Quantified Integer Programs with Polyhedral Uncertainty Set

Michael Hartisch[1(✉)], Thorsten Ederer[2], Ulf Lorenz[1], and Jan Wolf[1]

[1] Chair of Technology Management, University of Siegen, Siegen, Germany
{michael.hartisch,ulf.lorenz,jan.wolf}@uni-siegen.de
[2] Discrete Optimization, Technische Universität Darmstadt, Darmstadt, Germany
ederer@mathematik.tu-darmstadt.de

Abstract. Quantified Integer Programs (QIPs) are integer programs with variables being either existentially or universally quantified. They can be interpreted as a two-person zero-sum game with an existential and a universal player where the existential player tries to meet all constraints and the universal player intends to force at least one constraint to be not satisfied.

Originally, the universal player is only restricted to set the universal variables within their upper and lower bounds. We extend this idea by adding constraints for the universal variables, i.e., restricting the universal player to some polytope instead of the hypercube created by bounds. We also show how this extended structure can be polynomial-time reduced to a QIP.

1 Introduction

Integer linear programming has become a successful modeling and solution framework for a wide range of applications in the Operations Research community. Today, one can solve instances with thousands up to millions of variables and constraints. As problems get more complex, uncertainty becomes a relevant concern. Solutions to optimization problems can be sensitive to perturbations in the parameters, which can render them suboptimal or even infeasible in practice. Methods such as stochastic or robust programming are able to cope with parameter uncertainty and give average-case or worst-case optimal solutions, respectively.

A special class of optimization problems under uncertainty are quantified programs. Quantified Integer Programs (QIPs) are integer linear programs, where variables are either existentially or universally quantified. QIPs are PSPACE-complete [9, p. 92] and they can be interpreted as a two-person zero-sum game, where an existential player tries to stay feasible and a universal player tries to violate at least one constraint. In [1] it was shown that QIPs can be used to model and solve the game Gomoku.

In the original definition, a QIP is comparable to a multi-stage robust integer program with a cubic uncertainty set. This uncertainty set is rather conservative,

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since it allows for worst-case realizations of each universal variable at the same time. Therefore, we restrict the uncertainty set.

Such a restriction can also be seen from a gaming point of view: On the one hand, only a certain set of moves are legal moves for the opposing player according to the rules. On the other hand, when planning a move the aspect of opponent modeling [7] can be seen as restricting the response of the opponent by prohibiting unlikely moves during the analysis. This does not only help us to adapt more efficiently to a well known opponent but also shrinks the game tree of interest noticeably.

We will now generally and in an abstract manner define our problem. In contrast to the original QIP problem we restrict the universal player not only within some rigid bounds, but also dynamically, i.e., the permitted range of the variables depends on previous and possible future universal decisions. When setting a variable the universal player must check some conditions, depending only on own actions.

2 Previous and Related Work

Quantified Constraint Satisfaction Problems have been studied since at least 1995 [3]. In 2003, Subramani revived the idea of universal variables in Constraint Satisfaction Problems and coined the term Quantified Linear Program (QLP). His QLP did not have an objective function and the universal variables could only take values in their associated intervals. In the following year he extended this approach by integer variables and called them Quantified Integer Programs (QIPs) [6]. Later Wolf and Lorenz added a linear objective function [4] and enhanced the problem to: "Does a solution exist and if yes which one is the best." Within the scope of his dissertation [9], Wolf gave some theoretical results and adapted a solving procedure known from Stochastic Programming: With his implementation of Multistage Benders Decomposition it is possible to solve QLPs with millions of scenarios.

We will basically follow the notation used in [4]. Transposes are omitted when they are clear.

Definition 1 (Quantified Integer Program). *Let $x = (x_1, \dots, x_n)^\top \in \mathbb{Z}^n$ be a vector of $n \in \mathbb{N}$ integer variables and $l, u \in \mathbb{Z}^n$ lower and upper bounds. Let $\mathcal{D} = \{x \in \mathbb{Z}^n \mid x \in [l, u]\}$. Let $A \in \mathbb{Q}^{m \times n}$ be the coefficient matrix with rational entries, $b \in \mathbb{Q}^m$ the right-hand side vector and $Q = (Q_1, \dots, Q_n)^\top \in \{\exists, \forall\}^n$ a vector of quantifiers. The term $Q \circ x \in \mathcal{D}$ with the component wise binding operator $\circ$ denotes the quantification vector $(Q_1 x_1 \in [l_1, u_1] \cap \mathbb{Z}, \dots, Q_n x_n \in [l_n, u_n] \cap \mathbb{Z})^\top$ such that every quantifier Q_i binds the variable x_i ranging in the associated interval $[l_i, u_i]$. We call a maximal consecutive subsequence of Q consisting of identical quantifiers a quantifier block and denote the i-th corresponding subsequence of x by x^i and call it a variable block B_i. Let $\beta \in \mathbb{N}$ be the number of such blocks. Let $c \in \mathbb{Q}^n$ be a vector of objective coefficients and let c^i denote the segment of c associated with B_i.*

We call

$$\min_{B_1}\left(c^1x^1+\max_{B_2}\left(c^2x^2+\min_{B_3}\left(c^3x^3+\max_{B_4}\left(\ldots+\min_{B_\beta}c^\beta x^\beta\right)\right)\right)\right)$$

$$s.t.\quad Q\circ x\in\mathcal{D}: Ax\le b$$

a quantified integer program (QIP) and denote it with (c, Q, l, u, A, b).

Note that the universal variables are only restricted to be in their associated intervals. From now on the existential player will be referred to as "he" and the universal player as "she".

3 An Extension with Regard to the Uncertainty Set

We extend the idea of quantified variables by restricting the universal variables to a polytope that can be described through a second system $A^\forall x \le b^\forall$ with $A^\forall \in \mathbb{Q}^{m^\forall \times n}$ and $b^\forall \in \mathbb{Q}^{m^\forall}$ for $m^\forall \in \mathbb{N}$. For a given QIP (c, Q, l, u, A, b) we only restrict the universal variables in such way that their range only depends on other universal variables. In other words, we assume that existential variables have no influence on universal decisions. Thus, we demand

$$A^\forall_{i,j}=0\quad \forall\, i\in\{1,\ldots,m^\forall\}\ \forall j\in\{k\in\{1,\ldots,n\}\mid Q_k=\exists\}\,, \tag{1}$$

i.e., each entry of $A^\forall$ belonging to an existential variable is zero.

Definition 2 (QIP with Polyhedral Uncertainty Set (*QIP*$^{\pentagon}$)). *Let* (c, Q, l, u, A, b) *be a given QIP. Let* $b^\forall \in \mathbb{Q}^{m^\forall}$ *and* $A^\forall \in \mathbb{Q}^{m^\forall\times n}$ *with (1). Let* $\mathcal{D}^{\pentagon} = \{x \in \mathcal{D} \mid A^\forall x \le b^\forall\} \neq \emptyset$. *The quantified integer program with polyhedral uncertainty set (QIP*$^{\pentagon}$*) is given by* $(c, Q, l, u, A, b, A^\forall, b^\forall)$ *with*

$$\min_{B_1}\left(c^1x^1+\max_{B_2}\left(c^2x^2+\min_{B_3}\left(c^3x^3+\max_{B_4}\left(\ldots+\min_{B_\beta}c^\beta x^\beta\right)\right)\right)\right)$$

$$s.t.\ Q\circ x\in\mathcal{D}^{\pentagon}: Ax\le b\,.$$

Note that we forbid an empty domain $\mathcal{D}^{\pentagon}$ since it would complicate the following definitions.

Definition 3 (Legal Allocation). *A legal allocation of an existential variable* x_i *demands this variable to be integer and within its bounds* $[l_i, u_i]$. *The same is true for universal variables in standard QIPs. In a QIP*$^{\pentagon}$*, however, the legal allocation options also depend on the (legal) allocation of previous variables* $x_1, \ldots, x_{i-1}$. *Thus, when assigning a value to the universal variable* x_i *there must exist a series of legal moves* $x_{i+1}, \ldots, x_n$ *such that the resulting vector* x *fulfills* $A^\forall x \le b^\forall$. *The legal range* $[l_i^\forall, u_i^\forall]$ *of* x_i *can be determined by Fourier-Motzkin elimination [8] of the domain* $\mathcal{D}^{\pentagon}$ *and fixating the previous variable allocations.*

Definition 4 (Strategy). *A strategy $S = (V, E, c)$ is an edge-labeled finite arborescence*[1] *with a set of nodes $V = V_\exists \dot\cup V_\forall$, a set of edges E and a vector of edge labels $c \in \mathbb{Q}^{|E|}$. Each level of the tree consists either of only nodes from $V_\exists$ or only of nodes from $V_\forall$, with the root node at level 0 being from $V_\exists$. The i-th variable is represented by the inner nodes at depth $i-1$. Each edge connects a node at some level i to a node at level $i+1$. Outgoing edges represent moves of the player at the current node, the corresponding edge label encodes the variable allocation of the move. Each node $v_\exists \in V_\exists$ has exactly one child, and each node $v_\forall \in V_\forall$ has as many children as legal allocation options.*

A path from the root to a leaf represents a game sequence and the edge labels along this path encode the corresponding variable allocation. Such a leaf at the end of a path corresponding to x has the value $c^\top x$.

Definition 5 (Winning Strategy). *A strategy is called a winning strategy (for the existential player) if all paths from the root to a leaf represent a vector x such that $Ax \leq b$.*

Definition 6 (Optimal Winning Strategy). *A winning strategy is optimal if the minimax value of the root is smaller than or equal to the minimax values of all other winning strategies. The vector $\tilde{x}$ representing the path which obeys the minimax rule is called the principal variation (PV), i.e., it consists of the optimal moves when both players play perfectly. The optimal objective value is $c^\top \tilde{x}$.*

4 The Polynomial-Time Reduction to a QIP

Hereafter we provide an easy method to transform any given QIP$^{\pentagon}$ (with a polyhedral uncertainty set) into a QIP (only restricted by bounds). This enables us to use our solver Yasol, which is specialized in solving quantified programs [2]. Further, the deterministic equivalent program can be computed much more easily. It also enables us to model problems in a straightforward way (by stating both systems $A^\forall x \leq b^\forall$ and $Ax \leq b$) and transform them later into a QIP to solve them.

Our goal is to transfer the condition $A^\forall x \leq b^\forall$ out of the domain of the variables into the system of constraints. We rewrite the problem as a QIP as given in Definition 1. Note that we cannot simply add $A^\forall x \leq b^\forall$ to the constraint system. This would not restrict the universal player but tighten the conditions the existential player has to meet. Instead, the universal polyhedral constraints are not enforced a priori. We introduce helper constraints and variables that ensure that a violation of the universal constraints is detected, with the effect that the existential player's constraints are relaxed. That is, "the existential player wins by default if the universal player cheats". In addition to making all constraints feasible, the universal player is penalized via the objective function.

[1] An arborescence is a directed, rooted tree.

Let us consider the k-th row of the system $A^\forall x \leq b^\forall$ which is given by

$$\sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i \leq b^\forall_k \,. \tag{2}$$

It is solely the universal player's task to meet this condition, since the existential player cannot influence the left hand side because of (1). Thus

$$\sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i > b^\forall_k \tag{3}$$

$$\Longleftrightarrow \sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i \geq b^\forall_k + \epsilon_k \tag{4}$$

holds for some $\epsilon_k > 0$. To determine an assignment for the parameter ϵ_k we need to find the smallest possible gap between the sum of integral multiples of the coefficients $A^\forall_{k,i}$ and $b^\forall_k$. It is sufficient to underestimate this smallest possible gap in order to ensure (3) $\Leftrightarrow$ (4). This can be achieved by using the reciprocal of the (lowest) common multiplier of the denominators (LCD) of the universal polytope's coefficients. Let R^{LCD}_k be the reciprocal of the LCD of $b^\forall_k$ and of the coefficients $A^\forall_{k,i}$ for $i \in \{1, \ldots, n\}$. Then

$$\sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i \geq b^\forall_k + R^{LCD}_k \tag{5}$$

is fulfilled if and only if the original constraint (2) is not satisfied. Note, that $R^{LCD}_k = 1$ if all entries of row k are integer.

We now introduce a new binary existential variable $y_k \in \{0, 1\}$ with the property

$$y_k \begin{cases} = 0, & \text{if } \sum\limits_{i=1}^{n} A^\forall_{k,i} \cdot x_i \leq b^\forall_k \\ \in \{0,1\}, & \text{if } \sum\limits_{i=1}^{n} A^\forall_{k,i} \cdot x_i > b^\forall_k \end{cases} . \tag{6}$$

This is achieved by using the following constraint

$$\sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i \geq L_k + (-L_k + b^\forall_k + R^{LCD}_k) \cdot y_k \tag{7}$$

with

$$L_k = \sum_{\substack{1 \leq i \leq n \\ A^\forall_{k,i} < 0}} A^\forall_{k,i} \cdot u_i + \sum_{\substack{1 \leq i \leq n \\ A^\forall_{k,i} \geq 0}}^{n} A^\forall_{k,i} \cdot l_i \tag{8}$$

which is the smallest value the left hand side of the original universal constraint can take with respect to the bounds. Let us take a closer look at (7). If $y_k = 0$ the constraint is always fulfilled, since

$$\sum_{i=1}^{n} A^\forall_{k,i} \cdot x_i \geq L_k \tag{9}$$

is always true due to the definition of L_k. If and only if the original constraint is violated y_k also can take the value 1 since (5) is met. However, if the original constraint is satisfied y_k must be bound to zero. Thus, we embedded the variable y_k in a new constraint such that (6) is fulfilled. We now introduce the binary variable $p \in \{0,1\}$ with

$$p \begin{cases} = 0, & \text{if } \forall\ k \in \{1,\ldots,m^{\forall}\} : y_k = 0 \\ \in \{0,1\}, & \text{if } \exists\ k \in \{1,\ldots,m^{\forall}\} : y_k = 1 \end{cases} . \tag{10}$$

This variable can be embedded using the constraint

$$p \leq \sum_{k=1}^{m^{\forall}} y_k . \tag{11}$$

Thus, we introduced a variable that can indicate the violation of the system $A^{\forall} x \leq b^{\forall}$. If a universal constraint is violated we require each constraint of the systems $Ax \leq b$ to be trivially satisfied: If the universal player did not abide by her rules the existential player should not be punished for a violation of his system. Thus, the system is modified as follows

$$Ax - Mp \leq b \tag{12}$$

using the parameter vector $M \in \mathbb{Q}^m$ with

$$M_k = \max_{x \in \mathcal{D}} A_{k,*}x - b_k \tag{13}$$

$$= \sum_{\substack{1 \leq i \leq n \\ A_{k,i} < 0}} A_{k,i} \cdot l_i + \sum_{\substack{1 \leq i \leq n \\ A_{k,i} \geq 0}} A_{k,i} \cdot u_i - b_k \tag{14}$$

for each $k \in \{1,\ldots,m\}$. Hence, if $p = 1$ the inequality (12) is always satisfied.

The global indicator p is now used to punish the universal player by reducing the objective value massively. Since the universal player is trying to maximize the objective function we can penalize a violation of the universal constraints by subtracting this new variable p with a sufficiently large coefficient $\tilde{M}$ in the innermost term of the objective function. Note that this block is w.l.o.g. an existential block and thus the existential player will set this variable to 1 if possible, i.e., if the universal player did not meet her conditions. For the value of $\tilde{M}$ we choose

$$\tilde{M} = \sum_{\substack{1 \leq i \leq n \\ c_i < 0}} c_i \cdot (l_i - u_i) + \sum_{\substack{1 \leq i \leq n \\ c_i \geq 0}}^{n} c_i \cdot (u_i - l_i) + 1.$$

Note that

$$\max_{x \in \mathcal{D}} c^{\top} x - \tilde{M} < \min_{x \in \mathcal{D}} c^{\top} x \tag{15}$$

holds. Thus, when subtracting this value the objective function will definitely yield a better objective value for the existential player than he could have achieved without it. However, the universal player can counteract by meeting her system of equations and thus forcing p to be zero.

The final transformed problem looks as follows

$$\min_{B_1}\left(c^1x^1+\max_{B_2}\left(c^2x^2+\min_{B_3}\left(c^3x^3+\max_{B_4}\left(\ldots+\min_{B_\beta,y,p}\left(c^\beta x^\beta-\tilde{M}p\right)\right)\right)\right)\right)$$

$$\text{s.t. } Q\circ x\in\mathcal{D}\quad \exists y\in\{0,1\}^{m^\forall}\quad \exists p\in\{0,1\}:$$

$$Ax-Mp\leq b \tag{16}$$

$$-A^\forall x-(L-b^\forall-R^{LCD})y\leq -L \tag{17}$$

$$p-\sum_{k=1}^{m^\forall}y_k\leq 0 \tag{18}$$

Note, that $L\in\mathbb{Q}^{m^\forall}$ is a vector with entries according to (8) and $R^{LCD}\in\mathbb{Q}^{m^\forall}$ is the vector of the reciprocals of the lowest common multiplier of the denominators of the rows of $A^\forall$ and $b^\forall$. Further note, that the values for $\tilde{M}$, M and L can be calculated easily by using the upper and lower bound of x appropriately, depending on the sign of the corresponding entries in c and A, respectively. Also the number of auxiliary variables and constraints is linear in the input size. This problem has the structure of a QIP since the variables are only restricted to be within their bounds ($\mathcal{D}$ is a cubical integer lattice). For further investigations the PV of a solution (a strategy) of this transformed problem will be denoted by $z=(x,y,p)\in\mathcal{D}\times\{0,1\}^{m^\forall}\times\{0,1\}$.

In the following we show how the transformed QIP and the $\mathrm{QIP}^{\pentagon}$ are connected.

Theorem 1. *If* $\mathrm{QIP}^{\pentagon}$ *has an optimal winning strategy with PV* $\tilde{x}$ *and objective value* $v=c^\top\tilde{x}$ *the transformed QIP has an optimal winning strategy with PV* $\tilde{z}=(\tilde{x},\tilde{y},\tilde{p})$ *with* $\tilde{y}_i=0$ *for* $i=1,\ldots,m^\forall$ *and* $\tilde{p}=0$ *with objective value* v.

Proof. Since $\tilde{x}$ is the PV of an optimal winning strategy of $\mathrm{QIP}^{\pentagon}$ it satisfies $A\tilde{x}\leq b$ and $A^\forall\tilde{x}\leq b^\forall$. Thus, $\tilde{z}=(\tilde{x},\tilde{y},\tilde{p})$ with $\tilde{y}=0$ and $\tilde{p}=0$ is feasible for the transformed problem with objective value $c^\top\tilde{x}-M\tilde{p}=v$. Let $\hat{z}=(\hat{x},\hat{y},\hat{p})$ be the PV of the optimal winning strategy of the transformed problem and thus $c^\top\hat{x}-M\hat{p}\leq c^\top\tilde{x}$. If $\hat{x}\notin\mathcal{D}^{\pentagon}$ $\hat{z}$ would also fulfill $\hat{p}=1$, since at least one row of the system $A^\forall\hat{x}\leq b^\forall$ is violated. However, because of (15) the resulting value of the objective function is smaller than any other solution obeying $A^\forall\hat{x}\leq b^\forall$. This is a contradiction to the minimax optimality of $\hat{z}$ since the universal player can avoid this by assigning her variables such that $A^\forall\hat{x}\leq b^\forall$ holds. Thus, the assignment $\hat{x}\in\mathcal{D}^{\pentagon}$ is true and $A^\forall\hat{x}\leq b^\forall$. Further, $y=0$ and $p=0$ and $\hat{x}$ is also feasible for $\mathrm{QIP}^{\pentagon}$ with $c^\top\hat{x}\geq c^\top\tilde{x}$. Therefore, $c^\top\hat{x}=c^\top\tilde{x}=v$.

Theorem 2. *If* $\mathrm{QIP}^{\pentagon}$ *has no winning strategy, then the transformed QIP also has no feasible solution.*

Proof. Let $\text{QIP}^{\pentagon}$ have no winning strategy. Assume $S = (V, E, c)$ were a winning strategy for the transformed QIP, i.e., in each leaf the system of inequalities (16)-(18) is fulfilled. Note that this arborescence has a depth of $n + m^{\forall} + 1$. We consider the arborescence $\bar{S} = (\bar{V}, \bar{E}, \bar{c})$ with $\bar{V} \subseteq V$, $\bar{E} \subseteq E$ and $\bar{c}(e) = c(e)$ for each $e \in \bar{E}$. $\bar{V}$ contains no node of a level larger than n and $\bar{E}$ contains no edges leading to such nodes. Further, edges describing illegal allocations (see Definition 3) in terms of the $\text{QIP}^{\pentagon}$ are deleted as well as their whole underlying subtrees. This designed arborescence $\bar{S}$ describes a strategy for the underlying $\text{QIP}^{\pentagon}$, because

- the depth is n and thus for each variable a decision level exists,
- nodes of universal variables have only legal allocation options leading out,
- the remaining strategy properties are adopted from S.

This strategy $\bar{S}$ is also a winning strategy for $\text{QIP}^{\pentagon}$, since each path from the root to a leaf represents a vector x such that $Ax \leq b$; for each such path $A^{\forall}x \leq b^{\forall}$ holds, because illegal allocations were deleted.

Let us consider such a path $x_1, \ldots, x_n$ in $\bar{S}$ and the unique[2] associated overlying path $z = (x_1, \ldots, x_n, y_1, \ldots, y_{m^{\forall}}, p)$ in S. Since $A^{\forall}x \leq b^{\forall}$ and (16)-(18) we may conclude $p = 0$ and $y_i = 0$ for all $i \in \{1, \ldots, m^{\forall}\}$. Thus, because of (16), also $Ax \leq b$ holds for the leaf. Hence, we have found a winning strategy for $\text{QIP}^{\pentagon}$ which contradicts the assumption.

Note that the first-stage solution of the transformed QIP is identical[3] to the first-stage solution of the $\text{QIP}^{\pentagon}$.

Corollary 1. *$\text{QIP}^{\pentagon}$ is in PSPACE. Since the QIP with cubical uncertainty set is a special case of the $\text{QIP}^{\pentagon}$ it is even PSPACE-complete.*

5 Example

We consider a simple graph game where one player has to traverse a given graph while the opponent is allowed to erase some edges. However, the opponent is not allowed to erase edges arbitrarily but must obey some rules. This problem is closely related to the Dynamic Graph Reliability problem [5] with the difference that edges have weights and an objective function should be minimized. Further, edges are erased depending on the point in time instead of the location of the player. The underlying graph is given in Fig. 1.

The starting node is labeled with 0 and the target node with 7. The question is:

> Is there a strategy for the existential player which allows him to reach the target node no matter how the opponent acts? And if there are multiple strategies: Which one is the winning strategy with the shortest worst-case path to the target node (according to the weights of the edges).

[2] The path is unique, because all nodes with level $\geq n$ belong to existential variables and thus have only one successor in a strategy.

[3] except for auxiliary variable p in single-stage instances.

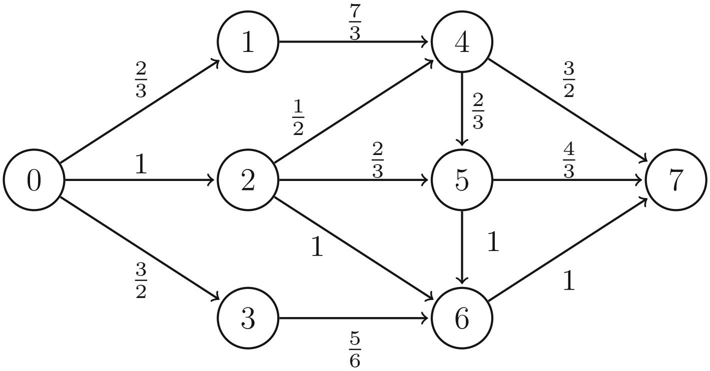

Fig. 1. Directed acyclic weighted graph with starting node 0 and target node 7

Let $G = (V, E, c)$ describe the graph given in Fig. 1 with V being the set of vertices, E the set of edges and $c : E \to \mathbb{Q}$ a function assigning weights to each edge. Let $x_{i,j} \in \{0, 1\}$ be variables indicating whether the existential player uses edge $(i, j) \in E$ or not. For each edge $(i, j) \in E$ with $i \neq 0$ let $d_{i,j} \in \{0, 1\}$ indicate whether the universal player deleted this edge or not. The turn order is given by the following quantifier string:

$$\exists x_{0,1}, x_{0,2}, x_{0,3}\ \forall d_{1,4}, d_{2,4}, d_{2,5}, d_{2,6}, d_{3,6}\ \exists x_{1,4}, x_{2,4} x_{2,5}, x_{2,6}, x_{3,6}$$
$$\forall d_{4,7}, d_{4,5}, d_{5,7}, d_{5,6}, d_{6,7}\ \exists x_{4,7}, x_{4,5}, x_{5,7}, x_{5,6}, x_{6,7}$$

Both players take turns while fixing some variables. The universal player is allowed to deactivate edges before the existential player is able to use them. In doing so the existential player wants to meet the system of equations given below.

$$\sum_{(0,j)\in E} x_{0,j} = 1 \tag{19}$$

$$\sum_{(i,7)\in E} x_{i,7} = 1 \tag{20}$$

$$\sum_{(i,k)\in E} x_{i,k} = \sum_{(k,j)\in E} x_{k,j} \qquad \forall k \in \{1, \ldots, 6\} \tag{21}$$

$$x_{i,j} \leq 1 - d_{i,j} \qquad \forall (i, j) \in E, i \neq 0 \tag{22}$$

It consists of constraints ensuring the flow from node 0 to 7 (viz. (19), (20) and (21)) and constraints forbidding to use edges that have been deleted by the universal player (22). However, the universal player is also restricted by her system $A^{\forall} x \leq b^{\forall}$ as follows:

$$\sum_{\substack{(i,j)\in E\\ i\neq 0}} d_{i,j} \leq 3\,, \quad \sum_{\substack{(i,j)\in E\\ i\neq 0}} c(i,j)\cdot d_{i,j} \geq \frac{3}{2}\,, \quad \sum_{\substack{(i,j)\in E\\ i\neq 0}} c(i,j)\cdot d_{i,j} \leq 2 \tag{23}$$

This system states that the universal player is allowed to delete at most 3 edges and the sum of the weights of the deleted edges must be between 1.5 and 2. Note, that we did not convert either system into a "less or equal" system in order to make their actual use more clear. Yet, this must be done to use the transformation described in Sect. 4. The final transformed QIP is displayed below. For convenience the repeating variable domains $\{0,1\}$ are omitted in the quantifier string.

$$\min_{B_1}\left(\frac{2}{3}x_{0,1}+x_{0,2}+\frac{3}{2}x_{0,3}+\max_{B_2}\left(\min_{B_3}\left(\frac{7}{3}x_{1,4}+\frac{1}{2}x_{2,4}+\frac{2}{3}x_{2,5}+x_{2,6}\right.\right.\right.$$
$$\left.\left.\left.\left.\left.+\frac{5}{6}x_{3,6}+\max_{B_4}\left(\min_{B_5,y,p}\left(\frac{3}{2}x_{4,7}+\frac{2}{3}x_{4,5}+\frac{4}{3}x_{5,7}+x_{5,6}+x_{6,7}-15p\right)\right)\right)\right)\right)\right)$$

s.t. $\exists x_{0,1},x_{0,2},x_{0,3}\ \forall d_{1,4},d_{2,4},d_{2,5},d_{2,6},d_{3,6}\ \exists x_{1,4},x_{2,4},x_{2,5},x_{2,6},x_{3,6}$
$\forall d_{4,7},d_{4,5},d_{5,7},d_{5,6},d_{6,7}\ \exists x_{4,7},x_{4,5},x_{5,7},x_{5,6},x_{6,7},y_1,y_2,y_3,p:$

$$-\sum_{(0,j)\in E} x_{0,j}-p\leq -1\ ,\quad \sum_{(0,j)\in E} x_{0,j}-2p\leq 1 \tag{24}$$

$$-\sum_{(i,7)\in E} x_{i,7}-p\leq -1\ ,\quad \sum_{(i,7)\in E} x_{i,7}-2p\leq 1 \tag{25}$$

$$\sum_{(i,k)\in E} x_{i,k}-\sum_{(k,j)\in E} x_{k,j}-\deg^{-}(k)\cdot p\leq 0 \qquad \forall k\in\{1,\ldots,6\} \tag{26}$$

$$\sum_{(k,j)\in E} x_{k,j}-\sum_{(i,k)\in E} x_{i,k}-\deg^{+}(k)\cdot p\leq 0 \qquad \forall k\in\{1,\ldots,6\} \tag{27}$$

$$x_{i,j}+d_{i,j}-p\leq 1 \qquad \forall (i,j)\in E, i\neq 0 \tag{28}$$

$$4y_1-\sum_{\substack{(i,j)\in E\\ i\neq 0}} d_{i,j}\leq 0 \tag{29}$$

$$\sum_{\substack{(i,j)\in E\\ i\neq 0}} c(i,j)\cdot d_{i,j}+9.5y_2\leq \frac{65}{6} \tag{30}$$

$$-\sum_{\substack{(i,j)\in E\\ i\neq 0}} c(i,j)\cdot d_{i,j}+\frac{13}{6}y_3\leq 0 \tag{31}$$

$$p-\sum_{k=1}^{3} y_k\leq 0 \tag{32}$$

Constraints (24)-(28) describe the transformed existential system (cf. (16)), (29)-(31) are the embedded universal constraints (cf. (17)), and (32) is similar to (18). In (26) and (27) the coefficients of p are the number of incoming edges $\deg(k)^{-}=|\{(i,j)\in E\mid j=k\}|$ and the number of outgoing edges of node k $\deg(k)^{+}=|\{(i,j)\in E\mid i=k\}|$, respectively. In (30) the coefficients result from

$L_2 = -\frac{65}{6}$, $R_2^{LCD} = \frac{1}{6}$ and $b_2^{\forall} = -\frac{3}{2}$. This standard QIP is easily solved by the QIP-solver Yasol. It turns out that there is a winning strategy for the existential player. The objective value of the PV is $\frac{11}{3}$ and the optimal first decision is moving from the starting node to node 2. The (perfect) universal player will then delete the edge between 2 and 4. The existential player then must move to node 5. After that the edge between node 5 and 7 is deleted and finally the target node is reached by passing node 6.

6 Conclusion

We extended the concept of quantified integer programs to a polyhedral uncertainty set. Thus, the universal variables can be restricted explicitly by using a second linear system of inequations $A^{\forall}x \leq b^{\forall}$. We also presented a general polynomial-time transformation of this new problem statement permitting us to solve a standard QIP instead of inventing new methods for solving QIP$^{\pentagon}$. Thus, the concept of QIPs can be put into practice in new areas of application in an easy and straightforward way. In particular, rules of games that must be obeyed by each player can be modeled easily. Therefore, the possibility of modeling and solving more complicated two-person zero-sum games with the help of quantified programming is provided.

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