Chapter 6
Theory of an Elastic Rod with Extension and Shear

"In this way, one arrives at the kinematical model of a rod consisting of a one-dimensional continuum M^1 and a set of two vector fields $\left(\underset{1}{\mathbf{d}}^i(\xi,t), \underset{2}{\mathbf{d}}^i(\xi,t)\right)$ in M^1 whose values fix a homogeneous deformation of the cross section of the rod through the point ξ."
R. A. Toupin's discussion in [348, Page 90] of a model for a rod as a material curve with a set of directors.

6.1 Introduction

For many of the applications analyzed using Kirchhoff's rod theory one cannot help but ask about the influence of extensibility of the centerline. This is particularly true for the telephone cord that is often used to demonstrate perversions. In attempting to use a rod theory to analyze the twisting and bending of a length of surgical tubing, the possibility for relaxing some of the assumptions associated with the deformation of the cross sections in Kirchhoff's rod theory also appears to be desirable. Two paths are available to develop the resulting rod theory. One avenue is to establish the theory as an approximate solution for a three-dimensional continuum. A second avenue, popularized by Ericksen and Truesdell's paper [100] on directors in 1958, is to model the rod as a directed (or Cosserat) curve - that is, as a material curve with a set of deformable vectors (directors) associated with each point on this curve. While the resulting rod theory stands alone as a separate theory, many of the parameters in the models produced by this theory, such as mass density per unit length and stiffnesses, are identified by directly comparing the predictions of the rod theory to corresponding problems from the three-dimensional theory. In the early 1970s several researchers, including Antman [10], Green and Laws [130], and Reissner [299, 300], extended Kirchhoff's theory to include extension of the centerline and

O.M. O'Reilly, *Modeling Nonlinear Problems in the Mechanics of Strings and Rods*,
Interaction of Mechanics and Mathematics, DOI 10.1007/978-3-319-50598-5_6

rotation of the cross sections relative to the centerline. The latter effect is known as transverse shear. As shall be shown in an exercise at the end of this chapter, a linearized version of the rod theory provides Stephen P. Timoshenko's (1878–1972) celebrated beam theory [344]. The extensions to Kirchhoff's original theory were neither free from controversy nor recriminations and many contributions on the topic have largely been forgotten.

Primarily because of Antman's seminal book [12] and papers, the most popular form of the theory in the recent literature is the one which he presents. His formulation has inspired a numerical implementation of the theory in an influential paper [327] written by Simo and Vu-Quoc. In addition, partially because both $\mathbf{n}$ and $\mathbf{m}$ are prescribed by constitutive relations involving six strains, the theory has enabled several teams of researcher to propose a range of Hamiltonian formulations using notions from geometric mechanics (cf. [86, 178, 238, 326]).

Antman [12] is the primary resource for the analyses of problems describing the rod theory discussed in this chapter. Our developments will be closely aligned with [12, Chapter 8] and include an explicit discussion of material momentum and recent treatments of material symmetry for elastic rods. We present a limited discussion of applications and refer the reader to [12] for examples and analyses. One application that we do consider is motivated by a remarkable series of recent works where strands of DNA molecules are subject to mechanical testing (see the review [40]). From these tests, it has become apparent that a twist-bend coupling [228] and a stretch-twist coupling [177, 227] is present. While the twist-bend coupling can be modeled using Kirchhoff's rod theory by incorporating the strains $v_1 v_3$ and $v_2 v_3$ in the strain energy function, the latter coupling requires an extensible rod theory of the type considered in this chapter. Further, as demonstrated by Healey [158], the theory discussed in the present chapter is ideal for wire ropes which possess an inherent helical symmetry. The second application we consider is an analysis of static solutions for the rod theory that is based on Ericksen's notion of uniform states for rods.

6.2 Kinematical Considerations

For the rod theory of interest in this chapter, the rod is modeled as a material curve $\mathscr{L}$ to which at each point a pair of directors $\mathbf{d}_1$ and $\mathbf{d}_2$ are defined. As in the previous chapter, the material curve with its associated directors is known as a Cosserat or directed curve. Variants of the rod theory discussed here were proposed by Antman [10] and Green and Laws [130], among others, and it is often known, following [12], as a special Cosserat rod theory.

In the reference configuration $\mathscr{R}_0$ and present configuration $\mathscr{R}$ of the directed curve, the directors are denoted by $\mathbf{D}_\alpha = \mathbf{D}_\alpha(\xi)$ and $\mathbf{d}_\alpha(\xi,t)$, where $\alpha = 1,2$. The locations of material points in the two configurations are defined by the vector-valued functions $\mathbf{R}(\xi)$ and $\mathbf{r}(\xi,t)$, respectively. The configuration $\mathscr{R}_0$ of the directed curve is defined by the triple $\mathbf{R}(\xi)$ and $\mathbf{D}_\alpha(\xi)$ and the configuration $\mathscr{R}$ of the directed curve is defined by the triple $\mathbf{r}(\xi,t)$ and $\mathbf{d}_\alpha(\xi,t)$.

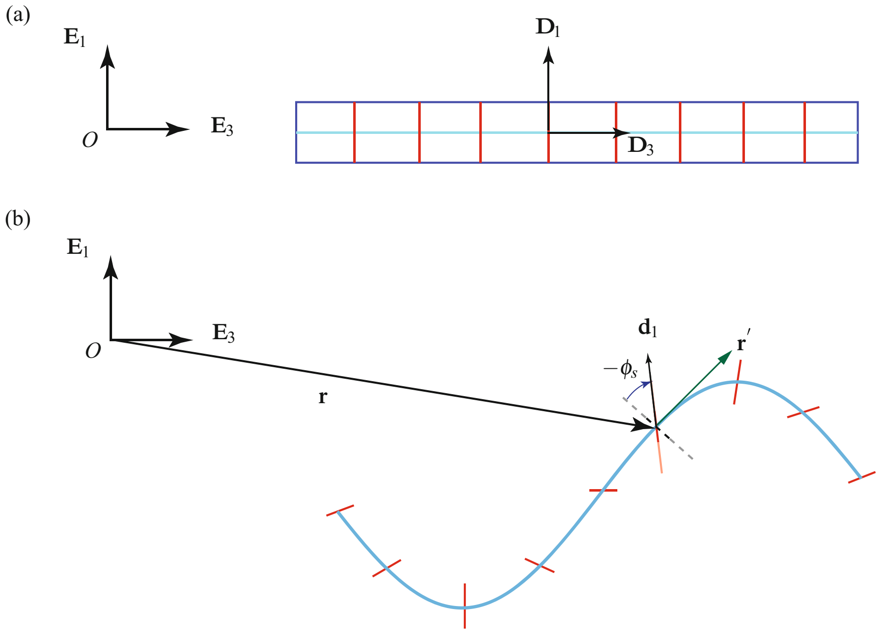

Fig. 6.1 (a) Schematic of the reference configuration for a straight rod. (b) The present configuration of the rod. Note that the cross sections of the rod are not constrained to remain normal to the tangent vector $\mathbf{r}'$.

For the rod theory of interest, the directors retain their magnitude and relative orientation:

$$\mathbf{d}_\alpha(\xi,t) = \mathbf{P}(\xi,t)\mathbf{D}_\alpha(\xi), \quad \mathbf{D}_\alpha(\xi) = \mathbf{P}_0(\xi)\mathbf{E}_\alpha. \tag{6.1}$$

Here, as in the previous chapter, $\mathbf{P}$ and $\mathbf{P}_0$ are rotation tensors and $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3\}$ are a set of fixed, right-handed orthonormal basis vectors. We recall the definitions of the skew-symmetric tensors

$$\mathbf{K} = \mathbf{P}^T\mathbf{P}', \qquad \mathbf{K}_0 = \mathbf{P}_0'\mathbf{P}_0^T, \tag{6.2}$$

and their respective axial vectors $\boldsymbol{\nu}$ and $\boldsymbol{\nu}_0$:

$$\boldsymbol{\nu} = \operatorname{ax}(\mathbf{K}), \qquad \boldsymbol{\nu}_0 = \operatorname{ax}(\mathbf{K}_0). \tag{6.3}$$

Differentiating Eqn. $(6.1)_1$ and using the identity (5.9), we find a familiar result:

$$\mathbf{d}_\alpha' = (\mathbf{P}(\boldsymbol{\nu}+\boldsymbol{\nu}_0)) \times \mathbf{d}_\alpha. \tag{6.4}$$

It is crucial to note here that we are not assuming that $\mathbf{d}_1 \times \mathbf{d}_2 \parallel \mathbf{r}'$. That is, we are allowing transverse shearing of the cross sections of the rod that may initially be normal to the tangent vector to the material curve $\mathscr{L}$ (cf. Figure 6.1).

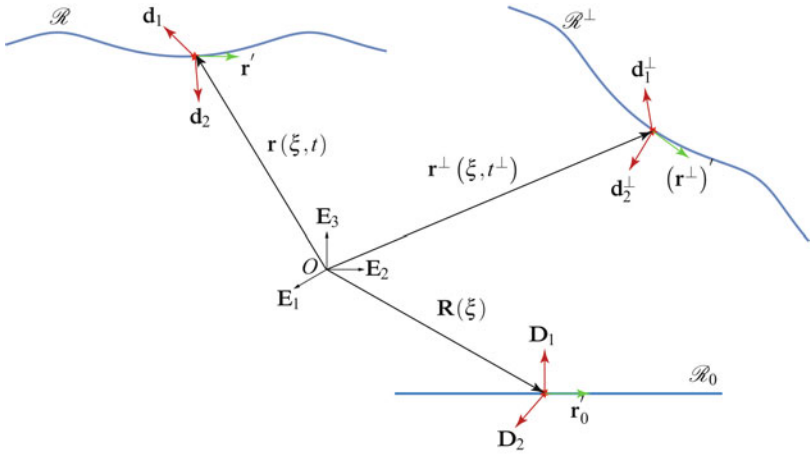

Fig. 6.2 Schematic of a pair of motions of a directed curve which differ by a rigid body motion. Observe that both motions are relative to the same reference configuration $\mathscr{R}_0$.

A pair of motions $(\mathbf{r}(\xi,t), \mathbf{d}_\alpha(\xi,t))$ and $\left(\mathbf{r}^\perp\left(\xi,t^\perp\right), \mathbf{d}_\alpha^\perp\left(\xi,t^\perp\right)\right)$ of a directed curve differ by a rigid body motion if

$$\begin{aligned}\mathbf{r}^\perp\left(\xi,t^\perp\right) &= \mathbf{Q}(t)\mathbf{r}(\xi,t)+\mathbf{q}(t),\\ \mathbf{d}_1^\perp\left(\xi,t^\perp\right) &= \mathbf{Q}(t)\mathbf{d}_1(\xi,t),\\ \mathbf{d}_2^\perp\left(\xi,t^\perp\right) &= \mathbf{Q}(t)\mathbf{d}_2(\xi,t).\end{aligned} \tag{6.5}$$

Here, $\mathbf{Q}$ is a proper-orthogonal tensor-valued function of time, $\mathbf{q}(t)$ is a vector-valued function of time, and $t^\perp = t + a$ with a being constant (cf. Figure 6.2). Observe that

$$\mathbf{P}^\perp\left(\xi,t^\perp\right) = \mathbf{Q}(t)\mathbf{P}(\xi,t), \qquad \left(\mathbf{P}^\perp\left(\xi,t^\perp\right)\right)' = \mathbf{Q}(t)\mathbf{P}'(\xi,t). \tag{6.6}$$

With the help of the identities $\mathbf{a}\cdot(\mathbf{A}\mathbf{b}) = \mathbf{b}\cdot\left(\mathbf{A}^T\mathbf{a}\right)$ for all second-order tensors $\mathbf{A}$ and $\mathbf{Q}(\mathbf{a}\times\mathbf{b}) = (\mathbf{Q}\mathbf{a})\times(\mathbf{Q}\mathbf{b})$ for all proper-orthogonal tensors $\mathbf{Q}$, it should be easy to see that the inner products $\mathbf{r}'\cdot\mathbf{r}'$, $\mathbf{r}'\cdot\mathbf{d}_\alpha$, and $\mathbf{d}_\alpha\cdot\mathbf{d}_\beta$ remain invariant under superposed rigid body motions whereas $\mathbf{r}\cdot\mathbf{r}$ or $\mathbf{r}\cdot\mathbf{d}_\alpha$ are not the same for a motion and another motion which differs from it by a rigid body motion.

The components of $\mathbf{P}\boldsymbol{\nu}$ and $\mathbf{r}' - \mathbf{P}\mathbf{R}'$ with respect to the basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 = \mathbf{d}_1\times\mathbf{d}_2\}$ define the six strain measures:

$$\begin{aligned}\boldsymbol{\eta} &= \mathbf{P}^T\mathbf{r}' - \mathbf{R}' = \eta_1\mathbf{D}_1 + \eta_2\mathbf{D}_2 + \eta_3\mathbf{D}_3,\\ \boldsymbol{\nu} &= \nu_1\mathbf{D}_1 + \nu_2\mathbf{D}_2 + \nu_3\mathbf{D}_3.\end{aligned} \tag{6.7}$$

Here, $\mathbf{D}_3 = \mathbf{D}_1 \times \mathbf{D}_2$ and neither $\mathbf{d}_3$ nor $\mathbf{D}_3$ should be confused with the unit tangent vector to the centerline of the rod. We leave it as an exercise to verify that

$$\boldsymbol{\eta}^{\perp}\left(\xi, t^{\perp}\right) = \boldsymbol{\eta}(\xi, t), \qquad \mathbf{v}^{\perp}\left(\xi, t^{\perp}\right) = \mathbf{v}(\xi, t). \tag{6.8}$$

That is, the vectors $\mathbf{v}$ and $\boldsymbol{\eta}$ are suitable candidates for strain measures for the rod theory. If we identify $\mathbf{d}_1$ and $\mathbf{d}_2$ with the cross section of the rod-like body that the directed curve is modeling, then v_1 and v_2 can be identified as flexural (bending) strains, and v_3 is the torsional strain of the cross section. The strains η_1 and η_2 measure the change in the $\mathbf{d}_1$ and $\mathbf{d}_2$ components of $\mathbf{r}'$ and are considered to be shearing strains:

$$\eta_\beta = \left(\mathbf{P}^T\mathbf{r}' - \mathbf{R}'\right) \cdot \mathbf{D}_\beta = \mathbf{r}' \cdot \mathbf{d}_\beta - \mathbf{R}' \cdot \mathbf{D}_\beta. \tag{6.9}$$

The sixth strain measure η_3 provides a measure of the change in the volume $\mathbf{r}' \cdot (\mathbf{d}_1 \times \mathbf{d}_2)$:

$$\eta_3 = \left(\mathbf{P}^T\mathbf{r}' - \mathbf{R}'\right) \cdot \mathbf{D}_3 = \mathbf{r}' \cdot \mathbf{d}_3 - \mathbf{R}' \cdot \mathbf{D}_3. \tag{6.10}$$

Only in exceptional instances is η_3 equal to the extension (stretch) squared of the centerline.

6.3 Summary of the Governing Equations for the Rod Theory

The balances laws for the theory are identical to those for the Kirchhoff rod theory that we presented in Section 5.7.2 of Chapter 5. The primary differences are the constitutive relations for $\mathbf{n}$ and $\mathbf{m}$ and the fact that $\mathbf{r}' \neq \mathbf{d}_1 \times \mathbf{d}_2$. These differences were anticipated in writing Chapter 5 so that much of the material could be recycled in the present chapter. As a consequence our discussion here is brief.

The conservation laws (5.75)–(5.79) are postulated and the local forms and jump conditions are established in the usual manner. Omitting details, we find the following jump conditions:

$$\begin{gathered}
[\![\mathbf{r}]\!]_\gamma = \mathbf{0}, \qquad [\![\mathbf{P}\mathbf{P}_0]\!]_\gamma = \mathbf{0}, \\
[\![\rho_0]\!]_\gamma \dot{\gamma} = 0, \qquad [\![\rho_0 y^{0\alpha}]\!]_\gamma \dot{\gamma} = 0, \qquad \left[\!\left[\rho_0 y^{\alpha\beta}\right]\!\right]_\gamma \dot{\gamma} = 0, \\
[\![\mathbf{n}]\!]_\gamma + [\![\mathbf{G}]\!]_\gamma \dot{\gamma} + \mathbf{F}_\gamma = \mathbf{0}, \\
[\![\mathsf{C}]\!]_\gamma + [\![\mathsf{P}]\!]_\gamma \dot{\gamma} + \mathsf{B}_\gamma = 0, \\
[\![\mathbf{m}]\!]_\gamma + \left[\!\left[\mathbf{d}_1 \times \mathbf{L}^1 + \mathbf{d}_2 \times \mathbf{L}^2\right]\!\right]_\gamma \dot{\gamma} + \mathbf{M}_{O_\gamma} - \mathbf{r}(\gamma, t) \times \mathbf{F} = \mathbf{0}.
\end{gathered} \tag{6.11}$$

Here, the linear momentum $\mathbf{G}$ and director momenta $\mathbf{L}^\beta$ have the familiar forms

$$\mathbf{G} = \rho_0 \dot{\mathbf{r}} + \rho_0 \sum_{\alpha=1}^{2} y^{0\alpha} \dot{\mathbf{d}}_\alpha, \qquad \mathbf{L}^\beta = \rho_0 y^{0\beta} \dot{\mathbf{r}} + \rho_0 \sum_{\alpha=1}^{2} y^{\beta\alpha} \dot{\mathbf{d}}_\alpha. \tag{6.12}$$

The jump conditions are supplemented by the pair of partial differential equations and the local form of the balance of energy:

$$\dot{\mathbf{G}} = \rho_0 \mathbf{f} + \frac{\partial \mathbf{n}}{\partial \xi},$$

$$\rho_0 \left(\sum_{\alpha=1}^{2} \mathbf{d}_\alpha \times y^{0\alpha} \ddot{\mathbf{r}} + \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \mathbf{d}_\alpha \times y^{\alpha\beta} \ddot{\mathbf{d}}_\beta \right) = \mathbf{m}_a + \frac{\partial \mathbf{m}}{\partial \xi} + \frac{\partial \mathbf{r}}{\partial \xi} \times \mathbf{n},$$

$$\rho_0 \dot{\psi} = \mathbf{n} \cdot \left(\frac{\partial \mathbf{v}}{\partial \xi} - \boldsymbol{\omega} \times \frac{\partial \mathbf{r}}{\partial \xi} \right) + \mathbf{m} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \xi}. \tag{6.13}$$

Observe that $\frac{\partial \mathbf{v}}{\partial \xi} - \boldsymbol{\omega} \times \frac{\partial \mathbf{r}}{\partial \xi}$ is the corotational rate of $\frac{\partial \mathbf{r}}{\partial \xi}$.

The jump condition arising from the energy balance (5.79) is not listed above. As with the theories of strings and rods we have discussed previously, this jump condition is used to relate the mechanical powers of the singular supplies:

$$\mathsf{B}_\gamma \dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma + \mathbf{M}_\gamma \cdot \boldsymbol{\omega}_\gamma = \Phi_{\mathrm{E}_\gamma}, \tag{6.14}$$

where the resultant moment $\mathbf{M}_\gamma$ is defined by the identity

$$\mathbf{M}_\gamma = \mathbf{M}_{O_\gamma} - \mathbf{r}(\gamma, t) \times \mathbf{F}_\gamma. \tag{6.15}$$

Given the appropriate boundary conditions, constitutive relations, and initial conditions, the preceding equations serve to enable the calculation of $\mathbf{r}$ and $\mathbf{P}$ (or equivalently $\mathbf{d}_\alpha$) for a directed curve.

6.3.1 Constitutive Relations for n *and* m

Using the strains $\mathbf{v}$ and $\boldsymbol{\eta}$, the strain energy function of the rod is assumed to have the representations

$$\begin{aligned} \rho_0 \psi &= \rho_0 \psi \left(\boldsymbol{\eta}(\xi), \mathbf{v}(\xi), \mathbf{v}_0(\xi), \mathbf{R}'(\xi), \xi \right) \\ &= \rho_0 \hat{\psi}(\eta_i, v_k, \xi). \end{aligned} \tag{6.16}$$

In the second of these representations, $\eta_i = \boldsymbol{\eta}(\xi) \cdot \mathbf{D}_i(\xi)$ and $v_k = \mathbf{v}(\xi) \cdot \mathbf{D}_k(\xi)$. Observe that both representations of $\rho_0 \psi$ are invariant under superposed rigid body motions of the directed curve: $\psi^\perp = \psi$. Later, we shall use the local form of the balance of energy to specify constitutive relations for $\mathbf{n}$ and $\mathbf{m}$ using ψ.

We can parallel our earlier developments in Section 5.4 of Chapter 5 to find that

$$\dot{\mathbf{d}}_\alpha = \boldsymbol{\omega} \times \mathbf{d}_\alpha, \qquad \boldsymbol{\omega} = \mathrm{ax}\left(\dot{\mathbf{P}} \mathbf{P}^T \right). \tag{6.17}$$

In addition, the derivatives of $\mathbf{P}$ with respect to ξ and t are related (cf. Eqn. (5.40)):

$$\boldsymbol{\omega}' = \mathbf{P} \dot{\mathbf{v}}. \tag{6.18}$$

Given any vector **b**, say, where

$$\mathbf{b} = b_1\mathbf{d}_1 + b_2\mathbf{d}_2 + b_3\left(\mathbf{d}_1 \times \mathbf{d}_2\right), \tag{6.19}$$

then the corotational rate of **b** has the representations

$$\dot{\mathbf{b}} - \boldsymbol{\omega} \times \mathbf{b} = \dot{b}_1\mathbf{d}_1 + \dot{b}_2\mathbf{d}_2 + \dot{b}_3\left(\mathbf{d}_1 \times \mathbf{d}_2\right). \tag{6.20}$$

This identity is helpful for interpreting representations for the derivatives of strain energy functions. For instance, differentiating $\boldsymbol{\eta}$ with respect to time and noting that the axial vector of $\mathbf{P}\dot{\mathbf{P}}^T$ is $-\boldsymbol{\omega}$, we find that

$$\dot{\boldsymbol{\eta}} = \mathbf{P}^T\left(\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{r}'\right). \tag{6.21}$$

The identities (6.18) and (6.21) can also be considered as compatibility equations for the material time and ξ derivatives.

The material time derivative of $\rho_0\psi$ has several representations:

$$\begin{aligned}\rho_0\dot{\psi} &= \rho_0\sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial\eta_k}\dot{\eta}_k + \rho_0\sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial v_k}\dot{v}_k \\ &= \left(\rho_0\sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial\eta_k}\mathbf{D}_k\right)\cdot\dot{\boldsymbol{\eta}} + \left(\rho_0\sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial v_k}\mathbf{D}_k\right)\cdot\dot{\mathbf{v}} \\ &= \mathbf{P}\rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{v}}\cdot\boldsymbol{\omega}' + \mathbf{P}\rho_0\frac{\partial\hat{\psi}}{\partial\boldsymbol{\eta}}\cdot\left(\mathbf{v}' - \boldsymbol{\omega}\times\mathbf{r}'\right),\end{aligned} \tag{6.22}$$

where we used the identities (6.18) and (6.21) and defined the following pair of vectors:

$$\frac{\partial\hat{\psi}}{\partial\mathbf{v}} = \sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial v_k}\mathbf{D}_k, \qquad \frac{\partial\hat{\psi}}{\partial\boldsymbol{\eta}} = \sum_{k=1}^{3}\frac{\partial\hat{\psi}}{\partial\eta_k}\mathbf{D}_k. \tag{6.23}$$

The role played by **P** in establishing the last of the relations (6.22) is to change the basis from $\mathbf{D}_k$ to $\mathbf{d}_k$ in the vectors $\frac{\partial\hat{\psi}}{\partial\mathbf{v}}$ and $\frac{\partial\hat{\psi}}{\partial\boldsymbol{\eta}}$.

The constitutive relations for the rod are found by considering the local form of the balance of energy:

$$\rho_0\dot{\psi} = \mathbf{m}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\xi} + \mathbf{n}\cdot\left(\frac{\partial\mathbf{v}}{\partial\xi} - \boldsymbol{\omega}\times\frac{\partial\mathbf{r}}{\partial\xi}\right). \tag{6.24}$$

Following the procedure we have used several times previously, we now introduce the representation $(6.22)_3$ for $\rho_0\dot{\psi}$ and rearrange the local form of the balance of energy:

$$\left(\mathbf{m} - \mathbf{P}\rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{v}}\right)\cdot\boldsymbol{\omega}' + \left(\mathbf{n} - \mathbf{P}\rho_0\frac{\partial\hat{\psi}}{\partial\boldsymbol{\eta}}\right)\cdot\left(\mathbf{v}' - \boldsymbol{\omega}\times\mathbf{r}'\right) = 0. \tag{6.25}$$

Assuming this equation holds for all motions of the rod and that $\mathbf{n}$ and $\mathbf{m}$ are independent of $\dot{\eta}_k$ and $\dot{v}_i$, it follows that

$$\begin{aligned}
\mathbf{n} &= \mathbf{P}\rho_0 \frac{\partial \hat{\psi}}{\partial \boldsymbol{\eta}} \\
&= \rho_0 \frac{\partial \hat{\psi}}{\partial \eta_1}\mathbf{d}_1 + \rho_0 \frac{\partial \hat{\psi}}{\partial \eta_2}\mathbf{d}_2 + \rho_0 \frac{\partial \hat{\psi}}{\partial \eta_3}\left(\mathbf{d}_1 \times \mathbf{d}_2\right), \\
\mathbf{m} &= \mathbf{P}\rho_0 \frac{\partial \hat{\psi}}{\partial \mathbf{v}} \\
&= \rho_0 \frac{\partial \hat{\psi}}{\partial v_1}\mathbf{d}_1 + \rho_0 \frac{\partial \hat{\psi}}{\partial v_2}\mathbf{d}_2 + \rho_0 \frac{\partial \hat{\psi}}{\partial v_3}\left(\mathbf{d}_1 \times \mathbf{d}_2\right).
\end{aligned} \tag{6.26}$$

Thus, in contrast to the constitutive relations Eqn. (5.91) for the Kirchhoff theory, $\mathbf{n}$ is completely described by constitutive relations. It may be helpful to note that we are no longer imposing the constraint $\mathbf{r}' = \mathbf{d}_1 \times \mathbf{d}_2$. Consequently, the shear forces $\mathbf{n} \cdot \mathbf{d}_\alpha$ can be loosely interpreted as being responsible for deforming (shearing) $\mathbf{d}_\alpha$ in the tangential direction to the material curve.

The reference configuration $\mathscr{R}_0$ is said to be a natural configuration if $\mathbf{n} = \mathbf{0}$ and $\mathbf{m} = \mathbf{0}$ when $\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\eta} = \mathbf{0}$:

$$\frac{\partial \hat{\psi}}{\partial \mathbf{v}}\left(\mathbf{v} = \mathbf{0}, \boldsymbol{\eta} = \mathbf{0}, \xi\right) = \mathbf{0}, \qquad \frac{\partial \hat{\psi}}{\partial \boldsymbol{\eta}}\left(\mathbf{v} = \mathbf{0}, \boldsymbol{\eta} = \mathbf{0}, \xi\right) = \mathbf{0}. \tag{6.27}$$

Such a directed curve can be held in equilibrium without the application of external forces $\rho_0\mathbf{f}$ or moments $\mathbf{m}_a$, or terminal forces and moments at its boundary.

We consider linearizations of $\rho_0\psi$ so these functions are expressed as quadratic functions of the components of $\mathbf{v}$ and $\boldsymbol{\eta}$. Thus, if the reference configuration is a natural configuration, we find the canonical form

$$\rho_0\psi = \frac{1}{2}\mathsf{g}^T\mathsf{A}\mathsf{g} + \mathsf{v}^T\mathsf{C}\mathsf{g} + \frac{1}{2}\mathsf{v}^T\mathsf{B}\mathsf{v}, \tag{6.28}$$

where

$$\mathsf{g} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}, \qquad \mathsf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{6.29}$$

and

$$\mathsf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}, \quad \mathsf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}. \tag{6.30}$$

The coefficients a_{ik}, b_{ik}, and c_{ik} may depend on ξ, the intrinsic strains v_{0_j}, and $\mathbf{R} \cdot \mathbf{E}_k$. Material symmetry conditions are often imposed to reduce the number of coefficients a_{ik}, b_{ik}, and c_{ik} from 21. These conditions will be discussed shortly and the most dramatic of them will reduce the number of coefficients from 21 to four. In a geometrically nonlinear theory with a quadratic strain energy function, the constitutive relations for **n** and **m** have the forms

$$\mathbf{n} = \sum_{i=1}^{3} n_i \mathbf{d}_i, \qquad \mathbf{m} = \sum_{i=1}^{3} m_i \mathbf{d}_i, \tag{6.31}$$

where

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \mathsf{Ag} + \mathsf{Cv}, \qquad \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \mathsf{Cg} + \mathsf{Bv}. \tag{6.32}$$

For a linearized theory, the strains η_i and ν_i are linearized about a reference configuration, and **n** and **m** are expressed in terms of the $\mathbf{D}_i$ basis, e.g., $\mathbf{n} \approx \sum_{k=1}^{3} n_k \mathbf{D}_k$.

6.4 Treatments of Material Symmetry

In contrast to three-dimensional continua, the strain energy function of an elastic rod depends not only on the constitution of the rod but also on its geometry. For Kirchhoff's rod theory, the most well-known example of the geometry dependence lies in the simplification to the strain energy function $\rho_0 \psi = \frac{EI_1}{2}\nu_1^2 + \frac{EI_2}{2}\nu_2^2 + \frac{\mathscr{D}}{2}\nu_3^2$ that occurs when the cross section of the rod is either circular or square (i.e., $I_1 = I_2$). In the more elaborate rod theory under consideration in this chapter the number of strains has risen to six and it is natural to ask if there are conditions under which the strain energy function contains coupling terms between, say, torsion ν_3 and dilation η_3? To explore this question, it is necessary to establish restrictions on the function $\rho_0 \psi$ that manifest because of material symmetry.

The notion of material symmetry employed here is broader than the one used in three-dimensional continuum mechanics because it must account not only for the constitution of the continuum composing the rod but also for the geometry of the rod. For instance, consider a rod composed of an isotropic linearly elastic material. The material symmetry of the rod will depend on the geometry of the cross section. For instance, a rod with a rectangular cross section is expected to behave differently than one with a circular cross section and this will be reflected in the strain energy function of the rod and the inertial coefficients $y^{\alpha\beta}$.

Several treatments of material symmetry for rods can be found in the literature. The first class of treatments considers the invariance of a strain energy function under specific orthogonal transformations of a reference configuration (see, e.g., Antman [12, Section 8.11], Cohen [62], Green and Naghdi [133, Section 9], Green, Naghdi, and Wenner [138, Section 8], and Lauderdale and O'Reilly [198]). The resulting set of transformations forms what is known as the material symmetry group of the rod. Two generalizations of these treatments can also be found in the literature. The first generalization includes transformations of the coordinate ξ in parallel with an orthogonal transformation. This treatment is motivated by the use of rod theories to model rod-like bodies, such as wire rope and DNA strands, that possess

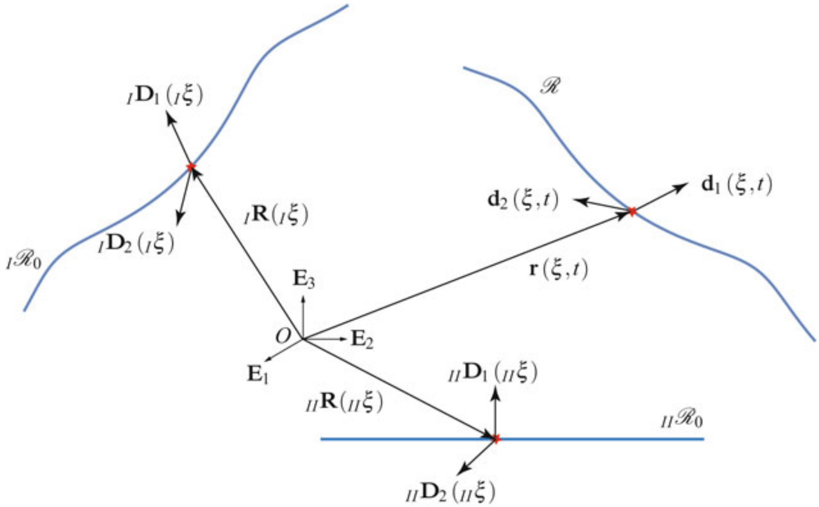

Fig. 6.3 The pair of reference configurations $_I\mathscr{R}_0$ and $_{II}\mathscr{R}_0$ of a directed curve and the present configuration $\mathscr{R}$.

helical microstructures. The seminal work on this type of treatment is a recent paper by Healey [158]. The second generalization considers orthogonal transformations which are functions of ξ and transformations of the coordinate ξ and can be found in the recent papers [199, 214]. In this chapter, we only consider the simplest treatment and refer the reader to the literature for details on the more general treatments.

6.4.1 The Case of a Constant Transformation Q

For the treatment of material symmetry that we present, we consider a pair of reference configurations of the directed curve (cf. Figure 6.3). The pair of configurations are denoted by $_I\mathscr{R}_0$ and $_{II}\mathscr{R}_0$, respectively. The material coordinates in these configurations are denoted by $_I\xi$ and $_{II}\xi$, respectively, and are presumed to be identical:

$$\xi = {}_{II}\xi = {}_I\xi. \tag{6.33}$$

We suppose that the directors and tangent vectors in $_I\mathscr{R}_0$ and $_{II}\mathscr{R}_0$ can be related by a *constant* orthogonal transformation $\mathbf{Q}$:

$$\begin{aligned}
{}_{II}\mathbf{D}_\beta\,({}_{II}\xi) &= \mathbf{Q}\,{}_I\mathbf{D}_\beta\,({}_I\xi)\,,\\
\frac{\partial_{II}\mathbf{R}}{\partial_{II}\xi}\,({}_{II}\xi) &= \mathbf{Q}\,{}_I\mathbf{R}'\,({}_I\xi)\,,\\
{}_{II}\mathbf{P}_0\,({}_{II}\xi) &= \mathbf{Q}\,{}_I\mathbf{P}_0\,({}_I\xi)\,.
\end{aligned} \tag{6.34}$$

Here, the prime denotes the partial derivative with respect to ${}_{I}\xi$. Representative examples of proper- and improper-orthogonal transformations elements of the material symmetry groups are shown in Figure 6.4.

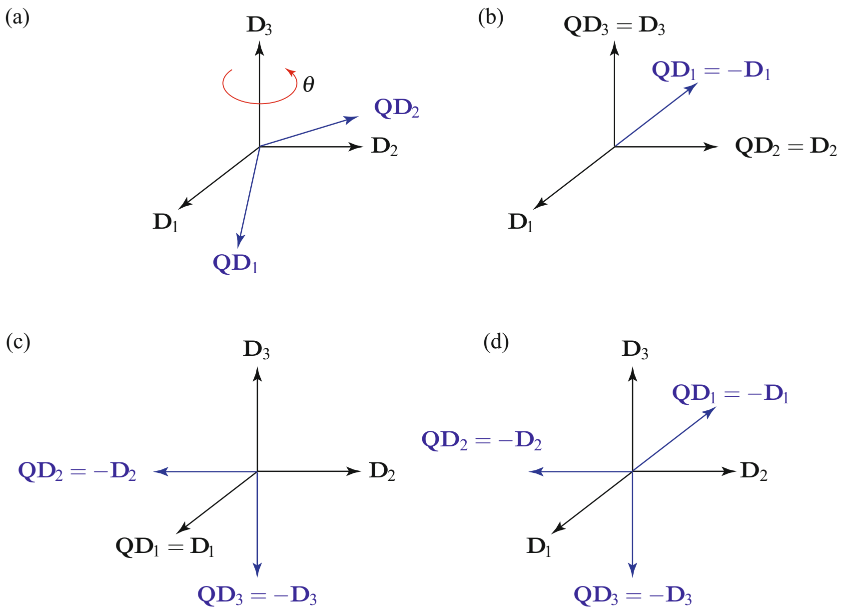

Fig. 6.4 The orthogonal transformations that are elements in the transverse isotropy material symmetry groups: (a), a rotation about $\mathbf{D}_3$: $\mathbf{Q} = \mathbf{Q}_{\mathbf{E}}(\theta, \mathbf{D}_3)$; (b), a reflection in the $\mathbf{D}_2 - \mathbf{D}_3$ plane: $\mathbf{Q} = \mathbf{Re}_{\mathbf{D}_1}$; (c), a rotation about $\mathbf{D}_1$ through an angle of 180°: $\mathbf{Q} = -\mathbf{Re}_{\mathbf{D}_1}$; and (d), an inversion: $\mathbf{Q} = -\mathbf{I}$.

It follows from the relations (6.34) that

$$ {}_{II}\mathbf{P}\left({}_{II}\xi\right) = {}_{I}\mathbf{P}\left({}_{I}\xi\right)\mathbf{Q}^T, \qquad {}_{II}\boldsymbol{\eta}\left({}_{II}\xi\right) = \mathbf{Q}\,{}_{I}\boldsymbol{\eta}\left({}_{I}\xi\right), \tag{6.35} $$

and

$$ {}_{II}\mathbf{K}\left({}_{II}\xi\right) = \mathbf{Q}\,{}_{I}\mathbf{K}\left({}_{I}\xi\right)\mathbf{Q}^T, \qquad {}_{II}\mathbf{v}\left({}_{II}\xi\right) = \mathbf{Q}^A\,{}_{I}\mathbf{v}\left({}_{I}\xi\right). \tag{6.36} $$

In these equations, $\mathbf{Q}^A = \det(\mathbf{Q})\,\mathbf{Q}$ is the adjugate of $\mathbf{Q}$. We recall that the adjugate $\mathbf{B}^A$ of a tensor $\mathbf{B}$ has the property that $\mathbf{B}^A(\mathbf{a}\times\mathbf{b}) = \mathbf{Ba}\times\mathbf{Bb}$. For an invertible tensor $\mathbf{B}$, $\mathbf{B}^A = \det(\mathbf{B})\left(\mathbf{B}^{-1}\right)^T$. Using the adjugate, it can be shown that

$$ \left(\mathbf{B}^A\mathbf{c}\right)\times\mathbf{b} = \left(\mathbf{BCB}^T\right)\mathbf{b}, \tag{6.37} $$

for all vectors $\mathbf{b}$ and skew-symmetric tensors $\mathbf{C} = -\mathbf{C}^T$ where $\mathbf{c} = \mathrm{ax}(\mathbf{C})$. This property of the adjugate is used to establish Eqn. $(6.36)_2$ from Eqn. $(6.36)_1$ and it is repeatedly used throughout Section 6.6.

For the same present configuration $(\mathbf{r}, \mathbf{d}_\alpha)$ and the *same* material point, the strain energy functions of the rod relative to the two reference configurations are, respectively,

$$\begin{aligned}\rho_0\psi &= {}_I\rho_{0I}\psi\left({}_I\boldsymbol{\eta}\left({}_I\xi\right), {}_I\mathbf{v}\left({}_I\xi\right), {}_I\xi\right)\\ &= {}_{II}\rho_{0II}\psi\left({}_{II}\boldsymbol{\eta}\left({}_{II}\xi\right), {}_{II}\mathbf{v}\left({}_{II}\xi\right), {}_{II}\xi\right).\end{aligned} \tag{6.38}$$

For notational simplicity, the possible dependency of the strain energy functions on $\mathbf{v}_0(\xi)$ and $\mathbf{R}'(\xi)$ have been lumped into the dependency on ξ. We find, with the help of the relations (6.34)–(6.36), that

$$\begin{aligned}\psi &= {}_{II}\rho_{0II}\psi\left({}_{II}\boldsymbol{\eta}\left({}_{II}\xi\right), {}_{II}\mathbf{v}\left({}_{II}\xi\right), {}_{II}\xi\right)\\ &= {}_I\rho_{0I}\psi\left(\mathbf{Q}^T{}_{II}\boldsymbol{\eta}\left({}_B\xi\right), \left(\mathbf{Q}^A\right)^{-1}{}_{II}\mathbf{v}\left({}_{II}\xi\right), {}_{II}\xi\right).\end{aligned} \tag{6.39}$$

The specific point where ${}_{II}\mathbf{v}$ and ${}_{II}\boldsymbol{\eta}$ are evaluated in ${}_I\psi$ is important to note.

We define two reference configurations of the rod to be material symmetry related if, for all motions with the same $\mathbf{v}$ and $\boldsymbol{\eta}$, and the same value of the material coordinates, ${}_I\xi = \xi$ and ${}_{II}\xi = \xi$,

$${}_I\rho_{0I}\psi\left(\boldsymbol{\eta}(\xi), \mathbf{v}(\xi), \xi\right) = {}_{II}\rho_{0II}\psi\left(\boldsymbol{\eta}(\xi), \mathbf{v}(\xi), \xi\right). \tag{6.40}$$

With the assistance of Eqn. (6.39), we can express the condition (6.40) in terms of the strain energy function ${}_I\psi$:

$$\begin{aligned}\rho_0\psi\left(\boldsymbol{\eta}(\xi), \mathbf{v}(\xi), \xi\right) &= \rho_0\psi\left(\mathbf{Q}^T\boldsymbol{\eta}(\xi), \det(\mathbf{Q})\mathbf{Q}^T\mathbf{v}(\xi), \xi\right),\\ &= \rho_0\tilde{\psi}\left(\mathbf{Q}^T\boldsymbol{\eta}(\xi), \mathbf{Q}^T\mathbf{K}(\xi)\mathbf{Q}, \xi\right).\end{aligned} \tag{6.41}$$

For convenience, we have dropped the left-subscript I and made some other obvious simplifications in notation. The function $\tilde{\psi}$ has been introduced for future convenience so we can easily exploit results from the literature.

In part because the product of two orthogonal transformations is an orthogonal transformation and the identity $\mathbf{I}$ is an orthogonal transformation, the group structure associated with the symmetry condition (6.41) can readily be developed. Elements $\mathfrak{g}$ of the group $\mathfrak{G}$ will be orthogonal transformations, $\mathfrak{g} = (\mathbf{Q})$. The group operation $\mathfrak{g}_2 \circ \mathfrak{g}_1 = (\mathbf{Q}_2\mathbf{Q}_1)$ yields another element $\mathbf{Q}_2\mathbf{Q}_1$ of the group. It is easy to see that the group operation is associative: $\mathfrak{g}_3 \circ (\mathfrak{g}_2 \circ \mathfrak{g}_1) = (\mathfrak{g}_3 \circ \mathfrak{g}_2) \circ \mathfrak{g}_1 = (\mathbf{Q}_3\mathbf{Q}_2\mathbf{Q}_1)$. In addition, the identity and inverse elements are

$$\mathfrak{i} = (\mathbf{I}), \qquad \mathfrak{g}^{-1} = \left(\mathbf{Q}^T\right). \tag{6.42}$$

The group $\mathfrak{G}$ is known as the material symmetry group of the rod.

Table 6.1 Invariants in irreducible function bases for the five different transverse isotropies. The function bases are taken from Zheng [371, Tables 12 and 14] for scalar functions of a vector $\boldsymbol{\eta}$ and a skew-symmetric tensor $\mathbf{K} = \text{skew}(\mathbf{v})$.

Material Symmetry Group $\mathfrak{G}$	Elements $\mathfrak{g}$ of $\mathfrak{G}$	Irreducible function bases
$\mathscr{C}_\infty$	$\{\mathbf{Q_E}(\theta,\mathbf{D}_3), \theta \in [0,2\pi]\}$	$\mathbf{v}\cdot\mathbf{v}$, $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$, $\boldsymbol{\eta}\cdot\mathbf{D}_3$, $\mathbf{v}\cdot\mathbf{D}_3$, $\mathbf{v}\cdot\boldsymbol{\eta}$, $\mathbf{D}_3\cdot(\boldsymbol{\eta}\times\mathbf{v})$.
$\mathscr{C}_{\infty h}$	$\{-\mathbf{I}, \mathbf{Q_E}(\theta,\mathbf{D}_3), \theta \in [0,2\pi]\}$	$\mathbf{v}\cdot\mathbf{v}$, $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$, $(\boldsymbol{\eta}\cdot\mathbf{D}_3)^2$, $\mathbf{v}\cdot\mathbf{D}_3$, $(\mathbf{v}\times\boldsymbol{\eta})\cdot(\mathbf{v}\times\boldsymbol{\eta})$, $\mathbf{D}_3\cdot(\boldsymbol{\eta}\times(\mathbf{v}\times\boldsymbol{\eta}))$, $\mathbf{D}_3\cdot(\boldsymbol{\eta}\times(\mathbf{v}\times(\mathbf{v}\times\boldsymbol{\eta})))$, $(\mathbf{D}_3\cdot\boldsymbol{\eta})(\mathbf{D}_3\cdot(\boldsymbol{\eta}\times\mathbf{v}))$.
$\mathscr{C}_{\infty v}$	$\{\mathbf{Re}_{\mathbf{D}_1}, \mathbf{Q_E}(\theta,\mathbf{D}_3), \theta \in [0,2\pi]\}$	$\mathbf{v}\cdot\mathbf{v}$, $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$, $\boldsymbol{\eta}\cdot\mathbf{D}_3$, $(\mathbf{v}\cdot\mathbf{D}_3)^2$, $\mathbf{D}_3\cdot(\boldsymbol{\eta}\times\mathbf{v})$, $\mathbf{D}_3\cdot(\mathbf{v}\times(\mathbf{v}\times\boldsymbol{\eta}))$.
$\mathscr{D}_{\infty h}$	$\{-\mathbf{Re}_{\mathbf{D}_1}, \mathbf{Q_E}(\theta,\mathbf{D}_3), \theta \in [0,2\pi]\}$	$\mathbf{v}\cdot\mathbf{v}$, $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$, $(\boldsymbol{\eta}\cdot\mathbf{D}_3)^2$, $(\mathbf{v}\cdot\mathbf{D}_3)^2$, $(\mathbf{v}\times\boldsymbol{\eta})\cdot(\mathbf{v}\times\boldsymbol{\eta})$, $(\mathbf{D}_3\cdot\boldsymbol{\eta})(\mathbf{D}_3\cdot(\mathbf{v}\times\boldsymbol{\eta}))$, $(\mathbf{D}_3\cdot(\mathbf{v}\times\boldsymbol{\eta}))(\mathbf{D}_3\cdot(\mathbf{v}\times(\mathbf{v}\times\boldsymbol{\eta})))$.
$\mathscr{D}_\infty$	$\{-\mathbf{I}, \mathbf{Re}_{\mathbf{D}_1}, \mathbf{Q_E}(\theta,\mathbf{D}_3), \theta \in [0,2\pi]\}$	$\mathbf{v}\cdot\mathbf{v}$, $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$, $(\boldsymbol{\eta}\cdot\mathbf{D}_3)^2$, $(\mathbf{v}\cdot\mathbf{D}_3)^2$, $\mathbf{v}\cdot\boldsymbol{\eta}$, $(\mathbf{D}_3\cdot\boldsymbol{\eta})(\mathbf{D}_3\cdot\mathbf{v})$, $(\mathbf{D}_3\cdot(\boldsymbol{\eta}\times\mathbf{v}))(\mathbf{D}_3\cdot\mathbf{v})$, $(\mathbf{D}_3\cdot\boldsymbol{\eta})(\mathbf{D}_3\cdot(\mathbf{v}\times\boldsymbol{\eta}))$.

6.4.2 Transverse Isotropy and Transverse Hemitropy

For many rod-like bodies modeled using rod theory, the structure of the continuum has a material symmetry and this symmetry is reflected in the constitutive relations for the rod. One prominent example arises when the body has a unique preferred direction and the material is said to be transversely isotropic. If this body is modeled as a rod with a circular cross section and the preferred direction is aligned with the axis of the rod, then the rod inherits this material symmetry. An example of this situation lies in modeling an insulated electrical cord using a rod theory. A second example arises when the elastic rod-like body is isotropic. If the cross sections are circular, then the resulting rod is said to be transversely isotropic or, more commonly, isotropic.

For a three-dimensional continua there are five distinct types of transverse isotropy and, depending on the rod's strain energy function, some of them may be indistinguishable.[1] The five groups, along with their elements, are as follows:

$$\begin{aligned}
\text{Rotational symmetry} : \mathscr{C}_\infty &= \{\mathbf{Q_E}(\theta, \mathbf{D}_3), \theta \in [0, 2\pi]\}, \\
\text{Rotational symmetry} : \mathscr{C}_{\infty h} &= \mathscr{C}_\infty \cup \{-\mathbf{I}\}, \\
\text{Rotational symmetry} : \mathscr{C}_{\infty v} &= \mathscr{C}_\infty \cup \{\mathbf{Re}_{\mathbf{D}_1}\}, \\
\text{Transverse hemitropy} : \mathscr{D}_{\infty h} &= \mathscr{C}_\infty \cup \{-\mathbf{Re}_{\mathbf{D}_1}\}, \\
\text{Transverse isotropy} : \mathscr{D}_\infty &= \mathscr{C}_\infty \cup \{-\mathbf{I}, \mathbf{Re}_{\mathbf{D}_1}\}.
\end{aligned}$$

In these groups,

$$\mathbf{Re}_{\mathbf{D}_1} = \mathbf{I} - 2\mathbf{D}_1 \otimes \mathbf{D}_1 \tag{6.43}$$

is a reflection about the plane perpendicular to $\mathbf{D}_1$ and $\mathbf{Q_E}(\theta, \mathbf{D}_3)$ is a rotation about the axis $\mathbf{D}_3$ through an angle θ (cf. Figure 6.4). We also note that

$$-\mathbf{Re}_{\mathbf{D}_1} = 2\mathbf{D}_1 \otimes \mathbf{D}_1 - \mathbf{I} = \mathbf{Q_E}(\pi, \mathbf{D}_1). \tag{6.44}$$

For the rod-like body, we assume that the preferred direction is chosen to coincide with the axis $\mathbf{D}_3$ in the reference configuration $\mathscr{R}_0$. In addition, we assume that this configuration is straight and natural with $\mathbf{v}_0 = \mathbf{0}$. Motivated by the correspondence between the rod theory and the three-dimensional theory of a continuum, we adopt the definitions above for the rod theory. We next seek the most general forms of the strain energy function for the rod which has one of the aforementioned five material symmetry groups.

The most general form of a function compatible with the material symmetry condition (6.41) is determined by an irreducible set of functions. The members of the irreducible set of functions are invariant under the elements of the rod's material symmetry group. For example, $\mathbf{v} \cdot \mathbf{v} = (\mathbf{Qv} \cdot \mathbf{Qv})$ and so the function $\mathbf{v} \cdot \mathbf{v}$ is invariant under orthogonal transformations while $\mathbf{v} \cdot \mathbf{E}_3$ (which transforms to $\mathbf{Qv} \cdot \mathbf{E}_3$) does not possess this invariance. Following seminal work by Ronald S. Rivlin (1905–1995) and others, Zheng et al. (see [369–371] and references therein) have compiled the smallest (irreducible) sets of invariant functions for a wide range of strain energy functions. Their results for functions of a vector and a skew-symmetric tensor are presented in Table 6.1. For the five material symmetry groups of the rod, we use these results to establish the simplest functional forms of the strain energy function $\rho_0 \psi$ which satisfy the material symmetry condition (6.41) for each of the groups. Specifically, using the results from [371, Tables 12 and 14] which are summarized in Table 6.1, we find that

$$\rho_0 \psi = F(A, \mathbf{v}_0(\xi), \xi), \tag{6.45}$$

[1] By way of additional background, we also note that if a function is invariant only under proper-orthogonal transformations (i.e., rotations) then it is said to be hemitropic. The adjective isotropic pertains to the case where the function is invariant under orthogonal transformations.

where the arguments A can be read from Table 6.1. The linearized (quadratic) strain energy function can also be readily determined by restricting attention to quadratic elements of the irreducible function basis.

For instance, consider the transverse isotropy $\mathscr{D}_{\infty h}$. As we shall see, this type of symmetry is synonymous with the notion of an isotropic rod. We can infer from Table 6.1 that the most general functional form of $\rho_0\psi$ that is compatible with the material symmetry condition for all of the elements of $\mathscr{D}_{\infty h}$ is

$$\rho_0\psi = F\left(\nu_1^2+\nu_2^2,\nu_3^2,\eta_1^2+\eta_2^2,\eta_3^2,i_1,i_2,i_3,\mathbf{v}_0\left(\xi\right),\xi\right), \tag{6.46}$$

where the quartic, cubic, and quintic terms are

$$\begin{aligned} i_1 &= (\boldsymbol{\nu}\times\boldsymbol{\eta})\cdot(\boldsymbol{\nu}\times\boldsymbol{\eta}), \\ i_2 &= (\mathbf{D}_3\cdot\boldsymbol{\eta})\left(\mathbf{D}_3\cdot(\boldsymbol{\nu}\times\boldsymbol{\eta})\right), \\ i_3 &= \left(\mathbf{D}_3\cdot(\boldsymbol{\nu}\times\boldsymbol{\eta})\right)\left(\mathbf{D}_3\cdot(\boldsymbol{\nu}\times(\boldsymbol{\nu}\times\boldsymbol{\eta}))\right). \end{aligned} \tag{6.47}$$

For the quadratic strain energy function (6.28) these restrictions imply that the most general quadratic strain energy function that is compatible with the transverse isotropy material symmetry group $\mathscr{D}_{\infty h}$ is

$$\rho_0\psi = \frac{a_{11}}{2}\left(\eta_1^2+\eta_2^2\right)+\frac{b_{11}}{2}\left(\nu_1^2+\nu_2^2\right)+\frac{a_{33}}{2}\eta_3^2+\frac{b_{33}}{2}\nu_3^2. \tag{6.48}$$

Whence, the number of coefficients has been reduced from 21 to four. We leave it as an exercise for the reader to show that the quadratic strain energy functions for the material symmetry groups associated with $\mathscr{C}_{\infty h}$ and $\mathscr{D}_{\infty h}$ are identical to the function (6.48).

A rod whose material symmetry group is $\mathscr{C}_{\infty}$ is aptly termed transversely hemitropic by Healey [158]. For such a rod, we can infer the most general form of the strain energy function that is compatible with the material symmetry group using Table 6.1:

$$\rho_0\psi = F\left(\nu_1^2+\nu_2^2,\nu_3,\eta_1^2+\eta_2^2,\eta_3,\eta_1\nu_1+\eta_2\nu_2,(\eta_1\nu_2-\nu_1\eta_2)\right). \tag{6.49}$$

The quadratic strain energy function for such a rod is readily inferred:

$$\begin{aligned} \rho_0\psi = {} & \frac{a_{11}}{2}\left(\eta_1^2+\eta_2^2\right)+\frac{b_{11}}{2}\left(\nu_1^2+\nu_2^2\right)+\frac{a_{33}}{2}\eta_3^2+\frac{b_{33}}{2}\nu_3^2 \\ & +\underbrace{c_{11}\left(\eta_1\nu_1+\eta_2\nu_2\right)+c_{33}\eta_3\nu_3+c_{12}\left(\eta_1\nu_2-\nu_1\eta_2\right)}. \end{aligned} \tag{6.50}$$

The underbraced terms in this equation are the differences between the quadratic strain energy functions for the isotropic and hemitropic cases. We also note that a nonvanishing c_{33} in this expression implies that there will be coupling between extension and torsion in a straight rod. As discussed in Healey [158], the strain energy function (6.50) with $c_{12}=0$ is suitable for modeling some rods whose microstructure has a helical symmetry.

6.4.3 Application to Kirchhoff's Rod Theory

The strain energy function for a homogenous rod modeled using Kirchhoff's rod theory with $\mathbf{v}_0 = \mathbf{0}$ has the functional form

$$\rho_0 \psi = \rho_0 \psi(\mathbf{v}). \tag{6.51}$$

It is straightforward to develop a material symmetry condition akin to the condition (6.41) for such a rod. For the material symmetry groups $\mathscr{C}_\infty$ and $\mathscr{C}_{\infty h}$, we can use the previous developments to show that the strain energy function has the invariant form

$$\rho_0 \psi = F\left(v_1^2 + v_2^2, v_3\right). \tag{6.52}$$

By way of contrast, for the material symmetry groups $\mathscr{D}_\infty$, $\mathscr{D}_{\infty h}$, and $\mathscr{C}_{\infty v}$, the invariant form of the strain energy function is

$$\rho_0 \psi = F\left(v_1^2 + v_2^2, v_3^2\right). \tag{6.53}$$

Thus, these three material symmetry groups are commonly associated with the notion of an isotropic Kirchhoff rod. Furthermore, for all five material symmetry groups, the quadratic form of the strain energy function is the familiar

$$\rho_0 \psi = \frac{b_{11}}{2}\left(v_1^2 + v_2^2\right) + \frac{b_{33}}{2} v_3^2. \tag{6.54}$$

By comparing solutions of the Kirchhoff rod theory to the solutions to the corresponding problems in three-dimensional linear elasticity, the identifications $b_{11} = EI$ and $b_{33} = \mathscr{D}$ can be made.

6.5 Application to Torsion and Extension

In many recent experiments on segments of double-stranded DNA, one end of the single molecule of DNA is attached to a fixed surface, while the other end is subject to a force $F_\ell \mathbf{E}_3$ and a moment $M_\ell \mathbf{E}_3$. These effects are simulated using optical tweezers, hydrodynamic drag, or magnetic fields. A schematic of one such experiment is shown in Figure 6.5(a) and reviews of the experiments are presented in [40, 120]. A coupling between stretching and twisting is observed in many of these experiments. In the sequel, we explore how this effect can be explained by modeling the strand of DNA as an initially straight rod with a helical microstructure. As emphasized in Đuričković et al. [89], one can also model the DNA strand as a helical spring and observe the same couplings reported in the biophysics literature.

Referring to Figures 5.7(b) and 6.5(b), we use the material curve to model the duplex (molecular) axis and choose the directors to follow the phosphate backbone strands of DNA:

$$\mathbf{D}_1 = \cos(\phi_h)\mathbf{E}_1 + \sin(\phi_h)\mathbf{E}_2, \qquad \mathbf{D}_2 = \cos(\phi_h)\mathbf{E}_2 - \sin(\phi_h)\mathbf{E}_1, \qquad \mathbf{D}_3 = \mathbf{E}_3, \tag{6.55}$$

where the angle ϕ_h is a function of the coordinate ξ. Hence,

$$\mathbf{v}_0 = v_{0_3}\mathbf{E}_3 = \frac{\partial \phi_h}{\partial \xi}\mathbf{E}_3, \tag{6.56}$$

and $\mathbf{v}_0$ is constant.

We will model the single molecule of DNA using a rod whose strain energy function $\rho_0\psi$ is

$$2\rho_0\psi = b_{11}v_1^2 + b_{22}v_2^2 + b_{33}v_3^2 + a_{11}\left(\eta_1^2 + \eta_2^2\right) + a_{33}\eta_3^2 + 2c_{33}\eta_3 v_3. \tag{6.57}$$

Here c_{33} is a coefficient responsible for the coupling of torsion and extension. The constants in this strain energy function must be identified by experiments or comparison to models based on worm-like chains used in the biophysics literature. By way of illustration, the radius of the rod will be approximately 10Å and $v_{0_3} \approx 0.185$ radians/Å [228]. Using data from [35], the bending moduli $b_{11} = b_{22} \approx 2\times 10^{-28}$ Nm^2 and the torsional modulus $b_{33} \approx 2-4\times 10^{-28}$ Nm^2. Gore et al. [120] have measured $c_{33} \approx -90\times 10^{-21}$ Nm. With the help of the identities (6.26), we conclude that

$$\begin{aligned}\mathbf{n} &= a_{11}\left(\eta_1\mathbf{d}_1 + \eta_2\mathbf{d}_2\right) + \left(a_{33}\eta_3 + c_{33}v_3\right)\mathbf{d}_3,\\ \mathbf{m} &= b_{11}v_1\mathbf{d}_1 + b_{22}v_2\mathbf{d}_2 + \left(b_{33}v_3 + c_{33}\eta_3\right)\mathbf{d}_3.\end{aligned} \tag{6.58}$$

From this pair of constitutive relations observe that, if $\mathbf{v} = \mathbf{0}$, a nonzero η_3 will induce a moment in the rod and, if $\boldsymbol{\eta} = \mathbf{0}$, a nonzero torsion v_3 will induce a force in the rod.

We suppose that the DNA strands are subject to negligible external body forces and negligible surface tractions on the lateral surfaces, $\rho_0\mathbf{f} = \mathbf{0}$ and $\mathbf{l}^\alpha = \mathbf{0}$, and we restrict attention to static solutions. Starting from the balance laws,

$$\frac{\partial \mathbf{n}}{\partial \xi} = \mathbf{0}, \qquad \frac{\partial \mathbf{m}}{\partial \xi} + \frac{\partial \mathbf{r}}{\partial \xi}\times\mathbf{n} = \mathbf{0}, \tag{6.59}$$

using the appropriate constitutive relations for $\mathbf{n}$ and $\mathbf{m}$, and taking the components of the balance laws (6.59) relative to the basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_1\times\mathbf{d}_2\}$, we can establish the ordinary differential equations governing the shape of the material curve and the behavior of $\mathbf{d}_\alpha$.

The boundary conditions for the problem of interest are

$$\begin{aligned}&\mathbf{r}(0,t) = \mathbf{0}, \qquad \mathbf{d}_1(0,t) = \mathbf{E}_1, \qquad \mathbf{d}_2(0,t) = \mathbf{E}_2,\\ &\mathbf{n}(\ell,t) = F_\ell\mathbf{E}_3, \qquad \mathbf{m}(\ell,t) = M_\ell\mathbf{E}_3,\end{aligned} \tag{6.60}$$

where F_ℓ and M_ℓ are constants.

From the balance laws, we find that $\mathbf{n}$ is constant. Assuming that the centerline remains straight, $\mathbf{r}' \parallel \mathbf{E}_3$, then we also find that $\mathbf{m}$ is constant. Consequently,

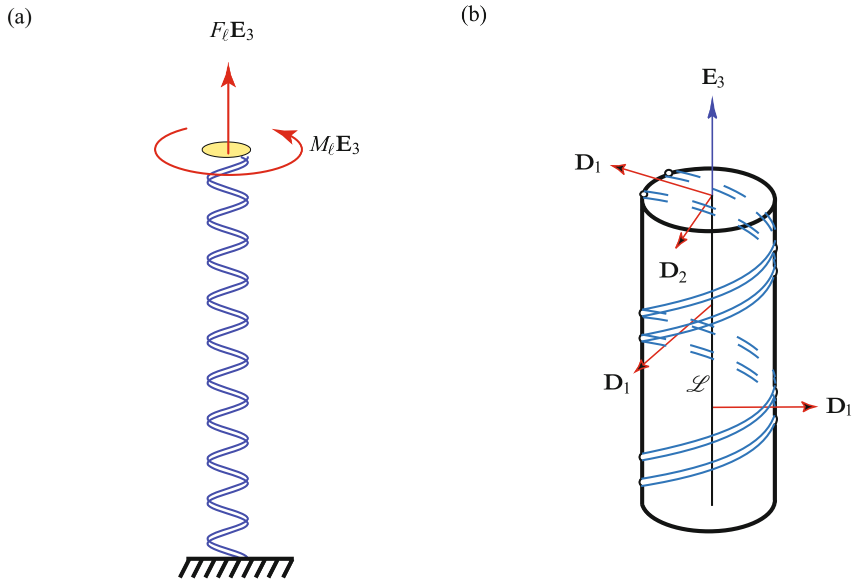

Fig. 6.5 (a) Schematic of a strand of DNA being stretched and torqued. One end of the strand is fixed while the other end is affixed to a bead which is used to transmit the applied torque and force. In experiments on DNA in the literature, the force F_ℓ ranges from 0 to 70 picoNewtons. (b) A portion of the reference configuration $\mathscr{R}_0$ for a rod model of the DNA strand.

$$\begin{bmatrix} F_\ell \\ M_\ell \end{bmatrix} = \begin{bmatrix} a_{33} & c_{33} \\ c_{33} & b_{33} \end{bmatrix} \begin{bmatrix} \eta_3 \\ \nu_3 \end{bmatrix}. \tag{6.61}$$

For the simple deformation being considered here, we can express η_3 and ν_3 in terms of the displacement $\Delta z \approx \mathbf{r}(\ell)\cdot\mathbf{E}_3 - \mathbf{r}(0)\cdot\mathbf{E}_3 - \ell$ and the change in the angle of twist at the top of the structure: $\Delta\theta \approx \nu_3 \ell$. In these expressions,

$$\ell = \mathbf{R}(\ell)\cdot\mathbf{E}_3 - \mathbf{R}(0)\cdot\mathbf{E}_3 \tag{6.62}$$

is the initial height of the helical structure. Inverting the linear equations (6.61), we find that

$$\begin{aligned} \frac{\Delta z}{\ell} &\approx \eta_3 = \frac{1}{a_{33}b_{33} - c_{33}^2}\left(b_{33}F_\ell - c_{33}M_\ell\right), \\ \frac{\Delta\theta}{\ell} &\approx \nu_3 = \frac{1}{a_{33}b_{33} - c_{33}^2}\left(a_{33}M_\ell - c_{33}F_\ell\right). \end{aligned} \tag{6.63}$$

After assuming that the strain energy function is positive definite,[2] we observe that if $c_{33} > 0\,(< 0)$ then application of a clockwise moment alone can stretch (compress)

[2] Necessary conditions for the positive definiteness of the strain energy function (6.57) include $a_{33} > 0$, $b_{33} > 0$, and $a_{33}b_{33} - c_{33}^2 > 0$.

the rod and application of a tensile force alone can cause it to rotate in the clockwise (counterclockwise) direction. The linear relations (6.63) can be used to identify the parameters a_{33}, b_{33}, and c_{33} from tests where F_ℓ and M_ℓ are controlled and $\frac{\Delta z}{\ell}$ and $\frac{\Delta\theta}{\ell}$ are measured.

We take this opportunity to note that relations which are similar to (6.61) appear in studies on the extension and twist of wire ropes [73, Chapter 4]. Indeed, we encountered related work earlier in Chapter 5 when we examined the coupling between twist and extension in a helical spring (cf. Eqn. (5.228)). If we now identify the stiffnesses for both models, we will find the identifications

$$\begin{aligned} a_{33} &= \frac{\mathscr{D}\kappa_0}{R} + \frac{EI}{R}\frac{\tau_0^2}{\kappa_0}, \\ b_{33} &= \mathscr{D}R\tau_0^2 + EIR\kappa_0, \\ c_{33} &= (\mathscr{D} - EI)\,\tau_0. \end{aligned} \tag{6.64}$$

We note that the coupling coefficient $c_{33} < 0$ for right-handed helices which agrees with the experimental results of Gore et al. [120] who examined the twist-stretch coupling of strands of DNA.

6.6 Ericksen's Uniform States

In a remarkable paper, Ericksen [97] proposed a static solution for a wide range of rod theories where the centerline of the rod describes a helical space curve, a straight line, or a circular arc, and the directors form constant angles with the normal and binormal vectors to this curve (cf. Figure 6.6). We now explore Ericksen's so-called uniform states for initially straight homogeneous rods where $\mathbf{D}_i = \mathbf{E}_i$. The deformed shape of the rod in a uniform state is specified by a rotation tensor $\mathbf{Q}$ as follows:

$$\begin{aligned} \mathbf{r}'(\xi) &= \mathbf{Q}(\xi)\,\mathbf{r}'(0), \\ \mathbf{d}_\beta(\xi) &= \mathbf{Q}(\xi)\,\mathbf{d}_\beta(0), \\ \mathbf{d}'_\alpha(\xi) &= \mathbf{Q}(\xi)\,\mathbf{d}'_\alpha(0). \end{aligned} \tag{6.65}$$

The axial vector associated with the skew-symmetric tensor $\mathbf{Q}'\mathbf{Q}^T$ is denoted by $\mathbf{v}_\mathbf{Q}$ and we shall find that it is a constant throughout the length of the rod.[3] We also note that Ericksen's analysis provides a transparent proof that wrenches are needed to maintain the deformed rod and the axis of the wrench coincides with the axis of the helical space curve. His analysis can be applied to rods modeled using Kirchhoff's rod theory and, to this end, we invite the reader to revisit Sections 5.14 and 5.16.2 of the previous chapter.

[3] Our use of the symbol $\mathbf{Q}$ to denote the rotation tensor associated with the uniform state should not be confused with the use of the same symbol to denote an orthogonal transformation in an earlier section of this chapter.

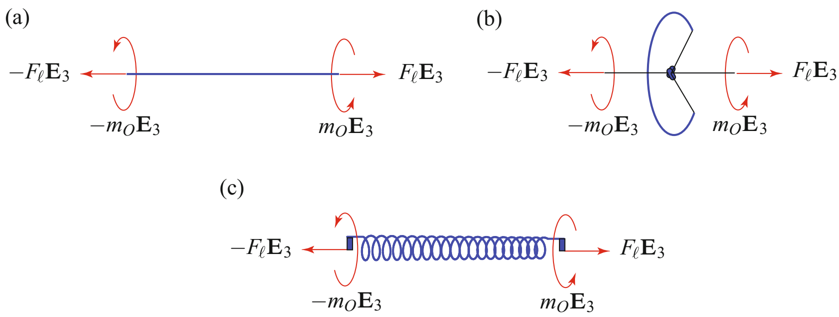

Fig. 6.6 Ericksen's uniform states of a rod where the centerline $\mathbf{r}(\xi)$ takes the form of (a), a straight line, (b), a circular arc, and (c), a circular helix. The deformed state of the rod is maintained by a wrench in all three cases. Rigid appendages are added to the ends of the rod to enable the wrench loading.

6.6.1 Kinematical Considerations

Taking the derivative of Eqn. (6.65)$_2$ and comparing the result to Eqn. (6.65)$_3$, we are lead to the compatibility condition

$$\mathbf{Q}'(\xi)\mathbf{Q}^T(\xi) = \mathbf{Q}(\xi)\left(\mathbf{Q}'(0)\mathbf{Q}^T(0)\right)\mathbf{Q}^T(\xi). \tag{6.66}$$

Whence,

$$\mathbf{v_Q}(\xi) = \mathbf{Q}(\xi)\mathbf{v_Q}(0). \tag{6.67}$$

It follows from this relation that the vector $\mathbf{v_Q}(\xi)$ is constant throughout the rod:

$$\mathbf{v}'_\mathbf{Q}(\xi) = \mathbf{Q}'(\xi)\mathbf{v_Q}(0) = \mathbf{Q}'(\xi)\mathbf{Q}^T(\xi)\mathbf{v_Q}(\xi) = \mathbf{v_Q}(\xi)\times\mathbf{v_Q}(\xi) = \mathbf{0}. \tag{6.68}$$

That is, the rotation $\mathbf{Q}$ has a constant angular rate. Because $\mathbf{Q}(\xi = 0) = \mathbf{I}$, the axis of rotation of $\mathbf{Q}$ is parallel to $\mathbf{v_Q}(\xi)$.[4] We choose the axis of rotation to be $\mathbf{E}_3$ without loss in generality. We also define the scalar v_Q:

$$\mathbf{v_Q}(\xi) = v_Q\mathbf{E}_3. \tag{6.69}$$

The identity $\mathbf{QE}_3 = \mathbf{E}_3$ will be exploited numerous times in the sequel.

Differentiating the equation $\mathbf{r}'(\xi) = \mathbf{Q}(\xi)\mathbf{r}'(0)$, we find that

$$\mathbf{r}''(\xi) = \mathbf{v_Q}(\xi)\times\mathbf{r}'(\xi). \tag{6.70}$$

Integrating this equation, we find the results

$$\mathbf{r}'(\xi) = \mathbf{v_Q}(\xi)\times\mathbf{r}(\xi) + \mathbf{c}_h, \qquad \mathbf{r}(\xi) = \mathbf{Q}(\xi)\mathbf{r}(0) + \xi\mathbf{c}_h, \tag{6.71}$$

[4] A proof of this result can be found in [267].

where $\mathbf{c}_h$ is a constant. It is convenient to choose the origin so that

$$\mathbf{c}_h = \Upsilon \nu_Q \mathbf{E}_3, \tag{6.72}$$

where Υ is a constant. Thus, the centerline of the rod has the shape of a circular helix, or, if $\Upsilon = 0$, a circle, or, if $\nu_Q = 0$, a straight line. The centerline is in a state of uniform stretch:

$$\mu = \left\| \mathbf{r}'(\xi) \right\| = \left\| \mathbf{r}'(0) \right\|. \tag{6.73}$$

Consequently, the arc-length parameter s for the centerline and the material coordinate ξ are not identical:

$$\mu = \frac{\partial s}{\partial \xi}. \tag{6.74}$$

With the help of Eqn. (1.33), we can identify the parameters of the circular helix:

$$R = \sqrt{(\mathbf{r}(0)\cdot\mathbf{E}_1)^2 + (\mathbf{r}(0)\cdot\mathbf{E}_2)^2}, \qquad \tau = \frac{\Upsilon}{R^2+\Upsilon^2}, \qquad \kappa = \frac{R}{R^2+\Upsilon^2}. \tag{6.75}$$

When $\Upsilon = 0$, the centerline is a circular arc of radius R.

With regards to the rotation tensor $\mathbf{P}(\xi)$, we find that

$$\mathbf{Q}(\xi) = \mathbf{P}(\xi)\mathbf{P}^T(0), \qquad \mathbf{P}(\xi) = \mathbf{Q}(\xi)\mathbf{P}(0). \tag{6.76}$$

Consequently,

$$\begin{aligned} \mathbf{v}(\xi) &= \mathrm{ax}\left(\mathbf{P}^T(\xi)\mathbf{P}'(\xi)\right) \\ &= \mathrm{ax}\left(\mathbf{P}^T(0)\mathbf{Q}^T(\xi)\mathbf{Q}'(\xi)\mathbf{P}(0)\right) \\ &= \mathbf{P}^T(0)\mathbf{Q}^T(\xi)\mathbf{v}_{\mathbf{Q}}(\xi) \\ &= \mathbf{P}^T(0)\mathbf{v}_{\mathbf{Q}}(0). \end{aligned} \tag{6.77}$$

For the uniform states, we can quickly find that the strain measure $\boldsymbol{\eta}$ is also a constant throughout the rod:

$$\begin{aligned} \boldsymbol{\eta}(\xi) &= \mathbf{P}^T(\xi)\mathbf{r}'(\xi) - \mathbf{E}_3 \\ &= \mathbf{P}^T(0)\mathbf{r}'(0) - \mathbf{E}_3 \\ &= \boldsymbol{\eta}(0). \end{aligned} \tag{6.78}$$

The constancy of the strains $\mathbf{v}$ and $\boldsymbol{\eta}$ throughout the rod is the motivation for Ericksen's choice of the term "uniform state." Because the rotation tensor for the tangent vector is identical to those for the directors, the deformed state of the rod will be twistless.

6.6.2 Forces and Moments

With the help of the constitutive relations (6.26) for $\mathbf{n}$ and $\mathbf{m}$ and the assumptions (6.65), it is straightforward to see that

$$\mathbf{n}(\xi) = \mathbf{Q}(\xi)\mathbf{n}(0), \qquad \mathbf{m}(\xi) = \mathbf{Q}(\xi)\mathbf{m}(0). \tag{6.79}$$

We assume the rod is maintained in equilibrium solely by terminal loadings:

$$\mathbf{n}\left(0^{+}\right) = -\mathbf{F}_{0}, \quad \mathbf{n}\left(\ell^{-}\right) = \mathbf{F}_{\ell}, \quad \mathbf{m}\left(0^{+}\right) = -\mathbf{M}_{0}, \quad \mathbf{m}\left(\ell^{-}\right) = \mathbf{M}_{\ell}, \tag{6.80}$$

where we will drop the $+$ ornamenting 0 and $-$ ornamenting ℓ in the sequel. The equilibrium equations yield

$$\mathbf{n}^{'} = \mathbf{0}, \qquad (\mathbf{m} + \mathbf{r} \times \mathbf{n})^{'} = \mathbf{0}. \tag{6.81}$$

It follows that $\mathbf{F}_0$ and $\mathbf{F}_\ell$ are equal and opposite and both are parallel to the axis of rotation of $\mathbf{Q}$:

$$\mathbf{F}_{\ell} = -\mathbf{F}_{0} = F_{\ell}\mathbf{E}_{3}. \tag{6.82}$$

The second conservation implies that

$$\mathbf{m}(\xi) + \mathbf{r}(\xi) \times \mathbf{n}(\xi) = \mathbf{m}_{O}, \tag{6.83}$$

where $\mathbf{m}_O$ is a constant. Substituting for $\mathbf{r}(\xi)$ and using the fact that $\mathbf{n} \parallel \mathbf{E}_3$, we find that

$$\mathbf{m}(\xi) + \mathbf{r}(\xi) \times \mathbf{n}(\xi) = \mathbf{Q}(\xi)\left(\mathbf{m}(0) + \mathbf{r}(0) \times \mathbf{n}(0)\right). \tag{6.84}$$

This implies that $\mathbf{Q}(\xi)\mathbf{m}_O = \mathbf{m}_O$. Consequently, either $\mathbf{m}_O = \mathbf{0}$ or $\mathbf{m}_O = m_O\mathbf{E}_3$ where m_O is a scalar. We can now conclude that the terminal moments on the rod are

$$\mathbf{M}_{0} = -m_{O}\mathbf{E}_{3} + \mathbf{r}(0) \times F_{\ell}\mathbf{E}_{3}, \qquad \mathbf{M}_{\ell} = m_{O}\mathbf{E}_{3} - \mathbf{r}(\ell) \times F_{\ell}\mathbf{E}_{3}. \tag{6.85}$$

The fact that $\mathbf{F}_\ell$ and the moment relative to O, $\mathbf{m}_O$, are parallel to $\mathbf{E}_3$ constitutes a type of loading known in the literature as a wrench (cf. Figure 6.6).

Thus Ericksen's ingenious perspective demonstrates the ubiquitous nature of helical forms and shows how they are supported by a wrench in the rod theory of interest here. An alternative derivation of this result can be found in Antman [12, Section 9.2]. It remains to solve for the deformed shape of the rod and we refer the interested reader to [12, Section 9.2] for a discussion of the solution procedure.

6.7 Closing Comments

The rod theory we have just discussed is capable of accommodating extensibility, transverse shear, torsion, and flexure. It is also the first theory we have considered since the elastic string where the contact forces and moments are completely prescribed by constitutive relations. Consequently, the governing equations lead to partial differential equations for the components of $\mathbf{r}$ and $\mathbf{d}_\alpha$. For most problems, these governing equations are provided in a noncontroversial manner by the balance of linear momentum and balance of angular momentum. If we relax the assumption that the directors are unit vectors and allow their magnitudes to vary, then we can in principle capture the contraction and expansion of the cross sections of the rod. However, the source of the extra equations needed to describe the evolution of these magnitudes is not obvious. Before turning to an exploration of one solution to this problem, we note that many of the problems analyzed using Kirchhoff's rod theory remain to be explored using the more sophisticated theory discussed in the present chapter. Some progress towards this end has been made. For instance, Stump analyzed the hocking problem in [334]. However, many problems remain to be examined.

6.8 Exercises

Exercise 6.1: Consider a rod whose strain energy function is given by the expression

$$2\rho_0\psi = EI_1\nu_1^2 + EI_2\nu_2^2 + \mathscr{D}\nu_3^2 + \frac{kEA}{2(1+\nu)}\left(\eta_1^2+\eta_2^2\right) + EA\eta_3^2, \tag{6.86}$$

where k is known as the shear correction factor.[5] Numerically determine the static equilibria of a uniform, homogeneous rod of length ℓ which is subject to equal and opposite end forces $\mathbf{n}_0 = -\mathbf{n}(0,t) = \mathbf{n}(\ell,t)$, and end moments $\mathbf{m}(0,t)$ and $\mathbf{m}(\ell,t)$. You should assume that there are no body forces and no surface tractions on the rod-like body that the rod is modeling. For the material properties, use those for steel or aluminum.

Exercise 6.2: Consider an infinitely long, homogeneous rod undergoing a steady axial motion. Show that the equations governing the deformed shape of the rod are similar to those governing a static equilibrium.[6]

Exercise 6.3: Consider the static equilibrium of a homogeneous rod in the absence of assigned forces and moments. Under which conditions is the material contact force C constant throughout the rod?

---

[5] The shear correction factor is a constant in beam theory that is used to match static and dynamic solutions of the three-dimensional theory to those for the rod theory. The factor depends on the geometry of the cross section and the type of comparisons used (cf. [74, 93, 143, 310] and references therein). For a square cross section of a linearly elastic isotropic rod-like body with $\nu = 0.3$, $k \approx 0.85\,(0.822)$ if a comparison based on a static (dynamic) solution is employed (cf. [93, Table 3]).

[6] An analysis of the resulting equations can be found in Antman and Liu [13]. We also refer the reader to the paper by Coleman et al. [66] for a related analysis for a planar rod theory.

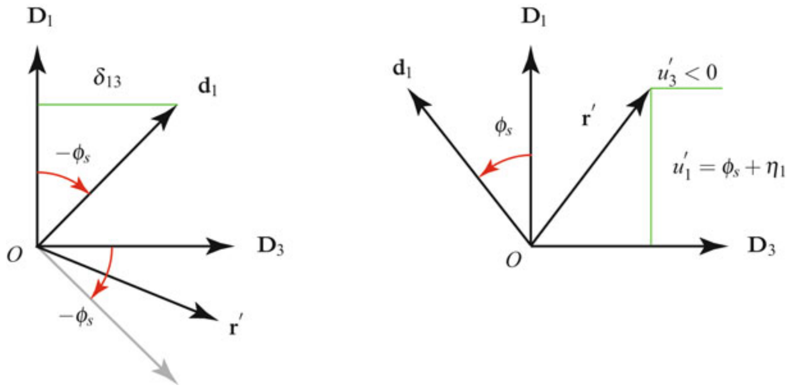

Fig. 6.7 The kinematic measures u_1, u_3, ϕ_s, and δ_{13} associated with a linearized theory known as Timoshenko beam theory [344]. It is important to notice that $\mathbf{r}'$ and $\mathbf{d}_1$ are not constrained to be orthogonal in the rod theory of interest here.

Exercise 6.4: Consider the homogeneous rod which is straight in an undeformed natural configuration, $\mathbf{P}_0 = \mathbf{I}$, that is shown in Figure 6.1. The strain energy function of the rod has the representation

$$\rho_0 \psi = \frac{EI_1}{2}\nu_1^2 + \frac{EI_2}{2}\nu_2^2 + \frac{\mathscr{D}}{2}\nu_2^3 + \frac{kEA}{4(1+\nu)}\left(\eta_1^2 + \eta_2^2\right) + \frac{EA}{2}\eta_2^3, \quad (6.87)$$

where k is known as the shear correction factor. We consider small amplitude, planar, flexural deformations of this rod[7]:

$$\begin{aligned} \mathbf{R} &= z\mathbf{E}_3, \\ \mathbf{r} &= u_1\mathbf{E}_1 + u_3\mathbf{E}_3 + \mathbf{R} + \mathscr{O}\left(\varepsilon^2\right), \\ \mathbf{d}_1 &= \mathbf{E}_1 + \delta_{13}\mathbf{E}_3 + \mathscr{O}\left(\varepsilon^2\right), \\ \mathbf{d}_2 &= \mathbf{E}_2, \end{aligned} \quad (6.88)$$

where ε is a small number.

(a) For the deformations of interest, the rotation tensor $\mathbf{P}$ has the representation $\mathbf{P} = \mathbf{Q}_{\mathbf{E}}\left(\phi_s, \mathbf{E}_2\right)$. Show that

$$\mathbf{P}\boldsymbol{\nu} = \frac{\partial \phi_s}{\partial z}\mathbf{E}_2. \quad (6.89)$$

With the help of the identity $\mathbf{d}_1 = \mathbf{P}\mathbf{D}_1$ and Figure 6.7, show that

$$\phi_s \approx -\delta_{13}. \quad (6.90)$$

[7] The reader is also referred to the closely related Exercises 5.6 and 7.4.

(b) Show that the nontrivial strains of the rod are, to $\mathscr{O}\left(\varepsilon^2\right)$,

$$\eta_1 = \frac{\partial u_1}{\partial z} + \delta_{13}, \qquad \eta_3 = \frac{\partial u_3}{\partial z}, \qquad \nu_2 = \frac{\partial \phi_s}{\partial z} = -\frac{\partial \delta_{13}}{\partial z}. \tag{6.91}$$

Explain why $\frac{\partial u_1}{\partial z} + \delta_{13}$ is known as the transverse shearing strain of the rod. Show that the contact force $\mathbf{n}$ and contact moment $\mathbf{m}$ have the representations

$$\mathbf{n} = \left(\frac{kEA}{2(1+\nu)}\right)\eta_1\mathbf{E}_1 + EA\eta_3\mathbf{E}_3, \qquad \mathbf{m} = EI_2\nu_2\mathbf{E}_2. \tag{6.92}$$

(c) Argue that the equations governing the motion of the rod reduce to the three differential equations

$$\left.\frac{\partial}{\partial z}\left(EA\frac{\partial u_3}{\partial z}\right) + \rho_0\mathbf{f}\cdot\mathbf{E}_3 = \rho_0\frac{\partial^2 u_3}{\partial t^2},\right\} \longrightarrow \text{Extension/Contraction}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial z}\left(\frac{kEA}{2(1+\nu)}\left(\frac{\partial u_1}{\partial z} - \phi_s\right)\right) + \rho_0\mathbf{f}\cdot\mathbf{E}_1 = \rho_0\frac{\partial^2 u_1}{\partial t^2}, \\ &\frac{\partial}{\partial z}\left(EI_2\frac{\partial \phi_s}{\partial z}\right) + \frac{kEA}{2(1+\nu)}\left(\frac{\partial u_1}{\partial z} - \phi_s\right) + \mathbf{m}_a\cdot\mathbf{E}_2 = \rho_0 y^{11}\frac{\partial^2 \phi_s}{\partial t^2}. \end{aligned}\right\} \longrightarrow \text{Flexure} \tag{6.93}$$

Here, $y^{11} = I_2/A$, $\mathbf{n}\cdot\mathbf{E}_1$ is known as a shear force, and $\mathbf{m}\cdot\mathbf{E}_2$ is known as the bending moment.

(d) Show that the flexural equations $(6.93)_{2,3}$ correspond to those for a Timoshenko beam [344] that can be found in the literature (cf. [152, 204, 275, 310] and references therein). You should observe that the extensional equations are identical to those for the longitudinal vibration of a bar. We also note that an alternative derivation of the Timoshenko beam equations is discussed in the forthcoming Exercise 7.4.