Chapter 4
Passive State Space Systems

In this chapter we focus on passive systems as an outstanding subclass of dissipative systems, firmly rooted in the mathematical modeling of physical systems.

4.1 Characterization of Passive State Space Systems

Recall from Chap. 3 the definitions of (input and/or output strict) passivity of a state space system, cf. Definition 3.1.4.

Definition 4.1.1 A *state space system* Σ with equal number of inputs and outputs

$$\begin{aligned} \dot{x} &= f(x,u), \quad x \in \mathcal{X},\ u \in U = \mathbb{R}^m \\ y &= h(x,u), \quad y \in Y = \mathbb{R}^m \end{aligned} \tag{4.1}$$

is *passive* if it is dissipative with respect to the supply rate $s(u, y) = u^T y$. Furthermore, Σ is called *cyclo-passive* if the storage function is *not* necessarily satisfying the nonnegativity condition. Σ is called *lossless* if it is conservative with respect to $s(u, y) = u^T y$. The system Σ is *input strictly passive* if there exists $\delta > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \delta||u||^2$ (also called δ-input strictly passive). Σ is *output strictly passive* if there exists $\varepsilon > 0$ such that Σ is dissipative with respect to $s(u, y) = u^T y - \varepsilon||y||^2$ (ϵ-output strictly passive).

Also recall from Chap. 3 that there is a minimal storage function S_a (the *available storage*), and under a reachability condition, a storage function S_r (the *required supply* from x^*), which is maximal in the sense of (3.26); see also Corollary 3.1.21. The storage function in the case of the passivity supply rate often has the interpretation of a (generalized) energy function, and $S_a(x)$ equals the *maximal* energy that can be extracted from the system being in state x, while $S_r(x)$ is the *minimal* energy that is needed to bring the system toward state x, while starting from a ground state x^*.

A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*,
Communications and Control Engineering, DOI 10.1007/978-3-319-49992-5_4

In physical examples, the true physical energy usually will be "somewhere in the middle" between S_a and S_r.

Assuming differentiability of the storage function (as will be done throughout this section), passivity, respectively input or output strict passivity, can be characterized through the differential dissipation inequalities (3.36). These take a particularly explicit form for systems which are *affine* in the input u (as often encountered in applications), and given as

$$\Sigma_a^{ft}: \begin{array}{l} \dot{x} = f(x) + g(x)u \\ y = h(x) + j(x)u, \end{array} \tag{4.2}$$

with $g(x)$ an $n \times m$ matrix, and $j(x)$ an $m \times m$ matrix. In case of the passivity supply rate $s(u, y) = u^T y$ the differential dissipation inequality then takes the form

$$\frac{d}{dt}S = S_x(x)[f(x) + g(x)u] \leq u^T[h(x) + j(x)u], \quad \forall x, u, \tag{4.3}$$

where, as before, the notation $S_x(x)$ stands for the row vector of partial derivatives of the function $S : \mathcal{X} \to \mathbb{R}$. Note that

$$\begin{aligned} &S_x(x)[f(x) + g(x)u] - u^T[h(x) + j(x)u] = \\ &\tfrac{1}{2}\begin{bmatrix} 1 & u^T \end{bmatrix}\begin{bmatrix} 2S_x(x)f(x) & S_x(x)g(x) - h^T(x) \\ g^T(x)S_x^T(x) - h(x) & -(j(x) + j^T(x)) \end{bmatrix}\begin{bmatrix} 1 \\ u \end{bmatrix} \end{aligned} \tag{4.4}$$

while similar expressions are obtained in the case of the output and input strict passivity supply rates.

This leads to the following characterizations.

Proposition 4.1.2 *Consider the system Σ_a^{ft} given by (4.2). Then:*
(i) Σ_a^{ft} is passive with C^1 storage function S if and only if for all x

$$\begin{bmatrix} 2S_x(x)f(x) & S_x(x)g(x) - h^T(x) \\ g^T(x)S_x^T(x) - h(x) & -(j(x) + j^T(x)) \end{bmatrix} \leq 0 \tag{4.5}$$

(ii) Σ_a^{ft} is ε-output strictly passive with C^1 storage function S if and only if for all x

$$\begin{bmatrix} 2S_x(x)f(x) + 2\varepsilon h^T(x)h(x) & S_x(x)g(x) - h^T(x) + k^T(x) \\ g^T(x)S_x^T(x) - h(x) + k(x) & \ell(x) - (j(x) + j^T(x)) \end{bmatrix} \leq 0, \tag{4.6}$$

where $k(x) := 4\varepsilon h^T(x)j(x)$, $\ell(x) := 2\varepsilon j(x)j^T(x)$.
(iii) Σ_a^{ft} is δ-input strictly passive with C^1 storage function S if and only if for all x

$$\begin{bmatrix} 2S_x(x)f(x) & S_x(x)g(x) - h^T(x) \\ g^T(x)S_x^T(x) - h(x) & 2\delta I_m - (j(x) + j^T(x)) \end{bmatrix} \leq 0 \tag{4.7}$$

The proof of this proposition is based on the following basic lemma.

Lemma 4.1.3 *Let $R = R^T$ be an $m \times m$ matrix, q an m-vector, and p a scalar. Then*

$$u^T R u + 2u^T q + p \leq 0\,, \text{ for all } u \in \mathbb{R}^m\,, \tag{4.8}$$

if and only if

$$\begin{bmatrix} p & q^T \\ q & R \end{bmatrix} \leq 0 \tag{4.9}$$

Proof (*of Lemma* 4.1.3) Obviously, the inequality (4.9) implies

$$u^T R u + 2u^T q + p = \begin{bmatrix} 1 & u^T \end{bmatrix} \begin{bmatrix} p & q^T \\ q & R \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix} \leq 0, \quad \forall u \in \mathbb{R}^m \tag{4.10}$$

In order to prove[1] the converse implication assume that

$$v^T \begin{bmatrix} p & q^T \\ q & R \end{bmatrix} v > 0 \tag{4.11}$$

for some $(m+1)$-dimensional vector v. If the first component of v is different from zero we can directly scale the vector v to a vector of the form $\begin{bmatrix} 1 \\ u \end{bmatrix}$ while still (4.11) holds, leading to a contradiction. If the first component of v equals zero then we can consider a small perturbation of v for which the first component of v is nonzero while still (4.11) holds, and we use the previous argument. □

Proof (*of Proposition* 4.1.2) Write out the dissipation inequalities in the form $u^T R(x) u + 2u^T q(x) + p(x) \leq 0$, and apply Lemma 4.1.3 with R, q, p additionally depending on x. □

Example 4.1.4 It follows from (4.7) that an input strictly passive system necessarily has a nonzero feedthrough term $j(x)u$. An example is provided by a proportional–integral (PI) controller

$$\begin{aligned} \dot{x}_c &= u_c \\ y_c &= k_I x_c + k_P u_c \end{aligned} \tag{4.12}$$

with $k_P, k_I \geq 0$ the proportional, respectively integral, control coefficients. This is a k_P-input strictly system with storage function is $\frac{1}{2} k_I x_c^2$, since

$$\frac{d}{dt} \frac{1}{2} k_I x_c^2 = u_c y_c - k_P u_c^2 \tag{4.13}$$

A drastic simplification of the conditions for (output strict) passivity occurs for systems *without feedthrough term* ($j(x) = 0$) given as

[1]With thanks to Anders Rantzer for a useful conversation.

$$\Sigma_a : \begin{array}{l} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{array} \tag{4.14}$$

Corollary 4.1.5 *Consider the system Σ_a given by (4.14). Then:*
(i) Σ_a is passive with C^1 storage function S if and only if for all x

$$\begin{array}{l} S_x(x) f(x) \leq 0 \\ S_x(x)g(x) = h^T(x) \end{array} \tag{4.15}$$

(ii) Σ_a is ε-output strictly passive with C^1 storage function S if and only if for all x

$$\begin{array}{l} S_x(x) f(x) \leq -\varepsilon h^T(x)h(x) \\ S_x(x)g(x) = h^T(x) \end{array} \tag{4.16}$$

(iii) Σ_a is not δ-input strictly passive for any $\delta > 0$.

Proof Use the well-known fact that $\begin{bmatrix} k & q^T \\ q & 0_m \end{bmatrix} \leq 0$ (with 0_m denoting the $m \times m$ zero matrix) if and only if $q = 0$ and $k \leq 0$. □

Remark 4.1.6 For a *linear* system

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \tag{4.17}$$

with quadratic storage function $S(x) = \frac{1}{2}x^T Qx$, $Q = Q^T \geq 0$, the passivity condition (4.5) amounts to the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T Q + QA & QB - C^T \\ B^T Q - C & -D - D^T \end{bmatrix} \leq 0 \tag{4.18}$$

Obvious extensions to input/output strict passivity are left to the reader. In case $D = 0$ (no feedthrough) the conditions (4.18) simplify to the LMI

$$A^T Q + QA \leq 0, \quad B^T Q = C \tag{4.19}$$

The relation of these LMIs to frequency-domain conditions is known as the *Kalman–Yakubovich–Popov Lemma*; see the Notes at the end of this chapter for references.

The inequalities in Proposition 4.1.2 and Corollary 4.1.5, as well as the resulting LMIs (4.18) and (4.19) in the linear system case, admit the following *factorization* perspective. Given a matrix inequality $P(x) \leq 0$, where $P(x)$ is an $k \times k$ symmetric matrix depending smoothly on x, we may always, by standard linear-algebraic factorization for every constant x, construct an $\ell \times k$ matrix $F(x)$ such that $P(x) = -F^T(x)F(x)$, where ℓ is equal to the maximal rank of $P(x)$ (over x).

Furthermore, by an application of the implicit function theorem, locally on a neighborhood where the rank of $P(x)$ is constant, this can be done in such a way that $F(x)$ is depending smoothly on x. Applied to (minus) the matrices appearing in Proposition 4.1.2 and Corollary 4.1.5 this leads to the following result. For concreteness, focus on the inequality (4.5); similar statements hold for the other cases. Inequality 4.5 holds if and only if

$$\begin{bmatrix} 2S_x(x)f(x) & S_x(x)g(x) - h^T(x) \\ g^T(x)S_x^T(x) - h(x) & -(j(x) + j^T(x)) \end{bmatrix} = -F^T(x)F(x) \leq 0 \tag{4.20}$$

for a certain matrix

$$F(x) = \begin{bmatrix} \phi(x) & \Psi(x) \end{bmatrix} \tag{4.21}$$

with $\phi(x)$ an ℓ-dimensional column vector, and $\psi(x)$ an $\ell \times m$ matrix, with ℓ the (local) rank of the matrix in (4.5). Writing out (4.20) yields

$$\begin{aligned} 2S_x(x)f(x) &= -\phi^T(x)\phi(x) \\ S_x(x)g(x) - h^T(x) &= -\phi^T(x)\Psi(x) \\ j(x) + j^T(x) &= \Psi^T(x)\Psi(x) \end{aligned} \tag{4.22}$$

It follows that by defining the new, artificial, output equation

$$\bar{y} = \phi(x) + \Psi(x)u \tag{4.23}$$

one obtains

$$S_x(x)[f(x) + g(x)u] - u^T[h(x) + j(x)u] = -\frac{1}{2}\|\bar{y}\|^2, \tag{4.24}$$

and therefore

$$\frac{d}{dt}S = u^T y - \frac{1}{2}\|\bar{y}\|^2. \tag{4.25}$$

Hence, by factorization we have turned the dissipativity of the system Σ_a^{ft} with respect to the passivity supply rate $s(u, y) = u^T y$ into the fact that Σ_a^{ft} is *conservative* with respect to the *new* supply rate

$$s_{\text{new}}(u, y) = u^T y - \frac{1}{2}\|\bar{y}\|^2, \tag{4.26}$$

defined in terms of the inputs u, outputs y, *as well as* the new outputs $\bar{y}$ defined by (4.23). The same can be done for the output and input strict passivity supply rates; in fact, for any supply rate which is quadratic in u, y. Within the context of the L_2-gain supply rate this[2] will be exploited in Chap. 9; see especially Sect. 9.4.

[2] In fact, in Sect. 9.4 we will see how this can be extended to *general* systems Σ.

Let us briefly focus on the *linear* passive system case, corresponding to the LMIs (4.18). As was already mentioned in Remark 3.1.22 for general supply rates, the available storage S_a of a linear passive system (4.17) with $D = 0$ is given as $\frac{1}{2}x^T Q_a x$ where Q_a is the *minimal* solution to the LMI (4.18), while the required supply is $\frac{1}{2}x^T Q_r x$ where Q_r is the *maximal* solution to this same LMI.

Although in general (4.18) has a convex *set* of solutions $Q \geq 0$, this set may sometimes reduce to a *unique* solution; even for systems with nonzero internal energy dissipation. This is illustrated by the following simple physical example.

Example 4.1.7 Consider the ubiquitous mass–spring–damper system

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad u = \text{ force} \\ y &= \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \text{ velocity} \end{aligned} \tag{4.27}$$

with physical energy $H(q, p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$ (q extension of the linear spring with spring constant k, p momentum of mass m), and internal energy dissipation corresponding to a linear damper with damping coefficient $d > 0$. The LMI (4.19) takes the form

$$\begin{aligned} \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{d}{m} \end{bmatrix} &\leq 0 \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} \end{aligned} \tag{4.28}$$

The last equation yields $q_{12} = 0$ as well as $q_{22} = \frac{1}{m}$. Substituted in the inequality this yields the unique solution $q_{11} = k$, corresponding to the *unique* quadratic storage function $H(q, p)$, which is equal to $S_a = S_r$. The explanation for the perhaps surprising equality of S_a and S_r in this case is the fact that the definitions of S_a and S_r involve sup and inf (instead of max and min).

We note for later use that passivity of a *static* nonlinear map $y = F(u)$, with $F : \mathbb{R}^m \to \mathbb{R}^m$, amounts to requiring that

$$u^T F(u) \geq 0, \quad \text{for all } u \in \mathbb{R}^m, \tag{4.29}$$

which for $m = 1$ reduces to the condition that the graph of the function F is in the first and third quadrant. This definition immediately extends to *relations* instead of mappings.

Furthermore, passivity of the dynamical system Σ implies the following *static* passivity property of the steady-state values of its inputs and outputs. Let Σ be an input-state-output system in the general form (4.1). For any constant input $\bar{u}$ consider the existence of a steady-state $\bar{x}$, and corresponding steady-state output value $\bar{y}$, satisfying

$$0 = f(\bar{x}, \bar{u}), \ \ \bar{y} = h(\bar{x}, \bar{u}) \tag{4.30}$$

This defines the following relation between $\bar{u}$ and $\bar{y}$, called the *steady-state input–output relation* Σ_{ss} corresponding to Σ:

$$\Sigma_{ss} := \{(\bar{u}, \bar{y}) \mid \exists \bar{x} \text{ s.t. (4.30) holds } \} \tag{4.31}$$

In case of a cyclo-passive system (4.1) with storage function S satisfying $\frac{d}{dt}S \leq u^T y$ it follows that

$$0 = \frac{d}{dt}S(\bar{x}) \leq \bar{u}^T \bar{y}, \quad \text{for any } (\bar{u}, \bar{y}) \in \Sigma_{ss}, \tag{4.32}$$

with the obvious interpretation that at its steady states every cyclo-passive system necessarily dissipates energy.

Note that in general Σ_{ss} need not be the graph of a *mapping* from $\bar{u}$ to $\bar{y}$. For example, Σ_{ss} corresponding to the (multi-dimensional) nonlinear integrator

$$\dot{x} = u, \ \ y = \frac{\partial H}{\partial x}(x), \quad x, u, y \in \mathbb{R}^m \tag{4.33}$$

(which is a cyclo-passive system with, possibly indefinite, storage function H), is given as

$$\Sigma_{ss} = \left\{ (\bar{u} = 0, \bar{y}) \mid \exists \bar{x} \text{ s.t. } \bar{y} = \frac{\partial H}{\partial x}(\bar{x}) \right\} \tag{4.34}$$

This will be further explored within the context of port-Hamiltonian systems in Chap. 6, Sect. 6.5.

4.2 Stability and Stabilization of Passive Systems

Many of the stability results as established in Chap. 3 for dissipative systems involving additional conditions on the supply rate directly apply to the passivity supply rate. In particular Propositions 3.2.7, 3.2.9 (see Remark 3.2.10) and Proposition 3.2.12 (see Remark 3.2.14) hold for passive systems. Moreover, Propositions 3.2.15 and 3.2.19 apply to output strictly passive systems; see Remark 3.2.20.

Loosely speaking, equilibria of passive systems are typically *stable*, but not necessarily *asymptotically stable*. On the other hand, there is no obvious relation between passivity and stability of the *input–output maps*. This is already illustrated by the simplest example of a passive (in fact, lossless) system; namely the *integrator*

$$\dot{x} = u, \ \ y = x, \quad x, \ u, \ y \in \mathbb{R}$$

Obviously, 0 is a stable equilibrium with Lyapunov function $\frac{1}{2}x^2$, while the input–output mappings of this system map $L_{2e}(\mathbb{R})$ into $L_{2e}(\mathbb{R})$, but *not* $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$. The same applies to a *nonlinear* integrator, with output equation $y = x$ replaced by

$y = S_x(x)$ for some nonnegative function S having its minimum at 0. The situation becomes different by changing $\dot{x} = u$, $y = x$ into $\dot{x} = -x + u$, $y = x$, leading to a system with asymptotically stable equilibrium 0 and finite L_2-gain input–output map. On the other hand, the minor modification $\dot{x} = -x^3 + u$ displays 0 as an asymptotically stable equilibrium, but does *not* define a mapping from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$. To explain the differences, notice that of the three preceding examples only $\dot{x} = -x + u$, $y = x$ is *output strictly* passive. Indeed, output strict passivity implies *finite L_2-gain*, as formulated in the following state space version of Theorem 2.2.13.

Proposition 4.2.1 *If Σ is ε-output strictly passive, then it has L_2-gain $\leq \frac{1}{\varepsilon}$.*

Proof If Σ is ε-output strictly passive there exists $S \geq 0$ such that for all $t_1 \geq t_0$ and all u

$$S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} (u^T(t)y(t) - \varepsilon ||y(t)||^2)dt \tag{4.35}$$

Therefore

$$\begin{aligned} &\varepsilon \int_{t_0}^{t_1} ||y(t)||^2)dt \;\leq\; \int_{t_0}^{t_1} u^T(t)y(t)dt - S(x(t_1)) + S(x(t_0)) \;\leq \\ &\int_{t_0}^{t_1} (u^T(t)y(t) + \tfrac{1}{2}||\tfrac{1}{\sqrt{\varepsilon}}u(t) - \sqrt{\varepsilon}y(t)||^2)dt - S(x(t_1)) + S(x(t_0)) = \\ &\int_{t_0}^{t_1} (\tfrac{1}{2\varepsilon}||u(t)||^2 + \tfrac{\varepsilon}{2}||y(t)||^2)dt - S(x(t_1)) + S(x(t_0)) \;, \end{aligned}$$

whence

$$S(x(t_1)) - S(x(t_0)) \;\leq\; \int_{t_0}^{t_1} \left(\frac{1}{2\varepsilon}||u(t||^2 - \frac{\varepsilon}{2}||y(t)||^2 \right) dt \;, \tag{4.36}$$

implying that Σ has L_2-gain $\leq \frac{1}{\varepsilon}$ (with storage function $\frac{1}{\varepsilon}S$). □

Further implications of output strict passivity for the input–output stability will be discussed in the context of L_2-gain analysis of state space systems in Chap. 8.

The importance of output strict passivity for asymptotic and input–output stability directly motivates the consideration of the following simple class of feedbacks which *render* a passive system output strictly passive. Indeed, consider a passive system Σ as given in (4.1) with C^1 storage function S, that is

$$\frac{d}{dt}S \leq u^T y \tag{4.37}$$

If the system is not output strictly passive, then an obvious way to *render* the system output strictly passive is to apply a *static output feedback*

$$u = -dy + v, \quad d > 0, \tag{4.38}$$

with $v \in \mathbb{R}^m$ the new input, and d a positive scalar.[3] Then the closed-loop system satisfies

$$\frac{d}{dt} S \leq v^T y - d||y||^2, \tag{4.39}$$

and thus is d-output strictly passive, and has L_2-gain $\leq \frac{1}{d}$ (from v to y). Hence, we obtain the following corollary of Propositions 3.2.16 and 3.2.19.

Corollary 4.2.2 *Consider the passive system Σ with storage function S satisfying $S(0) = 0$. Assume that S is positive definite at 0 and that the system $\dot{x} = f(x, 0), \ y = h(x, 0)$, is zero-state detectable. Alternatively, assume 0 is an asymptotically stable equilibrium of $\dot{x} = f(x, 0)$ conditionally to $\{x \mid h(x, 0) = 0\}$. In both cases the feedback $u = -dy, \ d > 0$, asymptotically stabilizes the system around the equilibrium 0.*

Finally, we remark that in certain cases the verification of the property of zero-state detectability or asymptotic stability conditionally to $y = h(x, 0) = 0$ can be reduced to the verification of the same property for a *lower-dimensional* system. Consider as a typical case the feedback interconnection of Σ_1 and Σ_2 as in Fig. 1.1 with $e_2 = 0$ (see Fig. 4.1 later on). Suppose that Σ_1 satisfies the property

$$y_1(t) = 0, \quad t \geq 0 \Rightarrow x_1(t) = 0, \quad t \geq 0 \text{ and } u_1(t) = 0, \quad t \geq 0 \tag{4.40}$$

(This is a strong zero-state observability property.) Now, let $y_1(t) = 0, \quad t \geq 0$, and $e_1(t) = 0, \quad t \geq 0$. Then $u_2(t) = 0, \quad t \geq 0$, and by (4.40), $y_2(t) = 0, \quad t \geq 0$. Hence, checking zero-state detectability or asymptotic stability conditionally to $y_1 = h_1(x_1) = 0$ for the closed-loop system is the same as checking the same property for Σ_2. Summarizing, we have obtained the following.

Proposition 4.2.3 *Consider the closed-loop system $\Sigma_1 \|_f \Sigma_2$ with $e_2 = 0$, having input e_1 and output y_1. Suppose that Σ_1 satisfies property (4.40). Then the closed-loop system is zero-state detectable, respectively asymptotically stable conditionally to $y_1 = 0$, if and only if Σ_2 is zero-state detectable, respectively, asymptotically stable conditionally to $y_2 = 0$.*

Example 4.2.4 (*Euler's equations*) Euler's equations of the dynamics of the angular velocities of a fully actuated rigid body, spinning around its center of mass (in the absence of gravity), are given by

$$I\dot{\omega} = -S(\omega)I\omega + u \tag{4.41}$$

Here I is the positive diagonal inertia matrix, $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the vector of angular velocities in body coordinates, $u = (u_1, u_2, u_3)^T$ is the vector of inputs, while the skew-symmetric matrix $S(\omega)$ is given as

[3] This can be extended to $u = -Dy + v$, with D a matrix satisfying $D + D^T > 0$.

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4.42}$$

Since $\frac{d}{dt}\frac{1}{2}\omega^T I\omega = u^T\omega$ it follows that the system (4.41) with output $y = \omega$ is passive (in fact, lossless). Stabilization to $\omega = 0$ is achieved by output feedback $u = -Dy$ for any positive matrix D. In Sect. 7.1 we will see how this can be extended to the underactuated case by making use of the underlying Hamiltonian structure of (4.41).

Example 4.2.5 (*Rigid body kinematics*) The dynamics of the orientation of a rigid body around its center of mass is described as

$$\dot{R} = RS(\omega) \tag{4.43}$$

where $R \in SO(3)$ is an orthonormal rotation matrix describing the orientation of the body with respect to an inertial frame, $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the vector of angular velocities as in the previous example, and $S(\omega)$ is given by (4.42). The rotation matrix $R \in SO(3)$ can be parameterized by a rotation φ around a unit vector k as follows:

$$R = I_3 + \sin\varphi\, S(k) + (1 - \cos\varphi)S^2(k) \tag{4.44}$$

The *Euler parameters* (ε, η) corresponding to R are now defined as

$$\varepsilon = \sin\left(\frac{\varphi}{2}\right)k, \qquad \eta = \cos\left(\frac{\varphi}{2}\right), \tag{4.45}$$

and satisfy

$$\varepsilon^T\varepsilon + \eta^2 = 1 \tag{4.46}$$

It follows that

$$R = (\eta^2 - \varepsilon^T\varepsilon)I_3 + 2\varepsilon\varepsilon^T + 2\eta S(\varepsilon), \tag{4.47}$$

and thus R can be represented as an element (ε, η) of the three-dimensional unit sphere S^3 in $\mathbb{R}^4$. Note that (ε, η) and $(-\varepsilon, -\eta)$ correspond to the same matrix R. In particular, $(0, 1)$ and $(0, -1)$ both correspond to $R = I_3$. Thus the unit sphere S^3 defines a double covering of the matrix group $SO(3)$. In this representation the dynamics (4.43) is given as

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\eta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \eta I_3 + S(\varepsilon) \\ -\varepsilon^T \end{bmatrix}\omega, \tag{4.48}$$

evolving on S^3 in $\mathbb{R}^4$. Define the function $V : S^3 \to \mathbb{R}$ as

$$V(\varepsilon, \eta) = \varepsilon^T\varepsilon + (1 - \eta)^2, \tag{4.49}$$

which by (4.46) is equal to $V(\varepsilon, \eta) = 2(1-\eta)$. Differentiating V along (4.48) yields

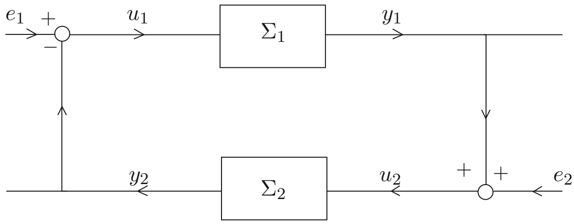

Fig. 4.1 Standard feedback configuration $\Sigma_1 \|_f \Sigma_2$

$$\frac{d}{dt} V = \omega^T \varepsilon \tag{4.50}$$

Hence the dynamics (4.48), with inputs ω and outputs ε, is passive (in fact, lossless) with storage function[4] V. As a consequence, the feedback control $\omega = -\varepsilon$ will asymptotically stabilize the system (4.48) toward $(0, \pm 1)$, that is, $R = I_3$. In Chap. 7 we will see how Examples 4.2.4 and 4.2.5 can be *combined* for the control of the total dynamics of the rigid body described by (4.43), (4.41) with inputs u.

4.3 The Passivity Theorems Revisited

The state space version of the passivity theorems as derived for passive input–output maps in Chap. 2, see in particular Theorem 2.2.11, follows the lines of the general theory of interconnection of dissipative systems as treated in Chap. 3, Sect. 3.3. Let us consider the standard feedback closed-loop system $\Sigma_1 \|_f \Sigma_2$ of Fig. 4.1, which is the same as Fig. 1.1 with the input–output maps G_1 and G_2 replaced by the state space systems

$$\Sigma_i : \begin{array}{ll} \dot{x}_i = f_i(x_i, u_i), & x_i \in \mathcal{X}_i, \quad u_i \in U_i \\ y_i = h_i(x_i, u_i), & \qquad\qquad y_i \in Y_i \end{array} \quad i = 1, 2, \tag{4.51}$$

with $U_1 = Y_2$, $U_2 = Y_1$. Suppose that both Σ_1 and Σ_2 in (4.51) (with $U_1 = U_2 = Y_1 = Y_2$) are *passive* or *output strictly passive*, with storage functions $S_1(x_1)$, respectively $S_2(x_2)$, i.e.,

$$\begin{aligned} S_1(x_1(t_1)) &\leq S_1(x_1(t_0)) + \int_{t_0}^{t_1} (u_1^T(t) y_1(t) - \varepsilon_1 \|y_1(t)\|^2) dt \\ S_2(x_2(t_1)) &\leq S_2(x_2(t_0)) + \int_{t_0}^{t_1} (u_2^T(t) y_2(t) - \varepsilon_2 \|y_2(t)\|^2) dt, \end{aligned} \tag{4.52}$$

with $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ in case of output strict passivity, and $\varepsilon_1 = \varepsilon_2 = 0$ in case of mere passivity. Substituting the standard feedback interconnection equations (see (1.30))

[4] Note that this storage function does *not* have an interpretation in terms of physical energy. It is instead a function that is directly related to the *geometry* of the dynamics (4.48) on S^3, integrating ω.

$$\begin{aligned} u_1 &= e_1 - y_2, \\ u_2 &= e_2 + y_1, \end{aligned} \tag{4.53}$$

the addition of the two inequalities (4.52) results in

$$\begin{aligned} S_1(x_1(t_1)) + S_2(x_2(t_1)) \leq\ & S_1(x_1(t_0)) + S_2(x_2(t_0)) + \\ \int_{t_0}^{t_1} (e_1^T(t)y_1(t) + e_2^T(t)y_2(t)\ -\ & \varepsilon_1 ||y_1(t)||^2 - \varepsilon_2 ||y_2(t)||^2)\, dt \\ \leq\ & S_1(x_1(t_0)) + S_2(x_2(t_0)) + \\ \int_{t_0}^{t_1} (e_1^T(t)y_1(t) + e_2^T(t)y_2(t)\ -\ & \varepsilon[||y_1(t)||^2 + ||y_2(t)||^2])\, dt \end{aligned} \tag{4.54}$$

with $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Hence the closed-loop system with inputs (e_1, e_2) and outputs (y_1, y_2) is output strictly passive if $\varepsilon > 0$, respectively, passive if $\varepsilon = 0$, with storage function

$$S(x_1, x_2) = S_1(x_1) + S_2(x_2)\ , \qquad (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 \tag{4.55}$$

Using Lemmas 3.2.9 and 3.2.16 we arrive at the following proposition, which can be regarded as the state space version of Theorems 2.2.6 and 2.2.11.

Proposition 4.3.1 (Passivity theorem) *Assume that for every pair of allowed external input functions $e_1(\cdot), e_2(\cdot)$ there exist allowed input functions $u_1(\cdot), u_2(\cdot)$ of the closed-loop system $\Sigma_1 \|_f \Sigma_2$.*
(i) Suppose Σ_1 and Σ_2 are passive or output strictly passive. Then $\Sigma_1 \|_f \Sigma_2$ with inputs (e_1, e_2) and outputs (y_1, y_2) is passive, and output strictly passive if both Σ_1 and Σ_2 are output strictly passive.
(ii) Suppose Σ_1 is passive and Σ_2 is input strictly passive, or Σ_1 is output strictly passive and Σ_2 is passive, then $\Sigma_1 \|_f \Sigma_2$ with $e_2 = 0$ and input e_1 and output y_1 is output strictly passive.
(iii) Suppose that S_1, S_2 satisfying (4.52) are C^1 and have strict local minima at x_1^*, respectively x_2^*. Then (x_1^*, x_2^*) is a stable equilibrium of $\Sigma_1 \|_f \Sigma_2$ with $e_1 = e_2 = 0$.
(iv) Suppose that Σ_1 and Σ_2 are output strictly passive and zero-state detectable, and that S_1, S_2 satisfying (4.52) are C^1 and have strict local minima at $x_1^* = 0$, respectively $x_2^* = 0$. Then $(0, 0)$ is an asymptotically stable equilibrium of $\Sigma^f_{\Sigma_1,\Sigma_2}$ with $e_1 = e_2 = 0$. If additionally S_1, S_2 have global minima at $x_1^* = 0$, respectively $x_2^* = 0$, and are proper, then $(0, 0)$ is a globally asymptotically stable equilibrium.

Proof (i) has been proved above, cf. (4.54), while (ii) follows similarly. (iii) results from application of Lemma 3.2.9 to $\Sigma_1 \|_f \Sigma_2$ with inputs (e_1, e_2) and outputs (y_1, y_2). (iv) follows from Proposition 3.2.16 applied to $\Sigma_1 \|_f \Sigma_2$. □

Remark 4.3.2 The standard negative feedback interconnection $u_1 = -y_2 + e_1$, $u_2 = y_1$ for $e_2 = 0$ has the following alternative interpretation. It can be also regarded as the *series interconnection* $u_2 = y_1$ of Σ_1 and Σ_2, together with the additional negative unit feedback loop $u_1 = -y_2 + e_1$. This interpretation will be used in Chap. 5, Theorem 5.2.1.

Remark 4.3.3 Note the inherent *robustness* property expressed in Proposition 4.3.1: the statements continue to hold for perturbed systems Σ_1 and Σ_2, as long as they remain (output strictly) passive and their storage functions satisfy the required properties.

Remark 4.3.4 As in Lemma 3.2.12 the strict positivity of S_1 and S_2 outside $x_1^* = 0, x_2^* = 0$ can be ensured by zero-state observability of Σ_1 and Σ_2.

In case $S_1(x_1) - S_1(x_1^*)$ and/or $S_2(x_2) - S_2(x_2^*)$ are not positive definite but only positive *semi*definite at x_1^*, respectively x_2^*, then Proposition 4.3.1 can be refined as in Theorem 3.2.19. We leave the details to the reader; see also [312].

In Theorem 2.2.18, see also Remark 2.2.19, we have seen how "lack of passivity" of one of the output maps G_1, G_2 can be compensated by "surplus of passivity" of the other. The argument generalizes to the state space setting as follows.

Corollary 4.3.5 *Suppose the systems $\Sigma_i, i = 1, 2$, are dissipative with respect to the supply rates*

$$s_i(u_i, y_i) = u_i^T y_i - \varepsilon_i \|y_i\|^2 - \delta_i \|u_i\|^2, \quad i = 1, 2, \tag{4.56}$$

where the constants $\varepsilon_i, \delta_i, i = 1, 2$, satisfy

$$\varepsilon_1 + \delta_2 > 0, \ \varepsilon_2 + \delta_1 > 0 \tag{4.57}$$

Then the standard feedback interconnection $\Sigma_1 \|_f \Sigma_2$ has finite L_2-gain from inputs e_1, e_2 to outputs y_1, y_2.

Proof Since Σ_i is dissipative with respect to the supply rates s_i we have

$$\dot{S}_i \leq u_i^T y_i - \varepsilon_i \|y_i\|^2 - \delta_i \|u_i\|^2, \quad i = 1, 2 \tag{4.58}$$

for certain storage functions $S_i, i = 1, 2$ (assumed to be differentiable; otherwise use the integral version of the dissipation inequalities). Substitution of $u_1 = e_1 - y_2$, $u_2 = e_2 + y_1$ into the sum of these two inequalities yields

$$\begin{aligned} \dot{S}_1 + \dot{S}_2 \leq\ & e_1^T y_1 + e_2^T y_2 \\ & -\varepsilon_1 \|y_1\|^2 - \delta_1 \|e_1 - y_2\|^2 - \varepsilon_2 \|y_2\|^2 - \delta_2 \|e_2 + y_1\|^2 \end{aligned} \tag{4.59}$$

which, multiplying both sides by -1, can be rearranged as

$$\begin{aligned} & -\delta_1 \|e_1\|^2 - \delta_2 \|e_2\|^2 - \dot{S}_1 - \dot{S}_2 \geq \\ & (\varepsilon_1 + \delta_2)\|y_1\|^2 + (\varepsilon_2 + \delta_1)\|y_2\|^2 - 2\delta_1 e_1^T y_2 - 2\delta_2 e_2^T y_1 - e_1^T y_1 - e_2^T y_2 \end{aligned} \tag{4.60}$$

Then, completely similar to the proof of Theorem 2.2.18, by the positivity assumption on $\alpha_1^2 := \varepsilon_1 + \delta_2, \alpha_2^2 := \varepsilon_2 + \delta_1$ we can perform "completion of the squares" on the right-hand side of the inequality (4.60), to obtain an expression of the form

$$\| \begin{bmatrix} \alpha_1 y_1 \\ \alpha_2 y_2 \end{bmatrix} - A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \|^2 \leq c^2 \| \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \|^2 - \dot{S}_1 - \dot{S}_2, \tag{4.61}$$

for a certain 2×2 matrix A and constant c. In combination with the triangle inequality (2.29) this gives the desired result. □

This corollary is illustrated by the following example, which contains a further interesting extension.

Example 4.3.6 (*Lur'e functions*) Consider an input-state-output system

$$\Sigma_1 : \begin{array}{l} \dot{x}_1 = f(x_1, u_1) \\ y_1 = h(x_1) \end{array} \qquad u_1, y_1 \in \mathbb{R}, \tag{4.62}$$

and a system Σ_2 given by a *static nonlinearity*

$$\Sigma_2 : \; y_2 = F(u_2), \qquad u_2, y_2 \in \mathbb{R}, \tag{4.63}$$

interconnected by negative feedback $u_1 = -y_2$, $u_2 = y_1$.

Suppose the static nonlinearity F is passive in the sense of (4.29), that is, $uF(u) \geq 0$ for all $u \in \mathbb{R}$ (its graph is in the first and third quadrant). Obviously, if Σ_1 is passive with C^1 storage function $S_1(x_1)$, then by a direct application of the passivity theorem (Proposition 4.3.1) the closed-loop system satisfies $\dot{S}_1 \leq 0$.

Now suppose that Σ_1 is *not* passive, but only dissipative with respect to the supply rate

$$s_1(u_1, y_1) = u_1 y_1 + \frac{u_1^2}{k}, \tag{4.64}$$

for some $k > 0$, having C^1 storage function S_1. On the other hand, suppose that F is $\frac{1}{k}$-*output strictly* passive; that is, dissipative with respect to the supply rate

$$s_2(u_2, y_2) = u_2 y_2 - \frac{y_2^2}{k} \tag{4.65}$$

for the same k as above. Then by application of Corollary 4.3.5 the closed-loop system satisfies $\dot{S}_1 \leq 0$. Note that dissipativity of F with respect to s_2 can be equivalently expressed by the *sector condition*

$$0 \leq \frac{F(u_2)}{u_2} \leq k \tag{4.66}$$

The story can be continued as follows. Suppose that Σ_1 is *not* dissipative with respect to s_1, but that instead $\Sigma_{1\alpha}$, defined as

$$\Sigma_{1\alpha} : \begin{array}{l} \dot{x}_1 = f(x_1, u_1) \\ \widehat{y}_1 := y_1 + \alpha \dot{y}_1 = h(x_1) + \alpha \frac{dh}{dx_1}(x_1) f_1(x_1, u_1) \end{array} \tag{4.67}$$

is dissipative with respect to s_1 for some $\alpha > 0$. Suppose as above that the static nonlinearity F satisfies (4.66) (and thus is output strictly passive). Then consider instead of the static nonlinearity Σ_2 defined by F the *dynamical* system

$$\Sigma_{2\alpha} : \begin{array}{rl} \alpha \dot{x}_2 &= -x_2 + u_2, \quad x_2 \in \mathbb{R} \\ y_2 &= F(x_2) \end{array} \tag{4.68}$$

It readily follows that $\Sigma_{2\alpha}$ is dissipative with respect to s_2, with storage function

$$S_2(x_2) := \alpha \int_0^{x_2} F(\sigma) d\sigma \geq 0 \tag{4.69}$$

Indeed, by (4.66)

$$\dot{S}_2 = \alpha F(x_2)\dot{x}_2 = F(x_2)(-x_2 + u_2) \leq u_2 F(x_2) - \frac{F^2(x_2)}{k}$$

Hence, (again by Corollary 4.3.5) the closed-loop system of $\Sigma_{1\alpha}$ and $\Sigma_{2\alpha}$ satisfies $\dot{S}_1 + \dot{S}_2 \leq 0$. Finally note that

$$\alpha \dot{x}_2 + x_2 = u_2 = y_1 + \alpha \dot{y}_1,$$

and thus $\alpha(\dot{x}_2 - \dot{y}_1) = -(x_2 - y_1)$, implying that the level set $x_2 = h(x_1)$ is an (attractive) invariant set. Hence, we can *restrict* the closed-loop system to the level set $x_2 = h(x_1)$, where the system has total storage function

$$S(x_1) := S_1(x_1) + \alpha \int_0^{h_1(x_1)} F(\sigma) d\sigma$$

satisfying $\dot{S} \leq 0$. In case of a *linear* system Σ_1 with quadratic storage function S_1 the obtained function S is called a *Lur'e function*. Depending on the properties of S, we may derive stability, and under strengthened conditions, (global) asymptotic stability, for Σ_1 with the static nonlinearity F in the negative feedback loop. This yields the *Popov criterion*; see the references in the Notes at the end of Chap. 2.

Example 4.3.7 Consider the system

$$\begin{array}{rll} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2, & u_1 \in \mathbb{R}^{m_1}, u_2 \in \mathbb{R}^{m_2} \\ y_1 &= h_1(x), & y_1 \in \mathbb{R}^{m_1} \\ y_2 &= h_2(x), & y_2 \in \mathbb{R}^{m_2} \end{array} \tag{4.70}$$

which is passive with respect to the inputs u_1, u_2 and outputs y_1, y_2, with storage function $S(x)$. Consider the static nonlinearity

$$\perp := \{(v, z) \in \mathbb{R}^{m_2} \times \mathbb{R}^{m_2} \mid v \geq 0,\ z \geq 0,\ v^T z = 0\}, \tag{4.71}$$

where $v \geq 0, z \geq 0$ means that all elements of v, z are nonnegative. Clearly this is a passive static system. Interconnect $\perp$ to the system by setting $u_2 = -z$, $y_2 = v$. The resulting system satisfies

$$\frac{d}{dt} S \leq u_1^T y_1, \tag{4.72}$$

and thus defines a passive system (although not of a standard input-state-output type). This type of systems occurs, e.g., in electrical circuits with ideal diodes; see the Notes at the end of this chapter.

The passivity theorems given so far are one-way: the feedback interconnection of two passive systems is again passive. As we will now see, the *converse* also holds: if the feedback interconnection of two systems is passive then necessarily these systems are passive. This will be shown to have immediate consequences for the set of storage functions of the interconnected system, which always contains an *additive* one.

Proposition 4.3.8 (Converse passivity theorem) *Consider Σ_i with state spaces $\mathcal{X}_1, i = 1, 2$, and with allowed input functions $u_1(\cdot), u_2(\cdot)$, in standard feedback configuration $u_1 = e_1 - y_2, u_2 = e_2 + y_2$. Assume that for every pair of allowed external input functions $e_1(\cdot), e_2(\cdot)$ there exist allowed input functions $u_1(\cdot), u_2(\cdot)$ of the closed-loop system $\Sigma_1 \|_f \Sigma_2$. Conversely, assume that for all allowed input functions $u_1(\cdot), u_2(\cdot)$ there exist allowed external input functions $e_1(\cdot), e_2(\cdot)$ satisfying at any time-instant $u_1 = e_1 - y_2, u_2 = e_2 + y_2$. Then $\Sigma_1 \|_f \Sigma_2$ with inputs e_1, e_2 and outputs y_1, y_2 is passive if and only if both Σ_1 and Σ_2 are passive. Furthermore, the available storage S_a and required supply S_r of $\Sigma_1 \|_f \Sigma_2$ (assuming Σ_i is reachable from some $x_i^*, i = 1, 2$) are additive, that is*

$$\begin{aligned} S_a(x_1, x_2) &= S_{a1}(x_1) + S_{a2}(x_2) \\ S_r(x_1, x_2) &= S_{r1}(x_1) + S_{r2}(x_2) \end{aligned} \tag{4.73}$$

with S_{ai}, S_{ri} denoting the available storage, respectively required supply, of Σ_i, $i = 1, 2$.

Proof The "if" part is Proposition 4.3.1. For the converse statement we note that $\Sigma_1 \|_f \Sigma_2$ is passive if and only

$$S_a(x_1, x_2) := \sup_{e_1(\cdot), e_2(\cdot), T \geq 0} - \int_0^T \left(e_1^T(t) y_1(t) + e_2^T(t) y_2(t) \right) dt < \infty \tag{4.74}$$

for all $(x_1, x_2) \in \mathcal{X}$. Substituting the "inverse" interconnection equations $e_1 = u_1 + y_2$ and $e_2 = u_2 - y_1$ this is equivalent to

$$\sup_{e_1(\cdot), e_2(\cdot), T \geq 0} - \int_0^T \left(u_1^T(t) y_1(t) + u_2^T(t) y_2(t) \right) dt < \infty \tag{4.75}$$

for all (x_1, x_2). Using the assumption that for all allowed $u_1(\cdot), u_2(\cdot)$ there exist allowed external input functions $e_1(\cdot), e_2(\cdot)$ this is equal to

$$\begin{aligned}&\sup_{u_1(\cdot),\, u_2(\cdot),\, T\geq 0} -\int_0^T \left(u_1^T(t)y_1(t) + u_2^T(t)y_2(t)\right) = \\ &\sup_{u_1(\cdot),\, T\geq 0} -\int_0^T u_1^T(t)y_1(t)dt + \sup_{u_2(\cdot),\, T\geq 0} -\int_0^T u_2^T(t)y_2(t)dt < \infty\end{aligned}$$

for all (x_1, x_2). Hence $\Sigma_1\|_f\Sigma_2$ is passive iff Σ_1 and Σ_2 are passive, in which case $S_a(x_1, x_2) = S_{a1}(x_1) + S_{a2}(x_2)$. The same reasoning leads to the second equality of (4.73). □

A similar statement, for any storage function of $\Sigma_1\|_f\Sigma_2$, can be obtained from the differential dissipation inequality as follows.

Proposition 4.3.9 *Consider $\Sigma_i, i = 1, 2$, of the form (4.14) with equilibria $x_i^* \in \mathcal{X}_i$ satisfying $f_i(x_i^*) = 0, i = 1, 2$. Assume that $\Sigma_1\|_f\Sigma_2$ is passive (lossless) with C^1 storage function $S(x_1, x_2)$. Then also $\Sigma_i, i = 1, 2$, are passive (lossless) with storage functions $S_1(x_1) := S(x_1, x_2^*)$, $S_2(x_2) := S(x_1^*, x_2)$.*

Proof We will only prove the passive case; the same arguments apply to the lossless case. $\Sigma_1\|_f\Sigma_2$ being passive is equivalent to the existence of $S : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}^+$ satisfying

$$\begin{aligned}&S_{x_1}(x_1, x_2)\,[f_1(x_1) - g_1(x_1)h_2(x_2)] \\ &+S_{x_2}(x_1, x_2)\,[f_2(x_2) + g_2(x_2)h_1(x_1)] \leq 0 \\ &S_{x_1}(x_1, x_2)g_1(x_1) = h_1^T(x_1) \\ &S_{x_2}(x_1, x_2)g_2(x_2) = h_2^T(x_2)\end{aligned} \tag{4.76}$$

This results in

$$\begin{aligned}&S_{x_1}(x_1, x_2)f_1(x_1) - \underbrace{S_{x_1}(x_1, x_2)g_1(x_1)}_{=h_1^T(x_1)}h_2(x_2) \\ &+ S_{x_2}(x_1, x_2)f_2(x_2) + \underbrace{S_{x_2}(x_1, x_2)g_2(x_2)}_{=h_2^T(x_2)}h_1(x_1) \\ &= S_{x_1}(x_1, x_2)f_1(x_1) + S_{x_2}(x_1, x_2)f_2(x_2) \leq 0\end{aligned} \tag{4.77}$$

For $x_2 = x_2^*$, (4.77) amounts to

$$\begin{aligned}&S_{x_1}(x_1, x_2^*)f_1(x_1) + S_{x_2}(x_1, x_2^*)f_2(x_2^*) \\ &= S_{x_1}(x_1, x_2^*)f_1(x_1) = S_{1x_1}(x_1)f_1(x_1) \leq 0\end{aligned} \tag{4.78}$$

since $f_2(x_2^*) = 0$. Furthermore, the second line of (4.76) becomes

$$S_{1x_1}(x_1)g_1(x_1) = S_{x_1}(x_1, x_2^*)g_1(x_1) = h_1^T(x_1) \tag{4.79}$$

Hence, $S_1(x_1)$ is a storage function for Σ_1. In the same way $S_2(x_2)$ is a storage function for Σ_2. □

An important consequence of Propositions 4.3.8 and 4.3.9 is the fact that among the storage functions of the passive system $\Sigma_1 \|_f \Sigma_2$ there always exist *additive* storage functions $S(x_1, x_2) = S_1(x_1) + S_2(x_2)$. In fact, the available storage and required supply functions are additive by Proposition 4.3.8, while by Proposition 4.3.9 an arbitrary storage function $S(x_1, x_2)$ for $\Sigma_1 \|_f \Sigma_2$ can be replaced by the additive storage function $S(x_1, x_2^*) + S(x_1^*, x_2)$.

4.4 Network Interconnection of Passive Systems

In many complex network systems—from mass–spring–damper systems, electrical circuits, communication networks, chemical reaction networks, and transportation networks to power networks—the passivity of the overall network system naturally arises from the properties of the network interconnection structure and the passivity of the subsystems. In this section this will be illustrated by three different scenarios of network interconnection of passive systems.

The interconnection structure of a network system can be advantageously encoded by a directed *graph*. Recall the following standard notions and facts from (algebraic) graph theory; see [48, 114] and the Notes at the end of the chapter for further information. A *graph* $\mathcal{G}$, is defined by a set $\mathcal{V}$ of N *vertices* (nodes) and a set $\mathcal{E}$ of M *edges* (links, branches), where $\mathcal{E}$ is identified with a set of unordered pairs $\{i, j\}$ of vertices $i, j \in \mathcal{V}$. We allow for multiple edges between vertices, but not for self-loops $\{i, i\}$. By endowing the edges with an orientation, turning the unordered pairs $\{i, j\}$ into ordered pairs (i, j), we obtain a *directed graph*. In the following "graph" will throughout mean "directed graph." A directed graph with N vertices and M edges is specified by its $N \times M$ *incidence matrix*, denoted by D. Every column of D corresponds to an edge of the graph, and contains one -1 at the row corresponding to its tail vertex and one $+1$ at the row corresponding to its head vertex, while all other elements are 0. In particular, $\mathbb{1}^T D = 0$ where $\mathbb{1}$ is the vector of all ones. Furthermore, $\ker D^T = \operatorname{span} \mathbb{1}$ if and only if the graph is *connected* (any vertex can be reached from any other vertex by a sequence of—undirected—edges). In general, the dimension of $\ker D^T$ is equal to the number of connected components of the graph. A directed graph is *strongly connected* if any vertex can be reached from any other vertex by a sequence of directed edges.

The *first* case of network interconnection of passive systems concerns the interconnection of passive systems which are partly associated to the *vertices*, and partly to the *edges* of an underlying graph. As illustrated later on, this is a common case in many physical networks. Thus to each i-th vertex there corresponds a passive system with scalar inputs and outputs (see Remark 4.4.2 for generalizations)

$$\begin{aligned} \dot{x}_i^v &= f_i^v(x_i^v, u_i^v), \quad && x_i^v \in \mathcal{X}_i^v, \; u_i^v \in \mathbb{R} \\ y_i^v &= h_i^v(x_i^v, u_i^v), \quad && y_i^v \in \mathbb{R} \end{aligned} \tag{4.80}$$

with storage function S_i^v, $i = 1, \ldots, N$, and to each j-th edge (branch) there corresponds a passive single-input single-output system

$$\begin{aligned} \dot{x}_i^b &= f_i^b(x_i^b, u_i^b), \quad x_i^b \in \mathcal{X}_i^b,\ u_i^b \in \mathbb{R} \\ y_i^b &= h_i^b(x_i^b, u_i^b), \quad y_i^b \in \mathbb{R} \end{aligned} \tag{4.81}$$

with storage function S_i^b, $i = 1, \ldots, M$. Collecting the scalar inputs and outputs into vectors

$$\begin{aligned} u^v &= \left[u_1^v, \ldots, u_N^v\right]^T, \quad & y^v &= \left[y_1^v, \ldots, y_N^v\right]^T \\ u^b &= \left[u_1^b, \ldots, u_M^b\right]^T, \quad & y^b &= \left[y_1^b, \ldots, y_M^b\right]^T \end{aligned} \tag{4.82}$$

these passive systems are interconnected to each other by the interconnection equations

$$\begin{aligned} u^v &= -Dy^b + e^v \\ u^b &= D^T y^v + e^b \end{aligned} \tag{4.83}$$

where $e^v \in \mathbb{R}^N$ and $e^b \in \mathbb{R}^M$ are external inputs. Since the interconnection (4.83) satisfies

$$(u^v)^T y^v + (u^b)^T y^b = (e^v)^T y^v + (e^b)^T y^b$$

the following result directly follows.

Proposition 4.4.1 *Consider a graph with incidence matrix D, with passive systems (4.80) with storage functions S_i^v associated to the vertices and passive systems (4.81) with storage functions S_i^b associated to the edges, interconnected by (4.83). Then the interconnected system is again passive with inputs e^v, e^b and outputs y^v, y^b, with total storage function*

$$S_1^v(x_1^v) + \cdots + S_N^v(x_N^v) + S_1^b(x_1^b) + \cdots + S_1^b(x_M^b) \tag{4.84}$$

Remark 4.4.2 The setup can be generalized to multi-input multi-output systems with $u_i^v, y_i^v, u_j^b, y_j^b$ all in $\mathbb{R}^m$ by replacing the incidence matrix D in the above by the Kronecker product $D \otimes I_m$ and D^T by $D^T \otimes I_m$, with I_m denoting the $m \times m$ identity matrix.

Remark 4.4.3 Proposition 4.4.1 continues to hold in cases where some of the edges or vertices correspond to *static* passive systems. Simply define the total storage function as the sum of the storage functions of the *dynamic* passive systems.

Example 4.4.4 (*Power networks*) Consider a power system of synchronous machines, interconnected by a network of purely inductive transmission lines. Modeling the synchronous machines by swing equations, and assuming that all voltage and current signals are sinusoidal of the same frequency and all voltages have constant amplitude one arrives at the following model. Associated to the N vertices each i-th synchronous machine is described by the passive system

$$\begin{aligned} \dot{p}_i &= -A_i \omega_i + u_i^v \\ y_i^v &= \omega_i \end{aligned} \tag{4.85}$$

where ω_i is the frequency deviation from nominal frequency (e.g., 50 Hz), $p_i = J_i \omega_i$ is the momentum deviation (with J_i related to the inertia of the synchronous machine), A_i the damping constant, and u_i^v is the incoming power, $i = 1, \ldots, N$. Furthermore, denoting the phase differences across the j-th line by q_j, the dynamics of the j-th line (associated to the j-th edge of the graph) is given by the passive system

$$\begin{aligned} \dot{q}_j &= u_j^b \\ y_j^b &= \gamma_j \sin q_j \end{aligned} \tag{4.86}$$

with the constant γ_j determined by the susceptance of the line and the voltage amplitude at the adjacent vertices, $j = 1, \ldots, M$. Here y_j^b equals the (average or active) power through the line. Denoting $p = (p_1, \ldots, p_N)^T$, $\omega = (\omega_1, \ldots, \omega_N)^T$, and $q = (q_1, \ldots, q_M)^T$, the final system resulting from the interconnection (4.83) is given as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \Gamma \operatorname{Sin} q \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = J\omega \\ y &= \omega, \end{aligned} \tag{4.87}$$

with A and J denoting diagonal matrices with elements A_i, J_i, $i = 1, \ldots, N$, and Γ the diagonal matrix with elements γ_j, $j = 1, \ldots, M$. Furthermore $\operatorname{Sin} : \mathbb{R}^M \to \mathbb{R}^M$ denotes the element-wise sinus function, i.e., $\operatorname{Sin} q = (\sin q_1, \ldots, \sin q_M)$. Finally, the input u denotes the vector of generated/consumed power and the output y the vector of frequency deviations, both associated to the vertices. The final system (4.87) is a passive system with additive storage function

$$H(q, p) := \frac{1}{2} p^T J^{-1} p - \sum_{j=1}^{M} \gamma_j \cos q_j \tag{4.88}$$

Example 4.4.5 (*Mass-spring systems*) Consider N masses moving in one-dimensional space interconnected by M springs. Associate the masses to the vertices of a graph with incidence matrix D, and the springs to the edges. Furthermore, let $p_1, \ldots, p_N$ be the momenta of the masses, and $q_1, \ldots, q_M$ the extensions of the springs. Then the equations of motion of the total system are given as

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -D \\ D^T & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial K}{\partial p}(p) \\ \frac{\partial P}{\partial q}(q) \end{bmatrix} + \begin{bmatrix} e^v \\ e^b \end{bmatrix}, \tag{4.89}$$

where $p = (p_1, \ldots, p_N)^T$ and $q = (q_1, \ldots, q_M)^T$, and where $K(p) = \sum \frac{1}{2m_i} p_i^2$ is the total kinetic energy of the masses, and $P(q)$ the total potential energy of the springs. This defines a passive system with inputs e^v, e^b (external forces, respectively, external velocity flows) and outputs $\frac{\partial K}{\partial p}(p)$, $\frac{\partial P}{\partial q}(q)$ (velocities, respectively, spring forces), and additive storage function $K(p) + P(q)$.

Similar to Remark 4.4.2 this can be generalized to a mass-spring system in $\mathbb{R}^3$, by considering $p_i, q_j \in \mathbb{R}^3$, and replacing the incidence matrix D by the Kronecker product $D \otimes I_3$ and D^T by $D^T \otimes I_3$. Furthermore, by Remark 4.4.3 the setup can be extended to *mass–spring–damper systems*, in which case part of the edges correspond to *dampers*.

In Chap. 6 we will see how Examples 4.4.4 and 4.4.5 actually define passive *port-Hamiltonian* systems.

A *second* case of network interconnection of passive systems is that of a multi-agent system, where the input of each passive agent system depends on the outputs of the other systems and of itself. Thus consider N passive systems Σ_i associated to the vertices of a graph, given by

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i), \quad x_i \in \mathcal{X}_i, \ u_i \in \mathbb{R} \\ y_i &= h_i(x_i, u_i), \quad y_i \in \mathbb{R} \end{aligned} \tag{4.90}$$

with storage functions $S_i, i = 1, \dots, N$. Collecting the inputs into the vector $u = (u_1, \dots, u_N)^T$ and the outputs into $y = (y_1, \dots, y_N)^T$ we consider interconnection equations

$$u = -Ly + e \tag{4.91}$$

where e is a vector of external inputs, and L is a *Laplacian matrix*, defined as follows.

Definition 4.4.6 A Laplacian matrix of a graph with N vertices is defined as an $N \times N$ matrix L with positive diagonal elements, and non-positive off-diagonal elements, with either the row sums of L equal to zero (a *communication Laplacian* matrix) or the column sums equal to zero (*flow Laplacian* matrix). If both the row and sums are zero then L is called a *balanced Laplacian* matrix.

This means that any communication Laplacian L_c satisfies $L_c\mathbb{1} = 0$, and can be written as $L_c = -K_c D^T$ for an incidence matrix D of the communication graph, and a matrix K_c of nonnegative elements. In fact, the nonzero elements of the i-th row of K_c are the weights of the edges incoming to vertex i. Dually, any flow Laplacian L_f satisfies $\mathbb{1}^T L_f = 0$, and can be written as $L_f = -DK_f$ for a certain incidence matrix, and a matrix K_f of nonnegative elements. The nonzero elements of the i-th column of K_f are the weights of the edges originating from vertex i.

A communication Laplacian matrix L_c, respectively flow Laplacian matrix L_f is balanced if and only [70]

$$L_c + L_c^T \geq 0, \ \text{respectively,} \ L_f + L_f^T \geq 0 \tag{4.92}$$

Remark 4.4.7 A special case of a balanced Laplacian matrix is a *symmetric* balanced Laplacian matrix L, which can be written as $L = DKD^T$, where D is the incidence matrix and K is an $M \times M$ diagonal matrix of positive weights corresponding to the M edges of the graph.

Remark 4.4.8 The interconnection (4.91) with L a communication Laplacian matrix corresponds to feeding back the *differences* of the output values

$$u_i = -\sum_k a_{ik}(y_i - y_k), \quad i = 1, \dots, N, \tag{4.93}$$

where the summation index k is running over all vertices that are connected to the i-th vertex by an edge directed toward i, and a_{ik} is the positive weight of this edge. On the other hand, the interconnection (4.91) with L a flow Laplacian matrix corresponds to an output feedback satisfying $\mathbb{1}^T u = 0$, corresponding to a distribution of the material flow through the network. This occurs for transportation and distribution networks, including chemical reaction networks.

Proposition 4.4.9 *Consider the passive systems (4.90) interconnected by (4.91), where L is a balanced Laplacian matrix. Then the interconnected system is passive with additive storage function $S_1(x_1) + \cdots + S_N(x_N)$.*

Proof Follows from the fact that by (4.92)

$$u^T y = -(Ly + e)^T y = -\frac{1}{2} y^T (L + L^T) y + e^T y \leq e^T y \qquad \square$$

Proposition 4.4.9 can be generalized to flow and communication Laplacian matrices that are *not balanced* by additionally assuming that the connected components of the underlying graph are *strongly connected*.[5] In fact, under this assumption, any flow or communication Laplacian matrix can be *transformed* into a balanced one. Furthermore, this can be done in a constructive way by employing a general form of *Kirchhoff's Matrix Tree theorem*, which for our purposes can be described as follows (see the Notes at the end of this chapter).

Let L be a *flow* Laplacian matrix, and assume for simplicity that the graph is connected, implying that $\dim \ker L = 1$. Denote the (i, j)-th cofactor of L by $C_{ij} = (-1)^{i+j} M_{i,j}$, where $M_{i,j}$ is the determinant of the (i, j)-th minor of L, which is the matrix obtained from L by deleting its i-th row and j-th column. Define the adjoint matrix $\mathrm{adj}(L)$ as the matrix with (i, j)-th element given by C_{ji}. It is well known that

$$L \cdot \mathrm{adj}(L) = (\det L) I_N = 0 \tag{4.94}$$

[5] In fact, balancedness of a communication or flow Laplacian matrix *implies* that all connected components are strongly connected; cf. [70].

Furthermore, since $\mathbb{1}^T L = 0$ the sum of the rows of L is zero, and hence by the properties of the determinant function the quantities C_{ij} do not depend on i, implying that $C_{ij} = \gamma_j,\ i = 1, \dots, N$. Hence, by defining $\gamma := (\gamma_1, \dots, \gamma_N)^T$, it follows from (4.94) that $L\gamma = 0$. Moreover, γ_i is equal to *the sum of the products of weights of all the spanning trees of $\mathcal{G}$ directed toward* vertex i. In particular, it follows that $\gamma_j \geq 0,\ j = 1, \dots, N$. In fact, $\gamma \neq 0$ if and only if $\mathcal{G}$ has a spanning tree. Since for every vertex i there exists at least one spanning tree directed toward i if and only if the graph is strongly connected, we conclude that $\gamma \in \mathbb{R}^N_+$ if and only if the graph is strongly connected.

In case the graph $\mathcal{G}$ is not connected the same analysis can be performed on each of its connected components. Hence, if all connected components of $\mathcal{G}$ are strongly connected, Kirchhoff's matrix tree theorem provides us with a vector $\gamma \in \mathbb{R}^N_+$ such that $L\gamma = 0$. It immediately follows that the transformed matrix $L\Gamma$, where Γ is the positive $N \times N$-dimensional diagonal matrix with diagonal elements $\gamma_1, \dots, \gamma_N$, is a balanced Laplacian matrix.

Dually, if L is a communication Laplacian matrix and the connected components of the graph are strongly connected, then there exist a positive $N \times N$ diagonal matrix Γ such that ΓL is balanced. Summarizing, we obtain the following.

Proposition 4.4.10 *Consider a flow Laplacian matrix L_f (communication Laplacian matrix L_c). Then there exists a positive diagonal matrix Γ_f (Γ_c) such that $L_f \Gamma_f$ ($\Gamma_c L_c$) is balanced if and only if the connected components of the graph are all strongly connected.*

This has the following consequence for the passivity of the interconnection of passive systems Σ_i, $i = 1, \dots, N$, under the interconnection (4.91).

Proposition 4.4.11 *Consider passive systems $\Sigma_1, \dots, \Sigma_N$ with storage functions $S_1, \dots, S_N$, interconnected by $u = -Ly + e$, where L is either a flow Laplacian L_f or a communication Laplacian L_c, and assume that the connected components of the interconnection graph are strongly connected. Let L_f be a flow Laplacian, and consider a positive diagonal matrix $\Gamma_f = \mathrm{diag}(\gamma_1^f, \dots, \gamma_N^f)$ such that $L_f \Gamma_f$ is balanced. Then the interconnected system with inputs e and scaled outputs $\frac{1}{\gamma_1^f} y_1, \dots, \frac{1}{\gamma_N^f} y_N$ is passive with storage function*

$$S^f(x_1, \dots, x_N) := \frac{1}{\gamma_1^f} S_1(x_1) + \cdots + \frac{1}{\gamma_N^f} S_N(x_N) \tag{4.95}$$

Alternatively, let L_c be a communication Laplacian, and consider a positive diagonal matrix $\Gamma_c = \mathrm{diag}(\gamma_1^c, \dots, \gamma_N^c)$ such that $\Gamma_c L_c$ is balanced. Then the interconnected system with inputs e and scaled outputs $\gamma_1^c y_1, \dots, \gamma_N^c y_N$, is passive with storage function

$$S^c(x_1, \dots, x_N) := \gamma_1^c S_1(x_1) + \cdots + \gamma_N^c S_N(x_N) \tag{4.96}$$

Proof The first statement follows by passivity from

$$\begin{aligned}\tfrac{d}{dt}S^f &\leq y^T\Gamma_f^{-1}u = -y^T\Gamma_f^{-1}L_f y + y^T\Gamma_f^{-1}e \\ &= -(\Gamma_f^{-1}y)^T L_f\Gamma_f(\Gamma_f^{-1}y) + (\Gamma_f^{-1}y)^T e\end{aligned} \tag{4.97}$$

and balancedness of $L_f\Gamma_f$. Similarly, the second statement follows from

$$\frac{d}{dt}S^c \leq y^T\Gamma_c u = -y^T\Gamma_c L_c y + y^T\Gamma_c e \tag{4.98}$$

and balancedness of $\Gamma_c L_c$. □

Remark 4.4.12 The result continues to hold in case some of the systems Σ_i are *static* passive nonlinearities. Indeed, since for each j-th static passive nonlinearity $u_j y_j \geq 0$, the same inequalities continue to hold, with the storage functions S^f or S^c now being the weighted sum of the storage functions of the *dynamical* passive systems Σ_i.

Remark 4.4.13 The notion of a balanced Laplacian matrix is also instrumental in defining the *effective resistance* from one vertex of the connected network to another. In fact, let L be a balanced Laplacian matrix. For any vertex i and j note that $e_i - e_j \in \text{im}\, L$, where e_i and e_j are the standard basis vectors with 1 at the i-th or j-th element, and 0 everywhere else. Thus there exists a vector v satisfying

$$Lv = e_i - e_j, \tag{4.99}$$

which is moreover unique up to addition of a multiple of the vector $\mathbb{1}$ of all ones. This means that the quantity

$$R_{ji} := v_i - v_j, \tag{4.100}$$

is independent of the choice of v satisfying (4.99). It is called the *effective resistance* of the network from vertex j to vertex i.

The same idea of taking *weighted combinations* of storage functions is used in the following *third* case of interconnection of passive systems. Consider again a multi-agent system, composed of N passive agent systems Σ_i with scalar inputs and outputs u_i, y_i, and storage functions $S_i(x_i)$, $i = 1, \ldots, N$. These are interconnected by

$$u = Ky + e \tag{4.101}$$

where $u = (u_1, \ldots, u_N)^T$, $y = (y_1, \ldots, y_N)^T$, and the $N \times N$ matrix K has the following special structure:

$$K = \begin{bmatrix} -\alpha_1 & 0 & \cdot & \cdot & 0 & -\beta_N \\ \beta_1 & -\alpha_2 & \cdot & \cdot & 0 & 0 \\ 0 & \beta_2 & -\alpha_3 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \beta_{N-2} & -\alpha_{N-1} & 0 \\ 0 & 0 & \cdot & 0 & \beta_{N-1} & -\alpha_N \end{bmatrix} \tag{4.102}$$

for positive constants $\alpha_i, \beta_i, i = 1, \ldots, N$. This represents a circular graph, where the first $N-1$ gains $\beta_1, \ldots, \beta_{N-1}$ are *positive*, but the last interconnection gain $-\beta_N$ (from vertex N to vertex 1) is *negative*.

The main differences with the case $u = -Ly + e$ considered before, where L is either a flow or communication Laplacian matrix, are the special structure of the graph (a circular graph instead of a general graph), the fact that the right-upper element of K, given by $-\beta_N$, is *negative*, and the fact neither the row or column sums of K are zero. Nevertheless, also the matrix K can be transformed by a diagonal matrix into a matrix satisfying a property similar to (4.92), provided the constants $\alpha_i, \beta_i, i = 1, \ldots, N$, satisfy the following condition.

Theorem 4.4.14 ([12]) *Consider the $N \times N$ matrix K given in (4.102). There exists a positive $N \times N$ diagonal matrix Γ such that $\Gamma K + K^T \Gamma < 0$ if and only the positive constants $\alpha_i, \beta_i, i = 1, \ldots, N$, satisfy*[6]

$$\frac{\beta_1 \cdots \beta_N}{\alpha_1 \cdots \alpha_N} < \sec\left(\frac{\pi}{N}\right)^N \tag{4.103}$$

The condition (4.103) is referred to as the *secant condition*. Proceeding in the same way as for the Laplacian matrix interconnection case we obtain the following interconnection result.

Proposition 4.4.15 *Consider passive systems $\Sigma_1, \ldots, \Sigma_N$ with storage functions $S_1, \ldots, S_N$, interconnected by $u = -Ky + e$, where K is given by (4.102) with $\alpha_i, \beta_i, i = 1, \ldots, N$, satisfying (4.103). Take any positive diagonal matrix $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_N)$ such that $\Gamma K + K^T \Gamma < 0$. Then the interconnected system with inputs e and scaled outputs $\gamma_1 y_1, \ldots, \gamma_N y_N$ is output strictly passive with storage function*

$$S^K(x_1, \ldots, x_N) := \gamma_1 S_1(x_1) + \cdots + \gamma_N S_N(x_N) \tag{4.104}$$

Proof This follows from

$$\frac{d}{dt} S^K \leq y^T \Gamma u = y^T \Gamma K y + y^T \Gamma e = y^T \Gamma K y + y^T \Gamma e \tag{4.105}$$

and $\Gamma K + K^T \Gamma < 0$. □

Remark 4.4.16 The stability of the interconnected system can be alternatively considered from the *small-gain* point of view; cf. Chaps. 2 and 8. Indeed, the interconnected system can be also formulated as the circular interconnection, with gains $+1$ for the first $N-1$ interconnections and gain -1 for the interconnection from vertex N to vertex 1, of the *modified* systems $\widehat{\Sigma}_i$ with inputs v_i and outputs $\widehat{y}_i$ obtained from Σ_i by substituting $u_i = -\alpha_i y_i + v_i$, $\widehat{y}_i = \beta_i y_i$, $i = 1, \ldots, N$. Then by output strict

[6] Note that the secant function is given as $\sec\phi = \frac{1}{\cos\phi}$.

passivity of $\widehat{\Sigma}_i$ the L_2-gain of $\widehat{\Sigma}_i$ is $\leq \frac{\alpha_i}{\beta_i}$. Application of the small-gain condition, cf. Chap. 8, then yields stability for all $\alpha_i, \beta_i, i = 1, \ldots, N$, satisfying (4.103) with the right-hand side replaced by 1. This latter condition is however (much) *stronger* than (4.103). For instance, $\sec(\frac{\pi}{N})^N = 8$ for $N = 3$.

4.5 Passivity of Euler–Lagrange Equations

A standard method for deriving the equations of motion for physical systems is via the *Euler–Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \tau, \tag{4.106}$$

where $q = (q_1, \ldots, q_n)^T$ are generalized configuration coordinates for the system with n degrees of freedom, L is the Lagrangian function,[7] and $\tau = (\tau_1 \ldots, \tau_n)^T$ is the vector of generalized forces acting on the system. Furthermore, $\frac{\partial L}{\partial \dot{q}}(q, \dot{q})$ denotes the column vector of partial derivatives of $L(q, \dot{q})$ with respect to the generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$, and similarly for $\frac{\partial L}{\partial q}(q, \dot{q})$.

By defining the vector of generalized *momenta* $p = (p_1, \ldots, p_n)^T$ as

$$p := \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \tag{4.107}$$

and assuming that the map $\dot{q} \mapsto p$ is invertible for every q, this defines the $2n$-dimensional state vector $(q_1, \ldots, q_n, p_1, \ldots, p_n)^T$, in which case the n *second-order* equations (4.106) transform into $2n$ *first-order* equations

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q, p) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q, p) + \tau, \end{aligned} \tag{4.108}$$

where the *Hamiltonian function* H is the Legendre transform of L, defined implicitly as

$$H(q, p) = p^T \dot{q} - L(q, \dot{q}), \quad p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \tag{4.109}$$

Equation (4.108) are called the *Hamiltonian equations* of motion. In physical systems the Hamiltonian H usually can be identified with the total energy of the system. It immediately follows from (4.108) that

[7] Not to be confused with the Laplacian matrix of the previous section; too many mathematicians with a name starting with "L."

$$\begin{aligned}\frac{d}{dt}H &= \frac{\partial^T H}{\partial q}(q,p)\dot{q} + \frac{\partial^T H}{\partial p}(q,p)\dot{p} \\ &= \frac{\partial^T H}{\partial p}(q,p)\tau \;=\; \dot{q}^T\tau,\end{aligned} \tag{4.110}$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*). This directly translates into the following statement regarding passivity (in fact, losslessness) of the Hamiltonian and Euler–Lagrange equations.

Proposition 4.5.1 *Assume the Hamiltonian H is bounded from below, i.e., $\exists\, C > -\infty$ such that $H(q,p) \geq C$. Then (4.106) with state vector $(q,\dot{q})$, and (4.108) with state vector (q,p), are lossless systems with respect to the supply rate $y^T\tau$, with output $y = \dot{q}$ and storage function $E(q,\dot{q}) := H(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q})) - C$, respectively $H(q,p) - C$.*

Proof Clearly $H(q,p) - C \geq 0$. The property of being lossless directly follows from (4.110). □

Remark 4.5.2 If the map from $\dot{q}$ to p is *not* invertible this means that there are algebraic constraints $\phi_i(q,p) = 0, i = 1,\dots,k$, relating the momenta p, and that the Hamiltonian $H(q,p)$ is only defined up to addition with an arbitrary combination of the constraint functions $\phi_i(q,p), i = 1,\dots,k$. This leads to a *constrained* Hamiltonian representation; see the Notes at the end of this chapter for further information.

The Euler–Lagrange equations (4.106) describe dynamics without internal energy dissipation, resulting in losslessness. The equations can be extended to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) + \frac{\partial R}{\partial \dot{q}}(\dot{q}) = \tau, \tag{4.111}$$

where $R(\dot{q})$ is a *Rayleigh dissipation* function, satisfying

$$\dot{q}^T\frac{\partial R}{\partial \dot{q}}(\dot{q}) \;\geq\; 0, \quad \text{for all } \dot{q} \tag{4.112}$$

Then the time evolution of $H(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}))$ satisfies

$$\frac{d}{dt}H = -\dot{q}^T\frac{\partial R}{\partial \dot{q}}(\dot{q}) + \dot{q}^T\tau \tag{4.113}$$

Hence if H is bounded from below, then, similar to Proposition 4.5.1, the systems (4.111) and (4.112) with inputs τ and outputs $\dot{q}$ are passive.

We may interpret (4.111) as the closed-loop system depicted in Fig. 4.2. Equation (4.111) thus can be seen as the feedback interconnection of the lossless system Σ_1 given by the Euler–Lagrange equations (4.106) with input τ', and the static passive system Σ_2 given by the map $\dot{q} \mapsto \frac{\partial R}{\partial \dot{q}}(\dot{q})$. If (4.112) is strengthened to

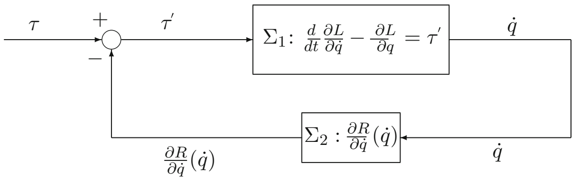

Fig. 4.2 Feedback representation of (4.111)

$$\dot q^T \frac{\partial R}{\partial \dot q}(\dot q) \;\geq\; \delta||\dot q||^2 \tag{4.114}$$

(assuming an inner product structure on the output space of generalized velocities) for some $\delta > 0$, then the nonlinearity (4.114) defines an δ-input strictly passive map from $\dot q$ to $\frac{\partial R}{\partial \dot q}(\dot q)$, and (4.111) with output $\dot q$ becomes output strictly passive; as also follows from Proposition 4.3.1(ii).

Furthermore, we can apply Theorem 2.2.15 as follows. Consider any initial condition $(q(0), \dot q(0))$, and the corresponding input–output map of the system Σ_1. Assume that for any $\tau \in L_{2e}(\mathbb{R}^n)$ there are solutions $\tau' = \frac{\partial R}{\partial \dot q}(\dot q), \dot q \in L_{2e}(\mathbb{R}^n)$. Then the map $\tau \mapsto \dot q$ has L_2-gain $\leq \frac{1}{\delta}$. In particular, if $\tau \in L_2(\mathbb{R}^n)$ then $\dot q \in L_2(\mathbb{R}^n)$. Note that not necessarily the signal $\frac{\partial R}{\partial \dot q}(\dot q)$ will be in $L_2(\mathbb{R}^n)$; in fact this will depend on the properties of the Rayleigh function R.

Finally, (4.113) for $\tau = 0$ yields

$$\frac{d}{dt}H = -\dot q^T \frac{\partial R}{\partial \dot q}(\dot q) \tag{4.115}$$

Hence, if we assume that H has a *strict minimum* at some some point $(q_0, 0)$, and by (4.114) and La Salle's invariance principle, $(q_0, 0)$ will be an asymptotically stable equilibrium of the system whenever R is such that $\dot q^T \frac{\partial R}{\partial \dot q}(\dot q) =$ if and only if $\dot q = 0$ (in particular, if (4.114) holds).

4.6 Passivity of Second-Order Systems and Riemannian Geometry

In standard mechanical systems the Lagrangian function $L(q, \dot q)$ is given by the difference

$$L(q, \dot q) = \frac{1}{2}\dot q^T M(q)\dot q - P(q) \tag{4.116}$$

of the *kinetic energy* $\frac{1}{2}\dot q^T M(q)\dot q$ and the *potential energy* $P(q)$. Here $M(q)$ is an $n \times n$ inertia (generalized mass) matrix, which is symmetric and positive definite for all q. It follows that the vector of generalized momenta is given as $p = M(q)\dot q$, and thus that the map from $\dot q$ to $p = M(q)\dot q$ is invertible. Furthermore, the resulting

Hamiltonian H is given as

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + P(q), \tag{4.117}$$

which equals the *total energy* (kinetic energy plus potential energy).

It turns out to be of interest to work out the Euler–Lagrange equations (4.106) and the property of conservation of total energy in more detail for this important case. This will lead to a direct connection to the passivity of a "*virtual system*" that can be associated to the Euler–Lagrange equations, and which has a clear geometric interpretation.

Let $m_{ij}(q)$ be the (i, j)-th element of $M(q)$. Writing out

$$\frac{\partial L}{\partial \dot{q}_k}(q, \dot{q}) = \sum_j m_{kj}(q)\dot{q}_j$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}(q, \dot{q})\right) &= \sum_j m_{kj}(q)\ddot{q}_j + \sum_j \frac{d}{dt} m_{kj}(q)\dot{q}_j \\ &= \sum_j m_{kj}(q)\ddot{q}_j + \sum_{i,j} \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j, \end{aligned}$$

as well as

$$\frac{\partial L}{\partial q_k}(q, \dot{q}) = \frac{1}{2}\sum_{i,j} \frac{\partial m_{ij}}{\partial q_k}(q)\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}(q),$$

the Euler–Lagrange equations (4.106) for $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q)$ take the form

$$\sum_j m_{kj}(q)\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial m_{kj}}{\partial q_i}(q) - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right\}(q)\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}(q) = \tau_k,$$

for $k = 1, \dots, n$. Furthermore, since

$$\sum_{i,j} \frac{\partial m_{kj}}{\partial q_i}(q)\dot{q}_i\dot{q}_j = \sum_{i,j} \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial q_i}(q) + \frac{\partial m_{ki}}{\partial q_j}\right\}(q)\dot{q}_i\dot{q}_j\,,$$

by defining the *Christoffel symbols* of the first kind

$$c_{ijk}(q) \;:=\; \frac{1}{2}\left\{\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right\}(q)\,, \tag{4.118}$$

we can further rewrite the Euler–Lagrange equations as

$$\sum_j m_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k}(q) = \tau_k\,, \qquad k = 1, \ldots, n,$$

or, more compactly,

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial P}{\partial q}(q) = \tau\,, \tag{4.119}$$

where the (k, j)-th element of the matrix $C(q, \dot{q})$ is defined as

$$c_{kj}(q) = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\,. \tag{4.120}$$

In a mechanical system context the forces $C(q, \dot{q})\dot{q}$ in (4.119) correspond to the *centrifugal and Coriolis forces*.

The definition of the Christoffel symbols leads to the following important observation. Adopt the notation $\dot{M}(q)$ for the $n \times n$ matrix with (i, j)-th element given by $\dot{m}_{ij}(q) = \frac{d}{dt}m_{ij}(q) = \sum_k \frac{\partial m_{ij}}{\partial q_k}(q)\dot{q}_k$.

Lemma 4.6.1 *The matrix*

$$\dot{M}(q) - 2C(q, \dot{q}) \tag{4.121}$$

is skew-symmetric for every $q, \dot{q}$.

Proof Leaving out the argument q, the (k, j)-th element of (4.121) is given as

$$\begin{aligned} \dot{m}_{kj} - 2c_{kj} &= \sum_{i=1}^{n}\left[\frac{\partial m_{kj}}{\partial q_i} - \left\{\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right\}\right]\dot{q}_i \\ &= \sum_{i=1}^{n}\left[\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j}\right]\dot{q}_i \end{aligned}$$

which changes sign if we interchange k and j. □

The skew-symmetry of $\dot{M}(q) - 2C(q, \dot{q})$ is another manifestation of the fact that the forces $C(q, \dot{q})\dot{q}$ in (4.119) are *workless*. Indeed by direct differentiation of the total energy $E(q, \dot{q}) := \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ along (4.119) one obtains

$$\begin{aligned} \tfrac{d}{dt}H &= \dot{q}^T M(q)\ddot{q} + \tfrac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T \tfrac{\partial P}{\partial q}(q) \\ &= \dot{q}^T\tau + \tfrac{1}{2}\dot{q}^T\big(\dot{M}(q) - 2C(q, \dot{q})\big)\dot{q} \;=\; \dot{q}^T\tau, \end{aligned} \tag{4.122}$$

in accordance with (4.110).

However, skew-symmetry of $\dot{M}(q) - 2C(q, \dot{q})$ is actually a *stronger* property than energy conservation. In fact, if we choose the matrix $C(q, \dot{q})$ different from the matrix of Christoffel symbols (4.116), i.e., as some other matrix $\tilde{C}(q, \dot{q})$ such that

$$\tilde{C}(q, \dot{q})\dot{q} = C(q, \dot{q})\dot{q} , \qquad \text{for all } q, \dot{q} , \tag{4.123}$$

then still $\dot{q}^T(\dot{M}(q) - 2\tilde{C}(q, \dot{q}))\dot{q} = 0$ (conservation of energy), but in general $\dot{M}(q) - 2\tilde{C}(q, \dot{q})$ will *not* be skew-symmetric anymore.

This observation is underlying the following developments. Start out from Eq. (4.119) for zero potential energy P and the vector of external forces τ denoted by u, that is

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = u \tag{4.124}$$

Definition 4.6.2 The *virtual system* associated to (4.124) is defined as the *first-order* system in the state vector $s \in \mathbb{R}^n$

$$\begin{aligned} M(q)\dot{s} + C(q, \dot{q})s &= u \\ y &= s \end{aligned} \tag{4.125}$$

with inputs $u \in \mathbb{R}^n$ and outputs $y \in \mathbb{R}^n$, *parametrized* by the vector $q \in \mathbb{R}^n$ and its time-derivative $\dot{q} \in \mathbb{R}^n$.

Thus for *any* curve $q(\cdot)$ and corresponding values $q(t)$, $\dot{q}(t)$ for all t, we may consider the time-varying system (4.125) with state vector s. Clearly, any solution $q(\cdot)$ of the Euler–Lagrange equations (4.124) for a certain input function $\tau(\cdot)$ generates the solution $s(t) := \dot{q}(t)$ to the virtual system (4.125) for $u = \tau$, but on the other hand *not* every pair $q(t), s(t)$, with $s(t)$ a solution of (4.125) parametrized by $q(t)$, corresponds to a solution of (4.124). In fact, this is only the case if additionally $s(t) = \dot{q}(t)$. This explains the name *virtual system*.

Remarkably, not only the Euler–Lagrange equations (4.124) are lossless with respect to the output $y = \dot{q}$, but also the virtual system (4.125) turns out to be lossless with respect to the output $y = s$, *for every time-function* $q(\cdot)$. This follows from the following computation, crucially relying on the skew-symmetry of $\dot{M}(q) - 2C(q, \dot{q})$. Define the storage function of the virtual system (4.125) as the following function of s, parametrized by q

$$S(s, q) := \frac{1}{2} s^T M(q) s \tag{4.126}$$

Then, by skew-symmetry of $\dot{M} - 2C$, along (4.125)

$$\begin{aligned} \tfrac{d}{dt} S(s, q) &= s^T M(q)\dot{s} + \tfrac{1}{2} s^T \dot{M}(q) s \\ &= -s^T C(q, \dot{q}) s + \tfrac{1}{2} s^T \dot{M}(q) s + s^T u = s^T u \end{aligned} \tag{4.127}$$

This is summarized in the following proposition.

Proposition 4.6.3 *For any curve $q(\cdot)$ the virtual system (4.125) with input u and output y is lossless, with parametrized storage function $S(s, q) = \frac{1}{2}s^T M(q)s$.*

This can be directly extended to

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial R}{\partial \dot{q}}(\dot{q}) = \tau, \tag{4.128}$$

with Rayleigh dissipation function $R(\dot{q})$ satisfying $\dot{q}^T \frac{\partial R}{\partial \dot{q}}(\dot{q}) \geq 0$, leading to the associated virtual system

$$\begin{aligned} \dot{s} &= -M^{-1}(q)C(q, \dot{q})s - M^{-1}(q)\tfrac{\partial R}{\partial s}(s) + M^{-1}(q)u \\ y &= s. \end{aligned} \tag{4.129}$$

Corollary 4.6.4 *For any curve $q(\cdot)$ the virtual system (4.129) is passive with parametrized storage function $S(s, q) := \frac{1}{2}s^T M(q)s$, satisfying $\frac{d}{dt}S(s, q) = -s^T \frac{\partial R}{\partial s}(s) + s^T u \leq s^T u$.*

Example 4.6.5 As an application of Proposition 4.6.3 suppose one wants to asymptotically track a given reference trajectory $q_d(\cdot)$ for a mechanical system (e.g., robot manipulator) with dynamics (4.119). Consider first the preliminary feedback

$$\tau = M(q)\dot{\xi} + C(q, \dot{q})\xi + \frac{\partial P}{\partial q}(q) + \nu \tag{4.130}$$

where

$$\xi := \dot{q}_d - \Lambda(q - q_d) \tag{4.131}$$

for some matrix $\Lambda = \Lambda^T > 0$. Substitution of (4.130) into (4.119) yields the virtual dynamics

$$M(q)\dot{s} + C(q, \dot{q})s = \nu \tag{4.132}$$

with $s := \dot{q} - \xi$. Define the additional feedback

$$\nu = -\hat{\nu} + \tau_e := -Ks + \tau_e, \qquad K = K^T > 0\,, \tag{4.133}$$

corresponding to an input strictly passive map $s \mapsto \hat{\nu}$.

Then by Theorem 2.2.15, part (b), for every $\tau_e \in L_2(\mathbb{R}^n)$ such that s (and thus ν) are in L_{2e}^n (see Fig. 4.3), actually the signal s will be in $L_2(\mathbb{R}^n)$. This fact has an important consequence, since by (4.131) and $s = \dot{q} - \xi$ the error $e = q - q_d$ satisfies

$$\dot{e} = -\Lambda e + s. \tag{4.134}$$

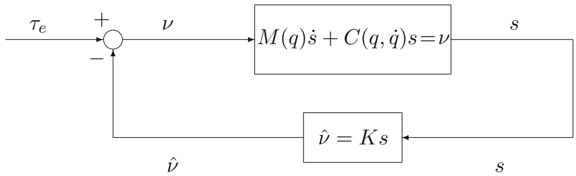

Fig. 4.3 Feedback configuration for tracking

Because we took $\Lambda = \Lambda^T > 0$ it follows from linear systems theory that also $e \in L_2(\mathbb{R}^n)$, and therefore by (4.134) that $\dot{e} \in L_2(\mathbb{R}^n)$. It is well known (see e.g., [83], pp. 186, 237) that this implies[8] $e(t) \to 0$ for $t \to \infty$.

An intrinsic *geometric* interpretation of the skew-symmetry of $\dot{M} - 2C$ and the virtual system (4.125) can be given as follows, within the framework of *Riemannian geometry*. The configuration space $\mathcal{Q}$ of the mechanical system is assumed to be a *manifold* with local coordinates $(q_1, \ldots, q_n)$. Then the generalized mass matrix $M(q) > 0$ defines a *Riemannian metric* $< , >$ on $\mathcal{Q}$ by setting

$$< v, w > := v^T M(q) w \tag{4.135}$$

for v, w tangent vectors to $\mathcal{Q}$ at the point q. The manifold $\mathcal{Q}$ endowed with the Riemannian metric is called a *Riemannian manifold*.

Furthermore, an affine *connection* ∇ on an arbitrary manifold $\mathcal{Q}$ is a map that assigns to each pair of vector fields X and Y on $\mathcal{Q}$ another vector field $\nabla_X Y$ on $\mathcal{Q}$ such that

(a) $\nabla_X Y$ is bilinear in X and Y
(b) $\nabla_{fX} Y = f \nabla_X Y$
(c) $\nabla_X fY = f \nabla_X Y + (L_X f) Y$

for every smooth function f, where $L_X f$ denotes the directional derivative of f along $\dot{q} = X(q)$, that is, in local coordinates $q = (q_1, \ldots, q_n)$ for $\mathcal{Q}$, $L_X f(q) = \sum_k \frac{\partial f}{\partial q_k}(q) X_k(q)$, where X_k is the k-th component of the vector field X. In particular, as will turn out to be important later on, Property (b) implies that $\nabla_X Y$ at $q \in \mathcal{Q}$ depends on the vector field X only through its value $X(q)$ at q.

In local coordinates q for Q an affine connection on Q is determined by n^3 smooth functions

$$\Gamma_{ij}^{\ell}(q), \quad i, j, \ell = 1, \ldots, n, \tag{4.136}$$

such that the ℓ-th component of $\nabla_X Y$, $\ell = 1, \ldots, n$, is given as

[8] A simple proof runs as follows (with thanks to J.W. Polderman and I.M.Y. Mareels). Take for simplicity $n = 1$. Then, since $\frac{d}{dt} e^2(t) = 2e(t)\dot{e}(t)$, $e^2(t_2) - e^2(t_1) = 2\int_{t_1}^{t_2} e(t)\dot{e}(t)dt \le \int_{t_1}^{t_2} [e^2(t) + \dot{e}^2(t)]dt \to 0$ for $t_1, t_2 \to \infty$. Thus for any sequence of time instants $t_1, t_2, \ldots, t_k, \ldots$ with $t_k \to \infty$ for $k \to \infty$ the sequence $e^2(t_i)$ is a Cauchy sequence, implying that $e^2(t_i)$ and thus $e^2(t)$ converges to some finite value for $t_i, t \to \infty$, which has to be zero since $e \in L_2(\mathbb{R})$.

$$(\nabla_X Y)_\ell = \sum_j \frac{\partial Y_\ell}{\partial q_j} X_j + \sum_{i,j} \Gamma^\ell_{ij} X_i Y_j, \tag{4.137}$$

with subscripts denoting the components of the vector fields involved.

The Riemannian metric $< , >$ on $\mathcal{Q}$ obtained from $M(q)$ defines a unique affine connection ∇^M on $\mathcal{Q}$ (called the *Levi-Civita connection*), which in local coordinates is determined by the n^3 *Christoffel symbols* (of the second kind)

$$\Gamma^\ell_{ij}(q) := \sum_{k=1}^{n} m^{\ell k}(q) c_{ijk}(q), \tag{4.138}$$

with $m^{\ell k}(q)$ the (ℓ, k)-th element of the inverse matrix $M^{-1}(q)$, and $c_{ijk}(q)$ the Christoffel symbols of the first kind as defined in (4.118). Thus in vector notation the affine connection ∇^M is given as

$$\nabla^M_X Y(q) = DY(q)(q)X(q) + M^{-1}(q)C(q, X)Y(q) \tag{4.139}$$

with $DY(q)$ the $n \times n$ Jacobian matrix of Y.

Identifying $s \in \mathbb{R}^n$ with a tangent vector at $q \in \mathcal{Q}$, we conclude that the coordinate-free description of the virtual system (4.125) is given by

$$\begin{aligned} \nabla^M_{\dot q(t)} s(t) &= M^{-1}(q(t))u(t) \\ y(t) &= s(t) \end{aligned} \tag{4.140}$$

Thus the state s of the virtual system at any moment t is an element of $T_{q(t)}\mathcal{Q}$. (Recall that $\nabla^M_X s(q)$ depends on the vector field X only through its value $X(q)$. Hence at every time t the expression in the left-hand side of (4.140) depends on the curve $q(\cdot)$ only through the value $\dot q(t) \in T_{q(t)}\mathcal{Q}$.)

With regard to the last term $M^{-1}(q)u$ we note that from a geometric point of view, the force u is an element of the *cotangent* space of $\mathcal{Q}$ at q. Since $M^{-1}(q)$ defines a map from the cotangent space to the tangent space, this yields $M^{-1}(q)u \in T_q\mathcal{Q}$. In terms of the Riemannian metric $< , >$ the tangent vector $Z = M^{-1}(q)u \in T_q\mathcal{Q}$ is determined by the requirement that the cotangent vector $< Z, \cdot >$ equals u. This is summarized in the following.

Proposition 4.6.6 *Consider a configuration manifold $\mathcal{Q}$ with Riemannian metric determined by the generalized mass matrix $M(q)$. Let ∇^M be the Levi-Civita connection on $\mathcal{Q}$. Then the virtual system is given by (4.140), where $q(\cdot)$ is any curve on $\mathcal{Q}$ and $s(t) \in T_{q(t)}\mathcal{Q}$ for all t. The virtual system is lossless with parametrized storage function $S(s, q) = \frac{1}{2} < s, s > (q)$.*

Remark 4.6.7 The expression $\nabla^M_{\dot q(t)} s(t)$ on the left-hand side of (4.140) is also called the *covariant derivative* of $s(t)$ (with respect to the affine connection ∇^M); sometimes denoted as $\frac{Ds}{dt}(t)$.

We emphasize that one can take *any* curve $q(t)$ in $\mathcal{Q}$ with corresponding velocity vector field $\dot{q}(t) = X(q(t))$, and consider the dynamics (4.140) of *any* vector field s *along* this curve $q(t)$ (that is, $s(t)$ being a tangent vector to Q at $q(t)$). If we take s to be *equal to* $\dot{q}$, then (4.140) reduces to

$$\nabla^M_{\dot{q}} \dot{q} = M^{-1}(q)\nu \tag{4.141}$$

which is nothing else than the second-order equations (4.124).

Finally, let us come back to the crucial property of skew-symmetry of $\dot{M} - 2C$. This property has the following geometric interpretation. First we note the following obvious lemma.

Lemma 4.6.8 *$\dot{M} - 2C$ is skew-symmetric if and only if $\dot{M} = C + C^T$*

Proof $(\dot{M} - 2C) = -(\dot{M} - 2C)^T$ iff $2\dot{M} = 2C + 2C^T$. □

Given an arbitrary Riemannian metric $<,>$ on $\mathcal{Q}$, an affine connection ∇ on $\mathcal{Q}$ is said to be *compatible* with $<,>$ if the following property holds:

$$L_X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z > \tag{4.142}$$

for all vector fields X, Y, Z on $\mathcal{Q}$.

Consider now the Riemannian metric $<,>$ determined by the mass matrix M as in (4.135). Furthermore, consider local coordinates $q = (q_1, \ldots, q_n)$ for $\mathcal{Q}$, and let $Y = \frac{\partial}{\partial q_i}$, $Z = \frac{\partial}{\partial q_j}$. Then (4.142) reduces to (see (4.137))

$$L_X m_{ij} = < \nabla_X \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} > + < \frac{\partial}{\partial q_i}, \nabla_X \frac{\partial}{\partial q_j} > \tag{4.143}$$

with m_{ij} the (i, j)-th element of the mass matrix M. Furthermore, by (4.139) we have

$$\begin{aligned} \nabla_X \tfrac{\partial}{\partial q_i} &= M^{-1}(q)C(q, X)e_i \\ \nabla_X \tfrac{\partial}{\partial q_j} &= M^{-1}(q)C(q, X)e_j \end{aligned} \tag{4.144}$$

with e_i, e_j denoting the i-th, respectively j-th, basis vector. Therefore, taking into account the definition of $<,>$ in (4.135), we obtain from (4.143)

$$L_X m_{ij} = (C^T(q, X))_{ij} + (C(q, X))_{ij}, \tag{4.145}$$

which we write (replacing L_X by the $\dot{}$ operator) as

$$\dot{M}(q) = C^T(q, \dot{q}) + C(q, \dot{q}). \tag{4.146}$$

Thus, in view of Lemma 4.6.8, the property of skew-symmetry of the matrix $\dot{M} - 2C$ is nothing else than the *compatibility* of the Levi-Civita connection ∇^M defined by the Christoffel symbols (4.138) with the Riemannian metric $<, >$ defined by $M(q)$.

This observation also implies that one may take *any* other affine connection ∇ (different from the Levi-Civita connection ∇^M), which is compatible with $<, >$ defined by M in order to obtain a lossless virtual system (4.140) (with ∇^M replaced by ∇).

Finally, we note that the Levi-Civita connection ∇^M defined by the Christoffel symbols (4.138) is the *unique* affine connection that is *compatible* with $<, >$ defined by M, *as well as* is *torsion-free* in the sense that

$$\nabla_X Y - \nabla_X Y = [X, Y] \tag{4.147}$$

for any two vector fields X, Y on $\mathcal{Q}$, where $[X, Y]$ denotes the Lie bracket of X and Y. In terms of the Christoffel symbols (4.138) the condition (4.147) amounts to the *symmetry* condition $\Gamma^\ell_{ij} = \Gamma^\ell_{ji}$ for all i, j, ℓ, or equivalently, with C_{kj} related to Γ^ℓ_{ij} by (4.138) and (4.120), that

$$C(q, X)Y = C(q, Y)X \tag{4.148}$$

for every pair of tangent vectors X, Y.

4.7 Incremental and Shifted Passivity

Recall the definition of *incremental passivity* as given in Definition 2.2.20. A state space version can be given as follows.

Definition 4.7.1 Consider a system as given in (4.1), with input and output spaces $U = Y = \mathbb{R}^m$ and state space $\mathcal{X}$. The system Σ is called *incrementally passive* if there exists a function, called the *incremental storage function*,

$$S : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+ \tag{4.149}$$

such that

$$\begin{aligned} S(x_1(T), x_2(T)) \leq\ & S(x_1(0), x_2(0)) \\ & + \int_0^T (u_1(t) - u_2(t))^T (y_1(t) - y_2(t)) dt \end{aligned} \tag{4.150}$$

for all $T \geq 0$, and for all pairs of input functions $u_1, u_2 : [0, T] \to \mathbb{R}^m$ and all pairs of initial conditions $x_1(0), x_2(0)$, with resulting pairs of state and output trajectories $x_1, x_2 : [0, T] \to \mathcal{X}$, $y_1, y_2 : [0, T] \to \mathbb{R}^m$.

Remark 4.7.2 Note that if $S(x_1, x_2)$ satisfies (4.150) then so does the function $\frac{1}{2}(S(x_1, x_2) + S(x_2, x_1))$. Hence, without loss of generality, we may assume that the storage function $S(x_1, x_2)$ satisfies $S(x_1, x_2) = S(x_2, x_1)$. Extensions of Definition 4.7.1 to incremental output strict or incremental input strict passivity are immediate.

Definition 4.7.1 directly implies incremental passivity of the input–output map $G_{\bar{x}}$ defined by Σ, for every initial state $\bar{x} \in \mathcal{X}$. This follows from (4.150) by taking *identical* initial conditions $x_1(0) = x_2(0) = \bar{x}$. Hence, the property of incremental passivity defined in Definition 4.7.1 for state space systems is in principle *stronger* than the property defined in Definition 2.2.20 for input–output maps.

As a direct corollary of Theorem 3.1.11 we obtain the following.

Corollary 4.7.3 *The system (4.1) is incrementally passive if and only if*

$$\sup_{u_1(\cdot), u_2(\cdot), T \geq 0} - \int_0^T (u_1(t) - u_2(t))^T (y_1(t) - y_2(t)) dt < \infty \tag{4.151}$$

for all initial conditions $(x_1(0), x_2(0)) \in \mathcal{X} \times \mathcal{X}$.

The *differential* version of the incremental dissipation inequality (4.149) takes the form

$$S_{x_1}(x_1, x_2) f(x_1, u_1) + S_{x_2}(x_1, x_2) f(x_2, u_2) \leq (u_1 - u_2)^T (y_1 - y_2) \tag{4.152}$$

for all $x_1, x_2, u_1, u_2, y_1 = h(x_1, u_1), y_2 = h(x_2, u_2)$, where $S_{x_1}(x_1, x_2)$ and $S_{x_2}(x_1, x_2)$ denote row vectors of partial derivatives with respect to x_1, respectively x_2.

An obvious example of an incrementally passive system is a *linear* passive system with quadratic storage function $\frac{1}{2} x^T Q x$. In this case, $S(x_1, x_2) := \frac{1}{2}(x_1 - x_2)^T Q (x_1 - x_2)$ define an incremental storage function, satisfying (4.149). Another example of an incrementally passive system is the *virtual* system defined in (4.125), with incremental storage function given by the parametrized expression (compare with (4.126)) $S(s_1, s_2, q) = \frac{1}{2}(s_1 - s_2)^T M(q)(s_1 - s_2)$. Furthermore, in both cases the system remains incrementally passive in the presence of an extra external (disturbance) input. For example, passivity of $\dot{x} = Ax + Bu, \; y = Cx$ implies incremental passivity of the disturbed system

$$\dot{x} = Ax + Bu + Gd, \; \dot{d} = Fd, \; y = Cx \tag{4.153}$$

for any F, G.

A different type of example of incremental passivity, relying on *convexity*, is given next.

Example 4.7.4 (*Primal–dual gradient algorithm*) Consider the constrained optimization problem

$$\min_{q;\, Aq = b} C(q), \tag{4.154}$$

where $C : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $Aq = b$ are affine constraints, for some $k \times n$ matrix A and vector $b \in \mathbb{R}^k$. The corresponding Lagrangian function is defined as

$$L(q,\lambda) := C(q) + \lambda^T (Aq - b), \quad \lambda \in \mathbb{R}^k, \tag{4.155}$$

which is convex in q and concave in λ. The primal–dual gradient algorithm for solving the optimization problem in continuous time is given as

$$\begin{aligned} \tau_q \dot{q} &= -\tfrac{\partial L}{\partial q}(q,\lambda) = -\tfrac{\partial C}{\partial q}(q) - A^T\lambda + u \\ \tau_\lambda \dot{\lambda} &= \tfrac{\partial L}{\partial \lambda}(q,\lambda) \quad = Aq - b \\ y &= q\,, \end{aligned} \tag{4.156}$$

where τ_q, τ_λ are diagonal positive matrices (determining the time-scales of the algorithm). Furthermore, we have added an input vector $u \in \mathbb{R}^n$ representing possible interaction with other algorithms or dynamics (e.g., if the primal–dual gradient algorithm is carried out in a distributed fashion). The output vector is defined as $y = q \in \mathbb{R}^n$. This defines an incrementally passive system with incremental storage function

$$S(q_1,\lambda_1,q_2,\lambda_2) := \frac{1}{2}(q_1-q_2)^T\tau_q(q_1-q_2) + \frac{1}{2}(\lambda_1-\lambda_2)^T\tau_\lambda(\lambda_1-\lambda_2) \tag{4.157}$$

Indeed

$$\begin{aligned} \frac{d}{dt}S &= (q_1-q_2)^T\tau_q(\dot{q}_1-\dot{q}_2) + (\lambda_1-\lambda_2)^T\tau_\lambda(\dot{\lambda}_1-\dot{\lambda}_2) \\ &= -(q_1-q_2)^T\left(\frac{\partial C}{\partial q}(q_1) - \frac{\partial C}{\partial q}(q_2)\right) + (u_1-u_2)^T(y_1-y_2) \\ &\le (u_1-u_2)^T(y_1-y_2) \end{aligned} \tag{4.158}$$

since $(q_1-q_2)^T\left(\frac{\partial C}{\partial q}(q_1) - \frac{\partial C}{\partial q}(q_2)\right) \ge 0$ for all q_1, q_2, by convexity of C.

Finally, a special case of incremental passivity is obtained by letting u_2 to be a *constant* input $\bar{u}$, and x_2 a corresponding *steady-state* $\bar{x}$ satisfying $f(\bar{x},\bar{u}) = 0$. Defining the corresponding constant output $\bar{y} = h(\bar{x},\bar{u})$ and denoting u_1, x_1, y_1 simply by u, x, y, this leads to requiring the existence of a storage function $S_{\bar{x}}(x)$ (parametrized[9] by $\bar{x}$) satisfying

$$S_{\bar{x}}(x(T)) \le S_{\bar{x}}(x(0)) + \int_0^T (u(t)-\bar{u})^T(y(t)-\bar{y})dt \tag{4.159}$$

This existence of a function $S_{\bar{x}}(x) \ge 0$ satisfying (4.159) is called *shifted passivity* (with respect to the steady-state values $\bar{u}, \bar{x}, \bar{y}$). We shall return to the notion of shifted passivity more closely in the treatment of port-Hamiltonian systems in Chap. 6, see especially Sect. 6.5.

[9]Note that in this case the subscript $\bar{x}$ does *not* refer to differentiation.

4.8 Notes for Chapter 4

1. The Kalman–Yakubovich–Popov Lemma is concerned with the equivalence between the frequency-domain condition of *positive realness* of the transfer matrix of a linear system and the existence of a solution to the LMI (4.18) or (4.19), and thus to the passivity of a (minimal) input-state-output realization. It was derived by Kalman [154], also bringing together results of Yakubovich and Popov. See Willems [351], Rantzer [257], Brogliato, Lozano, Maschke & Egeland [52]. For the uncontrollable case, see especially Rantzer [257], Camlibel, Belur & Willems [58].
2. Example 4.1.7 is taken from van der Schaft [283].
3. The factorization approach mentioned in Sect. 4.1 is due to Hill & Moylan [123, 126, 225]; see these papers for further developments along these lines.
4. Example 4.2.5 is taken from Dalsmo & Egeland [75, 76].
5. Corollary 4.3.5 is based on Vidyasagar [343], Sastry [267] (in the input–output map setting; see Chap. 2). See also Hill & Moylan [124, 125], Moylan [225] for further developments and generalizations.
6. The treatment of Example 4.3.6 is from Willems [352].
7. Example 4.3.7 is based on van der Schaft & Schumacher [302], where also applications are discussed. For further developments on passive *complementarity* systems see Camlibel, Iannelli & Vasca [59] and the references quoted therein.
8. Proposition 4.3.9 is taken from Kerber & van der Schaft [158].
9. Another interesting extension to the converse passivity theorems discussed in Sect. 4.3 concerns the following scenario. Suppose Σ_1 is such that $\Sigma_1 \|_f \Sigma_2$ is *stable* (in some sense) *for every* passive system Σ_2. Then under appropriate conditions this implies that also Σ_1 is necessarily passive. This is proved, using the Nyquist criterion, for single-input single-output linear systems in Colgate & Hogan [69], and for general nonlinear input–output maps, using the S-procedure lossless theorem, in Khong & van der Schaft [163]. Within a general state space setting the result is formulated and derived in Stramigioli [329], where also other important extensions are discussed. The result is of particular interest for robotic applications, where the "environment" Σ_2 of a controlled robot Σ_1 is usually unknown, but can be assumed to be passive. Hence, overall stability is only guaranteed if Σ_1 is passive; see e.g., Colgate & Hogan [69], Stramigioli [328, 329].
10. The first scenario of network interconnection of passive systems discussed in Sect. 4.4 is emphasized and discussed much more extensively in the textbook Bai, Arcak & Wen [18]. Here also a broad range of applications can be found, continuing on the seminal paper Arcak [10]. See also Arcak, Meissen & Packard

[11] for further developments, as well as Bürger, Zelazo & Allgöwer [55] for a network flow optimization perspective.

11. Example 4.4.4 can be found in Arcak [10]. See also van der Schaft & Stegink [303] for a generalization to "structure-preserving" networks of generators and loads.

12. Kirchhoff's matrix tree theorem goes back to the classical work of Kirchhoff on resistive electrical circuits [164]; see Bollobas [48] for a succinct treatment (see especially Theorem 14 on p. 58), and Mirzaev & Gunawardena [220] and van der Schaft, Rao & Jayawardhana [301] for an account in the context of chemical reaction networks.
    The *existence* (not the explicit *construction*) of $\gamma \in \mathbb{R}^N_+$ satisfying $L\gamma = 0$ already follows from the Perron–Frobenius theorem, exploiting the fact that the off-diagonal elements of $-L := DK$ are all nonnegative; see Sontag [320] (Lemma V.2).

13. The idea to assemble Lyapunov functions from a weighted sum of Lyapunov functions of component systems is well known in the literature on large-scale systems, see e.g., Michel & Miller [219], Siljak [315], and is sometimes referred to as the use of vector Lyapunov functions. Closely related developments to the second scenario discussed in Sect. 4.4 can be found in Zhang, Lewis & Qu [364]. The exposition here, distinguishing between flow and communication Laplacian matrices, is largely based on van der Schaft [287]. The interconnection of passive systems through a symmetric Laplacian matrix can be already found in Chopra & Spong [66].

14. Remark 4.4.13 generalizes the definition of *effective resistance* for symmetric Laplacians, which is well known; see e.g., Bollobas [48]. Note that in case of a symmetric Laplacian $R_{ij} = R_{ji}$.

15. The third scenario of network interconnection of passive systems as discussed in Sect. 4.4 is based on Arcak & Sontag [12], to which we refer for additional references and developments on the secant condition.

16. Section 4.5, as well as the first part of Sect. 4.5 is mainly based on the survey paper Ortega & Spong [243], for which we refer to additional references. See also the book Ortega, Loria, Nicklasson & Sira-Ramirez [239], as well as Arimoto [13]. Example 4.6.5 is due to Slotine & Li [316].

17. (Cf. Remark 4.5.2). If the map from $\dot{q}$ to p is not invertible one is led to constrained Hamiltonian dynamics as considered by Dirac [81, 82]. Under regularity conditions the constrained Hamiltonian dynamics is Hamiltonian with respect to the Poisson structure defined as the *Dirac bracket*. See van der Schaft [271] for an input–output decoupling perspective.

18. Background on the Riemannian geometry in Sect. 4.6 can be found, e.g., in Boothby [49], Abraham & Marsden [1]. For related work, see Li & Horowitz [180].

19. The concept of the *virtual system* defined in Definition 4.6.2 and the proof of its passivity (in fact, losslessness) is due to Slotine and coworkers, see e.g., Wang & Slotine [344], Jouffroy & Slotine [153], Manchester & Slotine [193].

20. Incremental passivity is also closely related to *differential passivity*, as explored in Forni & Sepulchre [100], Forni, Sepulchre & van der Schaft [101], van der Schaft [285]. Following the last reference, the notion of differential passivity involves the notion of the *variational systems* of Σ, defined as follows (cf. Crouch & van der Schaft [73]). Consider a one-parameter family of input-state-output trajectories $(x(t,\epsilon), u(t,\epsilon), y(t,\epsilon))$, $t \in [0,T]$, of Σ parametrized by $\epsilon \in (-c, c)$, for some constant $c > 0$. Denote the nominal trajectory by $x(t,0) = x(t)$, $u(t,0) = u(t)$ and $y(t,0) = y(t)$, $t \in [0,T]$. Then the infinitesimal *variations*

$$\delta x(t) = \frac{\partial x}{\partial \epsilon}(t,0)\,, \quad \delta u(t) = \frac{\partial u}{\partial \epsilon}(t,0)\,, \quad \delta y(t) = \frac{\partial y}{\partial \epsilon}(t,0)$$

satisfy

$$\begin{aligned} \dot{\delta x}(t) &= \tfrac{\partial f}{\partial x}(x(t),u(t))\delta x(t) + \tfrac{\partial f}{\partial x}(x(t),u(t))\delta u(t) \\ \delta y(t) &= \tfrac{\partial h}{\partial x}(x(t),u(t))\delta x(t) + \tfrac{\partial f}{\partial x}(x(t),u(t))\delta u(t) \end{aligned} \tag{4.160}$$

The system (4.160) (parametrized by $u(\cdot), x(\cdot), y(\cdot)$) is called the *variational system*, with variational state $\delta x(t) \in T_{x(t)}\mathcal{X}$, variational inputs $\delta u \in \mathbb{R}^m$, and variational outputs $\delta y \in \mathbb{R}^m$.
Suppose now that the original system Σ is *incrementally passive*. Identify $u(\cdot), x(\cdot), y(\cdot)$ with $u_2(\cdot), x_2(\cdot), y_2(\cdot)$ in (4.150), and $(x(t,\epsilon), u(t,\epsilon), y(t,\epsilon))$ for $\epsilon \neq 0$ with $u_1(\cdot), x_1(\cdot), y_1(\cdot)$. Dividing both sides of (4.150) by ϵ^2, and taking the limit for $\epsilon \to 0$, yields under appropriate assumptions

$$\bar{S}(x(T), \delta x(T)) \leq \bar{S}(x(0), \delta x(0)) + \int_0^T (\delta u(t))^T \delta y(t) dt \tag{4.161}$$

where

$$\bar{S}(x(t), \delta x(t)) := \lim_{\epsilon \to 0} \frac{S(x(t,\epsilon), x(t))}{\epsilon^2} \tag{4.162}$$

The thus obtained Eq. (4.161) amounts to the *definition* of differential passivity adopted in Forni & Sepulchre [100], van der Schaft [285].

21. For the numerous applications of the theory of passive systems to *adaptive control* we refer, e.g., to Brogliato, Lozano, Maschke & Egeland [52], and Astolfi, Karagiannis & Ortega [16], and the references quoted therein.