Constructions of Multivariate Copulas

Xiaonan Zhu, Tonghui Wang and Varith Pipitpojanakarn

Abstract In this chapter, several general methods of constructions of multivariate copulas are presented, which are generalizations of some existing constructions in bivariate copulas. Dependence properties of new families are explored and examples are given for illustration of our results.

1 Introduction

In recent years, copulas are hot topics in probability and statistics. By *Sklar theorem* [16], the importance of copulas comes from two aspects, (1) describing dependence properties of random variables, such as Joe [6], Nelsen [11], Siburg [15], Tasena [17], Shan [14], Wei [20]; and (2) constructing the joint distributions of random variables. In the second direction, there are many papers devoting to the constructions of bivariate copulas, such as Rodríguez-Lallena [12], Kim [7], Durante [4], Mesiar [9], Aguilo [1], Mesiar [10], but few of constructions of multivariate copulas, such as Liebscher [8], Durante [3].

In this paper, we discussed several general methods of constructing multivariate copulas, which are generalizations of some bivariate results. The paper is organized as follows: In Sect. 2, we introduce some necessary definitions and existing results. Several general methods for constructing multivariate copulas are provided in Sect. 3 and their dependence properties are discussed in Sect. 4. Finally, two examples are given in Sect. 5.

X. Zhu · T. Wang (✉)
Department of Mathematical Sciences, New Mexico State University, Las Cruces, USA
e-mail: xzhu@nmsu.edu
e-mail: twang@nmsu.edu

V. Pipitpojanakarn
Faculty of Economics, Chiang Mai University, Chiang Mai, Thailand
e-mail: Varithp@cmu.ac.th

V. Kreinovich et al. (eds.), *Robustness in Econometrics*,
Studies in Computational Intelligence 692, DOI 10.1007/978-3-319-50742-2_15

2 Definitions and Existing Results

A function $C : I^n \to I$ is called an *n-copula* [11], where $I = [0, 1]$, if C satisfies the following properties:

(i) C is *grounded*, i.e., for any $\mathbf{u} = (u_1, \cdots, u_n)' \in I^n$, if at least one $u_i = 0$, then $C(\mathbf{u}) = 0$,
(ii) One-dimensional marginals of C are uniformly distributed, i.e., for any $u_i \in I$, $i = 1, \cdots, n$,

$$C(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i,$$

(iii) C is *n-increasing*, i.e., for any $\mathbf{u}, \mathbf{v} \in I^n$ such that $\mathbf{u} \leq \mathbf{v}$, we have

$$V_C([\mathbf{u}, \mathbf{v}]) = \sum sgn(\mathbf{a})C(\mathbf{a}) \geq 0,$$

where the sum is taken over all vertices $\mathbf{a}$ of the n-box $[\mathbf{u}, \mathbf{v}] = [u_1, v_1] \times \cdots \times [u_n, v_n]$, and

$$sgn(\mathbf{a}) = \begin{cases} 1, & \text{if } a_i = u_i \text{ for an even number of } i's, \\ -1, & \text{if } a_i = u_i \text{ for an odd number of } i's. \end{cases}$$

Equivalently,

$$V_C([\mathbf{u}, \mathbf{v}]) = \Delta_{\mathbf{u}}^{\mathbf{v}} C(\mathbf{t}) = \Delta_{u_n}^{v_n} \cdots \Delta_{u_1}^{v_1} C(\mathbf{t}),$$

where $\Delta_{u_k}^{v_k} C(\mathbf{t}) = C(t_1, \cdots, t_{k-1}, v_k, t_{k+1}, \cdots, t_n) - C(t_1, \cdots, t_{k-1}, u_k, t_{k+1}, \cdots, t_n)$, $k = 1, \cdots, n$.

Note that above three conditions ensure that the range of C is I. By Sklar's theorem [16], any n random variables $X_1, \cdots, X_n$ can be connected by an n-copula via the equation

$$F(x_1, \cdots, x_n) = C(F_1(x_1), \cdots, F_n(x_n)),$$

where F is the joint distribution function of $X_1, \cdots, X_n$, F_i is the marginal distribution functions of X_i, $i = 1, \cdots, n$. In addition, if $X_1, \cdots, X_n$ are continuous, then the copula C is unique.

There are three important functions for n-copulas defined respectively by

$$M_n(\mathbf{u}) = \min\{u_1, \cdots, u_n\},$$

$$\Pi_n(\mathbf{u}) = \prod_{i=1}^{n} u_i,$$

and

$$W_n(\mathbf{u}) = \max\{u_1 + \cdots + u_n - n + 1, 0\},$$

for all $\mathbf{u} \in I^n$. Functions M_n and Π_n are n-copulas for all $n \geq 2$, but W_n is not an n-copula for any $n \geq 3$. M_n and W_n are called the *Fréchert-Hoeffding upper bound* and *lower bound* of n-copulas respectively since for any n-copula C, we have $W_n \leq C \leq M_n$.

Let $H : I^n \to \mathbb{R}$ be a function. The functions $H_{i_1 i_2 \cdots i_k}$: $I^k \to \mathbb{R}$ are called *k-dimensional marginals* of H defined by

$$H_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) = H(v_1, \cdots, v_n),$$

where $v_j = u_{i_l}$ if $j = i_l$ for some $l = 1, 2, \cdots, k$, otherwise, $v_j = 1$.

Any n-copula C defines a function $\overline{C}$: $I^n \to I$ by

$$\overline{C}(\mathbf{u}) = 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} C_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}). \tag{1}$$

It is called the *survival function*of C. For more details about copulas theory, see Nelsen's book [11].

Now let's recall some existing results. In 2004, Rodríguez-Lallena and Úbeda-Flores [12] considered the following family of bivariate copulas,

$$C_\theta(u, v) = uv + \theta f(u)g(v), \tag{2}$$

where $f, g : [0, 1] \to \mathbb{R}$ are two functions, $\theta \in \mathbb{R}$ is a parameter. This family is a generalization of the well-known bivariate *Farlie-Gumble-Morgenstern* (or FGM, for short) family,

$$C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v),$$

where $u, v \in [0, 1]$ and $\theta \in [-1, 1]$. In 2011, Kim et al. [7] extended Rodríguez-Lallena and Úbeda-Flores's work to the family,

$$C(u, v) = C^*(u, v) + \theta f(u)g(v), \tag{3}$$

where C^* is a known bivariate copula, $f, g : [0, 1] \to \mathbb{R}$ are two functions, θ is a parameter. In 2013 and 2015, Durante et al. [4] and Mesiar et al. [10] considered more general cases,

$$C(u, v) = C^*(u, v) + H(u, v), \tag{4}$$

where C^* is a known bivariate copula, $H : [0, 1] \times [0, 1] \to \mathbb{R}$ is a function.

3 Constructions of Multivariate Copulas

The constructions of all above results are adding some *perturbation functions* to a given bivariate copula. In fact, any n-copula C can be represented by a perturbation of the *independent* copula Π_n [19]. Based on this idea, we are going to extend these bivariate results to multivariate cases.

Firstly, for any given n-copula $C^* : I^n \to [0, 1]$, we consider the construction,

$$C(u_1, u_2, \cdots, u_n) = C^*(u_1, u_2, \cdots, u_n) + H(u_1, u_2, \cdots, u_n), \tag{5}$$

where $H : I^n \to \mathbb{R}$ is a function, called a *perturbation function*. C is called a *perturbation* of C^* by H.

Theorem 1 *Let C^* be an n-copula, $H : I^n \to \mathbb{R}$ be a function. C is defined by (5). Then C is an n-copula if and only if H satisfies the following three conditions,*

(i) $H(0, u_2, \cdots, u_n) = \cdots = H(u_1, \cdots, u_{n-1}, 0) = 0$ *for all* $(u_1, \cdots, u_n) \in I^n$,
(ii) *There are* $1 \leq i < j \leq n$ *such that*

$$H(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_n) = H(u_1, \cdots, u_{j-1}, 1, u_{j+1}, \cdots, u_n) = 0,$$

(iii) $V_{C^*}([\boldsymbol{u}, \boldsymbol{v}]) + V_H([\boldsymbol{u}, \boldsymbol{v}]) \geq 0$ *for all n-box* $[\boldsymbol{u}, \boldsymbol{v}]$ *in* I^n.

Proof The conditions (i) and (ii) ensure that C is grounded, and its one-dimensional marginals are uniform distributed, respectively. The n-increasing property of C is guaranteed by the condition (iii). □

Next we provide a necessary and sufficient condition on H under which C defined by (5) is an absolutely continuous n-copula.

Theorem 2 *Let C^* be an absolutely continuous n-copula with the density c^*, $H : I^n \to \mathbb{R}$ be a non-zero absolutely continuous function with the Radon-Nikodym derivative h with respect to the Lebesgue measure on I^n. C is defined by (5) is an absolutely continuous n-copula if and only if H satisfies the following conditions.*

(i) $H(0, u_2, \cdots, u_n) = \cdots = H(u_1, \cdots, u_{n-1}, 0) = 0$ *for all* $(u_i, \cdots, u_n) \in I^n$,
(ii) *There are* $1 \leq i < j \leq n$ *such that*

$$H(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_n) = H(u_1, \cdots, u_{j-1}, 1, u_{j+1}, \cdots, u_n) = 0,$$

(iii) $c^* + h \geq 0$ *almost surely.*

Proof Firstly, the boundary conditions of copulas are ensured by the condition (i) and (ii).

Next, we show that the condition (iii) is equivalent to the n-increasing property of C. On the one hand, suppose that C is n-increasing. If $c^* + h$ is not nonnegative almost surely, then there exist $\mathbf{u} < \mathbf{v} \in I^n$ such that $c^* + h < 0$ on $[\mathbf{u}, \mathbf{v}]$. Note that $V_C([\mathbf{u}, \mathbf{v}]) = \int_{[\mathbf{u},\mathbf{v}]}(c^* + h)(\mathbf{t})d\mathbf{t}$. So $V_C([\mathbf{u}, \mathbf{v}]) < 0$. It contradicts the n-increasing property of C. On the other hand, if $c^* + h \geq 0$ almost surely, we must have $V_C([\mathbf{u}, \mathbf{v}]) \geq 0$ for all $\mathbf{u}, \mathbf{v} \in I^n$ with $\mathbf{u} \leq \mathbf{v}$. □

Now let's consider a special case of (5) as follows, which are multivariate extensions of the result in [7].

$$C(u_1, u_2, \cdots, u_n) = C^*(u_1, u_2, \cdots, u_n) + \prod_{i=1}^{n} f_i(u_i), \tag{6}$$

where C^* is an n-copula, $f_i : [0, 1] \to \mathbb{R}$ is a function, $i = 1, 2, \cdots, n$.

The following theorem give us a sufficient condition under which C defined by (6) is an n-copula.

Theorem 3 *Let C^* be an n-copula, $f_i : [0, 1] \to \mathbb{R}$ be a function, $i = 1, 2, \cdots, n$. $C : [0, 1]^n \to \mathbb{R}$ is defined by (6) is an n-copula if $f_1, \cdots, f_n$ satisfy the following conditions,*

(i) $f_1(0) = \cdots = f_n(0) = 0$, *and there exist at least two functions f_i and f_j such that $f_i(1) = f_j(1) = 0$, $1 \leq i, j \leq n$,*
(ii) *f_i is absolutely continuous,*
(iii) $\min(B) \geq \sup\left\{-\dfrac{V_{C^*}([\boldsymbol{u}, \boldsymbol{v}])}{\Delta(\boldsymbol{u}, \boldsymbol{v})} : \boldsymbol{u}, \boldsymbol{v} \in [0, 1]^n, \boldsymbol{u} < \boldsymbol{v}\right\}$,
where $B = \{\alpha_{i_1} \cdots \alpha_{i_k}\beta_{j_1} \cdots \beta_{j_{n-k}} : 1 \leq k \leq n, k\, is\, odd, i_1, \cdots, i_k\, and\, j_1, \cdots, j_{n-k}$ *are pairwise distinct*$\}$, $\alpha_i = \inf\{f_i'(u_i) : u_i \in A_i\} < 0$, $\beta_i = \sup\{f_i'(u_i) : u_i \in A_i\} > 0$, $A_i = \{u_i \in [0, 1] : f'(u_i)$ *exists*$\}$, $i = 1, \cdots, n$, *and* $\Delta(\boldsymbol{u}, \boldsymbol{v}) = (v_1 - u_1) \cdots (v_n - u_n)$.

Proof Firstly, if there is $f_i = 0$, then $C = C^*$ is an n-copula. So without loss of generality, we may assume that f_i is non-zero, $i = 1, \cdots, n$.

Since C^* is an n-copula, C is grounded and its marginals are uniformly distributed if and only if C satisfies the above condition (i). Next we are going to show that C is n-increasing if C satisfies the conditions (ii) and (iii).

Suppose that C satisfies conditions (ii) and (iii). By Lemma 2.2 in [12], it holds that for any $\mathbf{u}, \mathbf{v} \in I^n$ with $\mathbf{u} < \mathbf{v}$,

$$\frac{(f_1(v_1) - f_1(u_1)) \cdots (f_n(v_n) - f_n(u_n))}{(v_1 - u_1) \cdots (v_n - u_n)} \geq -\frac{V_{C^*}([\mathbf{u}, \mathbf{v}])}{\Delta(\mathbf{u}, \mathbf{v})},$$

i.e.,

$$V_C([\mathbf{u},\mathbf{v}]) = V_{C^*}([\mathbf{u},\mathbf{v}]) + (f_1(v_1) - f_1(u_1))\cdots(f_n(v_n) - f_n(u_n)) \geq 0,$$

so C is n-increasing. □

Based on the construction (6), we introduce the following parametric families of n-copulas, which is a multivariate extension of (3).

$$C(u_1, u_2, \cdots, u_n) = C^*(u_1, u_2, \cdots, u_n) + \theta \prod_{i=1}^{n} f_i(u_i), \tag{7}$$

where C^* is an n-copula, $f_i : [0, 1] \to \mathbb{R}$ is a function, $i = 1, 2, \cdots, n$, $\theta \in \mathbb{R}$.

Corollary 1 *Let C^* be an n-copula, $f_i : [0, 1] \to \mathbb{R}$ be a function, $i = 1, 2, \cdots, n$. $C : [0, 1]^n \to \mathbb{R}$ is defined by (6) is an n-copula if $f_1, \cdots, f_n$ and θ satisfy the following conditions,*

(i) *$f_1(0) = \cdots = f_n(0) = 0$, and there exist at least two functions f_i and f_j such that $f_i(1) = f_j(1) = 0$, $1 \leq i, j \leq n$,*
(ii) *f_i is absolutely continuous,*
(iii) $\sup\{-\frac{V_{C^*}([\boldsymbol{u},\boldsymbol{v}])}{\Delta(\boldsymbol{u},\boldsymbol{v})} : \boldsymbol{u},\boldsymbol{v} \in [0,1]^n, \boldsymbol{u} < \boldsymbol{v}\}\frac{1}{\max(B')} \leq \theta \leq \sup\{-\frac{V_{C^*}([\boldsymbol{u},\boldsymbol{v}])}{\Delta(\boldsymbol{u},\boldsymbol{v})} : \boldsymbol{u},\boldsymbol{v} \in [0,1]^n, \boldsymbol{u} < \boldsymbol{v}\}\frac{1}{\min(B)}$,
where B is the same as Theorem 3, $B' = \{\alpha_{i_1}\cdots\alpha_{i_k}\beta_{j_1}\cdots\beta_{j_{n-k}} : 1 \leq k \leq n$, k is even, $i_1, \cdots, i_k$ and $j_1, \cdots, j_{n-k}$ are pairwise distinct$\}$, $\alpha_i = \inf\{f_i'(u_i) : u_i \in A_i\} < 0$, $\beta_i = \sup\{f_i'(u_i) : u_i \in A_i\} > 0$, $A_i = \{u_i \in [0, 1] : f'(u_i)$ exists$\}$, $i = 1, \cdots, n$, and $\Delta(\boldsymbol{u},\boldsymbol{v}) = (v_1 - u_1)\cdots(v_n - u_n)$.

Remark 1 Conditions in Theorem 3 and Corollary 1 are sufficient but may not be necessary. Consider the Fréchert-Hoeffding upper bound of n-copulas, $M_n(u_1, \cdots, u_n) = \min\{u_1, \cdots, u_n\}$. For any $\mathbf{u}, \mathbf{v} \in [0, 1]^n$ such that $\mathbf{u} < \mathbf{v}$, it can be shown that

$$V_{M_n}([\mathbf{u},\mathbf{v}]) = \max\{\min\{v_1, \cdots, v_n\} - \max\{u_1, \cdots, u_n\}, 0\}.$$

Thus,

$$\sup\left\{-\frac{V_{C^*}([\mathbf{u},\mathbf{v}])}{\Delta(\mathbf{u},\mathbf{v})} : \mathbf{u},\mathbf{v} \in [0,1]^n, \mathbf{u} < \mathbf{v}\right\} = 0.$$

So functions $f_1, \cdots, f_n$ that satisfy conditions in Theorem 3 or Corollary 1 for M_n must be zero, i.e., $f_1 = \cdots = f_n = 0$.

Next we provide a stronger sufficient condition on $f_1, \cdots, f_n$ to ensure that C defined by (6) is an n-copula. Example 2.1 in [12] shows that the condition is not necessary.

Theorem 4 *Let C be defined by (6). C is an n-copula if $f_1, \cdots, f_n$ satisfy the following conditions,*

(i) $f_1(0) = \cdots = f_n(0) = 0$, and there exist at least two functions f_i and f_j such that $f_i(1) = f_j(1) = 0$, $1 \le i, j \le n$,

(ii) f_i satisfies the Lipschitz condition,

$$|f_i(v) - f_i(u)| \le M_i |v - u|,$$

for all $u, v \in I$, such that $M_i > 0$, $i = 1, \cdots, n$, and

$$\prod_1^n M_i \le \inf \left\{ \frac{V_{C^*}([\boldsymbol{u}, \boldsymbol{v}])}{\Delta(\boldsymbol{u}, \boldsymbol{v})} : \boldsymbol{u}, \boldsymbol{v} \in [0, 1]^n, \boldsymbol{u} \le \boldsymbol{v} \right\}.$$

Proof By the condition (i), C is grounded and one-dimensional marginals of C are uniformly distributed. For any $\mathbf{u}, \mathbf{v} \in I^n$ with $\mathbf{u} < \mathbf{v}$, by the condition (ii), we have

$$\begin{aligned} -\frac{(f_1(v_1) - f_1(u_1)) \cdots (f_n(v_n) - f_n(u_n))}{(v_1 - u_1) \cdots (v_n - u_n)} &\le \frac{|f_1(v_1) - f_1(u_1)| \cdots |f_n(v_n) - f_n(u_n)|}{|v_1 - u_1| \cdots |v_n - u_n|} \\ &\le \prod_1^n M_i \\ &\le \inf \left\{ \frac{V_{C^*}([\mathbf{u}, \mathbf{v}])}{\Delta(\mathbf{u}, \mathbf{v})} : \mathbf{u}, \mathbf{v} \in [0, 1]^n, \mathbf{u} \le \mathbf{v} \right\}. \end{aligned}$$

So

$$\frac{(f_1(v_1) - f_1(u_1)) \cdots (f_n(v_n) - f_n(u_n))}{(v_1 - u_1) \cdots (v_n - u_n)} \ge \sup \left\{ -\frac{V_{C^*}([\mathbf{u}, \mathbf{v}])}{\Delta(\mathbf{u}, \mathbf{v})} : \mathbf{u}, \mathbf{v} \in [0, 1]^n, \mathbf{u} \le \mathbf{v} \right\}.$$

Thus, as the proof of Theorem 3, C is n-increasing. □

4 Properties of New Families

In this section, we are going to study some non-parametric copula-based measures of multivariate association, some dependence concepts for copulas defined in Sect. 3 and some properties of those families.

Firstly, recall that the multivariate generalizations of *Kendall's tau*, *Spearman's rho*, and *Blomqvist's beta* (see [13, 18] for details) are given by

$$\tau_n(C) = \frac{1}{2^{n-1} - 1} \left[2^n \int_{I^n} C(\mathbf{u}) dC(\mathbf{u}) - 1 \right], \tag{8}$$

$$\rho_n(C) = \frac{n+1}{2^n - n - 1}\left[2^{n-1}\left(\int_{I^n} C(\mathbf{u})d\Pi_n(\mathbf{u}) + \int_{I^n} \Pi_n(\mathbf{u})dC(\mathbf{u})\right) - 1\right], \quad (9)$$

$$\beta_n(C) = \frac{2^{n-1}\left[C\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{C}\left(\frac{1}{2}\mathbf{1}_n\right)\right] - 1}{2^{n-1} - 1}, \quad (10)$$

where $\mathbf{1}_n$ is the vector $(1, \cdots, 1)' \in \mathbb{R}^n$.

Theorem 5 *Let C be an n-copula defined by (5), then the Kendall's tau, Spearman's rho, and Blomqvist's beta of C are given by*

$$\tau_n(C) = \tau_n(C^*) + \tau_n(H) + a_1, \quad (11)$$

$$\rho_n(C) = \rho_n(C^*) + \rho_n(H) + \frac{n+1}{2^n - n - 1}, \quad (12)$$

$$\beta_n(C) = \beta_n(C^*) + \beta_n(H) + \frac{1 - 2^{n-1}}{2^n - 1}, \quad (13)$$

where $a_1 = \frac{1}{2^{n-1} - 1}\left[2^n \int_{I^n} C^*(\boldsymbol{u})dH(\boldsymbol{u}) + 2^n \int_{I^n} H(\boldsymbol{u})dC^*(\boldsymbol{u}) + 1\right]$.

Proof Firstly, by the definition of τ_n,

$$\begin{aligned}
\tau_n(C) &= \frac{1}{2^{n-1} - 1}\left[2^n \int_{I^n} C(\mathbf{u})dC(\mathbf{u}) - 1\right] \\
&= \frac{1}{2^{n-1} - 1}\left[2^n \int_{I^n} C^*(\mathbf{u}) + H(\mathbf{u})d(C^*(\mathbf{u}) + H(\mathbf{u})) - 1\right] \\
&= \frac{1}{2^{n-1} - 1}[2^n \int_{I^n} C^*(\mathbf{u})dC^*(\mathbf{u}) + 2^n \int_{I^n} H(\mathbf{u})dH(\mathbf{u}) \\
&\quad + 2^n \int_{I^n} C^*(\mathbf{u})dH(\mathbf{u}) + 2^n \int_{I^n} H(\mathbf{u})dC^*(\mathbf{u}) - 1] \\
&= \tau_n(C^*) + \tau_n(H) + a_1.
\end{aligned}$$

Secondly, by the definition of ρ_n,

$$\begin{aligned}
\rho_n(C) &= \frac{n+1}{2^n - n - 1}\left\{2^{n-1}\left[\int_{I^n} C(\mathbf{u})d\Pi_n(\mathbf{u}) + \int_{I^n} \Pi_n(\mathbf{u})dC(\mathbf{u})\right] - 1\right\} \\
&= \frac{n+1}{2^n - n - 1}\left\{2^{n-1}\left[\int_{I^n} C^*(\mathbf{u}) + H(\mathbf{u})d\Pi_n(\mathbf{u}) + \int_{I^n} \Pi_n(\mathbf{u})d(C^*(\mathbf{u}) + H(\mathbf{u}))\right] - 1\right\} \\
&= \frac{n+1}{2^n - n - 1}\{2^{n-1}[\int_{I^n} C^*(\mathbf{u})d\Pi_n(\mathbf{u}) + \int_{I^n} H(\mathbf{u})d\Pi_n(\mathbf{u}) + \int_{I^n} \Pi_n(\mathbf{u})dC^*(\mathbf{u}) \\
&\quad + \int_{I^n} \Pi_n(\mathbf{u})dH(\mathbf{u})] - 1\} \\
&= \rho_n(C^*) + \rho_n(H) + \frac{n+1}{2^n - n - 1}.
\end{aligned}$$

Lastly, by the definition of survival functions, for any $\mathbf{u} \in I^n$,

$$\begin{aligned}
\overline{C}(\mathbf{u}) &= 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} C_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) \\
&= 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} \left[C^*_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) + H_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) \right] \\
&= 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} C^*_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) \\
&\quad + 1 + \sum_{k=1}^{n} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} H_{i_1 i_2 \cdots i_k}(u_{i_1}, \cdots, u_{i_k}) - 1 \\
&= \overline{C^*}(\mathbf{u}) + \overline{H}(\mathbf{u}) - 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\beta_n(C) &= \frac{2^{n-1}\left[C\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{C}\left(\frac{1}{2}\mathbf{1}_n\right)\right] - 1}{2^{n-1} - 1} \\
&= \frac{2^{n-1}\left[C^*\left(\frac{1}{2}\mathbf{1}_n\right) + H\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{C^*}\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{H}\left(\frac{1}{2}\mathbf{1}_n\right) - 1\right] - 1}{2^{n-1} - 1} \\
&= \frac{2^{n-1}\left[C^*\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{C^*}\left(\frac{1}{2}\mathbf{1}_n\right)\right] - 1 + 2^{n-1}\left[H\left(\frac{1}{2}\mathbf{1}_n\right) + \overline{H}\left(\frac{1}{2}\mathbf{1}_n\right)\right] - 1 + 1 - 2^{n-1}}{2^{n-1} - 1} \\
&= \beta_n(C^*) + \beta_n(H) + \frac{1 - 2^{n-1}}{2^n - 1}.
\end{aligned}$$

□

Remark 2 In the above theorem, although the perturbation function H is not a copula, we still use $m(H)$ to denote the corresponding values of H, where $m = \tau_n, \rho_n$, or β_n, and use $\overline{H}$ to denoted the corresponding function of H defined by (1). The similar notations are used in the following context.

Remark 3 As n increasing, we can see that

$$\tau_n(C) \approx \tau_n(C^*) + \tau_n(H) + 2\int_{I^n} C^*(\mathbf{u}) dH(\mathbf{u}) + 2\int_{I^n} H(\mathbf{u}) dC^*(\mathbf{u}),$$

$$\rho_n(C) \approx \rho_n(C^*) + \rho_n(H),$$

and

$$\beta_n(C) \approx \beta_n(C^*) + \beta_n(H).$$

Corollary 2 *Let C be an n-copula defined by (7), then the Kendall's tau, Spearman's rho, and Blomqvist's beta of C are given by*

$$\tau_n(C) = \tau_n(C^*) + \tau_n(\theta\prod_{i=1}^{n} f_i) + a_2, \tag{14}$$

$$\rho_n(C) = \rho_n(C^*) + \rho_n(\theta\prod_{i=1}^{n} f_i) + \frac{n+1}{2^n - n - 1}, \tag{15}$$

$$\beta_n(C) = \beta_n(C^*) + \beta_n(\theta\prod_{i=1}^{n} f_i) + \frac{1-2^{n-1}}{2^n - 1}, \tag{16}$$

where $a_2 = \frac{1}{2^{n-1}-1}\left[2^n \int_{I^n} \theta C^*(\boldsymbol{u})\prod_{i=1}^{n} f_i{}'(u_i)d\boldsymbol{u} + 2^n \int_{I^n} \theta\prod_{i=1}^{n} f_i(u_i)dC^*(\boldsymbol{u}) + 1\right].$

In 2013, Tasena et al. [17] defined a measure of *multivariate complete dependence* as follows. Let C be an n-copula of random variables $X_1, \cdots, X_n$. Define

$$\delta_i(X_1, \cdots, X_n) = \delta_i(C) = \frac{\int(\partial_i C - \pi_i C)^2}{\int \pi_i C(1-\pi_i C)},$$

where $\pi_i C : I^{n-1} \to I$ is defined by

$$\pi_i C(u_1, \cdots, u_{n-1}) = C(u_1, \cdots, u_{i-1}, 1, u_i, \cdots, u_{n-1}), \qquad i = 1, 2, \cdots, n.$$

By Theorem 3.6 in [17], δ_i satisfies following properties,

(i) $0 \le \delta_i(C) \le 1$,
(ii) $\delta_i(C) = 1$ if and only if $(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ is a function of X_i. For details, see [17].

Theorem 6 *Let C be an n-copula defined by (5). If*

$$H(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_{n_1}) = 0,$$

then

$$\delta_i(C) = \delta(C^*) - \frac{\int 2\partial_i H(\partial_i C^* - \pi_i C^*) - (\partial_i H)^2}{\int \pi_i C^*(1-\pi_i C^*)}.$$

Proof By the definition,

$$\begin{aligned}\delta_i(C) &= \frac{\int(\partial_i C - \pi_i C)^2}{\int \pi_i C(1-\pi_i C)}\\ &= \frac{\int(\partial_i(C^*+H) - \pi_i(C^*+H))^2}{\int \pi_i(C^*+H)[1-\pi_i(C^*+H)]}\\ &= \frac{\int(\partial_i C^* + \partial_i H - \pi_i C^* - \pi_i H)^2}{\int(\pi_i C^* + \pi_i H)(1-\pi_i C^* - \pi_i H)}.\end{aligned}$$

If $\pi_i H(u_1, \cdots, u_{n_1}) = H(u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_{n_1}) = 0$, then

$$\begin{aligned}\delta_i(C) &= \frac{\int(\partial_i C^* + \partial_i H - \pi_i C^* - \pi_i H)^2}{\int(\pi_i C^* + \pi_i H)(1-\pi_i C^* \pi_i H)}\\ &= \frac{\int(\partial_i C^* + \partial_i H - \pi_i C^*)^2}{\int \pi_i C^*(1-\pi_i C^*)}\\ &= \frac{\int\left[(\partial_i C^* - \pi_i C^*)^2 - 2\partial_i H(\partial_i C^* - \pi_i C^*) + (\partial_i H)^2\right]}{\int \pi_i C^*(1-\pi_i C^*)}\\ &= \frac{\int(\partial_i C^* - \pi_i C^*)^2}{\int \pi_i C^*(1-\pi_i C^*)} - \frac{\int 2\partial_i H(\partial_i C^* - \pi_i C^*) - (\partial_i H)^2}{\int \pi_i C^*(1-\pi_i C^*)}\\ &= \delta(C^*) - \frac{\int 2\partial_i H(\partial_i C^* - \pi_i C^*) - (\partial_i H)^2}{\int \pi_i C^*(1-\pi_i C^*)}.\end{aligned}$$

□

Corollary 3 *Let C be an n-copula defined by (7). If $f_i(1) = 0$, then*

$$\delta_i(C) = \delta_i(C^*) - \frac{\int 2\theta f_i' \prod_{j\neq i} f_j(\partial_i C^* - \pi_i C^*) - (\theta f_i' \prod_{j\neq i} f_j)^2}{\int \pi_i C^*(1-\pi_i C^*)}.$$

Now, let's recall some dependence concepts of copulas. For details, see [6, 11]. Let C_1 and C_2 be two n-copulas. If $C_1 \geq C_2$ ($\overline{C}_1 \geq \overline{C}_2$ resp.), i.e., $C_1(\mathbf{u}) \geq C_2(\mathbf{u})$ ($\overline{C}_1(\mathbf{u}) \geq \overline{C}_2(\mathbf{u})$ resp.) for all $\mathbf{u} \in I^n$, then we say that C_1 is *more positive lower* (*upper* resp.) *orthant dependent* (PLOD) (PUOD resp.) than C_2. C_1 is *more positive orthant dependent* (POD) than C_2 if $C_1 \geq C_2$ and $\overline{C}_1 \geq \overline{C}_2$ hold.

The following results give us some dependence relations between C and C^*. The proof is trivial.

Proposition 1 *Let C_1 and C_2 be two n-copulas defined by (5). If they share the same n-copula C^* and may have different perturbation functions H_i $i = 1, 2$, then*

(i) C_1 more PLOD than C_2 if and only if $H_1 \geq H_2$,
(ii) C_1 more PUOD than C_2 if and only if $\overline{H}_1 \geq \overline{H}_2$,
(iii) C_1 more POD than C_2 if and only if $H_1 \geq H_2$ and $\overline{H}_1 \geq \overline{H}_2$.

Proposition 2 *Let C_1 and C_2 are two n-copulas defined by (7). If they share the same known n-copula C^* and may have different perturbation functions $f_{j1}, \cdots, f_{jn}$, and parameters θ_j, $j = 1, 2$ respectively, then*

(i) C_1 more PLOD than C_2 if and only if $\theta_1 \prod_{i=1}^{n} f_{1i} \geq \theta_2 \prod_{i=1}^{n} f_{2i}$,

(ii) C_1 more PUOD than C_2 if and only if $\overline{\theta_1 \prod_{i=1}^{n} f_{1i}} \geq \overline{\theta_2 \prod_{i=1}^{n} f_{2i}}$,

(iii) C_1 more POD than C_2 if and only if $\theta_1 \prod_{i=1}^{n} f_{1i} \geq \theta_2 \prod_{i=1}^{n} f_{2i}$ *and* $\overline{\theta_1 \prod_{i=1}^{n} f_{1i}} \geq \overline{\theta_2 \prod_{i=1}^{n} f_{2i}}$.

The next theorem give us a property of the construction (6).

Theorem 7 *Let $(U_1^*, \cdots, U_n^*)$ and $(U_1, \cdots, U_n)$ be random vectors with uniform marginals on $[0, 1]$ and connected by copulas C^* and C respectively. C and C^* satisfy conditions of Theorem 3. Suppose that $f_i(1) = f_j(1) = 0$, $1 \leq i < j \leq n$.*

(i) If there is $1 \leq l \leq n$ such that $l \neq i, j$ and $f_l(1) = 0$, then $P\{U_i < U_j\} = P\{U_i^* < U_j^*\}$,

(ii) If $f_l(1) \neq 0$ for all $l \neq i, j$ and $f_i = f_j$, then $P\{U_i < U_j\} = P\{U_i^* < U_j^*\}$.

Proof (i) Let c and c^* be the densities of C and C^* respectively, then we have

$$c(\mathbf{u}) = \frac{\partial^n C(\mathbf{u})}{\partial u_1 \cdots \partial u_n} = c^*(\mathbf{u}) + \prod_{i=1}^{n} f_i{}'(u_i).$$

Then

$$\begin{aligned}
P\{U_i < U_j\} &= \int_0^1 \cdots \int_0^{u_j} \cdots \int_0^1 c(u_1, \cdots, u_i, \cdots, u_n) du_1 \cdots du_i \cdots du_n \\
&= \int_0^1 \cdots \int_0^{u_j} \cdots \int_0^1 c^*(u_1, \cdots, u_i, \cdots, u_n) du_1 \cdots du_i \cdots du_n \\
&\quad + \int_0^1 \cdots \int_0^{u_j} \cdots \int_0^1 f_1'(u_i) \cdots f_i'(u_i) \cdots f_n'(u_n) du_1 \cdots du_i \cdots du_n \\
&= P\{U_i^* < U_j^*\} + \prod_{k \neq i,j}^{n} (f_k(1) - f_k(0)) \int_0^1 \int_0^{u_j} f_j'(u_j) f_i'(u_i) du_i du_j \\
&= P\{U_i^* < U_j^*\},
\end{aligned}$$

since $f_l(0) = f_l(1) = 0$ and $l \neq i, j$.

(ii) Similarly, if $f_l(1) \neq 0$ for all $l \neq i, j$,

$$P\{U_i < U_j\} = P\{U_i^* < U_j^*\} + \prod_{k\neq i,j}^{n} (f_k(1) - f_k(0)) \int_0^1 \int_0^{u_j} f_j'(u_j) f_i'(u_i) du_i du_j$$
$$= P\{U_i^* < U_j^*\} + \prod_{k\neq i,j}^{n} f_k(1) \int_0^1 \int_0^{u_j} f_j'(u_j) f_i'(u_i) du_i du_j$$
$$= P\{U_i^* < U_j^*\} + \prod_{k\neq i,j}^{n} f_k(1) \int_0^1 f_i(u_j) f_j'(u_j) du_j.$$

Since $f_i = f_j$,

$$\int_0^1 f_i(u_j) f_j'(u_j) du_j = f_j(1) f_i(1) - f_j(0) f_i(0) - \int_0^1 f_j(u_j) f_i'(u_j) du_j$$
$$= -\int_0^1 f_j(u_j) f_i'(u_j) du_j = -\int_0^1 f_i(u_j) f_j'(u_j) du_j,$$

and hence $\int_0^1 f_i(u_j) f_j'(u_j) du_j = 0$. So $P\{U_i < U_j\} = P\{U_i^* < U_j^*\}$. □

The following example shows that the converse of the above result (ii) in Theorem 7 may not hold in general. Moreover, it shows that Theorem 3 in [7] is incorrect.

Example 1 Let (U^*, V^*) and (U, V) be random vectors with uniform marginals on [0, 1]. Suppose that (U^*, V^*) is connected by the independent copula, i.e., $C^*(u, v) = uv$, and (U, V) is connected by $C(u, v) = C^*(u, v) + f(u)g(v)$, where $f(u) = u(1 - u)$, $g(v) = \frac{1}{2}v(1 - v)$. Then f and g satisfy the conditions in Theorem 3. In fact, C belongs to the bivariate FGM family.

As the proof of the above theorem, we have

$$P\{U < V\} = \int_0^1 \int_0^v c(u, v) du dv = \int_0^1 \int_0^v c^*(u, v) + f'(u) g'(v) du dv$$
$$= P\{U^* < V^*\} + \int_0^1 \int_0^v f'(u) g'(v) du dv = P\{U^* < V^*\} + \int_0^1 f(v) g'(v) dv.$$

where

$$\int_0^1 f(v) g'(v) dv = \int_0^1 \frac{1}{2} v(1 - v)(1 - 2v) dv = 0.$$

Thus $P\{U < V\} = P\{U^* < V^*\}$, but $f \neq g$.

5 Examples

In this section, we provide two examples. The given copula C^* in the first example is the simplest one, the independent copula. To emphasis multivariate and for simplicity, we will only consider 3-copulas, but results could be extended to n-copulas. In the second example, C^* is nontrivial. Also for simplicity, we will only consider 2-copulas.

Example 2 Let C^* be the independent 3-copula, i.e., $C^*(u, v, w) = uvw$. Let $f(x) = x(1 - x^k)$, where $u, v, w, x \in I$, $k \in \mathbb{N}$, the set of all positive integers. Consider the 3-copula family,

$$
\begin{aligned}
C(u, v, w) &= C^*(u, v, w) + \theta f(u) f(v) f(w) \\
&= uvw + \theta uvw(1 - u^k)(1 - v^k)(1 - w^k),
\end{aligned}
$$

where $\theta \in \mathbb{R}$.

It is clear that $f(x)$ satisfies the conditions (i) and (ii) of Corollary 1. Next we will use the condition (iii) of Corollary 1 to find the range of the parameter θ for each k. Firstly, it is easy to see that $\dfrac{V_{C^*}([\mathbf{u}, \mathbf{v}])}{\Delta(\mathbf{u}, \mathbf{v})} = 1$ for any $\mathbf{u}, \mathbf{v} \in [0, 1]^3$ with $\mathbf{u} < \mathbf{v}$. Secondly, $f'(x) = 1 - (k + 1)x^k$, so

$$
\alpha = \inf\{f'(x) : x \in I\} = f'(1) = 1 - (k + 1) = -k,
$$

and

$$
\beta = \sup\{f'(x) : x \in I\} = f'(0) = 1.
$$

Thus, as the notations in Theorem 3, $B = \{-k, -k^3\}$, $B' = \{k^2\}$. So by the condition (iii) of Corollary 1, the range of θ is

$$
-\frac{1}{\max(B')} \leq \theta \leq -\frac{1}{\min(B)},
$$

i.e.,

$$
-\frac{1}{k^2} \leq \theta \leq \frac{1}{k^3}.
$$

So, we can see that the range of θ is shrinking as k increasing. Specifically, if $k = 1$, $-1 \leq \theta \leq 1$. If $k = 2$, $-\frac{1}{4} \leq \theta \leq \frac{1}{8}$. If $k = 3$, $-\frac{1}{9} \leq \theta \leq \frac{1}{27}$.

Next, let's compute three measures discussed in Sect. 4 for these 3-copulas. By the definition of τ_n,

$$
\tau_3\left(\theta f(u) f(v) f(w)\right) = \frac{1}{3}\left[8\theta\left(\frac{k+1}{2k+3} - \frac{k+2}{k+3} + \frac{1}{3}\right) - 1\right].
$$

$$a_2 = \frac{1}{3}\left[8\int_{I^3}\theta C^*(\mathbf{u})\prod_{i=1}^{3} f_i{}'(u_i)d\mathbf{u} + 8\int_{I^3}\theta\prod_{i=1}^{3} f_i(u_i)dC^*(\mathbf{u}) + 1\right]$$
$$= \frac{1}{3}\left[-\frac{\theta k^3}{(k+2)^3} + \frac{\theta k^3}{(k+2)^3} + 1\right] = \frac{1}{3}.$$

So by Corollary 2,

$$\begin{aligned}\tau_3(C) &= \tau_3(C^*) + \tau_3(\theta f(u)f(v)f(w)) + a_2\\ &= 0 + \frac{1}{3}\left[8\theta\left(\frac{k+1}{2k+3} - \frac{k+2}{k+3} + \frac{1}{3}\right) - 1\right] + \frac{1}{3}\\ &= \frac{8\theta}{3}\left(\frac{k+1}{2k+3} - \frac{k+2}{k+3} + \frac{1}{3}\right).\end{aligned}$$

So the range of $\tau_3(C)$ is

$$\frac{8}{3k^3}\left(\frac{k+1}{2k+3} - \frac{k+2}{k+3} + \frac{1}{3}\right) \le \tau_3(C) \le -\frac{8}{3k^2}\left(\frac{k+1}{2k+3} - \frac{k+2}{k+3} + \frac{1}{3}\right).$$

By the definition of ρ_n,

$$\rho_3(\theta f(u)f(v)f(w)) = 4\left[\frac{\theta k^3}{8(k+2)^3} - \frac{\theta k^3}{8(k+2)^3}\right] - 1 = -1.$$

So

$$\begin{aligned}\rho_3(C) &= \rho_3(C^*) + \rho_3(\theta f(u)f(v)f(w)) + \frac{3+1}{2^3-3-1}\\ &= 0 - 1 + 1 = 0.\end{aligned}$$

By the definition of survival function (1),

$$\overline{\theta\prod_{i=1}^{3} f_i}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = 1 - \frac{\theta}{8}\left(1-\frac{1}{2^k}\right)^3.$$

So

$$\begin{aligned}\beta_3(\theta f(u)f(v)f(w)) &= \frac{2^2\left[\theta\prod_{i=1}^{3} f_i\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \overline{\theta\prod_{i=1}^{3} f_i}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right] - 1}{2^2-1}\\ &= \frac{4\left[\frac{\theta}{8}\left(1-\frac{1}{2^k}\right)^3 + 1 - \frac{\theta}{8}\left(1-\frac{1}{2^k}\right)^3\right] - 1}{3} = 1.\end{aligned}$$

Thus,

$$\beta_3(C) = \beta_3(C^*) + \beta_3(\theta f(u) f(v) f(w)) + \frac{1-2^{3-1}}{2^3-1}$$
$$= 0 + 1 - \frac{3}{7} = \frac{4}{7}.$$

Lastly, since $f(u)f(v)f(w) = uvw(1-u^k)(1-v^k)(1-w^k) \geq 0$ for all $(u, v, w) \in I^3$, we have that C is more PLOD than Π_3 if and only if $\theta \geq 0$ and Π_3 is more PLOD than C if and only if $\theta \leq 0$.

Remark 4 From the above example, we can see that this 3-copulas family, $C(u, v, w) = C^*(u, v, w) + \theta uvw(1-u^k)(1-v^k)(1-w^k)$, is interesting. As long as this C is a 3-copula, $\rho_3(C)$ and $\beta_3(C)$ are free of θ. Specifically, we always have $\rho_3(C) = \rho_3(C^*)$ and $\beta_3(C) = \beta_3(C^*) + \frac{4}{7}$.

Example 3 Let C^* be a *Frank's* copula [2, 5] defined by

$$C^*(u, v) = \ln\left[1 + \frac{(e^u-1)(e^v-1)}{e-1}\right].$$

Let

$$H = \theta(1-u)(1-e^u)(1-v)(1-e^v),$$

where $\theta \geq 0$. Define a bivariate function C by $C = C^* + H$. We will use Theorem 2 to find the range of θ such that C is a copula.

Firstly, it is easy to see that $H(0, v) = H(u, 0) = H(1, v) = H(u, 1) = 0$.

Secondly, we can find that

$$c^*(u, v) = \frac{(e-1)(u+v)}{[e-1+(e^u-1)(e^v-1)]^2},$$

and

$$h(u, v) = \theta(ue^u - 1)(ve^v - 1).$$

It can be shown that minimum values of $c = c^* + h$ occur at (0, 1) and (1, 0). So $c \geq 0$ if and only if $c(0, 1) = c(1, 0) = c^*(0, 1) + h(0, 1) = \frac{1}{e-1} - \theta(e-1) \geq 0$. Thus $C = C^* + H$ is a copula if $\theta \leq \frac{1}{(e-1)^2}$.

Acknowledgements The authors would like to thank referees for their valuable comments.

References

1. Aguiló I, Suñer J, Torrens J (2013) A construction method of semicopulas from fuzzy negations. Fuzzy Set Syst 226:99–114
2. Balakrishnan N, Lai CD (2009) Continuous bivariate distributions, 2nd edn. Springer, New York
3. Durante F, Foscolo E, Rodríguez-Lallena JA, Úbeda-Flores M (2012) A method for constructing higher-dimensional copulas. Statistics 46(3):387–404
4. Durante F, Sánchez JF, Úbeda-Flores M (2013) Bivariate copulas generated by perturbations. Fuzzy Set Syst 228:137–144
5. Frank M (1979) On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. Aequationes Math 19(1):194–226
6. Joe H (1997) Multivariate models and dependence concepts. CRC Press, Boca Raton
7. Kim JM, Sungur EA, Choi T, Heo TY (2011) Generalized bivariate copulas and their properties. Model Assist Stat Appl 6:127–136
8. Liebscher E (2008) Construction of asymmetric multivariate copulas. J Multivar Anal 99: 2234–2250
9. Mesiar R, Komorník J, Komorníková M (2013) On some construction methods for bivariate copulas. In: Aggregation functions in theory and in practise, pp 39–45
10. Mesiar R, Komorníková M, Komorník J (2015) Perturbation of bivariate copulas. Fuzzy Set Syst 268:127–140
11. Nelsen RB (2006) An introduction to copulas, 2nd edn. Springer, New York
12. Rodríguez-Lallena JA, Úbeda-Flores M (2004) A new class of bivariate copulas. Stat Probab Lett 66(3):315–325
13. Schmid F, Schmidt R, Blumentritt T, Gaißer S, Ruppert M (2010) Copula-based measures of multivariate association. In: Copula theory and its applications, pp 209–236
14. Shan Q, Wongyang T, Wang T, Tasena S (2015) A measure of mutual complete dependence in discrete variables through subcopula. Int J Apporx Reason 65:11–23
15. Siburg KF, Stoimenov PA (2009) A measure of mutual complete dependence. Metrika 71: 239–251
16. Sklar A (1959) Fonctions de répartition á n dimensions et leurs marges. Publ Inst Statist Univ Paris 8:229–231
17. Tasena S, Dhompongsa S (2013) A measure of multivariate mutual complete dependence. Int J Apporx Reason 54:748–761
18. Úbeda-Flores M (2005) Multivariate versions of Blomqvist's beta and Spearman's footrule. Ann Inst Statist Math 57(4):781–788
19. Victor H, Ibragimov R, Sharakhmetov S (2006) Characterizations of joint distributions, copulas, information, dependence and decoupling, with applications to time series. In: Optimality, pp 183–209
20. Wei Z, Wang T, Nguyen PA (2015) Multivariate dependence concepts through copulas. Int J Apporx Reason 65:24–33