Abstractions for Transition Systems with Applications to Stubborn Sets

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Abstract. *Partial order reduction* covers a range of techniques based on eliminating unnecessary transitions when generating a state space. On the other hand, *abstractions* replace sets of states of a system with abstract representatives in order to create a smaller state space. This article explores how stubborn sets and abstraction can be combined. We provide examples to provide intuition and expand on some recent results. We provide a classification of abstractions and give some novel results on what is needed to combine abstraction and partial order reduction in a sound way.

1 Introduction

The term *partial order reduction* refers to methods that combat state explosion by eliminating unnecessary transitions. This article focuses on *stubborn sets* [20]. The theory as presented here, mostly applies to also *ample* [17] and *persistent* [9] sets. We use the term "stubborn set method" or "partial order reduction" to mean any method that attempts to reduce the size of a state space by exploring some subset of enabled transitions in each state of a state.

The term *abstraction* [3] refers to methods that eliminate some features of a system, by mapping the states of a to a smaller set. The goal of abstraction is to preserve counterexamples to specifications while bringing down the complexity of model checking. Abstractions can be thought of as equivalence relations over states and an abstract state space is generated by expanding the relevant transitions that are enabled in the equivalence class. In this sense abstraction includes also methods such as symmetry [5]. Abstractions have been combined with partial order reduction methods both in the early literature and more recently. Significant synergistic benefits between abstraction and reduction was gained with a relaxed zone abstraction of timed automata [11]. In [1], partial order reduction was combined with an abstraction that replaces bisimilar states with a common representative. We discuss the result in this article.

We take the view in this article that *transitions* of systems are inherently *deterministic*, i.e., each transition has a unique outcome. We use the term *firing* for the execution of a single transition. Abstraction may then result in *nondeterminism*, because an abstraction may not distinguish between two states from which a given transition is fired, while still differentiating between the states that result when the transition is fired.

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The interpretation of the transitions from a semantic standpoint is done by associating transitions with *actions*. Several transitions may be associated with she same action, and this gives rise to the concept of *operational* (non) determinism, which refers to whether the external behavior of the system is deterministic. The relationship between operational determinism and abstraction is a complicated one, and we provide some insights on the issue in this article.

This article is organized so that we first explore a general theory of transition systems in Sect. 2, which gives the ground theory and semantic models, and also some results regarding determinism. Section 3 defines abstractions and abstract state spaces, and we prove that general abstractions behave monotonously only with linear time semantic models that are not significantly stronger than traces and divergence traces. Then, a state-of-the art version of stubborn sets for preservation of safety properties and some divergence properties are given in Sect. 4. We also discuss some static relations of transitions that can be used for the computation of stubborn sets.

Section 5 provides results about how stubborn sets can be combined with abstraction. We provide a few theorems for certain classes of abstractions, to show how relations for computing the stubborn sets generalize for the abstract state spaces. We also provide some examples that show that the results do not apply for abstractions in general.

The last section provides some concluding remarks and outlines future work.

2 Theoretical Preliminaries

We consider a system where transitions operate on a collection of n variables with domains $X_1, \ldots, X_n$. The domains will in most cases be numerable, but this need not be the case in general. For example, in the case of timed automata, clocks can assume non-negative real values. We denote the set of syntactically possible states by $X = X_1 \times \cdots \times X_n$.

A *transition* is a pair (g, e), where $g : X \to \{\mathit{true}, \mathit{false}\}$ is called a *guard* and $e : X \to X$ is called an *effect*. The set of transitions of the system is denoted with $\mathcal{T}$.

The *initial value* of a system is denoted $\hat{x} \in X$. We call the tuple $(X, \mathcal{T}, \hat{x})$ a *system description*, or simply a *system*.

The execution semantics of a system are defined over *labeled transition systems* (*LTS*s).

Definition 1 (LTS-unfolding). *An LTS is a 4-tuple $(S, \mathcal{T}, \Delta, \hat{s})$, where S is a set of* states*, $\Delta \subseteq S \times \mathcal{T} \times S$ is a set of* semantic transitions*, and $\hat{s} \in S$ is the* initial state*.*

Given a system $M = (X, \mathcal{T}, \hat{x})$, the LTS-unfolding *of the system M is given as $L = (S, \mathcal{T}, \Delta, \hat{s})$ where*

- *$\hat{s} = \hat{x}$.*
- *Δ and S are defined as the minimal sets such that*

1. *$\hat{x} \in S$, and*
2. *whenever $(g, e) = t$ and $t \in \mathcal{T}$, $x \in S$ and $g(x) = true$, then $e(x) \in S$ and $(x, t, e(x)) \in \Delta$.*

We refer to the LTS-unfolding of the system as the concrete state space *of M.*

Given an LTS, we define the following notation for convenience. We write $s \xrightarrow{t} s'$, when $(s, t, s') \in \Delta$. $s \xrightarrow{t}$ means that $\exists s' : s \xrightarrow{t} s'$. $s \not\xrightarrow{t}$ means that $s \xrightarrow{t}$ does not hold. For a sequence $t_1 t_2 \cdots t_n \in \mathcal{T}^*$, $s \xrightarrow{t_1 \cdots t_n} s'$ means that $\exists s_0, \ldots, s_n$ such that $s = s_0 \wedge s' = s_n$ and, for $0 < i \leq n$, $s_{i-1} \xrightarrow{t_i} s_i$. When we write $s_0 \xrightarrow{t_1 \cdots t_n} s_n$, we let s_i for $0 < i \leq n$, denote the state such that $s_0 \xrightarrow{t_1 \cdots t_i} s_i$. The set $en(s) = \{t \in \mathcal{T} | s \xrightarrow{t}\}$ refers to transitions enabled at s and $dis(s) = \mathcal{T} \backslash en(s)$ refers to actions disabled at s.

Because an effect of a transition is a function, a concrete state space is always *deterministic*, i.e., if $s \xrightarrow{t} s'$ and $s \xrightarrow{t} s''$, then $s' = s''$.

The properties of the system are determined as behaviours interpreted over a set of symbols Σ, called *actions* or *events*. We assume that a mapping $l : \mathcal{T} \to \Sigma \cup \{\epsilon\}$ is given and fixed, and extended to Σ^* in the usual manner by concatenating, i.e., for $l(t_1 \cdots t_n) = l(t_1)l(t_2) \cdots l(t_n)$, and ϵ denotes the empty string. We write $s \overset{\sigma}{\Rightarrow} s'$ when there are transitions $t_1, \ldots, t_n$ such that $s \xrightarrow{t_1 \cdots t_n} s'$ $l(t_1 \cdots t_n) = \sigma$.

In keeping with tradition, when there is some $t \in \mathcal{T}$ such that $l(t) = a$ and $s \xrightarrow{t} s'$, we also write $s \xrightarrow{a} s'$, except when $a = \epsilon$, when we write $s \xrightarrow{\tau} s'$. We refer to such transitions as τ-transitions or invisible transitions. We write $\sigma \leq \rho$ if σ is a prefix of ρ and $\sigma < \rho$ if it is a proper prefix.

A *semantic model* is an equivalence or pre-order relation for systems or LTSs. For the purpose of this article, we consider semantics over LTSs. Two systems are considered equivalent if and only if their concrete state spaces are equivalent.

There are several semantic models which can be considered, we shall consider only a few most relevant. Note that we restrict this study to linear time properties. The sets of *traces*, *divergence traces*, *failures*, and *stable failures* of an LTS are defined as:

$$\begin{aligned}
&\mathsf{Tr}(L) = \{\sigma \in \Sigma^* \mid \hat{s} \overset{\sigma}{\Rightarrow}\}\\
&\mathsf{Divtr}(L) = \{\sigma \in \Sigma^* \mid \exists s : \hat{s} \overset{\sigma}{\Rightarrow} s \wedge s \xrightarrow{\tau^\omega}\}\\
&\mathsf{Mindiv}(L) = \{\sigma \in \mathsf{Divtr}(L) \mid \forall \rho < \sigma : \rho \notin \mathsf{Divtr}(L)\}\\
&\mathsf{Fail}(L) = \{(\sigma, F) \in \Sigma^* \times 2^\Sigma \mid \exists s : \hat{s} \overset{\sigma}{\Rightarrow} s \wedge \forall a \in F : \neg(s \overset{a}{\Rightarrow})\}\\
&\mathsf{Sfail}(L) = \{(\sigma, F) \in \Sigma^* \times 2^\Sigma \mid \exists s : \hat{s} \overset{\sigma}{\Rightarrow} s \wedge \forall a \in F \cup \{\tau\} : \neg(s \xrightarrow{a})\}\\
&\mathsf{CSP}(L) = \mathsf{Sfail}(L) \cup \{(\sigma, X) \mid \exists \rho : \rho \leq \sigma \wedge \rho \in \mathsf{Divtr}(L) \wedge X \in 2^\Sigma\}
\end{aligned}$$

The CSP-set is known as the *failures-divergences*-model, which is named like this as it is commonly associated with the process-algebra CSP [18]. It preserves Sfail up to minimal divergence traces, and all divergence traces are extended with maximal behaviour in terms of Sfail. It is also worth to mention the so called CFFD-equivalence [24], which preserves infinite traces, Sfail, and Divtr. It also preserves, under suitable interpretation, all linear temporal logic properties of a system. A comprehensive survey of different semantic models and epistemological considerations behind them can be found in [26].

Let L_1 and L_2 be LTSs. We write $L_1 \sqsubseteq_{\mathsf{Tr}} L_2$ if and only if $\mathsf{Tr}(L_1) \subseteq (L_2)$. For the other semantic sets $L_1 \sqsubseteq_X L_2$ is defined analogously. We also write $L_1 \sqsubseteq_{X,Y} L_2$ to mean $L_1 \sqsubseteq_X L_2 \wedge L_1 \sqsubseteq_Y L_2$ If $L_1 \sqsubseteq_X L_2$ and $L_2 \sqsubseteq_X L_1$, we say that L_1 and L_2 are X-equivalent, and write $L_1 \equiv_X L_2$. We also write $M_1 \sqsubseteq_X M_2$ if and only $L_1 \sqsubseteq_X L_2$, where L_1 and L_2 are concrete state spaces of M_1 and M_2. We abuse the notation slightly by writing $\mathsf{Tr}(s) = \{\sigma \mid s \overset{\sigma}{\Rightarrow}\}$ and $s \sqsubseteq_X s'$ for states analogously.

We distinguish between determinism of the transition system, and *operational determinism*, i.e. determinism from the point of view of the actions it performs. Operational determinism, also known as *determinacy* [12,16] is defined as follows.

Definition 2. *An LTS* $L = (S, \mathcal{T}, \Delta, \hat{s})$ *is* operationally deterministic *if and only if for all traces* σ, *if* $\hat{s} \overset{\sigma}{\Rightarrow} s_1$ *and* $\hat{s} \overset{\sigma}{\Rightarrow} s_2$, *then*

1. *For each* $a \in \Sigma$, $s_1 \overset{a}{\Rightarrow}$ *if and only if* $s_2 \overset{a}{\Rightarrow}$, *and*
2. $s_1 \xrightarrow{\tau^\omega}$ *if and only if* $s_2 \xrightarrow{\tau^\omega}$.

Except for the treatment of divergences, all the semantic equivalences between trace + divergence trace equivalence and divergence sensitive branching bisimulation collapse into one equivalence for operationally deterministic LTSs [6,12].

The following theorem is evident.

Theorem 1. *For every LTS* L, *there exists an operationally deterministic* L_D *such that* $L \equiv_{\mathsf{Tr,Divtr}} L_D$.

For trace equivalence the theorem is well-known, and for finite LTSs a simple variant of the block-splitting algorithm produces exactly the equivalent LTS. For divergence traces, it is sufficient to store a local τ-loop in diverging states.

We provide a significant strengthening of [12, Corollary 1]. The following lemma applies in the absence of divergences.

Lemma 1. *Assume there are no divergences and* $L_1 \equiv_{\mathsf{Fail}} L_2$. *Then* L_1 *is operationally deterministic if and only if* L_2 *is.*

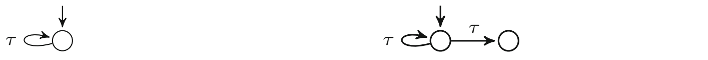
Proof. Let $\hat{s}_1 \overset{\sigma}{\Rightarrow} s_1$ in L_1. Now, there must be a state s_2 of L_2 such that $\hat{s}_2 \overset{\sigma}{\Rightarrow} s_2$, due to trace equivalence, which is implied by Fail-equivalence. Assume that L_1 is operationally deterministic and denote by $F_1 = \{a \in \Sigma \mid \neg(s_1 \overset{a}{\Rightarrow})\}$ and $F_2 = \{a \in \Sigma \mid \neg(s_2 \overset{a}{\Rightarrow})\}$. Firstly, assume $a \notin F_1$. If $\neg(s_2 \overset{a}{\Rightarrow})$, $F_2 \neq F_1$, which contradicts Fail-equivalence. Secondly, assume $a \in F_1$. Then σa is not a trace of L_1 because L_1 is operationally deterministic, so that $a \in F_2$ must also hold. This implies F_2 must be the same for every such state and hence L_2 is operationally deterministic. □

Theorem 2. *Let* $L_1 \equiv_{\mathsf{Divtr,Sfail}} L_2$. *Then* L_1 *is operationally deterministic if and only if* L_2 *is.*

Proof. Assume L_1 is operationally deterministic. Let $\hat{s}_1 \overset{\sigma}{\Rightarrow} s_1$ in L_1 and $\hat{s}_2 \overset{\sigma}{\Rightarrow} s_2$ in L_2. If s_1 is not diverging, s_2 cannot be diverging either due to Divtr-equivalence. Lemma 1 guarantees then that s_1 and s_2 must agree on failures.

If s_1 is diverging, then $\sigma \in \mathsf{Divtr}(L_1)$, and $(\sigma, X) \notin \mathsf{Sfail}(L_1)$ for any X. Thus s_2 must also be diverging, or there would be $(\sigma, X) \in \mathsf{Sfail}(L_2)$ for some X, which would contradict equivalence. □

Theorem 2 is strong enough for the purpose of this article, but we conjecture that a stronger theorem would hold. In process-algebra terms, with a reasonable set of operators, if $\equiv_P$ is a "reasonable" congruence and $L_1 \equiv_P L_2$ for any nondeterministic L_1 and L_2, then P is not stronger than $\mathsf{Tr}, \mathsf{Divtr}$. We leave the exact formulation of "reasonable" and research of the theory for future research.



(a) Operationally deterministic (b) Operationally nondeterministic

Fig. 1. Two CSP-equivalent processes

The failures/divergences theory associated with CSP considers divergence as *chaos*, or maximally nondeterministic behaviour. Neither Theorem 1 nor 2 holds for CSP-equivalence, the counterexample to the latter is shown in Fig. 1. An operationally deterministic LTS may be CSP-equivalent with a nondeterministic one. It should be noted, however, that the interpretation of operational determinism we use is different from the one usually associated with CSP, as it is customary to assume that diverging processes are not deterministic in the context of CSP.

A system is *specified* in terms of some semantic model P by giving a set $\mathcal{L}$, or, alternatively, an LTS that has the requires semantics. We say that the system M satisfies the specification if $P(L) \subseteq \mathcal{L}$ where L is the concrete state space of M. If M does not satisfy the specification, then there exists some behaviour that is not in $\mathcal{L}$. For example, if we specify in terms of traces, then $\hat{s} \overset{\sigma}{\Rightarrow}$ such that $\sigma \notin \mathcal{L}$. The execution $\hat{s} \xrightarrow{t_1 \cdots t_n} s$ such that $l(t_1 \cdots t_n) = \sigma$ is called a (concrete) *counterexample*. Similarly, a counterexample with respect to Divtr or Sfail is execution of L that diverges or has a stable failure not specified by $\mathcal{L}$.

3 Abstraction

Definition 3. *Let $M = (X, \mathcal{T}, \hat{x})$ be a system and X' be any set. An* abstraction *of the system α is a mapping $\alpha : X \to X'$, and an* abstract unfolding *or α-unfolding of M is the LTS $L^\alpha = (S^\alpha, \mathcal{T}, \Delta^\alpha, \hat{s}^\alpha)$, which is the minimal LTS satisfying*

1. $S^\alpha \subseteq X'$.
2. $\hat{s}^\alpha = \alpha(\hat{x}) \in S^\alpha$.

3. $(x_1, t, x_2) \in \Delta^\alpha$ *only if* $x_1 \in S^\alpha$, *and* $\exists x, y \in X : \alpha(x) = x_1$ *and* $g(x) = true$, $y = e(x)$ *and* $\alpha(y) = x_2$.
4. *If* $x_1 \in S^\alpha$, $\exists x, y \in X : \alpha(x) = x_1$ *and* $g(x) = true$, $y = e(x)$ *and* $\alpha(y) = x_2$, *and* $x_2 \neq x_1$ *or* $l(t) \in \Sigma$, *then* $(x_1, t, x_2) \in \Delta^\alpha$.

We refer to the α-unfolding as an *abstract state space*. Please note that as per the definition, L^α is not required to preserve such τ-transitions of L that firing the transition results in the same abstract state, though it is allowed to do so.

We use otherwise the same notation for abstract state spaces, but to avoid confusion we write $\xrightarrow{t}_\alpha$ and $\overset{a}{\Rightarrow}_\alpha$ for transitions and executions. An abstract unfolding gives rise to a different semantic sets than the concrete unfolding. For example, if $\mathcal{L}$ is a specification in terms of traces and given an abstraction α, an *abstract counterexample* is an execution $\hat{s}^\alpha \xrightarrow{t_1 \cdots t_n}_\alpha$ such that $l(t_1 \cdots t_n) \notin \mathcal{L}$.

If α is the identity mapping on X, then L^α is simply the unfolding of the system. Each α with range X' induces an equivalence relation for the states of the concrete state space, such that $s, s' \in S$ are equivalent if $\alpha(s) = \alpha(s')$. We denote by $X_\alpha = \{[s]_\alpha \mid s \in S\}$ where $[s]_\alpha = \{x \in X : \alpha(x) = \alpha(s)\}$. We mostly use this notation and write $[s]_\alpha$ for the states of L^α.

If α_1 and α_2 are abstractions, we write $\alpha_1 \prec \alpha_2$ iff for every $s \in X$ we have $[s]_{\alpha_1} \subseteq [s]_{\alpha_2}$. We say in such a case that α_1 is a *refinement* of α_2, and α_2 is said to be *coarser* than α_1. The identity mapping on X is a refinement of every abstraction, and a mapping that maps the whole of X into a single element set is the coarsest possible abstraction.

It is possible that the abstract unfolding is nondeterministic: if there are two states u, v such that $\alpha(u) = \alpha(v)$, and (u, t, w_1) and (v, t, w_2) are two (concrete) transitions, it may still be that $\alpha(w_1) \neq \alpha(w_2)$. To complicate matters further, it may be that $\alpha_1 \prec \alpha_2$ such that L^{α_1} is nondeterministic and L^{α_2} is deterministic. Consider Fig. 2. The fist LTS is a concrete state space, and we have two abstractions, $\alpha_1 \prec \alpha_2$. L^{α_1} is nondeterministic, while L^{α_2} is deterministic.

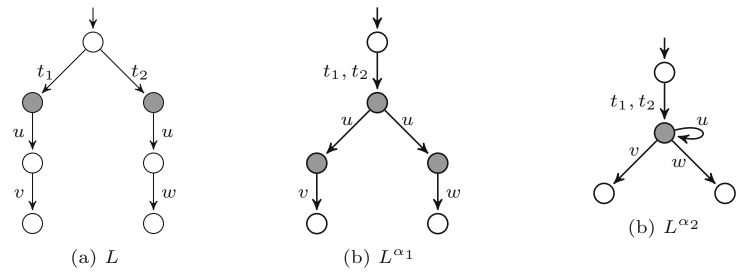


Fig. 2. Determinism may be both destroyed and introduced

We give here some properties of abstractions that can be used firstly, to overcome the limitation imposed by Theorem 3, and secondly, to deduce some important properties later when we combine abstractions with reductions.

Definition 4. *Let $M = (X, \mathcal{T}, \hat{x})$, where $X = X_1 \times \cdots \times X_n$. Let $\mathcal{A}$ be a collection of abstractions for M We say that $\mathcal{A}$*

– respects divergences *if and only if for every $\alpha \in \mathcal{A}$, $[s]_\alpha \xrightarrow{\tau}_\alpha [s]_\alpha$ implies that there is some $s' \in [s]_\alpha$ such that $s' \xrightarrow{\tau^\omega}$,*
– preserves divergences *if and only if for every $\alpha \in \mathcal{A}$ if there are $s_0, \ldots, s_{n-1} \in [s]_\alpha$ and $s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} s_{n-1} \xrightarrow{\tau} s_0$, then $[s]_\alpha \xrightarrow{\tau}_\alpha [s]_\alpha$,*
– *is* monotonous with respect to *the semantic model P if and only if for every $\alpha_1, \alpha_2 \in \mathcal{A}$, if $\alpha_1 \prec \alpha_2$, then $L^{\alpha_1} \sqsubseteq_P L^{\alpha_2}$,*
– preserves *the semantic model P if and only if for every $\alpha \in \mathcal{A}$, $s \in S$ $\alpha(s) \equiv_P s$,*
– respects *the semantic model P if and only if for every $\alpha \in \mathcal{A}$, $s_1, s_2 \in S$, $s_1 \sqsubseteq_P \alpha(s_1)$.*
– *is a collection of* 1-simulations *if and only if for every $\alpha \in \mathcal{A}$, $s_1, s_2, s_3 \in S$, and $t \in \mathcal{T}$, if $s_1 \xrightarrow{t} s_2$, $\alpha(s_1) = \alpha(s_3)$ and $s_3 \xrightarrow{t}$, then there exists $s_4 \in S$ such that $s_3 \xrightarrow{t} s_4$ and $\alpha(s_3) = \alpha(s_4)$,*
– preserves (operational) determinism *if and only if for every $\alpha_1, \alpha_2 \in \mathcal{A}$ such that $\alpha_1 \prec \alpha_2$, L^{α_2} is (operationally) deterministic if L^{α_1} is (operationally) deterministic,*
– respects enabling *if and only if, for every $\alpha \in \mathcal{A}$, if $[s]_\alpha = [s']_\alpha$, then $en(s) = en(s')$,*
– $\mathcal{A}$ respects stability *if and only if, for every $\alpha \in \mathcal{A}$, if $[s]_\alpha = [s']_\alpha$, then either (1) neither s, s' or $[s]_\alpha$ is stable, or (2) there exist stable states $s_1, s_1' \in [s]_\alpha$ such that $s \overset{\epsilon}{\Rightarrow} s_1$, $s' \overset{\epsilon}{\Rightarrow} s_1'$ and $en(s_1) = en(s_1') = en([s]_\alpha)$,*
– separable *if and only if, for every $\alpha \in \mathcal{A}$ where $\alpha : X \to X'$, $X' = X_1' \times \cdots X_n'$, and there exists $\alpha_1, \ldots \alpha_n$, $\alpha_i : X_i \to X_i'$ for $i = 1, \ldots, n$, such that for $x = (x_1, \ldots, x_n) \in X$, $\alpha(x) = (\alpha_1(x_1), \ldots, \alpha_n(x_n))$.*

From these definitions, the following proposition is self-evident.

Proposition 1. *An arbitrary collection of abstractions is monotonous with respect to* Tr*. If it preserves divergences, it is also monotonous with respect to* Divtr*.*

As a corollary, every concrete counterexample to Tr or Divtr specification has a corresponding abstract counterexample regardless of abstraction. Proposition 1 is the main reason why we adopt the trace model as our canonical model, as other semantic models are less robust with respect to abstraction.

Theorem 3. *Assume P is a semantic model that preserves* Fail*. There exists a family of abstractions $\mathcal{A}$ such that $\mathcal{A}$ preserves divergences, and there are $\alpha_1, \alpha_2 \in \mathcal{A}$, $\alpha_1 \prec \alpha_2$ and some L such that $L^{\alpha_1} \not\sqsubseteq_P L^{\alpha_1}$.*

Proof. Theorem 2 and the example in Fig. 2 prove this. □

Theorem 3 together with Theorem 2 mean that except for Tr and Divtr, abstraction *in general* is not guaranteed to preserve counterexamples.

Proposition 2. *If $\mathcal{A}$ is a collection of 1-simulations that preserves and respects divergences and respects stability, then $\mathcal{A}$ preserves operational determinism.*

Proof. Assume L is operationally deterministic. Then there exists a D-relation as described in [12, Definition 8] on the states of L. Denote this D-relation with $\sim$. It is simple to show that if α is a 1-simulation that preserves and respects divergences and stability, then $\alpha(s_1) = \alpha(s_2)$ implies $s_1 \sim s_2$. $\square$

Theorem 4. *Assume $\mathcal{A}$ respects stability and preserves divergences. Then $\mathcal{A}$ is monotonous with respect to* CSP.

Proof. Let $\alpha, \beta \in \mathcal{A}$ and $\mathcal{A}$ respects stability, and $\alpha \prec \beta$. Let $(\sigma, A) \in \mathsf{Sfail}(L^\alpha)$. Then there is a stable state $[s]_\alpha$ such that $[\hat{s}]_\alpha \overset{\sigma}{\Rightarrow}_\alpha [s]_\alpha$, and $A \cap en(s) = \emptyset$. Then $[\hat{s}]_\beta \overset{\sigma}{\Rightarrow} [s]_\beta$ holds, and either $[s]_\beta$ is diverging or it is stable and $en([s]_\beta) = en([s]_\alpha)$, so that $(\sigma, A) \in \mathsf{Sfail}(L^\beta)$ $\square$

Example 1. Consider the mapping $h(s)$ which returns a representative of s in the equivalence class under a strong bisimulation relation. The quotient L^h where $S^h = \{h(s) \mid s \in S\}$ and Δ^h is Δ restricted to S^h. L^h is strongly bisimilar to L. When $\mathcal{A}$ consists of strong bisimulations the abstractions preserve virtually all reasonable semantic models. Such abstractions were used in [1].

Example 2. In *Predicate abstraction*, a set of predicates over states is defined, and an abstract state consists of the set of states which agree on the truth value of all predicates. Predicate abstraction is used for example in the so-called CEGAR approach [2]. Predicate abstraction is not separable or monotonous in general, nor does it respect enabling or stability.

The abstractions described in [2], however, do result in $\mathcal{A}$ that respects enabledness, because the coarsest abstraction h has the property that for every guard g, $h(s) = h(s')$ if and only if $g(s) = g(s')$.

Example 3. *Data abstraction* or *value abstraction* is a general term for abstractions that replace the set of values of a variable with a smaller set. Data abstractions are easily expanded into separable sets of abstractions, where each variable is abstracted separately.

4 Stubborn Set Reductions

4.1 State-of-the Art for Finite Traces

In this section we start with definition of stubborn sets that is as such applicable to any LTSs.

Definition 5 (Reduced LTS). *Given an LTS $(S, \mathcal{T}, \Delta, \hat{s})$, a function $T : S \mapsto 2^{\mathcal{T}}$ is a* reduction function. *We define the* reduced LTS *of L by T, as $L_T = (S_T, \mathcal{T}, \Delta_T, \hat{s})$, where S_T and Δ_T are minimal subsets of S and Δ, respectively, that satisfy the following conditions:*

– *$\hat{s} \in S_T$, and*
– *if $s \in S_T$, $s \xrightarrow{t} s'$ and $t \in T(s)$, then $s' \in S_T$ and $(s, t, s') \in \Delta_T$.*

By definition, a reduced LTS is a sub-LTS of the original LTS. For the remainder of this section we shall refer to L_T an L as the reduced and full LTSs, respectively. The properties preserved in the reduction depend on the properties of the reduction function. The term *stubborn set* in this article is a collective term for the sets produced by reduction functions. We describe the stubborn set reduction functions by giving various conditions.

Such conditions can be given towards one of two goals. The first goal, adopted in most of the literature on partial order reduction, is to provide a set of conditions sufficient for preserving the semantics of the system in the fullest. In the case of traces, this would mean that the reduced LTS should be trace-equivalent to the original. That is, the reduced LTS should satisfy $\mathsf{Tr}(L) = \mathsf{Tr}(L_T)$. We will explore such conditions in what follows, but they will be of secondary importance.

The second goal, one which we shall be primarily aiming for, is the preservation of the existence of a counterexample. For example, with traces, given a specification $\mathcal{L}$, the reduction should satisfy $\mathsf{Tr}(L) \cap \mathcal{L} \neq \emptyset$ if and only if $\mathsf{Tr}(L_T) \cap \mathcal{L} \neq \emptyset$.

We introduce the conditions incrementally to make them more understandable. The first two (or their equivalents in the case for abstract state spaces) are common to all versions. We also give a third condition, which is more restrictive, but it is used in practical analysis. Every condition is given for a state s.

D1. For every $t \in T(s)$ and $t_1, \ldots, t_k \notin T(s)$, if $s \xrightarrow{t_1 \cdots t_k t} s'$, then $s \xrightarrow{t t_1 \cdots t_k} s'$.

N. There exists $t \in T(s)$ such that for every $t_1, \ldots, t_k \notin T(s)$, if $s \xrightarrow{t_1 \cdots t_k}$ and $s \xrightarrow{t} s'$, then $s' \xrightarrow{t_1 \cdots t_k}$. Such a transition is called a *neutral transition.*

D2. For every $t \in T(s)$ and every $t_1, \ldots, t_k \notin T(s)$, if $s \xrightarrow{t}$, then $s' \xrightarrow{t_1 \cdots t_k t}$.

The classical stubborn sets are defined using **D1** and **D2**; **D1** and **D2** clearly imply **N**. We give our theoretical treatment for **N** as it theoretically has the potential to overcome the optimality result of [22]. The conditions above are not sufficient for preservation of properties such as traces. This stems from two issues. The first one is that they allow the reduction to ignore permutations of transitions in such a way that the order of symbols in the trace is not preserved.

The trace-preserving version requires the concept of *visible transition.* A transition t is visible if and only if $l(t) \neq \epsilon$, i.e., if its occurrence in an execution has an effect on the trace.

V. If there is some visible transition $t \in T(s)$ such that $t \in en(s)$, then $T(s)$ contains all visible transitions, including the disabled ones.

But for preserving counterexamples, we do not need to be quite as strict. Assume $\hat{s} \overset{\sigma}{\Rightarrow} s$ and $\sigma\rho \notin \mathcal{L}$. Let $u_1, \ldots, u_n \in \mathcal{T}$ such that $\rho = l(u_1 \cdots u_n)$. We define the following conditions:

Va. If $u_i \in T(s)$ for some i and $u_j \notin T(s)$ for $1 \leq j < i$, then for some prefix of $\rho' \leq l(u_i u_1 \cdots u_{i-1} u_{i+1} \cdots u_n)$, $\sigma\rho' \notin \mathcal{L}$.

Vb. If $u_i \notin T(s)$ for each i, then there is a neutral transition $t \in T(s)$ such that for some prefix $\rho' \leq l(tu_1 \cdot \cdots \cdot u_n)$, $\sigma\rho' \notin \mathcal{L}$.

Verbally, the condition **Va** states that if we take a transition that is a part of a counterexample, then commuting the said transition to the front of the execution will also result in a counterexample. **Vb** states that if we explore a neutral transition and ignore transitions remaining of a counterexample, we can still continue and find a counterexample. Note that **V** trivially implies **Va** by forcing all visible transitions in the stubborn set and **Vb** is implied when the stubborn set contains only τ-transitions.

We still need to solve the so-called ignoring problem, where neutral transitions are taken indefinitely. Such a scenario is possible if the system contains, for example, a cycle consisting of neutral transitions. To ensure progress, the literature suggests rather crude rules such as those requiring that the stubborn set contains all transitions if a cycle is closed. We forgo such rules for a more nuanced approach.

We need to define the set of *interesting* transitions at a state s. If we set out to preserve all traces, the set of interesting transitions will be the set of visible transitions in every state. When we set out to preserve only the existence of counterexamples, we can choose the set of interesting transitions in several ways, as long as it has the property that all possible ways of completing a counterexample will contain at least one interesting transition.

We say that a set U of transitions is *interesting* at state s for every relevant execution $\hat{s} \overset{\sigma}{\Rightarrow}_T s \xrightarrow{t_1 \cdots t_n}$ there is some $1 \leq i \leq n$ such that $t_i \in U$. Note that "relevant execution" may mean one of several things. If the reduction must preserve all traces, then $U = \{t \mid l(t) \in \Sigma\}$, the set of visible transitions must be interesting. If it must preserve counterexamples of some type, then the set must guarantee that every counterexample that visits s, requires the firing of at least one interesting transition at s.

We say that a set $W \subseteq \mathcal{T}$ is *closed under enabling* at state s, if for every $t \in W$, $s \xrightarrow{t_1 \cdots t_n t}$ implies that either $s \xrightarrow{t}$ or $t_i \in W$ for some $1 \leq i \leq n$.

S. There exists a set W that is closed under enabling at s and contains all interesting transitions, and for every $u \in W \cap en(s)$ there exists a sequence $s \xrightarrow{t_1 \cdots t_n} s'$ of neutral transitions such that $u \in T(s')$.

The proof of the following theorem was given in [13].

Theorem 5. *Let $\mathcal{L}$ be a trace specification. If T is a reduction function that satisfies conditions* **D1***,* **N***,* **Va***,* **Vb***, and* **S***. Then L_T has a trace $\sigma \notin \mathcal{L}$ if and only if L some trace $\rho \notin \mathcal{L}$.*

Proof. If L_T has an illegal trace, then L trivially has an illegal trace, because all traces of L_T are traces of L. The other direction is by induction on the unexplored part of an illegal trace. Let s be a state of L_T. The situation where

$\hat{s} \overset{\sigma}{\Rightarrow}_T s$ and $\sigma \notin \mathcal{L}$ is the trivial base case. Let $\sigma\rho \notin \mathcal{L}$ be some trace of L_T and let $\hat{s} \overset{\sigma}{\Rightarrow}_T s$ be such that $s \overset{\rho}{\Rightarrow}$. Let $u_1, \ldots, u_n \in \mathcal{T}$ be transitions such that $s \xrightarrow{u_1 \cdots u_n}$ and $l(u_1 \cdots u_n) = \rho$.

Firstly, assume $u_i \in T(s)$ for some $1 \leq i \leq n$, and without loss of generality, assume i is chosen as the minimal such i. Let s_i be such that $s \xrightarrow{u_1 \cdots u_i} s_i$ and Now, **D1** guarantees that u_i must be enabled and that $s \xrightarrow{u_i u_1 \cdots u_{i-1}} s_i$. Then $s \xrightarrow{u_i}_T s'$ holds for some state. **Va** then guarantees that some prefix of $s' \xrightarrow{u_1 \cdots u_{i-1} u_{i+1} \cdots u_n}$ completes the illegal trace, and we have an inductive step.

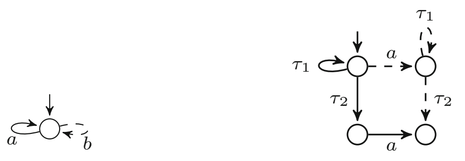
Secondly, assume that $t_i \notin T(s)$ for every $1 \leq i \leq n$. Then **N** guarantees there is a neutral transition $t \in T$ such that $s \xrightarrow{t}_T s'$ and $s' \xrightarrow{u_1 \cdots u_n}$. **Vb** then guarantees that some prefix of $l(tu_1 \cdots u_n)$ will complete the illegal trace, so that $\sigma l(t)\rho' \notin \mathcal{L}$, where ρ' is some prefix of ρ. If the prefix is shorter than ρ, this constitutes an inductive step.

If $\rho = \rho'$, we need to employ **S**. Note that because $s \xrightarrow{u_1 \cdots u_n}$ completes an illegal trace, it is guaranteed that u_i is interesting in s for some $1 \leq i \leq n$. Let u_k be interesting. Let W be the set stipulated by **S** that is closed under enabling. Because it W is closed under enabling, then some $u_i \in W$ such that u_i is enabled at s and $i < k$. Then there exists $s_0 \xrightarrow{t_1}_T s_1 \xrightarrow{t_2}_T \cdots \xrightarrow{t_m}_T s_m$ of neutral transitions such that $s_0 = s$ and $u_i \in T(s_m)$. **N** guarantees that in each of these states s_l, as long as $u_j \notin T(s_l)$ for $1 \leq j \leq n$, $s_{l+1} \xrightarrow{u_1 \cdots u_n}$ holds, and **Vb** guarantees some prefix $u_1 \cdots u_n$ of completes the illegal trace from s_{l+1}. If $u_j \in T(s_l)$ for some $1 \leq j \leq n$ and $1 \leq l < m$, the first inductive case materializes. At the latest in s_m this happens, because $u_i \in T(s_m)$. □

4.2 Stable Failures, Divergences, and Branching Time

The condition **S**, **Va** and **Vb** are sufficient for finite counterexamples, but not in general for infinite traces; we cannot just extend **Va** to infinite traces. Consider the language over $\Sigma = \{a, b\}$ that contains all infinite traces such that either a or b (but not both) can appear infinitely many times. Counterexamples consist of infinite sequences where both a and b appear infinitely many times. Consider Fig. 3(a). We label the transitions directly with elements of Σ. **Va** and **Vb** are satisfied by exploring only a in the initial state, because if ρ is an infinite counterexample of any kind, then $a\rho$ is also. This holds no matter how many finite steps we have taken, both a and b are needed to complete counterexamples, i.e., every counterexample contains at least an a-transition, so that $\{a\}$ is interesting and it is immediately chosen as the stubborn set. It is also neutral with respect to b. **V** would be sufficient with **S** to preserve traces, and this implies for finite systems that infinite traces are preserved.

Consider then Fig. 3(b) and the requirement that $\mathsf{Divtr} \subseteq \{\epsilon\}$. Counterexamples to this include any divergences after any visible transitions. In the initial state $\{\tau_1, \tau_2\}$ is a stubborn set that satisfies **V** in addition to all the conditions in Theorem 5. Because $a\tau_1^\omega$ is a counterexample, τ_1 is interesting because it is needed before any counterexample is finished. τ_2 is neutral, as $\xrightarrow{a}$ is the only



(a) Infinite trace (b) Divergence trace

Fig. 3. Counterexamples to infinite properties

execution consisting of transitions that are not in the stubborn set, and it is preserved.

The best conditions for preserving *all* divergence traces all require conditions such as the following. It has recently been discussed in [23].

L. For every visible transition t, every infinite execution $s_0 \xrightarrow{t_1}_T s_1 \xrightarrow{t_2}_T \cdots$ in the reduced state space contains a state s_i such that $t \in T(s_i)$.

For finite state spaces, this is called a *cycle condition* because all infinite executions are cyclic. We shall not explore their use in this article.

For CSP-semantics the problem in Fig. 3 does not manifest, because CSP does not require us to preserve *all* divergences, only minimal ones, which then are extended with otherwise maximal behaviour. The traditional stubborn sets as described, for example, in [21,23], require the conditions **V** and a condition called **I**, which unfortunately loses its meaning when we use the condition **N**.

IN. $T(s)$ contains either a neutral τ-transition or all τ-transitions.

Lemma 2. *If T is a reduction function that satisfies* **D1**, **N** *and* **IN** *in every state of L_T, and if $s \xrightarrow{\tau^\omega}$, then $s \xrightarrow{\tau^\omega}_T$.*

Proof. Let s be a state of L_T and let $s \xrightarrow{t_1 t_2 \cdots}$ be a diverging execution starting from s. If none of t_i is in $T(s)$, then **IN** guarantees there is a neutral τ-transition t and s' such that $s \xrightarrow{t}_T s'$ and $s' \xrightarrow{t_1 t_2 \cdots}$. On the other hand, if $t_i \in T(s)$ for some i, we choose the minimal i such that this holds. then let $s \xrightarrow{t_1 t_2 \cdots t_i} s_i$. **D1** guarantees that $s \xrightarrow{t_i}_T s'$ for some s' such that $s' \xrightarrow{t_1 t_2 \cdots t_{i-1} t_{i+1} \cdots}$.

This gives an infinite sequence of τ-transitions in L_T whenever a state is diverging. □

Unfortunately **D1**, **N** and **IN** are not enough, even with **V**, to preserve stable failures, as is witnessed by Fig. 4, but **D1**, **D2**, **V** and **IN** are. In the presence of **D2** all transitions are neutral, so **IN** is equivalent to **I**. Various solutions that use conditions more restrictive than **N** have been used in practice with good empirical results when preserving traces or CSP [8,14].

For completeness we restate the important results that are well-known about preservation of the semantic models discussed earlier, as well as one for branching properties. We need two further conditions.

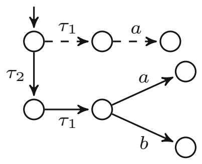


Fig. 4. Counterexample to stable failures

D0. $en(s) \cap T(s) = \emptyset$ if and only if $en(s) = \emptyset$.
B. $en(s) \subseteq T(s)$ or $T(s) \cap en(s)$ contains a single invisible transition.

Theorem 6. *Assume T satisfies* **D1** *at every state of L_T. Then the following hold:*

1. *If T satisfies* **D0** *and* **D2** *at every state of L_T, then L_T contains all reachable states s of L such that $en(s) = \emptyset$.*
2. *If all visible transitions are interesting and T satisfies* **N**, **IN**, *and* **V** *at every state of L_T, then $L_T \equiv_{\mathsf{Mindiv}} L$.*
3. *If all visible transitions are interesting and T satisfies* **D2**, **S**, **IN**, *and* **V** *at every state of L_T, then $L_T \equiv_{\mathsf{CSP}} L$.*
4. *If all visible transitions are interesting and T satisfies* **D0**, **D2**, **V**, *and* **L** *at every state of L_T, then $L_T \equiv_{\mathsf{Sfail,Divtr}} L$.*
5. *If L is deterministic and T(s) satisfies* **D2**, **V**, **B**, *and* **S** *in every state of L_T, then L_T is branching bisimilar to L.*

Mostly the theorem was proven in [21], albeit with a slightly different set of rules which, nevertheless, for deterministic transition systems are implied by the given conditions. The second statement of the theorem is novel, and follows from Lemma 2.

4.3 Considerations for Computing Stubborn Sets

Various methods for actually computing stubborn sets as defined in the earlier parts of this section have been proposed. Most commonly they include a form of *dependency relation*, or *dependency graph*, such as in [8,10,14]. Several authors discuss strategies based on shared variables and they range from forbidding changes of variables [4] to more nuanced approaches such as using write-up sets [19], analysis of guards [15]. It was proven in [22] that the classic stubborn sets are optimal in a model-theoretic sense with respect to symmetric dependency relations such as those used in [7]. We discuss some relations that can be defined by the rather coarse level of analysis [7], based on so-called effect sets. Unfortunately, these sets do not make it possible to employ the theoretical benefits afforded by the use of condition **N** instead of **D2**.

Definition 6. *Let L be an LTS. We define the following relations.*

- *A* left dependency *relation $\rightsquigarrow$ over $\mathcal{T}$ is any relation such that if either $t \rightsquigarrow u$ or for every state s, if $s \xrightarrow{ut} s_2$ then there is a state s_3 such that $s_1 \xrightarrow{u} s_3$ and $s_3 \xrightarrow{t} s_2$. We write $t \not\rightsquigarrow u$ when $t \rightsquigarrow u$ does not hold.*
- *A* dependency *relation $\risingdotseq$ over $\mathcal{T}$ is any relation such that if either $t \risingdotseq u$ or for every state s, if $s \xrightarrow{u} s_1$ and $s \xrightarrow{t} s_2$ then there is a state s_3 such that $s_1 \xrightarrow{t} s_3$ and $s_2 \xrightarrow{u} s_3$. We write $t \not\risingdotseq u$ when $t \risingdotseq u$ does not hold.*

The following lemma is given for the left-dependency and dependency relations

Lemma 3. *Let $s_0, s_n \in S$, $t, t_1, \ldots, t_n \in \mathcal{T}$ and $s_0 \xrightarrow{t_1 \cdots t_n} s_n$*

- *If $t \not\rightsquigarrow t_i$ for $1 \leq i \leq n$, and $s_n \xrightarrow{t} s'_n$ for some s'_n then there is a state s'_0 such that $s_0 \xrightarrow{t} s'_0$ and $s'_0 \xrightarrow{t_1 \cdots t_n} s'_n$.*
- *If $t \not\risingdotseq t_i$ for $1 \leq i \leq n$, and $s_0 \xrightarrow{t}$ then $s_n \xrightarrow{t}$.*

Recall that a *guard* is a binary function $X \to \{true, false\}$. Recall that transitions are of the form (g, e), where g is a guard.

Definition 7. *Let G be a set of guards. A* guard relation *for state s is a relation $\hookrightarrow$ over $\mathcal{T} \times G \cup G \times \mathcal{T}$ that has the following properties.*

1. *If $t = (g, e) \in \mathcal{T}$ and $g(s) = false$, then $t \hookrightarrow g_i$ if $g_i \in G$ such that $g_i(s) = false$ and for all $x \in X$, $g(x) = true$ implies $g_i(x) = true$.*
2. *If $g \in G$ and $g(s) = false$, then $g \hookrightarrow t$ if $t \in \mathcal{T}$ and there exists some states s_1 and s_2 such that $s_1 \xrightarrow{t} s_2$, $g(s_1) = false$ and $g(s_2) = true$.*

The following lemma is useful in calculation of stubborn sets:

Lemma 4. *Let $U \subseteq \mathcal{T}$ be a set of transitions and G be a set of guards. The set U is closed under enabling at state s if there exists a guard relation $\hookrightarrow$ for s and a subset $G' \subseteq G$ such that*

1. *For every $t \in dis(s) \cup U$, there is some $g \in G'$ such that $t \hookrightarrow g$.*
2. *For every $g \in G'$ and $t \in \mathcal{T}$, if $g \hookrightarrow t$ then $t \in U$.*

Lemma 4 was proven, for example, in [14]. Lemmas 3 and 4 are useful in the computation of stubborn sets; We do not go into details about particular algorithms, they have been discussed in [7,8,10,15,22,25] to name a few. We give the theorem that the computation of stubborn sets is based on.

Theorem 7. *Let $\risingdotseq$ be a dependency relation and $\hookrightarrow$ be a guard relation for s. The set $T(s)$ satisfies* **D1** *and* **D2** *if there exists a set of guards G such that*

1. *For every $t \in T(s) \cap en(s)$ and $u \in \mathcal{T}$, if $t \risingdotseq u$ then $u \in T(s)$.*
2. *For every $t \in T(s) \cap dis(s)$, there exists some $g \in G$ such that $t \hookrightarrow g$.*
3. *For every $g \in G$ and $t \in \mathcal{T}$, if $g \hookrightarrow t$ then $t \in T(s)$.*

Recall that $X = X_1 \times \cdots \times X_n$. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We write $\delta(x, y) = \{i \mid x_i \neq y_i\}$ Given a transition $t = (g, e)$ we define the *effect* sets as:

- The *guard set* of t as $Gd(t) = \{i \mid \exists x, y \in X : \delta(x, y) = i \wedge g(x) \neq g(y)\}$,
- the *write set* of t as $Wr(t) = \{i \mid \exists x \in X : i \in \delta(x, e(x)\}$, and
- the *read set* of t as $Rr(t) = \{i \mid \exists x, y \in X : \delta(x, y) = i \wedge \exists j \in Wr(t) : j \in \delta(e(x), e(y))\}$.

The union of these sets, $Vr(t) = Gd(t) \cup Wr(t) \cup Rr(t)$ is called *variable set* of t. Intuitively, the guard set consists of variables whose value has an effect on the guard, the write set is the set of variables whose value is subject to change when the transition is fired, and the read set is the set of variables whose value has an effect on the resulting state, i.e. if a variable in a read set changes its value, then firing the transition will result in some change in some variable in the write set.

Given $t_1, t_2 \in \mathcal{T}$, if we define the relation $\leftrightharpoons^G$ so that $t_1 \not\leftrightharpoons^G t_2$ implies $Wr(t_1) \cap Vr(t_2) = Wr(t_2) \cap Vr(t_1) = \emptyset$, then this will result in a dependency relation. It is also a left dependency relation.

We can define $\hookrightarrow^G$ using the guard of each transition, or, if the guard is given as a conjunction of clauses, for example so that $g = g_1 \wedge \cdots \wedge g_k$, we can use the conjuncts in G and have $g_i \hookrightarrow^G t$ if $Wr(t)$ contains some variable appearing in g_i, for example.

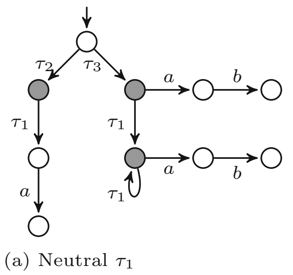
5 Stubborn Sets and Abstraction

As we saw in Sect. 3, abstraction may lead to nondeterminism. We also saw in Sect. 4 that the state-of-the art stubborn sets do not preserve all counterexamples if transitions may be nondeterministic. In this section we discuss some problems with combining stubborn sets and abstractions. We discuss the results of [1], and show that the approach applies only to a narrow class of abstractions.

The proof of Theorem 5 does not require the system is deterministic, but abstraction nevertheless gives rise problem. Consider the following hypothesis: given an abstraction α, the set U satisfies **N** at that state $[s]_\alpha$ in L^α if for every state $s' \in [s]_\alpha$, U satisfies **N** in L. Figure 5(a) and (b) demonstrate a counterexample to the hypothesis; abstractions do not in general satisfy this property. The transition τ_1 is neutral in all the states of the equivalence class (indicated by the gray states).

Firstly we note that we could define a dependency relation $\leftrightharpoons^\alpha$ for the transitions of L^α directly, analogously to Definition 6. Then Theorem 6 would hold for all other parts, except part 5 which assumes transitions are deterministic, and this is not true for L^α in general. The following lemma is trivial, but we state it in any case.

Lemma 5. *If α is separable, then the relations $\leftrightharpoons^G$ and $\hookrightarrow^G$ are dependency and guard relations for L^α.*



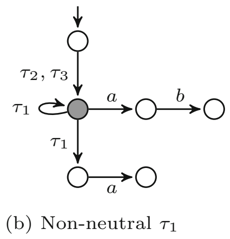


Fig. 5. Troublesome cases for abstraction and stubborn sets

This lemma is important in practice, as it applies to several methods that are used in practice. For example in [4], the analysis is carried out using this type of dependency. Thus, all linear time stubborn set methods based on these relations are in fact robust with respect to separable abstractions.

Definition 8. *Let α be an abstraction. The relation $\leftrightharpoons_\alpha$ over $\mathcal{T}$ is an* abstract dependency relation, *or α*-dependency, *if for every s, if $t, u \in en(s)$ either $t \leftrightharpoons_\alpha u$ or the following hold:*

1. *For for every s' such that $s \xrightarrow{t} s'$ we have $s' \xrightarrow{u}$ (and symmetrically for u), and*
2. *For every state s_1 such that $s \xrightarrow{tu} s_1$ there is a state s_2 such that $s \xrightarrow{ut} s_2$ and $[s_1]_\alpha = [s_2]_\alpha$ (and symmetrically).*

Lemma 6. *Assume $\mathcal{A}$ respects enabledness and $\alpha \in \mathcal{A}$. For every $t_1, \ldots, t_n$ such that $t_i \not\leftrightharpoons_\alpha t$ for $1 \le i \le n$, if $[s]_\alpha \xrightarrow{t_1 \cdots t_n}_\alpha [s_n]_\alpha$ and $[s]_\alpha \xrightarrow{t} [s']_\alpha$, then there is a state $[s'_n]_\alpha$ such that $[s']_\alpha \xrightarrow{t_1 \cdots t_n}_\alpha [s'_n]_\alpha$ and $[s_n]_\alpha \xrightarrow{t}_\alpha [s_n]'$.*

Furthermore, if $\mathcal{A}$ is a collection of 1-simulations, then for every $[s'_n]_\alpha$ such that $[s_n]_\alpha \xrightarrow{t}_\alpha [s'_n]_\alpha$, $[s]_\alpha \xrightarrow{tt_1 \cdots t_n}_\alpha [s'_n]_\alpha$ holds.

Proof. We prove the claims by induction. If $n = 0$, both claims holds trivially. Assume as inductive hypothesis that the first claim holds for $n - 1$. Let $[s_0]_\alpha \xrightarrow{t_1 \cdots t_n}_\alpha [s_n]_\alpha$ and $[s_0]_\alpha \xrightarrow{t}_\alpha [s'_0]_\alpha$. By inductive hypothesis we have $[s'_0]_\alpha \xrightarrow{t_1 \cdots t_{n-1}}_\alpha [s'_{n-1}]_\alpha$ and $[s_{n-1}]_\alpha \xrightarrow{t}_\alpha [s'_{n-1}]_\alpha$. Let s_{n-1} be one of the states in $[s_{n-1}]_\alpha$ such that $s_{n-1} \xrightarrow{t} s'_{n-1}$.

Because α respects enabledness, $s_{n-1} \xrightarrow{t_n} s_n$ for some state s_n. And because $\leftrightharpoons_\alpha$ is an α-dependency relation, we must have $s_n \xrightarrow{t} s'_n$ for some state s'_n and $s'_{n-1} \xrightarrow{t_n} s^*_n$ for some state $s^*_n \in [s'_n]_\alpha$.

Hence $[s_{n-1}]_\alpha \xrightarrow{t_n t}_\alpha [s'_n]_\alpha$ and $[s'_{n-1}]_\alpha \xrightarrow{t_n}_\alpha [s_n]'_\alpha$, finishing the inductive step for the first part of the lemma.

Assume then that α is a 1-simulation and the second claim holds for $n - 1$. Let $[s_n]_\alpha \xrightarrow{t}_\alpha [s'_n]_\alpha$. Then, for every state $s_n \in [s_n]_\alpha$ there exists some state

$s'_n \in [s'_n]_\alpha$ such that $s_n \xrightarrow{t} s'_n$. For at least one of them we have $s_{n-1} \xrightarrow{t_n} s_n \xrightarrow{t} s'_n$, and again, $s_{n-1} \xrightarrow{t} s'_{n-1}$ because α respects enabledness. And because of α-dependency, we have some state $s^*_n \in [s'_n]_\alpha$ such that $s'_{n-1} \xrightarrow{t_n} s^*_n$, finishing the inductive step for the second part of the lemma. □

Definition 9. *Let α be an abstraction. Let G be a set of of Boolean functions $\alpha(X) \rightarrow \{true, false\}$. The relation $\hookrightarrow_\alpha$ over $\mathcal{T} \times G \cup G \times \mathcal{T}$ is an* abstract guard relation *for $[s]_\alpha$ if*

1. *For every $t \in dis_\alpha([s]_\alpha)$ there exists a guard $g \in G$ such that $g([s]_\alpha) = false$ and $t \hookrightarrow_\alpha g$.*
2. *For every state $g \in G$ and $t \in \mathcal{T}$, if there is some states s', s'' such that $s' \xrightarrow{t} s''$, $g([s']_\alpha) = false$ and $g([s'']_\alpha) = true$ then $g \hookrightarrow_\alpha t$.*

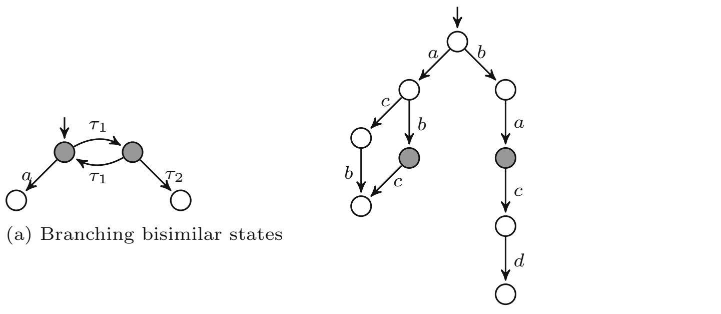
Theorem 8. *Assume $\mathcal{A}$ is a collection of 1-simulations and that it respects enabledness. Let $\alpha \in \mathcal{A}$. Let $[s]_\alpha$ be an abstract state. Then the set U satisfies* **D1** *and* **D2** *if there exist some abstract guard relation $\hookrightarrow_\alpha$, a set of guards G, and an abstract dependency relation $\leftrightharpoons_\alpha$ such that*

1. *For every $t \in en_\alpha([s]_\alpha)$ and $u \in \mathcal{T}$, if $t \in U$ and $t \leftrightharpoons_\alpha u$ then $u \in U$.*
2. *For every $t \in dis_\alpha([s]_\alpha)$ there is some $g \in G$ such that $t \hookrightarrow_\alpha g$ and for every $[s']_\alpha$.*
3. *For every $g \in G$ and $t \in \mathcal{T}$, if $g \hookrightarrow_\alpha t$, then $t \in U$.*

Proof. Because Definition 9 is strictly analogous to Definition 7 and Lemma 4 applies, we skip the proof that U will be closed under enabling. Lemma 6 applied to the first condition proves **D2**. Because α is a 1-simulation, Lemma 6 guarantees that also **D1** holds. □

The restriction that $\mathcal{A}$ must respect both enabledness and determinism is a severe one. The result of [1] merits discussion in light of the weakness of the above theorem. The result of using α-dependency when α is a strong bisimulation is sound, because two states cannot be strongly bisimilar unless they have the same enabled transitions. It does not hold, however, for weaker equivalences. Consider Fig. 6(a). The grey states are branching bisimilar, but not strongly bisimilar. α-dependency would declare the transitions labeled a and τ_2 as independent regardless of what happens after they are fired, because they are not enabled together. In fact, simple as it is, Fig. 6(a) leaves little hope for developing a method that is based on dependencies significantly less restrictive than those that consider the whole equivalence class, i.e., dependency in Definition 6 applied to the whole abstract state space.

The counterexample to abstractions that are not 1-simulations is given in Fig. 6(b). The abstraction equates the gray states in the figure. We can then define an abstract dependency relation that declares a and b independent. c and b are likewise independent. The set $\{a\}$ would satisfy the conditions of Theorem 8 under this abstraction. The abstraction also respects enabledness. In the state that follows the execution of a, $\{c\}$ is a set that likewise satisfies the conditions



(a) Branching bisimilar states

(b) Nondeterminism destroys correctness

Fig. 6. Counterexamples to α-dependency

of the theorem, and the execution of d is missed. The example leaves open the possibility that forbids possibly nondeterministic transitions (such as c) from being stubborn.

6 Discussion and Future Work

As this article is to appear in a collection to honor professor Bill Roscoe, I break the convention and write in first person. I do so out of respect for the community and the person, as I hope to explain in a more personal manner what has been written here. I also wish to express my gratitude and sense of honor to have been invited to write this article.

The main theorem in [11] states that for timed automata, relaxing zone abstractions and applying an abstract dependency very similar to the one in Theorem 8 will preserve the existence of counterexamples. In fact, it not only does this, but sometimes it is able to reduce away abstract counterexamples that are spurious. When I started writing this article, I started with a hypothesis stronger than Theorem 8, one that would replicate the same powerful results.

A series of counterexamples, given in the previous sections, emerged while I was trying to prove a similar result, and in the end, the result was not significantly stronger than the main theorem of [1]. Instead of this article providing groundbreaking results as I had hoped, it thus is more of a document of how such a result does not hold for the abstract dependency relation as defined here. I hope that a careful analysis of the counterexamples to weaker hypothesis might prove fruitful in the pursuit of a more general theorem, one that may still be out there. I am still haunted by the intuition that there is a hidden diamond amid the ashes of this failed theorem.

For other parts, the results in this article are mostly re-stated facts that have been known separately with a couple of minor improvements to existing results. I hope my analysis may serve as a starting point for a more careful analysis of properties of abstractions and how they combine with the myriad other methods.

For future work, there are two important avenues. Firstly, the classification of abstractions is in its own right an important topic, and this rather truncated treatment merely scratches the surface. Results pertaining to preservation of determinism and monotony with respect to semantic models is something that we plan to pursue further.

Secondly, the combination of stubborn set methods with abstractions. There are probably good reasons why the relaxed zone abstraction combine in a synergistic manner with stubborn sets but abstractions labeled transition systems in general do not. Understanding of the said reasons may lead to powerful methods, or at least understanding, in the never ending battle against state explosion. The guard relation, for example, was simply lifted verbatim to abstract state spaces. It remains a possibility that some version of such a relation might be the key to unlock a more powerful theory.

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References

1. Bošnački, D., Scheffer, M.: Partial order reduction and symmetry with multiple representatives. In: Havelund, K., Holzmann, G., Joshi, R. (eds.) NFM 2015. LNCS, vol. 9058, pp. 97–111. Springer, Heidelberg (2015). doi:10.1007/978-3-319-17524-9_8
2. Clarke, E., Grumberg, O., Jha, S., Lu, Y., Veith, H.: Counterexample-guided abstraction refinement. In: Emerson, E.A., Sistla, A.P. (eds.) CAV 2000. LNCS, vol. 1855, pp. 154–169. Springer, Heidelberg (2000). doi:10.1007/10722167_15
3. Clarke, E.M., Grumberg, O., Long, D.E.: Model checking and abstraction. ACM Trans. Programm. Lang. Syst. (TOPLAS) **16**(5), 1512–1542 (1994)
4. Clarke, E.M., Grumberg, O., Peled, D.: Model Checking. MIT press, Cambridge (1999)
5. Emerson, E.A., Sistla, A.P.: Symmetry and model checking. In: Courcoubetis, C. (ed.) CAV 1993. LNCS, vol. 697, pp. 463–478. Springer, Heidelberg (1993). doi:10.1007/3-540-56922-7_38
6. Engelfriet, J.: Determinancy → (observation equivalence = trace equivalence). Theor. Comput. Sci. **36**, 21–25 (1985)
7. Geldenhuys, J., Hansen, H., Valmari, A.: Exploring the scope for partial order reduction. In: Liu, Z., Ravn, A.P. (eds.) ATVA 2009. LNCS, vol. 5799, pp. 39–53. Springer, Heidelberg (2009). doi:10.1007/978-3-642-04761-9_4
8. Gibson-Robinson, T., Hansen, H., Roscoe, A.W., Wang, X.: Practical partial order reduction for CSP. In: Havelund, K., Holzmann, G., Joshi, R. (eds.) NFM 2015. LNCS, vol. 9058, pp. 188–203. Springer, Heidelberg (2015). doi:10.1007/978-3-319-17524-9_14

9. Godefroid, P. (ed.): Partial-Order Methods for the Verification of Concurrent Systems. LNCS, vol. 1032. Springer, Heidelberg (1996). doi:10.1007/3-540-60761-7
10. Hansen, H., Kwiatkowska, M., Qu, H.: Partial order reduction for model checking Markov decision processes under unconditional fairness. In: Quantitative Evaluation of Systems (QEST 2011), pp. 203–212. IEEE (2011)
11. Hansen, H., Lin, S.-W., Liu, Y., Nguyen, T.K., Sun, J.: Diamonds are a girl's best friend: partial order reduction for timed automata with abstractions. In: Biere, A., Bloem, R. (eds.) CAV 2014. LNCS, vol. 8559, pp. 391–406. Springer, Heidelberg (2014). doi:10.1007/978-3-319-08867-9_26
12. Hansen, H., Valmari, A.: Operational determinism and fast algorithms. In: Baier, C., Hermanns, H. (eds.) CONCUR 2006. LNCS, vol. 4137, pp. 188–202. Springer, Heidelberg (2006). doi:10.1007/11817949_13
13. Hansen, H., Valmari, A.: Safety property-driven stubborn sets. In: Larsen, K.G., Potapov, I., Srba, J. (eds.) RP 2016. LNCS, vol. 9899, pp. 90–103. Springer, Heidelberg (2016). doi:10.1007/978-3-319-45994-3_7
14. Hansen, H., Wang, X.: Compositional analysis for weak stubborn sets. In: 2011 11th International Conference on Application of Concurrency to System Design (ACSD), pp. 36–43. IEEE (2011)
15. Laarman, A., Pater, E., van de Pol, J., Hansen, H.: Guard-based partial-order reduction. Int. J. Softw. Tools Technol. Transfer **18**(4), 427–448 (2016)
16. Milner, R. (ed.): A Calculus of Communicating Systems. LNCS, vol. 92. Springer, Heidelberg (1980). doi:10.1007/3-540-10235-3
17. Peled, D.: All from one, one for all: on model checking using representatives. In: Courcoubetis, C. (ed.) CAV 1993. LNCS, vol. 697, pp. 409–423. Springer, Heidelberg (1993). doi:10.1007/3-540-56922-7_34
18. Roscoe, A.W.: The Theory and Practice of Concurrency. Prentice-Hall, Upper Saddle River (1997)
19. Valmari, A.: A stubborn attack on state explosion. In: Clarke, E.M., Kurshan, R.P. (eds.) CAV 1990. LNCS, vol. 531, pp. 156–165. Springer, Heidelberg (1991). doi:10.1007/BFb0023729
20. Valmari, A.: Stubborn sets for reduced state space generation. In: Rozenberg, G. (ed.) ICATPN 1989. LNCS, vol. 483, pp. 491–515. Springer, Heidelberg (1991). doi:10.1007/3-540-53863-1_36
21. Valmari, A.: Stubborn set methods for process algebras. In: Proceedings of the DIMACS Workshop on Partial Order Methods in Verification (1997)
22. Valmari, A., Hansen, H.: Can stubborn sets be optimal? Fundamenta Informaticae **113**(3–4), 377–397 (2011)
23. Valmari, A., Hansen, H.: Stubborn set intuition explained. In: International Workshop on Petri Nets and Software Engineering 2016, pp. 213–232 (2016)
24. Valmari, A., Tienari, M.: Compositional failure-based semantic models for basic lotos. Formal Aspects Comput. **7**(4), 440–468 (1995)
25. Valmari, A., Vogler, W.: Fair testing and stubborn sets. In: Bošnački, D., Wijs, A. (eds.) SPIN 2016. LNCS, vol. 9641, pp. 225–243. Springer, Heidelberg (2016). doi:10.1007/978-3-319-32582-8_16
26. Glabbeek, R.J.: The linear time — branching time spectrum II. In: Best, E. (ed.) CONCUR 1993. LNCS, vol. 715, pp. 66–81. Springer, Heidelberg (1993). doi:10.1007/3-540-57208-2_6