A Multi-linguistic-Valued Modal Logic

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Abstract. This paper develops a multi-valued modal logic, in which a logic formula takes a value of truth in linguistic terms. In other words, *truth* in our logic is regarded as a linguistic variable and its values are linguistic terms, which can be modelled as fuzzy sets. In particular, we define negation and implication on linguistic truth values such that their truth tables accord with those in conventional three-valued logic. Moreover, we also prove the soundness and completeness of our logic.

Keywords: Fuzzy logic · Linguistic variable · Multi-valued logic · Modal logic

1 Introduction

In our daily life, when talking about the age of somebody, we do not always give a statement with a precise number. Instead, often we simply say 'pretty young', which, unlike a precise number (*e.g.*, 23), is vague in its meaning. The prevalent phenomenon that we use words with fuzzy meaning imposes a challenge on characterising human reasoning. To take this challenge, Zadeh [17] introduces the concept of linguistic variable, which takes values in the forms of words or sentences in a natural language. For example, *age* could be a linguistic variable, taking values of *young, very young, not young, quite young, very old*, *not very old*, and so on, rather than 20, 21, 22, and so on. A linguistic value of *age* like *young* can be modelled as a fuzzy set with the numerical age as domain.

The reason why *truth* can be seen as linguistic variable is from the observation on our daily discourse that we frequently use expressions such as *very true, quite true, essentially true* to characterise a degree to which a statement is true. The phrases like *very true, quite true*, and *essentially true* are the linguistic values of *truth*. Thus, with *truth* as linguistic variable, it is required to define a fuzzy linguistic logic in which the truth value of every logic formula is a linguistic value. In fact, there are some efforts in this direction. However, few work is about fuzzy multi-valued modal logic (see Sect. 5 for detailed discussion).

So, in this paper, we will propose a modal logic with five linguistic truth values. In particular, we define the implication on linguistic truth values in a way that its truth table is consistent with that in conventional three-valued logic. Moreover, we prove the soundness and completeness of our logic system.

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This paper has the following structure: Sects. 2 and 3 present the syntax and semantics of our logic. Section 4 proves its soundness and completeness. Section 5 discusses the related work. Finally, Sect. 6 concludes our work.

2 Syntax

This section presents the syntax of our fuzzy multi-valued logic (denoted as F_5).

Definition 1. *The well-formed formulas of F_5 are given by:*

$$\phi ::= p \mid \neg\phi \mid \varphi \rightarrow \phi \mid \Diamond\phi, \tag{1}$$

where p ranges over elements of the proposition letter set Φ.

The possible interpretations of $\Diamond$ are *'is possible that'*, *'is permitted that'*, *'believe that'* and so on [3]. In our logic, we also need the following abbreviations:

$$\overline{\phi} := (\phi \rightarrow \neg\phi), \tag{2}$$

$$\Box\phi := \neg\Diamond\neg\phi. \tag{3}$$

Definition 2. *Suppose that φ_1, φ_2 and φ_3 are formulas in F_5. The axioms of F_5 are as follows:*

A1: $\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1)$,
A2: $(\varphi_1 \rightarrow \varphi_2) \rightarrow ((\varphi_2 \rightarrow \varphi_3) \rightarrow (\varphi_1 \rightarrow \varphi_3))$,
A3: $(\overline{\varphi_1} \rightarrow \varphi_1) \rightarrow \varphi_1$,
A4: $(\neg\varphi_1 \rightarrow \neg\varphi_2) \rightarrow (\varphi_2 \rightarrow \varphi_1)$,
K: $\Box(\varphi_1 \rightarrow \varphi_2) \rightarrow (\Box\varphi_1 \rightarrow \Box\varphi_2)$,
D: $\Diamond\varphi_1 \leftrightarrow \neg\Box\neg\varphi_1$.

Among the above axioms, A1 to A4 are those use in [4], which with the rules of proof below ensures that we can make use of results in [4].

1. Modus ponens: if ϕ and $\phi \rightarrow \varphi$, then φ.
2. Uniform substitution: if ϕ, then φ, if φ is obtained from ϕ by uniformly replacing proposition letters in ϕ by arbitrary formulas.
3. generalisation: if ϕ, then $\Box\phi$.
4. Diamond generalisation: if $\Diamond\phi$, then $\Box\phi$.

3 Semantics

In this section, we present the semantics of our logic.

Definition 3 (Truth as a linguistic variable). *Truth is a linguistic variable characterized by a triple $(truth, T(truth), [0,1])$, where*

$$T(truth) = \{very\text{-}true, fairly\text{-}true, fairly\text{-}false, very\text{-}false, undecided\}$$

and $[0,1]$ is a numerical interval of truth values.

Table 1. The truth table of $\phi \to \varphi$ and $\neg\varphi$ on linguistic truth values

ϕ \ φ	F_1	F	N	T	T_1
F_1	T_1	T_1	T_1	T_1	T_1
F	T	T	T	T	T_1
N	N	N	T	T	T_1
T	F	F	N	T	T_1
T_1	F_1	F	N	T	T_1

φ	$\neg\varphi$
F_1	T_1
F	T
N	N
T	F
T_1	F_1

For convenience, we use some abbreviation such that T_1 stands for *very true*, T for *fairly true*, F for *fairly false*, F_1 for *very false*, and N for *undecided*. Thus, $T(truth) = \{T_1, T, F, F_1, N\}$. It is noted that it accords with our daily use that the values of *truth* are fuzzy sets with numerical truth values as domain.

Definition 4 (Membership functions of linguistic truth values).

$$\mu_{T_1}(x) = x^6,\ \ \mu_T(x) = \sqrt{x},\ \ \mu_F(x) = \sqrt{1-x},\ \ \mu_{F_1}(x) = (1-x)^6,$$
$$\mu_N(x) = \mathrm{e}^{-26(x-0.5)^4}.$$

What we are going to do next is to use the extension principle (denoted as operator $\otimes$) [16] and the linguistic approximation method (denoted as operator $\odot$) [14] in fuzzy set theory to define negation and implication operating on linguistic truth values based on the operation on intervals.

Definition 5 (Negation on linguistic truth values). *Suppose $\tau \in T(truth)$, the negation of τ, denoted as $\neg\tau$, is a fuzzy set defined as*

$$\mu_{\neg\tau}(x) = \odot(\otimes(\tau, 1-x)). \tag{4}$$

Next, we extend Lucasiewicz implication to linguistic truth values.

Definition 6 (Implication on linguistic truth values). *Suppose $\tau_1, \tau_2 \in T(truth)$, the value of τ_1 implying τ_2 is a fuzzy set defined as*

$$\mu_{\tau_1 \to \tau_2}(x) = \odot(\otimes(\tau_1(x_1), \tau_2(x_2), \min\{1, 1 - x_1 + x_2\})). \tag{5}$$

By (4) and (5), we can calculate the negation and implication on linguistic truth values through MATLAB and then the result is showed in Table 1.

With the definition of truth as linguistic variable and operation on it, we can now introduce the concept of model satisfaction in our logic F_5 as follows:

Definition 7 (Model satisfaction). *Suppose w is a state in a model $\mathcal{M} = (W, R, V)$. Then we recursively define the notion of a formula being satisfied in $\mathcal{M}$ at a possible world w as follows:*

1. $\mathcal{M}, w \models p$ iff $V(p, w) \in \{T_1, T\}$, where $p \in \Phi$.

2. $\mathcal{M}, w \models \neg\phi$ *iff* $\overline{V}(\phi, w) \in \{F_1, F\}$.
3. $\mathcal{M}, w \models \phi \to \varphi$ *iff* $\overline{V}(\phi, w) \in \{F_1, F\}$ *or* $\overline{V}(\varphi, w) \in \{T_1, T\}$ *or* $\overline{V}(\phi, w) = \overline{V}(\varphi, w) = N$.
4. $\mathcal{M}, w \models \Diamond\phi$ *iff* $\exists v(wRv \wedge \mathcal{M}, v \models \phi)$.
5. $\mathcal{M}, w \models \Box\phi$ *iff* $\forall v(wRv \to \mathcal{M}, v \models \phi)$.

Table 1 shows the result of implication and negation on linguistic truth values, which corresponds to our definition of truth assignment for formulas and model satisfaction.

4 Soundness and Completeness

In this section, we present the soundness and completeness of our logic F_5.

Intuitively, the soundness of a logic system means that for a formula of a logic, if it is correct in the sense of syntax, then it is correct in the sense of semantics. The completeness of a logic system means that for a logic formula, if it is correct in the sense of semantics, then it is correct in the sense of syntax[1].

Theorem 1 (Soundness of F_5). *F_5 is sound with respect to the class of frames with the property that $\forall y(Rxy \to \forall z(Rxz \to y = z))$,* i.e., *$\vdash \phi$ implies $(W, R, V) \models \phi$, where (W, R) is a frame with the required property and V is arbitrary.*

Definition 8 (Canonical model). *The canonical model $\mathcal{M}^*_{F_5}$ of F_5 is the triple (W^*, R^*, V^*), where:*

1. *W^* is the set of all maximally consistent sets of F_5;*
2. *R^* is the binary relation on W^* such that wR^*u if for all formulas ψ, $u \vdash \psi$ implies $w \vdash \Diamond\psi$ or $w \nvdash \neg\Diamond\psi$; and*
3. *the valuation V is defined as follow:*
   (a) $V^*(p, w) = T$ *if* $w \vdash p$,
   (b) $V^*(p, w) = F$ *if* $w \vdash \neg p$,
   (c) $V^*(p, w) = N$ *if* $w \nvdash p$ *and* $w \nvdash \neg p$.

The canonical model we propose here is different from the classic one in two aspects: the definition of canonical relation and valuation. The change in canonical relation can help us prove the completeness of our logic. The classic canonical valuation is defined as follows: $V^-(p, w) = T$ if $p \in w$, and $V^-(p, w) = F$ if $\neg p \in w$. Since we have three categories of truth values in our logic, the classic canonical valuation does not work here.

Theorem 2 (Completeness). *Suppose that ϕ is a formula in F_5 and Γ is a set of formulas in F_5. If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

[1] For the sake of page limit, we cannot present the details of our theorems' proofs in this paper, but we will present them in the extended version of this paper.

5 Related Work

Recently, there have been many studies about modal logic. Kontinen *et al.* [9] introduce a logic called modal independence logic that can explicitly talk about independence among propositional variables. Formulas of their logic are evaluated in sets of worlds, rather than a single world. Herzig and Lorini [6] present a logic, which can reason about the relationship between an agent's belief and the information that the agent obtains. Bozzelli *et al.* [1] present a multi-agent refinement modal logic, which contains an operator $\forall$, and standard box modalities for each agent. A refinement is like a bisimulation where only the 'atoms' and 'back' requirements need to be satisfied. Operator $\forall$ is a quantifier over the set of all refinements of a given model. Sack and Hoek [13] present a modal logic for games that allow mixed strategies and demonstrate its soundness and strong completeness. However, our logic can reflect the fuzziness of modal formulas, but all of them above cannot. In addition, Jung *et al.* [8] set up a Kripke semantics for a modal expansion of bilattice logic, which is based on a four-valued logic. They prove the soundness and completeness of their logic with respect to four-valued Kripke frames. In our work, we set up Kripke frames of five truth values, which can be divided into three categories. Besides, we also prove the soundness and completeness of our logic.

There are also some studies on fuzzy modal logic, because this kind of logic is a powerful tool for dealing with fuzzy information and has been applied in many areas. Hájek [5] studies a fuzzy variant of classical modal logic *S5*. He proposes a recursively axiomatised logic. He also proposes three kinds of Kripke models and the corresponding systems. His work mainly focuses on the fuzzy extension of the classical modal logic, while our work focuses on the multi-valued extension of modal logic, the truth values of which are fuzzy linguistic terms. Rodrigues and Godo [12] define modal uncertainty logics with fuzzy neighborhood semantics. Different from them, the semantics we use is Kripke semantics. Pan and Xu [11] deal with a propositional fuzzy modal logic with evaluated syntax based on MV-algebras and they show its application to fuzzy decision implications. However, the syntax of our logic is different from theirs. Cintula *et al.* [2] explore a more general semantics of fuzzy modal logics, namely a fuzzified version of the classical neighborhood semantics. While they use neighborhood semantics, we use the relational semantics. Vidal *et al.* [15] study the modal extension of product fuzzy logic with both algebraic semantics and relational semantics based on Kripke structures with crisp accessibility relations. They prove the completeness for both kinds of semantics. While they study the modal extension of product fuzzy logic, we study fuzzy extension of the classic three-value logic. Jing *et al.* [7] propose a fuzzy modal logic on the basis of the work of Luo *et al.* [10]. However, our logic is different from theirs. Ours is a five-valued modal logic, but theirs is a nine-valued one; and our modal operator does not have a fixed meaning, while theirs mean 'know' and 'believe'.

6 Conclusion

Linguistic variable is a powerful tool to characterize human reasoning. In this paper, to reflect the fact that in real life people regard *truth* as a linguistic variable with its values being fuzzy sets, we define the implication on linguistic truth values, and thus develop a linguistic multi-valued modal logic. Moreover, after presenting the syntax and semantics of our logic, we prove its soundness and completeness. Since modal logic has been used as the base for epistemic logic, deontic logic, temporal logic and so on [3], we believe that the idea behind our logic can be applied to these fields and produces more interesting results in the future.

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References

1. Bozzelli, L., van Ditmarsch, H., French, T., Hales, J., Pinchinat, S.: Refinement modal logic. Inf. Comput. **239**, 303–339 (2014)
2. Cintula, P., Noguera, C., Rogger, J.: From Kripke to neighborhood semantics for modal fuzzy logics. In: Carvalho, J.P., Lesot, M.-J., Kaymak, U., Vieira, S., Bouchon-Meunier, B., Yager, R.R. (eds.) IPMU 2016. Communications in Computer and Information Science, vol. 611, pp. 95–107. Springer, Switzerland (2016). doi:10.1007/978-3-319-40581-0_9
3. Goble, L.: The Blackwell Guide to Philosophical Logic. Blackwell, Malden (2001)
4. Goldberg, H., Leblanc, H., Weaver, G.: A strong completeness theorem for 3-valued logic. Notre Dame J. Form. Logic **15**(2), 325–330 (1974)
5. Hájek, P.: On fuzzy modal logics S5($\mathscr{C}$). Fuzzy Sets Syst. **161**(18), 2389–2396 (2010)
6. Herzig, A., Lorini, E.: A modal logic of perceptual belief. In: Lihoreau, F., Rebuschi, M. (eds.) Epistemology, Context, and Formalism. Synthese Library, vol. 369, pp. 197–211. Springer, Switzerland (2014). doi:10.1007/978-3-319-02943-6_12
7. Jing, X., Luo, X., Zhang, Y.: A fuzzy dynamic belief logic system. Int. J. Intell. Syst. **29**(7), 687–711 (2014)
8. Jung, A., Rivieccio, U.: Kripke semantics for modal bilattice logic. In: 28th Annual IEEE/ACM Symposium on Logic in Computer Science, pp. 438–447 (2013)
9. Kontinen, J., Müller, J.-S., Schnoor, H., Vollmer, H.: Modal independence logic. arXiv preprint arXiv:1404.0144 (2014)
10. Luo, X., Zhang, C., Jennings, N.R.: A hybrid model for sharing information between fuzzy, uncertain and default reasoning models in multi-agent systems. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **10**(04), 401–450 (2002)
11. Pan, X., Xu, Y.: Semantics of propositional fuzzy modal logic with evaluated syntax and its application to fuzzy decision implications. Int. J. Comput. Intell. Syst. **8**(sup1), 85–93 (2015)
12. Rodriguez, R.O., Godo, L.: Modal uncertainty logics with fuzzy neighborhood semantics. In: IJCAI-13 Workshop on Weighted Logics for Artificial Intelligence, pp. 79–86 (2013)

13. Sack, J., van der Hoek, W.: A modal logic for mixed strategies. Studia Logica **102**(2), 339–360 (2014)
14. Schmucke, K.J.: Fuzzy Sets: Natural Language Computations, and Risk Analysis. Computer Science Press, Incorporated (1984)
15. Vidal, A., Esteva, F., Godo, L.: On modal extensions of product fuzzy logic. J. Log. Comput. (2015). doi:10.1093/logcom/exv046
16. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
17. Zadeh, L.A.: The concept of a linguistic variable, its application to approximate reasoning. Inf. Sci. Part I **8**(4), 199–249 (1975). Part II **8**(4), 301–357 (1975). Part III **9**(4). 43–80 (1975)