

11 × 11 Domineering Is Solved: The First Player Wins

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Abstract. We have developed a program called MUDoS (Maastricht University Domineering Solver) that solves Domineering positions in a very efficient way. It enables the solution of known positions (up to the 10×10 board) to be much quicker.

More importantly, it enables the solution of 11×11 Domineering, a board size that up till now was far out of the reach of previous Domineering solvers. The solution needed the investigation of 259,689,994,008 nodes, using almost half a year of computation time on a single simple desktop computer. The results show that under optimal play the first player wins 11×11 Domineering, irrespective whether Vertical or Horizontal starts.

In addition, several other new boards were solved. Using the convention that Vertical starts, the 8×15 , 11×9 , 12×8 , 12×15 , 14×8 , and 17×6 boards are all won by Vertical, whereas the 6×17 , 8×12 , 9×11 , and 11×10 boards are all won by Horizontal.

1 Introduction

Domineering is a two-player perfect-information game invented by Göran Andersson around 1973. It was popularized to the general public in an article by Martin Gardner [12]. It can be played on any subset of a square lattice, though mostly it is restricted to rectangular $m \times n$ boards, where m denotes the number of rows and n the number of columns. The version introduced by Andersson and Gardner was the 8×8 board.

Play consists of the two players alternately placing a 1×2 tile (domino) on the board, where the first player may place the tile only in a vertical alignment, the second player only horizontally. The first player being unable to move loses the game, his opponent (who made the last move) being declared the winner. Since the board is gradually filled, i.e., Domineering is a converging game, the game always ends, and ties are impossible. With these rules the game belongs to the category of *combinatorial games*, for which a whole theory (the Combinatorial Game Theory, or CGT in short) has been developed.

Among combinatorial game theorists Domineering received quite some attention, but this was limited to rather small or irregular boards [1, 3, 4, 10, 14, 23]. Larger (rectangular) boards were solved using α - β search [15], leading to solving

all boards up to the standard 8×8 board [7], later extended to the 9×9 board [13, 17], and finally extended to larger boards up to 10×10 [8, 9].

2 Three Approaches

The following subsections give a rough characterization of the two main programs used to systematically solve Domineering positions so far, and of the program used to obtain the new results, as described in this paper.

2.1 A Brute-Force Approach: DOMI

The first systematic analysis of rectangular Domineering boards was performed by Breuker et al. [7, 13, 17]. They developed the program DOMI, using a straightforward variant of the α - β technique [15], enhanced with a transposition table. The algorithm did not use perfect domain knowledge for classifying positions as wins or losses and hence can be characterized as a pure brute-force approach. Transposition tables with $2M$ (2^{21}) entries were used with a two-level replacement scheme called *TwoBig* [6], in which each entry can store two table positions. Mirror symmetries are taken into account. The newest position is always stored, overwriting the less important position in terms of nodes investigated.

2.2 A Knowledge-Based Approach: OBSEQUI

A few years later Nathan Bullock published results on solving Domineering boards up to the 10×10 board [9]. His program OBSEQUI used a sophisticated evaluation function which can determine statically the winner at a shallower point in the search tree than DOMI did. This allowed the elimination of large portions of the search space, resulting in much more efficient solving of Domineering boards. OBSEQUI used a transposition table (taking mirror symmetries into account) with 2^{23} entries with either a two-level *TwoBig* replacement scheme or a one-level replacement scheme called *FindFirst* [8]. Also, a much better move-ordering heuristic was used, plus the use of a dominance relation to prune provably irrelevant moves. Since the main advantage of Bullock's program is based on game-specific knowledge, we can characterize his approach as a knowledge-based approach.

2.3 A Knowledge-Intensive Approach: MUDoS

Uiterwijk continued using game-specific knowledge to an even more detailed extent. His program MUDoS incorporated deep knowledge of Domineering positions with known result. These knowledge rules are so intense, that it even enables solving many game boards without any search at all (i.e., investigating a single node, the empty board under consideration). This was called *perfectly solving* [18]. The most important feature of these knowledge rules is the number

of safe moves that a player provably can reach in a position [19–21]. The transposition table used (again taking mirror symmetries into account) contained 2^{26} entries, with a simple one-level *Deep* replacement scheme. Due to the heavy use of very knowledge-intense rules based on game-specific properties we can characterize his approach as a knowledge-intensive approach.

3 New Results

After almost half a year of computation time, 11 × 11 Domineering was solved. We give some data in Sect. 3.1. As a sidetrack, we solved several other new boards. Data are given in Sect. 3.2. An overview of updated combinatorial-game-theoretic values of Domineering boards is given in Sect. 3.3.

3.1 The Solution of 11 × 11 Domineering

The solution of 11 × 11 Domineering took 174 days and 15h on a standard desktop computer (a HP with duo core Intel E8400 3.00 GHz CPU with a 64-bit Windows 7 operating system and 4 GB internal memory). The MUDoS program is written in C#.

The result is that the first player under optimal play wins the game. Since the board is square, this is irrespective of Vertical or Horizontal moving first.

To put the solution of the 11 × 11 board into perspective, we show in Table 1 the results and number of nodes investigated to solve square boards up to 11 × 11 by the three programs mentioned in the previous section.

Table 1. Results and number of nodes investigated to solve square Domineering boards. Vertical always starts. A “1” and “2” in the results column indicate a first-player (Vertical) and second-player (Horizontal) win, respectively. A “–” in a column indicates that the program was unable to solve the position.

Board	Result	DOMI [7]	OBSEQUI [9]	MUDoS
2 × 2	1	1	1	1
3 × 3	1	1	1	1
4 × 4	1	40	23	1
5 × 5	2	604	259	17
6 × 6	1	17,232	908	1
7 × 7	1	408,260	31,440	1
8 × 8	1	441,990,070	2,023,301	24,147
9 × 9	1	~25,000,000,000 ^a	1,657,032,906	4,917,736
10 × 10	1	–	3,541,685,253,370	13,506,805
11 × 11	1	–	–	259,689,994,008

^aThis result was obtained with an improved version of DOMI, around 2000 [5]. The exact number of nodes investigated was lost.

For the result the investigation of 259,689,994,008 nodes was needed, with an average speed of 17,211 nodes/sec. While this is some ten times slower than OBSEQUI's speed, this decrease in speed is by far compensated by the much higher pruning efficiency, as evidenced by the ratio's of the number of nodes investigated by MUDoS and OBSEQUI. For the 8×8 , 9×9 and 10×10 boards these are 1.19%, 0.30%, and 0.00038%, respectively. Of course the latest number is so low, since OBSEQUI solved the 10×10 board on a distributed network of several computers (no further details given), without memory sharing, by which transposition tables will be far less effective. But as a striking fact, whereas OBSEQUI needed several months of computation time on this network, MUDoS needs only 21 min on a single computer to solve the 10×10 board.

3.2 The Solution of New Other Domineering Boards

Besides 11×11 Domineering we were able to solve several other new Domineering boards. The results are given in Table 2.

Table 2. Results and number of nodes investigated to solve other new Domineering boards. Vertical always starts. A “1” and “2” in the results column indicate a first-player (Vertical) and second-player (Horizontal) win, respectively. The 10×11 board was solved before (see below). A “–” in a column indicates that the program was unable to solve the position and hence the game-theoretic value is still unknown.

Board	Result	# nodes	Board	Result	# nodes
10×11	1	1	11×10	2	1
9×11	2	84,145,153	11×9	1	23,183,077
6×17	2	25,670,138,842	17×6	1	810,774,495
8×12	2	273,559,795	12×8	1	11,960,354
8×14	–	–	14×8	1	490,146,677
8×15	1	1	15×8	–	–
12×15	1	1	15×12	–	–

The most notable results and their consequences are given below. We there use the notion of outcome class [1, 4, 10] of an $m \times n$ board, denoted by $[m \times n]$, where an outcome class is N, P, V, or H, where N stands for a Next player win (i.e., a win for the player to move), P for a Previous player win (i.e., a loss for the player to move), V for a Vertical win (irrespective of who starts), and H for a Horizontal win (irrespective of who starts).

Other Boards with Width or Height 11. Although the 10×11 board was already solved (Vertical wins), using the translational symmetry rules of Lachmann c.s. [16], and even perfectly solved [18], the 11×10 board was not. However, MUDoS solves it investigating just 1 node, showing that Horizontal

wins.¹ As a result $[10 \times 11] = V$ (and $[11 \times 10] = H$). Further, with some more work, we were able to solve the 9×11 board (Horizontal wins) and the 11×9 board (Vertical wins). Consequently, $[9 \times 11] = H$ (and $[11 \times 9] = V$).

Boards with Width or Height 6. The 6×17 and 17×6 boards were also solved (wins for Horizontal and Vertical, respectively). Consequently, $[6 \times 17] = H$. Moreover, using the translational symmetry rules of Lachmann c.s. [16] and the facts that $[6 \times 4] = N$ and $[6 \times n]$ with $n = 8, 12,$ and 14 are H , it follows that $[6 \times 21] (17 + 4) = N$ or H , and $[6 \times 25] (17 + 8) = H$, $[6 \times 29] (17 + 12) = H$, and $[6 \times 31] (17 + 14) = H$. Moreover, in [11] it was shown that $[6 \times n]$ for $n > 31 = N$ or H for widths $33, 35, 37, 39, 43, 45, 47, 51,$ and 59 . Using the result for $[6 \times 17]$ all these values analogously are determined to be H , the only exception being width 35 (still N or H). This shows that the holes in the results for boards of height 6 have considerably been filled. The outcome classes for all $6 \times n$ boards are known now, the only exceptions being the $6 \times 18, 6 \times 21, 6 \times 23, 6 \times 27,$ and 6×35 boards, all five having outcome classes N or H , which means that Horizontal at least wins as first player. Of course the results for $[m \times 6]$ can similarly be updated, replacing H by V .

Boards with Width or Height 8. The 8×12 and 12×8 boards were also solved (wins for Horizontal and Vertical, respectively). Consequently, $[8 \times 12] = H$, but also, using the translational symmetry rules and the facts that $[8 \times 10]$ and $[8 \times 16]$ are H , it follows that $[8 \times 22] (12 + 10) = H$, $[8 \times 24] (12 + 12) = H$, and $[8 \times 28] (12 + 16) = H$. Moreover, since $[8 \times 10] = H$ and all $[8 \times n]$ for even n from 20 – 28 are H , it follows that all $[8 \times n]$ with even $n \geq 20$ are H . This makes the entries in the $8 \times n$ row completely regular for even n from $n = 20$ onwards, in contrast to [11], were (in an irregular way) some of those were determined to be H , the others as N or H . We also were able to solve the 14×8 board (Vertical wins), but not the 8×14 board yet. It means that $[8 \times 14] = N$ or H . This leaves the 8×14 and 8×18 boards as the only holes in this row for even width. Finally, the 8×15 (and 12×15) board is trivially solved to be a Vertical win (so outcome class N or V), but the rotated 15×8 (and 15×12) board could not yet be determined. Again, of course the results for $[m \times 8]$ can similarly be updated, replacing H by V , including that all $[m \times 8]$ with even $m \geq 20$ are V .

3.3 Updated Table of CGT Values of Domineering

In Table 3 we give a complete updated overview of all results for solved Domineering boards, as outcome classes. The results are taken from [11] and includes

¹ We note that solving a board investigating a single node is not exactly the same as perfectly solving a board, since in the latter the board is solved using characteristics of the board solely, without generating the possible moves, whereas in the former the possible moves are generated, but immediately proven to contain at least one winning move or only losing moves.

results from [3,4,7,9–11,13,16,17].² In addition, our new results have been added. This table is also available at [22], where any future updates will be made public.

In this table the following notes apply: (1) the outcome classes for all $n > 31$ are H, except that the outcome class for $n = 35$ is N or H; (2) the outcome classes for all even $n \geq 20$ are H; (3) the outcome classes are alternating H (even n) and N or H (odd n); (4)–(6): equivalent to notes (1)–(3) by replacing n with m and H with V.

For boards with one or both dimensions larger than 31, besides the results in the notes above, nothing is known about their outcome classes, except of course that $m \times m$ boards have outcome classes N or P, that $m \times 2km$ boards have outcome classes H, and that $2kn \times n$ boards have outcome classes V.

4 Conclusions and Future Work

As can be seen from the results it is clear that MUDoS is a very efficient Domineering solver. All boards solved before are solved in an equal amount (for the trivial boards) or far smaller (for the more complex boards) number of investigated nodes than by previous solvers.

The efficiency of our solver enabled the solution of the 11×11 Domineering board. The result indicates that the first player wins. Moreover, several new rather complex boards have been solved. Applying these together with the use of the translational symmetry rules updated the Domineering outcome class landscape considerably.

Regarding future work, foremost this condensed overview will be extended to a full publication. This will include a detailed description of MUDoS' knowledge rules and heuristics employed. Moreover, the impact of the rules and heuristics on solving performance, separately and in combination, will be illustrated with experiments.

As a follow-up we moreover intend as a last step to enhance the solving power of our Domineering program by incorporating knowledge from Combinatorial Game Theory into our solver. A preliminary experiment using endgame databases up to 16 squares filled with CGT values, combined with a very simplistic α - β solver showed reductions up to 99% for boards up to 7×7 [2].

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² Although Drummond-Cole determined the outcome classes for 8×26 and 26×8 (H and V), these results were not included in his table of known outcome classes for Domineering [11].

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