# Re-embedding a 1-Plane Graph into a Straight-Line Drawing in Linear Time

Seok-Hee Hong[1(✉)] and Hiroshi Nagamochi[2]

[1] University of Sydney, Sydney, Australia
seokhee.hong@sydney.edu.au
[2] Kyoto University, Kyoto, Japan
nag@amp.i.kyoto-u.ac.jp

**Abstract.** Thomassen characterized some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph is drawable in straight-lines if and only if it does not contain the configuration [C. Thomassen, Rectilinear drawings of graphs, J. Graph Theory, 10(3), 335–341, 1988].

In this paper, we characterize some 1-plane embedding as the forbidden configuration such that a given 1-plane embedding of a graph can be re-embedded into a straight-line drawable 1-plane embedding of the same graph if and only if it does not contain the configuration. Re-embedding of a 1-plane embedding preserves the same set of pairs of crossing edges. We give a linear-time algorithm for finding a straight-line drawable 1-plane re-embedding or the forbidden configuration.

## 1 Introduction

Since the 1930s, a number of researchers have investigated *planar* graphs. In particular, a beautiful and classical result, known as *Fáry's Theorem*, asserts that every plane graph admits a *straight-line drawing* [5]. Indeed, a straight-line drawing is the most popular drawing convention in Graph Drawing.

More recently, researchers have investigated *1-planar graphs* (i.e., graphs that can be embedded in the plane with at most one crossing per edge), introduced by Ringel [13]. Subsequently, the structure of 1-planar graphs has been investigated [4,12]. In particular, Pach and Toth [12] proved that a 1-planar graph with $n$ vertices has at most $4n-8$ edges, which is a tight upper bound. Unfortunately, testing the 1-planarity of a graph is NP-complete [6,11], however linear-time algorithms are available for special subclasses of 1-planar graphs [1,3,7].

Thomassen [14] proved that every 1-plane graph (i.e., a 1-planar graph embedded with a given *1-plane embedding*) admits a straight-line drawing if and only if it does not contain any of two special 1-plane graphs, called the *B-configuration* or *W-configuration*, see Fig. 1.

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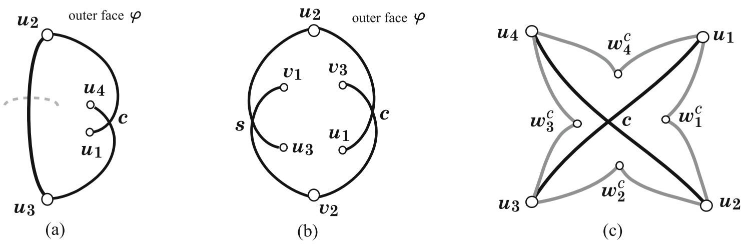


**Fig. 1.** (a) B-configuration with three edges $u_1u_2, u_2u_3$ and $u_3u_4$ and one crossing $c$ made by an edge pair $\{u_1u_2, u_3u_4\}$, where edge $u_2u_3$ may have a crossing when the configuration is part of a 1-plane embedding; (b) W-configuration with four edges $u_1u_2$, $u_2u_3$, $v_1v_2$ and $v_2v_3$ and two crossings $c$ and $s$ made by edge pairs $\{u_1u_2, v_2v_3\}$ and $\{u_2u_3, v_1v_2\}$, where possibly $u_1 = v_1$ and $u_3 = v_3$; (c) Augmenting a crossing $c \in \chi$ made by edges $u_1u_3$ and $u_2u_4$ with a new cycle $Q_c = (u_1, w_1^c, u_2, w_2^c, u_3, w_3^c, u_4, w_4^c)$ depicted by gray lines.

Recently, Hong et al. [8] gave an alternative constructive proof, with a linear-time testing algorithm and a drawing algorithm. They also showed that some 1-planar graphs need an exponential area with straight-line drawing.

We call a 1-plane embedding *straight-line drawable* (*SLD* for short) if it admits a straight-line drawing, i.e., it does not contain a B- or W-configuration by Thomassen [14]. In this paper, we investigate a problem of "re-embedding" a given non-SLD 1-plane embedding $\gamma$ into an SLD 1-plane embedding $\gamma'$. For a given 1-plane embedding $\gamma$ of a graph $G$, we call another 1-plane embedding $\gamma'$ of $G$ a *cross-preserving embedding* of $\gamma$ if exactly the same set of edge pairs make the same crossings in $\gamma'$.

More specifically, we first characterize the *forbidden configuration* of 1-plane embeddings that cannot admit an SLD cross-preserving 1-plane embedding. Based on the characterization, we present a linear-time algorithm that either detects the forbidden configuration in $\gamma$ or computes an SLD cross-preserving 1-plane embedding $\gamma'$.

Formally, the main problem considered in this paper is defined as follows.

**Re-embedding a 1-Plane Graph into a Straight-line Drawing**
**Input:** A 1-planar graph $G$ and a 1-plane embedding $\gamma$ of $G$.
**Output:** Test whether $\gamma$ admits an SLD cross-preserving 1-plane embedding $\gamma'$, and construct such an embedding $\gamma'$ if one exists, or report the forbidden configuration.

To design a linear-time implementation of our algorithm in this paper, we introduce a *rooted-forest representation of non-intersecting cycles* and an efficient procedure of flipping subgraphs in a plane graph. Since these data structure and procedure can be easily implemented, it has advantage over the complicated decomposition of biconnected graphs into triconnected components [10] or the SPQR tree [2].

## 2 Plane Embeddings and Inclusion Forests

Let $U$ be a set of $n$ elements, and let $\mathcal{S}$ be a family of subsets $S \subseteq U$. We say that two subsets $S, S' \subseteq U$ are *intersecting* if none of $S \cap S'$, $S - S'$ and $S' - S$ is empty. We call $\mathcal{S}$ a *laminar* if no two subsets in $\mathcal{S}$ are intersecting. For a laminar $\mathcal{S}$, the *inclusion-forest* of $\mathcal{S}$ is defined to be a forest $\mathcal{I} = (\mathcal{S}, \mathcal{E})$ of a disjoint union of rooted trees such that (i) the sets in $\mathcal{S}$ are regarded as the vertices of $\mathcal{I}$, and (ii) a set $S$ is an ancestor of a set $S'$ in $\mathcal{I}$ if and only if $S' \subseteq S$.

**Lemma 1.** *For a cyclic sequence $(u_1, u_2, \ldots, u_\delta)$ of $\delta \geq 2$ elements, define an interval $(i, j)$ to be the set of elements $u_k$ with $i \leq k \leq j$ if $i \leq j$ and $(i, j) = (i, \delta) \cup (1, j)$ if $i > j$. Let $\mathcal{S}$ be a set of intervals. A pair of two intersecting intervals in $\mathcal{S}$ (when $\mathcal{S}$ is not a laminar) or the inclusion-forest of $\mathcal{S}$ (when $\mathcal{S}$ is a laminar) can be obtained in $O(\delta + |\mathcal{S}|)$ time.*

Throughout the paper, a graph $G = (V, E)$ stands for a simple undirected graph. The set of vertices and the set of edges of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v$, let $E(v)$ be the set of edges incident to $v$, $N(v)$ be the set of neighbors of $v$, and $\deg(v)$ denote the degree $|N(v)|$ of $v$. A simple path with end vertices $u$ and $v$ is called a *$u, v$-path*. For a subset $X \subseteq V$, let $G - X$ denote the graph obtained from $G$ by removing the vertices in $X$ together with the edges in $\cup_{v \in X} E(v)$.

A *drawing* $D$ of a graph $G$ is a geometric representation of the graph in the plane, such that each vertex of $G$ is mapped to a point in the plane, and each edge of $G$ is drawn as a curve. A drawing $D$ of a graph $G = (V, E)$ is called *planar* if there is no edge crossing. A planar drawing $D$ of a graph $G$ divides the plane into several connected regions, called *faces*, where a face enclosed by a closed walk of the graph is called an *inner face* and the face not enclosed by any closed walk is called the *outer face*.

A planar drawing $D$ *induces* a plane embedding $\gamma$ of $G$, which is defined to be a pair $(\rho, \varphi)$ of the *rotation system* (i.e., the circular ordering of edges for each vertex) $\rho$, and the outer face $\varphi$ whose facial cycle $C_\varphi$ gives the outer boundary of $D$. Let $\gamma = (\rho, \varphi)$ be a plane embedding of a graph $G = (V, E)$. We denote by $F(\gamma)$ the set of faces in $\gamma$, and by $C_f$ the facial cycle determined by a face $f \in F$, where we call a subpath of $C_f$ a *boundary path* of $f$. For a simple cycle $C$ of $G$, the plane is divided by $C$ in two regions, one containing only inner faces and the other containing the outer area, where we say that the former is *enclosed* by $C$ or the *interior* of $C$, while the latter is called the *exterior* of $C$. We denote by $F_{\text{in}}(C)$ the set of inner faces in the interior of $C$, by $E_{\text{in}}(C)$ the set of edges in $E(C_f)$ with $f \in F_{\text{in}}(C)$, and by $V_{\text{in}}(C)$ the set of end-vertices of edges in $E_{\text{in}}(C)$. Analogously define $F_{\text{ex}}(C)$, $E_{\text{ex}}(C)$ and $V_{\text{ex}}(C)$ in the exterior of $C$. Note that $E(C) = E_{\text{in}}(C) \cap E_{\text{ex}}(C)$ and $V(C) = V_{\text{in}}(C) \cap V_{\text{ex}}(C)$.

For a subgraph $H$ of $G$, we define the embedding $\gamma|_H$ of $\gamma$ induced by $H$ to be a sub-embedding of $\gamma$ obtained by removing the vertices/edges not in $H$, keeping the same rotation system around each of the remaining vertices/crossings and the same outer face.

### 2.1 Inclusion Forests of Inclusive Set of Cycles

In this and next subsections, let $(G, \gamma)$ stand for a plane embedding of $\gamma = (\rho, \varphi)$ of a biconnected simple graph $G = (V, E)$ with $n = |V| \geq 3$.

Let $C$ be a simple cycle in $G$. We define the *direction* of $C$ to be an ordered pair $(u, v)$ with $uv \in E(C)$ such that the inner faces in $F_{\rm in}(C)$ appear on the right hand side when we traverse $C$ in the order that we start $u$ and next visit $v$.

For simplicity, we say that two simple cycles $C$ and $C'$ are *intersecting* if $F_{\rm in}(C)$ and $F_{\rm in}(C')$ are intersecting.

Let $\mathcal{C}$ be a set of simple cycles in $G$. We call $\mathcal{C}$ *inclusive* if no two cycles in $\mathcal{C}$ are intersecting, i.e., $\{F_{\rm in}(C) \mid C \in \mathcal{C}\}$ is a laminar. When $\mathcal{C}$ is inclusive, the *inclusion-forest* of $\mathcal{C}$ is defined to be a forest $\mathcal{I} = (\mathcal{C}, \mathcal{E})$ of a disjoint union of rooted trees such that:

(i) the cycles in $\mathcal{C}$ are regarded as the vertices of $\mathcal{I}$, and
(ii) a cycle $C$ is an ancestor of a cycle $C'$ in $\mathcal{I}$ if and only if $F_{\rm in}(C') \subseteq F_{\rm in}(C)$.

Let $\mathcal{I}(\mathcal{C})$ denote the inclusion-forest of $\mathcal{C}$. For a vertex subset $X \subseteq V$, let $\mathcal{C}(X)$ denote the set of cycles $C \in \mathcal{C}$ such that $x \in V(C)$ for some vertex $x \in X$, where we denote $\mathcal{C}(\{v\})$ by $\mathcal{C}(v)$ for short.

**Lemma 2.** *For $(G, \gamma)$, let $\mathcal{C}$ be a set of simple cycles of $G$. Then any of the following tasks can be executed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time.*

(i) *Decision of the directions of all cycles in $\mathcal{C}$;*
(ii) *Detection of a pair of two intersecting cycles in $\mathcal{C}$ when $\mathcal{C}$ is not inclusive, and construction of the inclusion-forests $\mathcal{I}(\mathcal{C}(v))$ for all vertices $v \in V$ when $\mathcal{C}$ is inclusive; and*
(iii) *Construction of the inclusion-forest $\mathcal{I}(\mathcal{C})$ when $\mathcal{C}$ is inclusive.*

### 2.2 Flipping Spindles

A simple cycle $C$ of $G$ is called a *spindle* (or a *$u, v$-spindle*) of $\gamma$ if there are two vertices $u, v \in V(C)$ such that no vertex in $V(C) - \{u, v\}$ is adjacent to any vertex in the exterior of $C$, where we call vertices $u$ and $v$ the *junctions* of $C$. Note that each of the two subpaths of $C$ between $u$ and $v$ is a boundary path of some face in $F(\gamma)$.

Given $(G, \gamma)$, we denote the rotation system around a vertex $v \in V$ by $\rho_\gamma(v)$. For a spindle $C$ in $\gamma$, let $J(C)$ denote the set of the two junctions of $C$.

*Flipping a $u, v$-spindle $C$* means to modify the rotation system of vertices in $V_{\rm in}(C)$ as follows:

(i) For each vertex $w \in V_{\rm in}(C) - J(C)$, reverse the cyclic order of $\rho_\gamma(w)$; and
(ii) For each vertex $u \in J(C)$, reverse the order of subsequence of $\rho_\gamma(u)$ that consists of vertices $N(u) \cap V_{\rm in}(C)$.

Every two distinct spindles $C$ and $C'$ in $\gamma$ are non-intersecting, and they always satisfy one of $E_{\rm in}(C) \cap E_{\rm in}(C') = \emptyset$, $E_{\rm in}(C) \subseteq E_{\rm in}(C')$, and $E_{\rm in}(C') \subseteq E_{\rm in}(C)$. Let $\mathcal{C}$ be a set of spindles in $\gamma$, which is always inclusive, and let $\mathcal{I}(\mathcal{C})$ denote the inclusion-forest of $\mathcal{C}$.

When we modify the current embedding $\gamma$ by flipping each spindle in $\mathcal{C}$, the resulting embedding $\gamma_{\mathcal{C}}$ is the same, independent from the ordering of the flipping operation to the spindles, since for two spindles $C$ and $C'$ which share a common junction vertex $u \in J(C) \cap J(C')$, the sets $N(u) \cap V_{\rm in}(C)$ and $N(u) \cap V_{\rm in}(C')$ do not intersect, i.e., they are disjoint or one is contained in the other.

Define the *depth* of a vertex $v \in V$ in $\mathcal{I}$ to be the number of spindles $C \in \mathcal{C}$ such that $v \in V_{\rm in}(C) - J(C)$, and denote by $\mathrm{p}(v)$ the parity of depth of vertex $v$, i.e., $\mathrm{p}(v) = 1$ if the depth is odd and $\mathrm{p}(v) = -1$ otherwise.

For a vertex $v \in V$, let $\mathcal{C}[v]$ denote the set of spindles $C \in \mathcal{C}$ such that $v \in J(C)$, and let $\gamma_{\mathcal{C}[v]}$ be the embedding obtained from $\gamma$ by flipping all spindles in $\mathcal{C}[v]$. Let $\mathrm{rev}\langle\sigma\rangle$ mean the reverse of a sequence $\sigma$. Then we see that $\rho_{\gamma_{\mathcal{C}}}(v) = \rho_{\gamma_{\mathcal{C}[v]}}(v)$ if $\mathrm{p}(v) = 1$; and $\rho_{\gamma_{\mathcal{C}}}(v) = \mathrm{rev}\langle\rho_{\gamma_{\mathcal{C}[v]}}(v)\rangle$ otherwise. To obtain the embedding $\gamma_{\mathcal{C}}$ from the current embedding $\gamma$ by flipping each spindle in $\mathcal{C}$, it suffices to show how to compute each of $\mathrm{p}(v)$ and $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.

**Lemma 3.** *Given $(G, \gamma)$, let $\mathcal{C}$ be a set of spindles of $\gamma$. Then any of the following tasks can be executed in $O(n + \sum_{C \in \mathcal{C}} |E(C)|)$ time.*

(i) *Decision of parity $\mathrm{p}(v)$ of all vertices $v \in V$; and*
(ii) *Computation of $\rho_{\gamma_{\mathcal{C}[v]}}(v)$ for all vertices $v \in V$.*

## 3 Re-embedding 1-Plane Graph and Forbidden Configuration

A drawing $D$ of a graph $G = (V, E)$ is called a *1-planar drawing* if each edge has at most one crossing. A 1-planar drawing $D$ of graph $G$ induces a *1-plane embedding* $\gamma$ of $G$, which is defined to be a tuple $(\chi, \rho, \varphi)$ of the *crossing system* $\chi$ of $E$, the rotation system $\rho$ of $V$, and the outer face $\varphi$ of $D$. The *planarization* $\mathcal{G}(G, \gamma)$ of a 1-plane embedding $\gamma$ of graph $G$ is the plane embedding obtained from $\gamma$ by regarding crossings also as graph vertices, called crossing-vertices. The set of vertices in $\mathcal{G}(G, \gamma)$ is given by $V \cup \chi$. For a notational convenience, we refer to a subgraph/face of $\mathcal{G}(G, \gamma)$ as a subgraph/face in $\gamma$.

Let $\gamma = (\chi, \rho, \varphi)$ be a 1-plane embedding of graph $G$. We call another 1-plane embedding $\gamma' = (\chi', \rho', \varphi')$ of graph $G$ a *cross-preserving* 1-plane embedding of $\gamma$ when the same set of edge pairs makes crossings, i.e., $\chi = \chi'$. In other words, the planarization $\mathcal{G}(G, \gamma')$ is another plane embedding of $\mathcal{G}(G, \gamma)$ such that the alternating order of edges incident to each crossing-vertex $c \in \chi$ is preserved.

To eliminate the additional constraint on the rotation system on each crossing-vertex $c \in \chi$, we introduce "circular instances." We call an instance $(G, \gamma)$ of 1-plane embedding *circular* when for each crossing $c \in \chi$, the four end-vertices of the two crossing edges $u_1u_3$ and $u_2u_4$ that create $c$ (where $u_1, u_2, u_3$ and $u_4$ appear in the clockwise order around $c$) are contained in a

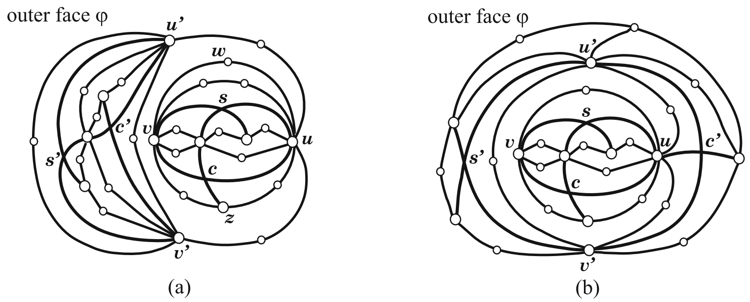


**Fig. 2.** Circular instances $(G, \gamma)$ with a cut-vertex $u$ of $\mathcal{G}$, where the crossing edges are depicted by slightly thicker lines: (a) hard B-cycles $C = (u, c, v, s)$ and $C' = (u', c', v', s')$, (b) hard B-cycle $C = (u, c, v, s)$ and a nega-cycle $C' = (u', c', v', s')$ whose reversal is a hard B-cycle, where vertices $u, v, u', v' \in V$ and crossings $c, s, c', s' \in \chi$.

cycle $Q_c = (u_1, w_1^c, u_2, w_2^c, u_3, w_3^c, u_4, w_4^c)$ of eight crossing-free edges for some vertices $w_i^c$, $i = 1, 2, 3, 4$ of degree 2, as shown in Fig. 1(c). By definition, $c$ and each $w_i^c$ not necessarily appear along the same facial cycle in the planarization $\mathcal{G}(G, \gamma)$. For example, path $(v, w, u)$ is part of such a cycle $Q_s$ for the crossing $s$ in the circular instance in Fig. 2(a), but $c$ and $w$ are not on the same facial cycle in the planarization.

A given instance can be easily converted into a circular instance by augmenting the end-vertices of each pair of crossing edges as follows. In the plane graph, $\mathcal{G}(G, \gamma)$, for each crossing-vertex $c \in \chi$ and its neighbors $u_1, u_2, u_3$ and $u_4$ that appear in the clockwise order around $c$, we add a new vertex $w_i^c$, $i = 1, 2, 3, 4$ and eight new edges $u_i w_i^c$ and $w_i^c u_{i+1}$, $i = 1, 2, 3, 4$ (where $u_5$ means $u_1$) to form a cycle $Q_c$ of length 8 whose interior contains no other vertex than $c$.

Let $H$ be the resulting graph augmented from $G$, and let $\Gamma$ be the resulting 1-plane embedding of $H$ augmented from $\gamma$. Note that $|V(H)| \leq |V(G)| + 4|\chi|$ holds. We easily see that if $\gamma$ admits an SLD cross-preserving embedding $\gamma'$ then $\Gamma$ admits an SLD cross-preserving embedding $\Gamma'$. This is because a straight-line drawing $D_{\gamma'}$ of $\gamma'$ can be changed into a straight-line drawing $D_{\Gamma'}$ of some cross-preserving embedding $\Gamma'$ of $\Gamma$ by placing the newly introduced vertices $w_i^c$ within the region sufficiently close to the position of $c$. We here see that cycle $Q_c$ can be drawn by straight-line segments without intersecting with other straight-line segments in $D_{\gamma'}$.

Note that the instance $(G, \gamma')$ remains circular for any cross-preserving embedding $\gamma'$ of $\gamma$. In the rest of paper, let $(G, \gamma)$ stand for a circular instance $(G = (V, E), \gamma = (\chi, \rho, \varphi))$ with $n \geq 3$ vertices and let $\mathcal{G}$ denote its planarization $\mathcal{G}(G, \gamma)$. Figure 2 shows examples of circular instances $(G, \gamma)$, where the vertex-connectivity of $\mathcal{G}$ is 1.

As an important property of a circular instance, the subgraph $G_{(0)}$ with crossing-free edges is a spanning subgraph of $G$ and the four end-vertices of any two crossing edges are contained in the same block of the graph $G_{(0)}$. The biconnectivity is necessary to detect certain types of cycles by applying Lemma 2.

### 3.1 Candidate Cycles, B/W Cycle, Posi/Nega Cycle, Hard/Soft Cycle

For a circular instance $(G, \gamma)$, finding a cross-preserving embedding of $\gamma$ is effectively equivalent to finding another plane embedding of $\mathcal{G}$ so that all the current B- and W-configurations are eliminated and no new B- or W-configurations are introduced. To detect the cycles that can be the boundary of a B- or W-configuration in changing the plane embedding of $\mathcal{G}$, we categorize cycles containing crossing vertices in $\mathcal{G}$.

A *candidate posi-cycle* (resp., *candidate nega-cycle*) in $\mathcal{G}$ is defined to be a cycle $C = (u, c, v)$ or $C = (u, c, v, s)$ in $\mathcal{G}$ with $u, v \in V$ and $c, s \in \chi$ such that the interior (resp., exterior) of $C$ does not contain a crossing-free edge $uv \in E$ and any other crossing vertex $c'$ adjacent to both $u$ and $v$.

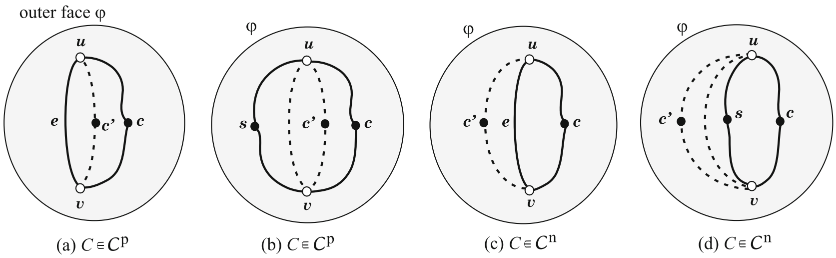


**Fig. 3.** Candidate posi- and nega-cycles $C = (u, c, v)$ and $C = (u, c, v, s)$ in $\mathcal{G}$, where white circles represent vertices in $V$ while black ones represent crossings in $\chi$: (a) candidate posi-cycle of length 3, (b) candidate posi-cycle of length 4, (c) candidate nega-cycle of length 3, and (d) candidate nega-cycle of length 4.

Figure 3(a)–(b) and (c)–(d) illustrate candidate posi-cycles and candidate nega-cycles, respectively. Let $\mathcal{C}^{\mathrm{p}}$ and $\mathcal{C}^{\mathrm{n}}$ be the sets of candidate posi-cycles and candidate nega-cycles, respectively. By definition we see that the set $\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{n}} \cup \{C_f \mid f \in F(\gamma)\}$ is inclusive, and hence $|\mathcal{C}^{\mathrm{p}} \cup \mathcal{C}^{\mathrm{n}} \cup \{C_f \mid f \in F(\gamma)\}| = O(n)$.

A candidate posi-cycle $C$ with $C = (u, c, v)$ (resp., $C = (u, c, v, s)$) is called a *B-cycle* if

**(a)-(B):** the exterior of $C$ contains no vertices in $V - \{u, v\}$ adjacent to $c$ (resp., contains exactly one vertex in $V - \{u, v\}$ adjacent to $c$ or $s$).

Note that $uv \in E$ when $C = (u, c, v)$ is a B-cycle, as shown in Fig. 4(a). Figure 4(b) and (d) illustrate the other types of B-cycles.

A candidate posi-cycle $C = (u, c, v, s)$ is called a *W-cycle* if

**(a)-(W):** the exterior of $C$ contains no vertices in $V - \{u, v\}$ adjacent to $c$ or $s$.

Figure 4(c) and (e) illustrate W-cycles.

Let $\mathcal{C}_{\mathrm{W}}$ (resp., $\mathcal{C}_{\mathrm{B}}$) be the set of W-cycles (resp., B-cycles) in $\gamma$. Clearly a W-cycle (resp., B-cycle) gives rise to a W-configuration (resp., B-configuration). Conversely, by choosing a W-configuration (resp., B-configuration) so that the interior is minimal, we obtain a W-cycle (resp., B-cycle). Hence we observe that the current embedding $\gamma$ admits a straight-line drawing if and only if $\mathcal{C}_{\mathrm{W}} = \mathcal{C}_{\mathrm{B}} = \emptyset$.

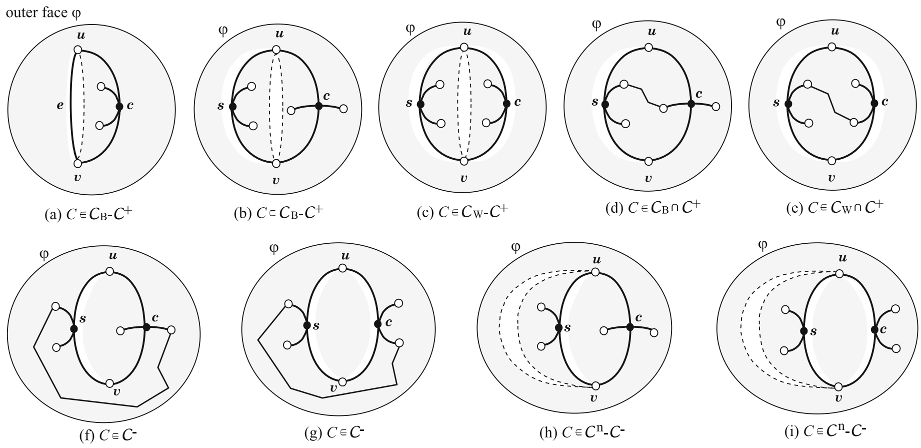


**Fig. 4.** Illustration of types of cycles $C = (u, c, v)$ and $C = (u, c, v, s)$ in $\mathcal{G}$, where white circles represent vertices in $V$ while black ones represent crossings in $\chi$: (a) B-cycle of length 3, which is always soft, (b) soft B-cycle of length 4, (c) soft W-cycle, (d) hard B-cycle of length 4, (e) hard W-cycle, (f) nega-cycle whose reversal is a hard B-cycle, (g) nega-cycle whose reversal is a hard W-cycle, (h) candidate nega-cycle of length 4 that is not a nega-cycle whose reversal is a hard B-cycle, and (i) candidate nega-cycle of length 4 that is not a nega-cycle whose reversal is a hard W-cycle.

A W- or B-cycle $C$ is called *hard* if

**(b):** length of $C$ is 4, and the interior of $C = (u, c, v, s)$ contains no inner face $f$ whose facial cycle $C_f$ contains both vertices $u$ and $v$, i.e., some path connects $c$ and $s$ without passing through $u$ or $v$.

On the other hand, a W- or B-cycle $C = (u, c, v, s)$ of length 4 that does not satisfy condition (b) or a B-cycle of length 3 is called *soft*. We also call a hard B- or W-cycle a *posi-cycle*.

Figure 4(d) and (e) illustrate a hard B-cycle and a hard W-cycles, respectively, whereas Fig. 4(a) and (b) (resp., (c)) illustrate soft B-cycles (resp., a soft W-cycle).

A cycle $C = (u, c, v, s)$ is called a *nega-cycle* if it becomes a posi-cycle when an inner face in the interior of $C$ is chosen as the outer face. In other words, a nega-cycle is a candidate nega-cycle $C = (u, c, v, s)$ of length 4 that satisfies the following conditions (a') and (b'), where (a') (resp., (b')) is obtained from the above conditions (a)-(B) and (a)-(W) (resp., (b)) by exchanging the roles of "interior" and "exterior":

**(a'):** the interior of $C$ contains at most one vertex in $V - \{u, v\}$ adjacent to $c$ or $s$; and

**(b'):** the exterior of $C$ contains no face $f$ whose facial cycle $C_f$ contains both vertices $u$ and $v$.

Figure 4(f) and (g) illustrate nega-cycles, whereas Fig. 4(h) and (i) illustrate candidate nega-cycles that are not nega-cycles.

Let $\mathcal{C}^+$ (resp., $\mathcal{C}^-$) denote the set of posi-cycles (resp., nega-cycles) in $\gamma$. By definition, it holds that $\mathcal{C}^+ \subseteq \mathcal{C}_{\rm W} \cup \mathcal{C}_{\rm B} \subseteq \mathcal{C}^{\rm p}$ and $\mathcal{C}^- \subseteq \mathcal{C}^{\rm n}$.

## 3.2 Forbidden Cycle Pairs

We define a forbidden configuration that characterizes 1-plane embeddings, which cannot be re-embedded into SLD ones. A *forbidden cycle pair* is defined to be a pair $\{C, C'\}$ of a posi-cycle $C = (u, c, v, s)$ and a posi- or nega-cycle $C' = (u', c', v', s')$ in $\mathcal{G}$ with $u, v, u', v' \in V$ and $c, s, c', s' \in \chi$ to which $\mathcal{G}$ has a $u, u'$-path $P_1$ and a $v, v'$-path $P_2$ such that:

(i) when $C' \in \mathcal{C}^+$, paths $P_1$ and $P_2$ are in the exterior of $C$ and $C'$, i.e., $V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\rm ex}(C) \cap V_{\rm ex}(C')$, where $C$ and $C'$ cannot have any common inner face; and
(ii) when $C' \in \mathcal{C}^-$, paths $P_1$ and $P_2$ are in the exterior of $C$ and the interior of $C'$, i.e., $V(P_1) - \{u, u'\}, V(P_2) - \{v, v'\} \subseteq V_{\rm ex}(C) \cap V_{\rm in}(C')$, where $C$ is enclosed by $C'$.

In (i) and (ii), $P_1$ and $P_2$ are not necessary disjoint, and possibly one of them consists of a single vertex, i.e., $u = u'$ or $v = v'$.

The pair of cycles $C$ and $C'$ in Fig. 5(a) (resp., Fig. 5(b)) is a forbidden cycle pair, because there is a pair of a $u, u'$-path $P_1 = (u, x, z, y, u')$ and a $v, v'$-path $P_2 = (v, x', z, y', v')$ that satisfy the above conditions (i) (resp., (ii)). Note that the pair of cycles $C$ and $C'$ in Fig. 2(a)–(b) is not forbidden cycle pair, because there are no such paths.

Our main result of this paper is as follows.

**Theorem 1.** *A circular instance $(G, \gamma)$ admits an SLD cross-preserving embedding if and only if it has no forbidden cycle pair. Finding an SLD cross-preserving embedding of $\gamma$ or a forbidden cycle pair in $\mathcal{G}$ can be computed in linear time.*

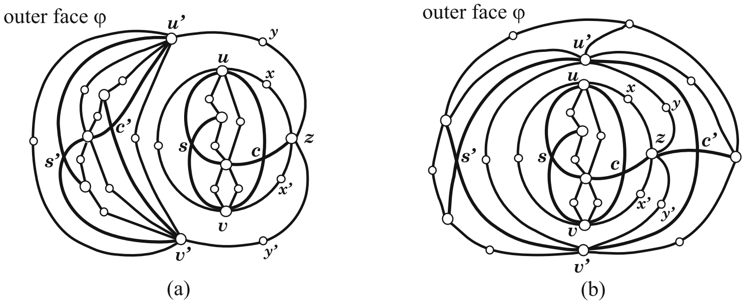


**Fig. 5.** Illustration of circular instances $(G, \gamma)$ with a cut-vertex $z$ of $\mathcal{G}$, where the crossing edges are depicted by slightly thicker lines: (a) forbidden cycle pair with hard B-cycles $C = (u, c, v, s)$ and $C' = (u', c', v', s')$ (b) forbidden cycle pair with a hard B-cycle $C = (u, c, v, s)$ and a nega-cycle $C' = (u', c', v', s')$ whose reversal is a hard B-cycle, where vertices $u, v, u', v' \in V$ and crossings $c, s, c', s' \in \chi$.

**Proof of necessity:** The necessity of the theorem follows from the next lemma.

For a cycle $C = (u, c, v, s) \in \mathcal{C}^+$ (resp., $\mathcal{C}^-$) with $u, v \in V$ and $c, s \in \chi$ in $\mathcal{G}$, we call a vertex $z \in V$ an *in-factor* of $C$ if the exterior of $C \in \mathcal{C}^+$ (resp., the interior of $C \in \mathcal{C}^-$) has a $z, u$-path $P_{z,u}$ and a $z, v$-path $P_{z,v}$, i.e., $V(P_{z,u} - \{u\}) \cup V(P_{z,v} - \{v\})$ is in $V_{\text{ex}}(C)$ (resp., $V_{\text{in}}(C)$). Paths $P_{z,u}$ and $P_{z,v}$ are not necessarily disjoint.

**Lemma 4.** *Given $\mathcal{G} = \mathcal{G}(G, \gamma)$, let $\gamma'$ be a cross-preserving embedding of $\gamma$. Then:*

(i) *Let $z \in V$ be an in-factor of a cycle $C \in \mathcal{C}^+ \cup \mathcal{C}^-$ in $\mathcal{G}$. Then cycle $C$ is a posi-cycle* (*resp., a nega-cycle*) *in $\mathcal{G}(G, \gamma')$ if and only if $z$ is in the exterior* (*resp., interior*) *of $C$ in $\gamma'$;*
(ii) *For a forbidden cycle pair $\{C, C'\}$, one of $C$ and $C'$ is a posi-cycle in $\mathcal{G}(G, \gamma')$* (*hence any cross-preserving embedding of $\gamma$ contains a B- or W-configuration and $(G, \gamma)$ admits no SLD cross-preserving embedding*).

**Proof of sufficiency:** In the rest of paper, we prove the sufficiency of Theorem 1 by designing a linear-time algorithm that constructs an SLD cross-preserving embedding of an instance without a forbidden cycle pair.

## 4 Biconnected Case

In this section, $(G, \gamma)$ stands for a circular instance such that the vertex-connectivity of the plane graph $\mathcal{G}$ is at least 2. In a biconnected graph $\mathcal{G}$, any two posi-cycles $C = (u, c, v, s)$, $C' = (u', c', v', s') \in \mathcal{C}^+$with $u, v, u', v' \in V$ give a forbidden cycle pair if they do not share an inner face, because there is a pair of $u, u'$-path and $v, v'$-path in the exterior of $C$ and $C'$. Analogously any pair of a posi-cycle $C$ and a nega-cycle $C'$ such that $C'$ encloses $C$ is also a forbidden cycle pair in a biconnected graph $\mathcal{G}$.

To detect such a forbidden pair in $\mathcal{G}$ in linear time, we first compute the sets $\mathcal{C}_{\text{p}}, \mathcal{C}_{\text{n}}, \mathcal{C}_{\text{W}}, \mathcal{C}_{\text{B}}, \mathcal{C}^+$ and $\mathcal{C}^-$ in $\gamma$ in linear time by using the inclusion-forest from Lemma 2.

**Lemma 5.** *Given $(G, \gamma)$, the following in (i)–(iv) can be computed in $O(n)$ time.*

(i) *The sets $\mathcal{C}_{\text{p}}$, $\mathcal{C}_{\text{n}}$ and the inclusion-forest $\mathcal{I}$ of $\mathcal{C}_{\text{p}} \cup \mathcal{C}_{\text{n}} \cup \{C_f \mid f \in F(\gamma)\}$;*
(ii) *The sets $\mathcal{C}_{\text{W}}$ and $\mathcal{C}_{\text{B}}$;*
(iii) *The sets $\mathcal{C}^+$, $\mathcal{C}^-$ and the inclusion-forest $\mathcal{I}^*$ of $\mathcal{C}^+ \cup \mathcal{C}^-$; and*
(iv) *A set $\{f_C \mid C \in (\mathcal{C}_{\text{W}} \cup \mathcal{C}_{\text{B}}) - \mathcal{C}^+\}$ such that $f_C$ is an inner face in the interior of a soft B- or W-cycle $C$ with $V(C_f) \supseteq V(C)$.*

Given $(G, \gamma)$, a face $f \in F(\gamma)$ is called *admissible* if all posi-cycles enclose $f$ but no nega-cycle encloses $f$. Let $A(\gamma)$ denote the set of all admissible faces in $F(\gamma)$.

**Lemma 6.** *Given* $(G, \gamma)$*, it holds* $A(\gamma) \neq \emptyset$ *if and only if no forbidden cycle pair exists in* $\gamma$*. A forbidden cycle pair, if one exists, and* $A(\gamma)$ *can be obtained in* $O(n)$ *time.*

By the lemma, if $(G, \gamma)$ has no forbidden cycle pair, i.e., $A(\gamma) \neq \emptyset$, then any new embedding obtained from $\gamma$ by changing the outer face with a face in $A(\gamma)$ is a cross-preserving embedding of $\gamma$ which has no hard B- or W-cycle.

### 4.1 Eliminating Soft B- and W-cycles

Suppose that we are given a circular instance $(G, \gamma)$ such that $\mathcal{G}$ is biconnected and $\mathcal{C}^+ = \emptyset$. We now show how to eliminate all soft B- and W-cycles in $\mathcal{G}$ in linear time using the inclusion-forest from Lemma 2 and the spindles from Lemma 3.

**Lemma 7.** *Given* $(G, \gamma)$ *with* $\mathcal{C}^+ = \emptyset$*, there exists an SLD cross-preserving embedding* $\gamma' = (\chi, \rho', \varphi')$ *of* $\gamma$ *such that* $V(C_{\varphi'}) \supseteq V(C_\varphi)$ *for the facial cycle* $C_\varphi$ *(resp.,* $C_{\varphi'}$*) of the outer face* $\varphi$ *(resp.,* $\varphi'$*), which can be constructed in* $O(n)$ *time.*

Given an instance $(G, \gamma)$ with a biconnected graph $\mathcal{G}$, we can test whether it has either a forbidden cycle pair or an admissible face by Lemmas 5 and 6. In the former, it cannot have an SLD cross-preserving embedding by Lemma 4. In the latter, we can eliminate all hard B- and W-cycles by choosing an admissible face as a new outer face, and then eliminate all soft B- and W-cycles by a flipping procedure based on Lemma 7. All the above can be done in linear time.

To treat the case where the vertex-connectivity of $\mathcal{G}$ is 1 in the next section, we now characterize 1-plane embeddings that can have an SLD cross-preserving embedding such that a specified vertex appears along the outer boundary. For a vertex $z \in V$ in a graph $G$, we call a 1-plane embedding $\gamma$ of $G$ *z-exposed* if vertex $z$ appears along the outer boundary of $\gamma$. We call $(G, \gamma)$ *z-feasible* if it admits a $z$-exposed SLD cross-preserving embedding $\gamma'$ of $\gamma$.

**Lemma 8.** *Given* $(G, \gamma)$ *such that* $A(\gamma) \neq \emptyset$*, let* $z$ *be a vertex in* $V$*. Then:*

(i) *The following conditions are equivalent:*
  (a) $\gamma$ *admits no* $z$*-exposed SLD cross-preserving embedding;*
  (b) $A(\gamma)$ *contains no face* $f$ *with* $z \in V(C_f)$*; and*
  (c) $\mathcal{G}$ *has a posi- or nega-cycle* $C$ *to which* $z$ *is an in-factor;*

(ii) *A* $z$*-exposed SLD cross-preserving embedding or a posi- or nega-cycle* $C$ *to which* $z$ *is an in-factor can be computed in* $O(n)$ *time.*

## 5 One-Connected Case

In this section, we prove the sufficiency of Theorem 1 by designing a linear-time algorithm claimed in the theorem. Given a circular instance $(G, \gamma)$, where $\mathcal{G}$ may be disconnected, obviously we only need to test each connected component of

$\mathcal{G}$ separately to find a forbidden cycle pair. Thus we first consider a circular instance $(G, \gamma)$ such that the vertex-connectivity of $\mathcal{G}$ is 1; i.e., $\mathcal{G}$ is connected and has some cut-vertices.

A block $B$ of $\mathcal{G}$ is a maximal biconnected subgraph of $\mathcal{G}$. For a biconnected graph $\mathcal{G}$, we already know how to find a forbidden cycle pair or an SLD cross-preserving embedding from the previous section. For a trivial block $B$ with $|V(B)| = 2$, there is nothing to do. If some block $B$ of $\mathcal{G}$ with $|V(B)| \geq 3$ contains a forbidden cycle pair, then $(G, \gamma)$ cannot admit any SLD cross-preserving embedding by Lemma 4.

We now observe that $\mathcal{G}$ may contain a forbidden cycle pair even if no single block of $\mathcal{G}$ has a forbidden cycle pair.

**Lemma 9.** *For a circular instance $(G, \gamma)$ such that the vertex-connectivity of $\mathcal{G}$ is 1, let $B_1$ and $B_2$ be blocks of $\mathcal{G}$ and let $P_{1,2}$ be a $z_1, z_2$-path of $\mathcal{G}$ with the minimum number of edges, where $V(B_i) \cap V(P_{1,2}) = \{z_i\}$ for each $i = 1, 2$. If $\gamma|_{B_i}$ has a posi- or nega-cycle $C_i$ to which $z_i$ is an in-factor for each $i = 1, 2$, then $\{C_1, C_2\}$ is a forbidden cycle pair in $\mathcal{G}$.*

For a linear-time implementation, we do not apply the lemma for all pairs of blocks in $\mathcal{B}$. A block of $\mathcal{G}$ is called a *leaf block* if it contains only one cut-vertex of $\mathcal{G}$, where we denote the cut-vertex in a leaf block $B$ by $v_B$. Without directly searching for a forbidden cycle pair in $\mathcal{G}$, we use the next lemma to reduce a given embedding by repeatedly removing leaf blocks.

**Lemma 10.** *For a circular instance $(G, \gamma)$ such that the vertex-connectivity of $\mathcal{G} = \mathcal{G}(G, \gamma)$ is 1 and a leaf block $B$ of $\mathcal{G}$ such that $\gamma|_B$ is $v_B$-feasible, let $H = G-(V(B)-\{v_B\})$ be the graph obtained by removing the vertices in $V(B)-\{v_B\}$. Then*

(i) *The instance $(H, \gamma|_H)$ is circular; and*
(ii) *If $(H, \gamma|_H)$ admits an SLD cross-preserving embedding $\gamma^*_H$, then an SLD cross-preserving embedding $\gamma^*$ of $\gamma$ can be obtained by placing a $v_B$-exposed SLD cross-preserving embedding $\gamma^*_B$ of $\gamma|_B$ within a space next to the cut-vertex $v_B$ in $\gamma^*_H$.*

Given a circular instance $(G, \gamma)$ such that $\mathcal{G} = \mathcal{G}(G, \gamma)$ is connected, an algorithm **Algorithm Re-Embed-1-Plane** for Theorem 1 is designed by the following three steps.

The first step tests whether $\mathcal{G}$ has a block $B$ such that $\gamma|_B$ has a forbidden cycle pair, based on Lemma 8. If one exists, the algorithm outputs a forbidden cycle pair and halts.

After the first step, no block has a forbidden cycle pair. In the current circular instance $(G, \gamma)$, one of the following holds:

(i) the number of blocks in $\mathcal{G}$ is at least two and there is at most one leaf block $B$ such that $\gamma|_B$ is not $v_B$-feasible;
(ii) $\mathcal{G}$ has two leaf blocks $B$ and $B'$ such that $\gamma|_B$ is not $v_B$-feasible and $\gamma|_{B'}$ is not $v_{B'}$-feasible; and
(iii) the number of blocks in $\mathcal{G}$ is at most one.

In (ii), $v_B$ is an in-factor of a cycle $C$ in $\gamma|_B$ and $v_{B'}$ is an in-factor of a cycle $C'$ in $\gamma|_{B'}$ by Lemma 8, and we obtain a forbidden cycle pair $\{C, C'\}$ by Lemma 9. Otherwise if (i) holds, then we can remove all leaf blocks $B$ such that $\gamma|_B$ is not $v_B$-feasible by Lemma 10. The second step keeps removing all leaf blocks $B$ such that $\gamma|_B$ is not $v_B$-feasible until (ii) or (iii) holds to the resulting embedding. If (i) occurs, then the algorithm outputs a forbidden cycle pair and halts.

When all the blocks of $\mathcal{G}$ can be removed successfully, say in an order of $B^1, B^2, \ldots, B^m$, the third step constructs an embedding with no B- or W-cycles by starting with such an SLD embedding of $B^m$ and by adding an SLD embedding of $B^i$ to the current embedding in the order of $i = m-1, m-2, \ldots, 1$. By Lemma 10, this results in an SLD cross-preserving embedding of the input instance $(G, \gamma)$.

Note that we can obtain an SLD cross-preserving embedding $\gamma^*_{H^1}$ of $\gamma$ in the third step when the first and second step did not find any forbidden cycle pair. Thus the algorithm finds either an SLD cross-preserving embedding of $\gamma$ or a forbidden cycle pair. This proves the sufficiency of Theorem 1.

By the time complexity result from Lemma 8, we see that the algorithm can be implemented in linear time.

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