Spectral Bounds for the k-Regular Induced Subgraph Problem

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Abstract Many optimization problems on graphs are reduced to the determination of a subset of vertices of maximum cardinality which induces a k-regular subgraph. For example, a maximum independent set, a maximum induced matching and a maximum clique is a maximum cardinality 0-regular, 1-regular and $(\omega(G) - 1)$-regular induced subgraph, respectively, were $\omega(G)$ denotes the clique number of the graph G. The determination of the order of a k-regular induced subgraph of highest order is in general an NP-hard problem. This paper is devoted to the study of spectral upper bounds on the order of these subgraphs which are determined in polynomial time and in many cases are good approximations of the respective optimal solutions. The introduced upper bounds are deduced based on adjacency, Laplacian and signless Laplacian spectra. Some analytical comparisons between them are presented. Finally, all of the studied upper bounds are tested and compared through several computational experiments.

Keywords Spectral graph theory · Maximum k-regular induced subgraphs · Combinatorial optimization

1 Introduction

Throughout the paper, we deal with simple undirected graphs G, with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G) \neq \emptyset$. Since this graph has n vertices, we say that the graph has *order* n. We write $u \sim v$ whenever the vertices u and v are adjacent. The neighborhood of a vertex $i \in V(G)$, that is, the set of vertices adjacent to i, is denoted by $N_G(i)$, the degree of i is $d_G(i) = |N_G(i)|$, $\Delta(G) = \max_{i \in V(G)} d_G(i)$ and

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N. Bebiano (ed.), *Applied and Computational Matrix Analysis*,
Springer Proceedings in Mathematics & Statistics 192,
DOI 10.1007/978-3-319-49984-0_7

$\delta(G) = \min_{i \in V(G)} d_G(i)$. The subgraph of G induced by the vertex subset $S \subset V(G)$ is denoted by $G[S]$. The graph G is p-regular when all vertices have the same degree equal to p. A vertex subset $S \subseteq V(G)$ is (k, τ)-regular if it induces a k-regular subgraph and $\forall v \notin S$, $|N_G(v) \cap S| = \tau$. The adjacency matrix $A_G = (a_{i,j})$ is the $n \times n$ matrix defined by

$$a_{i,j} = \begin{cases} 1 \text{ if } i \sim j, \\ 0 \text{ otherwise.} \end{cases}$$

The Laplacian matrix $L_G = (l_{i,j})$ and the signless Laplacian matrix $Q_G = (q_{i,j})$ of the graph G, are the matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$, respectively, where D_G stands for the diagonal matrix of order n with the i-th entry equal to the vertex degree $d_G(i)$. Therefore, A_G, L_G and Q_G are real symmetric matrices and then all their eigenvalues are real. These eigenvalues are herein denoted, in nonincreasing order, respectively by $\lambda_1 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \cdots \geq \mu_n$ and $q_1 \geq \cdots \geq q_n$. If G has at least one edge, then $\lambda_1 > 0 > \lambda_n$. From now on we consider only simple undirected graphs with at least one edge which will be called graphs.

Each adjacency eigenvalue of a graph G is main if the corresponding eigenspace contains an eigenvector which is not orthogonal to the all ones vector, otherwise is non-main. From Geršgorin's theorem, the eigenvalues of L_G and Q_G are nonnegative real numbers and since the entries of each row of L_G sum 0, then the eigenvalue $\mu_n = 0$ is associated to the all ones eigenvector $\hat{e}$. The multiplicity of 0, as an eigenvalue of L_G, is equal to the number of connected components of G. Furthermore, G is bipartite if and only if $q_n = 0$. Further basic details about graph spectra can be found in [6, 8]. A vertex subset inducing a 0-regular subgraph is called an independent (or stable) set. A maximum independent set is an independent set of maximum cardinality and its cardinality is called independence number and it is denoted by $\alpha(G)$.

In [3] it was proved that the problem of finding a maximum cardinality subset of vertices inducing a k-regular subgraph is NP-hard. Throughout this paper, this maximum is denoted by $\alpha_k(G)$. Note that in the particular case of $k = 0$, $\alpha_0(G) = \alpha(G)$.

The study of spectral upper bounds on the order of k-regular induced subgraphs (it should be noted that the independent sets are 0-regular induced subgraphs) appear in [3–5]. In [1] (see also [11]) an upper bound on the order of induced subgraphs with average degree d (based on adjacency eigenvalues) was obtained for regular graphs, extending the ratio bound (7) to the general case of maximum k-regular induced subgraphs (when $k = 0$, this bound coincide with the ratio bound). A similar result was obtained in [3], using convex quadratic programming techniques. In [4, 5] the arbitrary graph case is analyzed and upper bounds on the order of k-regular induced subgraphs are presented. In [4], the upper bounds are obtained using adjacency eigenvalues and eigenvectors, namely the least eigenvalue (whether it is non-main) and the corresponding eigenspace. In [5], the upper bound is obtained using a quadratic programming technique jointly with the main angles (see [8] for details) and the induced subgraph just must have average degree d.

The main goal of this paper is to introduce some new spectral upper bounds on the order of k-regular induced subgraphs, making an analytic comparison between them when possible. These new upper bounds are based on adjacency, Laplacian and signless Laplacian eigenvalues. Finally, a few computational experiments are presented.

2 Concepts and Fundamental Results

In this section, we introduce some definitions and we recall the previously obtained results needed for the deductions in the next section. In particular, we survey results concerning to spectral upper bounds on the order of k-regular induced subgraphs.

For arbitrary graphs, consider a graph G of order n with $V(G) = S \cup S^c$, where $S \subseteq V(G)$ denotes a vertex subset inducing a k-regular subgraph and S^c is its complement. The set of edges with just one end vertex in S, that is, the cut set defined by S is denoted $\partial(S)$. Hence, $|\partial(S)| = |S|(\bar{d}_S - k)$, where $\bar{d}_S = \frac{1}{|S|}\sum_{i \in S} d_G(i)$.

The next result relates the cardinality of the cut set $\partial(S)$ to the largest eigenvalue of the Laplacian matrix of a graph G.

Lemma 1 ([16]) *Let G be a graph of order n and $S \subseteq V(G)$. Then*

$$|\partial(S)| \leq \mu_1 \frac{|S|(n - |S|)}{n}. \tag{1}$$

Another relationship involving the largest Laplacian eigenvalue and the least adjacency eingenvalue of a graph G is (see [8]).

$$\delta(G) - \lambda_n \leq \mu_1 \leq \Delta(G) - \lambda_n. \tag{2}$$

Now we consider some relationships involving signless Laplacian eigenvalues. Assuming that G is a connected graph of order n, according to [7], the least eigenvalue of Q_G is zero if and only if G is bipartite and, in that case, zero is a simple eigenvalue. They also proved that

$$2\delta(G) \leq q_1 \leq 2\Delta(G). \tag{3}$$

Moreover, according to [9],

$$q_n < \delta(G). \tag{4}$$

From Weyl's inequalities we have an improvement of inequalities (3) and we state relationships between signless Laplacian and adjacency eigenvalues.

$$\delta(G) + \lambda_1 \leq q_1 \leq \Delta(G) + \lambda_1 \tag{5}$$

and

$$\delta(G) + \lambda_n \leq q_n \leq \Delta(G) + \lambda_n. \tag{6}$$

We now present some spectral upper bounds on the size of k-regular induced subgraphs starting with the particular case of $k = 0$, for which we consider only the ones most related with this work.

2.1 Bounds on $\alpha(G)$

In the case of regular graphs, the well known ratio bound, obtained by Hoffman (unpublished) and presented by Lovász in [14] can be stated by the following theorem where, for the last statement, the necessary condition was proved in [12] and the sufficient condition was proved in [2].

Theorem 1 ([2, 12, 14]) *If G is a regular graph of order n, then*

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}. \tag{7}$$

Furthermore, the cardinality of an independent set S attains the upper bound if and only if S is $(0, \tau)$-regular, with $\tau = -\lambda_n$.

The ratio bound (7) was extended by Haemers for arbitrary graphs, according to the following theorem.

Theorem 2 ([11]) *If G is a graph of order n, then*

$$\alpha(G) \leq \frac{-n\ \lambda_n\ \lambda_1}{\delta^2(G) - \lambda_n\ \lambda_1}. \tag{8}$$

The next spectral upper bound based on the largest Laplacian eigenvalue was independently deduced in [10, 15].

Theorem 3 ([10, 15]) *If G is a graph of order n, then*

$$\alpha(G) \leq n \frac{\mu_1 - \delta(G)}{\mu_1}. \tag{9}$$

2.2 Bounds on $\alpha_k(G)$

Cardoso, Kamińsky and Lozin in [3] introduced the following family of convex quadratic programming problems:

$$\upsilon_k(G) = \max_{x \geq 0} 2\hat{e}^T x - \frac{\tau}{k + \tau} x^T \left(\frac{A_G}{\tau} + I_n \right) x, \tag{10}$$

where $\hat{e}$ is the all ones vector, I_n the identity matrix of order n, $k \in \mathbb{N} \cup \{0\}$ and $\tau = -\lambda_n$ and they proved that $\alpha_k(G) \leq \upsilon_k(G)$, where $\alpha_k(G)$ is the cardinality of a vertex subset inducing a k-regular subgraph of maximum order. In fact, in [3], the obtained result was stated as follows.

Theorem 4 ([3]) *Let G be a graph and k a non-negative integer. If $S \subseteq V(G)$ induces a subgraph of G with average degree k, then $|S| \leq \upsilon_k(G)$. The equality holds if and only if $\tau + k \leq |N_G(v) \cap S| \ \ \forall v \notin S$.*

Considering the particular case of regular graphs we have the following theorem, where the upper bound was obtained in [11] and the last statement was proved in [3].

Theorem 5 ([3, 11]) *If G is a p-regular graph of order n, then*

$$\alpha_k(G) \leq n\frac{k-\lambda_n}{p-\lambda_n}. \tag{11}$$

Furthermore, the equality holds if and only if there exists $S \subseteq V(G)$ which $(k, k+\tau)$-regular, with $\tau = -\lambda_n$. In this case, $\alpha_k(G) = |S| = n\frac{k-\lambda_n}{p-\lambda_n}$.

In [4], considering the quadratic program not necessary convex (10), with $\tau > 0$, it was proved that

$$\alpha_k(G) \leq \lambda_{max}(A_{G^c}) + k + 1, \tag{12}$$

where G^c denotes the complement of the graph G, that is, the graph such that $V(G^c) = V(G)$ and $E(G^c) = \{ij : ij \notin E(G)\}$. Furthermore, the following upper bound was obtained.

Theorem 6 ([4]) *Consider a graph G such that $\lambda_{min}(A_G) = \lambda_n = \cdots = \lambda_{n-(p-1)}$ is a non-main eigenvalue with multiplicity p. Assuming that the eigenvectors $\hat{u}_1, \ldots, \hat{u}_n$, associated to the eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, are unitary and pairwise orthogonal, then*

$$\alpha_k(G) \leq \sum_{j=1}^{n-p} \frac{-\lambda_n + k}{-\lambda_n + \lambda_j}(\hat{e}^T \hat{u}_j)^2. \tag{13}$$

Later, in [5], using a quadratic programming technique jointly with the main angles of G, the upper bound (13) was improved as follows.

Theorem 7 ([5]) *Let G be a graph of order n, and let S be a set of vertices which induces a k-regular subgraph of G $(0 \leq k \leq n-1)$. If $t > -\lambda_n$ then*

$$\alpha_k(G) \leq h_k^G(t), \tag{14}$$

where $h_k^G(t) = (k+t)\left(1 - \frac{P_{G^c}(t-1)}{(-1)^n P_G(-t)}\right)$ and $P_G(x) = \det(xI - A)$.

3 Upper Bounds Based on the Spectrum of A_G, L_G and Q_G

Now it is worth to recall the following theorem obtained by Haemers.

Theorem 8 ([11]) *Let G be a graph on n vertices of average degree d and let the vertex set of G be partitioned into two sets such that G_1 and G_2 are the subgraphs induced by these two sets. For $i = 1, 2$ let n_i be the number of vertices of G_i, d_i be the average of vertex degrees of G_i and let $\bar{d}_i$ be the average of vertex degrees in G over the vertices of G_i. Then*

(i) $\lambda_1\lambda_2 \geq \frac{nd_id-n_i\bar{d}_i^2}{n-n_i} \geq \lambda_1\lambda_n$.
(ii) If the equality holds on one of the sides, then G_1 and G_2 are regular and also the degrees in G are constant over the vertices of G_1 and G_2.

As a consequence of this theorem, we have the following corollary.

Corollary 1 *If G is a graph of order n, then,*

$$\alpha_k(G) \leq \frac{2k|E(G)| - n\lambda_1\lambda_n}{\delta(G)^2 - \lambda_1\lambda_n}. \tag{15}$$

Proof Let us consider the vertex partition $V(G) = S \cup S^c$, where S induces a k regular subgraph of G. Applying Theorem 8-(i), setting $n_1 = |S|$ and $d_1 = k$, we have,

$$\begin{aligned}\frac{nkd - \bar{d}_1^{\,2}|S|}{n-|S|} \geq \lambda_1\lambda_n &\Leftrightarrow \lambda_1\lambda_n(n-|S|) \leq nkd - \bar{d}_1^{\,2}|S| \\ &\Leftrightarrow |S|(\bar{d}_1^{\,2} - \lambda_1\lambda_n) \leq nkd - n\lambda_1\lambda_n \\ &\Leftrightarrow |S| \leq \frac{nkd - n\lambda_1\lambda_n}{\bar{d}_1^{\,2} - \lambda_1\lambda_n}.\end{aligned}$$

Since $\bar{d}_1 \geq \delta$ and $d = \frac{2|E(G)|}{n}$, the inequality (15) follows. □

Notice that, when G is p-regular, $\lambda_1 = \delta(G)$ and $|E(G)| = \frac{np}{2}$ whereby the upper bound (15) is equal to (11).

The next corollary is a consequence of Lemma 1.

Corollary 2 *If G is a graph of order n, then*

$$\alpha_k(G) \leq n\frac{k + \mu_1 - \delta(G)}{\mu_1}. \tag{16}$$

Proof Considering a vertex subset $S \subseteq V(G)$ inducing a k-regular subgraph and taking into account that (as defined before) $\overline{d_S} = \frac{1}{|S|}\sum_{i\in S} d_G(i)$, it follows that $|\partial(S)| = |S|(\bar{d}_S - k)$. Then applying Lemma 1 we have

$$|S|(\bar{d}_S - k) \leq \mu_1 \frac{|S|(n-|S|)}{n} \Leftrightarrow \frac{n(\bar{d}_S - k)}{n-|S|} \leq \mu_1$$
$$\Leftrightarrow \mu_1|S| \leq n\mu_1 - n(\bar{d}_S - k)$$
$$\Leftrightarrow |S| \leq n\,\frac{k+\mu_1-\bar{d}_S}{\mu_1}.$$

Since $\bar{d}_S \geq \delta(G)$, the inequality (16) follows. □

If a graph G is p-regular, from (2) $\mu_1 + \lambda_n = p$ and we may conclude that the upper bound (16) is equal to (11).

Before the introduction of a new upper bound on the order of k-regular induced subgraphs in function of the largest and the least eigenvalues of the signless Laplacian matrix, it is worth to introduce the following lemma.

Lemma 2 *Let G be a graph of order n without isolated vertices. If G is bipartite or $\delta(G) \geq \frac{\Delta(G)}{2}$ or $q_1 < 4\delta(G)$, then $4\delta(G)^2 - q_n q_1 > 0$.*

Proof Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

1. If G is bipartite without isolated vertices, then $q_n = 0$, $\delta > 0$ and therefore, $4\delta^2 - q_n q_1 > 0$.
2. If $\delta \geq \frac{\Delta}{2}$, we have $\delta^2 \geq \frac{\delta\Delta}{2} \Leftrightarrow 4\delta^2 \geq 2\delta\Delta$ and, taking into account (3) and (4), since $q_1 \leq 2\Delta$ and $\delta > q_n$ it follows $4\delta^2 - q_n q_1 > 0$.
3. Finally, if $q_1 < 4\delta$, then $q_1 q_n \leq 4\delta q_n < 4\delta^2$, that is, $q_1 q_n < 4\delta^2$ and so $4\delta^2 - q_n q_1 > 0$.

□

Notice that there are graphs G, with $\delta = \delta(G)$, such that $4\delta^2 - q_n q_1 \leq 0$, as it is the case of the graph depicted in Fig. 1 which has $\delta = 2$, $q_n = 1.4991$ and $q_1 = 10.8517$.

Theorem 9 *Let G be a graph of order n such that $4\delta^2(G) - q_n q_1 > 0$. Then*

$$\frac{2k|E(G)| - n\lambda_1\lambda_n}{\delta^2(G) - \lambda_1\lambda_n} \leq \frac{4|E(G)|(\Delta(G)+k) - nq_nq_1}{4\delta^2(G) - q_nq_1}. \tag{17}$$

Proof Considering $\varepsilon = |E(G)|$, $\delta = \delta(G)$, $\Delta = \Delta(G)$ and assuming that the inequality of (17) holds, we have

Fig. 1 Graph G, with $4\delta(G)^2 - q_n q_1 \leq 0$

$$\frac{2k\varepsilon - n\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n} - \frac{\varepsilon(\Delta + k) - n\frac{q_1}{4}q_n}{\delta^2 - \frac{q_1}{4}q_n} \leq 0$$

$$\Updownarrow$$

$$2k\varepsilon\delta^2 - \frac{q_1}{2}q_nk\varepsilon - n\delta^2\lambda_1\lambda_n - \delta^2\Delta\varepsilon - \delta^2k\varepsilon + n\delta^2\frac{q_1}{4}q_n + \lambda_1\lambda_n\varepsilon\Delta + \lambda_1\lambda_n\varepsilon k \leq 0$$

$$\Updownarrow$$

$$k(\delta^2\varepsilon - \frac{q_1}{2}q_n\varepsilon + \lambda_1\lambda_n\varepsilon) - n\delta^2\lambda_1\lambda_n - \delta^2\Delta\varepsilon + n\delta^2\frac{q_1}{4}q_n + \lambda_1\lambda_n\varepsilon\Delta \leq 0$$

Let $f(k) = k(\delta^2\varepsilon - \frac{q_1}{2}q_n\varepsilon + \lambda_1\lambda_n\varepsilon) - n\delta^2\lambda_1\lambda_n - \delta^2\Delta\varepsilon + n\delta^2\frac{q_1}{4}q_n + \lambda_1\lambda_n\varepsilon\Delta$. Then,

$$\begin{aligned} f'(k) &= \delta^2\varepsilon - \frac{q_1}{2}q_n\varepsilon + \lambda_1\lambda_n\varepsilon \\ &= \varepsilon(\delta^2 - \frac{q_1}{2}q_n + \lambda_1\lambda_n). \end{aligned}$$

From (6),

$$\delta + \lambda_n < q_n \Leftrightarrow \delta^2 + \delta\lambda_n < \delta q_n \Leftrightarrow \delta^2 - \delta q_n + \delta\lambda_n < 0.$$

Since, from (3), $\frac{q_1}{2} \geq \delta$ and, as it is well known, $\lambda_1 \geq \delta$, it follows that $\delta^2 - \frac{q_1}{2}q_n + \lambda_1\lambda_n \leq \delta^2 - \delta q_n + \delta\lambda_n < 0$, that is, $f'(k) < 0$. Therefore, $f(k)$ is a decreasing function.

Considering the function $f(k)$ and setting $k = 0$ and $\Delta = \delta + \xi$ with ξ a nonnegative integer we may define the function

$$g(\delta, \xi) = -n\delta^2\lambda_1\lambda_n - \delta^2(\delta + \xi)\varepsilon + n\delta^2\frac{q_1}{4}q_n + \lambda_1\lambda_n\varepsilon(\delta + \xi).$$

Then

$$\begin{aligned} \frac{\partial g(\delta, \xi)}{\partial \xi} &= -\delta^2\varepsilon + \lambda_1\lambda_n\varepsilon \\ &= \varepsilon(-\delta^2 + \lambda_1\lambda_n) \\ &< 0. \end{aligned}$$

Therefore, $g(\delta, \xi)$ is a decreasing function with respect to ξ. Since $g(\delta, 0) = -n\delta^2\lambda_1\lambda_n - \delta^3\varepsilon + n\delta^2\frac{q_1}{4}q_n + \lambda_1\lambda_n\varepsilon\delta$ and $\delta = \Delta$ it follows that $\lambda_1 = \delta$. Furthermore, from (3), $\frac{q_1}{2} = \delta$ and from (6), $q_n = \delta + \lambda_n$. Therefore,

$$\begin{aligned} g(\delta, 0) &= -n\delta^3\lambda_n - \delta^3\varepsilon + n\frac{\delta^3}{2}(\delta + \lambda_n) + \lambda_n\varepsilon\delta^2 \\ &= -n\delta^3\lambda_n - \delta^3\varepsilon + n\frac{\delta^4}{2} + n\frac{\delta^3}{2}\lambda_n + \lambda_n\varepsilon\delta^2. \end{aligned}$$

Finally, since $\varepsilon = \frac{n\delta}{2}$ we obtain $g(\delta, 0) = -n\delta^3\lambda_n - n\frac{\delta^4}{2} + n\frac{\delta^4}{2} + n\frac{\delta^3}{2}\lambda_n + n\frac{\delta^3}{2}\lambda_n = 0$ and thus, for all nonnegative integers δ and ξ, $g(\delta, \xi) \leq 0$. Therefore, $f(0) \leq 0$ and, since $f(k)$ is a decreasing function, we may conclude that $f(k) \leq 0$ for all k. □

As immediate consequence of Corollary 1 and Theorem 9 we have the following corollary.

Corollary 3 *If G is a graph of order n, ε edges, $\Delta = \Delta(G)$ and $\delta = \delta(G)$, such that $4\delta^2 - q_n q_1 > 0$, then*

$$\alpha_k(G) \leq \frac{4\varepsilon(\Delta + k) - nq_n q_1}{4\delta^2 - q_n q_1}. \tag{18}$$

According to [7], a graph G with n vertices and ε edges is regular if and only if $4\varepsilon = nq_1$. Furthermore, when G is regular its degree is equal to $\frac{q_1}{2}$. Thus, assuming that G is p-regular, has n vertices and ε edges, by Lemma 2 the hypothesis of Corollary 3 is fulfilled and then we may write

$$\begin{aligned}\alpha_k(G) &\leq \frac{nq_1(p + k - q_n)}{2pq_1 - q_n q_1} \quad (\text{since } \Delta(G) = \delta(G) = p = \frac{q_1}{2} \text{ and } 4\varepsilon = nq_1)\\ &= \frac{n(p + k - q_n)}{2p - q_n} = n\frac{k - \lambda_n}{p - \lambda_n} \quad (\text{since } q_n - \lambda_n = p).\end{aligned}$$

Therefore, for regular graphs, all the upper bounds (11) (15), (16) and (18) are equal. Notice that there are graphs for which these upper bounds are tight. For instance, if $G = K_n$ (a complete graph of order n), then $\lambda_1 = n - 1$ and $\lambda_n = -1$. Thus, if $S \subseteq V(K_n)$ induces a k-regular subgraph, then $n\frac{k-\lambda_n}{\lambda_1 - \lambda_n} = k + 1 = |S|$. Therefore, when G is a complete graph, for each k, the upper bounds (15), (16) and (18) on the cardinality of vertex subsets inducing k-regular subgraphs are all reached. More generally, according to Theorem 5, if G is a regular graph and $S \subset V(G)$ is a $(k, k + \tau)$-regular set, with $\tau = -\lambda_n$, then all the above referred upper bounds are reached.

Throughout the paper, in all the proofs of the presented results, only the average degree in S is used and then, in all the obtained results we may replace k-regular induced subgraph by induced subgraph with average degree k. Moreover, all the obtained results remain valid when we consider positive weights on the edges, assuming in that case that the degree of a vertex v is then the sum of the weights of the edges incident to v.

4 Computational Experiments and Conclusions

In this section, several computational experiments with the upper bounds (15), (16) and (18) are presented in Table 1. In each row of this table appears the order n, the maximum degree Δ, the minimum degree δ, the degree of a regular induced subgraph

Table 1 Computational experiments with the upper bounds (15), (16) and (18)

Graph	n	$\Delta(G)$	$\delta(G)$	k	(15)	(16)	(18)
c-fat200-1	200	17	14	0	**74.01**	82.31	97.27
				1	**83.87**	90.72	109.28
				2	**93.73**	99.13	121.29
				6	133.18	**132.75**	169.33
				7	143.04	**141.16**	181.34
c-fat200-2	200	34	32	0	**55.72**	57.29	63.19
				1	**60.28**	61.75	67.86
				2	**64.83**	66.21	72.53
				16	128.65	**128.65**	137.88
				17	133.21	**133.10**	142.55
c-fat200-5	200	86	83	0	**45.85**	48.56	50.10
				1	**47.74**	50.39	52.06
				2	**49.64**	52.21	54.01
				39	119.79	**119.72**	126.41
				40	121.69	**121.55**	128.36
MANN-a9	45	41	40	0	**3.76**	4.46	4.23
				1	**4.81**	5.47	5.32
				2	**5.86**	6.48	6.41
				18	**22.69**	22.70	23.84
				19	23.74	**23.72**	24.93
MANN-a27	378	374	364	0	**5.17**	13.43	13.19
				1	**6.22**	14.43	14.27
				2	**7.27**	15.43	15.36
				3	**8.32**	16.44	16.45
				4	**9.37**	17.44	17.53
Keller4	171	124	102	0	**34.76**	45.74	109.56
				1	**36.20**	46.96	110.51
				2	**37.65**	48.19	111.47
				51	108.46	**108.37**	158.11
brock200-1	200	165	130	0	**20.25**	44.83	75.10
				1	**21.82**	46.02	77.09
				2	**23.40**	47.22	79.08
				64	**121.22**	**121.22**	202.28
				65	122.80	**122.41**	204.26
brock200-2	200	114	78	0	**37.48**	69.19	161.29
				1	**40.12**	70.87	165.49
				2	**42.75**	72.54	169.69
				33	124.54	**124.53**	300.04
				34	127.18	**126.21**	304.24

(continued)

Table 1 (continued)

Graph	n	$\Delta(G)$	$\delta(G)$	k	(15)	(16)	(18)
brock200-3	200	134	99	0	**29.41**	57.79	113.35
				1	**31.51**	59.23	116.37
				2	**33.61**	60.66	119.39
				43	119.58	**119.56**	243.18
				44	121.68	**121.00**	246.20
brock200-4	200	147	112	0	**24.94**	51.73	91.73
				1	**26.76**	53.05	94.15
				2	**28.59**	54.38	96.58
				54	123.58	**123.22**	222.61
				55	125.40	**124.54**	225.03

k and the computed upper bounds on the order of this induced subgraphs for some of the graphs of the family considered in the Second DIMACS Implementation Challenge (see [13]).

Notice that for the particular case of regular graphs the upper bounds (15), (16) and (18) are all equal. Moreover since, according to the Theorem 9, the upper bound (15) is less or equal than the upper bound (18), it follows that

$$\frac{4|E(G)|(\Delta(G)+k)-nq_nq_1}{4\delta(G)^2-q_nq_1} \geq \min\left\{\frac{2k|E(G)|-n\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n}, n\frac{k+\mu_1-\delta}{\mu_1}\right\}.$$

Concerning the comparison between the upper bounds (15) and (16) and also between (16) and (18), the computational results presented in the Table 1 show that none of them is always better than the others.

In fact, regarding the upper bounds (15) and (16), for $k = 0, 1, 2$, the former is better than the later. However, for much greater values of k, there are several graphs for which the upper bound (16) is better than (15). Finally, it should be noted that for the graphs MANN-a9 and MANN-a27 for $k = 0, 1, 2$ the upper bound (18) is better than the upper bound (16).

Acknowledgements The authors would like to thank Willem Haemers for several insightful comments and suggestions on this work that have helped us to improve the content of this paper. This research was supported by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through the CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013. We are also indebted to the anonymous referee for her/his careful reading and suggestions which have improved the text.

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