A Stronger Derived Torelli Theorem for K3 Surfaces

Max Lieblich and Martin Olsson

Abstract In an earlier paper the notion of a filtered derived equivalence was introduced, and it was shown that if two K3 surfaces admit such an equivalence, then they are isomorphic. In this paper we study more refined aspects of filtered derived equivalences related to the action on the cohomological realizations of the Mukai motive. It is shown that if a filtered derived equivalence between K3 surfaces also preserves ample cones then one can find an isomorphism that induces the same map as the equivalence on the cohomological realizations.

1 Introduction

1.1 Let k be an algebraically closed field of odd positive characteristic and let X and Y be K3 surfaces over k. Let

$$\Phi : D(X) \to D(Y)$$

be an equivalence between their bounded triangulated categories of coherent sheaves given by a Fourier–Mukai kernel $P \in D(X \times Y)$, so Φ is the functor given by sending $M \in D(X)$ to

$$R\mathrm{pr}_{2*}(L\mathrm{pr}_1^* M \otimes^{\mathbb{L}} P).$$

As discussed in [16, 2.9] the kernel P also induces an isomorphism on rational Chow groups modulo numerical equivalence

$$\Phi_P^{A^*} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}.$$

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We can consider how a given equivalence Φ interacts with the codimension filtration on A^*, or how it acts on the ample cone of X inside $A^1(X)$. The underlying philosophy of this work is that tracking filtrations and ample cones (in ways we will make precise in Sect. 2) gives a semi-linear algebraic gadget that behaves a lot like a Hodge structure. In Sect. 2 we will define a notion of *strongly filtered* for an equivalence Φ that imposes conditions reminiscent of the classical Torelli theorem for K3 surfaces.

With this in mind, the purpose of this paper is to prove the following result.

Theorem 1.2 *If $\Phi_P : D(X) \to D(Y)$ is a strongly filtered equivalence, then there exists an isomorphism $\sigma : X \to Y$ such that the maps on the crystalline and étale realizations of the Mukai motive induced by Φ_P and σ agree.*

For the definition of the realizations of the Mukai motive see [16, Sect. 2]. In [16, Proof of 6.2] it is shown that any filtered equivalence can be modified to be strongly filtered. As a consequence, we get a new proof of the following result.

Theorem 1.3 ([16, 6.1]). *If $\Phi_P^{A^*}$ preserves the codimension filtrations on $A^*(X)_{num,\mathbb{Q}}$ and $A^*(Y)_{num,\mathbb{Q}}$ then X and Y are isomorphic.*

Whereas the original proof of Theorem 1.3 relied heavily on liftings to characteristic 0 and Hodge theory, the proof presented here works primarily in positive characteristic using algebraic methods.

In Sect. 8 we present a proof of Theorem 1.2 using certain results about "Kulikov models" in positive characteristic (see Sect. 5). This argument implicitly uses Hodge theory which is an ingredient in the proof of Theorem 5.3. In Sect. 9 we discuss a characteristic 0 variant of Theorem 1.2, and finally in the last Sect. 10 we explain how to bypass the use of the Hodge theory ingredient of Theorem 5.3. This makes the argument entirely algebraic, except for the Hodge theory aspects of the proof of the Tate conjecture. This also gives a different algebraic perspective on the statement that any Fourier–Mukai partner of a K3 surface is a moduli space of sheaves, essentially inverting the methods of [16].

The bulk of this paper is devoted to proving Theorem 1.2. The basic idea is to consider a certain moduli stack $\mathscr{S}_d$ classifying data $((X, \lambda), Y, P)$ consisting of a primitively polarized K3 surface (X, λ) with polarization of some degree d, a second K3 surface Y, and a complex $P \in D(X \times Y)$ defining a strongly filtered Fourier–Mukai equivalence $\Phi_P : D(X) \to D(Y)$. The precise definition is given in Sect. 3, where it is shown that $\mathscr{S}_d$ is an algebraic stack which is naturally a $\mathbb{G}_m$-gerbe over a Deligne–Mumford stack $\overline{\mathscr{S}}_d$ étale over the stack $\mathscr{M}_d$ classifying primitively polarized K3 surfaces of degree d. The map $\overline{\mathscr{S}}_d \to \mathscr{M}_d$ is induced by the map sending a collection $((X, \lambda), Y, P)$ to (X, λ). We then study the locus of points in $\mathscr{S}_d$ where Theorem 1.2 holds showing that it is stable under both generization and specialization. From this it follows that it suffices to consider the case when X and Y are supersingular where we can use Ogus' crystalline Torelli theorem [24, Theorem I].

Remark 1.4 Our restriction to odd characteristic is because we appeal to the Tate conjecture for K3 surfaces, proven in odd characteristics by Charles, Maulik, and Madapusi Pera [7, 21, 26], which at present is not known in characteristic 2.

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2 Strongly Filtered Equivalences

2.1 Let X and Y be K3 surfaces over an algebraically closed field k and let $P \in D(X \times Y)$ be an object defining an equivalence

$$\Phi_P : D(X) \to D(Y),$$

and let

$$\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}$$

denote the induced map on Chow groups modulo numerical equivalence and tensored with $\mathbb{Q}$. We say that Φ_P is *filtered* (resp. *strongly filtered*, resp. *Torelli*) if $\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}}$ preserves the codimension filtration (resp. is filtered, sends $(1, 0, 0)$ to $(1, 0, 0)$, and sends the ample cone of X to plus or minus the ample cone of Y; resp. is filtered, sends $(1, 0, 0)$ to $\pm(1, 0, 0)$, and sends the ample cone of X to the ample cone of Y).

Remark 2.2 Note that if P is strongly filtered then either P or $P[1]$ is Torelli. If P is Torelli then either P or $P[1]$ is strongly filtered.

Remark 2.3 Note that $A^1(X)$ is the orthogonal complement of $A^0(X) \oplus A^2(X)$ and similarly for Y. This implies that if Φ_P is filtered and sends $(1, 0, 0)$ to $\pm(1, 0, 0)$ then $\Phi_P(A^1(X)_{\mathrm{num},\mathbb{Q}}) \subset A^1(Y)_{\mathrm{num},\mathbb{Q}}$.

Remark 2.4 It is shown in [16, 6.2] that if $\Phi_P : D(X) \to D(Y)$ is a filtered equivalence, then there exists a strongly filtered equivalence $\Phi : D(X) \to D(Y)$. In fact it is shown there that Φ can be obtained from Φ_P by composing with a sequence of shifts, twists by line bundles, and spherical twists along (-2)-curves.

2.5 As noted in [16, 2.11] an equivalence Φ_P is filtered if and only if the induced map on Chow groups

$$\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}$$

sends $A^2(X)_{\mathrm{num},\mathbb{Q}}$ to $A^2(X)_{\mathrm{num},\mathbb{Q}}$.

Lemma 2.6 *Let ℓ be a prime invertible in k, let $\widetilde{H}(X, \mathbb{Q}_\ell)$ (resp. $\widetilde{H}(Y, \mathbb{Q}_\ell)$) denote the $\mathbb{Q}_\ell$-realization of the Mukai motive of X (resp. Y) as defined in [16, 2.4], and let*

$$\Phi_P^{\acute{e}t} : \widetilde{H}(X, \mathbb{Q}_\ell) \to \widetilde{H}(Y, \mathbb{Q}_\ell)$$

denote the isomorphism defined by P. Then Φ_P is filtered if and only if $\Phi_P^{\acute{e}t}$ preserves the filtrations by degree on $\widetilde{H}(X, \mathbb{Q}_\ell)$ and $\widetilde{H}(Y, \mathbb{Q}_\ell)$.

Proof By the same reasoning as in [16, 2.4] the map $\Phi_P^{\text{ét}}$ is filtered if and only if

$$\Phi_P^{\text{ét}}(H^4(X, \mathbb{Q}_\ell)) = H^4(Y, \mathbb{Q}_\ell).$$

Since the cycle class maps

$$A^2(X)_{\mathrm{num},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^4(X, \mathbb{Q}_\ell), \quad A^2(Y)_{\mathrm{num},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^4(Y, \mathbb{Q}_\ell)$$

are isomorphisms and the maps Φ_P and $\Phi_P^{\text{ét}}$ are compatible in the sense of [16, 2.10] it follows that if Φ_P is filtered then so is $\Phi_P^{\text{ét}}$. Conversely if $\Phi_P^{\text{ét}}$ is filtered then since the cycle class maps

$$A^*(X)_{\mathrm{num},\mathbb{Q}} \to \widetilde{H}(X, \mathbb{Q}_\ell), \quad A^*(Y)_{\mathrm{num},\mathbb{Q}} \to \widetilde{H}(Y, \mathbb{Q}_\ell)$$

are injective it follows that Φ_P is also filtered. □

Remark 2.7 The same proof as in Lemma 2.6 gives variant results for crystalline cohomology and in characteristic 0 de Rham cohomology.

The condition that Φ_P takes the ample cone to plus or minus the ample cone appears more subtle. A useful observation in this regard is the following.

Lemma 2.8 *Let $P \in D(X \times Y)$ be an object defining a filtered equivalence $\Phi_P : D(X) \to D(Y)$ such that $\Phi_P^{A^*_{num}}$ sends $(1, 0, 0)$ to $(1, 0, 0)$. Then Φ_P is strongly filtered if and only if for some ample invertible sheaf L on X the class $\Phi_P^{A^*_{num}}(L) \in NS(Y)_{\mathbb{Q}}$ is plus or minus an ample class.*

Proof Following [24, p. 366] define

$$V_X := \{x \in NS(X)_{\mathbb{R}} | x^2 > 0, \text{ and } \langle x, \delta \rangle \neq 0 \text{ for all } \delta \in NS(X) \text{ with } \delta^2 = -2\},$$

and define V_Y similarly. Since $\Phi_P^{A^*_{\mathrm{num}}}$ is an isometry it induces an isomorphism

$$\sigma : V_X \to V_Y.$$

By [24, Proposition 1.10 and Remark 1.10.9] the ample cone C_X (resp. C_Y) of X (resp. Y) is a connected component of V_X (resp. V_Y) and therefore either $\sigma(C_X) \cap C_Y = \emptyset$ or $\sigma(C_X) = C_Y$, and similarly $\sigma(-C_X) \cap C_Y = \emptyset$ or $\sigma(-C_X) = C_Y$. □

Proposition 2.9 *Let X and Y be K3-surfaces over a scheme S and let $P \in D(X \times_S Y)$ be a relatively perfect complex. Assume that X/S is projective. Then the set of points $s \in S$ for which the induced transformation on the derived category of the geometric fibers*

$$\Phi_{P_{\bar{s}}} : D(X_{\bar{s}}) \to D(Y_{\bar{s}})$$

is a strongly filtered equivalence is an open subset of S.

Proof By a standard reduction we may assume that S is of finite type over $\mathbb{Z}$.

First note that the condition that $\Phi_{P_{\bar{s}}}$ is an equivalence is an open condition. Indeed by [12, 5.9] if $P^\vee$ denotes $\mathscr{R}Hom(P, \mathscr{O}_{X\times_S Y})$ and Q denotes $P^\vee[2]$, then for every $s \in S$ the functor $\Phi_{Q_{\bar{s}}}$ is both left and right adjoint to $\Phi_{P_{\bar{s}}}$, and the adjunction maps are induced by morphisms of complexes

$$Rp_{13*}(Lp_{12}^*P \otimes^{\mathbb{L}} Lp_{23}^*Q) \to \Delta_{X*}\mathscr{O}_X, \quad Rq_{13*}(Lq_{12}^*Q \otimes Lq_{23}^*P) \to \Delta_{Y*}\mathscr{O}_Y,$$

where we write p_{ij} (resp. q_{ij}) for the projection from $X \times_S Y \times_S X$ (resp. $Y \times_S X \times_S Y$) to the product of the i-th and j-th factor. Since the condition that these adjunction maps are quasi-isomorphisms is clearly an open condition on S it follows that the condition that $\Phi_{P_{\bar{s}}}$ is an equivalence is open.

Replacing S by an open set we may therefore assume that $\Phi_{P_{\bar{s}}}$ is an equivalence in every fiber.

Next we show that the condition that Φ_P is filtered is an open and closed condition. For this we may assume we have a prime ℓ invertible in S. Let $f_X : X \to S$ (resp. $f_Y : Y \to S$) be the structure morphism. Define $\widetilde{\mathscr{H}}_{X/S}$ to be the lisse $\mathbb{Q}_\ell$-sheaf on S given by

$$\widetilde{\mathscr{H}}_{X/S} := (R^0 f_{X*}\mathbb{Q}_\ell(-1)) \oplus (R^2 f_{X*}\mathbb{Q}_\ell) \oplus (R^4 f_{X*}\mathbb{Q}_\ell)(1),$$

and define $\widetilde{\mathscr{H}}_{Y/S}$ similarly. The kernel P then induces a morphism of lisse sheaves

$$\Phi_{P/S}^{\text{ét},\ell} : \widetilde{\mathscr{H}}_{X/S} \to \widetilde{\mathscr{H}}_{Y/S}$$

whose restriction to each geometric fiber is the map on the $\mathbb{Q}_\ell$-realization of the Mukai motive as in [16, 2.4]. In particular, $\Phi_{P/S}^{\text{ét},\ell}$ is an isomorphism. By Lemma 2.6 for every geometric point $\bar{s} \to S$ the map $\Phi_{P_{\bar{s}}}$ is filtered if and only if the stalk $\Phi_{P/S,\bar{s}}^{\text{ét},\ell}$ preserves the filtrations on $\widetilde{\mathscr{H}}_{X/S}$ and $\widetilde{\mathscr{H}}_{Y/S}$. In particular this is an open and closed condition on S. Shrinking on S if necessary we may therefore further assume that $\Phi_{P_{\bar{s}}}$ is filtered for every geometric point $\bar{s} \to S$.

It remains to show that in this case the set of points s for which Φ_P takes the ample cone $C_{X_{\bar{s}}}$ of $X_{\bar{s}}$ to $\pm C_{Y_{\bar{s}}}$ is an open subset of S. For this we can choose, by our assumption that X/S is projective, a relatively ample invertible sheaf L on X. Define

$$M := \det(R\mathrm{pr}_{2*}(L\mathrm{pr}_1^*(L) \otimes P)),$$

an invertible sheaf on Y. Then by Lemma 2.8 for a point $s \in S$ the transformation $\Phi_{P_{\bar{s}}}$ is strongly filtered if and only if the restriction of M to the fiber $Y_{\bar{s}}$ is plus or minus the class of an ample divisor. By openness of the ample locus [9, III, 4.7.1] we get that being strongly filtered is an open condition. □

3 Moduli Spaces of K3 Surfaces

3.1 For an integer d invertible in k let $\mathscr{M}_d$ denote the stack over k whose fiber over a scheme T is the groupoid of pairs (X, λ) where X/T is a proper smooth algebraic space all of whose geometric fibers are K3 surfaces and $\lambda : T \to \mathrm{Pic}_{X/T}$ is a morphism to the relative Picard functor such that in every geometric fiber λ is given by a primitive ample line bundle L_λ whose self-intersection is $2d$. The following theorem summarizes the properties of the stack $\mathscr{M}_d$ that we will need (see [17, 2.10] for a more detailed discussion considering also the case when the characteristic of k divides d).

Theorem 3.2 *(i) $\mathscr{M}_d$ is a Deligne–Mumford stack, smooth over k of relative dimension* 19.

(ii) The geometric fiber of $\mathscr{M}_d$ is irreducible (here we use that d is invertible in k).

(iii) The locus $\mathscr{M}_{d,\infty} \subset \mathscr{M}_d$ classifying supersingular K3 surfaces is closed of dimension ≥ 9.

Proof A review of (i) and (iii) can be found in [23, p. 1]. Statement (ii) can be found in [17, 2.10 (3)]. □

Remark 3.3 The stack $\mathscr{M}_d$ is defined over $\mathbb{Z}$, and it follows from (ii) that the geometric generic fiber of $\mathscr{M}_d$ is irreducible (this follows also from the Torelli theorem over $\mathbb{C}$ and the resulting description of $\mathscr{M}_{d,\mathbb{C}}$ as a period space). Furthermore over $\mathbb{Z}[1/d]$ the stack $\mathscr{M}_d$ is smooth. In what follows we denote this stack over $\mathbb{Z}[1/d]$ by $\mathscr{M}_{d,\mathbb{Z}[1/d]}$ and reserve the notation $\mathscr{M}_d$ for its reduction to k.

Remark 3.4 Note that in the definition of $\mathscr{M}_d$ we consider ample invertible sheaves, and don't allow contractions in the corresponding morphism to projective space.

3.5 Let $\mathscr{S}_d$ denote the fibered category over k whose fiber over a scheme S is the groupoid of collections of data

$$((X, \lambda), Y, P), \tag{3.5.1}$$

where $(X, \lambda) \in \mathscr{M}_d(S)$ is a polarized K3 surface, Y/S is a second K3 surface over S, and $P \in D(X \times_S Y)$ is an S-perfect complex such that for every geometric point $\bar{s} \to S$ the induced functor

$$\Phi_{P_{\bar{s}}} : D(X_{\bar{s}}) \to D(Y_{\bar{s}})$$

is strongly filtered.

Theorem 3.6 *The fibered category $\mathscr{S}_d$ is an algebraic stack locally of finite type over k.*

Proof By fppf descent for algebraic spaces we have descent for both polarized and unpolarized K3 surfaces.

To verify descent for the kernels P, consider an object (3.5.1) over a scheme S. Let $P^\vee$ denote $\mathscr{R}Hom(P, \mathscr{O}_X)$. Since P is a perfect complex we have $\mathscr{R}Hom(P, P) \simeq P^\vee \otimes P$. By [15, 2.1.10] it suffices to show that for all geometric points $\bar{s} \to S$ we have $H^i(X_{\bar{s}} \times Y_{\bar{s}}, P_{\bar{s}}^\vee \otimes P_{\bar{s}}) = 0$ for $i < 0$. This follows from the following result (we discuss Hochschild cohomology further in Sect. 4 below):

Lemma 3.7 ([28, 5.6], [11, 5.1.8]). *Let X and Y be K3 surfaces over an algebraically closed field k, and let $P \in D(X \times Y)$ be a complex defining a Fourier–Mukai equivalence $\Phi_P : D(X) \to D(Y)$. Denote by $HH^*(X)$ the Hochschild cohomology of X defined as*

$$RHom_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X).$$

(i) There is a canonical isomorphism $Ext^*_{X\times Y}(P, P) \simeq HH^*(X)$.
(ii) $Ext^i_{X\times Y}(P, P) = 0$ for $i < 0$ and $i = 1$.
(iii) The natural map $k \to Ext^0_{X\times Y}(P, P)$ is an isomorphism.

Proof Statement (i) is [28, 5.6]. Statements (ii) and (iii) follow immediately from this, since $HH^1(X) = 0$ for a K3 surface. □

Next we show that for an object (3.5.1) the polarization λ on X induces a polarization λ_Y on Y. To define λ_Y we may work étale locally on S so may assume there exists an ample invertible sheaf L on X defining λ. The complex

$$\Phi_P(L) := R\mathrm{pr}_{2*}(\mathrm{pr}_1^*L \otimes^{\mathbb{L}} P)$$

is S-perfect, and therefore a perfect complex on Y. Let M denote the determinant of $\Phi_P(L)$, so M is an invertible sheaf on Y. By our assumption that Φ^{P_s} is strongly filtered for all $s \in S$, the restriction of M to any fiber is either ample or antiample. It follows that either M or $M^\vee$ is a relatively ample invertible sheaf and we define λ_Y to be the resulting polarization on Y. Note that this does not depend on the choice of line bundle L representing λ and therefore by descent λ_Y is defined even when no such L exists.

The degree of λ_Y is equal to d. Indeed if $s \in S$ is a point then since Φ^{P_s} is strongly filtered the induced map $NS(X_{\bar{s}}) \to NS(Y_{\bar{s}})$ is compatible with the intersection pairings and therefore $\lambda_Y^2 = \lambda^2 = 2d$.

From this we deduce that $\mathscr{S}_d$ is algebraic as follows. We have a morphism

$$\mathscr{S}_d \to \mathscr{M}_d \times \mathscr{M}_d, \quad ((X, \lambda), Y, P) \mapsto ((X, \lambda), (Y, \lambda_Y)), \tag{3.7.1}$$

and $\mathscr{M}_d \times \mathscr{M}_d$ is an algebraic stack. Let $\mathscr{X}$ (resp. $\mathscr{Y}$) denote the pullback to $\mathscr{M}_d \times \mathscr{M}_d$ of the universal family over the first factor (resp. second factor). Sending a triple $((X, \lambda), Y, P)$ to P then realizes $\mathscr{S}_d$ as an open substack of the stack over $\mathscr{M}_d \times \mathscr{M}_d$ of simple universally gluable complexes on $\mathscr{X} \times_{\mathscr{M}_d \times \mathscr{M}_d} \mathscr{Y}$ (see for example [16, Sect. 5]). □

3.8 Observe that for any object $((X, \lambda), Y, P) \in \mathscr{S}_d$ over a scheme S there is an inclusion

$$\mathbb{G}_m \hookrightarrow \underline{\mathrm{Aut}}_{\mathscr{S}_d}((X, \lambda), Y, P)$$

giving by scalar multiplication by P. We can therefore form the rigidification of $\mathscr{S}_d$ with respect to $\mathbb{G}_m$ (see for example [2, Sect. 5]) to get a morphism

$$g : \mathscr{S}_d \to \overline{\mathscr{S}}_d$$

realizing $\mathscr{S}_d$ as a $\mathbb{G}_m$-gerbe over another algebraic stack $\overline{\mathscr{S}}_d$. By the universal property of rigidification the map $\mathscr{S}_d \to \mathscr{M}_d$ sending $((X, \lambda), Y, P)$ to (X, λ) induces a morphism

$$\pi : \overline{\mathscr{S}}_d \to \mathscr{M}_d. \tag{3.8.1}$$

Theorem 3.9 *The stack $\overline{\mathscr{S}}_d$ is Deligne–Mumford and the map (3.8.1) is étale.*

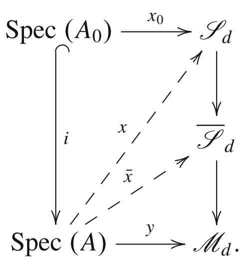
Proof Consider the map (3.7.1). By the universal property of rigidification this induces a morphism

$$q : \overline{\mathscr{S}}_d \to \mathscr{M}_d \times \mathscr{M}_d.$$

Since $\mathscr{M}_d \times \mathscr{M}_d$ is Deligne–Mumford, to prove that $\overline{\mathscr{S}}_d$ is a Deligne–Mumford stack it suffices to show that q is representable. This follows from Lemma 3.7 (iii) which implies that for any object $((X, \lambda), Y, P)$ over a scheme S the automorphisms of this object which map under q to the identity are given by scalar multiplication on P by elements of $\mathscr{O}_S^*$.

It remains to show that the map (3.8.1) is étale, and for this it suffices to show that it is formally étale.

Let $A \to A_0$ be a surjective map of artinian local rings with kernel I annhilated by the maximal ideal of A, and let k denote the residue field of A_0 so I can be viewed as a k-vector space. Let $((X_0, \lambda_0), Y_0, P_0) \in \mathscr{S}_d(A_0)$ be an object and let $(X, \lambda) \in \mathscr{M}_d(A)$ be a lifting of (X_0, λ_0) so we have a commutative diagram of solid arrows



Since $\mathscr{S}_d$ is a $\mathbb{G}_m$-gerbe over $\overline{\mathscr{S}}_d$, the obstruction to lifting a map $\bar{x}$ as indicated to a morphism x is given by a class in $H^2(\text{Spec }(A), \widetilde{I}) = 0$, and therefore any such map $\bar{x}$ can be lifted to a map x. Furthermore, the set of isomorphism classes of such liftings x of $\bar{x}$ is given by $H^1(\text{Spec }(A), \widetilde{I}) = 0$ so in fact the lifting x is unique up to isomorphism. The isomorphism is not unique but determined up to the action of

$$\text{Ker}(A^* \to A_0^*) \simeq I.$$

From this it follows that it suffices to show the following:

(i) The lifting (X, λ) of (X_0, λ_0) can be extended to a lifting $((X, \lambda), Y, P)$ of $((X_0, \lambda_0), Y_0, P_0)$.
(ii) This extension $((X, \lambda), Y, P)$ of (X, λ) is unique up to isomorphism.
(iii) The automorphisms of the triple $((X, \lambda), Y, P)$ which are the identity on (X, λ) and reduce to the identity over A_0 are all given by scalar multiplication on P by elements of $1 + I \subset A^*$.

Statement (i) is shown in [16, 6.3].

Next we prove the uniqueness statements in (ii) and (iii). Following the notation of [16, Discussion preceding 5.2], let $s\mathscr{D}_{X/A}$ denote the stack of simple, universally gluable, relatively perfect complexes on X, and let $sD_{X/A}$ denote its rigidification with respect to the $\mathbb{G}_m$-action given by scalar multiplication. The complex P_0 on $X_0 \times_{A_0} Y_0$ defines a morphism

$$Y_0 \to sD_{X/A} \otimes_A A_0$$

which by [16, 5.2 (ii)] is an open imbedding. Any extension of (X, λ) to a lifting $((X, \lambda), Y, P)$ defines an open imbedding $Y \hookrightarrow sD_{X/A}$. This implies that Y, viewed as a deformation of Y_0 for which there exists a lifting P of P_0 to $X \times_A Y$, is unique up to unique isomorphism.

Let Y denote the unique lifting of Y_0 to an open subspace of $sD_{X/A}$. By [15, 3.1.1 (2)] the set of isomorphism classes of liftings of P_0 to $X \times_A Y$ is a torsor under

$$\text{Ext}^1_{X_k \times Y_k}(P_k, P_k) \otimes I,$$

which is 0 by Lemma 3.7 (ii). From this it follows that P is unique up to isomorphism, and also by Lemma 3.7 (iii) we get the statement that the only infinitesimal automorphisms of the triple $((X, \lambda), Y, P)$ are given by scalar multiplication by elements of $1 + I$. □

3.10 There is an automorphism

$$\sigma : \mathscr{S}_d \to \mathscr{S}_d$$

satisfying $\sigma^2 = \mathrm{id}$. This automorphism is defined by sending a triple $((X, \lambda), Y, P)$ to $((Y, \lambda_Y), X, P^\vee[2])$. This automorphism induces an involution $\bar{\sigma} : \overline{\mathscr{S}}_d \to \overline{\mathscr{S}}_d$ over the involution $\gamma : \mathscr{M}_d \times \mathscr{M}_d \to \mathscr{M}_d \times \mathscr{M}_d$ switching the factors.

Remark 3.11 In fact the stack $\mathscr{S}_d$ is defined over $\mathbb{Z}[1/d]$ and Theorems 3.6 and 3.9 also hold over $\mathbb{Z}[1/d]$. In what follows we write $\mathscr{S}_{d,\mathbb{Z}[1/d]}$ for this stack over $\mathbb{Z}[1/d]$.

4 Deformations of Autoequivalences

In this section, we describe the obstructions to deforming Fourier–Mukai equivalences. The requisite technical machinery for this is worked out in [13, 14]. The results of this section will play a crucial role in Sect. 6.

Throughout this section let k be a perfect field of positive characteristic p and ring of Witt vectors W. For an integer n let R_n denote the ring $k[t]/(t^{n+1})$, and let R denote the ring $k[[t]]$.

4.1 Let X_{n+1}/R_{n+1} be a smooth proper scheme over R_{n+1} with reduction X_n to R_n. We then have the associated *relative Kodaira–Spencer class*, defined in [13, p. 486], which is the morphism in $D(X_n)$

$$\kappa_{X_n/X_{n+1}} : \Omega^1_{X_n/R_n} \to \mathscr{O}_{X_n}[1]$$

defined as the morphism corresponding to the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X_n} \xrightarrow{\cdot dt} \Omega^1_{X_{n+1}/k}|_{X_n} \longrightarrow \Omega^1_{X_n/R_n} \longrightarrow 0.$$

4.2 We also have the *relative universal Atiyah class* which is a morphism

$$\alpha_n : \mathscr{O}_{\Delta_n} \to i_{n*}\Omega^1_{X_n/R_n}[1]$$

in $D(X_n \times_{R_n} X_n)$, where $i_n : X_n \to X_n \times_{R_n} X_n$ is the diagonal morphism and $\mathscr{O}_{\Delta_n}$ denotes $i_{n*}\mathscr{O}_{X_n}$.

This map α_n is given by the class of the short exact sequence

$$0 \to I/I^2 \to \mathscr{O}_{X_n \times_{R_n} X_n}/I^2 \to \mathscr{O}_{\Delta_n} \to 0,$$

where $I \subset \mathscr{O}_{X_n \times_{R_n} X_n}$ is the ideal of the diagonal. Note that to get the morphism α_n we need to make a choice of isomorphism $I/I^2 \simeq \Omega^1_{X_n/R_n}$, which implies that the relative universal Atiyah class is not invariant under the map switching the factors, but rather changes by -1.

4.3 Define the *relative Hochschild cohomology* of X_n/R_n by

$$HH^*(X_n/R_n) := \mathrm{Ext}^*_{X_n \times_{R_n} X_n}(\mathscr{O}_{\Delta_n}, \mathscr{O}_{\Delta_n}).$$

The composition

$$\mathscr{O}_{\Delta_n} \xrightarrow{\alpha_n} i_{n*}\Omega^1_{X_n/R_n}[1] \xrightarrow{i_{n*}\kappa_{X_n/X_{n+1}}} \mathscr{O}_{\Delta_n}[2]$$

is a class

$$\nu_{X_n/X_{n+1}} \in HH^2(X_n/R_n).$$

4.4 Suppose now that Y_n/R_n is a second smooth proper scheme with a smooth lifting Y_{n+1}/R_{n+1} and that $E_n \in D(X_n \times_{R_n} Y_n)$ is a R_n-perfect complex.

Consider the class

$$\nu := \nu_{X_n \times_{R_n} Y_n/X_{n+1} \times_{R_{n+1}} Y_{n+1}} : \mathscr{O}_{\Delta_{n, X_n \times_{R_n} Y_n}} \to \mathscr{O}_{\Delta_{n, X_n \times_{R_n} Y_n}}[2].$$

Viewing this is a morphism of Fourier–Mukai kernels

$$D(X_n \times_{R_n} Y_n) \to D(X_n \times_{R_n} Y_n)$$

and applying it to E_n we get a class

$$\omega(E_n) \in \mathrm{Ext}^2_{X_n \times_{R_n} Y_n}(E_n, E_n).$$

In the case when

$$\mathrm{Ext}^1_{X_0 \times Y_0}(E_0, E_0) = 0,$$

which will hold in the cases of interest in this paper, we know by [13, Lemma 3.2] that the class $\omega(E_n)$ is 0 if and only if E_n lifts to a perfect complex on $X_{n+1} \times_{R_{n+1}} Y_{n+1}$.

4.5 To analyze the class $\omega(E_n)$ it is useful to translate it into a statement about classes in $HH^2(Y_n/R_n)$. This is done using Toda's argument [28, Proof of 5.6]. Denote by p_{ij} the various projections from $X_n \times_{R_n} X_n \times_{R_n} Y_n$, and let

$$E_n \circ : D(X_n \times_{R_n} X_n) \to D(X_n \times_{R_n} Y_n)$$

denote the Fourier–Mukai functor defined by the pushforward of $p_{23}^* E_n$ along the morphism

$$\mathrm{id}_{X_n} \times \Delta_{X_n} \times \mathrm{id}_{Y_n} : X_n \times_{R_n} \times X_n \times_{R_n} Y_n \to (X_n \times_{R_n} X_n) \times_{R_n} (X_n \times_{R_n} Y_n).$$

So for an object $K \in D(X_n \times_{R_n} X_n)$ the complex $E_n \circ K \in D(X_n \times_{R_n} Y_n)$ represents the Fourier–Mukai transform $\Phi_{E_n} \circ \Phi_K$, and is given explicitly by the complex

$$p_{13*}(p_{12}^* K \otimes p_{23}^* E_n).$$

As in loc. cit. the diagram

$$\begin{array}{ccc} D(X_n) & \xrightarrow{p_1^*(-)\otimes E_n} & \\ \downarrow{\scriptstyle i_{n*}} & \searrow & \\ D(X_n \times_{R_n} X_n) & \xrightarrow{E_n \circ} & D(X_n \times_{R_n} Y_n) \end{array}$$

commutes.

In particular we get a morphism

$$\eta_X^* : HH^*(X_n/R_n) \to \mathrm{Ext}^*_{X_n \times_{R_n} Y_n}(E_n, E_n).$$

Now assume that both X_n and Y_n have relative dimension d over R_n and that the relative canonical sheaves of X_n and Y_n over R_n are trivial. Let $E_n^\vee$ denote $\mathscr{R}Hom(E_n, \mathscr{O}_{X_n \times_{R_n} Y_n})$ viewed as an object of $D(Y_n \times_{R_n} X_n)$. In this case the functor

$$\Phi_{E_n^\vee[d]} : D(Y_n) \to D(X_n)$$

is both a right and left adjoint of Φ_{E_n} [4, 4.5]. By the same argument, the functor

$$\circ E_n^\vee[d] : D(X_n \times_{R_n} Y_n) \to D(Y_n \times_{R_n} Y_n),$$

defined in the same manner as $E_n \circ$ has left and right adjoint given by

$$\circ E_n : D(Y_n \times_{R_n} Y_n) \to D(X_n \times_{R_n} Y_n).$$

Composing with the adjunction maps

$$\alpha : \mathrm{id} \to \circ E_n \circ E_n^\vee[d], \quad \beta : \circ E_n \circ E_n^\vee[d] \to \mathrm{id} \tag{4.5.1}$$

applied to the diagonal $\mathscr{O}_{\Delta_{Y_n}}$ we get a morphism

$$\eta_{Y*} : \mathrm{Ext}^*_{X_n \times_{R_n} Y_n}(E_n, E_n) \to HH^*(Y_n/R_n).$$

We denote the composition

$$\eta_{Y*}\eta_X^* : HH^*(X_n/R_n) \to HH^*(Y_n/R_n)$$

by $\Phi_{E_n}^{HH^*}$. In the case when E_n defines a Fourier–Mukai equivalence this agrees with the standard definition (see for example [28]).

4.6 Evaluating the adjunction maps (4.5.1) on $\mathscr{O}_{\Delta_{Y_n}}$ we get a morphism

$$\mathscr{O}_{\Delta_{Y_n}} \xrightarrow{\alpha} \mathscr{O}_{\Delta_{Y_n}} \circ E_n \circ E_n^\vee[d] \xrightarrow{\beta} \mathscr{O}_{\Delta_{Y_n}}. \tag{4.6.1}$$

We say that E_n is *admissible* if this composition is the identity map.

If E_n is a Fourier–Mukai equivalence, then it is clear that E_n is admissible. Another example is if there exists a lifting $(\mathscr{X}, \mathscr{Y}, \mathscr{E})$ of (X_n, Y_n, E_n) to R, where $\mathscr{X}$ and $\mathscr{Y}$ are smooth proper R-schemes with trivial relative canonical bundles and $\mathscr{E}$ is a R-perfect complex on $\mathscr{X} \times_R \mathscr{Y}$, such that the restriction $\mathscr{E}$ to the generic fiber defines a Fourier–Mukai equivalence. Indeed in this case the map (4.6.1) is the reduction of the corresponding map $\mathscr{O}_{\Delta_{\mathscr{Y}}} \to \mathscr{O}_{\Delta_{\mathscr{Y}}}$ defined over R, which in turn is determined by its restriction to the generic fiber.

4.7 Consider Hochschild homology

$$HH_i(X_n/R_n) := H^{-i}(X_n, Li_n^*\mathscr{O}_{\Delta_n}).$$

As explained for example in [5, Sect. 5] we also get a map

$$\Phi_{E_n}^{HH_*} : HH_*(X_n/R_n) \to HH_*(Y_n/R_n).$$

Hochschild homology is a module over Hochschild cohomology, and an exercise (that we do not write out here) shows that the following diagram

$$\begin{array}{ccc} HH^*(X_n/R_n) \times HH_*(X_n/R_n) & \xrightarrow{\Phi_{E_n}^{HH^*} \times \Phi_{E_n}^{HH_*}} & HH^*(Y_n/R_n) \times HH_*(Y_n/R_n) \\ \downarrow \text{mult} & & \downarrow \text{mult} \\ HH_*(X_n/R_n) & \xrightarrow{\Phi_{E_n}^{HH_*}} & HH_*(Y_n/R_n) \end{array}$$

commutes.

Remark 4.8 We thank the referee for pointing out that the necessary functoriality of Hochschild homology was settled by Keller, Swan, and Weibel by the mid-1990s. See for example [10], [30, 9.8.19], and [27]. The Ref. [5] provides a very readable account of the results we need here.

4.9 Using this we can describe the obstruction $\omega(E_n)$ in a different way, assuming that E_n is admissible. First note that viewing the relative Atiyah class of $X_n \times_{R_n} Y_n$ as a morphism of Fourier–Mukai kernels we get the Atiyah class of E_n which is a morphism

$$A(E_n) : E_n \to E_n \otimes \Omega^1_{X_n \times_{R_n} Y_n/R_n}[1]$$

in $D(X_n \times_{R_n} Y_n)$. There is a natural decomposition

$$\Omega^1_{X_n \times_{R_n} Y_n/R_n} \simeq p_1^*\Omega^1_{X_n/R_n} \oplus p_2^*\Omega^1_{Y_n/R_n},$$

so we can write $A(E_n)$ as a sum of two maps

$$A(E_n)_X : E_n \to E_n \otimes p_1^*\Omega^1_{X_n/R_n}[1], \quad A(E_n)_Y : E_n \to E_n \otimes p_2^*\Omega^1_{Y_n/R_n}[1].$$

Similarly the Kodaira–Spencer class of $X_n \times_{R_n} Y_n$ can be written as the sum of the two pullbacks

$$p_1^*\kappa_{X_n/X_{n+1}} : p_1^*\Omega^1_{X_n/R_n} \to p_1^*\mathscr{O}_{X_n}[1], \quad p_2^*\kappa_{Y_n/Y_{n+1}} : p_2^*\Omega^1_{Y_n/R_n} \to p_2^*\mathscr{O}_{Y_n}[1].$$

It follows that the obstruction $\omega(E_n)$ can be written as a sum

$$\omega(E_n) = (p_1^*\kappa_{X_n/X_{n+1}} \circ A(E_n)_X) + (p_2^*\kappa(Y_n/Y_{n+1}) \circ A(E_n)_Y).$$

Now by construction we have

$$\eta^*_{X_n}(\nu_{X_n/X_{n+1}}) = p_1^*\kappa_{X_n/X_{n+1}} \circ A(E_n)_X,$$

and

$$\eta_{Y_n*}(p_2^*\kappa(Y_n/Y_{n+1}) \circ A(E_n)_Y) = -\nu_{Y_n/Y_{n+1}},$$

the sign coming from the asymmetry in the definition of the relative Atiyah class (it is in the verification of this second formula that we use the assumption that E_n is admissible). Summarizing we find the formula

$$\eta_{Y_n*}(\omega(E_n)) = \Phi^{HH^*}_{E_n}(\nu_{X_n/X_{n+1}}) - \nu_{Y_n/Y_{n+1}}. \tag{4.9.1}$$

In the case when Φ_{E_n} is an equivalence the maps η_{Y_n*} and $\eta^*_{X_n}$ are isomorphisms, so the obstruction $\omega(E_n)$ vanishes if and only if we have

$$\Phi^{HH^*}_{E_n}(\nu_{X_n/X_{n+1}}) - \nu_{Y_n/Y_{n+1}} = 0.$$

Remark 4.10 By [13, Remark 2.3 (iii)], the functor Φ_{E_n} is an equivalence if and only if $\Phi_{E_0} : D(X_0) \to D(Y_0)$ is an equivalence.

Corollary 4.11 *Suppose $F_n \in D(X_n \times_{R_n} Y_n)$ defines a Fourier–Mukai equivalence, and that $E_n \in D(X_n \times_{R_n} Y_n)$ is another admissible R_n-perfect complex such that $\Phi^{HH^*}_{F_n} = \Phi^{HH^*}_{E_n}$. If E_n lifts to a R_{n+1}-perfect complex $E_{n+1} \in D(X_{n+1} \times_{R_{n+1}} Y_{n+1})$ then so does F_n.*

Proof Indeed the condition that $\Phi^{HH^*}_{F_n} = \Phi^{HH^*}_{E_n}$ ensures that

$$\eta_{Y_n*}(\omega(E_n)) = \eta_{Y_n*}(\omega(F_n)),$$

and since $\omega(E_n) = 0$ we conclude that $\omega(F_n) = 0$. □

4.12 The next step is to understand the relationship between $\Phi^{HH^*}_{E_n}$ and the action of Φ_{E_n} on the cohomological realizations of the Mukai motive.

Assuming that the characteristic p is bigger than the dimension of X_0 (which in our case will be a K3 surface so we just need $p > 2$) we can exponentiate the relative Atiyah class to get a map

$$\exp(\alpha_n) : \mathscr{O}_{\Delta_n} \to \oplus_i i_{n*}\Omega^i_{X_n/R_n}$$

which by adjunction defines a morphism

$$Li_n^*\mathscr{O}_{\Delta_n} \to \oplus_i \Omega^i_{X_n/R_n} \tag{4.12.1}$$

in $D(X_n)$. By [1, Theorem 0.7], which also holds in positive characteristic subject to the bounds on dimension, this map is an isomorphism. We therefore get an isomorphism

$$I^{HKR} : HH^*(X_n/R_n) \to HT^*(X_n/R_n),$$

where we write

$$HT^*(X_n/R_n) := \oplus_{p+q=*} H^p(X_n, \bigwedge^q T_{X_n/R_n}).$$

We write

$$I^K_{X_n} : HH^*(X_n/R_n) \to HT^*(X_n/R_n)$$

for the composition of I^{HKR} with multiplication by the inverse square root of the Todd class of X_n/R_n, as in [6, 1.7].

The isomorphism (4.12.1) also defines an isomorphism

$$I_{HKR} : HH_*(X_n/R_n) \to H\Omega_*(X_n/R_n),$$

where

$$H\Omega_*(X_n/R_n) := \oplus_{q-p=*} H^p(X_n, \Omega^q_{X_n/R_n}).$$

We write

$$I_K^{X_n} : HH_*(X_n/R_n) \to H\Omega_*(X_n/R_n)$$

for the composition of I_{HKR} with multiplication by the square root of the Todd class of X_n/R_n.

We will consider the following condition $(\star)$ on a R_n-perfect complex $E_n \in D(X_n \times_{R_n} Y_n)$:

$(\star)$ The diagram

$$\begin{array}{ccc} HH_*(X_n/R_n) & \xrightarrow{\Phi^{HH_*}_{E_n}} & HH_*(Y_n/R_n) \\ \downarrow{\scriptstyle I_K^{X_n}} & & \downarrow{\scriptstyle I_K^{Y_n}} \\ H\Omega_*(X_n/R_n) & \xrightarrow{\Phi^{H\Omega_*}_{E_n}} & H\Omega_*(Y_n/R_n) \end{array}$$

commutes.

Remark 4.13 We expect this condition to hold in general. Over a field of characteristic 0 this is shown in [19, 1.2]. We expect that a careful analysis of denominators occurring of their proof will verify $(\star)$ quite generally with some conditions on the characteristic relative to the dimension of the schemes. However, we will not discuss this further in this paper.

4.14 There are two cases we will consider in this paper were $(\star)$ is known to hold:

(i) If $E_n = \mathscr{O}_{\Gamma_n}$ is the structure sheaf of the graph of an isomorphism $\gamma_n : X_n \to Y_n$. In this case the induced maps on Hochschild cohomology and $H\Omega_*$ are simply given by pushforward γ_{n*} and condition $(\star)$ immediately holds.

(ii) Suppose $B \to R_n$ is a morphism from an integral domain B which is flat over W and that there exists a lifting $(\mathscr{X}, \mathscr{Y}, \mathscr{E})$ of (X_n, Y_n, E_n) to B, where $\mathscr{X}$ and $\mathscr{Y}$ are proper and smooth over B and $\mathscr{E} \in D(\mathscr{X} \times_B \mathscr{Y})$ is a B-perfect complex pulling back to E_n. Suppose further that the groups $HH^*(\mathscr{X}/B)$ and $HH^*(\mathscr{Y}/B)$ are flat over B and their formation commutes with base change (this holds for example if $\mathscr{X}$ and $\mathscr{Y}$ are K3 surfaces). Then $(\star)$ holds. Indeed it suffices to verify the commutativity of the corresponding diagram over B, and this in turn can be verified after passing to the field of fractions of B. In this case the result holds by [19, 1.2].

Lemma 4.15 *Let $E_n, F_n \in D(X_n \times_{R_n} Y_n)$ be two R_n-perfect complexes satisfying condition $(\star)$. Suppose further that the maps $\Phi^{crys}_{E_0}$ and $\Phi^{crys}_{F_0}$ on the crystalline realizations $\widetilde{H}(X_0/W) \to \widetilde{H}(Y_0/W)$ of the Mukai motive are equal. Then the maps $\Phi^{HH_*}_{E_n}$ and $\Phi^{HH_*}_{F_n}$ are also equal. Furthermore if the maps on the crystalline realizations are isomorphisms then $\Phi^{HH_*}_{E_n}$ and $\Phi^{HH_*}_{F_n}$ are also isomorphisms.*

Proof Since $H\Omega_*(X_n/R_n)$ (resp. $H\Omega_*(Y_n/R_n)$) is obtained from the de Rham realization $\widetilde{H}_{\mathrm{dR}}(X_n/R_n)$ (resp. $\widetilde{H}_{\mathrm{dR}}(X_n/R_n)$) of the Mukai motive of X_n/R_n (resp. Y_n/R_n) by passing to the associated graded, it suffices to show that the two maps

$$\Phi^{\mathrm{dR}}_{E_n}, \Phi^{\mathrm{dR}}_{F_n} : \widetilde{H}_{\mathrm{dR}}(X_n/R_n) \to \widetilde{H}_{\mathrm{dR}}(Y_n/R_n)$$

are equal, and isomorphisms when the crystalline realizations are isomorphisms. By the comparison between crystalline and de Rham cohomology it suffices in turn to show that the two maps on the crystalline realizations

$$\Phi^{\mathrm{crys}}_{E_n}, \Phi^{\mathrm{crys}}_{F_n} : \widetilde{H}_{\mathrm{crys}}(X_n/W[t]/(t^{n+1})) \to \widetilde{H}_{\mathrm{crys}}(Y_n/W[t]/(t^{n+1}))$$

are equal. Via the Berthelot-Ogus isomorphism [3, 2.2], which is compatible with Chern classes, these maps are identified after tensoring with $\mathbb{Q}$ with the two maps obtained by base change from

$$\Phi^{\mathrm{crys}}_{E_0}, \Phi^{\mathrm{crys}}_{F_0} : \widetilde{H}_{\mathrm{crys}}(X_0/W) \to \widetilde{H}_{\mathrm{crys}}(Y_0/W).$$

The result follows. □

4.16 In the case when X_n and Y_n are K3 surfaces the action of $HH^*(X_n/R_n)$ on $HH_*(X_n/R_n)$ is faithful. Therefore from Lemma 4.15 we obtain the following.

Corollary 4.17 *Assume that X_n and Y_n are K3 surfaces and that $E_n, F_n \in D(X_n \times_{R_n} Y_n)$ are two R_n-perfect complexes satisfying condition $(\star)$. Suppose further that $\Phi^{crys}_{E_0}$ and $\Phi^{crys}_{F_0}$ are equal on the crystalline realizations of the Mukai motives of the reductions. Then $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{F_n}$ are equal.*

Proof Indeed since homology is a faithful module over cohomology the maps $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{F_n}$ are determined by the maps on Hochschild homology which are equal by Lemma 4.15. □

Corollary 4.18 *Let X_{n+1} and Y_{n+1} be K3 surfaces over R_{n+1} and assume given an admissible R_{n+1}-perfect complex E_{n+1} on $X_{n+1} \times_{R_{n+1}} Y_{n+1}$ such that E_n satisfies condition $(\star)$. Assume given an isomorphism $\sigma_n : X_n \to Y_n$ over R_n such that the induced map $\sigma_0 : X_0 \to Y_0$ defines the same map on crystalline realizations of the Mukai motive as E_0. Then σ_n lifts to an isomorphism $\sigma_{n+1} : X_{n+1} \to Y_{n+1}$.*

Proof Indeed by (4.9.1) and the fact that $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{\Gamma_{\sigma_n}}$ are equal by Corollary 4.17, we see that the obstruction to lifting σ_n is equal to the obstruction to lifting E_n, which is zero by assumption. □

Corollary 4.19 *Assume the hypotheses of Corollary 4.18 with the following exception: Φ_{E_0} is strongly filtered and σ_0 and Φ_{E_0} are only assumed to induce the same map*

$$H^2_{\text{crys}}(X_0/W) \to H^2_{\text{crys}}(Y_0/W).$$

(In other words, we do not require the same action on all crystalline cohomology, just on the middle.) Then the same conclusion holds: σ_n lifts to some σ_{n+1}.

Proof The assumption that Φ_{E_0} is strongly filtered implies that the map $I^K_{Y_n} \circ \Phi^{HH^*}_{E_n} \circ (I^K_{X_n})^{-1}$ sends $H^1(X_n, T_{X_n})$ to $H^1(Y_n, T_{Y_n})$. By construction we have $I^K_{X_n}(\nu_{X_n/X_{n+1}}) \in H^1(X_n, T_{X_n})$; in fact, this class is given by the relative Kodaira-Spencer class multiplied by the square root of the Todd class. Now by looking at the module structure of $H\Omega_*$ over HT^* one sees that the map $H^1(X_n, T_{X_n}) \to H^1(Y_n, T_{Y_n})$ is determined by the action on H^2_{crys}. From this the conclusion follows. □

Remark 4.20 Applying the argument of Corollary 4.19 with $E_n[1]$ we see that if σ_0 induces $\pm\Phi_{E_0}$ on H^2_{crys} then σ_n can be lifted to R_{n+1}.

5 A Remark on Reduction Types

5.1 In the proof of Theorem 1.2 we need the following Theorem 5.3, whose proof relies on known characteristic 0 results obtained from Hodge theory. In Sect. 10 below we give a different algebraic argument for Theorem 5.3 in a special case which suffices for the proof of Theorem 1.2.

5.2 Let V be a complete discrete valuation ring with field of fractions K and residue field k. Let X/V be a projective K3 surface with generic fiber X_K, and let Y_K be a second K3 surface over K such that the geometric fibers $X_{\overline{K}}$ and $Y_{\overline{K}}$ are derived equivalent.

Theorem 5.3 *Under these assumptions the K3 surface Y_K has potentially good reduction.*

Remark 5.4 Here potentially good reduction means that after possibly replacing V be a finite extension there exists a K3 surface Y/V whose generic fiber is Y_K.

Proof of Theorem 5.3 We use [16, 1.1 (1)] which implies that after replacing V by a finite extension Y_K is isomorphic to a moduli space of sheaves on X_K.

After replacing V by a finite extension we may assume that we have a complex $P \in D(X \times Y)$ defining an equivalence

$$\Phi_P : D(X_{\overline{K}}) \to D(Y_{\overline{K}}).$$

Let $E \in D(Y \times X)$ be the complex defining the inverse equivalence

$$\Phi_E : D(Y_{\overline{K}}) \to D(X_{\overline{K}})$$

to Φ_P. Let $\nu := \Phi_E(0, 0, 1) \in A^*(X_{\overline{K}})_{\text{num},\mathbb{Q}}$ be the Mukai vector of a fiber of E at a closed point $y \in Y_{\overline{K}}$ and write

$$\nu = (r, [L_X], s) \in A^0(X_{\overline{K}})_{\text{num},\mathbb{Q}} \oplus A^1(X_{\overline{K}})_{\text{num},\mathbb{Q}} \oplus A^2(X_{\overline{K}})_{\text{num},\mathbb{Q}}.$$

By [16, 8.1] we may after possibly changing our choice of P, which may involve another extension of V, assume that r is prime to p and that L_X is very ample. Making another extension of V we may assume that ν is defined over K, and therefore by specialization also defines an element, which we denote by the same letter,

$$\nu = (r, [L_X], s) \in \mathbb{Z} \oplus \text{Pic}(X) \oplus \mathbb{Z}.$$

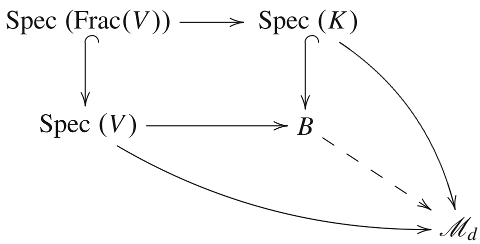
This class has the property that r is prime to p and that there exists another class ν' such that $\langle \nu, \nu' \rangle = 1$. This implies in particular that ν restricts to a primitive class on the closed fiber. Fix an ample class h on X, and let $\mathscr{M}_h(\nu)$ denote the moduli space of semistable sheaves on X with Mukai vector ν. By [16, 3.16] the stack $\mathscr{M}_h(\nu)$ is a μ_r-gerbe over a relative K3 surface $M_h(\nu)/V$, and by [16, 8.2] we have $Y_{\overline{K}} \simeq M_h(\nu)_{\overline{K}}$. In particular, Y has potentially good reduction. □

Remark 5.5 As discussed in [18, p. 2], to obtain Theorem 5.3 it suffices to know that every K3 surface Z_K over K has potentially semistable reduction and this would follow from standard conjectures on resolution of singularities and toroidization of morphisms in mixed and positive characteristic. In the setting of Theorem 5.3, once we know that Y_K has potentially semistable reduction then by [18, Theorem on bottom of p. 2] we obtain that Y_K has good reduction since the Galois representation $H^2(Y_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified being isomorphic to direct summand of the ℓ-adic realization $\widetilde{H}(X_{\overline{K}}, \mathbb{Q}_\ell)$ of the Mukai motive of X_K.

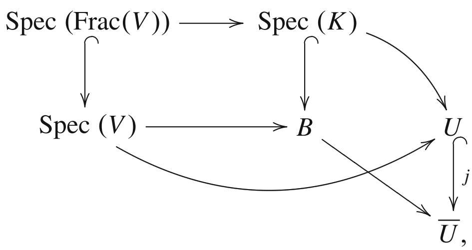
5.6 One can also consider the problem of extending Y_K over a higher dimensional base. Let B denote a normal finite type k-scheme with a point $s \in B(k)$ and let X/B be a projective family of K3 surfaces. Let K be the function field of B and let Y_K be a second K3 surface over K Fourier–Mukai equivalent to X_K. Dominating $\mathscr{O}_{B,s}$ by a suitable complete discrete valuation ring V we can find a morphism

$$\rho : \text{ Spec } (V) \to B$$

sending the closed point of Spec (V) to s and an extension Y_V of $\rho^* Y_K$ to a smooth projective K3 surface over V. In particular, after replacing B by its normalization in a finite extension of K we can find a primitive polarization λ_K on Y_K of degree prime to the characteristic such that $\rho^*\lambda_K$ extends to a polarization on Y_V. We then have a commutative diagram of solid arrows



for a suitable integer d. Base changing to a suitable étale neighborhood $U \to \mathscr{M}_d$ of the image of the closed point of Spec (V), with U an affine scheme, we can after shrinking and possibly replacing B by an alteration find a commutative diagram



where j is a dense open imbedding and $\overline{U}$ is projective over k. It follows that the image of s in $\overline{U}$ in fact lands in U which gives an extension of Y_K to a neighborhood of s. This discussion implies the following:

Corollary 5.7 *In the setup of Paragraph 5.6, we can, after replacing (B, s) by a neighborhood of a point in the preimage of s in an alteration of B, find an extension of Y_K to a K3 surface over B.*

6 Supersingular Reduction

6.1 Let B be a normal scheme of finite type over an algebraically closed field k of characteristic p at least 5, and let W denote the ring of Witt vectors of k. Let K denote the function field of B and let $s \in B$ be a closed point. Let $f : X \to B$ be a projective K3 surface over B and let Y_K/K be a second K3 surface over K such that there exists a strongly filtered Fourier–Mukai equivalence

$$\Phi_P : D(X_K) \to D(Y_K)$$

defined by an object $P \in D(X_K \times_K Y_K)$. Assume further that the fiber X_s of X over s is a supersingular K3 surface.

6.2 Using Corollary 5.7 we can, after possibly replacing B by a neighborhood of a point over s in an alteration, assume that we have a smooth K3 surface Y/B extending Y_K and an extension of the complex P to a B-perfect complex $\mathscr{P}$ on $X \times_B Y$, and furthermore that the complex Q defining the inverse of Φ_P also extends to a complex $\mathscr{Q}$ on $X \times_B Y$. Let $f_{\mathscr{X}} : \mathscr{X} \to B$ (resp. $f_{\mathscr{Y}} : \mathscr{Y} \to B$) be the structure morphism, and let $\mathscr{H}^i_{\text{crys}}(X/B)$ (resp. $\mathscr{H}^i_{\text{crys}}(Y/B)$) denote the F-crystal $R^i f_{\mathscr{X}*}\mathscr{O}_{\mathscr{X}/W}$ (resp. $R^i f_{\mathscr{Y}*}\mathscr{O}_{\mathscr{Y}/W}$) on B/W obtained by forming the i-th higher direct image of the structure sheaf on the crystalline site of $\mathscr{X}/W$ (resp. $\mathscr{Y}/W$). Because $\Phi_{\mathscr{P}}$ is strongly filtered, it induces an isomorphism of F-crystals

$$\Phi^{\text{crys},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{crys}}(X/B) \to \mathscr{H}^i_{\text{crys}}(Y/B)$$

for all i, with inverse defined by $\Phi_{\mathscr{Q}}$. Note that since we are working here with K3 surfaces these morphisms are defined integrally.

We also have the de Rham realizations $\mathscr{H}^i_{\text{dR}}(X/B)$ and $\mathscr{H}^i_{\text{dR}}(Y/B)$ which are filtered modules with integrable connection on B equipped with filtered isomorphisms compatible with the connections

$$\Phi^{\text{dR},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{dR}}(X/B) \to \mathscr{H}^i_{\text{dR}}(Y/B). \tag{6.2.1}$$

as well as étale realizations $\mathscr{H}^i_{\text{ét}}(X/B)$ and $\mathscr{H}^i_{\text{ét}}(Y/B)$ equipped with isomorphisms

$$\Phi^{\text{ét},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{ét}}(X/B) \to \mathscr{H}^i_{\text{ét}}(Y/B). \tag{6.2.2}$$

6.3 Let $H^i_{\text{crys}}(X_s/W)$ (resp. $H^i_{\text{crys}}(Y_s/W)$) denote the crystalline cohomology of the fibers over s. The isomorphism $\Phi^{\text{crys},2}_{\mathscr{P}}$ induces an isomorphism

$$\theta_2 : H^2_{\text{crys}}(X_s/W) \to H^2_{\text{crys}}(Y_s/W)$$

of F-crystals. By [24, Theorem I] this implies that X_s and Y_s are isomorphic. However, we may not necessarily have an isomorphism which induces θ_2 on cohomology.

6.4 Recall that as discussed in [12, 10.9 (iii)] if $C \subset X_K$ is a (-2)-curve then we can perform a spherical twist

$$T_{\mathscr{O}_C} : D(X_K) \to D(X_K)$$

whose action on $NS(X_K)$ is the reflection

$$r_C(a) := a + \langle a, C\rangle C.$$

Proposition 6.5 *After possibly changing our choice of model Y for Y_K, replacing (B, s) by a neighborhood of a point in an alteration over s, and composing with a sequence of spherical twists $T_{\mathscr{O}_C}$ along (-2)-curves in the generic fiber Y_K, there exists an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_2$ on the second crystalline cohomology group. If θ_2 preserves the ample cone of the generic fiber then we can find an isomorphism σ inducing θ_2.*

Proof By [25, 4.4 and 4.5] there exists an isomorphism $\theta_0 : NS(X_s) \to NS(Y_s)$ compatible with θ_2 in the sense that the diagram

$$\begin{array}{ccc} NS(X_s) & \xrightarrow{\theta_0} & NS(Y_s) \\ \downarrow c_1 & & \downarrow c_1 \\ H^2_{\text{crys}}(X_s/W) & \xrightarrow{\theta^2} & H^2_{\text{crys}}(Y_s/W) \end{array} \tag{6.5.1}$$

commutes. Note that as discussed in [17, 4.8] the map θ_0 determines θ_2 by the Tate conjecture for K3 surfaces, proven by Charles, Maulik, and Pera [7, 21, 26]. In particular, if we have an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_0$ on Néron-Severi groups then σ also induces $\pm\theta_2$ on crystalline cohomology. We therefore have to study the problem of finding an isomorphism σ compatible with θ_0.

Ogus shows in [24, Theorem II] that there exists such an isomorphism σ if and only if the map θ_0 takes the ample cone to the ample cone. So our problem is to choose a model of Y in such a way that $\pm\theta_0$ preserves ample cones. Set

$$V_{X_s} := \{x \in NS(X_s)_{\mathbb{R}} | x^2 > 0 \text{ and } \langle x, \delta\rangle \neq 0 \text{ for all } \delta \in NS(X_s) \text{ with } \delta^2 = -2\},$$

and define V_{Y_s} similarly. Being an isometry the map θ_0 then induces an isomorphism $V_{X_s} \to V_{Y_s}$, which we again denote by θ_0. Let R_{Y_s} denote the group of automorphisms of V_{Y_s} generated by reflections in (-2)-curves and multiplication by -1. By [24, Proposition 1.10 and Remark 1.10.9] the ample cone of Y_s is a connected component of V_{Y_s} and the group R_{Y_s} acts simply transitively on the set of connected components of V_{Y_s}.

Let us show how to change model to account for reflections by (-2)-curves in Y_s. We show that after replacing (P, Y) by a new pair (P', Y') consisting of the complex $P \in D(X_K \times_K Y_K)$ obtained by composing Φ_P with a sequence of spherical twists along (-2)-curves in Y_K and replacing Y by a new model Y' there exists an isomorphism $\gamma : Y'_s \to Y_s$ such that the composition

$$NS(X_s) \xrightarrow{\theta_0} NS(Y_s) \xrightarrow{r_C} NS(Y_s) \xrightarrow{\gamma^*} NS(Y'_s)$$

is equal to the map θ'_0 defined as for θ_0 but using the model Y'.

Let $C \subset Y_s$ be a (-2)-curve, and let

$$r_C : NS(Y_s) \to NS(Y_s)$$

be the reflection in the (-2)-curve. If C lifts to a curve in the family Y we get a (-2)-curve in the generic fiber and so by replacing our P by the complex P' obtained by composition with the spherical twist by this curve in Y_K (see [12, 10.9 (iii)]) and setting $Y' = Y$ we get the desired new pair. If C does not lift to Y, then we take $P' = P$ but now replace Y by the flop of Y along C as explained in [24, 2.8].

Thus after making a sequence of replacements $(P, Y) \mapsto (P', Y')$ we can arrange that θ_0 sends the ample cone of X_s to plus or minus the ample cone of Y_s, and therefore we get our desired isomorphism σ.

To see the last statement, note that we have modified the generic fiber by composing with reflections along (-2)-curves. Therefore if λ is an ample class on X with restriction λ_K to X_K, and for a general ample divisor H we have $\langle \Phi_P(\lambda), H\rangle > 0$, then the same holds on the closed fiber. This implies that the ample cone of X_s gets sent to the ample cone of Y_s and not its negative. □

Remark 6.6 One can also consider étale or de Rham cohomology in Proposition 6.5. Assume we have applied suitable spherical twists and chosen a model Y such that we have an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_0$. We claim that the maps

$$\theta_{\mathrm{dR}} : H^i_{\mathrm{dR}}(X_s/k) \to H^i_{\mathrm{dR}}(Y_s/k), \quad \theta_{\text{ét}} : H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(Y_s, \mathbb{Q}_\ell)$$

induced by the maps (6.2.1) and (6.2.2) also agree with the maps defined by $\pm\sigma$. For de Rham cohomology this is clear using the comparison with crystalline cohomology, and for the étale cohomology it follows from compatibility with the cycle class map.

6.7 With notation and assumptions as in Proposition 6.5 assume further that B is a curve or a complete discrete valuation ring, and that we have chosen a model Y such that each of the reductions satisfies the condition $(\star)$ of Paragraph 4.12 and such that the map θ_0 in (6.5.1) preserves plus or minus the ample cones. Let $\sigma : X_s \to Y_s$ be an isomorphism inducing $\pm\theta_0$.

Lemma 6.8 *The isomorphism σ lifts to an isomorphism $\tilde{\sigma} : X \to Y$ over the completion $\widehat{B}$ at s inducing the maps defined by $\pm\Phi_{\mathscr{P}}^{crys,i}$.*

Proof By Remark 4.20 σ lifts uniquely to each infinitesimal neighborhood of s in B, and therefore by the Grothendieck existence theorem we get a lifting $\tilde{\sigma}$ over $\widehat{B}$. That the realization of $\tilde{\sigma}$ on cohomology agrees with $\pm\Phi_{\mathscr{P}}^{\mathrm{crys},i}$ can be verified on the closed fiber where it holds by assumption. □

Lemma 6.9 *With notation and assumptions as in Lemma 6.8 the map $\Phi_P^{A^*_{num,\mathbb{Q}}}$ preserves the ample cones of the generic fibers.*

Proof The statement can be verified after making a field extension of the function field of B. If Φ_P does not preserve the ample cone, then by Lemma 6.8 we get an isomorphism $\sigma : X_K \to Y_K$ over some field extension K of $k(B)$ such that $\sigma^* \circ \Phi_P$ acts by $\mathrm{id}_{H^0} \bigoplus -\mathrm{id}_{H^2} \bigoplus \mathrm{id}_{H^4}$ on any cohomological realization. Lifting the associated derived equivalence to characteristic 0 (for example, using Deligne's liftability theorem [8] and Theorem 3.9) and applying [13, 4.1] we get a contradiction. □

Remark 6.10 In the case when the original Φ_P preserves the ample cones of the geometric generic fibers, no reflections along (-2)–curves in the generic fiber are needed. Indeed, by the above argument we get an isomorphism $\sigma_K : X_K \to Y_K$ such that the induced map on crystalline and étale cohomology agrees with $\Phi_P \circ \alpha$ for some sequence α of spherical twists along (-2)-curves in X_K (also using Lemma 6.9). Since both σ and Φ_P preserve ample cones it follows that α also preserves the ample cone of $X_{\overline{K}}$. By [24, 1.10] it follows that α acts trivially on the Néron-Severi group of $X_{\overline{K}}$. We claim that this implies that α also acts trivially on any of the cohomological realizations. We give the proof in the case of étale cohomology $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ (for a prime ℓ invertible in k) leaving slight modifications for the other cohomology theories to the reader. Let $\widetilde{R}_X$ denote the subgroup of $GL(H^2(X_{\overline{K}}, \mathbb{Q}_\ell))$ generated by -1 and the action induced by spherical twists along (-2)-curves in $X_{\overline{K}}$, and consider the inclusion of $\mathbb{Q}_\ell$-vector spaces with inner products

$$NS(X_{\overline{K}})_{\mathbb{Q}_\ell} \hookrightarrow H^2(X_{\overline{K}}, \mathbb{Q}_\ell).$$

By [12, Lemma 8.12] the action of the spherical twists along (-2)-curves in $X_{\overline{K}}$ on $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is by reflection across classes in the image of $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$. From this (and Gram-Schmidt!) it follows that the the group $\widetilde{R}_X$ preserves $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$, acts trivially on the quotient of $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ by $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$, and that the restriction map

$$\widetilde{R}_X \to GL(NS(X_{\overline{K}})_{\mathbb{Q}_\ell})$$

is injective. In particular, if an element $\alpha \in \widetilde{R}_X$ acts trivially on $NS(X_{\overline{K}})$ then it also acts trivially on étale cohomology. It follows that σ and Φ_P induce the same map on realizations.

7 Specialization

7.1 We consider again the setup of Paragraph 6.1, but now we don't assume that the closed fiber X_s is supersingular. Further we restrict attention to the case when B is a smooth curve, and assume we are given a smooth model Y/B of Y_K and a B-perfect complex $\mathscr{P} \in D(X \times_B Y)$ such that for all geometric points $\bar{z} \to B$ the induced complex $\mathscr{P}_{\bar{z}}$ on $X_{\bar{z}} \times Y_{\bar{z}}$ defines a strongly filtered equivalence $D(X_{\bar{z}}) \to D(Y_{\bar{z}})$.

Let $\mathscr{H}^i(X/B)$ (resp. $\mathscr{H}^i(Y/B)$) denote either $\mathscr{H}^i_{\text{ét}}(X/B)$ (resp. $\mathscr{H}^i_{\text{ét}}(Y/B)$) for some prime $\ell \neq p$ or $\mathscr{H}^i_{\text{crys}}(X/B)$ (resp. $\mathscr{H}^i_{\text{crys}}(Y/B)$). Assume further given an isomorphism

$$\sigma_K : X_K \to Y_K$$

inducing the map given by restricting

$$\Phi^i_{\mathscr{P}} : \mathscr{H}^i(X/B) \to \mathscr{H}^i(Y/B)$$

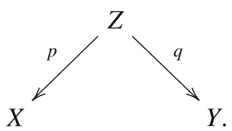
to the generic point.

Remark 7.2 If we work with étale cohomology in this setup we could also consider the spectrum of a complete discrete valuation ring instead of B, and in particular also a mixed characteristic discrete valuation ring.

Proposition 7.3 *The isomorphism σ_K extends to an isomorphism $\sigma : X \to Y$.*

Proof We give the argument here for étale cohomology in the case when B is the spectrum of a discrete valuation ring. The other cases require minor modifications which we do not include here.

Let $Z \subset X \times_B Y$ be the closure of the graph of σ_K, so Z is an irreducible flat V-scheme of dimension 3 and we have a correspondence



Fix an ample line bundle L on X and consider the line bundle $M := \det(Rq_* p^* L)$ on Y. The restriction of M to Y_K is simply $\sigma_{K*}L$, and in particular the étale cohomology class of M is equal to the class of $\Phi_{\mathscr{P}}(L)$. By our assumption that $\Phi_{\mathscr{P}}$ is strongly filtered in the fibers the line bundle M is ample on Y. Note also that by our assumption that $\Phi_{\mathscr{P}}$ is strongly filtered in every fiber we have

$$\Phi_{\mathscr{P}}(L^{\otimes n}) \simeq \Phi^{\mathscr{P}}(L)^{\otimes n}.$$

In particular we can choose L very ample in such a way that M is also very ample. The result then follows from Matsusaka-Mumford [20, Theorem 2]. □

Remark 7.4 One can also prove a variant of Proposition 7.3 when k has characteristic 0 using de Rham cohomology instead of étale cohomology and similar techniques. However, we will not discuss this further here.

8 Proof of Theorem 1.2

In this section we prove Theorem 1.2 when the characteristic is ≥ 5. Characteristic 3 is treated in Remark 8.5

8.1 Let K be an algebraically closed field extension of k and let X and Y be K3 surfaces over K equipped with a complex $P \in D(X \times_K Y)$ defining a strongly filtered Fourier–Mukai equivalence

$$\Phi_P : D(X) \to D(Y).$$

We can then choose a primitive polarization λ on X of degree prime to p such that the triple $((X, \lambda), Y, P)$ defines a K-point of $\mathscr{S}_d$. In this way the proof of Theorem 1.2 is reformulated into showing the following: For any algebraically closed field K and point $((X, \lambda), Y, P) \in \mathscr{S}_d(K)$ there exists an isomorphism $\sigma : X \to Y$ such that the maps on crystalline and étale realizations defined by σ and Φ_P agree.

8.2 To prove this it suffices to show that there exists such an isomorphism after replacing K by a field extension. To see this let I denote the scheme of isomorphisms between X and Y, which is a locally closed subscheme of the Hilbert scheme of $X \times_K Y$, and is thus locally of finite type over K. Over I we have a tautological isomorphism $\sigma^u : X_I \to Y_I$. The condition that the induced action on ℓ-adic étale cohomology agrees with Φ_P is an open and closed condition on I. It follows that there exists a locally closed subscheme $I' \subset I$ classifying isomorphisms σ as in the theorem, and this I' is thus also locally of finite type over K. This implies that if we can find an isomorphism σ over a field extension of K then such an isomorphism also exists over K, since K is algebraically closed and any scheme locally of finite type over such a field K has a K-point if and only if it is nonempty (i.e., has a point over some extension of K).

8.3 By Proposition 7.3 it suffices to show that the result holds for each generic point of $\mathscr{S}_d$. By Theorem 3.9 any such generic point maps to a generic point of $\mathscr{M}_d$ which by Theorem 3.2 admits a specialization to a supersingular point $x \in \mathscr{M}_d(k)$ given by a family $(X_R, \lambda_R)/R$, where R is a complete discrete valuation ring over k with residue field Ω, for some algebraically closed field Ω. By Theorem 5.3 the point $(Y, \lambda_Y) \in \mathscr{M}_d(K)$ also has a limit $y \in \mathscr{M}_d(\Omega)$ given by a second family $(Y_R, \lambda_R)/R$. Let P' be the complex on $X \times Y$ giving the composition of Φ_P with suitable twists by (-2)-curves such that after replacing Y_R by a sequence of flops the map $\Phi_{P'}$ induces an isomorphism on crystalline cohomology on the closed fiber preserving plus or minus the ample cone. By the Cohen structure theorem we have $R \simeq \Omega[[t]]$, and $((X, \lambda), Y, P')$ defines a point of $\mathscr{S}_d(\Omega((t)))$.

Let B denote the completion of the strict henselization of $\mathscr{M}_{\mathbb{Z}[1/d]} \times \mathscr{M}_{\mathbb{Z}[1/d]}$ at the point (x, y). So B is a regular complete local ring with residue field Ω. Let B' denote the formal completion of the strict henselization of $\overline{\mathscr{S}}_{d,\mathbb{Z}[1/d]}$ at the $\Omega((t))$-point given by $((X, \lambda), Y, P')$. So we obtain a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & \Omega[[t]] \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \Omega((t)). \end{array} \tag{8.3.1}$$

Over B we have a universal families $\mathscr{X}_B$ and $\mathscr{Y}_B$, and over the base changes to B' we have, after trivializing the pullback of the gerbe $\mathscr{S}_{d,\mathbb{Z}[1/d]} \to \overline{\mathscr{S}}_{d,\mathbb{Z}[1/d]}$, a complex $\mathscr{P}'_{B'}$ on $\mathscr{X}_{B'} \times_{B'} \mathscr{Y}_{B'}$, which reduces to the triple (X, Y, P') over $\Omega((t))$. The map $B \to B'$ is a filtering direct limit of étale morphisms. We can therefore replace B' by a finite type étale B-subalgebra over which all the data is defined and we still have the diagram (8.3.1). Let $\overline{B}$ denote the integral closure of B in B' so we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(B') & \hookrightarrow & \operatorname{Spec}(\overline{B}) \\ & \searrow & \downarrow \\ & & \operatorname{Spec}(B), \end{array}$$

where $\overline{B}$ is flat over $\mathbb{Z}[1/d]$ and normal. Let $Y \to \operatorname{Spec}(\overline{B})$ be an alteration with Y regular and flat over $\mathbb{Z}[1/d]$, and let $Y' \subset Y$ be the preimage of $\operatorname{Spec}(B')$. Lifting the map $B \to \Omega[[t]]$ to a map $\operatorname{Spec}(\widetilde{R}) \to Y$ for some finite extension of complete discrete valuation rings $\widetilde{R}/R$ and letting C denote the completion of the local ring of Y at the image of the closed point of $\operatorname{Spec}(\widetilde{R})$ we obtain a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & \Omega[[t]] \\ \downarrow & & \downarrow \\ C' & \longrightarrow & \Omega((t)), \end{array}$$

where $C \to C'$ is a localization, we have K3-surfaces $\mathscr{X}_C$ and $\mathscr{Y}_C$ over C and a perfect complex $\mathscr{P}'_{C'}$ on $\mathscr{X}_{C'} \times_{C'} \mathscr{Y}_{C'}$ defining a Fourier–Mukai equivalence and the triple $(\mathscr{X}_{C'}, \mathscr{Y}_{C'}, \mathscr{P}'_{C'})$ reducing to (X, Y, P) over $\Omega((t))$. By [29, 5.2.2] we can extend the complex $\mathscr{P}'_{C'}$ to a C-perfect complex $\mathscr{P}'_C$ on $\mathscr{X}_C \times_C \mathscr{Y}_C$ (here we use that C is regular). It follows that the base change $(X_{\Omega[[t]]}, Y_{\Omega[[t]]}, P'_{\Omega[[t]]})$ gives an extension of (X, Y, P) to $\Omega[[t]]$ all of whose reductions satisfy our condition $(\star)$ of Paragraph 4.12.

This puts us in the setting of Lemma 6.8, and we conclude that there exists an isomorphism $\sigma : X \to Y$ (over $\Omega((t))$, but as noted above we are allowed to make a field extension of K) such that the induced map on crystalline and étale cohomology agrees with $\Phi_P \circ \alpha$ for some sequence α of spherical twists along (-2)-curves in X (using also Lemma 6.9). By the same argument as in Remark 6.10 it follows that σ and Φ_P induce the same map on realizations which concludes the proof of Theorem 1.2. □

Remark 8.4 One consequence of the proof is that in fact any strongly filtered equivalence automatically takes the ample cone to the ample cone, and not its negative. This is closely related to [13, 4.1].

Remark 8.5 Given (X, Y, P) as in Theorem 1.2 and an isomorphism $\sigma : X \to Y$ inducing the same action as P on the ℓ-adic realization of the Mukai motive for a fixed prime ℓ invertible in k, we have that in fact σ and P define the same action on all the étale and crystalline realizations. This follows from the same specialization argument to supersingular K3s and [25, 3.20].

This observation can be used, in particular, to prove Theorem 1.2 in characteristic 3: With notation as in Theorem 1.2 fix a prime $\ell \neq 3$ and lift the triple (X, Y, P) to a triple $(\mathscr{X}, \mathscr{Y}, \mathscr{P})$ over a finite extension of $\mathbb{Z}_3$. By the specialization argument of Sect. 7 it then suffices to exhibit an isomorphism between the generic fibers inducing the same map as P on the ℓ-adic realizations. This reduces the result in characteristic 3 to the characteristic 0 result proven in Theorem 9.1 below.

9 Characteristic 0

From our discussion of positive characteristic results one can also deduce the following result in characteristic 0.

Theorem 9.1 *Let K be an algebraically closed field of characteristic* 0*, let X and Y be K3 surfaces over K, and let $\Phi_P : D(X) \to D(Y)$ be a strongly filtered Fourier–Mukai equivalence defined by an object $P \in D(X \times Y)$. Then there exists an isomorphism $\sigma : X \to Y$ whose action on ℓ-adic and de Rham cohomology agrees with the action of Φ_P.*

Proof It suffices to show that we can find an isomorphism σ which induces the same map on ℓ-adic cohomology as Φ_P for a single prime ℓ. For then by compatibility of the comparison isomorphisms with Φ_P, discussed in [16, Sect. 2], it follows that σ and Φ_P also define the same action on the other realizations of the Mukai motive.

Furthermore as in Paragraph 8.2 it suffices to prove the existence of σ after making a field extension of K.

As in Paragraph 8.2 let I' denote the scheme of isomorphisms $\sigma : X \to Y$ as in the theorem. Note that since the action of such σ on the ample cone is fixed, the scheme I' is in fact of finite type.

Since X, Y, and P are all locally finitely presented over K we can find a finite type integral $\mathbb{Z}$-algebra A, K3 surfaces X_A and Y_A over A, and an A-perfect complex $P_A \in D(X_A \times_A Y_A)$ defining a strongly filtered Fourier–Mukai equivalence in every fiber, and such that (X, Y, P) is obtained from (X_A, Y_A, P_A) by base change along a map $A \to K$. The scheme I' then also extends to a finite type A-scheme I'_A over A. Since I' is of finite type over A to prove that I' is nonempty it suffices to show that I'_A has nonempty fiber over $\mathbb{F}_p$ for infinitely many primes p. This holds by Theorem 1.2. □

10 Bypassing Hodge Theory

10.1 The appeal to analytic techniques implicit in the results of Sect. 5, where characteristic 0 results based on Hodge theory are used to deduce Theorem 5.3, can be bypassed in the following way using results of [18, 21].

10.2 Let R be a complete discrete valuation ring of equicharacteristic $p > 0$ with residue field k and fraction field K. Let X/R be a smooth K3 surface with supersingular closed fiber. Let Y_K be a K3 surface over K and $P_K \in D(X_K \times Y_K)$ a perfect complex defining a Fourier–Mukai equivalence $\Phi_{P_K} : D(X_{\overline{K}}) \to D(Y_{\overline{K}})$.

Theorem 10.3 *Assume that X admits an ample invertible sheaf L such that $p > L^2 + 4$. Then after replacing R by a finite extension there exists a smooth projective K3 surface Y/R with generic fiber Y_K.*

Proof Changing our choice of Fourier–Mukai equivalence P_K, we may assume that P_K is strongly filtered. Setting M_K equal to $\det(\Phi_{P_K}(L))$ or its dual, depending on whether Φ_{P_K} preserves ample cones, we get an ample invertible sheaf on Y_K of degree L^2. By [18, 2.2], building on Maulik's work [21, Discussion preceding 4.9] we get a smooth K3 surface Y/R with Y an algebraic space. Now after replacing P_K by the composition with twists along (-2)-curves and the model Y by a sequence of flops, we can arrange that the map on crystalline cohomology of the closed fibers induced by Φ_{P_K} preserves ample cones. Let $P \in D(X \times_R Y)$ be an extension of P_K and let M denote $\det(\Phi_P(L))$. Then M is a line bundle on Y whose reduction is ample on the closed fiber. It follows that M is also ample on Y so Y is a projective scheme. □

10.4 We use this to prove Theorem 1.2 in the case of étale realization in the following way. First observe that using the same argument as in Sect. 8, but now replacing the appeal to Theorem 5.3 by the above Theorem 10.3, we get Theorem 1.2 under the additional assumption that X admits an ample invertible sheaf L with $p > L^2 + 4$. By the argument of Sect. 9 this suffices to get Theorem 1.2 in characteristic 0, and by the specialization argument of Sect. 7 we then get also the result in arbitrary characteristic.

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