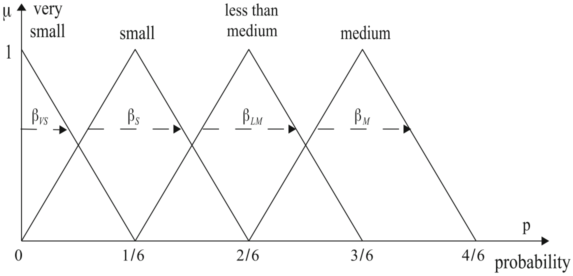


Fig. 8 Membership values of linguistic values of probability

$$\begin{aligned}
&x_R = (-8+4\mu) + \alpha_R(8-8\mu)\\
&x_{ST} = (-4+4\mu) + \alpha_{ST}(8-8\mu)\\
&x_{MG} = 4\mu + \alpha_{MG}(8-8\mu)\\
&x_{SG} = (4+4\mu) + \alpha_{SG}(8-8\mu)\\
&\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG} \in [0, 1]
\end{aligned} \tag{46}$$

$$\begin{aligned}
&p_{VS} = \left[\beta_{VS}\,(1-\mu)\right]/6\\
&p_S = \left[\mu + \beta_S\,(2-2\mu)\right]/6\\
&p_{LM} = \left[(1+\mu) + \beta_{LM}\,(2-2\mu)\right]/6\\
&p_M = \left[(2+\mu) + \beta_M\,(2-2\mu)\right]/6\\
&\mu, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{47}$$

Precise numerical values p_{VS}, p_S, p_{LM}, p_M are not known. However, it is known that their values have to sum to 1, condition (48).

$$p_{VS} + p_S + p_{LM} + p_M = 1 \tag{48}$$

Also, experts are of the opinion that between probability values order relations (49) exist.

$$p_S \geq p_{VS},\ p_{LM} \geq p_S,\ p_M \geq p_{LM} \tag{49}$$

After inserting in condition (48) horizontal MFs given by formulas (47) a new RDM-form (50) of this condition is achieved.

$$\begin{aligned}
&\beta_{VS} + 2\,(\beta_S + \beta_{LM} + \beta_M) = 3\\
&\beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{50}$$

After inserting in conditions (49) horizontal MFs given by (47) a new RDM-form (51) of these conditions is achieved.

$$\begin{aligned}
&\beta_{VS} - 2\beta_S \leq \mu/\,(1-\mu)\\
&\beta_S - \beta_{LM} \leq 0.5/\,(1-\mu)\\
&\beta_{LM} - \beta_M \leq 0.5/\,(1-\mu)\\
&\mu, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{51}$$

Expected value of the economic growth expresses formula (52).

$$x_{exp}^{gr} = x_R p_{VS} + x_{ST} p_S + x_{MG} p_{LM} + x_{SG} p_M \tag{52}$$

After inserting in (52) horizontal MFs given by (46) and (47) formula (53) has been achieved. It expresses the granular form of the expected growth value x_{exp}^{gr} with use of RDM-variables.

$$\begin{aligned}
&x_{\exp}^{gr} = 1/6\{[(-8+4\mu)+\alpha_R(8-8\mu)][\beta_{VS}\,(1-\mu)] \\
&+[(-4+4\mu)+\alpha_{ST}(8-8\mu)][\mu+\beta_S\,(2-2\mu)] \\
&+[4\mu+\alpha_{MG}(8-8\mu)][(1+\mu)+\beta_{LM}\,(2-2\mu)] \\
&+[(4+4\mu)+\alpha_{SG}(8-8\mu)][(2+\mu)+\beta_M\,(2-2\mu)]\} \\
&\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{53}$$

Formula (53) shows how simply calculations with horizontal MFs can be made. They are inserted in mathematical formulas as usual numbers in formulas of the classic mathematics. Formula (53) defines a multidimensional information granule $x_{exp}^{gr} = f(\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M)$ or with other words, the full, multidimensional set of possible values of the expected growth x. They can be generated only by values of RDM-variables which satisfy condition $\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]$ and conditions (50), (51). For example, set of RDM-values $\alpha_R = \alpha_{ST} = \alpha_{MG} = \alpha_{SG} = 1/8$ and $\beta_{VS} = \beta_S = \beta_M = 1/3$ and $\beta_{LM} = 2/3$ satisfies the required conditions and generates for the level $\mu = 0$ one of possible growth value $x = 1.9$. Another value set of $\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M$ will generate, in the general case, another value of possible growth x. The set of possible point-solutions $x_{exp}^{gr} = f(\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M)$ determined by (53) and conditions (50), (51) creates in the space information granule of irregular form which is not a hyper-cubicoid. This multidimensional granule is difficult to imagine. Hence, scientists usually try to determine its simplified 2D-representation $s(x_{exp}^{gr})$ called "span" or "spread" being the widest 2D-cut through the solution granule or its widest 2D-shadow. It is expressed by (54).

$$\begin{aligned}
&s\left(x_{\exp}^{gr}\right) = [\min x_{\exp}^{gr}(\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M), \\
&\max x_{\exp}^{gr}(\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M)] \\
&\mu, \alpha_R, \alpha_{ST}, \alpha_{MG}, \alpha_{SG}, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{54}$$

Values $\min x_{\exp}^{gr}$ and $\max x_{\exp}^{gr}$ have to be determined with taking into account conditions (50) and (51). Analysis of (53) allows for detection that $\min x_{\exp}^{gr}$ occurs only for $\alpha_R = \alpha_{ST} = \alpha_{MG} = \alpha_{SG} = 0$ and $\max x_{\exp}^{gr}$ for $\alpha_R = \alpha_{ST} = \alpha_{MG} = \alpha_{SG} = 1$. Hence, formula (54) can be simplified to (55).

$$\begin{aligned}
&s\left(x_{\exp}^{gr}\right) = [\min x_{\exp}^{gr}(\mu, 0, 0, 0, 0, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M), \\
&\max x_{\exp}^{gr}(\mu, 1, 1, 1, 1, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M)] \\
&\mu, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M \in [0, 1]
\end{aligned} \tag{55}$$

Determining optimal values of RDM-variables $\beta_{VS}, \beta_S, \beta_{LM}, \beta_M$ is not always easy because they are coupled (correlated) by conditions (50) and (51). Determining of their optimal values can be realized analytically (in this case extremes occur on borders of the solution domain). However, analytical domain examination for various levels of membership μ would take too much work. Therefore, an easier way can be used: numerical examination of the domain with MATLAB for particular variables $\mu, \beta_{VS}, \beta_S, \beta_{LM}, \beta_M$ changed with appropriately small descretization step $\Delta \leq 0.01$.

In the result of such a numerical searching the span-MF has been found as shown in Fig. 9.

For example, on the membership level $\mu = 0$ the minimal value of $x = -1$ and it occurs for $\alpha_R = \alpha_{ST} = \alpha_{MG} = \alpha_{SG} = 0$ and $\beta_{VS} = 1$, $\beta_S = 0.75$, $\beta_{LM} = 0.25$, $\beta_M = 0$ and the maximal value of $x = 10.67$ and it occurs for $\alpha_R = \alpha_{ST} = \alpha_{MG} = \alpha_{SG} = 1$ and $\beta_{VS} = \beta_S = 0, \beta_{LM} = 0.5, \beta_M = 1$. Another simple 1D-representation of the solution span can be position of its center of gravity $x_{CoG} = 4.96$ or position $x_{\mu=1} = 5.33$ of the highest membership of the span (Fig. 9) and of the solution granule x^{gr}_{exp}. However, one should always realize that a simplified representation of the solution is not the solution itself and that without the full-dimensional solution set x^{gr}_{exp} it is not possible (in the general case) to precisely determine solution representations. Figure 10 shows span $s(x^{gr}_{exp})$ of the solution granule x^{gr}_{exp} determined without constraint conditions (50) and (51).

One can see (compare supports in Figs. 9 and 10) that uncertainty of the span $s(x^{gr}_{exp})$ determined without taking into account constraint conditions (50) and (51) is greater than when these conditions are taken into account (for $\mu = 0$ supports' widths are correspondingly equal to 14.66 and 11.67). Thus, application of RDM-variables allow for decreasing of solutions uncertainty because they allow for modeling and taking into account dependences existing between variables occurring in problems.

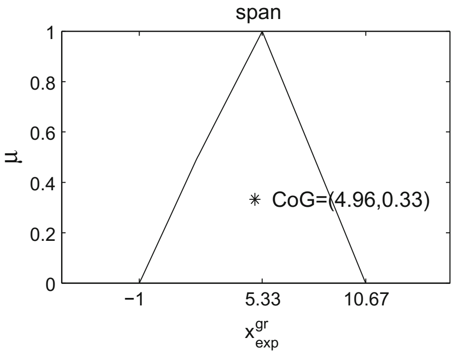


Fig. 9 Membership function of the span $s(x^{gr}_{exp})$ of the multidimensional solution granule x^{gr}_{exp} of the expected economic growth $x[\%]$ achieved with taking into account constraint conditions (50) and (51), CoG—center of gravity

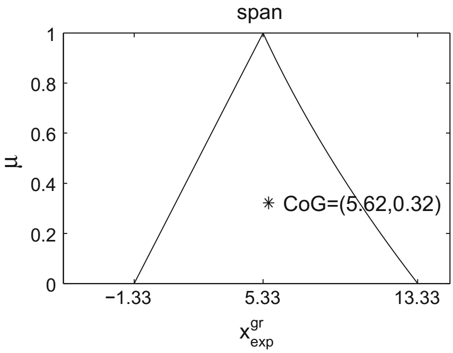


Fig. 10 Membership function of the span $s(x^{gr}_{exp})$ of the multidimensional solution granule x^{gr}_{exp} of the expected economic growth $x[\%]$ achieved without taking into account constraint conditions (50) and (51)

6 Conclusions

The paper shows how in terms of MD-RDM-F arithmetic basic operations as addition subtraction, multiplication and division should be carried out. It also shows certain significant properties of this arithmetic. Arithmetic operations are carried out with use of horizontal MFs, in an easy way, without the extension principle. New MD-RDM-F arithmetic has many advantages. Not all of them could be presented in this chapter because of its volume limitation. MD-RDM-F arithmetic delivers full-dimensional, precise problems' solutions that are multidimensional information granules. Without possessing this full-dimensional solution one cannot precisely and uniquely determine the solution span that is measure of the solution uncertainty. Instead, the span is what standard methods of fuzzy arithmetic try directly to determine. Unfortunately, in the general case it is not possible. Therefore the present fuzzy arithmetic is characterized by many paradoxes, as e.g. paradox of Hukuhara difference. On the basis of the full-dimensional solution achieved with MD-RDM-F arithmetic not only precise and unique solution span can be determined but also other solution representations as 2D cardinality distribution of all possible solutions, the core of the solution, center of gravity (*CofG*) of the solution, etc. MD-RDM-F arithmetic has such important mathematical properties as inverse additive element, inverse multiplicative element. In this arithmetic hold such very important laws as sub-distributive law $X(Y + Z) = XY + XZ$ and cancellation law $IF(XZ = YZ)THEN(X = Y)$. With use of MD-RDM-F arithmetic both linear and nonlinear multidimensional equations and equation systems can relatively easily be solved. In solving them very helpful are RDM-variables that allow for taking into account couplings existing between particular intervals. These possibilities will be shown in next publications of authors.

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Copula as a Bridge Between Probability Theory and Fuzzy Logic

Germano Resconi and Boris Kovalerchuk

Abstract This work shows how dependence in many-valued logic and probability theory can be fused into one concept by using copulas and marginal probabilities. It also shows that the t-norm concept used in fuzzy logic is covered by this approach. This leads to a more general statement that axiomatic probability theory covers logic structure of fuzzy logic. This paper shows the benefits of using structures that go beyond the simple concepts of classical logic and set theory for the modeling of dependences.

1 Introduction

Modern approaches for modeling uncertainty include Probability Theory (PT), Many-Valued Logic (MVL), Fuzzy Logic (FL), and others. PT is more connected with the classical set theory, classical propositional logic, and FL is more connected with fuzzy sets and MVL. The link of the concept of probability with the classical set theory and classical propositional logic is relatively straightforward for the case of a *single variable* that is based on the use of set operations and logic propositions. In contrast, modeling joint probability for *multiple variables* is more challenging because it involves complex *relations* and *dependencies* between variables. One of the goals of this work is to clarify the *logical structure* of a joint probability for multiple variables. Another goal is to *clarify links* between different logics and joint probability for multiple variables.

Differences between concepts of probability, fuzzy sets, possibility and other uncertainty measures have been studied for a long time [1]. Recently several new attempts have been made to connect the concepts of probability, possibility, and

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fuzzy logic into a single theory. Dubois, Prade and others [2–7] discuss conditional probability and subjective probability as a mixed concept that includes both fuzzy and probability concepts.

In [8–11] we proposed the Agent-based Uncertainty Theory (AUT) with a model for fuzzy logic and many-valued logic that involves agents and conflicts among agents. Later we extended AUT, providing a more formal description of uncertainty as the conflict process among agents with the introduction of a: new concept denoted as an *active set* [12]. This agent-based approach to uncertainty has a connection with the recent works in the theory of team semantics [13], and dependence/independence logic [13, 14].

The *dependence logic* [13–21] assumes that the logic truth value of a particular proposition *depends* on the truth value of other propositions. Dependence logic is based on *team semantics*, in which the truth of formulas is evaluated in sets of assignments provided by a team, instead of single assignments, which is common in the classical propositional logic, which is *truth functional*. The basic idea of dependence logic is that certain properties of sets *cannot be expressed* merely in terms of properties satisfied by each set individually. This situation is more complex than that which is considered in the classical logic approach. It is closer to the agent-based uncertainty theory [8], and the active sets [12] that deal with the sets of assignments, generated by teams of agents, which can conflict and produce uncertainty of the truth value, leading to a many-valued logic. Dependence logic is a special type of the first order predicate logic. While the goal to go beyond truth-functionality is quite old in fuzzy logic [22] and the dependence logic is not the first attempt outside of the fuzzy logic, the major issue is that to reach this goal in practice we need more data than truth-functional algorithms use. The current survey of attempts to go beyond truth-functionality is presented in [23].

In PT and statistics, a *copula* is a multivariate probability distribution function for "normalized" variables [24]. In other words, copula is a multivariate composition of marginal probabilities. The copula values represent degrees of dependence among values of these variables when the variables are "cleared" from differences in scales and frequencies of their occurrence. This "clearing" is done by "normalizing" arguments of copula to uniform distributions in [0, 1] interval.

Quite often the probabilistic approach is applied to study the frequency of independent phenomena. In the case of dependent variables, we cannot derive a joint probability $p(x_1, x_2)$ as a product of independent probabilities, $p(x_1)p(x_2)$ and must use the multidimensional probability distribution with dependent valuables. The common technique for modeling it is a Bayesian network. In the Bayesian approach, the evidence about the true state of the world is expressed in terms of degrees of belief in the form of Bayesian conditional probabilities. These probabilities can be causal or just correlational.

In the probability theory, the conditional probability is the main element to express the dependence or inseparability of the two states x_1 and x_2. The joint probability $p(x_1, x_2, \ldots, x_n)$ is represented via multiple conditional probabilities to express the dependence between variables. The *copula approach* introduces a *single* function $c(u_1, u_2)$ denoted as *density of copula* [25] as a way to model the *dependence* or *inseparability* of the variables with the following property in the case of two variables. The copula allows representing the joint probability $p(x_1, x_2)$ as a combination (product) of single dependent part $c(u_1, u_2)$ and independent parts: probabilities $p_1(x_1)$ and $p_2(x_2)$.

It is important to note that probabilities $p_1(x_1)$ and $p_2(x_2)$ belong to two different pdfs which is expressed by their indexes in p_1 and p_2. Below to simplify notation we will omit these indexes in similar formulas.

The investigation of copulas and their applications is a rather recent subject of mathematics. From one point of view, copulas are functions that join or 'couple' one-dimensional distribution functions u_1 and u_2 and the corresponding joint distribution function.

Copulas are of interest not only in probability theory and statistics, but also in many other fields requiring the aggregation of incoming data, such as multi-criteria decision making [26], and probabilistic metric spaces [27, 28]. Associative copulas are special continuous triangular norms (t-norms for short, [29, 30]). They are studied in several domains such as many-valued logic [31], fuzzy logic and sets [32], agent-based uncertainty theory and active sets, along with t-conorms to model the uncertainty.

2 Copula: Notation, Definitions, Properties, and Examples

2.1 Notation, Definitions and Properties

Definitions and properties below are based on [24, 33–36].

A *joint probability distribution* is

$$p(x_1 x_2, \ldots x_n) =$$
$$p(x_1)p(x_2|x_1)p(x_3|x_1, x_2)\ldots p(x_n|x_1, x_2, \ldots, x_{n-1}).$$

A function $c(u_1, u_2)$ is a *density of copula* for $p(x_1, x_2)$ if

$$p(x_1, x_2) = c(u_1 u_2)p(x_1)p(x_2) = p(x_1)p(x_2|x_1)$$

where u_1 and u_2 are *inverse functions*,

$$\frac{du_1(x_1)}{dx_1} = p(x_1), u_1(x_1) = \int_{-\infty}^{x_1} p(s)ds,$$
$$\frac{du_2(x_2)}{dx_2} = p(x_2), u_2(x_2) = \int_{-\infty}^{x_2} p(r)dr$$

To simplify notation of $\mathrm{u}_1(\mathrm{x}_1), \mathrm{u}_2(\mathrm{x}_2), \ldots, \mathrm{u}_\mathrm{n}(\mathrm{x}_\mathrm{n})$ sometimes we will write u_1, $\mathrm{u}_2, \ldots, \mathrm{u}_\mathrm{n}$ omitting arguments.

A *cumulative function* $C(u_1(x_1), u_2(x_2))$ with inverse functions u_i as arguments is called a *copula* and its derivative $c(u_1, u_2)$ is called a *density of copula*:

$$C(u_1(x_1), u_2(x_2)) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} c(u_1(s), u_2(r)p(s)p(r)dsdr =$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} c(u_1(s), u_2(r)du_1(s)du_2(r)$$

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}$$

Copula properties for 2-D case:

$$\begin{aligned} p(x_1)p(x_2|x_1) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} p(x_1)p(x_2) \\ p(x_2|x_1) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} p(x_2) \end{aligned},$$

Copula properties for 3-D case:

Joint density function:

$$p(x_1, x_2, x_n) = p(x_1)p(x_2|x_1)p(x_3|x_1, x_2) =$$
$$c(u_1, u_2, u_3)p(x_1)p(x_2)p(x_3)$$

Copula, copula density, joint and conditional probabilities.

Below in $c(u_1, u_2, u_3)$ we omitted arguments s_1, s_2, s_3 of u_1, u_2, u_3 for simplicity.

$$C(u_1(x_1), u_2(x_2), u_3(x_3)) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} p(s_1, s_2, s_3)ds_1ds_2ds_3 =$$

$$\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} c(u_1, u_2, u_3)p(s_1)p(s_2)p(s_3)ds_1ds_2ds_3 =$$

$$\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} p(s_1)p(s_2|s_1)p(s_3|s_1, s_2)ds_1ds_2ds_3$$

$$p(x_1) = \frac{du_1}{dx_1}, p(x_2) = \frac{du_2}{dx_2}, p(x_3) = \frac{du_3}{dx_2}$$

$$C(u_1, u_2, u_3) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} c(u_1, u_2, u_3) p(s_1) p(s_2) p(s_3) ds_1 ds_2 ds_3$$

$$c(u_1, u_2, u_3) = \frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2 \partial u_3}$$

$$\begin{aligned}
&p(x_1, x_2, x_n) = p(x_1) p(x_2|x_1) p(x_3|x_1, x_2) = \\
&\frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2 \partial u_3} p(x_1) p(x_2) p(x_3) \\
&p(x_2|x_1) p(x_3|x_1, x_2) = \frac{\partial^2 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2} p(x_2) p(x_3|x_1, x_2) = \\
&\frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2 \partial u_3} p(x_2) p(x_3)
\end{aligned}$$

$$\frac{\partial^2 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2} p(x_3|x_1, x_2) = \frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2 \partial u_3} p(x_3)$$

$$p(x_3|x_1, x_2) = \frac{\partial^3 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2 \partial u_3} \frac{1}{\frac{\partial^2 C(u_1, u_2, u_3)}{\partial u_1 \partial u_2}} p(x_3)$$

Copula properties and definitions for general n-D case:

$$\begin{aligned}
&p(x_n|x_1, x_2, \ldots, x_{n-1}) = \\
&= \frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n} \frac{1}{\frac{\partial^{n-1} C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_{n-1}}} p(x_n)
\end{aligned}$$

Conditional copula:

$$\begin{aligned}
&c(x_n|x_1, x_2, \ldots, x_{n-1}) = \\
&= \frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n} \frac{1}{\frac{\partial^{n-1} C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_{n-1}}}
\end{aligned}$$

Density of copula:

$$\frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n} = c(u_1, u_2, \ldots, u_n)$$

2.2 Examples

When $u(x)$ is a marginal probability $F_{\mathrm{i}}(x), u(x) = F_i(x)$ and u is uniformly distributed, then the inverse function $x(u)$ is not uniformly distributed, but has values concentrated in the central part as in the Gaussian distribution. The inverse process is represented graphically in Figs. 1 and 2.

Consider another example where a joint probability density function p is defined in the two dimensional interval [0, 2]×[0, 1] as follows,

$$p(x, y) = \frac{x + y}{3}$$

Then the cumulative function in this interval is

$$C(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p(s, r) ds dr = \int_{-\infty}^{x} \int_{-\infty}^{y} \frac{s + r}{3} ds dr = \frac{xy(x + y)}{6} \qquad (1)$$

Next we change the reference (x, y) into the reference (u_1, u_2). For

$$p(x_1) = \frac{du_1}{dx_1}, \quad p(x_2) = \frac{du_2}{dx_2}$$

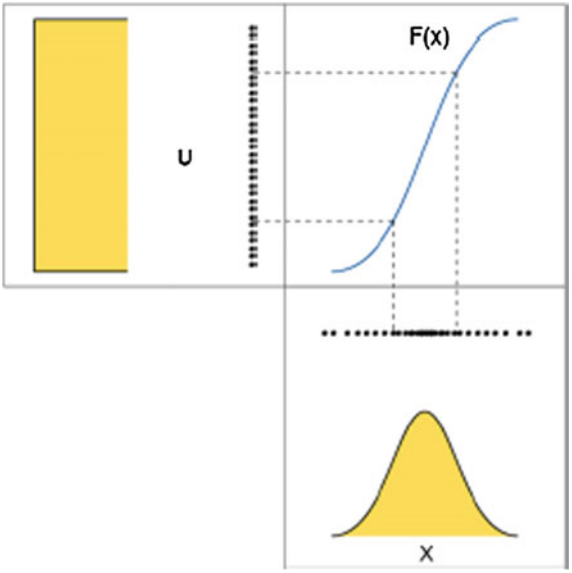

Fig. 1 Relation between marginal probability $F_{\mathrm{i}}(x)$ and the random variable x

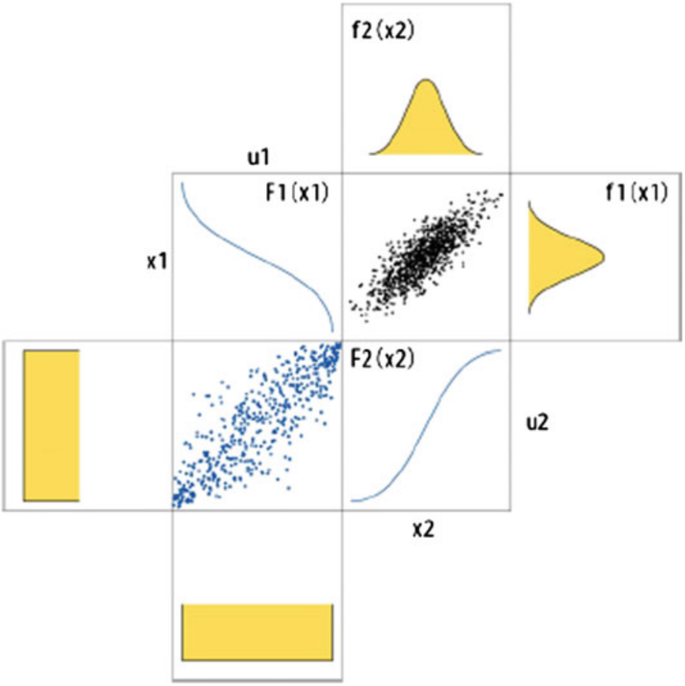

Fig. 2 Symmetric joint probability and copula

and use the marginal probabilities

$$u_1(x) = \int_{-\infty}^{x} p(s)ds, u_2(y) = \int_{-\infty}^{y} p(r)dr$$

to get

$$u_1(x) = C(x, 1) = \frac{x(x+1)}{6},\ x \in [0, 2]$$
$$u_2 = C(2, y) = \frac{y(y+2)}{6}\ y \in [0, 1]$$

This allows us to compute the inverse function to identify the variables x and y as functions of the marginal functions u_1 and u_2:

$$x(u_1) = \frac{\sqrt{1 + 24u_1} - 1}{2} \quad y(u_2) = \sqrt{1 + 3u_2} - 1$$

Then these values are used to compute the copula C in function (1),

$$C(u_1, u_2) = \frac{\sqrt{1+24u_1}-1}{2}(\sqrt{1+3u_2}-1)\frac{\frac{\sqrt{1+24u_1}-1}{2}+(\sqrt{1+3u_2}-1)}{6} = \frac{\sqrt{1+24u_1}-1}{2}(\sqrt{1+3u_2}-1)\frac{\frac{\sqrt{1+24u_1}-1}{2}+(\sqrt{1+3u_2}-1)}{6} \tag{2}$$

3 Approach

3.1 Meaning of Copulas

The main idea of the copula approach is *transforming a multivariate probability distribution* $F(x_1, x_2, \ldots, x_\mathrm{n})$ into a "normalized" multivariate distribution called a *copula*. The normalization includes transforming each of its arguments x_i into the [0, 1] interval and transforming the marginal distributions $F_\mathrm{i}(x_\mathrm{i})$ into uniform distributions. The main result of the copula approach (Sklar's Theorem [24, 35, 36]) is the basis for this. It splits modeling the marginal distributions $F_\mathrm{i}(x)$ from the modeling of the *dependence structure* that is presented by a normalized multivariate distribution (copula).

What is benefit of splitting? The copula is invariant under a strictly increasing transformation. Thus variability of copulas is less than the variability of n-D probability distributions. Therefore the selection of the copula can be simpler than the selection of a generic n-D probability distribution.

Next building a multivariate probability distribution, using the copula, is relatively simple in two steps [37, 38]:

(1) Selecting the univariate margins by transforming each of the one-factor distributions to be a uniform by setting $u_\mathrm{i} = F_\mathrm{i}(x_\mathrm{i})$ where the random variable u_i is from [0, 1].
(2) Selecting a copula connecting them by using the formula (3).

$$C(u_1, u_2, \ldots, u_\mathrm{n}) = F^{-1}(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_\mathrm{n}^{-1}(u_\mathrm{n})) \tag{3}$$

These selections are done by using available data and a collection of parametric models for the margins $F_\mathrm{i}(x_\mathrm{i})$ and the copula to fit the margins and the copula parameters with that data.

Next the property $\dfrac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n} = c(u_1, u_2, \ldots, u_n)$ is used to produce pdf

$$p(x_1, x_2, \ldots, x_\mathrm{n}) = c(u_1, u_2, \ldots, u_\mathrm{n})p(x_1)p(x_2)\ldots p(x_\mathrm{n})$$

where c is a copula density. Another positive aspect of using copulas is the abilities to generate a significant number of samples of n-D vectors that satisfy the copula probability distribution. After a copula in a "good" analytical form is approximated from the limited given data, a copula distribution can be simulated to generate more samples of data distributed according to it.

This is a known practical value of copulas. Does it ensure the quality/value of the produced multivariate distribution? It is noted in [24] that copulas have limited capabilities for constructing *useful* and *well–understood* multivariate pdfs, and much less for multivariate stochastic processes, quoting [39]: "Religious Copularians have *unshakable faith* in the value of transforming a multivariate distribution to its copula". These authors expect future techniques that will go beyond copulas. Arguments of heated discussions on copulas can be found in [40]. The question "which copula to use?" has no obvious answer [24]. This discussion reminds us a similar discussion in the fuzzy logic community on justification of t-norms in fuzzy logic.

3.2 Concept of NTF-Logical Dependence

Dependences traditionally are modeled by two distinct types of techniques based on the probability theory or logic (classical and/or many-valued). It is commonly assumed that dependencies are either stochastic or non-stochastic. Respectively it is assumed that if the dependence is not stochastic then it must be modeled by methods based on the classical and many-valued logic not by methods based on the probability theory.

This is an unfortunate and *too narrow* interpretation of probabilistic methods. In fact Kolmogorov's measures (K-measures) from the probability theory allow modeling both stochastic and non-stochastic "logical" dependencies (both classical and multi-valued). The basis for this is simply considering these measures as general measures of dependence, not as the frequency-based valuations of dependence. For instance, we can measure the dependence between two circles by computing a ratio S/T, where S is the area of their overlap and T is the total area of two circles. This ratio satisfies all of Kolmogorov's axioms of the probability theory, which is a "deterministic" method to measure the dependence in this example without any frequency.

Moreover, many dependencies are a mixture of stochastic and non-stochastic components. These dependencies also can be modeled by K-measures. Below we show this. Consider a dependency that is a mixture of stochastic and non-stochastic components. We start from a frequency-based joint probability distribution $F(x_1, \ldots, x_n)$ and *extract* from it a *dependence* that is not a *stochastic* dependence, but is a "*logical",* "deterministic" dependence. How do we define a logical dependence?

We will call dependence between variables $x_1, \ldots, x_n$ a Non-Truth-Functional *Logical Dependence* (***NTF-logical dependence***) if it is derived from F, but *does not* depend on the frequencies of occurrence of $x_1, \ldots, x_n$.

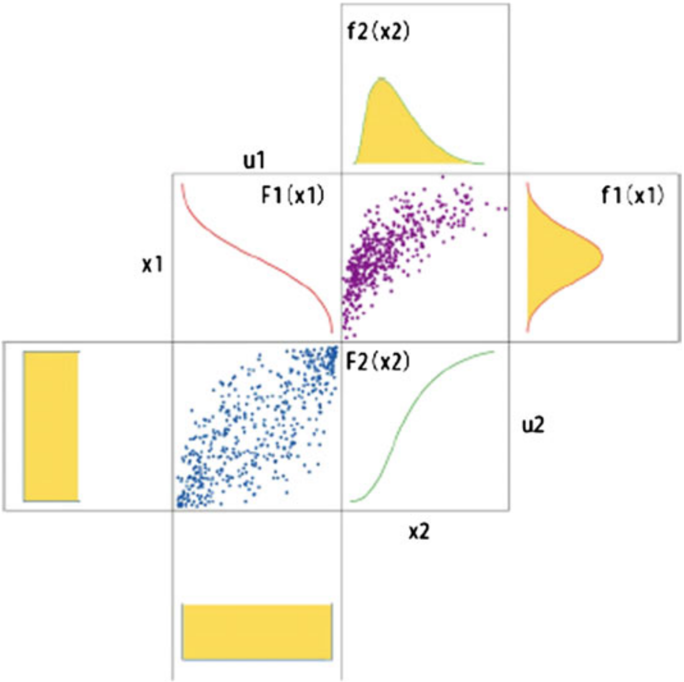

Fig. 3 Asymmetric joint probability and symmetric copula

How do we capture this logical dependence? We use a copula approach outlined above. First we convert (*normalize*) all marginal distributions $F_i(x_i)$ of F to uniform distributions $u_1, \ldots, u_n$ in [0, 1]. This will “eliminate” the dependence from frequencies of variables and differences in scales of variables. Next we build a copula: dependence C between $u_1, \ldots, u_n$, C: $\{(u_1, \ldots, u_n)\} \rightarrow [0, 1]$.

Figure 3 illustrates the difference between two dependencies. The first one is captured by F and the second one is captured by C. Copula C tells us that the logical dependence between x_1 and x_2 is symmetric, but differences in frequencies make the dependence asymmetric that is captured by F.

A similar idea is used in *machine learning* and *data mining* for discovering dependencies in data with *imbalanced classes*. For instance, in cancer diagnostics we can get many thousands of benign training cases and fortunately a much less number of malignant training cases. The actual discovery of the dependency (diagnostic rule) is conducted by selecting a similar number of training cases from both classes [41] which can be done by “normalizing” the marginal distributions of training data to the uniform distributions in each class in a coordinated way.

If F is not based on the frequencies then the interpretation of the marginal distributions is not that obvious. We still can formally convert marginal distributions $F_i(x_i)$ to uniform distributions and analyze two distributions F and C. In this case both distributions can be considered as logical dependencies. Here the copula will indicate dependence where some properties F are eliminated (lost). The usefulness of such elimination must be justified in each particular task. In essence we should show that eliminated properties are not about the mutual dependence of the variables, but are about other aspects of the variables, or even completely irrelevant, insignificant or "noise".

Note that the defined logical dependence is not expressed in the traditional logic form, e.g., as a propositional or first order logic expression, but as a copula, that is a specialized probability distribution. In fact, the copula corresponds to a set of logical expressions

$$\{(x_1 = a_1)\&(x_2 = a_2)\&\ldots\&(x_\mathrm{n} = a_\mathrm{n}) \text{ with probability } c(x_1, x_2, \ldots, x_\mathrm{n})\}$$

where c is a copula density.

The *interpretation* of copula as NTF-dependence is one of the key new points of this work showing how dependence in many-valued logic and probability theory can be fused in one concept by using copulas and marginal probabilities to clarify the relations between PT, FL and MVL. The next sections elaborate this approach.

3.3 Copula and Conditional Probability

Conditional probabilities capture some aspects of dependence between variables that include both frequency-based and logic based aspects. The relations between copula density and conditional probability are as follows as has been shown in Sect. 2:

$$\begin{array}{ll} p(x_2|x_1) = c(u_1, u_2)p(x_2) & p(x_1|x_2) = c(u_1, u_2)p(x_1) \\ p(x_2|x_1)/p(x_2) = c(u_1, u_2) & p(x_1|x_2)/p(x_1) = c(u_1, u_2) \end{array}$$

This shows that copula density as a "frequency-free" dependence can be obtained from frequency-dependent conditional probabilities $p(x_1|x_2)$ by "normalizing" it. We also can see that

$$p(x_2|x_1) \le \mathrm{c}(u_1, u_2)$$
$$p(x_1|x_2) \le \mathrm{c}(u_1, u_2)$$

because $p(x_2) \le 1$. In other words, frequency-free measure of dependence is no less than max of conditional probabilities:

$$\max(p(x_1|x_2), p(x_2|x_1)) \le \mathrm{c}(u_1, u_2)$$

In a logical form copula density represents a measure $c(u_1, u_2)$ of dependence for the conjunction,

$$(x_1 = a_1) \; \& \; (x_2 = a_2)$$

In contrast the conditional probabilities $p(x_1|x_2)$ and $p(x_2|x_1)$ represent measures of dependence for the implications:

$$(x_1 = a_1) \Rightarrow (x_2 = a_2), \qquad (x_2 = a_2) \Rightarrow (x_1 = a_1).$$

In a general case the relation between conditional probability and copula involves conditional density of copula as shown in Sect. 2.

$$p(x_\text{n}|x_1, \ldots, x_{\text{n}-1}) = c(x_\text{n}|x_1, \ldots, x_{\text{n}-1})p(x_\text{n})$$

This conditional density of copula also represents a frequency-free dependence measure.

3.4 Copulas, t-norms and Fuzzy Logic

Many copulas are t-norms [42], which are the basis of many-valued logics and fuzzy logic [31, 32]. Many-valued logics, classical logic, and fuzzy logic are *truth-functional*: "…the truth of a compound sentence is determined by the truth values of its component sentences (and so remains unaffected, when one of its component sentences is replaced by another sentence with the same truth value)" [31]. Only in very specific and simple for modeling cases, probabilities and copulas are made truth-functional. This is a major difference from t-norms that are truth-functional. The impact of the truth functionality on fuzzy logic is shown in the Bellman-Giertz theorem [43]. Probability theory, fuzzy logic and many-values logics all used t-norms, but not in the same way.

The concept of copula and t-norm are not identical. Not every copula is a t-norm [42] and only t-norms that satisfy the 1-Lipschitz condition are copulas [44]. Equating t-norms with copulas would be equating truth-functional fuzzy logic with probability theory that is not truth-functional. A joint probability density $p(x_1, x_2)$ in general cannot be made a function of only $p(x_1)$ and $p(x_2)$. A copula density or a conditional probability is required in addition due to properties $p(x_1, x_2) = c(x_1, x_2) \quad p(x_1)p(x_1)$ and $p(x_1, x_2) = p(x_1|x_2)p(x_2)$. In this sense the *probability theory is more general than fuzzy logic*, because it covers both truth-functional and non-truth functional dependences. Thus, in some sense fuzzy logic with t-norms represents the "*logic of independence*". On the other side, the copula gives a probabilistic interpretation of the fuzzy logic t-norms as well as interpretation of fuzzy logic theory as a many-valued logic.

Postulating a t-norm is another important aspect of the differences in the usage of copulas in FL and PT. In FL and possibility theory, a t-norm, e.g., $\min(x_1, x_2)$ is often postulated as a *"universal"* one for multiple data sets, without the empirical justification for specific data.

In PT according to Sklar's Theorem [24, 35, 36], the copula $C(\text{u}_1, \ldots, \text{u}_\text{n})$ is *specific/unique* for each probability distribution F with uniform marginals. If this copula happened to be a t-norm, then it is a unique t-norm, and is also *specific* for that probability distribution.

This important difference was also pointed out in [45]: “In contrast to fuzzy techniques, where the same “and”-operation (t-norm) is applied for combining all pieces of information, the vine copulas allow the use of different “and”-operations (copulas) to combine information about different variables.” Note that the copula approach not only allows different “and”-operations but it will actually get different “and”-operations when underlying joint probability distributions differ. Only for identical distributions the same “and”-operations can be expected.

The t-norm $\min(x_1, x_2)$ is a dominant t-norm in FL and possibility theory. This t-norm is known in the copula theory as a *comonotonicity copula* [46]. It satisfies a property

$$\mathrm{P}(\mathrm{U}_1 \le \mathrm{u}_1, \ldots, \mathrm{U}_\mathrm{d} \le \mathrm{u}_\mathrm{d}) = \mathrm{P}(\mathrm{U} \le \min\{\mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{d}\}) = \min\{\mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{d}\}, \mathrm{u}_1, \ldots, \mathrm{u}_\mathrm{d} \in [0, 1]$$

where a *single* random uniformly distributed variable $\mathrm{U} \sim \mathrm{U}[0, 1]$ produced the random vector $(U_1, \ldots, U_\mathrm{d}) \in [0, 1]^\mathrm{d}$. The variables of the comonotonicity copula are called *comonotonic* or *perfectly positively dependent variables* [47].

3.5 Forms of Dependence Based on Common Cause and Direct Interaction

Forms of dependence. Dependence can take multiple forms. One of them is dependence with *common source/cause*, i.e., a third object S is a cause of the dependence between A and B. We will call it as *third-party dependence*. In dancing music, S serves as the third party that synchronizes dancers A and B. In physics such a type of dependence is often apparent in synchronization of events, e.g., in laser photon synchronization the source for the photon synchronization is interaction of the photons in a crystal.

Technically the dependencies between A and B in these cases can be discovered by using the correlation methods, which includes computing the copulas and the conditional probabilities $p(A|B)$, $p(B|A)$. However, correlation methods do not uncover the *type of dependency*, namely *third-party* dependence or dependence as a result of *direct local interaction* of objects A and B. The last one is a main type of laws in the classical physics.

Historically correlation based dependencies (e.g., Kepler’s laws) were augmented later by more causal laws (e.g., Newton’s laws). It was done via accumulation of more knowledge in the physics area and much less by more sophisticated mathematical methods for discovering the causal dependencies. This is an apparent now in constructing Bayesian networks (BN) where causal links are mostly built manually by experts in the respective field. Links build by discovering correlations in data are still may or may not be causal.

4 Physics and Expanding Uncertainty Modeling Approaches

This section provides physics arguments for expanding uncertainty modeling approaches beyond current probability theory and classical and non-classical logics to deal with dependent/related evens under uncertainty. After that we propose a way of expanding uncertainty modeling approaches. The first argument for expansion is from the area of duality of particles and waves and the second one is from the area of Bell's inequality [48, 49]. The dependencies studied in the classical physics can be described using the classical logic and the classical set theory.

Logic operates with true/false values assigned to states of particles. To establish the true state of a particle without involving fields it is sufficient to know the state of the individual particle without considering all the others particles. In this sense particles are independent one from the others. In this sense the classical logic and the associated set theory are the conceptual instruments to study classical physics. In quantum physics with the "entanglement" the states of all particles are "glued" in one global entity where any particle is sensitive or dependent to all the others. In this sense the information of any individual particle (local information) is not sufficient to know the state of the particle. We must know the state of all the other particles (global information) to know the state of one particle. The physical phenomena that are studied in the quantum physics have more complex (non-local) dependencies where one physical phenomenon is under the influence of *all* the other phenomena both local and non-local. These dependencies have both stochastic probabilistic component (that can be expressed by a joint probability) and a logic component that we express by copula.

4.1 Beyond Propositional Logic and Probability Theory

Particles and waves are classically considered as distinct and incompatible concepts because particles are localized and waves are not [50]. In other words, particles interact locally or have *local not global dependence*, but waves interact *non-locally* possibly with all particles of its media [51]. The classical propositional logic and set theory have no tools to deal with this issue and new mechanisms are needed to deal with it. The global dependence (interaction with non-local elements of the media) is a property of the structure of the media (whole system). Feynman suggested a concept of the negative probability that later was developed by Suppes and others [51–54] to address such issues.

Limitations of probability theory. Kolmogorov's axioms of PT deal only with *one type of dependence*: dependence of sets elementary events in the form of their *overlap* that is presence of the common elements in subsets. This is reflected in the formula for probability of the union A ∪ B:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B).$$

where dependence is captured by computing $p(A \cap B)$ and subtracting it to get $p(A \cup B)$.

While capturing overlap is a huge *advantage* of PT over truth-functional logics (including fuzzy logic and possibility theory) that do not take into account overlap $A \cap B$. However, axioms of the probability theory do not include other dependencies. In fact, PT assumes *independence* of each elementary event from other elementary events, requiring their overlap to be empty, $\mathrm{e_a} \cap \mathrm{e_b} = \emptyset$.

What are the options to deal with the restriction of the probability theory in the tasks where more dependencies must be taken into account?

We have two options:

- The first option is to *choose other elementary events* that will be really independent.
- The second one is *incorporating relations* into a new theory.

The first option has difficulty in quantum physics. Particles are dependent and cannot serve as independent elementary events. This creates difficulties for building a classical probabilistic theory of particles. Next, at a particular time, we simply may have no other candidates for elementary events that would be more elementary than particles.

Generalization of Probability The previous analysis shows that we need to explore the second option: adding relations between elementary events and their sets. Below we propose a generalization of the probability for dependent events that we call **relation-aware probability**.

Let $\{R_i^+(A, B)\}$ be a set of "positive relations" and $\{R_j^-(A, B)\}$ be a set of "negative relations" between sets A and B. Positive relations increase probability $p(A \cup B)$ and negative relations decreases probability $p(A \cup B)$.

Next we introduce probability for relations $\{p(R_i^+(A, B))\}$, $\{p(R_j^-(A, B))\}$ and define the **relation-aware probability** of the union:

$$p\,(A \cup B) = p\,(A) + p\,(B) + \sum_{i=1}^{n} p(R_i^+\,(A, B)) - \sum_{j}^{m} p(R_j^-(A, B))$$

The classical formula

$$p\,(A \cup B) = p\,(A) + p\,(B) - p(A \cap B)$$

is a special case of this probability with the empty set of positive relations and with a single negative relation, $R\,(A, B) \equiv A \cap B \neq \emptyset$ that is intersection of A and B is not empty. A negative relation can work as "annihilator". A positive relation between A and B can work as a catalyst to increase A or B. For instance, we can get

$$p\,(A \cup B) = p\,(A) + p\,(B) + 0.2p\,(A) + 0.1p(B) - p(A \cap B).$$

As a result, the relation-aware probability space can differ from Kolmogorov's probability space having the more complex additivity axioms.

Another potential approach is associated with changing the classical logic into logics that allow conflicts, e.g., fuzzy logic, agent model of uncertainty [8], and active sets [12]. This will make the paradox more *apparent* due to abilities to embed the conflict via gradual measures of uncertainty such as known in fuzzy logic, and many-valued logic.

4.2 Beyond the Set Theory: Bell Inequality

Consider sets S_1-S_8 formed by three circles A, B and C as shown in Fig. 4, e.g., $S_1 = A \cap B \cap C$ and $S_8 = A^C \cap B^C \cap C^C$ for complements of A, B and C. These sets have the following classical set theory properties that include the *Bell's inequality*

$$\begin{array}{l} (A \cap B^C) \cup (B \cap C^C) = S_1 \cup S_7 \cup S_4 \cup S_6 \\ A \cap C^C = S_7 \cup S_4 \\ A \cap C^C \subseteq (A \cap B^C) \cup (B \cap C^C) \\ \left| A \cap C^C \right| \leq \left| (A \cap B^C) \cup (B \cap C^C) \right| \end{array}$$

Next consider the *correlated* (*dependent*) events, for instance we can have an event with the property A, and another event with a negated property A^C. This is a type of dependence that takes place for particles. Consider an event with property $A \cap C^C$ that is with both properties A and C^C at the same time. We cannot measure the two properties by using one instrument at the same time, but we can use the correlation to measure the second property, if the two properties are correlated. We can also view an event with property $A \cap C^C$ as two events: event e_1 with property A and e_2 with the property C in the opposite state (negated). The number of pairs of events (e_1, e_2) is the same as the number of events with the superposition of A and C^C, $A \cap C^C$. In [49] d'Espagnat explains the connection between the set theory

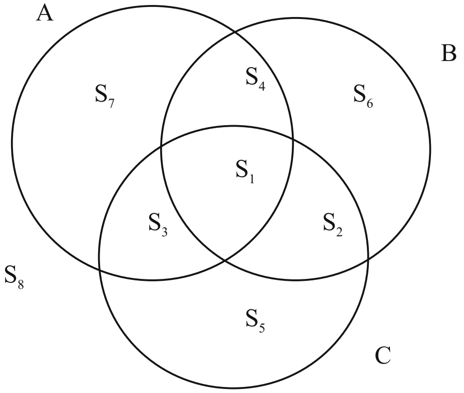

Fig. 4 Set theory intersections or elements

and Bell's inequality. It is known that the Bell's inequality that gives us the reality condition is violated [48].

How to reconcile the logic Bell's inequality and its violation in physics? The Bell inequality is based on the classical set theory that is consistent with the classical logic.

The set theory assumes *empty overlap* (as a form of *independence*) of elementary units. For dependent events the Bell's inequality does not hold. Thus the logic of dependence (logic of dependent events) must differ from the logic of independence (logic of independent events) to resolve this inconsistence. It means that we should use a theory that is *beyond* the classical set theory or at least use the classical set theory differently. The dependence logic [14, 15] is one of the attempts to deal with these issues.

4.3 Dependence and Independence in the Double Slit Experiment

The double slit experiment involves an electron gun that emits electrons through a double slit to the observation screen [55, 56]. Figure 5 [51] shows theoretical result of the double slit experiment when only the set theory is used to combine events: one event e_1 for one slid and another event e_2 for the second slid. In this set-theoretical approach it is assumed that events e_1 and e_2 are *elementary events* that do not overlap, $(e_1 \cap e_2) = \emptyset$, independent). In this case, the probability $p\,(e_1 \cap e_2) = 0$ and the probability that either one of these two events will occur is $p\,(e_1 \cup e_2) = p\,(e_1) + p(e_2)$. The actual distribution differs from distribution shown in Fig. 5 [51].

As an attempt to overcome this dependence difficulty and related non-monotonicity in quantum mechanics the use of *upper probabilities*, with the axiom subadditivity, has been proposed [51–54].

Below we explore whether the *copula* approach and *relation-aware probability* can be helpful in making this dependence *evident* and *explicit*. The relation-aware probability is helpful, because it explicitly uses the relations that are behind the dependences. This is its advantage relative to negative probabilities. The axioms of negative probability have no such components.

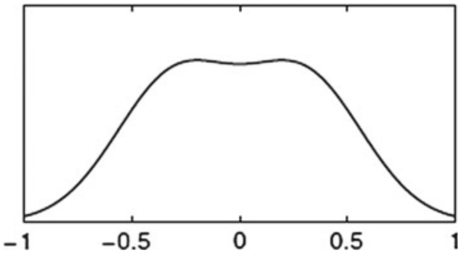

Fig. 5 [51]. Distribution of independent particles (events)

The value of the copula approach for this task needs more explorations. Copula allows representing the actual distribution F as a composition that includes the copula. The actual distribution is [51]

$$p(\alpha_1, \alpha_2) = k\cos^2(\alpha_1 - \alpha_2)$$

Respectively its copula is

$$C(F(\alpha_1), F(\alpha_2)) = k\sin^2[A \cdot B - C \cdot D] = C(u_1, u_2) \tag{4}$$

$$A = \frac{1}{2}\left(\frac{\frac{\pi}{2} \pm \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_1(\alpha_1)})}}{2}\right)^2$$

$$B = \frac{\frac{\pi}{2} \mp \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_2(\alpha_2)})}}{2}$$

$$C = \left(\frac{\frac{\pi}{2} \mp \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_2(\alpha_2)})}}{2}\right)^2$$

$$D = \frac{\frac{\pi}{2} \pm \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_1(\alpha_1)})}}{2}$$

This copula represents the dependence with the eliminated impact of marginal distributions via conversion of them into the uniform distribution. In this way the copula has the meaning of *graduate dependence between variables* resembling the many-valued logic, and t-norm compositions of variables. This is a valuable property. Thus while the copula is still a probability distribution it also represents a gradual degree of truth and falseness. The copula (5) is tabulated as follows:

$$\mathrm{M1} = \begin{pmatrix} 1 & 0.846 & 0.7 & 0.561 & 0.43 & 0.307 & 0.192 & 0.088 & 0 \\ 0.846 & 1 & 0.969 & 0.899 & 0.805 & 0.692 & 0.56 & 0.401 & 0.125 \\ 0.7 & 0.969 & 1 & 0.979 & 0.923 & 0.839 & 0.725 & 0.572 & 0.25 \\ 0.561 & 0.899 & 0.979 & 1 & 0.982 & 0.93 & 0.843 & 0.708 & 0.375 \\ 0.43 & 0.805 & 0.923 & 0.982 & 1 & 0.982 & 0.927 & 0.82 & 0.5 \\ 0.307 & 0.692 & 0.839 & 0.93 & 0.982 & 1 & 0.98 & 0.909 & 0.625 \\ 0.192 & 0.56 & 0.725 & 0.843 & 0.927 & 0.98 & 1 & 0.973 & 0.75 \\ 0.088 & 0.401 & 0.572 & 0.708 & 0.82 & 0.909 & 0.973 & 1 & 0.875 \\ 0 & 0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1 \end{pmatrix}.$$

It has the same form in its extreme values as classical logic equivalence relation ($\equiv$).

$$y = x_1 \equiv x_2 = \begin{bmatrix} & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This link is also evident from [42, p. 182]:

$$\mathrm{C(x, 0) = C(0, x) = 0, C(x, 1) = C(1, x) = x}$$

which is true for all x from [0, 1] including x = 1.

5 Conclusion and Future Work

This paper had shown the need for a theory *beyond* the classical set theory, classical logic, probability theory, and fuzzy logic to deal with dependent events including the physics paradox of particles and waves and the contradiction associated with the Bell inequality.

It is shown that the *copula* approach and *relation-aware probability* approach are promising approaches to go beyond the limitations of known approaches of PT, FL, MVL, and the classical set theory. Specifically *copula* and *relation-aware probability* approaches allow making dependences between variables in multidimensional probability distributions more *evident* with abilities to model the dependencies and relations between multiple variables more e*xplisit*.

It is pointed out that the concept of copula is *more general* than the concept of t-norm. Therefore the probability theory in some sense is *more general* than fuzzy logic, because it covers both truth-functional and non-truth functional dependences.

The *third-party dependence, direct local interaction* and causal dependence are analyzed concluding that discovering causal dependencies is far beyond current approaches in PT, FL, and MVL.

We introduced the concept of a Non-Truth-Functional *Logical Dependence* (***NTF-logical dependence***) derived as a copula from multidimensional probability distribution that does not depend on the frequencies of occurrence of variables. The interpretation of copula as NTF-dependence is one of the key of new points of this work. It shows how dependence in many-valued logic and probability theory can be fused in one concept by using the copulas and marginal probabilities to clarify the relations between PT, FL, and MVL to give a “frequency-free” dependence measure. It shows how, starting from probability and specifically from conditional probability, we introduce “logic” dependence.

We conclude that presence of t-norms in both probability theory (in copulas), and in fuzzy logic, creates a new opportunity to join them using t-norm copulas as a bridge.

Future directions are associated with linking the proposed approaches with dependence logic. Dependence logic proposed in 2007 [14] was later expanded to the contexts of modal, intuitionistic and probabilistic logic. Its goal is modeling the dependencies and interaction in dynamical scenarios. It is a type of first-order logic, and a fragment of second-order logic. While it claims applications in a wide range of domains from quantum mechanics to social choice theory, it still is in earlier stages of applications.

Another future opportunity to build the more realistic uncertainty modeling approaches is associated with the involvement of agents. Agents and teams of agents often are a source of input data, and the values of probabilities, copulas, and t-norms. The situation, when these agents are in a total agreement, can be modeled as a *total dependence* of their valuations, graduated or discrete (True/False). When a team of agents is in a conflict with each other in the valuation of the same attribute, they provide conflicting degrees of truth/falseness, which require adequate modeling approaches [8]. Active sets [12] that represent sets of agents can be a bridge between the many-valued logic, and fuzzy sets for providing the interpretable input data (valuations).

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Note to the Polemics Surrounding the Second Gödel's Theorem

Klimentij Samokhvalov

Abstract An extensive philosophical debate continues for years about impact of arithmetization of metamathematics on the Hilbert program with many papers published. Many of these works claim a death-blow to Hilbert's Program by Gödel's Second Incompleteness Theorem. This note provides a short and understandable argument that in fact Gödel's Second Incompleteness Theorem does not deliver the notorious death-blow (the *coup de grace*) to Hilbert's Program.

Keywords Primitive recursive arithmetic · Gödel's numbering · Gödel's canonical Consis · Second Gödel's incompleteness theorem · Hilbert's program

1 Introduction

A philosophical debate about the impact of arithmetization of metamathematics on the Hilbert program intensified after publishing "Arithmetization of metamathematics in a general setting" by Feferman [1] in 1960. The general content and the course of this debate are presented in excellent reviews [2, 3] that include an extensive bibliography. However, in our opinion, this "drama of ideas" requires the additional analysis that is provided below.

Let S be a formal system in the first order language with signature

$$(+, \cdot, ', 0, \approx).$$

It is assumed that S includes Primitive Recursive Arithmetic (PRA). We say that S is *consistent* if and only if an arithmetic sentence X exists, for which it is not true that X is provable in S.

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Symbolically,

$$\mathrm{CON}(S) \Leftrightarrow \exists X \text{ (it is not true that } \vdash_S X), \tag{1}$$

where CON(S) is an acronym for "S is consistent", and $\vdash_S \ldots$ is an abbreviation for "… is provable in S". Due to (1), it is clear that for any particular arithmetical formula P, the implication

$$\text{(It is not true that } \vdash_S P) \Rightarrow \mathrm{CON}(S) \tag{2}$$

takes place. It is also clear that the reverse implications, generally speaking, are not true. For example, the implication

$$\mathrm{CON}(S) \Rightarrow \text{(it is not true that } \vdash_S 0 \approx 0)$$

is not true. In fact, the sentence $0 \approx 0$ is provable in PRA, and S includes PRA.

It is also clear that the second Gödel's incompleteness theorem can effectively find a certain special case, wherein the reverse implication is true. In fact, this theorem says:

$$\mathrm{CON}(S) \Leftrightarrow \text{(it is not true that } \vdash_S \mathrm{Con}_S), \tag{3}$$

where Con_S is a canonical Gödel's *consis* (definite arithmetical formula of some special type).

Note that only finite provable statements are credible for staunch Hilbertians. Therefore, if the second Gödel's theorem is generally intended to have at least some value for them, it must be acknowledged that the equivalence (3) is proved by finite means.

Moreover, in terms of Hilbertians, this finitely establishing equivalence (3) fully covers the content of the given theorem. So, for Hilbertians everything else that is usually attributed to Gödel's second theorem is idle talk.

This includes the following usual formulation of Gödel's second theorem:

> if the system S is consistent, then the arithmetic sentence Con_S that *expresses consistency of S in S* is not provable in S.

In this formulation, the phrase in italics is redundant. For, whatever it meant, the actual finite proof of the theorem is independent of it.

In this finite proof, Con_S appears simply as a specific formal formula in language S, with respect to which it is important to know in advance only one thing: whether this formula is provable or not provable in S. It does not matter in this case at all, whether it expresses something or does not express anything, whether it is true or false, understandable or not, etc.

2 What Is "Expressed" by Gödel's Theorem?

When somebody says: "A formula Con_S 'expresses' consistency of S in S", we can ask: "What is this specific formula 'expresses' in a different choice of Gödel's numbering?". Then it becomes clear that in fact Con_S "expresses" not a property of the system S, which is the consistency of S, but the certain property of an ordered pair (g, S), where g is Gödel's numbering. Substituting one for the other is logically incorrect. In this circumstance one can see some inadequacy of *arithmetized* metamathematics to metamathematics envisioned by Hilbert initially.

On the other hand, finitely established equivalence (3) does not say anything about whether non-arithmetic meta-statement $\mathrm{CON}(S)$ or non-arithmetic meta-statement (It is not true that $\vdash_S \mathrm{Con}_S$) are finitely provable or not.

Therefore, the second Gödel's theorem is not "a fatal blow to Hilbert's program".

The natural context of both Gödel's incompleteness theorems is just the incompleteness of sufficiently strong consistent formal systems, rather than the question of the absence or presence of the finite proof of their consistence.

3 Afterword

Sadly, of course, a finite proof of the consistency of arithmetic is still not found. However, it should be remembered that Hilbert described the concept of a finite proof somewhat vaguely and never gave a precise definition of this concept, relying on the ability to immediately recognize, as soon as some reasoning is there, whether it is finite, or not [4].

It is not hard to guess here that Hilbert, like all rational people, was well aware that even a well-deserved trust in something is always an empirical psychological fact, which can be established in advance only as probable, and only within some empirical hypothesis. However, it was certainly distasteful for Hilbert, who was prone to some form of Kantian philosophy to base mathematics on the empirical hypotheses. Naturally, therefore, he described the concept of finite proof without giving exact definition.

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Ontological Data Mining

Evgenii Vityaev and Boris Kovalerchuk

Abstract We propose the ontological approach to Data Mining that is based on: (1) the analysis of subject domain ontology, (2) information in data that are interpretable in terms of ontology, and (3) interpretability of Data Mining methods and their results in ontology. Respectively concepts of Data Ontology and Data Mining Method Ontology are introduced. These concepts lead us to a new Data Mining approach—Ontological Data Mining (ODM). ODM uses the information extracted from data which is interpretable in the subject domain ontology instead of raw data. Next we present the theoretical and practical advantages of this approach and the Discovery system that implements this approach. The value of ODM is demonstrated by solutions of the tasks from the areas of financial forecasting, bioinformatics and medicine.

1 Introduction

At the International Workshop on Philosophies and Methodologies for Knowledge Discovery (22–26 August 2005, Copenhagen, Denmark) it was pointed out that any KDD&DM (Knowledge Discovery in Data Bases and Data Mining) ***method*** has its own ***ontology*** [1–4]. Any KDD&DM method explicitly or implicitly assume:

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1. some types of input data;
2. some language for data interpretation and hypothesis class construction (knowledge space of the KDD&DM method);
3. confirmed hypothesis in this language.

By the ontology of KDD&DM & ML method we mean the language for constructing hypotheses. At the same time any ***subject domain*** also has its own ***ontology*** that includes the system of notions for description of objects. If we apply some KDD&DM & ML method for solving a task in the given subject domain, we want to get results that are interpretable in the ontology of that subject domain. For that purpose we need a following requirement.

Requirement. *For the interpretability of the KDD&DM & ML results, the ontology of KDD&DM & ML method must be interpretable in the ontology of the subject domain.*

For example, to apply a classification method that uses a language of spherical shapes for hypotheses, we need interpret spherical shapes in the ontology of the subject domain.

The knowledge extracted by a KDD&DM method on some data is a set of confirmed hypothesis that are interpretable in the ontology of the KDD&DM method and at the same time in the ontology of the subject domain.

The subject domain ontology induces ***data ontology***. We need to emphasize that quantities in data are not numbers themselves, but numbers with their ***interpretation***. For example, abstract numbers 200, 3400, 300, 500 have three different interpretations shown in Table 1.

For every quantity there are relations and operations that are meaningful for this quantity. This interpretation of quantities is a core approach of the Representational Measurement Theory (RMT) [5–7]. The RMT interprets quantities as *empirical*

Table 1 Values and their interpretation

Interpretation	Values	Meaningful operations
Abstract numbers	5, 3400, 360, 500	Meaning of 360 > 5 is not clear. These numbers can be just labels
Abstract angles	5, 3400, 360, 500	360 meaningfully greater that 5. It is implicitly assumed that angles are rotational angles
Azimuth angles	5, 3400, 360, 500	Azimuth operations. 360 meaningfully less that 5, because Azimuth 360 = Azimuth 0
Rotational angles	5, 3400, 360, 500	Rotational angle operations. 360 meaningfully greater that 5 if the angle represents the rotation time

systems—algebraic structures defined on objects of subject domain with a set of relations and operations interpretable in the subject domain ontology.

More specifically, main statements of the measurement theory relative to data mining issues are as following [5–7]:

- numerical representations (scales) of quantities and data are determined by the corresponding empirical systems;
- scales are unique up to a certain sets of permissible transformations such as changing measurement units from meters to kilometers for ratio-scales;
- KDD&DM & ML methods need to be invariant relative to the sets of permissible transformations of quantities in data.

To obtain data ontology in accordance with the measurement theory, we need to transform data into many-sorted empirical systems. This transformation is described in [3, 8] for such data types as pair comparisons, binary matrices, matrices of orderings, matrices of proximity and an attribute-based matrix. Such transformation faces the following problem. Many physical quantities possess an interpretable operation • which has all formal properties of the usual addition operation. However, medicine and other areas may have no empirical interpretation for that operation. For example, empirical system of physical pressure, measured by a tonometer, have operation • as physical quantity, but have no this operation as medical quantity, because it is not interpretable in medicine. In that case, the • operation should be removed from the empirical system of pressure in medicine, and the corresponding scale should be reconsidered. The order relation for pressure is obviously interpretable in medicine, and it can be included in the empirical system of the pressure. Thus, data ontology and scales strongly depend on the subject domain ontology.

Consider another example from the area of finance. What is the ontology of financial time series? It also can be considered from points of view (onthologies) of: (1) a trader-expert, (2) one of the mathematical disciplines, (3) technical analysis, (4) trade indexes etc. We need to specify the ontology of a subject domain before determining the data ontology.

If a KDD&DM & ML method uses operations or relations that are not interpretable in the data ontology and hence in the subject domain ontology, then it may obtain non-interpretable results. For example, the average patients' temperature in the hospital has no medical interpretation. If all relations and operations, which are used in the method, are included in the data ontology, then the algorithm results will be interpretable in the subject domain ontology and hence invariant relative to the permissible transformations of scales of the used data.

Thus, to avoid the non-invariance of the method and non-interpretability of its results, we need to use only relations and operations from the data ontology. It means that hypotheses tested by the method must include only these relations and operations. However, as we pointed out above, scales of quantities depend on the data ontology. Hence, the invariance of the method cannot be established before we revise the scales of all quantities based on the data ontology.

2 Relational (Ontological) Approach to Data Mining

Since in any application of KDD&DM & ML methods we need to established data ontology and transform data into many-sorted empirical systems, we proposed a Relational Data Mining (RDM) approach [1–3, 9] to knowledge discovery that is working directly with the data presented as many-sorted empirical systems. We also developed a software system "Discovery" that implements this approach in a rather general form and extracts knowledge from the many-sorted empirical systems for various classes of hypothesis.

Thus, Relational Data Mining approach consists of the following stages:

1. transform data into many-sorted empirical systems according to data ontology;
2. use background knowledge expressed in the first-order logic for learning and forecasting;
3. determine, for corresponding Data Mining task, the hypothesis class in terms of many-sorted empirical system;
4. acquire knowledge, by testing hypotheses from the hypothesis class on many-sorted empirical systems;
5. use acquired knowledge for predictions, classifications, pattern recognition etc.

Relational Data Mining approach provides following possibilities that cannot be performed by other Data Mining methods:

- analyze data with unusual scales. Traditionally Data Mining methods use only few scale types, but in the measurement theory, there are known hundreds scale types;
- perform data exploration—when we can simultaneously vary data ontology, information extracted from data and hypothesis classes. The appropriate version of the system discovery was developed and applied for several tasks [10];
- acquire knowledge for data onthologies for which there are no Data Mining methods with appropriate set of relations and operations. Below we give examples of such data onthologies and corresponding hypotheses.

Relational Data Mining approach and Discovery system use logic-probabilistic techniques related to the Probabilistic Logic Programming for knowledge acquisition. However, there is a problem of probability and logic synthesis for this approach. This problem was discussed at the series of conferences with the title Progic (Probability+Logic, 2006–2013) [11]. In the framework of our approach, we propose a new solution to this problem based on special semantic probabilistic inference [9, 12–14]. The software system Discovery implements the semantic probabilistic inference in the process of hypotheses testing and prediction. We demonstrated in series of experiments that, in contrast with the Probabilistic Logic Programming the Discovery system can acquire knowledge from data with a significant noise level, for example, in financial forecasting [8]. Moreover, the Discovery system acquire maximal specific rules, for which we prove that they are predict without contradictions [9, 13].

3 Applications of Relational Data Mining

In the frame of Relational Data Mining approach, some problems were solved in the area of financial forecasting, bioinformatics, medicine, and other areas [15]. Here we present data onthologies and hypotheses classes for these tasks.

3.1 Financial Forecasting

As data ontology for the S&P500 (close) forecasting, we used information about local maxima and minima and their order interconnections. In some experiments we used information about weekdays with second and third differences between prices for these days [8, 16].

As objects, we used the periods for five consequent days for which the following relations and operations where determined:

1. function wd(**a**) that display five consequent days, for example wd(**a**) = $\langle 1, 2, 3, 4, 5\rangle$ means 5 days beginning from Monday;
2. first order difference:

$$\Delta_{ij}(a_t) = (SP500(a_t^j) - SP500(a_t^i))/SP500(a_t^i),\ i < j,\ i,j = 1,\dots,5,$$

   where a_t—5 days, a_t^j, a_t^i—corresponding i-th and j-th day of the 5 days. This function represents the difference of SP500C between the i-th and j-th days, normalized relative the i-th day;
3. second order difference, i.e., the difference between two first order differences

$$\Delta_{ijk}(a_t) = \Delta_{jk}(a_t) - \Delta_{ij}(a_t).$$

In this ontology, we stated following classes of hypotheses:

1. $(wd(a) = wd(b) = \langle d_1,\dots,d_5\rangle)\&(\Delta(a) \le \Delta(b))^{\varepsilon 1} \Rightarrow (SP500(a^5) \le SP500(b^5))^{\varepsilon 0}$, where $\Delta(a)$, $\Delta(b)$—arbitrary first or second order differences for 5 days periods **a** and **b** with the same sequences of days; $\varepsilon 1,\ \varepsilon 0 = 1$ (0) if relation has no (has) negation, for example A^1 means simply A, while A^0 means "not A".
   The example of the discovered regularity on real SP500C data for that hypotheses class is:
   IF for any 5-day objects **a** and **b** beginning from Tuesday,
   AND the SP500 difference $\Delta_{12}(a_t)$ is smaller the difference $\Delta_{12}(b_t)$,
   THEN the target stock for the last day of **a** will be greater than for the last day of **b**.

2. At the same time, hypotheses with two differences in the premise where tested:

$$(wd(a) = wd(b) = \langle d_1, \ldots, d_5\rangle)\&(\Delta(a) \le \Delta(b))^{\varepsilon 1}\&(\Delta(a) \le \Delta(b))^{\varepsilon 2}$$
$$\Rightarrow (SP500(a^5) \le SP500(b^5))^{\varepsilon 0},$$

3. The hypotheses with three and more relations at the premise also where tested:

$$(wd(a) = wd(b) = \langle d_1, \ldots, d_5\rangle)\&(\Delta(a) \le \Delta(b))^{\varepsilon 1}\&\ldots\&(\Delta(a) \le \Delta(b))^{\varepsilon k}$$
$$\Rightarrow (SP500(a^5) \le SP500(b^5))^{\varepsilon 0}.$$

For example, one of the tested statements was:
IF for any 5-day objects **a** and **b** with weekdays $\langle d_1, \ldots, d_5\rangle$,
the SP500C difference $\Delta_{12}(a_t)$ is smaller than $\Delta_{12}(b_t)$
AND the SP500C difference $\Delta_{23}(a_t)$ is greater than $\Delta_{23}(b_t)$
AND the SP500C difference $\Delta_{123}(a_t)$ is greater than $\Delta_{123}(b_t)$
AND ...
THEN the target stock for the last day of **a** will be greater than for the last day of **b**.

The example of rules discovered by the system Discovery is as follows:

IF Current 5 days end on Monday and there are some other ("old") 5 days (from the history of years 1984–1996) that end on Monday too

AND the relative SP500C difference between Tuesday and Thursday for the old 5 days is not greater than between Tuesday and Thursday for the current 5 days

AND the relative SP500C difference between Tuesday and Monday for the old 5 days is greater than between Tuesday and Monday for the current 5 days

AND the relative difference between SP500C differences for Tuesday, Wednesday and Wednesday, Thursday for the old 5 days is not greater than for the pairs of days for the current 5 days

AND we omit linguistic description of $(\Delta_{245}(a_t) > \Delta_{245}(b_t))$, which is similar to previous one

THEN the target value for Monday from the current 5 days should be not greater than the target value for the Monday from the old 5 days, i.e., we forecast that a target stock 5 days ahead from the current Monday will grow not greater than it was 5 days ahead from the old Monday.

3.2 DNA Regulatory Regions Analysis

We asked experts of the Institute of Cytology and Genetics SD RAS about the information that is needed for the solution of the analysis of regulatory regions. From their point of view this information is the following: distance between valuable DNA signals that may vary in some range, repetition of signals, some DNA intervals which

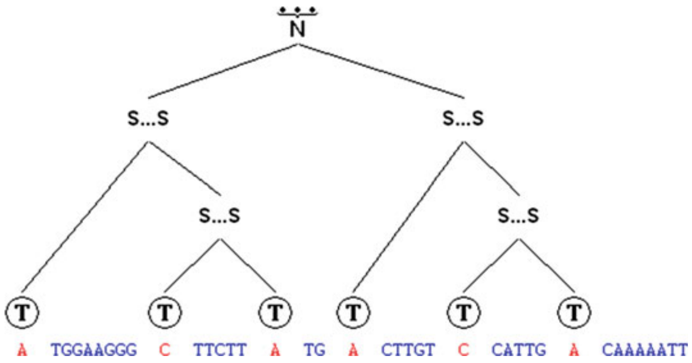

Fig. 1 Structure of complex signal

have a specific place relative to transcription start, and hierarchy of signals. Corresponding relations and operations where defined and hypothesis in these terms where formulated as a notion of a Complex Signals (CSs), which is defined recursively based on elementary signals and operations applied to them, see Fig. 1 [17, 18]. The recursive definition of the Complex Signal is following:

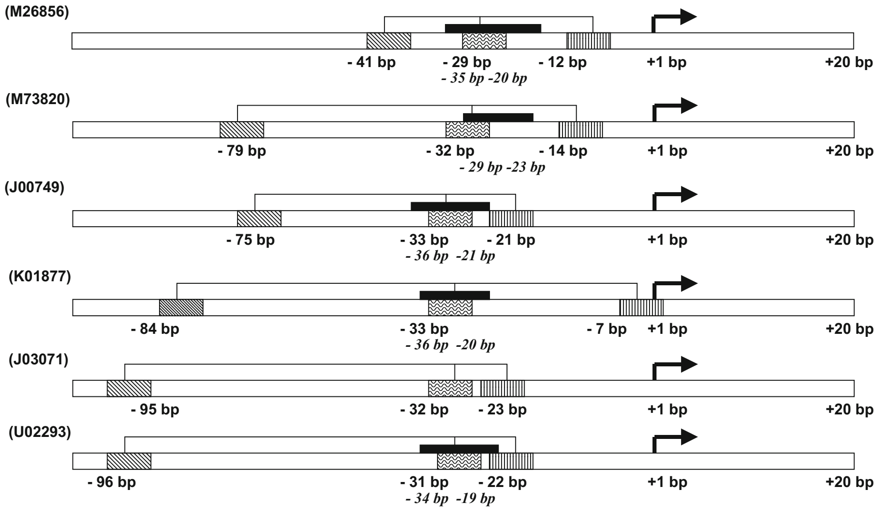

Fig. 2 Schematic localization of the complex signal $CWGNRGCN < NGSYMTAM < MAGKSHCN$ in promoters of endocrine system genes. The promoter sequences are aligned relative to the transcription start (position +1 bp), indicated by *arrows*. The EMBL identifiers of the promoters studied are given in parentheses to the *left*. The eight-bp oligonucleotide motifs composing the complex signal are shown as *dashed rectangles*; positions of the first nucleotides are indicated relative to the transcription start. The *black rectangles* mark experimentally defined positions of the TATA-box indicated in the TRRD database. Positions of its first and last nucleotides are italicized

1. the elementary signal is CS;
2. the result of the "repetition" or "interval" operation applied to CS is CS;
3. the result of the "distance" operation applied to a pair of CSs is CS.

Elementary signals are indivisible signals which are characterized by a name and location in the sequence. It may be nucleotide, motive, transcription factor binding site and any other signal. Figure 1 illustrates such signals. The examples of the complex signals are presented on the Fig. 2 [17, 18]. As elementary signals there where motifs of length 8 bp that were pre-selected as specific for the set of promoters by the program ARGO [19].

3.3 Breast Cancer Diagnostic System

Expert J.Ruiz (Baton Rouge Women hospital) defined 11 features, for developing the breast cancer diagnostic system [2, 20]. These features constituted the data ontology.

Table 2 Discovered breast cancer diagnostic rules

Diagnostic rule	F-criteria	Value of F-criteria			Diagnosis Round-Robin test (%)
		0.01	0.05	0.1	
IF NUMber of calcifications per cm^2 between 10 and 20	0.0029	+	+	+	93.3
AND VOLume > $5\,\text{cm}^3$	0.0040	+	+	+	
THEN Malignant					
IF TOTal # of calcifications >30	0.0229	–	+	+	100.0
AND VOLume > $5\,\text{cm}^3$	0.0124	–	+	+	
AND DENSITY of calcifications is moderate	0.0325	–	+	+	
THEN Malignant					
IF VARiation in shape of calcifications is marked	0.0044	+	+	+	100.0
AND NUMber of calcifications is between 10 and 20	0.0039	+	+	+	
AND IRRegularity in shape of calcifications is moderate	0.0254	–	+	+	
THEN Malignant					
IF variation in SIZE of calcifications is moderate	0.0150	–	+	+	92.86
AND variation in SHAPE of calcifications is mild	0.0114	–	+	+	
AND IRRegularity in shape of calcifications is mild	0.0878	–	–	+	
THEN Benign					

Using this ontology and data in terms of this ontology the set of regularities was discovered. Table 2 some of these regularities are presented.

4 Representative Measurement Theory

In accordance with the measurement theory, numerical representations of quantities and data are determined by empirical systems. In this section we present main definitions of the measurement theory [5–7].

An *empirical system* is a relational structure that contains a set of objects A, $k\ (i)$—ary relations $P_1, \ldots, P_n$ and $k\ (j)$—ary operations $\rho_1, \ldots, \rho_m$ defined on A,

$$\mathbf{A} = \langle A, P_1, \ldots, P_n, \rho_1, \ldots, \rho_m \rangle.$$

Every relation P_i is a Boolean function (a predicate) with $k\ (i)$ arguments from A, and ρ_j is the $k\ (j)$ argument operation on A. System **R**

$$\mathbf{R} = \langle R, T_1, \ldots, T_n, S_1, \ldots, S_m \rangle$$

is called a *numerical system of the same type as a system* **A**, if R is a subset of Re^m, m $\geq$ 1, Re^m is a set of m-tuples of real numbers, every relation T$_i$ has the same arity k(i) as the corresponding relation P$_i$, and every real-value function S_j has the same arity k(j) as the corresponding operation ρ_j.

A numerical system **R** is called a *numerical representation* of the empirical system **A**, if a (strong) homomorphism $\phi : A \to R$ exists such that:

$$P_i(a_1, \ldots, a_{k(i)}) \Rightarrow T_i(\phi(a_1), \ldots, \phi(a_{k(i)})),\ i = 1, \ldots, n;$$

$$\phi(\rho_j(a_1, \ldots, a_{k(j)})) = S_j(\phi(a_1), \ldots, \phi(a_{k(j)})), j = 1, \ldots, m.$$

The strong homomorphism means that, if predicate $T_i(\phi(a_1), \ldots, \phi(a_{k(i)}))$ is true on $\langle \phi(a_1), \ldots, \phi(a_{k(i)}) \rangle$, then there exists tuple $\langle b_1, \ldots, b_{k(i)} \rangle$ in A, such that $P_i(b_1, \ldots, b_{k(i)})$ is true and $\phi(b_1) = \phi(a_1), \ldots, \phi(b_{k(i)}) = \phi(a_{k(i)})$. We will denote such homomorphism between the empirical system **A** and numerical system **R** as $\Phi : \mathbf{A} \to \mathbf{R}$. Thus, the numerical system **R** represents a relational structure in computationally tractable form with a complete retention of all the properties of the relational structure.

In the measurement theory, the following problems are considered:

(1) find a numerical representation **R** for an empirical system **A**;
(2) prove a theorem that homomorphism $\Phi : \mathbf{A} \to \mathbf{R}$ exists;
(3) define the set of all possible automorphisms $f : R \to R$ (the uniqueness theorems), such that fΦ is also homomorphism $f\Phi : \mathbf{A} \to \mathbf{R}$.

Example A relational structure $A = \langle A, P\rangle$ is called a semi-ordering, if for all a, b, c, d ∈ A the following axioms are satisfied:

$$\begin{aligned}&\neg P(a,a);\\&P(a,b)\&P(c,d) \Rightarrow (P(a,d) \vee P(c,b));\\&P(a,b)\&P(b,c) \Rightarrow \forall d \in A(P(a,d) \vee P(d,c)).\end{aligned}$$

Theorem 1 *If* $A = \langle A, P\rangle$ *is a semi-ordering and* $A/\approx$ *is finite, then there exists a function* $U : A \to Re$*, such that:*

$$P(a,b) \Leftrightarrow U(a) + 1 < U(b).$$

There are hundreds of numerical representations known in the measurement theory with few most commonly used. The strongest one is called the absolute data type (*absolute scale*). The weakest numerical data type is the nominal data type (*nominal scale*). There is a spectrum of data types between them. They allow us comparing, ordering, adding, multiplying, dividing values and so on. The classification of these data types is presented in Table 3. The basis of this classification is a transformation group. The strongest absolute data type does not permit transformations of data at all, and the weakest nominal data type permits any one-to-one transformation. Intermediate data types permit different transformations such as positive affine, linear and others (see Table 3).

The transformation groups are used to determine the invariance of a regularity. The regularity expression must be invariant to the transformation group; otherwise it will depend not only on the nature, but on the subjective choice of the measurement units.

Table 3 Classification of data types

Transformation	Transformation group	Data type (scale)
$X \to \quad f(x)$,	F:Re → (onto) Re, 1 → 1 transformation group	Nominal
$X \to \quad f(x)$,	F:Re → (onto) Re monotone transformation group	Order
$X \to rx + s, r > 0$	Positive affine group	Interval
$X \to tx^r, t, r > 0$	Power group	Log-interval
$X \to x + s$	Translation group	Difference
$X \to tx, t > 0$	Similarity group	Ratio
$X \to x$	Identity group	Absolute

5 Data Ontology in Different Subject Domains

From the measurement theory point of view, data is a many-sorted empirical system **A** with the sets of relations and operations interpretable in the domain theory. For instance, a "stock price" data type can be represented as a relational structure $\mathbf{A} = \langle A; \{\leq, =, \geq\}\rangle$ with nodes A as individual stock prices and arcs as their relations $\{\leq, =, \geq\}$. As we pointed out scales are strongly depend on the subject domain ontology.

Let us consider the specificity of domain anthologies for different subject domains. We consider six different cases:

Physical data in physical domains.
Physical data in non-physical domains.
Non-physical data in non-physical domains.
Nominal discrete data.
Non-quantitative and non-discrete data.
Mix of data.

1. ***Physical data in physical domains***. Data contain only physical quantities, and the subject domain is physics. This is a realm of physics with well-developed data ontology and measurement procedures. In this case, the measurement theory [6] provides formalized relational structures for all physical quantities and KDD&DM methods can be correctly applied.

2. ***Physical data for non-physical domains***. The data contain physical quantities, but the subject domain is not physics. The ontology of the subject domain may refer to finance, geology, medicine, and other areas. In these cases data ontology is not known, as we pointed out above for the pressure in medicine. If the quantity is physical, then we can define the relational structure using the measurement theory. However, the physically interpretable relations of the relational structure are not necessarily interpretable in ontology of subject domain. If a relation is not interpretable, it should be removed from the relational structure.

3. ***Non-physical data in non-physical domains***. For non-physical quantities, data ontology is virtually unknown. There are two sub-cases:

Non-numerical data types. In [3, 8] it was developed a procedure for transformation the following representations of data into data ontology (many-sorted empirical systems): pair-wise and multiple comparison data types, attribute-based data types, and order, and coherence matrixes.

Numerical data types. Here, we have a measurer x(a), which produces a number as a result of a measurement procedure applied to an object *a*. Examples of measurers are psychological tests, stock market indicators, questionnaires, and physical measuring instruments used in non-physical areas.

For this case let us define a data ontology as the set of *relations and operations* for the measurer x(a). For any numerical relation $R(y_1, \ldots, y_k) \subset Re^k$ and operation $S(x_1, \ldots, x_m) : Re^m \to Re$, where Re is the set of real numbers, an *relation* P^R on A^k and an *operation* $\rho^S : A^m \to A$ can be defined as follows

$$P^R(a_1,\ldots,a_k) \Leftrightarrow R(x(a_1),\ldots,x(a_k)),\ \rho^S(a_1,\ldots,a_m) = s(x(a_1),\ldots,x(a_m)).$$

We should find such relations R and operations S that have interpretation in the subject domain ontology. The set of obtained interpretable relations is not empty, because at least one relation ($P^=$) has an empirical interpretation: $P^=(a_1,a_2) \Leftrightarrow x(a_1) = x(a_2)$. In the measurement theory, many sets of axioms were developed for data, having only ordering and equivalence relations. For instance, given weak order relation $<_y$ (for the attribute y) and n equivalence relations $\approx_{x_1}, \ \ldots, \ \approx_{x_n}$ for the attributes $x_1,\ldots,x_n$ one can construct a complex relation $G(y,x_1,\ldots,x_n) \Leftrightarrow y = f(x_1,\ldots,x_n)$ (defined by the axiomatic system) between y and $x_1,\ldots,x_n$, such that $f(x_1,\ldots,x_n)$ is a polynomial [6]. A polynomial function uses multiplication, power and sum operations. Hence, these operations can be defined for $y,\ x_1,\ldots,x_n$ using only relations $<_y, \approx_{x_1}, \ \ldots, \ \approx_{x_n}$. Ordering and equivalence relations are usually empirically interpretable in the ontology of various subject domains.

4. ***Nominal discrete data types***. Here, all numbers can be considered as names, and can be easily represented as predicates with a single variable. So, data are interpretable in the corresponding relational structures, because there is no difference between the numerical and empirical systems.

5. ***Non-quantitative and non-discrete data types***. Data contain no quantities and discrete variables, but do contain ranks, orders and other non-numerical data types. This case is similar to the above item 3a.

6. ***Mix of data types***. All the mentioned difficulties arise in this case.

6 Invariance of the KDD&DM Methods

The results of the KDD&DM & ML methods must not depend on the subjective choice of the measurement units, but usually it is not the case. Let us define the notion of invariance of a KDD&DM & ML method. To that end, we will use the common (attribute-based) representation of a supervised learning [21, 22] (see Fig. 3), where:

W = {w} is a training sample;

X(w) = ($x_1,\ldots,x_n$) is the tuple of values of n variables (attributes) for training sample w;

Y(w) is the target function assigning the target value for each training example w;

The result of the learning of some KDD&DM & ML method M on the training sample $\{\langle X(w), Y(w)\rangle\}$, $w \in W$ is a rule J

$$J = M\left(\{\langle X(w), Y(w)\rangle\}\right),$$

that predicts values of the target function Y(w). For example, consider w with unknown value Y(w), but with known values of all attributes X(w), then

$$J(X(w)) = Y(w),$$

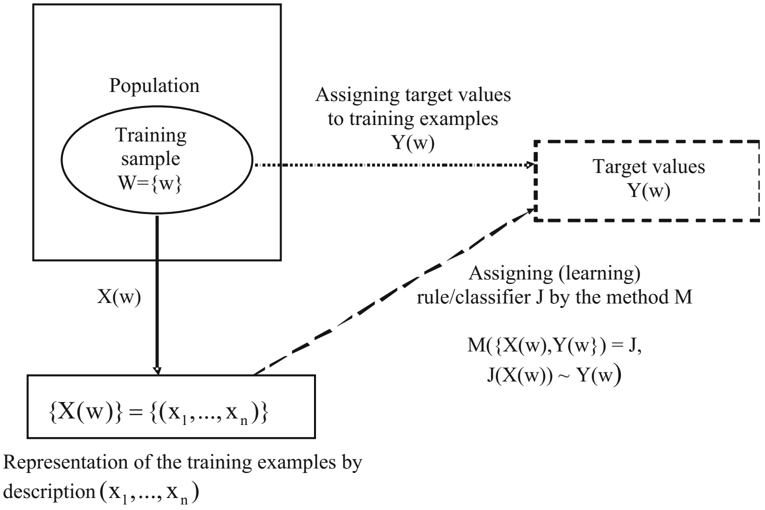
Population
Training sample W={w}
Assigning target values to training examples Y(w)
Target values Y(w)
X(w)
Assigning (learning) rule/classifier J by the method M
M({X(w),Y(w}) = J,
J(X(w)) ~ Y(w)
{X(w)} = {(x1,...,xn)}
Representation of the training examples by description (x1,...,xn)

Fig. 3 Representation of a supervised learning

where $J(X(w))$ is a value generated by the rule J. The resulting rule J can be an algebraic expression, a logic expression, a decision tree, a neural network, a complex algorithm, or a combination of these models.

If attributes $(x_1, \ldots, x_n, y)$ are determined by the empirical systems $A_1, \ldots, A_n$, B having the transformation groups $g_1, \ldots, g_n$, g respectively, then the transformation group G for all attributes is a product $G = g_1 \times \cdots \times g_n \times g$.

The KDD&DM & ML method M is invariant relative to the transformation group G iff for any g $\in$ G rules

$$J = M(\langle X(w), Y(w)\rangle),\ J_g = M(\langle gX(w), gY(w)\rangle),$$

produced by the method M, generate the same results for any w $\in$ W

$$gJ\,(X\,(w))\ = J_g(g\,(X\,(w))).$$

If the method is not invariant (that is the case for majority of the methods), then predictions generated by the method depend on the subjective choice of the measurement units.

The invariance of the method is closely connected to the interpretability of its results. The numerical KDD&DM & ML methods assume that operations such as +, –, *, / can be used in an algorithm despite their possible non-interpretability. In this case, the method can be non-invariant and can deliver non-interpretable results. In contrast a KDD&DM & ML method M is invariant if it uses only information from empirical systems A$_1$, …, A$_n$,B as data and produces rules J that are logical expressions in terms of these empirical systems. This approach is a core of the Relational Data Mining.

7 Relational Methodology for the Analysis of KDD&DM Methods

A non invariant KDD&DM & ML method $M : \{X(w)\} \to J$ can be analyzed and another invariant method can be created based on M. Let us define a many-sorted empirical system A(W) that is a product of empirical systems $A_1, \ldots, A_n, B$ bound on the set W. Next we define transformation $W \to A(W)$ of data W into a many-sorted empirical system $A(W)$, and replace the numeric representation

$$W \to \{\langle X(w), Y(w)\rangle\}$$

by the transformation

$$W \to A(W) \to \{\langle X(w), Y(w)\rangle\}.$$

Based on method $M : \{\langle X(w), Y(w)\rangle\} \to J$, we define a new method $ML : A(W) \to J$ such that

$$ML(A(W)) = M(\{\langle X(w), Y(w)\rangle\}) = J,$$

using transformation $W \to A(W) \to \{\langle X(w), Y(w)\rangle\}$. Thus, method ML uses only interpretable information from data $A(W)$ and produces the rule J using method M.

Let us analyze the transformation of the interpretable information $A(W)$ into the rule J through the method M. If we apply only interpretable relations and operations from the method M (and change non interpretable ones to appropriate interpretable) to the interpretable information $A(W)$, we may extract some logical rule JL from rule J. This rule JL will contain only interpretable information from the rule J, expressed in terms of empirical system $A(W)$.

Let us define the next method

$$MLogic : A(W) \to JL,$$

where rule JL is a set of logical rules in terms of empirical system $A(W)$, produced by method M, and interpretable information $A(W)$.

The method MLogic is obviously invariant. If we consider all possible data for the method M, and all rules JL, that may be produced by the MLogic method, then we will obtain a class of rules (hypotheses) {JL} (knowledge space) of the method M.

As a result we obtain the ontology of the particular KDD&DM & ML method M as:

1. empirical system A(W) as a types of input data;
2. the set of relations and operations of the empirical system $A(W)$ as a language for data interpretation and hypothesis class $\{\text{JL}\}$ construction (knowledge space);
3. discovered regularities as confirmed hypotheses.

Relational (ontological) approach to data mining includes (1) using an empirical system $A(W)$ as a form of the types of input data, (2) testing any hypotheses class $\{\text{JL}\}$ in terms of empirical system $A(W)$ and (3) produce regularities as confirmed hypotheses. In this sense Relational (ontological) approach to data mining approximates any other Data Mining method.

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