Chapter 6
Dams and Inventories

6.1 Introduction

In this chapter we analyze several models of dams and inventories with state space $\boldsymbol{S} \subseteq \mathbb{R}$, using LC. When the content in a dam, or stock on hand in an inventory, is positive-valued, it can decline at varying instantaneous rates in accordance with a general release rule specified in the model. Thus the efflux differs from the virtual wait or workload in M/G/1 queues, which *decreases* at rate 1 when positive, or the extended age in G/M/c queues, which *increases* at rate 1.

Section 6.2 describes a dam with general release rule, denoted by M/G/$r(\cdot)$ (or 'M/G/1 dam'). The function $r(x)$, $x \geq 0$, denotes the *efflux rate* when the content is at level x, having dimension [(*Content unit)/Time*]. We discuss sample-path and SP transitions in the time-state space, and derive *integro-differential* equations for the *transient* (time-dependent) distribution of the content. The subscript "t" is used to indicate transience. Integral equations for the *steady-state* (limiting) distribution of content are then obtained by taking limits as $t \to \infty$.

Sections 6.3–6.9 apply SPLC to analyze several models of dams and inventories in steady state.

6.2 M/G/r(·) Dam

6.2.1 Model Description

Consider a dam with state space $\boldsymbol{S} = [0, \infty)$. Denote the content at instant t by $W(t), t \geq 0$. Assume inputs occur at a Poisson rate λ. Denote the instants of input by τ_n, $n = 1, 2, \ldots,$ where $0 \equiv \tau_0 < \tau_1 < \tau_2 < \cdots$. Denote the input size at τ_n by S_n. We assume $\{S_n\}_{n=1,2,\ldots}$ are i.i.d. positive r.v.s independent of n, with $S_n \underset{dis}{\equiv} S$. Let $B(x) = P(S \leq x)$, and $\overline{B}(x) = 1 - B(x)$.

P.H. Brill, *Level Crossing Methods in Stochastic Models*,
International Series in Operations Research & Management Science 250,
DOI 10.1007/978-3-319-50332-5_6

In some state-dependent model variants, the input size may depend on the content $W(\tau_n^-)$ just before input instant τ_n (denoted by $S(W(\tau_n^-))$, or on a Markovian environment (e.g., denoted by $S_{(i)}$ where i is a state of a continuous-time Markov chain describing the environment). Other input-time dependencies are possible.

If S depends on the current content only, the conditional cdf of $S(W(\tau_n^-))$ given $W(\tau_n^-) = y$, is denoted as

$$B_y(x) = P(S(W(\tau_n^-)) \leq x | W(\tau_n^-) = y), y \geq 0, n = 1, 2, \ldots .$$

The *efflux rate* of content out of the dam, is denoted by $r(W(t))$, defined in Sect. 6.2.2 below. Generally, the efflux rate depends on the current content (see Sect. 5 in [77]).

In M/G/r(·), we assume that the entire input amount goes into the dam instantaneously at an input instant. Under this assumption the model applies to some real-world situations, e.g., systems involving torrential rainfalls, repeated shocks, bolus injections of a prescription medication in pharmacokinetics, instillment of certain eye drops, consumer response to a particular product when exposed to repeated non-uniform advertising in marketing-science models (e.g., [40, 47]), etc.

6.2.2 General Efflux Rate

Let $r(W(t))$ denote the instantaneous efflux rate at which the content decreases (flows out of the dam) at instant t, when the content is $W(t)$. Assume $r(W(t))$ is finite and

$$\left.\begin{array}{l} r(x) > 0 \text{ if } x > 0, \\ r(x) = 0 \text{ if } x = 0. \end{array}\right\} \tag{6.1}$$

The rate of decline of $W(t)$ between input instants is (see Sect. 5 in [77])

$$\frac{dW(t)}{dt} = -r(W(t)), \tau_n \leq t < \tau_{n+1}, n = 0, 1, 2, \ldots, \tag{6.2}$$

independent of n. The variable $r(W(t))$ has 'physical' dimension (unit) $\frac{[content\ unit]}{[Time]}$, e.g., $\frac{[Volume]}{[Time]} = [L^3 T^{-1}]$, where $L := Length$ and $T := Time$.

This section assumes that $r(x)$, $x \in \boldsymbol{S}$ is a time-homogeneous piecewise right-continuous function, except at level 0. Usually, $r(0) \neq r(0^+) = \lim_{x \downarrow 0} r(x)$. However, equality of $r(0)$ and $r(0^+)$ is possible in some models.

Example 6.1 Suppose $r(x) = (x + 1)^2, x > 0, r(0) = 0$. Then $r(0^+) = 1 \neq r(0)$. On the other hand, suppose $r(x) = x^2, x > 0$, and $r(0) = 0$. Then $r(0^+) = r(0)$.

In some model variants, $r(x), x \geq 0$, may have different functional forms on separate state space intervals. In such cases, consider a state-space partition $\{x_j\}_{j=0,1,\ldots,n+1}$ where $0 \equiv x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} \equiv \infty$. Let $\boldsymbol{I}_1 := (x_0, x_1)$, and

$\boldsymbol{I}_j := \left[x_{j-1}, x_j\right), j = 2, \ldots, n+1$. Define $\left\{r_j(x)\right\}_{j=0,1\ldots,n+1}$ by

$$r(x) = \begin{cases} r_0(0) = 0 \\ r_1(x), x \in (0, x_1) \equiv \boldsymbol{I}_1 \\ r_2(x), x \in [x_1, x_2) \equiv \boldsymbol{I}_2 \\ \quad \cdots\cdots \\ r_n(x), x \in [x_{n-1}, x_n) \equiv \boldsymbol{I}_n \\ r_{n+1}(x), x \in [x_n, \infty) \equiv \boldsymbol{I}_{n+1}, \end{cases} \tag{6.3}$$

where $r_j(x), x \in \boldsymbol{I}_j$ is positive and continuous, $j = 1, 2, \ldots, n+1$. (See, e.g., Sects. 3.1 and 3.2 in [19] for examples using state-space partitions.)

Remark 6.1 In some model generalizations $r(W(t))$ may also depend on t. We would then append a subscript t, i.e., denote the efflux rate as $r_t(W(t))$.

6.2.3 *Sample Paths*

We use the symbol '$W(t)$' to denote the content of the dam, and also to denote the ordinate of a sample path of the content at instant t (unless specified otherwise), for economy of notation, and because the usage will be clear from the context.

A sample path of $\{W(t)\}_{t\geq 0}$ is a piecewise *deterministic* function plotted in the *time-state* plane $\boldsymbol{T} \times \boldsymbol{S}$, where $\boldsymbol{T} := \{t | t \geq 0\}$ (Fig. 6.1).

6.2.4 *Time for $\{W(t)\}_{t\geq 0}$ to Decrease to a Level*

In Eq. (6.2), separating variables gives the differential equation

$$\frac{dW(t)}{r(W(t))} = -dt, \tau_n \leq t < \tau_{n+1}, n = 0, 1, 2, \ldots;$$

integrating both sides gives

$$\int_{W(t_x)}^{W(t_y)} \frac{1}{r(W(t))} dW(t) = -\int_{t_x}^{t_y} dt = t_x - t_y.$$

(See Fig. 6.2). The *time* required for a sample path of $\{W(t)\}_{t\geq 0}$ to descend from level $W(t_y) = y$ at instant t_y to a lower level $W(t_x) = x \geq 0$ at instant t_x, if no inputs to the dam intervene, i.e., if

$$W(\tau_n) > y > x \geq W(\tau_{n+1}^-) \geq 0, \text{ for some fixed } n,$$

is

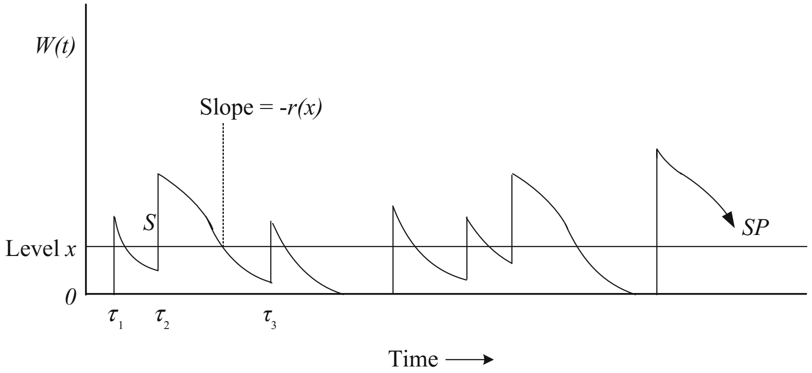

Fig. 6.1 Sample path of $\{W(t)\}_{t\geq 0}$ in $M/G/r(\cdot)$ dam

$$t_x - t_y = \int_{z=x}^{y} \frac{1}{r(z)} dz. \tag{6.4}$$

upon substituting $W(t) = z$.

Formula (6.4) is useful when analyzing models of dams and inventories in continuous time (as in this chapter), or when analyzing a dam via the *embedded LC method* (see Sect. 8.2 in Chap. 8).

6.2.5 Condition for $\{W(t)\}_{t\geq 0}$ to Return to Level 0

Formula (6.4) implies that a necessary and sufficient condition for a return by $\{W(t)\}_{t\geq 0}$ to level 0, is

$$\lim_{x\downarrow 0} \int_{z=x}^{y} \frac{1}{r(z)} dz < \infty \text{ for every finite } y > 0, \tag{6.5}$$

(see pp. 116–117 in [77]).

6.2.6 Transient Probability Distribution of Content

Transient CDF and PDF

Denote the transient cdf of $W(t)$ by $F_t(x)$, $x \geq 0$, and let $F_t(0) := P_0(t)$. Let $f_t(x) := \partial F_t(x)/\partial x$, $x > 0$, wherever the derivative exists. We denote the transient pdf of $W(t)$ by $\{P_0(t), f_t(x)\}_{t\geq 0}$. Assume $F_t(x)$, $f_t(x)$ are right continuous in x. We use $f_t(0^+)$ and $f_t(0)$ interchangeably for notational convenience since $f_t(0)$ adds zero probability to $P_0(t)$. The function $f_t(x)$ may have jump discontinuities depending on the distribution of the input r.v.s. (See, e.g., Sects. 3.10 and 3.11 regarding the pdf of wait in M/D/1 and M/Discrete/1 queues.)

For each $t \geq 0$,

$$F_t(x) = P_0(t) + \int_{y=0}^{x} f_t(y)dy,$$

and the normalizing condition is

$$F_t(\infty) = P_0(t) + \int_{y=0}^{\infty} f_t(y)dy = 1.$$

Steady-State Probability Distribution

We mention the steady-state cdf and pdf now because we will derive them in Sect. 6.2.11, immediately after the discussion of the transient cdf and pdf below. The steady-state cdf and pdf of content are denoted as $F(x), x \geq 0$, and $\{P_0, f(x)\}_{x>0}$ respectively, and are obtained by letting $t \to \infty$, i.e.,

$$F(x) = \lim_{t\to\infty} F_t(x), x \geq 0, f(x) = \lim_{t\to\infty} f_t(x), x > 0, P_0 = \lim_{t\to\infty} P_0(t).$$

Remark 6.2 P_0 exists if and only if a sample path of $\{W(t)\}_{t\geq 0}$ returns to level 0 with probability 1. However, some forms of $r(W(t))$ make returns to level 0 impossible (see pp. 116–117 in [77], and Sect. 6.2.5).

6.2.7 *Sample-Path and SP Downcrossings*

Consider a sample path of $\{W(t)\}_{t\geq 0}$ (Fig. 6.1). Fix level $x \in \boldsymbol{S}$. Let $\mathcal{D}_t(x)$ denote the number of SP downcrossings of level x during $(0, t)$. The SP traces the sample path during piecewise continuous segments between input instants. At sample-path discontinuities, the SP makes an upward jump, *not in Time* (see Sects. 2.4.3 and 2.4.4). Let $\mathcal{D}_t^c(x)$ and $\mathcal{D}_t^j(x)$ denote respectively the number of SP *left-continuous* downcrossings and SP *jump* downcrossings of level x during $(0, t)$. Then

$$\mathcal{D}_t(x) = \mathcal{D}_t^c(x) + \mathcal{D}_t^j(x), x \geq 0, t \geq 0.$$

In the basic M/G/r(·) dam of this section, $\mathcal{D}_t^j(x) \equiv 0$, $t \geq 0$. In variations of the basic model, however, SP downward jumps can indeed occur. Both SP left-continuous downcrossings and SP jump downcrossings also occur in a vast number of *inventory* and *production-inventory* models. Thus, we shall distinguish $\mathcal{D}_t(x)$ from $\mathcal{D}_t^c(x)$ in Theorem 6.1 in Sect. 6.2.8. Note that $\mathcal{D}_t^c(0)$ may equal 0 in certain cases of $r(W(t))$ (see Sect. 6.2.5).

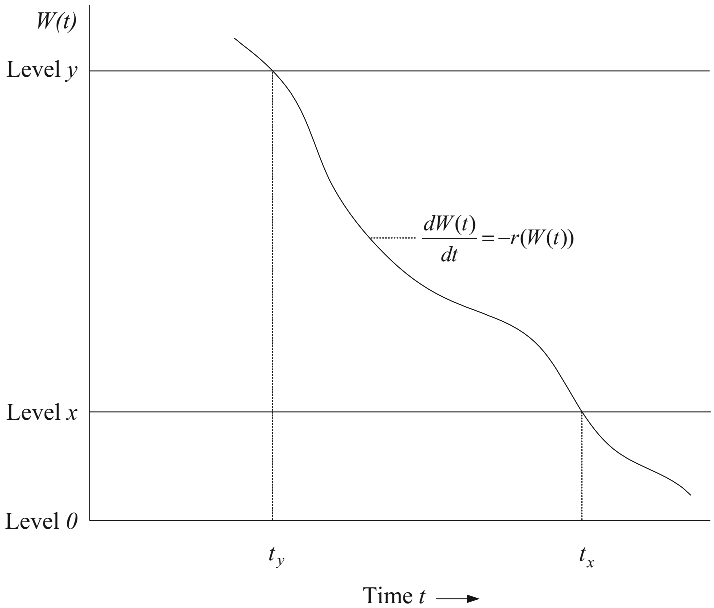

Fig. 6.2 M/G/r(·) dam: time to descend from level y to level $x > 0$ is $t_x - t_y = \int_{z=x}^{y} \frac{1}{r(z)} dz$

6.2.8 Level Crossings and Transient PDF of Content

In a sample path of $\{W(t)\}_{t \geq 0}$, fix level $x \in \boldsymbol{S}$ (Fig. 6.1). Let $\mathcal{U}_t(x) :=$ *number of SP upcrossings of level x during* $(0, t)$. It can be shown, along the lines of Sects. 3.2.1 and 3.2.2, that $\frac{\partial}{\partial t} E(\mathcal{D}_t^c(x))$, $\frac{\partial}{\partial t} E(\mathcal{U}_t(x))$ exist and are positive.

Theorems 6.1 and 6.2 were originally proved using LC in [23].

Downcrossings

Theorem 6.1 For the M/G/r(·) dam

$$\frac{\partial}{\partial t} E(\mathcal{D}_t^c(x)) = r(x) f_t(x), x > 0, \tag{6.6}$$

$$\frac{\partial}{\partial t} E(\mathcal{D}_t^c(0)) = r(0^+) f_t(0). \tag{6.7}$$

Proof Consider a sample path of $\{W(t)\}_{t \geq 0}$, and fix state-space level $x \in \boldsymbol{I}_j$ for some $j \in \{1, \ldots, n+1\}$ in (6.3). Fix instant t. Consider $t + h$, $(h > 0)$ and define $\delta > 0$ by

$$\int_{z=x}^{x+\delta} \frac{1}{r(z)} dz = h. \tag{6.8}$$

Assume h is sufficiently small so that level $x + \delta \in \boldsymbol{I}_j$; h is the time for the content to decrease from level $x + \delta$ to level x if there are no inputs during $(t, t + h)$ (see formula (6.4)). Applying the (first) mean value theorem for integrals with continuous integrand (see, e.g., Problems 27–28, p. 237 in [137]) to Eq. (6.8) yields

$$h = \frac{1}{r(z^*)}\delta \iff \delta = r(z^*)h \tag{6.9}$$

for some z^* such that $x < z^* < x + \delta$.

The event $\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x) = 1$ occurs iff $W(t) \in (x, x + \delta)$ and there is no input in a time subinterval $(t, t + \xi) \subseteq (t, t + h)$, or an event with probability $o(h)$ occurs. From (6.9)

$$\begin{aligned} P(\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x) = 1) &= f_t(x) \cdot \delta \cdot (1 - \lambda h) + o(h) \\ &= f_t(x) \cdot r(z^*) \cdot h \cdot (1 - \lambda h) + o(h). \end{aligned}$$

The value $\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x) = 0$ has no affect on $E(\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x))$. Due to the Poisson input stream, $P(\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x) \geq 2) = o(h)$. Hence the expected value

$$\begin{aligned} E(\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x)) &= 1 \cdot P(\mathcal{D}^c_{t+h}(x) - \mathcal{D}^c_t(x) = 1) + o(h), \\ E(\mathcal{D}^c_{t+h}(x)) - E(\mathcal{D}^c_t(x)) &= f_t(x) \cdot r(z^*) \cdot h \cdot (1 - \lambda h) + o(h). \end{aligned} \tag{6.10}$$

Dividing both sides of (6.10) by h and letting $h \downarrow 0$ gives (6.6) since $z^* \downarrow x$ and $r(z^*) \downarrow r(x^+) = r(x)$, $x > 0$, as $h \downarrow 0$. Then letting $x \downarrow 0$ in (6.6) gives (6.7). ■

Corollary 6.1 For each $t \geq 0$,

$$\begin{aligned} E(\mathcal{D}^c_t(x)) &= r(x) \int_{s=0}^{t} f_s(x) ds, \; x > 0, \\ E(\mathcal{D}^c_t(0)) &= r(0^+) \int_{s=0}^{t} f_s(0) ds. \end{aligned}$$

Proof In (6.6) and (6.7) set $t = s$, integrate with respect to $s \in [0, t]$, and apply the initial condition $E(\mathcal{D}^c_0(x)) = 0$, $x \geq 0$. ■

Corollary 6.2 The steady-state pdf of $\{W(t)\}_{t\geq 0}$ as $t \to \infty$ is given in terms of downcrossing rates by

$$\lim_{t\to\infty} \frac{E(\mathcal{D}^c_t(x))}{t} = r(x)f(x), \; x > 0, \tag{6.11}$$

$$\lim_{t\to\infty} \frac{E(\mathcal{D}^c_t(0))}{t} = r(0^+)f(0). \tag{6.12}$$

Proof In Corollary 6.1, since $\lim_{s\to\infty} f_s(x) = f(x)$, for every $\varepsilon > 0$ there exists t_ε such that $|f_s(x) - f(x)| < \varepsilon$ for $s > t_\varepsilon$, implying

$$\int_{s=0}^{t} f_s(x)ds < C_\varepsilon + \int_{t_\varepsilon}^{t} (f(x) + \varepsilon)\, ds = C_\varepsilon + (t - t_\varepsilon)\,(f(x) + \varepsilon)\,,$$
$$\int_{s=0}^{t} f_s(x)ds > C_\varepsilon + \int_{t_\varepsilon}^{t} (f(x) + \varepsilon)\, ds = C_\varepsilon + (t - t_\varepsilon)\,(f(x) - \varepsilon)\,,$$

where the constant $C_\varepsilon := \int_{s=0}^{t_\varepsilon} f_s(x)ds$ and $t > t_\varepsilon$. Combining both inequalities yields

$$C_\varepsilon + (t - t_\varepsilon)\,(f(x) - \varepsilon) < \int_{s=0}^{t} f_s(x)ds < C_\varepsilon + (t - t_\varepsilon)\,(f(x) + \varepsilon)\,.$$

Dividing throughout by t gives

$$\frac{C_\varepsilon}{t} + \left(1 - \frac{t_\varepsilon}{t}\right)(f(x) - \varepsilon) < \frac{1}{t}\int_{s=0}^{t} f_s(x)ds < \frac{C_\varepsilon}{t} + \left(1 - \frac{t_\varepsilon}{t}\right)(f(x) + \varepsilon)\,.$$

Letting $t \to \infty$ gives

$$f(x) - \varepsilon \quad < \quad \lim_{t\to\infty} \tfrac{1}{t} \int_{s=0}^{t} f_s(x)ds < f(x) + \varepsilon$$
$$\Longrightarrow \quad \lim_{t\to\infty} \tfrac{1}{t} \int_{s=0}^{t} f_s(x)ds = f(x)$$

since $\varepsilon > 0$ is arbitrarily small, thus yielding (6.11); then setting $x = 0$ gives (6.12). ■

Upcrossings

Theorem 6.2 For the M/G/r(·) dam

$$\frac{\partial}{\partial t} E(\mathcal{U}_t(x)) = \lambda \int_{z=0}^{x} \overline{B}(x - z)dF_t(z)$$
$$= \lambda P_0(t)\overline{B}(x) + \lambda \int_{z=0}^{x} \overline{B}(x - z)f_t(z)dz, x > 0, \tag{6.13}$$
$$\frac{\partial}{\partial t} E(\mathcal{U}_t(0)) = \lambda P_0(t). \tag{6.14}$$

Proof Fix instants t and $t + h, t \geq 0$, $h > 0$ (h small). Fix level $x > 0$. Then $\mathcal{U}_{t+h}(x) - \mathcal{U}_t(x) = 1$ iff $W(s) = z < x$ at an instant $s \in (t, t+h)$ at which there is an input of size $S > x - z$, or an event having probability $o(h)$ occurs. The value $\mathcal{U}_{t+h}(x) - \mathcal{U}_t(x) = 0$ does not contribute to $E(\mathcal{U}_{t+h}(x) - \mathcal{U}_t(x))$. Also $P(\mathcal{U}_{t+h}(x) - \mathcal{U}_t(x) \geq 2) = o(h)$.

$$\begin{aligned}E(\mathcal{U}_{t+h}(x)-\mathcal{U}_t(x)) &= E(\mathcal{U}_{t+h}(x))-E(\mathcal{U}_t(x))\\ &= \lambda\int_{z=0}^{x}\int_{s=0}^{h}\overline{B}(x-z)dsdF_{t+s}(z)+o(h)\\ &= \lambda h\int_{z=0}^{x}\overline{B}(x-z)dF_{t+s^*}(z)+o(h) \qquad (6.15)\end{aligned}$$

where $0<s^*<h$. Dividing both sides of (6.15) by h and letting $h\downarrow 0$ gives (6.13) since $s^*\downarrow 0$ as $h\downarrow 0$, and $F_t(\cdot)$ is right-continuous in t. Then letting $x\downarrow 0$ in (6.13) gives (6.14).

Note: If $r(0^+)=0$ then $P_0(t)=0,\ t>0$ (see Sect. 6.2.5 and Remark 6.2) in Sect. 6.2.6. ■

Corollary 6.3

$$\begin{aligned}E(\mathcal{U}_t(x)) &= \lambda\int_{s=0}^{t}\int_{z=0}^{x}\overline{B}(x-z)dF_s(z)ds\\ &= \lambda\int_{s=0}^{t}P_0(s)\overline{B}(x)ds+\lambda\int_{s=0}^{t}\left[\int_{z=0}^{x}\overline{B}(x-z)f_s(z)dz\right]ds,\ x>0,\\ E(\mathcal{U}_t(0)) &= \lambda\int_{s=0}^{t}P_0(s)\overline{B}(x)ds.\end{aligned}$$

Proof Set $t=s$ in (6.13) and (6.14), integrate with respect to $s\in[0,t]$ and apply the initial condition $E(\mathcal{U}_0(x))=0,\ x\geq 0$. ■

Corollary 6.4

$$\begin{aligned}\lim_{t\to\infty}\frac{E(\mathcal{U}_t(x))}{t} &= \lambda\int_{z=0}^{x}\overline{B}(x-z)dF(z)dz\\ &= \lambda P_0\overline{B}(x)+\lambda\int_{z=0}^{x}\overline{B}(x-z)f(z)dz,\ x>0,\\ \lim_{t\to\infty}\frac{E(\mathcal{U}_t(0))}{t} &= \lambda P_0\overline{B}(x).\end{aligned}$$

Proof In Corollary 6.3, interchange the order of integration. Divide both sides by t and let $t\to\infty$. The result follows since $\lim_{s\to\infty}f_s(z)=f(z)$ implying $\lim_{t\to\infty}(1/t)\int_{s=0}^{t}f_s(z)ds=f(z)$. (See the proof of Corollary 6.2 above in this Section. Also see the **Note** immediately after Theorem 6.2 above, regarding the condition ensuring $P_0>0$). ■

6.2.9 Equation for Transient Distribution of Content

The following theorem has been proved using classical methods by various authors (see, e.g., Eq. (5.4) in [77]). Here we prove it using LC (based on [23]).

Theorem 6.3 In the M/G/r(·) dam, the transient pdf of content, $f_t(x), x > 0$, satisfies the **integro-differential equation**

$$\begin{aligned} r(x)f_t(x) &= \tfrac{\partial}{\partial t}F_t(x) + \lambda \int_{z=0}^{x} \overline{B}(x-z)dF_t(z) \\ &= \tfrac{\partial}{\partial t}F_t(x) + \lambda\overline{B}(x)P_0(t) \\ &\quad +\lambda \int_{z=0}^{x} \overline{B}(x-z)f_t(z)dz, x > 0, \end{aligned} \tag{6.16}$$

and $P_0(t)$ satisfies the **differential equation**

$$\frac{d}{dt}P_0(t) + \lambda P_0(t) = r(0^+)f_t(0). \tag{6.17}$$

Proof In Theorem 4.1 (i.e., Theorem B in Sect. 4.2), substitute set $[0, x] = \boldsymbol{A}$, $\mathcal{D}_t^c(x) = \mathcal{I}_t(x), \mathcal{U}_t(x) = \mathcal{O}_t(x)$. This gives

$$\frac{\partial}{\partial t}E(\mathcal{D}_t^c(x)) = \frac{\partial}{\partial t}F_t(x) + \frac{\partial}{\partial t}E(\mathcal{U}_t(x)) \tag{6.18}$$

Substituting from (6.6) and (6.13) into (6.18) gives (6.16). Equation (6.17) then follows by letting $x \downarrow 0$ in (6.16), noting that $F_t(0) = P_0(t)$.

See the **Note** at the end of Theorem 6.2 in Sect. 6.2.8. ■

Remark 6.3 The dimension of $r(x)$ is $\left[\frac{\textit{content unit}}{\textit{Time}}\right]$. The dimension of $f_t(x)$ is $\left[\frac{1}{\textit{content unit}}\right]$. The dimension of the left sides of (6.16) and of (6.17), is

$$\left[r(x)f_t(x)\right] = \left[\frac{\textit{content unitt}}{\textit{Time}}\right] \cdot \left[\frac{1}{\textit{content unit}}\right] = \frac{1}{[\textit{Time}]}, x \geq 0,$$

which matches the dimensional unit of the right side.

6.2.10 Estimate of Transient Probability $P_0(t)$

We briefly outline an '***LC estimation***' procedure for the *transient* probability $P_0(t)$, $t \geq 0$, *assuming $P_0(t)$ exists for all $t > 0$*, which occurs provided returns to level 0 are regenerative points (i.e., $r(0^+) > 0$). (See Sect. 6.2.5 and Remark 6.2 in Sect. 6.2.6.) We also call this procedure LCE, or LC computation. LCE to compute a pdf $f_t(x)$, $x > 0$, would be similar. We do not detail LCE for *transient pdfs* elsewhere in this monograph. See Remark 9.2 in Sect. 9.2 in Chap. 9. We detail LCE for *limiting distributions* in Chap. 9.

To solve differential equation (6.17) multiply by the integrating factor $e^{\lambda t}$, and integrate with respect to t, yielding

$$P_0(t) = \left[\int_{s=0}^{t} e^{\lambda s}\frac{\partial}{\partial s}E(\mathcal{D}_s^c(0))ds + P_0(0)\right]e^{-\lambda t}, \tag{6.19}$$

where

$$P_0(0) = \begin{cases} 1 \text{ if } W(0) = 0, \\ 0 \text{ if } W(0) \neq 0. \end{cases}$$

Formula (6.19) connects $P_0(t)$ and $\partial E(\mathcal{D}_s^c(0))/ds$, $0 < s < t$, which appears as a factor in the integrand. This connection leads to an *estimation method* for $P_0(t)$, by estimating the integral in (6.19).

The idea is to first simulate N *independent* sample paths of $\{W(t)\}_{t\geq 0}$ denoted as $\{W_n(s)\}_{s\geq 0, n=1,\ldots,N}$ on the same time interval $[0, t_M + r]$, where t_M is the maximum finite time of interest, r is an "extra" finite time which ensures that t_M is not the right end point of the simulated time interval. N is a large positive integer. A reasonable value of N would be in the range [400, 1,000]. Due to the high speed of today's computers, N may be considerably larger than 1,000. Let $h = t_M/m$ be small, where m is a positive integer. We can use, e.g., $h = 0.001$ or 0.0001, or any small value $h < r$. The accuracy of the estimation of $P_0(t)$, $t \in [0, T_M]$, improves with larger values of N and smaller values of h.

We then compute the number of SP *left-continuous downcrossings* (hits of level 0) denoted by $\mathcal{D}^c_{ih,n}(0)$, $i = 0, \ldots, m$, for each sample path, $\{W_n(s)\}_{n=1,\ldots,N}$. For fixed i and n, the $\mathcal{D}^c_{ih,n}(0)$s are independent since the N sample paths are independent. We compute point estimates of the true downcrossing rates $\mathcal{D}^c_{ih,n}(0)$ and $\mathcal{D}^c_{(i+1)h,n}(0)$ at times ih and $(i+1)\,h$ respectively by averaging over the N sample paths. Then we compute estimates of $E(\mathcal{D}^c_{ih}(0))$ and $E(\mathcal{D}^c_{(i+1)h}(0))$ using

$$\widehat{E}(\mathcal{D}^c_{ih}(0)) = \frac{1}{N}\sum_{n=1}^{N} \mathcal{D}^c_{ih,n}(0), \widehat{E}(\mathcal{D}^c_{(i+1)h}(0)) = \frac{1}{N}\sum_{n=1}^{N} \mathcal{D}^c_{(i+1)h,n}(0).$$

An estimate of the *derivative* $\partial E(\mathcal{D}^c_{ih}(0))/dt$ is then given by the difference quotient

$$\widehat{\frac{\partial}{\partial t}} E(\mathcal{D}^c_{ih}(0)) = \frac{\widehat{E}(\mathcal{D}_{(i+1)h}(0)) - \widehat{E}(\mathcal{D}^c_{ih}(0))}{h}, i = 0, \ldots, m.$$

Finally, we approximate the integral $\int_{s=0}^{kh} e^{\lambda s} \frac{\partial}{\partial s} E(\mathcal{D}^c_s(0)) ds$ as a finite Riemann sum

$$h\sum_{i=0}^{k} e^{\lambda ih} \widehat{\frac{\partial}{\partial t}} E(\mathcal{D}^c_{ih}(0)), k = 1, \ldots, m$$

A point estimate of $P_0(kh)$ is

$$\widehat{P}_0(kh) = \left[h \sum_{i=0}^{k} e^{\lambda ih} \widehat{\frac{\partial}{\partial t}} E(\mathcal{D}_{ih}^c(0)) + P_0(0) \right] e^{-\lambda t}, k = 1, \ldots, m, \tag{6.20}$$

where $mh = t_M$. This technique results in estimates of $P_0(h)$, $P_0(2h)$, ..., $P_0(mh)$. Thus, we estimate $P_0(t)$, $t = 0, h, 2h, \ldots, t_M$. Smoothing techniques can be applied to estimate intermediate values. Then we can plot $\widehat{P}_0(t)$, $0 < t < t_M$. (We can also develop interval estimates for $P_0(kh)$, $k = 1, \ldots, m$.)

Generalizations and variations of this technique can be used to estimate transient distributions of state variables in many stochastic models having a continuous time parameter.

The foregoing example of **LCE** relates to Chap. 9, which describes LCE for steady-state distributions. LCE has also been discussed in [17] and [24]. (Also, see Remark 9.2 in Sect. 9.2 in Chap. 9.)

Remark 6.4 Future computer speeds will undoubtedly increase. Thus the computational method described above will achieve better and better accuracy. It will be possible to increase N and decrease h, while completing the computations in a much shorter amount of real time.

Remark 6.5 In the M/G/r(·) dam, possibly $P_0(t) = 0$ for all $t \geq \tau_1$ (instant of first input). For example, if $r(x) = kx$, $x > 0$, $k > 0$, the decline of the sample-path has a negative exponential form between inputs. In theory the content will never reach level zero after the first input at τ_1. If the inter-input time is very long, the content eventually declines below any preassigned level $\varepsilon > 0$ however small, but never reaches level 0. In that case we may use downcrossings of an arbitrary level $\varepsilon > 0$ as regeneration points of a regenerative process. $\{W(t)\}_{t\geq 0}$ will then move along level ε until the next arrival. We may then use '$P_t(\varepsilon)$' like '$P_0(t)$'. Alternatively, we just estimate $f_t(x)$, $x > 0$ (see Remark 6.2 in Sect. 6.2.6).

6.2.11 Equation for Steady-State PDF of Content

Assume the system is stable *and* $\{W(t)\}_{t\geq 0}$ returns to level 0 with probability 1. Then

$$F(x) = \lim_{t\to\infty} F_t(x), \qquad f(x) = \lim_{t\to\infty} f_t(x), \qquad P_0 = F(0) = \lim_{t\to\infty} P_0(t)$$

all exist, and $\lim_{t\to\infty} \frac{\partial}{\partial t} F_t(x) = 0$. In Eq. (6.16), taking limits of all terms as $t \to \infty$ yields

$$\begin{aligned} r(x)f(x) &= \lambda \textstyle\int_{y=0}^{x} \overline{B}(x-y)dF(y), x > 0, \\ r(x)f(x) &= \lambda P_0 \overline{B}(x) + \lambda \textstyle\int_{y=0}^{x} \overline{B}(x-y)f(y)dy, x > 0, \\ r(0^+)f(0) &= \lambda P_0. \end{aligned} \tag{6.21}$$

Alternative Forms of Equation for Steady-State PDF

Two alternative forms of the integral equation in (6.21) are

$$r(x)f(x) = \lambda F(x) - \lambda \int_{y=0}^{x} B(x-y)f(y)dy, x > 0; \tag{6.22}$$

$$r(x)f(x) = \lambda F(x) - \lambda \int_{y=0}^{x} F(x-y)b(y)dy, x > 0, \tag{6.23}$$

where $b(y) = dB(y)/dy$.

Explanation of (6.22) and (6.23). In each equation the left side is $\lim \mathcal{D}_t(x)/t$, the SP *downcrossing* rate of level x. On the right side, the first term $\lambda F(x)$ is the rate of inputs when the content is $\leq x$; these inputs generate upward jumps that *start* in state-space interval $[0, x]$. The second term subtracts off the rate of such jumps that do not upcross level x. Hence the right side is $\lim \mathcal{U}_t(x)/t$, the *upcrossing* rate of level x. Applying rate balance $\lim \mathcal{D}_t(x)/t = \lim \mathcal{U}_t(x)/t$, gives the alternative equations.

Equations (6.22) and (6.23) are analogous to Eqs. (3.43) and (3.44) in Sect. 3.3.1 for the M/G/1 queue.

Stability

A condition for stability of the M/G/r(·) dam is

$$\lambda E(S) < \lim_{x\to\infty} r(x). \tag{6.24}$$

Formula (6.24) asserts the rate at which the content increases is less than the efflux rate when the content is at high levels. So the content is prevented from increasing to indefinitely high amounts. Condition (6.24) guarantees the return of $\{W(t)\}_{t\geq 0}$ *to every level* $x > 0$ in a finite time (see pp. 116–117 in [77]; Theorem 2 in [134].).

A condition that guarantees the content *will return to level* 0, therefore implying $P_0 > 0$, is Eq. (6.5) in Sect. 6.2.5 above.

Example 6.2 The M/G/1 queue is a special case of the M/G/r(·) dam with $r(x) \equiv 1$, $x > 0$, and $r(0) = 0$. Stability holds iff $\lambda E(S) < \lim_{x\to\infty} r(x) = 1$, the well-known stability condition for M/G/1 queues; if stability holds $\{W(t)\}_{t\geq 0}$ returns to level 0 (a.s.) since for all **finite** $x > 0$

$$\lim_{u\downarrow 0} \int_{y=u}^{x} \frac{1}{r(y)} dy = \lim_{u\downarrow 0} \int_{y=u}^{x} 1 \cdot dy = \lim_{u\downarrow 0} (x-u) = x < \infty$$

if no arrivals intervene.

Example 6.3 In the M/G/r(·) dam with $\lambda > 0$, $E(S) < \infty$, and $r(x) = kx$, $k > 0$,

$$\lim_{u\downarrow 0} \int_{y=u}^{x} \frac{1}{ky} dy = \frac{1}{k} \lim_{u\downarrow 0} \left(\ln\left(\frac{x}{u}\right)\right) = \infty,$$

for every finite $x > 0$. Hence the content **does not return to level** 0**,** implying $P_0 = 0$. On the other hand, this dam is **stable** for every $k > 0$ because

$$\lambda E(S) < \lim_{x\to\infty} r(x) = \lim_{x\to\infty} kx = \infty.$$

6.2.12 Sojourn Times Related to State-Space Level x

Consider a sample path of $\{W(t)\}_{t\geq 0}$. Fix level $x > 0$. Due to Poisson arrivals and the level-dependent slope of the efflux, $\{\mathcal{D}_t(x)\}_{t\geq 0}$ (same as $\left\{\mathcal{D}_t^c(x)\right\}_{t\geq 0}$) is a renewal counting process. The times between successive downcrossings of level x (renewals) are i.i.d. r.v.s. The instants of SP downcrossings of level x are regenerative points with respect to the process $\{W(t)\}_{t\geq 0}$, where $\{W(t)\}_{t\geq 0}$ restarts independent of the past.

Let d_x := *time between successive downcrossings of level x*. Let a_x, b_x denote sojourn times above and below level x, respectively. A sojourn a_x begins with an upcrossing of x and ends with the first downcrossing of x thereafter. A sojourn b_x begins with a downcrossing of x and ends with the first upcrossing of x thereafter. Thus $d_x = b_x + a_x$.

Inter-downcrossing Time d_x
For the process $\{\mathcal{D}_t(x)\}_{t\geq 0}$, using (6.11) and the elementary renewal theorem, the renewal rate is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t^c(x))}{t} = \lim_{t\to\infty} \frac{\mathcal{D}_t^c(x)}{t} \underset{a.s.}{=} r(x)f(x) = \frac{1}{E(d_x)}, x > 0.$$

Hence

$$E(d_x) = \frac{1}{r(x)f(x)}, x > 0. \tag{6.25}$$

Sojourn a_x Above Level x
From the renewal reward theorem

$$\frac{E(a_x)}{E(d_x)} = \frac{\lim_{t\to\infty}(\textit{time SP is above level x during (0,t)})}{t} = 1 - F(x),$$

$$E(a_x) = (1 - F(x)) \cdot E(d_x) = \frac{1 - F(x)}{r(x)f(x)}, x > 0. \tag{6.26}$$

From (6.26)

$$\frac{f(x)}{1 - F(x)} = \frac{1}{r(x)E(a_x)}, x > 0, \tag{6.27}$$

$$\frac{d}{dx}\ln(1 - F(x)) = \frac{-1}{r(x)E(a_x)}, x > 0. \tag{6.28}$$

Integrating on both sides of (6.28) with respect to x and computing the constant of integration by letting $x \downarrow 0$, gives

$$\begin{aligned} F(x) &= 1 - (1 - P_0)e^{-\int_{y=0}^{x} \frac{1}{r(y)E(a_y)}dy}, x \geq 0, \\ f(x) &= \tfrac{1-P_0}{r(x)E(a_x)} e^{-\int_{y=0}^{x} \frac{1}{r(y)E(a_y)}dy}, x > 0. \end{aligned} \tag{6.29}$$

(Possibly $0 < P_0 < 1$ or $P_0 = 0$.) The normalizing condition $F(\infty) = 1$, is

$$1 - (1 - P_0)e^{-\int_{y=0}^{\infty} \frac{1}{r(y)E(a_y)}dy} = 1,$$

which implies that $e^{-\int_{y=0}^{\infty} \frac{1}{r(y)E(a_y)}dy} = 0$ if $0 < P_0 < 1$.

Hazard Rate of PDF of Content at x

The left side of (6.27) is the *hazard rate* of the steady-state pdf of content at x. An inverse relation holds between it and the product $r(x)E(a_x)$. The hazard rate has the same dimension as $f(x)$, i.e., 1/[*content unit*]. (See Sect. 3.4.18 for definition and discussion of hazard rate.)

Sojourn b_x Below Level x

By the renewal reward theorem, $E(b_x)/E(d_x) = F(x)$. Thus

$$E(b_x) = F(x) \cdot E(d_x) = \frac{F(x)}{r(x)f(x)}, x > 0, \tag{6.30}$$

implying

$$\frac{f(x)}{F(x)} = \frac{d}{dx}\ln F(x) = \frac{1}{r(x)E(b_x)}, x > 0,$$

and

$$\begin{aligned} F(x) &= P_0 e^{\int_{y=0}^{x} \frac{1}{r(y)E(b_y)}dy}, x \geq 0, \\ f(x) &= \frac{P_0}{r(x)E(b_x)} e^{\int_{y=0}^{x} \frac{1}{r(y)E(b_y)}dy}, x > 0, \end{aligned} \tag{6.31}$$

using $F(0^+) = P_0$.

Interestingly, formulas (6.29) and (6.31) give two different expressions for $F(x)$ and $f(x)$, in terms of $E(a_x)$ and $E(b_x)$, respectively.

If $r(x) \equiv 1, x > 0$, the right side of the second equation in (6.31) reduces to the pdf of wait in the M/G/1 queue, i.e., since $E(b_x) = F(x)/f(x), x \geq 0$,

$$\begin{aligned} \frac{P_0}{1 \cdot E(b_x)} e^{\int_{y=0}^{x} \frac{1}{1 \cdot E(b_y)}dy} &= \frac{P_0 \cdot f(x)}{F(x)} e^{\int_{y=0}^{x} \frac{f(y)}{F(y)}dy} \\ = \frac{P_0 \cdot f(x)}{F(x)} e^{(\ln F(x) - \ln F(0))} &= \frac{P_0 \cdot f(x)}{F(x)} F(x) P_0^{-1} = f(x). \end{aligned}$$

as in formula (3.95).

As a mild check on (6.31), we compute $f(x)$ for the M/M/1 queue in which

$$E(b_x) = \frac{F(x)}{f(x)} = \frac{1 - (1 - (1 - \frac{\lambda}{\mu}))e^{-(\mu-\lambda)x}}{\lambda(1 - \frac{\lambda}{\mu})e^{-(\mu-\lambda)x}}, x \geq 0,$$

and $F(0) = P_0 = 1 - \frac{\lambda}{\mu}$. Substituting these values directly for $E(b_x)$ and P_0 in (6.31) leads to $f(x) = \lambda(1 - \frac{\lambda}{\mu})e^{-(\mu-\lambda)x}$, $x \geq 0$, the steady-state pdf of wait in M/M/1 (formula (3.112) in Sect. 3.5).

6.2.13 CDF and PDF of Excess of Jump over Level x

Let $\gamma_x :=$ excess of an input upcrossing of x (jump starts below x). Let $G_x(z), z > 0$, $g_x(z) = \partial G_x(z)/\partial z$, $z > 0$, denote the cdf and pdf of γ_x, respectively. We determine these quantities by means of an argument analogous to that in the proof of Theorem (3.7) in Sect. 3.7. In steady state,

$$\lim_{t\to\infty} \mathcal{U}_t(x)/t = \lim_{t\to\infty} \mathcal{D}_t(x)/t = r(x)f(x).$$

The rate at which the SP *upcrosses level* $x+z$ *whenever an input amount jump-upcrosses level* x is

$$r(x)f(\dot{x})\left[1 - G_x(z)\right].$$

A different expression for the *upcrossing rate of level* $x+z$, whenever the input amount jump-upcrosses level x is

$$\lambda P_0\overline{B}(x+z) + \lambda\int_{y=0}^{x}\overline{B}(x+z-y)f(y)dy,$$

which is the rate of jumps that start below level x, having excesses over x that upcross level $x+z$.

Therefore

$$r(x)f(x)\left[1-G_x(z)\right] = \lambda P_0\overline{B}(x+z) + \lambda\int_{y=0}^{x}\overline{B}(x+z-y)f(y)dy,$$

$$1 - G_x(z) = \frac{\lambda P_0\overline{B}(x+z) + \lambda\int_{y=0}^{x}\overline{B}(x+z-y)f(y)dy}{r(x)f(x)}$$

$$= \frac{\lambda P_0\overline{B}(x+z) + \lambda\int_{y=0}^{x}\overline{B}(x+z-y)f(y)dy}{\lambda P_0\overline{B}(x) + \lambda\int_{y=0}^{x}\overline{B}(x-y)f(y)dy},$$

and

$$G_x(z) = 1 - \frac{\lambda P_0\overline{B}(x+z) + \lambda\int_{y=0}^{x}\overline{B}(x+z-y)f(y)dy}{\lambda P_0\overline{B}(x) + \lambda\int_{y=0}^{x}\overline{B}(x-y)f(y)dy}, \quad (6.32)$$

$$g_x(z) = \frac{\lambda P_0\overline{b}(x+z) + \lambda\int_{y=0}^{x}\overline{b}(x+z-y)f(y)dy + \lambda\overline{B}(z)f(x)}{\lambda P_0\overline{B}(x) + \lambda\int_{y=0}^{x}\overline{B}(x-y)f(y)dy}. \quad (6.33)$$

6.2.14 Expected Nonempty Period

Let $\mathcal{B}_D$:= *nonempty period of the dam*. Then $\mathcal{B}_D = a_0$. Generally, the structure of $\mathcal{B}_D$ differs from that of the busy period $\mathcal{B}$ in the M/G/1 queue given in (3.83), because in M/G/r(·) the efflux rate $r(x)$ varies as x varies. This variation causes the sub-nonempty periods to depend on the beginning ordinate of their initial inputs. For example, in M/G/r(·), a_0 is infinite if $P_0 = 0$, corresponding to the case $r(x) = kx, x > 0, k > 0$, since sample paths decay exponentially between inputs and never decay completely to level 0 (see Example 6.3, Sect. 6.2.11).

Constant Efflux Rate $k > 0$
In the particular case where there is some constant $k > 0$ such that $r(x) \equiv k, x > 0$, the structure of $\mathcal{B}$ given by (3.83) and Fig. 3.6, Sect. 3.4.12, is preserved for $\mathcal{B}_D$, except that the slope of the sample path between inputs is $-k$. Then $0 < P_0 < 1$. Let S := *input size*. In particular, S is the size of the first input of a nonempty period. Let N_S := *number of inputs during the time required for the first S to deplete*, i.e., during a time $\int_{y=0}^{S} \frac{1}{r(y)} dy = \int_{y=0}^{S} \frac{1}{k} dy = S/k$ time units. Then

$$\mathcal{B}_D = \frac{S}{k} + \sum_{i=1}^{N_S} \mathcal{B}_{D,i}, \tag{6.34}$$

where $\mathcal{B}_{D,i}, i = 1, \ldots, N_S$ are sub-nonempty periods $\underset{dis}{=} \mathcal{B}_D$, independent of N_S. Taking expected values on both sides of (6.34) gives

$$E(\mathcal{B}_D) = \frac{E(S)}{k} + E(N_S)E(\mathcal{B}_D) = \frac{E(S)}{k} + \lambda \frac{E(S)}{k} E(\mathcal{B}_D), \tag{6.35}$$

since $E(N_S) = \lambda\,(E(S)/k)$. Equation (6.35) gives

$$E(\mathcal{B}_D) = E(a_0) = \frac{E(S)}{k\left(1 - \frac{\lambda}{k}E(S)\right)}. \tag{6.36}$$

Alternative Derivation of $E(\mathcal{B}_D)$
We can obtain P_0 directly from formula (6.21) when $r(x) \equiv k, x > 0$, by dividing by k and integrating both sides with respect to $x \in (0, \infty)$. Since $1 - P_0 = \int_{x=0}^{\infty} f(x)dx$, we get

$$P_0 = 1 - \frac{\lambda}{k} E(S). \tag{6.37}$$

We now use P_0 in (6.37) and the renewal reward theorem. Since E(*nonempty cycle)* := $E(d_0) = 1/\left(r(0^+)f(0)\right) = 1/\,(\lambda P_0)$, we get

$$\frac{E(\mathcal{B}_D)}{E(d_0)} = 1 - P_0$$
$$E(\mathcal{B}_D) = \frac{1-P_0}{r(0^+)f(0)} = \frac{1-P_0}{\lambda P_0}.$$

Substituting for P_0 from (6.37) gives

$$E(\mathcal{B}_D) = E(a_0) = \frac{E(S)}{k\left(1 - \frac{\lambda}{k}E(S)\right)}. \tag{6.38}$$

Formula (6.38), derived by LC and the renewal reward theorem, illustrates the usefulness of the formula

$$E(a_0) = \frac{1-P_0}{\lambda P_0}, \tag{6.39}$$

which also applies to $E(\mathcal{B})$ in M/G/1 queues, as well as to the nonempty period in M/G/r(·) dams where $0 < P_0 < 1$.

6.3 M/M/r(·) Dam

Assume inputs are of size $S \underset{dis}{=} \text{Exp}_\mu$ occurring at a Poisson rate λ. Assume the dam is stable, i.e., $\lambda E(S) < \lim_{x\to\infty} r(x)$ (see formula (6.24)), so the steady-state distribution of content exists.

6.3.1 Equation for Steady-State PDF of Content

Substitute $\overline{B}(x-y) = e^{-\mu(x-y)}$, $0 \le y < x$, in Eq. (6.21), resulting in the integral equation for the steady-state pdf of content $f(x)$,

$$r(x)f(x) = \lambda P_0 e^{-\mu x} + \lambda \int_{y=0}^{x} e^{-\mu(x-y)} f(y)dy, x > 0, \tag{6.40}$$

$$f(x) = \frac{\lambda}{r(x)}\left(P_0 e^{-\mu x} + \int_{y=0}^{x} e^{-\mu(x-y)} f(y)dy\right), x > 0. \tag{6.41}$$

6.3.2 Solution of Equation (6.40) for PDF of Content

Assume $P_0 > 0$. (Recall $P_0 > 0$ iff $\{W(t)\}_{t\ge 0}$ returns to 0, i.e., (6.5) holds.) Applying differential operator $\langle D + \mu\rangle$ to both sides of (6.40), leads to the differential equation for $f(x)$,

$$\frac{f'(x)}{f(x)} = -\frac{r'(x) + \mu r(x) - \lambda}{r(x)}, x > 0,$$
$$\frac{d}{dx}\ln\left(r(x)f(x)\right) = -\mu + \frac{\lambda}{r(x)}, x > 0, \tag{6.42}$$

by transposing $r'(x)/r(x)$ $(=d\ln r(x)/dx)$ to the left side and using well-known properties of derivatives and logarithms. The solution of (6.42) is

$$f(x) = \frac{\lambda P_0}{r(x)} e^{-\left(\mu x - \lambda \int_{y=0}^{x} \frac{dy}{r(y)}\right)}, x > 0, \tag{6.43}$$

upon applying the initial condition $r(0^+)f(0) = \lambda P_0$.

Substituting $f(x)$ from (6.43) into the normalizing condition $P_0 + \int_{x=0}^{\infty} f(x)dx = 1$ gives

$$P_0 = \frac{1}{1 + \lambda \int_{x=0}^{\infty} \frac{1}{r(x)} e^{-\left(\mu x - \lambda \int_{y=0}^{x} \frac{1}{r(y)} dy\right)} dx}. \tag{6.44}$$

As a mild check, let $r(x) = k > 0$. From (6.44)

$$P_0 = 1/\left(1 + \lambda/\left(k\left(\mu - \lambda/k\right)\right)\right) = \left(k\mu - \lambda\right)/\left(k\mu\right) = 1 - \lambda/\left(k\mu\right),$$

which agrees with (6.37), since $E(S) = 1/\mu$. In the M/M/1 queue, $k = 1$, so $r(x) \equiv 1$, $x > 0$. Substituting $r(x) \equiv 1$ in (6.43) and (6.44) gives (3.112) and (3.113) respectively, agreeing with the analogous results for M/M/1.

6.3.3 Sojourn Times and State-Space Levels

Assume $P_0 > 0$. From (6.25) and (6.26) with $x = 0$, we get $E($*nonempty* ***cycle***$)$ and $E($*nonempty* ***period***$)$ as

$$E(d_0) = \frac{1}{r(0^+)f(0)} = \frac{1}{\lambda P_0},$$
$$\text{and } E(a_0) = E(\mathcal{B}_D) = (1 - P_0)E(d_0) = \frac{1 - P_0}{\lambda P_0},$$

respectively, with P_0 given in (6.44).

In M/M/r(·), all upward jumps are $\underset{dis}{=}\text{Exp}_\mu$. By the memoryless property, the excess of a jump over any level x is also $\underset{dis}{=}\text{Exp}_\mu$. But, generally a_x depends on x. This differs from the M/M/1 queue or M/M/r(·) dam with $r(x) = k > 0$, $x > 0$, where a_x is *independent of* x, and $E(a_x) \equiv E(\mathcal{B})$, and $E(a_x) \equiv E(\mathcal{B}_D)$; respectively. The structure of $\mathcal{B}$ and $\mathcal{B}_D$ guarantees this independence (see formula (3.83)). However, generally In M/M/r(·), $r(x)$ varies with x; so a_x depends on the values of $r(y)$, $y > x$, $x \geq 0$. Nevertheless, we can still determine $E(a_x)$, $E(b_x)$ and $E(d_x)$ as long as we can solve for $\{P_0, f((x)\}_{x>0}$ as in Sects. 6.3.1–6.3.2.

Constant Efflux Rate

When $r(x) \equiv k$, $k > 0$, $x > 0$, the structure of $\mathcal{B}_D$ is similar to that of $\mathcal{B}$ in M/G/1. Thus, from (6.37) and (6.38),

$$P_0 = 1 - \frac{\lambda}{k\mu}$$

$$E(a_x) = E(B_D) = \frac{\frac{1}{\mu}}{k\left(1 - \frac{\lambda}{k\mu}\right)} = \frac{1}{k\mu - \lambda}, x \geq 0.$$

6.4 M/M/r(·) Dam with $r(x) = kx$

When the efflux rate ***varies directly with content***, $r(x) = kx$, $x > 0$, for some fixed $k > 0$, and $P_0 = 0$ (i.e., the efflux rate is proportional to content). (See Example 6.3 in Sect. 6.2.11).The sample path of $\{W(t)\}_{t\geq 0}$ has a negative exponential shape between input instants, because $r(W(t)) = kW(t) = -dW(t)/dt$, implying that $dW(t)/W(t) = -kdt$, with solution $W(t) = W(\tau_n)e^{-k(t-\tau_n)}$, $\tau_n \leq t < \tau_{n+1}$, $n = 0, 1, 2, \ldots$(see Formula (6.2) in Sect. 6.2.2).

6.4.1 PDF of Content and Its Laplace Transform

Upon substituting $r(x) = kx$ in (6.41) with $P_0 = 0$, we solve for $f(x)$ using Laplace transforms (see Sect. 3.4.4). The Laplace transform of $f(x)$ is

$$\widetilde{f}(s) \equiv \int_{x=0}^{\infty} e^{-sx} f(x)dx, s > 0.$$

In (6.41), multiplying both sides by e^{-sx}, and integrating on $x \in (0, \infty)$ yields

$$\widetilde{f}(s) = \lambda \int_{x=0}^{\infty} e^{-sx} \frac{1}{kx} \int_{y=0}^{x} e^{-\mu(x-y)} f(y) dy dx. \tag{6.45}$$

Taking d/ds on both sides of (6.45) and interchanging the order of integration gives

$$\frac{d}{ds}\widetilde{f}(s) = -\frac{\lambda}{k} \int_{y=0}^{\infty} e^{-sy} f(y) \int_{x=y}^{\infty} e^{-(s+\mu)(x-y)} dx dy.$$

The inner integral is $1/(\mu + s)$, implying the right side is $-(\lambda/k)\widetilde{f}(s)/(\mu + s)$, yielding differential equation

$$\frac{d}{ds}\widetilde{f}(s) + \frac{\lambda}{k}\left(\frac{1}{\mu + s}\right)\widetilde{f}(s) = 0. \tag{6.46}$$

Separation of variables in (6.46), and integration with respect to s gives

$$\widetilde{f}(s) = A(\mu + s)^{-\frac{\lambda}{k}},$$

for some constant A. The identity $\widetilde{f}(s) \equiv \int_{x=0}^{\infty} e^{-sx} f(x) dx$ implies $\widetilde{f}(0^+) = \int_{x=0}^{\infty} f(x) dx = 1$ (normalizing condition since $P_0 = 0$). Thus

$$\widetilde{f}(0^+) = A\mu^{-\frac{\lambda}{k}} = 1 \text{ and } A = \mu^{\frac{\lambda}{k}}.$$

Hence

$$\widetilde{f}(s) = \left(\frac{\mu}{\mu + s}\right)^{\frac{\lambda}{k}} = \left(\frac{1}{1 + \frac{s}{\mu}}\right)^{\frac{\lambda}{k}} = \left(1 + \frac{s}{\mu}\right)^{-\frac{\lambda}{k}}, s > 0. \tag{6.47}$$

In (6.47) $\widetilde{f}(s)$ is the Laplace transform of a Gamma pdf (e.g., p. 128, Sect. 3.3.1, in [84]); p. 166ff in [97]; p. 109 in [75]), namely

$$f(x) = \frac{1}{\Gamma\left(\frac{\lambda}{k}\right) \mu^{-\frac{\lambda}{k}}} x^{\left(\frac{\lambda}{k} - 1\right)} e^{-\mu x} = \frac{1}{\Gamma\left(\frac{\lambda}{k}\right)} (\mu x)^{\frac{\lambda}{k} - 1} e^{-\mu x} \mu, x > 0. \tag{6.48}$$

In (6.48), letting $u = \mu x$ gives

$$\int_{x=0}^{\infty} (\mu x)^{\frac{\lambda}{k} - 1} e^{-\mu x} \mu dx = \int_{0}^{\infty} u^{\left(\frac{\lambda}{k} - 1\right)} e^{-u} du = \Gamma\left(\frac{\lambda}{k}\right),$$

implying $\int_{x=0}^{\infty} f(x)dx = 1$. The uniqueness of $f(x)$ in (6.48) is guaranteed due to a one-to-one correspondence between $\widetilde{f}(s)$ and its inverse, up to a set of measure 0 (see pp. 13–14 in [95]).

The statistical moments of $f(x)$ about 0, are

$$E(W^n) = (-1)^n \left. \frac{d^n}{ds^n}\widetilde{f}(s)\right|_{s=0}, n = 1, 2, \ldots .$$

The first and second moments are

$$E(W) = \frac{\lambda}{k\mu}, E(W^2) = \frac{\lambda}{k\mu^2}\left(\frac{\lambda}{k} + 1\right).$$

The variance is

$$Var(W) = E(W^2) - (E(W))^2 = \frac{\lambda}{k\mu^2}.$$

Remark 6.6 In the literature the formula for a standard Gamma pdf is often

$$g(x) = \frac{1}{b\Gamma(c)}\left(\frac{x}{b}\right)^{c-1} e^{-\frac{x}{b}}, x > 0,$$

where $b > 0$, $c > 1$ (see p. 109 in [75]), having Laplace transform

$$\widetilde{g}(s) = (1 + bs)^{-c}, s > -\frac{1}{b}.$$

Since $b > 0$, it is sufficient to take $s > 0$. (The significance of $s > 0$ is discussed on pp. 13–14ff in [95].) Setting $b = \frac{1}{\mu}$, $c = \frac{\lambda}{k}$ gives $\widetilde{g}(s) = \widetilde{f}(s)$ in (6.47).

6.4.2 CDF of Content

The steady-state cdf $F(x)$ and pdf $f(x)$ of the content are

$$F(x) = \int_{y=0}^{x} f(y)dy = \frac{1}{\Gamma\left(\frac{\lambda}{k}\right)}\int_{y=0}^{x} (\mu y)^{\left(\frac{\lambda}{k}-1\right)} e^{-\mu y}\mu\, dy = \frac{\Gamma(\frac{\lambda}{k}, x)}{\Gamma\left(\frac{\lambda}{k}\right)}, x > 0, \quad (6.49)$$

where $\Gamma(\frac{\lambda}{k}, x) = \int_{y=0}^{x} u^{\left(\frac{\lambda}{k}-1\right)} e^{-u}du$ is the *incomplete* Gamma function (e.g., p. 15 in [138]). Thus $F(\infty) = \Gamma(\frac{\lambda}{k}, \infty)/\Gamma\left(\frac{\lambda}{k}\right) = 1$. Generally, $F(x)$ in (6.49) cannot be expressed in closed form for finite $x > 0$, but can be evaluated numerically.

6.4.3 Sojourns with Respect to a Level x

We examine next the inter-downcrossing time d_x, and sojourns a_x and b_x. Consider a sample path of $\{W(t)\}_{t\geq 0}$. Referring to Eqs. (6.25), (6.26), and (6.30) above, we get

$$E(d_x) = \frac{1}{r(x)f(x)} = \frac{1}{kxf(x)} = \frac{\Gamma\left(\frac{\lambda}{k}\right)}{kx(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}\mu}, x > 0; \tag{6.50}$$

$$E(a_x) = (1 - F(x))E(d_x) = \frac{\frac{1}{\Gamma(\frac{\lambda}{k})}\int_{y=x}^{\infty}(\mu y)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu y}\mu dy}{kx\frac{1}{\Gamma(\frac{\lambda}{k})}(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}\mu}$$

$$= \frac{\int_{y=x}^{\infty}\mu(\mu y)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu y}dy}{kx(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}\mu}, x > 0; \tag{6.51}$$

$$E(b_x) = F(x)E(d_x) = \frac{\int_{y=0}^{x}(\mu y)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu y}\mu dy}{kx(\mu x)^{\left(\frac{\lambda}{k}-1\right)}e^{-\mu x}\mu}, x > 0. \tag{6.52}$$

Naturally, $E(a_x) + E(b_x) = E(d_x)$. Quantities $E(d_x), E(a_x), E(b_x)$ can be evaluated numerically and plotted over a range of x in the state space, for any valid triplet of model parameters $\{\lambda, k, \mu\}$ (see Figs. 6.3, 6.4, 6.5, 6.6, and 6.7).

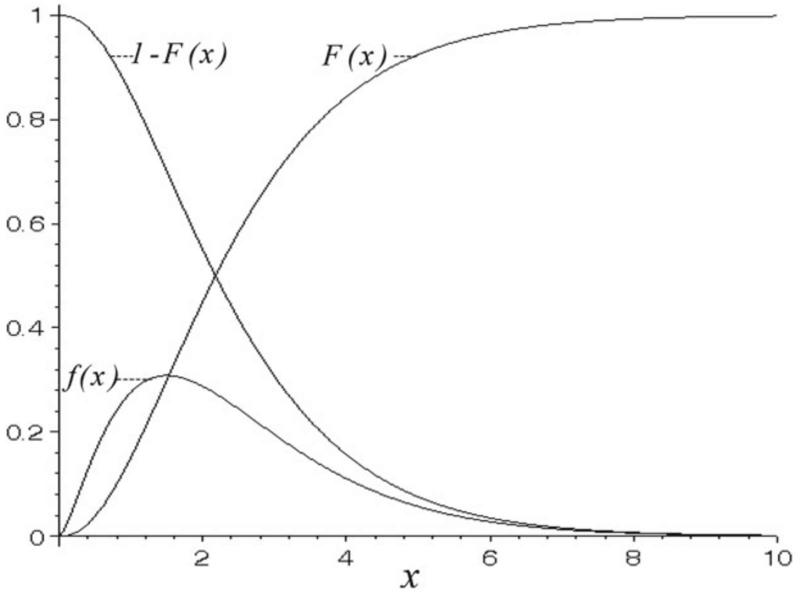

Fig. 6.3 Steady-state pdf $f(x)$, cdf $F(x)$, and complementary cdf $1 - F(x)$, in M/M/r(·) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$

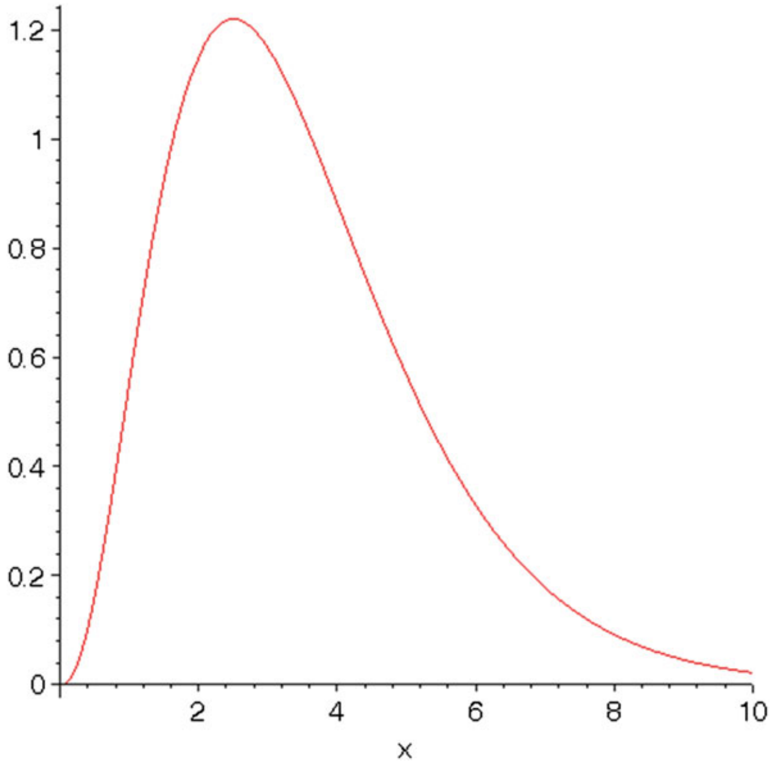

Fig. 6.4 $\text{Lim}_{t\to\infty}\mathcal{D}_t(x)/t$ versus x in M/M/r(·) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$. $\text{Lim}_{t\to\infty}\mathcal{D}_t(x)/t = \text{Lim}_{t\to\infty}\mathcal{U}_t(x)/t$

Example 6.4 Consider an M/M/r(·) dam with $r(x) = kx$, $x > 0$. (See Figs. 6.3, 6.4, 6.5, 6.6, 6.7 and 6.8.) Set $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$. The steady-state pdf of content is

$$f(x) = 0.752253x^{1.5}e^{-x}, x > 0.$$

The cdf of content is, for $x > 0$,

$$F(x) = -0.188063\left(4.0x^{3/2} + 6.0x^{1/2} - 5.317362 \cdot erf(x^{1/2}) \cdot e^{x}\right) \cdot e^{-x},$$

where $erf(x) := \left(2/\sqrt{\pi}\right)\int_0^x e^{-t^2}dt$, the **error function** (see p. 262 in [1]). Because $\mu = 1$ in this example,

$$E(W) = \frac{\lambda}{k\mu} = Var(W) = \frac{\lambda}{k\mu^2} = 2.5.$$

The hazard rate of $f(x)$ is plotted for values of x in Fig. 6.8.

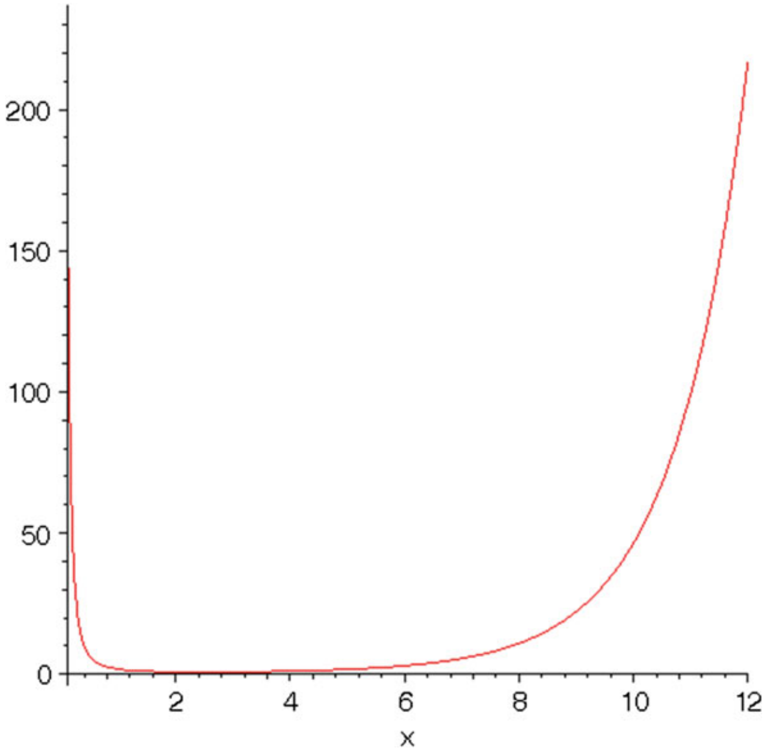

Fig. 6.5 $E(d_x)$ versus x, in M/M/r($\cdot$) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$. The "bathtub" shape of $E(d_x)$ is intuitive

6.5 M/M/$r(\cdot)$ with Special Zero-Content Inputs

Assume $P_0 > 0$, and inputs when the dam is empty, have a special size S_0 having cdf $B_0(x), x > 0$, pdf $b_0(x), x > 0$, and $\overline{B_0}(x) = 1 - B_0(x), x \geq 0$. Rate balance across level x, and the law of total probability (normalizing condition), imply

$$\begin{gathered} r(x)f(x) = \lambda P_0\overline{B_0}(x) + \lambda \int_{y=0}^{x} \overline{B}(x-y)f(y)dy, x > 0, \\ P_0 + \int_{y=0}^{\infty} f(x)dx = 1. \end{gathered} \tag{6.53}$$

We can solve (6.53) for $\{P_0, f(x)\}_{x>0}$ (analytically or numerically); then obtain $F(x)$, and $E(\mathcal{C}_D)$ $(=E(d_0)) = 1/\left(r(0^+)f(0)\right)$ using equality $r(0^+)f(0) = \lambda P_0$. Applying the renewal reward theorem, we obtain $E(a_x) = (1 - F(x))/(\lambda P_0)$ and $E(b_x) = F(x)/(\lambda P_0), x \geq 0$. Thus $E(\mathcal{B}_D) = (1 - P_0)/(\lambda P_0)$. In particular, the first 'input' of

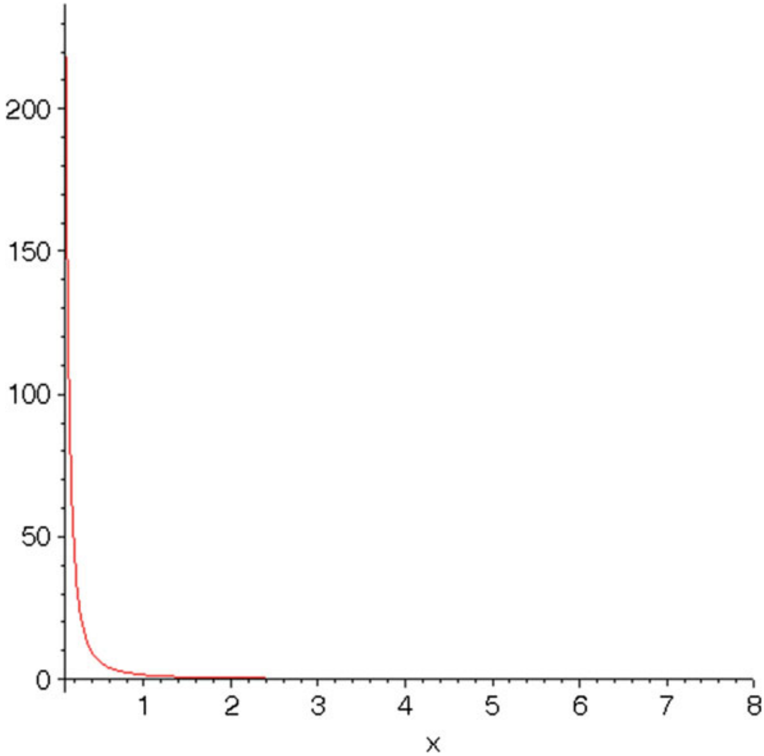

Fig. 6.6 $E(a_x)$ versus x in M/M/r(·) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$

a_x is $\underset{dis}{=}\gamma_x$, the excess of a jump over x, distributed differently from all other jumps during a_x. In that case the cdf of γ_x, denoted $G_x(\cdot)$, is given by formula (6.32) in Sect. 6.2.13.

An interesting inference about the structure of $\mathcal{B}_D$ (including a_x and b_x) follows because $\{\mathcal{D}_t(x)\}_{t\geq 0}$ is a renewal process. *Although the structure of* $\mathcal{B}_D$ *generally differs from that of* $\mathcal{B}$ *in M/G/1, or* $\mathcal{B}_D$ *in M/G/r(·)* ($r(x) = k > 0, x > 0$)*, we can always derive* $E(a_x)$ and $E(b_x)$ *once* $F(x)$*,* $x \geq 0$ *and* P_0 are known. Thus, the LC-connected derivation of $E(a_x)$ or $E(b_x)$, is more general than the derivation based directly on structure.

6.6 Generalization of M/G/r(·) Dam

We discuss a generalization of the M/G/r(·) dam considered in Sects. 6.2–6.3.The generalized model allows for SP *downward jumps due to exogenous* events; *state-dependent prescribed jumps* just after the SP *hits or jump-crosses designated state-*

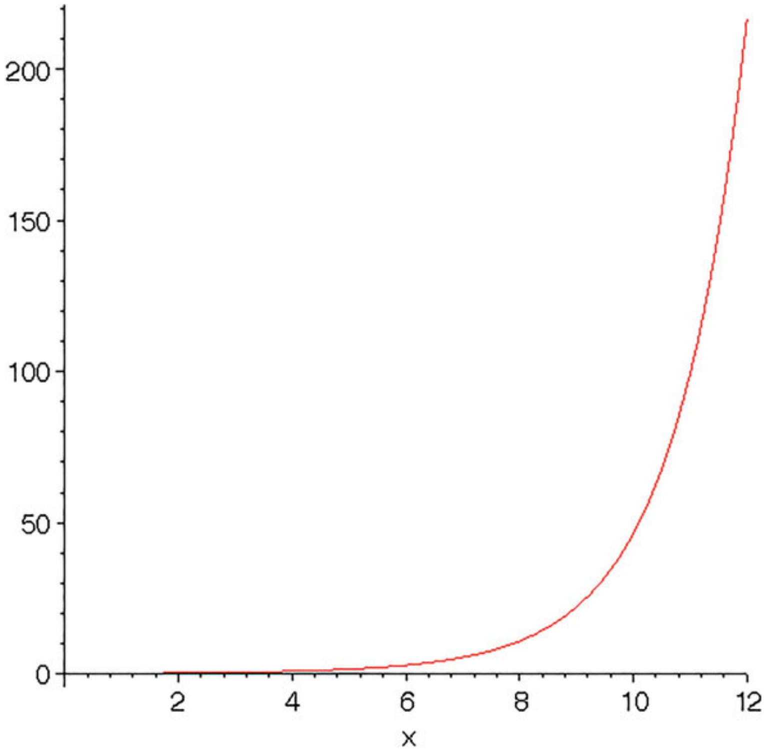

Fig. 6.7 $E(b_x)$ versus x in M/M/r(·) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$

space levels, e.g., thresholds or barriers; *specialized state-dependent jumps if an exogenous event occurs when the SP is in a designated state-space interval*; *etc.*

For example, in Marketing Science a target population of repeated advertisements for a product may develop a "rebound" effect against purchasing the product due to "overselling" (i.e., over-advertising). Let $\{W(t)\}_{t\geq 0}$ represent the *consumer response* process for the product, where high measures are favorable, and low measures are unfavorable. The SP may take a sudden jump downward if a new advertisement occurs while the SP is above a 'tolerance' threshold. A sample path of $\{W(t)\}_{t\geq 0}$ would increase in a roughly "saw-tooth", possibly non-linear, pattern, but make exceptional downward jumps from levels above the threshold. (See, e.g., [40].)

A related model applies in multiple dosing of a medication in pharmacokinetics. Suppose the control of a patient's illness depends on lowering to a therapeutic range, systolic blood pressure denoted by $\{BP(t)\}_{t\geq 0}$. The goal of the dosing regime is to maintain $BP(t)$ within a specified finite range, say (L, H) measured in millimeters of mercury (mmHg). This implies the concentration in the blood of the medication, denoted by $\{BC(t)\}_{t\geq 0}$, should be in a corresponding therapeutic range, say (α, β) measured in milligrams per liter (mg/L). If $BC(t)$ upcrosses threshold β, then $BP(t)$

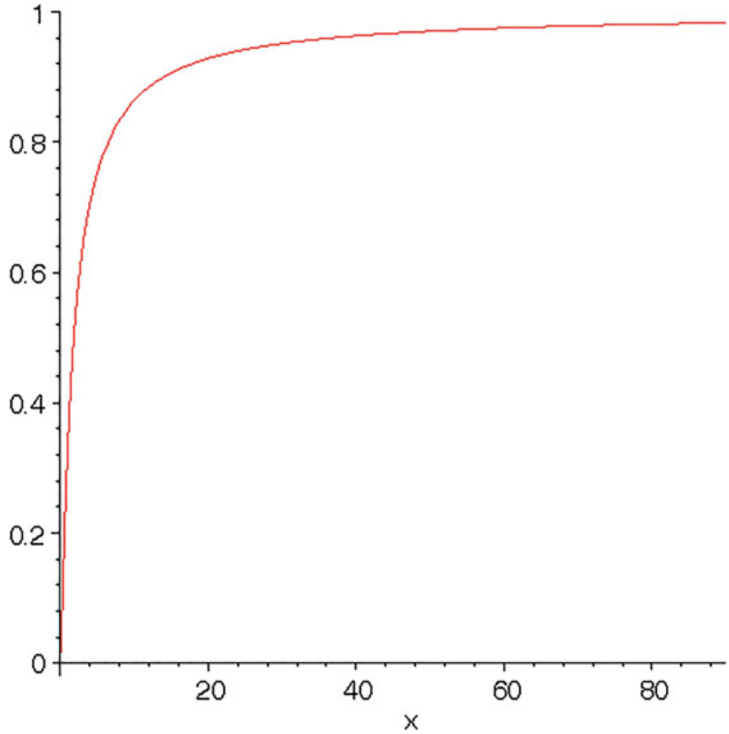

Fig. 6.8 ***Hazard rate*** $\frac{f(x)}{1-F(x)}$ for steady-state distribution of content in M/M/r(·) dam: $r(x) = kx$, $\lambda = 5.0$, $\mu = 1.0$, $k = 2.0$. Note the inverse relation with $E(a_x)$

will drop below threshold L. If $BC(t)$ downcrosses threshold α, then $BP(t)$ will upcross threshold H. A sample path of $\{BP(t)\}_{t\geq 0}$ would move in a roughly "saw-tooth" pattern, inversely emulating the pattern of $\{BC(t)\}_{t\geq 0}$. Similar remarks apply to illnesses depending on blood-thinning medications to protect against strokes (see Sect. 11.6; also [47]).

6.6.1 Model and Steady-State Distribution of Content

Let $\{W(t)\}_{t\geq 0}$ denote the content of a dam with "wide-sense" state space $\boldsymbol{S} \subseteq \mathbb{R}$, which may contain sets having probability 0 (see Sect. 2.3.1). For example, in the standard $\langle s, S\rangle$ inventory, the usual state space is interval $(s, S]$, which supports the probability distribution of inventory. The wide-sense state space is $(-\infty, S]$ because some demands propel the sample path of $\{W(t)\}_{t\geq 0}$ below the reorder point s. Prescribed replenishments then cause the SP to jump immediately up to level S (double jump), so $\{W(t)\}_{t\geq 0}$ spends *zero time* below level s (see Example 2.2 and

Fig. 2.2 in Sect. 2.2.2). The proportion of time $\{W(t)\}_{t\geq 0}$ spends below level s is zero, so the probability of $(-\infty, s]$ is zero.

A particular model may permit jumps due to exogenous events or by *prescription*. Assume that the SP makes *upward* and *downward* jumps at *exogenous* Poisson rates λ_u, λ_d respectively, which are independent of each other and of the current state of the system. Let the corresponding upward and downward jump magnitudes have cdfs $B_u(\cdot)$, $B_d(\cdot)$, and complementary cdfs $\overline{B_u}(\cdot)$, $\overline{B_d}(\cdot)$, respectively. Let $F(\cdot), f(\cdot)$ denote, respectively, the steady-state cdf and pdf of $W(t)$ as $t \to \infty$. Our immediate aim is to derive an integral equation for $f(x)$.

Let the downward jumps occur at instants $0 \equiv \tau_{d,1} < \tau_{d,2} < \cdots$, and upward jumps at instants $0 \equiv \tau_{u,1} < \tau_{u,2} < \cdots$, respectively. Possibly, the SP makes both an upward and downward jump at the same instant (see Sect. 2.3). Without loss of generality, we assume the initial state is $W(0) > 0$. Let $\{\tau_n\}_{n=1,\ldots} = \{\tau_{d,i}\}_{i=1,2,\ldots} \cup \{\tau_{u,i}\}_{i=1,2,\ldots}$.Thus $\{\tau_n\}_{n=1,2,\ldots}$ is a refinement of $\{\tau_{d,i}\}_{i=1,2,\ldots}$ and $\{\tau_{u,i}\}_{i=1,2,\ldots}$. The SP jumps occur at instants $0 < \tau_1 < \tau_2 < \cdots$.

Efflux Rate
The efflux rate $r(W(t)) := dW(t)/dt$ is specified by Eqs. (6.2) and (6.3), above.

Sample Path
A typical sample path of $\{W(t)\}_{t\geq 0}$ is a piecewise continuous function in the time-state plane, which decreases continuously between jumps (see Definition 2.1 in Sect. 2.2.1).

6.6.2 SP Downcrossings

Consider the following types of downcrossings of level x, and their number during $(0, t)$.

$\mathcal{D}_t^c(x)$:= *number of left-continuous downcrossings of level* x.

$\mathcal{D}_{t,d}^j(x)$:= number of *jump* downcrossings of level x *at exogenous Poisson rate* λ_d.

$\mathcal{D}_{t,p}^j(x)$:= *number of state-dependent policy (i.e., prescribed) jump* downcrossings of x, e.g., following hits of a threshold or barrier above x.

$\mathcal{D}_t^j(x)$:= *total* number of SP downward jumps of x.

Then $\mathcal{D}_t^j(x) = \mathcal{D}_{t,d}^j(x) + \mathcal{D}_{t,p}^j(x)$.

Theorem 6.4

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t^c(x))}{t} = \lim_{t\to\infty} \frac{\mathcal{D}_t^c(x)}{t} \underset{a.s.}{=} r(x)f(x), x \in \boldsymbol{S}, \tag{6.54}$$

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_{t,d}^j(x))}{t} = \lim_{t\to\infty} \frac{\mathcal{D}_{t,d}^j(x)}{t} \underset{a.s.}{=} \lambda_d \int_{y=x}^{\infty} \overline{B_d}(y-x)f(y)dy, x \in \boldsymbol{S}. \tag{6.55}$$

Proof Formula (6.54) follows similarly as in Theorem 6.1 and Corollary 6.2 in Sect. 6.2.8. Formula (6.55) follows as in Theorem 6.2 in Sect. 6.2.8. ■

6.6.3 SP Upcrossings

Consider the following types of upcrossings of level x and their number during $(0, t)$.
$\mathcal{U}^{j}_{t,u}(x) :=$ *number of jump upcrossings of level x due to the exogenous Poisson rate* λ_u.
$\mathcal{U}^{j}_{t,p}(x) :=$ *number of prescribed or policy state-dependent jump upcrossings of level* x.
$\mathcal{U}^{j}_{t}(x) :=$ *total number of SP jump upcrossings.*

Then $\mathcal{U}^{j}_{t}(x) = \mathcal{U}^{j}_{t,u}(x) + \mathcal{U}^{j}_{t,p}(x)$. In this model, every upcrossing is a ***jump upcrossing***.

Theorem 6.5

$$\lim_{t\to\infty} \frac{E(\mathcal{U}^{j}_{t,u}(x))}{t} = \lim_{t\to\infty} \frac{\mathcal{U}^{j}_{t,u}(x)}{t} \underset{a.s.}{=} \lambda_u \int_{y=-\infty}^{x} \overline{B_u}(x-y)f(y)dy, x \in \boldsymbol{S}, \tag{6.56}$$

Proof Similar to proof of Theorem 6.2 in Sect. 6.2.8. ■

Remark 6.7 All three terms in Theorem 6.5 represent the long-run rate of SP upward jumps **due to Poisson rate** λ_u, from state-space set $(-\infty, x]$ into (x, ∞).

6.6.4 Integral Equation for PDF of Content

Applying rate balance across level x, *total downcrossing rate* of x = *total upcrossing rate of* x. Thus

$$\lim_{t\to\infty} \frac{E(\mathcal{D}^{c}_{t}(x))}{t} + \lim_{t\to\infty} \frac{E(\mathcal{D}^{j}_{t}(x))}{t} = \lim_{t\to\infty} \frac{E(\mathcal{U}^{j}_{t,u}(x))}{t} + \lim_{t\to\infty} \frac{E(\mathcal{U}^{j}_{t,p}(x))}{t}. \tag{6.57}$$

Substituting from Theorems 6.4 and 6.5 gives

$$\begin{aligned} &r(x)f(x) + \lambda_d \textstyle\int_{y=x}^{\infty} \overline{B_d}(y-x)f(y)dy + \lim_{t\to\infty} \frac{E(\mathcal{D}^{j}_{t,p}(x))}{t} \\ &= \lambda_u \textstyle\int_{y=-\infty}^{x} \overline{B_u}(x-y)f(y)dy + \lim_{t\to\infty} \frac{E(\mathcal{U}^{j}_{t,p}(x))}{t}, x \in \boldsymbol{S}. \end{aligned} \tag{6.58}$$

In models where Eq. (6.58) applies, the terms

$$\lim_{t\to\infty} \frac{E(\mathcal{D}^{j}_{t,p}(x))}{t} \text{ and } \lim_{t\to\infty} \frac{E(\mathcal{U}^{j}_{t,p}(x))}{t}$$

are usually expressed in terms of $f(x)$, or as constants. For example, in a standard $\langle s, S\rangle$ inventory model,

$$\lambda_u = 0, \lim_{t\to\infty} \frac{E(\mathcal{D}^{j}_{t,p}(x))}{t} = 0,$$

$$\text{and } \lim_{t\to\infty} \frac{E(\mathcal{U}^j_{t,p}(x))}{t} = r(s)f(s) + \lambda_d \int_{y=s}^{S} \overline{B}(y-s)f(y)dy,$$

where λ_d is the demand rate (see Sect. 6.8, in which $\lambda_d \equiv \lambda$).

Remark 6.8 Integral Eq. (6.58) can serve as a **template** for variants of the M/G/r(·) dam. We do not solve the equation here. In any related variant, Eq. (6.58) will have a particular form, depending on the model parameters. It can then be solved for $f(x)$ (see Sect. 6.8 below).

$\langle S, S\rangle$ **Inventory**
The $\langle s, S\rangle$ continuous review inventory system is a special case of this model. If there is no lead time and no backlogging, then $r(x) > 0$ for all $x \in (s, S]$. If there is a lead time and backlogging is allowed, then the regular state space and wide-sense state space are both equal to the interval $(-\infty, S]$; also $r(x) = 0$ for $x < s$ (see, e.g., [4]).

In the $\langle s, S\rangle$ model, **prescribed (i.e., policy)** jump upcrossings occur, due to replenishments up to level S whenever the inventory jumps to or below level s, or makes a left-continuous hit of level s from above.

6.7 r(·)/G/M Dam

Consider a dam with a continuous *influx* when the content is positive. The influx is interrupted by "demands" for content (i.e., instantaneous outputs), which occur in an independent Poisson process. The demand sizes are i.i.d. positive random variables, having a common general distribution. If a demand exceeds the current content, the dam becomes empty. Empty periods are exponentially distributed with a common mean, independent of other factors. We may regard an empty period as "setup time" needed before starting a new influx cycle. We call a dam having these properties as an '**r(·)/G/M**' dam, to emphasize the continuous influx rate $r(\cdot)$. The r(·)/G/M dam generalizes the "extended age" process for a G/M/1 queue (Sect. 5.1.1).

The r(·)/G/M dam can be regarded as a template for a variety of production-inventory models where the production rate depends on the current stock level. There are many related variations. We can include: a fixed upper bound on content; thresholds indicating changes in influx rate; several fixed levels at which production may pause for a time; lost sales; backlogging, etc. The r(·)/G/M model is related to the surplus (risk reserve) in a risk model in actuarial science, where the influx is the rate of increase of surplus due to premium payments, and the outputs correspond to claim amounts (see Fig. 2.5 in Sect. 2.2.2, Table 2.1 in Sect. 2.4; Sect. 11.1).

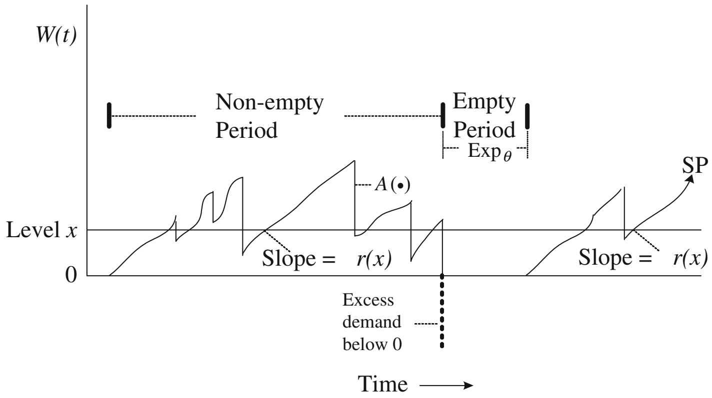

Fig. 6.9 Sample path of standard r(·)/G/M dam

6.7.1 *Model Specification and Notation*

Let $\{W(t)\}_{t\geq 0}$ denote the content of the dam at time $t \geq 0$. The influx goes on continuously at a positive rate $dW(t)/dt = r(W(t))$, when $W(t) > 0$. Demands for content occur at a Poisson rate μ, and are filled immediately (e.g., a sudden demand for water from a reservoir, oil from a storage tank; a rush order for a product; etc.). The demand sizes are positive with common cdf $A(\cdot)$ and complementary cdf $\overline{A}(\cdot)$. If a demand at t_0^- exceeds the current content, the resulting "content" would be negative. The corresponding end point of the SP downward jump would be below level 0 (Fig. 6.9). Only part of the demand is filled. Various policies can be used regarding the excess demand (e.g., backlogging). To focus on the LC analysis, we shall assume 'no backlogging'. Then the content at t_0 would be $W(t_0) = 0$. It *remains at level* 0 for a time $=$ Exp$_\theta$ independent of the unfilled excess demand below 0. During an empty period $dW(t)/dt = 0$. At the end of an empty period, the content begins to rise from level 0 at rate $r(0^+)$, and continues to rise in a roughly 'saw-tooth pattern' until some future demand takes the content 'below level 0' (see *pass by* in Fig. 2.16). The content alternates between nonempty and empty periods (Fig. 6.9).

If the dam is stable then the content will return to level 0 (state $\{0\}$ is positive recurrent). Denote the *transient* pdf and cdf of content, $t \geq 0$, by $\{P_0(t), f_t(x)\}_{x>0}$ and $F_t(x), x \geq 0$, respectively. Then $P_0(t) = F_t(0)$. Denote the *steady-state* pdf and cdf of content by $\{P_0, f(x)\}_{x,>0}$ and $F(x), x \geq 0$, respectively.

6.7.2 *Equation for Transient PDF of Content*

Consider a sample path of $\{W(t)\}_{t\geq 0}$. Let $\mathcal{U}_t(x), \mathcal{D}_t(x)$ denote the number of up- and downcrossings of x during $(0, t)$, respectively. In r(·)/G/M sample paths rise steadily at rate $r(x)$ depending on x. Even so we can readily modify the proofs in Theorem 3.4 in Sect. 3.2.7 and Theorem 3.3, in Sect. 3.2.5, getting

$$\begin{aligned}
&\tfrac{\partial}{\partial t}E(\mathcal{U}_t(x)) = r(x)f_t(x), x > 0, t > 0,\\
&\tfrac{\partial}{\partial t}E(\mathcal{U}_t(0)) = r(0^+)f_t(0) = \theta P_0(t), t > 0,\\
&\tfrac{\partial}{\partial t}E(\mathcal{D}_t(x)) = \mu \textstyle\int_{y=x}^{\infty} \overline{A}(y-x)f_t(y)dy, x \geq 0, t > 0.
\end{aligned} \tag{6.59}$$

Consider set $\boldsymbol{A} = [0, x]$, $x \geq 0$, in the state space. Theorem B (see Theorem 4.2 in Sect. 4.2.1 and Theorem 3.2 in Sect. 3.2.4) gives

$$\frac{\partial}{\partial t}E(\mathcal{I}_t(\boldsymbol{A})) = \frac{\partial}{\partial t}E(\mathcal{O}_t(\boldsymbol{A})) + \frac{\partial}{\partial t}P_t(\boldsymbol{A}), \tag{6.60}$$

where $\mathcal{I}_t(\boldsymbol{A})$, $\mathcal{O}_t(\boldsymbol{A})$ are the number of SP entrances and exits of $\boldsymbol{A}$ during $(0, t)$, respectively. We assume that a 'pass by' of level 0 due to a downcrossing of level 0 results immediately in an entrance of $\{0\}$. Thus $\mathcal{I}_t(\boldsymbol{A}) = \mathcal{D}_t(x)$, $\mathcal{O}_t(\boldsymbol{A}) = \mathcal{U}_t(x)$, $P_t(\boldsymbol{A}) = F_t(x)$. Substitution from (6.59) into (6.60) results in an *integro-differential* equation for $f_t(x)$ and a *differential* equation for $P_0(t)$, namely

$$\begin{aligned}
\mu \int_{y=x}^{\infty} \overline{A}(y-x)f_t(y)dy &= r(x)f_t(x) + \frac{\partial}{\partial t}F_t(x), x > 0,\\
&= r(0^+)f_t(0) + \frac{\partial}{\partial t}P_0(t) = \theta P_0(t) + \frac{\partial}{\partial t}P_0(t).
\end{aligned} \tag{6.61}$$

The normalizing condition is

$$P_0(t) + \int_{x=0}^{\infty} f_t(x)dx = 1, \text{ for each } t \geq 0.$$

Remark 6.9 In Eq. (6.61), the terms $\partial F_t(x)/\partial t$, $\partial P_0(t)/\partial t$ appear on the opposite side from the integrals, in contrast to Eqs. (6.16) and (6.17). The reason is that the sample path of content **increases** in r(·)/G/M, whereas it **decreases** in the M/G/r(·) dam discussed in Sect. 6.2.9.

6.7.3 Equation for Steady-State PDF of Content

If the dam is stable

$$\lim_{t\to\infty} f_t(x) = f(x), \quad \lim_{t\to\infty} \frac{\partial}{\partial t}F_t(x) = 0, x \geq 0, \quad \lim_{t\to\infty} P_0(t) = P_0.$$

We get an integral equation for $f(x)$ by letting $t \to \infty$ in (6.61), implying also a rate-balance equation for state $\{0\}$, viz.,

$$\begin{aligned} r(x)f(x) &= \mu \int_{y=x}^{\infty} \overline{A}(y-x)f(y)dy, \, x > 0, \\ r(0^+)f(0) &= \mu \int_{y=0}^{\infty} \overline{A}(y)f(y)dy = \theta P_0. \end{aligned} \tag{6.62}$$

The normalizing condition is

$$P_0 + \int_{x=0}^{\infty} f(x)dx = 1. \tag{6.63}$$

We can also derive (6.62) directly by considering a sample path of $\{W(t)\}_{t\geq 0}$ (Fig. 6.9). Fix level $x > 0$. The upcrossing rate of level x is $r(x)f(x)$. The downcrossing rate of x is $\mu \int_{y=x}^{\infty} \overline{A}(y-x)f(y)dy$. Rate balance across level x gives the first equation in (6.62); the second equation follows by balancing the SP entrance and exit rates of state $\{0\}$.

Remark 6.10 In r(·)/G/M, Eq. (6.62) for the steady-state pdf of content generalizes that for the pdf of "extended" age in the G/M/1 queue, i.e., $r(x)f(x)$ replaces $f(x)$ on the left side of Eq. (5.7) in Sect. 5.1.3.

6.7.4 Sojourn Times Above and Below a Level

Let a_x := *sojourn time above level* x, b_x := *sojourn time at or below level* x. Due to Poisson arrivals, the upcrossing instants of level x form a renewal process with d_x := *interarrival time*, between successive upcrossings, and $E(d_x) = 1/(r(x)f(x)), x \geq 0$.

***E*(*a*$_x$)**

By the renewal reward theorem

$$\frac{E(a_x)}{E(d_x)} = 1 - F(x).$$

Also,

$$\frac{E(\mathcal{B}_D)}{E(d_0)} = \frac{E(a_0)}{E(d_0)} = 1 - F(0).$$

Thus,

$$E(a_x) = \frac{1 - F(x)}{r(x)f(x)}, x > 0, \tag{6.64}$$

$$E(\mathcal{B}_D) = E(a_0) == \frac{1 - P_0}{r(0^+)f(0)} = \frac{1 - P_0}{\theta P_0}. \tag{6.65}$$

where $f(x), f(0)$ and P_0 are the solutions of (6.62), and (6.63); and *empty period* $\underset{dis}{=}$ Exp_θ.

Relating $f(x)$ and $E(a_x)$

From (6.64), the hazard rate of $f(x)$ is

$$\frac{f(x)}{1-F(x)} = \frac{1}{r(x)E(a_x)}, x > 0,$$

$$\text{and } \frac{d}{dx}\ln(1-F(x)) = \frac{-1}{r(x)E(a_x)},$$

$$1-F(x) = Ce^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}, \tag{6.66}$$

where C is a constant, evaluated by letting $x \downarrow 0$ in (6.66), resulting in $C = 1 - P_0$. Thus,

$$F(x) = 1-(1-P_0)e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}, x \ge 0. \tag{6.67}$$

Taking d/dx in (6.67) gives the pdf

$$f(x) = \frac{1-P_0}{r(x)E(a_x)}e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}, x > 0. \tag{6.68}$$

From (6.65)

$$f(x) = \frac{E(a_0)\theta P_0}{r(x)E(a_x)}e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}, x > 0. \tag{6.69}$$

The normalizing condition $P_0 + \int_{x=0}^{\infty} f(x)dx = 1$, gives

$$P_0 = \frac{1}{1+E(a_0)\theta\int_{x=0}^{\infty}\left(\frac{e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}}{r(x)E(a_x)}\right)dx}; \tag{6.70}$$

from (6.65), another expression for P_0 is

$$P_0 = \frac{1}{1+\theta E(a_0).} \tag{6.71}$$

Formula (6.71) implies that the integral in the denominator of (6.70), $\int_{x=0}^{\infty}\left(\frac{e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy}}{r(x)E(a_x)}\right)dx = 1$. This equality is verified by letting $u = \int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy$, whence $du = 1/(r(x)E(a_x))\,dx$, resulting in

$$\int_{u=0}^{\infty} e^{-u}du = \left[-e^{-u}\right]_{u=0}^{\infty} = -0 + e^0 = 1.$$

$E(b_x)$

Similarly, we obtain (using (6.62)),

$$E(b_x) = \frac{F(x)}{r(x)f(x)} = \frac{F(x)}{\mu\int_{y=x}^{\infty}\overline{A}(y-x)f(y)dy}, x \ge 0, \tag{6.72}$$

$$E(b_0) = \frac{P_0}{r(0^+)f(0)} = \frac{P_0}{\mu\int_{y=0}^{\infty}\overline{A}(y-x)f(y)dy} = \frac{P_0}{\theta P_0} = \frac{1}{\theta}. \tag{6.73}$$

Hence,

$$\begin{aligned}
&\frac{f(x)}{F(x)} = \frac{1}{r(x)E(b_x)},\\
&\frac{d}{dx}\ln F(x) = \frac{1}{r(x)E(b_x)},\\
&F(x) = C_1 e^{\int_{y=0}^{x}\frac{1}{r(y)E(b_y)}dy},\\
&F(x) = P_0 e^{\int_{y=0}^{x}\frac{1}{r(y)E(b_y)}dy}, x \geq 0,\\
&f(x) = \frac{P_0}{r(x)E(b_x)} e^{\int_{y=0}^{x}\frac{1}{r(y)E(b_y)}dy}, x > 0.
\end{aligned} \tag{6.74}$$

In (6.74) $F(0) = P_0$ and $F(\infty) = 1$.

Example 6.5 As a mild check on (6.74) let $r(x) \equiv 1, x > 0, \overline{A}(x) = e^{-\lambda x}, x \geq 0$, inter-demand time $\underset{dis}{=}\text{Exp}_\lambda$ (i.e., $\theta = \lambda$), corresponding to an M/M/1 queue (i.e., age in G/M/1 specialized to M/M/1).

In M/M/1 the relevant quantities are: $F(x) = 1 - \frac{\lambda}{\mu}e^{-(\mu-\lambda)x}$, $x \geq 0$; $f(x) = \lambda P_0 e^{-(\mu-\lambda)x}$, $x > 0$; $P_0 = 1 - \lambda/\mu$. Then

$$E(b_x) = \frac{F(x)}{f(x)} = \frac{1 - \frac{\lambda}{\mu}e^{-(\mu-\lambda)x}}{\lambda P_0 e^{-(\mu-\lambda)x}}.$$

In (6.74), using $F(0) = P_0$,

$$\begin{aligned}
\int_{y=0}^{x}\frac{1}{r(y)E(b_y)}dy = \int_{y=0}^{x}\frac{1}{E(b_y)}dy = \int_{y=0}^{x}\frac{f(y)}{F(y)}dy = \ln\left(\frac{F(x)}{F(0)}\right),\\
F(x) = P_0 e^{\int_{y=0}^{x}\frac{1}{r(y)E(b_y)}dy} = P_0 e^{\ln\left(\frac{F(x)}{F(0)}\right)} = P_0\frac{F(x)}{F(0)}\\
= 1 - \frac{\lambda}{\mu}e^{-(\mu-\lambda)x}, x \geq 0.
\end{aligned}$$

6.7.5 k/G/M Dam

Let the influx rate be $r(x) = k, x > 0, k > 0$, and assume the *output sizes* are $\underset{dis}{=}\text{Exp}_\lambda$. Thus $\overline{A}(x) = e^{-\lambda x}$, $x > 0$, $\lambda > 0$ (see Fig. 6.10). Since the inter-output times when $W(t) > 0$ are $\underset{dis}{=}$ Exp_μ, the sojourn time $a_x \underset{dis}{=} a_0$ (nonempty period), and $E(a_x) = (1 - P_0)/(\theta P_0)$, $x > 0$. From (6.68)

$$f(x) = \frac{E(a_0)\theta P_0}{r(x)E(a_x)} e^{-\int_{y=0}^{x}\frac{1}{r(y)E(a_y)}dy} = \frac{\theta P_0}{k} e^{-\frac{\theta P_0}{k(1-P_0)}x}, x > 0. \tag{6.75}$$

From (6.70)

$$P_0 = \frac{1}{1 + \frac{\theta}{k}\int_{x=0}^{\infty} e^{-\int_{y=0}^{x} \frac{dy}{kE(a_0)}}\,dx} = \frac{\mu - \lambda k}{\mu - \lambda k + \theta} = \frac{1}{1 + \theta/(\mu - \lambda k)}, \tag{6.76}$$

where the term $(\mu - \lambda k)/(\mu - \lambda k + \theta)$ in (6.76) is derived in formula (6.79) in Sect. 6.7.7 below.

6.7.6 *E*(Nonempty Period)

Assume a nonempty (aka non-empty) period a_0 starts at time τ_0. Let $\tau_1 < \tau_2 < \cdots$, be the times of successive *outputs within* a_0, that occur after τ_0. Let $\tau_1^* = \tau_1$ and

$$\tau_{n+1}^* = \min\{\tau_i > \tau_n^* | 0 < W(\tau_i) < W(\tau_n^*)\}, n = 1, 2, \ldots.$$

The ordinates $\{W(\tau_n^*)\}_{n=1,2,\ldots}$ are *strictly descending ladder points* (Fig. 6.10). (See Example (d), p. 280 in [73]; Fig. 1, p. 192, and pp. 390–394 in [74])). Let I^* be the initial influx amount, i.e., up to the first output (decrement) at τ_1 ($I^* = W(\tau_1^-)$). Let N_{I^*} denote the number of descending ladder points during a_0. Since output sizes are $\underset{dis}{=}\text{Exp}_\lambda$, the memoryless property implies these ladder points are distributed as Poisson "arrivals" in a length I^*, where $E(I^*) = E(\tau_1 - \tau_0) \cdot k = k/\mu$. If the output at τ_1 should empty the dam, then $a_0 = \tau_1 - \tau_0 = I^*/k$. In general,

$$a_0 = \frac{I^*}{k} + \sum_{i=1}^{N_{I^*}} a_{0,i}, \tag{6.77}$$

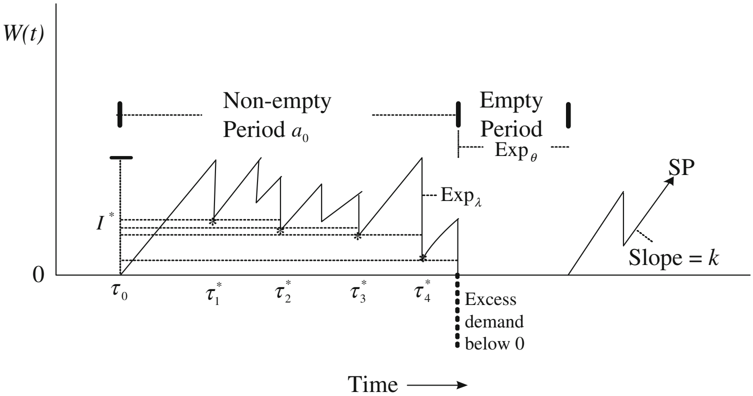

Fig. 6.10 Sample path for r(·)/G/M dam with $r(x) \equiv k$, $\overline{A}(x) = e^{-\lambda x}$. Shows $I^* = k(\tau_1 - \tau_0)$, and descending ladder points at $\tau_1^*, \ldots, \tau_4^*$. The indicated ladder point ordinates are equivalent to four Poisson arrivals (rate λ) within length I^*

where the $a_{0,i}$s are i.i.d. r.v.s $\underset{dis}{=} a_0$, independent of N_{I^*} (see Fig. 6.10 and Sect. 3.4.12).

From (6.77)

$$\begin{aligned} E(a_0) &= \frac{E(I^*)}{k} + E(N_{I^*}) \cdot E(a_0) \\ &= \frac{1}{\mu} + \lambda E(I^*) \cdot E(a_0) \\ &= \frac{1}{\mu} + \lambda \frac{k}{\mu} \cdot E(a_0), \\ E(a_0) &= \frac{\frac{1}{\mu}}{1 - \frac{\lambda k}{\mu}} = \frac{1}{\mu - \lambda k}. \end{aligned} \tag{6.78}$$

6.7.7 Probability of Emptiness and PDF of Content

We compute $\{P_0, f(x)\}_{x>0}$. Since $E(\textit{empty period}) = E(b_0) = 1/\theta$, from (6.78)

$$P_o = \frac{E(b_0)}{E(b_0) + E(a_0)} = \frac{\frac{1}{\theta}}{\frac{1}{\theta} + \frac{1}{\mu - \lambda k}} = \frac{\mu - \lambda k}{\mu - \lambda k + \theta}. \tag{6.79}$$

Substituting for P_0 into (6.75) gives

$$f(x) = \frac{\theta(\mu - \lambda k)}{k(\mu - \lambda k + \theta)} e^{-\frac{(\mu - \lambda k)}{k} x}, x > 0. \tag{6.80}$$

6.8 $\langle S, S \rangle$ Inventory with Product Decay

Consider a continuous review $\langle s, S \rangle$ inventory system with reorder point $s \geq 0$, and order-up-to level $S > s$. Assume that demands for stock occur at a Poisson rate λ. The demand quantities, $D_i, i = 1, 2, \ldots$, are i.i.d. random variables with common cdf $B(x)$, and $\overline{B}(x) = 1 - B(x), x \geq 0$. Denote the stock on hand process by $\{I(t)\}_{t \geq 0}$. Assume the stock decays at rate $dI(t)/dt = -r(I(t)) < 0$ when the stock is at level $I(t) \in (s, S]$. The ordering policy is as follows. If the stock either decays

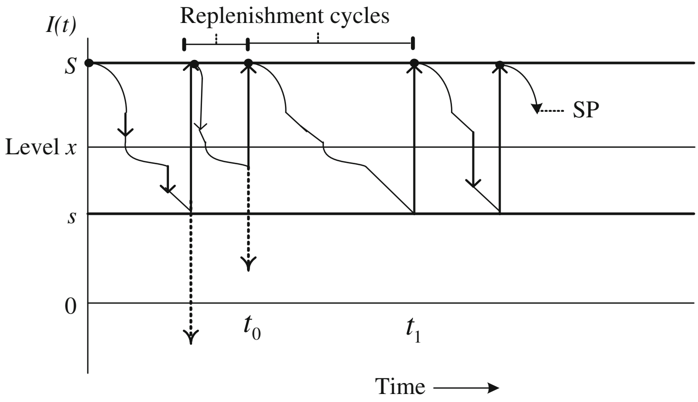

Fig. 6.11 Sample path $\{I(t)\}_{t\geq 0}$ in $\langle s, S\rangle$ inventory with general deacy

continuously to level s, or jumps downward to, or below level s due to a demand, then an order is placed and received immediately, replenishing the stock up to level S. All SP *upward jumps* end at level S. The *regular* state space is $(s, S]$ since all probability is concentrated on $(s, S]$. The *wide-sense* state space is $(-\infty, S]$, which accounts for SP downward jumps ending below s, and immediately jumping up to end at S (double jumps). The latter upward SP jumps correspond to replenishments (see Figs. 6.11 and 2.2 and Example 2.2). In order to focus on the LC analysis, we assume there is no lead time or backlogging. These extensions, as well as others, can be incorporated into the analysis (e.g., [4, 31]).

Let $f(x)$, $s < x \leq S$, and $F(x)$, $x \leq S$, denote, respectively, the steady-state pdf and cdf of $I(t)$ as $t \to \infty$. Assume each order size $\underset{dis}{=} \text{Exp}_\mu$.

6.8.1 PDF of Inventory with Constant Decay Rate k

To focus on the LC solution technique, we let the rate of decay be $r(x) := k > 0$, $x \in (s, S]$, and assume demand sizes are $\underset{dis}{=}\text{Exp}_\mu$. We derive an integral equation for $f(x), x \in (s, S]$ in the following section. (The decay rate is generalized to $r(x) = kx$ in [36].)

This $\langle s, S\rangle$ model is a special case of the generalized M/G/r(·) dam in Sect. 6.6, with demand rate $\lambda_d \equiv \lambda$, the rate of sample-path downward jumps. The cdf of the demand sizes is $B_d(x) \equiv B(x) = 1 - e^{-\mu x}$, $x > 0$. The upward jump rate due to *exogenous* factors is $\lambda_u \equiv 0$. In this standard $\langle s, S\rangle$ model all upward jumps are due to replenishments (*prescribed, i.e., policy* jumps). (The decay rate is generalized to $r(x) = kx$ in [36].)

6.8.2 Equation and Solution for PDF of Inventory

We develop an integral equation for $f(x)$, $x \in (s, S]$. Consider a sample path of $\{I(t)\}_{t\geq 0}$ (similar to Fig. 6.11 with slope $= -k$). Fix level $x \in (s, S)$. The rate at which the SP *decays* into level $x \in (s, S)$ from above (due to left-continuous strict downcrossings of x) is

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^c(x)\right)}{t} = r(x)f(x) = kf(x).$$

(We use the terms "rate" and "expected rate" synonymously when they are equal $a.s.$). The SP decay rate into level s is

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^c(s)\right)}{t} = r(s^+)f(s^+) \equiv r(s)f(s) = kf(s).$$

The rate at which the SP *jump-downcrosses* level $x \in [s, S)$ due to demands is

$$\lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^j(x)\right)}{t} = \lambda \int_{y=x}^{S} \overline{B}(y - xf(y)dy = \lambda \int_{y=x}^{S} e^{-\mu(y-x)} f(y)dy$$

(jumps start at $y \in (x, S)$ and are greater than $y - x$). The *total* SP *downcrossing* rate of level $x \in (s, S]$ is

$$\lim_{t\to\infty} \frac{E\left(D_t^c(x)\right)}{t} + \lim_{t\to\infty} \frac{E\left(\mathcal{D}_t^j(x)\right)}{t} = r(x)f(x) + \lambda \int_{y=x}^{S} \overline{B}(y - xf(y)dy$$
$$= kf(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} f(y)dy, x \in (s, S].$$

The total "downcrossing" rate of the reorder point s is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(s))}{t} = r(s)f(s) + \lambda \int_{y=s}^{S} \overline{B}(y - s)f(y)dy$$
$$= kf(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y)dy,$$

where we have counted a left-continuous hit of level s from above as a downcrossing of s.

The SP total downcrossing rate of level s is equal to the SP egress rate out of the order-up-to level S *below* (related to right limit tangent below in Fig. 2.15 in Chap. 2). This equality is due to the ordering policy, which replenishes stock on hand to level S with each left continuous hit, or jump downcrossing, of level s. (There is a one-to-one correspondence between SP *egresses from S below*, and *downcrossings of level s*.) Rate balance into and out of state $\{S\}$ results in the equality

$$\begin{aligned} r(s)f(s) + \lambda \textstyle\int_{y=s}^{S} \overline{B}(y-s)f(y)dy = r(S)f(S), \\ kf(s) + \lambda \textstyle\int_{y=s}^{S} e^{-\mu(y-s)}f(y)dy = kf(S), \end{aligned} \tag{6.81}$$

where $r(S) := r(S^-)$ and $f(S) := f(S^-)$.

A crucial simplifying feature of this model is: the total SP *upcrossing* rate of *every* level $x \in (s, S]$ *is equal to* the total *downcrossing* rate of level s (replenishment rate). Applying Eq. (6.81), and rate balance across level x, yields an integral equation for $f(x)$

$$\begin{aligned} r(x)f(x) + \lambda \int_{y=x}^{S} \overline{B}(y-x)f(y)dy &= r(s)f(s) + \lambda \int_{y=s}^{S} \overline{B}(y-s)f(y)dy \\ &= r(S)f(S), x \in (s, S], \end{aligned} \tag{6.82}$$

or, since $r(x) \equiv k$,

$$\begin{aligned} kf(x) + \lambda \int_{y=x}^{S} e^{-\mu(y-x)}f(y)dy &= kf(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)}f(y)dy \\ &= kf(S), x \in (s, S]. \end{aligned} \tag{6.83}$$

The state space has no atoms, i.e., there is no state in which the SP spends a positive time. The probability distribution of stock on hand is concentrated on $(s, S]$. The normalizing condition is

$$\int_{x=s}^{S} f(x)dx = 1. \tag{6.84}$$

6.8.3 Solution of Integral Equation (6.83)

Taking d/dx in (6.83) gives

$$kf'(x) + \mu\left[kf(S) - kf(x)\right] - \lambda f(x) = 0. \tag{6.85}$$

In (6.85), using the second equality in Eq. (6.83) we replace the term $\mu\lambda \int_{y=x}^{S} e^{-\mu(y-x)}f(y)dy$ by $\mu\left[kf(S) - kf(x)\right]$. The term $-\lambda f(x)$ results from taking the derivative under the integral sign. Simplifying (6.85) gives the differential equation

$$f'(x) - \left(\mu + \frac{\lambda}{k}\right) f(x) = -\mu f(S). \tag{6.86}$$

Multiplying both sides of (6.86) by the integrating factor $e^{-(\mu+\frac{\lambda}{k})x}$, integrating with respect to x, and simplifying gives

$$f(x) = \frac{\mu f(S)}{\mu + \frac{\lambda}{k}} + Ce^{-(\mu+\frac{\lambda}{k})x}, \tag{6.87}$$

where C is a constant. Setting $x = S$ in (6.87) yields

$$f(S) = \frac{\mu + \frac{\lambda}{k}}{\frac{\lambda}{k}} \cdot Ce^{-(\mu+\frac{\lambda}{k})S};$$

which substituted back into (6.87) gives

$$f(x) = \frac{\mu k}{\lambda} \cdot Ce^{-(\mu+\frac{\lambda}{k})S} + Ce^{-(\mu+\frac{\lambda}{k})x}, \; s < x \leq S. \tag{6.88}$$

Applying the normalizing condition (6.84) leads to

$$\frac{\mu k}{\lambda} \cdot Ce^{-(\mu+\frac{\lambda}{k})S}(S-s) + C\frac{e^{-\left(\mu+\frac{\lambda}{k}\right)S} - e^{-\left(\mu+\frac{\lambda}{k}\right)s}}{\mu+\frac{\lambda}{k}} = 1,$$

$$\text{or } \frac{1}{C\frac{\mu k}{\lambda}e^{-\left(\mu+\frac{\lambda}{k}\right)S}} = (S-s) + \frac{\lambda}{\mu k}\frac{1-e^{-\left(\mu+\frac{\lambda}{k}\right)(S-s)}}{\mu+\frac{\lambda}{k}}.$$

Factoring (6.88) results in

$$f(x) = C\frac{\mu k}{\lambda}e^{-(\mu+\frac{\lambda}{k})S}\left(1 + \frac{\lambda}{\mu k}e^{-(\mu+\frac{\lambda}{k})(S-x)}\right), \; s < x \leq S.$$

Letting $A := C\frac{\mu k}{\lambda}e^{-(\mu+\frac{\lambda}{k})S}$ gives

$$f(x) = A\left(1 + \frac{\lambda}{\mu k}e^{-(\mu+\frac{\lambda}{k})(S-x)}\right), \; s < x \leq S, \tag{6.89}$$

wherein (applying (6.84)),

$$A = \left[(S-s) + \frac{\lambda}{\mu k\left(\mu+\frac{\lambda}{k}\right)}\left(1 - e^{-\left(\mu+\frac{\lambda}{k}\right)(S-s)}\right)\right]^{-1}.$$

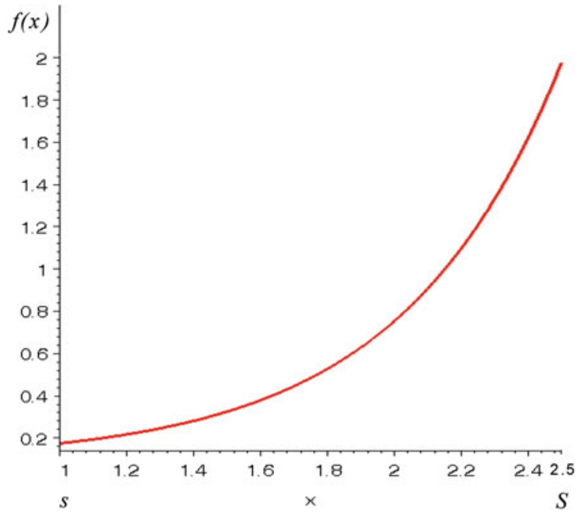

Fig. 6.12 PDF $f(x)$ in $\langle s, S\rangle$ inventory with decay rate k. $\lambda = 2$, $\mu = 0.10$, $k = 1$, $S = 2.5$, $s = 1$. Note that $f(x) = 0 \begin{cases} x < 1, \\ x > 2.5 \end{cases}$

In (6.89) $f(x)$ is convex and increasing on (s, S) (i.e., $f'(x) > 0, f''(x) > 0$). This property agrees with intuition which suggests that the stock resides a large proportion of time at high levels closer to S and a smaller proportion of time near s, for every $k > 0$. This accumulation of inventory near S is a consequence of the re-order policy, which instantaneously replenishes the stock up to level S at each replenishment instant since there is no lead time. (See the numerical example in Sect. 6.8.8 and Figs. 6.12 and 6.13.)

6.8.4 Sojourns Above and Below Level x

Let a_x := *sojourn time above* x, and b_x := *sojourn time at or below* x, $x \in (s, S]$, in $\{I(t)\}_{t\geq 0}$. Every instant $t \geq 0$ such that $I(t) = S$ is a regenerative point of $\{I(t)\}_{t\geq 0}$. The regenerative property holds whether replenishments up to S are due to SP smooth decays into level s from above, or due to SP jumps that end at or below level s as a result of a demand. For example, consider Fig. 6.11. At time points like t_1 the SP makes a left-continuous hit of level s from above, and jumps upward to level S. The Poisson arrival process for demands ensures that the excess arrival time until the next demand is $\underset{dis}{=} \text{Exp}_\lambda$ (memoryless property).

Time Between Successive Hits of Level S

The times between successive instants when $I(t) = S$, form a renewal process. Let d_S := *inter-level-S time*. From (6.83), the total SP downcrossing rate of level s is identical to the

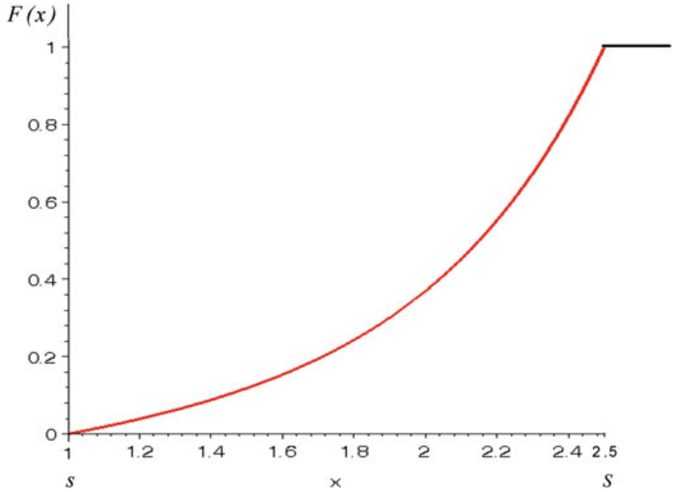

Fig. 6.13 CDF $F(x)$ in $\langle s, S\rangle$ inventory with decay rate k. $\lambda = 2$, $\mu = 0.10$, $k = 1$, $S = 2.5$, $s = 1$

$$\text{replenishment rate } = kf(s) + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y)dy = kf(S).$$

The value of $f(S)$ is obtained from (6.89). Thus, with A as in (6.89),

$$E(d_S) = \frac{1}{kf(S)} = \frac{\mu}{A(k\mu + \lambda)}, \tag{6.90}$$

***E*(a_x)**

The *proportion* of time the SP spends above level x is $1 - F(x)$, which is equal to $\frac{E(a_x)}{E(d_S)} = E(a_x)kf(S)$ (by the renewal reward theorem). Thus

$$E(a_x) = \frac{1 - F(x)}{kf(S)} = \frac{\mu(1 - F(x))}{A(k\mu + \lambda)}, \tag{6.91}$$

where $F(x) = \int_{y=s}^{x} f(y)dy$ is obtained from (6.89).

***E*(b_x)**

Similarly,

$$E(b_x) = \frac{F(x)}{kf(S)} = \frac{\mu F(x)}{A(k\mu + \lambda)}. \tag{6.92}$$

We can also obtain $E(b_x)$ using (6.91) and

$$E(a_x) + E(b_x) = E(d_S) = \frac{1}{kf(S)}.$$

A check on $E(a_x)$, $E(b_x)$ when $x = s$, using $F(s) = 0$, $1 - F(s) = 1$, and (6.90), is

$$E(a_s) = \frac{1-F(s)}{kf(S)} = \frac{1}{kf(S)} = \frac{\mu}{A(k\mu+\lambda)} = E(d_S),$$

the expected replenishment cycle, as intuitively expected. Also $E(b_s) = F(s)/kf(S) = 0$, which agrees with the SP spending zero time below level s. (State-space jumps, including jumps that end below s, occur *not in Time* because they are perpendicular to the time axis. See Remark 2.2 in Sect. 2.3).

6.8.5 Replenishments Due to Two Types of Signal

The replenishment rate (*total ordering*) is the SP total *downcrossing* rate of level s, which is the right side of (6.83), namely $kf(S) = A(k\mu + \lambda)/\mu$ (A as in (6.89)).

Replenishments are made up to level S whenever one of two types of signals occurs. A **type-*c*** signal denotes an SP *left continuous decay into level s from above* (time point t_1 in Fig. 6.11). A **type-*j*** signal denotes an SP *downward jump that ends at or below s*, *due to a demand* (time point t_0 in Fig. 6.11). (We use the term 'type-$\boldsymbol{k}$ order' to mean an order is due to a type-$\boldsymbol{k}$ signal, $\boldsymbol{k} = \boldsymbol{c}, \boldsymbol{j}$.) An **order cycle** (same as **replenishment** cycle) is the time between two successive instants when an order is *received*, namely, d_S. Due to Poisson arrivals of demands, the sequence $\{d_{S,i}\}_{i=1,2,...}$ with $d_{S,i} \underset{dis}{=} d_S$, is a renewal process. An order initiating d_S is either type-$\boldsymbol{c}$ or type-$\boldsymbol{j}$. Let $P_{\boldsymbol{c}} := P(\text{an order is type-}\boldsymbol{c})$, $P_{\boldsymbol{j}} := P(\text{an order is type-}\boldsymbol{j})$. Then $P_c + P_j = 1$.

We now determine $P_{\boldsymbol{c}}$ and $P_{\boldsymbol{j}}$. The counting process $\{\mathcal{D}_t^c(s)\}_{t\geq 0}$ is a renewal process due to Poisson arrivals of demand. Since there is exactly 1 type-$\boldsymbol{c}$ or exactly 1 type-$\boldsymbol{j}$ order in an ordering cycle,

$$E(\text{number of type-}\boldsymbol{c}\text{ orders in }d_S) = 1\cdot P_{\boldsymbol{c}} + 0\cdot P_{\boldsymbol{j}} = P_{\boldsymbol{c}},$$
$$E(\text{number of type-}\boldsymbol{j}\text{ orders in }d_S) = 0\cdot P_{\boldsymbol{c}} + 1\cdot P_{\boldsymbol{j}} = P_{\boldsymbol{j}}.$$

By the renewal reward theorem,

$$\frac{E(\text{number of type-}\boldsymbol{c}\text{ orders in }d_S)}{E(d_S)} = \lim_{t\to\infty}\frac{E(\mathcal{D}_t^c(s))}{t} = r(s)f(s)$$
$$\Longrightarrow \frac{P_{\boldsymbol{c}}}{E(d_S)} = \lim_{t\to\infty}\frac{E(\mathcal{D}_t^c(s))}{t} = r(s)f(s).$$

Since $E(d_S) = 1/\left(r(S)f(S)\right)$,

$$P_{\boldsymbol{c}} = r(s)f(s)\cdot E(d_S) = \frac{r(s)f(s)}{r(S)f(S)}. \tag{6.93}$$

Intuitively, in (6.93) the numerator is the rate of type-$\boldsymbol{c}$ orders; the denominator is the total rate of orders.

Also, $\{D_t^j(s)\}_{,\geq 0}$ is a renewal process (due to Poisson arrivals of demands). Therefore

$$\frac{E(\text{number of type-}\boldsymbol{j}\text{ orders in } d_S)}{E(d_S)} = \frac{1 \cdot P_j + 0 \cdot P_c}{E(d_S)}$$
$$= \frac{P_j}{E(d_S)} = \lim_{t \to \infty} \frac{E(\mathcal{D}_t^j(s))}{t},$$

and (see Sect. 6.6),

$$P_j = \lim_{t \to \infty} \frac{E(\mathcal{D}_t^j(s))}{t} E(d_S) = \frac{\lambda \int_{y=s}^{S} \overline{B}(y-s) f(y) dy}{r(S) f(S)}. \tag{6.94}$$

Intuitively, in (6.94) the numerator is the rate of type-$\boldsymbol{j}$ orders; the denominator is the total rate of orders.

If $r(x) \equiv k$, $x \in (s, S]$, and the demand sizes are $\underset{dis}{=}$ Exp_μ, then

$$P_c = \frac{kf(s)}{kf(S)} = \frac{f(s)}{f(S)}, P_j = \frac{\lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y) dy}{kf(S)}.$$

implying, with A as in (6.89),

$$P_c = \frac{k\mu + \lambda e^{-(\frac{k}{\lambda} + \mu)(S-s)}}{k\mu + \lambda}, \tag{6.95}$$

$$P_j = \frac{\mu \lambda A \int_{y=s}^{S} e^{-\mu(y-s)} \left(1 + \frac{\lambda e^{-\left(\frac{\lambda}{k} + \mu\right)(S-y)}}{k\mu}\right) dy}{k\mu + \lambda}. \tag{6.96}$$

6.8.6 Expected Order Size

Denote the replenishment order by R. If an order is type-$\boldsymbol{c}$ then $R = S - s$. If an order is type-$\boldsymbol{j}$ then $R = S - s + r_s$ where r_s := *excess demand below* s. If the demand sizes are $\underset{dis}{=}\text{Exp}_\mu$ then $r_s \underset{dis}{=} \text{Exp}_\mu$ (by memoryless property). Since $P_c + P_j = 1$, the expected order size is

$$E(R) = (S-s)P_c + \left(S - s + \frac{1}{\mu}\right) P_j = S - s + P_j \frac{1}{\mu}, \tag{6.97}$$

where P_j is given in (6.96).

6.8.7 Cost Rate

Since there is no backlogging or lead-time costs in the $\langle s, S\rangle$ model considered here, the cost function accounts only for costs of setup of placing orders, and for holding inventory. Let C := total average *cost rate*, $\mathcal{C}_s$:= setup cost per type-$\boldsymbol{c}$ order, $\mathcal{C}_j$, := setup cost per type-$\boldsymbol{j}$ order. Let $\mathcal{C}_H$ be the holding cost per unit per unit time. Then

$$\begin{aligned} C &= \mathcal{C}_s \cdot (\text{type-}\boldsymbol{c}\text{ ordering rate}) \\ &\quad + \mathcal{C}_j \cdot (\text{type-}\boldsymbol{j}\text{ ordering rate}) + \mathcal{C}_H \int_{x=s}^{S} x f(x) dx \\ &= \mathcal{C}_s k f(s) + \mathcal{C}_j \int_{x=s}^{S} e^{-(x-s)\mu} f(x) dx + \mathcal{C}_H \int_{x=s}^{S} x f(x) dx, \end{aligned} \tag{6.98}$$

where $f(x)$ is given by (6.89).

6.8.8 Numerical Example

In $\langle s, S\rangle$ with $r(x) \equiv k$ and all demand sizes $\underset{dis}{=} \text{Exp}_\mu$, assume $\lambda = 2$, $\mu = 0.10$, $k = 1, S = 2.5, s = 1$. Calculations give $A = 0.094200$ in (6.89). The pdf of inventory is

$$f(x) = 0.094200 + 1.88400e^{(-5.250+2.10x)}, \ 1 < x \leq 2.5,$$
$$f(1) = 0.1749, \qquad f(2.5) = 1.9782.$$

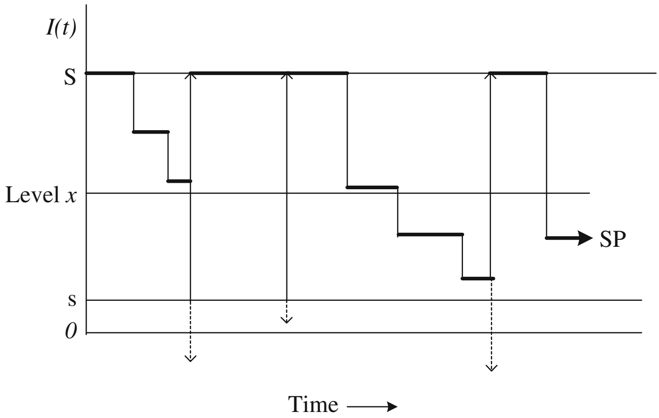

Fig. 6.14 Sample path of $\{I(t)\}_{t\geq 0}$ in $\langle s, S\rangle$ inventory with *no* decay. The SP stays at a level for a time $\underset{dis}{=} \text{Exp}_\lambda$

The cdf of inventory is

$$F(x) = -0.132645 + 0.094200x + 0.897144e^{(-5.25+2.1x)}, 1 < x \leq 2.5,$$
$$F(1) = 0, \qquad F(2.5) = 1.0.$$

Functions $f(x)$ and $F(x)$ are plotted in Figs. 6.12 and 6.13, which demonstrate convexity and the probability massed towards level S.

6.9 $\langle S, S\rangle$ Inventory with No Product Decay

Consider an $\langle s, S\rangle$ model as in Sect. 6.8, having demand sizes $\underset{dis}{=}$ Exp_μ and ***no decay of inventory***. Thus $r(x) \equiv 0$. Once the stock on hand enters a level in $(s, S]$, it remains at that level for a time $\underset{dis}{=}$ Exp_λ, until the next demand instant (see Figs. 2.6 and 6.14 above). The state space has an *atom* at level S (positive probability). Every state $\{x\} \in \{y|y \in (s, S)\}$ is continuous (not an atom), because the probability of entering and remaining in such $\{x\}$ for a positive time is 0, due to continuous demand sizes.

Let $\Pi_S = P(\{I(t)\}_{t\geq 0}$ is at level $S)$ in the steady state. Equating the SP down- and upcrossing rates of level $x \in (s, S)$ yields an integral equation for $f(x)$,

$$\lambda\Pi_S e^{-\mu(S-x)} + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} f(y)dy$$
$$= \lambda\Pi_S e^{-\mu(S-s)} + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y)dy = \lambda\Pi_S, \tag{6.99}$$

$$\lambda\Pi_S e^{-\mu(S-x)} + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} f(y)dy = \lambda\Pi_S, \; s < x < S. \tag{6.100}$$

The first equality in Eq. (6.99) indicates that the upcrossing rate of every $x \in (s, S)$ is equal to the downcrossing rate of level s.

Equation (6.100) employs the second equality in (6.99), which expresses the fact *SP rate into* $\{S\}$ = *SP rate out of* $\{S\}$. The normalizing condition is

$$\Pi_S + \int_{x=s}^{S} f(x)dx = 1. \tag{6.101}$$

6.9.1 PDF of Inventory

Some algebra using (6.100) and (6.101), shows that $\{\Pi_S, f(x)\}_{x\in(s,S)}$ is given by

$$\Pi_S = \frac{1}{1+\mu(S-s)}, f(x) = \frac{\mu}{1+\mu(S-s)}, x \in (s, S). \tag{6.102}$$

That is, $f(x)$ is uniformly distributed on state-space interval (s, S), and there is an atom at S. In (6.102), if $\mu < 1$ then $\Pi_S > f(x), x \in (s, S)$; if $\mu > 1$ then $\Pi_S < f(x)$, $x \in (s, S)$. If $\mu = 1$ then $\Pi_S = f(x), x \in (s, S)$.

6.9.2 Sojourn Times Above and Below a Level

The rate of replenishments up to S is the total SP downcrossing rate of level s, namely

$$\lambda \Pi_S e^{-\mu(S-s)} + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y)dy = \lambda \Pi_S,$$

since all replenishments are due to type-***j*** orders that include the excess s. Let $d_S :=$ *time between two successive replenishments up to level S* (same as an ordering cycle). Then

$$E(d_S) = \frac{1}{\lambda \Pi_S e^{-\mu(S-s)} + \lambda \int_{y=s}^{S} e^{-\mu(y-s)} f(y)dy} = \frac{1}{\lambda \Pi_S}. \tag{6.103}$$

Fix level $x \in (s, S]$. From the theory of regenerative processes (specifically the renewal reward theorem)

$$\frac{E(a_x)}{E(d_S)} = 1 - F(x), \qquad \frac{E(b_x)}{E(d_S)} = F(x), x \in [s, S],$$

$$E(a_x) = (1 - F(x)) E(d_S) = \frac{1 - F(x)}{\lambda \Pi_S}, \tag{6.104}$$

$$E(b_x) = F(x) E(d_S) = \frac{F(x)}{\lambda \Pi_S}, \tag{6.105}$$

where

$$F(x) = \int_{y=s}^{x} f(y)dy = \frac{\mu(x-s)}{1+\mu(S-s)}, s < x \leq S.$$

The law of total probability gives

$$F(S) = \frac{\mu(S-s)}{1+\mu(S-s)} + \Pi_S = \frac{\mu(S-s)}{1+\mu(S-s)} + \frac{1}{1+\mu(S-s)} = 1.$$

6.9.3 Ordering Characteristics

Ordering Rate

The total ordering rate is the right hand side of (6.100), i.e., $\lambda\Pi_S = \lambda/\,(1+\mu(S-s))$.

Expected Order Size

All orders are type-***j***, signalled by demands ending at or below s. Thus $P_c = 0$ and $P_j = 1$ (see Sect. 6.8.5 and 6.8.6). Hence the expected order size is

$$E(R) = S - s + \frac{1}{\mu}. \tag{6.106}$$

Expected Number of Demands in an Ordering Cycle

Let N_{d_S} := number of demands in an ordering cycle d_S. By the renewal reward theorem,

$$\frac{E(N_{d_S})}{E(d_S)} = \lambda,$$

$$E(N_{d_S}) = \lambda E(d_S) = \frac{\lambda}{\lambda\Pi_S} = \frac{1}{\Pi_S} = 1 + \mu(S-s). \tag{6.107}$$

An alternative derivation of (6.107) using stopping times, is of interest. First,

$$N_{d_S} = \min\left\{ n \middle| \sum_{i=1}^{n} D_i > S - s \right\}, \tag{6.108}$$

implying N_{d_S} is a stopping time for the sequence $\{D_i\}_{i=1,2,...}$. Also

$$d_S = \sum_{i=1}^{N_{d_S}} T_i,$$

where $T_i \underset{dis}{\equiv} T \underset{dis}{=} \text{Exp}_\lambda$. N_{d_S} is also a stopping time for the sequence $\{T_i\}_{i=1.2....}$ because there is a 1–1 correspondence between $\{T_i\}_{i=1.2....}$ and $\{D_i\}_{i=1,2,...}$. Thus

$$E(d_S) = E(N_{d_S}) \cdot E(T),$$

and

$$E(N_{d_S}) = \frac{E(d_S)}{E(T)} = \frac{\frac{1}{\lambda\Pi_S}}{\frac{1}{\lambda}} = \frac{1}{\Pi_S} = 1 + \mu(S-s), \tag{6.109}$$

corroborating (6.107).

6.9.4 Cost Rate

Let $\mathcal{C}_O, \mathcal{C}_H$ denote the setup cost per order and holding cost per order per unit time, respectively. The total average cost rate is

$$C = \mathcal{C}_O \cdot (\text{ordering rate}) + \mathcal{C}_H \int_{x=s}^{S^+} xf(x)dx.$$

The ordering rate is $\lambda\Pi_S = \lambda/\left(1 + \mu(S - s)\right)$. The average stock on hand is

$$\begin{aligned}\int_{x=s}^{S^+} xf(x)dx &= S\Pi_S + \int_{x=s}^{S} \frac{\mu x}{1 + \mu(S - s)}dx \\ &= \frac{S}{1 + \mu(S - s)} + \frac{\mu(S^2 - s^2)}{2(1 + \mu(S - s))} \\ &= \frac{2S + \mu(S^2 - s^2)}{2(1 + \mu(S - s))}.\end{aligned}$$

Thus

$$C = \frac{\lambda}{1 + \mu(S - s)} \cdot C_O + \frac{2S + \mu(S^2 - s^2)}{2(1 + \mu(S - s))} \cdot C_H. \tag{6.110}$$

Remark 6.11 In the standard $\langle s, S\rangle$ with **no decay**, suppose the inter-demand times form a **renewal process** (not necessarily a Poisson process). Then the results will be the same as in this section, except for the **arrival-point mixed pdf** denoted $\left\{\Pi_{S,\iota}, f_\iota(x)\right\}_{x\in(s,S)}$, because PASTA will not apply. The integral equation for the pdf $f_\iota(x)$ would be the same as (6.100), where λ represents the renewal rate of the demand process. The arrival rate λ cancels out of the equation. Thus the formulas for $\Pi_{S,\iota}$ and $f_\iota(x)$, $x \in (s, S)$ in (6.102) are independent of λ. The underlying reason for this property is that all orders are type-$\boldsymbol{j}$ at the ends (and starts) of inter-demand times. When the SP jumps up to level S, the time until the next demand is a **full inter-arrival time**. Hence $\left\{d_{S,i}\right\}$ is a renewal process, where $d_{S,i} \underset{dis}{\equiv} d_S$.

Remark 6.12 For exposition, we have applied LC to only two basic $\langle s, S\rangle$ inventory systems. We emphasize that LC equally applies to a vast array of other inventory systems as well, e.g., $\langle r, nQ\rangle$, variations of *EOQ* models, models with lead time and backlogging, production-inventory models of various complexity, models with a variety of state-dependent control policies (e.g., [3, 4]).