Chapter 8
A Rapid Review of Some Elements of Continuum Mechanics

"There are analogs of the energy-momentum tensor in branches of continuum mechanics other than the theory of elasticity. Some of them might repay investigation."
J. D. Eshelby [103, Page 113]. His remarks foreshadow the use of the material force C in one-dimensional continua.

8.1 Introduction

Our interest lies in the application of one-dimensional theories of matter to various problems and the analysis of the resulting models. One central issue that arises at the onset is the development of a suitable model from a wide range of possible choices. Partially as a result of historical and pedagogical developments, it is often not transparent how various beam, rod, and cable theories can be considered in the context of continuum mechanics of three-dimensional continua. One of the goals of this textbook is to make these connections transparent. Fortunately, we have plenty of help and guidance from the literature to achieve this goal (see, e.g., [12, 137, 147, 243, 309]).

In this chapter, we review some needed background from continuum mechanics. Most of the kinematics we cover are standard and can be found in many introductory texts on continuum mechanics (such as Gurtin [147]). We supplement the kinematics with details on convected coordinates from a seminal textbook by Green and Zerna [140]. These coordinates play a key role in deriving rod and string theories from three-dimensional continuum mechanics and they also serve to illuminate the role played by various stress tensors. We then turn to a discussion of the balance laws and constitutive relations for the stress tensors. The chapter closes with a discussion of superposed rigid body motions, constraints, and a material (configurational) force balance. As discussed in Chapters 2, 4, and 5, the counterpart of the material force

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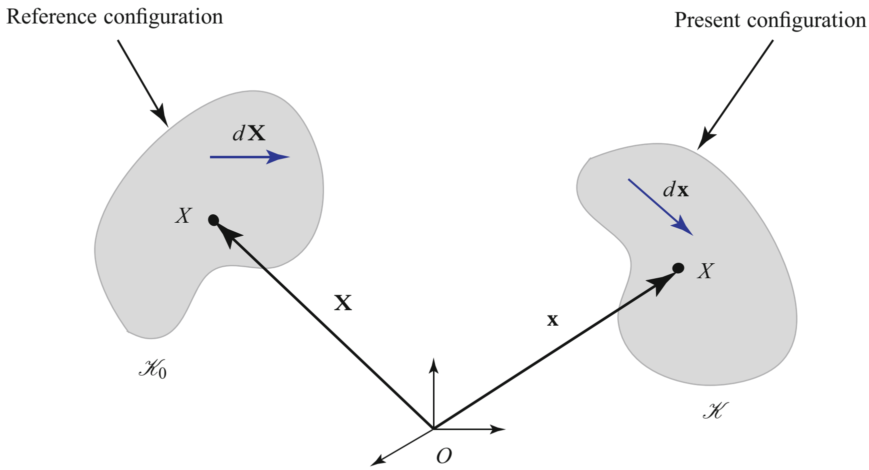

Fig. 8.1 The reference $\mathscr{K}_0$ and present $\mathscr{K}$ configurations of a body $\mathscr{B}$.

balance for one-dimensional continua, such as strings and rods, plays a key role in solving problems where a discontinuity is present in the motion and often yields a useful conservation law in others.

8.2 Some Kinematical Results

Consider a body $\mathscr{B}$ and let $\mathscr{K}_0$ and $\mathscr{K}$ denote its reference and present configurations, respectively (see Figure 8.1). Here, we define a body $\mathscr{B}$ to be a collection of material points X. For the present purposes, this collection of material points is fixed. The reference configuration occupies a fixed region of three-dimensional Euclidean space $\mathbb{E}^3$ and the position vector of a material point X in this configuration is denoted by $\mathbf{X} = \mathbf{R}^*$. The position vector of the same material point in the subset of $\mathbb{E}^3$ known as the present configuration $\mathscr{K}$ is denoted by $\mathbf{x} = \mathbf{r}^*$.

The motion of $\mathscr{B}$ is denoted by the vector-valued function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)\,. \tag{8.1}$$

We also have the deformation gradient $\mathbf{F}$ of this motion:

$$\mathbf{F} = \operatorname{Grad}(\boldsymbol{\chi})\,. \tag{8.2}$$

This second-order tensor can also be viewed as a linear transformation of vectors $d\mathbf{X}$ which are tangent to material curves in $\mathscr{K}_0$ to their counterparts $d\mathbf{x}$ in the present configuration $\mathscr{K}$:

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}. \tag{8.3}$$

We shall assume that **F** preserves orientation and is invertible:

$$J = \det(\mathbf{F}) > 0. \tag{8.4}$$

The positiveness of J ensures that the motion is orientation preserving.

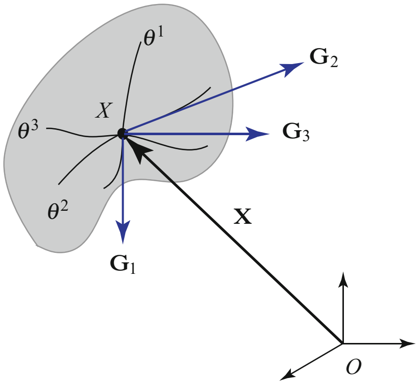

Fig. 8.2 The curvilinear coordinate system that is used to identify material points in the reference configuration.

8.2.1 Curvilinear Coordinates

Motivated by Green and Zerna [140], it is convenient to define a set of curvilinear coordinates $\left\{\theta^i\right\}$ which uniquely identify material points in $\mathscr{K}_0$. That is, we assume that the curvilinear coordinates and their Cartesian counterparts are related by invertible functions:

$$\theta^k = \theta^k(X_1, X_2, X_3), \tag{8.5}$$

where $X_i = \mathbf{X} \cdot \mathbf{E}_i$. It follows from this assumption that any function $f = f(X_1, X_2, X_3)$ can be expressed as an equivalent function of the curvilinear coordinates: $f = \hat{f}\left(\theta^1, \theta^2, \theta^3\right)$. In particular,

$$\mathbf{X} = \mathbf{R}^*\left(\theta^1, \theta^2, \theta^3\right) = \mathbf{R}^*\left(\theta^i\right). \tag{8.6}$$

A schematic of such a coordinate system and its coordinate curves is shown in Figure 8.2. Where confusion may arise, we ornament quantities associated with three-dimensional fields with an asterisk so as to distinguish them from their one-dimensional counterparts: e.g., $\mathbf{R}(\xi)$ and $\mathbf{R}^*\left(\theta^1, \theta^2, \theta^3\right)$.

We define the covariant vectors associated with this coordinate system in $\mathscr{K}_0$:

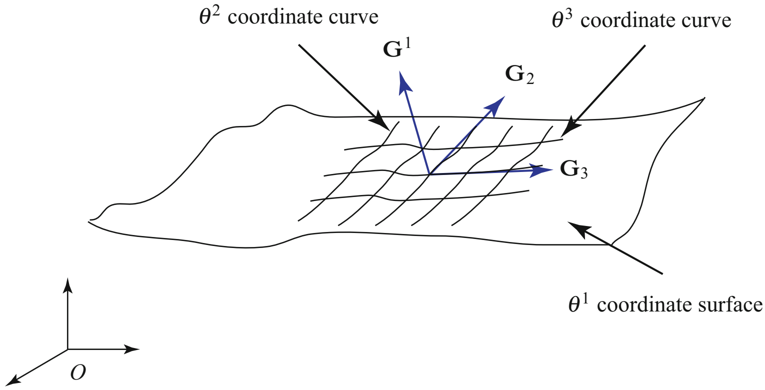

Fig. 8.3 Schematic of a θ^1 coordinate surface which is foliated by θ^2 and θ^3 coordinate curves. The vector $\mathbf{G}^1$ is normal to the θ^1 coordinate surface, while the vectors $\mathbf{G}_2$ and $\mathbf{G}_3$ are tangent to this surface.

$$\mathbf{G}_i = \frac{\partial \mathbf{R}^*}{\partial \theta^i} = \frac{\partial X_1}{\partial \theta^i}\mathbf{E}_1 + \frac{\partial X_2}{\partial \theta^i}\mathbf{E}_2 + \frac{\partial X_3}{\partial \theta^i}\mathbf{E}_3. \tag{8.7}$$

These vectors are tangent to their respective coordinate curves. For example, the vector $\mathbf{G}_2$ is tangent to a θ^2 coordinate curve. We also define the three dual or contravariant vectors $\mathbf{G}^k$:

$$\mathbf{G}^k = \text{Grad}\left(\theta^k\right) = \frac{\partial \theta^k}{\partial X_1}\mathbf{E}_1 + \frac{\partial \theta^k}{\partial X_2}\mathbf{E}_2 + \frac{\partial \theta^k}{\partial X_3}\mathbf{E}_3. \tag{8.8}$$

With the help of the chain rule, it is possible to show that

$$\mathbf{G}^k \cdot \mathbf{G}_i = \delta_i^k, \tag{8.9}$$

where δ_i^k is the Kronecker delta: $\delta_1^1 = \delta_2^2 = \delta_3^3 = 1$ and $\delta_2^1 = \delta_1^2 = \delta_2^3 = \ldots = 0$. It follows that $\mathbf{G}^k$ is normal to a θ^k coordinate surface (see Figure 8.3). In addition, solving the nine equations (8.9) for the nine components of the contravariant vectors, we find the well-known results

$$\mathbf{G}^1 = \frac{1}{\sqrt{G}}\mathbf{G}_2 \times \mathbf{G}_3, \quad \mathbf{G}^2 = \frac{1}{\sqrt{G}}\mathbf{G}_3 \times \mathbf{G}_1, \quad \mathbf{G}^3 = \frac{1}{\sqrt{G}}\mathbf{G}_1 \times \mathbf{G}_2, \tag{8.10}$$

where

$$\sqrt{G} = (\mathbf{G}_1 \times \mathbf{G}_2) \cdot \mathbf{G}_3. \tag{8.11}$$

It follows from the expressions (8.10) that

$$\frac{1}{\sqrt{G}} = \left(\mathbf{G}^1 \times \mathbf{G}^2\right) \cdot \mathbf{G}^3. \tag{8.12}$$

The set of vectors $\{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3\}$ form a basis for $\mathbb{E}^3$ which is known as a covariant basis. Similarly, the set $\{\mathbf{G}^1, \mathbf{G}^2, \mathbf{G}^3\}$ form a contravariant basis for $\mathbb{E}^3$. All of the six vectors in these sets are not necessarily of unit magnitude and some of the vectors may depend on θ^1, θ^2, and θ^3.

As illustrative examples, consider the Cartesian coordinate system $\theta^k = x^k$:

$$\mathbf{G}_k = \mathbf{E}_k, \qquad \mathbf{G}^j = \mathbf{E}_j, \qquad (j,k = 1,2,3). \tag{8.13}$$

A more interesting example is the spherical polar coordinate system (R, φ, ϑ):

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \tan(\varphi) = \frac{\sqrt{x_2^2 + x_1^2}}{x_3}, \qquad \tan(\vartheta) = \frac{x_2}{x_1}. \tag{8.14}$$

Whence,

$$\begin{aligned} \mathbf{G}_1 = \mathbf{e}_R &= \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}(x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3), \\ \mathbf{G}_2 = R\mathbf{e}_\varphi &= R\cos(\varphi)(\cos(\vartheta)\mathbf{E}_1 + \sin(\vartheta)\mathbf{E}_2) - R\sin(\varphi)\mathbf{E}_3, \\ \mathbf{G}_3 = R\sin(\varphi)\mathbf{e}_\vartheta &= R\sin(\varphi)(\cos(\vartheta)\mathbf{E}_2 - \sin(\vartheta)\mathbf{E}_1), \end{aligned} \tag{8.15}$$

and

$$\mathbf{G}^1 = \mathbf{G}_1, \qquad \mathbf{G}^2 = \frac{1}{R^2}\mathbf{G}_2, \qquad \mathbf{G}^3 = \frac{1}{R^2\sin^2(\varphi)}\mathbf{G}_3. \tag{8.16}$$

For this coordinate system, $G = R^4\sin^2(\varphi)$.

Any vector $\mathbf{b}$ can be expressed as a linear combination of either the covariant or contravariant basis vectors:

$$\mathbf{b} = \sum_{i=1}^{3} b^i\mathbf{G}_i = \sum_{k=1}^{3} b_k\mathbf{G}^k. \tag{8.17}$$

To calculate the components, we note that

$$b^k = \mathbf{b}\cdot\mathbf{G}^k, \qquad b_i = \mathbf{b}\cdot\mathbf{G}_i. \tag{8.18}$$

For the purposes of introducing the Christoffel symbols, it is useful to define

$$G_{ik} = \mathbf{G}_i\cdot\mathbf{G}_k, \qquad G^{ik} = \mathbf{G}^i\cdot\mathbf{G}^k. \tag{8.19}$$

It is straightforward to show that $\sum_{r=1}^{3} G_{ir}G^{rk} = \delta_i^k$.

Following [29, 59], we define the connection coefficients γ_{irk} and γ_{ir}^k:

$$\frac{\partial\mathbf{G}_i}{\partial\theta^r} = \sum_{k=1}^{3}\gamma_{irk}\mathbf{G}^k = \sum_{k=1}^{3}\gamma_{ir}^k\mathbf{G}_k. \tag{8.20}$$

As

$$\frac{\partial \mathbf{G}_i}{\partial \theta^r} = \frac{\partial^2 \mathbf{R}^*}{\partial \theta^r \partial \theta^i} = \frac{\partial \mathbf{G}_r}{\partial \theta^i}, \tag{8.21}$$

these coefficients are identical to the Christoffel symbols,

$$\begin{aligned} \Gamma_{irk} &= \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial \theta^r} + \frac{\partial G_{rk}}{\partial \theta^i} - \frac{\partial G_{ir}}{\partial \theta^k}\right), \\ \Gamma_{ir}^k &= \sum_{s=1}^{3} G^{ks}\Gamma_{irs}, \end{aligned} \tag{8.22}$$

respectively, that can be found in classic texts on differential geometry, such as [234, 325, 328], and texts on continuum mechanics, such as [140].[1]

The gradient of a scalar-valued function $f = f\left(\theta^1, \theta^2, \theta^3\right)$ and the gradient of a vector-valued function $\mathbf{f} = \mathbf{f}\left(\theta^1, \theta^2, \theta^3\right)$ are defined as

$$\operatorname{Grad}(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial \theta^i}\mathbf{G}^i, \qquad \operatorname{Grad}(\mathbf{f}) = \sum_{i=1}^{3} \frac{\partial \mathbf{f}}{\partial \theta^i} \otimes \mathbf{G}^i. \tag{8.23}$$

Expressing $\mathbf{f}$ in terms of its covariant or contravariant components and expanding the partial derivatives $\frac{\partial \mathbf{f}}{\partial \theta^i}$, an expression for $\operatorname{Grad}(\mathbf{f})$ containing the connection coefficients (8.20) can be established.

8.2.2 A Material Curve

Consider a curve $\mathscr{C}$ which is parameterized by $u \in [u_0, u_1]$ in $\mathscr{K}_0$. That is, on this curve

$$\theta^i = \theta^i(u). \tag{8.24}$$

The length of $\mathscr{C}$ is obtained by evaluating the following integral:

$$\begin{aligned} s(u_1) - s(u_0) &= \int_{u_0}^{u_1} \sqrt{\mathbf{G}\cdot\mathbf{G}}\,du \\ &= \int_{u_0}^{u_1} \sqrt{\sum_{i=1}^{3}\sum_{k=1}^{3} \frac{\partial \theta^i}{\partial u}\frac{\partial \theta^k}{\partial u}(\mathbf{G}_i\cdot\mathbf{G}_k)}\,du, \end{aligned} \tag{8.25}$$

where $\mathbf{G} = \sum_{i=1}^{3} \frac{\partial \theta^i}{\partial u}\mathbf{G}_i$ is a tangent vector to $\mathscr{C}$. As we shall shortly observe, we can use the coordinates θ^i to readily parameterize this material curve in the present configuration $\mathscr{K}$.

[1] For the Euler basis vectors and dual Euler basis vectors that are discussed in Section 5.3.1 of Chapter 5, because $\frac{\partial \mathbf{e}_1}{\partial \alpha^3} \neq \frac{\partial \mathbf{e}_3}{\partial \alpha^1}$, some of the associated connection coefficients may differ from the corresponding Christoffel symbols. This is a property that the 3-2-3 set of Euler angles has in common with all of the other sets of Euler angles.

8.2.3 Metric Tensors and Identities

The determinant G of the metric tensor $[G_{ij}] = [\mathbf{G}_i \cdot \mathbf{G}_j]$ can be calculated using the identity[2]

$$\sqrt{G} = [\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3] . \tag{8.26}$$

To see this result, you may wish to note that the identity tensor has the representations

$$\mathbf{I} = \sum_{i=1}^{3} \mathbf{G}_i \otimes \mathbf{G}^i = \sum_{k=1}^{3} \mathbf{G}^k \otimes \mathbf{G}_k = \sum_{i=1}^{3} \sum_{j=1}^{3} G^{ij} \mathbf{G}_i \otimes \mathbf{G}_j = \sum_{n=1}^{3} \sum_{m=1}^{3} G_{nm} \mathbf{G}^n \otimes \mathbf{G}^m , \tag{8.27}$$

and the determinant of any tensor $\mathbf{A}$ satisfies the identity

$$[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}] = \det(\mathbf{A}) [\mathbf{a}, \mathbf{b}, \mathbf{c}] , \tag{8.28}$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are any three vectors. The tensor product $\otimes$ that we use throughout this book is defined as follows:

$$(\mathbf{a} \otimes \mathbf{b}) \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) , \tag{8.29}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

8.2.4 Convected Coordinates

When the θ^i coordinate curves are identified with material curves in $\mathscr{B}$, then the θ^i coordinate system is said to be a convected (or material) coordinate system. We henceforth assume that this is the case. If a particle $\bar{X}$ has coordinates $\bar{\theta}^i$ in $\mathscr{K}_0$, then it can be identified with these same coordinates in $\mathscr{K}$:

$$\bar{\mathbf{x}} = \boldsymbol{\chi}(\bar{\mathbf{X}}, t) = \mathbf{r}^* \left(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, t\right) . \tag{8.30}$$

Needless to say the images of the θ^i coordinate curves in $\mathscr{K}$ can be very intricate and a simple example is shown in Figure 8.4.

Associated with the convected coordinates, we define the following sets of covariant $\mathbf{g}_i$ and contravariant $\mathbf{g}^k$ bases vectors:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}^*}{\partial \theta^i} , \tag{8.31}$$

and

$$\mathbf{g}^1 = \frac{1}{\sqrt{g}} \mathbf{g}_2 \times \mathbf{g}_3 , \quad \mathbf{g}^2 = \frac{1}{\sqrt{g}} \mathbf{g}_3 \times \mathbf{g}_1 , \quad \mathbf{g}^3 = \frac{1}{\sqrt{g}} \mathbf{g}_1 \times \mathbf{g}_2 , \tag{8.32}$$

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[2] Here, $[\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3]$ denotes the scalar triple product $\mathbf{G}_1 \cdot (\mathbf{G}_2 \times \mathbf{G}_3)$.

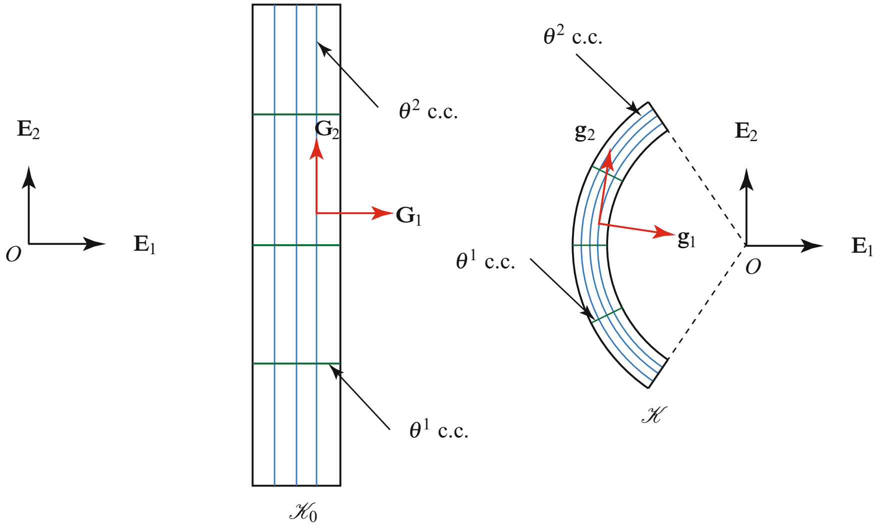

Fig. 8.4 Reference $\mathscr{K}_0$ and present $\mathscr{K}$ configurations of a parallelepiped that is being deformed into a state of pure flexure. The abbreviation c.c. stands for coordinate curve. This problem is discussed in a seminal work by Rivlin [302, Sections 14–16] and the text by Green and Zerna [140, Section 3.11].

where, paralleling the definition of $\sqrt{G}$,

$$\sqrt{g} = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3. \tag{8.33}$$

It is a good exercise to consider the parallels in the representations involving $\mathbf{g}^i$ and $\mathbf{g}_k$ for the arc-length of a material curve in $\mathscr{K}$ and the identity tensor $\mathbf{I}$ (cf. Eqns. (8.25) and (8.27)). In addition, one can define Christoffel symbols and connection coefficients for these basis vectors.

We now turn to some very useful representations for tensors that are widely employed in continuum mechanics. First, consider the deformation gradient tensor $\mathbf{F}$. Recall that $\mathbf{x} = \mathbf{r}^*\left(\theta^1, \theta^2, \theta^3, t\right)$. Now, with the help of the definition (8.23),

$$\begin{aligned}\mathbf{F} &= \text{Grad}\left(\mathbf{r}^*\right) \\ &= \sum_{i=1}^{3} \frac{\partial \mathbf{r}^*}{\partial \theta^i} \otimes \mathbf{G}^i \\ &= \sum_{i=1}^{3} \mathbf{g}_i \otimes \mathbf{G}^i. \end{aligned} \tag{8.34}$$

From this representation, it follows that

$$\mathbf{g}_i = \mathbf{F}\mathbf{G}_i, \qquad \mathbf{g}^i = \mathbf{F}^{-T}\mathbf{G}^i, \tag{8.35}$$

where $\mathbf{F}^{-1} = \mathbf{G}_1 \otimes \mathbf{g}^1 + \mathbf{G}_2 \otimes \mathbf{g}^2 + \mathbf{G}_3 \otimes \mathbf{g}^3$. A graphical summary of the transformations induced by $\mathbf{F}$ and $\mathbf{F}^{-T}$ is presented in Figure 8.5.

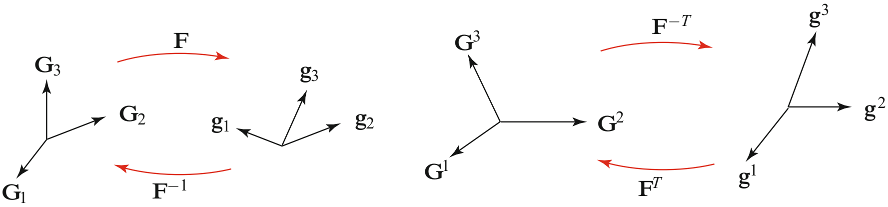

Fig. 8.5 Graphical representations of the transformations induced by $\mathbf{F}$ and $\mathbf{F}^{-T}$ and their inverses.

The representation for $\mathbf{F}$ results in the following representations for the right Cauchy-Green strain tensor $\mathbf{C}$, the left Cauchy-Green strain tensor $\mathbf{B}$, the Lagrangian strain tensor $\mathbf{E}$, the determinant of $\mathbf{F}$, and the adjugate of $\mathbf{F}$:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \sum_{i=1}^{3}\sum_{k=1}^{3} (\mathbf{g}_i \cdot \mathbf{g}_k)\, \mathbf{G}^i \otimes \mathbf{G}^k,$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \sum_{i=1}^{3}\sum_{k=1}^{3} \left(\mathbf{G}^i \cdot \mathbf{G}^k\right) \mathbf{g}_i \otimes \mathbf{g}_k,$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \sum_{i=1}^{3}\sum_{k=1}^{3} \frac{1}{2} (\mathbf{g}_i \cdot \mathbf{g}_k - \mathbf{G}_i \cdot \mathbf{G}_k)\, \mathbf{G}^i \otimes \mathbf{G}^k,$$

$$J = \det(\mathbf{F}) = \frac{\sqrt{g}}{\sqrt{G}},$$

$$\mathbf{F}^A = J\mathbf{F}^{-T} = \sum_{i=1}^{3} \frac{\sqrt{g}}{\sqrt{G}} \mathbf{g}^i \otimes \mathbf{G}_i. \tag{8.36}$$

In the last of these results, $\mathbf{F}^A$ denotes the adjugate of $\mathbf{F}$. It is easy to show from the representations listed above that $\det(\mathbf{C}) = \det(\mathbf{B}) = g/G$.

The adjugate of $\mathbf{F}$ plays a key role in Nanson's formula:

$$\mathsf{n}\,da = \mathbf{F}^A \mathsf{N}\,dA. \tag{8.37}$$

Here, n is the unit normal at a point on a material surface $F\left(\theta^1, \theta^2, \theta^3\right) = 0$ in $\mathscr{K}$, and N is the unit normal at the same point of the corresponding surface in $\mathscr{K}_0$. An example of such a material surface can be seen in Figure 8.6. For the material surface $\theta^3 = \xi_2$ in the reference configuration $\mathscr{K}_0$, the area element $\mathsf{N}dA$ has the representation

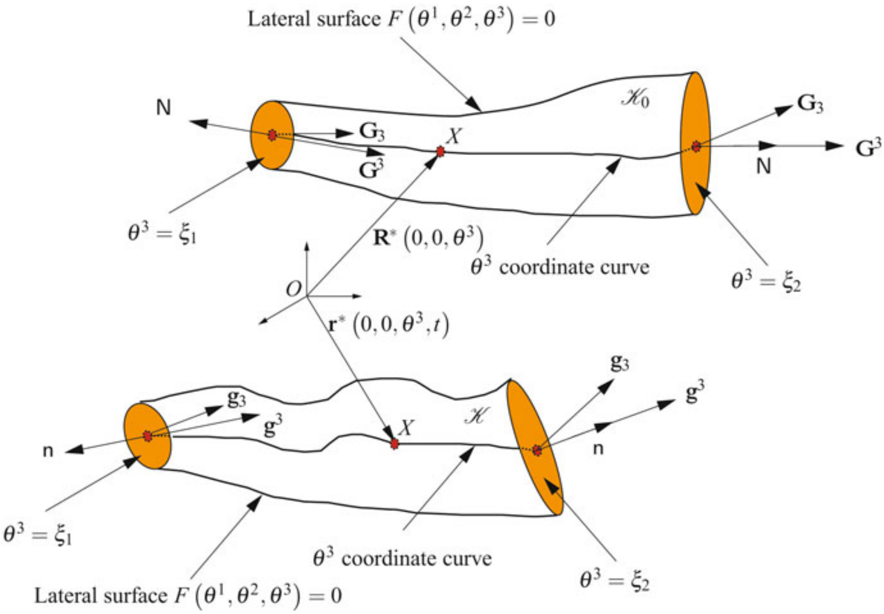

Fig. 8.6 Schematic of the reference $\mathscr{K}_0$ and present $\mathscr{K}$ configurations of a rod-like body whose reference configuration is parameterized using a curvilinear coordinate system. The ends of the body are described using θ^3 coordinate surfaces, the centerline is described as a θ^3 coordinate curve where $\theta^1 = \theta^2 = 0$, and the lateral surface of the body is described using the function $F(\theta^1, \theta^2, \theta^3) = 0$.

$$
\begin{aligned}
\mathsf{N} dA &= \frac{\partial \mathbf{R}^*}{\partial \theta^1}\left(\theta^1, \theta^2, \xi_2\right) d\theta^1 \times \frac{\partial \mathbf{R}^*}{\partial \theta^2}\left(\theta^1, \theta^2, \xi_2\right) d\theta^2 \\
&= \mathbf{G}_1\left(\theta^1, \theta^2, \xi_2\right) \times \mathbf{G}_2\left(\theta^1, \theta^2, \xi_2\right) \, d\theta^1 d\theta^2 \\
&= \mathbf{G}^3\left(\theta^1, \theta^2, \xi_2\right) \sqrt{G\left(\theta^1, \theta^2, \xi_2\right)} \, d\theta^1 d\theta^2. \qquad (8.38)
\end{aligned}
$$

Paralleling (8.38), the area element $\mathsf{n}da$ in the present configuration for the material surface $\theta^3 = \xi_2$ has the representation

$$
\mathsf{n} da = \mathbf{g}^3\left(\theta^1, \theta^2, \xi_2\right) \sqrt{g\left(\theta^1, \theta^2, \xi_2\right)} \, d\theta^1 d\theta^2. \qquad (8.39)
$$

It is a useful exercise to verify how these representations for $\mathsf{N}dA$ and $\mathsf{n}da$ are in compliance with Nanson's formula.

8.3 Stress Tensors and Divergences

We next recall four stress tensors: the Cauchy stress tensor $\mathbf{T}$, the first Piola-Kirchhoff stress tensor $\mathbf{P}$, the nominal stress tensor $\boldsymbol{\Sigma}$, and the second Piola-Kirchhoff stress tensor $\mathbf{S}$ (see, e.g., [55, 147]). These four tensors are related:

$$\begin{aligned}\mathbf{P} &= \mathbf{T}\mathbf{F}^{A},\\ \boldsymbol{\Sigma} &= \mathbf{F}^{-1}\mathbf{P}\mathbf{F}^{T} = \mathbf{P}^{T},\\ \mathbf{S} &= \mathbf{F}^{-1}\mathbf{P} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}.\end{aligned} \tag{8.40}$$

To see these definitions in a different light, let the Cauchy stress tensor have the representation

$$\mathbf{T} = \sum_{i=1}^{3}\sum_{k=1}^{3}\tau^{ik}\mathbf{g}_i\otimes\mathbf{g}_k. \tag{8.41}$$

Then,

$$J^{-1}\mathbf{P} = \sum_{i=1}^{3}\sum_{k=1}^{3}\tau^{ik}\mathbf{g}_i\otimes\mathbf{G}_k, \qquad J^{-1}\boldsymbol{\Sigma} = \sum_{i=1}^{3}\sum_{k=1}^{3}\tau^{ik}\mathbf{G}_k\otimes\mathbf{g}_i,$$
$$J^{-1}\mathbf{S} = \sum_{i=1}^{3}\sum_{k=1}^{3}\tau^{ik}\mathbf{G}_i\otimes\mathbf{G}_k. \tag{8.42}$$

Observe that the tensors have the same components and "legs" in different configurations: $\mathbf{S}$ has both "legs" in $\mathscr{K}_0$, $\mathbf{T}$ has both "legs" in $\mathscr{K}$, and $\mathbf{P}$ and $\boldsymbol{\Sigma}$ have one "leg" in $\mathscr{K}_0$ and the other in $\mathscr{K}$.

The representations of the stress tensors using the covariant basis vectors also illuminate the role played by a traction vector $\mathbf{T}^i$ that was first introduced by Green and Zerna [140][3]:

$$\begin{aligned}\mathbf{T}^i &= \sqrt{g}\mathbf{T}\mathbf{g}^i = \sum_{k=1}^{3}\sqrt{g}\tau^{ki}\mathbf{g}_k\\ &= \sqrt{G}\mathbf{P}\mathbf{G}^i\\ &= \sqrt{G}\mathbf{F}\mathbf{S}\mathbf{G}^i.\end{aligned} \tag{8.43}$$

To further elaborate on $\mathbf{T}^i$ and the role it plays in formulating the governing equations for rods and strings, consider the material surface $\mathscr{A}$ defined by $\theta^3 = \xi_2$ that is shown in Figure 8.6. Suppose that a traction vector $\mathbf{t}$ acts on this surface. From Cauchy's lemma [55, 147], we know that $\mathbf{t} = \mathbf{T}\mathsf{n}$, where n is the outward normal. Using Eqn. (8.39), the resultant force acting on this surface can be computed[4]:

[3] Green and Zerna use the notation $\mathbf{T}_i$ for these vectors. Our notation follows later papers by Green and Naghdi [133, 135, 137, 138]. As will shortly become apparent, the vector $\mathbf{T}^i$ has similarities to the vector $\mathbf{t} = \mathbf{T}\mathsf{n}$ acting on a surface whose unit outward normal is n.

[4] Observe that $da \neq d\theta^1 d\theta^2$.

$$\begin{aligned}\int_{\mathscr{A}} \mathbf{t} da &= \int_{\mathscr{A}} \mathbf{T}\mathsf{n} da \\ &= \int_{\mathscr{A}} \mathbf{T}\mathbf{g}^3\left(\theta^1, \theta^2, \xi_2\right) \sqrt{g\left(\theta^1, \theta^2, \xi_2\right)}\, d\theta^1 d\theta^2 \\ &= \int_{\mathscr{A}} \mathbf{T}^3\left(\theta^1, \theta^2, \xi_2\right)\, d\theta^1 d\theta^2. \end{aligned} \tag{8.44}$$

Concomitantly,

$$\mathbf{T}^1 = \mathbf{T}\left(\mathbf{g}_2 \times \mathbf{g}_3\right), \qquad \mathbf{T}^2 = \mathbf{T}\left(\mathbf{g}_3 \times \mathbf{g}_1\right). \tag{8.45}$$

The elegance of the representation (8.44) is remarkable and it is often used to establish a representation for the contact force $\mathbf{n}$ in rod and string theories. We leave it as an exercise to show that the corresponding representation for the referential traction vector $\mathbf{p} = \mathbf{P}\mathsf{N}$ is $\int_{\mathscr{A}} \mathbf{p} dA = \int_{\mathscr{A}} \mathbf{T}^3\left(\theta^1, \theta^2, \xi_2\right) d\theta^1 d\theta^2$.

8.3.1 Divergences

In the balance laws for a continuum, one finds a pair of distinct divergences of a tensor:

$$\mathrm{Div}\left(\mathbf{P}\right) = \sum_{k=1}^{3} \frac{\partial \mathbf{P}}{\partial \theta^k}\mathbf{G}^k, \qquad \mathrm{div}\left(\mathbf{T}\right) = \sum_{k=1}^{3} \frac{\partial \mathbf{T}}{\partial \theta^k}\mathbf{g}^k. \tag{8.46}$$

To motivate these representations, we first consider a pair of gradient operators:

$$\mathrm{grad}\left(a\right) = \nabla\left(a\right) = \sum_{r=1}^{3} \mathbf{g}^r \frac{\partial a}{\partial \theta^r}, \qquad \mathrm{Grad}\left(a\right) = \nabla_0\left(a\right) = \sum_{r=1}^{3} \mathbf{G}^r \frac{\partial a}{\partial \theta^r}, \tag{8.47}$$

where a is an arbitrary differentiable scalar-valued function. For any vector $\mathbf{c}$, we use the aforementioned gradient operators to define the divergences of a vector:

$$\mathrm{Div}\left(\mathbf{c}\right) = \nabla_0 \cdot \mathbf{c} = \sum_{k=1}^{3} \mathbf{G}^k \cdot \frac{\partial \mathbf{c}}{\partial \theta^k}, \qquad \mathrm{div}\left(\mathbf{c}\right) = \nabla \cdot \mathbf{c} = \sum_{k=1}^{3} \mathbf{g}^k \cdot \frac{\partial \mathbf{c}}{\partial \theta^k}. \tag{8.48}$$

Following the treatment in Gurtin [147, Section 4], we next employ the definition (8.48) to define the divergences of a tensor $\mathbf{H}$:

$$\mathbf{a} \cdot \mathrm{div}\left(\mathbf{H}\right) = \mathrm{div}\left(\mathbf{H}^T\mathbf{a}\right), \qquad \mathbf{a} \cdot \mathrm{Div}\left(\mathbf{H}\right) = \mathrm{Div}\left(\mathbf{H}^T\mathbf{a}\right), \tag{8.49}$$

where $\mathbf{a}$ is any constant vector. This final step sets the stage to use Eqn. (8.49) to establish Eqn. (8.46). For example,

$$\begin{aligned}\mathbf{a}\cdot\operatorname{div}(\mathbf{H}) &= \operatorname{div}\left(\mathbf{H}^T\mathbf{a}\right)\\ &= \sum_{r=1}^{3}\mathbf{g}^r\cdot\frac{\partial}{\partial\theta^r}\left(\mathbf{H}^T\mathbf{a}\right)\\ &= \sum_{r=1}^{3}\mathbf{g}^r\cdot\left(\frac{\partial\mathbf{H}^T}{\partial\theta^r}\mathbf{a}\right)\\ &= \sum_{r=1}^{3}\left(\frac{\partial\mathbf{H}}{\partial\theta^r}\mathbf{g}^r\right)\cdot\mathbf{a}. \end{aligned} \tag{8.50}$$

We used the property of the transpose of a second-order tensor $(\mathbf{Ab})\cdot\mathbf{a}=\mathbf{b}\cdot\left(\mathbf{A}^T\mathbf{a}\right)$ to manipulate the previous expression. As a consequence of the earlier manipulations, we can conclude that

$$\left(\operatorname{div}(\mathbf{H})-\sum_{r=1}^{3}\left(\frac{\partial\mathbf{H}}{\partial\theta^r}\mathbf{g}^r\right)\right)\cdot\mathbf{a}=0. \tag{8.51}$$

As this result is true for all $\mathbf{a}$ and the term inside the parentheses is independent of $\mathbf{a}$, we find that

$$\operatorname{div}(\mathbf{H})=\sum_{r=1}^{3}\frac{\partial\mathbf{H}}{\partial\theta^r}\mathbf{g}^r. \tag{8.52}$$

As expected, this result agrees with Eqn. $(8.46)_2$. A parallel derivation applies for Eqn. $(8.46)_1$.

8.3.2 The Traction Vector and a Divergence

Using $\mathbf{T}^r$, one finds very useful representations for the divergences of $\mathbf{T}$ and $\mathbf{P}$. To see these results, we need to perform some lengthy but straightforward manipulations:

$$\begin{aligned}\operatorname{div}(\mathbf{T}) &= \sum_{r=1}^{3}\sum_{i=1}^{3}\frac{\partial}{\partial\theta^r}\left(\frac{1}{\sqrt{g}}\mathbf{T}^i\otimes\mathbf{g}_i\right)\mathbf{g}^r\\ &= \sum_{r=1}^{3}\sum_{i=1}^{3}\frac{(\mathbf{g}_i\cdot\mathbf{g}^r)}{\sqrt{g}}\frac{\partial\mathbf{T}^i}{\partial\theta^r}+\sum_{i=1}^{3}\mathbf{T}^i\underbrace{\left(\sum_{r=1}^{3}\left(\frac{\partial}{\partial\theta^r}\left(\frac{\mathbf{g}_i}{\sqrt{g}}\right)\cdot\mathbf{g}^r\right)\right)}_{=0\text{ using Eqns. (8.56) and (8.59)}}\\ &= \sum_{r=1}^{3}\sum_{i=1}^{3}\frac{\delta_i^r}{\sqrt{g}}\frac{\partial\mathbf{T}^i}{\partial\theta^r}+\sum_{i=1}^{3}\mathbf{T}^i(0)\\ &= \sum_{r=1}^{3}\frac{1}{\sqrt{g}}\frac{\partial\mathbf{T}^r}{\partial\theta^r}. \end{aligned} \tag{8.53}$$

Similarly,

$$\mathrm{Div}\left(\mathbf{P}\right) = \sum_{r=1}^{3} \frac{1}{\sqrt{G}} \frac{\partial \mathbf{T}^r}{\partial \theta^r}. \tag{8.54}$$

In summary,

$$\sum_{r=1}^{3} \frac{\partial \mathbf{T}^r}{\partial \theta^r} = \sqrt{G}\mathrm{Div}\left(\mathbf{P}\right) = \sqrt{g}\mathrm{div}\left(\mathbf{T}\right). \tag{8.55}$$

To establish the representations (8.55), we used one of the following identities in Eqn. (8.53):

$$\sum_{r=1}^{3}\sum_{i=1}^{3}\left(\mathbf{T}^i \otimes \frac{\partial}{\partial \theta^r}\left(\frac{\mathbf{g}_i}{\sqrt{g}}\right)\right)\mathbf{g}^r = \mathbf{0}, \qquad \sum_{r=1}^{3}\sum_{i=1}^{3}\left(\mathbf{T}^i \otimes \frac{\partial}{\partial \theta^r}\left(\frac{\mathbf{G}_i}{\sqrt{G}}\right)\right)\mathbf{G}^r = \mathbf{0}. \tag{8.56}$$

It suffices to consider Eqn. $(8.56)_1$ in order to show how its referential counterpart $(8.56)_2$ can be established. The proof starts by examining the derivative of $\sqrt{g}$ and using the identities (8.32):

$$\begin{aligned} \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial \theta^r} &= \frac{1}{\sqrt{g}}\frac{\partial \mathbf{g}_1}{\partial \theta^r}\cdot\left(\mathbf{g}_2 \times \mathbf{g}_3\right) + \frac{1}{\sqrt{g}}\frac{\partial \mathbf{g}_2}{\partial \theta^r}\cdot\left(\mathbf{g}_3 \times \mathbf{g}_1\right) + \frac{1}{\sqrt{g}}\frac{\partial \mathbf{g}_3}{\partial \theta^r}\cdot\left(\mathbf{g}_1 \times \mathbf{g}_2\right) \\ &= \sum_{k=1}^{3} \frac{\partial \mathbf{g}_k}{\partial \theta^r}\cdot \mathbf{g}^k. \end{aligned} \tag{8.57}$$

Because $\mathbf{g}_k = \frac{\partial \mathbf{r}^*}{\partial \theta^k}$, the following identity for the mixed partial derivatives holds:

$$\frac{\partial \mathbf{g}_i}{\partial \theta^r} = \frac{\partial \mathbf{g}_r}{\partial \theta^i}. \tag{8.58}$$

Returning to the underbraced term in Eqn. (8.53), we can isolate the ith term and consider its expansion:

$$\begin{aligned} \sum_{r=1}^{3} \frac{\partial}{\partial \theta^r}\left(\frac{\mathbf{g}_i}{\sqrt{g}}\right)\cdot \mathbf{g}^r &= \sum_{r=1}^{3} \frac{\partial}{\partial \theta^r}\left(\frac{1}{\sqrt{g}}\right)\mathbf{g}_i\cdot\mathbf{g}^r + \frac{1}{\sqrt{g}}\sum_{r=1}^{3}\frac{\partial \mathbf{g}_i}{\partial \theta^r}\cdot\mathbf{g}^r \\ &= \frac{\partial}{\partial \theta^i}\left(\frac{1}{\sqrt{g}}\right) + \frac{1}{\sqrt{g}}\sum_{r=1}^{3}\underbrace{\frac{\partial \mathbf{g}_r}{\partial \theta^i}}_{\text{using Eqn. (8.58)}}\cdot\mathbf{g}^r \\ &= -\frac{1}{g}\frac{\partial \sqrt{g}}{\partial \theta^i} + \frac{1}{\sqrt{g}}\sum_{r=1}^{3}\frac{\partial \mathbf{g}_r}{\partial \theta^i}\cdot\mathbf{g}^r \\ &= -\underbrace{\frac{1}{\sqrt{g}}\sum_{k=1}^{3}\frac{\partial \mathbf{g}_k}{\partial \theta^i}\cdot\mathbf{g}^k}_{\text{using Eqn. (8.57)}} + \frac{1}{\sqrt{g}}\sum_{r=1}^{3}\frac{\partial \mathbf{g}_r}{\partial \theta^i}\cdot\mathbf{g}^r \\ &= 0. \end{aligned} \tag{8.59}$$

The identity $(8.56)_1$ now follows in a straightforward manner.

8.4 Balance Laws

We recall the local forms of the balance laws for mass, linear momentum, and angular momentum for a three-dimensional continuum:

$$\begin{aligned}\dot{\rho}^* + \rho^* \text{div}\,(\mathbf{v}^*) &= 0, \\ \text{div}\,(\mathbf{T}) + \rho^* \mathbf{b} &= \rho^* \dot{\mathbf{v}}^*, \\ \mathbf{T} &= \mathbf{T}^T. \end{aligned} \tag{8.60}$$

In these equations, the superposed dot denotes the material time derivative, $\mathbf{b}$ is the body force per unit mass, ρ^* is the mass density per unit volume of $\mathscr{B}$ in $\mathscr{K}$, and $\mathbf{v}^* = \dot{\mathbf{r}}^*$.

Mass conservation $(8.60)_1$ integrates to

$$J\rho^* = \rho_0^*, \tag{8.61}$$

where ρ_0^* is the mass density per unit volume of $\mathscr{B}$ in $\mathscr{K}_0$. As $J = \frac{\sqrt{g}}{\sqrt{G}}$, we can write the linear momentum balance $(8.60)_2$ as

$$\frac{1}{\sqrt{G}}\left(\sqrt{g}\,\text{div}\,(\mathbf{T})\right) + \rho_0^* \mathbf{b} = \rho_0^* \dot{\mathbf{v}}^*. \tag{8.62}$$

We next invoke Eqn. (8.55) to conclude that

$$\sum_{r=1}^{3} \frac{1}{\sqrt{G}} \frac{\partial \mathbf{T}^r}{\partial \theta^r} + \rho_0^* \mathbf{b} = \rho_0^* \dot{\mathbf{v}}^*. \tag{8.63}$$

It is easy to write this result in terms of the divergence of $\mathbf{P}$. We also remark that, because the representation (8.63) is convenient to integrate over a three-dimensional continuum, Eqn. (8.63) is used in Green and Naghdi's papers [133, 137] to establish the linear momentum and director momentum balance laws.

8.5 Invariance Requirements under Superposed Rigid Body Motions

Consider two motions of $\mathscr{B}$. The two motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^{\perp}$ differ by a rigid body motion if, and only if,

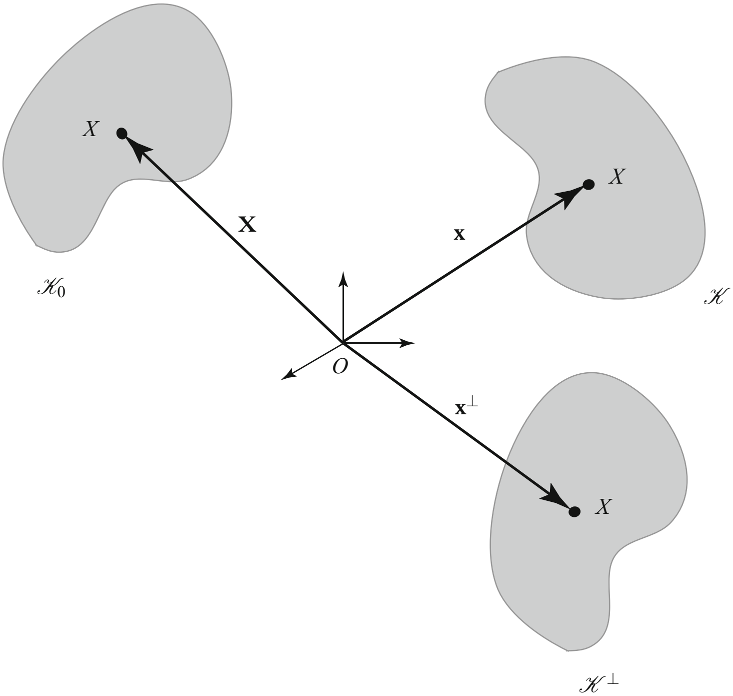

Fig. 8.7 Two configurations, $\mathscr{K}$ and $\mathscr{K}^{\perp}$, of a body $\mathscr{B}$ which differ by a superposed rigid body motion. The reference configuration $\mathscr{K}_0$ is also shown.

$$\begin{aligned}\mathbf{x}^{\perp} &= \boldsymbol{\chi}^{\perp}\left(\mathbf{X}, t^{\perp} = t + a\right) \\ &= \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X},t) + \mathbf{q}(t) \\ &= \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{q}(t),\end{aligned} \tag{8.64}$$

where $\mathbf{Q}$, which is a rotation tensor, and $\mathbf{q}$ are functions of t only and a is a constant. Notice that we are considering two distinct present configurations and a single reference configuration here (see Figure 8.7).[5] For the two motions, it can be shown from Eqn. (8.64) that

$$\mathbf{F}^{\perp} = \mathbf{QF}, \qquad \mathbf{C}^{\perp} = \mathbf{C}, \qquad \mathbf{D}^{\perp} = \mathbf{QDQ}^{T}. \tag{8.65}$$

Here, the stretching tensor $\mathbf{D}$ is the symmetric part of the tensor $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$:

$$\mathbf{D} = \frac{1}{2}\left(\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\right). \tag{8.66}$$

[5] This is in contrast to the framework required to establish restrictions on constitutive relations using the principle of material frame indifference [242].

You might have noticed that

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \sum_{k=1}^{3}\sum_{i=1}^{3} \dot{\mathbf{g}}_i \otimes \mathbf{G}^i \left(\mathbf{G}_k \otimes \mathbf{g}^k\right)$$
$$= \sum_{i=1}^{3} \dot{\mathbf{g}}_i \otimes \mathbf{g}^i. \tag{8.67}$$

Thus, the tensor $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ transforms $\mathbf{g}_k$ to $\dot{\mathbf{g}}_k$.

Supplementing the balance laws and response relations, it is necessary to impose invariance requirements under superposed rigid body motions. The invariance requirements we impose are standard:

$$(\psi^*)^{\perp} = \psi^*, \qquad (\phi^*)^{\perp} = \phi^*, \tag{8.68}$$

where ψ^* is the strain energy function per unit volume in $\mathscr{K}_0$ and ϕ^* is an internal constraint on the motion of $\mathscr{B}$. Examples of such constraints include incompressibility ($\phi^* = \det(\mathbf{F}) - 1$).

8.6 Constitutive Relations for Hyperelastic Bodies

For a hyperelastic (or Green) elastic body, a strain energy function ψ^* exists which is a function of $\mathbf{F}$:

$$\psi^* = \hat{\psi}^*(\mathbf{F}, \mathbf{X}, t). \tag{8.69}$$

However, because we are imposing the invariance requirement $(8.68)_1$, ψ^* cannot depend on t and can only depend on $\mathbf{F}$ through its invariant part. Consequently,

$$\psi^* = \psi^*(\mathbf{C}, \mathbf{X}). \tag{8.70}$$

We also assume that the body is subject to an internal constraint which is properly invariant:

$$\phi^* = \phi^*(\mathbf{C}, \mathbf{X}). \tag{8.71}$$

That is, $(\phi^*)^{\perp} = \phi^*$. We now seek constitutive relations for the stress tensor $\mathbf{T}$ of the constrained hyperelastic continuum. Our treatment follows Ericksen and Rivlin [99] and benefits from the insights of later works by Antman and Marlow [14], Carlson et al. [46, 47], Casey et al. [49, 50], Green et al. [136], and Truesdell and Noll [351, Section 30].

In what follows, we make frequent use of the fact that, for any pair of tensors,

$$\mathbf{A} = \sum_{i=1}^{3}\sum_{k=1}^{3} A_{ik}\mathbf{E}_i \otimes \mathbf{E}_k, \qquad \mathbf{B} = \sum_{i=1}^{3}\sum_{k=1}^{3} B_{ik}\mathbf{E}_i \otimes \mathbf{E}_k, \tag{8.72}$$

the trace operator,

$$\mathbf{A}\cdot\mathbf{B} = \operatorname{tr}\left(\mathbf{A}\mathbf{B}^T\right) = \sum_{i=1}^{3}\sum_{k=1}^{3}\left(A_{ik}B_{ik}\right), \tag{8.73}$$

provides an inner-product. We will also invoke the following result in the sequel: Assuming that $\mathbf{A}$ and $\mathbf{B}$ are independent of $\mathbf{Z}$, then the solution to the equation

$$\mathbf{A}\cdot\mathbf{Z} = 0 \text{ for all } \mathbf{Z} \text{ which satisfy } \mathbf{B}\cdot\mathbf{Z} = 0, \tag{8.74}$$

is $\mathbf{A} = \lambda\mathbf{B}$ where λ is a scalar. One proof of this result can be found in Green et al. [136, Page 902]. The result $\mathbf{A} = \lambda\mathbf{B}$, which also appears in Exercise 8.4, has a (well-known) geometric interpretation: $\mathbf{A}$ is parallel to $\mathbf{B}$.

To prescribe constitutive relations for $\mathbf{T}$, we require that the stress power is equal to the rate of change of strain energy for all motions which satisfy the constraint[6]:

$$\operatorname{tr}\left(\mathbf{T}\mathbf{L}^T\right) = \rho^*\dot{\psi}^* \text{ for all } \mathbf{C} \text{ which satisfy } \phi^*\left(\mathbf{C},\mathbf{X}\right) = 0. \tag{8.75}$$

There are several representations for $\dot{\psi}^*$:

$$\dot{\psi}^* = \operatorname{tr}\left(\frac{\partial\psi^*}{\partial\mathbf{C}}\dot{\mathbf{C}}\right) = \operatorname{tr}\left(\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial\psi^*}{\partial C_{ik}}\mathbf{E}_i\otimes\mathbf{E}_k\right)\dot{\mathbf{C}}\right) = \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial\psi^*}{\partial C_{ik}}\dot{C}_{ik}, \tag{8.76}$$

where

$$\mathbf{C} = \sum_{i=1}^{3}\sum_{k=1}^{3}C_{ik}\mathbf{E}_i\otimes\mathbf{E}_k, \qquad \frac{\partial\psi^*}{\partial\mathbf{C}} = \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial\psi^*}{\partial C_{ik}}\mathbf{E}_i\otimes\mathbf{E}_k. \tag{8.77}$$

Further,

$$\dot{\mathbf{C}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}. \tag{8.78}$$

Invoking the moment of momentum balance law, we note that $\mathbf{T} = \mathbf{T}^T$. However, for all skew-symmetric tensors $\mathbf{B}$ and symmetric tensors $\mathbf{A}$,

$$\operatorname{tr}\left(\mathbf{A}\mathbf{B}\right) = 0. \tag{8.79}$$

Thus, the symmetry of $\mathbf{T}$ implies that the expression $\operatorname{tr}\left(\mathbf{T}\mathbf{L}^T\right)$ can be simplified by removing the skew-symmetric part of $\mathbf{L}$:

$$\begin{aligned}\operatorname{tr}\left(\mathbf{T}\mathbf{L}^T\right) &= \operatorname{tr}\left(\mathbf{T}\left(\mathbf{D} = \frac{1}{2}\left(\mathbf{L}+\mathbf{L}^T\right)\right)\right) + \frac{1}{2}\operatorname{tr}\left(\mathbf{T}\left(\mathbf{L}^T - \mathbf{L}\right)\right)\\ &= \operatorname{tr}\left(\mathbf{T}\mathbf{D}\right).\end{aligned} \tag{8.80}$$

We are now in a position to rephrase Eqn. (8.75) as

$$\operatorname{tr}\left(\mathbf{T}\mathbf{D}\right) = \operatorname{tr}\left(2\rho^*\frac{\partial\psi^*}{\partial\mathbf{C}}\mathbf{F}^T\mathbf{D}\mathbf{F}\right) \text{ for all } \mathbf{C} \text{ which satisfy } \phi^*\left(\mathbf{C},\mathbf{X}\right) = 0. \tag{8.81}$$

[6] Observe that $\frac{1}{\sqrt{G}}\mathbf{T}^i\cdot\dot{\mathbf{g}}_i = \operatorname{tr}\left(\mathbf{P}\dot{\mathbf{F}}^T\right) = J\operatorname{tr}\left(\mathbf{T}\mathbf{F}^{-T}\dot{\mathbf{F}}^T\right) = J\operatorname{tr}\left(\mathbf{T}\mathbf{L}^T\right)$. The former representations for stress power are often more illuminating than $\operatorname{tr}\left(\mathbf{T}\mathbf{L}^T\right)$.

If $\mathbf{C}$ satisfies the constraint $\phi^* = 0$, then its derivative satisfies

$$\mathrm{tr}\left(\frac{\partial \phi^*}{\partial \mathbf{C}}\dot{\mathbf{C}}\right) = 0. \tag{8.82}$$

Or, equivalently, with the help of Eqn. (8.78),

$$\mathrm{tr}\left(2\frac{\partial \phi^*}{\partial \mathbf{C}}\mathbf{F}^T\mathbf{D}\mathbf{F}\right) = 0. \tag{8.83}$$

With this in mind, we rephrase Eqn. (8.81) as

$$\mathrm{tr}\left(\left(\mathbf{T} - 2\rho^*\mathbf{F}\frac{\partial \psi^*}{\partial \mathbf{C}}\mathbf{F}^T\right)\mathbf{D}\right) = 0 \text{ for all } \mathbf{D} \text{ which satisfy } \mathrm{tr}\left(2\frac{\partial \phi^*}{\partial \mathbf{C}}\mathbf{F}^T\mathbf{D}\mathbf{F}\right) = 0. \tag{8.84}$$

If we assume that $\mathbf{T}$ does not depend on $\mathbf{D}$ and that Eqn. (8.84) is true for all $\mathbf{D}$, then, appealing to the solution to Eqn. (8.74), we find the classic response function for the Cauchy stress tensor:

$$\mathbf{T} = 2\rho^*\mathbf{F}\frac{\partial \psi^*}{\partial \mathbf{C}}\mathbf{F}^T + 2\lambda\mathbf{F}\frac{\partial \phi^*}{\partial \mathbf{C}}\mathbf{F}^T. \tag{8.85}$$

The scalar-valued function $\lambda = \lambda(\mathbf{X}, t)$ and is an unknown that must be determined as part of the solution to the boundary-value problem associated with the continuum. This function is sometimes identified as a Lagrange multiplier (cf. Ericksen and Rivlin [99, Section 4]). Following Casey and Carroll [49], we do not assume that the function λ that enforces the constraint $\phi^* = 0$ is invariant under superposed rigid body motions. That is, $\lambda^\perp$ and λ are not necessarily identical. It is also important to note that the constitutive relations (8.85) guarantee that $\mathbf{T}$ is symmetric and automatically satisfies the balance of angular momentum: $\mathbf{T} = \mathbf{T}^T$.

One useful interpretation of Eqn. (8.85) is that, for a constrained material, the stress response can be considered as an additive decomposition of a part associated with the deformation of the material and a part needed to ensure that the constraint is satisfied. The former part, $2\rho^*\mathbf{F}\frac{\partial \psi^*}{\partial \mathbf{C}}\mathbf{F}^T$, is known as the active stress and the latter part, $2\lambda\mathbf{F}\frac{\partial \phi^*}{\partial \mathbf{C}}\mathbf{F}^T$, is known as the reactive stress. Indeed, the alternative derivation of Eqn. (8.85) in Truesdell and Noll [351, Section 30] postulates the decomposition of the stress into active and reactive parts, assumes that the reactive stress is workless in any motion of the continuum that satisfies the constraints, and assumes that the active part identically satisfies the local form of the energy balance.

With the help of the identities

$$\frac{\partial \psi^*}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \psi^*}{\partial \mathbf{C}}, \qquad \frac{\partial \phi^*}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \phi^*}{\partial \mathbf{C}}, \tag{8.86}$$

the constitutive relations for $\mathbf{T}$ can also be expressed in a manner that is convenient for representations of the other stress tensors that appear in these pages:

$$\mathbf{T} = \rho^* \frac{\partial \psi^*}{\partial \mathbf{F}} \mathbf{F}^T + \lambda \frac{\partial \phi^*}{\partial \mathbf{F}} \mathbf{F}^T. \tag{8.87}$$

We leave it as an exercise to write out the corresponding response functions for $\mathbf{T}^i$, $\mathbf{P}$, $\mathbf{\Sigma}$, and $\mathbf{S}$. The representation for $\mathbf{S}$ can be used to transparently demonstrate that the reactive stress is normal to the constraint manifold $\phi^*(\mathbf{C}) = 0$. This five-dimensional manifold corresponds to the set of all symmetric tensors $\mathbf{C}$ which satisfy the constraint $\phi^*(\mathbf{C}) = 0$ and is a subset of the space of all symmetric second-order tensors. For additional details and perspectives on the constraint manifold, we refer the reader to [46, 50]. The geometric perspective in these papers also enables one to see that the prescription for the reactive stress in Eqn. (8.85) is equivalent to the Lagrange prescription for constraint forces and moments in particle and rigid body dynamics.[7]

8.6.1 A Mooney-Rivlin Material

One of the most prominent examples of constitutive relations for an incompressible, isotropic elastic body is due to Mooney and Rivlin:

$$\mathbf{T} = -p\mathbf{I} + \beta_1 \mathbf{B} + \beta_{-1} \mathbf{B}^{-1}. \tag{8.88}$$

Here, p is the pressure associated with the incompressibility constraint,

$$\phi^*(\mathbf{C}) = \det(\mathbf{C}) - 1, \tag{8.89}$$

(i.e., $p = -\lambda$ in Eqn. (8.85)). Additionally, β_1 and β_{-1} are constants in the simplest Mooney-Rivlin material and, when $\beta_{-1} = 0$, the material is known, following Rivlin, as a neo-Hookean material. The strain energy function for the (simplest) Mooney-Rivlin material is

$$\rho_0^* \psi^* = \frac{1}{2}\beta_1 (I_{\mathbf{C}} - 3) - \frac{1}{2}\beta_{-1} (II_{\mathbf{C}} - 3), \tag{8.90}$$

where the two nontrivial invariants of $\mathbf{C}$ are

$$I_{\mathbf{C}} = \mathrm{tr}(\mathbf{C}), \qquad 2II_{\mathbf{C}} = \mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}\left(\mathbf{C}^2\right). \tag{8.91}$$

The third invariant $III_{\mathbf{C}} = \det(\mathbf{C}) = 1$ for an incompressible continuum. The derivation of the relations (8.88) from the strain energy function given by Eqn. (8.90) is outlined in Exercise 8.4 at the end of this chapter.

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[7] In the case of a single particle, the Lagrange prescription implies that the constraint force is normal to surface or curve that the particle is constrained to move on. For this reason, this prescription is sometimes known as the normality prescription. We refer the reader to [271, 283, 284] for additional background on constraint forces and constraint moments in classical mechanics.

8.6.2 Additional Remarks

It is important to note that the constitutive relations (8.85), the balance laws, and the constraint may be used to provide a determinate system of equations to determine the motion $\mathbf{r}^*$ of the body and λ:

$$\sum_{r=1}^{3}\left(\frac{\partial}{\partial\theta^r}\left(2\mathbf{F}\left(\rho_0^*\sqrt{G}\frac{\partial\psi^*}{\partial\mathbf{C}}\mathbf{G}^r+\sqrt{g}\lambda\frac{\partial\phi^*}{\partial\mathbf{C}}\mathbf{G}^r\right)\right)\right)+\sqrt{G}\rho_0^*\mathbf{b}=\sqrt{G}\rho_0^*\dot{\mathbf{v}}^*,$$

$$\operatorname{tr}\left(2\frac{\partial\phi^*}{\partial\mathbf{C}}\mathbf{F}^T\mathbf{D}\mathbf{F}\right)=0. \tag{8.92}$$

The corresponding set of equations for the unconstrained case were shown in Eqn. (8.63). Of course, both sets of equations need to be supplemented with boundary conditions and initial conditions.

8.7 Configurational, Material, or Eshelbian Forces

Following the seminal work of John D. Eshelby (1916–1981), it has become standard to consider the behavior of an energy-momentum tensor for hyperelastic bodies. For elastostatic problems, several alternative definitions of this tensor appear in the literature:

$$\boldsymbol{\sigma}_{\mathrm{C}}=\rho_0^*\psi^*\mathbf{I}-\mathbf{S}\mathbf{C},\ \text{proposed by Chadwick [54]},$$
$$\boldsymbol{\sigma}_{\mathrm{E}}=\rho_0^*\psi^*\mathbf{I}+\left(\mathbf{I}-\mathbf{F}^T\right)\boldsymbol{\Sigma}^T,\ \text{proposed by Eshelby (cf. [103, Eqn. (13)])},$$
$$\boldsymbol{\sigma}_{\mathrm{G}}=\rho_0^*\psi^*\mathbf{I}-\mathbf{F}^T\boldsymbol{\Sigma}^T,\ \text{proposed by Gurtin (cf. [149, Eqn. (5.14)])},$$
$$\boldsymbol{\sigma}_{\mathrm{M}}=\rho_0^*\psi^*\mathbf{I}-\mathbf{P}\mathbf{F},\ \text{proposed by Maugin and co-workers (cf. [231, Eqn. (3.7)])}.$$

The extension of these definitions to the dynamic case is obtained by subtracting the kinetic energy density from the strain energy function (cf., e.g., [103, Eqn. (53)]). For present purposes, we use a definition of the (dynamic) energy-momentum tensor that can be found in a variety of sources including [82, Eqn. (2.18)] and [149, Eqn. (7.8)]:

$$\boldsymbol{\sigma}=\left(\rho_0^*\psi^*-\frac{1}{2}\rho_0^*\mathbf{v}^*\cdot\mathbf{v}^*\right)\mathbf{I}-\mathbf{F}^T\mathbf{P}. \tag{8.93}$$

Concomitant with $\boldsymbol{\sigma}$, we follow Eshelby [103, Eqn. (55)] and define the material momentum P^*:

$$\mathsf{P}^*=-\rho_0^*\mathbf{F}^T\mathbf{v}^*. \tag{8.94}$$

Among other uses, the momentum P^*, which is alternatively referred to as the pseudomomentum or configurational momentum, and tensor $\boldsymbol{\sigma}$ can be used to establish a balance law for material momentum. This law provides a conservation law in certain instances and, as demonstrated by Eshelby [103], the energy release rate for a crack in others. The force $\rho_0^* \mathbf{b}_M$ in this balance law is known as an assigned material (or configurational) force. As we shall see below, if the material is homogeneous and there are no body forces, $\mathbf{b} = \mathbf{0}$, then $\rho_0^* \mathbf{b}_M$ is identically zero. The vanishing of $\mathbf{b}_M$ has been championed by Braun [36] as a test for the accuracy of numerical methods used in computational mechanics and we refer the reader to [240] for additional references and interesting examples.

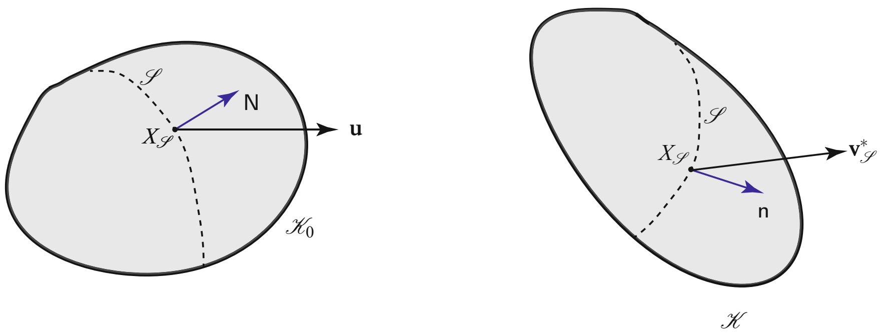

Fig. 8.8 The reference $\mathscr{K}_0$ and present $\mathscr{K}$ configurations of a body $\mathscr{B}$ showing a propagating surface of discontinuity $\mathscr{S}$ and the associated velocity vectors $\mathbf{u}$ and $\mathbf{v}^*_{\mathscr{S}}$ associated with a material point $X_{\mathscr{S}} \in \mathscr{S}$. The unit normal vectors to the shock surface in the reference and present configurations are N and n. The normal vectors are related by Nanson's formula (8.37) and the velocity vectors are related by Eqn. (8.95).

8.8 A Material Momentum Balance Law

The development of a balance law for material momentum follows from the aforementioned works by Gurtin (cf. [149] and references therein) and Maugin (cf. [232] and references therein). Referring to Figure 8.8, we allow the existence of a surface of discontinuity $\mathscr{S}$ in the body. A material point $X_{\mathscr{S}}$ on this surface has a position vector $\mathbf{X}_{\mathscr{S}}$ in a fixed reference configuration and a position vector $\mathbf{x}_{\mathscr{S}} = \boldsymbol{\chi}(\mathbf{X}_{\mathscr{S}}, t)$ in the present configuration. This material point has the velocity vector $\mathbf{u} = \dot{\mathbf{X}}_{\mathscr{S}}$ in the reference configuration and a velocity vector $\mathbf{v}^*_{\mathscr{S}}$ in the present configuration. These velocity vectors are related by compatibility conditions:

$$\mathbf{v}^*_{\mathscr{S}} = (\mathbf{v}^* + \mathbf{F}\mathbf{u})^+ = (\mathbf{v}^* + \mathbf{F}\mathbf{u})^- . \tag{8.95}$$

The normal velocity of the shock or discontinuity as it propagates through material points in the fixed reference configuration is

$$U_n = \mathbf{u} \cdot \mathsf{N}, \tag{8.96}$$

where the unit normal vector N is shown in Figure 8.8. We allow the existence of sources of material momentum $\mathsf{B}^*_{\mathscr{S}}$, linear momentum $\mathbf{F}^*_{\mathscr{S}}$, and power $\Phi^*_{\mathrm{E}_{\mathscr{S}}}$ on $\mathscr{S}$. The source $\mathsf{B}^*_{\mathscr{S}}$ is vector-valued in contrast to the scalar-valued supply B_γ in one-dimensional theories.

The integral form of the balance law is equivalent to a local form,

$$\mathrm{Div}\,(\boldsymbol{\sigma}) + \rho_0^* \mathbf{b}_{\mathrm{M}} = \dot{\mathsf{P}}^*, \tag{8.97}$$

and a companion jump condition,

$$[\![\boldsymbol{\sigma}\mathsf{N} + U_n \mathsf{P}^*]\!]_{\mathscr{S}} = -\mathsf{B}^*_{\mathscr{S}}. \tag{8.98}$$

We also note that the associated jump conditions for mass, linear momentum, and energy are

$$\begin{aligned}
[\![\rho_0^* U_n]\!]_{\mathscr{S}} &= 0, \\
[\![\mathbf{P}\mathsf{N} + U_n \rho_0^* \mathbf{v}^*]\!]_{\mathscr{S}} &= -\mathbf{F}^*_{\mathscr{S}}, \\
[\![\mathbf{P}\mathsf{N}\cdot\mathbf{v}^*]\!]_{\mathscr{S}} + \left[\!\!\left[\rho_0^*\psi^* + \frac{\rho_0^*}{2}\mathbf{v}^*\cdot\mathbf{v}^*\right]\!\!\right]_{\mathscr{S}} U_n &= -\Phi^*_{\mathrm{E}_{\mathscr{S}}}.
\end{aligned} \tag{8.99}$$

In the sequel, we shall elaborate on the local form of the balance of material momentum and comment on the relationship between the jump condition (8.98) and related treatments in the literature. One of our intentions is to give context to the one-dimensional material momentum balance law that is used in this book. For certain problems in elastostatics, the balance law (8.97) can lead to a conservation law. Dating to the works of Günther [146] and Knowles and Sternberg [187], it is known that the resulting conservation law can also be established using Noether's theorem.[8]

8.8.1 The Local Form

The local form of the balance of material momentum is $\mathrm{Div}\,(\boldsymbol{\sigma}) + \rho_0^* \mathbf{b}_{\mathrm{M}} = \dot{\mathsf{P}}^*$. By suitably specifying $\rho_0^* \mathbf{b}_{\mathrm{M}}$, this law will be identically satisfied. The procedure has obvious parallels to the one used for one-dimensional theories, but the algebra is somewhat more involved.

To elaborate on the prescription for $\rho_0^* \mathbf{b}_{\mathrm{M}}$, several preliminary results are needed.[9] First, with the help of the constitutive relations $\mathbf{P} = \rho_0^* \frac{\partial \psi^*}{\partial \mathbf{F}}$, we find the following intermediate results:

[8] Kinzler and Hermann [182, Chapter 1] and Olver [256] provide accessible treatments of infinitesimal transformations and their role in establishing conservation laws using Noether's theorem.

[9] The corresponding developments for an incompressible hyperelastic material are easily inferred from Chadwick's lucid paper [54].

$$\begin{aligned}\mathrm{Div}\left(\left(\rho_0^*\psi^* - \frac{1}{2}\rho_0^*\mathbf{v}^*\cdot\mathbf{v}^*\right)\mathbf{I}\right) &= \sum_{r=1}^{3}\frac{\partial}{\partial\theta^r}\left(\rho_0^*\psi^* - \frac{1}{2}\rho_0^*\mathbf{v}^*\cdot\mathbf{v}^*\right)\mathbf{G}^r \\ &= \sum_{r=1}^{3}\left(\mathrm{tr}\left(\mathbf{P}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\right)\right)\mathbf{G}^r + \rho_0^*\dot{\mathbf{F}}^T\mathbf{v}^* \\ &\quad + \nabla_{\mathrm{exp}}\left(\rho_0^*\psi^* - \frac{1}{2}\rho_0^*\mathbf{v}^*\cdot\mathbf{v}^*\right). \end{aligned} \tag{8.100}$$

The derivative ∇_{exp} is defined for a function $f\left(\theta^1,\theta^2,\theta^3,\mathbf{C},t\right)$ as

$$\nabla_{\mathrm{exp}}(f) = \sum_{r=1}^{3}\frac{\partial f}{\partial\theta^r}\mathbf{G}^r\bigg|_{\substack{\mathbf{C}=\mathrm{const.}\\ t=\mathrm{const.}}}. \tag{8.101}$$

For example, if a body is homogeneous, then ρ_0^* is independent of θ^k and $\nabla_{\mathrm{exp}}(\rho_0^*) = 0$. Invoking the balance of linear momentum, $\mathrm{Div}(\mathbf{P}) - \rho_0^*\dot{\mathbf{v}}^* = -\rho_0^*\mathbf{b}$, we find that

$$\begin{aligned}-\mathrm{Div}\left(\mathbf{F}^T\mathbf{P}\right) &= -\sum_{r=1}^{3}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\mathbf{P}\mathbf{G}^r - \mathbf{F}^T\mathrm{Div}(\mathbf{P}) \\ &= -\sum_{r=1}^{3}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\mathbf{P}\mathbf{G}^r + \rho_0^*\mathbf{F}^T\mathbf{b} - \rho_0^*\mathbf{F}^T\dot{\mathbf{v}}, \\ -\dot{\mathsf{P}}^* &= \rho_0^*\dot{\mathbf{F}}^T\mathbf{v}^* + \rho_0^*\mathbf{F}^T\dot{\mathbf{v}}^*. \end{aligned} \tag{8.102}$$

Choosing θ^r to be Cartesian coordinates in the reference configuration is the easiest method to see that

$$\sum_{r=1}^{3}\left(\mathrm{tr}\left(\mathbf{P}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\right)\mathbf{I} - \sum_{r=1}^{3}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\mathbf{P}\right)\mathbf{G}^r = \mathbf{0}. \tag{8.103}$$

That is,

$$\begin{aligned}\sum_{r=1}^{3}\mathrm{tr}\left(\mathbf{P}\frac{\partial\mathbf{F}^T}{\partial\theta^r}\right)\mathbf{G}^r - \sum_{s=1}^{3}\frac{\partial\mathbf{F}^T}{\partial\theta^s}\mathbf{P}\mathbf{G}^s &= \sum_{r=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}P_{kl}\frac{\partial F_{kl}}{\partial X_r}\mathbf{E}_r - \sum_{s=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}P_{ks}\frac{\partial F_{kl}}{\partial X_s}\mathbf{E}_l \\ &= \sum_{r=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}P_{kl}\left(\frac{\partial x_k}{\partial X_r\partial X_l} - \frac{\partial x_k}{\partial X_l\partial X_r}\right)\mathbf{E}_r \\ &= \mathbf{0}, \end{aligned} \tag{8.104}$$

where

$$\mathbf{F} = \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial x_i}{\partial X_k}\mathbf{E}_i\otimes\mathbf{E}_k, \qquad \mathbf{P} = \sum_{i=1}^{3}\sum_{k=1}^{3}P_{ik}\mathbf{E}_i\otimes\mathbf{E}_k. \tag{8.105}$$

It should be noted that the final step in the derivation above used the identity (8.58).

We now use the intermediate results (8.100), (8.102), and (8.103) to solve for a material force:

$$\rho_0^* \mathbf{b}_{\mathrm{M}} = -\mathrm{Div}\,(\boldsymbol{\sigma}) + \dot{\mathsf{P}}^*. \tag{8.106}$$

The force $\rho_0^* \mathbf{b}_{\mathrm{M}}$, which is known as the assigned material force, is given by

$$\rho_0^* \mathbf{b}_{\mathrm{M}} = -\nabla_{\exp}\left(\rho_0^* \psi^* - \frac{1}{2}\rho_0^* \mathbf{v}^* \cdot \mathbf{v}^*\right) - \rho_0^* \mathbf{F}^T \mathbf{b}. \tag{8.107}$$

This is the desired prescription for the assigned material force and it ensures that the balance law (8.97) is identically satisfied. For elastostatic problems where the body is homogeneous and the body force is zero, the balance law (8.97) immediately implies the conservation law $\mathrm{Div}\,(\boldsymbol{\sigma}) = \mathbf{0}$.

8.8.2 The Jump Condition

In the purely mechanical theory of interest, the primary singular supplies pertain to linear momentum and material momentum (cf. Figure 8.9). Further, the jump condition $[\![\boldsymbol{\sigma}\mathsf{N} + U_n \mathsf{P}^*]\!]_{\mathscr{S}} + \mathsf{B}^*_{\mathscr{S}} = \mathbf{0}$ that is associated with the balance of material momentum is related to a driving force (or driving force) on a discontinuity and Eshelby's notion of a force on a singularity. Before elaborating on these connections, we recall from [263] that the jump conditions for mass, material momentum, and linear momentum can be used in a straightforward manner to reduce the jump condition from the balance of energy to an identity for $\Phi^*_{\mathrm{E}_{\mathscr{S}}}$:

$$\Phi^*_{\mathrm{E}_{\mathscr{S}}} = \mathbf{F}^*_{\mathscr{S}} \cdot \mathbf{v}^*_{\mathscr{S}} + \mathsf{B}^*_{\mathscr{S}} \cdot \mathbf{u}. \tag{8.108}$$

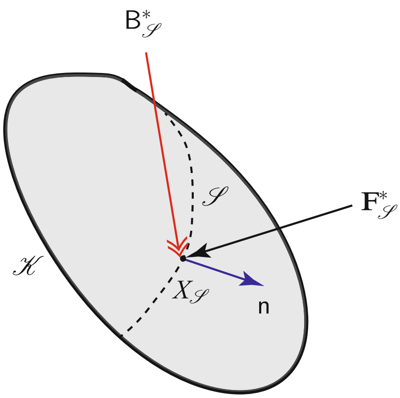

Fig. 8.9 Schematic of the singular supplies of linear momentum $\mathbf{F}^*_{\mathscr{S}}$ and material momentum acting at a point $X_{\mathscr{S}}$ on a surface of discontinuity.

Thus, the power supply $\Phi^*_{\mathrm{E}_{\mathscr{S}}}$ can be related to the power of the sources of momenta. This parallels the situation for the supply Φ_{E_γ} for one-dimensional media that is presented throughout this book.

The normal component of the supply of material momentum $\mathsf{B}^*_{\mathscr{S}}$ can be related to a quantity known as the driving force f that appears in works by Abeyaratne and Knowles [1, 4, 5] and Truskinovsky [353, 354] and the force on a singularity that appears in Eshelby [102]. Specifically, a driving force f is defined in [4, Eqn. (19)] (or [5, Eqn. (6.28)]),

$$f = [\![\rho_0^*\psi^*]\!]_{\mathscr{S}} - \{\mathbf{P}\}_{\mathscr{S}} \cdot [\![\mathbf{F}]\!]_{\mathscr{S}}. \tag{8.109}$$

Abeyaratne and Knowles interpret this force as "a normal traction applied to $\mathscr{S}$ by the body" [1, Page 353] and prescriptions for f play a key role in developing discontinuous solutions to boundary-value problems. Prescribing f is equivalent to prescribing the sources of material and linear momenta. To see this, we note that we can expand $[\![\boldsymbol{\sigma}\mathsf{N} + U_n\mathsf{P}^*]\!]_{\mathscr{S}} \cdot \mathsf{N}$ by substituting for the energy-momentum tensor $\boldsymbol{\sigma}$ and P^* in terms of ρ_0^*, ψ^*, $\mathbf{F}$, and $\mathbf{v}^*$:

$$\begin{aligned} \boldsymbol{\sigma} &= \left(\rho_0^*\psi^* - \frac{1}{2}\rho_0^*\mathbf{v}^* \cdot \mathbf{v}^*\right)\mathbf{I} - \mathbf{F}^T\mathbf{P}, \\ \mathsf{P}^* &= -\rho_0^*\mathbf{F}^T\mathbf{v}^*. \end{aligned} \tag{8.110}$$

With some rearranging and elimination of terms using the jump conditions (8.99), we find that[10]

$$[\![\boldsymbol{\sigma}\mathsf{N} + U_n\mathsf{P}^*]\!]_{\mathscr{S}} \cdot \mathsf{N} = \underbrace{[\![\rho_0^*\psi^*]\!]_{\mathscr{S}} - \{\mathbf{P}\}_{\mathscr{S}} \cdot [\![\mathbf{F}]\!]_{\mathscr{S}}} + \mathbf{F}_{\mathscr{S}}^* \cdot \{\mathbf{F}\mathsf{N}\}_{\mathscr{S}}. \tag{8.111}$$

After invoking the N component of the jump condition $[\![\boldsymbol{\sigma}\mathsf{N} + U_n\mathsf{P}^*]\!]_{\mathscr{S}} = -\mathsf{B}_{\mathscr{S}}^*$ and identifying the driving force as the underbraced term in (8.111), we conclude that

$$f = -\mathsf{B}_{\mathscr{S}}^* \cdot \mathsf{N} - \mathbf{F}_{\mathscr{S}}^* \cdot \{\mathbf{F}\mathsf{N}\}_{\mathscr{S}}. \tag{8.112}$$

In [4], source terms such as $\mathbf{F}_{\mathscr{S}}^*$ are set to zero, so the "normal traction" quoted above is $\mathsf{B}_{\mathscr{S}}^* \cdot \mathsf{N}$. However, the identity (8.112) implies that the driving force could also be supplied by $\mathbf{F}_{\mathscr{S}}^*$. The one-dimensional counterpart of the result (8.112) and its relation to Eshelby's force on a singularity is discussed in Section 1.8.1 of Chapter 1 and Exercise 1.7.

Throughout this book, we advocate for, and exploit, the notion that the jump condition associated with the material momentum balance is helpful in solving problems associated with strings and rods. In the problems of interest in this book, the material momentum is supplied by adhesion and impacts. We hope however to have supplied sufficient detail for the reader to be able to appreciate how the balance is related to one-dimensional phase transformation problems that are discussed in [2, 3, 5, 296] and references therein.

8.9 Closing Comments

The partial differential equations (8.92) are formidable: only a small collection of exact static solutions (and an even smaller set of exact dynamic solutions) are known. Consequently, most analyses of the equations use numerical methods.

[10] Further details of this calculation can be found in [263, Section 5].

The one-dimensional theories discussed in this book are designed to provide more tractable models compared to the three-dimensional theory. One important point to note as you explore these theories is that their structures are similar to that of the three-dimensional theory. Consequently, the one-dimensional theories do not necessarily have to be considered as a mishmash of assumptions that they often seem when one first encounters them in undergraduate courses.

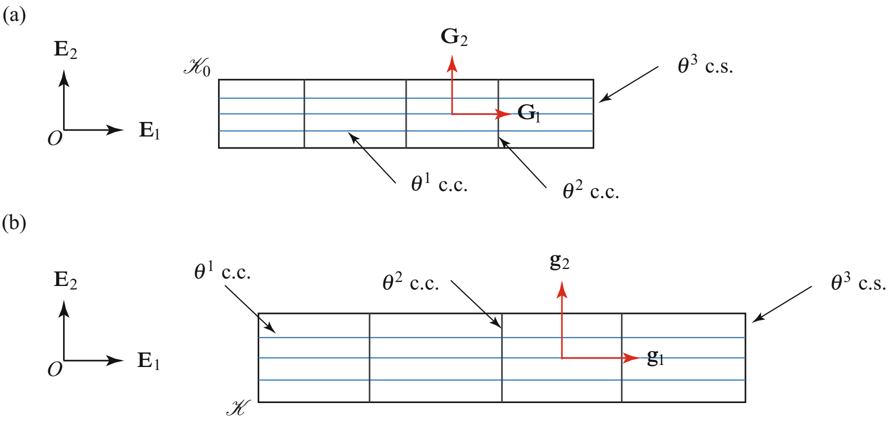

Fig. 8.10 An example of a θ^3 coordinate surface (c.s.) in (a) the reference configuration $\mathscr{K}_0$ and (b) the present $\mathscr{K}$ configurations of a parallelepiped that is being dilated. The abbreviation c.c. stands for coordinate curve.

8.10 Exercises

Exercise 8.1: The covariant and contravariant basis vectors are related by the nine equations

$$\mathbf{g}_k \cdot \mathbf{g}^i = \delta^i_k, \qquad (i, k = 1, 2, 3). \tag{8.113}$$

Assuming the contravariant basis vectors $\mathbf{g}^i$ are known, show that

$$\mathbf{g}_1 = \sqrt{g}\left(\mathbf{g}^2 \times \mathbf{g}^3\right), \quad \mathbf{g}_2 = \sqrt{g}\left(\mathbf{g}^3 \times \mathbf{g}^1\right), \quad \mathbf{g}_3 = \sqrt{g}\left(\mathbf{g}^1 \times \mathbf{g}^2\right), \tag{8.114}$$

where

$$\frac{1}{\sqrt{g}} = \left(\mathbf{g}^1 \times \mathbf{g}^2\right) \cdot \mathbf{g}^3. \tag{8.115}$$

Exercise 8.2: Consider the dilation of a parallelepiped shown in Figure 8.10. In the reference configuration, the convected coordinate system coincides with a Cartesian coordinate system:

$$\theta^k = x_k, \qquad \mathbf{R}^*\left(\theta^1, \theta^2, \theta^3\right) = \theta^1\mathbf{E}_1 + \theta^2\mathbf{E}_2 + \theta^3\mathbf{E}_3. \tag{8.116}$$

For the present configuration, the convected coordinate system can be described using Cartesian coordinates:

$$\theta^1 = f_1(x_1), \qquad \theta^2 = f_2(x_2), \qquad \theta^3 = f_3(x_3), \tag{8.117}$$

where f_k are smooth invertible functions.

(a) Show that

$$\mathbf{G}_1 = \mathbf{G}^1 = \mathbf{E}_1, \qquad \mathbf{G}_2 = \mathbf{G}^2 = \mathbf{E}_2, \qquad \mathbf{G}_3 = \mathbf{G}^3 = \mathbf{E}_3. \tag{8.118}$$

Compute $\sqrt{G}$.

(b) For the present configuration, show that

$$\mathbf{g}^1 = \frac{\partial f_1}{\partial x_1}\mathbf{E}_1, \qquad \mathbf{g}^2 = \frac{\partial f_2}{\partial x_2}\mathbf{E}_2, \qquad \mathbf{g}^3 = \frac{\partial f_3}{\partial x_3}\mathbf{E}_3. \tag{8.119}$$

(c) Verify that $\frac{1}{\sqrt{g}} = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}$.

(d) Establish the following representations for the covariant basis vectors $\mathbf{g}_k$:

$$\mathbf{g}_1 = \left(\frac{\partial f_1}{\partial x_1}\right)^{-1}\mathbf{E}_1, \qquad \mathbf{g}_2 = \left(\frac{\partial f_2}{\partial x_2}\right)^{-1}\mathbf{E}_2, \qquad \mathbf{g}_3 = \left(\frac{\partial f_3}{\partial x_3}\right)^{-1}\mathbf{E}_3. \tag{8.120}$$

(e) Show that the deformation gradient $\mathbf{F}$ associated with this problem has the representation

$$\mathbf{F} = \left(\frac{\partial f_1}{\partial x_1}\right)^{-1}\mathbf{E}_1\otimes\mathbf{E}_1 + \left(\frac{\partial f_2}{\partial x_2}\right)^{-1}\mathbf{E}_2\otimes\mathbf{E}_2 + \left(\frac{\partial f_3}{\partial x_3}\right)^{-1}\mathbf{E}_3\otimes\mathbf{E}_3. \tag{8.121}$$

Show that the deformation is homogeneous if $f_k = a_k x_k + c_k$ where a_1, a_2, a_3, c_1, c_2, and c_3 are constants. What are these functions if the body expands uniformly so that its volume in the present configuration is 8 times its volume in the reference configuration?

Exercise 8.3: Consider the flexure of a parallelepiped shown in Figure 8.4.[11] In the reference configuration, the convected coordinate system coincides with a Cartesian coordinate system:

$$\theta^k = x_k, \qquad \mathbf{R}^*\left(\theta^1, \theta^2, \theta^3\right) = \theta^1\mathbf{E}_1 + \theta^2\mathbf{E}_2 + \theta^3\mathbf{E}_3. \tag{8.122}$$

[11] Further details on the solution to this problem for specific constitutive equations can be found in [140]. The problem is of particular relevance because the rod theories we use give solutions to this flexure problem that are only approximations to the solution obtained using three-dimensional considerations.

For the present configuration, the convected coordinate system can be described using cylindrical polar coordinates (r, ϑ, z):

$$\theta^1 = f_1(r), \qquad \theta^2 = f_2(\vartheta), \qquad \theta^3 = f_3(z), \tag{8.123}$$

where f_k are smooth invertible functions.

(a) Show that

$$\mathbf{G}_1 = \mathbf{G}^1 = \mathbf{E}_1, \qquad \mathbf{G}_2 = \mathbf{G}^2 = \mathbf{E}_2, \qquad \mathbf{G}_3 = \mathbf{G}^3 = \mathbf{E}_3. \tag{8.124}$$

Compute $\sqrt{G}$.

(b) For the present configuration, show that

$$\mathbf{g}^1 = \frac{\partial f_1}{\partial r}\mathbf{e}_r, \qquad \mathbf{g}^2 = \frac{1}{r}\frac{\partial f_2}{\partial \vartheta}\mathbf{e}_\vartheta, \qquad \mathbf{g}^3 = \frac{\partial f_3}{\partial z}\mathbf{E}_3, \tag{8.125}$$

where

$$\mathbf{e}_r = \cos(\vartheta)\mathbf{E}_1 + \sin(\vartheta)\mathbf{E}_2, \qquad \mathbf{e}_\vartheta = \cos(\vartheta)\mathbf{E}_2 - \sin(\vartheta)\mathbf{E}_1. \tag{8.126}$$

(c) Verify that $\frac{1}{\sqrt{g}} = \frac{1}{r}\frac{\partial f_1}{\partial r}\frac{\partial f_2}{\partial \vartheta}\frac{\partial f_3}{\partial z}$.

(d) Establish the following representations for the covariant basis vectors $\mathbf{g}_k$:

$$\mathbf{g}_1 = \left(\frac{\partial f_1}{\partial r}\right)^{-1}\mathbf{e}_r, \qquad \mathbf{g}_2 = r\left(\frac{\partial f_2}{\partial \vartheta}\right)^{-1}\mathbf{e}_\vartheta, \qquad \mathbf{g}_3 = \left(\frac{\partial f_3}{\partial z}\right)^{-1}\mathbf{E}_3. \tag{8.127}$$

(e) Show that the deformation gradient $\mathbf{F}$ associated with this problem has the representation

$$\mathbf{F} = \left(\frac{\partial f_1}{\partial r}\right)^{-1}\mathbf{e}_r \otimes \mathbf{E}_1 + r\left(\frac{\partial f_2}{\partial \vartheta}\right)^{-1}\mathbf{e}_\vartheta \otimes \mathbf{E}_2 + \left(\frac{\partial f_3}{\partial z}\right)^{-1}\mathbf{E}_3 \otimes \mathbf{E}_3. \tag{8.128}$$

(f) If the parallelepiped is composed of an incompressible material, then show that

$$\frac{\partial f_3}{\partial z} = r\left(\frac{\partial f_1}{\partial r}\frac{\partial f_2}{\partial \vartheta}\right)^{-1}. \tag{8.129}$$

Exercise 8.4: This exercise is devoted to an exploration of constitutive relations for a constrained hyperelastic continuum. We note that there are several treatments of this topic, among them [46, 50, 351], and these treatments have some overlaps to one that is used in the mechanics of rigid bodies and particles.

(a) Consider solving the following equation for $\mathsf{f}(\mathsf{x})$:

$$\mathsf{f} \cdot \dot{\mathsf{x}} = 0, \tag{8.130}$$

for all $\dot{\mathsf{x}}$ which satisfy the equation

$$\mathsf{g} \cdot \dot{\mathsf{x}} = 0. \tag{8.131}$$

Here, $\mathsf{g} = \mathsf{g}(\mathsf{x})$, the vector x and vector-valued functions f and g are N-dimensional, and $\mathsf{a} \cdot \mathsf{b} = \sum_{K=1}^{N} a_K b_K$. Prove that the solution to Eqn. (8.130) is

$$\mathsf{f} = 0 + \lambda \mathsf{g}, \tag{8.132}$$

where λ is a scalar. That is, f is parallel to g.[12] Would this result hold if $\mathsf{f} = \mathsf{f}(\mathsf{x}, \dot{\mathsf{x}})$?

(b) Using the results of (a), show that the constitutive equations for a constrained hyperelastic material whose strain energy function is $\psi^*(\mathbf{C})$ and which is subject to a constraint $\phi^*(\mathbf{C}) = 0$ is

$$\mathbf{S} = 2\rho_0^* \frac{\partial \psi^*}{\partial \mathbf{C}} + \lambda \frac{\partial \phi^*}{\partial \mathbf{C}}. \tag{8.133}$$

Your starting point should be the identity $\mathbf{S} \cdot \dot{\mathbf{C}} = 2\dot{\overline{\rho_0^* \psi^*}}$, where the inner-product $\cdot$ of two tensors is defined using the trace operator: $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right)$.[13]

(c) The prescription $\lambda \frac{\partial \phi^*}{\partial \mathbf{C}}$ for the constraint response is sometimes known as the normality prescription. Why is this the case? Give a brief discussion of the identical satisfaction of the moment of momentum balance law by the constitutive relations (8.133).

(d) Show that the principal invariants of $\mathbf{B}$ and $\mathbf{C}$ are identical:

$$I_{\mathbf{B}} = I_{\mathbf{C}}, \qquad II_{\mathbf{B}} = II_{\mathbf{C}}, \qquad III_{\mathbf{B}} = III_{\mathbf{C}}, \tag{8.134}$$

where, for any second-order tensor $\mathbf{A}$,

$$I_{\mathbf{A}} = \mathrm{tr}(\mathbf{A}), \qquad II_{\mathbf{A}} = \frac{1}{2}\left(\mathrm{tr}(\mathbf{A})^2 - \mathrm{tr}\left(\mathbf{A}^2\right)\right), \qquad III_{\mathbf{A}} = \det(\mathbf{A}). \tag{8.135}$$

(e) Using the results of (b) show that the constitutive relations for an incompressible hyperelastic body are

$$\mathbf{T} = 2\rho^* \mathbf{F} \frac{\partial \psi^*}{\partial \mathbf{C}} \mathbf{F}^T - p\mathbf{I}. \tag{8.136}$$

Here, the scalar function $p = p(\mathbf{X}, t)$ is known as the pressure. For this exercise, you may need to use the following identity:

$$\frac{\partial \det(\mathbf{C})}{\partial \mathbf{C}} = \det(\mathbf{C})\mathbf{C}^{-1}. \tag{8.137}$$

(f) Apply the results of (e) to the case of an incompressible Mooney-Rivlin material where $\rho_0^* \psi^*$ is given by Eqn. (8.90). Your final expression for $\mathbf{T}$ should be

[12] For assistance with this exercise, it may be helpful to point out that a closely related proof can be found in Green et al. [136, Page 902].

[13] You may wish to notice that $\mathbf{T} \cdot \mathbf{L}^T = J\mathbf{P} \cdot \dot{\mathbf{F}} = J\mathbf{S} \cdot \dot{\mathbf{E}}$, where, among others, the moment of momentum balance law $\mathbf{T} = \mathbf{T}^T$ is used to establish the equivalence. As mentioned on Page 361, the tensor $\mathbf{L}$ has the representation $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \sum_{k=1}^{3} \dot{\mathbf{g}}_k \otimes \mathbf{g}^k$.

equivalent to Eqn. (8.88). In addition to the Cayley-Hamilton theorem for $\mathbf{B}$,

$$\mathbf{B}^3 - I_\mathbf{B}\mathbf{B}^2 + II_\mathbf{B}\mathbf{B} - III_\mathbf{B}\mathbf{I} = \mathbf{0}, \tag{8.138}$$

the following identities will be helpful:

$$\frac{\partial \mathrm{tr}(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{I}, \qquad \frac{1}{2}\frac{\partial}{\partial \mathbf{C}}\left(\mathrm{tr}\left(\mathbf{C}\right)^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right) = \mathrm{tr}(\mathbf{C})\mathbf{I} - \mathbf{C}. \tag{8.139}$$

(g) Consider the theory of a nonlinear elastic string that is discussed in Chapter 1. Suppose that the string has a strain energy function $\rho_0\psi = \rho_0(\xi)\psi(\mu,\xi)$ where $\mu = \left\|\mathbf{r}'\right\|$ is the stretch of the string and ξ is a convected coordinate. Starting from the energy theorem $(1.82)_2$ and the local form of the balance of angular momentum (1.84) for the string,

$$\rho_0\dot{\psi} = \mathbf{n}\cdot\mathbf{v}', \qquad \mathbf{r}'\times\mathbf{n} = \mathbf{0}, \tag{8.140}$$

using the results of (a), and the identity $\mu\dot{\mu} = \mathbf{r}'\cdot\mathbf{v}'$, determine the constitutive relations for the contact force $\mathbf{n}$ in the string. Specialize your results to the case of a strain energy function

$$\rho_0\psi = \frac{EA}{2}(\mu - 1)^2. \tag{8.141}$$

When would E and A depend on ξ? What is $\rho_0\psi$ for a linearly elastic string?

(h) Suppose that the string in (g) is inextensible. Using the results of (a), develop an expression for the constitutive relations for the string.