Chapter 10
Renewal Theory Using LC

In this chapter, Sect. 10.1 gives an LC analysis of a replacement model, which is structured using two interconnected renewal processes. We derive efficiently, via sample paths and LC, the *limiting* pdfs of the excess life , the age, and the total life, of both renewal processes. Section 10.2 gives an LC analysis of a classical renewal problem with a barrier. Section 10.3 uses LC to derive the *finite time-t probability distributions* of the excess, age and total life, of a renewal process.

10.1 Replacement Model via Renewal Theory

We first describe a replacement model, which is a variant of the GI/G/r(·) dam. (see Sect. 6.2 for a related M/G/r(·) dam.) Sects. 10.1.3 and 10.1.4 derive the ***steady-state (limiting)*** pdfs of the excess, age and total life of two connected, renewal processes in the model.

10.1.1 The Model

Let $\{X(t)\}_{t\geq 0}$ denote a continuous-time continuous-state stochastic process having upward jumps of i.i.d. sizes $X_n > 0$, all starting at level 0, at times τ_n^-, where $0 = \tau_0 < \tau_1 \cdots < \tau_n < \cdots$, such that $X(\tau_n) = X_n, n = 0, 1, 2, \ldots$ Let the state space be $\boldsymbol{S} = [0, \infty)$. Figure 10.1 shows a sample path of $\{X(t)\}_{t\geq 0}$ (we use $X(t)$ to denote both the state variable and a sample path, for

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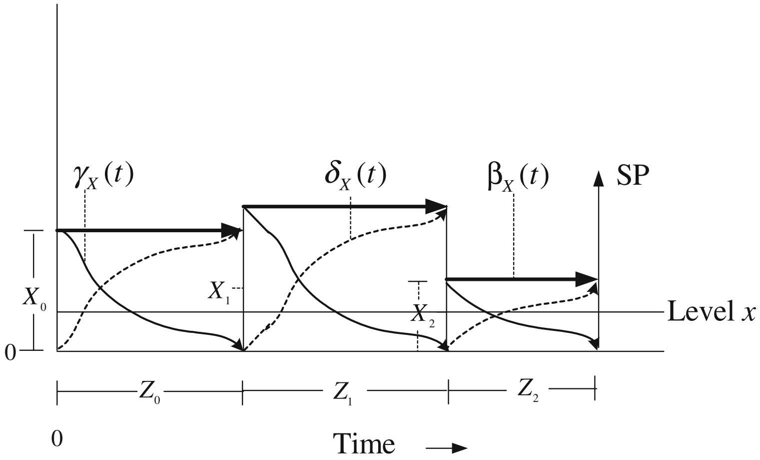

Fig. 10.1 Sample path of excess life $\gamma_X(t)$, age $\delta_X(t)$, total life $\beta_X(t)$. Also shows a state-space level x

economy of notation). Assume $dX(t)/dt = -r(X(t)), t \in [\tau_n, \tau_{n+1}), n = 0, 1, \ldots$, where $r(x) > 0, x > 0$. Thus $X(t)$ is a piecewise, decreasing deterministic function between jumps. Assume that for all $v > 0$,

$$\lim_{u\downarrow 0} \int_{y=u}^{v} \frac{1}{r(y)} dy < \infty, \tag{10.1}$$

which guarantees that a sample path $X(t)$, $t \geq 0$, starting from any level $v > 0$, returns to level 0 in a finite time. The process $\{X(t)\}_{t\geq 0}$ is a variant of the GI/G/r(·) dam subject to inputs $\{X_n\}_{n=0,1,\ldots}$ occurring at all instants τ_n when the dam becomes empty, and at no other time points. This mechanism can be thought of as that of a *replacement model*. New inputs replace the immediately preceding used-up inputs. Thus $\{X(t)\}_{t\geq 0}$ is never at level 0 for a positive duration, and $\tau_n, n = 1, 2, \ldots$, are replacement times.

Denote the inter-replacement times by $\{Z_n\}_{n=0,1,\ldots}$. The random variables Z_n and X_n are related by

$$Z_n = \int_{y=0}^{X_n} \frac{1}{r(y)} dy, n = 0, 1, \ldots. \tag{10.2}$$

From (10.2), Z_n is the *time required* for $\{X(t)\}_{t\geq 0}$ to descend from level X_n to level 0. The Z_n, $n = 0, 1,$ are i.i.d. because X_n, $n = 0, 1,$ are i.i.d.

10.1.2 Renewal Processes $\{Z_n\}_{n=0,1,...}$ and $\{X_n\}_{n=0,1,...}$

$\{Z_n\}_{n=0,1,...}$ is in one-to-one correspondence with $\{X_n\}_{n=0,1,...}$, and with the piecewise deterministic continuous efflux rate $r(X(t))$, $\tau_n < \tau_{n+1}$, $n = 0, 1, \ldots$.

Let $X_n \underset{dis}{=} X$ and $Z_n \underset{dis}{=} Z$, $n = 0, 1, \ldots$

Example 10.1 Consider a newly-installed battery at τ_0 with initial electrical charge $X_0 \underset{dis}{\equiv} X$, to power a device. Assume that the charge declines at a rate that depends on the present charge. That is, $dX(t)/dt = -r(X(t)) < 0$, $t \in [\tau_0, \tau_1)$. Suppose the battery's charge dissipates non-uniformly and descends to 0 after a time $\tau_1 = Z_0 \underset{dis}{\equiv} Z$. The battery is immediately replaced by a new fully-charged one. This procedure is repeated as batteries wear out, at times τ_n, $n = 2, 3, \ldots$. Thus $Z_n \underset{dis}{\equiv} Z$, and $X_n \underset{dis}{\equiv} X$, $n = 0, 1, 2, \ldots$, and

$$Z = \int_{y=0}^{X} \frac{1}{r(y)} dy, \tag{10.3}$$

is the inter-replacement time (see formula (6.4) in Sect. 6.2.4). The dimension of Z is $[Time]$. The dimension of X is $[Coulombs]$. The function $r\,(X(t))$ has dimension $[Coulomb][Time]^{-1}$.

10.1.3 The Renewal Process $\{X_n\}_{n=0,1,...}$

Let $\gamma_X(t) :=$ *excess life* of *content at instant* $t \geq 0$. The process $\{\gamma_X(t)\}_{t\geq 0}$ has the same sample path as $\{X(t)\}_{t\geq 0}$, since we assume that all input jumps start at level 0. Then $d(\gamma_X(t))/dt = -r(\gamma_X(t))$. Let $\delta_X(t) :=$ *age of the content at instant* $t \geq 0$, i.e., amount of content used up at instant t, from the last renewed amount prior to t. Then $d\,(\delta_X(t))\,/dt = +r(\delta_X(t))$. Let $\beta_X(t) :=$ *total life* (*span*) *of the latest renewed amount of content at* t (Fig. 10.1). (In Example 10.1, $\gamma_X(t))$, $\delta_X(t)$, $\beta_X(t)$ are respectively, the remaining charge, the charge used up, and the total charge, of the battery in use at time t.)

In the sample paths of the processes $\{\gamma(t)\}_{t\geq 0}$, $\{\delta(t)\}_{t\geq 0}$, $\{\beta(t)\}_{t\geq 0}$ all upward jumps start at level 0 and are $\underset{dis}{=} X$. All downward jumps start at a random level X and end at level 0.

Limiting Distributions in $\{X_n\}_{n=0,1,...}$ Using LC

We derive the *limiting* pdfs $f_{\gamma_X}(x)$, $f_{\delta_X}(x)$, $f_{\beta_X}(x)$, $x > 0$, of r.v.s $\gamma_X(t)$, $\delta_X(t)$, $\beta_X(t)$, as $t \to \infty$, assuming $E(X) < \infty$, which is the condition for their existence. Consider sample paths of $\{\gamma_X(t)\}$, $\{\delta_X(t)\}$, $\{\beta_X(t)\}$, $t \geq 0$ (Fig. 10.1).

Let $F_X(x)$, $f_X(x)$, μ_X be the cdf, pdf and expected value respectively of r.v. X. Let $\overline{F}_X(x) = 1 - F_X(x)$, $x \geq 0$.

Limiting PDF of Excess Life in $\{X_n\}_{n=0,1,...}$

Consider a sample path of $\{\gamma(t)\}_{t\geq 0}$. The long-run SP expected *downcrossing* rate of level $x > 0$, is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(x))}{t} = r(x) f_{\gamma_X}(x). \tag{10.4}$$

(as in Corollary 6.2 in Sect. 6.2.8).

The long-run SP expected *upcrossing* rate of level x is

$$\lim_{t\to\infty} \frac{E(\mathcal{U}_t(x))}{t} = \frac{1}{E(Z)} \cdot \overline{F}_X(x), \tag{10.5}$$

since the expected time between upward jumps starting from level 0 is $E(Z)(= E(\tau_{n+1} - \tau_n), n = 0, 1, \ldots)$, and $\overline{F}_X(x) = P(\textit{SP jump starting at level } 0 \textit{ is} > x)$. In (10.3), substituting from (10.2), and conditioning on $X = x$ gives

$$\begin{aligned} E(Z) &= \int_{x=0}^{\infty} \left(\int_{y=0}^{x} \frac{1}{r(y)} dy \right) f_X(x) dx = \int_{y=0}^{\infty} \int_{x=y}^{\infty} \frac{1}{r(y)} f_X(x) dx dy \\ &= \int_{y=0}^{\infty} \frac{1}{r(y)} \left(\int_{x=y}^{\infty} f_X(x) dx \right) dy = \int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy. \end{aligned} \tag{10.6}$$

Equating (10.4) and (10.5) by the principle of rate balance across level x, and using (10.6), yields the equation

$$r(x) f_{\gamma_X}(x) = \frac{\overline{F}_X(x)}{E(Z)} = \frac{\overline{F}_X(x)}{\int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}, \tag{10.7}$$

$$f_{\gamma_X}(x) = \frac{\overline{F}_X(x)}{r(x) \int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}. \tag{10.8}$$

The dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$ (= $[Coulomb]^{-1}$ in Example 10.1)).

Limiting PDF of Excess Life in $\{X_n\}_{n=0,1,...}$ when $r(x) \equiv 1$
If the efflux rate $r(x) \equiv 1$, formula (10.8) reduces to

$$f_{\gamma_X}(x) = \frac{\overline{F}_X(x)}{\int_{y=0}^{\infty} \overline{F_X}(y)dy} = \frac{\overline{F}_X(x)}{\mu_X}, \tag{10.9}$$

since $\int_{y=0}^{\infty} \overline{F}_X(y)dy = E(X) = \mu_X$. ($\gamma_X$ represents the limiting excess life of *content*, having pdf $f_{\gamma_X}(x)$.) Formula (10.9) is exactly the same as the well-known limiting pdf of the excess life in a "standard" renewal process. However, here the dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$ instead of $[Time]^{-1}$.

Limiting PDF of Age in $\{X_n\}_{n=0,1,...}$
For the age process $\{\delta_X(t)\}_{t\geq 0}$, the long-run SP expected *upcrossing rate of level* $x > 0$ is

$$\lim_{t\to\infty} \frac{E(\mathcal{U}_t(x))}{t} = +r(x) f_{\delta_X}(x), \tag{10.10}$$

(see Corollary 6.2 in Sect. 6.2.8). The long-run SP *expected downcrossing rate of level* x is

$$\lim_{t\to\infty} \frac{E(\mathcal{D}_t(x))}{t} = \frac{1}{E(Z)} \int_{y=x}^{\infty} f_X(y)dy = \frac{\overline{F}_X(x)}{E(Z)}, \tag{10.11}$$

because (1) downward jumps occur at rate $1/E(Z)$, (2) in order for the SP to downcross level x by a jump at some τ_n^-, the *upward* jump at τ_{n-1} starting from level 0 must have been such that $X_{n-1} > x$. Moreover, X_{n-1} at τ_{n-1} is also equal to the downward jump size at τ_n^- (see Fig. 10.1).

Equating (10.10) and (10.15) (rate balance across level x), gives

$$r(x) f_{\delta_X}(x) = \frac{\overline{F}_X(x)}{E(Z)} = \frac{\overline{F}_X(x)}{\int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy};$$

$$f_{\delta_X}(x) = \frac{\overline{F}_X(x)}{r(x) \int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}. \tag{10.12}$$

Comparing (10.8) with (10.12) shows that $f_{\delta_X}(x) = f_{\gamma_X}(x)$. The dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$.

Limiting PDF of Age in $\{X_n\}_{n=0,1,...}$ when $r(x) \equiv 1$
If $r(x) \equiv 1$, we obtain similarly as in (10.9), the limiting pdf

$$f_{\delta_X}(x) = \frac{\overline{F}_X(x)}{\mu_X}. \tag{10.13}$$

The dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$. It is well known that for an ordinary renewal process, the limiting distributions of the excess life and age are identical. In the variant of a GI/G/r(·) dam possessing the renewal structure outlined here, these distributions are also identical with respect to the content, even when the efflux rate has a general form $r(x)$, $x > 0$. That is, formulas (10.8) and (10.12) are identical.

Limiting PDF of Total Life in $\{X_n\}_{n=0,1,...}$
For the process $\{\beta_X(t)\}_{t\geq 0}$, the long-run SP expected *downcrossing rate of level* $x > 0$, is

$$\lim_{t\to\infty}\frac{E(\mathcal{D}_t(x))}{t} = \int_{y=x}^{\infty}\left(\frac{1}{\int_{u=0}^{y}\frac{1}{r(u)}du}\right) f_{\beta_X}(y)dy, \tag{10.14}$$

where we have conditioned on $\beta_X(t) = y > x$. In (10.14), $1/\int_{u=0}^{y}\frac{1}{r(u)}du$, *the reciprocal of the expected sojourn time of* $\{\beta_X(t)\}_{t\geq 0}$ *at level* $y > x$, is equal to the *downward jump rate* across level x starting from level y (Fig. 10.1). At the end of a level-y ($y > x$) sojourn time, the SP jumps downward to level 0, and *downcrosses every state-space level* in $(0, y)$, including level x.

The SP long-run (expected) *upcrossing* rate of level x is

$$\lim_{t\to\infty}\frac{E(\mathcal{U}_t(x))}{t} = \frac{\overline{F}_X(x)}{E(Z)} = \frac{1}{E(Z)}\int_{y=x}^{\infty} f_X(y)dy, \tag{10.15}$$

since the expected time between SP upward jumps *out of level* 0 is $E(Z)$, and the probability that such an SP jump exceeds level x is $\overline{F}_X(x)$. The SP makes a *double jump* in opposite directions at each renewal instant of the sequence $\{Z_n\}_{n=0,1,...}$. One jump is downward ending at level 0; the "opposite jump" is upward starting at level 0.

Equating (10.14) and (10.15) (rate balance across level x), results in the integral equation for $f_{\beta_X}(\cdot)$,

$$\int_{y=x}^{\infty}\frac{1}{\left(\int_{u=0}^{y}\frac{1}{r(u)}du\right)} f_{\beta_X}(y)dy = \frac{\overline{F_X}(x)}{E(Z)}. \tag{10.16}$$

In (10.16), taking d/dx on both sides yields

$$-\frac{1}{\left(\int_{u=0}^{x}\frac{1}{r(u)}du\right)} f_{\beta_X}(x) = -\frac{f_X(x)}{E(Z)}.$$

Hence

$$f_{\beta_X}(x) = \frac{\left(\int_{y=0}^{x} \frac{1}{r(y)} dy\right) f_X(x)}{E(Z)} = \frac{\left(\int_{y=0}^{x} \frac{1}{r(y)} dy\right) f_X(x)}{\int_{y=0}^{\infty} \frac{\overline{F}(y)}{r(y)} dy}. \tag{10.17}$$

The dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$.

Limiting PDF of Total Life in $\{X_n\}_{n=0,1,...}$ when $r(x) \equiv 1$
If $r(x) \equiv 1$, then $Z_n = X_n$ and $E(Z_n) = E(X_n) = \mu_X$ in *magnitude*. (However, their dimensions differ; since $[X_n] = [content unit]$ and $[Z_n] = [Time]$.) Formula (10.17) reduces to the well-known limiting pdf of total life (span) for an *ordinary* renewal process,

$$f_{\beta_X}(x) = \frac{x f_X(x)}{E(Z)} = \frac{x f_X(x)}{\mu_X}, \tag{10.18}$$

except that the dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$ instead of $[Time]^{-1}$. That is, in the variant of the GI/G/r(·) dam described here, the "life" is measured in 'content' units.

Remark 10.1 The foregoing variant of GI/G/r(·) exhibits the phenomenon of SP multiple jumps at the same (renewal) instant. Recall that SP jumps in the state space **do not occur in Time**. (See Examples 2.2, 2.3 in Sect. 2.3, regarding SP multiple jumps.)

Example 10.2 Suppose $r(x) = kx$, $x > 0$, where $k > 0$ is a constant. Then the inequality (10.1) does not hold because sample paths decay as a negative exponential function (see Sect. 6.4). The SP returns to every level $x > 0$, however small. Let us choose a small level $\varepsilon > 0$ to indicate that it is time for a new replenishment of content. Thus, whenever the content hits level ε from above, it increases by an amount $\underset{dis}{=} X$. We consider two cases.

Case 1: All replenishments start at level 0.
Whenever the content decays to level ε, a new replenishment starts at level 0. (We assume that the added amount is greater than ε; otherwise it is discarded.) Many systems are of this type. For example, heat and smoke alarms make a beep and/or show a red light, when the charge in the alarm's battery decays to a certain level. This signals that the battery needs replacing. In Example 10.1, this corresponds to replacing a battery with a new one when the preceding charge decreases to ε Coulombs.

Case 2: All replenishments start at level $\varepsilon > 0$
For each positive $v > \varepsilon$, the time to decay to level ε is

$$\int_{y=\varepsilon}^{v} \frac{1}{kx} dx = \frac{1}{k} \ln \frac{v}{\varepsilon} < \infty,$$

meaning the content returns to level ε in a finite time from any level $\varepsilon + X$. In this case, $\{X_n\}_{n=0,1,\ldots}$ is a delayed renewal process, where the first inter-renewal time is $X_0 \underset{dis}{=} X - \varepsilon$, because all future inputs (replenishments) will start at level ε. The inter-renewal amounts $X_1, X_2, \ldots$ are $\underset{dis}{=} X$. In the renewal process $\{X_n\}_{n=0,1,\ldots}$, the support of X_0 equals the support of X minus ε, because the remaining inputs start at level ε. This model structure is similar to that in the example in Sect. 10.3.8. We will not continue the analysis here; the limiting and time-t distributions of the quantities of interest can be derived from the reasoning given below in Sect. 10.3.8. This model would apply to cases where it is important to maintain the concentration of a solute in a solution above a certain level (say ε). Examples include: pharmacokinetics (see Sect. 11.6); consumer response to nonuniform advertisements (see [40]); and many others.

10.1.4 The Renewal Process $\{Z_n\}_{n=0,1,\ldots}$

Limiting Excess Life, Age, Total Life in $\{Z_n\}_{n=0,1,\ldots}$
Let $\gamma_Z(t), \delta_Z(t), \beta_Z(t)$ denote the excess life, age, and total life respectively, at a fixed time $t > 0$.

Define $\mathcal{G}(x) := \int_{y=0}^{x} \frac{1}{r(y)} dy :=$ *time for* $\{X(t)\}_{t \geq 0}$ *to decay from level* $x > 0$ *to level* 0. Then $\mathcal{G}(x)$ is an increasing differentiable function of x (since $d\mathcal{G}(x)/dx = 1/r(x)$), which implies the inverse $\mathcal{G}^{-1}(x)$ of $\mathcal{G}(x)$ exists, and

$$\frac{d}{dx}\mathcal{G}^{-1}(x) = \frac{1}{\frac{d}{dx}\mathcal{G}(x)} = \frac{1}{\frac{1}{r(x)}} = r(x), x > 0$$

(see, e.g., pp. 206–207 in [137], and other Calculus texts). Thus $\mathcal{G}^{-1}(x)$ is also an increasing (differentiable) function of x. Moreover, $\mathcal{G}^{-1}(x)$ is that level in the Time-state space $\boldsymbol{T} \times \lfloor 0, \infty)$ from which a descent to level 0 takes time period x, as $\mathcal{G}(\mathcal{G}^{-1}(x)) = x$. We now derive the pdfs of $\gamma_Z, \delta_Z, \beta_Z$ from the results for the pdfs of $\gamma_X, \delta_X, \beta_X$, respectively, given in Sect. 10.1.3.

Limiting PDF of Excess Life γ_Z in $\{Z_n\}_{n=0,1,\ldots}$
The relation between Z_n and $X(t)$ implies

$$\gamma_Z \leq x \text{ iff } \gamma_X \leq \mathcal{G}^{-1}(x)).$$

(see Fig. 10.1). Hence

$$F_{\gamma_Z}(x) = F_{\gamma_X}(\mathcal{G}^{-1}(x)). \tag{10.19}$$

Taking d/dx on both sides of (10.19) and substituting from (10.8) gives

$$\begin{aligned} f_{\gamma_Z}(x) &= f_{\gamma_X}(\mathcal{G}^{-1}(x)) \cdot \frac{d}{dx}\mathcal{G}^{-1}(x) = f_{\gamma_X}(\mathcal{G}^{-1}(x)) \cdot r(x) \\ &= \frac{r(x) \cdot \overline{F}_X(\mathcal{G}^{-1}(x))}{r(\mathcal{G}^{-1}(x)) \int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}. \end{aligned} \tag{10.20}$$

The dimension of $f_{\gamma_Z}(x)$ is $[Time]^{-1}$.
If $r(y) \equiv 1$, then $\mathcal{G}(x) = \mathcal{G}^{-1}(x) = x$ and

$$f_{\gamma_Z}(x) = \overline{F}_X(x) / \int_{y=0}^{\infty} \overline{F}_X(y) dy = \overline{F}_X(x)/\mu_X = f_{\gamma_X}(x).$$

The dimension of $f_{\gamma_Z}(x)$ is $[Time]^{-1}$, whereas the dimension of $f_{\gamma_X}(x)$ is $[content]^{-1}$.

Limiting PDF of Age δ_Z in $\{Z_n\}_{n=0,1,\ldots}$
In a similar manner as for the excess life, the age satisfies

$$\delta_Z \leq x \text{ iff } \delta_X \leq \mathcal{G}^{-1}(x).$$

Thus, $F_{\delta_Z}(x) = F_{\delta_X}(\mathcal{G}^{-1}(x))$. Taking $\frac{d}{dx}$ then yields

$$f_{\delta_Z}(x) = \frac{r(x)\overline{F}_X(\mathcal{G}^{-1}(x))}{r(\mathcal{G}^{-1}(x)) \int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}. \tag{10.21}$$

Thus $f_{\delta_Z}(x) = f_{\gamma_Z}(x)$, $x > 0$. The dimension of $f_{\delta_Z}(x)$ is $[Time]^{-1}$.
If $r(y) \equiv 1$ then $\mathcal{G}(x) = \mathcal{G}^{-1}(x) = x$, and

$$f_{\delta_Z}(x) = \overline{F}_X(x) / \int_{y=0}^{\infty} \overline{F}_X(y) dy = f_{\delta_X}(x).$$

The dimension of $f_{\delta_Z}(x)$ is $[Time]^{-1}$, whereas the dimension of $f_{\delta_X}(x)$ is $[content]^{-1}$.

Limiting PDF of Total Life β_Z in $\{Z_n\}_{n=0,1,...}$
Since $\beta_Z \leq x$ iff $\beta_X \leq \mathcal{G}^{-1}(x)$ then we obtain similarly as for $f_{\delta_Z}(x)$ and $f_{\gamma_X}(x)$ above,

$$f_{\beta_Z}(x) = f_{\beta_X}(\mathcal{G}^{-1}(x)) \cdot \frac{d}{dx}\mathcal{G}^{-1}(x) = f_{\beta_X}(\mathcal{G}^{-1}(x)) \cdot r(x).$$

From (10.17) we get

$$f_{\beta_Z}(x) = \frac{r(x) \cdot \left(\int_{y=0}^{x} \frac{1}{r(y)} dy\right) f_X(\mathcal{G}^{-1}(x))}{\int_{y=0}^{\infty} \frac{\overline{F}_X(y)}{r(y)} dy}. \tag{10.22}$$

The dimension of $f_{\beta_Z}(x)$ is $[Time]^{-1}$ whereas the dimension of $f_{\beta_X}(x)$ is $[content]^{-1}$. If $r(x) \equiv 1$ then

$$f_{\beta_Z}(x) = \frac{x f_X(x)}{\int_{y=0}^{\infty} \overline{F}_X(y) dy} = \frac{x f_X(x)}{\mu_X},$$

having dimension $[Time]^{-1}$.

10.1.5 Limiting PDFs in Ordinary Renewal Process

We now give the steady-state pdfs of excess, age and total life for the ordinary (i.e., standard) renewal process as a *special case* of those for the replacement model above. In the *ordinary* renewal process, we have $X_n = Z_n, n = 0, 1, 2, ...,$ since $r(X(t)) \equiv 1$ (see Fig. 10.2). The dimensions of X_n and Z_n are the same, usually $[Time]$. The pdfs of the excess, age and total life, i.e., $f_{\gamma_Z}(x)$, $f_{\delta_Z}(x)$, $f_{\beta_Z}(x)$, $x > 0$, are the same as formulas (10.9), (10.13), (10.18) respectively, and all have dimension $[Time]^{-1}$.

Direct Derivation of Limiting PDFs $f_{\gamma_Z}(x)$, $f_{\delta_Z}(x)$, $f_{\beta_Z}(x)$
We can derive these limiting pdfs very simply and directly in the ordinary renewal process. For example, to get $f_{\gamma_Z}(x)$, $X > 0$, we examine the sample path of $\gamma_Z(t)$, $t \geq 0$, in Fig. 10.2. The downcrossing rate of level x is $f_{\gamma_Z}(x)$; the upcrossing rate of level x is $\overline{F}_Z(x)/E(Z)$. Rate balance gives $f_{\gamma_Z}(x) = \overline{F}_Z(x)/E(Z)$. Similarly, examining the sample path of $\delta_Z(t)$, $t \geq 0$, gives $f_{\delta_Z}(x) = \overline{F}_Z(x)/E(Z)$, $x > 0$. To derive $f_{\beta_Z}(x)$, $x > 0$,

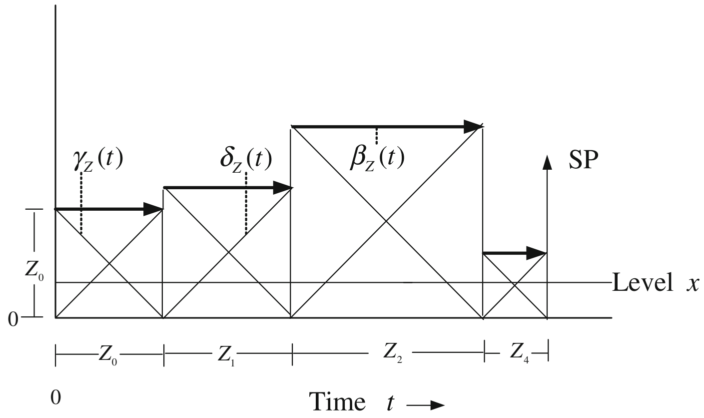

Fig. 10.2 Sample paths of excess $\gamma_Z(t)$, age $\delta_Z(t)$ and total life $\beta_Z(t)$, $t \geq 0$, in the ordinary (standard) renewal process

examine the sample path of $\beta_Z(t)$, $t \geq 0$. The downcrossing rate of level x is $\int_{y=x}^{\infty} (1/y)\, f_{\beta_Z}(y)dy$ and the upcrossing rate of level x is $\overline{F}_Z(x)/E(Z)$. Rate balance gives $\int_{y=x}^{\infty} (1/y)\, f_{\beta_Z}(y)dy = \overline{F}_Z(x)/E(Z)$. Taking d/dx of both sides yields $f_{\beta_Z}(x) = xf_Z(x)/E(Z)$.

Remark 10.2 The LC derivations of the limiting pdfs of excess life, age and total life, at time t as $t \to \infty$, are **relatively easy** in the replacement model, and are **much simpler** for the ordinary (standard) renewal process.

Remark 10.3 All the derivations in Sect. 10.1 are based directly on the author's unpublished notes of June 18–July 26, 1992 [28]. These notes were motivated by a talk on the ordinary renewal process by van Harn and Steutel (see Partial Bibliography below) at the 21st Conference on Stochastic Processes and their Applications at York University, Toronto, June 15–19, 1992. Their presented work differs completely from LC conceptually. Results for the **ordinary renewal process** using LC were published independently in Katayama (2002) (see Partial Bibliography).

10.2 A Renewal Problem with Barrier

Consider a renewal process $\{Z_n\}_{n=1,2,...}$, where $Z_n \underset{dis}{=} U_{(0,1)}$:= *uniform random variable on* (0, 1) (Fig. 10.3). Let N_K denote the number of renewals required to *first exceed* a barrier $K > 0$. In this section we derive the expected

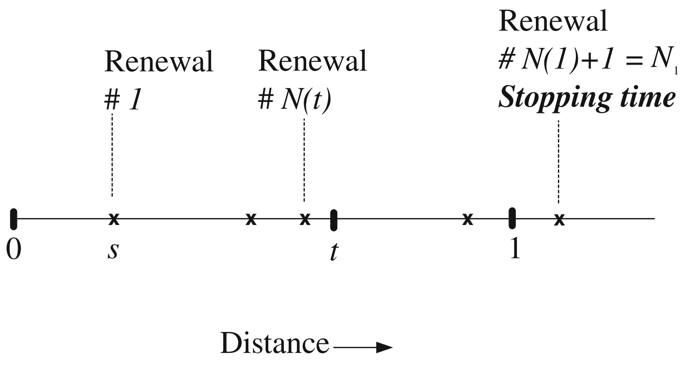

Fig. 10.3 Renewal process $\{Z_n\}_{n=1,2...}$ showing renewals. $N(t)$ is the number of renewals within $(0, t)$. $N_1 = N(1) + 1$ is number of renewals required to first exceed barrier $K = 1$. N_1 is a stopping time for the sequence $\{Z_n\}_{n=1,2,...}$ where $Z_n \underset{dis}{=} U_{(0,1)}$

value $E(N_K)$, $K = 1, 2, 3, \ldots$, and related quantities. It is well known that $E(N_1) = e$, the base of natural logarithms (see Problem 5, p. 485 in [125]). Usually, it is shown that $E(N_1) = e$ by a standard renewal argument, i.e., conditioning on the first renewal distance s (Fig. 10.3), deriving a renewal equation, and solving it. However, the general formula for $E(N_K)$, $K = 2, 3$, ... is not well known.

Here we derive $E(N_1)$ by an alternative method, which also applies to derive $E(N_K)$, $K = 2, 3$, ... The idea is to embed statistically independent replicas of the one-dimensional renewal process $\{Z_n\}_{n=1,2,....}$ into the cycles of a regenerative process such that the time axis of the embedded processes is perpendicular to the time axis of the regenerative process. Thus, the one-dimensional process $\{Z_n\}_{n=1,2,....}$ becomes transformed into an infinite sequence of statistically independent copies of $\{Z_n\}_{n=1,2,...}$, in a two-dimensional construct having two different perpendicular time axes. One time axis is for the regenerative process; the other is for $\{Z_n\}_{n=1,2,....}$. The type of construction in this alternative method, facilitates finding the expected number of renewals required to exceed a barrier or threshold, in other (seemingly unrelated) stochastic models as well.

10.2.1 Method for $E(N_K)$ Using a Regenerative Process

We construct a continuous-time continuous-state positive recurrent regenerative process

$$\{X(t)\}_{t\geq 0}, \qquad X(0)=0,$$

which embeds statistically independent reproductions of $\{Z_n\}_{n=1,2,\ldots}$ in all cycles of $\{X(t)\}_{t\geq 0}$ (Fig. 10.4). A sample path of $\{X(t)\}_{t\geq 0}$ is a non-decreasing step function, which makes SP *upward* jumps of size $\underset{dis}{=} U_{(0,1)}$ at an ***arbitrary Poisson rate*** λ. (*We select* $\lambda = 1$, for convenience.) The upward jumps are denoted by

$$b_n := Z_{n:} \underset{dis}{\equiv} U_{(0,1)}, n = 1, 2, \ldots .$$

(We replace symbol Z_n by b_n for generality beyond the threshold $K = 1$, and because of applicability to other models. See [33]).

Define random variable N_K by

$$N_K = \min\{n|\sum_{i=1}^{n} b_i > K\}, K = 1, 2, \ldots; \tag{10.23}$$

thus N_K is a *stopping time for the sequence* $\{b_n\}_{n=1,2,\ldots}$.

Let random variable $a \underset{dis}{=} \text{Exp}_\lambda = \text{Exp}_1$, implying $E(a) = 1$.

Define random variable c by

$$c = \sum_{i=1}^{N_K} a_i, \qquad \text{where each } a_i \underset{dis}{=} a, \tag{10.24}$$

and the a_is are i.i.d. r.v.s.

Let $\{c_n\}_{n=1,2,\ldots}$ be a renewal process where $c_n \underset{dis}{\equiv} c$; the c_ns are i.i.d. Then $\{c_n\}_{n=1,2,\ldots}$ are "compound" cycles of a regenerative process with sub-components $\{a_i\}_{n=1,2,\ldots}$. Since there is a one-to-one correspondence between a_n and b_n, $n = 1, 2, \ldots$, the random variable N_K is also a stopping time for the sequence $\{a_i\}_{n=1,2,\ldots}$. Taking the expected value in (10.24) yields

$$E(c) = E(N_K)E(\alpha) = E(N_K), \tag{10.25}$$

by Wald's equation (e.g., see Exercises 13–24, p. 486–489 in [125]).

Just after each instant when a sample path of $\{X(t)\}_{t\geq 0}$ upcrosses level K, *the SP jumps downward* (rebounds) *immediately to level* 0, *and the process* $\{X(t)\}_{t\geq 0}$ *restarts*. Our construction guarantees that the limiting distribution of $X(t)$ exists as $t \to \infty$ (see [132]). Random variable N_K equals the number of SP jumps required for $\{X(t)\}_{t\geq 0}$ to first exceed level K. A simple, but key observation, is that N_K is equal to the number of sub-intervals with lengths $\underset{dis}{=}$ a, comprising a cycle c. The state space of $\{X(t)\}_{t \geq 0}$ is $\boldsymbol{S} = [0, K + 1)$, because the excess of the jumps that exceed level K is less than 1 (due to jump sizes $\underset{dis}{=}$ $U_{(0,1)}$).

Relation to $\langle s, S \rangle$ Inventory with No Decay
Other stochastic models have a related structure. For example, the $\langle s, S \rangle$ inventory *with no decay* in Sect. 6.9 is the "flip" (like $\updownarrow$) of the $\{X(t)\}_{t \geq 0}$ process, where $K := S - s$, and the jump sizes are distributed as Exp_{μ}. In the $\langle s, S \rangle$ model $E(N_K)$ $(= E(N_{S-s}))$ is the expected number of demands in an ordering cycle.

10.2.2 Derivation of $E(N_1)$

Let the limiting mixed pdf of $\{X(t)\}_{t \geq 0}$ as $t \to \infty$, be $\{\pi_1, f_0(x)\}_{0<x<1}$. Consider a sample path of $\{X(t)\}_{t \geq 0}$; fix level $x \in (0, 1)$ (Fig. 10.4). SP upcrossings of level x are due to jumps $\underset{dis}{=}$ $U_{(0,1)}$ starting at level 0, or starting at some level $y \in (0, x)$. Thus, the SP upcrossing rate of level x is

$$1 \cdot \pi_1 \cdot P(b > x) + 1 \cdot \int_{y=0}^{x} P(b > x - y) \cdot f_0(y)dy$$

where $b \underset{dis}{\equiv} U_{(0,1)}$, and upward jumps occur at rate $1/E(a) = \lambda = 1$. This leads to the equation

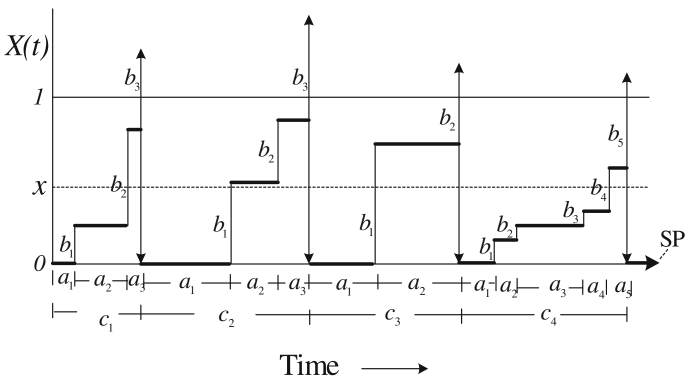

Fig. 10.4 Sample path of $\{X(t)\}_{t \geq 0}$, in renewal problem to determine $E(N_1)$ when renewal times are $\underset{dis}{=}$ $U_{(0,1)}$

$$1 \cdot \pi_{0.1} \cdot P(b > x) + 1 \cdot \int_{y=0}^{x} P(b > x - y) \cdot f_0(y)dy = \pi_1, x \in (0, 1)\,, \tag{10.26}$$

explained as follows. On the right side of (10.26), π_1 is the downcrossing rate of level x because the rate of SP downward jumps is the same as the rate of SP *entrances into state* $\{0\}$ *from above* (also called here 'downcrossings' of level 0). From the principle of *set balance*, this entrance rate is equal to the exit rate of $\{0\}$, namely $\lambda\pi_1 = 1 \cdot \pi_1 = \pi_1$.

The SP downcrossing rate of every level $x \in [0, 1)$ is equal to the total *upcrossing rate of level* 1. The SP rebounds into level 0 immediately after it upcrosses level 1. (The SP makes a *double jump*. Compare with the $\langle s, S\rangle$ inventory with no decay in Example 2.3 in Sect. 2.3; see also Sect. 2.3.) In the inventory model, whenever the stock on hand jumps below the reorder point s, it is replenished immediately up to level S.

Letting $x = 1$ in equation (10.26) gives

$$\pi_1 \cdot P(b > 1) + \int_{y=0}^{1} P(b > 1 - y) \cdot f_0(y)dy = \pi_1. \tag{10.27}$$

Since $b \underset{dis}{=} U_{(0,1)}$, we substitute $P(b > x) = 1 - x$, $0 < x < 1$, into (10.26), resulting in

$$\pi_{0,0}(1 - x) + \int_{y=0}^{x} (1 - x + y) f(y)dy = \pi_{0,0}, 0 < x < 1. \tag{10.28}$$

Taking d/dx twice in (10.28), and solving the resulting ordinary differential equation gives

$$f(x) = \pi_1 e^x, 0 < x < 1, \tag{10.29}$$

which we substitute into the normalizing condition $\pi_0 + \int_{x=0}^{1} f(x)dx = 1$, giving

$$\pi_1 = \frac{1}{e}. \tag{10.30}$$

The renewal rate of $\{c_n\}_{n=1,2,...}$ is $1/E(c)$ = SP entrance (or exit) rate of $\{0\} = \pi_1$. Thus $E(c) = 1/\pi_1$. From (10.25) and (10.30),

$$E(N_1) = E(c) \cdot E(a) = \frac{1}{\pi_1} \cdot 1 = e = 2.71828. \tag{10.31}$$

We have derived $E(N_1)$ in detail using the compound-cycle regenerative process structure, to fix ideas. The following values of $E(N_K)$, $K = 2, 3, \ldots$, in this Section are not well known.

10.2.3 Derivation of $E(N_2)$

Let $\pi_2 := \lim_{t\to\infty} P(X(t) = 0)$. Let the steady-state PDF of $\{X(t)\}_{t\geq 0}$ be

$$f(x) = f_0(x)\boldsymbol{I}_{(0,1]}(x) + f_1(x)\boldsymbol{I}_{[1,2)}(x), x \in (0, 2),$$

where $\boldsymbol{I}_{\boldsymbol{A}}(x) = 1$ if $x \in \boldsymbol{A}$, and $\boldsymbol{I}_{\boldsymbol{A}}(x) = 0$ if $x \notin \boldsymbol{A}$ (the characteristic function of set $\boldsymbol{A}$).

Consider a sample path of $\{X(t)\}_{t\geq 0}$ (Fig. 10.5), where the state space is $\boldsymbol{S} = [0, 3)$. Balancing SP up- and downcrossing rates of $x \in (0, 1)$, as in the case $K = 1$, gives

$$\pi_2(1 - x) + \int_{y=0}^{x} (1 - x + y) f_0(y)dy = \pi_2, x \in (0, 1). \tag{10.32}$$

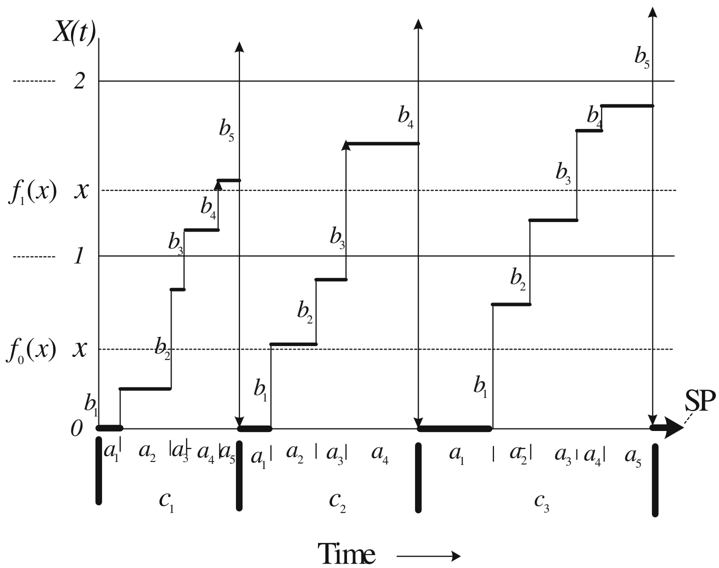

Fig. 10.5 Sample path of $\{X(t)\}_{t\geq 0}$ for renewal problem, with state space $\boldsymbol{S} = [0, 3)$. Facilitates solution for $E(N_2)$

Fix $x \in [1, 2)$ (see Fig. 10.5). Balancing SP up- and downcrossing rates of x, gives

$$\int_{y=x-1}^{1} (1 - x + y) f_0(y) dy + \int_{y=1}^{x} (1 - x + y) f_1(y) dy = \pi_2, x \in [1, 2) . \tag{10.33}$$

In (10.33), the first integral is the upcrossing rate of level x due to jumps starting in $(0, 1)$. The lower limit is $y = x - 1$ because an SP jump can upcross level $x \in (1, 2)$ only if it starts in interval $(x - 1, 1)$, which is a subset of $[0, 1)$. The second integral is the upcrossing rate of level x due to jumps starting in $[1, x)$.

Taking d/dx in (10.33) gives

$$-\int_{y=x-1}^{1} f_0(y) dy - 0 - \int_{y=1}^{x} f_1(y) dy + f_1(x) = 0, x \in [1, 2). \tag{10.34}$$

Substituting $\pi_2 e^y$ from (10.29), for $f_0(y)$ in (10.34) with π_1 replaced by π_2, and letting $x \downarrow 1$ yields

$$f_1(1^+) = \pi_2 (e - 1) = \pi_2 e - \pi_2 = f_0(1^-) - \pi_2.$$

which shows that $f(x)$ has a discontinuity at $x = 1$ of size π_2. (In $f_0(x)$, π_2 replaces π_1 because at this stage we are solving for $E(N_2)$). Taking d/dx in (10.34) gives

$$\begin{aligned} f_1'(x) - f_1(x) &= -f_0(x - 1) = -\pi_2 e^{(x-1)}, \\ \frac{d}{dx}(e^{-x} f_1(x)) &= -\pi_2 e^{-1}, \\ f_1(x) &= -\pi_2 e^{-1} x e^x + C e^x, x \in [1, 2), \end{aligned} \tag{10.35}$$

where C is a constant, evaluated by letting $x \downarrow 1$ in (10.35), resulting in

$$\begin{gathered} f_1(1^+) = -\pi_2 + Ce = \pi_2 e - \pi_2, \\ \text{and} \quad C = \pi_2. \end{gathered}$$

Substituting $C = \pi_2$ into (10.35) gives

$$f_1(x) = \pi_2 \left(-e^{-1} x + 1\right) e^x, x \in [1, 2).$$

Thus we obtain

$$f_0(x) = \pi_2 e^x, x \in (0, 1),$$
$$f_1(x) = \pi_2(1 - e^{-1}x)e^x, x \in [1, 2. \tag{10.36}$$

From (10.36) we check the discontinuity at $x = 1$,

$$f_1(1^+) = \pi_2 e - \pi_2 = f_0(1^-) - \pi_2,$$

Fig. 10.6 shows that the discontinuity stays in place at $x = 1$ for $K = 2, 3, \ldots$, but with the sizes π_K, $K = 2, 3, \ldots$ respectively.

The normalizing condition is

$$\pi_2 + \int_{x=0}^{1} f_0(x)dx + \int_{x=1}^{2} f(x)dx = 1. \tag{10.37}$$

Substituting from (10.36) into (10.37) gives

$$\pi_2 = \frac{1}{-e + e^{2.}}. \tag{10.38}$$

From (10.25),

$$E(N_2) = E(c)E(a) = \frac{1}{\pi_2} \cdot 1 = -e + e^2 = 4.67077. \tag{10.39}$$

10.2.4 Derivation of $E(N_3)$

We now explore further the pattern of $E(N_K)$, $K = 1, 2, \ldots$. For deriving $E(N_3)$, the state space is $\boldsymbol{S} = [0, 4)$. Let $\pi_3 := \lim_{t\to\infty} P(X(t) = 0)$. Let the steady state pdf of $\{X(t)\}_{t\geq 0}$ be

$$f(x) = f_0(x)\boldsymbol{I}_{(0,1]}(x) + f_1(x)\boldsymbol{I}_{[1,2)}(x) + f_2(x)\boldsymbol{I}_{[2,3)}(x), x \in (0, 3),$$

(plotted as $f(x)/\pi_3$ in Fig. 10.6 since π_3 is a factor of each $f_j(x)$, $j = 0, 1, 2$—see (10.43) below). We now balance SP up- and downcrossing rates across arbitrary levels $x \in (0, 1)$; $x \in [1, 2)$; $x \in [2, 3)$, which gives respectively, integral equations

$$\pi_3(1 - x) + \int_{y=0}^{x} (1 - x + y) f_0(y)dy = \pi_3, \tag{10.40}$$

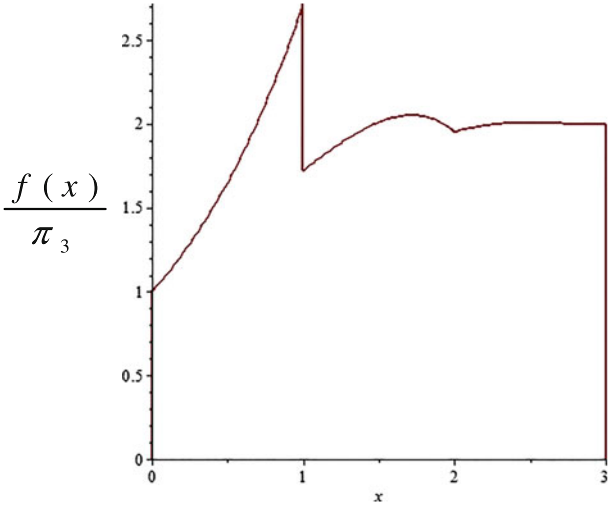

Fig. 10.6 Plot of $f(x)/\pi_3, x \in (0,3). E(N_3) = 1/\pi_3 = 1 + \int_0^3 f(x)dx = 1 + \sum_{j=0}^{2} \int_j^{j+1} f_j(x)dx$. The only discontinuity of $f(x)$ is at $x = 1$, of size $= \pi_3$; $f(x)$ is continuous for all $x \in (1,3)$

$$\int_{y=x-1}^{1} (1-x+y) f_0(y)dy + \int_{y=1}^{x} (1-x+y) f_1(y)dy = \pi_3, \quad (10.41)$$

$$\int_{y=x-1}^{2} (1-x+y) f_1(y)dy + \int_{y=2}^{x} (1-x+y) f_2(y)dy = \pi_3. \quad (10.42)$$

Solving integral equations (10.40), (10.41) in a similar manner as for $K = 1, 2$ above, gives

$$\begin{aligned} f_0(x) &= \pi_3 e^x, x \in (0,1), \\ f_1(x) &= \pi_3(1 - e^{-1}x)e^x, x \in [0,1, \\ f_2(x) &= \frac{1}{2}\pi_3(-2xe^{-2} + e^{-2}x^2 - 2xe^{-1} + 2)e^x, x \in [2,3). \end{aligned} \quad (10.43)$$

The normalizing condition is

$$\pi_3 + \int_{x=0}^{1} f_0(x)dx + \int_{x=1}^{2} f_1(x)dx + \int_{x=2}^{3} f_2(x)dx = 1, \quad (10.44)$$

yielding

$$\pi_3 = \frac{1}{\frac{1}{2}e - 2e^2 + e^3}.$$

Substituting from (10.43) into (10.44) gives

$$E(N_3) = \frac{1}{\pi_3} = \frac{1}{2}e - 2e^2 + e^3 = 6.66656563. \tag{10.45}$$

The form of $f(x)$ is reminiscent of the pdf of wait in the M/D/1 queue in Fig. 3.18 in Sect. 3.10.2. In both models, the pdf has a discontinuity at exactly one point: at $x = 1$ in the present renewal problem, and at $x = D$ in the M/D/1 queue.

10.2.5 Derivation of $E(N_K)f$ for General K

Repeating the foregoing procedure for several more values of K with the aid of mathematical software (e.g., Maple) gives, for $x \in [3, 4)$

$$f_3(x) = \pi_4 \left(1 - e^{-1}x - \frac{3}{2}e^{-3}x - e^{-2}x + \frac{1}{2}e^{-2}x^2 \right.$$
$$\left. - \frac{1}{6}e^{-3}x^3 + e^{-3}x^2e^x \right),$$

and for $x \in [4, 5)$

$$f_4(x) = \pi_5 \left(1 + \frac{1}{24}e^{-4}x^4 - \frac{1}{2}e^{-4}x^3 + 2e^{-4}x^2 - e^{-2}x - \frac{3}{2}e^{-3}x \right.$$
$$\left. - \frac{8}{3}e^{-4}x - \frac{1}{6}e^{-3}x^3 + e^{-3}x^2 + \frac{1}{2}e^{-2}x^2 + e^{-1}x \right) e^x.$$

Applying the normalizing conditions for $K = 4$ and 5 respectively then results in

$$E(N_4) = \frac{1}{\pi_4} = -\frac{1}{6}e + 2e^2 - 3e^3 + e^4,$$
$$E(N_5) = \frac{1}{\pi_5} = \frac{1}{24}e - \frac{4}{3}e^2 + \frac{9}{2}e^3 - 4e^4 + e^5.$$

The author hypothesized that $E(N_K)$ is the sum of powers of $e, e^2, \ldots, e^k$, with coefficients given in the series

$$\frac{1}{\pi_K} = E(N_K) = \sum_{j=1}^{K} \frac{(-j)^{K-i}}{(K-j)!} e^j, \; K = 1, 2, \ldots. \tag{10.46}$$

This can be verified by mathematical induction, carried out by first deriving the formulas for $f_j(x)$, $j = 0, \ldots, K$, in the same way as for those in (10.43). Then we obtain (10.46) in a similar manner as for the derivation of (10.45).

10.2.6 Asymptotic Formula for $E(N_K)$ as $K \to \infty$

We now show that in (10.46),

$$E(N_K) \approx 2K + 2/3,$$

i.e.,

$$\lim_{K\to\infty} \frac{E(N_K)}{2K + \frac{2}{3}} = 1. \tag{10.47}$$

For example, using (10.47) with the "large" number $K = 20$, we immediately have the approximation $E(N_{20}) \approx 2(20) + 2/3 = 40.6667$. The analytical value up to the same number of decimals using (10.46) is also 40.6667, whose accuracy depends on the number of digits carried, and on the computational algorithm used.

Remarkably, from the analytical values of $E(N_2)$ and $E(N_3)$ given in (10.39) and (10.45), the approximation (10.47) is very accurate for $K = 2, 3, \ldots$. Even for $K = 1$, the "asymptotic" approximation $2K + \frac{2}{3} = 2.6666$, which is within 1.90% of $e = 2.71828$.

Derivation of Asymptotic Formula (10.47)
Let γ denote the excess life at an arbitrary point $x \in \boldsymbol{S}$, as $x \to \infty$. Then $f_\gamma(y) = \frac{1}{\mu}(1 - B(y))$, $y > 0$, where $B(y)$ is the common cdf of the inter-renewal time, having mean μ (formula (10.9) above; see also Example 7.24, p. 453 in [125]. Here, the inter-renewal times are $\underset{dis}{=} U_{(0,1)}$. Thus $B(y) = y$, $0 < y < 1$, $\mu = 1/2$, and

$$E(\gamma) = \frac{1}{\mu} \int_{y=0}^{\infty} y \cdot f_\gamma(y) dy = 2 \int_{y=0}^{1} y\,(1-y)\,dy = \frac{1}{3}. \tag{10.48}$$

Let γ_K denote the excess life at K; if K is large then $E(\gamma_K) \approx \frac{1}{3}$. If K is finite then

$$K + \gamma_K = \sum_{j=1}^{N_K} Z_j, \tag{10.49}$$

where the Z_js are i.i.d. $\underset{dis}{\equiv} U_{(0,1)}$, and N_K is a stopping time for $\{Z_j\}_{j=1,2,\ldots}$. Taking expected values in (10.49) yields

$$K + E(\gamma_K) = E(N_K) \cdot \frac{1}{2},$$

$$\text{implying} \qquad E(N_K) = 2K + 2 \cdot E(\gamma_K).$$

If K is large, $E(\gamma_K) \approx E(\gamma)$; substituting from (10.48) gives

$$E(N_K) \approx 2K + \frac{2}{3},$$

which is equivalent to formula (10.47). Also, if $\alpha > 0$ is a "large" real number then $E(N_\alpha) \approx 2\alpha + \frac{2}{3}$, where N_α is the number of renewals required to first exceed α.

10.2.7 Number of Renewals Within an Arbitrary Interval

Let $N(\alpha, \beta)$ denote the number of renewals *within* an interval $(\alpha, \beta) \subseteq (0, K)$, during a single cycle c_n, $n \in \{1, 2, \ldots\}$, of $\{X(t)\}_{t\geq 0}$. Without loss of generality, $X(0) = 0$, and we stop after N_K subintervals of $\{a_n\}_{n=1,2,\ldots}$. Then

$$N(0, K) = N_K - 1, \quad \text{and} \quad E(N(0, K)) = E(N_K) - 1.$$

Thus the values of $E(N_1)$; $E(N_2)$; $E(N_3)$ lead to the expected number of renewals within intervals (0, 1); (0, 2), (0, 3), (1, 2); (2, 3), as follows:

$$\begin{aligned}
E(N(0,1)) &= E(N_1) - 1 = e - 1 \approx 1.7183, \\
E(N(0,2)) &= E(N_2) - 1 = -e + e^2 - 1 \approx 3.6708, \\
E(N(0,3)) &= E(N_3) - 1 = \frac{1}{2}e - 2e^2 + e^3 - 1 \approx 5.6666, \\
E(N(1,2)) &= E(N(0,2)) - E(N(0,1)) = E(N_2) - E(N_1) \approx 1.9525, \\
E(N(2,3)) &= E(N(0,3)) - E(N(0,2)) = E(N_3) - E(N_2) \approx 1.9958.
\end{aligned} \tag{10.50}$$

For large K,

$$E(N(K, K+1) = E(0, K+1) - E(0, K) = E(N_{K+1}) - E(N_K) \approx 2.0.$$

In (10.50), the values of $E(N(1, 2))$, $E(N(2, 3))$ are already within 2.38% and 1.40% of the limiting value 2.0, respectively.

If $0 < \alpha < \beta < 1$, where α and β are arbitrary real numbers then $E(N_\alpha) = e^\alpha$, and $E(N_\beta) = e^\beta$, obtained similarly as in the solution for $E(N_1)$. Hence, $E(N(0, \alpha)) = e^\alpha - 1$, $E(N(0, \beta)) = e^\beta - 1$, implying the expected number of renewals within (α, β) is

$$E(N(\alpha, \beta)) = E(N_\beta) - E(N_\alpha) = e^\beta - e^\alpha, 0 < \alpha < \beta < 1. \quad (10.51)$$

In particular,

$$E\left(N\left(\frac{2}{3}, 1\right)\right) = e - e^{\frac{2}{3}} \approx 0.77055,$$
$$E\left(N\left(\frac{1}{3}, \frac{2}{3}\right)\right) = e^{\frac{2}{3}} - e^{\frac{1}{3}} \approx 0.55212,$$
$$E\left(N\left(0, \frac{1}{3}\right)\right) = e^{\frac{1}{3}} - e^0 \approx 0.39561.$$

Thus, approximately 44.84% of the renewals occur *in the top third*, 32.13% *in the middle third* and 23.02% *in the bottom third*, of interval (0, 1), indicating renewal instants tend to accumulate *in the top portion* of (0, 1). For a possible intuitive explanation of this phenomenon, fix the length of a "sliding interval" $\boldsymbol{I}_h$ using $|\boldsymbol{I}_h| = h$, $0 < h < 1$. As we slide $\boldsymbol{I}_h$ steadily from position $(0, h)$ to position $(1 - h, 1)$, the expected number of renewals in $\boldsymbol{I}_h$ increases steadily.

We can extend the analysis to determine the expected number of renewals within an arbitrary interval (α, β), $0 \le \alpha < \beta < \infty$.

10.2.8 Discussion

We can apply the compound-cycle regenerative process model of this section, to an arbitrary renewal process $\{b_n\}_{n=1,2,...}$, where b_n is a non-lattice positive r.v. The analysis can also be extended to models where $-\infty < b_n < \infty$, so that $\{b_n\}_{n=1,2,...}$ is not a renewal process, but the cycles $\{c_n\}_{n=1,2,...}$ and subintervals $\{a_n\}_{n=1,2,...}$ are inter-renewal times of renewal processes.

Possible applications are to problems where it is required to determine the expected number of events until a stopping criterion is satisfied. Examples are the number of: customers served in a busy period of a queue; demands in an ordering cycle of an inventory system; inputs until overflow of a dam;

shocks until failure of a machine part; claims until ruin in an actuarial model; doses of a prescription drug until an overdose occurs; advertisements until a favorable consumer response occurs for a product.

10.3 The Time-*t* PDFs of a Renewal Process

We now apply LC to obtain the pdfs of the *excess life*, *age* and *total life at an arbitrary finite time* $t > 0$, based on concepts in [34]. The time-t probability distributions have been analyzed classically in [66, 99, 123, 135], and in many other studies.

Consider an ***ordinary renewal process*** $\{Z_n\}_{n=1,2,...}$ with continuous inter-arrival times $Z_n \underset{dis}{=} Z$ having cdf $B(\cdot)$, pdf $b(\cdot)$, ccdf (the complementary cdf) $\bar{B}(\cdot) = 1 - B(\cdot)$, and support $(0, \boldsymbol{U})$, $\boldsymbol{U} > 0$. We transform the one-dimensional process $\{Z_n\}_{n=1,2,...}$ into a regenerative process $\{X(s)\}_{s\geq 0}$ having state space $[0, \infty)$ with upward jumps $\underset{dis}{=} Z$ occurring at an arbitrary Poisson rate, and with regenerative epochs at instants of level-t exceedance (see the first subsection of Sect. 10.2; Sect. 10.2.1; Figs. 10.3 and 10.4). The limiting pdf of $\{X(s)\}_{s\geq 0}$ as $s \to \infty$, exists (see, e.g., [132]), and is concentrated on the state-space interval $[0, t)$. Knowledge of *this limiting pdf* leads directly to the finite time-t distributions of $\{Z_n\}_{n=1,2,...}$.

10.3.1 Structure of Regenerative Process $\{X(s)\}_{s\geq 0}$

The process $\{X(s)\}_{s\geq 0}$ is built up from i.i.d. replicas of $\{Z_n\}_{n=1,2,...}$ and an *independent* Poisson process of arbitrary rate λ ($\lambda := 1$ for simplicity). Let $X(0) = 0$. Sample paths of $\{X(s)\}_{s\geq 0}$ make upward jumps at Poisson rate 1, with inter-jump times $a_i \underset{dis}{=} a$ and $E(a) \equiv 1$. The jumps originate in state-space subset $[0, t)$ and are in one-to-one correspondence with the horizontal intervals $\{a_i\}_{i=1,2,...}$ (see Fig. 10.7). When a jump upcrosses level t the SP immediately jumps *downward to level* 0 (double jumps—see Example 2.2 in Sect. 2.3 and subsequent material therein). At that instant $\{X(s)\}_{s\geq 0}$ restarts (Fig. 10.7). (Fig. 10.8 depicts the regenerative process when $Z_n \underset{dis}{\equiv} U_{(0,1)}$, and $t \in (1, 2)$.)

Let $\left\{\pi^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ be the limiting mixed pdf of $\{X(s)\}_{s\geq 0}$ as $s \to \infty$, where $\pi^{(t)} = \lim_{s\to\infty} P(X(s) = 0)$. Then $\left\{\pi^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ is a time-average pdf since upward jumps occur at a Poisson rate, implying the arrival-

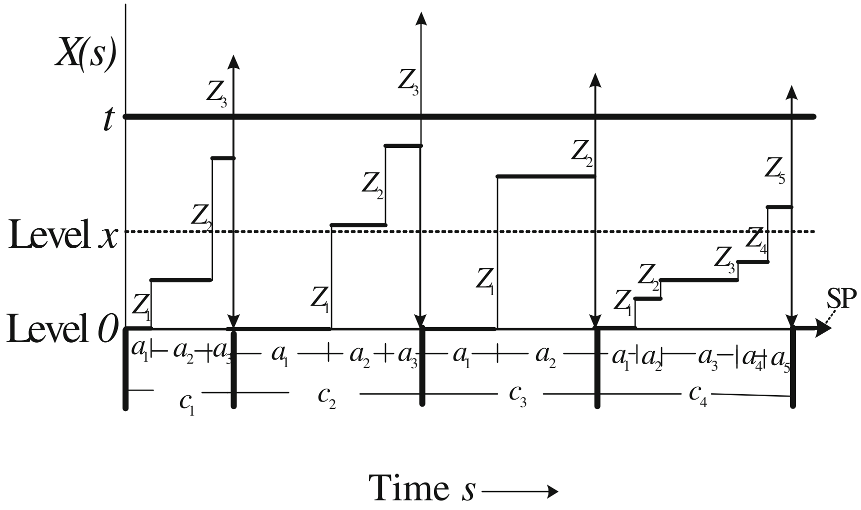

Fig. 10.7 Sample path of regenerative process $\{X(s)\}_{s\geq 0}$. Indicates embedded renewal process $\{Z_n\}_{n=1,2,,\ldots}$, the fixed time t, cycles c_1, c_2, ..., interarrival times between upward jumps $a_i \underset{dis}{=} \text{Exp}_1$, a fixed state-space level x, SP motion

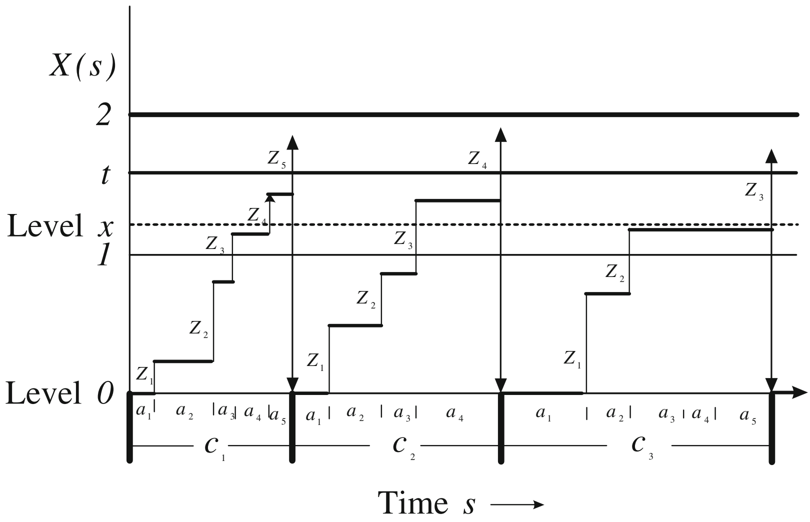

Fig. 10.8 Sample path of $\{X(s)\}_{s\geq 0}$, where $Z_n \underset{dis}{\equiv} U_{(0,1)}$ and $t \in (1, 2)$. In each cycle having at least j upward jumps $Z_j \underset{dis}{=} U_{(0,1)}$, but the Z_j's in different cycles have different sizes. (See Fig. 10.4.)

point pdf is the same as the time-average pdf (e.g., [145]). A rate-balance equation for $\left\{\pi^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ is

$$\lim_{s\to\infty}\frac{\mathcal{U}_s^j(x)}{s} = \lim_{s\to\infty}\frac{\mathcal{D}_s^j(x)}{s}, x \in [0, t). \tag{10.52}$$

Substituting the formulas for the rates in (10.52) gives

$$\pi^{(t)}\bar{B}(x) + \int_0^x \overline{B}(x-y)f^{(t)}(y)dy = \pi^{(t)}, x \in [0, t), \tag{10.53}$$

which is to be solved with the normalizing condition

$$\pi^{(t)} + \int_0^t f^{(t)}(y)dy = 1. \tag{10.54}$$

10.3.2 Solution of Equation for $\left\{\pi^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$

The study [34] shows that the solution of (10.53) and (10.54) is

$$f^{(t)}(x) = \frac{M'(x)}{M(t)+1}, 0 < x < t, \tag{10.55}$$

$$\pi^{(t)} = \frac{1}{M(t)+1}, \tag{10.56}$$

where

$$M(x) = \sum_{n=1}^{\infty} B^{*n}(x); \qquad M'(x) = \sum_{n=1}^{\infty} b^{*n}(x), x \in (0, t], \tag{10.57}$$

and $B^{*n}(x)$, $b^{*n}(x)$ are the n-fold convolutions of $B(\cdot)$ and $b(\cdot)$, respectively. $M(x)$ is the *renewal function* for $\{Z_n\}_{n=1,2,...}$; $M(x) = E($ *number of renewals up to time* $x)$—see pp. 167–169 in [99]. (Step 1 in Sect. 10.3.8 has a more detailed derivation of $\left\{\pi^{(t)}, f^{(t)}(x)\right\}_{0<x<t}$ in a particular *modified* renewal process.) From (10.56)

$$\frac{1}{\pi^{(t)}} = M(t) + 1 = E(number\ of\ renewals\ required\ to\ exceed\ x). \tag{10.58}$$

Formula (10.58) connects $M(t)$ to $1/\pi_1$ in formula (10.31) when $K = 1$ in Sect. 10.2.2 (and also to $1/\pi_K$ when $K = 2, 3, \ldots$, in Sects. 10.2.3–10.2.5).

10.3.3 Time-t Probability Distributions of $\{Z_n\}_{n=1,2,...}$

We look separately at the two cases: $t \le \boldsymbol{U}$, and $t > \boldsymbol{U}$, where $(0, \boldsymbol{U})$ is the support of the inter-renewal pdf $b(\cdot)$. (See Sect. 4, pp. 191–195, in [34]).

10.3.4 PDF of Excess Life γ_t

Case $t \le \boldsymbol{U}$

If $t \le \boldsymbol{U}$ then $\boldsymbol{U}$ is finite or infinite. Integral equation (10.59) below for the ccdf $\overline{F}_{\gamma_t}(x)$ equates two different upcrossing rates of level $t + x$ (see Sect. 3.16.5 for similar reasoning in an M/G/1 queue with bounded virtual wait). For $0 < x < \boldsymbol{U}$,

$$\pi^{(t)}\overline{F}_{\gamma_t}(x) = \pi^{(t)}\overline{B}(t+x) + \int_{y=(t+x-U,0)^+}^{t} \overline{B}(t+x-y)f^{(t)}(y)dy, \tag{10.59}$$

$$\overline{F}_{\gamma_t}(x) = \overline{B}(t+x) + \int_{y=(t+x-U,0)^+}^{t} \overline{B}(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy. \tag{10.60}$$

where $(t + x - \boldsymbol{U}, 0)^+ := \max(t + x - \boldsymbol{U}, 0)$.

Taking d/dx in (10.60) gives, since $\overline{B}(t+x-(t+x-U)) = \overline{B}(U) = 0$,

$$f_{\gamma_t}(x) = b(t+x) + \int_{y=(t+x-U,0)^+}^{t} b(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy, 0 < x < U. \tag{10.61}$$

From (10.61) we obtain

$$f_{\gamma_t}(x) = \begin{cases} b(t+x) + \int_{y=0}^{t} b(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy, 0 < x < \boldsymbol{U} - t, \\ \int_{y=t+x-U}^{t} b(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy, \boldsymbol{U} - t < x < \boldsymbol{U}. \end{cases} \tag{10.62}$$

If $\boldsymbol{U} = \infty$, only the first formula in (10.62) applies; if $\boldsymbol{U} < \infty$, both formulas in (10.62) apply.

Remark 10.4 If $t < \boldsymbol{U} < \infty$ then $f_{\gamma_t}(x)$ has a jump discontinuity at $x = \boldsymbol{U} - t$ of magnitude $f_{\gamma_t}\left((\boldsymbol{U}-t)^+\right) - f_{\gamma_t}\left((\boldsymbol{U}-t)^-\right) = -b(\boldsymbol{U}^-)$. This follows by letting $x \downarrow (\boldsymbol{U} - t)$ in the second formula of (10.62), and $x \uparrow (\boldsymbol{U} - t)$ in the first formula of (10.62), and subtracting.

Case $t > \boldsymbol{U}$
If $t > \boldsymbol{U}$, then $\boldsymbol{U}$ must be finite; assume $t \in [NU, (N+1)U)$ for some integer $N \geq 1$. Upward jumps starting below $t + x - \boldsymbol{U}$, cannot upcross level $t + x$. Thus, an equation analogous to (10.60) is

$$\overline{F}_{\gamma_t}(x) = \int_{y=t+x-\boldsymbol{U}}^{t} \overline{B}(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy,\ 0 < x < U. \tag{10.63}$$

Taking d/dx in (10.63) and noting that $\overline{B}(U) = 0$, gives

$$f_{\gamma_t}(x) = \int_{y=t+x-\boldsymbol{U}}^{t} b(t+x-y)\frac{f^{(t)}(y)}{\pi^{(t)}}dy,\ 0 < x < U. \tag{10.64}$$

From (10.64), with

$$f_n^{(t)}(y) := f^{(t)}(y) \cdot \boldsymbol{I}_{(n\boldsymbol{U},(n+1))\boldsymbol{U}}(y), y > 0, n = 0, \ldots, N,$$

we get

$$f_{\gamma_t}(x) = \begin{cases} \int_{y=t+x-\boldsymbol{U}}^{NU} b(t+x-y)\frac{f_{N-1}^{(t)}(y)}{\pi^{(t)}}dy \\ \qquad + \int_{y=N\boldsymbol{U}}^{t} b(t+x-y)\frac{f_N^{(t)}(y)}{\pi^{(t)}}dy, 0 < x < (N+1)U - t, \\ \int_{y=t+x-\boldsymbol{U}}^{t} b(t+x-y)\frac{f_N^{(t)}(y)}{\pi^{(t)}}dy, (N+1)U - t < x < U. \end{cases}$$

10.3.5 PDF of $\{X(s)\}_{s\geq 0}$ Just Before a Jump Over t

Let $X_{JB}^{(t)}$:= ordinate of $\{X(s)\}_{s\geq 0}$ just before the jump that first exceeds level t. Denote its mixed pdf by $\left\{\pi_{JB}^{(t)},\ f_{JB}^{(t)}(x)\right\}_{0<x<t}$ where $\pi_{JB}^{(t)} = P(X_{JB}^{(t)} = 0)$. ($X_{JB}^{(t)}$ is an important random variable in various stochastic models, such as actuarial ruin models – see, e.g., [79] and [54].) We now state the results for $\left\{\pi_{JB}^{(t)},\ f_{JB}^{(t)}(x)\right\}_{0<x<t}$. (For detailed derivations see [34].)

Case $t \leq \boldsymbol{U}$
Considering a sample path of $\{X(s)\}_{s\geq 0}$ (Fig. 10.7), and applying Bayes' rule, leads to

$$f_{JB}^{(t)}(x) = \overline{B}(t-x)\frac{f^{(t)}(x)}{\pi^{(t)}}, 0 < x < t\ . \tag{10.65}$$

$$\pi_{JB}^{(t)} = \overline{B}(t). \tag{10.66}$$

Case $t > U$

$$f_{JB}^{(t)}(x) = \begin{cases} \overline{B}(t-x)\dfrac{f_{N-1}^{(t)}(x)}{\pi^{(t)}}, t-U \le x < NU, \\ \overline{B}(t-x)\dfrac{f_{N}^{(t)}(x)}{\pi^{(t)}}, NU \le x < t. \end{cases} \tag{10.67}$$

Random variable $X_{JB}^{(t)}$ is related to the age δ_t.

10.3.6 PDF of Age δ_t

We get the mixed pdf $\left\{\pi_{\delta_t}, f_{\delta_t}(x)\right\}_{0<x<t}$ using (10.65)–(10.67). Since $\delta_t = t - X_{JB}^{(t)}$, $\pi_{\delta_t} = P(\delta_t = t) = P(X_{JB}^{(t)} = 0) = \pi_{JB}^{(t)}$, and $f_{\delta_t}(x) = f_{JB}^{(t)}(t-x)$, $0 < x < t$.

Case $t \le U$
Using (10.65) and (10.66) yields

$$\pi_{\delta_t} = \overline{B}(t), \qquad f_{\delta_t}(x) = \overline{B}(x)\frac{f^{(t)}(t-x)}{\pi^{(t)}}, 0 < x < t. \tag{10.68}$$

Case $t > U$
Probability $\pi_{\delta_t} = 0$ since $t > U$. Using (10.67) and applying $f_{\delta_t}(x) = f_{JB}^{(t)}(t-x)$, yields

$$f_{\delta_t}(x) = \begin{cases} \dfrac{\overline{B}(x) f_N^{(t)}(t-x)}{\pi^{(t)}}, 0 < x < t - NU, \\ \dfrac{\overline{B}(x) f_{N-1}^{(t)}(t-x)}{\pi^{(t)}}, t - NU < x < U. \end{cases} \tag{10.69}$$

10.3.7 PDF of Total life β_t

The total life is $\beta_t := \gamma_t + \delta_t$. Hence $P(\beta_t = x)dx = P(\gamma_t = x - \delta_t)dx$, $x > \delta_t$.

Case $t \leq \boldsymbol{U}$

$$f_{\beta_t}(x|\delta_t = y)dx = P(Z = x|Z > y)dx = \frac{P(Z = x)dx}{P(Z > y)} = \frac{b(x)dx}{\overline{B}(y)}.$$

Unconditioning $f_{\beta_t}(x|\delta_t = y)$ with respect to $f_{\delta_t}(y)$, and substituting for $f_{\delta_t}(y)$ from (10.68), gives

$$\begin{aligned} f_{\beta_t}(x) &= \int_{y=0}^{x} \frac{b(x)}{\overline{B}(y)} f_{\delta_t}(y)dy = \int_{y=0}^{x} \frac{b(x)}{\overline{B}(y)} \overline{B}(y) \frac{f^{(t)}(t-y)}{\pi^{(t)}} dy \\ &= b(x) \int_{y=0}^{x} \frac{f^{(t)}(t-y)}{\pi^{(t)}} dy, 0 < x < t. \end{aligned} \tag{10.70}$$

Similar reasoning yields

$$\begin{aligned} f_{\beta_t}(x) &= \frac{b(x)}{\overline{B}(t)} \pi_{\delta_t} + b(x) \int_{y=0}^{t} \frac{f^{(t)}(t-y)}{\pi^{(t)}} dy \\ &= b(x) \left(1 + \int_{y=0}^{t} \frac{f^{(t)}(t-y)}{\pi^{(t)}} dy \right), t < x < \boldsymbol{U}. \end{aligned} \tag{10.71}$$

Formulas (10.70) and (10.71) imply $f_{\beta_t}(x)$ has a jump discontinuity at $x = t$ of magnitude

$$f_{\beta_t}(t^+) - f_{\beta_t}(t^-) = b(t). \tag{10.72}$$

If $\boldsymbol{U} = \infty$ then $\lim_{t\to\infty} b(t) = 0$ in (10.72).

Check on $\lim_{t\to\infty} f_{\beta_t}(x)$
Formula (10.70) gives the known result $\lim_{t\to\infty} f_{\beta_t}(x) = xb(x)/E(Z), x > 0$, since (10.55) and (10.56) imply $f^{(t)}(t-y)/\pi^{(t)} = M'(t-y)$, and (10.70) gives

$$f_{\beta_t}(x) = b(x) \int_{y=0}^{x} M'(t-y)dy = b(x) \left(M(t) - M(t-x)\right), 0 < x < t.$$

Applying Blackwell's theorem (p. 191 in [99]) implies

$$\lim_{t\to\infty} (M(t) - M(t-x)) = \frac{x}{E(Z)}, x > 0 \implies \lim_{t\to\infty} f_{\beta_t}(x) = \frac{xb(x)}{E(Z)}, x > 0.$$

Case $t > \boldsymbol{U}$
Expressing $f_{\beta_t}(x)$ in terms of $f_{\beta_t}(x|\delta_t = y)$ and substituting for $f_{\delta_t}(y)$ from (10.69) yields

$$f_{\beta_t}(x) = b(x)\int_{y=0}^{x} \frac{f_N^{(t)}(t-y)}{\pi^{(t)}}dy,\ 0 < x < t - N\boldsymbol{U}. \tag{10.73}$$

Reasoning as for (10.73) and using also (10.69), yields

$$f_{\beta_t}(x) = b(x)\left(\int_{y=0}^{t-N U} \frac{f_N^{(t)}(t-y)}{\pi^{(t)}}dy + \int_{y=t-NU}^{x} \frac{f_{N-1}^{(t)}(t-y)}{\pi^{(t)}}dy\right),$$
$$t - N\boldsymbol{U} < x < \boldsymbol{U}. \tag{10.74}$$

10.3.8 Example—A Modified Renewal Process

Consider a ***modified renewal process*** $\{Z_n\}_{n=1,2,\ldots}$ where $Z_1 \underset{dis}{=} U_{(0,1)}$ and $Z_n \underset{dis}{=} \text{Exp}_\mu, n = 2, 3, \ldots$ This is also called a ***delayed renewal process*** (see pp. 197–199 in [99]; pp. 27–29 in [66]). Thus $\overline{B}_0(x) := P(Z_1 > x) = 1 - x, x \in (0, 1)$, and $\overline{B}_1(x) := P(Z_n > x) = e^{-\mu x}, x \in (0, \infty)$. We now derive $f_{\gamma_t}(\cdot)$, $f_{\delta_t}(\cdot)$ and $f_{\beta_t}(\cdot)$, for the case $t \in (0, 1)$. The support of Z_1 is $(0, \boldsymbol{U}_0)$, where $\boldsymbol{U}_0 = 1$, and that of $Z_n, n = 2, 3, \ldots$, is $(0, \boldsymbol{U}_1)$, where $\boldsymbol{U}_1 = \infty$. Therefore, this example deals with the case $0 < t\boldsymbol{U}_0 < \boldsymbol{U}_1$, and differs from the three examples in Sect. 5, pp. 195–200 in [34].

Step 1. Derive pdf $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x\in(0,t)}$ for process $\{X(s)\}_{s\geq 0}$.
Since $t \in (0, 1)$, equating up- and downcrossing rates of level $x \in (0, t)$ gives

$$\pi_0^{(t)}(1-x) + \int_0^x e^{-\mu(x-y)} f^{(t)}(y)dy = \pi_0^{(t)}, x \in (0, t)\,.$$

Taking d/dx results in

$$-\pi_0^{(t)} - \mu\left(\pi_0^{(t)} - \pi_0^{(t)}(1-x)\right) + f^{(t)}(x) = 0,$$
$$f^{(t)}(x) = \pi_0^{(t)}(\mu x + 1), x \in (0, t)\,. \tag{10.75}$$

The law of total probability (normalizing condition) (10.54) gives

$$\pi_0^{(t)} = \frac{2}{\mu t^2 + 2t + 2}. \tag{10.76}$$

Step 2. Derive $f_{\gamma_t}(z)$, $z > 0$, **using** (10.75) **and** (10.76)
Equating *two different expressions* for the upcrossing rate of level $t + z$ gives

$$\begin{aligned} \pi_0^{(t)} \overline{F}_{\gamma_t}(z) &= \pi_0^{(t)} \overline{B}_0(t+z) + \textstyle\int_0^t \overline{B}_1(t+z-y)\pi_0^{(t)}(\mu y + 1)\,dy \\ &= \pi_0^{(t)}(1-t-z) + \textstyle\int_0^t e^{-\mu(t+z-y)}\pi_0^{(t)}(\mu y+1)\,dy,\ z \in (0, 1-t), \end{aligned}$$

and

$$\begin{aligned} \pi_0^{(t)} \overline{F}_{\gamma_t}(z) &= \textstyle\int_0^t \overline{B}_1(t+z-y)\pi_0^{(t)}(\mu y+1)\,dy \\ &= \textstyle\int_0^t e^{-\mu(t+z-y)}\pi_0^{(t)}(\mu y+1)\,dy,\ z \in (1-t, \infty). \end{aligned}$$

Thus

$$\overline{F}_{\gamma_t}(z) = 1 - t - z + \int_0^t e^{-\mu(t+z-y)}(\mu y+1)\,dy,\ z \in (0, 1-t),$$
$$\overline{F}_{\gamma_t}(z) = \int_0^t e^{-\mu(t+z-y)}(\mu y+1)\,dy,\ z \in (1-t, \infty);$$

taking d/dz in both equations gives

$$f_{\gamma_t}(z) = 1 + \mu \int_0^t e^{-\mu(t+z-y)}(\mu y+1)\,dy,\ z \in (0, 1-t),$$
$$f_{\gamma_t}(z) = \mu \int_0^t e^{-\mu(t+z-y)}(\mu y+1)\,dy,\ z \in (1-t, \infty),$$

implying, respectively,

$$f_{\gamma_t}(z) = 1 + \mu t e^{-\mu z},\ z \in (0, 1-t), \tag{10.77}$$
$$f_{\gamma_t}(z) = \mu t e^{-\mu z},\ z \in (1-t, \infty), \tag{10.78}$$

which satisfies the normalizing condition

$$\int_0^{t-1} f_{\gamma_t}(z)dz + \int_{t-1}^{\infty} f_{\gamma_t}(z)dz = 1.$$

The pdf $f_{\gamma_t}(z)$ has a discontinuity at $z = 1 - t$ of size

$$f_{\gamma_t}((1-t)^+) - f_{\gamma_t}((1-t)^-) = -1 = -b_0\left(\boldsymbol{U}_0^-\right).$$

(See Remark 10.4 in Sect. 10.3.4 for a similar discontinuity in the ordinary renewal process.)

Step 3. Derive $\left\{\pi_{\delta_t}, f_{\delta_t}(x)\right\}_{x\in(0,t)}, t \in (0,1)$ **using** (10.75) **and** (10.76)
Substituting $\left\{\pi_0^{(t)}, f^{(t)}(x)\right\}_{x\in(0,t)}$ from Step 1 above into (10.68) gives

$$\pi_{\delta_t} = \overline{B}_0(t) = 1 - t, \tag{10.79}$$

$$f_{\delta_t}(x) = \overline{B}_1(x)\frac{f^{(t)}(t-x)}{\pi^{(t)}} = e^{-\mu x}\left(\mu\left(t-x\right)+1\right), x \in (0,t). \tag{10.80}$$

The normalizing condition $\pi_{\delta_t} + \int_0^t f_{\delta_t}(x)dx = 1$ is readily checked.

Step 4. Derive $f_{\beta_t}(x)$, $x > 0$, $t \in (0,1)$ **using** (10.75) **and** (10.76)
Using similar reasoning as for (10.70) in Sect. 10.3.7, we obtain

$$\begin{aligned}
f_{\beta_t}(x) &= b_1(x)\int_{y=0}^{x}\frac{1}{\overline{B}(y)}\overline{B}(y)\frac{f^{(t)}t-y)}{\pi^{(t)}}dy \\
&= \mu e^{-\mu x}\int_{y=0}^{x}\left(\mu\left(t-y\right)+1\right)dy \\
&= \mu e^{-\mu x}\left[\left(\mu t+1\right)x - \mu\frac{x^2}{2}\right], x \in (0,t).
\end{aligned} \tag{10.81}$$

Since $Z_1 \underset{dis}{=} U_{(0,1)}$ has support in (0, 1) (and substituting $\overline{B}_0(t) = \pi_{\delta_t}$ from (10.79)), we get

$$\begin{aligned}
f_{\beta_t}(x) &= \frac{b_0(x)}{\overline{B}_0(t)}\pi_{\delta_t} + b_1(x)\int_{y=0}^{t}\frac{1}{\overline{B}(y)}\overline{B}(y)\frac{f^{(t)}t-y)}{\pi^{(t)}}dy \\
&= b_0(x) + b_1(x)\int_{y=0}^{t}\frac{1}{\overline{B}(y)}\overline{B}(y)\frac{f^{(t)}t-y)}{\pi^{(t)}}dy, x \in (t,1),
\end{aligned}$$

which differs from formula (10.71) for the ordinary renewal process. Thus

$$\begin{aligned}
f_{\beta_t}(x) &= b_0(x) + e^{-\mu x}\left[\left(\mu t+1\right)t - \mu\frac{t^2}{2}\right] \\
&= 1 + e^{-\mu x}\left[\left(\mu t+1\right)t - \mu\frac{t^2}{2}\right], x \in (t,1).
\end{aligned} \tag{10.82}$$

Subtracting (10.81) at t^- from (10.82) at t^+ gives $f_{\beta_t}(t^+) - f_{\beta_t}(t^+) = 1 = b_0(\boldsymbol{U}_0^-)$. The magnitude of this jump discontinuity is similar to that in formula (10.72) for the ordinary renewal process, but in this particular delayed renewal process, only the support of Z_1 contributes to the size of the discontinuity.

Since jumps starting from level 0 cannot upcross level $x = 1$, we obtain

$$
\begin{aligned}
f_{\beta_t}(x) &= b_1(x) \int_{y=0}^{t} \frac{1}{\overline{B}(y)} \overline{B}(y) \frac{f^{(t)}t-y)}{\pi^{(t)}} dy \\
&= e^{-\mu x} \left[(\mu t + 1)\, t - \mu \frac{t^2}{2} \right] \\
&= e^{-\mu x} \left[(\mu t + 1)\, t - \mu \frac{t^2}{2} \right], x \in (1, \infty). \qquad (10.83)
\end{aligned}
$$

The formula for $f_{\beta_t}(x)$, $x > 0$, satisfies the normalizing condition, since

$$
\int_0^t f_{\beta_t}(x)dx = t - \frac{1}{2} e^{-\mu t} \mu t^2 - e^{-\mu t} t,
$$

$$
\int_t^1 f_{\beta_t}(x)dx + \int_1^{\infty} f_{\beta_t}(x)dx = 1 - t + \frac{1}{2} e^{-\mu t} \mu t^2 + e^{-\mu t} t,
$$

implying $\int_0^{\infty} f_{\beta_t}(x)dx = 1$.