Max Celebrity Games

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Abstract. We introduce *Max celebrity games* a new variant of Celebrity games defined in [4]. In both models players have weights and there is a critical distance β as well as a link cost α. In the max celebrity model the cost of a player depends on the cost of establishing links to other players and on the maximum of the weights of those nodes that are farther away than β (instead of the sum of weights as in celebrity games). The main results for $\beta > 1$ are that: computing a best response for a player is NP-hard; the optimal social cost of a celebrity game depends on the relation between α and w_{max}; NE always exist and NE graphs are either connected or a set of $r \geq 1$ connected components where at most one of them is not an isolated node; for the class of connected NE graphs we obtain a general upper bound of $2\beta+2$ for the diameter. We also analyze the price of anarchy (PoA) of connected NE graphs and we show that there exist games Γ such that PoA$(\Gamma) = \Theta(n/\beta)$; modifying the cost of a player we guarantee that all NE graphs are connected, but the diameter might be $n-1$. Finally, when $\beta = 1$, computing a best response for a player is polynomial time solvable and the PoA $= O(w_{max}/w_{min})$.

1 Introduction

The increasing use of Internet and social networks, has motivated a great interest to model theoretically their behavior. Fabrikant et al. [15] proposed a game-theoretic model of network creation (NCG) as a simple tool to analyze the creation of Internet as a decentralized and non-cooperative communication network among players (the network nodes).

In this model the goal of each player is to have, in the resulting network, all the other nodes as close as possible paying a minimum cost. It is assumed that: all the players have the same interest (all-to-all communication pattern with identical weights); the cost of being disconnected is infinite; and the links to other nodes paid by one node can be used by others. Formally, a game Γ in this seminal model is defined as a tuple $\Gamma = \langle V, \alpha \rangle$, where V is the set of n nodes and α the cost of establishing a link. A strategy for player $u \in V$ is a subset $S_u \subseteq V - \{u\}$, the set of players for which player u pays for establishing a link. The n players and their joint strategy choices $S = (S_u)_{u \in V}$ create an undirected graph $G[S]$. The cost function for each node u under strategy S is defined by $c_u(S) = \alpha|S_u| + \sum_{v \in V} d_{G[S]}(u, v)$

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where $d_{G[S]}(u,v)$ is the distance between nodes u and v in graph $G[S]$. By changing the cost function to $c_u(S) = \alpha|s_u| + \max\{d_{G[S]}(u,v)|v \in V\}$ as proposed in [13] one obtains the max game model.

From here on several versions have been considered to make the model a little more realistic. For different variants we refer the interested reader to [1–3,6,9–14,16,17,19] among others.

In Internet as well as in social networks not all the nodes have the same importance. It seems natural to consider nodes with different relevance weights. In such a setting, the cost of being far (even if connected) from high-weight nodes should be greater than the cost of being far from low-weight nodes. In [4] we introduce *celebrity games* with the aim to study the combined effect of having players with different weights that share a common distance bound.

In celebrity games the cost of a player has two components. The first one is the cost of the links established by the node. The second one is the sum of the weights of those nodes that are farther away than the critical distance. Formally, a celebrity game is defined by $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, where V is a set of nodes with weights $(w_u)_{u\in V}$, α is the cost of establishing a link and β establishes the desirable distance bound. The cost function for each node is defined by $c_u(S) = \alpha|S_u| + \sum_{\{v|d_{G[S]}(u,v)>\beta\}} w_v$.

In this paper we extend the study initiated in [4]: we define a max version of the celebrity games that we name *max celebrity games* and we analyze the structure and quality of their Nash equilibria. From now on, let us refer to celebrity games as *sum celebrity games*. In the max celebrity model the cost of a player takes into account the maximum of the weights (worst-case) of those nodes that are farther away than the critical distance, instead of the sum of weights (average-case). The cost function is formally defined by $c_u(S) = \alpha|S_u| + \max_{\{v|d_{G[S]}(u,v)>\beta\}} w_v$. Intuitively, the goal of each player in max celebrity games is to buy as few links as possible in order to have the high-weighted nodes closer to the given critical distance. Observe that if the cost of establishing links is higher than the benefit of having close a node (or set of nodes), players might rather prefer to stay either far or even disconnected from it.

Observe that the main feature of both, sum and max celebrity games, is the combination of bounded distance with players having different weights. Even though heterogeneous players have been considered in NCG under bilateral contracting [5,18], and the notion of bounded distance has been studied in [8], to the best of our knowledge sum celebrity games is the first model that studies how a common critical distance, different weights, and a link cost, altogether affect the individual preferences of the players. Furthermore, max celebrity games is the first model that focuses on how the maximum weight of those nodes that are farther than β affects the creation of graphs.

In this paper we analyze the structure of Nash equilibrium (NE) graphs of max celebrity games and their quality with respect to the optimal strategies. To do so we address the cases $\beta > 1$ and $\beta = 1$, separately. For $\beta > 1$, every player u has to choose for each non-edge (u,v) between paying the maximum of the weights of the nodes with distance to u greater than β, or buying the link (u,v) and paying α for the link minus the maximum of the weights of those nodes whose distance to u will become less or equal than the critical distance β. While for $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link or at least w_v (the weight of the non-adjacent node v).

For the general case $\beta > 1$ our results can be summarized as follows: computing a best response for a player is NP-hard; the optimal social cost of a celebrity game Γ depends on the relation between α and the maximum weight w_{max}; NE always exist and NE graphs are either connected or a set of $r \geq 1$ connected components where at most one of them is not an isolated node; for the class of connected NE graphs we obtain a general upper bound of $2\beta+2$ for the diameter; we also analyze the quality of connected NE graphs and we show that there exist max celebrity games such that $\mathrm{PoA}(\Gamma) = \Theta(n/\beta)$; we consider a variation of the cost of the player in order to avoid non-connected NE graphs.

Finally, for the particular case $\beta = 1$, we show that computing a best response for a player is polynomial time solvable and that the $\mathrm{PoA} = O(w_{max}/w_{min})$.

The paper is organized as follows. In Sect. 2 we introduce the basic definitions and the max celebrity model. In Sect. 3 we study the fundamental properties of optimal graphs and NE graphs. Section 4 studies for $\beta > 1$ the quality of connected NE graphs and considers a modification of the cost of a player in order to guarantee connected NE graphs. In Sect. 5 we study the complexity of the best response problem and the PoA for the case $\beta = 1$. Finally, in Sect. 6 we give an outline of the main differences between max and sum celebrity models.

2 The Model

We use standard notation for graphs and strategic games. All the graphs in the paper are undirected unless explicitly said otherwise. For a graph $G = (V, E)$ and $u, v \in E$, $d_G(u, v)$ denotes the distance, i.e. the length of a shortest path, from u to v in G. The *diameter* of a vertex $u \in V$, $diam_G(u)$, is defined as $diam_G(u) = \max_{v \in V}\{d_G(u, v)\}$ and the *diameter* of G, $diam(G)$, is defined as usual as $diam(G) = \max_{v \in V}\{diam_G(v)\}$. An *orientation* of an undirected graph is an assignment of a direction to each edge, turning the initial graph into a directed graph.

For a weighted set $(V, (w_u)_{u \in V})$ we extend the weight function to subsets in the following way. For $U \subseteq V$, $w(U) = \max_{u \in U}\{w_u\}$. Furthermore, we set $w_{max} = \max_{u \in V}\{w_u\}$ and $w_{min} = \min_{u \in V}\{w_u\}$.

Definition 1. *A* max celebrity game Γ *is defined by a tuple* $\langle V, (w_u)_{u \in V}, \alpha, \beta\rangle$ *where:* $V = \{1, \ldots, n\}$ *is the set of players, for each player* $u \in V$*;* $w_u > 0$ *is the* weight *of player* u*;* $\alpha > 0$ *is the cost of establishing a link and* β*,* $1 \leq \beta \leq n-1$*, is the* critical distance.

A strategy *for player* u *is a subset* $S_u \subseteq V - \{u\}$ *denoting the set of players for which player* u *pays for establishing a direct link. A* strategy profile *for* Γ *is a tuple* $S = (S_1, S_2, \ldots, S_n)$ *defining a strategy for each player. For a strategy profile* S*, the associated* outcome graph *is the undirected graph* $G[S]$ *which is defined by* $G[S] = (V, \{\{u, v\} | u \in S_v \vee v \in S_u\})$*.*

For a strategy profile $S = (S_1, S_2, \dots, S_n)$*, the cost function of player* u*, denoted by* c_u*, is defined as* $c_u(S) = \alpha|S_u| + W_u$ *where* $W_u = \max_{\{v|d_{G[S]}(u,v)>\beta\}}\{w_v\}$*. And as usual, the* social cost *of a strategy profile* S *in* Γ *is defined as* $C(S) = \sum_{u\in V} c_u(S)$*. The* social cost *of a graph* G *in* Γ *is defined analogously as* $C(G) = \alpha|E(G)| + \sum_{u\in V(G)} W_u$*.*

Observe that, even though a link might be established by only one player, we consider the outcome graph as an undirected graph, assuming that once a link is bought the link can be used in both directions. In our definition we have considered a general case in which players may have different weights and defined the cost function through properties of the undirected graph created by the strategy profile. The player's cost function takes into account two components: the cost of establishing links and the maximum of the weights of the players that are at a distance greater than the critical distance β.

In the remaining of the paper, we assume that, for $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, the parameters verify the required conditions. Furthermore, unless specifically stated, we consider $\beta > 1$, the case $\beta = 1$ will be studied in Sect. 5. We use the following notation, for a game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, $n = |V|$. We denote by $\mathcal{S}(u)$ the set of strategies for player u and by $\mathcal{S}(\Gamma)$ the set of strategy profiles of Γ.

As usual, for a strategy profile S and a strategy S'_u for player u, (S_{-u}, S'_u) represents the strategy profile in which S_u is replaced by S'_u while the strategies of the other players remain unchanged. The *cost difference* $\Delta(S_{-u}, S'_u)$ is defined as $\Delta(S_{-u}, S'_u) = c_u(S_{-u}, S'_u) - c_u(S)$. Observe that, if $\Delta(S_{-u}, S'_u) < 0$, player u has an incentive to deviate from S_u. A best response to $S \in \mathcal{S}(\Gamma)$ for player u is a strategy $S'_u \in \mathcal{S}(u)$ minimizing $\Delta(S_{-u}, S'_u)$. Let us remind the definition of Nash equilibrium.

Definition 2. *Let* $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ *be a max celebrity game. A strategy profile* $S = (S_1, S_2, \dots, S_n)$ *is a* Nash equilibria *of* Γ *if no player has an incentive to deviate from his strategy. Formally, for a player* u *and each strategy* $S'_u \in \mathcal{S}(u)$*,* $\Delta(S_{-u}, S'_u) \geq 0$*.*

We denote by $\mathrm{NE}(\Gamma)$ the set of Nash equilibria of a game Γ. We use the term NE to refer to a strategy profile $S \in \mathrm{NE}(\Gamma)$. We say that a graph G is a NE *graph* if there is $S \in \mathrm{NE}(\Gamma)$ so that $G = G[S]$.

We denote by $\mathsf{opt}(\Gamma)$ the minimum value of the social cost, i.e. $\mathsf{opt}(\Gamma) = \min_{S\in\mathcal{S}(\Gamma)} C(S)$. We denote by $\mathrm{OPT}(\Gamma)$ the set of optimum strategy profiles of Γ w.r.t. the social cost, that is, for $S \in \mathrm{OPT}(\Gamma)$, $C(S) = \mathsf{opt}(\Gamma)$. We use the term OPT *strategy profile* to refer to a $S \in \mathrm{OPT}(\Gamma)$.

It is worth observing that: for $S \in \mathrm{NE}(\Gamma)$, it never happens that $v \in S_u$ and $u \in S_v$, for any $u, v \in V$; a NE graph G can be the outcome of several strategy profiles and not all the orientations of a NE graph G are NE.

In the following we make use of some particular outcome graphs on n vertices: I_n, the independent set; and ST_n a star graph, i.e. a tree in which one of the vertices, the *central* vertex, is connected to all the other $n-1$ vertices.

We define the Price of Anarchy and the Price of Stability as usual.

Definition 3. *Let Γ be a max celebrity game. The* Price of Anarchy *of Γ is defined as $PoA(\Gamma) = \max_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$ and the* Price of Stability *of Γ is defined as $PoS(\Gamma) = \min_{S \in NE(\Gamma)} C(S)/\mathsf{opt}(\Gamma)$*

The explicit reference to Γ will be dropped whenever Γ is clear from the context. We will refer to $\text{NE}(\Gamma)$, $\mathsf{opt}(\Gamma)$, $PoA(\Gamma)$, and $PoS(\Gamma)$ by NE, opt, PoA and PoS respectively.

Our first result shows that the computation of a best response in max celebrity games is a NP-hard problem for $\beta \geq 2$. The proof consists in a reduction from the Dominating Set problem. The problem becomes polynomial time computable for $\beta = 1$ as we show in Sect. 5.

Proposition 1. *Computing a best response for a player to a strategy profile in a max celebrity game is NP-hard even when $\beta = 2$.*

3 Social Optimum and Nash Equilibrium

In this section we analyze some properties of the opt and NE strategy profiles in max celebrity games. We start by giving bounds for opt that depend on the existence of one or more than one connected components.

Proposition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. We have that $2\alpha(n-1) \geq \mathsf{opt}(\Gamma) \geq \min\{\alpha(n-1), w_{max}(n-1) + w_{min}\}$.*

Proof. Let $S \in \text{OPT}(\Gamma)$ and let $G = G[S]$. Let $G_1, ..., G_r$ be the connected components of G and let $V_i = V(G_i), k_i = |V_i|$, and $W_i = w(V_i)$, for $1 \leq i \leq r$. Assume w.l.o.g that $W_1 \geq W_2 \geq ... \geq W_r$. Observe that the social cost of a disconnected graph can be expressed as the sum of the social cost of the connected components plus the additional contribution of the pairs of vertices that lie in different components. Each connected component must be a tree of diameter at most β, otherwise a strategy profile with smaller social cost could be obtained by replacing the connections on V_i by such a tree. W.l.o.g we can assume that, for $1 \leq i \leq r$, the $i-$th connected component is a star graph ST_{k_i} on k_i vertices. Recall that $C(ST_{k_i}) = \alpha(k_i - 1)$, thus $C(G) = \sum_{i=1}^{r} \alpha(k_i - 1) + \sum_{i=1}^{r} k_i \left(\max_{j \neq i}\{W_j\}\right) = \alpha(n-r) + nW_1 - k_1(W_1 - W_2)$.

Notice that if for some $i > 1$, the i-th connected component is not an isolated node, then the node with maximum weight in this connected component can be moved to G_1. By preserving the connectivity and structure (a star) of G_1, the social cost of the resulting graph is strictly smaller than the cost of the original G. This implies that for every $i > 1$, $k_i = 1$. Hence, $C(G) = \alpha(n-r) + (r-1)W_1 + (n-r+1)W_2$.

If $r = 1$ then $C(G) = \alpha(n-1)$ and we are done. Otherwise, if $r > 1$, then we have the inequality $C(ST_n) \geq C(G)$. This implies that $\alpha \geq \frac{1}{r-1}(W_1(r-1) + W_2(n-r+1)) \geq W_1$.

Then we get the following results. First: $C(G) \geq W_1(n-r) + (r-1)W_1 + (n-r+1)W_2 \geq (n-1)W_1 + W_2 \geq (n-1)w_{max} + w_{min}$.

Secondly, using that $r > 1$: $C(G) = \alpha(n-r) + (r-1)W_1 + (n-r+1)W_2 \leq \alpha(n-r) + (r-1)\alpha + (n-r+1)\alpha \leq 2\alpha(n-1)$.

The relationship between α and w_{max} determines partially the topology of the NE graphs. As one can expect, if $\alpha > w_{max}$, no player has incentive to establish a link then the independent set is the unique NE graph. Otherwise, any NE graph can be connected or disconnected.

Proposition 3. *Every max celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ has a* NE*. Furthermore, when $\alpha \leq w_{max}$, ST_n is a* NE *graph, and when $\alpha \geq w_{max}$, I_n is a* NE *graph. If $\alpha > w_{max}$, then I_n is the unique* NE *graph.*

Proof. When $\alpha \leq w_{max}$ let us show that ST_n is a NE graph. Let u_{max} a node with maximum weight and we suppose that it is the central node of the star. If $S_{u_{max}} = \emptyset$ and for every node $v \neq u_{max}$, $S_v = \{u_{max}\}$, then u_{max} cannot improve its actual cost since it is exactly 0. Moreover, the other nodes can only delete the edge to u_{max}. Since such deviation has a cost increment of $-\alpha + w_{max} \geq 0$, then we are done.

When $\alpha \geq w_{max}$, let us show that I_n is a NE graph. Let S be the empty strategy profile, $I_n = G[S]$. Notice that for any player u, if $S'_u \neq \emptyset$, then $\Delta(S_{-u}, S'_u) \geq |S'_u|\alpha - w_{max} \geq (|S'_u| - 1)w_{max} \geq 0$. Finally, if $\alpha > w_{max}$ it is easy to see that the unique NE graph is I_n. Let us suppose that there exist $u, v \in V$ such that $v \in S_u$. If $S'_u = S_u - \{v\}$, then $\Delta(S_{-u}, S'_u) \leq -\alpha + w_{max} < 0$. Hence, if $G \neq I_n$, then G is not a NE graph.

Corollary 1. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a max celebrity game. Then, $PoS(\Gamma) = 1$ for $\alpha \leq w_{max}$ and $PoS(\Gamma) < 2$ for $\alpha > w_{max}$.*

In particular, even in the case that $\alpha < w_{max}$, it can be shown that there exist max celebrity games where I_n is a NE graph. Indeed consider a game with $n \geq 2$ and weights defined as $w_i = (i-1)\alpha$ for $i > 1$ and $w_1 = \alpha$. Then, clearly $\alpha < w_{max}$ and I_n is a NE graph.

Furthermore, for every integer $1 < r \leq n$, there exists non-connected max celebrity games with exactly r different connected components. Moreover, the only connected component that can have more than one node is the one that contains a node with weight w_{max}.

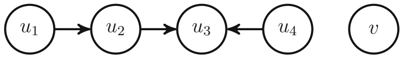
Proposition 4. *Any* NE *graph distinct from I_n has at most one non-trivial connected component. Moreover, for every integer $r \geq 2$ there exists a max celebrity game having a* NE *graph with exactly r connected components.*

Proof. (Sketch) For the first part, let $G_1, ..., G_r$ be the connected components of a NE graph. Assume that a node with the maximum weight is in G_1. If for some $i > 1$, $|G_i| > 1$, then there exist $u, v \in V(G_i)$ such that $u \in S_v$. In this case, v can strictly decrease its cost deleting this edge because the node with maximum weight is still at distance greater than β, contradicting the fact that G is a NE graph.

For the second part we distinguish two cases: $r \geq 3$ and $r = 2$. For the first case, let $n = r + 1$, $V = \{v_0, v_1, ..., v_r\}$, $E = \{\{v_0, v_1\}\}$ with $S_{v_1} = \{v_0\}$, $S_{v_i} = \emptyset$ for $i \neq 1$, as depicted in the figure below. For the weights consider $w_{v_0} = w_1$ and

$w_{v_i} = w_2$ for all $i \geq 1$, with $w_1 > w_2$, $w_1 - w_2 = \alpha$ and $\alpha \geq w_1/(n-1), w_2/(n-2)$. We have that this configuration is a NE.

For the case $r = 2$ see the figure below. It is not hard to see that this configuration is also a NE.



4 The Price of Anarchy

Observe that there exist max celebrity games Γ with $\alpha \leq w_{max}$ having disconnected NE graphs with high social cost in comparison with the optimum. Indeed, consider the example given in Proposition 4 with $w_2 = \frac{w_1(n-2)}{(n-1)}$. The cost of this NE graph is $w_1(n-1) + \frac{w_1(n-2)}{n-1}$ and combining it with $\mathsf{opt} \leq 2\alpha(n-1)$, we get the bound $\mathrm{PoA}(\Gamma) \geq (n-1)/2$. Hence, we focus on the study of the PoA for connected NE graphs. Since the restriction $\alpha \leq w_{max}$ by itself does not exclude the possibility of having non-connected NE graphs, we study the PoA of connected equilibria from two different perspectives: first, we analyze the worst case among all connected NE graphs; second, we introduce a slight modification of the player's cost function in order to guarantee connectivity in the class of NE graphs. Whenever we consider the class of connected NE graphs we compare the social cost of such equilibria with the optimum value among the connected graphs, $\mathsf{opt}(\Gamma) = \alpha(n-1)$.

4.1 PoA and Diameter of Connected NE Graphs

In this subsection we analyze the quality and structure of equilibria in terms of the parameters that define the max celebrity games. Our next result indicates that the price to pay for the anarchy is low when α is close to w_{max}.

Proposition 5. *For every max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$, $PoA(\Gamma) \leq 2(w_{max}/\alpha)$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. Then, no player has an incentive to deviate from S. In particular, for each $u \in V$ we have that $0 \leq \Delta(S_{-u}, \emptyset) = -\alpha|S_u| + W'_u - W_u$ where $W_u = \max_{\{x | d_G(u,x) > \beta\}} w_x$ and $W'_u = \max_{\{x | d_{G[(S_{-u}, \emptyset)]}(u,x) > \beta\}} w_x$. By adding for each $u \in V$ the corresponding inequalities, we have that $0 \leq \sum_{u \in V}(-\alpha|S_u| + W'_u - W_u) = -\alpha|E| + \sum_{u \in V} W'_u - \sum_{u \in V} W_u$.

Therefore, $C(G) = \alpha|E| + \sum_{u \in V} W_u \leq \sum_{u \in V} W'_u \leq n w_{max}$ and we can conclude that $\mathrm{PoA}(\Gamma) \leq \frac{n w_{max}}{\alpha(n-1)} \leq 2\frac{w_{max}}{\alpha}$.

The diameter of NE graphs depends directly on the critical distance β.

Proposition 6. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. In a* NE *graph G for Γ, $diam(G) \leq 2\beta + 2$.*

Proof. Let $S \in \text{NE}(\Gamma)$ such that $G = G[S]$ is connected. Assume that the node u satisfies that $d_G(u, u_{max}) > \beta$ and $|S_u| > 0$. Then u has incentive to break any of its bought links because after doing so, u_{max} will still remain inside the complementary of the ball of radius β centered at u. Next, assume that $diam(u_{max}) \geq \beta + 2$. Let $u_{max}, u_1, u_2, ..., u_{\beta+2}$ be a path. Then, either $u_{\beta+1} \in S_{u_{\beta+2}}$ or $u_{\beta+2} \in S_{u_{\beta+1}}$. Therefore, since both $u_{\beta+1}, u_{\beta+2}$ are at distances greater than β from u_{max}, G cannot be a NE. This proves that $diam(u_{max}) \leq \beta + 1$ in any connected NE and, as a consequence, that $diam(G) \leq 2\beta + 2$.

Let us provide for a NE graph G, a bound on the contribution of the *weight component* of the social cost of G, $W(G, \beta) = \sum_{\{u \in V(G)\}} W_u$. The following lemma is a reformulation of a similar result that can be found in [4] using a cleaner and simpler argument.

Lemma 1. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. In a* NE *graph G for Γ, $W(G, \beta) = O(\alpha n^2/\beta)$.*

Proof. Let S be a NE and $G = G[S]$ be a connected NE graph. Let $u \in V$ be any node in V. Consider the sets $A_i(u) = \{v \mid d_G(u, v) = i\}$. Define for $i = 1, ..., k$, $C_i = \{v \in V \mid (i-1)(\beta-1) \leq d_G(u, v) < i(\beta-1)\} = \cup_{(i-1)(\beta-1) \leq j < i(\beta-1)} A_j(u)$ with k such that $\cup_{i=1}^{k} C_i = V(G)$. By the pigeonhole principle, for each $i = 1, ..., k$ there exists at least one subindex, call it $j(i)$, for which $(i-1)(\beta-1) \leq j(i) < i(\beta-1)$ and $|A_{j(i)}(u)| \leq |C_i|/(\beta-1)$. In this way, for any node $v \in V(G)$, let $S'_v = (S_v \cup_{i=1}^{k} A_{j(i)}(u)) - \{v\}$ and let $G' = G[S_{-v}, S'_v]$. By construction, $diam_{G'}(v) \leq \beta$. Therefore, as S is a NE, we have $0 \leq \Delta(S_{-v}, S'_v) \leq \alpha \sum_{i=1}^{k} \frac{|C_i|}{\beta-1} - W_v = \alpha \left(\frac{n-1}{\beta-1}\right) - W_v$.

Thus, $W(G, \beta) \leq \frac{n(n-1)\alpha}{\beta-1} = O\left(\frac{\alpha n^2}{\beta}\right)$.

Using the same technique to provide a bound for the number of edges in NE graphs for a sum celebrity games (Proof of Lemma 4 of [4]), we obtain the following result.

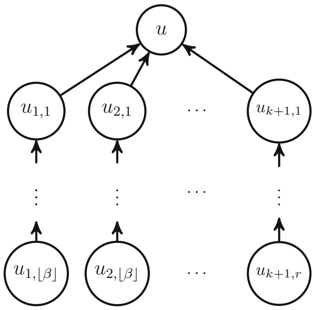
Lemma 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a max celebrity game. In a* NE *graph G for Γ, $|E(G)| \leq n - 1 + \frac{3n^2}{\beta}$.*

Corollary 2. *For every max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$, $PoA(\Gamma) = O(n/\beta)$*

Proposition 7. *For every $n > \beta > 1$, there exists a max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$ such that $PoA(\Gamma) = \Omega(n/\beta)$.*

Proof. Given n, let k and r be such that $n-1 = k\lfloor\beta\rfloor + r$, $k \geq 3$ and $0 \leq r < \lfloor\beta\rfloor$. Let $V = \{u\} \cup \left(\cup_{i=1}^{k} \{u_{i,j} \mid 1 \leq j \leq \lfloor\beta\rfloor\}\right) \cup \{u_{k+1,1}, u_{k+1,2}, ..., u_{k+1,r}\}$. We then define $w_u = W$ and $w_{u_{i,j}} = w$ with W, w such that $w = (k-2)\alpha$ and $W > n\alpha$.

In this way we consider the configuration S defined by the relations $S_u = \emptyset$, $S_{u_{i,j}} = \{u_{i,j-1}\}$ for $j \geq 2$ and $S_{u_{i,1}} = \{u\}$ for $i = 1, ..., k+1$, as depicted in the figure below. To prove that $S \in \text{NE}(\Gamma)$ we see that the cost difference associated to any deviation is not negative.



Clearly, u has no incentive of deviating his strategy because his cost is zero. Let us prove that any other node $u_{i,j}$ has no incentive in deviating from its current strategy. We say that a node v is covered with respect a node v' if v is at a distance at most β from v'. We have three cases:

1. *The deviation is such that all nodes are covered with respect* $u_{i,j}$. In this situation the cost difference is $l\alpha - w$. Notice that every node $u_{h,\lfloor\beta\rfloor}$ with $h \neq i$ can be reached only when a link from $u_{i,j}$ to the path formed by $u_{h,1}, u_{h,2}, ..., u_{h,\lfloor\beta\rfloor}$ is bought. Since initially $u_{i,j}$ has bought one link this leads to the inequality $l \geq k-2$. Therefore $l\alpha - w \geq (k-2)\alpha - (k-2)\alpha = 0$.
2. *The deviation is such that u is uncovered with respect* $u_{i,j}$. In this situation, since $W > n\alpha$, the cost difference is $l\alpha - w + W \geq 0$, for $-1 \leq l \leq n-1$.
3. *The deviation is such that u is covered with respect* $u_{i,j}$ *but there is at least one node node of weight w uncovered with respect* $u_{i,j}$. Then the cost difference is $l\alpha$ for some integer l. The only negative value that l can take is -1, but in such case the configuration leaves u uncovered with respect $u_{i,j}$, a contradiction. Therefore, $l\alpha \geq 0$.

Hence, $S \in \text{NE}(\Gamma)$ and $C(S) > (n-1)w = (n-1)(k-2)\alpha$. Using the bound for the social optimum $\mathsf{opt}(\Gamma) \leq 2\alpha(n-1)$ we have that $\text{PoA}(\Gamma) \geq (k-2)/2$.

Theorem 1. *For every $n > \beta > 1$, there exists a max celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$ such that $PoA(\Gamma) = \Theta(n/\beta)$.*

4.2 The PoA When the Connectivity of the NE Graphs Is Guaranteed

Let us consider a new cost function that excludes non-connected NE graphs. We define a *connected max celebrity game* Γ^{con} as a max celebrity game $\Gamma^{con} = \langle V, (w_u)_{u \in V}, \alpha, \beta, \rangle$, but now, the cost for each player $u \in V$ in strategy profile S is denoted by $c_u^{con}(S)$ and it is defined as follows: $c_u^{con}(S) = c_u(S)$, if $diam_{G[S]}(u) \leq n-1$; otherwise, $c_u^{con}(S) = \infty$. As usual, the social cost of a strategy profile S in Γ^{con} is defined as $C^{con}(S) = \Sigma_{u\in V} c_u^{con}(S)$. Since for any connected graph G, $C^{con}(G) = C(G) \geq \alpha(n-1)$, then we have that $\mathsf{opt}(\Gamma^{con}) = \alpha(n-1)$. Notice that the same tuple $\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle$ can define a max celebrity game as well as a connected max celebrity game. In order to distinguish one from the other, we denote by $\Gamma = \Gamma(\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle)$ the corresponding max celebrity game and by $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta, \rangle)$, the corresponding connected max celebrity game.

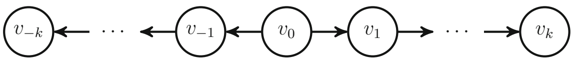
Proposition 8. *Let $\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a tuple defining $\Gamma = \Gamma(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$ and $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$. Then, $\text{NE}(\Gamma) \subsetneq \text{NE}(\Gamma^{con})$ when we consider $\text{NE}(\Gamma)$ restricted to connected graphs.*

Proof. Let $S \in \text{NE}(\Gamma)$ be such that $G = G[S]$ is connected. Let u be a player, let S'_u be a deviation, and let $G' = G[(S_{-u}, S'_u)]$. Let $\Delta(S_{-u}, S'_u)$ and $\Delta^{con}(S_{-u}, S'_u)$ be the corresponding increments in the games Γ and Γ^{con}, respectively. We have that $\Delta^{con}(S_{-u}, S'_u) = \Delta(S_{-u}, S'_u)$, if G' is connected. Otherwise, $\Delta^{con}(S_{-u}, S'_u) = \infty$, $\Delta(S_{-u}, S'_u) < \infty$. Therefore, $\Delta^{con}(S_{-u}, S'_u) \geq \Delta(S_{-u}, S'_u)$ and then, $\text{NE}(\Gamma) \subseteq \text{NE}(\Gamma^{con})$.

To see that the inclusion might be strict, let us consider that $V = \{u, v\}$, $v \in S_u$, and $S_v = \emptyset$. If $w_v > \alpha$, S is not a NE for Γ. On the other hand, independently of the weights of u, v, S is a NE for Γ^{con}.

Proposition 9. *There are connected max celebrity games that have NE graphs with diameter equal to $n-1$.*

Proof. Let $n = 2k+1$ be a positive integer and let $V = \{v, v_1, v_{-1}, v_2, v_{-2}, \ldots, v_k, v_{-k}\}$. Let S be the strategy profile defined by $v_1, v_{-1} \in S_v$ and $v_{i+1} \in S_{v_i}, v_{-(i+1)} \in S_{v_{-i}}$ for $i \leq k-1$ (see the figure below). Setting the weights $w_x \leq \alpha$ for all $x \in V$ and for any $\beta < (n-1)/4$ it is easy to see that the corresponding graph is indeed a NE.



The bounds on the PoA obtained for the class of connected NE graphs for max celebrity games also hold for connected max celebrity games. The proofs also work for this case.

Theorem 2. *The PoA for the connected max celebrity games satisfies:*

1. *For every connected max celebrity game $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$, $PoA(\Gamma^{con}) = O(n/\beta)$*
2. *For every $n > \beta > 1$, there exists a connected max celebrity game $\Gamma^{con} = \Gamma^{con}(\langle V, (w_u)_{u\in V}, \alpha, \beta\rangle)$ such that $PoA(\Gamma^{con}) = \Theta(n/\beta)$.*

5 Max Celebrity Games for $\beta = 1$

When $\beta = 1$, each player u has to decide for every non-edge (u, v) of the graph to pay either α for the link, or at least w_v. It is not difficult to show that the best response of a player can be computed by sorting the weights of the non-adjacent nodes and then, selecting the number of links to be added to the most weighted non-adjacent nodes.

Proposition 10. *The problem of computing a best response of a player to a strategy profile in max celebrity games is polynomial time solvable when* $\beta = 1$.

In the next result we show that the price to pay for the anarchy is low when w_{min} is close to w_{max}.

Theorem 3. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1\rangle$ *be a max celebrity game. Then,* $PoA(\Gamma) = O(w_{max}/w_{min})$.

Proof. Let $S \in \text{OPT}(\Gamma)$ and $G = G[S] = (V, A)$. Let $X = \{v \in V \mid deg(v) = n-1\}$ where $deg(v)$ means the degree of v in the undirected graph G. We have that $C(G) \geq \frac{1}{2}\alpha(n-1)|X| + (n - |X|)w_{min}$. Hence, $C(G) \geq nw_{min}$, if $w_{min} \leq \frac{(n-1)}{2}\alpha$ and $C(G) \geq \binom{n}{2}\alpha$, otherwise. To prove the result we distinguish three cases:

First we see that if $w_{min} \leq \alpha(n-1)/2$, then $\text{PoA}(\Gamma) \leq w_{max}/w_{min}$. Indeed, let S be a NE of Γ and let $G = G[S] = (V, E)$. Using the same reasoning as in Proposition 5 we have that $C(G) = \sum_{u \in V}(|S_u|\alpha + \max_{\{x \mid d(u,x)>1\}}\{w_x\}) \leq nw_{max}$. Therefore, if $w_{min} \leq \alpha(n-1)/2$, then $\text{PoA}(\Gamma) \leq w_{max}/w_{min}$, as we wanted to see.

Now, let us see that $PoA(\Gamma) = 1$ for $w_{min} > (n-1)\alpha$. This is because if $G \neq K_n$ then there exists some $v \in V$ with $diam_G(v) > 1$. Then considering the deviation for player v that consists in adding links to all the remaining nodes from the graph we get a cost increment of $k\alpha - w$ for some $k > 0$ and $w \geq w_{min}$. Since $k \leq (n-1)$ then $k\alpha - w \leq (n-1)\alpha - w_{min} < 0$, a contradiction for G being a NE. Thus $G = K_n$ and hence the result.

Finally, we see that for $\frac{n-1}{2}\alpha < w_{min} \leq (n-1)\alpha$ then $\text{PoA}(\Gamma) \leq 3$. Indeed, let S be a NE and $G = G[S] = (V, A)$. For a given $u \in V$ such that $diam_G(u) > 1$, let v be such that $w_v = W_u$. If $w_v > (n-1)\alpha$ then buying from u all the links to the remaining nodes from $V - \{x \mid d_G(u,x) \leq 1\}$ yields a cost increment of at most $(n-1)\alpha - w_v < 0$, a contradiction with G being a NE. Therefore $PoA(\Gamma) \leq (\binom{n}{2}\alpha + n(n-1)\alpha)/\binom{n}{2}\alpha = 3$.

6 Max Celebrity Games Vs Sum Celebrity Games

The main differences between max and sum celebrity games are that: for $\beta > 1$, in max model there exist other disconnected NE graphs than I_n; in connected NE graphs, PoA $= O(n/\beta)$ in both models, but this is tight for some max games; for $\beta = 1$, PoA $=$O(w_{max}/w_{min}) in max, while in sum PoA ≤ 2. Finally, max celebrity games are equivalent to the MaxBD games (see [7,8]) when $\alpha < w_{min}/(n-1)$ as they are sum celebrity games when $\alpha < w_{min}$. (See the proof of Proposition 8 in [4] and replace $\alpha < w_{min}$ by $\alpha < w_{min}/(n-1)$).

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