On the Kobayashi Pseudometric, Complex Automorphisms and Hyperkähler Manifolds

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Abstract We define the Kobayashi quotient of a complex variety by identifying points with vanishing Kobayashi pseudodistance between them and show that if a complex projective manifold has an automorphism whose order is infinite, then the fibers of this quotient map are nontrivial. We prove that the Kobayashi quotients associated to ergodic complex structures on a compact manifold are isomorphic. We also give a proof of Kobayashi's conjecture on the vanishing of the pseudodistance for hyperkähler manifolds having Lagrangian fibrations without multiple fibers in codimension one. For a hyperbolic automorphism of a hyperkähler manifold, we

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prove that its cohomology eigenvalues are determined by its Hodge numbers, compute its dynamical degree and show that its cohomological trace grows exponentially, giving estimates on the number of its periodic points.

1 Introduction

Kobayashi conjectured that a compact Kähler manifold with semipositive Ricci curvature has vanishing Kobayashi pseudometric. In a previous paper [16] Kamenova–Lu–Verbitsky have proved the conjecture for all K3 surfaces and for certain hyperkähler manifolds that are deformation equivalent to Lagrangian fibrations. Here we give an alternative proof of this conjecture for hyperkähler Lagrangian fibrations without multiple fibers in codimension one, see Sect. 3.

Theorem 1.1 Let $f : M \longrightarrow B = \mathbb{C}\mathbb{P}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then the Kobayashi pseudometric d_M vanishes identically on M and the Royden–Kobayashi pseudonorm $|\ |_M$ vanishes identically on a Zariski open subset of M.

In Sect. 4, we explore compact complex manifolds M having an automorphism of infinite order. If such a manifold is projective, we show that the Kobayashi pseudometric is everywhere degenerate. For each point $x \in M$ we define the subset $M_x \subset M$ of points in M whose pseudo-distance to x is zero. Define the relation $x \sim y$ on M given by $d_M(x, y) = 0$. There is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$, called **the Kobayashi quotient map**. We say that $|\ |_M$ is **Voisin-degenerate** at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that $\varphi_n(0) \to x,\ |\varphi_n'(0)|_h = 1$ and $r_n \to \infty$.

Theorem 1.2 Let M be a complex projective manifold with an automorphism f of infinite order. Then the Kobayashi pseudo-metric d_M is everywhere degenerate in the sense that $M_x \neq \{x\}$ for all $x \in M$. The Royden–Kobayashi pseudo-norm $|\ |_M$ is everywhere Voisin-degenerate. Moreover, every fiber of the map $\Psi : M \longrightarrow S$ constructed above contains a Brody curve and is connected.

Define **the Kobayashi quotient** M_K of M to be the space of all equivalence classes $\{x \sim y \mid d_M(x, y) = 0\}$ equipped with the metric induced from d_M.

In Sect. 5, we show that the Kobayashi quotients for ergodic complex structures are isometric, equipped with the natural quotient pseudometric. This generalizes the key technical result of [16] for the identical vanishing of d_M for ergodic complex structures on hyperkähler manifolds.

Theorem 1.3 Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Finally in Sect. 6, we prove that the cohomology eigenvalues of a hyperbolic automorphism of a hyperkähler manifold are determined by its Hodge numbers. We compute its dynamical degree in the even cases and give an upper bound in the odd cases.

Theorem 1.4 Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

As a corollary we obtain that the trace $\mathrm{Tr}(\gamma^N)$ grows asymptotically as α^{nN}. We also show that the number of k-periodic points grows as α^{nk}.

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2 Preliminaries

Definition 2.1 A *hyperkähler* (or *irreducible holomorphic symplectic*) manifold M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ where σ is everywhere non-degenerate.

Recall that a fibration is a connected surjective holomorphic map. On a hyperkähler manifold the structure of a fibration, if one exists, is limited by Matsushita's theorem.

Theorem 2.2 (Matsushita, [21]) Let M be a hyperkähler manifold and $f\colon M \longrightarrow B$ a fibration with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2} \dim M$ and the general fiber of f is a Lagrangian abelian variety. The base B has at worst $\mathbb{Q}$-factorial log-terminal singularities, has Picard number $\rho(B) = 1$ and $-K_B$ is ample.

Remark 2.3 B is smooth in all of the known examples. It is conjectured that B is always smooth.

Theorem 2.4 (Hwang [15]) In the settings above, if B is smooth then B is isomorphic to $\mathbb{CP}^n$, where $\dim_{\mathbb{C}} M = 2n$.

Definition 2.5 Given a hyperkähler manifold M, there is a non-degenerate integral quadratic form q on $H^2(M, \mathbb{Z})$, called the *Beauville–Bogomolov–Fujiki form* (BBK form for short), of signature $(3, b_2 - 3)$ and satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M, \mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. For a detailed description of the form we refer the reader to [2, 6, 13].

Remark 2.6 Given $f: M \longrightarrow \mathbb{CP}^n$, h the hyperplane class on $\mathbb{CP}^n$, and $\alpha = f^*h$, then α is nef and $q(\alpha) = 0$.

Conjecture 2.7 [SYZ] If L is a nontrivial nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, given as above.

Remark 2.8 This conjecture is known for deformations of Hilbert schemes of points on K3 surfaces (Bayer–Macrì [1]; Markman [20]), and for deformations of the generalized Kummer varieties $K_n(A)$ (Yoshioka [36]).

Definition 2.9 The *Kobayashi pseudometric* on M is the maximal pseudometric d_M such that all holomorphic maps $f: (D, \rho) \longrightarrow (M, d_M)$ are distance decreasing, where (D, ρ) is the unit disk with the Poincaré metric.

Definition 2.10 A manifold M is *Kobayashi hyperbolic* if d_M is a metric, otherwise it is called *Kobayashi non-hyperbolic*.

Remark 2.11 In [17], it is asked whether a compact Kähler manifold M of semi-positive Ricci curvature has identically vanishing pseudometric, which we denote by $d_M \equiv 0$. The question applies to hyperkähler manifolds but was unknown even for the case of surfaces outside the projective case. But Kamenova–Lu–Verbitsky (in [16]) have recently resolved completely the case of surfaces with the following affirmative results.

Theorem 2.12 [16] Let S be a K3 surface. Then $d_S \equiv 0$.

Remark 2.13 A birational version of a conjecture of Kobayashi [17] would state that a compact hyperbolic manifold be of general type if its Kobayashi pseudometric is nondegenerate somewhere (i.e. nondegenerate on some open set). This was open for surfaces but now resolved outside surfaces of class VII.

Theorem 2.14 [16] Let M be a hyperkähler manifold of non-maximal Picard rank and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 2.15 [16] Let M be a hyperkähler manifold with $b_2(M) \geq 7$ (expected to always hold) and with maximal Picard rank $\rho = b_2 - 2$. Assume the SYZ conjecture for deformations of M. Then $d_M \equiv 0$.

Remark 2.16 Except for the proof of Theorem 2.15, we indicate briefly a proof of these theorems below. Theorem 2.15 is proved in [16] using the existence of double Lagrangian fibrations on certain deformations of M. Here we give a different proof of vanishing of the Kobayashi pseudometric for certain hyperkähler Lagrangian fibrations without using double fibrations.

Definition 2.17 Let M be a compact complex manifold and $\mathrm{Diff}^0(M)$ the connected component to identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\mathrm{Teich} := \mathrm{Comp} / \mathrm{Diff}^0(M)$. The Teichmüller space is finite-dimensional for M Calabi–Yau [11]. Let $\mathrm{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group* $\Gamma := \mathrm{Diff}^+(M)/\mathrm{Diff}^0(M)$ acts on Teich. An element $I \in \mathrm{Teich}$ is called *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where

$$\Gamma \cdot I = \{I' \in \mathrm{Teich} : (M, I) \sim (M, I')\}.$$

Theorem 2.18 (Verbitsky, [32]) If M is hyperkähler and $I \in \mathrm{Teich}$, then I is ergodic if and only if $\rho(M, I) < b_2 - 2$.

Remark 2.19 For a K3 surface (M, I) not satisfying the above condition on the Picard rank ρ, it is easily seen to admit Lagrangian (elliptic) fibrations over $\mathbb{CP}^1$ without multiple fibers, and it is projective. Then $d_{(M,J)} \equiv 0$ by Theorem 3.2 below, for example.

Proposition 2.20 Let (M, J) be a compact complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in \mathrm{Teich}$ be an ergodic complex structure deformation equivalent to J. Then $d_{(M,I)} \equiv 0$.

Proof Here we shall reproduce the proof from [16]. Consider the diameter function $\mathsf{diam} : \mathrm{Teich} \longrightarrow \mathbb{R}_{\geqslant 0}$, the maximal distance between two points. It is upper semi-continuous (Corollary 1.23 in [16]). Since the complex structure J is in the limit set of the orbit of the ergodic structure I, by upper semi-continuity $0 \leqslant \mathsf{diam}(I) \leqslant \mathsf{diam}(J) = 0$. ■

3 (Royden–)Kobayashi Pseudometric on Abelian Fibrations

The following lemma is a generalization of Lemma 3.8 in [8] to the case of abelian fibrations. The generalization is given for example in the Appendix of [16]. Recall that an abelian fibration is a connected locally projective surjective Kähler morphism with abelian varieties as fibers.

Lemma 3.1 Let $\pi : T \longrightarrow C$ be an abelian fibration over a non-compact complex curve C which locally has sections and such that not all components of the fibers are multiple. Then T has an analytic section over C. This is the case if π has no multiple fibers.

Proof There is a Neron model N for T and a short exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}(L) \longrightarrow \mathcal{O}(N) \longrightarrow 0$$

where L is a vector bundle, F is a sheaf of groups $\mathbb{Z}^{2n}$ with degenerations, i.e., sheaf of discrete subgroups with generically maximal rank, and $\mathcal{O}(N)$ is the sheaf of local sections of N (whose general fibers are abelian varieties). Thus T corresponds to an element θ in $H^1(C, \mathcal{O}(N))$. There is an induced exact sequence of cohomologies: $H^1(C, \mathcal{O}(L)) \longrightarrow H^1(C, \mathcal{O}(N)) \longrightarrow H^2(C, F)$. Note that $H^1(C, \mathcal{O}(L)) = 0$ since C is Stein, and $H^2(C, F) = 0$ since it is topologically one-dimensional. Thus $\theta = 0$ and hence there is an analytic section. The last part of the lemma is given by Proposition 4.1 of [16]. ■

Theorem 3.2 Let $f: M \longrightarrow B = \mathbb{CP}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then $d_M \equiv 0$ and $| \, |_M$ vanishes on a nonempty Zariski open subset of M.

Proof The fibers of f are projective, and furthermore, there is a canonical polarization on them (see [25, 26], respectively). This also follows from [31], Theorem 1.10, which implies that the given fibration is diffeomorphic to another fibration $f: M' \longrightarrow B$ with holomorphically the same fibers and the same base, but with projective total space M'. Standard argument (via the integral lattice in the "local" Neron–Severi group) now shows that f is locally projective.

By assumption, there are no multiple fibers outside a codimension 2 subset $S \subset B$ whose complement U contains at most the smooth codimension-one part D_0 of the discriminant locus of f where multiplicity of fibers are defined locally generically. Since the pseudometric is unchanged after removing codimension 2 subsets [18], it is enough to restrict the fibration to that over U.

Let $C = \mathbb{P}^1$ be a line in $B = \mathbb{P}^n$ contained in U (and intersecting D_0 transversely). Then f restricts to an abelian fibration $X = f^{-1}(C)$ over C without multiple fibers and so Lemma 3.1 applies to give a section over the affine line $A^1 = C \setminus (\infty)$.

As S is codimension two or higher, we can connect any two general points in U by a chain of such A^1's in U. One can thus connect two general points x and y on M by a chain consisting of fibers and sections over the above A^1's. Since the Kobayashi pseudometric vanishes on each fiber and each such section, the triangle inequality implies $d_M(x, y) = 0$. Therefore d_M vanishes on a dense open subset of M and hence $d_M \equiv 0$ by the continuity of d_M.

The same argument gives the vanishing statement of $| \, |_M$ via Theorem A.2 of [16]. ■

Remark 3.3 In the theorem above, it is sufficient to assume that B is nonsingular and that $d_B \equiv 0$, true if B is rationally connected. In fact, if one assumes further the vanishing of $| \, |_B$ on a nonempty Zariski open, then the same is true for $| \, |_M$, generalizing the corresponding theorems in [16]. The reader should have no difficulty to see these by the obvious modifications of the above proof.

4 Automorphisms of Infinite Order

We first sketch the proof of Kobayashi's theorem that Kobayashi hyperbolic manifolds have only finite order automorphisms (Theorem 9.5 in [17]).

Theorem 4.1 Let M be a Kobayashi hyperbolic manifold. Then its group of birational transformations is finite.

Proof First, notice that a birational self-map is a composition of a blow-up, an automorphism and a blow-down. Since M contains no rational curves, any birational self-map is holomorphic, and we need to prove the finiteness of the automorphism group.

Observe that the automorphisms of a hyperbolic manifold are isometries of the Kobayashi metric. Also the group of isometries of a compact metric space is compact with respect to the compact open topology by a theorem of Dantzig and Van der Waerden, see for example [18, Theorem 5.4.1]. On the other hand, compact Kobayashi hyperbolic manifolds have no holomorphic vector fields, because each such vector field gives an orbit which is an entire curve. This means that the group of holomorphic automorphisms $\mathrm{Aut}(M)$ of M is discrete as it is a complex Lie group in the compact open topology acting holomorphically on M by the work of Bochner–Montgomery [4, 5]. Since $\mathrm{Aut}(M)$ is discrete and compact, this means it is finite. ■

Consider the pseudo-distance function $d_M : M \times M \longrightarrow \mathbb{R}$, defined by the Kobayashi pseudo-distance $d_M(x, y)$ on pairs (x, y). It is a symmetric continuous function which is bounded for compact M. Since it is symmetric, we can consider d_M as a function on the symmetric product $\mathrm{Sym}^2 M$ with $d_M = 0$ on the diagonal.

Lemma 4.2 There is a compact space S with a continuous map $\Psi : M \longrightarrow S$ and there is a distance function d_S on S making S into a compact metric space such that $d_M = d_S \circ \psi$, where $\psi : \mathrm{Sym}^2 M \longrightarrow \mathrm{Sym}^2 S$ is the map induced by Ψ.

Proof The subset $M_x \subset M$ of points $y \in M$ with $d_M(x, y) = 0$ is compact and connected. The relation $x \sim y$ on M given by $d_M(x, y) = 0$ is symmetric and transitive so that $M_x = M_y$ if and only if $x \sim y$. So there is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$. Note that the set S is equipped with a natural metric induced from d_M. Indeed, $d_M(x', y')$ is the same for any points $x' \in M_x$, $y' \in M_y$, and hence d_M induces a metric d_S on S. This metric provides a topology on S, and since the set $U_{x,\varepsilon} = \{y \in M \mid d_M(x, y) < \varepsilon\}$ is open, the map $\Psi : M \longrightarrow S$ is continuous. Thus the metric space S is also compact. This completes the proof of the lemma. ■

Remark 4.3 The natural quotient considered above was already proposed in [17] albeit little seems to be known about its possible structure. In particular, it is known that even when M is compact, S may not have the structure of a complex variety [14]. As we note in Remark 4.12, Campana conjectured that the Kobayashi metric quotient of a Kähler manifold has birational general type, and hence, a dense subset of the metric quotient should carry a complex (even quasi-projective) structure for such manifolds.

Remark 4.4 If there is a holomorphic family of varieties X_t smooth over a parameter space T of say dimension 1, then the relative construction also works by considering the problem via that of the total space over small disks in T. In particular, there is a monodromy action on the resulting family of compact metric spaces S_t by isometries over T, c.f. Sect. 5.

Let M be a complex manifold and h a hermitian metric on M with its associated norm $|\ |_h$.

Recall that a theorem of Royden says that the Kobayashi pseudo-metric d_M can be obtained by taking the infimum of path-integrals of the infinitesimal pseudonorm $|\ |_M$, where

$$|v|_M = \inf\left\{ \frac{1}{R} \mid f : D_R \to M \text{ holomorphic}, R > 0, f'(0) = v \right\}.$$

Here D_R is the disk of radius R centred at the origin. Recall also that $|\ |_M$ is upper-semicontinuous [29].

Definition 4.5 We say that $|\ |_M$ is *Voisin-degenerate* at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that

$$\varphi_n(0) \to x,\ |\varphi_n'(0)|_h = 1 \text{ and } r_n \to \infty.$$

Observe that the locus Z_M of M consisting of points where $|\ |_M$ is Voisin-degenerate is a closed set.

Remark 4.6 If $(x, v) \in T_x M$ is a point in the tangent bundle of M at x which is Voisin-degenerate, then it does not necessarily follow that $|v|_M = 0$, because the Kobayashi pseudometric is semicontinuous but might not be continuous at that point. However, the other implication is true: by upper semicontinuity, if $|v|_M = 0$, then for any sequence $(x_n, v_n) \longrightarrow (x, v)$ we have $|v_n|_M \longrightarrow 0$, i.e., the point x is Voisin degenerate in a strong sense.

The following theorem is essentially [35, Proposition 1.19].

Theorem 4.7 Consider the equivalence relation $x \sim y$ on M given by $d_M(x, y) = 0$ where d_M is the Kobayashi pseudo-metric on M. Then every non-trivial orbit (that is, a non-singleton equivalence class) of this relation consists of Voisin-degenerate points, and the union of such orbits is a closed set. If, further, M is compact, then each nontrivial orbit contains the image of a nontrivial holomorphic map $\mathbb{C} \to M$.

We also need the following theorem.

Theorem 4.8 Assume M is compact. Then each orbit of the equivalence relation given above is connected.

Proof Let M_x be the orbit passing through x as before and

$$M_x(n) = \left\{ y \in X \,\middle|\, d_X(x, y) \leqslant \frac{1}{n} \right\}.$$

Then each $M_x(n)$ is compact and connected and $M_x = \cap_n M_x(n)$. If M_x is not connected, then there are disjoint open sets U, V in M separating M_x leading to the contradiction

$$\emptyset = (U \cup V)^c \cap M_x = \cap_n[(U \cup V)^c \cap M_x(n)] \neq \emptyset,$$

each $(U \cup V)^c \cap M_x(n)$ being nonempty compact as $M_x(n)$ is connected. ■

We want to exploit the existence of an automorphism of an infinite order for the analysis of Kobayashi metric. The following conjecture provides with a necessary argument for a projective manifold.

The rest of this section contains several arguments which suggest a possible strategy to study the vanishing locus of Kobayashi metric on a projective manifold in the presence of an infinite order automorphism. We label them as “conjectures” to distinguish these suggestive arguments from the fully rigorous proofs. We plan to put rigour to these heuristic arguments at some later date.

Conjecture 4.9 Let X be a complex projective manifold and $[C]$ an ample class of curves on X. Let U be an open domain in X and w_h the volume form of a Kähler metric h on X. Then for a sufficently big n there is a curve $C_1 \in [nC]$ such that $\mathrm{Vol}_h(C \cap U) \geqslant (w_h(U)/w_h(X) - \varepsilon)\,\mathrm{Vol}_h(C)$ for arbitrary small ε.

Sketch of a possible proof of this result: The result evidently holds for P^n and Fubini-Study metric on P^n since P^n is homogeneous with respect to the Fubini-Study metric. In this case it follows from the integral volume formula for the family of projective lines, parametrized by the Grassmanian which surjects onto P^n. It immediately implies the existence of lines which satisfy the inequality.

Similar formula holds for the family of algebraic curves of any given degree. In particular we obtain an infinitesimal version of the formula which therefore holds for any metric on projective space. Using a finite map of an n-dimensional projective manifold X onto $\mathbb{C}P^n$ we can derive the same formula for the Kähler pseudometrics induced from $\mathbb{C}P^n$ and then use its local nature for any X. ■

Conjecture 4.10 Let f be an automorphism of infinite order on a complex projective manifold X of dimension n. Assume that there is a domain U in X, a smooth Kähler metric g on X and positive constants c, c' such that $cg \leqslant (f^m)^*g \leqslant c'g$ on U for all powers f^m of f. Then f is an isometry of (X, h) for some Kähler metric h on X and hence some power of f is contained in a connected component of the group of complex isometries of (X, h). In particular, X has a faithful holomorphic action by an abelian variety.

Sketch of a possible proof of this result: Let h be the pull back of the Fubini-Study metric on X of the embedding corresponding to a very ample line bundle L on X. Note that we can assume that $ag \leqslant (f^m)^*h \leqslant a'g$ on $\bar{U}$ for some positive constants a, a' which are independent of the parameter m. Note that $\int_X (f^m)^*h^n$ does not depend on m since the class of the volume h^n maps into itself. Therefore, we have

$$\mu' \int_X h^n < \int_{\bar{U}} (f^m)^*h^n < \mu \int_X h^n \tag{4.1}$$

for some μ and μ' independent of m. Let c be a class of ample (i.e., very movable) curves. Then, for a sufficiently big multiple Nc of the class c, there are curves $C \in Nc$ with $\int_{C \bigcap \bar{U}} h > \nu(h, c)$, and similarly we have $\int_{C \bigcap \bar{U}} (f^m)^*h > \nu((f^m)^*h, c)$, where (h, c) is a pairing of the homology class c and the class of kahler metric h. Since

$$a \int_{C \bigcap \bar{U}} g < \int_{C \bigcap \bar{U}} (f^m)^*h < a' \int_{C \bigcap \bar{U}} g,$$

we obtain that $((f^m)^*h, c)$ is bounded from above by $a'(g, c)$ and from below by by $a(g, c)$ for any ample class. Since ample classes generate the dual $\mathrm{N}_1(X)_{\mathbb{R}}$ of $\mathrm{NS}(X) \otimes \mathbb{R}$ we obtain that $(f^m)^*h$ as linear functional on $\mathrm{N}_1(X) \otimes \mathbb{R}$ is contained in a bounded subset.

A slightly more direct argument for this last boundedness is as follows. Since each $f^{m*}h$ represents a Kähler class, it is sufficient to bound them from above as linear functionals on ample classes c of curves. Note that the first inequality in Eq. 4.1 says that $w_{f^{m*}h}(U) > \mu' w_h(X)$ with μ' independent of m. By the previous lemma, one can therefore choose $C_m \in c$ such that

$$((f^m)^*h, c) = \mathrm{Vol}_{(f^m)^*h}(C_m) \leqslant \delta \int_{C_m \bigcap \bar{U}} (f^m)^*h < \delta a' \int_{C_m \bigcap \bar{U}} g \leqslant \gamma(g, c),$$

where δ is independent of m and $\gamma = \delta a'$.

Since there are only a finite number of integral classes in any bounded set in $\mathrm{NS}(X)_{\mathbb{R}}$, it follows that f^{m_0} leaves invariant the Káhler class of h for some $m_0 \neq 0$ and we may therefore assume that f itself leaves it invariant. In the case of Ricci-flat X, f must therefore be an isometry with respect to the unique Ricci-flat metric in the Kähler class given by Yau's solution to the Calabi conjecture. In general, if $H^1(X, C) = 0$, then f is induced from a projective action on P^N under a map $X \to P^N$. If $H^1(X, C) \neq 0$, then we have a map from $X \times \mathrm{Pic}^0(X)$ to a projective family of projective spaces $\{\mathbb{P}(H^0(L_t)^\vee), t \in \mathrm{Pic}^0(X)\}$ over $\mathrm{Pic}^0(X)$ and L_t defines a very ample invariant invertible sheaf on $X \times \mathrm{Pic}^0(X)$ over $\mathrm{Pic}^0(X)$. Hence, f has to be a complex isometry on X which completes the argument. ■

Conjecture 4.11 Let M be a projective manifold with an automorphism f of infinite order. Then the Kobayashi pseudo-metric d_M is everywhere degenerate in the sense that $M_x \neq \{x\}$ for all $x \in M$. Also the Kobayashi–Royden pseudo-norm $| \; |_M$

is everywhere Voisin-degenerate. Moreover, every fiber of the map $\Psi : M \longrightarrow S$ constructed above contains a Brody curve and is connected.

Sketch of a possible proof of this result: The map $f : M \longrightarrow M$ commutes with the projection onto S, and hence, induces an isometry on S. Since the action of f has infinite order on S, there is a sequence of powers f^{N_i} which converges to the identity on S by the compactness of the group Isom(S) of isometries of S (in the compact open topology) and by setting $N_i = n_i - n_{i-1}$ for a convergent subsequence f^{n_i} in Isom(S). We assume arguing by contradiction that d_M is non-degenerate at a point $x \in M$. Let U be the maximal subset in M where Ψ is a local isomorphism. Since the subsets M_x are connected, then U is exactly the subset where Ψ is an embedding. The set U is invariant under f and is open by Theorem 4.7. Hence, f^{N_i} converges to the trivial action on U. The boundary ∂U of U is a compact subset in M with $\partial U \neq \bar{U}$ and $d_M(x, \partial U) > 0$ for any point $x \in U$. Thus, a compact subset U_ε which consists of points $x \in U$ with $d_M(x, \partial U) \geqslant \varepsilon$ is f-invariant and the restriction of d_M on U_ε is a metric. It is also invariant under the action of f and by theorem of Royden ([28, Theorem 2]) we know that there are smooth Kähler metrics g, g' on X with the property that $g' > d_M \geqslant g$ on U_ε. Applying Conjecture 4.10 we obtain that f is an isometry on M with respect to some Kähler metric. Thus, either M has a nontrivial action of a connected algebraic group, and hence, trivial Kobayshi pseudometric, or f is of finite order which contradicts our assumption. Thus, we obtain a contradiction also with our initial assertion that d_M is metric on some open subset in M.

Note that a limit of Brody curves is again a nontrivial Brody curve by Brody's classical argument. By Theorem 4.7, this implies that the map $\Psi : M \longrightarrow S$ is everywhere degenerate, as it is degenerate in the complement of an everywhere dense open subset. ■

Remark 4.12 In [9, Conjecture 9.16], F. Campana conjectured that the Kobayashi quotient map of a complex projective manifold M should coincide (in the birational category) with the "core map" of M, with fibers which are "special" and the base which is a "general type" orbifold. Then Conjecture 4.11 would just follow, because the automorphism group of a general type variety is finite. Then a general fiber of the Kobayashi quotient map contains infinitely many points, hence its fibers are positively dimensional.

Remark 4.13 Note that both conditions of Conjecture 4.10 are sharp. It was shown by McMullen [23] that there are Kahler non-projective K3 surfaces with automorphisms of infinite order which contain invariant domains isomorphic to the two-dimensional ball. There are also examples by Bedford and Kim [3] of rational projective surfaces X with automorphisms of infinite order which contain an invariant ball. In this case there are no invariant volume forms on the variety X.

5 Metric Geometry of Kobayashi Quotients

Definition 5.1 Let M be a complex manifold, and d_M its Kobayashi pseudometric. Define **the Kobayashi quotient** M_K of M as the space of all equivalence classes $\{x \sim y \mid d_M(x, y) = 0\}$ equipped with the metric induced from d_M.

The main result of this section is the following theorem.

Theorem 5.2 Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Proof Consider the limit $\lim \nu_i(I) = J$, where ν_i is a sequence of diffeomorphisms of M. For each point $x \in (M, I)$, choose a limiting point $\nu(x) \in (M, J)$ of the sequence $\nu_i(x)$. Fix a dense countable subset $M_0 \subset M$ and replace the sequence ν_i by its subsequence in such a way that $\nu(m) := \lim \nu_i(m)$ is well defined for all $m \in M_0$.

By the upper-semicontinuity of the Kobayashi pseudometric, we have

$$d_{(M,J)}(\nu(x), \nu(y)) \geqslant d_{(M,I)}(x, y). \tag{5.1}$$

Let C_0 be the union of all $\nu(x)$ for all $x \in M_0$. Define a map $\psi : C_0 \longrightarrow (M, I)$ mapping $z = \nu(x)$ to x (if there are several choices of such x, choose one in arbitrary way). By (5.1), the map ψ is 1-Lipschitz with respect to the Kobayashi pseudometric. We extend it to a Lipschitz map on the closure C of C_0. For any $x \in (M, J)$, the Kobayashi distance between x and $\psi(\nu(x))$ is equal zero, also by (5.1). Therefore, ψ defines a surjective map on Kobayashi quotients: $\Psi : C_K \longrightarrow (M, I)_K$. Exchanging I and J, we obtain a 1-Lipshitz surjective map $\Phi : C'_K \longrightarrow (M, J)_K$, where C'_K is a subset of $(M, I)_K$. Taking a composition of Ψ and Φ, we obtain a 1-Lipschitz, surjective map from a subset of $(M, I)_K$ to $(M, I)_K$. The following proposition shows that such a map is always an isometry, finishing the proof of Theorem 5.2. ■

Proposition 5.3 Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then $C = M$ and f is an isometry.

Proposition 5.3 is implied by the following three lemmas, some which are exercises found in [7].

Lemma 5.4 Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then M is the closure of C.

Proof Suppose that M is not the closure $\bar{C}$ of C. Take $q \in M \backslash \bar{C}$, and let $\varepsilon = d(q, \bar{C})$. Define p_i inductively, $p_0 = q$, $f(p_{i+1}) = p_i$. Let $p \in \bar{C}$ be any limit point of the sequence $\{p_i\}$, with $\lim_i p_{n_i} = p$. Since $f^m(p_n) \in C$ for any $m < n$, one has $f^m(p) \in \bar{C}$.

Clearly, $f^{n_i}(p_{n_i}) = q$. Take n_i such that $d(p, p_{n_i}) < \varepsilon$. Then $d(f^{n_i}(p), q) < \varepsilon$. This is a contradiction, because $f^n(p) \in \bar{C}$ and $\varepsilon = d(q, \bar{C})$. ■

Lemma 5.5 Let M be a compact metric space, and $f : M \longrightarrow M$ an isometric embedding. Then f is bijective.

Proof Follows from Lemma 5.4 directly. ■

Lemma 5.6 Let M be a compact metric space, and $f : M \longrightarrow M$ a 1-Lipschitz, surjective map. Then f is an isometry.

Proof Let d be the diameter of M, and let K be the space of all 1-Lipschitz functions $\mu : M \longrightarrow [0, d]$ with the sup-metric. By the Arzela–Ascoli theorem, K is compact. Now, f^* defines an isometry from K to itself, $\mu \longrightarrow \mu \circ f$. For any $z \in M$, the function $d_z(x) = d(x, z)$ belongs to K. However, $d_{f(z)}$ does not belong to the image of f^* unless $d(z, x) = d(f(z), f(x))$ for all x, because if $d(z, x) < d(f(z), f(x))$, one has $(f^*)^{-1}(d_{f(z)})(f(x)) = d(z, x) > d(f(z), f(x))$, hence $(f^*)^{-1}(d_{f(z)})$ cannot be Lipschitz. This is impossible by Lemma 5.5, because an isometry from K to itself must be bijective. Therefore, the map $f : M \longrightarrow M$ is an isometry. ■

The proof of Proposition 5.3 easily follows from Lemmas 5.6 and 5.4. Indeed, by Lemma 5.4, f is a surjective, 1-Lipschitz map from M to itself, and by Lemma 5.6 it is an isometry. ■

6 Eigenvalues and Periodic Points of Hyperbolic Automorphisms

The following proposition follows from a simple linear-algebraic observation.

Proposition 6.1 Let T be a holomorphic automorphism of a hyperkähler manifold (M, I), and $\gamma : H^2(M) \longrightarrow H^2(M)$ the corresponding isometry of $H^2(M)$. Then γ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real.

Proof Since T is holomorphic, γ preserves the Hodge decomposition

$$H^2(M, \mathbb{R}) = H^{(2,0)+(0,2)}(M, \mathbb{R}) \oplus H^{1,1}(M, \mathbb{R}).$$

Since the BBF form is invariant under γ and is positive definite on $H^{(2,0)+(0,2)}(M, \mathbb{R})$, the eigenvalues of γ are $|\alpha_i| = 1$ on this space. On $H^{1,1}(M, \mathbb{R})$, the BBF form has signature $(+, -, -, ..., -)$, hence γ can be considered as an element of $O(1, n)$. However, it is well known that any element of $SO(1, n)$ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real. ■

Definition 6.2 An automorphism of a hyperkähler manifold (M, I) or an automorphism of its cohomology algebra preserving the Hodge type is called **hyperbolic** if it acts with an eigenvalue α, $|\alpha| > 1$ on $H^2(M, \mathbb{R})$.

In holomorphic dynamics, there are many uses for the **d-th dynamical degree of an automorphism**, which is defined as follows. Given an automorphism T of a manifold M, we consider the corresponding action on $H^d(M, \mathbb{R})$, and d-th dynamical degree is logarithm of the maximal absolute value of its eigenvalues. In [27], K. Oguiso has shown that the dynamical degree of a hyperbolic automorphism is positive for all even d, and computed it explicitly for automorphisms of Hilbert schemes of K3 which come from automorphisms of K3. For 3-dimensional Kähler manifolds, dynamical degree was computed by F. Lo Bianco [19].

We compute the dynamical degree and the maximal eigenvalue of the automorphism action on cohomology for all even d and give an upper bound for odd ones. We also compute asymptotical growth of the trace of the action of T^N in cohomology, which could allow one to prove that the number of quasi-periodic points grows polynomially as the period grows. One needs to be careful here, because there could be periodic and fixed subvarieties, and their contribution to the Lefschetz fixed point formula should be calculated separately.

Theorem 6.3 Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Remark 6.4 Since the Kähler cone of M is fixed by γ, α is positive; see e.g. [10].

Remark 6.5 From Theorem 6.3, it follows immediately that $\mathrm{Tr}(\gamma^N)$ grows asymptotically as α^{nN}.

We prove Theorem 6.3 at the end of this section.

Recall that the Hodge decomposition defines multiplicative action of $U(1)$ on cohomology $H^*(M)$, with $t \in U(1) \subset \mathbb{C}$ acting on $H^{p,q}(M)$ as t^{p-q}. In [34], the group generated by $U(1)$ for all complex structures on a hyperkähler manifold was computed explicitly, and it was found that it is isomorphic $G = \mathrm{Spin}^+(H^2(M, \mathbb{R}), q)$ (with center acting trivially on even-dimensional forms and as -1 on odd-dimensional forms; see [33]). Here Spin^+ denotes the connected component.

In [30], it was shown that the connected component of the group of automorphisms of $H^*(M)$ is mapped to G surjectively and with compact kernel ([30, Theorem 3.5]). Therefore, to study the eigenvalues of automorphisms of $H^*(M)$, we may always assume that they belong to G.

Now, the eigenvalues of $g \in G$ on its irreducible representations can always be computed using the Weyl character formula. The computation is time-consuming, and instead of using Weyl character formula, we use the following simple observation.

Claim 6.6 Let G be a group, and V its representation. Then the eigenvalues of g and xgx^{-1} are equal for all $x, g \in G$. ■

To prove Theorem 6.3, we replace one-parametric group containing the hyperbolic automorphism by another one-parametric group adjoint to it in G, and describe this second one-parametric group in terms of the Hodge decomposition.

Proposition 6.7 Let (M, I) be a hyperkähler manifold, and γ an automorphism of the ring $H^*(M)$. Assume that γ acts on $H^2(M)$ with an eigenvalue $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d (with eigenspace of dimension 1), and on odd cohomology $H^{2d+1}(M)$ it is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Proof Denote by G the group of automorphisms of $H^*(M)$. As shown above, its Lie algebra is $(so)(3, b_2(M) - 3)$, hence the connected component of G is a simple Lie group.

Write the polar decomposition $\gamma = \gamma_1 \circ \beta$, where $\gamma_1 \in G$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$, β belongs to the maximal compact subgroup, and they commute. Clearly, the eigenvalues of β on V are of absolute value 1, and absolute values of eigenvalues of γ and γ_1 are equal. Therefore, we can without restricting generality assume that $\gamma = \gamma_1$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$.

Consider now the following one-parametric subgroup of the complexification $G_{\mathbb{C}} \subset \mathrm{Aut}(H^*(M, \mathbb{C}))$: $\rho(t)$ acts on $H^{p,q}$ as t^{p-q}, $t \in \mathbb{R}$. The corresponding element of the Lie algebra has only two non-zero real eigenvalues in adjoint action. Clearly, all one-parametric subgroups of $G_{\mathbb{C}} = \mathrm{Spin}(H^2(M, \mathbb{C}))$ with this property are conjugate. This implies that γ is conjugate to an element $\rho(\alpha)$.

By Claim 6.6, γ and $\rho(\alpha)$ have the same eigenvalues, and $\rho(\alpha)$ clearly has eigenvalues $\alpha^{\frac{d-i}{2}}, \alpha^{\frac{d-i-1}{2}}, ...\alpha^{\frac{i-d}{2}}$ on $H^d(M)$. ■

Corollary 6.8

$$\lim_{n \longrightarrow \infty} \frac{\log \mathrm{Tr}(f^n)\big|_{H^*(M)}}{n} = d \log \alpha,$$

where $2d = \dim_{\mathbb{C}} M$. In particular, the number of k-periodic points grows as α^{nk}, assuming that they are isolated. ■

Remark 6.9 The case when f admits non-isolated periodic points is treated in [12], who prove that the number of isolated k-periodic points still grows no faster than α^{nk}; the lower bound is still unknown.

The same argument as in Proposition 6.7 also proves the following theorem.

Theorem 6.10 Let M be a hyperkähler manifold, and $\gamma \in \mathrm{Aut}(H^*(M))$ an automorphism of cohomology algebra preserving the Hodge decomposition and acting on $H^{1,1}(M)$ hyperbolically. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the eigenspace of eigenvalue $\alpha^{k/2}$ on $H^d(M)$ is isomorphic to $H^{\frac{(d+k)}{2}, \frac{(d-k)}{2}}(M)$. ■

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