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Geometry Over Nonclosed Fields

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Preface

Geometry Over Nonclosed Fields is a reference to an active area at the interface of classical algebraic geometry and arithmetic geometry. In recent years, there has been a rapid exchange of ideas between these domains: many questions concerning integral or rational solutions of Diophantine equations become accessible via complex or even symplectic geometric techniques. On the other hand, deep problems such as the rationality problem in algebraic geometry are now understood to be related to arithmetic properties of function fields.

The papers in this proceedings volume of a Simons Symposium that took place in 2015 are concerned with different aspects of this interaction.

The joint work of Bogomolov, Kamenova, Lu, and Verbitsky is devoted to the study of the Kobayashi metric on compact hyperkähler manifolds. This metric is defined via holomorphic maps of complex one-dimensional discs with standard metric into the manifold. For any compact complex manifold, the Kobayashi metric defines a unique compact metric space with a metric-continuous surjection of the manifold to that space which induces the Kobayashi metric. Conjecturally, this metric is trivial for compact hyperkähler manifolds. This would follow immediately from a version of the SYZ conjecture for such manifolds, i.e., from the existence of a smooth complex deformation to a hyperkähler manifold with a Lagrangian fibration. However, this is not yet known for all manifolds of this type. In this paper, the authors establish a partial result. Namely, they show that a compact hyperkähler manifold with an automorphism of infinite order has everywhere degenerate Kobayashi metric, i.e., the fibers of the projection to the compact metric space have positive dimension.

The article of Debarre, Laface, and Roulleau is devoted to a classical problem of describing lines on a cubic hypersurface. It is a well-known fact that a smooth cubic surface has exactly 27 lines. This also holds for cubic surfaces over arbitrary algebraically closed fields. However, there are examples of cubic surfaces over nonclosed fields without lines, in particular, over arbitrarily large finite fields. Here, the authors consider the following question: "Over which finite fields does every smooth cubic hypersurface of dimension at least three contain at least one line?" Their answer is close to optimal: they show that in dimension three a line exists when the finite field contains at least 11 elements and that there exist smooth cubic threefolds without lines for fields with 2, 3, 4 or 5 elements. For fields with 7, 8, and 9 elements, the situation is still unclear. Cubic fourfolds contain lines over fields with 2 or at least 5 elements, while there exist counterexamples over fields with 3 and 4 elements. All smooth cubics of dimension ≥ 5 contain lines.

Harder, Katzarkov, and Liu use long-standing rationality problems as an impetus to develop a theory of perverse sheaves of categories, inspired by recent progress in mathematical physics. This offers the prospect of connecting cohomological and cycle-theoretic techniques (like decomposition of the diagonal) with geometric structures arising from homological mirror symmetry, e.g., derived categories of coherent sheaves and Lagrangian fibrations. These connections are fleshed out in key examples like Fano threefolds and cubic hypersurfaces.

The contribution of de Jong and Starr is motivated by a desire to understand Kontsevich moduli spaces of genus zero stable maps to smooth projective varieties. These can have many irreducible components of varying dimensions but nevertheless carry a virtual fundamental class. When the moduli space happens to be irreducible of the expected dimension, it is important to understand its place in the Kodaira classification. Higher-order notions like rational simple connectedness hinge on the rational connectedness of the moduli spaces, which often holds when they admit negative canonical classes. This paper develops “virtual canonical classes”, i.e., formulas in terms of tautological divisors that make sense even when the moduli space is not integral.

Lieblich and Olsson explore modern formulations of the Torelli theorem for $K3$ surfaces. The original formulation states that two complex $K3$ surfaces with isomorphic Hodge structures are in fact isomorphic. But if only the transcendental cohomologies are isomorphic, then the $K3$ surfaces have equivalent derived categories of coherent sheaves. The notion of derived equivalence makes sense over more general algebraically closed fields. Lieblich and Olsson explore these, isolating a class of derived equivalences, strongly filtered equivalences, that suffice to recover isomorphism classes of $K3$ surfaces. This builds on dramatic recent progress on the Tate conjecture for $K3$ surfaces.

Liedtke’s manuscript explores the observation that Galois-invariant globally generated line bundles are associated with morphisms to Brauer–Severi varieties. He considers this over arbitrary fields and carefully analyzes the associated homomorphisms of Picard groups. He also revisits the classification of del Pezzo surfaces from the perspective of morphisms to Brauer–Severi varieties.

Várilly–Alvarado’s article provides a survey of results and conjectures in the arithmetic of $K3$ surfaces. The first topic concerns the structure and the computation of Picard groups of $K3$ surfaces defined over arithmetic fields and their behavior under reduction modulo primes. The second topic is Brauer groups of $K3$ surfaces over arithmetic fields, their relation to the Hasse principle and Brauer–Manin obstruction to the existence of rational points, and possible effective bounds on the transcendental parts of Brauer groups. The last problem is analogous to the Mazur–Merel theorem concerning effective uniform bounds for torsion of elliptic curves. An abundance of explicit examples will make this a great reference for researchers working in this area.

Zarhin considers the odd-dimensional étale cohomology of an algebraic variety, twisted by tensoring with a power of roots of unity. He generalizes a result of Serre: if an abelian variety defined over K contains a point of order precisely m over K, then K contains roots of unity of order m. This result follows from the existence of a non-degenerate pairing on the one-dimensional cohomology group of an abelian variety, modulo m, with values in the multiplicative group of m-th roots of unity. Zarhin found a similar pairing for arbitrary twisted odd-dimensional cohomology. Thus if the variety is defined over K and the Galois action on such cohomology modulo m is trivial, then K contains roots of unity of order m.

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Contents

On the Kobayashi Pseudometric, Complex Automorphisms and Hyperkähler Manifolds

Fedor Bogomolov, Ljudmila Kamenova, Steven Lu and Misha Verbitsky

Abstract We define the Kobayashi quotient of a complex variety by identifying points with vanishing Kobayashi pseudodistance between them and show that if a complex projective manifold has an automorphism whose order is infinite, then the fibers of this quotient map are nontrivial. We prove that the Kobayashi quotients associated to ergodic complex structures on a compact manifold are isomorphic. We also give a proof of Kobayashi's conjecture on the vanishing of the pseudodistance for hyperkähler manifolds having Lagrangian fibrations without multiple fibers in codimension one. For a hyperbolic automorphism of a hyperkähler manifold, we

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prove that its cohomology eigenvalues are determined by its Hodge numbers, compute its dynamical degree and show that its cohomological trace grows exponentially, giving estimates on the number of its periodic points.

1 Introduction

Kobayashi conjectured that a compact Kähler manifold with semipositive Ricci curvature has vanishing Kobayashi pseudometric. In a previous paper [16] Kamenova–Lu–Verbitsky have proved the conjecture for all K3 surfaces and for certain hyperkähler manifolds that are deformation equivalent to Lagrangian fibrations. Here we give an alternative proof of this conjecture for hyperkähler Lagrangian fibrations without multiple fibers in codimension one, see Sect. 3.

Theorem 1.1 Let $f: M \longrightarrow B = \mathbb{CP}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then the Kobayashi pseudometric d_M vanishes identically on M and the Royden–Kobayashi pseudonorm $|\ |_M$ vanishes identically on a Zariski open subset of M.

In Sect. 4, we explore compact complex manifolds M having an automorphism of infinite order. If such a manifold is projective, we show that the Kobayashi pseudometric is everywhere degenerate. For each point $x \in M$ we define the subset $M_x \subset M$ of points in M whose pseudo-distance to x is zero. Define the relation $x \sim y$ on M given by $d_M(x, y) = 0$. There is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$, called **the Kobayashi quotient map**. We say that $|\ |_M$ is **Voisin-degenerate** at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that $\varphi_n(0) \to x,\ |\varphi_n'(0)|_h = 1$ and $r_n \to \infty$.

Theorem 1.2 Let M be a complex projective manifold with an automorphism f of infinite order. Then the Kobayashi pseudo-metric d_M is everywhere degenerate in the sense that $M_x \neq \{x\}$ for all $x \in M$. The Royden–Kobayashi pseudo-norm $|\ |_M$ is everywhere Voisin-degenerate. Moreover, every fiber of the map $\Psi : M \longrightarrow S$ constructed above contains a Brody curve and is connected.

Define **the Kobayashi quotient** M_K of M to be the space of all equivalence classes $\{x \sim y \mid d_M(x, y) = 0\}$ equipped with the metric induced from d_M.

In Sect. 5, we show that the Kobayashi quotients for ergodic complex structures are isometric, equipped with the natural quotient pseudometric. This generalizes the key technical result of [16] for the identical vanishing of d_M for ergodic complex structures on hyperkähler manifolds.

Theorem 1.3 Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Finally in Sect. 6, we prove that the cohomology eigenvalues of a hyperbolic automorphism of a hyperkähler manifold are determined by its Hodge numbers. We compute its dynamical degree in the even cases and give an upper bound in the odd cases.

Theorem 1.4 Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

As a corollary we obtain that the trace $\mathrm{Tr}(\gamma^N)$ grows asymptotically as α^{nN}. We also show that the number of k-periodic points grows as α^{nk}.

The work on this paper started during the Simons Symposium "Geometry over nonclosed fields" held in March, 2015. The authors are grateful to the Simons Foundation for providing excellent research conditions.

2 Preliminaries

Definition 2.1 A *hyperkähler* (or *irreducible holomorphic symplectic*) manifold M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ where σ is everywhere non-degenerate.

Recall that a fibration is a connected surjective holomorphic map. On a hyperkähler manifold the structure of a fibration, if one exists, is limited by Matsushita's theorem.

Theorem 2.2 (Matsushita, [21]) Let M be a hyperkähler manifold and $f : M \longrightarrow B$ a fibration with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2} \dim M$ and the general fiber of f is a Lagrangian abelian variety. The base B has at worst $\mathbb{Q}$-factorial log-terminal singularities, has Picard number $\rho(B) = 1$ and $-K_B$ is ample.

Remark 2.3 B is smooth in all of the known examples. It is conjectured that B is always smooth.

Theorem 2.4 (Hwang [15]) In the settings above, if B is smooth then B is isomorphic to $\mathbb{CP}^n$, where $\dim_{\mathbb{C}} M = 2n$.

Definition 2.5 Given a hyperkähler manifold M, there is a non-degenerate integral quadratic form q on $H^2(M, \mathbb{Z})$, called the *Beauville–Bogomolov–Fujiki form* (BBK form for short), of signature $(3, b_2 - 3)$ and satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M, \mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. For a detailed description of the form we refer the reader to [2, 6, 13].

Remark 2.6 Given $f : M \longrightarrow \mathbb{CP}^n$, h the hyperplane class on $\mathbb{CP}^n$, and $\alpha = f^*h$, then α is nef and $q(\alpha) = 0$.

Conjecture 2.7 [SYZ] If L is a nontrivial nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, given as above.

Remark 2.8 This conjecture is known for deformations of Hilbert schemes of points on K3 surfaces (Bayer–Macrì [1]; Markman [20]), and for deformations of the generalized Kummer varieties $K_n(A)$ (Yoshioka [36]).

Definition 2.9 The *Kobayashi pseudometric* on M is the maximal pseudometric d_M such that all holomorphic maps $f : (D, \rho) \longrightarrow (M, d_M)$ are distance decreasing, where (D, ρ) is the unit disk with the Poincaré metric.

Definition 2.10 A manifold M is *Kobayashi hyperbolic* if d_M is a metric, otherwise it is called *Kobayashi non-hyperbolic*.

Remark 2.11 In [17], it is asked whether a compact Kähler manifold M of semi-positive Ricci curvature has identically vanishing pseudometric, which we denote by $d_M \equiv 0$. The question applies to hyperkähler manifolds but was unknown even for the case of surfaces outside the projective case. But Kamenova–Lu–Verbitsky (in [16]) have recently resolved completely the case of surfaces with the following affirmative results.

Theorem 2.12 [16] Let S be a K3 surface. Then $d_S \equiv 0$.

Remark 2.13 A birational version of a conjecture of Kobayashi [17] would state that a compact hyperbolic manifold be of general type if its Kobayashi pseudometric is nondegenerate somewhere (i.e. nondegenerate on some open set). This was open for surfaces but now resolved outside surfaces of class VII.

Theorem 2.14 [16] Let M be a hyperkähler manifold of non-maximal Picard rank and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 2.15 [16] Let M be a hyperkähler manifold with $b_2(M) \geq 7$ (expected to always hold) and with maximal Picard rank $\rho = b_2 - 2$. Assume the SYZ conjecture for deformations of M. Then $d_M \equiv 0$.

Remark 2.16 Except for the proof of Theorem 2.15, we indicate briefly a proof of these theorems below. Theorem 2.15 is proved in [16] using the existence of double Lagrangian fibrations on certain deformations of M. Here we give a different proof of vanishing of the Kobayashi pseudometric for certain hyperkähler Lagrangian fibrations without using double fibrations.

Definition 2.17 Let M be a compact complex manifold and $\mathrm{Diff}^0(M)$ the connected component to identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient Teich := Comp / $\mathrm{Diff}^0(M)$. The Teichmüller space is finite-dimensional for M Calabi–Yau [11]. Let $\mathrm{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group* $\Gamma := \mathrm{Diff}^+(M)/\mathrm{Diff}^0(M)$ acts on Teich. An element $I \in$ Teich is called *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where

$$\Gamma \cdot I = \{I' \in \mathrm{Teich} : (M, I) \sim (M, I')\}.$$

Theorem 2.18 (Verbitsky, [32]) If M is hyperkähler and $I \in$ Teich, then I is ergodic if and only if $\rho(M, I) < b_2 - 2$.

Remark 2.19 For a K3 surface (M, I) not satisfying the above condition on the Picard rank ρ, it is easily seen to admit Lagrangian (elliptic) fibrations over $\mathbb{CP}^1$ without multiple fibers, and it is projective. Then $d_{(M,J)} \equiv 0$ by Theorem 3.2 below, for example.

Proposition 2.20 Let (M, J) be a compact complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in$ Teich be an ergodic complex structure deformation equivalent to J. Then $d_{(M,I)} \equiv 0$.

Proof Here we shall reproduce the proof from [16]. Consider the diameter function diam : Teich $\longrightarrow \mathbb{R}_{\geqslant 0}$, the maximal distance between two points. It is upper semi-continuous (Corollary 1.23 in [16]). Since the complex structure J is in the limit set of the orbit of the ergodic structure I, by upper semi-continuity $0 \leqslant \mathsf{diam}(I) \leqslant \mathsf{diam}(J) = 0$. ■

3 (Royden–)Kobayashi Pseudometric on Abelian Fibrations

The following lemma is a generalization of Lemma 3.8 in [8] to the case of abelian fibrations. The generalization is given for example in the Appendix of [16]. Recall that an abelian fibration is a connected locally projective surjective Kähler morphism with abelian varieties as fibers.

Lemma 3.1 Let $\pi : T \longrightarrow C$ be an abelian fibration over a non-compact complex curve C which locally has sections and such that not all components of the fibers are multiple. Then T has an analytic section over C. This is the case if π has no multiple fibers.

Proof There is a Neron model N for T and a short exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}(L) \longrightarrow \mathcal{O}(N) \longrightarrow 0$$

where L is a vector bundle, F is a sheaf of groups $\mathbb{Z}^{2n}$ with degenerations, i.e., sheaf of discrete subgroups with generically maximal rank, and $\mathcal{O}(N)$ is the sheaf of local sections of N (whose general fibers are abelian varieties). Thus T corresponds to an element θ in $H^1(C, \mathcal{O}(N))$. There is an induced exact sequence of cohomologies: $H^1(C, \mathcal{O}(L)) \longrightarrow H^1(C, \mathcal{O}(N)) \longrightarrow H^2(C, F)$. Note that $H^1(C, \mathcal{O}(L)) = 0$ since C is Stein, and $H^2(C, F) = 0$ since it is topologically one-dimensional. Thus $\theta = 0$ and hence there is an analytic section. The last part of the lemma is given by Proposition 4.1 of [16]. ■

Theorem 3.2 Let $f : M \longrightarrow B = \mathbb{CP}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then $d_M \equiv 0$ and $|\ |_M$ vanishes on a nonempty Zariski open subset of M.

Proof The fibers of f are projective, and furthermore, there is a canonical polarization on them (see [25, 26], respectively). This also follows from [31], Theorem 1.10, which implies that the given fibration is diffeomorphic to another fibration $f : M' \longrightarrow B$ with holomorphically the same fibers and the same base, but with projective total space M'. Standard argument (via the integral lattice in the "local" Neron–Severi group) now shows that f is locally projective.

By assumption, there are no multiple fibers outside a codimension 2 subset $S \subset B$ whose complement U contains at most the smooth codimension-one part D_0 of the discriminant locus of f where multiplicity of fibers are defined locally generically. Since the pseudometric is unchanged after removing codimension 2 subsets [18], it is enough to restrict the fibration to that over U.

Let $C = \mathbb{P}^1$ be a line in $B = \mathbb{P}^n$ contained in U (and intersecting D_0 transversely). Then f restricts to an abelian fibration $X = f^{-1}(C)$ over C without multiple fibers and so Lemma 3.1 applies to give a section over the affine line $A^1 = C \setminus (\infty)$.

As S is codimension two or higher, we can connect any two general points in U by a chain of such A^1's in U. One can thus connect two general points x and y on M by a chain consisting of fibers and sections over the above A^1's. Since the Kobayashi pseudometric vanishes on each fiber and each such section, the triangle inequality implies $d_M(x, y) = 0$. Therefore d_M vanishes on a dense open subset of M and hence $d_M \equiv 0$ by the continuity of d_M.

The same argument gives the vanishing statement of $|\ |_M$ via Theorem A.2 of [16]. ■

Remark 3.3 In the theorem above, it is sufficient to assume that B is nonsingular and that $d_B \equiv 0$, true if B is rationally connected. In fact, if one assumes further the vanishing of $|\ |_B$ on a nonempty Zariski open, then the same is true for $|\ |_M$, generalizing the corresponding theorems in [16]. The reader should have no difficulty to see these by the obvious modifications of the above proof.

4 Automorphisms of Infinite Order

We first sketch the proof of Kobayashi's theorem that Kobayashi hyperbolic manifolds have only finite order automorphisms (Theorem 9.5 in [17]).

Theorem 4.1 Let M be a Kobayashi hyperbolic manifold. Then its group of birational transformations is finite.

Proof First, notice that a birational self-map is a composition of a blow-up, an automorphism and a blow-down. Since M contains no rational curves, any birational self-map is holomorphic, and we need to prove the finiteness of the automorphism group.

Observe that the automorphisms of a hyperbolic manifold are isometries of the Kobayashi metric. Also the group of isometries of a compact metric space is compact with respect to the compact open topology by a theorem of Dantzig and Van der Waerden, see for example [18, Theorem 5.4.1]. On the other hand, compact Kobayashi hyperbolic manifolds have no holomorphic vector fields, because each such vector field gives an orbit which is an entire curve. This means that the group of holomorphic automorphisms $\mathrm{Aut}(M)$ of M is discrete as it is a complex Lie group in the compact open topology acting holomorphically on M by the work of Bochner–Montgomery [4, 5]. Since $\mathrm{Aut}(M)$ is discrete and compact, this means it is finite. ■

Consider the pseudo-distance function $d_M : M \times M \longrightarrow \mathbb{R}$, defined by the Kobayashi pseudo-distance $d_M(x, y)$ on pairs (x, y). It is a symmetric continuous function which is bounded for compact M. Since it is symmetric, we can consider d_M as a function on the symmetric product $\mathrm{Sym}^2 M$ with $d_M = 0$ on the diagonal.

Lemma 4.2 There is a compact space S with a continuous map $\Psi : M \longrightarrow S$ and there is a distance function d_S on S making S into a compact metric space such that $d_M = d_S \circ \psi$, where $\psi : \mathrm{Sym}^2 M \longrightarrow \mathrm{Sym}^2 S$ is the map induced by Ψ.

Proof The subset $M_x \subset M$ of points $y \in M$ with $d_M(x, y) = 0$ is compact and connected. The relation $x \sim y$ on M given by $d_M(x, y) = 0$ is symmetric and transitive so that $M_x = M_y$ if and only if $x \sim y$. So there is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$. Note that the set S is equipped with a natural metric induced from d_M. Indeed, $d_M(x', y')$ is the same for any points $x' \in M_x$, $y' \in M_y$, and hence d_M induces a metric d_S on S. This metric provides a topology on S, and since the set $U_{x,\varepsilon} = \{y \in M \mid d_M(x, y) < \varepsilon\}$ is open, the map $\Psi : M \longrightarrow S$ is continuous. Thus the metric space S is also compact. This completes the proof of the lemma. ■

Remark 4.3 The natural quotient considered above was already proposed in [17] albeit little seems to be known about its possible structure. In particular, it is known that even when M is compact, S may not have the structure of a complex variety [14]. As we note in Remark 4.12, Campana conjectured that the Kobayashi metric quotient of a Kähler manifold has birational general type, and hence, a dense subset of the metric quotient should carry a complex (even quasi-projective) structure for such manifolds.

Remark 4.4 If there is a holomorphic family of varieties X_t smooth over a parameter space T of say dimension 1, then the relative construction also works by considering the problem via that of the total space over small disks in T. In particular, there is a monodromy action on the resulting family of compact metric spaces S_t by isometries over T, c.f. Sect. 5.

Let M be a complex manifold and h a hermitian metric on M with its associated norm $|\ |_h$.

Recall that a theorem of Royden says that the Kobayashi pseudo-metric d_M can be obtained by taking the infimum of path-integrals of the infinitesimal pseudonorm $|\ |_M$, where

$$|v|_M = \inf \left\{ \frac{1}{R} \mid f : D_R \to M \text{ holomorphic, } R > 0,\ f'(0) = v \right\}.$$

Here D_R is the disk of radius R centred at the origin. Recall also that $|\ |_M$ is upper-semicontinuous [29].

Definition 4.5 We say that $|\ |_M$ is *Voisin-degenerate* at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that

$$\varphi_n(0) \to x,\ |\varphi_n'(0)|_h = 1 \text{ and } r_n \to \infty.$$

Observe that the locus Z_M of M consisting of points where $|\ |_M$ is Voisin-degenerate is a closed set.

Remark 4.6 If $(x, v) \in T_x M$ is a point in the tangent bundle of M at x which is Voisin-degenerate, then it does not necessarily follow that $|v|_M = 0$, because the Kobayashi pseudometric is semicontinuous but might not be continuous at that point. However, the other implication is true: by upper semicontinuity, if $|v|_M = 0$, then for any sequence $(x_n, v_n) \longrightarrow (x, v)$ we have $|v_n|_M \longrightarrow 0$, i.e., the point x is Voisin degenerate in a strong sense.

The following theorem is essentially [35, Proposition 1.19].

Theorem 4.7 Consider the equivalence relation $x \sim y$ on M given by $d_M(x, y) = 0$ where d_M is the Kobayashi pseudo-metric on M. Then every non-trivial orbit (that is, a non-singleton equivalence class) of this relation consists of Voisin-degenerate points, and the union of such orbits is a closed set. If, further, M is compact, then each nontrivial orbit contains the image of a nontrivial holomorphic map $\mathbb{C} \to M$.

We also need the following theorem.

Theorem 4.8 Assume M is compact. Then each orbit of the equivalence relation given above is connected.

Proof Let M_x be the orbit passing through x as before and

$$M_x(n) = \left\{ y \in X \,\middle|\, d_X(x, y) \leqslant \frac{1}{n} \right\}.$$

Then each $M_x(n)$ is compact and connected and $M_x = \cap_n M_x(n)$. If M_x is not connected, then there are disjoint open sets U, V in M separating M_x leading to the contradiction

$$\emptyset = (U \cup V)^c \cap M_x = \cap_n[(U \cup V)^c \cap M_x(n)] \neq \emptyset,$$

each $(U \cup V)^c \cap M_x(n)$ being nonempty compact as $M_x(n)$ is connected. ■

We want to exploit the existence of an automorphism of an infinite order for the analysis of Kobayashi metric. The following conjecture provides with a necessary argument for a projective manifold.

The rest of this section contains several arguments which suggest a possible strategy to study the vanishing locus of Kobayashi metric on a projective manifold in the presence of an infinite order automorphism. We label them as "conjectures" to distinguish these suggestive arguments from the fully rigorous proofs. We plan to put rigour to these heuristic arguments at some later date.

Conjecture 4.9 Let X be a complex projective manifold and $[C]$ an ample class of curves on X. Let U be an open domain in X and w_h the volume form of a Kähler metric h on X. Then for a sufficently big n there is a curve $C_1 \in [nC]$ such that $\mathrm{Vol}_h(C \cap U) \geqslant (w_h(U)/w_h(X) - \varepsilon)\,\mathrm{Vol}_h(C)$ for arbitrary small ε.

Sketch of a possible proof of this result: The result evidently holds for P^n and Fubini-Study metric on P^n since P^n is homogeneous with respect to the Fubini-Study metric. In this case it follows from the integral volume formula for the family of projective lines, parametrized by the Grassmanian which surjects onto P^n. It immediately implies the existence of lines which satisfy the inequality.

Similar formula holds for the family of algebraic curves of any given degree. In particular we obtain an infinitesimal version of the formula which therefore holds for any metric on projective space. Using a finite map of an n-dimensional projective manifold X onto $\mathbb{C}P^n$ we can derive the same formula for the Kähler pseudometrics induced from $\mathbb{C}P^n$ and then use its local nature for any X. ■

Conjecture 4.10 Let f be an automorphism of infinite order on a complex projective manifold X of dimension n. Assume that there is a domain U in X, a smooth Kähler metric g on X and positive constants c, c' such that $cg \leqslant (f^m)^*g \leqslant c'g$ on U for all powers f^m of f. Then f is an isometry of (X, h) for some Kähler metric h on X and hence some power of f is contained in a connected component of the group of complex isometries of (X, h). In particular, X has a faithful holomorphic action by an abelian variety.

Sketch of a possible proof of this result: Let h be the pull back of the Fubini-Study metric on X of the embedding corresponding to a very ample line bundle L on X. Note that we can assume that $ag \leqslant (f^m)^*h \leqslant a'g$ on $\bar{U}$ for some positive constants a, a' which are independent of the parameter m. Note that $\int_X (f^m)^*h^n$ does not depend on m since the class of the volume h^n maps into itself. Therefore, we have

$$\mu' \int_X h^n < \int_{\bar{U}} (f^m)^*h^n < \mu \int_X h^n \tag{4.1}$$

for some μ and μ' independent of m. Let c be a class of ample (i.e., very movable) curves. Then, for a sufficiently big multiple Nc of the class c, there are curves $C \in Nc$ with $\int_{C \bigcap \bar{U}} h > \nu(h, c)$, and similarly we have $\int_{C \bigcap \bar{U}} (f^m)^*h > \nu((f^m)^*h, c)$, where (h, c) is a pairing of the homology class c and the class of kahler metric h. Since

$$a \int_{C \bigcap \bar{U}} g < \int_{C \bigcap \bar{U}} (f^m)^*h < a' \int_{C \bigcap \bar{U}} g,$$

we obtain that $((f^m)^*h, c)$ is bounded from above by $a'(g, c)$ and from below by by $a(g, c)$ for any ample class. Since ample classes generate the dual $\mathrm{N}_1(X)_{\mathbb{R}}$ of $\mathrm{NS}(X) \otimes \mathbb{R}$ we obtain that $(f^m)^*h$ as linear functional on $\mathrm{N}_1(X) \otimes \mathbb{R}$ is contained in a bounded subset.

A slightly more direct argument for this last boundedness is as follows. Since each $f^{m*}h$ represents a Kähler class, it is sufficient to bound them from above as linear functionals on ample classes c of curves. Note that the first inequality in Eq. 4.1 says that $w_{f^{m*}h}(U) > \mu' w_h(X)$ with μ' independent of m. By the previous lemma, one can therefore choose $C_m \in c$ such that

$$((f^m)^*h, c) = \mathrm{Vol}_{(f^m)^*h}(C_m) \leqslant \delta \int_{C_m \bigcap \bar{U}} (f^m)^*h < \delta a' \int_{C_m \bigcap \bar{U}} g \leqslant \gamma(g, c),$$

where δ is independent of m and $\gamma = \delta a'$.

Since there are only a finite number of integral classes in any bounded set in $\mathrm{NS}(X)_{\mathbb{R}}$, it follows that f^{m_0} leaves invariant the Káhler class of h for some $m_0 \neq 0$ and we may therefore assume that f itself leaves it invariant. In the case of Ricci-flat X, f must therefore be an isometry with respect to the unique Ricci-flat metric in the Kähler class given by Yau's solution to the Calabi conjecture. In general, if $H^1(X, C) = 0$, then f is induced from a projective action on P^N under a map $X \to P^N$. If $H^1(X, C) \neq 0$, then we have a map from $X \times \mathrm{Pic}^0(X)$ to a projective family of projective spaces $\{\mathbb{P}(H^0(L_t)^\vee), t \in \mathrm{Pic}^0(X)\}$ over $\mathrm{Pic}^0(X)$ and L_t defines a very ample invariant invertible sheaf on $X \times \mathrm{Pic}^0(X)$ over $\mathrm{Pic}^0(X)$. Hence, f has to be a complex isometry on X which completes the argument. ■

Conjecture 4.11 Let M be a projective manifold with an automorphism f of infinite order. Then the Kobayashi pseudo-metric d_M is everywhere degenerate in the sense that $M_x \neq \{x\}$ for all $x \in M$. Also the Kobayashi–Royden pseudo-norm $|\ |_M$

is everywhere Voisin-degenerate. Moreover, every fiber of the map $\Psi : M \longrightarrow S$ constructed above contains a Brody curve and is connected.

Sketch of a possible proof of this result: The map $f : M \longrightarrow M$ commutes with the projection onto S, and hence, induces an isometry on S. Since the action of f has infinite order on S, there is a sequence of powers f^{N_i} which converges to the identity on S by the compactness of the group Isom(S) of isometries of S (in the compact open topology) and by setting $N_i = n_i - n_{i-1}$ for a convergent subsequence f^{n_i} in Isom(S). We assume arguing by contradiction that d_M is non-degenerate at a point $x \in M$. Let U be the maximal subset in M where Ψ is a local isomorphism. Since the subsets M_x are connected, then U is exactly the subset where Ψ is an embedding. The set U is invariant under f and is open by Theorem 4.7. Hence, f^{N_i} converges to the trivial action on U. The boundary ∂U of U is a compact subset in M with $\partial U \neq \bar{U}$ and $d_M(x, \partial U) > 0$ for any point $x \in U$. Thus, a compact subset U_ε which consists of points $x \in U$ with $d_M(x, \partial U) \geqslant \varepsilon$ is f-invariant and the restriction of d_M on U_ε is a metric. It is also invariant under the action of f and by theorem of Royden ([28, Theorem 2]) we know that there are smooth Kähler metrics g, g' on X with the property that $g' > d_M \geqslant g$ on U_ε. Applying Conjecture 4.10 we obtain that f is an isometry on M with respect to some Kähler metric. Thus, either M has a nontrivial action of a connected algebraic group, and hence, trivial Kobayshi pseudometric, or f is of finite order which contradicts our assumption. Thus, we obtain a contradiction also with our initial assertion that d_M is metric on some open subset in M.

Note that a limit of Brody curves is again a nontrivial Brody curve by Brody's classical argument. By Theorem 4.7, this implies that the map $\Psi : M \longrightarrow S$ is everywhere degenerate, as it is degenerate in the complement of an everywhere dense open subset. ■

Remark 4.12 In [9, Conjecture 9.16], F. Campana conjectured that the Kobayashi quotient map of a complex projective manifold M should coincide (in the birational category) with the "core map" of M, with fibers which are "special" and the base which is a "general type" orbifold. Then Conjecture 4.11 would just follow, because the automorphism group of a general type variety is finite. Then a general fiber of the Kobayashi quotient map contains infinitely many points, hence its fibers are positively dimensional.

Remark 4.13 Note that both conditions of Conjecture 4.10 are sharp. It was shown by McMullen [23] that there are Kahler non-projective K3 surfaces with automorphisms of infinite order which contain invariant domains isomorphic to the two-dimensional ball. There are also examples by Bedford and Kim [3] of rational projective surfaces X with automorphisms of infinite order which contain an invariant ball. In this case there are no invariant volume forms on the variety X.

5 Metric Geometry of Kobayashi Quotients

Definition 5.1 Let M be a complex manifold, and d_M its Kobayashi pseudometric. Define **the Kobayashi quotient** M_K of M as the space of all equivalence classes $\{x \sim y \mid d_M(x, y) = 0\}$ equipped with the metric induced from d_M.

The main result of this section is the following theorem.

Theorem 5.2 Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Proof Consider the limit $\lim \nu_i(I) = J$, where ν_i is a sequence of diffeomorphisms of M. For each point $x \in (M, I)$, choose a limiting point $\nu(x) \in (M, J)$ of the sequence $\nu_i(x)$. Fix a dense countable subset $M_0 \subset M$ and replace the sequence ν_i by its subsequence in such a way that $\nu(m) := \lim \nu_i(m)$ is well defined for all $m \in M_0$.

By the upper-semicontinuity of the Kobayashi pseudometric, we have

$$d_{(M,J)}(\nu(x), \nu(y)) \geqslant d_{(M,I)}(x, y). \tag{5.1}$$

Let C_0 be the union of all $\nu(x)$ for all $x \in M_0$. Define a map $\psi : C_0 \longrightarrow (M, I)$ mapping $z = \nu(x)$ to x (if there are several choices of such x, choose one in arbitrary way). By (5.1), the map ψ is 1-Lipschitz with respect to the Kobayashi pseudometric. We extend it to a Lipschitz map on the closure C of C_0. For any $x \in (M, J)$, the Kobayashi distance between x and $\psi(\nu(x))$ is equal zero, also by (5.1). Therefore, ψ defines a surjective map on Kobayashi quotients: $\Psi : C_K \longrightarrow (M, I)_K$. Exchanging I and J, we obtain a 1-Lipshitz surjective map $\Phi : C'_K \longrightarrow (M, J)_K$, where C'_K is a subset of $(M, I)_K$. Taking a composition of Ψ and Φ, we obtain a 1-Lipschitz, surjective map from a subset of $(M, I)_K$ to $(M, I)_K$. The following proposition shows that such a map is always an isometry, finishing the proof of Theorem 5.2. ■

Proposition 5.3 Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then $C = M$ and f is an isometry.

Proposition 5.3 is implied by the following three lemmas, some which are exercises found in [7].

Lemma 5.4 Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then M is the closure of C.

Proof Suppose that M is not the closure $\bar{C}$ of C. Take $q \in M \backslash \bar{C}$, and let $\varepsilon = d(q, \bar{C})$. Define p_i inductively, $p_0 = q$, $f(p_{i+1}) = p_i$. Let $p \in \bar{C}$ be any limit point of the sequence $\{p_i\}$, with $\lim_i p_{n_i} = p$. Since $f^m(p_n) \in C$ for any $m < n$, one has $f^m(p) \in \bar{C}$.

Clearly, $f^{n_i}(p_{n_i}) = q$. Take n_i such that $d(p, p_{n_i}) < \varepsilon$. Then $d(f^{n_i}(p), q) < \varepsilon$. This is a contradiction, because $f^n(p) \in \bar{C}$ and $\varepsilon = d(q, \bar{C})$. ■

Lemma 5.5 Let M be a compact metric space, and $f : M \longrightarrow M$ an isometric embedding. Then f is bijective.

Proof Follows from Lemma 5.4 directly. ■

Lemma 5.6 Let M be a compact metric space, and $f : M \longrightarrow M$ a 1-Lipschitz, surjective map. Then f is an isometry.

Proof Let d be the diameter of M, and let K be the space of all 1-Lipschitz functions $\mu : M \longrightarrow [0, d]$ with the sup-metric. By the Arzela–Ascoli theorem, K is compact. Now, f^* defines an isometry from K to itself, $\mu \longrightarrow \mu \circ f$. For any $z \in M$, the function $d_z(x) = d(x, z)$ belongs to K. However, $d_{f(z)}$ does not belong to the image of f^* unless $d(z, x) = d(f(z), f(x))$ for all x, because if $d(z, x) < d(f(z), f(x))$, one has $(f^*)^{-1}(d_{f(z)})(f(x)) = d(z, x) > d(f(z), f(x))$, hence $(f^*)^{-1}(d_{f(z)})$ cannot be Lipschitz. This is impossible by Lemma 5.5, because an isometry from K to itself must be bijective. Therefore, the map $f : M \longrightarrow M$ is an isometry. ■

The proof of Proposition 5.3 easily follows from Lemmas 5.6 and 5.4. Indeed, by Lemma 5.4, f is a surjective, 1-Lipschitz map from M to itself, and by Lemma 5.6 it is an isometry. ■

6 Eigenvalues and Periodic Points of Hyperbolic Automorphisms

The following proposition follows from a simple linear-algebraic observation.

Proposition 6.1 Let T be a holomorphic automorphism of a hyperkähler manifold (M, I), and $\gamma : H^2(M) \longrightarrow H^2(M)$ the corresponding isometry of $H^2(M)$. Then γ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real.

Proof Since T is holomorphic, γ preserves the Hodge decomposition

$$H^2(M, \mathbb{R}) = H^{(2,0)+(0,2)}(M, \mathbb{R}) \oplus H^{1,1}(M, \mathbb{R}).$$

Since the BBF form is invariant under γ and is positive definite on $H^{(2,0)+(0,2)}(M, \mathbb{R})$, the eigenvalues of γ are $|\alpha_i| = 1$ on this space. On $H^{1,1}(M, \mathbb{R})$, the BBF form has signature $(+, -, -, ..., -)$, hence γ can be considered as an element of $O(1, n)$. However, it is well known that any element of $SO(1, n)$ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real. ■

Definition 6.2 An automorphism of a hyperkähler manifold (M, I) or an automorphism of its cohomology algebra preserving the Hodge type is called **hyperbolic** if it acts with an eigenvalue α, $|\alpha| > 1$ on $H^2(M, \mathbb{R})$.

In holomorphic dynamics, there are many uses for the **d-th dynamical degree of an automorphism**, which is defined as follows. Given an automorphism T of a manifold M, we consider the corresponding action on $H^d(M, \mathbb{R})$, and d-th dynamical degree is logarithm of the maximal absolute value of its eigenvalues. In [27], K. Oguiso has shown that the dynamical degree of a hyperbolic automorphism is positive for all even d, and computed it explicitly for automorphisms of Hilbert schemes of K3 which come from automorphisms of K3. For 3-dimensional Kähler manifolds, dynamical degree was computed by F. Lo Bianco [19].

We compute the dynamical degree and the maximal eigenvalue of the automorphism action on cohomology for all even d and give an upper bound for odd ones. We also compute asymptotical growth of the trace of the action of T^N in cohomology, which could allow one to prove that the number of quasi-periodic points grows polynomially as the period grows. One needs to be careful here, because there could be periodic and fixed subvarieties, and their contribution to the Lefschetz fixed point formula should be calculated separately.

Theorem 6.3 Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Remark 6.4 Since the Kähler cone of M is fixed by γ, α is positive; see e.g. [10].

Remark 6.5 From Theorem 6.3, it follows immediately that $\mathrm{Tr}(\gamma^N)$ grows asymptotically as α^{nN}.

We prove Theorem 6.3 at the end of this section.

Recall that the Hodge decomposition defines multiplicative action of $U(1)$ on cohomology $H^*(M)$, with $t \in U(1) \subset \mathbb{C}$ acting on $H^{p,q}(M)$ as t^{p-q}. In [34], the group generated by $U(1)$ for all complex structures on a hyperkähler manifold was computed explicitly, and it was found that it is isomorphic $G = \mathrm{Spin}^+(H^2(M, \mathbb{R}), q)$ (with center acting trivially on even-dimensional forms and as -1 on odd-dimensional forms; see [33]). Here Spin^+ denotes the connected component.

In [30], it was shown that the connected component of the group of automorphisms of $H^*(M)$ is mapped to G surjectively and with compact kernel ([30, Theorem 3.5]). Therefore, to study the eigenvalues of automorphisms of $H^*(M)$, we may always assume that they belong to G.

Now, the eigenvalues of $g \in G$ on its irreducible representations can always be computed using the Weyl character formula. The computation is time-consuming, and instead of using Weyl character formula, we use the following simple observation.

Claim 6.6 Let G be a group, and V its representation. Then the eigenvalues of g and xgx^{-1} are equal for all $x, g \in G$. ■

To prove Theorem 6.3, we replace one-parametric group containing the hyperbolic automorphism by another one-parametric group adjoint to it in G, and describe this second one-parametric group in terms of the Hodge decomposition.

Proposition 6.7 Let (M, I) be a hyperkähler manifold, and γ an automorphism of the ring $H^*(M)$. Assume that γ acts on $H^2(M)$ with an eigenvalue $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d (with eigenspace of dimension 1), and on odd cohomology $H^{2d+1}(M)$ it is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Proof Denote by G the group of automorphisms of $H^*(M)$. As shown above, its Lie algebra is $(so)(3, b_2(M) - 3)$, hence the connected component of G is a simple Lie group.

Write the polar decomposition $\gamma = \gamma_1 \circ \beta$, where $\gamma_1 \in G$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$, β belongs to the maximal compact subgroup, and they commute. Clearly, the eigenvalues of β on V are of absolute value 1, and absolute values of eigenvalues of γ and γ_1 are equal. Therefore, we can without restricting generality assume that $\gamma = \gamma_1$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$.

Consider now the following one-parametric subgroup of the complexification $G_{\mathbb{C}} \subset \operatorname{Aut}(H^*(M, \mathbb{C}))$: $\rho(t)$ acts on $H^{p,q}$ as t^{p-q}, $t \in \mathbb{R}$. The corresponding element of the Lie algebra has only two non-zero real eigenvalues in adjoint action. Clearly, all one-parametric subgroups of $G_{\mathbb{C}} = \operatorname{Spin}(H^2(M, \mathbb{C}))$ with this property are conjugate. This implies that γ is conjugate to an element $\rho(\alpha)$.

By Claim 6.6, γ and $\rho(\alpha)$ have the same eigenvalues, and $\rho(\alpha)$ clearly has eigenvalues $\alpha^{\frac{d-i}{2}}, \alpha^{\frac{d-i-1}{2}}, ...\alpha^{\frac{i-d}{2}}$ on $H^d(M)$. ■

Corollary 6.8

$$\lim_{n \longrightarrow \infty} \frac{\log \operatorname{Tr}(f^n)\big|_{H^*(M)}}{n} = d \log \alpha,$$

where $2d = \dim_{\mathbb{C}} M$. In particular, the number of k-periodic points grows as α^{nk}, assuming that they are isolated. ■

Remark 6.9 The case when f admits non-isolated periodic points is treated in [12], who prove that the number of isolated k-periodic points still grows no faster than α^{nk}; the lower bound is still unknown.

The same argument as in Proposition 6.7 also proves the following theorem.

Theorem 6.10 Let M be a hyperkähler manifold, and $\gamma \in \operatorname{Aut}(H^*(M))$ an automorphism of cohomology algebra preserving the Hodge decomposition and acting on $H^{1,1}(M)$ hyperbolically. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the eigenspace of eigenvalue $\alpha^{k/2}$ on $H^d(M)$ is isomorphic to $H^{\frac{(d+k)}{2}, \frac{(d-k)}{2}}(M)$. ■

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Lines on Cubic Hypersurfaces Over Finite Fields

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Abstract We show that smooth cubic hypersurfaces of dimension n defined over a finite field $\mathbf{F}_q$ contain a line defined over $\mathbf{F}_q$ in each of the following cases:

- $n = 3$ and $q \geq 11$;
- $n = 4$, and $q = 2$ or $q \geq 5$;
- $n \geq 5$.

For a smooth cubic threefold X, the variety of lines contained in X is a smooth projective surface $F(X)$ for which the Tate conjecture holds, and we obtain information about the Picard number of $F(X)$ and the 5-dimensional principally polarized Albanese variety $A(F(X))$.

Keywords Cubic hypersurfaces · Lines in hypersurfaces · Finite fields · Zeta functions · Tate conjecture · Fano varieties

1991 Mathematics Subject Classification 14G15 · 14J70 · 14F20 · 14G10

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1 Introduction

The study of rational points on hypersurfaces in the projective space defined over a finite field has a long history. Moreover, if $X \subset \mathbf{P}^{n+1}$ is a (smooth) cubic hypersurface, the (smooth) variety $F(X)$ parametrizing lines contained in X is an essential tool for the study of the geometry of X. Therefore, it seems natural to investigate $F(X)$ when X is a cubic hypersurface defined over a finite field $\mathbf{F}_q$ and the first question to ask is whether X contains a line defined over $\mathbf{F}_q$.

One easily finds smooth cubic surfaces defined over $\mathbf{F}_q$ containing no $\mathbf{F}_q$-lines, with q arbitrarily large. On the other hand, if $\dim(X) \geq 5$, the variety $F(X)$, when smooth, has ample anticanonical bundle, and it follows from powerful theorems of Esnault and Fakhruddin–Rajan that X always contains an $\mathbf{F}_q$-line (Sect. 6). So the interesting cases are when $\dim(X) = 3$ or 4.

When X is a smooth cubic threefold, $F(X)$ is a smooth surface of general type. Using a recent formula of Galkin–Shinder which relates the number of $\mathbf{F}_q$-points on $F(X)$ with the number of $\mathbf{F}_q$- and $\mathbf{F}_{q^2}$-points on X (Sect. 2.3), we find the zeta function of $F(X)$ (Theorem 4.1). Using the Weil conjectures, we obtain that a smooth X always contains $\mathbf{F}_q$-lines when $q \geq 11$ (Theorem 4.4). $\mathbf{F}_q$-lines using a computer, we produce examples of smooth cubic threefolds containing no lines for $q \in \{2, 3, 4, 5\}$ (Sect. 4.5.4), leaving only the cases where $q \in \{7, 8, 9\}$ open, at least when X is smooth.

Theorem 4.1 can also be used for explicit computations of the zeta function of $F(X)$. For that, one needs to know the number of $\mathbf{F}_{q^r}$-points of X for sufficiently many r. Direct computations are possible for small q or when X has symmetries (see Sect. 4.5.1 for Fermat hypersurfaces, Sect. 4.5.2 for the Klein threefold, and [19] for cyclic cubic threefolds). If X contains an $\mathbf{F}_q$-line, it is in general faster to use the structure of conic bundle on X induced by projection from this line, a method initiated by Bombieri and Swinnerton-Dyer in 1967 (Sect. 4.3). This is illustrated by an example in Sect. 4.5.3, where we compute the zeta function of a cubic X and of its Fano surface $F(X)$ in characteristics up to 31. In all these examples, once one knows the zeta function of $F(X)$, the Tate conjecture (known for Fano surfaces, see Remark 4.2) gives its Picard number. It is also easy to determine whether its 5-dimensional Albanese variety $A(F(X))$ is simple, ordinary, supersingular...

Singular cubics tend to contain more lines (Example 4.17). When X is a cubic threefold with a single node, the geometry of $F(X)$ is closely related to that of a smooth genus-4 curve ([9, 20]; see also [14, Example 5.8]). Using the results of [16] on pointless curves of genus 4, we prove that X always contains $\mathbf{F}_q$-lines when $q \geq 4$ (Corollary 4.8) and produce examples for $q \in \{2, 3\}$ where X contains no $\mathbf{F}_q$-lines (Sect. 4.5.5).

When X is a smooth cubic fourfold, $F(X)$ is a smooth fourfold with trivial canonical class. Using again the Galkin–Shinder formula, we compute the zeta function of $F(X)$ (Theorem 5.1) and deduce from the Weil conjectures that X contains an $\mathbf{F}_q$-line when $q \geq 5$ (Theorem 5.2). Since the cohomology of $\mathscr{O}_{F(X)}$ is very simple (it was determined by Altman and Kleiman; see Proposition 5.3), we apply the Katz trace formula and obtain that X still contains an $\mathbf{F}_q$-line when $q = 2$ (Corollary 5.4). This leaves the cases where $q \in \{3, 4\}$ open, at least when X is smooth. We suspect that any cubic fourfold defined over $\mathbf{F}_q$ should contain an $\mathbf{F}_q$-line.

2 Definitions and Tools

2.1 The Weil and Tate Conjectures

Let $\mathbf{F}_q$ be a finite field with q elements and let ℓ be a prime number prime to q.

Let Y be a projective variety of dimension n defined over $\mathbf{F}_q$. For every integer $r \geq 1$, set

$$N_r(Y) := \operatorname{Card}\bigl(Y(\mathbf{F}_{q^r})\bigr)$$

and define the *zeta function*

$$Z(Y, T) := \exp\Bigl(\sum_{r\geq 1} N_r(Y)\frac{T^r}{r}\Bigr).$$

Let $\overline{\mathbf{F}_q}$ be an algebraic closure of $\mathbf{F}_q$ and let $\overline{Y}$ be the variety obtained from Y by extension of scalars from $\mathbf{F}_q$ to $\overline{\mathbf{F}_q}$. The Frobenius morphism $F\colon \overline{Y} \to \overline{Y}$ acts on the étale cohomology $H^\bullet(\overline{Y}, \mathbf{Q}_\ell)$ by a $\mathbf{Q}_\ell$-linear map which we denote by F^*. We have Grothendieck's Lefschetz Trace formula ([22, Theorem 13.4, p. 292]): for all integers $r \geq 1$,

$$N_r(Y) = \sum_{0\leq i\leq 2n} (-1)^i \operatorname{Tr}\bigl(F^{*r}, H^i(\overline{Y}, \mathbf{Q}_\ell)\bigr). \tag{1}$$

If Y is moreover smooth, the Weil conjectures proved by Deligne in [10, Théorème (1.6)] say that for each i, the (monic) characteristic polynomial

$$Q_i(Y, T) := \det\bigl(T\operatorname{Id} - F^*, H^i(\overline{Y}, \mathbf{Q}_\ell)\bigr)$$

has integral coefficients and is independent of ℓ; in particular, so is its degree $b_i(Y) := h^i(\overline{Y}, \mathbf{Q}_\ell)$, called the *i-th Betti number* of Y. All the conjugates of its complex roots ω_{ij} have modulus $q^{i/2}$. Poincaré duality implies $b_{2n-i}(Y) = b_i(Y)$ and $\omega_{2n-i,j} = q^n/\omega_{ij}$ for all $1 \leq j \leq b_i(Y)$.

We can rewrite the trace formula (1) as

$$N_r(Y) = \sum_{0\leq i\leq 2n} (-1)^i \sum_{j=1}^{b_i(Y)} \omega_{ij}^r \tag{2}$$

or

$$Z(Y,T) = \prod_{0 \le i \le 2n} P_i(Y,T)^{(-1)^{i+1}}. \tag{3}$$

Finally, it is customary to introduce the polynomials

$$P_i(Y,T) := \det\bigl(\mathrm{Id} - TF^*, H^i(\overline{Y}, \mathbf{Q}_\ell)\bigr) = T^{b_i(Y)} Q_i\Bigl(Y, \frac{1}{T}\Bigr) = \prod_{j=1}^{b_i(Y)} (1 - \omega_{ij} T). \tag{4}$$

Whenever i is odd, the real roots of $Q_i(Y,T)$ have even multiplicities ([11, Theorem 1.1.(b)]), hence $b_i(Y)$ is even. We can therefore assume $\omega_{i,j+b_i(Y)/2} = \bar{\omega}_{ij}$ for all $1 \le j \le b_i(Y)/2$, or $T^{b_i(Y)} Q_i(Y, q^i/T) = q^{ib_i(Y)/2} Q_i(Y,T)$. If $m := b_1(Y)/2$, we will write

$$Q_1(Y,T) = T^{2m} + a_1 T^{2m-1} + \cdots + a_m T^m + q a_m T^{m+1} + \cdots + q^{m-1} a_1 T + q^m. \tag{5}$$

The Tate conjecture for divisors on Y states that the $\mathbf{Q}_\ell$-vector space in $H^2\bigl(\overline{Y}, \mathbf{Q}_\ell(1)\bigr)$ generated by classes of $\mathbf{F}_q$-divisors is equal to the space of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$-invariants classes and that its dimension is equal to the multiplicity of q as a root of the polynomial $Q_2(Y,T)$ ([29, Conjecture 2, p. 104]).

2.2 The Katz Trace Formula

Let Y be a proper scheme of dimension n over $\mathbf{F}_q$. The endomorphism $f \mapsto f^q$ of $\mathscr{O}_Y$ induces an $\mathbf{F}_q$-linear endomorphism $\mathfrak{F}_q$ of the $\mathbf{F}_q$-vector space $H^\bullet(Y, \mathscr{O}_Y)$ and for all $r \ge 1$, one has ([18], Corollaire 3.2)

$$N_r(Y) \cdot 1_{\mathbf{F}_q} \equiv \sum_{j=0}^{n} (-1)^j \,\mathrm{Tr}\bigl(\mathfrak{F}_q^r, H^j(Y, \mathscr{O}_Y)\bigr) \qquad \text{in } \mathbf{F}_q. \tag{6}$$

In particular, the right side, which is a priori in $\mathbf{F}_q$, is actually in the prime subfield of $\mathbf{F}_q$.

2.3 The Galkin–Shinder Formulas

Let $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ be a reduced cubic hypersurface defined over $\mathbf{F}_q$, with singular set $\mathrm{Sing}(X)$.

We let $F(X) \subset \mathrm{Gr}(1, \mathbf{P}^{n+1}_{\mathbf{F}_q})$ be the scheme of lines contained in X, also defined over $\mathbf{F}_q$. When $n \ge 3$ and $\mathrm{Sing}(X)$ is finite, $F(X)$ is a local complete intersection of dimension $2n-4$, smooth if X is smooth, and geometrically connected ([2, Theorem 1.3 and Corollary 1.12]).

In the Grothendieck ring of varieties over $\mathbf{F}_q$, one has the relation ([14, Theorem 5.1])

$$\mathbb{L}^2[F(X)] = [X^{(2)}] - (1+\mathbb{L}^n)[X] + \mathbb{L}^n[\mathrm{Sing}(X)], \tag{7}$$

where $X^{(2)} := X^2/\mathfrak{S}_2$ is the symmetric square of X and, as usual, $\mathbb{L}$ denotes the class of the affine line. Together with the relation [14, (2.5)], it implies that, for all $r \geq 1$, we have ([14, Corollary 5.2.3)])

$$N_r\big(F(X)\big) = \frac{N_r(X)^2 - 2(1+q^{nr})N_r(X) + N_{2r}(X)}{2q^{2r}} + q^{(n-2)r}N_r\big(\mathrm{Sing}(X)\big). \tag{8}$$

2.4 Abelian Varieties Over Finite Fields

Let A be an abelian variety of dimension n defined over a finite field $\mathbf{F}_q$ of characteristic p and let ℓ be a prime number prime to p. The $\mathbf{Z}_\ell$-module $H^1(\overline{A}, \mathbf{Z}_\ell)$ is free of rank $2n$ and there is an isomorphism

$$\textstyle\bigwedge^\bullet H^1(\overline{A}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^\bullet(\overline{A}, \mathbf{Q}_\ell) \tag{9}$$

of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$-modules.

An elliptic curve E defined over $\overline{\mathbf{F}_q}$ is *supersingular* if its only p-torsion point is 0. All supersingular elliptic curves are isogenous. The abelian variety A is *supersingular* if $A_{\overline{\mathbf{F}_q}}$ is isogenous to E^n, where E is a supersingular elliptic curve (in particular, any two supersingular abelian varieties are isogenous over $\overline{\mathbf{F}_q}$). The following conditions are equivalent ([15, Theorems 110, 111, and 112])

(i) A is supersingular;
(ii) $Q_1(A_{\mathbf{F}_{q^r}}, T) = (T \pm q^{r/2})^{2n}$ for some $r \geq 1$;
(iii) $\mathrm{Card}\big(A(\mathbf{F}_{q^r})\big) = (q^{r/2} \pm 1)^{2n}$ for some $r \geq 1$;
(iv) each complex root of $Q_1(A, T)$ is $\sqrt{q}$ times a root of unity;
(v) in the notation of (5), if $q = p^r$, one has $p^{\lceil rj/2 \rceil} \mid a_j$ for all $j \in \{1, \dots, n\}$.

If condition (ii) is satisfied, one has $Q_2(A_{\mathbf{F}_{q^r}}, T) = (T - q^r)^{n(2n-1)}$ and the Tate conjecture, which holds for divisors on abelian varieties, implies that the Picard number of $A_{\mathbf{F}_{q^r}}$, hence also the geometric Picard number of A, is $n(2n-1)$, the maximal possible value. Conversely, when $n > 1$, if $A_{\mathbf{F}_{q^r}}$ has maximal Picard number for some r, the abelian variety A is supersingular.

The abelian variety A is *ordinary* if it contains p^n (the maximal possible number) p-torsion $\overline{\mathbf{F}_q}$-points. This is equivalent to the coefficient a_n of T^n in $Q_1(A, T)$ being prime to p; if this is the case, A is simple (over $\mathbf{F}_q$) if and only if the polynomial $Q_1(A,T)$ is irreducible (see [17, Sect. 2]).

3 Cubic Surfaces

There exist smooth cubic surfaces defined over $\mathbf{F}_q$ containing no $\mathbf{F}_q$-lines, with q arbitrarily large. This is the case for example for the diagonal cubics defined by

$$x_1^3 + x_2^3 + x_3^3 + a x_4^3 = 0,$$

where $a \in \mathbf{F}_q$ is not a cube. If $q \equiv 1 \pmod 3$, there is such an a, since there are elements of order 3 in $\mathbf{F}_q^\times$, hence the morphism $\mathbf{F}_q^\times \to \mathbf{F}_q^\times$, $x \mapsto x^3$ is not injective, hence not surjective.

4 Cubic Threefolds

4.1 The Zeta Function of the Surface of Lines

Let $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ be a *smooth* cubic hypersurface defined over $\mathbf{F}_q$. Its Betti numbers are 1, 0, 1, 10, 1, 0, 1, and the eigenvalues of the Frobenius morphism acting on the 10-dimensional vector space $H^3(\overline{X}, \mathbf{Q}_\ell)$ are all divisible by q as algebraic integers ([18, Remark 5.1]). We can therefore write (1) as

$$N_r(X) = 1 + q^r + q^{2r} + q^{3r} - q^r \sum_{j=1}^{10} \omega_j^r,$$

where, by the Weil conjectures proved by Deligne (Sect. 2.1), the complex algebraic integers ω_j (and all their conjugates) have modulus $\sqrt{q}$. The trace formula (3) reads

$$Z(X,T) = \frac{P_3(X,T)}{(1-T)(1-qT)(1-q^2T)(1-q^3T)},$$

where $P_3(X,T) = \prod_{j=1}^{10}(1 - q\omega_j T)$. If we set

$$M_r(X) := \frac{1}{q^r}\big(N_r(X) - (1 + q^r + q^{2r} + q^{3r})\big) = -\sum_{j=1}^{10} \omega_j^r, \tag{10}$$

we obtain

$$P_3(X,T) = \exp\Big(\sum_{r\geq 1} M_r(X)\frac{(qT)^r}{r}\Big). \tag{11}$$

We will show in Sect. 4.3 that the numbers $M_r(X)$ have geometric significance.

Theorem 4.1 *Let $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ be a* smooth *cubic hypersurface defined over $\mathbf{F}_q$ and let $F(X)$ be the smooth surface of lines contained in X. With the notation (4), we have*

$$P_1(F(X),T) = P_3(X,T/q) =: \prod_{1\leq j\leq 10}(1-\omega_j T),$$

$$P_2(F(X),T) = \prod_{1\leq j<k\leq 10}(1-\omega_j\omega_k T),$$

$$P_3(F(X),T) = P_3(X,T) = \prod_{1\leq j\leq 10}(1-q\omega_j T),$$

where the complex numbers $\omega_1,\ldots,\omega_{10}$ have modulus $\sqrt{q}$. In particular,

$$Z(F(X),T) = \frac{\prod_{1\leq j\leq 10}(1-\omega_j T)\prod_{1\leq j\leq 10}(1-q\omega_j T)}{(1-T)(1-q^2T)\prod_{1\leq j<k\leq 10}(1-\omega_j\omega_k T)}. \tag{12}$$

Proof There are several ways to prove this statement. The first is to prove that there are isomorphisms

$$H^3(\overline{X},\mathbf{Q}_\ell) \xrightarrow{\sim} H^1\big(\overline{F(X)},\mathbf{Q}_\ell(-1)\big) \quad\text{and}\quad \textstyle\bigwedge^2 H^1(\overline{F(X)},\mathbf{Q}_\ell) \xrightarrow{\sim} H^2(\overline{F(X)},\mathbf{Q}_\ell)$$

of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$-modules. The first isomorphism holds with $\mathbf{Z}_\ell$-coefficients: if we introduce the incidence variety $I = \{(L,x)\in F(X)\times X \mid x\in L\}$ with its projections $\mathrm{pr}_1\colon I\to F(X)$ and $\mathrm{pr}_2\colon I\to X$, it is given by $\mathrm{pr}_{1*}\,\mathrm{pr}_2^*$ ([8, p. 256]). The second isomorphism follows, by standard arguments using smooth and proper base change, from the analogous statement in singular cohomology, over $\mathbf{C}$, which is proven in [25, Proposition 4].

These isomorphisms (and Poincaré duality) then imply the formulas for the polynomials $P_i(F(X),T)$ given in the theorem.

Alternatively, simply substituting in the definition of $Z(F(X),T)$ the values for $N_r(F(X))$ given by the Galkin–Shinder formula (8) directly gives (12), from which one deduces the formulas for the polynomials $P_i(F(X),T)$. □

Remark 4.2 (The Tate conjecture for $F(X)$) The Tate conjecture for the surface $F(X)$ (see Sect. 2.1) was proved in [25] over any field $\mathbf{k}$ of finite type over the prime field, *of characteristic other than* 2. This last restriction can in fact be lifted as follows: the proof in [25] rests on the following two facts

(a) $F(X)$ maps to its (5-dimensional) Albanese variety $A(F(X))$ onto a surface with class a multiple of θ^3, where θ is a principal polarization on $A(F(X))$;

(b) $b_2(A(F(X))) = b_2(F(X))$.

Item (a) is proved (in characteristic $\neq 2$) via the theory of Prym varieties ([4, Proposition 7]). For item (b), we have $\dim(A(F(X))) = h^1(F(X), \mathscr{O}_{F(X)}) = 5$ ([2, Proposition (1.15)]), hence $b_2(A(F(X))) = \binom{2\dim(A(X))}{2} = 45$, whereas $b_2(F(X)) = \deg\big(P_2(F(X), T)\big) = 45$ by Theorem 4.1.

To extend (a) to all characteristics, we consider X as the reduction modulo the maximal ideal $\mathfrak{m}$ of a smooth cubic $\mathscr{X}$ defined over a valuation ring of characteristic zero. There is a "difference morphism" $\delta_{F(X)}\colon F(X) \times F(X) \to A(F(X))$, defined over $\mathbf{k}$, which is the reduction modulo $\mathfrak{m}$ of the analogous morphism $\delta_{F(\mathscr{X})}\colon F(\mathscr{X}) \times F(\mathscr{X}) \to A\big(F(\mathscr{X})\big)$. By [4, Proposition 5], the image of $\delta_{F(\mathscr{X})}$ is a divisor which defines a principal polarization ϑ on $A\big(F(\mathscr{X})\big)$, hence the image of $\delta_{F(X)}$ is also a principal polarization on $A(F(X))$, defined over $\mathbf{k}$.

Since the validity of the Tate conjecture is not affected by passing to a finite extension of $\mathbf{k}$, we may assume that $F(X)$ has a $\mathbf{k}$-point, which we lift to $F(\mathscr{X})$. We can then define Albanese morphisms, and $a_{F(X)}\colon F(X) \to A(F(\mathscr{X}))$ is the reduction modulo $\mathfrak{m}$ of $a_{F(\mathscr{X})}\colon F(\mathscr{X}) \to A(F(\mathscr{X}))$. The image of $a_{\mathscr{X}}$ has class $\vartheta^3/3!$ ([4, Proposition 7]), hence the image of $a_{F(X)}$ also has class $(\vartheta|_{A(X)})^3/3!$ (this class is not divisible in $H^6(A(X), \mathbf{Z}_\ell)$, hence $a_{F(X)}$ is generically injective). This proves (a), hence the Tate conjecture for $F(X)$, in all characteristics.

Going back to the case where $\mathbf{k}$ is finite, Theorem 4.1 implies the equality $Q_2(F(X), T) = Q_2(A(F(X)), T)$. Since the Tate conjecture holds for divisors on abelian varieties, this proves that $F(X)$ and $A(F(X))$ have the same Picard numbers, whose maximal possible value is 45.

Corollary 4.3 *Let $2m_\pm$ be the multiplicity of the root $\pm\sqrt{q}$ of $Q_1(F(X), T)$ and let $m_1, \dots, m_c$ be the multiplicities of the pairs of non-real conjugate roots of $Q_1(F(X), T)$, so that $m_+ + m_- + \sum_{i=1}^c m_i = 5$. The Picard number of $F(X)$ is then*

$$\rho(F(X)) = m_+(2m_+ - 1) + m_-(2m_- - 1) + \sum_{i=1}^{c} m_i^2.$$

We have $\rho(F(X)) \geq 5$, with equality if and only if $Q_1(F(X), T)$ has no multiple roots.

If q is not a square, the possible Picard numbers are all odd numbers between 5 and 13, 17, and 25.

If q is a square, the possible Picard numbers are all odd numbers between 5 and 21, 25, 29, and 45. We have $\rho(F(X)) = 45$ if and only if $Q_1(F(X), T) = (T \pm \sqrt{q})^{10}$.

Proof The Tate conjecture holds for divisors on $F(X)$ (Remark 4.2). As explained at the end of Sect. 2.1, it says that the rank of the Picard group is the multiplicity of q as a root of $Q_2(F(X), T)$. The remaining statements then follow from Theorem 4.1 by inspection of all possible cases for the values of $m_+, m_-, m_1, \dots, m_c$. □

4.2 Existence of Lines on Smooth Cubic Threefolds Over Large Finite Fields

We can now bound the number of $\mathbf{F}_q$-lines on a smooth cubic threefold defined over $\mathbf{F}_q$.

Theorem 4.4 *Let X be a smooth cubic threefold defined over $\mathbf{F}_q$ and let $N_1(F(X))$ be the number of $\mathbf{F}_q$-lines contained in X. We have*

$$N_1(F(X)) \geq \begin{cases} 1 + 45q + q^2 - 10(q+1)\sqrt{q} & \text{if } q \geq 64; \\ 1 + 13q + q^2 - 6(q+1)\sqrt{q} & \text{if } 16 \leq q \leq 61; \\ 1 - 3q + q^2 - 2(q+1)\sqrt{q} & \text{if } q \leq 13. \end{cases}$$

In particular, X contains at least 10 $\mathbf{F}_q$-lines if $q \geq 11$.

Moreover, for all q,

$$N_1(F(X)) \leq 1 + 45q + q^2 + 10(q+1)\sqrt{q}.$$

Proof As we saw in Sect. 2.1, we can write the roots of $Q_1(F(X), T)$ as $\omega_1, \dots, \omega_5, \overline{\omega}_1, \dots, \overline{\omega}_5$. The $r_j := \omega_j + \overline{\omega}_j$ are then real numbers in $[-2\sqrt{q}, 2\sqrt{q}]$ and, by (2) and Theorem 4.1, we have

$$\begin{aligned} N_1\big(F(X)\big) &= 1 - \sum_{1\leq j\leq 5} r_j + 5q + \sum_{1\leq j<k\leq 5} (\omega_j\omega_k + \overline{\omega}_j\omega_k + \omega_j\overline{\omega}_k + \overline{\omega}_j\overline{\omega}_k) - \sum_{1\leq j\leq 5} qr_j + q^2 \\ &= 1 + 5q + q^2 - (q+1) \sum_{1\leq j\leq 5} r_j + \sum_{1\leq j<k\leq 5} r_j r_k \\ &=: F_q(r_1, \dots, r_5). \end{aligned}$$

Since the real function $F_q \colon [-2\sqrt{q}, 2\sqrt{q}]^5 \to \mathbf{R}$ is *linear in each variable,* its extrema are reached on the boundary of its domain, i.e., at one of the points $2\sqrt{q}\,(\pm 1, \dots, \pm 1)$. At such a point $\mathbf{r}_l$ (with l positive coordinates), we have

$$F_q(\mathbf{r}_l) = 1 + 5q + q^2 - 2(2l-5)(q+1)\sqrt{q} + \frac{1}{2}\big(4q(2l-5)^2 - 20q\big).$$

The minimum is obviously reached for $l \in \{3, 4, 5\}$, the maximum for $l = 0$, and the rest is easy. □

4.3 Computing Techniques: The Bombieri–Swinnerton-Dyer Method

By Theorem 4.1, the zeta function of the surface $F(X)$ of lines contained in a smooth cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ defined over $\mathbf{F}_q$ is completely determined by the roots

$q\omega_1, \ldots, q\omega_{10}$ of the degree-10 characteristic polynomial of the Frobenius morphism acting on $H^3(\overline{X}, \mathbf{Q}_\ell)$. If one knows the numbers of points of X over sufficiently many finite extensions of $\mathbf{F}_q$, these roots can be computed from the relations

$$\exp\Big(\sum_{r\geq 1} M_r(X)\frac{T^r}{r}\Big) = P_3(X, T/q) = P_1(F(X), T) = \prod_{1\leq j\leq 10}(1-\omega_j T),$$

where $M_r(X) = \frac{1}{q^r}\big(N_r(X) - (1+q^r+q^{2r}+q^{3r})\big)$ was defined in (10).

The reciprocity relation (5) implies that the polynomial $P_1(F(X), T)$ is determined by the coefficients of $1, T, \ldots, T^5$, hence by the numbers $N_1(X), \ldots, N_5(X)$. The direct computation of these numbers is possible (with a computer) when q is small (see Sect. 4.5 for examples), but the amount of calculations quickly becomes very large.

We will explain a method for computing directly the numbers $M_1(X), \ldots, M_5(X)$. It was first introduced in [5] and uses a classical geometric construction which expresses the blow up of X along a line as a conic bundle. It is valid only in characteristics $\neq 2$ and requires X to contain an $\mathbf{F}_q$-line L.

Let $\widetilde{X} \to X$ be the blow up of L. Projecting from L induces a morphism $\pi_L\colon \widetilde{X} \to \mathbf{P}^2_{\mathbf{F}_q}$ which is a conic bundle and we denote by $\Gamma_L \subset \mathbf{P}^2_{\mathbf{F}_q}$ its discriminant curve, defined over $\mathbf{F}_q$. *Assume from now on that q is odd;* the curve Γ_L is then a nodal plane quintic curve and the associated double cover $\rho\colon \widetilde{\Gamma}_L \to \Gamma_L$ is admissible in the sense of [3, Définition 0.3.1] (the curve $\widetilde{\Gamma}_L$ is nodal and the fixed points of the involution associated with ρ are exactly the nodes of $\widetilde{\Gamma}_L$; [5, Lemma 2]).[1]

One can then define the Prym variety associated with ρ and it is isomorphic to the Albanese variety of the surface $F(X)$ ([24, Theorem 7] when Γ_L is smooth). The following is [5, Formula (18)].

Proposition 4.5 *Let $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ be a smooth cubic threefold defined over $\mathbf{F}_q$, with q odd, and assume that X contains an $\mathbf{F}_q$-line L. With the notation (10), we have, for all $r \geq 1$,*

$$M_r(X) = N_r(\widetilde{\Gamma}_L) - N_r(\Gamma_L).$$

Proof We will go quickly through the proof of [5] because it is the basis of our algorithm. A point $x \in \mathbf{P}^2(\mathbf{F}_q)$ corresponds to an $\mathbf{F}_q$-plane $P_x \supset L$ and the fiber $\pi_L^{-1}(x)$ is isomorphic to the conic C_x such that $X \cap P_x = L + C_x$. We have four cases:

(i) either C_x is geometrically irreducible, i.e., $x \notin \Gamma_L(\mathbf{F}_q)$, in which case $\pi_L^{-1}(x)(\mathbf{F}_q)$ consists of $q+1$ points;
(ii) or C_x is the union of two different $\mathbf{F}_q$-lines, i.e., x is smooth on Γ_L and the 2 points of $\rho^{-1}(x)$ are in $\widetilde{\Gamma}_L(\mathbf{F}_q)$, in which case $\pi_L^{-1}(x)(\mathbf{F}_q)$ consists of $2q+1$ points;

[1] In characteristic 2, the curves Γ_L and $\widetilde{\Gamma}_L$ might not be nodal (see Lemma 4.13).

(iii) or C_x is the union of two different conjugate $\mathbf{F}_{q^2}$-lines, i.e., x is smooth on Γ_L and the 2 points of $\rho^{-1}(x)$ are *not* in $\widetilde{\Gamma}_L(\mathbf{F}_q)$, in which case $\pi_L^{-1}(x)(\mathbf{F}_q)$ consists of 1 point;

(iv) or C_x is twice an $\mathbf{F}_q$-line, i.e., x is singular on Γ_L, in which case $\pi_L^{-1}(x)(\mathbf{F}_q)$ consists of $q+1$ points.

The total number of points of $\widetilde{\Gamma}_L(\mathbf{F}_q)$ lying on a degenerate conic C_x is therefore $qN_1(\widetilde{\Gamma}_L) + N_1(\Gamma_L)$ and we obtain

$$N_1(\widetilde{X}) = (q+1)\big(N_1(\mathbf{P}^2_{\mathbf{F}_q}) - N_1(\Gamma_L)\big) + qN_1(\widetilde{\Gamma}_L) + N_1(\Gamma_L).$$

Finally, since each point on $L \subset X$ is replaced by a $\mathbf{P}^1_{\mathbf{F}_q}$ on $\widetilde{X}$, we have

$$N_1(\widetilde{X}) = N_1(X) - (q+1) + (q+1)^2,$$

thus $N_1(X) = q^3 + q^2 + q + 1 + q\big(N_1(\widetilde{\Gamma}_L) - N_1(\Gamma_L)\big)$. Since the same conclusion holds upon replacing q with q^r, this proves the proposition. □

Let $x \in \Gamma_L(\mathbf{F}_q)$. In order to compute the numbers $N_1(\Gamma_L) - N_1(\widetilde{\Gamma}_L)$, we need to understand when the points of $\rho^{-1}(x)$ are defined over $\mathbf{F}_q$.

We follow [5, p. 6]. Take homogenous $\mathbf{F}_q$-coordinates $x_1, \ldots, x_5$ on $\mathbf{P}^4$ so that L is given by the equations $x_1 = x_2 = x_3 = 0$. The equation of the cubic X can then be written as

$$f + 2q_1x_4 + 2q_2x_5 + \ell_1x_4^2 + 2\ell_2x_4x_5 + \ell_3x_5^2 = 0,$$

where f is a cubic form, q_1, q_2 are quadratic forms, and ℓ_1, ℓ_2, ℓ_3 are linear forms in the variables x_1, x_2, x_3. We choose the plane $\mathbf{P}^2_{\mathbf{F}_q} \subset \mathbf{P}^4_{\mathbf{F}_q}$ defined by $x_4 = x_5 = 0$. If $x = (x_1, x_2, x_3, 0, 0) \in \mathbf{P}^2_{\mathbf{F}_q}$, the conic C_x considered above is defined by the equation

$$fy_1^2 + 2q_1y_1y_2 + 2q_2y_1y_3 + \ell_1y_2^2 + 2\ell_2y_2y_3 + \ell_3y_3^2 = 0$$

and the quintic $\Gamma_L \subset \mathbf{P}^2_{\mathbf{F}_q}$ is defined by the equation $\det(M_L) = 0$, where

$$M_L := \begin{pmatrix} f & q_1 & q_2 \\ q_1 & \ell_1 & \ell_2 \\ q_2 & \ell_2 & \ell_3 \end{pmatrix}. \tag{13}$$

For each $i \in \{1, 2, 3\}$, let $\delta_i \in H^0\big(\Gamma_L, \mathscr{O}(a_i)\big)$, where $a_i = 2, 4$, or 4, be the determinant of the submatrix of M_L obtained by deleting its ith row and ith column. The $-\delta_i$ are transition functions of an invertible sheaf $\mathscr{L}$ on Γ_L such that $\mathscr{L}^{\otimes 2} = \omega_{\Gamma_L}$ (a theta characteristic). It defines the double cover $\rho\colon \widetilde{\Gamma}_L \to \Gamma_L$.

A point $x \in \mathbf{P}^2_{\mathbf{F}_q}$ is singular on Γ_L if and only if $\delta_1(x) = \delta_2(x) = \delta_3(x) = 0$. These points do not contribute to M_r since the only point of $\rho^{-1}(x)$ is defined over the field of definition of x. This is the reason why we may assume that x is smooth in the next proposition.

Proposition 4.6 *Let x be a smooth $\mathbf{F}_q$-point of Γ_L. The curve $\widetilde{\Gamma}_L$ has two $\mathbf{F}_q$-points over $x \in \Gamma_L(\mathbf{F}_q)$ if and only if either $-\delta_1(x) \in (\mathbf{F}_q^\times)^2$, or $\delta_1(x) = 0$ and either $-\delta_2(x)$ or $-\delta_3(x)$ is in $(\mathbf{F}_q^\times)^2$.*

Proof With the notation above, the line $L = V(y_1) \subset \mathbf{P}^2_{\mathbf{F}_q}$ meets the conic $C_x \subset \mathbf{P}^2_{\mathbf{F}_q}$ at the points $(0, y_2, y_3)$ such that

$$\ell_1 y_2^2 + 2\ell_2 y_2 y_3 + \ell_3 y_3^2 = 0.$$

Therefore, if $-\delta_1(x) = \ell_2^2(x) - \ell_1(x)\ell_3(x)$ is non-zero, the curve $\widetilde{\Gamma}_L$ has two rational points over $x \in \Gamma_L(\mathbf{F}_q)$ if and only if $-\delta_1(x) \in (\mathbf{F}_q^\times)^2$.

When $\delta_1(x) = 0$, we have $C_x = L_1 + L_2$, where L_1 and L_2 are lines meeting in an $\mathbf{F}_q$-point z of L which we assume to be $(0, 0, 1)$. This means that there is no y_3 term in the equation of C_x, hence $\ell_2(x) = \ell_3(x) = q_2(x) = 0$. The conic C_x is defined by the equation

$$\ell_1(x) y_2^2 + 2q_1(x) y_1 y_2 + f(x) y_1^2 = 0$$

and the two lines L_1 and L_2 are defined over $\mathbf{F}_q$ if and only if $-\delta_3(x) = q_1^2(x) - \ell_1(x) f(x) \in (\mathbf{F}_q^\times)^2$ (since $\delta_1(x) = \delta_2(x) = 0$, this is necessarily non-zero because x is smooth on Γ_L).

For the general case: if $y_3(z) \neq 0$, we make a linear change of coordinates $y_1 = y_1'$, $y_2 = y_2' + t y_3'$, $y_3 = y_3'$ in order to obtain $y_2'(z) = 0$, and we check that $-\delta_3(x)$ is unchanged; if $z = (0, 1, 0)$, we obtain as above $\delta_1(x) = \delta_3(x) = 0$ and L_1 and L_2 are defined over $\mathbf{F}_q$ if and only if $-\delta_2(x) \in (\mathbf{F}_q^\times)^2$. This proves the proposition. □

We can now describe our algorithm for the computation of the numbers $M_r(X) = N_r(\widetilde{\Gamma}_L) - N_r(\Gamma_L)$.

The input data is a cubic threefold X over $\mathbf{F}_q$ containing an $\mathbf{F}_q$-line L. We choose coordinates as above and construct the matrix M_L of (13) whose determinant is the equation of the quintic $\Gamma_L \subset \mathbf{P}^2_{\mathbf{F}_q}$. We compute M_r with the following simple algorithm.

Input: (X, L, r)
Output: M_r
Compute the matrix M_L, the three minors $\delta_1, \delta_2, \delta_3$ and the curve Γ_L;
$M_r := 0$;
while $p \in \{p \,:\, p \in \Gamma_L(\mathbf{F}_{q^r}) \mid \Gamma_L$ *is smooth at* $p\}$ **do**
  if $-\delta_1(p) \in (\mathbf{F}_{q^r}^\times)^2$ *or* $(\delta_1(p) = 0$ *and* $(-\delta_2(p) \in (\mathbf{F}_{q^r}^\times)^2$ *or* $-\delta_3(p) \in (\mathbf{F}_{q^r}^\times)^2))$ **then**
    $M_r := M_r + 1$;
  else
    $M_r := M_r - 1$;
  end
end
return M_r;

Algorithm 1: Computing M_r

4.4 Lines on Mildly Singular Cubic Threefolds

We describe a method based on results of Clemens–Griffiths and Kouvidakis–van der Geer which reduces the computation of the number of $\mathbf{F}_q$-lines on a cubic threefold with a single singular point, of type A_1 or A_2,[2] to the computation of the number of points on a smooth curve of genus 4. One consequence is that there is always an $\mathbf{F}_q$-line when $q > 3$.

Let C be a smooth non-hyperelliptic curve of genus 4 defined over a perfect field $\mathbf{F}$. We denote by g^1_3 and $h^1_3 = K_C - g^1_3$ the (possibly equal) degree-3 pencils on C. The canonical curve $\phi_{K_C}(C) \subset \mathbf{P}^3_{\mathbf{F}}$ is contained in a unique geometrically integral quadric surface Q whose rulings cut out the degree-3 pencils on C; more precisely,

- either $Q \simeq \mathbf{P}^1_{\mathbf{F}} \times \mathbf{P}^1_{\mathbf{F}}$ and the two rulings of Q cut out distinct degree-3 pencils g^1_3 and $h^1_3 = K_C - g^1_3$ on C which are defined over $\mathbf{F}$;
- or Q is smooth but its two rulings are defined over a quadratic extension of $\mathbf{F}$ and are exchanged by the Galois action, and so are g^1_3 and h^1_3;
- or Q is singular and its ruling cuts out a degree-3 pencil g^1_3 on C which is defined over $\mathbf{F}$ and satisfies $K_C = 2g^1_3$.

Let $\rho\colon \mathbf{P}^3_{\mathbf{F}} \dashrightarrow \mathbf{P}^4_{\mathbf{F}}$ be the rational map defined by the linear system of cubics containing $\phi_{K_C}(C)$. The image of ρ is a cubic threefold X defined over $\mathbf{F}$; it has a single singular point, $\rho(Q)$, which is of type A_1 if Q is smooth, and of type A_2 otherwise. Conversely, every cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}}$ defined over $\mathbf{F}$ with a single singular point x, of type A_1 or A_2, is obtained in this fashion: the curve C is $\mathbf{T}_{X,x} \cap X$ and parametrizes the lines in X through x ([7, Corollary 3.3]).

The surface $F(X)$ is isomorphic to the non-normal surface obtained by gluing the images C_g and C_h of the morphisms $C \to C^{(2)}$ defined by $p \mapsto g^1_3 - p$ and $p \mapsto h^1_3 - p$ (when Q is singular, $F(X)$ has a cusp singularity along the curve $C_g = C_h$). This was proved in [9, Theorem 7.8] over $\mathbf{C}$ and in [20, Proposition 2.1] in general.

Proposition 4.7 *Let $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ be a cubic threefold defined over $\mathbf{F}_q$ with a single singular point, of type A_1 or A_2. Let C be the associated curve of genus 4, with degree-3 pencils g^1_3 and h^1_3. For any $r \geq 1$, set $n_r := \mathrm{Card}(C(\mathbf{F}_{q^r}))$. We have*

$$\mathrm{Card}\big(F(X)(\mathbf{F}_q)\big) = \begin{cases} \frac{1}{2}(n_1^2 - 2n_1 + n_2) & \textit{if } g^1_3 \textit{ and } h^1_3 \textit{ are distinct and defined over } \mathbf{F}_q; \\ \frac{1}{2}(n_1^2 + 2n_1 + n_2) & \textit{if } g^1_3 \textit{ and } h^1_3 \textit{ are not defined over } \mathbf{F}_q; \\ \frac{1}{2}(n_1^2 + n_2) & \textit{if } g^1_3 = h^1_3. \end{cases}$$

[2] A hypersurface singularity is of type A_j if it is, locally analytically, given by an equation $x_1^{j+1} + x_2^2 + \cdots + x_{n+1}^2 = 0$. Type A_1 is also called a node.

Proof Points of $C^{(2)}(\mathbf{F}_q)$ correspond to

- the $\frac{1}{2}(n_1^2 - n_1)$ pairs of distinct points of $C(\mathbf{F}_q)$,
- the n_1 $\mathbf{F}_q$-points on the diagonal,
- the $\frac{1}{2}(n_2 - n_1)$ pairs of distinct conjugate points of $C(\mathbf{F}_{q^2})$,

for a total of $\frac{1}{2}(n_1^2 + n_2)$ points (compare with [14, (2.5)]). When g_3^1 and h_3^1 are distinct and defined over $\mathbf{F}_q$, the gluing process eliminates n_1 $\mathbf{F}_q$-points. When g_3^1 and h_3^1 are not defined over $\mathbf{F}_q$, the curves C_g and C_h contain no pairs of conjugate points, and the gluing process creates n_1 new $\mathbf{F}_q$-points. Finally, when $g_3^1 = h_3^1$, the map $C^{(2)}(\mathbf{F}_q) \to F(X)(\mathbf{F}_q)$ is a bijection. □

Corollary 4.8 *When $q \geq 4$, any cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_q}$ defined over $\mathbf{F}_q$ with a single singular point, of type A_1 or A_2, contains an $\mathbf{F}_q$-line.*

For $q \in \{2, 3\}$, we produce in Sect. 4.5.5 explicit examples of cubic threefolds with a single singular point, of type A_1, but containing no $\mathbf{F}_q$-lines: the bound in the corollary is the best possible.

Proof Assume that X contains no $\mathbf{F}_q$-lines. Proposition 4.7 then implies that either $n_1 = n_2 = 0$, or $n_1 = n_2 = 1$ and g_3^1 and h_3^1 are distinct and defined over $\mathbf{F}_q$. The latter case cannot in fact occur: if $C(\mathbf{F}_q) = \{x\}$, we write $g_3^1 \equiv x + x' + x''$. Since g_3^1 is defined over $\mathbf{F}_q$, so is $x' + x''$, hence x' and x'' are both defined over $\mathbf{F}_{q^2}$. But $C(\mathbf{F}_{q^2}) = \{x\}$, hence $x' = x'' = x$ and $g_3^1 \equiv 3x$. We can do the same reasoning with h_3^1 to obtain $h_3^1 \equiv 3x \equiv g_3^1$, a contradiction.

Therefore, we have $n_1 = n_2 = 0$. According to [16, Theorem 1.2], every genus-4 curve over $\mathbf{F}_q$ with $q > 49$ has an $\mathbf{F}_q$-point so we obtain $q \leq 7$.

Because of the reciprocity relation (5), there is a monic degree-4 polynomial H with integral coefficients that satisfies $Q_1(C, T) = T^4 H(T + q/T)$. If $\omega_1, \dots, \omega_4, \bar{\omega}_1, \dots, \bar{\omega}_4$ are the roots of $Q_1(C, T)$ (see Sect. 2.4), with $|\omega_j| = \sqrt{q}$, the roots of H are the $r_j := \omega_j + \bar{\omega}_j$, and

$$q + 1 - n_1 = \sum_{1 \leq j \leq 4} r_j \quad , \quad q^2 + 1 - n_2 = \sum_{1 \leq j \leq 4} (\omega_j^2 + \bar{\omega}_j^2) = \sum_{1 \leq j \leq 4} (r_j^2 - 2q).$$

Since $n_1 = n_2 = 0$, we obtain $\sum_{1 \leq j \leq 4} r_j = q + 1$ and $\sum_{1 \leq j \leq 4} r_j^2 = q^2 + 8q + 1$, so that $\sum_{1 \leq i < j \leq 4} r_i r_j = -3q$; we can therefore write

$$H(T) = T^4 - (q + 1)T^3 - 3qT^2 + aT + b. \tag{14}$$

Finally, since $|r_j| \leq 2\sqrt{q}$ for each j, we also have $|b| = |r_1 r_2 r_3 r_4| \leq 16q^2$ and $|a| = |\sum_{j=1}^4 b/r_j| \leq 32q^{3/2}$. A computer search done with these bounds shows that polynomials of the form (14) with four real roots and $q \in \{2, 3, 4, 5, 7\}$ only exist for $q \leq 3$, which proves the corollary. □

Remark 4.9 For $q \in \{2, 3\}$, the computer gives a list of all possible polynomials

$$(q=2) \quad H(T) = \begin{cases} T^4-3T^3-6T^2+24T-16 \\ T^4-3T^3-6T^2+24T-15 \ (\star) \\ T^4-3T^3-6T^2+23T-13 \\ T^4-3T^3-6T^2+22T-10 \ (\star) \\ T^4-3T^3-6T^2+21T-7 \\ T^4-3T^3-6T^2+18T+1 \ \ (\star) \end{cases}, \quad (q=3) \quad H(T) = \begin{cases} T^4-4T^3-9T^2+48T-36 \ (?) \\ T^4-4T^3-9T^2+47T-32 \ (\star) \\ T^4-4T^3-9T^2+46T-29 \ (\star) \\ T^4-4T^3-9T^2+44T-22 \ (?) \end{cases}.$$

The nodal cubics of Sect. 4.5.5, defined over $\mathbf{F}_2$ and $\mathbf{F}_3$, correspond to the polynomials $T^4 - 3T^3 - 6T^2 + 24T - 15$ and $T^4 - 4T^3 - 9T^2 + 47T - 32$, respectively. Over $\mathbf{F}_2$, it is possible to list all genus-4 canonical curves and one obtains that only the polynomials marked with ($\star$) actually occur (all three are irreducible).

Over $\mathbf{F}_3$, our computer searches show that the two polynomials marked with ($\star$) actually occur (both are irreducible). We do not know whether the other two, $T^4 - 4T^3 - 9T^2 + 48T - 36 = (T-1)(T-3)(T^2-12)$ and $T^4 - 4T^3 - 9T^2 + 44T - 22 = (T^2 - 4T + 2)(T^2 - 11)$ (marked with (?)), actually occur.

4.5 *Examples of Cubic Threefolds*

In this section, we present some of our calculations and illustrate our techniques for some cubic threefolds. We begin with Fermat cubics (Sect. 4.5.1), which have good reduction in all characteristics but 3. The case of general Fermat hypersurfaces was worked out by Weil in [30] (and was an inspiration for his famous conjectures discussed in Sect. 2.1). We explain how Weil's calculations apply to the zeta function of Fermat cubics (Theorem 4.11) and we compute, in dimension 3, the zeta function of their surface of lines (Corollary 4.12).

The Fermat cubic threefold contains the line $L := \langle (1,-1,0,0,0), (0,0,1,-1,0) \rangle$ and we compute the discriminant quintic $\Gamma_L \subset \mathbf{P}^2$ defined in Sect. 4.3, exhibiting strange behavior in characteristic 2.

In Sect. 4.5.2, we turn our attention to the Klein cubic, which has good reduction in all characteristics but 11. It also contains an "obvious" line L' and we compute the discriminant quintic $\Gamma_{L'} \subset \mathbf{P}^2$, again exhibiting strange behavior in characteristic 2. Using the Bombieri–Swinnerton-Dyer method, we determine the zeta function of $F(X)$ over $\mathbf{F}_p$, for $p \le 13$. We also compute the geometric Picard numbers of the reduction of $F(X)$ modulo any prime, using the existence of an isogeny between $A(F(X))$ and the self-product of an elliptic curve.

In Sect. 4.5.3, we compute, using the same method, the zeta function of $F(X)$ of a "random" cubic threefold X containing a line, over the fields $\mathbf{F}_5$, $\mathbf{F}_7$, $\mathbf{F}_{23}$, $\mathbf{F}_{29}$, and $\mathbf{F}_{31}$. Note that existing programs are usually unable to perform calculations in such high characteristics.

In Sect. 4.5.4, we present examples, found by computer searches, of smooth cubic threefolds defined over $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, or $\mathbf{F}_5$ with no lines. We were unable to find examples over $\mathbf{F}_q$ for the remaining values $q \in \{7, 8, 9\}$ (by Theorem 4.4, there are always $\mathbf{F}_q$-lines for $q \geq 11$). For the example over $\mathbf{F}_2$, we compute directly the number of points over small extensions and deduce the polynomial P_1 for the Fano surface $F(X)$. For the example over $\mathbf{F}_3$, we obtain again the polynomial P_1 for the Fano surface $F(X)$ by applying the Bombieri–Swinnerton-Dyer method over $\mathbf{F}_9$.

Finally, in Sect. 4.5.5, we exhibit cubic threefolds with one node but no lines, defined over $\mathbf{F}_2$ or $\mathbf{F}_3$, thereby proving that the bound in Corollary 4.8 is optimal.

4.5.1 Fermat Cubics

The n-dimensional Fermat cubic $X^n \subset \mathbf{P}^{n+1}_{\mathbf{Z}}$ is defined by the equation

$$x_1^3 + \cdots + x_{n+2}^3 = 0. \tag{15}$$

It has good reduction at every prime $p \neq 3$.

Remark 4.10 In general, if $q \equiv 2 \pmod 3$ and $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ is a cyclic cubic hypersurface defined by the equation $f(x_1, \ldots, x_{n+1}) + x_{n+2}^3 = 0$, the projection $\pi \colon X \to \mathbf{P}^n_{\mathbf{F}_q}$ defined by $(x_1, \ldots, x_{n+2}) \mapsto (x_1, \ldots, x_{n+1})$ induces a bijection $X(\mathbf{F}_q) \to \mathbf{P}^n(\mathbf{F}_q)$, because the map $x \mapsto x^3$ is a bijection of $\mathbf{F}_q$ ([19, Observation 1.7.2]).

The remark gives in particular $\mathrm{Card}\big(X^n(\mathbf{F}_2)\big) = \mathrm{Card}\big(\mathbf{P}^n(\mathbf{F}_2)\big) = 2^{n+1} - 1$. For the number of points of $X^n(\mathbf{F}_4)$, observe that the cyclic cover π is 3-to-1 outside its branch divisor $V(f)$. Let $(x_1, \ldots, x_{n+1}) \in \mathbf{P}^n(\mathbf{F}_4)$. Since $x^3 \in \{0, 1\}$ for any $x \in \mathbf{F}_4$, either $x_1^3 + \cdots + x_{n+1}^3 = 0$ and the inverse image by π has one $\mathbf{F}_4$-point, or $x_1^3 + \cdots + x_{n+1}^3 = 1$ and the inverse image by π has three $\mathbf{F}_4$-points. One obtains the inductive formula

$$\mathrm{Card}\big(X^n(\mathbf{F}_4)\big) = \mathrm{Card}\big(X^{n-1}(\mathbf{F}_4)\big) + 3\Big(\mathrm{Card}\big(\mathbf{P}^n(\mathbf{F}_4)\big) - \mathrm{Card}\big(X^{n-1}(\mathbf{F}_4)\big)\Big).$$

Since $\mathrm{Card}\big(X^0(\mathbf{F}_4)\big) = 3$, we get

$$\mathrm{Card}\big(X^n(\mathbf{F}_4)\big) = \frac{1}{3}\left(2^{2n+3} - (-2)^{n+1} - 1\right).$$

Using (8), we see that the number of $\mathbf{F}_2$-lines on $X^n_{\mathbf{F}_2}$ is

$$\frac{(2^{n+1}-1)^2 - 2(1+2^n)(2^{n+1}-1) + \frac{1}{3}(2^{2n+3} - (-2)^{n+1} - 1)}{8} = \frac{2^{2n} + 1 + ((-1)^n - 9)2^{n-2}}{3}.$$

For example, the 15 $\mathbf{F}_2$-lines contained in $X^3_{\mathbf{F}_2}$ are the line $L_{\mathbf{F}_2}$ and its images by permutations of the coordinates.

In fact, general results are available in the literature on the zeta function of Fermat hypersurfaces over finite fields (starting with [30]; see also [26, Sect. 3]), although they do not seem to have been spelled out for cubics. Let us first define

$$P_n^0(X^n_{\mathbf{F}_p},T) = \begin{cases} P_n(X^n_{\mathbf{F}_p},T) & \text{if } n \text{ is odd,} \\ \frac{P_n(X^n_{\mathbf{F}_p},T)}{1-p^{n/2}T} & \text{if } n \text{ is even} \end{cases}$$

(this is the reciprocal characteristic polynomial of the Frobenius morphism acting on the *primitive* cohomology of $X^n_{\mathbf{F}_p}$) and set $b_n^0(X^n) := \deg(P_n^0)$; this is $b_n(X^n)$ if n is odd, and $b_n(X^n) - 1$ if n is even.

Theorem 4.11 (Weil) *Let $X^n \subset \mathbf{P}^{n+1}_{\mathbf{Z}}$ be the Fermat cubic hypersurface. Let p be a prime number other than* 3.

- *If $p \equiv 2 \pmod 3$, we have*

$$P_n^0(X^n_{\mathbf{F}_p},T) = (1-(-p)^nT^2)^{b_n^0(X^n)/2}.$$

- *If $p \equiv 1 \pmod 3$, one can write uniquely $4p = a^2 + 27b^2$ with $a \equiv 1 \pmod 3$ and $b > 0$, and*

$$P_n^0(X^n_{\mathbf{F}_p},T) = \begin{cases} 1 + aT + pT^2 & \textit{when } n = 1, \\ (1-pT)^6 & \textit{when } n = 2, \\ (1 + apT + p^3T^2)^5 & \textit{when } n = 3, \\ (1 + (2p - a^2)T + p^2T^2)(1-p^2T)^{20} & \textit{when } n = 4, \\ (1 + ap^2T + p^5T^2)^{21} & \textit{when } n = 5. \end{cases}$$

As will become clear from the proof, it would be possible to write down (complicated) formulas for all n in the case $p \equiv 1 \pmod 3$. We leave that exercise to the interested reader and restrict ourselves to the lower-dimensional cases.

Proof Assume first $p \equiv 2 \pmod 3$. It follows from Remark 4.10 that the polynomial $P_n^0(X^n_{\mathbf{F}_p},T)$ is even (this is explained by (19) and (20) when $n = 4$). It is therefore equivalent to prove $P_n^0(X^n_{\mathbf{F}_{p^2}},T) = (1-(-p)^nT)^{b_n^0(X^n)}$. We follow the geometric argument of [26].

It is well known that $P_1(X^1_{\mathbf{F}_p},T) = 1 + pT^2$, hence $P_1(X^1_{\mathbf{F}_{p^2}},T) = (1+pT)^2$. In other words, the Frobenius morphism of $\mathbf{F}_{p^2}$ acts on the middle cohomology of $X^1_{\mathbf{F}_{p^2}}$ by multiplication by $-p$. By the Künneth formula, it acts by multiplication by $(-p)^2$ on the middle cohomology of $X^1_{\mathbf{F}_{p^2}} \times X^1_{\mathbf{F}_{p^2}}$. The proof by induction on n of [26, Theorem 2.10] then applies and gives that the Frobenius morphism acts by multiplication by $(-p)^n$ on the middle cohomology of $X^n_{\mathbf{F}_{p^2}}$.

Assume now $p \equiv 1 \pmod 3$. The number of points of $X^1(\mathbf{F}_p)$ was computed by Gauss ([28, Theorem 4.2]): writing $4p = a^2 + 27b^2$ as in the theorem, one has $\mathrm{Card}(X^1(\mathbf{F}_p)) = p + 1 + a$, i.e., $P_1(X^1_{\mathbf{F}_p},T) = 1 + aT + pT^2 =: (1-\omega T)(1-\bar{\omega}T)$. In other words, the eigenvalues of the Frobenius morphism of $\mathbf{F}_p$ acting on the first cohomology group are ω and $\bar{\omega}$. They are therefore the Jacobi sums denoted by

$j(1,2)$ and $j(2,1)$ in [26, (3.1)], and also the generators of the prime ideals $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in $\mathbf{Z}[\zeta]$ ($\zeta = \exp(2i\pi/3)$) such that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$.

The eigenvalues of the Frobenius morphism acting on the primitive middle cohomology of $X^n_{\mathbf{F}_p}$ are denoted $j(\alpha)$ by Weil, where α runs over the set

$$\mathfrak{U}_n = \{(\alpha_0, \dots, \alpha_{n+1}) \in \{1,2\}^{n+2} \mid \alpha_0 + \cdots + \alpha_{n+1} \equiv 0 \pmod 3\}.$$

The ideal $(j(\alpha))$ in $\mathbf{Z}[\zeta]$ is invariant under permutations of the α_i and its decomposition is computed by Stickelberger (see [26, (3.10)]):

$$(j(\alpha)) = \mathfrak{p}^{A(\alpha)}\bar{\mathfrak{p}}^{A(\bar{\alpha})},$$

with $A(\alpha) = \left\lfloor \sum_{j=1}^{n+1} \frac{\alpha_j}{3} \right\rfloor$ and $\bar{\alpha}_j = 3 - \alpha_j$.

The elements of $\mathfrak{U}_1$ are $(1,1,1)$ and $(2,2,2)$, and the corresponding values of A are 0 and 1. The eigenvalues are therefore (up to multiplication by a unit of $\mathbf{Z}[\zeta]$), ω and $\bar{\omega}$. By Gauss' theorem, we know they are exactly ω and $\bar{\omega}$. By induction on n, it then follows from the embeddings [26, (2.17)] that

$$j(\alpha) = \omega^{A(\alpha)}\bar{\omega}^{A(\bar{\alpha})}.$$

The elements of $\mathfrak{U}_2$ are (up to permutations) $(1,1,2,2)$ and the corresponding value of A is 1. The only eigenvalue is therefore $\omega\bar{\omega} = p$, with multiplicity $\binom{4}{2}$.

The elements of $\mathfrak{U}_3$ are (up to permutations) $(1,1,1,1,2)$ and $(1,2,2,2,2)$, and the corresponding values of A are 1 and 2. The eigenvalues are therefore $\omega^2\bar{\omega} = p\omega$ and $p\bar{\omega}$, with multiplicity 5.

The elements of $\mathfrak{U}_4$ are (up to permutations) $(1,1,1,1,1,1)$, $(1,1,1,2,2,2)$, and $(2,2,2,2,2,2)$, and the corresponding values of A are 1, 2, and 3. The eigenvalues are therefore $p\omega^2$ and $p\bar{\omega}^2$, with multiplicity 1, and p^2, with multiplicity $\binom{6}{3}$.

The elements of $\mathfrak{U}_5$ are (up to permutations) $(1,1,1,1,1,2,2)$ and $(1,1,2,2,2,2,2)$, and the corresponding values of A are 2 and 3. The eigenvalues are therefore $p^2\omega$ and $p^2\bar{\omega}$, with multiplicity $\binom{7}{2}$. This finishes the proof of the theorem. □

Corollary 4.12 *Let $X \subset \mathbf{P}^4_{\mathbf{Z}}$ be the Fermat cubic threefold defined by the equation $x_1^3 + \cdots + x_5^3 = 0$ and let $F(X)$ be its surface of lines. Let p be a prime number other than* 3.

The Albanese variety $A(F(X))_{\mathbf{F}_p}$ is isogenous to $E^5_{\mathbf{F}_p}$, where E is the Fermat plane cubic curve. Moreover,

- *if $p \equiv 2 \pmod 3$, we have*

$$Z(F(X)_{\mathbf{F}_p}, T) = \frac{(1+pT^2)^5(1+p^3T^2)^5}{(1-T)(1-p^2T)(1+pT)^{20}(1-pT)^{25}},$$

  the Picard number of $F(X)_{\mathbf{F}_p}$ is 25 and that of $F(X)_{\mathbf{F}_{p^2}}$ is 45, and the abelian variety $A(F(X))_{\mathbf{F}_p}$ is supersingular;

- *if* $p \equiv 1 \pmod 3$, *we have (with the notation of Theorem 4.11)*

$$Z(F(X)_{\mathbf{F}_p}, T) = \frac{(1 + aT + pT^2)^5(1 + apT + p^3T^2)^5}{(1 - T)(1 - p^2T)(1 + (2p - a^2)T + p^2T^2)^{10}(1 - pT)^{25}},$$

the Picard and absolute Picard numbers of $F(X)_{\mathbf{F}_p}$ *are 25, and the abelian variety* $A(F(X))_{\mathbf{F}_p}$ *is ordinary.*

Proof Theorems 4.1 and 4.11 imply that the characteristic polynomials of the Frobenius morphisms acting on H^1 are the same for the abelian varieties $A(F(X))_{\mathbf{F}_p}$ and $E^5_{\mathbf{F}_p}$; they are therefore isogenous ([23, Appendix I, Theorem 2]). The statements about $A(F(X))_{\mathbf{F}_p}$ being supersingular or ordinary follow from the analogous statements about $E_{\mathbf{F}_p}$.

The values of the zeta functions also follow from Theorems 4.1 and 4.11, and the statements about the Picard numbers from Corollary 4.3. □

We now restrict ourselves to the Fermat cubic threefold $X \subset \mathbf{P}^4_{\mathbf{Z}}$ $(n = 3)$. We parametrize planes containing the line $L := \langle (1, -1, 0, 0, 0), (0, 0, 1, -1, 0) \rangle \subset X$ by the $\mathbf{P}^2$ defined by the equations $x_1 = x_3 = 0$ and determine the discriminant quintic $\Gamma_L \subset \mathbf{P}^2$ (see Sect. 4.3).

Lemma 4.13 *In the coordinates* x_2, x_4, x_5, *an equation of the discriminant quintic* $\Gamma_L \subset \mathbf{P}^2$ *is* $x_2x_4(x_2^3 + x_4^3 + 4x_5^3) = 0$. *Therefore,*

- *in characteristics other than 2 and 3, it is a nodal quintic which is the union of two lines and an elliptic curve, all defined over the prime field;*
- *in characteristic 2, it is the union of 5 lines meeting at the point* $(0, 0, 1)$*; 3 of them are defined over* $\mathbf{F}_2$, *the other 2 over* $\mathbf{F}_4$.

Proof We use the notation of the proof of Proposition 4.5 (although the choice of coordinates is different). If $x = (0, x_2, 0, x_4, x_5) \in \mathbf{P}^2$, the residual conic C_x is defined by the equation

$$\frac{1}{y_1}(y_2^3 + (x_2y_1 - y_2)^3 + y_3^3 + (x_4y_1 - y_3)^3 + y_1^3x_5^3) = y_1^2(x_2^2 + x_4^2 + x_5^2) - 3x_2^2y_1y_2 - 3x_4^2y_1y_3 + 3x_2y_2^2 + 3x_4y_3^2$$

in the coordinates (y_1, y_2, y_3). In characteristics other than 2 and 3, an equation of Γ_L is therefore given by

$$\begin{vmatrix} x_2^3+x_4^3+x_5^3 & -\frac{3}{2}x_2^2 & -\frac{3}{2}x_4^2 \\ -\frac{3}{2}x_2^2 & 3x_2 & 0 \\ -\frac{3}{2}x_4^2 & 0 & 3x_4 \end{vmatrix} = \frac{9}{4}x_2x_4 \begin{vmatrix} 4(x_2^3+x_4^3+x_5^3) & 3x_2 & 3x_4 \\ x_2^2 & 1 & 0 \\ x_4^2 & 0 & 1 \end{vmatrix} = \frac{9}{4}x_2x_4(x_2^3 + x_4^3 + 4x_5^3) = 0.$$

In characteristic 2, the Jacobian criterion says that the singular points of C_x must satisfy $y_1 = 0$ and $x_2y_2^2 + x_4y_3^2 = x_2^2y_2 + x_4^2y_3 = 0$. The curve Γ_L is therefore defined by $\begin{vmatrix} x_2^{1/2} & x_4^{1/2} \\ x_2^2 & x_4^2 \end{vmatrix} = 0$, or $x_2x_4(x_2^3 + x_4^3) = 0$. It is therefore the "same" equation reduced modulo 2. □

4.5.2 The Klein Threefold

This is the cubic threefold $X \subset \mathbf{P}^4_{\mathbf{Z}}$ defined by the equation

$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 = 0. \tag{16}$$

It has good reduction at every prime $p \neq 11$.

It contains the line $L' = \langle (1, 0, 0, 0, 0), (0, 0, 1, 0, 0) \rangle$ and we parametrize planes containing L' by the $\mathbf{P}^2$ defined by $x_1 = x_3 = 0$.

Lemma 4.14 *In the coordinates* x_2, x_4, x_5*, an equation of the discriminant quintic* $\Gamma_{L'} \subset \mathbf{P}^2$ *is* $x_2^5 + x_4 x_5^4 - 4x_2 x_4^3 x_5 = 0$*. Therefore,*

- *in characteristics other than 2 and 11, it is a geometrically irreducible quintic with a single singular point,* $(0, 1, 0)$*, which is a node;*
- *in characteristic 2, it is a geometrically irreducible rational quintic with a single singular point of multiplicity 4,* $(0, 1, 0)$.

Proof We proceed as in the proof of Lemma 4.13. If $x = (0, x_2, 0, x_4, x_5) \in \mathbf{P}^2$, an equation of the residual conic C_x is

$$\frac{1}{y_1}\left(y_2^2 x_2 y_1 + x_2^2 y_1^2 y_3 + y_3^2 x_4 y_1 + x_4^2 y_1^2 x_5 y_1 + x_5^2 y_1^2 y_2\right) = y_2^2 x_2 + x_2^2 y_1 y_3 + y_3^2 x_4 + x_4^2 y_1^2 x_5 + x_5^2 y_1 y_2$$

in the coordinates (y_1, y_2, y_3). In characteristic other than 2, an equation of $\Gamma_{L'}$ is therefore

$$\begin{vmatrix} x_4^2 x_5 & \frac{1}{2}x_5^2 & \frac{1}{2}x_2^2 \\ \frac{1}{2}x_5^2 & x_2 & 0 \\ \frac{1}{2}x_2^2 & 0 & x_4 \end{vmatrix} = \frac{1}{4}(x_2^5 + x_4 x_5^4 - 4x_2 x_4^3 x_5) = 0.$$

In characteristic 2, one checks that $\Gamma_{L'}$ is defined by the equation $x_2^5 + x_4 x_5^4 = 0$. In both cases, the singularities are easily determined. □

In characteristic 11, $X_{\mathbf{F}_{11}}$ has a unique singular point, $(1, 3, 3^2, 3^3, 3^4)$, which has type A_2. The quintic $\Gamma_{L'} \subset \mathbf{P}^2$ is still geometrically irreducible, with a node at $(0, 1, 0)$ and an ordinary cusp (type A_2) at $(5, 1, 3)$.

In characteristic 2, the isomorphism $(x_1, \ldots, x_5) \mapsto (x_1 + x_5, x_2 + x_5, x_3 + x_5, x_4 + x_5, x_1 + x_2 + x_3 + x_4 + x_5)$ maps $X_{\mathbf{F}_2}$ to the cyclic cubic defined by $x_5^3 + (x_1 + x_2 + x_3 + x_4)^3 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 = 0$. Thus $M_{2m+1}(X_{\mathbf{F}_2}) = 0$ for any $m \geq 0$ (reasoning as in Sect. 4.5.1). The computer gives $M_2(X_{\mathbf{F}_2}) = M_4(X_{\mathbf{F}_2}) = 0$. Using (8), we find that $X_{\mathbf{F}_2}$ contains 5 $\mathbf{F}_2$-lines; they are the line L' and its images by the cyclic permutations of the coordinates.

By the reciprocity property (5), we obtain

$$P_1(F(X)_{\mathbf{F}_2}, T) = P_3(X_{\mathbf{F}_2}, T/2) = 1 + 2^5 T^{10}.$$

Since this polynomial has simple roots, the Picard number of $F(X)_{\mathbf{F}_2}$ is 5 (Corollary 4.3). The eigenvalues of the Frobenius morphism F are $\omega \exp(2ik\pi/10)$, for $k \in \{0, \ldots, 9\}$, where $\omega^{10} = -2^5$; hence F^{10} acts by multiplication by -2^5. This implies $P_1(F(X)_{\mathbf{F}_{2^{10}}}, T) = (1 + 2^5 T)^{10}$. It follows that $F(X)_{\mathbf{F}_{2^{10}}}$ has maximal Picard number 45 (Corollary 4.3) and that $A(F(X))$ is isogenous to E^5 over $\mathbf{F}_{2^{10}}$, where E is the Fermat plane cubic defined in Sect. 4.5.1.

We also get $P_2(F(X)_{\mathbf{F}_2}, T) = (1 - 2^5 T^5)(1 - 2^{10} T^{10})^4 = (1 - 2^5 T^5)^5 (1 + 2^5 T^5)^4$ and

$$Z(F(X)_{\mathbf{F}_2}, T) = \frac{(1 + 2^5 T^{10})(1 + 2^{15} T^{10})}{(1 - T)(1 - 4T)(1 - 2^5 T^5)^5 (1 + 2^5 T^5)^4}.$$

Over other small fields, we find, using the Bombieri–Swinnerton-Dyer method (Proposition 4.5) and a computer,

$$\begin{aligned} P_1(F(X)_{\mathbf{F}_3}, T) &= 1 + 31T^5 + 3^5 T^{10} \\ P_1(F(X)_{\mathbf{F}_5}, T) &= 1 - 57T^5 + 5^5 T^{10} \\ P_1(F(X)_{\mathbf{F}_7}, T) &= 1 + 7^5 T^{10} \\ P_1(F(X)_{\mathbf{F}_{13}}, T) &= 1 + 13^5 T^{10}. \end{aligned}$$

Note that $A(F(X))$ is ordinary in the first two cases and supersingular with maximal Picard number in the other two cases. One can easily compute the Picard numbers and write down the corresponding zeta functions if desired. We compute the geometric Picard numbers by a different method. Note that -11 is a square modulo 3 or 5, but not modulo 7 or 13.

Proposition 4.15 *Let $X \subset \mathbf{P}^4_{\mathbf{Z}}$ be the Klein cubic threefold with Eq.* (16) *and let $F(X)$ be its surface of lines. Suppose $p \neq 2$. If -11 is a square modulo p, the reduction modulo p of $F(X)$ has geometric Picard number* 25, *otherwise it has geometric Picard number* 45.

Proof Set $\nu := \frac{-1+\sqrt{-11}}{2}$ and $E'_{\mathbf{C}} := \mathbf{C}/\mathbf{Z}[\nu]$. By [1, Corollary 4, p. 138], $A(F(X))_{\mathbf{C}}$ is isomorphic to $(E'_{\mathbf{C}})^5$. By [27, Appendix A3], the elliptic curve $E'_{\mathbf{C}}$ has a model defined by the equations

$$y^2 + y = x^3 - x^2 - 7x + 10 = 0$$

over $\mathbf{Q}$, which we denote by E'. Since $A(F(X))_{\mathbf{C}}$ and $E'^5_{\mathbf{C}}$ are isomorphic, $A(F(X))$ and E'^5 are isomorphic over some number field ([23, Appendix I, p. 240]).

We use Deuring's criterion [21, Chapter 13, Theorem 12)]: for odd $p \neq 11$, the reduction of E' modulo p is supersingular if and only if p is inert or ramified in $\mathbf{Z}[\nu]$. By classical results in number theory, an odd prime $p \neq 11$ is inert or ramified in $\mathbf{Z}[\nu]$ if and only if -11 is not a square modulo p. The geometric Picard number of the reduction modulo p of $A(F(X))$ is therefore 45 if -11 is not a square modulo p, and 25 otherwise. □

4.5.3 An Implementation of Our Algorithm

We use the notation of Sect. 4.3. Let $X \subset \mathbf{P}^4_{\mathbf{Z}}$ be the cubic threefold defined by the equation

$$f + 2q_1x_4 + 2q_2x_5 + x_1x_4^2 + 2x_2x_4x_5 + x_3x_5^2 = 0,$$

where

$$\begin{aligned} f &= x_2^2x_3 - (x_1^3 + 4x_1x_3^2 + 2x_3^3),\\ q_1 &= x_1^2 + 2x_2^2 + x_2x_3 + x_3^2,\\ q_2 &= x_1x_2 + 4x_2x_3 + x_3^2. \end{aligned}$$

It contains the line L defined by the equations $x_1 = x_2 = x_3 = 0$.

In characteristics ≤ 31, the cubic X is smooth except in characteristics 2 or 3 and the plane quintic curve Γ_L is smooth except in characteristics 2 or 5.

We implemented in `Sage` the algorithm described in Algorithm 1 (see [31]). Over $\mathbf{F}_5$, we get

$$P_1(F(X)_{\mathbf{F}_5}, T) = (1 + 5T^2)(1 + 2T^2 + 8T^3 - 6T^4 + 40T^5 + 50T^6 + 625T^8).$$

It follows that $A(F(X)_{\mathbf{F}_5})$ is not ordinary and not simple (it contains an elliptic curve).

Over the field $\mathbf{F}_7$, we compute that $P_1(F(X)_{\mathbf{F}_7}, T)$ is equal to

$$1 + 4T + 15\,T^2 + 46\,T^3 + 159\,T^4 + 460\,T^5 + 1\,113\,T^6 + 2\,254\,T^7 + 5\,145\,T^8 + 9\,604\,T^9 + 16\,807\,T^{10}.$$

This polynomial is irreducible over $\mathbf{Q}$; it follows that $A(F(X)_{\mathbf{F}_7})$ is ordinary and simple (Sect. 2.4). We can even get more by using a nice criterion from [17].

Proposition 4.16 *The abelian variety $A(F(X)_{\mathbf{F}_7})$ is absolutely simple, i.e., it remains simple over any field extension.*

Proof We want to apply the criterion [17, Proposition 3 (1)] to the abelian variety $A := A(F(X)_{\mathbf{F}_7})$. Let $d > 1$. Since the characteristic polynomial $Q_1(A, T)$ (which is also the minimal polynomial) of the Frobenius morphism F is not in $\mathbf{Z}[T^d]$, it is enough to check that, for any $d > 1$, there are no dth roots of unity ζ such that $\mathbf{Q}(F^d) \subsetneq \mathbf{Q}(F)$ and $\mathbf{Q}(F^d, \zeta) = \mathbf{Q}(F)$. If this is the case, $\mathbf{Q}(\zeta)$ is contained in $\mathbf{Q}(F)$, hence $\phi(d)$ (where ϕ is the Euler totient function) divides $\deg\big(Q_1(A, T)\big) = 10$. This implies $d \in \{2, 3, 4, 6, 11, 22\}$. But for these values of d, one computes that the characteristic polynomial $Q_1(A_{\mathbf{F}_{7^d}}, T)$ of F^d is irreducible (of degree 10), and this contradicts $\mathbf{Q}(F^d) \subsetneq \mathbf{Q}(F)$. Thus A is absolutely simple. $\square$

Here are some more computations in "high" characteristics:

$$P_1(F(X)_{\mathbf{F}_{23}},T) = 1+21T^2-35T^3+759T^4-890T^5+17\,457T^6-18\,515T^7+255\,507T^8+6\,436\,343T^{10},$$
$$P_1(F(X)_{\mathbf{F}_{29}},T) = 1+3T+5T^2+15T^3+352T^4+2\,828T^5+10\,208T^6+12\,615T^7+121\,945T^8+2\,121\,843T^9+20\,511\,149T^{10},$$
$$P_1(F(X)_{\mathbf{F}_{31}},T) = 1+2T+2T^2+72T^3+117T^4-812T^5+3\,627T^6+69\,192T^7+59\,582T^8+1\,847\,042T^9+28\,629\,151T^{10}.$$

4.5.4 Smooth Cubic Threefolds over $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$, or $\mathbf{F}_5$ with No Lines

Using a computer, it is easy to find many smooth cubic threefolds defined over $\mathbf{F}_2$ with no $\mathbf{F}_2$-lines (see Example 4.17). For example, the cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_2}$ defined by the equation

$$x_1^3+x_2^3+x_3^3+x_1^2x_2+x_2^2x_3+x_3^2x_1+x_1x_2x_3+x_1x_4^2+x_1^2x_4+x_2x_5^2+x_2^2x_5+x_4^2x_5=0$$

contains no $\mathbf{F}_2$-lines. We also have[3]

$$N_1(X)=9, N_2(X)=81, N_3(X)=657, N_4(X)=4\,225, N_5(X)=34\,049,$$

hence (see (10) for the definition of $M_r(X)$)

$$M_1(X)=-3, M_2(X)=-1, M_3(X)=9, M_4(X)=-9, M_5(X)=7.$$

The polynomial $P_1(F(X),T)=P_3(X,T/2)=\prod_{j=1}^{10}(1-\omega_jT)$ is then given by

$$\exp\Bigl(\sum_{r=1}^{5} M_r(X)\frac{T^r}{r}\Bigr)+O(T^6)=1-3T+4T^2-10T^4+20T^5+O(T^6).$$

Using the reciprocity property (5), we obtain

$$P_1(F(X),T)=1-3T+4T^2-10T^4+20T^5-10\cdot 2T^6+4\cdot 2^3T^8-3\cdot 2^4T^9+2^5T^{10}.$$

Since this polynomial has no multiple roots, the Picard number of $F(X)$ is 5 (Corollary 4.3).

We found by random computer search the smooth cubic threefold $X' \subset \mathbf{P}^4_{\mathbf{F}_3}$ defined by the equation

$$2x_1^3+2x_2^3+x_1x_3^2+x_2^2x_4+2x_3^2x_4+x_1^2x_5+x_2x_3x_5+2x_1x_4x_5+2x_2x_4x_5+2x_4^2x_5+2x_4x_5^2+x_5^3=0.$$

[3] Among smooth cubics in $\mathbf{P}^4_{\mathbf{F}_2}$ with no $\mathbf{F}_2$-lines, the computer found examples whose number of $\mathbf{F}_2$-points is any odd number between 3 and 13.

It contains no $\mathbf{F}_3$-lines and 25 $\mathbf{F}_3$-points. Computing directly the number of points on extensions of $\mathbf{F}_3$, as we did above for $\mathbf{F}_2$, takes too much time, and it is quicker to use the Bombieri–Swinnerton-Dyer method (Proposition 4.5) on $X'_{\mathbf{F}_9}$, which contains an $\mathbf{F}_9$-line. The result is that $P_1(F(X')_{\mathbf{F}_9}, T)$ is equal to

$$1 - 5T + 8T^2 + 10T^3 - 124T^4 + 515T^5 - 1\,116T^6 + 810T^7 + 5\,832T^8 - 32\,805T^9 + 59\,049T^{10}.$$

Using the fact that X' has 25 $\mathbf{F}_3$-points and that the roots of $P_1(F(X')_{\mathbf{F}_3}, T)$ are square roots of the roots of $P_1(F(X')_{\mathbf{F}_9}, T)$, one finds

$$P_1(F(X')_{\mathbf{F}_3}, T) = 1 - 5T + 10T^2 - 2T^3 - 36T^4 + 95T^5 - 108T^6 - 18T^7 + 270T^8 - 405T^9 + 243T^{10},$$

and the numbers of $\mathbf{F}_{3^r}$-lines in $X'_{\mathbf{F}_{3^r}}$, for $r \in \{1, \ldots, 5\}$, are 0, 40, 1 455, 5 740, 72 800, respectively.

Similarly, the smooth cubic threefold in $\mathbf{P}^4_{\mathbf{F}_4}$ defined by the equation

$$x_1^3 + x_1^2x_2 + x_2^3 + x_1^2x_3 + ux_1x_3^2 + ux_2x_3^2 + u^2x_1x_2x_4 + x_2^2x_4 + ux_4^3 + x_2^2x_5 + ux_2x_3x_5 + x_3^2x_5 + x_3x_5^2 + x_5^3 = 0,$$

where $u^2 + u + 1 = 0$, contains no $\mathbf{F}_4$-lines and 61 $\mathbf{F}_4$-points.

Finally, the smooth cubic threefold in $\mathbf{P}^4_{\mathbf{F}_5}$ defined by the equation

$$\begin{aligned} x_1^3 + 2x_2^3 + x_2^2x_3 + 3x_1x_3^2 + x_1^2x_4 + x_1x_2x_4 + x_1x_3x_4 + 3x_2x_3x_4 + 4x_3^2x_4 + x_2x_4^2 \\ + 4x_3x_4^2 + 3x_2^2x_5 + x_1x_3x_5 + 3x_2x_3x_5 + 3x_1x_4x_5 + 3x_4^2x_5 + x_2x_5^2 + 3x_5^3 = 0 \end{aligned}$$

contains no $\mathbf{F}_5$-lines and 126 $\mathbf{F}_5$-points.

We were unable to find smooth cubic threefolds defined over $\mathbf{F}_q$ with no $\mathbf{F}_q$-lines for the remaining values $q \in \{7, 8, 9\}$ (by Theorem 4.4, there are always $\mathbf{F}_q$-lines for $q \geq 11$).

4.5.5 Nodal Cubic Threefolds over $\mathbf{F}_2$ or $\mathbf{F}_3$ with No Lines

Regarding cubic threefolds with one node and no lines, we found the following examples.

The unique singular point of the cubic in $\mathbf{P}^4_{\mathbf{F}_2}$ defined by the equation

$$x_2^3 + x_2^2x_3 + x_3^3 + x_1x_2x_4 + x_3^2x_4 + x_4^3 + x_1^2x_5 + x_1x_3x_5 + x_2x_4x_5 = 0$$

is an ordinary double point at $x := (0, 1, 0, 0, 1)$ and this cubic contains no $\mathbf{F}_2$-lines. As we saw during the proof of Corollary 4.8, the base of the cone $\mathbf{T}_{X,x} \cap X$ is a smooth genus-4 curve defined over $\mathbf{F}_2$ with no $\mathbf{F}_4$-points. The pencils g_3^1 and h_3^1 are defined over $\mathbf{F}_2$.

The unique singular point of the cubic in $\mathbf{P}^4_{\mathbf{F}_3}$ defined by the equation

$$2x_1^3 + 2x_1^2x_2 + x_1x_2^2 + 2x_2x_3^2 + 2x_1x_2x_4 + x_2x_3x_4 \\ + x_1x_4^2 + 2x_4^3 + x_2x_3x_5 + 2x_3^2x_5 + x_2x_5^2 + x_5^3 = 0$$

is an ordinary double point at $x := (1, 0, 0, 0, 1)$ and this cubic contains no $\mathbf{F}_3$-lines. Again, the base of the cone $\mathbf{T}_{X,x} \cap X$ is a smooth genus-4 curve defined over $\mathbf{F}_3$ with no $\mathbf{F}_9$-points, and the pencils g_3^1 and h_3^1 are defined over $\mathbf{F}_3$.

4.6 Average Number of Lines

Consider the Grassmannian $G := \mathrm{Gr}(1, \mathbf{P}^{n+1}_{\mathbf{F}_q})$, the parameter space $\mathbf{P} = \mathbf{P}\big(H^0\big(\mathbf{P}^{n+1}_{\mathbf{F}_q}, \mathscr{O}_{\mathbf{P}^{n+1}_{\mathbf{F}_q}}(d)\big)\big)$ for all degree-d hypersurfaces in $\mathbf{P}^{n+1}_{\mathbf{F}_q}$, and the incidence variety $I = \{(L, X) \in G \times \mathbf{P} \mid L \subset X\}$. The first projection $I \to G$ is a projective bundle, hence it is easy to compute the number of $\mathbf{F}_q$-points of I. The fibers of the second projection $I \to \mathbf{P}$ are the varieties of lines. The average number of lines (on *all* degree-d n-folds) is therefore

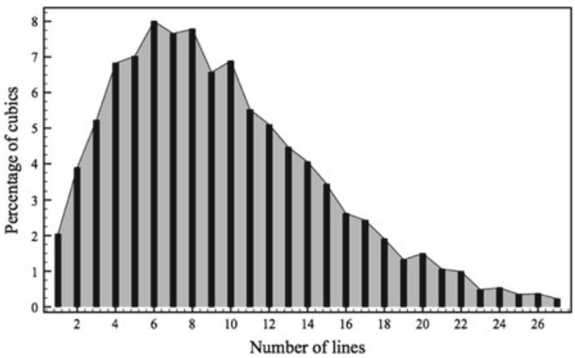
$$\frac{\mathrm{Card}\big(G(\mathbf{F}_q)\big)(q^{\dim(\mathbf{P})-d} - 1)}{q^{\dim(\mathbf{P})+1} - 1} \sim \mathrm{Card}\big(G(\mathbf{F}_q)\big)q^{\dim(\mathbf{P})-d-1}. \tag{17}$$

Recall that $\mathrm{Card}\big(G(\mathbf{F}_q)\big) = \sum_{0\le i<j\le n+1} q^{i+j-1}$. For cubic 3-folds, the right side of (17) is

$$q^2 + q + 2 + 2q^{-1} + 2q^{-2} + q^{-3} + q^{-4}.$$

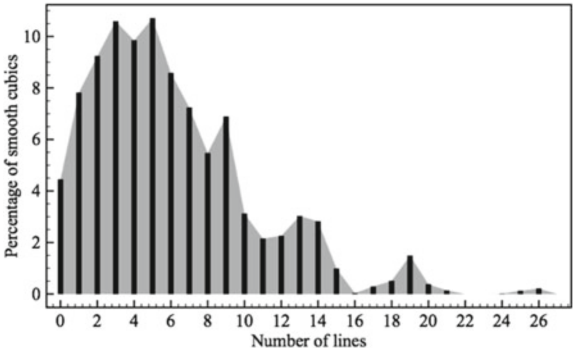
For $q = 2$, the average number of lines on a cubic threefold is therefore ~9.688 (compare with Example 4.17 below).

Example 4.17 (Computer experiments) For a random sample of $5 \cdot 10^4$ cubic threefolds defined over $\mathbf{F}_2$, we computed for each the number of $\mathbf{F}_2$-lines.



The average number of lines in this sample is ~9.651.

Smooth cubic threefolds contain less lines: here is the distribution of the numbers of $\mathbf{F}_2$-lines for a random sample of $5 \cdot 10^4$ *smooth* cubic threefolds defined over $\mathbf{F}_2$.



The average number of lines in this sample is $\sim$6.963.

5 Cubic Fourfolds

We now examine cubic fourfolds over $\mathbf{F}_q$. We expect them to contain "more" lines than cubic threefolds (indeed, all the examples we computed do contain $\mathbf{F}_q$-lines). Unfortunately, we cannot just take $\mathbf{F}_q$-hyperplane sections and apply our results from Sect. 4, because these results only concern mildly singular cubic threefolds, and there is no *a priori* reason why there would exist a hyperplane section defined over $\mathbf{F}_q$ with these suitable singularities.

We follow the same path as in Sect. 4. Recall that for any field $\mathbf{k}$, the scheme $F(X)$ of lines contained in a cubic fourfold $X \subset \mathbf{P}^5_{\mathbf{k}}$ with finite singular set is a geometrically connected local complete intersection fourfold (Sect. 2.3) with trivial canonical sheaf ([2, Proposition (1.8)]).

5.1 The Zeta Function of the Fourfold of Lines

Let $X \subset \mathbf{P}^5_{\mathbf{F}_q}$ be a *smooth* cubic hypersurface defined over $\mathbf{F}_q$. Its Betti numbers are 1, 0, 1, 0, 23, 0, 1, 0, 1, and the eigenvalues of the Frobenius morphism acting on $H^4(\overline{X}, \mathbf{Q}_\ell)$ are all divisible by q as algebraic integers ([18, Remark 5.1]). We write

$$N_r(X) = 1 + q^r + q^{3r} + q^{4r} + q^r \sum_{j=1}^{23} \omega_j^r,$$

where the complex algebraic integers ω_j (and all their conjugates) have modulus q, with $\omega_{23} = q$ (it corresponds to the part of the cohomology that comes from $H^4(\mathbf{P}^5_{\overline{\mathbf{F}}_q}, \mathbf{Q}_\ell)$). The trace formula (3) reads

$$Z(X,T) = \frac{1}{(1-T)(1-qT)(1-q^2T)(1-q^3T)(1-q^4T)P_4^0(X,T)},$$

where

$$P_4^0(X,T) := \frac{P_4(X,T)}{1-q^2T} = \prod_{j=1}^{22}(1-q\omega_j T). \tag{18}$$

If we set

$$M_r(X) := \frac{1}{q^r}\big(N_r(X) - (1+q^r+q^{2r}+q^{3r}+q^{4r})\big) = \sum_{j=1}^{22}\omega_j^r, \tag{19}$$

we obtain

$$P_4^0(X,T) = \exp\Big(\sum_{r\geq 1} M_r(X)\frac{(qT)^r}{r}\Big). \tag{20}$$

Theorem 5.1 *Let $X \subset \mathbf{P}^5_{\mathbf{F}_q}$ be a* smooth *cubic hypersurface defined over $\mathbf{F}_q$ and let $F(X)$ be the smooth fourfold of lines contained in X. With the notation above, we have $P_i(F(X),T) = 0$ for i odd and*

$$P_2(F(X),T) = P_6(F(X),T/q^2) = P_4(X,T/q) = \prod_{1\leq j\leq 23}(1-\omega_j T)$$
$$P_4(F(X),T) = \prod_{1\leq j\leq k\leq 23}(1-\omega_j\omega_k T),$$

where the complex numbers $\omega_1,\ldots,\omega_{22}$ have modulus q and $\omega_{23} = q$, and

$$Z(F(X),T) = \frac{1}{(1-T)(1-q^4T)\prod_{1\leq j\leq 23}\big((1-\omega_jT)(1-q^2\omega_jT)\big)\prod_{1\leq j\leq k\leq 23}(1-\omega_j\omega_kT)}. \tag{21}$$

Proof The various methods of proof described in the proof of Theorem 4.1 are still valid here. For example, one may deduce the theorem from the isomorphisms

$$H^4(\overline{X},\mathbf{Q}_\ell) \xrightarrow{\sim} H^2\big(\overline{F(X)},\mathbf{Q}_\ell(1)\big) \quad \text{and} \quad \mathrm{Sym}^2 H^2(\overline{F(X)},\mathbf{Q}_\ell) \xrightarrow{\sim} H^4(\overline{F(X)},\mathbf{Q}_\ell)$$

obtained from the Galkin–Shinder relation (7) ([14, Example 6.4]) or the analogous (known) statements in characteristic 0. We leave the details to the reader. □

5.2 Existence of Lines over Large Finite Fields

As we did for cubic threefolds, we use the Deligne–Weil estimates to find a lower bound for the number of $\mathbf{F}_q$-lines on a smooth cubic fourfold defined over $\mathbf{F}_q$.

Theorem 5.2 *Let X be a smooth cubic fourfold defined over $\mathbf{F}_q$ and let $N_1(F(X))$ be the number of $\mathbf{F}_q$-lines contained in X. For $q \geq 23$, we have*

$$N_1(F(X)) \geq q^4 - 21q^3 + 210q^2 - 21q + 1$$

and, for smaller values of q,

q	5	7	8	9	11	13	16	17	19
$N_1(F(X)) \geq$	26	638	1 337	2 350	5 930	12 338	29 937	38 438	61 010

In particular, X always contains an $\mathbf{F}_q$-line when $q \geq 5$.

When $q = 2$, we will see in Corollary 5.4 that X always contains an $\mathbf{F}_2$-line. These leaves only the cases $q = 3$ or 4 open (see Sect. 5.4.3).

Proof Write the roots of $Q_2(F(X), T)$ as q (with multiplicity a), $-q$ (with multiplicity b), $\omega_1, \dots, \omega_c, \overline{\omega}_1, \dots, \overline{\omega}_c$, with $a + b + 2c = 23$. The $r_j := \omega_j + \overline{\omega}_j$ are then real numbers in $[-2q, 2q]$ and, by (2) and Theorem 5.1, we have

$$\begin{aligned} N_1\big(F(X)\big) &= 1 + q^4 + \sum_{1\leq j\leq k\leq 23} \omega_j\omega_k + (1+q^2)\sum_{1\leq j\leq 23}\omega_j \\ &= 1 + q^4 + \Big(\frac{1}{2}\big(a(a+1) + b(b+1)\big) - ab\Big)q^2 + (a-b)q\sum_{1\leq j\leq c} r_j \\ &\quad + cq^2 + \sum_{1\leq j<k\leq c} r_j r_k + (1+q^2)\Big((a-b)q + \sum_{1\leq j\leq c} r_j\Big) \\ &= 1 + q^4 + \frac{1}{2}\big((a-b)^2 + 23\big)q^2 + (1+q^2)(a-b)q \\ &\quad + \sum_{1\leq j<k\leq c} r_j r_k + \big(1 + q^2 + (a-b)q\big)\sum_{1\leq j\leq c} r_j. \end{aligned}$$

Since $a + b = 23 - 2c$ is odd, it is enough to study the cases $a = 1$ and $b = 0$, or $a = 0$ and $b = 1$, since we can always consider pairs q, q, or $-q, -q$, as $\omega, \overline{\omega}$. We then have $c = 22$ and we set $\varepsilon := a - b \in \{-1, 1\}$.

As in the proof of Theorem 4.4, we note that this last expression $G_q^\varepsilon(\mathbf{r})$ is linear in each variable, hence its minimum is reached at a point on the boundary, when the r_j are all equal to $\pm 2q$. At such a point $\mathbf{r}_l$ (with l positive coordinates), we compute

$$G_q^\varepsilon(\mathbf{r}_l) = 1 + q^4 + 12q^2 + \varepsilon q(1+q^2) + 2q^2((2l-11)^2 - 11) + 2(2l-11)q(1+q^2+\varepsilon q).$$

Since q is always an eigenvalue, we must have $\varepsilon = 1$ when $l = 0$. As a function of l, the minimum is reached for $2l - 11 = -\frac{1+q^2+\varepsilon q}{2q}$. For $q \geq 23$, the allowable values for which $G_q^\varepsilon(\mathbf{r}_l)$ is smallest are $l = 0$ and $\varepsilon = 1$, and the minimum is $q^4 - 21q^3 + 210q^2 - 21q + 1 > 0$.

For $q \leq 19$, the numbers in the table follow from a longish comparison of the various functions G_q^ε. □

5.3 *Existence of Lines over Some Finite Fields*

The cohomology of the structure sheaf of the fourfold $F(X)$ is particularly simple and this can be used to prove congruences for its number of $\mathbf{F}_q$-points by using the Katz formula (6).

Proposition 5.3 (Altman–Kleiman) *Let $X \subset \mathbf{P}^5_{\mathbf{k}}$ be a cubic hypersurface defined over a field* $\mathbf{k}$*, with finite singular set. We have*

$$h^0(F(X), \mathscr{O}_{F(X)}) = h^2(F(X), \mathscr{O}_{F(X)}) = h^4(F(X), \mathscr{O}_{F(X)}) = 1$$
$$h^1(F(X), \mathscr{O}_{F(X)}) = h^3(F(X), \mathscr{O}_{F(X)}) = 0.$$

Proof The scheme $F(X)$ is the zero scheme of a section of the rank-4 vector bundle $\mathscr{E}^\vee := \mathrm{Sym}^3 \mathscr{S}^\vee$ on $G := \mathrm{Gr}(1, \mathbf{P}^5_{\mathbf{k}})$ and the Koszul complex

$$0 \to \textstyle\bigwedge^4 \mathscr{E} \to \bigwedge^3 \mathscr{E} \to \bigwedge^2 \mathscr{E} \to \mathscr{E} \to \mathscr{O}_G \to \mathscr{O}_{F(X)} \to 0 \tag{22}$$

is exact. By [2, Theorem (5.1)], the only non-zero cohomology groups of $\bigwedge^r \mathscr{E}$ are

$$H^8(G, \textstyle\bigwedge^4 \mathscr{E}) \simeq H^4(G, \bigwedge^2 \mathscr{E}) \simeq \mathbf{k}.$$

Chasing through the cohomology sequences associated with (22), we obtain $H^1(F(X), \mathscr{O}_{F(X)}) = H^3(F(X), \mathscr{O}_{F(X)}) = 0$ and

$$H^0(F(X), \mathscr{O}_{F(X)}) \simeq H^0(G, \mathscr{O}_G),$$
$$H^2(F(X), \mathscr{O}_{F(X)}) \simeq H^4(G, \textstyle\bigwedge^2 \mathscr{E}),$$
$$H^4(F(X), \mathscr{O}_{F(X)}) \simeq H^8(G, \textstyle\bigwedge^4 \mathscr{E}).$$

This proves the proposition. □

Since $\omega_{F(X)}$ is trivial, the multiplication product

$$H^2(F(X), \mathscr{O}_{F(X)}) \otimes H^2(F(X), \mathscr{O}_{F(X)}) \to H^4(F(X), \mathscr{O}_{F(X)}) \tag{23}$$

is the Serre duality pairing. It is therefore an isomorphism.

Corollary 5.4 *Let $X \subset \mathbf{P}^5_{\mathbf{F}_q}$ be a cubic hypersurface with finite singular set, defined over $\mathbf{F}_q$. If $q \equiv 2 \pmod 3$, the hypersurface X contains an $\mathbf{F}_q$-line.*

Proof The $\mathbf{F}_q$-linear map $\mathfrak{F}_q$ defined in Sect. 2.2 acts on the one-dimensional $\mathbf{F}_q$-vector space $H^2(F(X), \mathscr{O}_{F(X)})$ (Proposition 5.3) by multiplication by some $\lambda \in \mathbf{F}_q$; since (23) is an isomorphism, $\mathfrak{F}_q$ acts on $H^4(F(X), \mathscr{O}_{F(X)})$ by multiplication by λ^2. It then follows from the Katz formula (6) that we have

$$N_1(F(X)) \cdot 1_{\mathbf{F}_q} = 1 + \lambda + \lambda^2 \qquad \text{in } \mathbf{F}_q.$$

If $1 + \lambda + \lambda^2 = 0_{\mathbf{F}_q}$, we have $\lambda^3 = 1_{\mathbf{F}_q}$. Since $3 \nmid q - 1$, there are no elements of order 3 in $\mathbf{F}_q^\times$, hence the morphism $\mathbf{F}_q^\times \to \mathbf{F}_q^\times$, $x \mapsto x^3$ is injective. Therefore, $\lambda = 1_{\mathbf{F}_q}$, hence $3 \cdot 1_{\mathbf{F}_q} = 1_{\mathbf{F}_q}$, but this contradicts our hypothesis.

We thus have $1 + \lambda + \lambda^2 \neq 0_{\mathbf{F}_q}$, hence $N_1(F(X))$ is not divisible by the characteristic of $\mathbf{F}_q$ and the corollary is proved. □

5.4 *Examples of Cubic Fourfolds*

5.4.1 Fermat Cubics

If $X \subset \mathbf{P}^5_{\mathbf{F}_p}$ is the Fermat fourfold, it is a simple exercise to write down the zeta function of $F(X)$ using Theorems 4.11 and 5.1, as we did in dimension 3 in Corollary 4.12.

5.4.2 Cubic Fourfolds over $\mathbf{F}_2$ with only One Line

Smooth cubic fourfolds defined over $\mathbf{F}_2$ always contain an $\mathbf{F}_2$-line by Corollary 5.4. Random computer searches produce examples with exactly one $\mathbf{F}_2$-line: for example, the only $\mathbf{F}_2$-line contained in the smooth cubic fourfold defined by the equation

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_2^2x_1 + x_3^2x_1 + x_1x_2x_3 + x_1x_4^2 + x_1^2x_4 + x_2x_5^2 + x_2^2x_5 \\ + x_4^2x_5 + x_4x_5^2 + x_3x_6^2 + x_3^2x_6 + x_4^2x_6 + x_4x_6^2 + x_5^2x_6 + x_5x_6^2 + x_4x_5x_6 = 0 \end{aligned}$$

is the line $\langle(0, 0, 0, 0, 1, 1), (0, 0, 0, 1, 0, 1)\rangle$; the fourfold contains 13 $\mathbf{F}_2$-points.

5.4.3 Cubic Fourfolds over $\mathbf{F}_3$ or $\mathbf{F}_4$

Our results say nothing about the existence of lines in smooth cubic fourfolds defined over $\mathbf{F}_3$ or $\mathbf{F}_4$. Our computer searches only produced fourfolds containing lines (and over $\mathbf{F}_3$, both cases $N_1(F(X)) \equiv 0$ or $1 \pmod 3$ do occur), leading us to suspect that all (smooth) cubic fourfolds defined over $\mathbf{F}_3$ or $\mathbf{F}_4$ should contain lines.

6 Cubics of Dimensions 5 or More

In higher dimensions, the existence of lines is easy to settle.

Theorem 6.1 *Any cubic hypersurface $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ of dimension $n \geq 6$ defined over $\mathbf{F}_q$ contains $\mathbf{F}_q$-points and through any such point, there is an $\mathbf{F}_q$-line contained in X.*

Proof This is an immediate consequence of the Chevalley–Warning theorem: $X(\mathbf{F}_q)$ is non-empty because $n+2 > 3$ and given $x \in X(\mathbf{F}_q)$, lines through x and contained in X are parametrized by a subscheme of $\mathbf{P}^n_{\mathbf{F}_q}$ defined by equations of degrees 1, 2, and 3 and coefficients in $\mathbf{F}_q$. Since $n+1 > 1+2+3$, this subscheme contains an $\mathbf{F}_q$-point. □

The Chevalley–Warning theorem implies $N_1(X) \geq \frac{q^{n-1}-1}{q-1}$. When $n \geq 6$, we obtain from the theorem $N_1(F(X)) \geq \frac{q^{n-1}-1}{q^2-1}$; when X (hence also $F(X)$) is smooth, the Deligne–Weil estimates for $F(X)$ provide better bounds.

When $n \geq 5$, we may also use the fact that the scheme of lines contained in a smooth cubic hypersurface is a Fano variety (its anticanonical bundle $\mathscr{O}(4-n)$ is ample).

Theorem 6.2 *Assume $n \geq 5$ and let $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ be any cubic hypersurface defined over $\mathbf{F}_q$. The number of $\mathbf{F}_q$-lines contained in X is $\equiv 1 \pmod q$.*

Proof When X is smooth, the variety $F(X)$ is also smooth, connected, and a Fano variety. The result then follows from [12, Corollary 1.3].

To prove the result in general, we consider as in Sect. 4.6 the parameter space $\mathbf{P}$ for all cubic hypersurfaces in $\mathbf{P}^{n+1}_{\mathbf{F}_q}$ and the incidence variety $I = \{(L, X) \in G \times \mathbf{P} \mid L \subset X\}$. The latter is smooth and geometrically irreducible; the projection $\mathrm{pr}\colon I \to \mathbf{P}$ is dominant and its geometric generic fiber is a (smooth connected) Fano variety ([2, Theorem (3.3)(ii), Proposition (1.8), Corollary (1.12), Theorem (1.16)(i)]). It follows from [13, Corollary 1.2] that for any $x \in \mathbf{P}(\mathbf{F}_q)$ (corresponding to a cubic hypersurface $X \subset \mathbf{P}^{n+1}_{\mathbf{F}_q}$ defined over $\mathbf{F}_q$), one has $\mathrm{Card}\big(\mathrm{pr}^{-1}(x)\big) \equiv 1 \pmod q$. Since $\mathrm{pr}^{-1}(x) = F(X)$, this proves the theorem. □

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Perverse Sheaves of Categories and Non-rationality

Andrew Harder, Ludmil Katzarkov and Yijia Liu

1 Introduction

In this paper we take a new look at the classical notions of rationality and stable rationality from the perspective of sheaves of categories.

Our approach is based on three recent developments:

(1) The new striking approach to stable rationality introduced by Voisin and developed later by Colliot-Thélène and Pirutka, Totaro, Hassett, Kresch and Tschinkel.
(2) Recent breakthroughs made by Haiden, Katzarkov, Kontsevich, Pandit [15], who introduced an additional, to the Harder–Narasimhan, filtration on the semistable but not polystable objects.
(3) The theory of categorical linear systems and sheaves of categories developed by Katzarkov and Liu, [27]. The main outcome of this paper was a proposal of a new perverse category of sheaves analog of unramified cohomology.

An important part of our approach is the analogy between the theory of Higgs bundles and the theory of perverse sheaves of categories (PSC) initiated in [26, 27]. In the same way as the moduli spaces of Higgs bundles record the homotopy type of projective and quasi-projective varieties, sheaves of categories should record the information of the rationality of projective and quasi-projective varieties. It was demonstrated in [28, 30] that there is a correspondence between harmonic maps to

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Table 1 Higgs bundles ↔ Perverse sheaves of categories

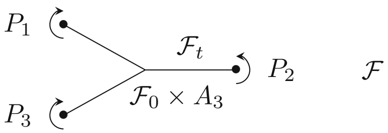
$\mathrm{Func}(\Pi_1^{\leqslant}(X,s),\mathrm{Vect})$ groupoid / category of vector spaces	$\mathrm{Func}(\Pi^{\leqslant\infty}(X,s),\mathrm{dg\ Cat})$ 2 category / dg category
Higgs bundles	Perverse sheaves of categories
Complex var. Hodge structures	Classical LG models

buildings and their singularities with stable networks and limiting stability conditions for degenerated categories, degenerated sheaves of categories. In this paper we take this correspondence further. We have described this correspondence in Table 1.

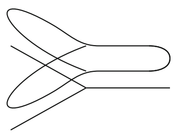
In this paper we describe a technology for finding such "good" flat families of perverse sheaves of categories. This is done by deforming LG models as sheaves of categories. The main geometric outcomes of our work are:

Classical	Categorical
$W = P$ equality for tropical varieties	"$W = P$" for perverse sheaves of categories
Voisin theory of deformations	Good flat deformations of PSC
Canonical deformations and compactification of moduli spaces	HN and additional filtrations of perverse sheaves of categories

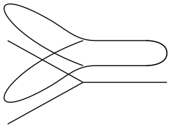
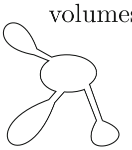
We will briefly discuss our procedure. We start with a perverse sheaf of categories (we will say more precisely what we mean by perverse sheaf of categories in the next section) $\mathcal{F}$ over $\mathbb{P}^1$ ($\mathbb{P}^2$, etc.). We then use a graph Γ (cell complex) in $\mathbb{P}^1$ ($\mathbb{P}^2$, etc.) to construct a semistable singular Lagrangian $\mathcal{L}$:



Here $\mathcal{F}_t$ are the fiber categories and P_i are categories equipped with spherical functors to $\mathcal{F}_t$. A global section in $\mathcal{F}$ defines a semistable object in the category of global sections of this PSC, which is analogous to the Lagrangian $\mathcal{L}$ shown in the following diagram.



Here, the fiber category is the category $A_1 \oplus A_1$ and the categories $P_1, \ldots, P_3$ are the category A_1 with the diagonal embedding into $A_1 \oplus A_1$ being the associated spherical functor. Observe that this object in the category of global sections can depend on the initial category or its degeneration. For most of the paper $\mathcal{L}$ will be a generator. We proceed with a correspondence:

PSC and degenerations	Deformations of categories
generators – graphs with sections 	Semistable objects
Flow $\longrightarrow$  volumes of necks $\longleftrightarrow$	Filtration on semistable generators

The filtration above is a refinement of the Harder–Narasimhan filtration. It will be defined in Sect. 5.

We formulate the main conjecture of the paper.

Conjecture 1.1 (**The main conjecture**).
The weights of semistable generators are birational invariants.

We will confirm this conjecture on some examples. We indicate that our technique contains Voisin's technique which uses CH_0-groups of degenerations. On the A side, these weights produce symplectic invariants.

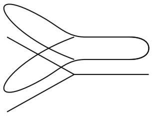
We briefly summarize our technique. We start with a perverse sheaf of categories (PSC) or its deformation. This produces a representation:

$$\rho : \pi_1(\mathbb{P}^1/\mathrm{pts}) \to \mathrm{Aut}(\mathcal{F}_t).$$

Observe that this gives us more possibilities than in the classical case, where only cohomology groups are acted upon,

$$\rho : \pi_1(\mathbb{P}^1/\mathrm{pts}) \to \mathrm{GL}(\oplus \mathrm{H}^*).$$

Our semistable objects (e.g. $\mathcal{L}$) correspond to global sections such as



$$\longleftrightarrow \qquad \rho : \pi_1(\mathbb{P}^1/\text{pts}) \to \text{Aut}(\mathcal{F}_t).$$

Observe that our filtration contains the filtrations

(1) Coming from degenerations of cohomologies. (Clemens' approach)
(2) Degenerations and nontrivial Brauer groups. (Voisin's approach)

We propose that our categorical method generalizes the methods of both Clemens and Voisin.

$$\begin{array}{|lcr|} \text{Clemens} & & \\ & \subset & \\ & & \text{Filtrations} \\ & \subset & \\ \text{Voisin} & & \end{array}$$

In a very general sense our filtration is a generalization of classical Hodge theory. There should be an analogy between nilpotent representations of $\mathbb{Z}$ (which correspond to degenerations over the punctured disc with nilpotent monodromy) and their associated weight filtrations and the filtrations on an Artinian category $\mathcal{A}$ obtained from a central charge $Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R}$,

$$\{\rho : \pi_1(Z) \to Nil\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{an Artinian category with} \\ Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{Artinian category of} \\ \text{nilpotent representations} \end{array} \right\} \quad \longleftrightarrow \quad \{\text{any Artinian category}\}$$

Based on this observation and based on many examples explained in this paper, we propose a correspondence:

Classical	Categorical
Unramified cohomologies	Hybrid models with filtrations

The paper is organized as follows:

In Sect. 2, we introduce briefly the theory of perverse sheaves of categories and their deformations. In Sect. 3, we give examples of deformations of categories. In Sect. 4, we show how our approach relates to Voisin's approach. In Sect. 5, we introduce hybrid models and explain how the examples given in this paper support our main conjecture. A more detailed treatment will appear in another paper.

This paper outlines a new approach. More details, examples and calculations will appear elsewhere.

2 Perverse Sheaves of Categories

2.1 Definitions

In this section we develop the theory of sheaves of categories and their deformations. First, we will explain what we mean when we talk about perverse sheaves of categories. Let M be a manifold with stratification S. Let K be a singular Lagrangian subspace of M so that each $K_i = K \cap S_i$ is a deformation retract of S_i. Furthermore, assume that the functor $\mathcal{R}$ which assigns to each perverse sheaf $\mathcal{F}$ on (M, S) a constructible sheaf on K with singularities in $S_K := K \cap S$ the sheaf $\mathbb{H}_K^{\dim M}(M, \mathcal{F})$ is faithful. Then a perverse sheaf of categories on M with singularities in K will be a constructible sheaf of categories on (M, S) is a constructible sheaf of categories on (K, S_K) which satisfies some appropriate conditions.

Such conditions are not known in general, and depend upon the singularities of K, but the general idea is that one should find appropriate translations of conditions which define the image of $\mathcal{R}$ into the language of pretriangulated dg categories. The most basic form of this condition is found in work of Kapranov–Schectman [25]. If one takes the stratified space $(\mathbb{C}, 0)$, then an appropriate skeleton K is the non-negative real line. The restriction functor expresses each perverse sheaf on $(\mathbb{C}, 0)$ as a constructible sheaf on the line K which has generic fiber a vector space ψ at any point on the positive real line and fiber ϕ at 0. There are two natural maps $\upsilon : \phi \to \psi$ and $u : \psi \to \phi$ which satisfy the condition that $\mathrm{Id}_\psi - \upsilon u$ is an automorphism, or equivalently $\mathrm{Id}_\phi - u\upsilon$ is invertible.

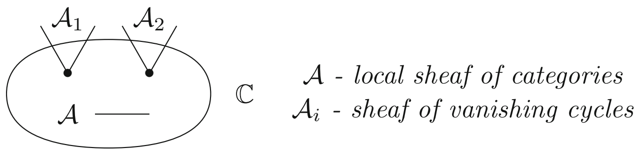
If we replace the vector spaces ϕ and ψ with pretriangulated dg categories Φ and Ψ then the map υ becomes a functor $F : \Phi \to \Psi$. The condition that the map υ exists and that $\mathrm{Id}_\phi - u\upsilon$ is an automorphism is analogous to claiming that F is spherical. The difference of morphisms becomes the cone of the unit $RF \to \mathrm{Id}_\Psi$, which is the twist of F. The sheaf Φ should be thought of as the "category of vanishing cycles" at 0, and Ψ should be thought of as the "category of nearby cycles".

A rough definition of perverse sheaf of categories is as follows.

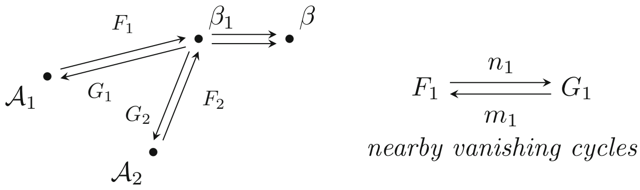
Definition 2.1 *A perverse sheaf of categories on (M, S) is a constructible sheaf of categories on an appropriate skeleton (K, S_K) so that there are functors between stalks of this constructible sheaf which have properties which emulate the structure of $\mathcal{R}(\mathcal{F})$ for $\mathcal{F}$ a perverse sheaf on (M, S).*

We start with a definition. We shuffle our definition in order to study deformations of perverse sheaf of categories. We localize (K, S_K) and several smaller skeleta $\mathrm{Sch}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$. Our definition now looks like:

Definition 2.2 (Sheaves of categories over $\mathrm{Sch}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$).



$\mathcal{A}$, $\mathcal{A}_i$ - 2 categories
F, G - 2 functors
n, m - 2 natural transformations



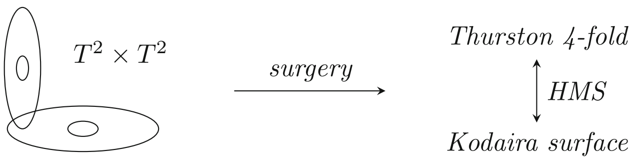
$$T_{G_1} = \beta_1 \operatorname{Id}_{G_1} - m_1 n_1$$

Theorem 2.3 *The deformations of* $\operatorname{Sch}(\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ *are described by:*

(1) Adding a new category β*;*
(2) Changes in natural transformations n_i, m_j.

We give some examples.

Example 2.4 *We start with a simple example* $T^2 \times T^2$ *- the product of two 2-dimensional tori.*

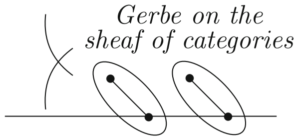
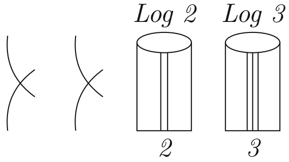


In [3] the following theorem is proven.

Theorem 2.5 *The following categories are equivalent:*

$$\mathrm{D^b}(T^2 \times T^2, \mathit{Gerbe}) \cong \operatorname{Fuk}(\mathit{Thurston}) \cong \mathrm{D^b}(\mathit{Kodaira}).$$

Example 2.6 *We generalize this construction to the case of LG models. The addition of gerbes should be an operation that is captured by perverse sheaves of categories and LG models, as described in the following example.*

LG model	$Dolg_{2,3}$ surface
	

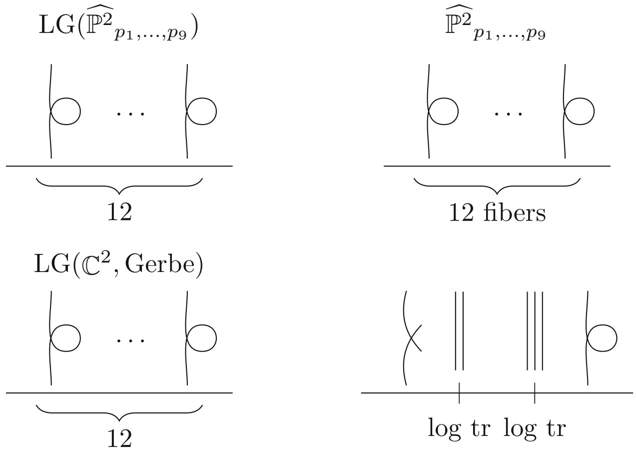
Recall: $Dolg_{2,3}$ is the Dolgachev surface with multiple fibers of multiplicities 2 and 3, and is obtained from the rational elliptic surface $\widehat{\mathbb{P}^2}_{p_1,\ldots,p_9}$ by applying 2 surgeries with order 2, 3.

The mirror of the surgery transforming the rational elliptic surface into the Dolgachev surface should be the addition of new fibers to the LG model of the mirror to the rational elliptic surface. The rational elliptic surface has mirror which is a rational elliptic surface with one smooth fiber removed and potential w the natural elliptic fibration over $\mathbb{C}$.

Theorem 2.7 *The mirror of $Dolg_{2,3}$ is obtained from the LG model of $\widehat{\mathbb{P}^2}_{p_1,\ldots,p_9}$ by adding a gerbe G on it corresponding to a log transform. In other words:*

$$\mathrm{D^b}(Dolg_{2,3}) = \mathrm{FS}(LG(\widehat{\mathbb{P}^2}_{p_1,\ldots,p_9}), G). \tag{2.1}$$

We indicate the proof of the theorem in the following diagram.



2.2 Some More Examples

Consider a fibration $\mathcal{F} \xrightarrow{f} \mathbb{C}$ with a multiple n-fiber over 0.

$$\begin{array}{ccc} E \times \mathbb{C} & \xrightarrow[(\times l, \times \mathcal{E})]{n:1} & \mathcal{F} \xrightarrow{f} \mathbb{C} \\ \Big\downarrow \mathbb{Z}^n & & \\ \mathbb{C} & & nl = 0,\ \mathcal{E}^n = 1 \end{array}$$

The idea is that the addition of smooth fibers with multiplicity greater than 1 (by surgery) into an elliptic fibration over $\mathbb{C}$ should introduce quasi-phantoms into the Fukaya–Seidel category of the associated elliptic fibration. This is summarized in the following theorem.

Theorem 2.8 $\mathrm{MF}(\mathcal{F} \xrightarrow{f} \mathbb{C})$ *contains a quasi-phantom.*

Proof Indeed $\mathrm{H}^*(\mathcal{F}, \text{vanishing cycles}) = 0$, since vanishing cycles are the elliptic curve E and $\mathrm{H}^*(E, L) = 0$, for any L - nontrivial rank 1 local system.

Also $\mathrm{K}(\mathrm{MF}(\mathcal{F} \to \mathbb{C})) = \mathbb{Z}_n$. □

One expects that phantom and quasi-phantom categories should be detectable via moduli spaces of objects. The following proposition provides evidence for this.

Proposition 2.9 *There exists a moduli space of stable objects on* $\mathrm{MF}(\mathcal{F} \to \mathbb{C})$.

Proof Indeed these are the $\mathbb{Z}_n$-equivalent objects on $E \times \mathbb{C}$. For example, we have $M^{\mathrm{stab}} = E'$, E' - multiple fiber. □

In the next section, we consider more examples of deformations of perverse sheaves of categories.

3 Deformations of Perverse Sheaves of Categories and Poisson Deformations

Recall that deformations of perverse sheaves of categories are determined by three different types of deformations,

1. Deformations of the Stasheff polytope,
2. Deformations of the fiber categories,
3. Deformation of natural transformations.

Here we will give several examples of deformations of sheaves of categories which come from the second piece of data. We will show that noncommutative deformations of $\mathbb{P}^2$ and $\mathbb{P}^3$ may be obtained as globalization of a deformation of perverse sheaves of categories. We will describe an explicit realization of the following correspondence.

$$\left\{ \begin{array}{c} \text{Deformation of} \\ \text{natural transformations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Quantization of} \\ \text{Poisson deformations} \end{array} \right\}$$

In our case, we will recover quantizations of Poisson deformations for the simple reason that the deformation of perverse sheaves of categories that we produce comes with a deformation of the fiber category.

The results described below will appear in forthcoming work of the first two named authors [18].

3.1 Warmup: Deformations of $\mathbb{P}^2$

As a warmup we can analyze the case of $\mathbb{P}^2$. Here we will recover the classical non-commutative deformations of $\mathbb{P}^2$ as the deformations of a perverse sheaf of categories which is obtained by deforming the spherical functors and fixing the fiber category.

Recall that the following data determines a noncommutative deformation of $\mathbb{P}^2$. Let E be a smooth curve of genus 0, $\mathscr{L}$ be a line bundle on E of degree 3 and σ a translation automorphism of E. Then, according to Artin–Tate–van den Bergh [6], the twisted coordinate ring of E associated to $(E, \mathscr{L}, \sigma)$ is the coordinate ring of a noncommutative deformation $\mathbb{P}^2_\mu$ of $\mathbb{P}^2$. Under the identification between $\bigwedge^2 T_{\mathbb{P}^2}$ and $-K_{\mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^2}(3)$, these deformations are associated to the choice of section of $\mathscr{O}_{\mathbb{P}^2}(3)$ which vanishes on the canonical image of E associated to $\mathscr{L}$.

The same data can be used to build a perverse schober on the disc Δ with three critical points p_1, p_2, p_3. This perverse schober can be used to reconstruct the derived category of the associated noncommutative deformation of $\mathbb{P}^2$. This fact was essentially noticed by Bondal–Polishchuk [7], but of course not in the language of perverse schobers.

We note that any line bundle $\mathscr{L}$ on E corresponds to a spherical functor $S_{\mathscr{L}}$: $D^b(k) \longrightarrow D^b(\mathrm{coh}\, E)$, and in particular, given the triple $(E, \mathscr{L}, \sigma)$, we can construct three spherical functors $S_0, S_{\mathscr{L}}, S_\sigma$ corresponding to line bundles $\mathscr{O}_E, \mathscr{L}$ and $\sigma^*\mathscr{L}^2$ respectively. If $\sigma = \mathrm{id}$ then this triple is precisely what one gets by restricting the strong exceptional collection $\mathscr{O}_{\mathbb{P}^2}, \mathscr{O}_{\mathbb{P}^2}(1), \mathscr{O}_{\mathbb{P}^2}(2)$ on $\mathbb{P}^2$ to the image of E under the embedding associated to $\mathscr{L}$.

We will let $\mathfrak{S}(E, \mathscr{L}, \sigma)$ be the perverse schober on the disc with three critical points associated to the spherical functors S_0, $S_{\mathscr{L}}$ and S_σ.

Proposition 3.1 (Harder–Katzarkov [18]). *The category of global sections of the perverse schober $\mathfrak{S}(E, \mathscr{L}, \sigma)$ is $D^b(\mathrm{coh}\, \mathbb{P}^2_\mu)$ where $\mathbb{P}^2_\mu$ is the noncommutative deformation of $\mathbb{P}^2$ associated to the triple $(E, \mathscr{L}, \sigma)$.*

The fact that makes this possible is that we can deform spherical objects on an elliptic curve. By definition, if S is a spherical object on E, then $\mathrm{Ext}^1(S, S) = \mathbb{C}$. These infinitesimal deformations are obtained by pullback along an automorphism of E, though of course, deformation may be obstructed. Whether the corresponding perverse schober recovers $\mathbb{P}^2$ or not can be detected using the "monodromy at infinity". In essence, the action of spherical functors $S_0, S_{\mathscr{L}}, S_\sigma$ should be interpreted as monodromy around the degenerate fibers of the perverse sheaf of categories at points p_1, p_2 and p_3. The composition of the three monodromy functors should be interpreted as monodromy around the loop encompassing all three degenerate points. We have the following proposition.

Proposition 3.2 *The global sections of the perverse schober* $\mathfrak{S}(E, \mathscr{L}, \sigma)$ *is* $D^b(\text{coh}\,\mathbb{P}^2)$ *if and only if* $S_0 \cdot S_{\mathscr{L}} \cdot S_\sigma$ *is the spherical twist associated to the line bundle* $\mathscr{L}^3$.

3.2 Noncommutative Deformations of $\mathbb{P}^3$

Here we exhibit noncommutative deformations of $\mathbb{P}^3$ as coming from the deformations of a perverse sheaf of categories over a 1-dimensional base by deforming the structure of the category of nearby cycles, or in terms of a PSC over a 2-dimensional base by deforming the sheaves of vanishing cycles.

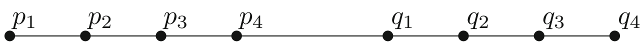
Polishchuk shows [39] that there exist Poisson structures on $\mathbb{P}^3$ so that there are Poisson divisors which look like

1. A normal crossings union of two quadrics
2. A normal crossings union of a hyperplane and a cubic.

Thus we should be able to perform the construction of noncommutative deformations of $\mathbb{P}^2$ by replacing the smooth elliptic curve by either a normal crossings pair of quadrics $X_{2,2}$ or a normal crossings union of a hyperplane and a cubic surface, denoted $X_{1,3}$. For the sake of notation, we will only look at the case of $X_{2,2}$ in what follows, but all results hold for $X_{1,3}$ as well. We then obtain a schober over the disc with four singular points whose general fiber is $\text{Perf}(X_{2,2})$. We then deform the schober, not by deforming the spherical functors S_0, S_1, S_2, S_3 and keeping $\text{Perf}(X_{2,2})$ constant, but by taking non-commutative deformations of $\text{Perf}(X_{2,2})$ which deform the spherical functors S_0, S_1, S_2, S_3.

Proposition 3.3 (Harder–Katzarkov [18]). *One may construct noncommutative deformations* $X_{2,2,\mu}$ *of* $X_{2,2}$ *which preserve the spherical functors* S_i *for* $i = 1, \dots, 4$ *corresponding to the data* $(E, \mathscr{M}, \tau)$ *where, as before,* E *is a smooth curve of genus 1 and* τ *is an automorphism of* E*, but now* $\mathscr{M}$ *is an ample line bundle on* E *of degree* 2*. There is a corresponding perverse schober over the disc with four singularities called* $\mathfrak{T}(E, \mathscr{M}, \tau)$*. This deformation has coordinate ring given by a quantization of the Sklyanin algebra of degree 4.*

The category $\text{Perf}(X_{2,2})$ itself appears as global sections of a constructible sheaf of categories on a 2-dimensional complex as well. Taking E to be the elliptic curve that forms the singular locus of the union of smooth quadrics $X_{2,2}$, we take the skeleton $K_{X_{2,2}}$ of Δ with singularities in eight points,



To an edge of the skeleton above we take a dg extension of $D^b(\text{coh}\,E)$. There is then a standard semiorthogonal decomposition of $D^b(\text{coh}\,Q)$ for Q a generic quadric,

$$\langle \mathscr{O}_Q, \mathscr{O}_Q(1,0), \mathscr{O}_Q(0,1), \mathscr{O}_Q(1,1)\rangle$$

where the bundles above are determined by the natural identification of Q with $\mathbb{P}^1 \times \mathbb{P}^1$. If we let Q_1 and Q_2 be the two quadrics so that $X_{2,2} = Q_1 \cup Q_2$, then there are spherical functors $S_{i,(j,k)}$ associated to the pullback of $\mathscr{O}_{Q_i}(j,k)$ to $E = Q_1 \cap Q_2$. To the strata p_1 and q_4, we associate categories $\langle \mathscr{O}_{Q_1}\rangle$ and $\langle \mathscr{O}_{Q_2}(1,1)\rangle$ with the appropriate spherical functors to $D^b(\mathrm{coh}\, E)$. To the remaining 0-dimensional strata of the skeleton $K_{X_{2,2}}$, the appropriate categories are a bit less obvious. If $\mathcal{D}(k)$ is a dg extension of $D^b(k{-}\mathrm{mod})$ and $\mathcal{D}(\mathrm{coh}\, E)$ is a dg extension of $D^b(\mathrm{coh}\, E)$, then we can define $\mathcal{D}(k) \times_{S_i,(j,k)} \mathcal{D}(\mathrm{coh}\, E)$ to be the gluing of $\mathcal{D}(k)$ to $\mathcal{D}(\mathrm{coh}\, E)$ along the dg bimodule which assigns

$$(A, B) \in \mathrm{Ob}(\mathcal{D}(\mathrm{coh}\, E)) \times \mathrm{Ob}(\mathcal{D}(k)) \mapsto \mathrm{Hom}_{\mathcal{D}(\mathrm{coh}\, E)}(A, S_{i,(j,k)}(B)).$$

(see [32] for definitions). To p_2, p_3 and p_4 we assign the categories $\mathcal{D}(k) \times_{S_{i,(j,k)}} \mathcal{D}(\mathrm{coh}\, E)$ for $i = 1$ and (j,k) equal to $(1,0)$, $(0,1)$ and $(1,1)$ respectively. Descriptions of appropriate functors will be given in [18].

Proposition 3.4 *The dg category of global sections of the constructible sheaf of categories of $\mathcal{S}_{X_{2,2}}$ on $K_{X_{2,2}}$ is equivalent to a dg extension of* $\mathrm{Perf}(X_{2,2})$.

Therefore, we have that there is a constructible sheaf of categories whose sheaf of global sections gives the generic fiber of the constructible sheaf of categories which reconstructs $D^b(\mathrm{coh}\,\mathbb{P}^3)$. This suggests that perhaps there is a perverse sheaf of categories over $\Delta \times \Delta$ whose sheaf of global sections is $D^b(\mathrm{coh}\,\mathbb{P}^3)$. It is a somewhat remarkable fact that such a perverse sheaf of categories is provided by mirror symmetry.

Recall that the Landau–Ginzburg mirror of $\mathbb{P}^3$ is given by the pair $((\mathbb{C}^\times)^3, \mathsf{w})$ where w is the Laurent polynomial

$$\mathsf{w} = x + y + z + \frac{1}{xyz}.$$

Each monomial in this expression corresponds to a boundary divisor in $\mathbb{P}^3$, and the sum of these divisors is $-K_{\mathbb{P}^3}$. The decomposition of $-K_{\mathbb{P}^3}$ into a union of smooth quadrics then informally can be traced to a decomposition of this potential into the sum of a pair of functions,

$$\mathsf{w}_1 = x + y, \qquad \mathsf{w}_2 = z + \frac{1}{xyz}$$

This pair of functions give a map from $(\mathbb{C}^\times)^3$ to $\mathbb{C}^2$. The generic fiber of this map is a punctured elliptic curve, and this elliptic curve degenerates along the curve C_{deg}

$$\mathsf{w}_1\mathsf{w}_2(\mathsf{w}_1^2\mathsf{w}_2^2 - 16) = 0$$

The composition of the map $(x, y, z) \in (\mathbb{C}^\times)^3 \mapsto (\mathsf{w}_1, \mathsf{w}_2) \in \mathbb{C}^2$ with the map $(\mathsf{w}_1, \mathsf{w}_2) \in \mathbb{C}^2 \mapsto \mathsf{w}_1 + \mathsf{w}_2 \in \mathbb{C}$ recovers the map w. The map w has critical points over $4\sqrt{-1}^i$ for $i = 0, 1, 2, 3$. We state the following theorem. Details will appear in [18].

Theorem 3.5 (Harder–Katzarkov [18]). *There is a singular (real) two dimensional skeleton K_2 of $\mathbb{C}^2$ whose singularities lie in $C_{\deg}$ which maps to a skeleton K of $\mathbb{C}$ with singularities at $4\sqrt{-1}^i$ for $i = 0, 1, 2, 3$ under the map $(\mathsf{w}_1, \mathsf{w}_2) \mapsto \mathsf{w}_1 + \mathsf{w}_2$. On this skeleton, there is a constructible sheaf of categories whose category of global sections is a dg extension of $D^b(\operatorname{coh} \mathbb{P}^3)$.*

The skeleton K is the skeleton associated to a perverse schober on $\mathbb{C}$ with four singular points. The structure of the skeleton K_2 is determined completely by the braid monodromy of the projection of the curve $C_{\deg}$ to $\mathbb{C}$ induced by the map $(\mathsf{w}_1, \mathsf{w}_2) \mapsto \mathsf{w}_1 + \mathsf{w}_2$. Finally, the following theorem holds.

Theorem 3.6 (Harder–Katzarkov [18]). *There are deformations of the constructible sheaf of categories in Theorem 3.5, and these deformations correspond to quantizations of the Poisson deformations of $D^b(\operatorname{coh} \mathbb{P}^3)$ for which $X_{2,2}$ is a Poisson divisor.*

A similar theorem holds for the Poisson deformations of $\mathbb{P}^3$ for which $X_{1,3}$ remains a Poisson divisor – their quantizations may be recovered from deformations of a natural two-dimensional constructible sheaf of categories on a skeleton of $\mathbb{C}^2$ with singularities in a curve $D_{\deg}$, which is distinct from $C_{\deg}$.

3.3 Perverse Sheaves of Categories and Elliptic Curves

Here we describe 2-dimensional perverse sheaves of categories associated to the LG model whose Fukaya–Seidel category is equivalent to $D^b(\operatorname{coh} E)$. This approach should generalize to allow us to compute the derived category of an arbitrary curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us take a curve of degree $(2, n)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, then we build the following LG model

$$\mathsf{w} = x_1 + \frac{x_2^2}{x_1} + x_3 + \frac{x_2^n}{x_3}$$

which is a map $(\mathbb{C}^\times)^3$. The associated potential can be thought of as the composition of two maps. The first map sends $(x_1, x_2, x_3) \mapsto (x_2, \mathsf{w})$ and the second is a projection onto the second coordinate. Therefore, w is a fibration over $\mathbb{C}$ whose fibers are LG models of $\mathbb{P}^1 \times \mathbb{P}^1$ except the fiber over 0. The map (x_2, w) is a fibration over $\mathbb{C}^2$ with generic fiber a punctured elliptic curve. This can be partially compactified to an elliptic fibration written in Weierstrass form written as

$$Y^2 = X^3 - (2w_1^2 - w_2^2)X^2 + 4(w_2 - w_1)(w_1 + w_2)w_1^2 X + 4w_1^4(2w_1^2 + w_2^2).$$

This fibration degenerates along the curve

$$(w_2 - 4w_1)(4w_1 + w_2)w_2w_1 = 0.$$

Blow up the base of this fibration at (0, 0) and call the result $\mathbb{C}^{\in}$. We can pull back the above elliptic fibration to get a fibration over $\widetilde{\mathbb{C}^2}$ with exceptional divisor E. We can resolve singularities of this fibration over $\widetilde{\mathbb{C}^2}$ to obtain a smooth elliptic fibration Y. We can choose a chart $C_1 = \mathbb{C}^2$ on $\widetilde{\mathbb{C}^2}$ so that the map onto $\mathbb{C}$ is given by a quadratic map given in coordinates (t, s) as the function ts. Restricting the fibration of Y over $\widetilde{\mathbb{C}^2}$ to the chart C_1 and calling this elliptic fibration Y_1. This is written in Weierstrass form as

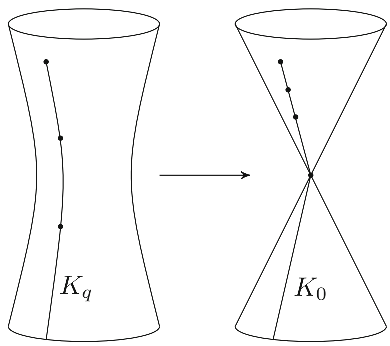
$$Y^2 = X^3 - (2t^2 - 4t + 1) - 4t(t - 2)(t - 1)^2X + 4(t - 1)^4(2t^2 - 4t + 3).$$

The discriminant curve of this fibration is given by the equation

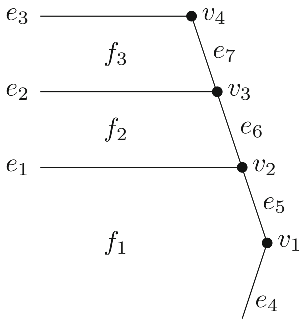
$$(4t - 5)(4t - 3)(t - 1) = 0$$

in terms of coordinates (t, s). The manifold Y_1 is then a smooth Calabi–Yau partial compactification of the LG model of the elliptic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ whose Fukaya–Seidel category should be equivalent to $D^b(\text{coh}E)$.

We can build a natural complex on $\mathbb{C}^2$ near the fiber $ts = 0$ and equip it with a perverse sheaf of categories whose global sections category should be $D^b(\text{coh}E)$. All of the data involved in this perverse sheaf of categories comes from the elliptic fibration above. This complex is built as follows. Take a straight path γ in a disc around 0 in $\mathbb{C}$ going from the boundary to 0. Fibers over points q in this path are smooth conics if $q \neq 0$, and a pair of copies of $\mathbb{C}$ meeting in a single point if $q = 0$. In each fiber over a point in γ, we can draw a skeleton K_q, and over the point 0 with singularities in the intersection of the discriminant curve in $\mathbb{C}^2$ with the fiber over q, we draw the skeleton K_0, which has singularities at (0, 0) as well as at the intersection of the fiber over 0 with the discriminant curve. These skeleta are drawn as in the following diagram



Putting all of these fiberwise complexes into a two-dimensional complex, we get a complex which looks as follows –



The constructible sheaf of categories on this complex is given by a set of categories assigned to each vertex, edge and face, and a sequence of functors $F_{s\to t} : \mathscr{C}_s \to \mathscr{C}_t$ for pairs of strata t and s so that $s \subseteq \bar{t}$ which satisfy the natural relations, i.e. that $F_{t\to q} \cdot F_{s\to t} = F_{s\to q}$. Our categories are:

$$\begin{aligned}
&\mathscr{C}_{f_1} = \mathscr{C}_{f_2} = \mathscr{C}_{f_3} = \mathscr{C}_{v_1} = \mathscr{D}(\mathrm{coh}E)\\
&\mathscr{C}_{e_4} = \mathscr{C}_{e_5} = \mathscr{C}_{e_6} = \mathscr{C}_{e_7} = \mathscr{C}_{v_2} = \mathscr{C}_{v_3} = \mathscr{C}_{v_4} = 0\\
&\mathscr{C}_{e_3} = \mathscr{C}_{k-\mathrm{dgm}}\\
&\mathscr{C}_{e_1} = \mathscr{C}_{k-\mathrm{dgm}} \times_{\Phi_2} \mathscr{D}(\mathrm{coh}E)\\
&\mathscr{C}_{e_2} = \mathscr{A} \times_{\Phi_1} \mathscr{D}(\mathrm{coh}E)
\end{aligned}$$

Here, $\mathscr{D}(\mathrm{coh}E)$ is a pretriangulated dg extension of $D^b(\mathrm{coh}E)$. If we have a functor $\Phi : \mathscr{C} \to \mathscr{D}(\mathrm{coh}E)$, then the category $\mathscr{C} \times_\Phi \mathscr{D}(\mathrm{coh}E)$ is the dg category of pairs (A, B, μ) where $A \in \mathscr{C}$ and $B \in \mathscr{D}(\mathrm{coh}E)$ and $\mu \in \mathrm{Hom}^0_{\mathscr{D}(\mathrm{coh}E)}(\Phi_2(A), B)$ and closed (see Kuznetsov–Lunts [32] for definition). There are functors Φ_0, Φ_1 and Φ_2 given as follows. There's a semiorthogonal decomposition

$$D^b(\mathrm{coh}\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathcal{O}, \mathcal{A}, \mathcal{O}(1, 1)\rangle$$

thus there are spherical functors

$$\begin{aligned}
&\phi_0 : D^b(k) \longrightarrow D^b(\mathrm{coh}E)\\
&\phi_1 : \mathcal{A} \longrightarrow D^b(\mathrm{coh}E)\\
&\phi_2 : D^b(k) \longrightarrow D^b(\mathrm{coh}E)
\end{aligned}$$

which have dg lifts of these functors

$$\begin{aligned}
&\Phi_0 : \mathscr{C}_{k-\mathrm{dgm}} \longrightarrow \mathscr{D}(\mathrm{coh}E)\\
&\Phi_1 : \mathscr{A} \longrightarrow \mathscr{D}(\mathrm{coh}E)
\end{aligned}$$

$$\Phi_2 : \mathscr{C}_{k-\text{dgm}} \longrightarrow \mathscr{D}(\text{coh}E).$$

There are two well-defined functors from $\mathscr{A} \times_\Phi \mathscr{D}(\text{coh}E)$ to $\mathscr{D}(\text{coh}E)$ then there are two functors F^+ and F^-, given by the map sending (A, B, μ) to B and Cone$(\mu)[1]$ respectively. The functor $F_{e_i \to f_i}$ is given by the corresponding functor F^- for $i = 1, 2$, the functor $F_{e_i \to f_{i+1}}$ is the corresponding functor F^+ for $i = 1, 2$. The functor $F_{e_3 \to f_3}$ is the functor Φ_0. The category $F_{v_1 \to f_1}$ is the identity functor. It's easy to check that the global sections of this sheaf of categories is $\mathscr{D}(\text{coh}E)$.

The restriction of this perverse sheaf of categories to the 1-dimensional skeleton K_q in each fiber for $q \neq 0$ has global sections category which is a dg extension of $D^b(\text{coh}\,\mathbb{P}^1 \times \mathbb{P}^1)$ [18], which is equivalent to the Fukaya–Seidel category of the Landau–Ginzburg model associated to each fiber of w over $q \neq 0$.

Remark 3.7 (Cubic fourfolds with two planes and K3 surfaces). A similar structure should arise in the case of the cubic fourfold containing a pair of planes. It is well known that the cubic containing two planes is $\mathbb{P}^4$ blown up at the transversal intersection S of a cubic and a quadric hypersurface.

According to mirror symmetry, there should be a pair of potentials w_1 and w_2 on the LG model of this cubic (defined on some open subset of the relatively compactified LG model) corresponding to the divisor classes D_1 and D_2 of the cubic and quadrics in $\mathbb{P}^4$ containing S. This gives rise to a fibration over $\mathbb{C}^2$ which can then be viewed as the composition of a fibration over $\mathbb{C}^2$ blown up at $(0, 0)$ and the contraction of the exceptional divisor. This fibration over the blown up plane should have relative dimension 2 and the fibers over a general point should be a K3 surface mirror to the intersection of a cubic and a quadric in $\mathbb{P}^4$. Furthermore, the fibers over the exceptional divisor should be generically smooth, as should the fibers over the locus $\mathsf{w}_1 + \mathsf{w}_2 = 0$.

Associated to this fibration, there should be a perverse sheaf of Fukaya categories over the blown-up plane. Near the intersection of the exceptional divisor and the proper transform of $\mathsf{w}_1 + \mathsf{w}_2 = 0$, this perverse sheaf of Fukaya categories should localize along a skeleton to look exactly like the construction above, except instead of having generic fibers the Fukaya category of an elliptic curve, we should have generic fibers the Fukaya category of the mirror to the complete intersection in $\mathbb{P}^4$ of a cubic and a quadric. The global section of this constructible sheaf of categories should have category of global sections equal to the derived category of the intersection of the cubic and the quadric in $\mathbb{P}^4$, which is reflected in the fact that the cubic fourfold containing two planes has, as a semiorthogonal summand of its derived category the derived category of the complete intersection of the cubic and the quadric in $\mathbb{P}^4$.

4 Landau–Ginzburg Model Computations for Threefolds

In this section we connect our program to birational geometry and the theory of LG models. The main goal of this section is to emphasize our program is connected to Voisin's approach. In terms of deformations of perverse sheaves of categories, the

LG model gives a PSC whose global sections recover the derived category of a Fano variety. We will degenerate one of the categories of vanishing cycles of this PSC in order to produce a category which has nontrivial "Brauer group". The approach to degeneration that we take is standard in symplectic geometry and goes back at least to Seidel [40], and involves removing closed subvarieties.

We recall some inspiration from birational geometry stemming from the work of Voisin [41], Colliot-Thélène and Pirutka [8]. A variety X is called stably non-rational if $X \times \mathbb{P}^n$ is non-rational for all n. It is known that if a variety over $\mathbb{C}$ is stably rational then for any field L containing $\mathbb{C}$, the Chow group $\mathrm{CH}_0(X_L)$ is isomorphic to $\mathbb{Z}$. Under this condition, $\mathrm{CH}_0(X)$ is said to be universally trivial. Voisin has shown that universal nontriviality of $\mathrm{CH}_0(X)$ can be detected by deformation arguments, in particular [41, Theorem 1.1] says that if we have a smooth variety $\mathcal{X}$ fibered over a smooth curve B so that a special fiber $\mathcal{X}_0$ has only mild singularities and a very general fiber $X := \mathcal{X}_b$ has universally trivial $\mathrm{CH}_0(X)$ then so does any projective model of $\mathcal{X}_0$. If V is a threefold, then one can detect failure of universal $\mathrm{CH}_0(X)$-triviality by showing that there exists torsion in $H^3(V, \mathbb{Z})$ (i.e. there exists torsion in the Brauer group). As an example, we may look at the classical Artin–Mumford example [5] which takes a degeneration of a quartic double solid to a variety which is a double cover of $\mathbb{P}^3$ ramified along a quartic with ten nodes. It is then proven in [5] that the resolution of singularities of this particular quartic double solid V has a $\mathbb{Z}/2$ in $H^3(V, \mathbb{Z})$. Voisin uses this to conclude that a general quartic double solid is not stably rational, whereas Artin and Mumford could only conclude from this that their specific quartic double solid is not rational.

The main idea that we explore in this section is that the approach of Voisin to stable non-rationality should have a generalization to deformations or degenerations of $\mathrm{D}^b(\mathrm{coh}\, X)$. Via mirror symmetry, this should translate to a question about deformations or degenerations of sheaves of categories associated to the corresponding LG model of X. Mirror symmetry for Fano threefolds should exchange

$$H^{\mathrm{even}}(X, \mathbb{Z}) \cong H^{\mathrm{odd}}(\mathrm{LG}(X), S; \mathbb{Z})$$
$$H^{\mathrm{odd}}(X, \mathbb{Z}) \cong H^{\mathrm{even}}(\mathrm{LG}(X), S; \mathbb{Z})$$

where S is a smooth generic fiber of the LG model of X. See [29] for some justification for this relationship. This is analogous to the case where X is a Calabi–Yau threefold (see [11, 12]). The degenerations of the sheaf of categories associated to $\mathrm{LG}(X)$ that we will produce are not necessarily degenerations of LG models in the usual geometric sense, but they are produced by blowing up or excising subvarieties from X, as described in Sect. 3. We then show that we find torsion in $H^2(U, S; \mathbb{Z})$ for U our topologically modified LG model. We propose that this torsion is mirror dual to torsion in the K_0 of some deformation of the corresponding category. By the relation above, the torsion groups appearing in the following subsections should be mirror categorical obstructions to stable rationality of the quartic double solid and the cubic threefold.

4.1 The LG Model of a Quartic Double Solid

Here we review a description of the LG models of several Fano threefolds in their broad strokes. We begin with the following situation. Let X be a Fano threefold of one of the following types. Recall that V_7 denotes the blow-up of $\mathbb{P}^3$ at a single point.

(1) X is a quartic double solid.
(2) X is a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$.
(3) X is a double cover of V_7 with branch locus an anticanonical divisor.
(4) X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with ramification locus of degree $(2, 2, 2)$.

Then the singular fibers of the LG model of X take a specific form which is independent of X. The construction described here appears in [9] for the case of quartic double solids. There are several fibers of each LG model which are simply nodal K3 surfaces, and there is one fiber which is a more complicated. We assume the complicated fiber is the fiber over 0 in $\mathbb{C}$ and we will denote it Y_0. Monodromy about this complicated fiber has order 2, and the fiber itself has a single smooth rational component with multiplicity 2 and a number of rational components with multiplicity 1. We will henceforward denote the LG model by Y, and it will be equipped with a regular function w.

A natural way to understand Y_0 is to take base-change along the map $t = s^2$ where s is a parameter on the base $\mathbb{C}_t$ of the original LG model Y. Performing this base-change and taking normalization, we obtain a (possibly) singular family of K3 surfaces $\widehat{Y}$ with a map $\widehat{\mathsf{w}} : \widehat{Y} \to \mathbb{C}_s$. The (possible) singularities of $\widehat{Y}$ are contained in the fiber $\widehat{\mathsf{w}}^{-1}(0) = \widehat{Y}_0$, which is a K3 surface with several A_1 singularities.

Furthermore, there is an involution ι on $\widehat{Y}$ from which we may recover the original LG model Y. This quotient map sends no fiber to itself except for $\widehat{Y}_0$. On this fiber, the automorphism ι acts as a non-symplectic involution on $\widehat{Y}_0$ and fixes a number of rational curves.

In the Landau–Ginzburg model Y, given as the resolved quotient of $\widehat{Y}/\iota$, the fiber Y_0 is described as follows. In the quotient $\widehat{Y}/\iota$, the fiber over 0 is scheme-theoretically 2 times the preimage of 0 under the natural map. Furthermore, there are a number of curves of cA_1 singularities. We resolve these singularities by blowing up along these loci in sequence, since there is nontrivial intersection between them. This blow-up procedure succeeds in resolving the singularities of $\widehat{Y}/\iota$ and that the relative canonical bundle of the resolved threefold is trivial. Let $E_1, \ldots, E_n$ denote the exceptional divisors obtained in Y under this resolution of singularities.

4.2 Torsion in Cohomology of the LG Model

We will now denote by U the manifold obtained from Y by removing components of Y_0 with multiplicity 1, in other words, $U = Y \setminus (\cup_{i=1}^n E_i)$ where $E_1, \ldots, E_n$ are the exceptional divisors described in the previous paragraph. Another way to describe this threefold is as follows. Take the threefold $\widehat{Y}$ described above, and excise the fixed locus of ι, calling the resulting threefold $\widehat{U}$. Note that this is the complement of a union of smooth codimension 2 subvarieties. The automorphism ι extends to a fixed-point free involution on $\widehat{U}$ and the quotient $\widehat{U}/\iota$ is U. Let us denote by w_U the restriction of w to U. Our goal is to show that if S is a generic smooth fiber of w_U, then there is $\mathbb{Z}/2$ torsion in $H^2(U, S; \mathbb{Z})$.

The group $H^2(U, S; \mathbb{Z})$ should be part of the K-theory of some quotient category of the Fukaya–Seidel category of $\mathrm{LG}(X)$ equipped with an appropriate integral structure.

Proposition 4.1 *The manifold $\widehat{U}$ is simply connected.*

Proof First, let $\widetilde{Y}$ be a small analytic resolution of singularities of $\widehat{Y}$ and let $\widetilde{\mathsf{w}}$ be the natural map $\widetilde{\mathsf{w}} : \widetilde{Y} \to \mathbb{A}^1_s$. Then, since the fixed curves of ι contain the singular points of $\widehat{Y}$, the variety $\widehat{U}$ can be written as the complement in $\widetilde{Y}$ of the union of the exceptional curves of the resolution $\widetilde{Y} \to \widehat{Y}$ and the proper transform of the fixed locus of the involution ι on $\widehat{Y}$. This is all to say that $\widehat{U}$ is the complement of a codimension 2 subvariety of the smooth variety $\widetilde{Y}$. Thus it follows by general theory that $\pi_1(\widehat{U}) = \pi_1(\widehat{Y})$, and so it is enough to show that $\pi_1(\widehat{Y})$ is simply connected.

At this point, we may carefully apply the van Kampen theorem and the fact that ADE singular K3 surfaces are simply connected to prove that $\widetilde{Y}$ is simply connected. Begin with a covering $\{V_i\}_{i=1}^m$ of $\mathbb{A}^1$ so that the following holds:

(1) Each V_i is contractible,
(2) Each $\widetilde{\mathsf{w}}^{-1}(V_i)$ contains at most one singular fiber of $\widetilde{\mathsf{w}}$,
(3) For each pair of indices i, j, the intersection $V_i \cap V_j$ is contractible, connected,
(4) For each triple of indices i, j, k, the intersection $V_i \cap V_j \cap V_k$ is empty.

(it is easy to check that such a covering can be found). Then the Clemens contraction theorem tells us that $Y_i := \widetilde{\mathsf{w}}^{-1}(V_i)$ is homotopic to the unique singular fiber (if V_i contains no critical point, then Y_i is homotopic to a smooth K3 surface). Since ADE singular K3 surfaces are simply connected, then Y_i is simply connected. The condition that $V_i \cap V_j$ is connected then allows us to use the Seifert–van Kampen theorem to conclude that $\widetilde{Y}$ is simply connected. □

As a corollary to this proposition, we have that

Corollary 4.2 *The free quotient $U = \widehat{U}/\iota$ has fundamental group $\mathbb{Z}/2$ and hence $H^2(U, \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}^n$ for some positive integer n.*

Now, finally, we show that this implies that there is torsion $\mathbb{Z}/2$ in the cohomology group $H^2(U, S; \mathbb{Z})$.

Theorem 4.3 *We have an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some positive integer m.*

Proof We compute using the long exact sequence in relative cohomology,

$$\cdots \to H^1(S, \mathbb{Z}) \to H^2(U, S; \mathbb{Z}) \to H^2(U, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to \dots$$

Since S is a smooth K3 surface, we know that $H^1(S, \mathbb{Z}) = 0$, and that the subgroup $\mathbb{Z}/2$ of $H^2(U, \mathbb{Z})$ must be in the kernel of the restriction map $H^2(U, \mathbb{Z}) \to H^2(S, \mathbb{Z})$. Thus it follows that there is a copy of $\mathbb{Z}/2$ in $H^2(U, S; \mathbb{Z})$, and furthermore, that $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^m$ for some integer m. □

4.3 The Cubic Threefold

A very similar construction can be performed in the case of the LG model of the cubic threefold with some minor modifications. The details of the construction of the LG model of the cubic threefold that are relevant are contained in [14].[1] There is a smooth log Calabi–Yau LG model of the cubic threefold, which we denote (Y, w) with the following properties:

(1) The generic fiber is a K3 surface with Picard lattice $\mathrm{M}_6 = E_8^2 \oplus U \oplus \langle -6 \rangle$.
(2) There are three fibers with nodes.
(3) The fiber over 0 which is a union of 6 rational surfaces whose configuration is described in [14]. Monodromy around this fiber is of order 3.

By taking base change of Y along the map $g : \mathbb{C} \to \mathbb{C}$ which assigns λ to μ^3, and resolving g^*Y, we obtain a threefold $\hat{Y}$ which is K3 fibered over $\mathbb{C}$, but now has only 6 singular fibers, each with only a node. This means that there is a birational automorphism ι on $\hat{Y}$ of order 3 so that $\hat{Y}/\iota$ is birational to Y. Explicitly, in [14] it is shown that the automorphism ι is undefined on nine pairs of rational curves, each pair intersecting in a single point and all of these pairs of curves are in the fiber of $\hat{Y}$ over 0. We can contract these A_2 configurations of rational curves to get a threefold $\widetilde{Y}$ on which ι acts as an automorphism, but which is singular. The automorphism ι fixes six rational curves in the fiber of $\widetilde{Y}$ over 0. After blowing up sequentially along these six rational curves to get $\widetilde{Y}'$, the automorphism ι continues to act biholomorphically, and no longer has fixed curves. The quotient $\widetilde{Y}'/\iota$ is smooth, according to [14], and there are seven components, the image of the six exceptional divisors, and a single component $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ of multiplicity three. The rational surfaces coming from exceptional divisors meet R along three vertical and three horizontal curves. The divisor R can be contracted onto either one of its $\mathbb{P}^1$ factors. Performing one of these two contractions, we recover Y.

Now let $U = (\widetilde{Y}'/\iota) \setminus \{S_1, \ldots, S_6\}$. Note that this can be obtained by blowing up Y in the curve which is the intersection of three components of the central fiber and removing all of the other components. Then a proof almost identical to that of Theorem 4.3 shows that, if S is a generic fiber of w, then

Theorem 4.4 *There is an isomorphism $H^2(U, S; \mathbb{Z}) \cong \mathbb{Z}/3 \oplus \mathbb{Z}^m$ for some positive integer m.*

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[1]In the most recent versions of [14], these details have been removed, so we direct the reader to versions 1 and 2 of [14] on the arXiv.

Therefore, if X is the cubic threefold, then there should exist a non-commutative deformation of $\mathrm{D}^b(\mathrm{coh}\, X)$ with torsion in its periodic cyclic cohomology obstructing stable rationality of X.

4.4 The Quartic Double Fourfold

Here we will look at the LG models of the quartic double fourfold. There is an analogy between the LG model of the quartic double fourfold and the LG model of the cubic threefold.

Here we will give a model which describes the LG model of the quartic double fourfold, which we call X. Recall that we may write such a variety as a hypersurface in $\mathbb{WP}(1, 1, 1, 1, 1, 2)$ of degree 4. Therefore, following the method of Givental, we may write the LG model of X as a hypersurface in $(\mathbb{C}^\times)^5$ cut out by the equation

$$x_4 + x_5 + \frac{1}{x_1x_2x_3x_4x_5^2} = 1$$

equipped with a superpotential

$$\mathsf{w} = x_1 + x_2 + x_3.$$

Call this hypersurface Y^0. We may write this superpotential as the sum of three superpotentials,

$$\mathsf{w}_\mathsf{i} = x_i \text{ for } i = 1, 2, 3.$$

There's then a map from $\mathrm{LG}(X)$ to $\mathbb{C}^3$ given by the restriction of the projection

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3).$$

The fibers of this projection map are open elliptic curves which can be compactified in $\mathbb{C}^2$ to

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3x_4x_5^2(x_4 + x_5 - 1) + 1 = 0$$

We may then write this threefold in Weierstrass form as

$$y^2 = x(x^2 + \mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3^2x + 16\mathsf{w}_1^3\mathsf{w}_2^3\mathsf{w}_3^3)$$

This elliptic fibration over $\mathbb{C}^3$ has smooth fibers away from the coordinate axes. We will resolve this threefold to get an appropriate smooth resolution of Y^0. We do this by blowing up the base of the elliptic fibration and pulling back until we can resolve singularities by blowing up the resulting fourfold in fibers.

First, we blow up $\mathbb{C}^3$ at $(0, 0, 0)$, and we call the resulting divisor E_0. Then we blow up the resulting threefold base at the intersection of E_0 and the strict transforms of $\{\mathsf{w}_i = 0\}$, calling the resulting exceptional divisors $E_{i,0}$. We then blow up the intersections of the strict transforms of $\mathsf{w}_i = \mathsf{w}_j = 0$ five times (in appropriate sequence) and call the resulting divisors $E_{ij,k}$, $k = 1, \ldots, 5$. There is now a naturally defined elliptic fibration over this blown-up threefold. Over an open piece in each divisor in the base, the fibers of this elliptic fibration and their resolutions can be described by Kodaira's classification. Identifying E_0 and $E_{i,0}$ with their proper transforms in R, we have:

- Fibers of type III over points in E_0.
- Fibers of type III* over points in $\{\mathsf{w}_i = 0\}$.
- Fibers of type I_0^* over points in $E_{ij,3}$.
- Fibers of type III over $E_{ij,2}$ and $E_{ij,4}$
- Fibers of type I_1 along some divisor which does not intersect any other divisor in the set above.

and smooth fibers everywhere else. We may now simply blow up appropriately to resolve most singularities in the resulting elliptic fourfold over R. We are left with singularities in fibers over $E_{ij,2} \cap E_{ij,3}$ and $E_{ij,4} \cap E_{ij,3}$. These singularities admit a small resolution by work of Miranda. Thus we obtain a smooth resolution of our elliptic fourfold.

We will call this resolved fourfold $\mathrm{LG}(X)$. The map w can be extended to a morphism from $\mathrm{LG}(X)$ to $\mathbb{C}$ by simply composing the elliptic fibration map from $\mathrm{LG}(X)$ to R with the contraction map from R onto $\mathbb{C}$ and the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$. The fiber over any point in $\mathbb{C}$ away from 0 is irreducible, and the fiber over 0 is composed of the preimages of E_0 and $E_{i,0}$ in the elliptic fibration, along with the strict transform of the preimage of $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3 = 0$ in Y^0, which is simply a smooth elliptically fibered threefold.

Therefore, the fiber over 0 is composed of 6 divisors with multiplicity 1. However, this is not normal crossings, since the preimage of E_0 in the elliptic fibration on $\mathrm{LG}(X)$ is a pair of divisors which intersect with multiplicity 2 in the fiber over each point in E_0.

4.5 Base Change and Torsion

Just as in the case of the cubic threefold, we may blow-up the LG model (Y, w) of the quartic double fourfold to get a fibration over $\mathbb{A}^1$ which we call $(\widetilde{Y}, \widetilde{\mathsf{w}})$ and remove divisors from $\widetilde{\mathsf{w}}^{-1}(0)$ to get a fibration over $\mathbb{A}^1$ which we denote $(Y_{np}, \mathsf{w}_{np})$ so that there is torsion in $H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z})$ for s a regular value of w.

We outline this construction, ignoring possible birational maps which are isomorphisms in codimension 1. We note that over the fibration E_0 in the LG model (Y, w) expressed as an elliptic fourfold over a blow-up of $\mathbb{C}^3$ as described in the previous section is a fibration by degenerate elliptic curves of Kodaira type III. Each fiber then, over a Zariski open subset of E_0 is a pair of rational curves meeting tangentially in a single point. The preimage of E_0 in Y is then a pair of divisors D_1 and D_2 in Y

which intersect with multiplicity 4 along a surface. Blowing up Y in this surface of intersection of D_1 and D_2 which is isomorphic to E_0 produces a rational threefold D' in the blow up (which we call $\widetilde{Y}$), whose multiplicity in the fiber over 0 of the inherited fibration over $\mathbb{C}$ is four.

Taking base change of $\widetilde{Y}$ along the map $t \mapsto s^4$ is the same as taking the fourfold cover of $\widetilde{Y}$ ramified along the fiber over 0. After doing this, the multiplicity of the preimage of D' is 1 and all components of the fiber over 0 except for the preimage of D' can be smoothly contracted to produce a fibration (Y', w') over $\mathbb{C}$.

The upshot of this all is that Y' admits a birational automorphism σ of order 4 so that Y'/σ is birational to $\widetilde{Y}$. In fact, if we excise the (codimension ≥ 2) fixed locus of σ and take the quotient, calling the resulting threefold Y_{np}, then Y_{np} is just $\widetilde{Y}$ with all components of the fiber over 0 which are not equal to D' removed. The fibration map on Y_{np} over $\mathbb{C}$ will be called w_{np}, and we claim that $H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z})$ has order four torsion. To do this, one uses arguments identical to those used in the case of the quartic double solid.

Proposition 4.5 *Letting Y_{np} and w_{np} be as above, and let s be a regular value of w. Then*

$$H^2(Y_{np}, \mathsf{w}_{np}^{-1}(s); \mathbb{Z}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}^a$$

for some positive integer a.

Therefore, the deformation of the Fukaya–Seidel category of $(\widetilde{Y}, \mathsf{w})$ obtained by removing cycles passing through the components of $\widetilde{\mathsf{w}}^{-1}(0)$ of multiplicity 1 should have 4-torsion in its K_0. This torsion class, under mirror symmetry should be an obstruction to the rationality of the quartic double fourfold.

4.6 Cubic Fourfolds and Their Mirrors

This section does not relate directly to deformations of perverse sheaves of categories, though it continues to explore the relationship between rationality and symplectic invariants of corresponding LG models.

In this section, we will look at the LG models of cubic fourfolds and cubic fourfolds containing one or two planes. Since cubic fourfolds containing one or two planes are still topologically equivalent to a generic cubic fourfold, this is a somewhat subtle problem which we avoid by instead obtaining LG models for cubic fourfolds containing planes which are blown up in the relevant copies of $\mathbb{P}^2$.

It is known (see [31]) that a general cubic has bounded derived category of coherent sheaves $\mathrm{D}^b(X)$ which admits a semi-orthogonal decomposition

$$\langle \mathcal{A}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{O}_X(3) \rangle.$$

When X contains a plane, $\mathcal{A}_X = \mathrm{D}^b(S, \beta)$ is the bounded derived category of β twisted sheaves on a K3 surface S for β an order 2 Brauer class. It is known

[21, Lemma 4.5] that the lattice T in $H^4(X, \mathbb{Z})$ orthogonal to the cycles $[H]^2$ and $[P]$ where H is the hyperplane class and P is the plane contained in X, is isomorphic to

$$E_8^2 \oplus U \oplus \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

which is not the transcendental lattice of any K3 surface. It is expected that such cubic fourfolds are non-rational. When X contains two planes, it is known that X is then rational. According to Kuznetsov [31], we then have that the category $\mathcal{A}$ is the derived category of a K3 surface S, and by work of Hassett [21], we have that the orthogonal complement of the classes $[H]^2$, $[P_1]$, $[P_2]$ where P_1 and P_2 are the planes contained in X is isomorphic to

$$U \oplus E_8^2 \oplus \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix},$$

which is the transcendental lattice of a K3 surface S, and generically $\mathcal{A}_X = \mathrm{D}^b(\operatorname{coh} S)$ and $S^{[2]}$ is the Fano variety of lines in X.

Our goal in this section is to describe the mirror side of this story. In particular, we want to observe in the three cases above, how rationality and non-rationality can be detected using symplectic characteristics of LG models. We will construct smooth models of smooth models of

(1) The LG model of a cubic fourfold (which we call Z_0).
(2) The LG model of a cubic fourfold containing a plane P blown up in P (which we call Z_1).
(3) The LG model of a cubic fourfold containing a pair of disjoint planes P_1 and P_2 blown up in $P_1 \cup P_2$ (which we call Z_2).

According to a theorem of Orlov [37], the bounded derived categories of Z_1 and Z_2 admit semi-orthogonal decompositions with summands equal to the underlying cubics. Therefore, homological mirror symmetry predicts that the derived categories of coherent sheaves of the underlying cubics should be visible in the Fukaya–Seidel (or directed Fukaya) categories of the LG models of Z_1 and Z_2. In particular, we should be able to see $\mathrm{D}^b(\operatorname{coh} S, \beta)$ in the Fukaya–Seidel category of $\mathrm{LG}(Z_1)$ and $\mathrm{D}^b(\operatorname{coh} Z_2)$ in the Fukaya–Seidel category of $\mathrm{LG}(Z_2)$.

It is conjectured by Kuznetsov [31] that a cubic fourfold X is rational if and only if $\mathcal{A}_X$ is the bounded derived category of a geometric K3 surface, thus in the case where X contains a single plane, the gerbe β is an obstruction to rationality of X. Such gerbes arise naturally in mirror symmetry quite commonly. If we have a special Lagrangian fibration on a manifold M over a base B, and assume that there is a special Lagrangian multisection of π and no special Lagrangian section, then mirror symmetry is expected assign to a pair (L, ∇) in the Fukaya category of M a complex of α-twisted sheaves on the mirror for α some nontrivial gerbe. We will see this structure clearly in the LG models of Z_0, Z_1 and Z_2.

4.7 The General Cubic Fourfold

Let us now describe the LG model of the general cubic fourfold in a such a way that a nice smooth resolution becomes possible. Givental [10] gives a description of constructions of mirrors of toric complete intersections. A more direct description of Givental's construction is described in [17].

We begin with the polytope Δ corresponding to $\mathbb{P}^5$ given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Using Givental's construction, we get a LG model with total space

$$Y^0 = \{z + w + u = 1\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu}.$$

We will express Y^0 as a fibration over $\mathbb{C}^3$ by elliptic curves. Then we will use work of Miranda [33] to resolve singularities of this fibration and thus obtain a smooth model of Y^0. This is necessary, since there are singularities "at infinity" in the LG model provided by Givental. A more uniform construction of smooth compactifications of the LG models constructed by Givental can be found in [16, Chap. 3].

To carry do this, we decompose w into three different functions

$$\mathsf{w}_1 = x, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

Then Y^0 is birational to a variety fibered by affine curves written as

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw(z + w - 1) - 1 = 0$$

where $\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3$ are treated as coordinates on $\mathbb{C}^3$. This is can be rearranged into Weierstrass form as

$$y^2 = x^3 + \mathsf{w}_2^2\mathsf{w}_1^2\mathsf{w}_3^2 x^2 + 8\mathsf{w}_3^3\mathsf{w}_2^3\mathsf{w}_1^3 x + 16\mathsf{w}_1^4\mathsf{w}_2^4\mathsf{w}_3^4.$$

The discriminant locus of this fibration over $\mathbb{C}^3$ has four components, and for a generic point in each component we can give a description of the structure of the resolution of singularities over that point in terms of Kodaira's classification of the singular fibers of elliptic fibrations.

- Singular fibers of type IV* along $\{\mathsf{w}_i = 0\}$ for $i = 1, 2, 3$,
- Singular fibers of type I_1 along the divisor cut out by the equation $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0$.

The loci $\mathsf{w}_i = 0$ intersect each other of course, but D_{I_1} does not intersect any $\{w_i = 0\}$, thus we must only worry about singularities at $(0, 0, 0)$ and $\mathsf{w}_i = \mathsf{w}_j = 0$ for $i, j = 1, 2, 3$ and $i \neq j$. We blow up sequentially at these loci and describe the fibers over the exceptional divisors. We will use Kodaira's conventions for describing the minimal resolution of singular fibers of an elliptic fibration.

- Blow up the base $\mathbb{C}^3$ at $(0, 0, 0)$. Call the associated blow-up map $f_1 : T_1 \to \mathbb{C}^3$ and call the exceptional divisor Q. As before, if π_1 is the induced elliptic fibration on T_1, then on Q there are just smooth fibers away from the intersection of the strict transform of $\{\mathsf{w}_i = 0\}$.
- Blow up the intersections $\{\mathsf{w}_i = \mathsf{w}_j = 0\}$ for $i, j = 1, 2, 3$ and $i \neq j$. Call the associated map $f_2 : T_2 \to T_1$ and call the exceptional divisors $E_{i,j}$. Let π_2 be the induced elliptic fibration on T_2. The fibration π_2 has fibers with resolutions of type IV over R_{ij}.
- Blow up at the intersections of R_{ij} and the strict transforms of $\{\mathsf{w}_i = 0\}$ and $\{\mathsf{w}_j = 0\}$. Call the associated map $f_3 : T_3 \to T_2$ and call the exceptional divisors $R_{ij,i}$ and $R_{ij,j}$ respectively. Let π_3 be the induced elliptic fibration over T_3, then the fibration π_3 has smooth fibers over the divisors $R_{ij,i}$ and $R_{ij,j}$.

Thus we have a fibration over T_3 with discriminant locus a union of divisors, and none of these divisors intersect one another. Thus we may resolve singularities of the resulting Weierstrass form elliptic fourfold by simply blowing up repeatedly the singularities along these loci. Call this fourfold $\mathrm{LG}(Z_0)$. By composing the elliptic fibration π_3 of $\mathrm{LG}(Z_0)$ over T_3 with the contraction of T_3 onto $\mathbb{C}^3$ we get a map which we call $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_2$ from $\mathrm{LG}(Z_0)$ to $\mathbb{C}^3$. We will describe explicitly the fibers over points of $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$.

- If p is a point in the complement of the strict transform of

$$\{\mathsf{w}_1 = 0\} \cup \{\mathsf{w}_2 = 0\} \cup \{\mathsf{w}_3 = 0\} \cup \{\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0\}$$

  then the fiber over p is smooth.
- If p is in $\{\mathsf{w}_1 = 0\}$, $\{\mathsf{w}_2 = 0\}$, or $\{\mathsf{w}_3 = 0\}$, then the fiber over p is of type IV*. If p is a point in $\{\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27 = 0\}$, then the fiber over p is a nodal elliptic curve.
- If $p \in \{\mathsf{w}_1 = \mathsf{w}_2 = 0\}$, $\{\mathsf{w}_1 = \mathsf{w}_3 = 0\}$ or $\{\mathsf{w}_2 = \mathsf{w}_3 = 0\}$, then the fiber over p is of dimension 2.
- If $p = (0, 0, 0)$, then the fiber is a threefold. This threefold is precisely the restriction of the fibration π_3 to the strict transform of the exceptional $\mathbb{P}^2$ obtained by blowing up $(0, 0, 0)$.

Now we will let $\mathrm{LG}(Z_0)$ be the smooth resolution of the elliptically fibered threefold over T_3 described above. We compose the fibration map π_3 with the map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from $\mathbb{C}^3$ to $\mathbb{C}$, then we recover the map w on the open set that $\mathrm{LG}(Z_0)$ and Y^0 have in common. Then we obtain a nice description of the fiber in $\mathrm{LG}(Z_0)$ of w over 0 as a union of two elliptically fibered threefolds, one component being the threefold fiber over (0, 0, 0) in Y, and the other being the natural elliptically fibered threefold obtained by taking the preimage of the line $\mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3 = 0$ in $\mathrm{LG}(Z_0)$ under the elliptic fibration map. These two threefolds intersect along a surface S which is naturally elliptically fibered. This surface can be described by taking the subvariety of the exceptional divisor $Q = \mathbb{P}^2$ given by a the natural fibration over a hyperplane in $\mathbb{P}^2$. This is an elliptically fibered surface over $\mathbb{P}^2$ with three singular fibers of type IV* and a order 3 torsion section.

Proposition 4.6 *The smooth K3 surface S of Picard rank 20 with transcendental lattice isomorphic to the (positive definite) root lattice A_2, which has Gram matrix*

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

This can be proved using the techniques described in [19].

4.8 Cubic Fourfolds Blown up in a Plane

We will apply a similar approach to describe the LG model of the cubic fourfold blown up in a plane. We start by expressing this as a toric hypersurface. Blowing up $\mathbb{P}^5$ in the intersection of three coordinate hyperplanes is again a smooth toric Fano variety $\mathbb{P}_\Delta$ which is determined by the polytope Δ with vertices given by points $\rho_1, \dots, \rho_7$ given by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The vertices of this polytope (determined by the columns of the above matrix) determine torus invariant Cartier divisors in $\mathbb{P}_\Delta$, and the cubic blown up in a plane is linearly equivalent to $D_{\rho_3} + D_{\rho_4} + D_{\rho_5}$. Thus, following the prescription of Givental [10] (or more precisely, [17]), one obtains the Landau–Ginzburg model with

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} = 1 \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with potential given by restriction of

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu}$$

to Y^0. We may decompose w into the three potentials

$$\mathsf{w}_1 = x, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

so that $\mathsf{w} = \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$. Therefore, if we take the map $\pi : Y^0 \to \mathbb{C}^3$ given by $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3)$, this can be compactified to a family of elliptic curves with fiber

$$\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw(z + w - 1) + 1 + a\mathsf{w}_3 w = 0.$$

This can be written as a family of elliptic curves in Weierstrass form as

$$y^2 = x^3 + \mathsf{w}_1\mathsf{w}_2^2\mathsf{w}_3(\mathsf{w}_1\mathsf{w}_3 - 4a)x^2 + 8\mathsf{w}_1^3\mathsf{w}_2^3\mathsf{w}_3^3 x + 16\mathsf{w}_1^4\mathsf{w}_2^4\mathsf{w}_3^4.$$

Away from $(0, 0, 0)$, the singularities of this fibration can be resolved.

- I_1^* along $\mathsf{w}_1 = 0$ and $\mathsf{w}_2 = 0$
- IV^* along $\mathsf{w}_3 = 0$
- I_1 along

$$(a\mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3^2 - 8a^2\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3^2 + \mathsf{w}_1^2\mathsf{w}_2^2\mathsf{w}_3 + 16a^3\mathsf{w}_3^2 - 36a\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 - 27\mathsf{w}_1\mathsf{w}_2) = 0$$

We first blow up the base $\mathbb{C}^3$ at $(0, 0, 0)$ to obtain a fibration with smooth fibers over the exceptional divisor. We cannot yet resolve singularities of this fibration, since the fibers over the intersection of any two coordinate hyperplanes do not have known resolutions. Following work of Miranda [33], we may blow up the base of this fibration again several times in order to produce a fibration over a threefold which has a fiber-wise blow-up which resolves singularities.

We blow up the base along the lines $R_{ij} = \{\mathsf{w}_i = \mathsf{w}_j = 0\}$ to get three exceptional surfaces R_{ij} over which there are singular fibers generically of type IV. Blowing up again in all lines of intersection between R_{ij} and $\mathsf{w}_j = 0$ and R_{ij} and $\mathsf{w}_i = 0$, calling the resulting exceptional divisors $R_{ij,j}$ and $R_{ij,i}$, we get an elliptic fibration over this blown up threefold so that:

- I_1^* along $\mathsf{w}_1 = 0$ and $\mathsf{w}_2 = 0$
- IV^* along $\mathsf{w}_3 = 0$
- IV along R_{ij}.
- I_0 (i.e. smooth) along $R_{ij,j}$ and $R_{ij,i}$.
- I_1 along some divisor which does not intersect $\mathsf{w}_1 = 0$, $\mathsf{w}_2 = 0$, $\mathsf{w}_3 = 0$ or $R_{ij} = 0$.

Therefore, one may simply resolve singularities of this fibration in the same way as one would in the case of surfaces – blowing up repeatedly in sections over divisors in the discriminant locus. Let us refer to this elliptically fibered fourfold as $\mathrm{LG}(Z_1)$.

There is an induced map from $\mathrm{LG}(Z_1)$ to $\mathbb{C}$ which we call w essentially comes from the composition of the fibration on $\mathrm{LG}(Z_1)$ by elliptic curves with its contraction onto $\mathbb{C}^3$ along with the addition map $(z_1, z_2, z_3) \mapsto z_1 + z_2 + z_3$ from $\mathbb{C}^3$ to $\mathbb{C}$. This is the superpotential on $\mathrm{LG}(Z_1)$, and $\mathrm{LG}(Z_1)$ is a partially compactified version of the Landau–Ginzburg model of the cubic fourfold blown up in a plane.

The fiber of w over 0 is the union of two elliptically fibered smooth threefolds, one being the induced elliptic fibration over the proper transform of the exceptional divisor obtained when we blew up $(0, 0, 0)$ in $\mathbb{C}^3$. The other is the proper transform in $\mathrm{LG}(Z_1)$ of the induced elliptic fibration over the surface $z_1 + z_2 + z_3 = 0$ in $\mathbb{C}^3$.

These two threefolds meet transversally along a smooth K3 surface S. This K3 surface is equipped naturally with an elliptic fibration structure over $\mathbb{P}^1$ and inherits two singular fibers of type I_1^*, a singular fiber of type IV^* and two singular fibers of type I_1.

Proposition 4.7 *The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

To prove this, one uses a concrete model of S and shows that there is another elliptic fibration on S so that the techniques in [19] can be applied to show that there is a lattice polarization on a generic such S by the lattice

$$E_8^2 \oplus \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}. \tag{4.1}$$

Then one shows that the complex structure on the surface S varies nontrivially as the parameter a varies, thus a generic such S has Picard lattice equal to exactly the lattice in Eq. (4.1). Then applying standard results of Nikulin [36], one obtains the proposition.

4.9 Cubic Threefolds Blown up in Two Planes

Here we begin with the toric variety $\mathbb{P}^5$ blown up at two disjoint planes, which is determined by the polytope Δ with vertices at the columns $\rho_1, \ldots, \rho_8$ of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

The cubic blown up along two disjoint planes is then linearly equivalent to the torus invariant divisor $D_{\rho_3} + D_{\rho_4} + D_{\rho_5} + D_{\rho_7}$, therefore, by the prescription of Givental, we may write the associated LG model as

$$Y^0 = \left\{ z + w + u + \frac{a}{xyz} \right\} \subseteq (\mathbb{C}^\times)^5$$

equipped with the function

$$\mathsf{w}(x, y, z, w, u) = x + y + \frac{1}{xyzwu} + bxyz.$$

We split this into the sum of three functions,

$$\mathsf{w}_1 = x + bxyz, \qquad \mathsf{w}_2 = y, \qquad \mathsf{w}_3 = \frac{1}{xyzwu}.$$

The fibers of the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3)$ from Y to $\mathbb{C}^3$ are written as a family of affine cubics

$$(z + w - 1)\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 zw + (1 + b\mathsf{w}_2 z)(1 + a\mathsf{w}_3 w) = 0$$

which are open elliptic curves. We may write this in Weierstrass form and use Tate's algorithm to show that, the singular fibers of this fibration are of types:

- I_1^* along $\mathsf{w}_3 = 0$ and $\mathsf{w}_2 = 0$
- I_5 along $\mathsf{w}_1 = 0$
- I_1 along a divisor determined by a complicated equation in w_1, w_2 and w_3.

Elsewhere, the fibers of this map can be compactified to smooth elliptic curves.

In order to obtain a smooth model of this fibration, we will first blow up $\mathbb{C}^3$ at $(0, 0, 0)$. The induced elliptic fibration is generically smooth over this exceptional divisor, which we call Q. In order to obtain a model of this elliptic fibration which we may resolve by sequentially blowing up in singular fibers, we must now blow up along the line $\mathsf{w}_2 = \mathsf{w}_3 = 0$. We will call the exceptional surface under this blow-up R_{23}. We obtain a singular elliptically fibered fourfold over this new threefold base so that the fibers over the divisor R_{23} are generically of Kodaira type IV. Blowing up again at the intersections of R_{23} and $\mathsf{w}_2 = 0$ and at the intersection of R_{23} and $\mathsf{w}_3 = 0$ (calling the exceptional divisors $R_{23,2}$ and $R_{23,3}$ respectively) we obtain a fibration which can be resolved by blowing up curves of divisors in the fibers over R_{23}, $\mathsf{w}_1 = 0$, $\mathsf{w}_2 = 0$ and $\mathsf{w}_3 = 0$, and by taking resolution over curves in $\mathsf{w}_1 = \mathsf{w}_2 = 0$ and $\mathsf{w}_1 = \mathsf{w}_3 = 0$

(following [33, Table 14.1]). Call the resulting fibration $\mathrm{LG}(Z_2)$ and let π be the fibration map onto the blown up threefold. We have singular fibers of types:

- I_1^* along $\mathsf{w}_3 = 0$ and $\mathsf{w}_2 = 0$
- I_5 along $\mathsf{w}_1 = 0$
- IV along R_{23}
- Fibers over $\mathsf{w}_1 = \mathsf{w}_2 = 0$ and $\mathsf{w}_1 = \mathsf{w}_3 = 0$ of the type determined by Miranda [33] and described explicitly in [33, Table 14.1].
- I_1 along a complicated divisor which does not intersect any of the divisors above.

and smooth fibers otherwise.

The variety $\mathrm{LG}(Z_2)$ admits a non-proper elliptic fibration over $\mathbb{C}^3$ obtained by composing π with the blow-up maps described above. Then the fiber in $\mathrm{LG}(Z_2)$ over $(0, 0, 0)$ is an elliptic threefold over a blown-up $\mathbb{P}^2$ base. Composing this non-proper elliptic fibration with the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$ from $\mathbb{C}^3$ to $\mathbb{C}$ recovers the potential w. The fiber over 0 of the map w from $\mathrm{LG}(Z_2)$ to $\mathbb{C}$ has two components, each an elliptically fibered threefold meeting along a smooth K3 surface. This K3 surface, which we call S, admits an elliptic fibration over $\mathbb{P}^1$ canonically with two singular fibers of type I_1^*, a singular fiber of type I_5 and five singular fibers of type I_1.

Proposition 4.8 *The orthogonal complement of the Picard lattice in $H^2(S, \mathbb{Z})$ is isomorphic to*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

for a generic K3 surface S appearing as in the computations above.

Again, this result is obtained by finding an appropriate alternative elliptic fibration on S and demonstrating that an appropriate lattice embeds into its Picard lattice, then combining results of Nikulin [36] and the fact that there is a non-trivial 2-dimensional deformation of S obtained by letting the parameters a and b vary to see that indeed, this is the transcendental lattice of a generic such S.

Remark 4.9 In the last three sections, we have glossed over the issue of providing an appropriate relative compactification of our LG models with respect to w. Indeed, one wants to produce a relatively compact partial compactification of the LG models above whose total space is smooth and has at least trivial canonical class. In the cases that we have described above, this can be done by taking a relative compactification of $\mathbb{C}^3$ with respect to the map $(\mathsf{w}_1, \mathsf{w}_2, \mathsf{w}_3) \mapsto \mathsf{w}_1 + \mathsf{w}_2 + \mathsf{w}_3$ and writing $\mathrm{LG}(Z_i)$ as an elliptically fibered fourfold over this variety. Performing the same procedure as above (blowing up the base of this fibration until a global resolution can be obtained) and then simply blowing up in fibers or taking small resolutions as described by Miranda [33], one can produce a partial compactification of $\mathrm{LG}(Z_i)$ so that the fibers of w are compact. Using the canonical bundle formula in [33], one can then show that this compactification is indeed appropriate. We note that, strictly speaking, Miranda's work only applies to three dimensional elliptic fibrations. However, since we do not have to deal with intersections of more than two divisors in our discriminant locus, and all of our intersections are transverse, the arguments of [33] still may be applied.

4.10 Special Lagrangian Fibrations

In the case of hyperkähler surfaces, special Lagrangian fibrations can be constructed with relatively little difficulty. The procedure is outlined in work of Gross and Wilson [13]. We review their work in the following section and apply it to our examples.

Definition 4.10 *A K3 surface S is lattice polarized by a lattice L if there is a primitive embedding of L into* $\mathrm{Pic}(S)$ *whose image contains a pseudo-ample class.*

For a given lattice L of signature $(1, \rho - 1)$ for $\rho \leq 20$ which may be embedded primitively into $H^2(S, \mathbb{Z})$ for a K3 surface, there is a $(20 - \rho)$-dimensional space of complex structures on S corresponding to K3 surfaces which admit polarization by L. A generic L-polarized K3 surface will then be a general enough choice of complex structure in this space.

We will follow the notation of Gross and Wilson [13] from here on. We choose I to be a complex structure on a K3 surface S and let g be a compatible Kähler–Einstein metric. Since S is hyperkähler, there is an S^2 of complex structures on S which are compatible with g. We will denote by I, J and K the complex structures from which all of these complex structures are obtained. The complex 2-form associated to the complex structure I is written as $\Omega(u, v) = g(J(u), v) + ig(K(u), v)$ for u and v sections of T_S. The associated Kähler form is given, as usual, by $\omega(u, v) = g(I(u), v)$. Similarly, one may give formulas for the holomorphic 2-form and Kähler forms associated to the complex structures J and K easily in terms of the real and imaginary parts of Ω and ω as described in [13, pp. 510].

A useful result that Gross and Wilson attribute to Harvey and Lawson [20, pp. 154] is:

Proposition 4.11 ([13, Proposition 1.2]). *A two-dimensional submanifold Y of S is a special Lagrangian submanifold of S with respect to the complex structure I if and only if it is a complex submanifold with respect to the complex structure K.*

Using the same notation as in [13], we will let S_K be the complex K3 surface with complex structure K, which then has holomorphic 2-form given by $\Omega_K = \mathrm{Im}\Omega + i\omega$ where ω and Ω are as before. If this vanishes when restricted to a submanifold E of S, then we must have $\omega_{|E} = 0$ as well. If ω is chosen generically enough in the Kähler cone of S (so that $\omega \cap L = 0$) then this forces E to be in $L^\perp$. One can show that a complex elliptic curve E on a K3 surface satisfies $[E]^2 = 0$ therefore, since $L^\perp$ has no isotropic elements, S_K cannot contain any complex elliptic curves and thus S has no special Lagrangian fibration. Therefore, we have proven that:

Proposition 4.12 *If L is a lattice so that $L^\perp$ contains no isotropic element, then a generic L-polarized K3 surface with a generic choice of Kähler–Einstein metric g has no special Lagrangian fibration.*

We will use this to prove a theorem regarding K3 surfaces which appeared in the previous sections. Let us recall that the transcendental lattices of the K3 surface appearing as the intersection of the pair of divisors in $\mathrm{LG}(Z_0)$, $\mathrm{LG}(Z_1)$ and $\mathrm{LG}(Z_2)$ are

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

In the first case, it is clear that the lattice is positive definite, therefore it cannot represent 0, and thus Proposition 4.12 shows that in this case there is no special Lagrangian fibration on this specific K3 surface. In the third case, we can use [13, Proposition 1.3] to see that there is a special Lagrangian fibration with numerical special Lagrangian section for a generic choice of Kähler–Einstein metric g.

In the second case, the discriminant of the lattice (which we will call M) is -8, and its discriminant group, which is just $M^\vee/M$, is isomorphic to $\mathbb{Z}/8$ and has generator with square $3/8$. Using a result of Nikulin [36], it follows that this is not equivalent to the lattice $\langle -8\rangle \oplus U$. At the same time, one can conclude that this is not the lattice $\langle -2\rangle \oplus U(2)$, and therefore, we cannot directly apply [13, Proposition 1.3] to obtain a special Lagrangian fibration on such a K3 surface.

However, applying the method used in the proofs of [13, Proposition 1.1] and [13, Proposition 1.3], one obtains a special Lagrangian fibration on S for a generic choice of g so that there is no special Lagrangian section, but there is a numerical special Lagrangian 2-section. To do this, we use the fact that $(1, -1, 1)$ is isotropic in this lattice.

Putting all of this together, we obtain the following theorem:

Theorem 4.13 *Let S be a generic K3 surface appearing as the intersection of the two components of the fiber over* 0 *of the LG models of a generic cubic Z_0, a cubic blown up in a plane Z_1, and a cubic blown up in two disjoint planes Z_2. Let ω be a generic Kähler class on S and Ω the corresponding holomorphic 2-form on S. Then:*

(1) In the case where $S \subseteq \mathrm{LG}(Z_0)$, then S admits no special Lagrangian torus fibration.
(2) In the case where $S \subseteq \mathrm{LG}(Z_1)$, then S admits a special Lagrangian torus fibration with no Lagrangian section but a (numerical) Lagrangian 2*-section.*
(3) In the case where $S \subseteq \mathrm{LG}(Z_2)$, then S admits a special Lagrangian torus fibration with a (numerical) Lagrangian section.

The first statement in Theorem 4.13 is mirror dual to the fact that the subcategory $\mathcal{A}_X$ of $\mathrm{D}^b(\mathrm{coh}\, X)$ for X a generic cubic fourfold is not the derived category of a K3 surface. The second statement corresponds to the fact that $\mathcal{A}_X \cong \mathrm{D}^b(S, \beta)$ for β an order 2 Brauer class on S for X a general cubic fourfold containing a plane. The third case corresponds to the fact that when X contains two disjoint planes, $\mathcal{A}_X \cong \mathrm{D}^b(S)$ for S a K3 surface.

According to [1, Corollary 7.8], there is an embedding of the (derived) Fukaya category of the K3 surface S appearing in Theorem 4.13 as a subcategory of the derived version of the Fukaya–Seidel category of the LG model of Z_0, Z_1 and Z_2 respectively. The objects in the Fukaya–Seidel category of an LG model are so-called admissible Lagrangians, which are, roughly, Lagrangian submanifolds L of the LG model with (possible) boundary in a fiber V of w. In the case where w is a Lefschetz fibration, it is well-known (see [40]) that such Lagrangians (so-called Lagrangian thimbles) can be produced by taking appropriate paths between V and p for p a critical value of w and tracing the image of the vanishing cycle at $\mathsf{w}^{-1}(p)$ along this path.

This embedding works as follows. The central fiber of our degeneration is simply a union of two smooth varieties meeting transversally in a K3 surface, so the vanishing cycle is simply an S^1 bundle over the critical locus of the degenerate fiber. In our case, this is simply an S^1 bundle over a K3 surface, which is then homotopic to $S^1 \times \mathrm{K3}$. Thus, along any straight path approaching 0 in $\mathbb{C}$, we have a vanishing thimble homotopic to $D^2 \times \mathrm{K3}$ where D^2 is the two-dimensional disc. This, of course, cannot be a Lagrangian in $\mathrm{LG}(Z_i)$ for dimension reasons, but if instead we take all points in $D^2 \times \mathrm{K3}$ which converge to a Lagrangian ℓ in the K3 surface (in some appropriate sense), then there exists a Lagrangian thimble L_ℓ whose restriction to $\mathsf{w}^{-1}(0)$ is ℓ. In this way, Lagrangians in S extend to admissible Lagrangians in $\mathrm{LG}(X)$ and in particular induce a faithful A_∞-functor from the Fukaya category of S into the Fukaya–Seidel category of $\mathrm{LG}(Z_i)$, both with appropriate symplectic forms. In particular, we have that

(1) There is no admissible Lagrangian L in $\mathrm{LG}(Z_0)$ so that $L_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian torus.
(2) There is no pair of admissible Lagrangians L_1 and L_2 in $\mathrm{LG}(Z_1)$ so that $(L_1)_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian torus and $(L_2)_{|\mathsf{w}^{-1}(0)}$ is a special Lagrangian section of a special Lagrangian fibration on S.

These statements should be viewed as interpretations of Theorem 4.13 in terms of the Fukaya–Seidel category of Z_0, Z_1 and Z_2. As claimed in Sect. 3, the non-existence of a family appropriate Lagrangians in the LG models of Z_0 and Z_1 therefore corresponds to the conjectural fact that Z_0 and Z_1 are non-rational.

5 Hybrid Models and Filtrations

In this section, we introduce a perverse sheaf of categories analog of unramified cohomology - hybrid models [38]. We will associate with this hybrid model a Hodge type filtration - this is the invariant discussed in the main conjecture. Our consideration can be considered as generalizations of classical degenerations in Hodge theory.

5.1 Filtration

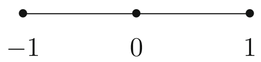
Let $\mathcal{A}$ be an Artinian category and $Y : \mathrm{K}^0(\mathcal{A}) \to \mathbb{R}$ an additive homomorphism.

Theorem 5.1 *For any object E in $\mathcal{A}$, there exists a filtration F_λ with the following properties:*

(1) $\bigcap_{\lambda\in\mathbb{R}} F_{\leqslant\lambda} = 0$*;*
(2) $\bigcup_{\lambda\in\mathbb{R}} F_{\leqslant\lambda} = E$*;*
(3) $F_{\lambda+1}/F_\lambda = \bigoplus G_\alpha$ *is semisimple and splits for every* λ.

Example 5.2

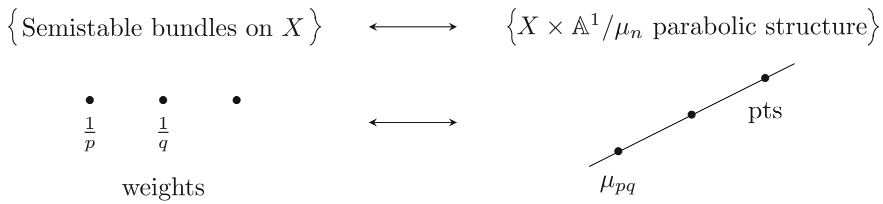
(1) (A_3) $Ob = \mathbb{C}[x]/x^3$.



The filtration here is $-1, 0, 1$.

(2) (A_7) *The filtration here is* $-3, -2, -1, 0, 1, 2, 3$.

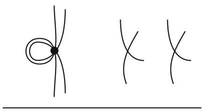
The above filtrations can be given the following interpretation by parabolic structures.



The multiplicity of the divisor over 0 is equal to the common multiple of all denominators. The points on this divisor determine the jumps of the filtration. This geometric interpretation suggests:

Theorem 5.3 *Let* $Cone(a \xrightarrow{\varphi} b)$ *be the cone of* a *and* b *with respect to the functor* φ, *then* $Filt(Cone(a \xrightarrow{\varphi} b)) = superposition(Filt\ a, Filt\ b)$.

One example with such filtration is the symplectic Lefschetz pencils.

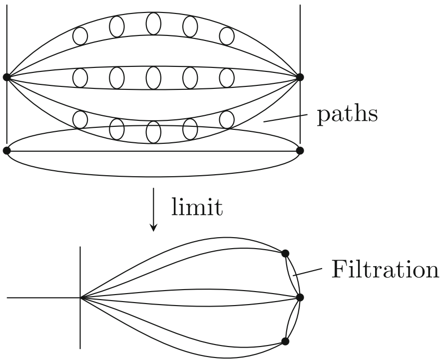


We have a symplectic pencil $(X, [\omega])$ for X a four dimensional compact symplectic manifold. Here $[\omega]$ is the symplectic form on the pencil. A symplectic Lefschetz pencil is defined by a word in the mapping class group.

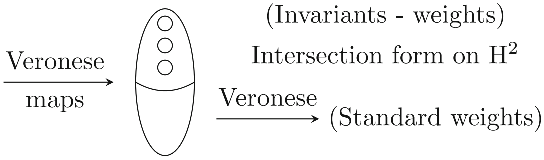
$$\mu : \pi_1(\mathbb{P}^1/\{p_1, \ldots, p_\upsilon\}) \to Map(g)$$

Here $Map(g)$ is the mapping class group of Riemann surfaces of genus g.

We consider a symplectic Lefschetz pencil as a perverse sheaf of categories over $\mathbb{P}^1$. An object in the Fukaya category of this symplectic pencil gives a graph Γ in the base along with a choice of singular Lagrangian in each smooth fiber over Γ. For example:

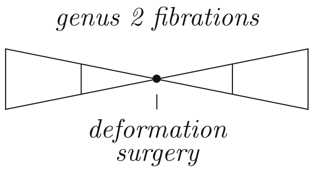


The asymptotic behavior of the above semistable Lagrangian under the mean curvature flow determines a filtration.



The asymptotic behavior of semistable Lagrangians added after the Veronese embedding reduces standard weights and does not affect initial symplectic invariants.

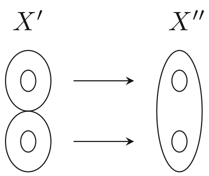
Conjecture 5.4 *The intersection form on* $\mathrm{H}^2(X)$ *determines the filtration.*



$$(\sigma_1, \ldots, \sigma_5)^{24} = 1 \qquad \longleftrightarrow \qquad (\sigma_1, \ldots, \sigma_6)^{20} = 1$$

Each equation determines a semistable Lagrangian. The filtrations associated with the two words in the mapping class group are different. This suggests that the above genus 2 Lefschetz pencils are not symplectomorphic. This is the A side application of our construction.

Our filtrations share many properties with classical weight filtrations. In particular we have the following strictness property.



Theorem 5.5

Consider a fully faithful functor

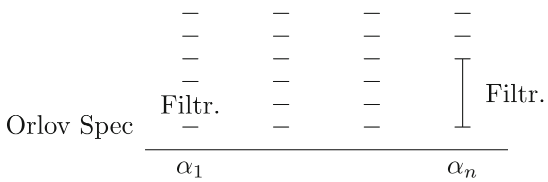
$$\mathcal{F} : \mathrm{FS}(X') \to \mathrm{FS}(X'')$$

so that the induced map $\mathrm{Ext}^1(o') \to \mathrm{Ext}^1(o'')$ *is injective. Here o' and o'' are semi-stable objects in* $\mathrm{FS}(X')$ *and* $\mathrm{FS}(X'')$*. Then we have a compatibility of filtrations under the functor $\mathcal{F}$. (Here* $\mathrm{FS}(X')$*,* $\mathrm{FS}(X'')$ *are 1-dimensional FS categories.)*

Corollary 5.6 *The filtration of $FS(X')$ determines the filtration of $FS(LP)$.*

As a consequence of Theorem 5.5, we can define a filtration for any generator of a category. In fact, we can associate a filtration with a generator corresponding to an element in the Orlov spectrum of a category.

For generator α, Cone($\alpha \xrightarrow{F} T$) $\longrightarrow$ sequence of filtrations on α.



Question 5.7 *Does this sequence of filtrations determine a categorical invariant?*

Now we consider a B side example, [34, 35]. Let X be a smooth projective variety and D a divisor on it. Following [34, 35], we define an object in $\mathrm{D^b}(X)$, $F_k\omega_x(*D)$.

$$j : U = X/D \hookrightarrow X$$

$$F_k\omega_x(*D) = \omega_x(k+1)D \otimes I_k(D) \quad \forall k >> 0.$$

This is an example of filtrations discussed above.

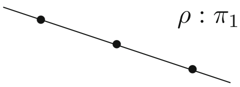
Theorem 5.8 *The above filtration satisfies the cone and functoriality properties.*

Indeed let $H \subset X$ be a hypersurface. So $I_k(D_H) \leq I_k(D) \cdot \mathcal{O}_H$. We also have the cone property:

$$I_k(D_1 + D_2) \subseteq \sum_{i+j=k} I_i(D_1) I_j(D_2) \cdot \mathcal{O}_X(-jD_1 - jD_2).$$

5.2 Hybrid Models

In this section, we take a brief look at the results of Pirutka [38]. Our considerations suggest that there are two new ways of constructing filtrations. Classically we can use the degenerations of cohomologies in order to obtain filtrations.



$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{GL}(\mathrm{H}^3)$$

Nilpotent degenerations produce classical filtrations.

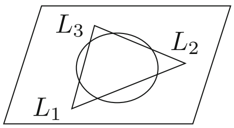
The examples of previous section suggest that we can extend the applications of this method from

$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{GL}(\mathrm{H}^3)$$

to

$$\rho : \pi_1(\mathbb{P}^1/_{p_1,\ldots,p_k}) \to \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathcal{F}_t)).$$

We propose a new possible way to create “interesting filtrations”. We generalize the procedure suggested by A. Pirutka [38].

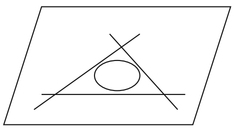


In her approach, Pirutka expresses the existence of nontrivial Brauer group via the combinatorics of the base of the nontrivial Del Pezzo fibration.

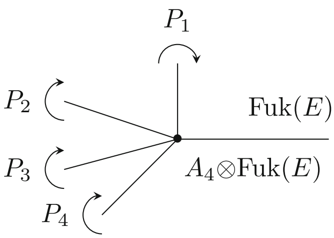
Our considerations in Sect. 4 suggests the following:

Proposition 5.9 *The Pirutka condition can be represented as a filtration on semi-stable objects.*

Now we will look at 4-dimensional quadric bundles. We have a base:



with trivial nonramified cohomology. On the fiber we have a perverse sheaf of cohomology groups (see Sect. 3).



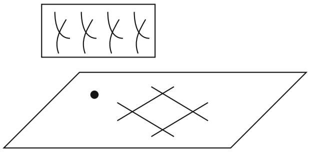
According to Sect. 3, noncommutative deformations can be determined by changing spherical functors. One way of approaching rationality of quadric bundles could be to take a noncommutative deformation of the quadric bundle and compute its invariants. To deform a quadric bundle, one might consider a noncommutative deformation of the quadrics themselves, as described in Sect. 4. We can then try to understand what the unramified cohomology of such an object looks like to deduce non-rationality of the original quadric bundle.

Question 5.10 *Can we find an example of sheaf of noncommutative quadrics such that*

(1) Pirutka's invariant (unramified cohomology) is trivial;
(2) We have nontrivial filtrations on some semistable generator.

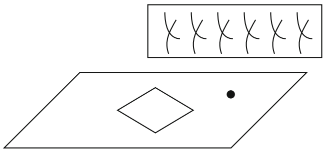
There are two important cases where this approach might bear fruit. These cases correspond to cubics containing extra algebraic cycles, for instance the quadric containing a plane described in Sect. 4.4.

(1) Sheaves of quadrics over $\mathbb{P}^2$.



Question 5.11 *Can we find a deformation of* $\mathrm{D}^{\mathrm{b}}(\mathcal{F}_t)$ *so that non-abelian Pirutka invariant is nontrivial?*

(2) Sheaves of Del Pezzo surfaces.



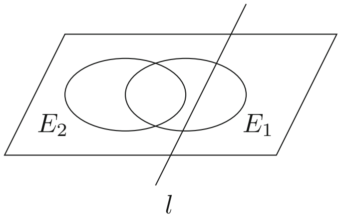
We get a hybrid model over $\mathbb{P}^2$ with fiber $\mathrm{D}^{\mathrm{b}}(\mathcal{F}_t)$ - category of Del Pezzo surfaces.

Question 5.12 *Can we find a deformation of* $\mathrm{D}^{\mathrm{b}}(\mathcal{F}_t)$ *so a noncommutative version of Pirutka's invariant is nontrivial?*

5.3 Artin–Mumford Example

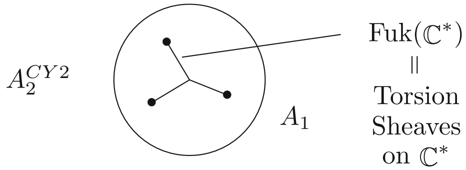
We can also look at the Artin–Mumford example [5] from the perspective of perverse sheaves of categories.

Recall that the classical Artin–Mumford example is a conic bundle over $\mathbb{P}^2$ with curves of degeneration $C = E_1 \cup E_2$, where E_1 and E_2 are smooth degree 3 curves.

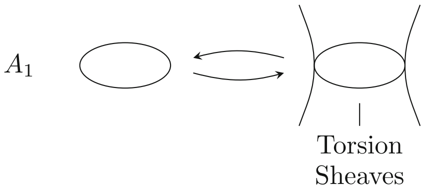


Let l be a line in $\mathbb{P}^2$. Over l we have a conic bundle. This conic bundle itself defines a perverse sheaf of categories as described below.

(⋆)



The spherical functors are functors from A_1 to $\text{Fuk}(\mathbb{C}^*)$, which is just the category of torsion sheaves on $\mathbb{C}^*$.



In terms of representations, we have classically:

$$\rho : \pi_1(\mathbb{P}^1 / \text{pts}) \to \text{GL}\,\text{H}^1(\mathbb{C}^*).$$

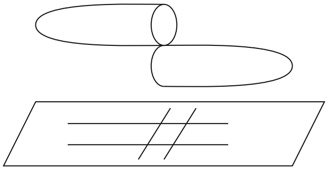
Categorically, our sets of spherical functors give

$$\rho : \pi_1(\mathbb{P}^1 / \text{pts}) \to \text{Aut}\,\text{Fuk}(\mathbb{C}^*).$$

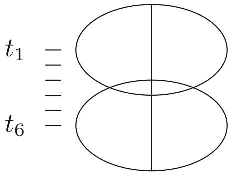
The second representation, along with the braid group representation of monodromy of the curve of degeneration of the Artin–Mumford threefold contains a wealth of information regarding the topology of the Artin–Mumford threefold. Since it is the topology of this threefold which determines its non-rationality, we should be able to recover the main theorem of [5] from this perverse sheaf of categories.

This gives us possibilities for non-commutative deformations. We start with:

(1) Classical Artin–Mumford example.



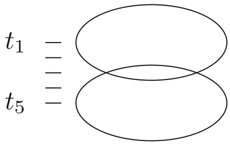
The Pirutka type configuration leads to nontrivial torsion in H^3, [38] (see also [4, 22–24]). Artin–Mumford's construction can be reproduced using the technique of PSC. Instead of the classical monodromy, we use the spherical functors in the PSC to construct a cycle with linking number $\frac{1}{2}$. Here A_3^{CY2} are 2-dimensional CY categories constructed in ($\star$).



$$A_3^{CY2} + A_3^{CY2}$$

This amounts to a semistable Lagrangian with strictly quasi-unipotent monodromy (asymptotics).

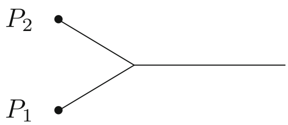
(2) Smooth cubic, compared with 4.3. In this case we start with the hybrid model described below:



$$A_3^{CY2} + A_2^{CY2}$$

The conic bundle has a curve of degeneration consisting of a quadric and a cubic in $\mathbb{P}^2$. The linking number is 0. So the monodromy is strictly unipotent.

(3) Let us consider now the PSC A_2^{CY2} associated with a conic.

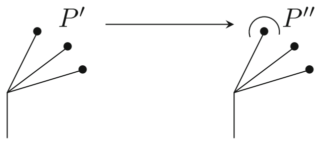


We deform this PSC so that the spherical functor in $\mathbb{P}^2$ does not belong to $GL(H^1)$. In such a way we produce a strictly non-unipotent filtration for the noncommutative deformation of the PSC associated with the quadric. This leads to a nontrivial torsion in H^3, compared with 4.3.

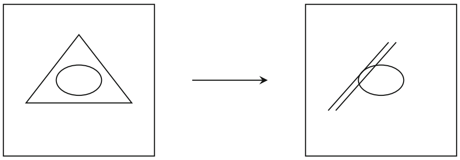
5.4 Conclusions

In conclusion, we can say the following: the construction of hybrid models gives new directions of deforming PSC.

(1) (Monodromy 1) Deforming PSC of the fiber of hybrid model, see Sect. 3.



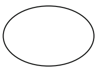
(2) (Monodromy 2) Changing the monodromy of hybrid models, see Sect. 4.



We have a categorical version of unramified cohomology - hybrid models with monodromies and filtrations. The main conjecture states that these filtrations produce new birational invariants. More details will be given elsewhere.

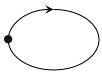
5.5 Final Example

We give one more example. It is known, see [2], that $F_{\text{wrap}}(\mathbb{C}^*) = D^b(\mathbb{C}^*)$. The object $\mathbb{C}(t)/(t-a)^n$ corresponds to a loop with holonomy:

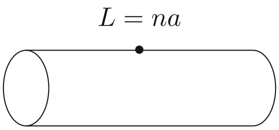


$$\begin{pmatrix} 0 & 1 & & \\ 0 & & 1 & \\ n & & & 0 \end{pmatrix}$$

On the B side we have a quiver with a relation $l^n = 1$:



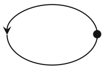
The flow of (E, h) creates a filtration of E.



Here $E = H^0(na)$ and the filtration on E is coming from the action of

$$\begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

On the A side we have

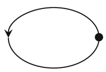


$$\mathrm{HF}(L, L') = \mathrm{H}(na)$$

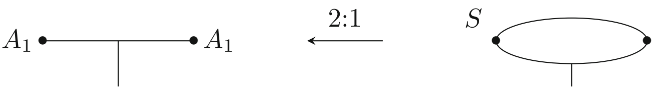
with holonomy

$$\begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}.$$

The above cycle

$S = \Big($  $l^n = 1$, $\begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \Big)$

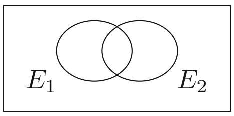
can be seen as a vanishing cycle of the base change of perverse sheaf of categories.



Instead of the vanishing cycle A_1, we have a vanishing cycle S.

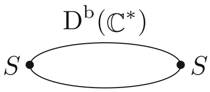
Based on that we propose now a hybrid model associated with the construction in 4.4 - 4-dimensional cubic containing a plane.

(1) We degenerate the sextic in $\mathbb{P}^2$ to the union of two elliptic curves $E_1 \cup E_2$.

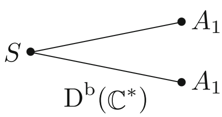


$$\deg E_i = 3$$

(2) We put a sheaf of categories over $\mathbb{P}^2$. Over a point on E_i we put the following category:



and over generic point we put the following:



It is rather clear that the hybrid model above produces nontrivial filtration. It is an intriguing question to use Artin–Mumford's idea in the case of the above hybrid model in order to prove the non-rationality of 4-dim cubics.

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Divisor Classes and the Virtual Canonical Bundle for Genus 0 Maps

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Abstract We prove divisor class relations for families of genus 0 curves and used them to compute the divisor class of the "virtual" canonical bundle of the Kontsevich space of genus 0 maps to a smooth target. This agrees with the canonical bundle in good cases. This work generalizes Pandharipande's results in the special case that the target is projective space, [7] (Pandharipande, Trans. Am. Math. Soc. 351(4), 1481–1505, 1999), [8] (Pandharipande, Trans. Am. Math. Soc. 351(4), 1481–1505, 1999). Our method is completely different from Pandharipande's.

1 Statement of Results

Much geometry of a higher-dimensional complex variety X is captured by the rational curves in X. For uniruled and rationally connected varieties the parameter spaces for rational curves in X are also interesting. These parameter spaces are rarely compact, but there are natural compactifications: the Chow variety, the Hilbert scheme and the Kontsevich moduli space. Of these, the most manageable is the Kontsevich space. For every integer $r \geq 0$, and for every curve class β on X, i.e., for every homomorphism of Abelian groups,

$$\langle -, \beta \rangle : \mathrm{Pic}(X) \to \mathbb{Z}, \;\; D \mapsto \langle D, \beta \rangle,$$

the moduli space $\overline{\mathcal{M}}_{0,r}(X, \beta)$ parametrizes all data $(C, p_1, \ldots, p_r, f)$ of a proper, connected, at-worst-nodal, arithmetic genus 0 curve C, a collection $p_1, \ldots, p_r$ of distinct, smooth points of C, and a morphism $f : C \to X$ with $\deg_C(f^*D) = \langle D, \beta \rangle$

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for all $D \in \mathrm{Pic}(X)$, i.e., $f_*[C]$ equals β, satisfying a natural stability condition: $\mathrm{Aut}(C, p_1, \dots, p_r, f)$ is finite.

One advantage of the Kontsevich space is that it has several natural invertible sheaves together with specified global section (a "pseudodivisor" a la Fulton). The union of the supports of these pseudodivisions is the "boundary" of the Kontsevich space, and each pseudodivisor has a description in terms of Kontsevich spaces with smaller curve classes or a smaller number of marked points. This boundary decomposition provides the basis for several induction arguments. Precisely, for every ordered pair $((\beta', A), (\beta'', B))$ of curve classes (β', β'') such that $\beta' + \beta'' = \beta$ and a partition $\{1, \dots, r\} = A \sqcup B$ such that for $r' = \#A$, $r'' = \#B$, both (β', r') and (β'', r'') equal neither $(0, 0)$ nor $(0, 1)$, there is an invertible sheaf $\mathcal{O}(\Delta_{(\beta',A),(\beta'',B)})$ on $\overline{\mathcal{M}}_{0,r}(X, \beta)$ and there is a global section of $\mathcal{O}(\Delta_{(\beta',A),(\beta'',B)})$ whose zero scheme, denoted $\Delta_{(\beta',A),(\beta'',B)}$ (slight abuse of notation), equals the image of an everywhere unramified morphism

$$\overline{\mathcal{M}}_{0,r'+1}(X, \beta') \times_X \overline{\mathcal{M}}_{0,r''+1}(X, \beta'') \to \overline{\mathcal{M}}_{0,r}(X, \beta),$$

that associates to every pair of stable maps $(C', (p_i)_{i\in A}, q', f')$ and $(C'', (p_j)_{j\in B}, q'', f'')$ with $f'(q') = f''(q'')$ the stable map with $C = C' \cup_{q'\sim q''} C''$ and with f the unique morphism whose restriction to C' equals f', resp. to C'' equals f''. Please note that the image $\Delta_{(\beta',A),(\beta'',B)}$ equals $\Delta_{(\beta'',B),(\beta',A)}$, and indeed, there are canonical isomorphisms of the invertible sheaves $\mathcal{O}(\Delta_{(\beta',A),(\beta'',B)}) \cong \mathcal{O}(\Delta_{(\beta'',B),(\beta',A)})$ that identify the canonical global sections. For this reason, the invertible sheaf and global section are associated to the "unordered" pair $\{(\beta', A), (\beta'', B)\}$. A key step in the proofs is the simple observation that each of these invertible sheaves and pseudodivisors can be defined on an Artin stack of all proper, flat families of *prestable* pointed curves, i.e., connected, at-worst-nodal, arithmetic genus 0 curves with an ordered r-tuple of smooth points with no stability hypothesis.

For every unordered pair $\{\beta', \beta''\}$ with $\beta' + \beta'' = \beta$ and $\beta' \neq 0$, resp., $\beta'' \neq 0$, denote by $\mathcal{O}(\Delta_{\beta',\beta''})$ the tensor product over the finitely many partitions $\{1, \dots, r\} = A \sqcup B$ of the invertible sheaf $\mathcal{O}(\Delta_{(\beta',A),(\beta'',B)})$. The tensor product of the canonical global sections is a canonical global section whose zero scheme has underlying closed set equal to the union of all of the closed sets $\Delta_{(\beta',A),(\beta'',B)}$ as (A, B) varies over all partitions. Similarly, for $r \geq 1$, for every partition $\{1, \dots, r\} = A \sqcup B$, denote by $\mathcal{O}(\Delta_{(A,B)})$ the tensor product over the finitely many pairs (β', β'') of effective curve classes with $\beta' + \beta'' = \beta$ and with $(\beta', r') \neq (0, 0)$, $(\beta'', r'') \neq (0, 0)$ of the invertible sheaf $\mathcal{O}(\Delta_{(\beta',A),(\beta'',B)})$. The tensor product of the canonical global sections is a canonical global section whose zero scheme has underlying closed set equal to the union of all of the closed sets $\Delta_{(\beta',A),(\beta'',B)}$ as (β', β'') varies over all partitions. Every partition is automatically ordered; there is a unique partition set that contains the element 1. We usually denote this partition set by A and denote by B the complementary partition set. Many sums occuring in divisor class relations are indexed by all unordered pairs $\{\beta', \beta''\}$ as above with $\beta' + \beta'' = \beta$. Some other sums are indexed by all partitions (A, B) of $\{1, \dots, r\}$.

Another advantage of the Kontsevich space is that the universal curve over $\overline{\mathcal{M}}_{0,r}(X,\beta)$ is canonically identified with $\overline{\mathcal{M}}_{0,r+1}(X,\beta)$ with the forgetful morphism $\pi : \overline{\mathcal{M}}_{0,r+1}(X,\beta) \to \overline{\mathcal{M}}_{0,r}(X,\beta)$ that forgets the last marked point. With this identification, the universal map f from the universal curve is the same as the evaluation morphism $\overline{\mathcal{M}}_{0,r+1}(X,\beta) \to X$ that evaluates a stable map at the final marked point. In particular, for a pair (β', β'') of curve classes with $\beta' + \beta'' = \beta$ and with $\beta', \beta'' \neq 0$, the pseudodivisor $\Delta_{(\beta',\{1\}),(\beta'',\emptyset)}$ in $\overline{\mathcal{M}}_{0,1}(X,\beta)$ is identified with a pseudodivisor $\widetilde{\Delta}_{\beta',\beta''}$ on the universal curve over $\overline{\mathcal{M}}_{0,0}(X,\beta)$; this pseudodivisor will be defined later in a more general context.

A final advantage of the Kontsevich space is the existence of a *perfect obstruction theory* and associated *virtual fundamental cycle*, cf. [2]. Briefly, the perfect obstruction theory on $M = \overline{\mathcal{M}}_{0,r}(X,\beta)$ is a complex $E^\bullet$ of $\mathcal{O}_M$-modules (for the étale topology) and a morphism in the derived category $\phi : E^\bullet \to L^\bullet_M$, where $L^\bullet_M$ is the cotangent complex. The complex $E^\bullet$ is required to be perfect of amplitude $[-1, 0]$, i.e., everywhere locally quasi-isomorphic to a 2-term complex with locally free terms F^i that are nonzero only in degrees 0 and -1. For the functor h^0, resp. h^{-1}, of cohomology sheaves in degree 0, resp. degree -1, $h^0(\phi)$ is required to be an isomorphism, resp. $h^{-1}(\phi)$ is required to be surjective. The *virtual dimension* of M is the (locally) constant function that equals the difference in the ranks of F^0 and F^{-1}. This is a lower bound on the dimension of every component of M, cf. [6, Theorems II.1.2, II.1.7], and it equals

$$\langle C_1(T_X), \beta\rangle + \dim(X) + r - 3.$$

Because $E^\bullet$ is perfect, via the det-div formalism of [5], there is an associated invertible sheaf $\det(E^\bullet)$ on M that is locally isomorphic to $\det(F^0) \otimes \det(F^{-1})^\vee$. This invertible sheaf is the *virtual canonical bundle*.

When $\overline{\mathcal{M}}_{0,r}(X,\beta)$ is integral and when the dimension equals the virtual dimension, then M is locally a complete intersection, and ϕ determines a unique isomorphism from the virtual canonical bundle to the usual canonical bundle $\det(L^\bullet_M)$. The perfect obstruction theory is amenable to computation, even in those cases when M is not "transverse". In the transverse case, we can ask more refined questions about the geometry of $\overline{\mathcal{M}}_{0,r}(X,\beta)$ and its canonical bundle, e.g., what is the Kodaira dimension? The first step in answering this and other questions is understanding the virtual canonical bundle.

In this article we give a formula for the virtual canonical bundle of $\overline{\mathcal{M}}_{0,r}(X,\beta)$ as a linear combination of more elementary tautological divisor classes. Moreover, we prove a number of divisor class relations among natural divisor classes on $\overline{\mathcal{M}}_{0,r}(X,\beta)$.

Theorem 1.1 *Assume that* $e := \langle C_1(T_X), \beta\rangle \neq 0$. *For* $\overline{\mathcal{M}}_{0,0}(X,\beta)$, *the virtual canonical bundle equals*

$$\frac{1}{2e}[2e\pi_* f^* C_2(T_X) - (e+1)\pi_* f^* C_1(T_X)^2 +$$

$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^* C_1(T_X), \beta' \rangle \langle f^* C_1(T_X), \beta'' \rangle - 4e)\Delta_{\beta',\beta''}].$$

For $\overline{\mathcal{M}}_{0,1}(X, \beta)$, the virtual canonical bundle equals

$$\frac{1}{2e}[2e\pi_* f^* C_2(T_X) - (e+1)\pi_* f^* C_1(T_X)^2 +$$

$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^* C_1(T_X), \beta' \rangle \langle f^* C_1(T_X), \beta'' \rangle - 4e)\Delta_{\beta',\beta''}] + \psi_1.$$

Finally, for $r \geq 2$, the virtual canonical bundle of $\overline{\mathcal{M}}_{0,r}(X, \beta)$ equals

$$\frac{1}{2e}[2e\pi_* f^* C_2(T_X) - (e+1)\pi_* f^* C_1(T_X)^2 +$$

$$\sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} (\langle f^* C_1(T_X), \beta' \rangle \langle f^* C_1(T_X), \beta'' \rangle - 4e)\Delta_{\beta',\beta''}] +$$

$$\frac{1}{r-1} \sum_{(A,B),1\in A} \#B(r - \#B)\Delta_{(A,B)}.$$

In order to prove these formulas, we need to prove some divisor class relations for families of genus 0 curves. These relations are of some independent interest.

Proposition 1.2 *Let $\pi : \mathcal{C} \to M$ be a proper, flat family of connected, at-worst-nodal, arithmetic genus 0 curves over a quasi-projective variety M or over a Deligne–Mumford stack M with quasi-projective coarse moduli space. Let D be a $\mathbb{Q}$-Cartier divisor class on $\mathcal{C}$.*

(i) There is an equality of $\mathbb{Q}$-divisor classes on M

$$\pi_*(D \cdot D) + \langle D, \beta \rangle \pi_*(D \cdot C_1(\omega_\pi)) = \sum_{\{\beta',\beta''\},\beta'+\beta''=\beta} \langle D, \beta' \rangle \langle D, \beta'' \rangle \Delta_{\beta',\beta''}.$$

(ii) Also, there is an equality of $\mathbb{Q}$-divisor classes on $\mathcal{C}$

$$2\langle D, \beta \rangle D - \pi^* \pi_*(D \cdot D) + \langle D, \beta \rangle^2 C_1(\omega_\pi) = \sum_{(\beta',\beta'')} \langle D, \beta'' \rangle^2 \widetilde{\Delta}_{\beta',\beta''}.$$

The pseudodivisors Δ, resp. $\widetilde{\Delta}$, constructed in Sect. 2, are the pseudodivisors on the Artin stack of prestable curves whose restriction to the Kontsevich stack of stable curves is the "usual" boundary pseudodivisor. The invertible sheaf ω_π on $\mathcal{C}$ is the relative dualizing sheaf of π. Finally the bundle ψ_i on M is the pullback of ω_π by the "i^{th} marked point" section.

One motivating problem is to extend these results to the case that the target X is allowed to be singular but with a specified perfect obstruction theory, e.g., X is itself a Kontsevich space. One result in this direction is the following, cf. [4].

Proposition 1.3 ([4, Lemma 2.2]) *Let C be a projective Cohen-Macaulay curve, let $B \subset C$ be a divisor along which C is smooth, and let $f : C \to \overline{\mathcal{M}}_{0,r}(X, \beta)$ be a 1-morphism. Assume that every generic point of C parametrizes a smooth, free curve in X. Then for $Y = \overline{\mathcal{M}}_{0,r}(X, \beta)$*

$$dim_{[f]}Hom(C, Y; f|_B) \geq \langle -K_Y^{virt}, f_*[C]\rangle + dim^{virt}(Y)(1 - p_a(C) - deg(B)).$$

1.1 Outline of the Article

There is a universal family of stable maps over $\overline{\mathcal{M}}_{0,r}(X, \beta)$

$$(\pi : \mathcal{C} \to \overline{\mathcal{M}}_{0,r}(X, \beta), (\sigma_1 : \overline{\mathcal{M}}_{0,r}(X, \beta) \to \mathcal{C})_{i=1,\ldots,r}, f : \mathcal{C} \to X).$$

The Behrend-Fantechi obstruction theory is defined in terms of total derived push-forwards under π of the relative cotangent sheaf of π and the pullback under f of the cotangent bundle of X. Thus the Grothendieck–Riemann–Roch theorem gives a formula for the virtual canonical bundle. Unfortunately it is not a very useful formula. For instance, using this formula it is difficult to determine whether the virtual canonical bundle is NEF, ample, etc. But combined with Proposition 1.2, Grothendieck–Riemann–Roch gives the formula from Theorem 1.1. The main work in this article is proving Proposition 1.2.

The proof reduces to local computations for the universal family over the Artin stack of all prestable curves of genus 0, cf. Sect. 4. Because of this, most results are stated for Artin stacks. This leads to one *ad hoc* construction: since there is as yet no theory of cycle class groups for Artin stacks admitting Chern classes for all perfect complexes of bounded amplitude, a Riemann–Roch theorem for all perfect morphisms relatively representable by proper algebraic spaces, and arbitrary pullbacks for all cycles coming from Chern classes, a stand-in Q_π is used, cf. Sect. 3. (Also by avoiding Riemann–Roch, this allows some relations to be proved "integrally" rather than "modulo torsion").

In the special case $X = \mathbb{P}^n_k$, Pandharipande proved both Theorem 1.1 and Proposition 1.2 in [7, 8]. Pandharipande's work was certainly our inspiration. But our proofs are completely different, yield a more general virtual canonical bundle formula, and hold modulo torsion (and sometimes "integrally") rather than modulo numerical equivalence.

2 Notation for Moduli Spaces and Boundary Divisor Classes

The Relative Picard of the Universal Family of Genus 0 Curves. Denote by $\mathfrak{M}_{0,0}$ the category whose objects are proper, flat families $\rho : C \to M$ of connected, at-worst-nodal, arithmetic genus 0 curves, and whose morphisms are Cartesian diagrams of such families. This category is a smooth Artin stack over Spec $\mathbb{Z}$ (with the flat topology), cf. [1]. Denote by $U_2 \subset \mathfrak{M}_{0,0}$ the open substack parameterizing families for which ρ is everywhere smooth. Denote by $\Delta \subset \mathfrak{M}_{0,0}$ the closed complement of U_2 with its reduced structure. Checking on a smooth atlas, Δ is everywhere locally a reduced normal crossings divisor in $\mathfrak{M}_{0,0}$. Denote by Δ' the singular locus in Δ, and denote by $U_1 \subset \mathfrak{M}_{0,0}$ the open complement of Δ'.

Denote by $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$ the universal family. This is a proper, flat, locally finitely presented 1-morphism of Artin stacks, representable by algebraic spaces. Moreover, π is perfect of Tor dimension [0, 1]. Thus, for every object $\rho : C \to M$ of $\mathfrak{M}_{0,0}$, for every perfect complex $F^\bullet$ of amplitude $[a, b]$, also $R\rho_* F^\bullet$ is perfect of amplitude $[a, b+1]$; in particular $R\rho_*\mathcal{L}$ is perfect of amplitude [0, 1] for every invertible sheaf $\mathcal{L}$ on C. For each perfect complex of bounded amplitude on M, there is a corresponding determinant invertible sheaf, and this is compatible with arbitrary base change of M, cf. [5].

Denote by Pic_π the stack parameterizing proper, flat families of connected, at-worst-nodal, arithmetic genus 0 curves together with a section of the relative Picard functor of the family. This is also an Artin stack, and the natural 1-morphism $\text{Pic}_\pi \to \mathfrak{M}_{0,0}$ is representable by (highly nonseparated) étale group schemes, cf. [9, Proposition 9.3.1].

The stack Pic_π is a countable union of connected open and closed substacks $\text{Pic}^e_\pi \subset \text{Pic}_\pi$ where e is the degree of $\mathcal{L}$ on fibers of ρ. By Riemann–Roch, $R\rho_*\mathcal{L}$ has virtual rank $h^0 - h^1$ equal to $e + 1$. In particular, the subgroup object $\text{Pic}^0_\pi \subset \text{Pic}_\pi$ is quasi-compact over $\mathfrak{M}_{0,0}$, and each $\text{Pic}^e_\pi \to \mathfrak{M}_{0,0}$ is a torsor for Pic^0_π. Moreover, tensoring by ω_π defines an isomorphism of torsors, $\text{Pic}^e_\pi \to \text{Pic}^{e-2}_\pi$. Thus every Pic^e_π is either naturally equivalent to Pic^0_π, or it is naturally equivalent to Pic^{-1}_π.

An Involution of the Relative Picard. The group inverse restricts to an involution ϵ_0 of Pic^0_π that associates to every 1-morphism $M \to \text{Pic}^0_\pi$ coming from a degree 0 invertible sheaf $\mathcal{L}$ on C the 1-morphism corresponding to $\mathcal{L}^\vee$. More generally, for every integer e there is an involution ϵ_e of Pic^e_π that associates to every 1-morphism of an invertible sheaf $\mathcal{L}$ the 1-morphism of the invertible sheaf $\omega_\rho^{-e} \otimes \mathcal{L}^\vee$. The involution ϵ_e is compatible with ϵ_0 for the action of Pic^0_π on Pic^e_π, and it is compatible with the "twisting by ω_π" isomorphisms $\text{Pic}^e_\pi \to \text{Pic}^{e-2}_\pi$ of Pic^0_π-torsors. In particular, on Pic^{-1}_π the involution ϵ_{-1} associates to every invertible sheaf $\mathcal{L}'$ the Serre dual invertible sheaf $\mathcal{L}'' = \omega_\pi \otimes (\mathcal{L}')^\vee$. Since Pic_π is the disjoint union of the components Pic^e_π, there exists a unique involution ϵ_{Pic} of Pic_π that restricts to ϵ_e on each Pic^e_π.

A Universal Invertible Sheaf of Degree Zero. Because a (local) "universal" invertible sheaf is only unique up to tensoring by the pullback of an invertible sheaf $\mathcal{A}$ from the base, there is no universal invertible sheaf on $\mathrm{Pic}^{-1}_{\pi} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$. Nonetheless, for every invertible sheaf $\mathcal{A}$ on Pic^0_{π}, relative to the projection

$$\mathrm{pr}_1 : \mathrm{Pic}^0_{\pi} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathrm{Pic}^0_{\pi},$$

there exists a unique invertible sheaf $\mathcal{L}$ inducing the universal section of the relative Picard and such that $\det(R\mathrm{pr}_{1,*}\mathcal{L})$ equals $\mathcal{A}$. Indeed, for every invertible sheaf $\mathcal{L}$ of degree 0, since $\det(R\mathrm{pr}_{1,*}(\mathcal{L} \otimes \mathrm{pr}_1^*\mathcal{A}))$ equals $\det(R\mathrm{pr}_{1,*}\mathcal{L}) \otimes \mathcal{A}$, this allows us to normalize the universal invertible sheaf.

To calibrate the universal invertible sheaf further, observe that pr_1 is representable, fppf, and, therefore, also universally open. There is a unique choice of universal invertible sheaf $\mathcal{O}(\mathcal{D})$ of relative degree 0 on $\mathrm{Pic}^0_{\pi} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ that comes from an effective Cartier divisor $\mathcal{D}$ whose open complement W has open image $\mathrm{pr}_1(W)$ containing the open substack $\mathrm{Pic}^0_{\pi} \times_{\mathfrak{M}_{0,0}} U_1$, i.e., $\mathcal{D}$ contains no fiber of $\mathfrak{C}$ over $\mathrm{Pic}^0_{\pi} \times_{\mathfrak{M}_{0,0}} U_1$ (it does contain fibers over some generic points of Δ').

A Basis for the Relative Picard. For every algebraically closed field k and for every at-worst-nodal genus 0 k-curve C, $\mathrm{Pic}(C)$ is a free Abelian group with a finite basis. There is an open substack of Pic_{π} that gives such a basis for every geometric point of $\mathfrak{M}_{0,0}$ as above. Denote by $\iota' : \widetilde{\mathfrak{M}_{0,0}} \subset \mathrm{Pic}_{\pi}$ the open substack with the following universal property. For every object $\rho : C \to M$ of $\mathfrak{M}_{0,0}$, for every invertible sheaf $\mathcal{L}'$ on C and the associated 1-morphism $M \to \mathrm{Pic}_{\pi}$, denoting by $\mathcal{L}''$ the Serre dual invertible sheaf, $\omega_{\rho} \otimes (\mathcal{L}')^{\vee}$, the inverse image of $\widetilde{\mathfrak{M}_{0,0}}$ is the maximal open subscheme of M on which each of the following coherent $\mathcal{O}_M$-modules is zero: $R^1\rho_*(\mathcal{L}')$, $R^1\rho_*(\mathcal{L}'')$, and $R^1\rho_*(\mathcal{L}'' \otimes (\mathcal{L}')^{\vee})$. Similarly, denote by $\iota'' : \widetilde{\mathfrak{M}_{0,0}} \to \mathrm{Pic}_{\pi}$ the 1-morphism that is Serre dual to ι', i.e., coming from $\mathcal{L}''$ instead of $\mathcal{L}'$. By Riemann–Roch and the classification of invertible sheaves on $\mathbb{P}^1_k$, both $\mathcal{L}'$ and $\mathcal{L}''$ are invertible sheaves of relative degree -1, i.e., both ι' and ι'' factor through Pic^{-1}_{π}.

Compatibility with ϵ_{-1} over U_1. Please note, even if the ordered pair of Serre dual invertible sheaves $(\mathcal{L}', \mathcal{L}'')$ satisfies the H^1-vanishing hypothesis above, typically the ordered pair $(\mathcal{L}'', \mathcal{L}')$ does not satisfy the H^1-vanishing hypothesis. However, this does hold over the open substack $\widetilde{U}_1 = \widetilde{\mathfrak{M}_{0,0}} \times_{\mathfrak{M}_{0,0}} U_1$, and the involution ϵ_{-1} above restricts to an involution ϵ on $\widetilde{U}_1$ pulling back $(\mathcal{L}', \mathcal{L}'')$ to $(\mathcal{L}'', \mathcal{L}')$ up to tensoring by pullbacks of invertible sheaves from the base. For the inverse image $\widetilde{U}_2$ of U_2 in $\widetilde{\mathfrak{M}_{0,0}}$, the projection $\widetilde{U}_2 \to U_2$ is an isomorphism, so we identify these two stacks. Since $\widetilde{\mathfrak{M}_{0,0}}$ is an open substack of Pic_{π}, which is itself étale over $\mathfrak{M}_{0,0}$, also $\widetilde{\mathfrak{M}_{0,0}} \to \mathfrak{M}_{0,0}$ is representable and étale (although highly non-separated). For every 1-morphism $T \to \mathfrak{M}_{0,0}$, denote by $\widetilde{T} \to \widetilde{\mathfrak{M}_{0,0}}$ the 2-fibered product of T with $\widetilde{\mathfrak{M}_{0,0}}$ over $\mathfrak{M}_{0,0}$. In particular, denote by $\widetilde{\pi} : \widetilde{\mathfrak{C}} \to \widetilde{\mathfrak{M}_{0,0}}$ the base change of $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$. Similarly, denote by $\widetilde{\Delta}$, the base change to $\widetilde{\mathfrak{M}_{0,0}}$ of Δ. The representable 1-morphism $\widetilde{U}_1 \to U_1$ is the universal categorical quotient by the associated action ϵ of $\mathfrak{S}_2$. The involution

ϵ does not extend over Δ'. The 1-morphism $\widetilde{U_1 \cap \Delta} \to U_1 \cap \Delta$ is finite and flat of degree 2, an $\mathfrak{S}_2$-torsor for the free action of ϵ.

Relation to Components of Fibers of π. The smooth locus of the universal curve gives a smooth atlas of $\widetilde{\mathfrak{M}_{0,0}}$ as follows. Denote by $\mathfrak{C}^o$ the open substack of $\mathfrak{C}$ that is the smooth locus of the 1-morphism π. Denote by $\pi^o : \mathfrak{C}^o \to \mathfrak{M}_{0,0}$ the restriction of π. The diagonal 1-morphism $\Delta_\pi^o : \mathfrak{C}^o \to \mathfrak{C}^o \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ is representable by closed immersions. The corresponding ideal sheaf $\mathcal{I}$ is an invertible sheaf that fits into a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathfrak{C}^o \times_{\mathfrak{M}_{0,0}} \mathfrak{C}} \longrightarrow \Delta_* \mathcal{O}_{\mathfrak{C}^o} \longrightarrow 0.$$

Applying the long exact sequence of higher direct images and relative duality, all of $R^1\mathrm{pr}_{1,*}\mathcal{I}$, $R^1\mathrm{pr}_{1,*}(\omega_\pi \otimes \mathcal{I}^\vee)$, and $R^1\mathrm{pr}_{1,*}(\omega_\pi \otimes (\mathcal{I}^\vee)^{\otimes 2})$ equal zero. Thus, for the invertible sheaf $\mathcal{I}$, there is an associated 1-morphism $\widetilde{\zeta} : \mathfrak{C}^o \to \widetilde{\mathfrak{M}_{0,0}}$ compatible with the given 1-morphisms to $\mathfrak{M}_{0,0}$ (each of these morphisms is representable by algebraic spaces, so the compatibility is strict). By Lemma 2.5, $\widetilde{\zeta}$ is smooth and faithfully flat, locally constant on geometric fibers of π^o, distinguishing distinct connected components of geometric fibers. Thus, $\widetilde{\mathfrak{M}_{0,0}} \subset \mathrm{Pic}_\pi$ is a "basis" for Pic_π as a group scheme over $\mathfrak{M}_{0,0}$. The 1-morphisms ι' and ι'', in particular, define two sections of $\widetilde{\mathrm{Pic}}_\pi = \mathrm{Pic}_{\widetilde{\pi}}$, and thus define a morphism of locally finitely presented, étale, commutative group objects over $\widetilde{\mathfrak{M}_{0,0}}$,

$$(\iota', \iota'') : (\mathbb{Z}^{\oplus 2}) \times \widetilde{\mathfrak{M}_{0,0}} \to \mathrm{Pic}_{\widetilde{\pi}}.$$

The section corresponding to $(1, 0) \in \mathbb{Z}^{\oplus 2}$ is ι', and the section corresponding to $(0, 1)$ is ι''. Thus, for example, the section corresponding to $(1, 1)$ comes from the invertible sheaf ω_π, and thus is the base change to $\widetilde{\mathfrak{M}_{0,0}}$ of a section of $\mathrm{Pic}_\pi \to \mathfrak{M}_{0,0}$. In particular, for integers (e', e''), the section corresponding to $(-e', -e'')$ is the base change of a section of Pic_π if and only if e'' equals e'.

Smooth Atlases for Some Opens. There are smooth atlases for $\widetilde{U}_1$, U_1, and U_2 as follows. For U_2, the family $\pi : \mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}\,\mathbb{Z}$ defines a 1-morphism $\zeta_2 : \mathrm{Spec}\,\mathbb{Z} \to U_2$. The 1-morphism ζ_2 is representable, smooth, and surjective. The 2-fibered product

$$\mathrm{Spec}\,\mathbb{Z} \times_{\zeta_2, U_2, \zeta_2} \mathrm{Spec}\,\mathbb{Z} = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{Z}})$$

is the group scheme $\mathbf{PGL}_2$ with its natural action on $\mathbb{P}^1_{\mathbb{Z}}$. Thus U_2 is isomorphic to the quotient stack $[\mathrm{Spec}\,\mathbb{Z}/\mathbf{PGL}_2]$. For this atlas, the unique lift to $\widetilde{\mathfrak{M}_{0,0}}$ comes from the pair $(\mathcal{L}', \mathcal{L}'') = (\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1), \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1))$. There is no compatible $\mathbf{PGL}_2$-linearization of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1)$, so that there is no invertible sheaf of relative degree -1 on $\widetilde{U}_2 \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$.

There is a similar atlas for U_1 and $\widetilde{U}_1$. Let $V = \mathbb{Z}\{\mathbf{e}_0, \mathbf{e}_1\}$ be a free module of rank 2. Choose dual coordinates y_0, y_1 for $V^\vee$. Let $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{P}(V)$ be the projective space with homogeneous coordinates y_0, y_1. Let $\mathbb{A}^1_{\mathbb{Z}}$ be the affine space with coordinate x.

Denote by $Z \subset \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ the closed subscheme $\mathbb{V}(x, y_1)$, i.e., the image of the section $(0, [1, 0])$. Let $\nu : C \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ be the blowing-up along Z. Denote by $E \subset C$ the exceptional divisor. Define $\pi : C \to \mathbb{A}^1_{\mathbb{Z}}$ to be $\text{pr}_{\mathbb{A}^1} \circ \nu$. This family defines a 1-morphism $\zeta_1 : \mathbb{A}^1_{\mathbb{Z}} \to U_1$. This 1-morphism is representable, smooth, and surjective. The 2-fibered product

$$\mathbb{A}^1_{\mathbb{Z}} \times_{\zeta_1, U_1, \zeta_1} \mathbb{A}^1_{\mathbb{Z}} = \text{Isom}_{\mathbb{A}^2_{\mathbb{Z}}}(\text{pr}_1^* C, \text{pr}_2^* C)$$

restricted over the open $\mathbb{G}_m \times \mathbb{G}_m \subset \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ is the group scheme $\mathbf{PGL}_2 \times (\mathbb{G}_m \times \mathbb{G}_m)$. The restriction of the 2-fibered product over the origin Spec $\mathbb{Z} \subset \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ is the wreath product $G_0 := (B \times B) \rtimes \mathfrak{S}_2$. Here $B \subset \mathbf{PGL}_2$ is a Borel subgroup, i.e., the stabilizer of a point in $\mathbb{P}^1$, and $\mathfrak{S}_2$ acts by interchanging the two components of C_0. Thus the stack $U_1 \cap \Delta$ is isomorphic to the quotient stack [Spec $\mathbb{Z}/G_0$]. Note that the invertible sheaf $\mathcal{L}' = \nu^* \text{pr}^*_{\mathbb{P}^1_{\mathbb{Z}}} \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(-1)$ and its Serre dual invertible sheaf $\mathcal{L}'' = \omega_\pi \otimes (\mathcal{L}')^\vee \cong \mathcal{L}'(\underline{E})$ have $H^1(C, \mathcal{L}')$, $H^1(C, \mathcal{L}'')$, and $H^1(C, \mathcal{L}'' \otimes (\mathcal{L}')^\vee)$ all equal to zero. Up to isomorphism, the only ordered pairs of Serre dual invertible sheaves with this property are $(\mathcal{L}', \mathcal{L}'')$ and $(\mathcal{L}'', \mathcal{L}')$. The action of $\mathfrak{S}_2$ interchanges these pairs.

The lifted 1-morphism $\widetilde{\zeta}_1 : \mathbb{A}^1_{\widetilde{\mathbb{Z}}} \to \widetilde{U}_1$ is again representable, smooth, and surjective. The 2-fibered product for $\widetilde{\zeta}_1$ agrees with the 2-fibered product of ζ_1 over $\mathbb{G}_m \times \mathbb{G}_m$, yet the restriction over the origin is the normal subgroup $B \times B$ of G_0. Thus the stack $\widetilde{U_1 \cap \Delta}$ is isomorphic to the quotient stack [Spec $\mathbb{Z}/(B \times B)$]. Although we shall never use this, in fact $\widetilde{U}_1$ is a global quotient stack $[\widehat{\mathbf{PGL}}_2/(\mathbf{PGL}_2 \times \mathbf{PGL}_2)]$, where $\widehat{\mathbf{PGL}}_2$ is the wonderful compactification, i.e., $\mathbb{P}^3_{\mathbb{Z}}$, and $\mathbf{PGL}_2 \times \mathbf{PGL}_2$ acts by both left and right multiplication. The "matrix adjugate" (which happens to be homogeneous of degree 1 for 2×2 matrices) defines an involution $\epsilon : \widehat{\mathbf{PGL}}_2 \to \widehat{\mathbf{PGL}}_2$ extending the involution $a \mapsto a^{-1}$ on $\mathbf{PGL}_2$. This involution induces an action on $\widehat{\mathbf{PGL}}_2$ of the wreath product $(\mathbf{PGL}_2 \times \mathbf{PGL}_2) \rtimes \mathfrak{S}_2$. The stack $U_1 \cap \Delta$ is the quotient stack associated to this action restricted to the boundary divisor of $\widehat{\mathbf{PGL}}_2$.

Description of the Relative Picard over $\widetilde{U}_1$. Although there is no universal invertible sheaf of degree -1 on $U_2 \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$, nonetheless, there is an isomorphism of $\text{Pic}_\pi \times_{\mathfrak{M}_{0,0}} U_2$ with the group object $\mathbb{Z} \times U_2$ (the constant étale group object with fiber group $\mathbb{Z}$). We normalize this isomorphism so that the section 1 of $\mathbb{Z}$ corresponds to the 1-morphism $\iota' : U_2 \to \text{Pic}_\pi^{-1}$.

Using the atlas, the group object $\text{Pic}_\pi \times_{\mathfrak{M}_{0,0}} (U_1 \cap \Delta)$ over $U_1 \cap \Delta$ pulls back to the rank 2 group object $(\mathbb{Z}^{\oplus 2}) \times$ Spec $\mathbb{Z}$ over Spec $\mathbb{Z}$, but with a nontrivial action of the wreath product $G_0 = (B \times B) \rtimes \mathfrak{S}_2$ on $\mathbb{Z}^{\oplus 2}$ where $\mathfrak{S}_2$ sends (e', e'') to (e'', e'). In particular, the pullback to $\widetilde{U_1 \cap \Delta}$ is a constant group object, and the 1-morphism

$$(\iota', \iota'') : (\mathbb{Z}^{\oplus 2}) \times \widetilde{U_1 \cap \Delta} \to \text{Pic}_{\widetilde{\pi}}$$

is an isomorphism of relative group schemes over $\widetilde{U_1 \cap \Delta}$. In particular, for every pair of integers (e', e''), the image $\Delta_{e',e''}$ of the section $(\iota', \iota'')(-e', -e'')$ is an effective Cartier divisor in $\mathrm{Pic}^e_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_2}$ for $e = e' + e''$.

Description of the Involution ϵ_e over $\widetilde{U}_1$. The involution ϵ on $\widetilde{U}_1$ and the involution ϵ_e on Pic^e_π are compatible as follows, for every pair of integers (e', e'') with $e' + e'' = e$, for the 1-morphism $(\iota', \iota'')(-e', -e'') : \widetilde{U}_1 \to \mathrm{Pic}^e_\pi$ over $\mathfrak{M}_{0,0}$, all of the following 1-morphisms over $\mathfrak{M}_{0,0}$ are equal,

$$(\iota', \iota'')(-e', -e'') \circ \epsilon = (\iota'', \iota')(-e', -e'') = (\iota', \iota'')(-e'', -e') = \epsilon_e \circ (\iota', \iota'')(-e', -e'').$$

Thus, both $\epsilon \times \mathrm{Id}$ and $\mathrm{Id} \times \epsilon_e$ define the same involution $\widetilde{\epsilon}_e$ of $\mathrm{Pic}^e_\pi \times_{\mathfrak{M}_{0,0}} \widetilde{U}_1 = \mathrm{Pic}^e_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U}_1$ that commutes both with ϵ via pr_1 and with ϵ_e via pr_2. Since $\mathrm{Pic}_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$ is the disjoint union of the components $\mathrm{Pic}^e_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$, there is a unique involution $\widetilde{\epsilon}_{\mathrm{Pic}}$ restricting to $\widetilde{\epsilon}_e$ on every component. In particular, the restriction $\mathrm{Pic}^e_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1 \cap \Delta} \to \mathrm{Pic}^e_\pi \times_{\mathfrak{M}_{0,0}} (U_1 \cap \Delta)$ is an $\mathfrak{S}_2$-torsor for the action of $\epsilon \times \mathrm{Id}$.

Extending and Descending Cartier Divisors. Since Pic^e_π, resp. $\mathrm{Pic}^e_{\widetilde{\pi}}$, is a smooth Artin stack, and since the complement of U_1, resp. $\widetilde{U_1}$, has codimension 2, every effective Cartier divisor on $\mathrm{Pic}^e_\pi \times_{\mathfrak{M}_{0,0}} U_1$, resp. on $\mathrm{Pic}^e_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$, extends uniquely to a Cartier divisor on all of Pic^e_π, resp. on all of $\mathrm{Pic}^e_{\widetilde{\pi}}$. Thus each of the effective divisors $\Delta_{e',e''}$ extends to an effective Cartier divisor on all of $\mathrm{Pic}^e_{\widetilde{\pi}}$, also denoted by $\Delta^e_{e',e''}$. Since $\mathrm{Pic}^e_{\widetilde{\pi}}$ is an open and closed substack of $\mathrm{Pic}_{\widetilde{\pi}}$, this is also an effective Cartier divisor on $\mathrm{Pic}_{\widetilde{\pi}}$ that is trivial on every $\mathrm{Pic}^d_{\widetilde{\pi}}$ with $d \neq e$. Also, an effective Cartier divisor on $\mathrm{Pic}_{\widetilde{\pi}}$ is the pullback of an effective Cartier divisor on Pic_π if and only if the restriction of the divisor to $\mathrm{Pic}_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$ is the pullback of an effective Cartier divisor on $\mathrm{Pic}_\pi \times_{\mathfrak{M}_{0,0}} U_1$. Since $\mathrm{Pic}_\pi \times_{\mathfrak{M}_{0,0}} U_1$ is the quotient of $\mathrm{Pic}_{\widetilde{\pi}} \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$ by the $\mathfrak{S}_2$-action, a Cartier divisor is a pullback if and only if it is invariant under this action.

The Tautological Boundary Divisors. The collection of Cartier divisors $(\Delta_{e',e''})$ for $(e', e'') \in \mathbb{Z}^{\oplus 2}$ is locally finite: every quasi-compact open subset of $\mathrm{Pic}_{\widetilde{\pi}}$ intersects the support of only finitely many of these divisors. Thus, for every function,

$$g : \mathbb{Z}^{\oplus 2} \to \mathbb{Z},$$

there is a well-defined Cartier divisor

$$\sum_{(e',e'') \in \mathbb{Z}^{\oplus 2}} g(e', e'') \Delta_{e',e''}$$

on $\mathrm{Pic}_{\widetilde{\pi}}$. The pullback of this divisor under $\widetilde{\epsilon}_{\mathrm{Pic}}$ equals $\sum_{(e',e'')} g(e'', e') \Delta_{e',e''}$. Thus the divisor equals the pullback of a divisor on Pic_π if and only if $g(e'', e')$ equals $g(e', e'')$ for every $(e', e'') \in \mathbb{Z}^{\oplus 2}$.

Concretely, for every object $\rho : C \to M$, for every invertible sheaf $\mathcal{L}' = \mathcal{L}$ defining a 1-morphism to $\widetilde{\mathfrak{M}_{0,0}}$, for the Serre dual invertible sheaf $\mathcal{L}'' = \omega_\rho \otimes \mathcal{L}^\vee$, and for every invertible sheaf $\mathcal{N}$ on C defining a morphism $M \to \mathrm{Pic}_{\widetilde{\pi}}$, the inverse image in M of $\Delta_{e',e''}$ is a pseudo-divisor whose support is the locus in Δ over which $\mathcal{N}$ is isomorphic to $(\mathcal{L}')^{\otimes -e'} \otimes (\mathcal{L}'')^{\otimes -e''}$ up to pullback of an invertible sheaf from M, i.e., the degree of $\mathcal{N}$ distributes as (e', e'') over the two connected subcurves of the fiber of ρ whose union equals the fiber and that intersect in a single node of the fiber.

The Universal Boundary Divisor on $\widetilde{\mathfrak{M}_{0,0}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$. The 1-morphism $(\iota', \iota'')(1,-1) : \widetilde{\mathfrak{M}_{0,0}} \to \mathrm{Pic}^0_\pi$ associates to every invertible sheaf $\mathcal{L}'$ of relative degree -1 the invertible sheaf $(\mathcal{L}'' \otimes (\mathcal{L}')^\vee)^\vee = \omega_\pi^\vee \otimes (\mathcal{L}')^{\otimes 2}$ of relative degree 0. The 2-fiber product of this 1-morphism with $\mathrm{Id}_{\mathfrak{C}}$ is a 1-morphism $\widetilde{\mathfrak{M}_{0,0}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$. In particular, the universal invertible sheaf $\mathcal{O}(\mathcal{D})$ and the associated Cartier divisor $\mathcal{D}$ on $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ pulls back via this 1-morphism to an invertible sheaf $\mathcal{O}(\mathcal{D}')$ and a Cartier divisor $\mathcal{D}'$ on $\widetilde{\mathfrak{M}_{0,0}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ of relative degree 0 over $\widetilde{\mathfrak{M}_{0,0}}$, resp. with support contained in $\widetilde{\Delta} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$. Similarly, the pullback by the 1-morphism $(\iota', \iota'')(-1, 1)$ gives a Cartier divisor $\mathcal{D}''$ with support contained in $\widetilde{\Delta} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$. In particular, $\mathcal{D}' + \mathcal{D}''$ equals $\widetilde{\Delta} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ as effective Cartier divisors on $\widetilde{\mathfrak{M}_{0,0}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$.

Ordered Tuples of Divisors. All of these constructions extend to r-tuples of divisor classes. For every integer $r \geq 0$, denote by $\Pi^r(\mathrm{Pic}_\pi)$, resp. by $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}})$, the r-fold 2-fibered product of Pic_π over $\mathfrak{M}_{0,0}$, resp. the r-fold 2-fibered product of $\mathrm{Pic}_{\widetilde{\pi}}$ with itself over $\widetilde{\mathfrak{M}_{0,0}}$. Equivalently, $\Pi^r(\mathrm{Pic}_\pi)$, resp. $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}})$, is the stack of families of genus 0 curves, and an ordered r-tuple of sections of the relative Picard functor of the family, resp., this data together with a lifting $\widetilde{\zeta}$ of the 1-morphism $\zeta : M \to \mathfrak{M}_{0,0}$ to $\widetilde{\mathfrak{M}_{0,0}}$. For every r-tuple of integers $(e_1, \dots, e_r)$, there is an open and closed substack of $\Pi^r(\mathrm{Pic}_\pi)$ that is the r-fold 2-fibered product of $\mathrm{Pic}^{e_1}_\pi, \dots, \mathrm{Pic}^{e_r}_\pi$. The homomorphism (ι', ι'') defines a morphism of representable group objects over $\widetilde{\mathfrak{M}_{0,0}}$,

$$(\iota', \iota'', \dots, \iota', \iota'') : \mathbb{Z}^{\oplus 2r} \times \widetilde{\mathfrak{M}_{0,0}} \to \Pi^r(\mathrm{Pic}_{\widetilde{\pi}}),$$

that maps $(-e'_1, e''_1, \dots, e'_r, e''_r)$ to the component with $(e_1, \dots, e_r) = (e'_1 + e''_1, \dots, e'_r + e''_r)$. The involution ϵ_{Pic} induces an involution of $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}})$ that corresponds to the involution of $\mathbb{Z}^{\oplus 2r}$ by $(e'_1, e''_1, \dots, e'_r, e''_r) \mapsto (e''_1, e'_1, \dots, e''_r, e'_r)$. With respect to this involution, $\Pi^r(\mathrm{Pic}_\pi) \times_{\mathfrak{M}_{0,0}} U_1$ is the universal geometric quotient of $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}}) \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1}$, and the 1-morphism $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}}) \times_{\widetilde{\mathfrak{M}_{0,0}}} \widetilde{U_1 \cap \Delta} \to \Pi^r(\mathrm{Pic}_\pi) \times_{\mathfrak{M}_{0,0}} (U_1 \cap \Delta)$ is an $\mathfrak{S}_2$-torsor under this involution.

For every $(e'_1, e''_1, \dots, e'_r, e''_r)$ in $\mathbb{Z}^{\oplus 2r}$ define $\Delta_{(e'_1, e''_1, \dots, e'_r, e''_r)}$ to be the iterated 2-fiber product

$$\Delta_{(e'_1, e''_1, \dots, e'_r, e''_r)} := \Delta_{(e'_1, e''_1)} \times_{\widetilde{\mathfrak{M}_{0,0}}} \cdots \times_{\widetilde{\mathfrak{M}_{0,0}}} \Delta_{(e'_r, e''_r)}$$

as an effective Cartier divisor in $\Pi^r(\mathrm{Pic}_{\widetilde{\pi}})$ contained in the component of $(e'_1 + e''_1, \dots, e'_r + e''_r)$. As above, for every function,

$$g : \mathbb{Z}^{\oplus 2r} \to \mathbb{Z},$$

there is a well-defined Cartier divisor

$$\sum_{(e'_1, e''_1, \dots, e'_r, e''_r) \in \mathbb{Z}^{\oplus 2r}} g(e'_1, e''_1, \dots, e'_r, e''_r) \Delta_{(e'_1, e''_1, \dots, e'_r, e''_r)}$$

on $\Pi^r(\text{Pic}_{\widetilde{\pi}})$. Also, this is the pullback of a Cartier divisor on $\Pi^r(\text{Pic}_\pi)$ if and only if g is invariant under the involution induced by ϵ_{Pic}, $(e'_1, e''_1, \dots, e'_r, e''_r) \mapsto (e''_1, e'_1, \dots, e''_r, e'_r)$. When g is invariant, denote by

$$\sum_{(e'_1, e''_1, \dots, e'_r, e''_r)}{}' g(e'_1, e''_1, \dots, e'_r, e''_r) \Delta_{(e'_1, e''_1, \dots, e'_r, e''_r)},$$

the corresponding divisor on $\Pi^r(\text{Pic}_\pi)$.

Now let $\rho : C \to M$ be a flat 1-morphism representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0, and assume that M is connected. This defines a 1-morphism $\xi_0 : M \to \mathfrak{M}_{0,0}$. For every invertible sheaf $\mathcal{L}$ on C, $R\rho_*\mathcal{L}$ is a perfect complex of amplitude $[0, 1]$ whose virtual rank $h^0 - h^1$ equals an integer denoted $1 + \langle c_1(\mathcal{L}), \beta \rangle$. The 1-morphism $M \to \text{Pic}_\pi$ associated to $\mathcal{L}$ has image in Pic^e_π for $e = \langle c_1(\mathcal{L}), \beta \rangle$. The rule $\mathcal{L} \mapsto \langle c_1(\mathcal{L}), \beta \rangle$ is a group homomorphism

$$\beta : \text{Pic}(C) \to \mathbb{Z},$$

i.e., it is a curve class on C. The pullback by ξ_0 of the Cartier divisor Δ is a pseudo-divisor $\xi_0^*\Delta$ on M whose support equals the image under ρ of the singular locus of ρ. For every connected component of the support, say $(\xi_0^*\Delta)_j$, there is an associated pseudodivisor, particularly an associated invertible sheaf $\xi_0^*\mathcal{O}(\Delta)_j$ on M. Let $\widetilde{\xi_0} : M \to \widetilde{\mathfrak{M}_{0,0}}$ be a lift of ξ_0. The inverse image $\widetilde{\xi_0}^*(\mathcal{D}')$ is a pseudodivisor on C that is contained in $\rho^*\xi_0^*\Delta$. For every connected component $(\xi_0^*\Delta)_j$, there is an associated connected component $\widetilde{\xi_0}^*(\mathcal{D}')_j$ (since $\mathcal{D}'$ has connected fibers over $\widetilde{\Delta}$). As above $R\rho_*(\mathcal{L} \otimes \mathcal{O}(\widetilde{\xi_0}^*(-\mathcal{D}')_j))$ is a perfect complex whose virtual rank is an integer $\langle c_1(\mathcal{L}), \beta \rangle - \langle c_1(\mathcal{L}), \beta'_j \rangle = e - e'$ for a unique integer e', also denoted by $\langle c_1(\mathcal{L}), \beta' \rangle$. Denote $e - e'$ by e''. Locally near $(\xi_0^*\Delta)_j$, the 1-morphism $M \to \text{Pic}^e_\pi$ associated to $\mathcal{L}$ factors as the composition of $\widetilde{\xi_0}$ and the section $(\iota', \iota'')(-e', -e'')$. The rule $\mathcal{L} \mapsto \langle c_1(\mathcal{L}), \beta'_j \rangle$ is again a group homomorphism, i.e., it is a curve class β'_j. Denote the curve class $\beta - \beta'_j$ by β''_j. For every pair of curve classes (β', β'') with $\beta' + \beta'' = \beta$, denote by $\Delta_{\beta', \beta''}$ the pseudodivisor on M that is the sum of the pseudo-divisors $(\xi_0^*\Delta)_j$ over precisely those connected components such that (β'_j, β''_j) equals (β', β''). The collection of pseudodivisors $(\Delta_{\beta', \beta''})$ as (β', β'') varies over all pairs of curve classes is locally finite. The pseudodivisor $\Delta_{\beta', \beta''}$ depends on the choice of lift $\widetilde{\xi_0}$; for a different lift, $(\xi_0^*\Delta)_j$ may become part of $\Delta_{\beta'', \beta'}$ rather than $\Delta_{\beta', \beta''}$. However, if β'' equals β', then $\Delta_{\beta', \beta'}$ is independent of the choice of lift. Similarly, the sum of the two pseudodivisors $\Delta_{\beta', \beta''} + \Delta_{\beta'', \beta'}$ is also independent of the choice of lift.

Let $D_1, \ldots, D_r$ be Cartier divisor classes on C (equivalently, invertible sheaves $\mathcal{O}(D_1), \ldots, \mathcal{O}(D_r)$ on C). For $i = 1, \ldots, r$, denote $\langle D_i, \beta \rangle$ by e_i. The r-tuple $(D_1, \ldots, D_r)$ defines a 1-morphism $\xi : M \to \Pi^r(\mathrm{Pic}_\pi)$ with image in the connected component with multidegree $(e_1, \ldots, e_r)$. The lifting $\widetilde{\xi}_0$ defines a lifting $\widetilde{\xi} : M \to \Pi^r(\mathrm{Pic}_{\widetilde{\pi}})$ of ξ. Let $g(e'_1, e''_1, \ldots, e'_r, e''_r)$ be a function on $\mathbb{Z}^{\oplus 2r}$ with values in $\mathbb{Z}$, resp. $\mathbb{Q}$.

Notation 2.1 Denote by

$$\sum_{(\beta',\beta'')} g(\langle D_1, \beta' \rangle, \langle D_1, \beta'' \rangle, \ldots, \langle D_r, \beta' \rangle, \langle D_r, \beta'' \rangle) \Delta_{\beta',\beta''}$$

the Cartier divisor class, resp. $\mathbb{Q}$-Cartier divisor class, that is the pullback by $\widetilde{\xi}$ of the Cartier divisor class, resp. $\mathbb{Q}$-Cartier divisor class,

$$\sum_{(e'_1,e''_1,\ldots,e'_r,e''_r)} g(e'_1, e''_1, \ldots, e'_r, e''_r) \Delta_{(e'_1,e''_1,\ldots,e'_r,e''_r)},$$

the summation over all sequences $(e'_1, e''_1, \ldots, e'_r, e''_r)$ in $\mathbb{Z}^{\oplus 2r}$. If g is invariant under the involution $(e'_1, e''_1, \ldots, e'_r, e''_r) \mapsto (e''_1, e'_1, \ldots, e''_r, e'_r)$, then denote by

$$\sum_{(\beta',\beta'')}{}' g(\langle D_1, \beta' \rangle, \langle D_1, \beta'' \rangle, \ldots, \langle D_r, \beta' \rangle, \langle D_r, \beta'' \rangle) \Delta_{\beta',\beta''}$$

the pullback by ξ of,

$$\sum_{(e'_1,e''_1,\ldots,e'_r,e''_r)}{}' g(e'_1, e''_1, \ldots, e'_r, e''_r) \Delta_{(e'_1,e''_1,\ldots,e'_r,e''_r)},$$

where the summation is over all orbits of sequences $(e'_1, e''_1, \ldots, e'_r, e''_r)$ in $\mathbb{Z}^{\oplus 2r}$ for the involution. This is independent of the choice of lift $\widetilde{\xi}_0$.

Example 2.2 (Boundary Divisor of a Partition of Marked Points) Let $n \geq 0$ be an integer and let (A, B) be a partition of $\{1, \ldots, n\}$. For the universal family over $\mathfrak{M}_{0,n}$, denote by $s_1, \ldots, s_n$ the universal sections. Then

$$\sum_{(\beta',\beta'')} \prod_{i \in A} \langle \mathrm{Image}(s_i), \beta' \rangle \cdot \prod_{j \in B} \langle \mathrm{Image}(s_j), \beta'' \rangle \Delta_{\beta',\beta''}$$

is the Cartier divisor of the boundary divisor $\Delta_{(A,B)}$. The corresponding invariant function is

$$g(e'_1, e''_1, \ldots, e'_n, e''_n) = \left(\prod_{i \in A} e'_i \right) \left(\prod_{j \in B} e''_j \right) + \left(\prod_{i \in A} e''_i \right) \left(\prod_{j \in B} e'_j \right).$$

Example 2.3 (The Q_π-Divisor Associated to a Degree 0 Divisor) Restrict now over the open substack Pic^0_π so that $e' + e''$ equals 0, i.e., $e'' = -e'$. Recall that there exists a unique effective Cartier divisor $\mathcal{D}$ on $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ representing the universal invertible sheaf and whose open complement W has image in Pic^0_π containing the open $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} U_1$. There is an associated effective divisor on Pic^0_π,

$$Q = \sum_{(e',e'')}{}' - e'e''\Delta_{(e',e'')} = \sum_{(\beta',\beta'')}{}' - \langle \mathcal{D}, \beta' \rangle \langle \mathcal{D}, \beta'' \rangle \Delta_{\beta',\beta''}$$

coming from the invariant function $g(e', e'') = -e'e''$. In this case, invariance also implies that the effective Cartier divisor Q pulls back to itself under the involution ϵ_0 of Pic^0_π.

Example 2.4 (Closed Image of the Degree 0 Divisor) For the proper 1-morphism $\mathrm{pr}_1 : \mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathrm{Pic}^0_\pi$, the closed image $\mathrm{pr}_1(\mathcal{D})$ is the minimal effective Cartier divisor supported on Δ and whose inverse image in $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ contains $\mathcal{D}$. This divisor is

$$\mathrm{pr}_1(\mathcal{D}) = \sum_{(e',e'')}{}' |e'| \Delta_{(e',e'')} = \sum_{(\beta',\beta'')}{}' |\langle \mathcal{D}, \beta' \rangle| \Delta_{\beta',\beta''},$$

coming from the invariant function $g(e', e'') = \sqrt{-e'e''}$, which also happens to equal the asymmetric expression $|e'| = |e''|$ since $e' = -e''$. For the associated involution $\epsilon_0 \times \mathrm{Id}_{\mathfrak{C}}$ of $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$, $\mathcal{D} + (\epsilon_0 \times \mathrm{Id}_{\mathfrak{C}})^*\mathcal{D}$ equals $\mathrm{pr}_1^*(\mathrm{pr}_1(\mathcal{D}))$ as effective Cartier divisors on $\mathrm{Pic}^0_\pi \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$.

The following lemma clarifies $\widetilde{\mathfrak{M}_{0,0}}$. Since we could not find an explicit reference, we go through the argument in some detail.

Lemma 2.5 *The* 1*-morphism* $\widetilde{\zeta} : \mathfrak{C}^o \to \widetilde{\mathfrak{M}_{0,0}}$ *is smooth and faithfully flat. The* 1*-morphism is constant on connected components of geometric fibers of* π^o*, and it identifies* $\widetilde{\mathfrak{M}_{0,0}} \to \mathfrak{M}_{0,0}$ *with the étale (highly non-separated)* 1*-morphism of connected components of fibers of* π^o.

Proof Since $\widetilde{\mathfrak{M}_{0,0}}$ is an open substack of Pic_π, and since Pic_π is étale over $\mathfrak{M}_{0,0}$, also $\widetilde{\mathfrak{M}_{0,0}}$ is étale over $\mathfrak{M}_{0,0}$. Thus, since $\mathfrak{C}^o$ is smooth over $\mathfrak{M}_{0,0}$, also $\mathfrak{C}^o$ is smooth over $\widetilde{\mathfrak{M}_{0,0}}$. On the level of geometric points, i.e., for an invertible sheaf $\mathcal{L}$ on a curve C_k over an algebraically closed field k, it is straightforward that the isomorphism class of $\mathcal{L}$ is uniquely determined by the data of the degree of $\mathcal{L}$ on each irreducible component C_i of C_k. Thus $\widetilde{\zeta}$ is constant on connected components of geometric fibers of π^o, and it does distinguish distinct irreducible components.

Since $\widetilde{\zeta}$ is smooth, it is flat. Thus, to prove that $\widetilde{\zeta}$ is faithfully flat, it suffices to prove that it is surjective on geometric points. Because $\widetilde{\mathfrak{M}_{0,0}}$ is étale over $\mathfrak{M}_{0,0}$, and because $\mathfrak{M}_{0,0}$ is smooth over Spec $(\mathbb{Z})$ with dense open U_2, every geometric point of $\widetilde{\mathfrak{M}_{0,0}}$ is a specialization of a point of U_2 that is “transversal” to the boundary Δ. More precisely, for every geometric point $\zeta'_k : \text{Spec}(k) \to \widetilde{\mathfrak{M}_{0,0}}$, there exists a strictly Henselian DVR R with residue field k, and there exists a 1-morphism Spec $(R) \to \mathfrak{M}_{0,0}$ extending the morphism on Spec(k) and corresponding to a morphism $\rho : C \to$ Spec (R) with C regular and with smooth generic fiber. Moreover, there exists an invertible sheaf $\mathcal{L}'$ on C with Serre dual $\mathcal{L}' = \omega_\rho \otimes (\mathcal{L}')^\vee$ such that all of the following sheaves are zero, and such that $\mathcal{L}'$ induces the 1-morphism ζ'_k: $R^1\rho_*(\mathcal{L}')$, $R^1\rho_*(\mathcal{L}'')$, and $R^1\rho_*(\mathcal{L}'' \otimes (\mathcal{L}')^\vee)$. From a deformation theory perspective, this holds because there is an effective versal deformation of C_k, and the versal deformation space is the (formal completion of the) product of smooth 1-dimensional factors corresponding to the deformation spaces of each node. For every deformation over a DVR R such that the induced morphism from Spec(R) to each deformation space of a node is formally unramified, the corresponding morphism $\rho : C \to$ Spec (R) has C regular and has smooth generic fiber. Similarly, there is no obstruction to deforming the invertible sheaf $\mathcal{L}'_k$ to an invertible sheaf on all of C.

The goal is to prove that there exists a section s : Spec $(R) \to C^o$ of ρ^o such that the ideal sheaf of the image is isomorphic to $\mathcal{L}'$. By the classification of invertible sheaves on $\mathbb{P}^1$, the vanishing conditions on h^1 of $\mathcal{L}'$ and $\mathcal{L}''$ imply that the invertible sheaf $\mathcal{L}'_\eta$ on the smooth generic fiber C_η has degree -1. Thus $H^0(C_\eta, (\mathcal{L}'_\eta)^\vee)$ is 2-dimensional as a vector space over the fraction field, and the zero scheme of every nonzero global section is the image of a section of ρ_η. If the fiber C_k is smooth, the closure of this section is a section of ρ, and the ideal sheaf of this section differs from $\mathcal{L}'$ by twisting by a multiple of the closed fiber C_k. Since the closed fiber C_k is principal in C, it follows that $\mathcal{L}'$ is isomorphic to the ideal sheaf of a section of ρ. Thus the lemma is proved when C_k is smooth.

Therefore, without loss of generality, assume that C_k is singular, i.e., reducible. Then C_k is a tree of smooth, genus 0 curves. The remainder of the proof proceeds by induction on the number of irreducible components of C_k. The key induction step is the analysis of the restriction of $\mathcal{L}$ to a "leaf of the tree".

Because C_k is a tree of smooth, genus 0 curves, it has at least two irreducible components C_i such that $C_i \cap \overline{C \setminus C_i}$ consists of a single (disconnecting) node, i.e., C_i is a leaf of the tree. In particular, $\omega_\rho|_{C_i}$ is an invertible sheaf of degree -1. For every invertible sheaf $\mathcal{K}$ on C, there is a short exact sequence of $\mathcal{O}_C$-modules,

$$0 \longrightarrow \mathcal{K}(-\underline{C}_i) \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}|_{C_i} \longrightarrow 0.$$

In particular, if $H^1(C, \mathcal{K})$ is zero, then also $H^1(C_i, \mathcal{K}|_{C_i})$ is zero. Since $H^1(C, \mathcal{L}')$ and $H^1(C, \mathcal{L}'')$ are both zero, it follows that $\mathcal{L}'|_{C_i}$ has degree $d' = 0$ or degree $d' = -1$. When d' equals 0, resp. when d' equals -1, then $\mathcal{L}''|_{C_i}$ has degree d'' equal to -1, resp. $d'' = 0$. When (d', d'') equals $(0, -1)$, resp. when (d', d'') equals $(-1, 0)$, then $\overline{\mathcal{L}}' := \mathcal{L}'$, resp. $\overline{\mathcal{L}}' := \mathcal{L}'(-\underline{C}_i)$, restricts to a trivial invertible sheaf on C_i, as does the invertible sheaf $\overline{\mathcal{L}}'' := \omega_\rho \otimes (\overline{\mathcal{L}}')^\vee(-\underline{C}_i)$. By Castelnuovo's contractibility criterion, there exists a morphism of R-schemes,

$$\nu : C \to \widehat{C},$$

such that $\widehat{\rho}: \widehat{C} \to \text{Spec}\,(R)$ is an object of $\mathfrak{M}_{0,0}$ satisfying the same hypotheses as ρ, and such that ν is equivalent to the blowing up of $\widehat{C}$ at a smooth k-point q of $\widehat{C}_k$. In particular, the number of irreducible components of C_k is one greater than the number of irreducible components of $\widehat{C}_k$. Since $\overline{\mathcal{L}}'|_{C_i}$ is trivial, $\widehat{\mathcal{L}}' := \nu_* \overline{\mathcal{L}}'$ is an invertible sheaf on $\widehat{C}$ such that the natural $\mathcal{O}_C$-module homomorphism $\nu^*\widehat{\mathcal{L}}' \to \overline{\mathcal{L}}'$ is an isomorphism of invertible sheaves on C. By adjunction, ω_ρ is isomorphic to $\nu^*\omega_{\widehat{\rho}}(\underline{C}_i)$. Thus, denoting by $\widehat{\mathcal{L}}'' = \omega_{\widehat{\rho}} \otimes (\widehat{\mathcal{L}}')^\vee$ the Serre dual of $\widehat{\mathcal{L}}'$, the sheaf $\overline{\mathcal{L}}''$ with degree 0 on C_i is isomorphic to $\nu^*(\widehat{\mathcal{L}}'')$.

The key to the induction step is proving that the invertible sheaves $\widehat{\mathcal{L}}'$ and $\widehat{\mathcal{L}}''$ on $\widehat{C}$ satisfying the same H^1-vanishing hypotheses as do $\mathcal{L}'$ and $\mathcal{L}''$ on C. For every invertible sheaf $\widehat{\mathcal{K}}$ on $\widehat{C}$ with pullback $\mathcal{K} = \nu^*\widehat{\mathcal{K}}$, observe that both of the following natural maps are isomorphisms,

$$H^1(\widehat{C}, \widehat{\mathcal{K}}) \to H^1(C, \mathcal{K}) \to H^1(C, \mathcal{K}(\underline{C}_i)).$$

Thus one of these three equals zero if and only if all of them equal zero. Also, for the natural short exact sequence of invertible sheaves on C,

$$0 \longrightarrow \mathcal{K}(-\underline{C}_i) \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}|_{C_i} \longrightarrow 0,$$

the third term $\mathcal{K}|_{C_i}$ is an invertible sheaf of degree 0 on C_i, hence it has vanishing H^1. Thus, by the long exact sequence of cohomology, if $H^1(C, \mathcal{K}(-\underline{C}_i))$ equals zero, then also $H^1(C, \mathcal{K})$ equals zero, so that also $H^1(\widehat{C}, \widehat{\mathcal{K}})$ equals zero. In the first case that (d', d'') equals $(0, -1)$, then $\mathcal{L}'$ equals $\nu^*\widehat{\mathcal{L}}'$, $\mathcal{L}''$ equals $\nu^*\widehat{\mathcal{L}}''(\underline{C}_i)$, and $\mathcal{L}'' \otimes (\mathcal{L}')^\vee$ equals $\nu^*(\widehat{\mathcal{L}}'' \otimes (\widehat{\mathcal{L}}')^\vee)(\underline{C}_i)$. Thus, in this case, the H^1-vanishing hypotheses for $\mathcal{L}'$ and $\mathcal{L}''$ on C is equivalent to the H^1-vanishing hypotheses for $\widehat{\mathcal{L}}'$ and $\widehat{\mathcal{L}}''$ on $\widehat{C}$. In the second case that (d', d'') equals $(-1, 0)$, then $\mathcal{L}'$ equals $\nu^*\widehat{\mathcal{L}}'(\underline{C}_i)$, $\mathcal{L}''$ equals $\nu^*\widehat{\mathcal{L}''}$, and $\mathcal{L}'' \otimes (\mathcal{L}')^\vee$ equals $\nu^*(\widehat{\mathcal{L}}'' \otimes (\widehat{\mathcal{L}}')^\vee)(-\underline{C}_i)$. Thus the H^1-vanishing hypotheses for $\mathcal{L}'$ and $\mathcal{L}''$ on C imply H^1-vanishing hypotheses for $\widehat{\mathcal{L}}'$ and $\widehat{\mathcal{L}}''$ on $\widehat{C}$ (but they are not always equivalent hypotheses). Thus, since $\mathcal{L}'$ and $\mathcal{L}''$ satisfy the H^1-vanishing hypotheses, so do $\widehat{\mathcal{L}}'$ and $\widehat{\mathcal{L}}''$. Since $\widehat{C}_k$ has fewer irreducible components than C_k, by the induction hypothesis, $\widehat{\mathcal{L}}'$ is the ideal sheaf of a section of $\widehat{\rho}$. Moving this section in its linear equivalence class, $\widehat{\mathcal{L}}'$ is the ideal sheaf of a section $\widehat{s}$ of $\widehat{\rho}$ whose image is disjoint from q. Thus the total transform s of $\widehat{s}$ is a section of ρ whose image is disjoint from C_i and such that $\nu^*\widehat{\mathcal{L}}'$ is the ideal sheaf of s. In case (d', d'') equals $(0, -1)$ for some leaf C_i of C_k, this proves that $\mathcal{L}$ is the ideal sheaf of a section s of ρ.

Finally, by way of contradiction, assume that $\mathcal{L}'|_{C_i}$ has degree -1 for every leaf C_i of C_k, i.e., $\mathcal{L}'$ equals $\nu^*\mathcal{L}(\underline{C}_i)$ for the ideal sheaf $\mathcal{L}$ of a section s. Since this sheaf has degree -1 on every leaf of C_k, it follows that there is precisely one leaf other than C_i, this leaf is the unique irreducible component of C_k that intersects s, and this leaf does not intersect C_i. Thus, C_k is a chain of genus 0 curves with at least 3 irreducible components and precisely two leaves. For the unique irreducible component C_j of C_k intersecting C_i and that is different from C_i, $\mathcal{L}'|_{C_j}$ has degree 1, so that $\mathcal{L}''$ has degree -1. Thus the invertible sheaf $\mathcal{L}'' \otimes (\mathcal{L}')^\vee$ has degree -2.

Since $H^1(C_j, \mathcal{L}'' \otimes (\mathcal{L}')^\vee|_{C_j})$ is nonzero, also $H^1(C, \mathcal{L}'' \otimes (\mathcal{L}')^\vee)$ is nonzero, contradicting the hypothesis. Therefore, by way of contradiction, there exists at least one leaf of C_k on which $\mathcal{L}'$ has degree 0. So $\mathcal{L}'$ is the ideal sheaf of a section s of ρ. Therefore the lemma is proved by induction on the number of irreducible components of C_k. □

3 The Functor Q_π

Let M be an Artin stack, and let $\pi : C \to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. There exists an invertible dualizing sheaf ω_π, and the relative trace map, $\mathrm{Tr}_\pi : R\pi_*\omega_\pi[1] \to \mathcal{O}_M$, is a quasi-isomorphism. In particular, $\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_\pi, \mathcal{O}_C)$ is canonically isomorphic to $H^0(M, \mathcal{O}_M)$. Therefore $1 \in H^0(M, \mathcal{O}_M)$ determines an extension class, i.e., a short exact sequence,

$$0 \longrightarrow \omega_\pi \longrightarrow E_\pi \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

The morphism π is perfect, so for every complex $F^\bullet$ perfect of bounded amplitude on C, $R\pi_* F^\bullet$ is a perfect complex of bounded amplitude on M. By [5], the determinant of a perfect complex of bounded amplitude is defined.

Definition 3.1 For every complex $F^\bullet$ perfect of bounded amplitude on C, define $Q_\pi(F^\bullet) = \det(R\pi_* E_\pi \otimes F^\bullet)$.

There is another interpretation of $Q_\pi(F^\bullet)$.

Lemma 3.2 *For every complex $F^\bullet$ perfect of bounded amplitude on C,*

$$Q_\pi(F^\bullet) \cong det(R\pi_*(F^\bullet)) \otimes det(R\pi_*((F^\bullet)^\vee))^\vee.$$

Proof By the short exact sequence for E_π, $Q_\pi(F^\bullet) \cong \det(R\pi_*(F^\bullet)) \otimes \det(R\pi_*(\omega_\pi \otimes F^\bullet))$. The lemma follows by duality. □

It is straightforward to compute $F^\bullet$ whenever there exist cycle class groups for C and M such that Chern classes are defined for all perfect complexes of bounded amplitude and such that Grothendieck–Riemann–Roch holds for π.

Lemma 3.3 *If there exist cycle class groups for C and M such that Chern classes exist for all perfect complexes of bounded amplitude and such that Grothendieck–Riemann–Roch holds for π, then modulo 2-power torsion, the first Chern class of $Q_\pi(F^\bullet)$ is $\pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet))$.*

Proof Denote the Todd class of π by $\tau = 1 + \tau_1 + \tau_2 + \dots$. Of course $2\tau_1 = -C_1(\omega_\pi)$. By GRR, $\mathrm{ch}(R\pi_*\mathcal{O}_C) = \pi_*(\tau)$. The canonical map $\mathcal{O}_M \to R\pi_*\mathcal{O}_C$ is a quasi-isomorphism. Therefore $\pi_*(\tau_2) = 0$, modulo 2-power torsion. By additivity of the Chern character, $\mathrm{ch}(E_\pi) = 2 + C_1(\omega_\pi) + \frac{1}{2}C_1(\omega_\pi)^2 + \dots$. Therefore,

$$\mathrm{ch}(E_\pi) \cdot \tau = 2 + 2\tau_2 + \dots$$

So for any complex $F^\bullet$ perfect of bounded amplitude,

$$\mathrm{ch}(E_\pi \otimes F^\bullet) \cdot \tau = \mathrm{ch}(F^\bullet) \cdot \mathrm{ch}(E_\pi) \cdot \tau =$$
$$(\mathrm{rk}(F^\bullet) + C_1(F^\bullet) + \tfrac{1}{2}(C_1(F^\bullet)^2 - 2C_2(F^\bullet)) + \dots)(2 + 2\tau_2 + \dots).$$

Applying π_* gives,

$$2\pi_*(C_1(F^\bullet)) + \pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet)) + \dots$$

Therefore the first Chern class of $\det(R\pi_*(E_\pi \otimes F^\bullet))$ is $\pi_*(C_1(F^\bullet)^2 - 2C_2(F^\bullet))$, modulo 2-power torsion. □

Remark 3.4 The point is this. In every reasonable case, Q_π is just $\pi_*(C_1^2 - 2C_2)$. Moreover Q_π is compatible with base-change by arbitrary 1-morphisms. This allows to reduce certain computations to the Artin stack of all genus 0 curves. As far as we are aware, no one has written a definition of cycle class groups for all locally finitely presented Artin stacks that has Chern classes for all perfect complexes of bounded amplitude, has pushforward maps and Grothendieck–Riemann–Roch for perfect 1-morphisms representable by proper algebraic spaces, and has pullback maps by arbitrary 1-morphisms for cycles coming from Chern classes. Doubtless such a theory exists; whatever it is, $Q_\pi = \pi_*(C_1^2 - 2C_2)$.

Let the following diagram be 2-Cartesian,

$$\begin{array}{ccc} C' & \xrightarrow{\zeta_C} & C \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{\zeta_M} & M \end{array}$$

together with a 2-equivalence $\theta : \pi \circ \zeta_C \Rightarrow \zeta_M \circ \pi'$.

Lemma 3.5 *For every complex $F^\bullet$ perfect of bounded amplitude on C, $\zeta_M^* Q_\pi(F^\bullet)$ is canonically isomorphic to $Q_{\pi'}(\zeta_C^* F^\bullet)$.*

Proof Of course $\zeta_C^* E_\pi$ is canonically isomorphic to $E_{\pi'}$ since $\zeta_C^*\omega_\pi$ is canonically isomorphic to $\omega_{\pi'}$. Also $\zeta_M^* R\pi_*$ is canonically equivalent to $R(\pi')_*\zeta_C^*$ for perfect complexes of bounded amplitude. Therefore we have the chain of equivalences,

$$\zeta_M^* Q_\pi(F^\bullet) = \det(\zeta_M^* R\pi_*(E_\pi \otimes F^\bullet)) = \det(R(\pi')_*\zeta_C^*(E_\pi \otimes F^\bullet)) =$$

$$\det(R(\pi')_* E_{\pi'} \otimes \zeta_C^* F^\bullet) = Q_{\pi'}(\zeta_C^* F^\bullet).$$

□

Lemma 3.6 *Let L be an invertible sheaf on C of relative degree e over M. For every invertible sheaf L' on M, $Q_\pi(L \otimes \pi^* L')$ is canonically isomorphic to $Q_\pi(L) \otimes (L')^{2e}$. In particular, if $e = 0$, then $Q_\pi(L \otimes \pi^* L')$ is canonically isomorphic to $Q_\pi(L)$ so that Q_π induces a well-defined map $Pic_\pi^0(M) \to Pic(M)$.*

Proof To compute the rank of $R\pi_*(E_\pi \otimes F^\bullet)$ over any connected component of M, it suffices to base-change to the spectrum of a field mapping to that component. Then, by Grothendieck–Riemann–Roch, the rank is $2\deg(C_1(F^\bullet))$. In particular, $R\pi_*(E_\pi \otimes L)$ has rank $2e$.

By the projection formula, $R\pi_*(E_\pi \otimes L \otimes \pi^* L') \cong R\pi_*(E_\pi \otimes L) \otimes L'$. Thus, also $\det(R\pi_*(E_\pi \otimes L) \otimes L')$ equals $Q_\pi(L) \otimes (L')^{\text{rank}}$. This follows from the uniqueness of det; for any invertible sheaf L' the association $F^\bullet \mapsto \det(F^\bullet \otimes L') \otimes (L')^{-\text{rank}(F^\bullet)}$ also satisfies the axioms for a determinant function, and, hence, it is canonically isomorphic to $\det(F^\bullet)$. Therefore $Q_\pi(L \otimes \pi^* L')$ equals $Q_\pi(L) \otimes (L')^{2e}$. In particular, when e equals 0, this gives a canonical isomorphism of $Q_\pi(L \otimes \pi^* L')$ with $Q_\pi(L)$. Since also Q_π is compatible with pullback, for every element of $\text{Pic}_\pi^0(M)$ that is étale locally represented by an invertible sheaf in $\text{Pic}(C)$, Q_π of these invertible sheaves satisfies the descent condition for an invertible sheaf on M relative to this étale cover. □

4 Local Computations

This section contains two computations: $Q_\pi(\omega_\pi)$ and $Q_\pi(L)$ for every invertible sheaf on C of relative degree 0. Because of Lemma 3.5 the first computation reduces to the universal case over $\mathfrak{M}_{0,0}$. Because of Lemmas 3.5 and 3.6, the second computation reduces to the universal case over Pic_π^0.

4.1 Computation of $Q_\pi(\omega_\pi)$

Associated to $\pi_C : C \to M$, there is a 1-morphism $\zeta_M : M \to \mathfrak{M}_{0,0}$, a 1-morphism $\zeta_C : C \to \mathcal{C}$, and a 2-equivalence $\theta : \pi_C \circ \zeta_C \Rightarrow \zeta_M \circ \pi_C$ such that the following diagram is 2-Cartesian,

$$\begin{array}{ccc} C & \xrightarrow{\zeta_C} & \mathcal{C} \\ {\scriptstyle \pi_C}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{\zeta_M} & \mathfrak{M}_{0,0} \end{array}$$

Of course ω_{π_C} is isomorphic to $\zeta_C^*\omega_{\pi_C}$. By Lemma 3.5, $Q_{\pi_C}(\omega_{\pi_C}) \cong \zeta_M^* Q_{\pi_C}(\omega_{\pi_C})$. So the computation of $Q_{\pi_C}(\omega_{\pi_C})$ is reduced to the universal family $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$. In what follows, denote by $j : U_2 \to U_1$ the open immersion.

Proposition 4.1 *(i) Over the open substack U_1, $\omega_\pi^\vee$ is π-relatively ample.*
(ii) Over U_1, $R^1\pi_*\omega_\pi^\vee|_{U_1} = (0)$ and $\pi_*\omega_\pi^\vee|_{U_1}$ is locally free of rank 3.
(iii) Over U_2, there is a canonical isomorphism $i : det(\pi_*\omega_\pi^\vee|_{U_2}) \to \mathcal{O}_{U_2}$.
(iv) The image of the homomorphism of quasi-coherent sheaves $det(\pi_*\omega_\pi^\vee|_{U_1}) \to j_*det(\pi_*\omega_\pi^\vee|_{U_2}) \xrightarrow{i} j_*\mathcal{O}_{U_2}$ equals the image of the canonical homomorphism of quasi-coherent sheaves $\mathcal{O}_{U_1}(-\Delta) \hookrightarrow \mathcal{O}_{U_1} \hookrightarrow j_*\mathcal{O}_{U_2}$.
(v) Over U_1, $Q_\pi(\omega_\pi)|_{U_1} \cong \mathcal{O}_{U_1}(-\Delta)$. Therefore on all of $\mathfrak{M}_{0,0}$, $Q_\pi(\omega_\pi) \cong \mathcal{O}_{\mathfrak{M}_{0,0}}(-\Delta)$.

Proof Recall the 1-morphism $\zeta_1 : \mathbb{A}^1_{\mathbb{Z}} \to U_1$ from Sect. 2. Because ζ_1 is smooth and surjective, (i) and (ii) can be checked after base-change by ζ_1. Also (iv) will reduce to a computation over $\mathbb{A}^1_{\mathbb{Z}}$ after base-change by ζ_1.

(i) and (ii): Denote by $\mathbb{P}^2_{\mathbb{Z}}$ the projective space with coordinates u_0, u_1, u_2. There is a rational transformation $f : \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}} \dashrightarrow \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ by

$$\begin{aligned} f^*x &= x, \\ f^*u_0 &= xy_0^2, \\ f^*u_1 &= y_0y_1, \\ f^*u_2 &= y_1^2. \end{aligned}$$

By local computation, this extends to a morphism $f : C \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$ that is a closed immersion and whose image is $\mathbb{V}(u_0u_2 - xu_1^2)$. By the adjunction formula, ω_π is the pullback of $\mathcal{O}_{\mathbb{P}^2}(-1)$. In particular, $\omega_\pi^\vee$ is very ample. Moreover, because $H^1(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^2}(1)) = H^2(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^2}(-1)) = (0)$, also $H^1(C, \omega_\pi^\vee) = (0)$. By cohomology and base-change results, $R^1\pi_*(\omega_\pi^\vee) = (0)$ and $\pi_*(\omega_\pi^\vee)$ is locally free of rank 3.

(iii): The curve $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{P}(V)$ determines a morphism $\eta : \text{Spec}\,(\mathbb{Z}) \to U_2$. This is smooth and surjective on geometric points. Moreover it gives a realization of U_2 as the classifying stack of the group scheme $\text{Aut}(\mathbb{P}(V)) = \mathbf{PGL}(V)$. Taking the exterior power of the Euler exact sequence, $\omega_{\mathbb{P}(V)/\mathbb{Z}} = \bigwedge^2(V^\vee) \otimes \mathcal{O}_{\mathbb{P}(V)}(-2)$. Therefore $H^0(\mathbb{P}(V), \omega^\vee_{\mathbb{P}(V)/\mathbb{Z}})$ equals $\bigwedge^2(V) \otimes \text{Sym}^2(V^\vee)$ as a representation of $\mathbf{GL}(V)$. The determinant of this representation is the trivial character of $\mathbf{GL}(V)$. Therefore it is the trivial character of $\mathbf{PGL}(V)$. This gives an isomorphism of $\det(\pi_*\omega_\pi|_{U_2})$ with $\mathcal{O}_{U_2}$.

(iv): This can be checked after pulling back by ζ_1. The pullback of U_2 is $\mathbb{G}_{m,\mathbb{Z}} \subset \mathbb{A}^1_{\mathbb{Z}}$. The pullback of i comes from the determinant of $H^0(\mathbb{G}_{m,\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}, \omega_\pi^\vee) = \bigwedge^2(V) \otimes \text{Sym}^2(V^\vee) \otimes \mathcal{O}_{\mathbb{G}_m}$. By the adjunction formula, $\omega_{C/\mathbb{A}^1} = \nu^*\omega_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}(E)$. Hence $\nu_*\omega^\vee_{C/\mathbb{A}^1} = I_Z\omega_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}$. Therefore the canonical map,

$$H^0(C, \omega^\vee_{C/\mathbb{A}^1}) \to H^0(\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}, \omega^\vee_{\mathbb{A}^1\times\mathbb{P}^1/\mathbb{A}^1}),$$

is given by,

$$\begin{aligned}
&\mathcal{O}_{\mathbb{A}^1}\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2\} \to \bigwedge^2(V) \otimes \mathrm{Sym}^2(V^\vee) \otimes \mathcal{O}_{\mathbb{A}^1}, \\
&\mathbf{f}_0 \mapsto x \cdot (\mathbf{e}_0 \wedge \mathbf{e}_1) \otimes y_0^2, \\
&\mathbf{f}_1 \mapsto \quad (\mathbf{e}_0 \wedge \mathbf{e}_1) \otimes y_0 y_1, \\
&\mathbf{f}_2 \mapsto \quad (\mathbf{e}_0 \wedge \mathbf{e}_1) \otimes y_1^2
\end{aligned}$$

It follows that $\det(\pi_*\omega_\pi^\vee) \to \mathcal{O}_{\mathbb{G}_m}$ has image $\langle x\rangle\mathcal{O}_{\mathbb{A}^1}$, i.e., $\zeta_1^*\mathcal{O}_{U_1}(-\Delta)$.

(v): By the short exact sequence for E_π, $Q_\pi(\omega_\pi) = \det(R\pi_*\omega_\pi) \otimes \det(R\pi_*\omega_\pi^2)$. Because the trace map is a quasi-isomorphism, $\det(R\pi_*\omega_\pi) = \mathcal{O}_{U_1}$. By (ii) and duality,

$$\det(R\pi_*\omega_\pi^2) \cong \det(R^1\pi_*\omega_\pi^2)^\vee \cong \det(\pi_*\omega_\pi^\vee).$$

By (iv), this is $\mathcal{O}_{U_1}(-\Delta)$. Therefore $Q_\pi(\omega_\pi) \cong \mathcal{O}_{U_1}(-\Delta)$ on U_1. Because $\mathfrak{M}_{0,0}$ is regular, and because the complement of U_1 has codimension 2, this isomorphism of invertible sheaves extends to all of $\mathfrak{M}_{0,0}$. □

The sheaf of relative differentials Ω_π is a pure coherent sheaf on $\mathcal{C}$ of rank 1, flat over $\mathfrak{M}_{0,0}$ and is quasi-isomorphic to a perfect complex of amplitude $[-1, 0]$.

Lemma 4.2 *The perfect complex $R\pi_*\Omega_\pi$ has rank -1 and determinant $\cong \mathcal{O}_{\mathfrak{M}_{0,0}}(-\Delta)$. The perfect complex $R\pi_*RHom_{\mathcal{O}_\mathcal{C}}(\Omega_\pi, \mathcal{O}_\mathcal{C})$ has rank 3 and determinant $\cong \mathcal{O}_{\mathfrak{M}_{0,0}}(-2\Delta)$.*

Proof There is a canonical injective sheaf homomorphism $\Omega_\pi \to \omega_\pi$ and the support of the cokernel, $Z \subset \mathcal{C}$, is a closed substack that is smooth and such that $\pi : Z \to \mathfrak{M}_{0,0}$ is unramified and is the normalization of Δ. Over U_1, the lemma immediately follows from this and the arguments in the proof of Proposition 4.1. As in that case, it suffices to establish the lemma over U_1. □

4.2 Computation of $Q_\pi(L)$ for Invertible Sheaves of Degree 0

Let M be an Artin stack, let $\pi : C \to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. Let L be an invertible sheaf on C of relative degree 0 over M. This determines a 1-morphism to the relative Picard of the universal family over $\mathfrak{M}_{0,0}$,

$$\zeta_M : M \to \mathrm{Pic}^0_\pi.$$

The pullback of the universal family $\mathfrak{C}$ is equivalent to C and the pullback of the universal bundle $\mathcal{O}_\mathfrak{C}(\mathcal{D})$ differs from L by π^*L' for an invertible sheaf L' on M. By Lemmas 3.5 and 3.6, $Q_\pi(L) \cong \zeta_M^* Q_\pi(\mathcal{O}_\mathfrak{C}(\mathcal{D}))$.

Proposition 4.3 *Over* Pic^0_π, $\pi_* E_\pi(\mathcal{D}) = (0)$ *and* $R^1\pi_* E_\pi(\mathcal{D})$ *is a sheaf supported on the inverse image of* Δ. *The stalk of* $R^1\pi_* E_\pi(\mathcal{D})$ *at the generic point of* $\Delta_{(a,-a)}$ *is a torsion sheaf of length* a^2. *The filtration by order of vanishing at the generic point has associated graded pieces of length* $2a-1, 2a-3, \ldots, 3, 1$.

Proof Over the open complement of Δ, the divisor $\mathcal{D}$ is 0. So the first part of the proposition reduces to the statement that $R\pi_* E_\pi$ is quasi-isomorphic to 0. By definition of E_π, there is an exact triangle,

$$R\pi_* E_\pi \longrightarrow R\pi_* \mathcal{O}_{\mathcal{C}} \xrightarrow{\delta} R\pi_* \omega_\pi[1] \longrightarrow R\pi_* E_\pi[1].$$

Of course the bundle E_π and the canonical isomorphism $R\pi_* \mathcal{O}_{\mathcal{C}} \cong \mathcal{O}_{\mathfrak{M}}$ were defined so that the composition of δ with the trace map, which is a quasi-isomorphism in this case, would be the identity. Therefore δ is a quasi-isomorphism, so $R\pi_* E_\pi$ is quasi-isomorphic to 0.

The second part can be proved (and perhaps only makes sense) after smooth base-change to a scheme. Let $\mathbb{P}^1_s$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates S_0, S_1. Let $\mathbb{P}^1_x$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates X_0, X_1. Let $\mathbb{P}^1_y$ be a copy of $\mathbb{P}^1$ with homogeneous coordinates Y_0, Y_1. Denote by $C \subset \mathbb{P}^1_s \times \mathbb{P}^1_x \times \mathbb{P}^1_y$ the divisor with defining equation $F = S_0X_0Y_0 - S_1X_1Y_1$. The projection $\mathrm{pr}_s : C \to \mathbb{P}^1_s$ is a proper, flat morphism whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0. Denote by $\zeta_0 : \mathbb{P}^1_s \to \mathfrak{M}_{0,0}$ the associated 1-morphism. The restriction $\mathcal{L}'$ to C of the invertible sheaf $\mathrm{pr}_x^* \mathcal{O}_{\mathbb{P}^1_x}(-1)$ defines a lifting $\widetilde{\zeta_0} : \mathbb{P}^1_s \to \widetilde{\mathfrak{M}_{0,0}}$. Denote by L the invertible sheaf on C that is the restriction of $\mathrm{pr}_x^* \mathcal{O}_{\mathbb{P}^1_x}(a) \otimes \mathrm{pr}_y^* \mathcal{O}_{\mathbb{P}^1_y}(-a)$. This is an invertible sheaf of relative degree 0. Therefore there is an induced 1-morphism $\zeta : \mathbb{P}^1_s \to \mathrm{Pic}^0_\pi$, and a lift $\widetilde{\zeta} : \mathbb{P}^1_s \to \mathrm{Pic}^0_{\widetilde{\pi}}$ over $\widetilde{\zeta_0}$ and ζ.

It is straightforward to check smoothness of ζ_0, and hence also $\widetilde{\zeta_0}$, ζ, and $\widetilde{\zeta}$. The image of $\widetilde{\zeta}$ intersects $\Delta_{(b,-b)}$ if and only if b equals $\pm a$. The divisor $\widetilde{\zeta}^*(\Delta_{(a,-a)} + \Delta_{(-a,a)})$ if $a \neq 0$, resp. $\widetilde{\zeta}^*(\Delta_{(0,0)})$ if $a = 0$, equals the Cartier divisor $\mathbb{V}(S_0S_1) \subset \mathbb{P}^1_s$. There is an obvious involution $i : \mathbb{P}^1_s \to \mathbb{P}^1_s$ by $i(S_0, S_1) = (S_1, S_0)$, and $\zeta \circ i$ is 2-equivalent to ζ. Therefore the total length of the torsion sheaf $R^1\mathrm{pr}_{s,*} E_{\mathrm{pr}_s} \otimes L$ is 2 times the length of the stalk of $R^1\pi_* E_\pi(\mathcal{D})$ at the generic point of the image of $\Delta_{(a,-a)}$. More precisely, the length of the stalk at each of $(1, 0), (0, 1) \in \mathbb{P}^1_s$ is the length of the stalk at the image of $\Delta_{(a,-a)}$. Similarly for the lengths of the associated graded pieces of the filtration by vanishing order.

Because E_{pr_s} is the extension class of the trace mapping, $R^1\mathrm{pr}_{s,*} E_{\mathrm{pr}_s} \otimes L$ is the cokernel of the $\mathcal{O}_{\mathbb{P}^1_s}$-homomorphisms,

$$\gamma : \mathrm{pr}_{s,*}(L) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1_s}}(\mathrm{pr}_{s,*}(L^\vee), \mathcal{O}_{\mathbb{P}^1_s}),$$

induced via adjointness from the multiplication map,

$$\mathrm{pr}_{s,*}(L) \otimes \mathrm{pr}_{s,*}(L^\vee) \to \mathrm{pr}_{s,*}(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1_s}.$$

On $\mathbb{P}^1_s \times \mathbb{P}^1_x \times \mathbb{P}^1_y$ there is a locally free resolution of the push-forward of L, resp. $L^\vee$,

$$0 \to \mathcal{O}_{\mathbb{P}^1_s}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(a-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(-a-1) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s}(0) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(-a) \to L \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^1_s}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(-a-1) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(a-1) \xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s}(0) \boxtimes \mathcal{O}_{\mathbb{P}^1_x}(-a) \boxtimes \mathcal{O}_{\mathbb{P}^1_y}(a) \to L^\vee \to 0$$

Hence $R\text{pr}_{s,*}L$ is the complex,

$$\begin{aligned} &\mathcal{O}_{\mathbb{P}^1_s}(-1) \otimes_k H^0(\mathbb{P}^1_x, \mathcal{O}_{\mathbb{P}^1_x}(a-1)) \otimes_k H^1(\mathbb{P}^1_y, \mathcal{O}_{\mathbb{P}^1_y}(-a-1)) \\ &\xrightarrow{F} \mathcal{O}_{\mathbb{P}^1_s} \otimes_k H^0(\mathbb{P}^1_x, \mathcal{O}_{\mathbb{P}^1_x}(a)) \otimes_k H^1(\mathbb{P}^1_y, \mathcal{O}_{\mathbb{P}^1_y}(-a)). \end{aligned}$$

A similar result holds for $R\text{pr}_{s,*}L^\vee$. It is possible to write out this map explicitly in terms of bases for H^0 and H^1, but for the main statement just observe the complex has rank 1 and degree $-a^2$. A similar result holds for $R\text{pr}_{s,*}L^\vee$. Therefore $R^1\pi_* E_\pi(L)$ is a torsion sheaf of length $2a^2$. Because it is equivariant for i, the localization at each of $(0, 1)$ and $(1, 0)$ has length a^2.

The lengths of the associated graded pieces of the filtration by order of vanishing at $\mathbb{V}(S_0 S_1)$ can be computed from the complexes for $R\text{pr}_{s,*}L$ and $R\text{pr}_{s,*}L^\vee$. This is left to the reader. □

Corollary 4.4 *In the universal case, $Q_\pi(\mathcal{D}) = -\sum_{a \geq 0} a^2 \Delta_a$. Therefore in the general case of $\pi : C \to M$ and an invertible sheaf L of relative degree* 0*,*

$$Q_\pi(L) = {\sum_{\beta', \beta''}}' \langle C_1(L), \beta' \rangle \langle C_1(L), \beta'' \rangle \Delta_{\beta', \beta''}.$$

5 Proof of Proposition 1.2

As usual, let M be an Artin stack and let $\pi : C \to M$ be a flat 1-morphism, relatively representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of genus 0.

Hypothesis 5.1 *There are cycle class groups for C and M admitting Chern classes for locally free sheaves, and such that Grothendieck–Riemann–Roch holds for π. In particular, this holds if M is a Deligne–Mumford stack whose coarse moduli space is quasi-projective.*

Proof of Proposition 1.2*(i).* Define $D' = 2D + \langle D, \beta \rangle C_1(\omega_\pi)$. This is a Cartier divisor class of relative degree 0. By Corollary 4.4,

$$Q_\pi(D') = {\sum_{\beta', \beta''}}' (\langle 2D, \beta' \rangle - \langle D, \beta \rangle)(\langle 2D, \beta'' \rangle - \langle D, \beta \rangle) \Delta_{\beta', \beta''}.$$

By Lemma 3.3 this is,

$$4\pi_*(D \cdot D) + 4\langle D, \beta\rangle \pi_*(D \cdot C_1(\omega_\pi) + (\langle D, \beta\rangle)^2 Q_\pi(C_1(\omega_\pi)) = \\ {\sum_{\beta',\beta''}}' (4\langle D, \beta'\rangle\langle D, \beta''\rangle - (\langle D, \beta\rangle)^2)\Delta_{\beta',\beta''}.$$

By Proposition 4.1, $Q_\pi(\omega_\pi) = -{\sum_{\beta',\beta''}}' \Delta_{\beta',\beta''}$. Substituting this into the equation, simplifying, and dividing by 4 gives the relation. □

Lemma 5.2 *For every pair, D_1, D_2, of Cartier divisor classes on C of relative degrees $\langle D_1, \beta\rangle$, resp. $\langle D_2, \beta\rangle$, modulo 2-power torsion,*

$$2\pi_*(D_1 \cdot D_2) + \langle D_1, \beta\rangle \pi_*(D_2 \cdot C_1(\omega_\pi)) + \langle D_2, \beta\rangle \pi_*(D_1 \cdot C_1(\omega_\pi)) =$$

$${\sum_{\beta',\beta''}}' (\langle D_1, \beta'\rangle\langle D_2, \beta''\rangle + \langle D_2, \beta'\rangle\langle D_1, \beta''\rangle)\Delta_{\beta',\beta''}.$$

Proof This follows from Proposition 1.2(i) and the polarization identity for quadratic forms. □

Lemma 5.3 *For every section of π, $s : M \to C$, whose image is contained in the smooth locus of π,*

$$s(M) \cdot s(M) + s(M) \cdot C_1(\omega_\pi) = 0.$$

Proof This follows by adjunction since the relative dualizing sheaf of $s(M) \to M$ is trivial. □

Lemma 5.4 *For every section of π, $s : M \to C$, whose image is contained in the smooth locus of π and for every Cartier divisor class D on C of relative degree $\langle D, \beta\rangle$ over M, modulo 2-power torsion,*

$$2\langle D, \beta\rangle s^* D - \pi_*(D \cdot D) - \langle D, \beta\rangle^2 \pi_*(s(M) \cdot s(M)) =$$

$${\sum_{\beta',\beta''}}' (\langle D, \beta'\rangle^2 \langle s(M), \beta''\rangle + \langle D, \beta''\rangle^2 \langle s(M), \beta'\rangle)\Delta_{\beta',\beta''}.$$

Proof By Lemma 5.2, with $D_1 = s(M)$ and with $D_2 = D$,

$$2s^* D + \pi_*(D \cdot C_1(\omega_\pi)) + \langle D, \beta\rangle \pi_*(s(M) \cdot C_1(\omega_\pi)) =$$

$${\sum}' (\langle D, \beta'\rangle\langle s(M), \beta''\rangle + \langle D, \beta''\rangle\langle s(M), \beta'\rangle)\Delta_{\beta',\beta''}.$$

Multiplying both sides by $\langle D, \beta\rangle$, we obtain,

$$2\langle D, \beta\rangle s^* D + \langle D, \beta\rangle \pi_*(D \cdot C_1(\omega_\pi)) + \langle D, \beta\rangle^2 \pi_*(s(M) \cdot C_1(\omega_\pi)) =$$

$${\sum}' (\langle D, \beta\rangle\langle D, \beta'\rangle\langle s(M), \beta''\rangle + \langle D, \beta\rangle\langle D, \beta''\rangle\langle s(M), \beta'\rangle)\Delta_{\beta',\beta''}.$$

First of all, by Lemma 5.3, $\langle D, \beta\rangle^2\pi_*(s(M)\cdot C_1(\omega_\pi)) = -\langle D, \beta\rangle^2\pi_*(s(M)\cdot s(M))$. Next, by Proposition 1.2(i),

$$\langle D, \beta\rangle\pi_*(D\cdot C_1(\omega_\pi)) = -\pi_*(D\cdot D) + {\sum}' \langle D, \beta'\rangle\langle D, \beta''\rangle\Delta_{\beta',\beta''}.$$

Finally,

$$\langle D, \beta\rangle\langle D, \beta'\rangle\langle s(M), \beta''\rangle + \langle D, \beta\rangle\langle D, \beta''\rangle\langle s(M), \beta'\rangle =$$

$$(\langle D, \beta'\rangle + \langle D, \beta''\rangle)\langle D, \beta'\rangle\langle s(M), \beta''\rangle + (\langle D, \beta'\rangle + \langle D, \beta''\rangle)\langle D, \beta''\rangle\langle s(M), \beta'\rangle =$$

$$\langle D, \beta'\rangle^2\langle s(M), \beta''\rangle + \langle D, \beta''\rangle^2\langle s(M), \beta'\rangle + \langle D, \beta'\rangle\langle D, \beta''\rangle(\langle s(M), \beta'\rangle + \langle s(M), \beta''\rangle) =$$

$$\langle D, \beta'\rangle^2\langle s(M), \beta''\rangle + \langle D, \beta''\rangle^2\langle s(M), \beta'\rangle + \langle D, \beta'\rangle\langle D, \beta''\rangle.$$

Plugging in these 3 identities and simplifying gives the relation. □

Proof of Proposition 1.2(ii). Let $\pi : \mathfrak{C} \to \mathfrak{M}_{0,0}$ denote the universal family. Let $\mathfrak{C}_{\text{smooth}}$ denote the smooth locus of π. The 2-fibered product $\text{pr}_1 : \mathfrak{C}_{\text{smooth}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathfrak{C}_{\text{smooth}}$ together with the diagonal $\Delta : \mathfrak{C}_{\text{smooth}} \to \mathfrak{C}_{\text{smooth}} \times_{\mathfrak{M}_{0,0}} \mathfrak{C}$ determine a 1-morphism $\mathfrak{C}_{\text{smooth}} \to \mathfrak{M}_{0,1}$. This extends to a 1-morphism $\mathfrak{C} \to \mathfrak{M}_{0,1}$. The pull-back of the universal curve is a 1-morphism $\pi' : \mathfrak{C}' \to \mathfrak{C}$ that factors through $\text{pr}_1 : \mathfrak{C} \times_{\mathfrak{M}_{0,0}} \mathfrak{C} \to \mathfrak{C}$. Denote the pullback of the universal section by $s : \mathfrak{C} \to \mathfrak{C}'$. Now $\mathfrak{C}$ is regular, and the complement of $\mathfrak{C}_{\text{smooth}}$ has codimension 2. Therefore $s^*\mathcal{O}_{\mathfrak{C}'}(s(\mathfrak{C}))$ can be computed on $\mathfrak{C}_{\text{smooth}}$. But the restriction to $\mathfrak{C}_{\text{smooth}}$ is clearly $\omega_\pi^\vee$. Therefore $s^*\mathcal{O}_{\mathfrak{C}'}(s(\mathfrak{C})) \cong \omega_\pi^\vee$ on all of $\mathfrak{C}$.

Pulling this back by $\zeta_C : C \to \mathfrak{C}$ gives a 1-morphism $\pi' : C' \to C$ that factors through $\text{pr}_1 : C \times_M C \to C$. The induced morphism $C' \to C \times_M C$ is Let D be a Cartier divisor class on C and consider the pullback to C' of $\text{pr}_2^* D$ on $C \times_M C$. This is a Cartier divisor class D' on C'. Of course $s^*D' = D$. Moreover, by the projection formula, since $C' \to C \times_M C$ is birational, the pushforward to $C \times_M C$ of $D' \cdot D'$ is $\text{pr}_2^*(D\cdot D)$. Therefore $(\pi')_*(D'\cdot D')$ is $(\text{pr}_1)_*\text{pr}_2^*(D\cdot D)$, i.e., $\pi^*\pi_*(D\cdot D)$. Finally, denote by,

$$\sum_{\beta',\beta''} \langle D, \beta''\rangle^2 \widetilde{\Delta}_{\beta',\beta''},$$

the divisor class on C,

$${\sum_{\beta',\beta''}}' (\langle D', \beta''\rangle^2\langle s(C), \beta'\rangle + \langle D', \beta'\rangle^2\langle s(C), \beta''\rangle)\Delta_{\beta',\beta''}.$$

The point is this: if π is smooth over every generic point of M, then the divisor class $\widetilde{\Delta}_{\beta',\beta''}$ is the irreducible component of $\pi^{-1}(\Delta_{\beta',\beta''})$ corresponding to the vertex v', i.e., the irreducible component with "curve class" β'. Therefore Proposition 1.2(ii) follows from Lemma 5.4. □

Remark 5.5 If $\langle D, \beta\rangle \neq 0$ then, at least up to torsion, Proposition 1.2(i) follows from Proposition 1.2(ii) by intersecting both sides of the relation by D and then applying π_*. This was pointed out by Pandharipande, who also proved Lemma 5.4 up to numerical equivalence in [8, Lemma 2.2.2] (by a very different method).

Lemma 5.6 *Let $s, s' : M \to C$ be sections with image in the smooth locus of π such that $s(M)$ and $s'(M)$ are disjoint. Then,*

$$\pi_*(s(M)\cdot s(M)) + \pi_*(s'(M)\cdot s'(M)) = -\sum_{\beta',\beta''}\langle s(M),\beta'\rangle\langle s'(M),\beta''\rangle\Delta_{\beta',\beta''}.$$

Proof Apply Lemma 5.2 and use $s(M)\cdot s'(M) = 0$ and Lemma 5.3. □

Lemma 5.7 *Let $r \geq 2$ and $s_1,\dots,s_r : M \to C$ be sections with image in the smooth locus of π and which are pairwise disjoint. Then,*

$$-\sum_{i=1}^{r}\pi_*(s_i(M)\cdot s_i(M)) = (r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle\langle s_2(M) + \cdots + s_r(M),\beta''\rangle\Delta_{\beta',\beta''}.$$

Proof This follows from Lemma 5.6 by induction. □

Lemma 5.8 *Let $r \geq 2$ and let $s_1,\dots,s_r : M \to C$ be sections with image in the smooth locus of π and which are pairwise disjoint. Then,*

$$-\sum_{i=1}^{r}\pi_*(s_i(M)\cdot s_i(M)) = r(r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle\langle s_2(M) + \cdots + s_r(M),\beta''\rangle^2\Delta_{\beta',\beta''}.$$

Combined with Lemma 5.7 this gives,

$$(r-1)(r-2)\pi_*(s_1(M)\cdot s_1(M)) = -\sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle\langle s_2(M) + \cdots + s_r(M),\beta''\rangle(\langle s_2(M) + \cdots + s_r(M),\beta''\rangle - 1)\Delta_{\beta',\beta''},$$

which in turn gives,

$$-(r-1)\textstyle\sum_{i=1}^{r}\pi_*(s_i(M)\cdot s_i(M)) = \sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle\langle s_2(M) + \cdots + s_r(M),\beta''\rangle(r - \langle s_2(M) + \cdots + s_r(M),\beta''\rangle)\Delta_{\beta',\beta''}.$$

In the notation of Example 2.2, this is,

$$-(r-1)(r-2)\pi_*(s_1(M)\cdot s_1(M)) = \sum_{(A,B),\ 1\in A} \#B(\#B-1)\Delta_{(A,B)},$$

and

$$-(r-1)\sum_{i=1}^{r}\pi_*(s_i(M)\cdot s_i(M)) = \sum_{(A,B),\ 1\in A} \#B(r-\#B)\Delta_{(A,B)}.$$

Proof Denote $D = \sum_{i=2}^{r} s_i(M)$. Apply Lemma 5.4 to get,

$$2(r-1)\cdot 0 - \textstyle\sum_{i=2}^{r}\pi_*(s_i(M)\cdot s_i(M)) - (r-1)^2\pi_*(s_1(M)\cdot s_1(M)) =$$

$$\textstyle\sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle\langle s_2(M)+\cdots+s_r(M),\beta''\rangle^2\Delta_{\beta',\beta''}.$$

Simplifying,

$$-\sum_{j=1}^{r}\pi_*(s_i(M)\cdot s_i(M)) = r(r-2)\pi_*(s_1(M)\cdot s_1(M)) + \sum\langle s_1(M),\beta'\rangle\langle s_2(M)$$
$$+\cdots+s_r(M),\beta''\rangle^2\Delta_{\beta',\beta''}.$$

Subtracting from the relation in Lemma 5.7 gives the relation for $(r-1)(r-2)\pi_*(s_1(M)\cdot s_1(M))$. Multiplying the first relation by $(r-1)$, plugging in the second relation and simplifying gives the third relation. □

Lemma 5.9 *Let $r \geq 2$, and let $s_1, \ldots, s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus. For every $1 \leq i < j \leq r$, using the notation from Example 2.2,*

$$\sum_{(A,B),\ i\in A} \#B(r-\#B)\Delta_{(A,B)} = \sum_{(A',B'),\, j\in A'} \#B'(r-\#B')\Delta_{(A',B')}.$$

Proof This follows from Lemma 5.8 by permuting the roles of 1 with i and j. □

Lemma 5.10 *Let $r \geq 2$, and let $s_1, \ldots, s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus of π. For every Cartier divisor class D on C of relative degree $\langle D, \beta\rangle$,*

$$2(r-1)(r-2)\langle D,\beta\rangle s_1^* D = (r-1)(r-2)\pi_*(D\cdot D)$$
$$+\sum_{\beta',\beta''}\langle s_1(M),\beta'\rangle a(D,\beta'')\Delta_{\beta',\beta''},$$

where,

$$a(D,\beta'') = (r-1)(r-2)\langle D,\beta''\rangle^2 - \langle D,\beta\rangle^2\langle s_2(M) + \cdots + s_r(M),\beta''\rangle(\langle s_2(M)+\cdots+s_r(M),\beta''\rangle - 1).$$

In particular, if $r \geq 3$, then modulo torsion s_i^*D is in the span of $\pi_*(D\cdot D)$ and boundary divisors for every $i = 1,\ldots,r$.

Proof This follows from Lemmas 5.4 and 5.8. □

Lemma 5.11 *Let $s_1,\ldots,s_r : M \to C$ be everywhere disjoint sections with image in the smooth locus of π, possibly with $r = 0$ (i.e., no sections specified). Consider the sheaf $\mathcal{E} = \Omega_\pi(s_1(M)+\cdots+s_r(M))$. The perfect complex $R\pi_* RHom_{\mathcal{O}_C}(\mathcal{E},\mathcal{O}_C)$ has rank $3-r$ and the first Chern class of the determinant is $-2\Delta - \sum_{i=1}^r \pi_*(s_i(M)\cdot s_i(M))$. In particular, if $r\geq 2$, up to torsion,*

$$C_1(detR\pi_* RHom_{\mathcal{O}_C}(\Omega_\pi(s_1(M)+\cdots+s_r(M)),\mathcal{O}_C)) =$$

$$-2\Delta + \tfrac{1}{r-1}\textstyle\sum_{(A,B),\ 1\in A}\#B(r-\#B)\Delta_{(A,B)}.$$

Proof There is a short exact sequence,

$$0 \longrightarrow \Omega_\pi \longrightarrow \Omega_\pi(s_1(M)+\cdots+s_r(M)) \longrightarrow \oplus_{i=1}^r (s_i)_*\mathcal{O}_M \longrightarrow 0.$$

Combining this with Lemmas 4.2, and 5.8, and chasing through exact sequences gives the lemma. □

6 Proof of Theorem 1.1

Let k be a field, let X be a connected, smooth algebraic space over k of dimension n, let M be an Artin stack over k, let $\pi : C \to M$ be a flat 1-morphism, representable by proper algebraic spaces whose geometric fibers are connected, at-worst-nodal curves of arithmetic genus 0, let $s_1,\ldots,s_r : M \to C$ be pairwise disjoint sections with image contained in the smooth locus of π (possibly $r=0$, i.e., there are no sections), and let $f : C \to X$ be a 1-morphism of k-stacks. In this setting, Behrend and Fantechi introduced a perfect complex $E^\bullet$ on M of amplitude $[-1,1]$ and a morphism to the cotangent complex, $\phi : E^\bullet \to L^\bullet_M$, [3]. If $\mathrm{char}(k) = 0$ and M is the Deligne–Mumford stack of stable maps to X, Behrend and Fantechi prove $E^\bullet$ has amplitude $[-1,0]$, $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. In many interesting cases, ϕ is a quasi-isomorphism. Then $\det(E^\bullet)$ is an invertible dualizing sheaf for M. Because of this, $\det(E^\bullet)$ is called the *virtual canonical bundle*. In this section the relations from Sect. 5 are used to give a formula for the divisor class of the virtual canonical bundle. Assume that Hypothesis 5.1 holds for π.

Denote by $L_{(\pi,f)}$ the cotangent complex of the morphism $(\pi, f) : C \to M \times X$. This is a perfect complex of amplitude $[-1, 0]$. There is a distinguished triangle,

$$L_\pi \longrightarrow L_{(\pi,f)} \longrightarrow f^*\Omega_X[1] \longrightarrow L_\pi[1].$$

There is a slight variation $L_{(\pi,f,s)}$ taking into account the sections which fits into a distinguished triangle,

$$L_\pi(s_1(M) + \cdots + s_r(M)) \longrightarrow L_{(\pi,f,s)} \longrightarrow f^*\Omega_X[1] \longrightarrow L_\pi(s_1(M) \\ + \cdots + s_r(M))[1].$$

The complex $E^\bullet$ is defined to be $(R\pi_*(L^\vee_{(\pi,f,s)})[1])^\vee$, where $(F^\bullet)^\vee$ is $RHom(F^\bullet, \mathcal{O})$. In particular, $\det(E^\bullet)$ is the determinant of $R\pi_*(L^\vee_{(\pi,f,s)})$. From the distinguished triangle, $\det(E^\bullet)$ is

$$\det(R\pi_* RHom_{\mathcal{O}_C}(\Omega_\pi(s_1(M) + \cdots + s_r(M)), \mathcal{O}_C)) \otimes \det(R\pi_* f^*T_X)^\vee.$$

By Lemma 5.11, the first term is known. The second term follows easily from Grothendieck–Riemann–Roch.

Lemma 6.1 *Assume that the relative degree of $f^*C_1(\Omega_X)$ is nonzero. Then $R\pi_* f^* T_X[-1]$ has rank $\langle -f^*C_1(\Omega_X), \beta\rangle + n$, and up to torsion the first Chern class of the determinant is,*

$$\tfrac{1}{2\langle -f^*C_1(\Omega_X),\beta\rangle} \Big[2\langle -f^*C_1(\Omega_X), \beta\rangle \pi_* f^*C_2(\Omega_X) \\ -(\langle -f^*C_1(\Omega_X), \beta\rangle + 1)\pi_* f^*C_1(\Omega_X)^2 + \\ \sum{}' \langle -f^*C_1(\Omega_X), \beta'\rangle \langle -f^*C_1(\Omega_X), \beta''\rangle \Delta_{\beta',\beta''}\Big].$$

Proof The Todd class τ_π of π is $1 - \frac{1}{2}C_1(\omega_\pi) + \tau_2 + \ldots$, where $\pi_*\tau_2 = 0$. The Chern character of f^*T_X is,

$$n - f^*C_1(\Omega_X) + \frac{1}{2}(f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X)) + \ldots$$

Therefore $\mathrm{ch}(f^*T_X) \cdot \tau_\pi$ equals,

$$n - \Big[f^*C_1(\Omega_X) + \frac{n}{2}C_1(\Omega_\pi)\Big] + \frac{1}{2}\big[f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X) \\ + f^*C_1(\Omega_X) \cdot C_1(\omega_\pi)\big] + n\tau_2 + \ldots$$

Applying π_* and using that $\pi_*\tau_2 = 0$, the rank is $n + \langle -f^*C_1(\Omega_X), \beta\rangle$, and the determinant has first Chern class,

$$\frac{1}{2}\pi_*\left[f^*C_1(\Omega_X)^2 - 2f^*C_2(\Omega_X)\right] + \frac{1}{2}\pi_*(f^*C_1(\Omega_X)\cdot C_1(\omega_\pi)).$$

Applying Proposition 1.2 and simplifying gives the relation. □

Putting the two terms together gives the formulas in Theorem 1.1.

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A Stronger Derived Torelli Theorem for K3 Surfaces

Max Lieblich and Martin Olsson

Abstract In an earlier paper the notion of a filtered derived equivalence was introduced, and it was shown that if two K3 surfaces admit such an equivalence, then they are isomorphic. In this paper we study more refined aspects of filtered derived equivalences related to the action on the cohomological realizations of the Mukai motive. It is shown that if a filtered derived equivalence between K3 surfaces also preserves ample cones then one can find an isomorphism that induces the same map as the equivalence on the cohomological realizations.

1 Introduction

1.1 Let k be an algebraically closed field of odd positive characteristic and let X and Y be K3 surfaces over k. Let

$$\Phi : D(X) \to D(Y)$$

be an equivalence between their bounded triangulated categories of coherent sheaves given by a Fourier–Mukai kernel $P \in D(X \times Y)$, so Φ is the functor given by sending $M \in D(X)$ to

$$R\mathrm{pr}_{2*}(L\mathrm{pr}_1^* M \otimes^{\mathbb{L}} P).$$

As discussed in [16, 2.9] the kernel P also induces an isomorphism on rational Chow groups modulo numerical equivalence

$$\Phi_P^{A^*} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}.$$

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We can consider how a given equivalence Φ interacts with the codimension filtration on A^*, or how it acts on the ample cone of X inside $A^1(X)$. The underlying philosophy of this work is that tracking filtrations and ample cones (in ways we will make precise in Sect. 2) gives a semi-linear algebraic gadget that behaves a lot like a Hodge structure. In Sect. 2 we will define a notion of *strongly filtered* for an equivalence Φ that imposes conditions reminiscent of the classical Torelli theorem for K3 surfaces.

With this in mind, the purpose of this paper is to prove the following result.

Theorem 1.2 *If $\Phi_P : D(X) \to D(Y)$ is a strongly filtered equivalence, then there exists an isomorphism $\sigma : X \to Y$ such that the maps on the crystalline and étale realizations of the Mukai motive induced by Φ_P and σ agree.*

For the definition of the realizations of the Mukai motive see [16, Sect. 2]. In [16, Proof of 6.2] it is shown that any filtered equivalence can be modified to be strongly filtered. As a consequence, we get a new proof of the following result.

Theorem 1.3 ([16, 6.1]). *If $\Phi_P^{A^*}$ preserves the codimension filtrations on $A^*(X)_{num,\mathbb{Q}}$ and $A^*(Y)_{num,\mathbb{Q}}$ then X and Y are isomorphic.*

Whereas the original proof of Theorem 1.3 relied heavily on liftings to characteristic 0 and Hodge theory, the proof presented here works primarily in positive characteristic using algebraic methods.

In Sect. 8 we present a proof of Theorem 1.2 using certain results about "Kulikov models" in positive characteristic (see Sect. 5). This argument implicitly uses Hodge theory which is an ingredient in the proof of Theorem 5.3. In Sect. 9 we discuss a characteristic 0 variant of Theorem 1.2, and finally in the last Sect. 10 we explain how to bypass the use of the Hodge theory ingredient of Theorem 5.3. This makes the argument entirely algebraic, except for the Hodge theory aspects of the proof of the Tate conjecture. This also gives a different algebraic perspective on the statement that any Fourier–Mukai partner of a K3 surface is a moduli space of sheaves, essentially inverting the methods of [16].

The bulk of this paper is devoted to proving Theorem 1.2. The basic idea is to consider a certain moduli stack $\mathscr{S}_d$ classifying data $((X, \lambda), Y, P)$ consisting of a primitively polarized K3 surface (X, λ) with polarization of some degree d, a second K3 surface Y, and a complex $P \in D(X \times Y)$ defining a strongly filtered Fourier–Mukai equivalence $\Phi_P : D(X) \to D(Y)$. The precise definition is given in Sect. 3, where it is shown that $\mathscr{S}_d$ is an algebraic stack which is naturally a $\mathbb{G}_m$-gerbe over a Deligne–Mumford stack $\overline{\mathscr{S}}_d$ étale over the stack $\mathscr{M}_d$ classifying primitively polarized K3 surfaces of degree d. The map $\overline{\mathscr{S}}_d \to \mathscr{M}_d$ is induced by the map sending a collection $((X, \lambda), Y, P)$ to (X, λ). We then study the locus of points in $\mathscr{S}_d$ where Theorem 1.2 holds showing that it is stable under both generization and specialization. From this it follows that it suffices to consider the case when X and Y are supersingular where we can use Ogus' crystalline Torelli theorem [24, Theorem I].

Remark 1.4 Our restriction to odd characteristic is because we appeal to the Tate conjecture for K3 surfaces, proven in odd characteristics by Charles, Maulik, and Madapusi Pera [7, 21, 26], which at present is not known in characteristic 2.

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2 Strongly Filtered Equivalences

2.1 Let X and Y be K3 surfaces over an algebraically closed field k and let $P \in D(X \times Y)$ be an object defining an equivalence

$$\Phi_P : D(X) \to D(Y),$$

and let

$$\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}$$

denote the induced map on Chow groups modulo numerical equivalence and tensored with $\mathbb{Q}$. We say that Φ_P is *filtered* (resp. *strongly filtered*, resp. *Torelli*) if $\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}}$ preserves the codimension filtration (resp. is filtered, sends $(1, 0, 0)$ to $(1, 0, 0)$, and sends the ample cone of X to plus or minus the ample cone of Y; resp. is filtered, sends $(1, 0, 0)$ to $\pm(1, 0, 0)$, and sends the ample cone of X to the ample cone of Y).

Remark 2.2 Note that if P is strongly filtered then either P or $P[1]$ is Torelli. If P is Torelli then either P or $P[1]$ is strongly filtered.

Remark 2.3 Note that $A^1(X)$ is the orthogonal complement of $A^0(X) \oplus A^2(X)$ and similarly for Y. This implies that if Φ_P is filtered and sends $(1, 0, 0)$ to $\pm(1, 0, 0)$ then $\Phi_P(A^1(X)_{\mathrm{num},\mathbb{Q}}) \subset A^1(Y)_{\mathrm{num},\mathbb{Q}}$.

Remark 2.4 It is shown in [16, 6.2] that if $\Phi_P : D(X) \to D(Y)$ is a filtered equivalence, then there exists a strongly filtered equivalence $\Phi : D(X) \to D(Y)$. In fact it is shown there that Φ can be obtained from Φ_P by composing with a sequence of shifts, twists by line bundles, and spherical twists along (-2)-curves.

2.5 As noted in [16, 2.11] an equivalence Φ_P is filtered if and only if the induced map on Chow groups

$$\Phi_P^{A^*_{\mathrm{num},\mathbb{Q}}} : A^*(X)_{\mathrm{num},\mathbb{Q}} \to A^*(Y)_{\mathrm{num},\mathbb{Q}}$$

sends $A^2(X)_{\mathrm{num},\mathbb{Q}}$ to $A^2(X)_{\mathrm{num},\mathbb{Q}}$.

Lemma 2.6 *Let ℓ be a prime invertible in k, let $\widetilde{H}(X, \mathbb{Q}_\ell)$ (resp. $\widetilde{H}(Y, \mathbb{Q}_\ell)$) denote the $\mathbb{Q}_\ell$-realization of the Mukai motive of X (resp. Y) as defined in [16, 2.4], and let*

$$\Phi_P^{\acute{e}t} : \widetilde{H}(X, \mathbb{Q}_\ell) \to \widetilde{H}(Y, \mathbb{Q}_\ell)$$

denote the isomorphism defined by P. Then Φ_P is filtered if and only if $\Phi_P^{\acute{e}t}$ preserves the filtrations by degree on $\widetilde{H}(X, \mathbb{Q}_\ell)$ and $\widetilde{H}(Y, \mathbb{Q}_\ell)$.

Proof By the same reasoning as in [16, 2.4] the map $\Phi_P^{\text{ét}}$ is filtered if and only if

$$\Phi_P^{\text{ét}}(H^4(X, \mathbb{Q}_\ell)) = H^4(Y, \mathbb{Q}_\ell).$$

Since the cycle class maps

$$A^2(X)_{\mathrm{num},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^4(X, \mathbb{Q}_\ell), \quad A^2(Y)_{\mathrm{num},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^4(Y, \mathbb{Q}_\ell)$$

are isomorphisms and the maps Φ_P and $\Phi_P^{\text{ét}}$ are compatible in the sense of [16, 2.10] it follows that if Φ_P is filtered then so is $\Phi_P^{\text{ét}}$. Conversely if $\Phi_P^{\text{ét}}$ is filtered then since the cycle class maps

$$A^*(X)_{\mathrm{num},\mathbb{Q}} \to \widetilde{H}(X, \mathbb{Q}_\ell), \quad A^*(Y)_{\mathrm{num},\mathbb{Q}} \to \widetilde{H}(Y, \mathbb{Q}_\ell)$$

are injective it follows that Φ_P is also filtered. □

Remark 2.7 The same proof as in Lemma 2.6 gives variant results for crystalline cohomology and in characteristic 0 de Rham cohomology.

The condition that Φ_P takes the ample cone to plus or minus the ample cone appears more subtle. A useful observation in this regard is the following.

Lemma 2.8 *Let $P \in D(X \times Y)$ be an object defining a filtered equivalence $\Phi_P : D(X) \to D(Y)$ such that $\Phi_P^{A^*_{num}}$ sends $(1, 0, 0)$ to $(1, 0, 0)$. Then Φ_P is strongly filtered if and only if for some ample invertible sheaf L on X the class $\Phi_P^{A^*_{num}}(L) \in NS(Y)_{\mathbb{Q}}$ is plus or minus an ample class.*

Proof Following [24, p. 366] define

$$V_X := \{x \in NS(X)_{\mathbb{R}} | x^2 > 0, \text{ and } \langle x, \delta \rangle \neq 0 \text{ for all } \delta \in NS(X) \text{ with } \delta^2 = -2\},$$

and define V_Y similarly. Since $\Phi_P^{A^*_{\mathrm{num}}}$ is an isometry it induces an isomorphism

$$\sigma : V_X \to V_Y.$$

By [24, Proposition 1.10 and Remark 1.10.9] the ample cone C_X (resp. C_Y) of X (resp. Y) is a connected component of V_X (resp. V_Y) and therefore either $\sigma(C_X) \cap C_Y = \emptyset$ or $\sigma(C_X) = C_Y$, and similarly $\sigma(-C_X) \cap C_Y = \emptyset$ or $\sigma(-C_X) = C_Y$. □

Proposition 2.9 *Let X and Y be K3-surfaces over a scheme S and let $P \in D(X \times_S Y)$ be a relatively perfect complex. Assume that X/S is projective. Then the set of points $s \in S$ for which the induced transformation on the derived category of the geometric fibers*

$$\Phi_{P_{\bar{s}}} : D(X_{\bar{s}}) \to D(Y_{\bar{s}})$$

is a strongly filtered equivalence is an open subset of S.

Proof By a standard reduction we may assume that S is of finite type over $\mathbb{Z}$.

First note that the condition that $\Phi_{P_{\bar{s}}}$ is an equivalence is an open condition. Indeed by [12, 5.9] if $P^\vee$ denotes $\mathscr{R}Hom(P, \mathscr{O}_{X\times_S Y})$ and Q denotes $P^\vee[2]$, then for every $s \in S$ the functor $\Phi_{Q_{\bar{s}}}$ is both left and right adjoint to $\Phi_{P_{\bar{s}}}$, and the adjunction maps are induced by morphisms of complexes

$$Rp_{13*}(Lp_{12}^* P \otimes^{\mathbb{L}} Lp_{23}^* Q) \to \Delta_{X*}\mathscr{O}_X, \quad Rq_{13*}(Lq_{12}^* Q \otimes Lq_{23}^* P) \to \Delta_{Y*}\mathscr{O}_Y,$$

where we write p_{ij} (resp. q_{ij}) for the projection from $X \times_S Y \times_S X$ (resp. $Y \times_S X \times_S Y$) to the product of the i-th and j-th factor. Since the condition that these adjunction maps are quasi-isomorphisms is clearly an open condition on S it follows that the condition that $\Phi_{P_{\bar{s}}}$ is an equivalence is open.

Replacing S by an open set we may therefore assume that $\Phi_{P_{\bar{s}}}$ is an equivalence in every fiber.

Next we show that the condition that Φ_P is filtered is an open and closed condition. For this we may assume we have a prime ℓ invertible in S. Let $f_X : X \to S$ (resp. $f_Y : Y \to S$) be the structure morphism. Define $\widetilde{\mathscr{H}}_{X/S}$ to be the lisse $\mathbb{Q}_\ell$-sheaf on S given by

$$\widetilde{\mathscr{H}}_{X/S} := (R^0 f_{X*}\mathbb{Q}_\ell(-1)) \oplus (R^2 f_{X*}\mathbb{Q}_\ell) \oplus (R^4 f_{X*}\mathbb{Q}_\ell)(1),$$

and define $\widetilde{\mathscr{H}}_{Y/S}$ similarly. The kernel P then induces a morphism of lisse sheaves

$$\Phi_{P/S}^{\text{ét},\ell} : \widetilde{\mathscr{H}}_{X/S} \to \widetilde{\mathscr{H}}_{Y/S}$$

whose restriction to each geometric fiber is the map on the $\mathbb{Q}_\ell$-realization of the Mukai motive as in [16, 2.4]. In particular, $\Phi_{P/S}^{\text{ét},\ell}$ is an isomorphism. By Lemma 2.6 for every geometric point $\bar{s} \to S$ the map $\Phi_{P_{\bar{s}}}$ is filtered if and only if the stalk $\Phi_{P/S,\bar{s}}^{\text{ét},\ell}$ preserves the filtrations on $\widetilde{\mathscr{H}}_{X/S}$ and $\widetilde{\mathscr{H}}_{Y/S}$. In particular this is an open and closed condition on S. Shrinking on S if necessary we may therefore further assume that $\Phi_{P_{\bar{s}}}$ is filtered for every geometric point $\bar{s} \to S$.

It remains to show that in this case the set of points s for which Φ_P takes the ample cone $C_{X_{\bar{s}}}$ of $X_{\bar{s}}$ to $\pm C_{Y_{\bar{s}}}$ is an open subset of S. For this we can choose, by our assumption that X/S is projective, a relatively ample invertible sheaf L on X. Define

$$M := \det(R\mathrm{pr}_{2*}(L\mathrm{pr}_1^*(L) \otimes P)),$$

an invertible sheaf on Y. Then by Lemma 2.8 for a point $s \in S$ the transformation $\Phi_{P_{\bar{s}}}$ is strongly filtered if and only if the restriction of M to the fiber $Y_{\bar{s}}$ is plus or minus the class of an ample divisor. By openness of the ample locus [9, III, 4.7.1] we get that being strongly filtered is an open condition. □

3 Moduli Spaces of K3 Surfaces

3.1 For an integer d invertible in k let $\mathscr{M}_d$ denote the stack over k whose fiber over a scheme T is the groupoid of pairs (X, λ) where X/T is a proper smooth algebraic space all of whose geometric fibers are K3 surfaces and $\lambda : T \to \mathrm{Pic}_{X/T}$ is a morphism to the relative Picard functor such that in every geometric fiber λ is given by a primitive ample line bundle L_λ whose self-intersection is $2d$. The following theorem summarizes the properties of the stack $\mathscr{M}_d$ that we will need (see [17, 2.10] for a more detailed discussion considering also the case when the characteristic of k divides d).

Theorem 3.2 *(i) $\mathscr{M}_d$ is a Deligne–Mumford stack, smooth over k of relative dimension* 19.

(ii) The geometric fiber of $\mathscr{M}_d$ is irreducible (here we use that d is invertible in k).

(iii) The locus $\mathscr{M}_{d,\infty} \subset \mathscr{M}_d$ classifying supersingular K3 surfaces is closed of dimension ≥ 9.

Proof A review of (i) and (iii) can be found in [23, p. 1]. Statement (ii) can be found in [17, 2.10 (3)]. □

Remark 3.3 The stack $\mathscr{M}_d$ is defined over $\mathbb{Z}$, and it follows from (ii) that the geometric generic fiber of $\mathscr{M}_d$ is irreducible (this follows also from the Torelli theorem over $\mathbb{C}$ and the resulting description of $\mathscr{M}_{d,\mathbb{C}}$ as a period space). Furthermore over $\mathbb{Z}[1/d]$ the stack $\mathscr{M}_d$ is smooth. In what follows we denote this stack over $\mathbb{Z}[1/d]$ by $\mathscr{M}_{d,\mathbb{Z}[1/d]}$ and reserve the notation $\mathscr{M}_d$ for its reduction to k.

Remark 3.4 Note that in the definition of $\mathscr{M}_d$ we consider ample invertible sheaves, and don't allow contractions in the corresponding morphism to projective space.

3.5 Let $\mathscr{S}_d$ denote the fibered category over k whose fiber over a scheme S is the groupoid of collections of data

$$((X, \lambda), Y, P), \tag{3.5.1}$$

where $(X, \lambda) \in \mathscr{M}_d(S)$ is a polarized K3 surface, Y/S is a second K3 surface over S, and $P \in D(X \times_S Y)$ is an S-perfect complex such that for every geometric point $\bar{s} \to S$ the induced functor

$$\Phi_{P_{\bar{s}}} : D(X_{\bar{s}}) \to D(Y_{\bar{s}})$$

is strongly filtered.

Theorem 3.6 *The fibered category $\mathscr{S}_d$ is an algebraic stack locally of finite type over k.*

Proof By fppf descent for algebraic spaces we have descent for both polarized and unpolarized K3 surfaces.

To verify descent for the kernels P, consider an object (3.5.1) over a scheme S. Let $P^\vee$ denote $\mathscr{R}Hom(P, \mathscr{O}_X)$. Since P is a perfect complex we have $\mathscr{R}Hom(P, P) \simeq P^\vee \otimes P$. By [15, 2.1.10] it suffices to show that for all geometric points $\bar{s} \to S$ we have $H^i(X_{\bar{s}} \times Y_{\bar{s}}, P_{\bar{s}}^\vee \otimes P_{\bar{s}}) = 0$ for $i < 0$. This follows from the following result (we discuss Hochschild cohomology further in Sect. 4 below):

Lemma 3.7 ([28, 5.6], [11, 5.1.8]). *Let X and Y be K3 surfaces over an algebraically closed field k, and let $P \in D(X \times Y)$ be a complex defining a Fourier–Mukai equivalence $\Phi_P : D(X) \to D(Y)$. Denote by $HH^*(X)$ the Hochschild cohomology of X defined as*

$$RHom_{X\times X}(\Delta_*\mathscr{O}_X, \Delta_*\mathscr{O}_X).$$

(i) There is a canonical isomorphism $Ext^*_{X\times Y}(P, P) \simeq HH^*(X)$.
(ii) $Ext^i_{X\times Y}(P, P) = 0$ for $i < 0$ and $i = 1$.
(iii) The natural map $k \to Ext^0_{X\times Y}(P, P)$ is an isomorphism.

Proof Statement (i) is [28, 5.6]. Statements (ii) and (iii) follow immediately from this, since $HH^1(X) = 0$ for a K3 surface. □

Next we show that for an object (3.5.1) the polarization λ on X induces a polarization λ_Y on Y. To define λ_Y we may work étale locally on S so may assume there exists an ample invertible sheaf L on X defining λ. The complex

$$\Phi_P(L) := R\mathrm{pr}_{2*}(\mathrm{pr}_1^* L \otimes^{\mathbb{L}} P)$$

is S-perfect, and therefore a perfect complex on Y. Let M denote the determinant of $\Phi_P(L)$, so M is an invertible sheaf on Y. By our assumption that Φ^{P_s} is strongly filtered for all $s \in S$, the restriction of M to any fiber is either ample or antiample. It follows that either M or $M^\vee$ is a relatively ample invertible sheaf and we define λ_Y to be the resulting polarization on Y. Note that this does not depend on the choice of line bundle L representing λ and therefore by descent λ_Y is defined even when no such L exists.

The degree of λ_Y is equal to d. Indeed if $s \in S$ is a point then since Φ^{P_s} is strongly filtered the induced map $NS(X_{\bar{s}}) \to NS(Y_{\bar{s}})$ is compatible with the intersection pairings and therefore $\lambda_Y^2 = \lambda^2 = 2d$.

From this we deduce that $\mathscr{S}_d$ is algebraic as follows. We have a morphism

$$\mathscr{S}_d \to \mathscr{M}_d \times \mathscr{M}_d, \quad ((X, \lambda), Y, P) \mapsto ((X, \lambda), (Y, \lambda_Y)), \qquad (3.7.1)$$

and $\mathscr{M}_d \times \mathscr{M}_d$ is an algebraic stack. Let $\mathscr{X}$ (resp. $\mathscr{Y}$) denote the pullback to $\mathscr{M}_d \times \mathscr{M}_d$ of the universal family over the first factor (resp. second factor). Sending a triple $((X, \lambda), Y, P)$ to P then realizes $\mathscr{S}_d$ as an open substack of the stack over $\mathscr{M}_d \times \mathscr{M}_d$ of simple universally gluable complexes on $\mathscr{X} \times_{\mathscr{M}_d \times \mathscr{M}_d} \mathscr{Y}$ (see for example [16, Sect. 5]). □

3.8 Observe that for any object $((X, \lambda), Y, P) \in \mathscr{S}_d$ over a scheme S there is an inclusion

$$\mathbb{G}_m \hookrightarrow \underline{\mathrm{Aut}}_{\mathscr{S}_d}((X, \lambda), Y, P)$$

giving by scalar multiplication by P. We can therefore form the rigidification of $\mathscr{S}_d$ with respect to $\mathbb{G}_m$ (see for example [2, Sect. 5]) to get a morphism

$$g : \mathscr{S}_d \to \overline{\mathscr{S}}_d$$

realizing $\mathscr{S}_d$ as a $\mathbb{G}_m$-gerbe over another algebraic stack $\overline{\mathscr{S}}_d$. By the universal property of rigidification the map $\mathscr{S}_d \to \mathscr{M}_d$ sending $((X, \lambda), Y, P)$ to (X, λ) induces a morphism

$$\pi : \overline{\mathscr{S}}_d \to \mathscr{M}_d. \tag{3.8.1}$$

Theorem 3.9 *The stack $\overline{\mathscr{S}}_d$ is Deligne–Mumford and the map (3.8.1) is étale.*

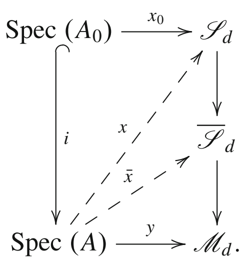
Proof Consider the map (3.7.1). By the universal property of rigidification this induces a morphism

$$q : \overline{\mathscr{S}}_d \to \mathscr{M}_d \times \mathscr{M}_d.$$

Since $\mathscr{M}_d \times \mathscr{M}_d$ is Deligne–Mumford, to prove that $\overline{\mathscr{S}}_d$ is a Deligne–Mumford stack it suffices to show that q is representable. This follows from Lemma 3.7 (iii) which implies that for any object $((X, \lambda), Y, P)$ over a scheme S the automorphisms of this object which map under q to the identity are given by scalar multiplication on P by elements of $\mathscr{O}_S^*$.

It remains to show that the map (3.8.1) is étale, and for this it suffices to show that it is formally étale.

Let $A \to A_0$ be a surjective map of artinian local rings with kernel I annhilated by the maximal ideal of A, and let k denote the residue field of A_0 so I can be viewed as a k-vector space. Let $((X_0, \lambda_0), Y_0, P_0) \in \mathscr{S}_d(A_0)$ be an object and let $(X, \lambda) \in \mathscr{M}_d(A)$ be a lifting of (X_0, λ_0) so we have a commutative diagram of solid arrows



Since $\mathscr{S}_d$ is a $\mathbb{G}_m$-gerbe over $\overline{\mathscr{S}}_d$, the obstruction to lifting a map $\bar{x}$ as indicated to a morphism x is given by a class in $H^2(\text{Spec }(A), \widetilde{I}) = 0$, and therefore any such map $\bar{x}$ can be lifted to a map x. Furthermore, the set of isomorphism classes of such liftings x of $\bar{x}$ is given by $H^1(\text{Spec }(A), \widetilde{I}) = 0$ so in fact the lifting x is unique up to isomorphism. The isomorphism is not unique but determined up to the action of

$$\text{Ker}(A^* \to A_0^*) \simeq I.$$

From this it follows that it suffices to show the following:

(i) The lifting (X, λ) of (X_0, λ_0) can be extended to a lifting $((X, \lambda), Y, P)$ of $((X_0, \lambda_0), Y_0, P_0)$.
(ii) This extension $((X, \lambda), Y, P)$ of (X, λ) is unique up to isomorphism.
(iii) The automorphisms of the triple $((X, \lambda), Y, P)$ which are the identity on (X, λ) and reduce to the identity over A_0 are all given by scalar multiplication on P by elements of $1 + I \subset A^*$.

Statement (i) is shown in [16, 6.3].

Next we prove the uniqueness statements in (ii) and (iii). Following the notation of [16, Discussion preceding 5.2], let $s\mathscr{D}_{X/A}$ denote the stack of simple, universally gluable, relatively perfect complexes on X, and let $sD_{X/A}$ denote its rigidification with respect to the $\mathbb{G}_m$-action given by scalar multiplication. The complex P_0 on $X_0 \times_{A_0} Y_0$ defines a morphism

$$Y_0 \to sD_{X/A} \otimes_A A_0$$

which by [16, 5.2 (ii)] is an open imbedding. Any extension of (X, λ) to a lifting $((X, \lambda), Y, P)$ defines an open imbedding $Y \hookrightarrow sD_{X/A}$. This implies that Y, viewed as a deformation of Y_0 for which there exists a lifting P of P_0 to $X \times_A Y$, is unique up to unique isomorphism.

Let Y denote the unique lifting of Y_0 to an open subspace of $sD_{X/A}$. By [15, 3.1.1 (2)] the set of isomorphism classes of liftings of P_0 to $X \times_A Y$ is a torsor under

$$\text{Ext}^1_{X_k \times Y_k}(P_k, P_k) \otimes I,$$

which is 0 by Lemma 3.7 (ii). From this it follows that P is unique up to isomorphism, and also by Lemma 3.7 (iii) we get the statement that the only infinitesimal automorphisms of the triple $((X, \lambda), Y, P)$ are given by scalar multiplication by elements of $1 + I$. □

3.10 There is an automorphism

$$\sigma : \mathscr{S}_d \to \mathscr{S}_d$$

satisfying $\sigma^2 = \mathrm{id}$. This automorphism is defined by sending a triple $((X, \lambda), Y, P)$ to $((Y, \lambda_Y), X, P^\vee[2])$. This automorphism induces an involution $\bar{\sigma} : \overline{\mathscr{S}}_d \to \overline{\mathscr{S}}_d$ over the involution $\gamma : \mathscr{M}_d \times \mathscr{M}_d \to \mathscr{M}_d \times \mathscr{M}_d$ switching the factors.

Remark 3.11 In fact the stack $\mathscr{S}_d$ is defined over $\mathbb{Z}[1/d]$ and Theorems 3.6 and 3.9 also hold over $\mathbb{Z}[1/d]$. In what follows we write $\mathscr{S}_{d,\mathbb{Z}[1/d]}$ for this stack over $\mathbb{Z}[1/d]$.

4 Deformations of Autoequivalences

In this section, we describe the obstructions to deforming Fourier–Mukai equivalences. The requisite technical machinery for this is worked out in [13, 14]. The results of this section will play a crucial role in Sect. 6.

Throughout this section let k be a perfect field of positive characteristic p and ring of Witt vectors W. For an integer n let R_n denote the ring $k[t]/(t^{n+1})$, and let R denote the ring $k[[t]]$.

4.1 Let X_{n+1}/R_{n+1} be a smooth proper scheme over R_{n+1} with reduction X_n to R_n. We then have the associated *relative Kodaira–Spencer class*, defined in [13, p. 486], which is the morphism in $D(X_n)$

$$\kappa_{X_n/X_{n+1}} : \Omega^1_{X_n/R_n} \to \mathscr{O}_{X_n}[1]$$

defined as the morphism corresponding to the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X_n} \xrightarrow{\cdot dt} \Omega^1_{X_{n+1}/k}|_{X_n} \longrightarrow \Omega^1_{X_n/R_n} \longrightarrow 0.$$

4.2 We also have the *relative universal Atiyah class* which is a morphism

$$\alpha_n : \mathscr{O}_{\Delta_n} \to i_{n*}\Omega^1_{X_n/R_n}[1]$$

in $D(X_n \times_{R_n} X_n)$, where $i_n : X_n \to X_n \times_{R_n} X_n$ is the diagonal morphism and $\mathscr{O}_{\Delta_n}$ denotes $i_{n*}\mathscr{O}_{X_n}$.

This map α_n is given by the class of the short exact sequence

$$0 \to I/I^2 \to \mathscr{O}_{X_n \times_{R_n} X_n}/I^2 \to \mathscr{O}_{\Delta_n} \to 0,$$

where $I \subset \mathscr{O}_{X_n \times_{R_n} X_n}$ is the ideal of the diagonal. Note that to get the morphism α_n we need to make a choice of isomorphism $I/I^2 \simeq \Omega^1_{X_n/R_n}$, which implies that the relative universal Atiyah class is not invariant under the map switching the factors, but rather changes by -1.

4.3 Define the *relative Hochschild cohomology* of X_n/R_n by

$$HH^*(X_n/R_n) := \mathrm{Ext}^*_{X_n \times_{R_n} X_n}(\mathscr{O}_{\Delta_n}, \mathscr{O}_{\Delta_n}).$$

The composition

$$\mathscr{O}_{\Delta_n} \xrightarrow{\alpha_n} i_{n*}\Omega^1_{X_n/R_n}[1] \xrightarrow{i_{n*}\kappa_{X_n/X_{n+1}}} \mathscr{O}_{\Delta_n}[2]$$

is a class

$$\nu_{X_n/X_{n+1}} \in HH^2(X_n/R_n).$$

4.4 Suppose now that Y_n/R_n is a second smooth proper scheme with a smooth lifting Y_{n+1}/R_{n+1} and that $E_n \in D(X_n \times_{R_n} Y_n)$ is a R_n-perfect complex.

Consider the class

$$\nu := \nu_{X_n \times_{R_n} Y_n / X_{n+1} \times_{R_{n+1}} Y_{n+1}} : \mathscr{O}_{\Delta_{n, X_n \times_{R_n} Y_n}} \to \mathscr{O}_{\Delta_{n, X_n \times_{R_n} Y_n}}[2].$$

Viewing this is a morphism of Fourier–Mukai kernels

$$D(X_n \times_{R_n} Y_n) \to D(X_n \times_{R_n} Y_n)$$

and applying it to E_n we get a class

$$\omega(E_n) \in \mathrm{Ext}^2_{X_n \times_{R_n} Y_n}(E_n, E_n).$$

In the case when

$$\mathrm{Ext}^1_{X_0 \times Y_0}(E_0, E_0) = 0,$$

which will hold in the cases of interest in this paper, we know by [13, Lemma 3.2] that the class $\omega(E_n)$ is 0 if and only if E_n lifts to a perfect complex on $X_{n+1} \times_{R_{n+1}} Y_{n+1}$.

4.5 To analyze the class $\omega(E_n)$ it is useful to translate it into a statement about classes in $HH^2(Y_n/R_n)$. This is done using Toda's argument [28, Proof of 5.6]. Denote by p_{ij} the various projections from $X_n \times_{R_n} X_n \times_{R_n} Y_n$, and let

$$E_n \circ : D(X_n \times_{R_n} X_n) \to D(X_n \times_{R_n} Y_n)$$

denote the Fourier–Mukai functor defined by the pushforward of $p_{23}^* E_n$ along the morphism

$$\mathrm{id}_{X_n} \times \Delta_{X_n} \times \mathrm{id}_{Y_n} : X_n \times_{R_n} \times X_n \times_{R_n} Y_n \to (X_n \times_{R_n} X_n) \times_{R_n} (X_n \times_{R_n} Y_n).$$

So for an object $K \in D(X_n \times_{R_n} X_n)$ the complex $E_n \circ K \in D(X_n \times_{R_n} Y_n)$ represents the Fourier–Mukai transform $\Phi_{E_n} \circ \Phi_K$, and is given explicitly by the complex

$$p_{13*}(p_{12}^* K \otimes p_{23}^* E_n).$$

As in loc. cit. the diagram

$$\begin{array}{ccc} D(X_n) & \xrightarrow{p_1^*(-)\otimes E_n} & \\ \downarrow{\scriptstyle i_{n*}} & \searrow & \\ D(X_n \times_{R_n} X_n) & \xrightarrow{E_n\circ} & D(X_n \times_{R_n} Y_n) \end{array}$$

commutes.

In particular we get a morphism

$$\eta_X^* : HH^*(X_n/R_n) \to \mathrm{Ext}^*_{X_n \times_{R_n} Y_n}(E_n, E_n).$$

Now assume that both X_n and Y_n have relative dimension d over R_n and that the relative canonical sheaves of X_n and Y_n over R_n are trivial. Let $E_n^\vee$ denote $\mathscr{R}Hom(E_n, \mathscr{O}_{X_n \times_{R_n} Y_n})$ viewed as an object of $D(Y_n \times_{R_n} X_n)$. In this case the functor

$$\Phi_{E_n^\vee[d]} : D(Y_n) \to D(X_n)$$

is both a right and left adjoint of Φ_{E_n} [4, 4.5]. By the same argument, the functor

$$\circ E_n^\vee[d] : D(X_n \times_{R_n} Y_n) \to D(Y_n \times_{R_n} Y_n),$$

defined in the same manner as $E_n\circ$ has left and right adjoint given by

$$\circ E_n : D(Y_n \times_{R_n} Y_n) \to D(X_n \times_{R_n} Y_n).$$

Composing with the adjunction maps

$$\alpha : \mathrm{id} \to \circ E_n \circ E_n^\vee[d], \quad \beta : \circ E_n \circ E_n^\vee[d] \to \mathrm{id} \tag{4.5.1}$$

applied to the diagonal $\mathscr{O}_{\Delta_{Y_n}}$ we get a morphism

$$\eta_{Y*} : \mathrm{Ext}^*_{X_n \times_{R_n} Y_n}(E_n, E_n) \to HH^*(Y_n/R_n).$$

We denote the composition

$$\eta_{Y*}\eta_X^* : HH^*(X_n/R_n) \to HH^*(Y_n/R_n)$$

by $\Phi_{E_n}^{HH^*}$. In the case when E_n defines a Fourier–Mukai equivalence this agrees with the standard definition (see for example [28]).

4.6 Evaluating the adjunction maps (4.5.1) on $\mathscr{O}_{\Delta_{Y_n}}$ we get a morphism

$$\mathscr{O}_{\Delta_{Y_n}} \xrightarrow{\alpha} \mathscr{O}_{\Delta_{Y_n}} \circ E_n \circ E_n^\vee[d] \xrightarrow{\beta} \mathscr{O}_{\Delta_{Y_n}}. \tag{4.6.1}$$

We say that E_n is *admissible* if this composition is the identity map.

If E_n is a Fourier–Mukai equivalence, then it is clear that E_n is admissible. Another example is if there exists a lifting $(\mathscr{X}, \mathscr{Y}, \mathscr{E})$ of (X_n, Y_n, E_n) to R, where $\mathscr{X}$ and $\mathscr{Y}$ are smooth proper R-schemes with trivial relative canonical bundles and $\mathscr{E}$ is a R-perfect complex on $\mathscr{X} \times_R \mathscr{Y}$, such that the restriction $\mathscr{E}$ to the generic fiber defines a Fourier–Mukai equivalence. Indeed in this case the map (4.6.1) is the reduction of the corresponding map $\mathscr{O}_{\Delta_{\mathscr{Y}}} \to \mathscr{O}_{\Delta_{\mathscr{Y}}}$ defined over R, which in turn is determined by its restriction to the generic fiber.

4.7 Consider Hochschild homology

$$HH_i(X_n/R_n) := H^{-i}(X_n, Li_n^*\mathscr{O}_{\Delta_n}).$$

As explained for example in [5, Sect. 5] we also get a map

$$\Phi_{E_n}^{HH_*} : HH_*(X_n/R_n) \to HH_*(Y_n/R_n).$$

Hochschild homology is a module over Hochschild cohomology, and an exercise (that we do not write out here) shows that the following diagram

$$\begin{array}{ccc} HH^*(X_n/R_n) \times HH_*(X_n/R_n) & \xrightarrow{\Phi_{E_n}^{HH^*} \times \Phi_{E_n}^{HH_*}} & HH^*(Y_n/R_n) \times HH_*(Y_n/R_n) \\ \downarrow \text{mult} & & \downarrow \text{mult} \\ HH_*(X_n/R_n) & \xrightarrow{\Phi_{E_n}^{HH_*}} & HH_*(Y_n/R_n) \end{array}$$

commutes.

Remark 4.8 We thank the referee for pointing out that the necessary functoriality of Hochschild homology was settled by Keller, Swan, and Weibel by the mid-1990s. See for example [10], [30, 9.8.19], and [27]. The Ref. [5] provides a very readable account of the results we need here.

4.9 Using this we can describe the obstruction $\omega(E_n)$ in a different way, assuming that E_n is admissible. First note that viewing the relative Atiyah class of $X_n \times_{R_n} Y_n$ as a morphism of Fourier–Mukai kernels we get the Atiyah class of E_n which is a morphism

$$A(E_n) : E_n \to E_n \otimes \Omega^1_{X_n \times_{R_n} Y_n/R_n}[1]$$

in $D(X_n \times_{R_n} Y_n)$. There is a natural decomposition

$$\Omega^1_{X_n \times_{R_n} Y_n/R_n} \simeq p_1^*\Omega^1_{X_n/R_n} \oplus p_2^*\Omega^1_{Y_n/R_n},$$

so we can write $A(E_n)$ as a sum of two maps

$$A(E_n)_X : E_n \to E_n \otimes p_1^*\Omega^1_{X_n/R_n}[1], \quad A(E_n)_Y : E_n \to E_n \otimes p_2^*\Omega^1_{Y_n/R_n}[1].$$

Similarly the Kodaira–Spencer class of $X_n \times_{R_n} Y_n$ can be written as the sum of the two pullbacks

$$p_1^*\kappa_{X_n/X_{n+1}} : p_1^*\Omega^1_{X_n/R_n} \to p_1^*\mathcal{O}_{X_n}[1], \quad p_2^*\kappa_{Y_n/Y_{n+1}} : p_2^*\Omega^1_{Y_n/R_n} \to p_2^*\mathcal{O}_{Y_n}[1].$$

It follows that the obstruction $\omega(E_n)$ can be written as a sum

$$\omega(E_n) = (p_1^*\kappa_{X_n/X_{n+1}} \circ A(E_n)_X) + (p_2^*\kappa(Y_n/Y_{n+1}) \circ A(E_n)_Y).$$

Now by construction we have

$$\eta^*_{X_n}(\nu_{X_n/X_{n+1}}) = p_1^*\kappa_{X_n/X_{n+1}} \circ A(E_n)_X,$$

and

$$\eta_{Y_n*}(p_2^*\kappa(Y_n/Y_{n+1}) \circ A(E_n)_Y) = -\nu_{Y_n/Y_{n+1}},$$

the sign coming from the asymmetry in the definition of the relative Atiyah class (it is in the verification of this second formula that we use the assumption that E_n is admissible). Summarizing we find the formula

$$\eta_{Y_n*}(\omega(E_n)) = \Phi^{HH^*}_{E_n}(\nu_{X_n/X_{n+1}}) - \nu_{Y_n/Y_{n+1}}. \tag{4.9.1}$$

In the case when Φ_{E_n} is an equivalence the maps η_{Y_n*} and $\eta^*_{X_n}$ are isomorphisms, so the obstruction $\omega(E_n)$ vanishes if and only if we have

$$\Phi^{HH^*}_{E_n}(\nu_{X_n/X_{n+1}}) - \nu_{Y_n/Y_{n+1}} = 0.$$

Remark 4.10 By [13, Remark 2.3 (iii)], the functor Φ_{E_n} is an equivalence if and only if $\Phi_{E_0} : D(X_0) \to D(Y_0)$ is an equivalence.

Corollary 4.11 *Suppose $F_n \in D(X_n \times_{R_n} Y_n)$ defines a Fourier–Mukai equivalence, and that $E_n \in D(X_n \times_{R_n} Y_n)$ is another admissible R_n-perfect complex such that $\Phi^{HH^*}_{F_n} = \Phi^{HH^*}_{E_n}$. If E_n lifts to a R_{n+1}-perfect complex $E_{n+1} \in D(X_{n+1} \times_{R_{n+1}} Y_{n+1})$ then so does F_n.*

Proof Indeed the condition that $\Phi^{HH^*}_{F_n} = \Phi^{HH^*}_{E_n}$ ensures that

$$\eta_{Y_n*}(\omega(E_n)) = \eta_{Y_n*}(\omega(F_n)),$$

and since $\omega(E_n) = 0$ we conclude that $\omega(F_n) = 0$. □

4.12 The next step is to understand the relationship between $\Phi^{HH^*}_{E_n}$ and the action of Φ_{E_n} on the cohomological realizations of the Mukai motive.

Assuming that the characteristic p is bigger than the dimension of X_0 (which in our case will be a K3 surface so we just need $p > 2$) we can exponentiate the relative Atiyah class to get a map

$$\exp(\alpha_n) : \mathcal{O}_{\Delta_n} \to \oplus_i i_{n*}\Omega^i_{X_n/R_n}$$

which by adjunction defines a morphism

$$Li_n^* \mathcal{O}_{\Delta_n} \to \oplus_i \Omega^i_{X_n/R_n} \tag{4.12.1}$$

in $D(X_n)$. By [1, Theorem 0.7], which also holds in positive characteristic subject to the bounds on dimension, this map is an isomorphism. We therefore get an isomorphism

$$I^{HKR} : HH^*(X_n/R_n) \to HT^*(X_n/R_n),$$

where we write

$$HT^*(X_n/R_n) := \oplus_{p+q=*} H^p(X_n, \bigwedge^q T_{X_n/R_n}).$$

We write

$$I^K_{X_n} : HH^*(X_n/R_n) \to HT^*(X_n/R_n)$$

for the composition of I^{HKR} with multiplication by the inverse square root of the Todd class of X_n/R_n, as in [6, 1.7].

The isomorphism (4.12.1) also defines an isomorphism

$$I_{HKR} : HH_*(X_n/R_n) \to H\Omega_*(X_n/R_n),$$

where

$$H\Omega_*(X_n/R_n) := \oplus_{q-p=*} H^p(X_n, \Omega^q_{X_n/R_n}).$$

We write

$$I_K^{X_n} : HH_*(X_n/R_n) \to H\Omega_*(X_n/R_n)$$

for the composition of I_{HKR} with multiplication by the square root of the Todd class of X_n/R_n.

We will consider the following condition ($\star$) on a R_n-perfect complex $E_n \in D(X_n \times_{R_n} Y_n)$:

($\star$) The diagram

$$\begin{array}{ccc} HH_*(X_n/R_n) & \xrightarrow{\Phi^{HH_*}_{E_n}} & HH_*(Y_n/R_n) \\ \downarrow{\scriptstyle I_K^{X_n}} & & \downarrow{\scriptstyle I_K^{Y_n}} \\ H\Omega_*(X_n/R_n) & \xrightarrow{\Phi^{H\Omega_*}_{E_n}} & H\Omega_*(Y_n/R_n) \end{array}$$

commutes.

Remark 4.13 We expect this condition to hold in general. Over a field of characteristic 0 this is shown in [19, 1.2]. We expect that a careful analysis of denominators occurring of their proof will verify ($\star$) quite generally with some conditions on the characteristic relative to the dimension of the schemes. However, we will not discuss this further in this paper.

4.14 There are two cases we will consider in this paper were ($\star$) is known to hold:

(i) If $E_n = \mathscr{O}_{\Gamma_n}$ is the structure sheaf of the graph of an isomorphism $\gamma_n : X_n \to Y_n$. In this case the induced maps on Hochschild cohomology and $H\Omega_*$ are simply given by pushforward γ_{n*} and condition ($\star$) immediately holds.

(ii) Suppose $B \to R_n$ is a morphism from an integral domain B which is flat over W and that there exists a lifting $(\mathscr{X}, \mathscr{Y}, \mathscr{E})$ of (X_n, Y_n, E_n) to B, where $\mathscr{X}$ and $\mathscr{Y}$ are proper and smooth over B and $\mathscr{E} \in D(\mathscr{X} \times_B \mathscr{Y})$ is a B-perfect complex pulling back to E_n. Suppose further that the groups $HH^*(\mathscr{X}/B)$ and $HH^*(\mathscr{Y}/B)$ are flat over B and their formation commutes with base change (this holds for example if $\mathscr{X}$ and $\mathscr{Y}$ are K3 surfaces). Then ($\star$) holds. Indeed it suffices to verify the commutativity of the corresponding diagram over B, and this in turn can be verified after passing to the field of fractions of B. In this case the result holds by [19, 1.2].

Lemma 4.15 *Let $E_n, F_n \in D(X_n \times_{R_n} Y_n)$ be two R_n-perfect complexes satisfying condition* ($\star$). *Suppose further that the maps $\Phi^{crys}_{E_0}$ and $\Phi^{crys}_{F_0}$ on the crystalline realizations $\widetilde{H}(X_0/W) \to \widetilde{H}(Y_0/W)$ of the Mukai motive are equal. Then the maps $\Phi^{HH_*}_{E_n}$ and $\Phi^{HH_*}_{F_n}$ are also equal. Furthermore if the maps on the crystalline realizations are isomorphisms then $\Phi^{HH_*}_{E_n}$ and $\Phi^{HH_*}_{F_n}$ are also isomorphisms.*

Proof Since $H\Omega_*(X_n/R_n)$ (resp. $H\Omega_*(Y_n/R_n)$) is obtained from the de Rham realization $\widetilde{H}_{\mathrm{dR}}(X_n/R_n)$ (resp. $\widetilde{H}_{\mathrm{dR}}(X_n/R_n)$) of the Mukai motive of X_n/R_n (resp. Y_n/R_n) by passing to the associated graded, it suffices to show that the two maps

$$\Phi^{\mathrm{dR}}_{E_n}, \Phi^{\mathrm{dR}}_{F_n} : \widetilde{H}_{\mathrm{dR}}(X_n/R_n) \to \widetilde{H}_{\mathrm{dR}}(Y_n/R_n)$$

are equal, and isomorphisms when the crystalline realizations are isomorphisms. By the comparison between crystalline and de Rham cohomology it suffices in turn to show that the two maps on the crystalline realizations

$$\Phi^{\mathrm{crys}}_{E_n}, \Phi^{\mathrm{crys}}_{F_n} : \widetilde{H}_{\mathrm{crys}}(X_n/W[t]/(t^{n+1})) \to \widetilde{H}_{\mathrm{crys}}(Y_n/W[t]/(t^{n+1}))$$

are equal. Via the Berthelot-Ogus isomorphism [3, 2.2], which is compatible with Chern classes, these maps are identified after tensoring with $\mathbb{Q}$ with the two maps obtained by base change from

$$\Phi^{\mathrm{crys}}_{E_0}, \Phi^{\mathrm{crys}}_{F_0} : \widetilde{H}_{\mathrm{crys}}(X_0/W) \to \widetilde{H}_{\mathrm{crys}}(Y_0/W).$$

The result follows. □

4.16 In the case when X_n and Y_n are K3 surfaces the action of $HH^*(X_n/R_n)$ on $HH_*(X_n/R_n)$ is faithful. Therefore from Lemma 4.15 we obtain the following.

Corollary 4.17 *Assume that X_n and Y_n are K3 surfaces and that $E_n, F_n \in D(X_n \times_{R_n} Y_n)$ are two R_n-perfect complexes satisfying condition $(\star)$. Suppose further that $\Phi^{crys}_{E_0}$ and $\Phi^{crys}_{F_0}$ are equal on the crystalline realizations of the Mukai motives of the reductions. Then $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{F_n}$ are equal.*

Proof Indeed since homology is a faithful module over cohomology the maps $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{F_n}$ are determined by the maps on Hochschild homology which are equal by Lemma 4.15. □

Corollary 4.18 *Let X_{n+1} and Y_{n+1} be K3 surfaces over R_{n+1} and assume given an admissible R_{n+1}-perfect complex E_{n+1} on $X_{n+1} \times_{R_{n+1}} Y_{n+1}$ such that E_n satisfies condition $(\star)$. Assume given an isomorphism $\sigma_n : X_n \to Y_n$ over R_n such that the induced map $\sigma_0 : X_0 \to Y_0$ defines the same map on crystalline realizations of the Mukai motive as E_0. Then σ_n lifts to an isomorphism $\sigma_{n+1} : X_{n+1} \to Y_{n+1}$.*

Proof Indeed by (4.9.1) and the fact that $\Phi^{HH^*}_{E_n}$ and $\Phi^{HH^*}_{\Gamma_{\sigma_n}}$ are equal by Corollary 4.17, we see that the obstruction to lifting σ_n is equal to the obstruction to lifting E_n, which is zero by assumption. □

Corollary 4.19 *Assume the hypotheses of Corollary 4.18 with the following exception: Φ_{E_0} is strongly filtered and σ_0 and Φ_{E_0} are only assumed to induce the same map*

$$H^2_{\text{crys}}(X_0/W) \to H^2_{\text{crys}}(Y_0/W).$$

(In other words, we do not require the same action on all crystalline cohomology, just on the middle.) Then the same conclusion holds: σ_n lifts to some σ_{n+1}.

Proof The assumption that Φ_{E_0} is strongly filtered implies that the map $I^K_{Y_n} \circ \Phi^{HH^*}_{E_n} \circ (I^K_{X_n})^{-1}$ sends $H^1(X_n, T_{X_n})$ to $H^1(Y_n, T_{Y_n})$. By construction we have $I^K_{X_n}(\nu_{X_n/X_{n+1}}) \in H^1(X_n, T_{X_n})$; in fact, this class is given by the relative Kodaira-Spencer class multiplied by the square root of the Todd class. Now by looking at the module structure of $H\Omega_*$ over HT^* one sees that the map $H^1(X_n, T_{X_n}) \to H^1(Y_n, T_{Y_n})$ is determined by the action on H^2_{crys}. From this the conclusion follows. □

Remark 4.20 Applying the argument of Corollary 4.19 with $E_n[1]$ we see that if σ_0 induces $\pm\Phi_{E_0}$ on H^2_{crys} then σ_n can be lifted to R_{n+1}.

5 A Remark on Reduction Types

5.1 In the proof of Theorem 1.2 we need the following Theorem 5.3, whose proof relies on known characteristic 0 results obtained from Hodge theory. In Sect. 10 below we give a different algebraic argument for Theorem 5.3 in a special case which suffices for the proof of Theorem 1.2.

5.2 Let V be a complete discrete valuation ring with field of fractions K and residue field k. Let X/V be a projective K3 surface with generic fiber X_K, and let Y_K be a second K3 surface over K such that the geometric fibers $X_{\overline{K}}$ and $Y_{\overline{K}}$ are derived equivalent.

Theorem 5.3 *Under these assumptions the K3 surface Y_K has potentially good reduction.*

Remark 5.4 Here potentially good reduction means that after possibly replacing V be a finite extension there exists a K3 surface Y/V whose generic fiber is Y_K.

Proof of Theorem 5.3 We use [16, 1.1 (1)] which implies that after replacing V by a finite extension Y_K is isomorphic to a moduli space of sheaves on X_K.

After replacing V by a finite extension we may assume that we have a complex $P \in D(X \times Y)$ defining an equivalence

$$\Phi_P : D(X_{\overline{K}}) \to D(Y_{\overline{K}}).$$

Let $E \in D(Y \times X)$ be the complex defining the inverse equivalence

$$\Phi_E : D(Y_{\overline{K}}) \to D(X_{\overline{K}})$$

to Φ_P. Let $\nu := \Phi_E(0, 0, 1) \in A^*(X_{\overline{K}})_{\mathrm{num},\mathbb{Q}}$ be the Mukai vector of a fiber of E at a closed point $y \in Y_{\overline{K}}$ and write

$$\nu = (r, [L_X], s) \in A^0(X_{\overline{K}})_{\mathrm{num},\mathbb{Q}} \oplus A^1(X_{\overline{K}})_{\mathrm{num},\mathbb{Q}} \oplus A^2(X_{\overline{K}})_{\mathrm{num},\mathbb{Q}}.$$

By [16, 8.1] we may after possibly changing our choice of P, which may involve another extension of V, assume that r is prime to p and that L_X is very ample. Making another extension of V we may assume that ν is defined over K, and therefore by specialization also defines an element, which we denote by the same letter,

$$\nu = (r, [L_X], s) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}.$$

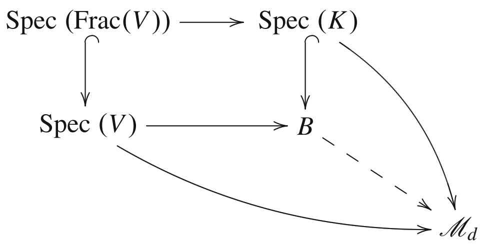
This class has the property that r is prime to p and that there exists another class ν' such that $\langle \nu, \nu' \rangle = 1$. This implies in particular that ν restricts to a primitive class on the closed fiber. Fix an ample class h on X, and let $\mathscr{M}_h(\nu)$ denote the moduli space of semistable sheaves on X with Mukai vector ν. By [16, 3.16] the stack $\mathscr{M}_h(\nu)$ is a μ_r-gerbe over a relative K3 surface $M_h(\nu)/V$, and by [16, 8.2] we have $Y_{\overline{K}} \simeq M_h(\nu)_{\overline{K}}$. In particular, Y has potentially good reduction. □

Remark 5.5 As discussed in [18, p. 2], to obtain Theorem 5.3 it suffices to know that every K3 surface Z_K over K has potentially semistable reduction and this would follow from standard conjectures on resolution of singularities and toroidization of morphisms in mixed and positive characteristic. In the setting of Theorem 5.3, once we know that Y_K has potentially semistable reduction then by [18, Theorem on bottom of p. 2] we obtain that Y_K has good reduction since the Galois representation $H^2(Y_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified being isomorphic to direct summand of the ℓ-adic realization $\widetilde{H}(X_{\overline{K}}, \mathbb{Q}_\ell)$ of the Mukai motive of X_K.

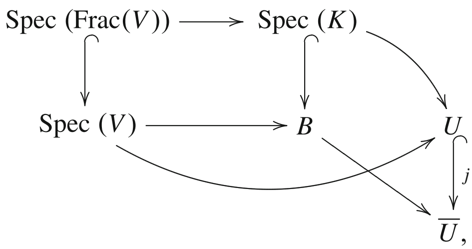
5.6 One can also consider the problem of extending Y_K over a higher dimensional base. Let B denote a normal finite type k-scheme with a point $s \in B(k)$ and let X/B be a projective family of K3 surfaces. Let K be the function field of B and let Y_K be a second K3 surface over K Fourier–Mukai equivalent to X_K. Dominating $\mathscr{O}_{B,s}$ by a suitable complete discrete valuation ring V we can find a morphism

$$\rho : \text{Spec}\,(V) \to B$$

sending the closed point of Spec (V) to s and an extension Y_V of $\rho^* Y_K$ to a smooth projective K3 surface over V. In particular, after replacing B by its normalization in a finite extension of K we can find a primitive polarization λ_K on Y_K of degree prime to the characteristic such that $\rho^*\lambda_K$ extends to a polarization on Y_V. We then have a commutative diagram of solid arrows



for a suitable integer d. Base changing to a suitable étale neighborhood $U \to \mathscr{M}_d$ of the image of the closed point of Spec (V), with U an affine scheme, we can after shrinking and possibly replacing B by an alteration find a commutative diagram



where j is a dense open imbedding and $\overline{U}$ is projective over k. It follows that the image of s in $\overline{U}$ in fact lands in U which gives an extension of Y_K to a neighborhood of s. This discussion implies the following:

Corollary 5.7 *In the setup of Paragraph 5.6, we can, after replacing (B, s) by a neighborhood of a point in the preimage of s in an alteration of B, find an extension of Y_K to a K3 surface over B.*

6 Supersingular Reduction

6.1 Let B be a normal scheme of finite type over an algebraically closed field k of characteristic p at least 5, and let W denote the ring of Witt vectors of k. Let K denote the function field of B and let $s \in B$ be a closed point. Let $f : X \to B$ be a projective K3 surface over B and let Y_K/K be a second K3 surface over K such that there exists a strongly filtered Fourier–Mukai equivalence

$$\Phi_P : D(X_K) \to D(Y_K)$$

defined by an object $P \in D(X_K \times_K Y_K)$. Assume further that the fiber X_s of X over s is a supersingular K3 surface.

6.2 Using Corollary 5.7 we can, after possibly replacing B by a neighborhood of a point over s in an alteration, assume that we have a smooth K3 surface Y/B extending Y_K and an extension of the complex P to a B-perfect complex $\mathscr{P}$ on $X \times_B Y$, and furthermore that the complex Q defining the inverse of Φ_P also extends to a complex $\mathscr{Q}$ on $X \times_B Y$. Let $f_{\mathscr{X}} : \mathscr{X} \to B$ (resp. $f_{\mathscr{Y}} : \mathscr{Y} \to B$) be the structure morphism, and let $\mathscr{H}^i_{\text{crys}}(X/B)$ (resp. $\mathscr{H}^i_{\text{crys}}(Y/B)$) denote the F-crystal $R^i f_{\mathscr{X}*}\mathscr{O}_{\mathscr{X}/W}$ (resp. $R^i f_{\mathscr{Y}*}\mathscr{O}_{\mathscr{Y}/W}$) on B/W obtained by forming the i-th higher direct image of the structure sheaf on the crystalline site of $\mathscr{X}/W$ (resp. $\mathscr{Y}/W$). Because $\Phi_{\mathscr{P}}$ is strongly filtered, it induces an isomorphism of F-crystals

$$\Phi^{\text{crys},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{crys}}(X/B) \to \mathscr{H}^i_{\text{crys}}(Y/B)$$

for all i, with inverse defined by $\Phi_{\mathscr{Q}}$. Note that since we are working here with K3 surfaces these morphisms are defined integrally.

We also have the de Rham realizations $\mathscr{H}^i_{\text{dR}}(X/B)$ and $\mathscr{H}^i_{\text{dR}}(Y/B)$ which are filtered modules with integrable connection on B equipped with filtered isomorphisms compatible with the connections

$$\Phi^{\text{dR},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{dR}}(X/B) \to \mathscr{H}^i_{\text{dR}}(Y/B). \tag{6.2.1}$$

as well as étale realizations $\mathscr{H}^i_{\text{ét}}(X/B)$ and $\mathscr{H}^i_{\text{ét}}(Y/B)$ equipped with isomorphisms

$$\Phi^{\text{ét},i}_{\mathscr{P}} : \mathscr{H}^i_{\text{ét}}(X/B) \to \mathscr{H}^i_{\text{ét}}(Y/B). \tag{6.2.2}$$

6.3 Let $H^i_{\text{crys}}(X_s/W)$ (resp. $H^i_{\text{crys}}(Y_s/W)$) denote the crystalline cohomology of the fibers over s. The isomorphism $\Phi^{\text{crys},2}_{\mathscr{P}}$ induces an isomorphism

$$\theta_2 : H^2_{\text{crys}}(X_s/W) \to H^2_{\text{crys}}(Y_s/W)$$

of F-crystals. By [24, Theorem I] this implies that X_s and Y_s are isomorphic. However, we may not necessarily have an isomorphism which induces θ_2 on cohomology.

6.4 Recall that as discussed in [12, 10.9 (iii)] if $C \subset X_K$ is a (-2)-curve then we can perform a spherical twist

$$T_{\mathscr{O}_C} : D(X_K) \to D(X_K)$$

whose action on $NS(X_K)$ is the reflection

$$r_C(a) := a + \langle a, C\rangle C.$$

Proposition 6.5 *After possibly changing our choice of model Y for Y_K, replacing (B, s) by a neighborhood of a point in an alteration over s, and composing with a sequence of spherical twists $T_{\mathscr{O}_C}$ along (-2)-curves in the generic fiber Y_K, there exists an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_2$ on the second crystalline cohomology group. If θ_2 preserves the ample cone of the generic fiber then we can find an isomorphism σ inducing θ_2.*

Proof By [25, 4.4 and 4.5] there exists an isomorphism $\theta_0 : NS(X_s) \to NS(Y_s)$ compatible with θ_2 in the sense that the diagram

$$\begin{array}{ccc} NS(X_s) & \xrightarrow{\theta_0} & NS(Y_s) \\ \downarrow c_1 & & \downarrow c_1 \\ H^2_{\text{crys}}(X_s/W) & \xrightarrow{\theta^2} & H^2_{\text{crys}}(Y_s/W) \end{array} \qquad (6.5.1)$$

commutes. Note that as discussed in [17, 4.8] the map θ_0 determines θ_2 by the Tate conjecture for K3 surfaces, proven by Charles, Maulik, and Pera [7, 21, 26]. In particular, if we have an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_0$ on Néron-Severi groups then σ also induces $\pm\theta_2$ on crystalline cohomology. We therefore have to study the problem of finding an isomorphism σ compatible with θ_0.

Ogus shows in [24, Theorem II] that there exists such an isomorphism σ if and only if the map θ_0 takes the ample cone to the ample cone. So our problem is to choose a model of Y in such a way that $\pm\theta_0$ preserves ample cones. Set

$$V_{X_s} := \{x \in NS(X_s)_{\mathbb{R}} | x^2 > 0 \text{ and } \langle x, \delta\rangle \neq 0 \text{ for all } \delta \in NS(X_s) \text{ with } \delta^2 = -2\},$$

and define V_{Y_s} similarly. Being an isometry the map θ_0 then induces an isomorphism $V_{X_s} \to V_{Y_s}$, which we again denote by θ_0. Let R_{Y_s} denote the group of automorphisms of V_{Y_s} generated by reflections in (-2)-curves and multiplication by -1. By [24, Proposition 1.10 and Remark 1.10.9] the ample cone of Y_s is a connected component of V_{Y_s} and the group R_{Y_s} acts simply transitively on the set of connected components of V_{Y_s}.

Let us show how to change model to account for reflections by (-2)-curves in Y_s. We show that after replacing (P, Y) by a new pair (P', Y') consisting of the complex $P \in D(X_K \times_K Y_K)$ obtained by composing Φ_P with a sequence of spherical twists along (-2)-curves in Y_K and replacing Y by a new model Y' there exists an isomorphism $\gamma : Y'_s \to Y_s$ such that the composition

$$NS(X_s) \xrightarrow{\theta_0} NS(Y_s) \xrightarrow{r_C} NS(Y_s) \xrightarrow{\gamma^*} NS(Y'_s)$$

is equal to the map θ'_0 defined as for θ_0 but using the model Y'.

Let $C \subset Y_s$ be a (-2)-curve, and let

$$r_C : NS(Y_s) \to NS(Y_s)$$

be the reflection in the (-2)-curve. If C lifts to a curve in the family Y we get a (-2)-curve in the generic fiber and so by replacing our P by the complex P' obtained by composition with the spherical twist by this curve in Y_K (see [12, 10.9 (iii)]) and setting $Y' = Y$ we get the desired new pair. If C does not lift to Y, then we take $P' = P$ but now replace Y by the flop of Y along C as explained in [24, 2.8].

Thus after making a sequence of replacements $(P, Y) \mapsto (P', Y')$ we can arrange that θ_0 sends the ample cone of X_s to plus or minus the ample cone of Y_s, and therefore we get our desired isomorphism σ.

To see the last statement, note that we have modified the generic fiber by composing with reflections along (-2)-curves. Therefore if λ is an ample class on X with restriction λ_K to X_K, and for a general ample divisor H we have $\langle \Phi_P(\lambda), H \rangle > 0$, then the same holds on the closed fiber. This implies that the ample cone of X_s gets sent to the ample cone of Y_s and not its negative. □

Remark 6.6 One can also consider étale or de Rham cohomology in Proposition 6.5. Assume we have applied suitable spherical twists and chosen a model Y such that we have an isomorphism $\sigma : X_s \to Y_s$ inducing $\pm\theta_0$. We claim that the maps

$$\theta_{\mathrm{dR}} : H^i_{\mathrm{dR}}(X_s/k) \to H^i_{\mathrm{dR}}(Y_s/k), \quad \theta_{\text{ét}} : H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \to H^i_{\text{ét}}(Y_s, \mathbb{Q}_\ell)$$

induced by the maps (6.2.1) and (6.2.2) also agree with the maps defined by $\pm\sigma$. For de Rham cohomology this is clear using the comparison with crystalline cohomology, and for the étale cohomology it follows from compatibility with the cycle class map.

6.7 With notation and assumptions as in Proposition 6.5 assume further that B is a curve or a complete discrete valuation ring, and that we have chosen a model Y such that each of the reductions satisfies the condition $(\star)$ of Paragraph 4.12 and such that the map θ_0 in (6.5.1) preserves plus or minus the ample cones. Let $\sigma : X_s \to Y_s$ be an isomorphism inducing $\pm\theta_0$.

Lemma 6.8 *The isomorphism σ lifts to an isomorphism $\tilde{\sigma} : X \to Y$ over the completion $\widehat{B}$ at s inducing the maps defined by $\pm\Phi_{\mathscr{P}}^{crys,i}$.*

Proof By Remark 4.20 σ lifts uniquely to each infinitesimal neighborhood of s in B, and therefore by the Grothendieck existence theorem we get a lifting $\tilde{\sigma}$ over $\widehat{B}$. That the realization of $\tilde{\sigma}$ on cohomology agrees with $\pm\Phi_{\mathscr{P}}^{\mathrm{crys},i}$ can be verified on the closed fiber where it holds by assumption. □

Lemma 6.9 *With notation and assumptions as in Lemma 6.8 the map $\Phi_P^{A^*_{num,\mathbb{Q}}}$ preserves the ample cones of the generic fibers.*

Proof The statement can be verified after making a field extension of the function field of B. If Φ_P does not preserve the ample cone, then by Lemma 6.8 we get an isomorphism $\sigma : X_K \to Y_K$ over some field extension K of $k(B)$ such that $\sigma^* \circ \Phi_P$ acts by $\mathrm{id}_{H^0} \bigoplus -\mathrm{id}_{H^2} \bigoplus \mathrm{id}_{H^4}$ on any cohomological realization. Lifting the associated derived equivalence to characteristic 0 (for example, using Deligne's liftability theorem [8] and Theorem 3.9) and applying [13, 4.1] we get a contradiction. □

Remark 6.10 In the case when the original Φ_P preserves the ample cones of the geometric generic fibers, no reflections along (−2)–curves in the generic fiber are needed. Indeed, by the above argument we get an isomorphism $\sigma_K : X_K \to Y_K$ such that the induced map on crystalline and étale cohomology agrees with $\Phi_P \circ \alpha$ for some sequence α of spherical twists along (−2)-curves in X_K (also using Lemma 6.9). Since both σ and Φ_P preserve ample cones it follows that α also preserves the ample cone of $X_{\overline{K}}$. By [24, 1.10] it follows that α acts trivially on the Néron-Severi group of $X_{\overline{K}}$. We claim that this implies that α also acts trivially on any of the cohomological realizations. We give the proof in the case of étale cohomology $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ (for a prime ℓ invertible in k) leaving slight modifications for the other cohomology theories to the reader. Let $\widetilde{R}_X$ denote the subgroup of $GL(H^2(X_{\overline{K}}, \mathbb{Q}_\ell))$ generated by −1 and the action induced by spherical twists along (−2)-curves in $X_{\overline{K}}$, and consider the inclusion of $\mathbb{Q}_\ell$-vector spaces with inner products

$$NS(X_{\overline{K}})_{\mathbb{Q}_\ell} \hookrightarrow H^2(X_{\overline{K}}, \mathbb{Q}_\ell).$$

By [12, Lemma 8.12] the action of the spherical twists along (−2)-curves in $X_{\overline{K}}$ on $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ is by reflection across classes in the image of $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$. From this (and Gram-Schmidt!) it follows that the the group $\widetilde{R}_X$ preserves $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$, acts trivially on the quotient of $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ by $NS(X_{\overline{K}})_{\mathbb{Q}_\ell}$, and that the restriction map

$$\widetilde{R}_X \to GL(NS(X_{\overline{K}})_{\mathbb{Q}_\ell})$$

is injective. In particular, if an element $\alpha \in \widetilde{R}_X$ acts trivially on $NS(X_{\overline{K}})$ then it also acts trivially on étale cohomology. It follows that σ and Φ_P induce the same map on realizations.

7 Specialization

7.1 We consider again the setup of Paragraph 6.1, but now we don't assume that the closed fiber X_s is supersingular. Further we restrict attention to the case when B is a smooth curve, and assume we are given a smooth model Y/B of Y_K and a B-perfect complex $\mathscr{P} \in D(X \times_B Y)$ such that for all geometric points $\bar{z} \to B$ the induced complex $\mathscr{P}_{\bar{z}}$ on $X_{\bar{z}} \times Y_{\bar{z}}$ defines a strongly filtered equivalence $D(X_{\bar{z}}) \to D(Y_{\bar{z}})$.

Let $\mathscr{H}^i(X/B)$ (resp. $\mathscr{H}^i(Y/B)$) denote either $\mathscr{H}^i_{\text{ét}}(X/B)$ (resp. $\mathscr{H}^i_{\text{ét}}(Y/B)$) for some prime $\ell \neq p$ or $\mathscr{H}^i_{\text{crys}}(X/B)$ (resp. $\mathscr{H}^i_{\text{crys}}(Y/B)$). Assume further given an isomorphism

$$\sigma_K : X_K \to Y_K$$

inducing the map given by restricting

$$\Phi^i_{\mathscr{P}} : \mathscr{H}^i(X/B) \to \mathscr{H}^i(Y/B)$$

to the generic point.

Remark 7.2 If we work with étale cohomology in this setup we could also consider the spectrum of a complete discrete valuation ring instead of B, and in particular also a mixed characteristic discrete valuation ring.

Proposition 7.3 *The isomorphism σ_K extends to an isomorphism $\sigma : X \to Y$.*

Proof We give the argument here for étale cohomology in the case when B is the spectrum of a discrete valuation ring. The other cases require minor modifications which we do not include here.

Let $Z \subset X \times_B Y$ be the closure of the graph of σ_K, so Z is an irreducible flat V-scheme of dimension 3 and we have a correspondence

$$\begin{array}{ccc} & Z & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & Y. \end{array}$$

Fix an ample line bundle L on X and consider the line bundle $M := \det(Rq_* p^* L)$ on Y. The restriction of M to Y_K is simply $\sigma_{K*}L$, and in particular the étale cohomology class of M is equal to the class of $\Phi_{\mathscr{P}}(L)$. By our assumption that $\Phi_{\mathscr{P}}$ is strongly filtered in the fibers the line bundle M is ample on Y. Note also that by our assumption that $\Phi_{\mathscr{P}}$ is strongly filtered in every fiber we have

$$\Phi_{\mathscr{P}}(L^{\otimes n}) \simeq \Phi^{\mathscr{P}}(L)^{\otimes n}.$$

In particular we can choose L very ample in such a way that M is also very ample. The result then follows from Matsusaka-Mumford [20, Theorem 2]. □

Remark 7.4 One can also prove a variant of Proposition 7.3 when k has characteristic 0 using de Rham cohomology instead of étale cohomology and similar techniques. However, we will not discuss this further here.

8 Proof of Theorem 1.2

In this section we prove Theorem 1.2 when the characteristic is ≥ 5. Characteristic 3 is treated in Remark 8.5

8.1 Let K be an algebraically closed field extension of k and let X and Y be K3 surfaces over K equipped with a complex $P \in D(X \times_K Y)$ defining a strongly filtered Fourier–Mukai equivalence

$$\Phi_P : D(X) \to D(Y).$$

We can then choose a primitive polarization λ on X of degree prime to p such that the triple $((X, \lambda), Y, P)$ defines a K-point of $\mathscr{S}_d$. In this way the proof of Theorem 1.2 is reformulated into showing the following: For any algebraically closed field K and point $((X, \lambda), Y, P) \in \mathscr{S}_d(K)$ there exists an isomorphism $\sigma : X \to Y$ such that the maps on crystalline and étale realizations defined by σ and Φ_P agree.

8.2 To prove this it suffices to show that there exists such an isomorphism after replacing K by a field extension. To see this let I denote the scheme of isomorphisms between X and Y, which is a locally closed subscheme of the Hilbert scheme of $X \times_K Y$, and is thus locally of finite type over K. Over I we have a tautological isomorphism $\sigma^u : X_I \to Y_I$. The condition that the induced action on ℓ-adic étale cohomology agrees with Φ_P is an open and closed condition on I. It follows that there exists a locally closed subscheme $I' \subset I$ classifying isomorphisms σ as in the theorem, and this I' is thus also locally of finite type over K. This implies that if we can find an isomorphism σ over a field extension of K then such an isomorphism also exists over K, since K is algebraically closed and any scheme locally of finite type over such a field K has a K-point if and only if it is nonempty (i.e., has a point over some extension of K).

8.3 By Proposition 7.3 it suffices to show that the result holds for each generic point of $\mathscr{S}_d$. By Theorem 3.9 any such generic point maps to a generic point of $\mathscr{M}_d$ which by Theorem 3.2 admits a specialization to a supersingular point $x \in \mathscr{M}_d(k)$ given by a family $(X_R, \lambda_R)/R$, where R is a complete discrete valuation ring over k with residue field Ω, for some algebraically closed field Ω. By Theorem 5.3 the point $(Y, \lambda_Y) \in \mathscr{M}_d(K)$ also has a limit $y \in \mathscr{M}_d(\Omega)$ given by a second family $(Y_R, \lambda_R)/R$. Let P' be the complex on $X \times Y$ giving the composition of Φ_P with suitable twists by (-2)-curves such that after replacing Y_R by a sequence of flops the map $\Phi_{P'}$ induces an isomorphism on crystalline cohomology on the closed fiber preserving plus or minus the ample cone. By the Cohen structure theorem we have $R \simeq \Omega[[t]]$, and $((X, \lambda), Y, P')$ defines a point of $\mathscr{S}_d(\Omega((t)))$.

Let B denote the completion of the strict henselization of $\mathscr{M}_{\mathbb{Z}[1/d]} \times \mathscr{M}_{\mathbb{Z}[1/d]}$ at the point (x, y). So B is a regular complete local ring with residue field Ω. Let B' denote the formal completion of the strict henselization of $\overline{\mathscr{S}}_{d,\mathbb{Z}[1/d]}$ at the $\Omega((t))$-point given by $((X, \lambda), Y, P')$. So we obtain a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & \Omega[[t]] \\ \downarrow & & \downarrow \\ B' & \longrightarrow & \Omega((t)). \end{array} \tag{8.3.1}$$

Over B we have a universal families $\mathscr{X}_B$ and $\mathscr{Y}_B$, and over the base changes to B' we have, after trivializing the pullback of the gerbe $\mathscr{S}_{d,\mathbb{Z}[1/d]} \to \overline{\mathscr{S}}_{d,\mathbb{Z}[1/d]}$, a complex $\mathscr{P}'_{B'}$ on $\mathscr{X}_{B'} \times_{B'} \mathscr{Y}_{B'}$, which reduces to the triple (X, Y, P') over $\Omega((t))$. The map $B \to B'$ is a filtering direct limit of étale morphisms. We can therefore replace B' by a finite type étale B-subalgebra over which all the data is defined and we still have the diagram (8.3.1). Let $\overline{B}$ denote the integral closure of B in B' so we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(B') & \hookleftarrow & \operatorname{Spec}(\overline{B}) \\ & \searrow & \downarrow \\ & & \operatorname{Spec}(B), \end{array}$$

where $\overline{B}$ is flat over $\mathbb{Z}[1/d]$ and normal. Let $Y \to \operatorname{Spec}(\overline{B})$ be an alteration with Y regular and flat over $\mathbb{Z}[1/d]$, and let $Y' \subset Y$ be the preimage of $\operatorname{Spec}(B')$. Lifting the map $B \to \Omega[[t]]$ to a map $\operatorname{Spec}(\widetilde{R}) \to Y$ for some finite extension of complete discrete valuation rings $\widetilde{R}/R$ and letting C denote the completion of the local ring of Y at the image of the closed point of $\operatorname{Spec}(\widetilde{R})$ we obtain a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & \Omega[[t]] \\ \downarrow & & \downarrow \\ C' & \longrightarrow & \Omega((t)), \end{array}$$

where $C \to C'$ is a localization, we have K3-surfaces $\mathscr{X}_C$ and $\mathscr{Y}_C$ over C and a perfect complex $\mathscr{P}'_{C'}$ on $\mathscr{X}_{C'} \times_{C'} \mathscr{Y}_{C'}$ defining a Fourier–Mukai equivalence and the triple $(\mathscr{X}_{C'}, \mathscr{Y}_{C'}, \mathscr{P}'_{C'})$ reducing to (X, Y, P) over $\Omega((t))$. By [29, 5.2.2] we can extend the complex $\mathscr{P}'_{C'}$ to a C-perfect complex $\mathscr{P}'_C$ on $\mathscr{X}_C \times_C \mathscr{Y}_C$ (here we use that C is regular). It follows that the base change $(X_{\Omega[[t]]}, Y_{\Omega[[t]]}, P'_{\Omega[[t]]})$ gives an extension of (X, Y, P) to $\Omega[[t]]$ all of whose reductions satisfy our condition $(\star)$ of Paragraph 4.12.

This puts us in the setting of Lemma 6.8, and we conclude that there exists an isomorphism $\sigma : X \to Y$ (over $\Omega((t))$, but as noted above we are allowed to make a field extension of K) such that the induced map on crystalline and étale cohomology agrees with $\Phi_P \circ \alpha$ for some sequence α of spherical twists along (-2)-curves in X (using also Lemma 6.9). By the same argument as in Remark 6.10 it follows that σ and Φ_P induce the same map on realizations which concludes the proof of Theorem 1.2. □

Remark 8.4 One consequence of the proof is that in fact any strongly filtered equivalence automatically takes the ample cone to the ample cone, and not its negative. This is closely related to [13, 4.1].

Remark 8.5 Given (X, Y, P) as in Theorem 1.2 and an isomorphism $\sigma : X \to Y$ inducing the same action as P on the ℓ-adic realization of the Mukai motive for a fixed prime ℓ invertible in k, we have that in fact σ and P define the same action on all the étale and crystalline realizations. This follows from the same specialization argument to supersingular K3s and [25, 3.20].

This observation can be used, in particular, to prove Theorem 1.2 in characteristic 3: With notation as in Theorem 1.2 fix a prime $\ell \neq 3$ and lift the triple (X, Y, P) to a triple $(\mathscr{X}, \mathscr{Y}, \mathscr{P})$ over a finite extension of $\mathbb{Z}_3$. By the specialization argument of Sect. 7 it then suffices to exhibit an isomorphism between the generic fibers inducing the same map as P on the ℓ-adic realizations. This reduces the result in characteristic 3 to the characteristic 0 result proven in Theorem 9.1 below.

9 Characteristic 0

From our discussion of positive characteristic results one can also deduce the following result in characteristic 0.

Theorem 9.1 *Let K be an algebraically closed field of characteristic* 0*, let X and Y be K3 surfaces over K, and let $\Phi_P : D(X) \to D(Y)$ be a strongly filtered Fourier–Mukai equivalence defined by an object $P \in D(X \times Y)$. Then there exists an isomorphism $\sigma : X \to Y$ whose action on ℓ-adic and de Rham cohomology agrees with the action of Φ_P.*

Proof It suffices to show that we can find an isomorphism σ which induces the same map on ℓ-adic cohomology as Φ_P for a single prime ℓ. For then by compatibility of the comparison isomorphisms with Φ_P, discussed in [16, Sect. 2], it follows that σ and Φ_P also define the same action on the other realizations of the Mukai motive.

Furthermore as in Paragraph 8.2 it suffices to prove the existence of σ after making a field extension of K.

As in Paragraph 8.2 let I' denote the scheme of isomorphisms $\sigma : X \to Y$ as in the theorem. Note that since the action of such σ on the ample cone is fixed, the scheme I' is in fact of finite type.

Since X, Y, and P are all locally finitely presented over K we can find a finite type integral $\mathbb{Z}$-algebra A, K3 surfaces X_A and Y_A over A, and an A-perfect complex $P_A \in D(X_A \times_A Y_A)$ defining a strongly filtered Fourier–Mukai equivalence in every fiber, and such that (X, Y, P) is obtained from (X_A, Y_A, P_A) by base change along a map $A \to K$. The scheme I' then also extends to a finite type A-scheme I'_A over A. Since I' is of finite type over A to prove that I' is nonempty it suffices to show that I'_A has nonempty fiber over $\mathbb{F}_p$ for infinitely many primes p. This holds by Theorem 1.2. □

10 Bypassing Hodge Theory

10.1 The appeal to analytic techniques implicit in the results of Sect. 5, where characteristic 0 results based on Hodge theory are used to deduce Theorem 5.3, can be bypassed in the following way using results of [18, 21].

10.2 Let R be a complete discrete valuation ring of equicharacteristic $p > 0$ with residue field k and fraction field K. Let X/R be a smooth K3 surface with supersingular closed fiber. Let Y_K be a K3 surface over K and $P_K \in D(X_K \times Y_K)$ a perfect complex defining a Fourier–Mukai equivalence $\Phi_{P_K} : D(X_{\overline{K}}) \to D(Y_{\overline{K}})$.

Theorem 10.3 *Assume that X admits an ample invertible sheaf L such that $p > L^2 + 4$. Then after replacing R by a finite extension there exists a smooth projective K3 surface Y/R with generic fiber Y_K.*

Proof Changing our choice of Fourier–Mukai equivalence P_K, we may assume that P_K is strongly filtered. Setting M_K equal to $\det(\Phi_{P_K}(L))$ or its dual, depending on whether Φ_{P_K} preserves ample cones, we get an ample invertible sheaf on Y_K of degree L^2. By [18, 2.2], building on Maulik's work [21, Discussion preceding 4.9] we get a smooth K3 surface Y/R with Y an algebraic space. Now after replacing P_K by the composition with twists along (-2)-curves and the model Y by a sequence of flops, we can arrange that the map on crystalline cohomology of the closed fibers induced by Φ_{P_K} preserves ample cones. Let $P \in D(X \times_R Y)$ be an extension of P_K and let M denote $\det(\Phi_P(L))$. Then M is a line bundle on Y whose reduction is ample on the closed fiber. It follows that M is also ample on Y so Y is a projective scheme. □

10.4 We use this to prove Theorem 1.2 in the case of étale realization in the following way. First observe that using the same argument as in Sect. 8, but now replacing the appeal to Theorem 5.3 by the above Theorem 10.3, we get Theorem 1.2 under the additional assumption that X admits an ample invertible sheaf L with $p > L^2 + 4$. By the argument of Sect. 9 this suffices to get Theorem 1.2 in characteristic 0, and by the specialization argument of Sect. 7 we then get also the result in arbitrary characteristic.

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Morphisms to Brauer–Severi Varieties, with Applications to Del Pezzo Surfaces

Christian Liedtke

Abstract We classify morphisms from proper varieties to Brauer–Severi varieties, which generalizes the classical correspondence between morphisms to projective space and globally generated invertible sheaves. As an application, we study del Pezzo surfaces of large degree with a view towards Brauer–Severi varieties, and recover classical results on rational points, the Hasse principle, and weak approximation.

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1 Introduction

1.1 Overview

The goal of this article is the study of morphisms $X \to P$ from a proper variety X over a field k to a Brauer–Severi variety P over k, i.e., P is isomorphic to projective space over the algebraic closure $\overline{k}$ of k, but not necessarily over k. If X has a k-rational point, then so has P, and then, P is isomorphic to projective space already over k. In this case, there exists a well-known description of morphisms $X \to P$ in terms of globally generated invertible sheaves on X. However, if X has no k-rational point, then we establish in this article a correspondence between globally generated classes of $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, whose obstruction to coming from an invertible sheaf on X is measured by some class β in the Brauer group $\mathrm{Br}(k)$, and morphisms to Brauer–Severi varieties of class β over k.

As an application of this correspondence, we study del Pezzo surfaces over k in terms of Brauer–Severi varieties, and recover many known results about their geometry and their arithmetic. If k is a global field, then we obtain applications concerning the Hasse principle and weak approximation. Our approach has the advantage of

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being elementary, self-contained, and that we sometimes obtain natural reasons for the existence of k-rational points.

1.2 Morphisms to Brauer–Severi Varieties

Let X be a proper variety over a field k, and let $\overline{k}$ be the algebraic closure of k. When studying invertible sheaves on X, there are inclusions and equalities of abelian groups

$$\mathrm{Pic}(X) \subseteq \mathrm{Pic}_{(X/k)(\text{ét})}(k) = \mathrm{Pic}_{(X/k)(\text{fppf})}(k) \subseteq \mathrm{Pic}(X_{\overline{k}}).$$

On the left (resp. right), we have invertible sheaves on X (resp. $X_{\overline{k}}$) up to isomorphism, whereas in the middle, we have sections of the sheafified relative Picard functor over k (with respect to the étale and fppf topology, respectively). Moreover, the first inclusion is part of an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}_{(X/k)(\text{ét})}(k) \xrightarrow{\delta} \mathrm{Br}(k),$$

where $\mathrm{Br}(k)$ denotes the Brauer group of the field k, and we refer to Remark 3.3 for explicit descriptions of δ. If X has a k-rational point, then δ is the zero map, i.e., the first inclusion is a bijection.

By definition, a *Brauer–Severi variety* is a variety P over k, such that $P_{\overline{k}} \cong \mathbb{P}^N_{\overline{k}}$ for some N, i.e., P is a twisted form of projective space. Associated to P, there exists a Brauer class $[P] \in \mathrm{Br}(k)$ and by a theorem of Châtelet, P is trivial, i.e., isomorphic to projective space over k, if and only if $[P] = 0$. This is also equivalent to P having a k-rational point. In any case, we have a class $\mathcal{O}_P(1) \in \mathrm{Pic}_{(P/k)(\text{fppf})}(k)$, in general not arising from an invertible sheaf on P, which becomes isomorphic to $\mathcal{O}_{\mathbb{P}^N}(1)$ over $\overline{k}$, see Definition 2.17.

In this article, we extend the notion of a *linear system* to classes in $\mathrm{Pic}_{(X/k)(\text{fppf})}(k)$ that do not necessarily come from invertible sheaves. More precisely, we extend the notions of being *globally generated*, *ample*, and *very ample* to such classes, see Definition 3.1. Then, we set up a dictionary between globally generated classes in $\mathrm{Pic}_{(X/k)(\text{fppf})}(k)$ and morphisms from X to Brauer–Severi varieties over k. In case X has a k-rational point, then we recover the well-known correspondence between globally generated invertible sheaves and morphisms to projective space. Here is an easy version of our correspondence and we refer to Theorem 3.4 and Remark 3.5 for details.

Theorem 1.1 *Let X be a proper variety over a field k.*

(1) Let $\varphi : X \to P$ be a morphism to a Brauer–Severi variety P over k. If we set $\mathcal{L} := \varphi^*\mathcal{O}_P(1) \in \mathrm{Pic}_{(X/k)(\text{fppf})}(k)$, then $\mathcal{L}$ is a globally generated class and

$$\delta(\mathcal{L}) = [P] \in \mathrm{Br}(k).$$

(2) If $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ is globally generated, then $\mathcal{L} \otimes_k \overline{k}$ corresponds to a unique invertible sheaf $\mathcal{M}$ on $X_{\overline{k}}$ and the morphism associated to the complete linear system $|\mathcal{M}|$ descends to a morphism over k

$$|\mathcal{L}| \ : \ X \ \to \ P,$$

where P is a Brauer–Severi variety over k with $\delta(\mathcal{L}) = [P]$.

We note that our result is inspired by a geometric construction of Brauer–Severi varieties of Grothendieck, see [21, Sect. 5.4], and it seems that it is known to the experts. As immediate corollaries, we recover two classical theorems about Brauer–Severi varieties due to Châtelet and Kang, see Corollaries 3.6 and 3.8.

1.3 Del Pezzo Surfaces

In the second part, we apply this machinery to the geometry and arithmetic of del Pezzo surfaces over arbitrary ground fields. I would like to stress that most, if not all, of the results of this second part are well-known. To the best of my knowledge, I have tried to give the original references. However, my organization of the material and the hopefully more geometric approach to del Pezzo surfaces via morphisms to Brauer–Severi varieties is new.

By definition, a *del Pezzo surface* is a smooth and proper surface X over a field k, whose anti-canonical invertible sheaf ω_X^{-1} is ample. The *degree* of a del Pezzo surface is the self-intersection number of ω_X. The classification of del Pezzo surfaces over $\overline{k}$ is well-known: The degree d satisfies $1 \leq d \leq 9$, and they are isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to the blow-up of $\mathbb{P}^2$ in $(9 - d)$ points in general position.

As an application of Theorem 1.1, we obtain the following.

(1) If $d = 8$ and $X_{\overline{k}} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, then there exists an embedding

$$| - \tfrac{1}{2} K_X | : X \hookrightarrow P$$

into a Brauer–Severi threefold P. Moreover, X is either isomorphic to a product of two Brauer–Severi curves or to a quadratic twist of the self-product of a Brauer–Severi curve. We refer to Theorem 5.1 and Proposition 5.2 for details.

(2) If $d \geq 7$ and $X_{\overline{k}} \not\cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, then there exists a birational morphism

$$f : X \to P$$

to a Brauer–Severi surface P over k that is the blow-up in a closed and zero-dimensional subscheme of length $(9 - d)$ over k. We refer to Theorem 6.1 for details.

(3) If $d = 6$, then there exist two finite field extensions $k \subseteq K$ and $k \subseteq L$ with $[K : k]|2$ and $[L : k]|3$ such that there exists a birational morphism $f : X \to P$ to a Brauer–Severi surface P over k that is the blow-up in a closed and zero-dimensional subscheme of length 3 over k if and only $k = K$. On the other hand, there exists a birational morphism $X \to Y$ onto a degree 8 del Pezzo surface Y of product type if and only if $k = L$. We refer to Theorem 7.1 for details.
(4) For partial results if $d \leq 5$, as well as birationality criteria for when a del Pezzo surface is birationally equivalent to a Brauer–Severi surface, we refer to Sect. 8.

As further applications, we recover well-known results about rationality, unirationality, existence of k-rational points, Galois cohomology, the Hasse principle, and weak approximation for del Pezzo surfaces.

Notations and Conventions

In this article, k denotes an arbitrary field, $\overline{k}$ (resp. k^{sep}) its algebraic (resp. separable) closure, and $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ its absolute Galois group. By a variety over k we mean a scheme X that is of finite type, separated, and geometrically integral over k. If K is a field extension of k, then we define $X_K := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$.

2 Picard Functors and Brauer Groups

This section, we recall a couple of definitions and general results about the various relative Picard functors, about Brauer groups of fields and schemes, as well as Brauer–Severi varieties.

2.1 Relative Picard Functors

Let us first recall a couple of generalities about the several Picard functors. Our main references are [22, 23], as well as the surveys [3, Chap. 8] and [30].

For a scheme X, we define its *Picard group* $\mathrm{Pic}(X)$ to be the abelian group of invertible sheaves on X modulo isomorphism. If $f : X \to S$ is a separated morphism of finite type over a Noetherian base scheme S, then we define the *absolute Picard functor* to be the functor that associates to each Noetherian $T \to S$ the abelian group $\mathrm{Pic}_X(T) := \mathrm{Pic}(X_T)$, where $X_T := X \times_S T$. Now, as explained, for example in [30, Sect. 9.2], the absolute Picard functor is a separated presheaf for the Zariski, étale, and the fppf topologies, but it is never a sheaf for the Zariski topology. In particular, the absolute Picard functor is never representable by a scheme or by an algebraic space. This leads to the introduction of the *relative Picard functor* $\mathrm{Pic}_{X/S}$ by setting $\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$, and then, we have the associated sheaves for the Zariski, étale, and fppf topologies

$$\mathrm{Pic}_{(X/S)(\mathrm{zar})}, \quad \mathrm{Pic}_{(X/S)(\text{ét})}, \quad \text{and} \quad \mathrm{Pic}_{(X/S)(\mathrm{fppf})}.$$

In many important cases, these sheaves are representable by schemes or algebraic spaces over S. For our purposes, it suffices to work with the sheaves so that we will not address representability questions here, but refer the interested reader to [3, Chap. 8.2] and [30, Chap. 9.4] instead. Having introduced these sheaves, let us recall the following easy facts, see, for example, [30, Exercise 9.2.3].

Proposition 2.1 *Let $X \to S$ be a scheme that is separated and of finite type over a Noetherian scheme S. Let L be a field with a morphism* Spec $L \to S$.

(1) Then, the following natural maps are isomorphisms:

$$\mathrm{Pic}_X(L) \xrightarrow{\cong} \mathrm{Pic}_{X/S}(L) \xrightarrow{\cong} \mathrm{Pic}_{(X/S)(\mathrm{zar})}(L).$$

(2) If L is algebraically closed, then also the following natural maps are isomorphisms:

$$\mathrm{Pic}_X(L) \xrightarrow{\cong} \mathrm{Pic}_{(X/S)(\text{ét})}(L) \xrightarrow{\cong} \mathrm{Pic}_{(X/S)(\mathrm{fppf})}(L).$$

It is important to note that if L is not algebraically closed, then the natural map $\mathrm{Pic}_X(L) \to \mathrm{Pic}_{(X/S)(\text{ét})}(L)$ is usually not an isomorphism, i.e., not every section of $\mathrm{Pic}_{(X/S)(\text{ét})}$ over L arises from an invertible sheaf on X_L. The following example, taken from [30, Exercise 9.2.4], is crucial to everything that follows and illustrates this.

Example 2.2 Let X be the smooth plane conic over $\mathbb{R}$ defined by

$$X := \{ x_0^2 + x_1^2 + x_2^2 = 0 \} \subset \mathbb{P}^2_{\mathbb{R}}.$$

Then, X is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$ since $X(\mathbb{R}) = \emptyset$, but there exists an isomorphism $X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. In particular, X is an example of a non-trivial Brauer–Severi variety (see Definition 2.14).

Next, if $x \in X$ is a closed point, then $\kappa(x) \cong \mathbb{C}$, that is, x is a zero-cycle of degree 2. Moreover, $\mathcal{O}_X(x)$ generates $\mathrm{Pic}_X(\mathbb{R})$, for if there was an invertible sheaf of odd degree on X, then there would exist an invertible sheaf of degree 1 on X and then, Riemann–Roch would imply $X(\mathbb{R}) \neq \emptyset$, a contradiction.

On the other hand, x splits on $X_{\mathbb{C}}$ into two closed points, say x_1 and x_2. Since $\mathcal{O}_{X_{\mathbb{C}}}(x_1)$ and $\mathcal{O}_{X_{\mathbb{C}}}(x_2)$ are isomorphic as invertible sheaves on $X_{\mathbb{C}}$, it follows that $\mathcal{O}_{X_{\mathbb{C}}}(x_1)$ descends from a class in $\mathrm{Pic}_{(X/\mathbb{R})(\text{ét})}(\mathbb{C})$ to a class in $\mathrm{Pic}_{(X/\mathbb{R})(\text{ét})}(\mathbb{R})$.

These observations show that the natural map $\mathrm{Pic}_X(\mathbb{R}) \to \mathrm{Pic}_{(X/\mathbb{R})(\text{ét})}(\mathbb{R})$ is not surjective.

In this example, we have $X(\mathbb{R}) = \emptyset$, i.e., the structure morphism $X \to \mathrm{Spec}\,\mathbb{R}$ has no section. Quite generally, we have the following comparison theorem for the several relative Picard functors, and refer, for example, to [30, Theorem 9.2.5] for details and proofs.

Theorem 2.3 (Grothendieck) *Let $f : X \to S$ be a scheme that is separated and of finite type over a Noetherian scheme S, and assume that $\mathcal{O}_S \xrightarrow{\cong} f_*\mathcal{O}_X$ holds universally.*

(1) Then, the natural maps

$$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)(\mathrm{zar})} \hookrightarrow \mathrm{Pic}_{(X/S)(\text{ét})} \hookrightarrow \mathrm{Pic}_{(X/S)(\mathrm{fppf})}$$

are injections.

(2) If f has a section, then all three maps are isomorphisms. If f has a section locally in the Zariski topology, then the latter two maps are isomorphisms, and if f has a section locally in the étale topology, then the last map is an isomorphism.

To understand the obstruction to realizing a section of $\mathrm{Pic}_{(X/S)(\text{ét})}$ or $\mathrm{Pic}_{(X/S)(\mathrm{fppf})}$ over S by an invertible sheaf on X in case there is no section of $X \to S$, we recall the following definition.

Definition 2.4 For a scheme T, the étale cohomology group $H^2_{\text{ét}}(T, \mathbb{G}_m)$ is called the *cohomological Brauer group*, and is denoted $\mathrm{Br}'(T)$. The set of sheaves of Azumaya algebras on T modulo Brauer equivalence also forms a group, the *Brauer group* of T, and is denoted $\mathrm{Br}(T)$.

We will not discuss sheaves of Azumaya algebras on schemes in the sequel, but only remark that these generalize central simple algebras over fields (see Sect. 2.3 for the latter), and refer the interested reader to [20] and [37, Chap. IV] for details and references, as well as to [41] for a survey.

Using that $\mathbb{G}_m$ is a smooth group scheme, Grothendieck [21] showed that the natural map $H^2_{\text{ét}}(T, \mathbb{G}_m) \to H^2_{\mathrm{fppf}}(T, \mathbb{G}_m)$ is an isomorphism, i.e., it does not matter whether the cohomological Brauer group $\mathrm{Br}'(T)$ is defined with respect to the étale or the fppf topology. Next, there exists a natural injective group homomorphism $\mathrm{Br}(T) \to \mathrm{Br}'(T)$, whose image is contained in the torsion subgroup of $\mathrm{Br}'(T)$. If T is the spectrum of a field k, then this injection is even an isomorphism, i.e., $\mathrm{Br}(k) = \mathrm{Br}'(k)$, see, for example, [18, 21], and [37, Chap. IV] for details and references.

The connection between Brauer groups, Proposition 2.1, and Theorem 2.3 is as follows, see, for example [3, Chap. 8.1] or [30, Sect. 9.2].

Proposition 2.5 *Let $f : X \to S$ be a scheme that is separated and of finite type over a Noetherian scheme S, and assume that $\mathcal{O}_S \xrightarrow{\cong} f_*\mathcal{O}_X$ holds universally. Then, for each S-scheme T there exists a canonical exact sequence*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)(\mathrm{fppf})}(T) \xrightarrow{\delta} \mathrm{Br}'(T) \to \mathrm{Br}'(X_T)\,.$$

If f has a section, then δ is the zero-map. □

2.2 Varieties and the Amitsur Subgroup

By our conventions above, a variety over a field k is a scheme X that is of finite type, separated, and geometrically integral over k. In this situation, the conditions of Proposition 2.5 are fulfilled, as the following remark shows.

Remark 2.6 If X is a proper variety over a field k, then

(1) the structure morphism $f : X \to \operatorname{Spec} k$ is separated, of finite type, and $\mathcal{O}_{\operatorname{Spec} k} \cong f_*\mathcal{O}_X$ holds universally.
(2) The morphism f has sections locally in the étale topology (see, for example, [18, Appendix A]).
(3) Since the base scheme is a field k, we have $\operatorname{Br}(k) = \operatorname{Br}'(k)$.

In Remark 3.3, we will give an explicit description of δ in this case.

In Example 2.2, the obstruction to representing the class of $\mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ in $\operatorname{Pic}_{(X/\mathbb{R})(\mathrm{fppf})}(\mathbb{R})$ by an invertible sheaf on X can be explained via δ, which maps $\mathcal{L}$ to the non-zero element of $\operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. In terms of Azumaya algebras (since the base is $\operatorname{Spec} \mathbb{R}$, these are central simple $\mathbb{R}$-algebras), this Brauer class corresponds the $\mathbb{R}$-algebra $\mathbb{H}$ of quaternions, but we will not pursue this point of view in the sequel.

Proposition 2.7 *Let X be a proper variety over a field k. Then, there exist natural isomorphisms of abelian groups*

$$\operatorname{Pic}_{X/k}(k^{\mathrm{sep}})^{G_k} \xrightarrow{\cong} \operatorname{Pic}_{(X/k)(\text{ét})}(k) \xrightarrow{\cong} \operatorname{Pic}_{(X/k)(\mathrm{fppf})}(k),$$

where the $-^{G_k}$ denotes Galois invariants.

Proof The first isomorphism follows from Galois theory and sheaf axioms and the second isomorphism follows from Theorem 2.3 and Remark 2.6. □

The Brauer group $\operatorname{Br}(k)$ of a field k is an abelian torsion group, see, for example, [18, Corollary 4.4.8]. Motivated by Proposition 2.5, we introduce the following subgroup of $\operatorname{Br}(k)$ that measures the deviation between $\operatorname{Pic}_{(X/k)(\mathrm{fppf})}(k)$ and $\operatorname{Pic}(X)$.

Definition 2.8 Let X be a proper variety over a field k. Then, the *Amitsur subgroup* of X in $\operatorname{Br}(k)$ is the subgroup

$$\operatorname{Am}(X) := \delta(\operatorname{Pic}_{(X/k)(\mathrm{fppf})}(k)) \subseteq \operatorname{Br}(k).$$

By the previous remarks, it is an abelian torsion group.

The following lemma gives bounds for the order of torsion in $\operatorname{Am}(X)$.

Lemma 2.9 *Let X be a proper variety over a field k. If there exists a closed point on X, whose residue field is of degree n over k, then every element of* $\operatorname{Am}(X)$ *has an order dividing n.*

Proof Let $x \in X$ be a closed point, say, with residue field K/k that is of degree n over k. Since X_K has a K-rational point, the map δ of X_K is identically zero by Proposition 2.5. Thus, we have an inclusion $\mathrm{Am}(X) \subseteq \mathrm{Br}(K|k) := \ker(\mathrm{Br}(k) \to \mathrm{Br}(K))$, where $\mathrm{Br}(k) \to \mathrm{Br}(K)$ is the restriction homomorphism.

If K is separable over k, then $\mathrm{Br}(K|k)$ is contained in the n-torsion of $\mathrm{Br}(k)$, which follows from the fact that the composition of restriction and corestriction is multiplication by n, see [18, Proposition 4.2.10].

If K is a purely inseparable extension of k, generated by p^r-th roots, then $\mathrm{Br}(K|k)$ is p^r-torsion (which yields even stronger bounds on the torsion than claimed), see for example, Hochschild's Theorem [18, Theorem 9.1.1] for an explicit description for this group.

In general, we can factor the extension K/k into a separable and a purely inseparable extension, and by combining the previous two special cases, the statement follows. □

Using Proposition 2.5, we can give two alternative definitions of $\mathrm{Am}(X)$. In fact, the birational invariance of this group for Brauer–Severi varieties is a classical result of Amitsur, probably known to Châtelet and Witt in some form or another, see also Theorem 2.19 below.

Proposition 2.10 *Let X be a smooth and proper variety over k. Then,*

$$\mathrm{Am}(X) \;=\; \ker\left(\mathrm{Br}(k) \;\to\; \mathrm{Br}'(X)\right) \;=\; \ker\left(\mathrm{Br}(k) \;\to\; \mathrm{Br}(k(X))\right).$$

In particular, $\mathrm{Am}(X)$ *is a birational invariant of smooth and proper varieties over k.*

Proof The first equality follows from the exact sequence of Proposition 2.5. Since X is smooth over k, the natural map $\mathrm{Br}'(X) \to \mathrm{Br}(k(X))$ is injective, see, for example, [37, Example III.2.22], and then, the second equality follows. From this last description, it is clear that $\mathrm{Am}(X)$ is a birational invariant. □

Remark 2.11 In [10, Sect. 5], the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(X))$ was denoted $\mathrm{Br}(k(X)/k)$. Thus, if X is smooth and proper over k, then this latter group coincides with $\mathrm{Am}(X)$. However, this group should not be confused with $\mathrm{Br}(k(X))/\mathrm{Br}(k)$, which is related to another important birational invariant that we will introduce in Sect. 4.2.

If X has a k-rational point, then $\mathrm{Am}(X) = 0$ by Proposition 2.5. On the other hand, there exist proper varieties X with trivial Amitsur subgroup without k-rational points (some degree 8 del Pezzo surfaces of product type with $\rho = 1$ provide examples, see Corollary 5.4). Let us recall that a *zero-cycle* on X is a formal finite sum $\sum_i n_i Z_i$, where the $n_i \in \mathbb{Z}$ and where the Z_i are closed points of X. It is called *effective* if $n_i \geq 0$ for all i. The *degree* is defined to be $\deg(Z) := \sum_i n_i [\kappa(Z_i) : k]$, where $\kappa(Z_i)$ denotes the residue field of the point Z_i.

Corollary 2.12 *Let X be a proper variety over a field k. If there exists a zero cycle of degree 1 on X, then* $\mathrm{Am}(X) = 0$. □

If X is a projective variety over k, then $\mathrm{Pic}_{(X/k)(\text{ét})}$ and $\mathrm{Pic}_{(X/k)(\text{fppf})}$ are representable by a group scheme $\mathrm{Pic}_{X/k}$ over k, the *Picard scheme*. The connected component of the identity is denoted $\mathrm{Pic}^0_{X/k}$, and the quotient

$$\mathrm{NS}_{X/k}(\overline{k}) \;:=\; \mathrm{Pic}_{X_{\overline{k}}/\overline{k}}(\overline{k}) \,/\, \mathrm{Pic}^0_{X_{\overline{k}}/\overline{k}}(\overline{k}),$$

the *Néron–Severi group*, is a finitely generated abelian group, whose rank is denoted $\rho(X_{\overline{k}})$. We refer to [3, Sect. 8.4] for further discussion. Moreover, if X is smooth over k, then $\mathrm{Pic}^0_{X/k}$ is of dimension $\frac{1}{2}b_1(X)$, where b_1 denotes the first ℓ-adic Betti number.

Lemma 2.13 *Let X be a smooth and projective variety over a field k with $b_1(X) = 0$. Then,* $\mathrm{Pic}_{(X/k)(\text{fppf})}(k)$ *is a finitely generated abelian group,*

$$\operatorname{rank} \mathrm{Pic}(X) \;=\; \operatorname{rank} \mathrm{Pic}_{(X/k)(\text{fppf})}(k) \;\leq\; \rho(X_{\overline{k}}),$$

and $\mathrm{Am}(X)$ *is a finite abelian group.*

Proof If $b_1(X) = 0$, then, by the previous discussion, $\mathrm{Pic}(X_{\overline{k}})$ is a finitely generated abelian group of rank $\rho(X_{\overline{k}})$. Since $\mathrm{Pic}(X)$ and $\mathrm{Pic}_{(X/k)(\text{fppf})}(k)$ are contained in $\mathrm{Pic}(X_{\overline{k}})$, they are also finitely generated of rank at most $\rho(X_{\overline{k}})$. Since $\mathrm{Am}(X) = \delta(\mathrm{Pic}_{(X/k)(\text{fppf})}(k))$ is a torsion subgroup of $\mathrm{Br}(k)$, Proposition 2.5 implies the stated equality of ranks. Moreover, being torsion and a finitely generated abelian group, $\mathrm{Am}(X)$ is finite. □

2.3 Brauer–Severi Varieties

Next, we recall a couple of results about Brauer–Severi varieties, and refer the interested reader to [18, Chap. 5] and the surveys [27, 41] for details, proofs, and further references.

Definition 2.14 A *Brauer–Severi variety* over a field k is a proper variety P over k, such that there exists a finite field extension K of k and an isomorphism $P_K \cong \mathbb{P}^n_K$ over K.

In case P is of dimension one (resp. two, resp. three), we will also refer to it as a Brauer–Severi curve (resp. Brauer–Severi surface, resp. Brauer–Severi threefold). Any field extension K of k such that P_K is isomorphic to projective space over K is called a *splitting field* for P, and P is said to *split* over K. By a theorem of Châtelet, a Brauer–Severi variety P over k is *trivial*, i.e., splits over k, i.e., is k-isomorphic to projective space over k, if and only if it possesses a k-rational point. Since a geometrically integral variety over a field k always has points over k^{sep}, it follows that a Brauer–Severi variety can be split over a finite and separable extension of k, which we may also assume to be Galois if we want.

For a finite field extension K of k that is Galois with Galois group G, the set of all Brauer–Severi varieties of dimension n over k that split over K, can be interpreted as the set of all G-twisted forms of $\mathbb{P}^n_K$, which is in bijection to the cohomology group $H^1(G, \mathrm{Aut}(\mathbb{P}^n_K))$. Using $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}$, and taking cohomology in the short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1,$$

the boundary map associates to the class of a Brauer–Severi variety P of dimension n in $H^1(G, \mathrm{PGL}_{n+1}(K))$ a class in

$$\mathrm{Br}(K|k) := \ker\left(\mathrm{Br}(k) \to \mathrm{Br}(K)\right) = \ker\left(H^2_{\text{ét}}(k, \mathbb{G}_m) \to H^2_{\text{ét}}(K, \mathbb{G}_m)\right).$$

Taking the limit over all finite Galois extensions of k, we obtain for every Brauer–Severi variety P over k a class $[P] \in \mathrm{Br}(k)$. This cohomology class is torsion and its order is called the *period* of P, denoted $\mathrm{per}(P)$. By a theorem of Châtelet, a Brauer–Severi variety is trivial if and only if the class $[P] \in \mathrm{Br}(k)$ is zero, i.e., if and only if $\mathrm{per}(P) = 1$. We will say that two Brauer–Severi varieties over k are *Brauer equivalent* if their associated classes in $\mathrm{Br}(k)$ are the same.

To say more about Brauer classes associated to Brauer–Severi varieties, we will shortly digress on non-commutative k-algebras, and refer to [18, Sect. 2] and [26] for details: We recall that a *central simple k-algebra* is a k-algebra A, whose center is equal to k (i.e., A is central), and whose only two-sided ideals are (0) and A (i.e., A is simple). If A is moreover finite-dimensional over k, then by theorems of Noether, Köthe, and Wedderburn, there exists a finite and separable field extension $k \subseteq K$ that *splits* A, i.e., $A \otimes_k K \cong \mathrm{Mat}_{n\times n}(K)$. In particular, the dimension of A over k is always a square, and we set the *degree* of A to be $\deg(A) := \sqrt{\dim_k(A)}$. Two central simple k-algebras A_1 and A_2 are said to be *Brauer equivalent* if there exist integers $a_1, a_2 \geq 1$ such that $A_1 \otimes_k \mathrm{Mat}_{a_1\times a_1}(k) \cong A_2 \otimes_k \mathrm{Mat}_{a_2\times a_2}(k)$.

The connection between central simple algebras and Brauer–Severi varieties is the following dictionary, see [18, Theorem 2.4.3].

Theorem 2.15 *Let $k \subseteq K$ be a field extension that is Galois with Galois group G. Then, there is a natural bijection of sets between*

(1) Brauer–Severi varieties of dimension n over k that split over K,
(2) $H^1(G, \mathrm{PGL}_{n+1}(K))$, and
(3) central simple k-algebras of degree $n+1$ over k that split over K.

Under this bijection, Brauer equivalence of (1) and (3) coincide.

We also recall that a *division algebra* is a k-algebra in which every non-zero element has a two-sided multiplicative inverse. For example, field extensions of k are division algebras, and a non-commutative example is provided by the quaternions over $\mathbb{R}$. Given a simple and finite-dimensional k-algebra A, a theorem of Wedderburn states that there exists a unique division algebra D over k and a unique integer $m \geq 1$ and an isomorphism of k-algebras $A \cong \mathrm{Mat}_{m\times m}(D)$, see [18, Theorem 2.1.3].

Corollary 2.16 *If two Brauer–Severi varieties over k of the same dimension are Brauer equivalent, then they are isomorphic as schemes over k.*

Proof By Theorem 2.15, it suffices to show that two Brauer equivalent central simple k-algebras A_1, A_2 of the same dimension are isomorphic. By Wedderburn's theorem, there exist division algebras D_i and integers $m_i \geq 1$ such that $A_i \cong \mathrm{Mat}_{m_i \times m_i}(D_i)$ for $i = 1, 2$. By definition of Brauer-equivalence, there exist integers $a_i \geq 1$ and an isomorphism of k-algebras

$$A_1 \otimes_k \mathrm{Mat}_{a_1 \times a_1}(k) \;\cong\; A_2 \otimes_k \mathrm{Mat}_{a_2 \times a_2}(k).$$

Together with the k-algebras isomorphisms

$$\begin{aligned} A_i \otimes_k \mathrm{Mat}_{a_i \times a_i}(k) &\cong \mathrm{Mat}_{m_i \times m_i}(D_i) \otimes_k \mathrm{Mat}_{a_1 \times a_1}(k) \\ &\cong \mathrm{Mat}_{a_i m_i \times a_i m_i}(D_i) \end{aligned}$$

and the uniqueness part in Wedderburn's theorem, we conclude $D_1 \cong D_2$, as well as $a_1 = a_2$, whence $A_1 \cong A_2$, see also [18, Remark 2.4.7]. □

For Brauer–Severi varieties over k that are of different dimension, we refer to Châtelet's theorem (Corollary 3.8) below. On the other hand, for Brauer–Severi varieties over k that are of the same dimension, Amitsur conjectured that they are birationally equivalent if and only if their classes generate the same cyclic subgroup of $\mathrm{Br}(k)$, see also Remark 2.20.

For projective space, the degree map $\deg : \mathrm{Pic}(\mathbb{P}^n_k) \to \mathbb{Z}$, which sends $\mathcal{O}_{\mathbb{P}^n_k}(1)$ to 1, is an isomorphism. Thus, if P is a Brauer–Severi variety over k and $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$, then there are isomorphisms

$$\begin{aligned} \mathrm{Pic}_{(P/k)(\mathrm{fppf})}(k) &\cong \mathrm{Pic}_{(P/k)}(k^{\mathrm{sep}})^{G_k} \quad \cong \quad \mathrm{Pic}_{(P/k)}(k^{\mathrm{sep}}) \\ &\cong \mathrm{Pic}(\mathbb{P}^{\dim(P)}_{k^{\mathrm{sep}}}) \quad \xrightarrow{\deg} \mathbb{Z}. \end{aligned}$$

The first isomorphism is Proposition 2.7, and the second follows from the fact that the G_k-action must send the unique ample generator of $\mathrm{Pic}_{(P/k)}(k^{\mathrm{sep}})$ to an ample generator, showing that G_k acts trivially. The third isomorphism follows from the fact that P splits over a separable extension.

Definition 2.17 For a Brauer–Severi variety P over k, we denote the unique ample generator of $\mathrm{Pic}_{(P/k)(\mathrm{fppf})}(k)$ by $\mathcal{O}_P(1)$.

We stress that $\mathcal{O}_P(1)$ is a class in $\mathrm{Pic}_{(P/k)(\mathrm{fppf})}(k)$ that usually does not come from an invertible sheaf on P - in fact this happens if and only if P is a trivial Brauer–Severi variety, i.e., split over k. For a Brauer–Severi variety, the short exact sequence from Proposition 2.5 becomes the following.

Theorem 2.18 (Lichtenbaum) *Let P be a Brauer–Severi variety over k. Then, there exists an exact sequence*

$$0 \to \operatorname{Pic}(P) \to \underbrace{\operatorname{Pic}_{(P/k)(\mathrm{fppf})}(k)}_{\cong \mathbb{Z}} \xrightarrow{\delta} \operatorname{Br}(k) \to \operatorname{Br}(k(P)).$$

More precisely, we have

$$\begin{aligned} \delta(\mathcal{O}_P(1)) &= [P], \quad \textit{and} \\ \operatorname{Pic}(P) &= \mathcal{O}_P(\operatorname{per}(P)) \cdot \mathbb{Z}. \end{aligned}$$

Since $\omega_P \cong \mathcal{O}_P(-\dim(P) - 1)$, the period $\operatorname{per}(P)$ *divides* $\dim(P) + 1$.

Again, we refer to [18, Theorem 5.4.5] for details and proofs. Using Proposition 2.10, we immediately obtain the following classical result of Amitsur [1] as corollary.

Theorem 2.19 (Amitsur) *If P is a Brauer–Severi variety over k, then* $\operatorname{Am}(P) \cong \mathbb{Z}/\operatorname{per}(P)\mathbb{Z}$. *If two Brauer–Severi varieties are birationally equivalent over k, then the have the same Amitsur subgroups inside* $\operatorname{Br}(k)$ *and in particular, the same period.* □

Remark 2.20 In general, it is not true that two Brauer–Severi varieties of the same dimension and the same Amitsur subgroup are isomorphic. We refer to Remark 7.2 for an example arising from a Cremona transformation of Brauer–Severi surfaces. However, Amitsur asked whether two Brauer–Severi varieties of the same dimension with the same Amitsur subgroup are birationally equivalent.

In our applications to del Pezzo surfaces below, we will only need the following easy and probably well-known corollary.

Corollary 2.21 *Let P be a Brauer–Severi variety over k. If there exists a zero-cycle on P, whose degree is prime to* $(\dim(P) + 1)$, *then P is is trivial.*

PROOF Since $\operatorname{Am}(P) \cong \mathbb{Z}/\operatorname{per}(P)\mathbb{Z}$ and its order divides $(\dim(P) + 1)$, Lemma 2.9 and the assumptions imply $\operatorname{Am}(P) = 0$. Thus, $\operatorname{per}(P) = 1$, and then, P is trivial. □

We end this section by mentioning another important invariant of a Brauer–Severi variety P over k, namely, its *index*, denoted $\operatorname{ind}(P)$. We refer to [18, Chap. 4.5] for the precise definition and note that it is equal to the smallest degree of a finite separable field extension K/k such that P_K is trivial, as well as to the greatest common divisor of the degrees of all finite separable field extensions K/k such that P_K is trivial. By a theorem of Brauer, the period divides the index, and they have the same prime factors, see [18, Proposition 4.5.13].

3 Morphisms to Brauer–Severi Varieties

This section contains Theorem 3.4, the main observation of this article that describes morphisms from a proper variety X over a field k to Brauer–Severi varieties in terms of classes in of $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$. We start by extending classical notions for invertible sheaves to such classes, and then, use these notions to phrase and prove Theorem 3.4. As immediate corollaries, we obtain two classical results of Kang and Châtelet on the geometry of Brauer–Severi varieties.

3.1 Splitting Fields, Globally Generated and Ample Classes

Before coming to the main result of this section, we introduce the following.

Definition 3.1 Let X be a proper variety over k and $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$.

(1) A *splitting field* for $\mathcal{L}$ is a field extension K/k such that $\mathcal{L} \otimes_k K$ lies in $\mathrm{Pic}(X_K)$, i.e., arises from an invertible sheaf on X_K.
(2) The class $\mathcal{L}$ is called *globally generated* (resp. *ample*, resp. *very ample*) if there exists a splitting field K for $\mathcal{L}$ such that $\mathcal{L} \otimes_k K$ is globally generated (resp. ample, resp. very ample) as an invertible sheaf on X_K.

From the short exact sequence in Proposition 2.5, it follows that if K is a splitting field for the class $\mathcal{L}$, then there exists precisely one invertible sheaf on X_K up to isomorphism that corresponds to this class. The following lemma shows that these notions are independent of the choice of a splitting field of the class $\mathcal{L}$.

Lemma 3.2 *Let X be a proper variety over k and $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$.*

(1) There exists a splitting field for $\mathcal{L}$ that is a finite and separable extension k, and it can also chosen to be Galois over k.
(2) Let K and K' be splitting fields for $\mathcal{L}$. Then $\mathcal{L} \otimes_k K \in \mathrm{Pic}(X_K)$ is globally generated (resp. ample, resp. very ample) if and only if $\mathcal{L} \otimes_k K' \in \mathrm{Pic}(X_{K'})$ is globally generated (resp. ample, resp. very ample).

PROOF To simplify notation in this proof, we set $\mathcal{L}_K := \mathcal{L} \otimes_k K$.

Let K be a finite and separable extension of k, such that $\delta(\mathcal{L}) \in \mathrm{Br}(k)$ lies in $\mathrm{Br}(K|k)$, where δ is as in Proposition 2.5. Then, $\delta(\mathcal{L}_K) = 0$, i.e., $\mathcal{L}_K$ comes from an invertible sheaf on X_K. In particular, K is a splitting field for $\mathcal{L}$, which is a finite and separable extension of k. Passing to the Galois closure of K/k, we obtain a splitting field for $\mathcal{L}$ that is a finite Galois extension of k. This establishes claim (1).

Claim (2) is a well-known application of flat base change, but let us recall the arguments for the reader's convenience: By choosing a field extension of k that contains both K and K', we reduce to the case $k \subseteq K \subseteq K'$. We have $H^0(X_K, \mathcal{L}_K) \otimes_K K' \cong H^0(X_{K'}, \mathcal{L}_{K'})$ by flat base change for cohomology, from which it is easy to see that $\mathcal{L}_K$ is globally generated if and only if $\mathcal{L}_{K'}$ is so. Next, if $\mathcal{L}_K$ is very ample, then its global sections give rise to a closed immersion $X_K \to \mathbb{P}^n_K$ for some n. After base change to K', we obtain a closed embedding $X_{K'} \to \mathbb{P}^n_{K'}$ which corresponds to the global sections of $\mathcal{L}_{K'}$, and so, also $\mathcal{L}_{K'}$ is very ample. Conversely, if $\mathcal{L}_{K'}$ is very ample, then it is globally generated, and thus, $\mathcal{L}_K$ is globally generated by what we just established, and thus, gives rise to a morphism $\varphi_K : X_K \to \mathbb{P}^n_K$. By assumption and flat base change, $\varphi_{K'}$ is a closed embedding, and thus, φ_K is a closed embedding, and $\mathcal{L}_K$ is very ample. From this, it also follows that $\mathcal{L}_K$ is ample if and only if $\mathcal{L}_{K'}$ is. □

Remark 3.3 Let X be a proper variety over k and let

$$\delta : \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k) \longrightarrow \mathrm{Br}(k)$$

be as in Proposition 2.5. We are now in a position to describe δ explicitly.

(1) First, and more abstractly: given a class $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, we can choose a splitting field K that is a finite extension k. Thus, $\mathrm{Spec}\, K \to \mathrm{Spec}\, k$ is an fppf cover, the class $\mathcal{L} \otimes_k K$ comes with an fppf descent datum, and it arises from an invertible sheaf $\mathcal{M}$ on X_K. The crucial point is that the descent datum is for a class in $\mathrm{Pic}(X_K)$, where isomorphism classes of invertible sheaves are identified. In order to turn this into a descent datum for the invertible sheaf $\mathcal{M}$, we have to choose isomorphisms, which are only unique up to a $\mathbb{G}_m = \mathrm{Aut}(\mathcal{M})$-action, and we obtain a $\mathbb{G}_m$-gerbe that is of class $\delta(\mathcal{L}) \in H^2_{\mathrm{fppf}}(\mathrm{Spec}\, k, \mathbb{G}_m) = \mathrm{Br}(k)$. This gerbe is neutral if and only if $\delta(\mathcal{L}) = 0$. This is equivalent to being able to extend the descent datum for the class $\mathcal{L} \otimes_k K$ to a descent datum for the invertible sheaf $\mathcal{M}$.

(2) Second, and more concretely: given a class $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, we can choose a splitting field K that is a finite Galois extension of k, say with Galois group G. Thus, the class $\mathcal{L} \otimes_k K$ arises from an invertible sheaf $\mathcal{M}$ on X_K and lies in $\mathrm{Pic}_X(K)^G$ and we can choose isomorphisms

$$\iota_g : g^*\mathcal{M} \xrightarrow{\cong} \mathcal{M},$$

which are unique up to a $\mathbb{G}_m$-action. In particular, they may fail to form a Galois descent datum for $\mathcal{M}$, and the failure of turning $\{\iota_g\}_{g\in G}$ into a Galois descent datum for $\mathcal{M}$ gives rise to a cohomology class $\delta(\mathcal{L}) \in H^2_{\text{ét}}(\mathrm{Spec}\, k, \mathbb{G}_m) = \mathrm{Br}(k)$. More precisely, this class lies in the subgroup $\mathrm{Br}(K|k)$ of $\mathrm{Br}(k)$.

The following is an analog for Brauer–Severi varieties of the classical correspondence between morphisms to projective space and globally generated invertible sheaves as explained, for example, in [24, Theorem II.7.1], see also Remark 3.5 below.

Theorem 3.4 *Let X be a proper variety over a field k.*

(1) Let $\varphi : X \to P$ be a morphism to a Brauer–Severi variety P over k, and consider the induced homomorphism of abelian groups

$$\varphi^* : \mathrm{Pic}_{(P/k)(\mathrm{fppf})}(k) \to \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k).$$

Then, $\mathcal{L} := \varphi^*\mathcal{O}_P(1)$ is a globally generated class with

$$\delta(\mathcal{L}) = [P] \in \mathrm{Br}(k),$$

where δ is as in Proposition 2.5. If φ is a closed immersion, then $\mathcal{L}$ is very ample.

(2) Let $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ be a globally generated class. If K is a splitting field, then the morphism to projective space over K associated to the complete linear system $|\mathcal{L} \otimes_k K|$ descends to morphism over k

$$|\mathcal{L}| : X \to P,$$

where P is a Brauer–Severi variety over k with $\delta(\mathcal{L}) = [P]$. If $\mathcal{L}$ is very ample, then $|\mathcal{L}|$ is a closed immersion.

PROOF Let $\varphi : X \to P$ and $\mathcal{L}$ be as in (1). Then, we have $\delta(\mathcal{L}) = \delta(\mathcal{O}_P(1)) = [P] \in \mathrm{Br}(k)$, where the first equality follows from functoriality of the exact sequence in Proposition 2.5, and the second from Theorem 2.18. Let K be a splitting field for $\mathcal{L}$, and let $\mathcal{M}$ be the invertible sheaf corresponding to $\mathcal{L} \otimes_k K$ on X_K. Being an invertible sheaf, we have $\delta(\mathcal{M}) = 0 \in \mathrm{Br}(K)$, which implies that the morphism $\varphi_K : X_K \to P_K$ maps to a Brauer–Severi variety of class $[P_K] = \delta(\mathcal{M}) = 0$, i.e., $P_K \cong \mathbb{P}^n_K$. By definition and base change, we obtain $\mathcal{M} \cong \varphi_K^*(\mathcal{O}_{\mathbb{P}^n_K}(1))$. Thus, $\mathcal{M}$ is globally generated (as an invertible sheaf), which implies that $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ is globally generated in the sense of Definition 3.1. Moreover, if φ is a closed immersion, then so is φ_K, which implies that $\mathcal{M} \in \mathrm{Pic}(X_K)$ is very ample (as an invertible sheaf), and thus, $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ is very ample in the sense of Definition 3.1. This establishes claim (1).

To establish claim (2), let $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ be globally generated. By Lemma 3.2, there exists a splitting field K' for $\mathcal{L}$ that is a finite Galois extension of k, say with Galois group G. Thus, $\mathcal{L} \otimes_k K'$ corresponds to an invertible sheaf $\mathcal{M}$ on $X_{K'}$, whose isomorphism class lies in $\mathrm{Pic}_X(K')^G$, see Proposition 2.7.

If $f : X \to \mathrm{Spec}\, k$ is the structure morphism, then $(f_{K'})_*\mathcal{M}$ is a finite-dimensional K'-vector space. By our assumptions on global generation we obtain a morphism over K'

$$|\mathcal{M}| : X_{K'} \to \mathbb{P}((f_{K'})_*\mathcal{M}).$$

As explained in Remark 3.3.(2), there exist isomorphisms $\{\iota_g : g^*\mathcal{M} \to \mathcal{M}\}_{g\in G}$ that are unique up to a $\mathbb{G}_m$-action. In particular, we obtain a well-defined G-action on $\mathbb{P}((f_{K'})_*\mathcal{M})$, and the morphism defined by $|\mathcal{M}|$ is G-equivariant. Taking the quotient by G, we obtain a morphism over k

$$|\mathcal{L}| : X \to P.$$

Since P_K is isomorphic to $\mathbb{P}((f_{K'})_*\mathcal{M})$, we see that P is a Brauer–Severi variety over k and, as observed by Grothendieck in [21, Sect. 5.4], we have $\delta(\mathcal{L}) = [P]$ in $\mathrm{Br}(k)$.

Finally, let K be an arbitrary splitting field for $\mathcal{L}$. Let $\varphi : X \to P$ be the previously constructed morphism and choose an extension field Ω of k that contains K and K'. Then, $\mathcal{L} \otimes_k \Omega$ is an invertible sheaf on X_Ω, globally generated by Lemma 3.2, and, since $k \subseteq K' \subseteq \Omega$, the morphism associated to $|\mathcal{L} \otimes_k \Omega|$ is equal to $\varphi_\Omega = (\varphi_{K'})_\Omega : X_\Omega \to P_\Omega$. Since K is a splitting field for $\mathcal{L}$, it is also a splitting field for P_K (see the argument in the proof of claim (1)), and in particular, $P_{K'}$ is a trivial Brauer–Severi variety. We have $\mathcal{L} \otimes_k \Omega \cong \varphi_\Omega^* \mathcal{O}_{P_\Omega}(1)$, from which we deduce $\mathcal{L} \otimes_k K \cong \varphi_K^* \mathcal{O}_{P_K}(1)$, as well as that φ_K is the morphism associated to $|\mathcal{L} \otimes_k K|$. In particular, the morphism associated to $|\mathcal{L} \otimes_k K|$ descends to $\varphi : X \to P$, where P is a Brauer–Severi variety of class $\delta(\mathcal{L})$. This establishes claim (2). □

Remark 3.5 Let us note the following.

(1) The construction of a Brauer–Severi variety over k from a globally generated class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ (in our terminology) is due to Grothendieck in [21, Sect. 5.4].
(2) In Theorem 3.4.(2), we only considered complete linear systems. We leave it to the reader to show the following generalization: Given a class $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, a splitting field K that is finite and Galois over k with Galois group G, and $V \subseteq H^0(X_K, \mathcal{L} \otimes_k K)$ a G-stable K-linear subspace, whose global sections generate $\mathcal{L} \otimes_k K$, we can descend the morphism $X_K \to \mathbb{P}(V)$ to a morphism $X \to P'$, where P' is a Brauer–Severi variety over k of class $[P'] = \delta(\mathcal{L}) \in \mathrm{Br}(k)$.
(3) If X in Theorem 3.4 has a k-rational point, i.e., $X(k) \neq \emptyset$, then we recover the well-known correspondence between morphisms to projective space and globally generated invertible sheaves:
    (a) Then, $\delta \equiv 0$ and every class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ comes from an invertible sheaf on X by Proposition 2.5,
    (b) and since every morphism $\varphi : X \to P$ gives rise to a k-rational point on P, i.e., P is a trivial Brauer–Severi variety.

3.2 Two Classical Results on Brauer–Severi Varieties

As our first corollary and application, we recover the following theorem of Kang [29], see also [18, Theorem 5.2.2], which is a Brauer–Severi variety analog of Veronese embeddings of projective spaces.

Corollary 3.6 (Kang) *Let P be a Brauer–Severi variety of period* $\mathrm{per}(P)$ *over k. Then, the class of $\mathcal{O}_P(\mathrm{per}(P))$ arises from a very ample invertible sheaf on P and gives rise to an embedding*

$$|\mathcal{O}_P(\mathrm{per}(P))| \;:\; P \;\to\; \mathbb{P}^N_k, \quad \textit{where} \quad N \;=\; \binom{\dim(P)+\mathrm{per}(P)}{\mathrm{per}(P)}.$$

After base change to a splitting field K of P, this embedding becomes the per(P)*-uple Veronese embedding of* $\mathbb{P}_K^{\dim(P)}$ *into* $\mathbb{P}^N_K$.

Proof If $n \geq 1$, then $\mathcal{O}_P(n)$ is very ample in the sense of Definition 3.1, and thus, defines an embedding into a Brauer–Severi variety P' over k. Over a splitting field of P, this embedding becomes the n-uple Veronese embedding. Since $\delta(\mathcal{O}_P(1)) = [P] \in \mathrm{Br}(k)$ and this element of order per(P), we see that if per(P) divides n, then $\mathcal{O}_P(n)$ is an invertible sheaf on P and P' is a trivial Brauer–Severi variety. □

Example 3.7 Let X be a smooth and proper variety of dimension one over k. If ω_X^{-1} is ample, then it is a curve of genus $g(X) = h^0(X, \omega_X) = 0$. Thus, X is isomorphic to $\mathbb{P}^1$ over $\overline{k}$, i.e., X is a Brauer–Severi curve. There exists a unique class $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ with $\mathcal{L}^{\otimes 2} \cong \omega_X^{-1}$, and it gives rise to an isomorphism $|\mathcal{L}| : X \to P$, where P is a Brauer–Severi curve with $\delta(\mathcal{L}) = [P] \in \mathrm{Br}(k)$. Moreover, $\mathcal{L}^{\otimes 2} \cong \omega_X^{-1}$ is an invertible sheaf on X that defines an embedding $|\omega_X^{-1}| : X \to \mathbb{P}^2_k$ as a plane conic.

A subvariety $X \subseteq P$ of a Brauer–Severi variety P over k is called *twisted linear* if $X_{\overline{k}}$ is a linear subspace of $P_{\overline{k}}$. As second application, we recover the following theorem of Châtelet, see [18, Sect. 5.3], and it follows from a Brauer–Severi variety analog of Segre embeddings of products of projective spaces.

Corollary 3.8 (Châtelet) *Let* P_1 *and* P_2 *be two Brauer–Severi varieties over k of dimension* d_1 *and* d_2*, respectively.*

(1) If P_1 *is a twisted linear subvariety of* P_2*, then* $[P_1] = [P_2] \in \mathrm{Br}(k)$.
(2) If $[P_1] = [P_2] \in \mathrm{Br}(k)$*, then there exists a Brauer–Severi variety P over k, such that* P_1 *and* P_2 *can be embedded as twisted-linear subvarieties into P.*

Proof If $\varphi : P_1 \hookrightarrow P_2$ is a twisted-linear subvariety, then $\varphi^*\mathcal{O}_{P_2}(1) = \mathcal{O}_{P_1}(1) \in \mathrm{Pic}_{(P_1/k)(\mathrm{fppf})}(k)$. We find $[P_1] = \delta(\mathcal{O}_{P_1}(1)) = \delta(\mathcal{O}_{P_2}(1)) = [P_2]$ by functoriality of the exact sequence of Proposition 2.5, and (1) follows.

Next, we show (2). By Theorem 3.4, there exists an embedding φ of $P_1 \times \mathbb{P}^{d_2}_k$ into a Brauer–Severi variety P of dimension $N := (d_1+1)(d_2+1) - 1 = d_1 d_2 + d_1 + d_2$ over k associated to the class $\mathcal{O}_{P_1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{d_2}_k}(1)$. Over a splitting field of P_1, this embedding becomes the Segre embedding of $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$ into $\mathbb{P}^N$. If x is a k-rational point of $\mathbb{P}^{d_2}_k$, then $\varphi(P_1 \times \{x\})$ realizes P_1 as twisted-linear subvariety of P and we have $[P] = [P_1] \in \mathrm{Br}(k)$ by claim (1). Similarly, we obtain an embedding of P_2 as twisted-linear subvariety into a Brauer–Severi variety P' of dimension N over k of class $[P'] = [P_2] \in \mathrm{Br}(k)$. Since $[P] = [P'] \in \mathrm{Br}(k)$ and $\dim(P) = \dim(P')$, we find $P \cong P'$ by Corollary 2.16 and (2) follows. □

4 Del Pezzo Surfaces

For the remainder of this article, we study del Pezzo surfaces with a view towards Brauer–Severi varieties. Most, if not all, results of these sections are known in some form or another to the experts. However, our more geometric approach, as well as some of the proofs, are new.

Let us first recall some classical results about del Pezzo surfaces, and refer the reader to [35, Chap. IV] or the surveys [7, 41, 47] for details, proofs, and references. For more results about the classification of geometrically rational surfaces, see [25, 34].

Definition 4.1 A *del Pezzo surface* is a smooth and proper variety X of dimension two over a field k such that ω_X^{-1} is ample. The *degree* of a del Pezzo surface is the self-intersection number of ω_X.

In arbitrary dimension, smooth and proper varieties X over k with ample ω_X^{-1} are called *Fano varieties*. As discussed in Example 3.7, Fano varieties of dimension one over k are the same as Brauer–Severi curves over k.

4.1 Geometry

The degree d of a del Pezzo surface X over a field k satisfies $1 \leq d \leq 9$. Set $\overline{X} := X_{\overline{k}}$. We will say that X is *of product type* if

$$\overline{X} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}},$$

in which case we have $d = 8$. If X is not of product type, then there exists a birational morphism

$$\overline{f} : \overline{X} \to \mathbb{P}^2_{\overline{k}}$$

that is a blow-up of $(9 - d)$ closed points $P_1, ..., P_{9-d}$ in general position, i.e., no 3 of them lie on a line, no 6 of them lie on a conic, and there is no cubic through all these points having a double point in one of them. In particular, if $d = 9$, then $\overline{f}$ is an isomorphism and X is a Brauer–Severi surface over k.

4.2 Arithmetic

By the previous discussion and Lemma 2.13, the *Néron–Severi rank* of a del Pezzo surface X of degree d over k satisfies

$$1 \;\leq\; \rho(X) \;:=\; \operatorname{rank} \operatorname{Pic}(X) \;=\; \operatorname{rank} \operatorname{Pic}_{(X/k)(\mathrm{fppf})}(k) \;\leq\; 10 - d,$$

and $\rho(X_{\overline{k}}) = 10 - d$.

The following result about geometrically rational surfaces allows using methods from Galois theory even if the ground field k is not perfect. This result is particularly useful in proofs, see also the discussion in [47, Sect. 1.4]. In particular, it applies to del Pezzo surfaces.

Theorem 4.2 (Coombes+ε) *Let X be a smooth and proper variety over k such that $X_{\overline{k}}$ is birational to $\mathbb{P}^2_{\overline{k}}$. Then,*

(1) $X_{k^{\mathrm{sep}}}$ is birationally equivalent to $\mathbb{P}^2_{k^{\mathrm{sep}}}$ via a sequence of blow-ups in points in k^{sep}-rational points and their inverses.
(2) The natural map $\mathrm{Pic}_X(k^{\mathrm{sep}}) \to \mathrm{Pic}_X(\overline{k})$ is an isomorphism.

Proof Assertion (1) is the main result of [11]. Clearly, assertion (2) holds for projective space over any field. Next, let Y be a variety that is smooth and proper over k^{sep}, $\widetilde{Y} \to Y$ be the blow-up of a k^{sep}-rational point, and let $E \subset \widetilde{Y}$ be the exceptional divisor. Then, $\mathrm{Pic}_{\widetilde{Y}}(K) = \mathrm{Pic}_Y(K) \oplus \mathbb{Z} \cdot E$ for $K = k^{\mathrm{sep}}$, as well as for $K = \overline{k}$. Using (1) and these two observations, assertion (2) follows. □

We will also need the following useful observation, due to Lang [33] and Nishimura [39], which implies that having a k-rational point is a birational invariant of smooth and proper varieties over k. We refer to [47, Sect. 1.2] for details and proof.

Lemma 4.3 (Lang–Nishimura) *Let $X \dashrightarrow Y$ be a rational map of varieties over k, such that X is smooth over k, and such that Y is proper over k. If X has a k-rational point, then so has Y.* □

Moreover, we have already seen that a Brauer–Severi variety P over k is isomorphic to projective space over k if and only if P has a k-rational point, and we refer the interested reader to [14] for an algorithm to decide whether a Brauer–Severi surface has a k-rational point. In Definition 2.8, we defined the Amitsur group and showed its birational invariance in Proposition 2.10. Using Iskovskih's classification [25] of geometrically rational surfaces, we obtain the following list and refer to [10, Proposition 5.2] for details and proof.

Theorem 4.4 (Colliot-Thélène–Karpenko–Merkurjev) *Let X be a smooth and proper variety over a perfect field k such that $X_{\overline{k}}$ is birationally equivalent to $\mathbb{P}^2_{\overline{k}}$. Then, $\mathrm{Am}(X)$ is one of the following groups*

$$0, \quad \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{Z}/2\mathbb{Z})^2, \quad \textit{and} \quad \mathbb{Z}/3\mathbb{Z}.$$

We will see explicit examples of all these groups arising as Amitsur groups of del Pezzo surfaces in the next sections.

We now introduce another important invariant. Namely, if G_k denotes the absolute Galois group of k, and $H \subseteq G_k$ is a closed subgroup, then we consider for a smooth and projective variety X over k the group cohomology

$$H^1\left(H,\ \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})\right),$$

which is an abelian torsion group. If $b_1(X) = 0$, then $\mathrm{Pic}_{X/k}(k^{\mathrm{sep}})$ is finitely generated by Lemma 2.13 and then, $H^1(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}}))$ is a finite abelian group. Moreover, if $X_{k^{\mathrm{sep}}}$ is a rational surface, then $\mathrm{Br}'(X_{k^{\mathrm{sep}}}) = 0$ (see, for example, [35, Theorem 42.8] or [36]) and an appropriate Hochschild–Serre spectral sequence yields an exact sequence

$$0 \to \mathrm{Br}'(X)/\mathrm{Br}(k) \xrightarrow{\alpha} H^1\left(G_k, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})\right) \to H^3(G_k, (k^{\mathrm{sep}})^\times).$$

Moreover, if k is a global field, then the term on the right is zero by a theorem of Tate (see, for example, [38, Chap. VIII.3]), thus, α is an isomorphism, and we obtain an interpretation of this cohomology group in terms of Brauer groups, see [47, Sect. 3.4].

Lemma 4.5 *If P is a Brauer–Severi variety over k, then*

$$H^1\left(H, \mathrm{Pic}_{P/k}(k^{\mathrm{sep}})\right) = 0$$

for all closed subgroups $H \subseteq G_k$.

Proof Since $\mathrm{Pic}_{P/k}(k^{\mathrm{sep}}) \cong \mathbb{Z} \cdot \mathcal{O}_P(1)$ and since G_k acts trivially on the class $\mathcal{O}_P(1)$, the desired H^1 is isomorphic to $\mathrm{Hom}(H, \mathbb{Z})$, see [4, Chap. III.1, Exercise 2], for example. This is zero since H is a profinite group and the homomorphisms to $\mathbb{Z}$ are required to be continuous. □

In Proposition 2.10, we established birational invariance of $\mathrm{Am}(X)$. The following result of Manin [35, Sect. 1 of the Appendix] shows that also the above group cohomology groups are a birational invariants.

Theorem 4.6 (Manin) *For every closed subgroup $H \subseteq G_k$, the group*

$$H^1\left(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})\right)$$

is a birational invariant of smooth and projective varieties over k. □

Remark 4.7 Every birational map between smooth and projective surfaces can be factored into a sequence of blow-ups in closed points, see [35, Chap. III]. Using this, one can give very explicit proofs of Proposition 2.10 and Theorem 4.6 in dimension 2. (For such a proof of Theorem 4.6 in dimension 2, see the proof of [35, Theorem 29.1].)

4.3 Hasse Principle and Weak Approximation

For a global field K, i.e., a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$, we denote by Ω_K the set of its places, including the infinite ones if K is of characteristic zero. A class $\mathcal{C}$ of varieties over K satisfies

(1) the *Hasse principle*, if for every $X \in \mathcal{C}$ we have $X(K) \neq \emptyset$ if and only if $X(K_\nu) \neq \emptyset$ for all $\nu \in \Omega_K$. Moreover, $\mathcal{C}$ satisfies

(2) *weak approximation*, if the diagonal embedding

$$X(K) \to \prod_{\nu \in \Omega_K} X(K_\nu)$$

is dense for the product of the ν-adic topologies.

If $\mathcal{C}$ satisfies weak approximation, then it obviously also satisfies the Hasse principle, but the converse need not hold. For example, Brauer–Severi varieties over K satisfy the Hasse principle by a theorem of Châtelet [5], as well as weak approximation. However, both properties may fail for del Pezzo surfaces over K, and we refer to [47] for an introduction to this topic. We end this section by noting that the obstruction to a class $\mathrm{Pic}_{(X/K)(\mathrm{fppf})}(K)$ coming from $\mathrm{Pic}_X(K)$ satisfies the Hasse principle.

Lemma 4.8 *Let X a proper variety over a global field K and let $\mathcal{L} \in \mathrm{Pic}_{(X/K)(\mathrm{fppf})}(K)$. Then, the following are equivalent*

(1) $0 = \delta(\mathcal{L}) \in \mathrm{Br}(K)$*, and*
(2) $0 = \delta(\mathcal{L} \otimes_K K_\nu) \in \mathrm{Br}(K_\nu)$ *for all* $\nu \in \Omega_K$.

Proof A class in $\mathrm{Br}(K)$ is zero if and only if its image in $\mathrm{Br}(K_\nu)$ is zero for all $\nu \in \Omega_K$ by the Hasse principle for the Brauer group. From this, and functoriality of the exact sequence from Proposition 2.5, the assertion follows. □

For example, if $X(K_\nu) \neq \emptyset$ for all $\nu \in \Omega_X$, then δ is the zero map by Proposition 2.5 and this lemma. In this case, every class in $\mathrm{Pic}_{(X/K)(\mathrm{fppf})}(K)$ comes from an invertible sheaf on X.

5 Del Pezzo Surfaces of Product Type

In this section, we classify degree 8 del Pezzo surfaces of product type over k, i.e., surfaces X over k with $X_{\overline{k}} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, in terms of Brauer–Severi varieties.

First, for $\mathbb{P}^1_k \times \mathbb{P}^1_k$, the anti-canonical embedding can be written as composition of Veronese- and Segre-maps as follows

$$|-K_{\mathbb{P}^1_k \times \mathbb{P}^1_k}| : \mathbb{P}^1_k \times \mathbb{P}^1_k \stackrel{\nu_2 \times \nu_2}{\longrightarrow} \mathbb{P}^2_k \times \mathbb{P}^2_k \stackrel{\sigma}{\longrightarrow} \mathbb{P}^8_k .$$

Next, the invertible sheaf $\omega^{-1}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}$ is uniquely 2-divisible in the Picard group, and we obtain an embedding as a smooth quadric

$$|-\tfrac{1}{2}K_{\mathbb{P}^1_k \times \mathbb{P}^1_k}| : \mathbb{P}^1_k \times \mathbb{P}^1_k \stackrel{\sigma}{\longrightarrow} \mathbb{P}^3_k .$$

Now, let X be a degree 8 del Pezzo surface of product type over k. Then, the anti-canonical linear system yields an embedding of X as a surface of degree 8 into $\mathbb{P}^8_k$.

However, the "half-anti-canonical linear system" exists in general only as a morphism to a Brauer–Severi threefold as the following result shows.

Theorem 5.1 *Let X be a degree 8 del Pezzo surface of product type over a field k. Then, there exist a unique class $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ with $\mathcal{L}^{\otimes 2} \cong \omega_X^{-1}$ and an embedding*

$$|\mathcal{L}| : X \hookrightarrow P$$

into a Brauer–Severi threefold P over k with Brauer class

$$\delta(\mathcal{L}) = [P] \in \mathrm{Br}(k),$$

and such that $X_{\overline{k}}$ is a smooth quadric in $P_{\overline{k}} \cong \mathbb{P}^3_{\overline{k}}$. Moreover, X is rational if and only if X has a k-rational point. In this case, we have $P \cong \mathbb{P}^3_k$.

Proof To simplify notation, set $L := k^{\mathrm{sep}}$. We have $X(L) \neq \emptyset$, for example, by [18, Proposition A.1.1], as well as $\mathrm{Pic}(X_L) \cong \mathrm{Pic}(X_{\overline{k}}) \cong \mathbb{Z}^2$ by Theorem 4.2. The classes $(1, 0)$ and $(0, 1)$ of $\mathrm{Pic}(X_L)$ give rise to two morphisms $X_L \to \mathbb{P}^1_L$, and we obtain an isomorphism $X_L \cong \mathbb{P}^1_L \times \mathbb{P}^1_L$. By abuse of notation, we re-define $\overline{X}$ to be X_L. Next, the absolute Galois group G_k acts trivially on the canonical class $(-2, -2)$, and thus, the G_k-action on $\mathbb{Z}(1, 1) \subset \mathbb{Z}^2$ is trivial. By Proposition 2.7, we have $\mathrm{Pic}_{X/k}(K)^{G_k} \cong \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, and, since $(1, 1) \in \mathbb{Z}^2$ is G_k-invariant, the unique invertible sheaf $\mathcal{L}$ on $\overline{X}$ with $\mathcal{L}^{\otimes 2} \cong \omega_{\overline{X}}^{-1}$ descends to a class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$. Over L, the class $\mathcal{L}$ is very ample and defines an embedding of $\overline{X}$ as smooth quadric surface into $\mathbb{P}^3_L$. Thus, by Theorem 3.4, we obtain an embedding $|\mathcal{L}| : X \hookrightarrow P$, where P is a Brauer–Severi threefold over k with $\delta(\mathcal{L}) = [P] \in \mathrm{Br}(k)$.

Finally, if X is rational, then it has a k-rational point, and then, also P has a k-rational point, i.e., $P \cong \mathbb{P}^3_k$. Conversely, if there exists a k-rational point $x \in X$, then X is a quadric in $\mathbb{P}^3_k$, and projection away from x induces a birational map $X \dashrightarrow \mathbb{P}^2_k$. □

Next, we establish an explicit classification of degree 8 del Pezzo surfaces of product type in terms of the Néron–Severi rank ρ and Brauer–Severi curves. To simplify notation in the sequel, let us recall the definition of contracted products. If a finite group G acts on a scheme X from the right and it acts on a scheme Y from the left and all schemes and actions are over $\mathrm{Spec}\, k$ for some field k, then we denote the quotient of $X \times_{\mathrm{Spec}\, k} Y$ by the diagonal G-action defined by $(x, y) \mapsto (xg, g^{-1}y)$ for all $g \in G$ by

$$X \wedge^G Y := (X \times_{\mathrm{Spec}\, k} Y)/G.$$

We refer to [19, Chap. III.1.3] for details and applications.

Proposition 5.2 *Let X and $X \subset P$ be as in Theorem 5.1.*

(1) if $\rho(X) = 2$, then

$$X \cong P' \times P'',$$

where P' and P'' are Brauer–Severi curves over k, whose Brauer classes satisfy $[P] = [P'] + [P''] \in \mathrm{Br}(k)$. In particular, $P \cong \mathbb{P}^3_k$ if and only if $P' \cong P''$.

(2) *If $\rho(X) = 1$, then there exist a Brauer–Severi curve P' over k and a finite Galois extension K/k with Galois group $H := \mathbb{Z}/2\mathbb{Z}$, such that X arises as twisted self-product*

$$X \cong (P' \times P')_K/H = \mathrm{Spec}\, K \wedge^H (P' \times P'),$$

where the H-action permutes the factors of $P'_K \times P'_K$. Moreover, $P \cong \mathbb{P}^3_k$ and P' is a hyperplane section of $X \subset \mathbb{P}^3_k$.

Proof We keep the notations and assumptions from the proof of Theorem 5.1. The G_k-action fixes the class $(1, 1)$. Since the G_k-action preserves the intersection pairing on $\mathrm{Pic}_{X/k}(k^{\mathrm{sep}})$, it follows that G_k acts on $\mathbb{Z}(1, -1)$ either trivially, or by sign changes. We have $\rho(X) = 2$ in the first case, and $\rho(X) = 1$ in the latter.

First, assume that $\rho(X) = 2$. By Theorem 3.4, the classes $(1, 0)$ and $(0, 1)$ give rise to morphisms to Brauer–Severi curves $X \to P'$ and $X \to P''$ of class $[P'] = \delta((1, 0))$ and $[P''] = \delta((0, 1))$ in $\mathrm{Br}(k)$, respectively. Thus, we obtain a morphism $X \to P' \times P''$, which is an isomorphism because it is an isomorphism over k^{sep}. Since δ is a homomorphism, we find $[P] = \delta(\mathcal{L}) = \delta((1, 1)) = \delta((1, 0)) + \delta((0, 1)) = [P'] + [P'']$. Using that P' and P'' are of period 2, we find that $P \cong \mathbb{P}^3_k$ if and only if $[P] = 0$, i.e., if and only if $[P'] = [P'']$. By Corollary 2.16, the latter is equivalent to $P' \cong P''$.

Second, assume that $\rho(X) = 1$. Then, the G_k-action permutes $(0, 1)$ and $(1, 0)$, i.e., it permutes the factors of $\mathbb{P}^1_{k^{\mathrm{sep}}} \times \mathbb{P}^1_{k^{\mathrm{sep}}}$. Thus, there exists a unique quadratic Galois extension K/k, such that $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ acts trivially on $\mathrm{Pic}_{X/k}(k^{\mathrm{sep}})$ and by the previous analysis we have $X_K := Q'' \times Q'''$ for two Brauer–Severi curves Q'', Q''' over K. Using these and the $H := \mathrm{Gal}(K/k)$-action, we obtain a H-stable diagonal embedding $Q' \subset X_K$ of a Brauer–Severi curve over K, and then, the two projections induce isomorphisms $Q' \cong Q''$ and $Q' \cong Q'''$ over K. Taking the quotient by H, we obtain a Brauer–Severi curve $P' := Q'/H \subset X$ over k. Clearly, $P'_K \cong Q'$ and we obtain the description of X as twisted self-product. On X, the curve P' is a section of the class $(1, 1)$, which implies that this class comes from an invertible sheaf, and thus, $0 = \delta((1, 1)) \in \mathrm{Br}(k)$ by Proposition 2.5. Since $\delta((1, 1)) = [P]$, we conclude $P \cong \mathbb{P}^3_k$. □

Remark 5.3 In the case of quadrics in $\mathbb{P}^3$, similar results were already established in [9]. A related, but somewhat different view on degree 8 del Pezzo surfaces of product type was taken in (the proof of) [10, Proposition 5.2]: If X is such a surface, then there exists a quadratic Galois extension K/k and a Brauer–Severi curve C over K, such that $X \cong \mathrm{Res}_{K/k}C$, where $\mathrm{Res}_{K/k}$ denotes Weil restriction, see also [41].

Corollary 5.4 *Let X be as in Theorem 5.1. Then,*

$$H^1\left(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})\right) = 0$$

for all closed subgroups $H \subseteq G_k$, and

$$\mathrm{Am}(X) \cong \begin{cases} 0 & \textit{if } \rho = 1 \textit{ or if } X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \textit{if } \rho = 2 \textit{ and } \mathbb{P}^1_k \not\cong P' \not\cong P'' \not\cong \mathbb{P}^1_k, \\ (\mathbb{Z}/2\mathbb{Z}) & \textit{in the remaining } \rho = 2\textit{-cases.} \end{cases}$$

PROOF Set $H^1(H) := H^1(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}}))$. If $\rho = 2$, then the G_k-action on $\mathrm{Pic}_{X/k}(k^{\mathrm{sep}})$ is trivial, and we find $H^1(H) = 0$ as in the proof of Lemma 4.5. Moreover, $\mathrm{Am}(X)$ is generated by $\delta((0, 1)$ and $\delta((1, 0))$, i.e., by $[P']$ and $[P'']$ in $\mathrm{Br}(k)$. From this, the assertions on $\mathrm{Am}(X)$ follow in case $\rho = 2$.

If $\rho = 1$, then there exists an isomorphism $\mathrm{Pic}_{X/k}(k^{\mathrm{sep}}) \cong \mathbb{Z}^2$, such that the G_k-action factors through a surjective homomorphism $G_k \to \mathbb{Z}/2\mathbb{Z}$ and acts on $\mathbb{Z}^2$ via $(a, b) \mapsto (b, a)$. In particular, we find $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}^2) = 0$ with respect to this action, see, for example, [4, Chap. III.1, Example 2]. From this, we deduce $H^1(H) = 0$ using inflation maps. Moreover, $\mathrm{Am}(X)$ is generated by $\delta((1, 1))$, which is zero, since $(1, 1)$ is the class of an invertible sheaf. □

Corollary 5.5 *If X is as in Theorem 5.1, then the following are equivalent*

(1) X is birationally equivalent to a Brauer–Severi surface,
(2) X is rational,
(3) X has a k-rational point, and
(4) X is isomorphic to

$$X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k \quad \textit{or to} \quad X \cong \mathrm{Spec}\, K \wedge (\mathbb{P}^1_k \times \mathbb{P}^1_k).$$

PROOF The implications (2) ⇒ (1) and (2) ⇒ (3) are trivial, and we established (3) ⇒ (2) in Theorem 5.1. Moreover, if X is birationally equivalent to a Brauer–Severi surface P, then $\mathrm{Am}(P) = \mathrm{Am}(X)$ is cyclic of order 1 or 3 by Lemma 4.5 and Theorem 4.6. Together with Corollary 5.4, we conclude $\mathrm{Am}(P) = \mathrm{Am}(X) = 0$, i.e., $P \cong \mathbb{P}^2_k$, which establishes (1) ⇒ (2).

Since (4) ⇒ (3) is trivial, it remains to establish (3) ⇒ (4). Thus, we assume $X(k) \neq \emptyset$. If $\rho = 2$, then $X \cong P' \times P''$ and both Brauer–Severi curves P' and P'' have k-rational points, i.e., $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$. If $\rho = 1$, we have an embedding $X \subset \mathbb{P}^3_k$ and $X \cong \mathrm{Spec}\, K \wedge (P' \times P')$. Since $X(k) \neq \emptyset$, we have $X(K) \neq \emptyset$, which yields $P'(K) \neq \emptyset$, and thus $P'_K \cong \mathbb{P}^1_K$. A k-rational point on X gives rise to a K-rational and $\mathrm{Gal}(K/k)$-stable point on $X_K \cong \mathbb{P}^1_K \times \mathbb{P}^1_K$. In particular, this point lies on some diagonal $\mathbb{P}^1_K \subset X_K$, and thus, lies on some diagonal $P'' \subseteq X$ with $X \cong \mathrm{Spec}\, K \wedge (P'' \times P'')$. Since $P''(k) \neq \emptyset$, we find $P'' \cong \mathbb{P}^1_k$. □

We refer to Sect. 6.1 for more applications of these results to the arithmetic and geometry of these surfaces.

6 Del Pezzo Surfaces of Large Degree

Let X be a del Pezzo surface of degree d over a field k that is not of product type. Then, there exists a birational morphism

$$\overline{f} : \overline{X} \to \mathbb{P}^2_{\overline{k}}$$

that is a blow-up in $(9-d)$ closed points $P_1, ..., P_{9-d}$ in general position. We set $H := \overline{f}^* \mathcal{O}_{\mathbb{P}^2_{\overline{k}}}(1)$ and let $E_i := \overline{f}^{-1}(P_i)$ be the exceptional divisors of $\overline{f}$. Then, there exists an isomorphism of abelian groups

$$\operatorname{Pic}(\overline{X}) \cong \mathbb{Z}H \oplus \bigoplus_{i=1}^{9-d} \mathbb{Z}E_i.$$

The (-1)-curves of $\overline{X}$ consist of the E_i, of preimages under $\overline{f}$ of lines through two distinct points P_i, of preimages under $\overline{f}$ of quadrics through five distinct points P_i, etc., and we refer to [35, Theorem 26.2] for details. Let $K_{\overline{X}}$ be the canonical divisor class of $\overline{X}$, and let $\widetilde{E}$ be the sum of all (-1)-curves on $\overline{X}$. We leave it to the reader to verify the following table.

d	class of $\widetilde{E}$ in $\operatorname{Pic}(\overline{X})$	relations
9	0	$3H = -K_{\overline{X}}$
8	E_1	$3H = -K_{\overline{X}} + \widetilde{E}$
7	H	$H = \widetilde{E}$
6	$3H - \sum_{i=1}^{3} E_i$	$0 = -K_{\overline{X}} - \widetilde{E}$
5	$6H - 2\sum_{i=1}^{4} E_i$	$0 = -2K_{\overline{X}} - \widetilde{E}$
4	$12H - 4\sum_{i=1}^{5} E_i$	$0 = -4K_{\overline{X}} - \widetilde{E}$
3	$27H - 9\sum_{i=1}^{6} E_i$	$0 = -9K_{\overline{X}} - \widetilde{E}$
2	$84H - 28\sum_{i=1}^{7} E_i$	$0 = -28K_{\overline{X}} - \widetilde{E}$
1	$720H - 240\sum_{i=1}^{8} E_i$	$0 = -240K_{\overline{X}} - \widetilde{E}$

Together with Theorem 3.4, we obtain the following result.

Theorem 6.1 *Let X be a del Pezzo surface of degree $d \geq 7$ over a field k that is not of product type. Then, $\overline{f}$ descends to a birational morphism*

$$f : X \to P$$

to a Brauer–Severi surface P over k, where

$$\delta(H) = [P] \in \operatorname{Br}(k) \quad \textit{and} \quad \operatorname{Am}(X) \cong \mathbb{Z}/\operatorname{per}(P)\mathbb{Z}.$$

Moreover, X is rational if and only if $P \cong \mathbb{P}^2_k$. This is equivalent to X having a k-rational point.

Proof By Theorem 4.2, the invertible sheaf H on $X_{\overline{k}}$ defining $\overline{f}$ already lies in $\operatorname{Pic}_X(k^{\mathrm{sep}})$, i.e., $\overline{f}$ descends to k^{sep}, and by abuse of notation, we re-define $\overline{X}$ to be $X_{k^{\mathrm{sep}}}$. Clearly, the canonical divisor class $K_{\overline{X}}$ is G_k-invariant, and since G_k permutes the (-1)-curves of $\overline{X}$, also the class of $\widetilde{E}$ is G_k-invariant. In particular, $K_{\overline{X}}$ and $\widetilde{E}$ define classes in $\operatorname{Pic}_{X/k}(k^{\mathrm{sep}})^{G_k} \cong \operatorname{Pic}_{(X/k)(\mathrm{fppf})}(k)$. If $d \geq 7$, then the above table shows that there exist positive multiples of H that are integral linear combinations of $K_{\overline{X}}$ and $\widetilde{E}$. Thus, $H \in \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})$ descends to a class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$. By Theorem 3.4, $\overline{f}$ descends to a birational morphism $f : X \to P$, where P is a Brauer–Severi surface of class $\delta(H) \in \mathrm{Br}(k)$. The assertion on $\mathrm{Am}(X)$ follows from Proposition 2.10 and Theorem 2.19.

If X has a k-rational point, then so has P, and then $P \cong \mathbb{P}^2_k$. Since f is a birational morphism, $P \cong \mathbb{P}^2_k$ implies that X is rational. And if X is rational, then it has a k-rational point by Lemma 4.3. □

As an immediate consequence, we obtain rationality and the existence of k-rational points in some cases.

Corollary 6.2 *Let X be as in Theorem 6.1. If $d \in \{7, 8\}$, then X has a k-rational point and $\overline{f}$ descends to a birational morphism $f : X \to \mathbb{P}^2_k$.*

Proof By Theorem 6.1, there exists a birational morphism $X \to P$ that is a blow-up in a closed subscheme $Z \subset P$ of length $(9 - d)$. By Corollary 2.21, we have $P \cong \mathbb{P}^2_k$ if 3 and $(9 - d)$ are coprime. In particular, we have $X(k) \neq \emptyset$ in these cases by Theorem 6.1 and Lemma 4.3. □

Since a del Pezzo surface of degree 9 is a Brauer–Severi surface, it has rational points if and only if it is trivial. In particular, Corollary 6.2 does not hold for $d = 9$.

6.1 Applications to Arithmetic Geometry

We now give a couple of applications of the just established results. Again, we stress that most if not all of these applications are well-known, and merely illustrate the usefulness of studying varieties via Brauer–Severi varieties.

Corollary 6.3 *If X is a del Pezzo surface of degree ≥ 7 over k, then*

$$H^1\left(H,\ \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})\right) = 0.$$

for all closed subgroups $H \subseteq G_k$

PROOF If X is not of product type, then it is birationally equivalent to a Brauer–Severi surface P by Theorem 6.1, and then the statement follows from Theorem 4.6 and Lemma 4.5. If X is of product type, then this is Corollary 5.4. □

For the next application, let us recall that a surface is called *rational* if it is birationally equivalent to $\mathbb{P}^2_k$, and that it is called *unirational* if there exists a dominant and rational map from $\mathbb{P}^2_k$ onto it. The following result is a special case of [35, Theorem 29.4].

Corollary 6.4 *Let X be a del Pezzo surface of degree ≥ 7 over a field k. Then, the following are equivalent:*

(1) X is rational,
(2) X is unirational, and
(3) X has a k-rational point.

PROOF Clearly, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3), whereas (3) $\Rightarrow$ (1) follows from Corollary 5.5 and Theorem 6.1. □

This leads us to the question whether a del Pezzo surface necessarily has a k-rational point. Over finite fields, this is true and follows from the Weil conjectures, which we will recall in Theorem 8.1 below. By a theorem of Wedderburn, finite fields have trivial Brauer groups, and thus, the following corollary gives existence of k-rational points for more general fields.

Corollary 6.5 *Let X be a del Pezzo surface of degree ≥ 7 over a field k with* $\mathrm{Br}(k) = 0$*. Then, X has a k-rational point, and thus, is rational.*

Proof If X is not of product type, then there exists a birational morphism $f : X \to P$ to a Brauer–Severi surface by Theorem 6.1. Since $\mathrm{Br}(k) = 0$, we have $P \cong \mathbb{P}^2_k$, and Theorem 6.1 gives $X(k) \neq \emptyset$.

Thus, let X be of product type. By Proposition 5.2, X is a product of Brauer–Severi curves ($\rho = 2$), or contains at least a Brauer–Severi curve ($\rho = 1$). Since $\mathrm{Br}(k) = 0$, all Brauer–Severi curves are isomorphic to $\mathbb{P}^1_k$, and thus, contain k-rational points. In particular, we find $X(k) \neq \emptyset$. □

In Sect. 4.3, we discussed the Hasse principle and weak approximation for varieties over global fields. Here, we establish the following.

Corollary 6.6 *Del Pezzo surfaces of degree ≥ 7 over global fields satisfy weak approximation and the Hasse principle.*

Proof If X is not of product type, then it is birationally equivalent to a Brauer–Severi surface by Theorem 6.1, and since the two claimed properties are preserved under birational maps and hold for Brauer–Severi varieties, the assertion follows in this case.

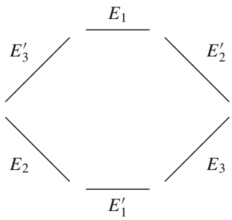
If X is of product type, then there are two cases by Proposition 5.2. If $\rho = 2$, then X is a product of two Brauer–Severi curves, and we conclude as before.

Thus, we may assume $\rho = 1$. Let us first establish the Hasse principle: there exists a quadratic Galois extension L/K, such that $\rho(X_L) = 2$. From $X(K_\nu) \neq \emptyset$ for all $\nu \in \Omega_K$, we find $X_{L_\mu} \cong \mathbb{P}^1_{L_\mu} \times \mathbb{P}^1_{L_\mu}$ for all $\mu \in \Omega_L$, and thus, $X_L \cong \mathbb{P}^1_L \times \mathbb{P}^1_L$ by the Hasse principle for Brauer–Severi curves. As in the proof of Corollary 5.5, we exhibit X as twisted self-product of $\mathbb{P}^1_k$, which has a k-rational point and establishes the Hasse principle. Thus, to establish weak approximation, we may assume that X has a k-rational point. But then, X is rational by Corollary 5.5, and since weak approximation is a birational invariant, the assertion follows. □

7 Del Pezzo Surfaces of Degree 6

In the previous sections, we have seen a close connection between Brauer–Severi varieties and del Pezzo surfaces of degree ≥7. In this section, we discuss del Pezzo surfaces of degree 6, which are not so directly linked to Brauer–Severi varieties.

For the geometry and the arithmetic of these surfaces, we refer the interested reader to [6, 35], and the survey [47, Sect. 2.4]. We keep the notation introduced in Sect. 6: If X is a degree 6 del Pezzo surface over a field k, then there exists a blow-up $f_{\overline{k}} : \overline{X} \to \mathbb{P}^2_{\overline{k}}$ in three points in general position with exceptional (-1)-curves E_1, E_2, and E_3. Then, there are six (-1)-curves on X, namely the three exceptional curves E_i, $i = 1, 2, 3$ of $\overline{f}$, as well as the three curves $E'_i := H - E_j - E_k$, $i = 1, 2, 3$ where $\{j, k\} = \{1, 2, 3\} \backslash \{i\}$ and where $H = \overline{f}^* \mathcal{O}_{\mathbb{P}^2}(1)$ as in Sect. 6. These curves intersect in a hexagon as follows.



The absolute Galois group G_k acts on these six (-1)-curves on $X_{k^{\mathrm{sep}}}$, and associated to this action, we have following field extensions of k.

(1) Since G_k acts on the two sets $\{E_1, E_2, E_3\}$ and $\{E'_1, E'_2, E'_3\}$, there is a group homomorphism
$$\varphi_1 \,:\, G_k \,\to\, S_2 \,\cong\, \mathbb{Z}/2\mathbb{Z}.$$
The fixed field of either of the two sets is a finite separable extension $k \subseteq K$ with $[K : k]|2$, and $k \neq K$ if and only if φ_1 is surjective.

(2) Since G_k acts on the three sets $\{E_i, E'_i\}$, $i = 1, 2, 3$, there is a group homomorphism
$$\varphi_2 \,:\, G_k \,\to\, S_3.$$
There exists a finite separable extension $k \subseteq L$ with $[L : k]|3$, unique up to conjugation in k^{sep}, over which at least one of these three sets is defined. We have $k \neq L$ if and only if 3 divides the order of $\varphi_2(G_k)$. Next, there exists a finite and separable extension $L \subseteq M$ with $[M : L]|2$, over which all three sets are defined.

Combining φ_1 and φ_2, we obtain a group homomorphism

$$G_k \xrightarrow{\varphi_1 \times \varphi_2} \mathbb{Z}/2\mathbb{Z} \times S_3 \,\cong\, D_{2\cdot 6},$$

where $D_{2\cdot 6}$ denotes the dihedral group of order 12, i.e., the automorphism group of the hexagon. Using these field extensions, we obtain the following classification, which uses and slightly extends a classical result of Manin from [35] in case (3).

Theorem 7.1 *Let X be a del Pezzo surface of degree 6 over a field k.*

(1) The morphism $\overline{f}$ descends to a birational morphism

$$f : X \to P$$

to a Brauer–Severi surface P if and only if $k = K$. In this case, $\rho(X) \geq 2$ and $\mathrm{Am}(X) = \mathrm{Am}(P)$.

(2) There exists a birational morphism $X \to Y$ onto a degree 8 del Pezzo surface Y of product type if and only if $k = L$. In this case,

	$\rho(X)$	Y
$k \neq M$	3	$\mathrm{Spec}\, M \wedge (\mathbb{P}^1_k \times \mathbb{P}^1_k)$
$k = M$	4	$\mathbb{P}^1_k \times \mathbb{P}^1_k$

X has a k-rational point, and $\mathrm{Am}(X) = 0$.

(3) If $k \neq K$ and $k \neq L$, then $\rho(X) = 1$, $\mathrm{Am}(X) = 0$, *and the following are equivalent.*

(a) X is birationally equivalent to a Brauer–Severi surface,
(b) X is birationally equivalent to a product of two Brauer–Severi curves,
(c) X is rational, and
(d) X has a k-rational point.

PROOF Let us first show (1). If $k = K$, then $F := E_1 + E_2 + E_3$ descends to a class in $\mathrm{Pic}(X_{k^{\mathrm{sep}}})^{G_k} = \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ and we find $\rho(X) \geq 2$. Thus, also $H = \frac{1}{3}(-K_X + F)$ descends to a class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$, and by Theorem 3.4, we obtain a birational morphism $|H| : X \to P$ to a Brauer–Severi surface, which coincides with $\overline{f}$ over $\overline{k}$. Conversely, if $\overline{f}$ descends to a birational morphism $f : X \to P$, then the exceptional divisor of f is of class F or $E'_1 + E'_2 + E'_3$, and we find $k = K$. Moreover, we have $\mathrm{Am}(X) = \mathrm{Am}(P)$ by Theorem 4.6.

If $k = L$, then, say $E_1 + E'_1$, descends to a class in $\mathrm{Pic}(X_{k^{\mathrm{sep}}})^{G_k}$. Moreover, we find that the classes $\frac{1}{2}(-K_X + E_1 + E'_1) = 2H - E_2 - E_3$ as well as $\frac{1}{2}(-K_X - E_1 - E'_1) = H - E_1$, and thus, the classes H, E_1, and $E'_1 = H - E_2 - E_3$ lie in $\mathrm{Pic}(X_{k^{\mathrm{sep}}})^{G_k}$. The G_k-action is trivial on H and E_1, whereas it is either trivial on the set $\{E_2, E_3\}$ (if $k = M$) or permutes the two (if $k \neq M$). Since the class of E_1 is G_k-invariant and there is a unique effective divisor in this linear system, we find that $\mathbb{P}^1_k \cong E_1 \subset X$. In particular, X has a k-rational point and $\mathrm{Am}(X) = 0$. Using Theorem 3.4 and the fact that X has a k-rational point, we obtain a birational morphism

$$|\frac{1}{2}(-K_X + E_1 + E'_1)| : X \to Y \subset \mathbb{P}^3_k$$

onto a smooth quadric Y with a k-rational point. In particular, Y is a degree 8 del Pezzo surface of product type. Over k^{sep}, this morphism contracts E_1 and E_1' and thus, we find

$$\mathrm{Pic}(Y_{k^{\mathrm{sep}}}) \cong \left(\mathbb{Z}H \oplus \bigoplus_{i=1}^{3} \mathbb{Z}E_i\right) / \langle E_1, E_1' \rangle \cong \mathbb{Z}\overline{E}_2 \oplus \mathbb{Z}\overline{E}_3.$$

The G_k-action on it is either trivial ($k = M$) or permutes the two summands ($k \neq M$). Using $Y(k) \neq \emptyset$ and Corollary 5.5, we find $\rho(X) = 4$ and $Y \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ in the first case, and $\rho(X) = 3$ and $Y \cong \mathrm{Spec}\, M \wedge (\mathbb{P}^1_k \times \mathbb{P}^1_k)$ in the latter. Conversely, if there exists a birational morphism $X \to Y$ onto a degree 8 del Pezzo surface of product type, then the exceptional divisor is of class $E_i + E_i'$ for some i, and thus, $k = L$. This establishes (2).

Finally, assume that $k \neq K$ and $k \neq L$. Then, φ_1 is surjective, and $\varphi_2(G_k)$ contains all 3-cycles of S_3. From this, it is not difficult to see that $\mathrm{Pic}(X_{\overline{k}})^{G_k}$ is of rank 1 and generated by the class of K_X. Since this latter class is an invertible sheaf, we find $\mathrm{Am}(X) = 0$. Thus, if X is birationally equivalent to a Brauer–Severi surface P, then $\mathrm{Am}(X) = 0$ together with Lemma 4.5 and Theorem 4.6 implies that $P \cong \mathbb{P}^2_k$. Similarly, if X is birationally equivalent to the product $P' \times P''$ of two Brauer–Severi curves, then $P' \cong P'' \cong \mathbb{P}^1_k$. From this, we obtain the implications $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d)$. The implication $(d) \Rightarrow (c)$ is due to Manin [35, Theorem 29.4]. □

Remark 7.2 In case (1) of the above theorem it is important to note that P need not be unique, but that $\mathrm{Am}(P)$ is well-defined. More precisely, if we set $F := E_1 + E_2 + E_3$ and $F' = E_1' + E_2' + E_3'$, then Theorem 3.4 provides us with two morphisms to Brauer–Severi surfaces P_1 and P_2

$$\begin{array}{rl} |H| \;=\; |\tfrac{1}{3}(-K_X + F)| \;:\; X & \to\; P_1 \\ |H'| := |\tfrac{1}{3}(-K_X + F')| \;:\; X & \to\; P_2 \end{array}$$

Since $H + H' = -K_X$ and $\delta(K_X) = 0$, we find

$$[P_1] = \delta(H) = \delta(-K_X - H') = -\delta(H') = -[P_2] \in \mathrm{Br}(k),$$

and thus, $P_1 \cong P_2$ if and only if both are isomorphic to $\mathbb{P}^2_k$. On the other hand, P_1 and P_2 are birationally equivalent, since we have birational morphisms

$$P_1 \stackrel{|H|}{\longleftarrow} X \stackrel{|H'|}{\longrightarrow} P_2 \,.$$

Over $\overline{k}$, this becomes the blow-up of three closed points Z followed by the blow-down of the three (-1)-curves that are the strict transforms of lines through any two of the points in Z. This is an example of a *Cremona transformation*.

We remark that a surface of case (3) and without k-rational points is neither birationally equivalent to a Brauer–Severi surface nor to the product of two Brauer–Severi curves. For finer and more detailed classification results for degree 6 del Pezzo surfaces, we refer the interested reader to [2, 10, 13]. Finally, the sum $\widetilde{E}$ of all (-1)-curves on $X_{k^{\text{sep}}}$ is a G_k-invariant divisor, and thus, descends to a curve on X. By [35, Theorem 30.3.1], the complement $X\backslash\widetilde{E}$ is isomorphic to a torsor under a two-dimensional torus over k, which can be used to study the arithmetic and geometry of these surfaces, see also [43].

8 Del Pezzo Surfaces of Small Degree

For the remainder of this article, our results will be less complete and less self-contained. We will circle around questions of birationality of a del Pezzo surface X of degree ≤ 5 to Brauer–Severi surfaces, and about descending the morphism $\overline{f} : \overline{X} \to \mathbb{P}^2_{\overline{k}}$ to k.

8.1 Birationality to Brauer–Severi Surfaces

Let $k = \mathbb{F}_q$ be a finite field of characteristic p, and let X be a smooth and projective surface over k such that $X_{\overline{k}}$ is birationally equivalent to $\mathbb{P}^2$. Then, it follows from the Weil conjectures (in this case already a theorem of Weil himself) that the number of k-rational points is congruent to 1 modulo q, see [35, Chap. IV.27]. In particular, we obtain that

Theorem 8.1 (Weil) *If X is a del Pezzo surface over a finite field $\mathbb{F}_q$, then X has a $\mathbb{F}_q$-rational point.*

Since $\mathrm{Br}(\mathbb{F}_q) = 0$ by a theorem of Wedderburn, there are no non-trivial Brauer–Severi varieties over $\mathbb{F}_q$.

Remark 8.2 Let X be a del Pezzo surface of degree ≥ 5 over a field k. Manin [35, Theorem 29.4] showed that X is rational if and only if it contains a k-rational point. Even if X has no k-rational point, Manin [35, Theorem 29.3] showed that

$$H^1\left(H,\ \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k^{\mathrm{sep}})\right) = 0$$

for all closed subgroups $H \subseteq G_k$. We refer to [8, Théorème 2.B.1] for a general principle explaining this vanishing of cohomology.

In this section, we give a partial generalization to birational maps to Brauer–Severi surfaces.

Lemma 8.3 *Let X be a degree d del Pezzo surface over k. Then,*

(1) *There exists an effective zero-cycle Z of degree d on X. If $d \neq 2$ or if $\mathrm{char}(k) \neq 2$, then there exists such a zero-cycle Z, whose closed points have residue fields that are separable over k.*
(2) *The abelian group $\mathrm{Am}(X)$ is finite and every element has an order dividing d.*

PROOF If $d \geq 3$, then ω_X^{-1} is very ample, and $|\omega_X^{-1}|$ embeds X as a surface of degree d into $\mathbb{P}_k^d$. Intersecting X with a linear subspace of codimension 2, we obtain an effective zero-cycle Z of degree d on X. The closed points of Z have automatically separable residue fields if k is finite. Otherwise, k is infinite, and then, the intersection with a generic linear subspace of codimension 2 yields a Z that is smooth over k by [28, Théorème I.6.3]. Thus, in any case, we obtain a Z, whose closed points have residue fields that are separable over k. If $d = 2$, then $|\omega_X^{-1}|$ defines a double cover $X \to \mathbb{P}_k^2$, and the pre-image of a k-rational point yields an effective zero-cycle Z of degree 2 on X. If $\mathrm{char}(k) \neq 2$, then residue fields of closed points of Z are separable over k. If $d = 1$, then $|-K_X|$ has a unique-base point, and in particular, X has a k-rational point. This establishes (1). Since $b_1(X) = 0$, the group $\mathrm{Am}(X)$ is finite by Lemma 2.13. Then, assertion (2) follows from Lemma 2.9. □

Corollary 8.4 *Let X be a del Pezzo surface of degree d over a field k.*

(1) *If $d \in \{1, 2, 4, 5, 7, 8\}$ and X is birationally equivalent to a Brauer–Severi surface P, then $P \cong \mathbb{P}_k^2$ and X has a k-rational point.*
(2) *If $d \in \{1, 3, 5, 7, 9\}$ and X is birationally equivalent to a product $P' \times P''$ of two Brauer–Severi curves, then $P' \cong P'' \cong \mathbb{P}_k^1$ and X has a k-rational point.*

PROOF Let X and d be as in (1). Then, every element of $\mathrm{Am}(X)$ is of order dividing d by Lemma 8.3, but also of order dividing 3 by Theorems 2.18 and 4.6. By our assumptions on d, we find $\mathrm{Am}(P) = 0$, and thus, $P \cong \mathbb{P}_k^2$. Since the latter has a k-rational point, so has X by Lemma 4.3. This shows (1). The proof of (2) is similar and we leave it to the reader. □

Combining this with a result of Coray [12], we obtain the following.

Theorem 8.5 *Let X be a del Pezzo surface of degree $d \in \{5, 7, 8\}$ over a perfect field k. Then, the following are equivalent*

(1) *There exists a dominant and rational map $P \dashrightarrow X$ from a Brauer–Severi surface P over k,*
(2) *X is birationally equivalent to a Brauer–Severi surface,*
(3) *X is rational, and*
(4) *X has a k-rational point.*

PROOF The implications (3) ⇒ (2) ⇒ (1) are trivial.

Let $\varphi : P \dashrightarrow X$ be as in (1). By Lemma 8.3, there exists a zero-cycle of degree 9 on P, and another one of degree d on X. Using φ, we obtain a zero-cycle of degree dividing 9 on X. By assumption, d is coprime to 9, and thus, there exists a zero-cycle of degree 1 on X. By [12], this implies that X has a k-rational point and establishes (1) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is a result of Manin [35, Theorem 29.4]. $\square$

Now, if a del Pezzo surface X over a field k is birationally equivalent to a Brauer–Severi surface, then $H^1(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})) = 0$ for all closed subgroups $H \subseteq G_k$ by Theorem 4.6. Moreover, this vanishing holds for all del Pezzo surfaces of degree ≥ 5, see Remark 8.2. However, for del Pezzo surfaces of degree ≤ 4, these cohomology groups may be non-zero, see [35, Sect. 31], [32, 45, 46]. In particular, del Pezzo surfaces of degree ≤ 4 are in general *not* birationally equivalent to Brauer–Severi surfaces.

For further information concerning geometrically rational surfaces, unirationality, central simple algebras, and connections with cohomological dimension, we refer the interested reader to [10].

8.2 Del Pezzo Surfaces of Degree 5

In order to decide whether a birational map $f_{\overline{k}} : X_{\overline{k}} \to \mathbb{P}^2_{\overline{k}}$ as in Sect. 6 descends to k for a degree 5 del Pezzo surface X over k, we introduce the following notion.

Definition 8.6 Let X be a del Pezzo surface over a field k. A *conic* on X is a geometrically integral curve C on X with $C^2 = 0$ and $-K_X \cdot C = 2$. An element $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ is called a *conic class* if $\mathcal{L} \otimes_k \overline{k} \cong \mathcal{O}_{X_{\overline{k}}}(\overline{C})$ for some conic $\overline{C}$ on $X_{\overline{k}}$.

The following is an analogue of Theorem 6.1 for degree 5 del Pezzo surfaces.

Theorem 8.7 *Let X be a del Pezzo surface of degree 5 over a field k. Then, the following are equivalent:*

(1) There exists a birational morphism $f : X \to P$ to a Brauer–Severi surface, such that $f_{\overline{k}}$ is the blow-up of 4 points in general position.
(2) There exists a birational morphism $f : X \to \mathbb{P}^2_k$, such that $f_{\overline{k}}$ is the blow-up of 4 points in general position.
(3) There exists a class $F \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ such that

$$F_{\overline{k}} \cong \mathcal{O}_{\overline{X}}(E_1 + E_2 + E_3 + E_4),$$

where the E_i are disjoint (-1)-curves on $\overline{X}$.
(4) There exists a conic class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$.

If these equivalent conditions hold, then X has a k-rational point.

Proof If f is as in (1), then X has a k-rational point by Corollary 8.4. Thus, $P \cong \mathbb{P}^2_k$, and we obtain (1) $\Rightarrow$ (2).

If f is as in (2), then the exceptional divisor of f is a class F as stated in (3), and we obtain (2) $\Rightarrow$ (3).

If f is as in (3), then, using Theorem 3.4, there exists a birational morphism $|\frac{1}{3}(-K_X - F)| : X \to P$ to a Brauer–Severi surface P as in (1), which establishes (3) $\Rightarrow$ (1).

If f is as in (2), let $Z \subset \mathbb{P}^2_k$ be the degree 4 cycle blown up by f. Then $f^*(\mathcal{O}_{\mathbb{P}^2_k}(2)(-Z))$, i.e., the pullback of the pencil of conics through Z, is a conic class on X and establishes (2) $\Rightarrow$ (4).

Finally, if C is a conic class on X, then, using Theorem 3.4, there exists a birational morphism $|-K_X + C| : X \to P$ to a Brauer–Severi surface P as in (1), which establishes (4) $\Rightarrow$ (1). □

Remark 8.8 By theorems of Enriques, Swinnerton-Dyer, Skorobogatov, Shepherd-Barron, Kollár, and Hassett (see [47, Theorem 2.5] for precise references and overview), a degree 5 del Pezzo X over a field k always has a k-rational point. Thus, X is rational by [35, Theorem 29.4], and we have

$$\mathrm{Am}(X) = 0, \quad \text{as well as} \quad H^1(H, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}})) = 0$$

for every closed subgroup $H \subseteq G_k$ by Corollary 2.12, Theorem 4.6, and Lemma 4.5.

8.3 Del Pezzo Surfaces of Degree 4

A classical theorem of Manin [35, Theorem 29.4] states that a del Pezzo surface of degree 4 over a sufficiently large field k is unirational if and only if it contains a k-rational point. Here, we have the following analogue in our setting.

Proposition 8.9 *Let X be a del Pezzo surface of degree 4 over a perfect field k. Then, the following are equivalent*

(1) There exists a dominant rational map $P \dashrightarrow X$ from a Brauer–Severi surface P over k.
(2) X is unirational,
(3) X has a k-rational point,

Proof The implications (2) $\Rightarrow$ (1) is trivial and (2) $\Rightarrow$ (3) is Lemma 4.3.

The implication (3) $\Rightarrow$ (2) is shown in [35, Theorem 29.4] and [35, Theorem 30.1] if k has at least 23 elements and in [31, Theorem 2.1] and [40, Proposition 5.19] in the remaining cases.

To show (1) $\Rightarrow$ (3), we argue as in the proof of the implication (1) $\Rightarrow$ (4) of Theorem 8.5 by first exhibiting a degree 1 zero-cycle on X, and then, using [12] to deduce the existence of a k-rational point on X. We leave the details to the reader. □

If a field k is finite or perfect of characteristic 2, then a degree 4 del Pezzo surface over k always has a k-rational point, see [35, Theorem 27.1] and [16]. In this case, we also have $\mathrm{Am}(X) = 0$. From Lemma 8.3, we infer that $\mathrm{Am}(X)$ is at most 4-torsion for degree 4 del Pezzo surfaces. For the possibilities of $H^1(G_k, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}}))$, see [45].

The following is an analog of Theorem 6.1 for degree 4 del Pezzo surfaces.

Theorem 8.10 *Let X be a del Pezzo surface of degree 4 over a field k. Then, the following are equivalent:*

(1) There exists a birational morphism $f : X \to P$ to a Brauer–Severi surface, such that $f_{\overline{k}}$ is the blow-up of 5 points in general position.
(2) There exists a birational morphism $f : X \to \mathbb{P}^2_k$, such that $f_{\overline{k}}$ is the blow-up of 5 points in general position.
(3) There exists a curve $\mathbb{P}^1_k \cong E \subset X$ with $E^2 = -1$.
(4) There exists a class $E \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ with $E^2 = K_X \cdot E = -1$.

If these equivalent conditions hold, then X has a k-rational point.

PROOF The implication (2) $\Rightarrow$ (1) is trivial. If f is as in (1), then X has a k-rational point by Corollary 8.4. Thus, $P \cong \mathbb{P}^2_k$, and we obtain (1) $\Rightarrow$ (2).

If f is as in (2), let $Z \subset \mathbb{P}^2_k$ be the degree 5 cycle blown up by f. Then $f^*(\mathcal{O}_{\mathbb{P}^2_k}(2)(-Z))$, i.e., the pullback of the class of the unique conic through Z, is a class E as stated in (4) on X and establishes (2) $\Rightarrow$ (4).

If E is a class as in (4), then, using Theorem 3.4, there exists a birational morphism $|-K_X - E| : X \to P$ to a Brauer–Severi surface P as in (1), which establishes (4) $\Rightarrow$ (1).

The implication (3) $\Rightarrow$ (4) is trivial, and if E is a class as in (4), then there exists a unique section of the associated invertible sheaf on k^{sep}. This is necessarily G_k-invariant, thus, descends to a curve on X, and establishes (4) $\Rightarrow$ (3). □

Remark 8.11 In [44], Skorobogatov called del Pezzo surfaces of degree 4 that satisfy condition (3) above *quasi-split*.

Before proceeding, let us recall a couple of classical results on the geometry of degree 4 del Pezzo surfaces, and refer the interested reader to [44] and [15, Chap. 8.6] for details. The anti-canonical linear system embeds X as a complete intersection of two quadrics in $\mathbb{P}^4_k$, i.e., X is given by $Q_0 = Q_1 = 0$, where Q_0 and Q_1 are two quadratic forms in five variables over k. The *degeneracy locus* of this pencil of quadrics

$$\mathrm{Deg}_X := \{\det(t_0 Q_0 + t_1 Q_1) = 0\} \subset \mathbb{P}^1_k = \mathrm{Proj}\, k[t_0, t_1]$$

is a zero-dimensional subscheme, which is étale and of length 5 over k. Over $\overline{k}$, its points correspond to the singular quadrics containing X, all of which are cones over smooth quadric surfaces. Let $\nu_2 : \mathbb{P}^1_k \to \mathbb{P}^2_k$ be the 2-uple Veronese embedding and set

$$Z := \nu_2(\mathrm{Deg}_X) \subset C := \nu_2(\mathbb{P}^1_k) \subset \mathbb{P}^2_k.$$

If X contains a k-rational (-1)-curve, i.e., if X is quasi-split, then X is the blow-up of $\mathbb{P}^2_k$ in Z, see Theorem 8.10 and [44, Theorem 2.3].

Proposition 8.12 *Let X be a del Pezzo surface of degree 4 over a field k of characteristic $\neq 2$ with at least 5 elements. Then, the following are equivalent:*

(1) The degeneracy scheme Deg_X has a k-rational point.
(2) There exists a finite morphism $\psi : X \to S$ of degree 2, where S is a del Pezzo surface of degree 8 of product type.

Moreover, if ψ is as in (2), then S is isomorphic to a quadric in $\mathbb{P}^3_k$.

Proof To show (1) $\Rightarrow$ (2), assume that Deg_X has a k-rational point. Thus, there exists degenerate quadric Q with $X \subset Q \subset \mathbb{P}^4_k$. As explained in the proof of [15, Theorem 8.6.8], Q is a cone over a smooth quadric surface, and the projection away from its vertex $\mathbb{P}^4_k \dashrightarrow \mathbb{P}^3_k$ induces a morphism $X \to \mathbb{P}^3_k$ that is finite of degree 2 onto a smooth quadric surface S. In particular, S is a del Pezzo surface of degree 8 of product type.

To show (2) $\Rightarrow$ (1), let $\psi : X \to S$ be as in the statement. Then, we have a short exact sequence (which even splits since $\mathrm{char}(k) \neq 2$)

$$0 \to \mathcal{O}_S \to \psi_*\mathcal{O}_X \to \mathcal{L}^{-1} \to 0,$$

where $\mathcal{L}$ is an invertible sheaf on S, which is of type $(1, 1)$ on $S_{\overline{k}} \cong \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. In particular, $|\mathcal{L}|$ defines an embedding $\iota : S \to \mathbb{P}^3_k$ as a quadric, and establishes the final assertion. Now, $\iota \circ \psi$ arises from a 4-dimensional subspace V inside the linear system $(\iota \circ \psi)^*\mathcal{O}_{\mathbb{P}^3_k}(1) \cong \omega_X^{-1}$. Thus, $\iota \circ \psi$ is the composition of the anti-canonical embedding $X \to \mathbb{P}^4_k$, followed by a projection $\mathbb{P}^4_k \dashrightarrow \mathbb{P}^3_k$. As explained in the proof of [15, Theorem 8.6.8], such a projection induces a degree 2 morphism onto a quadric if and only if the point of projection is the vertex of a singular quadric in $\mathbb{P}^4_k$ containing X. In particular, this vertex and the corresponding quadric are defined over k, giving rise to a k-rational point of Deg_X. □

In order to refine Proposition 8.12, we will use conic classes as introduced in Definition 8.6.

Proposition 8.13 *Let X be a del Pezzo surface of degree 4 over a field k. Then, the following are equivalent:*

(1) There exists a conic class in $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$.
(2) There exists a finite morphism $\psi : X \to P' \times P''$ of degree 2, where P' and P'' are a Brauer–Severi curves over k.

Moreover, if ψ is as in (2), then $P' \cong P''$.

Proof Let $\mathcal{L} \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ be a conic class. By Theorem 3.4, there exist morphisms $|\mathcal{L}| : X \to P'$ and $|\omega_X^{-1} \otimes \mathcal{L}^{-1}| : X \to P''$, where P' and P'' are Brauer–Severi curves over k. Combining them, we obtain a finite morphism $X \to P' \times P''$

of degree 2. As in the proof of (2) $\Rightarrow$ (1) of Proposition 8.12 we find that $P' \times P''$ embeds into $\mathbb{P}^3$, and thus, $0 = [\mathbb{P}^3_k] = [P'] + [P''] \in \mathrm{Br}(k)$ by Proposition 5.2. This implies $[P'] = [P'']$ since these classes are 2-torsion, and thus, $P' \cong P''$ by Corollary 2.16. This establishes (1) $\Rightarrow$ (2).

Conversely, let $\psi : X \to P' \times P''$ be as in (2). Then, $\psi^*(\mathcal{O}_{P'}(1) \boxtimes \mathcal{O}_{P''}(1))$ is a conic class, and (1) follows. □

8.4 Del Pezzo Surfaces of Degree 3

For these surfaces, we have the following analogue of Theorem 6.1.

Theorem 8.14 *Let X be a del Pezzo surface of degree 3 over a field k. Then, the following are equivalent:*

(1) There exists a birational morphism $f : X \to P$ to a Brauer–Severi surface, such that $f_{\overline{k}}$ is the blow-up of 6 points in general position.
(2) There exists a class $F \in \mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k)$ such that

$$F_{\overline{k}} \cong \mathcal{O}_{\overline{X}}(E_1 + E_2 + E_3 + E_4 + E_5 + E_6),$$

where the E_i are disjoint (-1)-curves on $\overline{X}$.

Proof The proof is analogous to that of Theorem 8.7, and we leave the details to the reader. □

Note that if the equivalent conditions of this theorem are fulfilled, then X is not minimal. But the converse does not hold in general: If Y is a unirational, but not rational del Pezzo surface of degree 4 over k, and $y \in Y$ is a k-rational point not lying on an exceptional curve, then the blow-up $X \to Y$ in y is a non-minimal degree 3 del Pezzo surface over k with k-rational points that is not birationally equivalent to a Brauer–Severi surface over k.

By [35, Theorem 28.1], a degree 3 del Pezzo surface X is minimal if and only if $\rho(X) = 1$, i.e., $\mathrm{Pic}_{(X/k)(\mathrm{fppf})}(k) = \mathbb{Z} \cdot \omega_X$. In this case, we have $\mathrm{Am}(X) = 0$. In particular, if such a surface is birationally equivalent to a Brauer–Severi surface P, then $P \cong \mathbb{P}^2_k$ by Proposition 2.10 and Theorem 2.19. In particular, X is rational and has a k-rational point in this case.

8.5 Del Pezzo Surfaces of Degree 2

Arguing as in the proof of Theorem 8.5, it follows that if there exists a dominant and rational map $P \dashrightarrow X$ from a Brauer–Severi surface P onto a degree 2 del Pezzo surface over a perfect field k, then X has a k-rational point, and thus $\mathrm{Am}(X) = 0$.

In particular, if X is birationally equivalent to a Brauer–Severi surface, then it is rational, see also Corollary 8.4.

By work of Manin [35, Theorem 29.4], a degree 2 del Pezzo surface over a field k is unirational if it has a k-rational point not lying on an exceptional curve. Together with non-trivial refinements of [17, 42], such surfaces over finite fields are always unirational.

By Lemma 8.3, we have that $\mathrm{Am}(X)$ is at most 2-torsion for degree 2 del Pezzo surfaces. For the possibilities of $H^1(G_k, \mathrm{Pic}_{X/k}(k^{\mathrm{sep}}))$, as well as further information concerning arithmetic questions, we refer to [32].

8.6 Del Pezzo Surfaces of Degree 1

If X is a del Pezzo surface of degree 1, then it has a k-rational point, namely the unique base point of $|-K_X|$. Thus, we have $\mathrm{Am}(X) = 0$, and there are no morphisms or rational maps to non-trivial Brauer–Severi varieties.

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Arithmetic of K3 Surfaces

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1 Introduction

Being surfaces of intermediate type, i.e., neither geometrically rational or ruled, nor of general type, K3 surfaces have a rich yet accessible arithmetic theory, which has started to come into focus over the last fifteen years or so. These notes, written to accompany a 4-hour lecture series at the 2015 Arizona Winter School, survey some of these developments, with an emphasis on explicit methods and examples. They are mostly expository, though I have included at the end two admittedly optimistic conjectures on uniform boundedness of Brauer groups (modulo constants) for lattice polarized K3 surfaces over number fields, which to my knowledge have not appeared in print before (Conjectures 5.5 and 5.6). The topics treated in these notes are as follows.

Geometry of K3 surfaces. We start with a crash course, light on proofs, on the geometry of K3 surfaces: topological properties, including the lattice structure of $\mathrm{H}^2(X, \mathbb{Z})$ and simple connectivity; the period point of K3 surface, the Torelli theorem and surjectivity of the period map.

Picard groups. Over a number field k, the geometric Picard group $\mathrm{Pic}(\overline{X})$ of a projective K3 surface X/k is a free $\mathbb{Z}$-module of rank $1 \leq \rho(\overline{X}) \leq 20$. Determining $\rho(\overline{X})$ for a given K3 surface is a difficult task; we explain how work of van Luijk, Kloosterman, Elsenhans-Jahnel and Charles [16, 29, 60, 109] solves this problem.

Brauer groups. The Galois module structure of $\mathrm{Pic}(\overline{X})$ allows one to compute an important piece of the Brauer group $\mathrm{Br}(X) = \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ of a locally solvable K3 surface X, consisting of the classes of $\mathrm{Br}(X)$ that are killed by passage to an algebraic closure, modulo Brauer classes coming from the ground field. These algebraic classes can be used to construct counter-examples to the Hasse principle on K3 surfaces via Brauer–Manin obstructions.

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For surfaces of negative Kodaira dimension (e.g., cubic surfaces) the Brauer group consists entirely of algebraic classes. In contrast, for K3 surfaces we know that $\mathrm{Br}(X(\mathbb{C})) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. However, a remarkable theorem of Skorobogatov and Zarhin [102] says that over a number field the quotient of $\mathrm{Br}(X)$ by the subgroup of constant classes is finite. We explain work by several authors on the computation of the transcendental Brauer classes on K3 surfaces, and their impact on the arithmetic of such surfaces [43, 44, 72].

Uniform boundedness questions. Finally, we explain in broad strokes an analogy between Brauer classes on K3 surfaces and torsion points on elliptic curves; the later are known to be uniformly bounded over a fixed number field, by work of Merel [73]. It is our hope that analogous statements could be true for K3 surfaces.

Results from AWS. As part of the Arizona Winter School, a number of students were assigned to work on projects related to material of these notes. The experience was successful beyond reasonable expectations, and several members of the resulting three group projects continued working together long after the school. We briefly report on their findings.

I omitted several active research topics due to time constraints, notably rational curves on K3 surfaces, modularity questions, and Mordell-Weil ranks of elliptic K3 surfaces over number fields. I have resisted the temptation to add these topics so that the notes remain a faithful, detailed transcription of the four lectures that gave rise to them.[1]

Prerequisites. The departure point for these notes is working knowledge of the core chapters of Hartshorne's text [39, I-III], as well as a certain familiarity with the basic theory of algebraic surfaces, as presented in [39, V Sects. 1, 3, 5] or [8]. I also assume the reader is familiar with basic algebraic number theory (including group cohomology and Brauer groups of fields), and basic algebraic topology, at the level usually covered in first-year graduate courses in the United States. More advanced parts of the notes use étale cohomology as a tool; Milne's excellent book [75] will come in handy as a reference. Many of the topics treated here have not percolated to advanced textbooks yet. For this reason, I provide detailed references throughout for readers seeking more depth on particular topics.

2 Geometry of K3 Surfaces

References: [7, 47, 64, 76].

Huybrechts' notes [47] are quite detailed and superbly written, and will soon appear in book form. Our presentation of the material in this section owes a lot to them.

[1] Videos of the lectures can be found at http://swc.math.arizona.edu/aws/2015/index.html.

2.1 Examples of K3 Surfaces

By a variety X over an arbitrary field k we mean a separated scheme of finite type over k. Unless otherwise stated, we shall assume varieties to be geometrically integral. For a smooth variety, we write ω_X for the canonical sheaf of X and K_X for its class in Pic X.

Definition 2.1 An algebraic K3 surface is a smooth projective 2-dimensional variety over a field k such that $\omega_X \simeq \mathscr{O}_X$ and $\mathrm{H}^1(X, \mathscr{O}_X) = 0$. A polarized K3 surface is a pair (X, h), where X is an algebraic K3 surface and $h \in \mathrm{H}^2(X, \mathbb{Z})$ is an ample class. The degree of a polarized K3 surface is the self-intersection h^2.

Example 2.2 (K3 surfaces of degrees 4, 6, and 8). Let X be a smooth complete intersection of type $(d_1, \ldots, d_r)$ in $\mathbb{P}^n_k$, i.e., $X \subseteq \mathbb{P}^n$ has codimension r and $X = H_1 \cap \cdots \cap H_r$, where H_i is a hypersurface of degree $d_i \geq 1$ for $i = 1, \ldots, r$. Then $\omega_X \simeq \mathscr{O}_X(\sum d_i - n - 1)$ [39, Exercise II.8.4]. To be a K3 surface, such an X must satisfy $r = n - 2$ and $\sum d_i = n + 1$. It does not hurt to assume that $d_i \geq 2$ for each i. This leaves only a few possibilities for X (check this!):

(1) $n = 3$ and $(d_1) = (4)$, i.e., X is a smooth quartic surface in $\mathbb{P}^3_k$.
(2) $n = 4$ and $(d_1, d_2) = (2, 3)$, i.e., X is a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4_k$.
(3) $n = 5$ and $(d_1, d_2, d_3) = (2, 2, 2)$, i.e., X is a smooth complete intersection of three quadrics in $\mathbb{P}^5_k$.

In each case, taking h to be the restriction to X of a hyperplane class in the ambient projective space, we obtain a polarized K3 surface whose degree coincides with the degree of X as a variety embedded in projective space.

Exercise 2.3 For each of the three types X of complete intersections in Example 2.2 prove that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$.

Example 2.4 (K3 surfaces of degree 2). Suppose for simplicity that char $k \neq 2$. Let $\pi \colon X \to \mathbb{P}^2_k$ be a double cover branched along a smooth sextic curve $C \subseteq \mathbb{P}^2_k$. Note that X is smooth if and only if C is smooth. By the Hurwitz formula [7, I.17.1], we have $\omega_X \simeq \pi^*(\omega_{\mathbb{P}^2_k} \otimes \mathscr{O}_{\mathbb{P}^2_k}(6)^{\otimes 1/2}) \simeq \mathscr{O}_X$, and since $\pi_*\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^2_k} \oplus \mathscr{O}_{\mathbb{P}^2_k}(-3)$, we deduce that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$; see [24, Chap. 0, Sect. 1] for details. Hence X is a K3 surface if it is smooth. Letting $h = \pi^*(\ell)$ be the pull-back of a line, we obtain a polarized K3 surface of degree 2.

Example 2.5 (Kummer surfaces). Let A be an abelian surface over a field k of characteristic $\neq 2$. The involution $\iota \colon A \to A$ given by $x \mapsto -x$ has sixteen $\bar{k}$-fixed points (the 2-torsion points of A). Let $\widetilde{A} \to A$ be the blow-up of A along the k-scheme defined by these fixed points. The involution ι lifts to an involution $\tilde{\iota} \colon \widetilde{A} \to \widetilde{A}$; the quotient $\pi \colon \widetilde{A} \to \widetilde{A}/\tilde{\iota} =: X$ is a double cover ramified along the geometric components of the exceptional divisors of the blow-up $E_1, \ldots, E_{16}$. Let $\overline{E}_i$ be the image of E_i in X, for $i = 1, \ldots, 16$.

We have $\omega_{\widetilde{A}} \simeq \mathscr{O}_{\widetilde{A}}(\sum E_i)$, and the Hurwitz formula implies that $\omega_{\widetilde{A}} \simeq \pi^*\omega_X \otimes \mathscr{O}_{\widetilde{A}}(\sum E_i)$. Hence $\mathscr{O}_{\widetilde{A}} \simeq \pi^*\omega_X$. The projection formula [39, Exercise II.5.1] then gives

$$\omega_X \otimes \pi_*\mathscr{O}_{\widetilde{A}} \simeq \pi_*\mathscr{O}_{\widetilde{A}}. \tag{1}$$

Since $\pi_*\mathscr{O}_{\widetilde{A}} \simeq \mathscr{O}_X \oplus L^{\otimes -1}$, where L is the square root of $\mathscr{O}_X(\sum \overline{E}_i)$, taking determinants of both sides of (1) gives $\omega_X^{\otimes 2} \simeq \mathscr{O}_X$. We conclude that $K_X \in \operatorname{Pic} X$ is numerically trivial (i.e., its image in Num X is zero—see Sect. 2.3), and thus $h^0(X, \omega_X) = 0$ if $\omega_X \not\simeq \mathscr{O}_X$. Suppose this is the case. Then since $h^0(X, \pi_*\mathscr{O}_{\widetilde{A}}) = 1$, (1) implies that $h^0(X, \omega_X \otimes \pi_*\mathscr{O}_{\widetilde{A}}) = 1$, and hence $h^0(X, \omega_X \otimes L^{\otimes -1}) = 1$. Fix an ample divisor A on X; our discussion above implies that $(A, K_X - [L])_X > 0$, where $(\ ,\ )_X$ denotes the intersection pairing on X. On the other hand, $L \sim \frac{1}{2}\sum \overline{E}_i$, so $(A, [L])_X > 0$. But then $(A, K_X) > 0$, which contradicts the numerical triviality of K_X. Hence we must have $\omega_X \simeq \mathscr{O}_X$.

Exercise 2.6 Prove that $\mathrm{H}^1(X, \mathscr{O}_X) = 0$ for the surfaces in Example 2.5.

2.2 *Euler Characteristic*

If X is an algebraic K3 surface, then by definition we have $h^0(X, \mathscr{O}_X) = 1$ and $h^1(X, \mathscr{O}_X) = 0$. Serre duality then gives $h^2(X, \mathscr{O}_X) = h^0(X, \mathscr{O}_X) = 1$, so

$$X \text{ an algebraic K3 surface} \implies \chi(X, \mathscr{O}_X) = 2.$$

2.3 *Linear, Algebraic, and Numerical Equivalence*

Let X be a smooth surface over a field k, and write Div X for its group of Weil divisors. Let $(\ ,\ )_X\colon \operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$ denote the intersection pairing on X [39, Sect. V.1]. Recall three basic equivalence relations one can put on Div X:

(1) **Linear equivalence**: $C, D \in \operatorname{Div} X$ are linearly equivalent if $C = D + \operatorname{div}(f)$ for some $f \in \mathbf{k}(X)$ (the function field of X).
(2) **Algebraic equivalence**: $C, D \in \operatorname{Div} X$ are algebraically equivalent if there is a connected curve T, two closed points 0 and $1 \in T$, and a divisor E in $X \times T$, flat over T, such that $E|_{X\times\{0\}} - E|_{X\times\{1\}} = C - D$.
(3) **Numerical equivalence**: $C, D \in \operatorname{Div} X$ are numerically equivalent if $(C, E)_X = (D, E)_X$ for all $E \in \operatorname{Div} X$.

These relations obey the following hierarchy:

$$\text{Linear equivalence} \implies \text{Algebraic equivalence} \implies \text{Numerical equivalence}.$$

Briefly, here is why these implications hold. For the first implication: if $C = D + \operatorname{div}(f)$, then we can take $T = \mathbb{P}^1_k = \operatorname{Proj} k[t, u]$ and $E = \operatorname{div}(tf - u)$ in $X \times \mathbb{P}^1_k$ to see that C and D are algebraically equivalent. For the second implication: suppose that an algebraically equivalence between C and D is witnessed by $E \subseteq X \times T$. Let H be a very ample divisor on X, and let $X \hookrightarrow \mathbb{P}^n_k$ be the embedding induced by H. This allows us to embed $X \times T$ (and hence E) in $\mathbb{P}^n_T$. The Hilbert polynomials of the fibers of $E \to T$ above closed points are constant, by flatness (and connectedness of T). Since $(C, H)_X$ is the degree of C in the embedding induced by H, we conclude that $(C, H)_X = (D, H)_X$. Now use the fact that any divisor on X can be written as a difference of ample divisors [39, p. 359]—this decomposition need not happen over the ground field of course, but intersection numbers are preserved by base extension of the ground field, so we may work over an algebraically closed field to begin with.

Write, as usual, $\operatorname{Pic} X$ for the quotient of $\operatorname{Div} X$ by the linear equivalence relation; let $\operatorname{Pic}^\tau X \subseteq \operatorname{Pic} X$ be the set of numerically trivial classes, i.e.,

$$\operatorname{Pic}^\tau X = \{L \in \operatorname{Pic} X : (L, L')_X = 0 \text{ for all } L' \in \operatorname{Pic} X\}.$$

Finally, let $\operatorname{Pic}^0 X \subseteq \operatorname{Pic}^\tau X$ be the set of classes algebraically equivalent to zero. Let $\operatorname{NS} X = \operatorname{Pic} X / \operatorname{Pic}^0 X$ be the Néron-Severi group of X, and let $\operatorname{Num} X = \operatorname{Pic} X / \operatorname{Pic}^\tau X$.

Lemma 2.7 *Let X be an algebraic K3 surface, and let $L \in \operatorname{Pic} X$. Then*

$$\chi(X, L) = \frac{L^2}{2} + 2.$$

Proof This is just the Riemann–Roch theorem for surfaces [39, Theorem V.1.6], taking into account that $K_X = 0$ and $\chi(X, \mathscr{O}_X) = 2$. □

Proposition 2.8 *Let X be an algebraic K3 surface over a field. Then the natural surjections*

$$\operatorname{Pic} X \to \operatorname{NS} X \to \operatorname{Num} X$$

are isomorphisms.

Proof Since X is projective, there is an ample sheaf L' on X. If $L \in \ker(\operatorname{Pic} X \to \operatorname{Num} X)$, then $(L, L')_X = 0$, and thus if $L \neq \mathscr{O}_X$ then $\mathrm{H}^0(X, L) = 0$. Serre duality implies that $\mathrm{H}^2(X, L) \simeq \mathrm{H}^0(X, L^{\otimes -1})^\vee = 0$. Hence $\chi(X, L) \leq 0$; on the other hand, by Lemma 2.7 we have $\chi(X, L) = \frac{1}{2}L^2 + 2$, and hence $L^2 < 0$, which means L cannot be numerically trivial. □

2.4 Complex K3 Surfaces

Over $k = \mathbb{C}$, there is a notion of K3 surfaces as complex manifolds that includes algebraic K3 surfaces over $\mathbb{C}$, although most complex K3 surfaces are not projective. This more flexible theory is crucial in proving important results for K3 surfaces, such as the Torelli Theorem [12, 64, 84]. It also allows us to study K3 surfaces via singular cohomology.

Definition 2.9 A complex K3 surface is a compact connected 2-dimensional complex manifold X such that $\omega_X := \Omega_X^2 \simeq \mathscr{O}_X$ and $\mathrm{H}^1(X, \mathscr{O}_X) = 0$.

Let us explain the sense in which an algebraic K3 surface is also a complex K3 surface. To a separated scheme X locally of finite type over $\mathbb{C}$ one can associate a complex space X^{an}, whose underlying space consists of $X(\mathbb{C})$, and a map $\phi\colon X^{\mathrm{an}} \to X$ of locally ringed spaces in $\mathbb{C}$-algebras. For a ringed space Y, let $\mathfrak{Coh}(Y)$ denote the category of coherent sheaves on Y. To $\mathscr{F} \in \mathfrak{Coh}(X)$ one can then associate $\mathscr{F}^{\mathrm{an}} := \phi^*\mathscr{F} \in \mathfrak{Coh}(X^{\mathrm{an}})$; we have $\Omega_{X/\mathbb{C}}^{\mathrm{an}} \simeq \Omega_{X^{\mathrm{an}}}$. If X is a projective variety, then the functor

$$\Phi\colon \mathfrak{Coh}(X) \to \mathfrak{Coh}(X^{\mathrm{an}}) \qquad \mathscr{F} \to \mathscr{F}^{\mathrm{an}}$$

is an equivalence of abelian categories. This is known as Serre's GAGA principle [94]. In the course of proving this equivalence, Serre shows that for $\mathscr{F} \in \mathfrak{Coh}(X)$, certain functorial maps

$$\epsilon\colon \mathrm{H}^q(X, \mathscr{F}) \to \mathrm{H}^q(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$$

are bijective for all $q \geq 0$ [94, Théorème 1]. Hence:

Proposition 2.10 *Let X be an algebraic K3 surface over $k = \mathbb{C}$. Then X^{an} is a complex K3 surface.* □

2.5 Singular Cohomology of Complex K3 Surfaces

In this section X denotes a complex K3 surface, $e(\cdot)$ is the topological Euler characteristic of a space, and $c_i(X)$ is the i-th Chern class of (the tangent bundle of) X for $i = 1$ and 2. As in Sect. 2.2, one can show that $\chi(X, \mathscr{O}_X) = 2$. Noether's formula states that

$$\chi(X, \mathscr{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X));$$

see [7, Theorem I.5.5] and the references cited therein. Since $\omega_X \simeq \mathscr{O}_X$, we have $c_1(X)^2 = 0$, and hence $e(X) = c_2(X) = 24$.

For the singular cohomology groups of X, we have

$$\mathrm{H}^0(X,\mathbb{Z}) \cong \mathbb{Z} \text{ because } X \text{ is connected, and}$$
$$\mathrm{H}^4(X,\mathbb{Z}) \cong \mathbb{Z} \text{ because } X \text{ is oriented.}$$

The exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^\times \to 0$$

gives rise to a long exact sequence in sheaf cohomology

$$\begin{aligned} 0 &\to \mathrm{H}^0(X,\mathbb{Z}) \to \mathrm{H}^0(X,\mathscr{O}_X) \to \mathrm{H}^0(X,\mathscr{O}_X^\times) \to \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^1(X,\mathscr{O}_X) \to \\ &\to \mathrm{H}^1(X,\mathscr{O}_X^\times) \xrightarrow{c_1} \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathscr{O}_X) \to \mathrm{H}^2(X,\mathscr{O}_X^\times) \to \mathrm{H}^3(X,\mathbb{Z}) \end{aligned} \tag{2}$$

Since $\mathrm{H}^0(X,\mathscr{O}_X) \to \mathrm{H}^0(X,\mathscr{O}_X^\times)$ is surjective and $\mathrm{H}^1(X,\mathscr{O}_X) = 0$, we have $\mathrm{H}^1(X,\mathbb{Z}) = 0$. Poincaré duality then gives

$$0 = \operatorname{rk} \mathrm{H}^1(X,\mathbb{Z}) = \operatorname{rk} \mathrm{H}_1(X,\mathbb{Z}) \overset{\mathrm{PD}}{=} \operatorname{rk} \mathrm{H}^3(X,\mathbb{Z}),$$

so $\mathrm{H}^3(X,\mathbb{Z})$ is a torsion abelian group, and $\mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}} \cong \mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$. The universal coefficients short exact sequence

$$0 \to \mathrm{Ext}^1(\mathrm{H}_1(X,\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_2(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

shows that $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$ is dual to $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tors}}$ (fill in the details!).

Proposition 2.11 *Let X be a complex K3 surface. Then* $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}} = 0$.

Proof An element of order n in $\mathrm{H}_1(X,\mathbb{Z})_{\mathrm{tors}}$ gives a surjection $\mathrm{H}_1(X,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, hence a surjection $\pi_1(X,x) \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$, which corresponds to an unramified cover $Y \to X$ of degree n, and we must have $e(Y) = ne(X) = 24n$. The Hurwitz formula tells us that $\omega_Y \simeq \pi^*\omega_X$, so $\omega_Y \simeq \mathscr{O}_Y$, which implies $h^2(Y,\mathscr{O}_Y) = h^0(Y,\omega_Y) = 1$. Noether's formula tells us that $\chi(Y,\mathscr{O}_Y) = \frac{1}{12}(c_1(Y)^2 + c_2(Y))$. So $2 - h^1(\mathscr{O}_Y) = \frac{1}{12} \cdot 24n$ and hence $h^1(\mathscr{O}_Y) = 2 - 2n$. We conclude that $n = 1$. □

Proposition 2.11 and the discussion preceding it shows that $\mathrm{H}^3(X,\mathbb{Z}) = 0$ and $\mathrm{H}^2(X,\mathbb{Z})$ is a free abelian group. Since $e(X) = 24$, we deduce that $\operatorname{rk} \mathrm{H}^2(X,\mathbb{Z}) = 24 - 1 - 1 = 22$. Poincaré duality thus tells us that the cup product induces a perfect bilinear pairing:

$$B\colon \mathrm{H}^2(X,\mathbb{Z}) \times \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}.$$

Proposition 2.12 ([7, VIII.3.1]). *The pairing B is even, i.e., $B(x,x) \in 2\mathbb{Z}$ for all $x \in \mathrm{H}^2(X,\mathbb{Z})$.* □

The bilinear form B thus gives rise to an even integral quadratic form

$$q\colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathbb{Z}, \qquad x \mapsto B(x,x).$$

Extend q by $\mathbb{R}$-linearity to a form $q_{\mathbb{R}}\colon \mathrm{H}^2(X,\mathbb{Z}) \otimes \mathbb{R} \to \mathbb{R}$, and let b_+ (resp. b_-) denote the number of positive (resp. negative) eigenvalues of q. The Thom–Hirzebruch index theorem [45, p. 86] says that

$$b_+ - b_- = \frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16.$$

On the other hand, we know that

$$b_+ + b_- = 22,$$

so we conclude that $b_+ = 3$ and $b_- = 19$. In sum, $\mathrm{H}^2(X,\mathbb{Z})$ equipped with the cup-product is an indefinite even integral lattice of signature (3, 19). Perfectness of the pairing B tells us that the lattice $\mathrm{H}^2(X,\mathbb{Z})$ is unimodular, i.e., the absolute value of the determinant of a Gram matrix is 1. This is enough information to pin down the lattice $\mathrm{H}^2(X,\mathbb{Z})$, up to isometry. To state a precise theorem, recall that the hyperbolic plane U is the rank 2 lattice, which under a suitable choice of $\mathbb{Z}$-basis has Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $E_8(-1)$ denotes the rank 8 lattice, which under a suitable choice of $\mathbb{Z}$-basis has Gram matrix

$$\begin{pmatrix}
-2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}.$$

Theorem 2.13 ([95, Sect. V.2.2]). *Let L be a an even indefinite unimodular lattice of signature (r,s), and suppose that $s - r \geq 0$. Then $r \cong s \bmod 8$ and L is isometric to*

$$U^{\oplus r} \oplus E_8(-1)^{\oplus (s-r)/8}. \qquad \square$$

The above discussion can thus be summarized in the following theorem.

Theorem 2.14 *Let X be a complex K3 surface. The singular cohomology group $\mathrm{H}^2(X, \mathbb{Z})$, equipped with the cup-product, is an even indefinite unimodular lattice of signature $(3, 19)$, isometric to the* K3 lattice

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}. \qquad \square$$

2.6 Complex K3 Surfaces Are Simply Connected

Theorem 2.15 *Every complex K3 surface is simply connected.*

Sketch of the proof: The key ingredient is that all complex K3 surfaces are diffeomorphic to each other [7, VIII Corollary 8.6]; this theorem takes a fair amount of work: first, (complex) Kummer surfaces are diffeomorphic, because any two 2-tori are isomorphic as real Lie groups. Second, there is an open set in the period domain around the period point of a K3 surface where the K3 surface can be deformed. Third, projective Kummer surfaces are dense in the period domain. Putting these three ideas together shows all complex K3 surfaces are diffeomorphic. It thus suffices to compute $\pi_1(X, x)$ for a single K3 surface. We will pick X a smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ and apply the following proposition.

Proposition 2.16 *Any smooth quartic in $\mathbb{P}^3_{\mathbb{C}}$ is simply connected.*

Proof Let $\nu : \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^{34}_{\mathbb{C}}$ be the 4-uple embedding. Any smooth quartic $X \subset \mathbb{P}^3_{\mathbb{C}}$ is embedded under ν as $\nu(\mathbb{P}^3_{\mathbb{C}}) \cap H$ for some hyperplane $H \subset \mathbb{P}^{34}_{\mathbb{C}}$. By the Lefschetz hyperplane theorem $\pi_1(\nu(\mathbb{P}^3) \cap H)$ is isomorphic to $\pi_1(\nu(\mathbb{P}^3)) = \pi_1(\mathbb{P}^3) = 0$. $\square$

2.7 Differential Geometry of Complex K3 Surfaces

A theorem of Siu [97] (see also [7, Sect. IV.3]) asserts that complex K3 surfaces are Kähler; thus there is a Hodge decomposition on $\mathrm{H}^k(X, \mathbb{C}) \simeq \mathrm{H}^n_{\mathrm{dR}}(X)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ (here $\mathrm{H}^n_{\mathrm{dR}}(X)_{\mathbb{R}}$ denotes de Rham cohomology on the underlying real manifold X):

$$\mathrm{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X), \tag{3}$$

where $\mathrm{H}^{p,q}(X)$ denotes the Dolbeault cohomology group of complex differential forms of type (p, q) (isomorphic by Dolbeaut's theorem to $\mathrm{H}^q(X, \Omega^p_X)$), which satisfy:

$$\mathrm{H}^{p,q}(X) = \overline{\mathrm{H}^{q,p}(X)} \qquad \text{and} \qquad \sum_{p+q=k} h^{p,q}(X) = b_k,$$

where $h^{p,q}(X) = \dim_{\mathbb{C}} \mathrm{H}^{p,q}(X)$, and $b_k = \mathrm{rk}(\mathrm{H}^k(X, \mathbb{Z})) = \dim_{\mathbb{C}} \mathrm{H}^k(X, \mathbb{C})$ denotes the k-th Betti number of X; see [115, Chap. 6].

Proposition 2.17 *Let X be a complex K3 surface. The Hodge diamond of X is given by*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Proof From $\mathrm{H}^1(X, \mathbb{Z}) = \mathrm{H}^3(X, \mathbb{Z}) = 0$ and the Hodge decomposition (3) applied to the complexification of these groups for $k = 1$ and 3 we get the vanishing of the second and fourth rows. We have $h^{0,0} = h^0(X, \mathscr{O}_X) = 1$, and from $\omega_X \simeq \mathscr{O}_X$ we get $h^{2,0} = 1$. Serre duality and $\omega_X \simeq \mathscr{O}_X$ together give $h^{0,2} = h^{0,0} = h^{2,2}$. Since $b_2 = h^{2,0} + h^{1,1} + h^{0,2} = 22$ we obtain $h^{1,1} = 20$. Finally, the $h^{p,q}$ "outside" this diamond vanish by Serre duality and dimension reasons. □

The lattice $\mathrm{H}^2(X, \mathbb{Z})$ can be endowed with a Hodge structure of weight 2. We review what this means; for more details see [47, Chap. 3] and [115, Chap. 7]

Definition 2.18 Let $\mathrm{H}_{\mathbb{Z}}$ be a free abelian group of finite rank. An integral Hodge structure of weight n on $\mathrm{H}_{\mathbb{Z}}$ is a decomposition, called the Hodge decomposition,

$$\mathrm{H}_{\mathbb{C}} := \mathrm{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$

such that $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$ and $\mathrm{H}^{p,q} = 0$ for $p < 0$.

When X is a complex K3 surface, the middle cohomology decomposes as

$$\mathrm{H}^2(X, \mathbb{C}) \cong \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X),$$

and the outer pieces are 1-dimensional. The cup product on $\mathrm{H}^2(X, \mathbb{Z})$ extends to a symmetric pairing on $\mathrm{H}^2(X, \mathbb{C})$, equal to the bilinear form $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. Write $\mathrm{H}^{2,0}(X) = \mathbb{C}\omega_X$. Then the Hodge–Riemann relations assert that

(1) $(\omega_X, \omega_X) = 0$;
(2) $(\omega_X, \overline{\omega_X}) > 0$;
(3) $V := \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X)$ is orthogonal to $\mathrm{H}^{1,1}(X)$.

Exercise 2.19 Check the Hodge–Riemann relations above.

Thus $\mathbb{C}\omega_X = \mathrm{H}^{2,0}(X)$ determines the Hodge decomposition on $\mathrm{H}^2(X, \mathbb{C})$. Let

$$V_{\mathbb{R}} = \{v \in V : v = \overline{v}\} = \mathbb{R} \cdot \{\omega_X + \overline{\omega_X}, i(\omega_X - \overline{\omega_X})\},$$

so that $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The intersection form restricted to $V_{\mathbb{R}}$ is positive definite and diagonal on the basis given above. Hence, the cup product restricted to $\mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X, \mathbb{R})$ has signature $(1, 19)$.

2.8 The Néron-Severi Lattice of a Complex K3 Surface

For a complex K3 surface, the long exact sequence (2) associated to the exponential sequence and the vanishing $\mathrm{H}^1(X, \mathscr{O}_X) = 0$ give an injection

$$c_1 : \mathrm{Pic}(X) \cong \mathrm{H}^1(X, \mathscr{O}_X^*) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z}).$$

which is also called the first Chern class. Let $i_*\colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{C})$ be the canonical map. The Lefschetz (1,1)-theorem says that the image of $i_* \circ c_1$ is $\mathrm{H}^{1,1}(X) \cap i_*\mathrm{H}^2(X, \mathbb{Z})$. It is called the Néron-Severi lattice $\mathrm{NS}\, X$. When X is an algebraic K3 surface, this lattice coincides with the Néron-Severi group previously defined in Sect. 2.3 by Proposition 2.8 and the GAGA principle [94, Proposition 18 and the remarks that follow].

In words, the Néron-Severi lattice consists of the integral classes in $\mathrm{H}^2(X, \mathbb{Z})$ that are closed (1,1)-forms. In particular, the Picard number $\rho(X) = \mathrm{rk}\, \mathrm{NS}(X) = \mathrm{rk}\, \mathrm{Pic}(X)$ is at most the dimension of $\mathrm{H}^{1,1}(X)$.

Proposition 2.20 *Let X be a complex K3 surface. Then $0 \le \rho(X) \le 20$. If X is algebraic, then the signature of* $\mathrm{NS}\, X \otimes \mathbb{R}$ *is* $(1, \rho(X) - 1)$. □

2.9 The Torelli Theorem

A marking on a complex K3 surface X is an isometry, i.e., an isomorphism of lattices,

$$\Phi\colon \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}.$$

A marked complex K3 surface is a pair (X, Φ) as above. We denote the complexification of Φ by $\Phi_{\mathbb{C}}$. The period point of (X, Φ) is $\Phi_{\mathbb{C}}(\mathbb{C}\omega_X) \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$. By the Hodge–Riemann relations, the period point lies in an open subset Ω (in the complex topology) of a 20-dimensional quadric inside $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$:

$$\Omega = \{x \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) : (x, x) = 0, (x, \overline{x}) > 0\};$$

here $(\ ,\ )$ denotes the bilinear form on $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$. We call Ω the period domain of complex K3 surfaces.

Exercise 2.21 Check that Ω is indeed an open subset of a quadric in $\mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C}) \simeq \mathbb{P}^{21}_{\mathbb{C}}$.

Theorem 2.22 (Weak Torelli theorem [12, 64, 84]). *Two complex K3 surfaces X and X' are isomorphic if and only if there are markings*

$$\Phi \colon \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}} \xleftarrow{\sim} \mathrm{H}^2(X',\mathbb{Z}) \colon \Phi'$$

whose period points in Ω coincide. □

The weak Torelli theorem follows from the strong Torelli theorem. We briefly explain the statement of the latter. Since the intersection form on $\mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{R})$ is indefinite, the set $\{x \in \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{R}) : (x,x) > 0\}$ has two connected components. Exactly one of these components contains Kähler classes[2]; we call this component the positive cone. A class $x \in \mathrm{NS}\, X$ is effective if there is an effective divisor D on X such that $x = i_* \circ c_1(\mathscr{O}_X(D))$.

Theorem 2.23 (Strong Torelli Theorem). *Let (X, Φ) and (X', Φ') be marked complex K3 surfaces whose period points on Ω coincide. Suppose that*

$$f^* = (\Phi')^{-1} \circ \Phi \colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X',\mathbb{Z})$$

takes the positive cone of X to the positive cone of X', and induces a bijection between the respective sets of effective classes. Then there is a unique isomorphism $f \colon X' \to X$ inducing f^*. □

2.10 Surjectivity of the Period Map

A point $\omega \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$ gives a 1-dimensional $\mathbb{C}$-linear subspace $\mathrm{H}^{2,0} \subseteq \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$. Let $\mathrm{H}^{0,2} = \overline{\mathrm{H}^{2,0}} \subseteq \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ be the conjugate linear subspace, and let $\mathrm{H}^{1,1}$ be the orthogonal complement of $\mathrm{H}^{2,0} \oplus \mathrm{H}^{0,2}$, with respect to the $\mathbb{C}$-linear extension of the bilinear form on Λ_{K3}. We say $\mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$ is a decomposition of K3 type for $\Lambda_{\mathrm{K3}} \otimes \mathbb{C}$.

Theorem 2.24 (Surjectivity of the period map [107]). *Given a point $\omega \in \Omega$ inducing a decomposition $\Lambda_{\mathrm{K3}} \otimes \mathbb{C} = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$ of K3 type there exists a complex K3 surface X and a marking $\Phi : \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ whose $\mathbb{C}$-linear extension preserves Hodge decompositions.* □

---

[2] A Kähler class $h \in \mathrm{H}^2(X,\mathbb{R})$ is a class that can be represented by a real $(1,1)$-form which in local coordinates (z_1, z_2) can be written as $i \sum \alpha_{ij} z_i \wedge \bar{z}_j$, where the hermitian matrix $(\alpha_{ij}(p))$ is positive definite for every $p \in X$.

2.11 Lattices and Discriminant Groups

To give an application of the above results, we need a few facts about lattices; the objects introduced here will also play a decisive role in identifying nontrivial elements of the Brauer group of a complex K3 surface.

Although we have already been using the concept of lattice in previous sections, we start here from scratch, for the sake of clarity and completeness. A lattice L is a free abelian group of finite rank endowed with a symmetric nondegenerate integral bilinear form

$$\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}.$$

We say L is even if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. We may extend $\langle\ ,\ \rangle$ $\mathbb{Q}$-linearly to $L \otimes \mathbb{Q}$, and define the dual abelian group

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}) \simeq \{x \in L \otimes \mathbb{Q} : \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in L\}.$$

There is an injective map of abelian groups $L \to L^\vee$ sending x to $\phi_x\colon y \mapsto \langle x, y\rangle$. The discriminant group is $L^\vee/L$, which is finite since $\langle\ ,\ \rangle$ is nondegenerate. Its order is the absolute value of the discriminant of L. For an even lattice L we define the discriminant form by

$$q_L\colon L^\vee/L \to \mathbb{Q}/2\mathbb{Z} \qquad x + L \mapsto \langle x, x\rangle \bmod 2\mathbb{Z}.$$

Let $\ell(L)$ be the minimal number of generators of $L^\vee/L$ as an abelian group.

Theorem 2.25 ([80, Corollary 1.13.3]). *If a lattice L is even and indefinite (when tensored with $\mathbb{R}$), and* $\mathrm{rk}\, L \geq \ell(L) + 2$ *then L is determined up to isometry by its rank, signature and its discriminant form.* □

An embedding of lattices $L \hookrightarrow M$ is primitive if it has saturated image, i.e., if the cokernel M/L is torsion-free.

Exercise 2.26 Let $L \hookrightarrow M$ be an embedding of lattices, and write let $L^\perp = \{x \in M : \langle x, y\rangle = 0 \text{ for all } y \in L\}$.

(1) Show that $L^\perp$ is a primitive sublattice of M.
(2) Show that if L is primitive, then $(L^\perp)^\perp = L$.

Theorem 2.27 ([80, Corollary 1.12.3]). *There exists a primitive embedding $L \hookrightarrow \Lambda_{\mathrm{K3}}$ of an even lattice L of rank r and signature $(p, r-p)$ into the K3 lattice Λ_{K3} if $p \leq 3$, $r - p \leq 19$, and $\ell(L) \leq 22 - r$.* □

2.12 K3 Surfaces Out of Lattices

We conclude our discussion of the geometry of complex K3 surfaces with an application of the foregoing results, in the spirit of [76, Sect. 12].

Question: Is there a complex K3 surface X with Pic X a rank 2 lattice with the following intersection form?

	H	C
H	4	8
C	8	4

(A better question would be: does there exist a smooth quartic surface $X \subset \mathbb{P}^3$ containing a smooth curve C of genus 3 and degree 8? Such a surface would contain the above lattice in its Picard group. The answer to this question is yes, but it would take a little more technology than we've developed to answer this better question.)

Let $L = \mathbb{Z}H + \mathbb{Z}C$, with an intersection pairing given by the above Gram matrix. By Theorem 2.27, we know there is a primitive embedding $L \hookrightarrow \Lambda_{\mathrm{K3}}$; fix such an embedding. Our next move is to construct a Hodge structure of weight two on Λ_{K3}

$$\Lambda_{\mathrm{K3}} \otimes \mathbb{C} = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2}$$

such that $\mathrm{H}^{1,1} \cap \Lambda_{\mathrm{K3}} = L$. For this, choose $\omega \in \Lambda_{\mathrm{K3}} \otimes \mathbb{C}$ satisfying $(\omega, \omega) = 0$, $(\omega, \overline{\omega}) > 0$, in such a way that $L^{\perp} \otimes \mathbb{Q}$ is the smallest $\mathbb{Q}$-vector space of $\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}$ whose complexification contains ω. Essentially, this means that we want to set $\omega = \sum \alpha_i x_i$ where $\{x_i\}$ is a basis for $L^{\perp} \otimes \mathbb{Q}$ and the α_i are algebraically independent transcendental numbers except for the conditions imposed by the relation $(\omega, \omega) = 0$. Then:

$$\mathrm{H}^{1,1} \cap (\Lambda_{\mathrm{K3}} \otimes \mathbb{Q}) = (L^{\perp})^{\perp} \otimes \mathbb{Q} = L \otimes \mathbb{Q},$$

which by the saturatedness of L implies that $\mathrm{H}^{1,1} \cap \Lambda_{\mathrm{K3}} = L$. By Theorem 2.24, there exists a K3 surface X and a marking $\Phi \colon \mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ such that $\mathrm{NS}(X) \cong L$. Using stronger versions of Theorem 2.24 (e.g. [76, p. 70]), one can show that $h = \Phi^{-1}(H)$ is ample. Furthermore, Reider's method can be used to show that h is very ample.

3 Picard Numbers of K3 Surfaces

References: [16, 26–30, 41, 60, 86, 92, 93, 106, 109]

In this section, all K3 surfaces considered are algebraic. Let X be a K3 surface over a field K. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{X} = X \times_K \overline{K}$. Let $\rho(\overline{X})$ denote the rank of the Néron-Severi group NS $\overline{X}$ of $\overline{X}$. The goal of this section is to give an account of the explicit computation of $\rho(\overline{X})$ in the case when K is a number field. One of the key tools is reduction modulo a finite prime $\mathfrak{p}$ of K. We will see that whenever X has good reduction at $\mathfrak{p}$, there is an injective specialization homomorphism NS $\overline{X} \hookrightarrow$ NS $\overline{X}_{\mathfrak{p}}$. For a prime ℓ different from the residue characteristic of $\mathfrak{p}$ there is in turn an injective cycle class map NS $\overline{X}_{\mathfrak{p}} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ of Galois modules. The basic idea is to use the composition of these two maps (after tensoring the first one by

$\mathbb{Q}_\ell$) for *several* finite primes $\mathfrak{p}$ to establish tight upper bounds on $\rho(\overline{X})$. We begin by explaining what good reduction means, and where the two maps above come from.

3.1 Good Reduction

Definition 3.1 Let R be a Dedekind domain, set $K = \operatorname{Frac} R$, and let $\mathfrak{p} \subseteq R$ be a nonzero prime ideal. Let X be a smooth proper K-variety. We say X has good reduction at $\mathfrak{p}$ if X has a smooth proper $R_\mathfrak{p}$-model, i.e., if there exists a smooth proper morphism $\mathcal{X} \to \operatorname{Spec} R_\mathfrak{p}$, such that $\mathcal{X} \times_{R_\mathfrak{p}} K \simeq X$ as K-schemes.

Remark 3.2 Let $k = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ be the residue field at $\mathfrak{p}$. The special fiber $\mathcal{X} \times_{R_\mathfrak{p}} k$ is a smooth proper k-scheme.

Remark 3.3 The ring $R_\mathfrak{p}$ is always a discrete valuation ring [5, Theorem 9.3].

Example 3.4 Let p be a rational prime and let

$$R = \mathbb{Z}_{(p)} = \left\{ m/n \in \mathbb{Q} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \text{ and } p \nmid n \right\}.$$

Set $\mathfrak{p} = p\mathbb{Z}_{(p)}$. In this case $K = \mathbb{Q}$ and $R_\mathfrak{p} = R$. Let $X \subseteq \mathbb{P}^3 = \operatorname{Proj} \mathbb{Q}[x, y, z, w]$ be the K3 surface over $\mathbb{Q}$ given by

$$x^4 + 2y^4 = z^4 + 4w^4.$$

Let $\mathcal{X} = \operatorname{Proj} \mathbb{Z}_{(p)}[x, y, z, w]/(x^4 + 2y^4 - z^4 - 4w^4)$. Note that if $p \neq 2$, then $\mathcal{X}$ is smooth and proper over R, and $\mathcal{X} \times_R \mathbb{Q} \simeq X$. Hence X has good reduction at primes $p \neq 2$.

Exercise 3.5 Prove that the conic $X := \operatorname{Proj} \mathbb{Q}[x, y, z]/(xy - 19z^2)$ has good reduction at $p = 19$. Naively, we might think that p is not a prime of good reduction if reducing the equations of X mod p gives a singular variety over the residue field. This example is meant to illustrate that this intuition can be wrong.

3.2 Specialization

In this section, we follow the exposition in [70, Sect. 3]; the reader is urged to consult this paper and the references contained therein for a more in-depth treatment of specialization of Néron-Severi groups.

Let R be a discrete valuation ring with fraction field K and residue field k. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{R}$ be the integral closure of R in $\overline{K}$. Choose a nonzero prime $\mathfrak{p} \in \overline{R}$ so that $\bar{k} = \overline{R}/\mathfrak{p}$ is an algebraic closure of k. For each finite extension L/K contained in $\overline{K}$, we let R_L be the integral closure of R in L. This is a Dedekind domain, and thus the localization of R_L at $\mathfrak{p} \cap R_L$ is a discrete valuation ring R'_L; call its residue field k'.

Let $\mathcal{X}$ be a smooth proper R-scheme. Restriction of Weil divisors, for example, gives natural group homomorphisms

$$\operatorname{Pic} \mathcal{X}_L \leftarrow \operatorname{Pic} \mathcal{X}_{R'_L} \rightarrow \operatorname{Pic} \mathcal{X}_{k'}, \tag{4}$$

and the map $\operatorname{Pic} \mathcal{X}_{R'_L} \rightarrow \operatorname{Pic} \mathcal{X}_L$ is an isomorphism (see the proof of [9, Sect. 8.4 Theorem 3]). If $\mathcal{X} \rightarrow \operatorname{Spec} R$ has relative dimension 2, then the induced map[3] $\operatorname{Pic} \mathcal{X}_L \rightarrow \operatorname{Pic} \mathcal{X}_{k'}$ preserves the intersection product on surfaces [33, Corollary 20.3]. Taking the direct limit over L of the maps (4) gives a homomorphism

$$\operatorname{Pic} \mathcal{X}_{\bar{K}} \rightarrow \operatorname{Pic} \mathcal{X}_{\bar{k}}$$

that preserves intersection products of surfaces when $\mathcal{X} \rightarrow \operatorname{Spec} R$ has relative dimension 2.

Proposition 3.6 *With notation as above, if $\mathcal{X} \rightarrow \operatorname{Spec} R$ is a proper, smooth morphism of relative dimension 2, then $\rho(\mathcal{X}_{\bar{K}}) \leq \rho(\mathcal{X}_{\bar{k}})$.*

Proof Since the map $\operatorname{Pic} \mathcal{X}_{\bar{K}} \rightarrow \operatorname{Pic} \mathcal{X}_{\bar{k}}$ preserves intersection products, it induces an injection

$$\operatorname{Pic} \mathcal{X}_{\bar{K}} / \operatorname{Pic}^{\tau} \mathcal{X}_{\bar{K}} \hookrightarrow \operatorname{Pic} \mathcal{X}_{\bar{k}} / \operatorname{Pic}^{\tau} \mathcal{X}_{\bar{k}}.$$

The claim now follows from the isomorphism $\operatorname{Pic} \overline{Y} / \operatorname{Pic}^{\tau} \overline{Y} \simeq \operatorname{NS} \overline{Y} / (\operatorname{NS} \overline{Y})_{\text{tors}}$ [105, p. 98], applied to $Y = \mathcal{X}_K$ and $\mathcal{X}_k$. ☐

Remark 3.7 The hypothesis that $\mathcal{X} \rightarrow \operatorname{Spec} R$ has relative dimension 2 in Proposition 3.6 is not necessary, but it simplifies the exposition. See [33, Example 20.3.6].

We can do a little better than Proposition 3.6. Indeed, without any assumption on the relative dimension of $\mathcal{X} \rightarrow \operatorname{Spec} R$, the map $\operatorname{Pic} \mathcal{X}_{\bar{K}} \rightarrow \operatorname{Pic} \mathcal{X}_{\bar{k}}$ gives rise to a specialization homomorphism

$$\operatorname{sp}_{\bar{K},\bar{k}} \colon \operatorname{NS} \mathcal{X}_{\bar{K}} \rightarrow \operatorname{NS} \mathcal{X}_{\bar{k}};$$

see [70, Proposition 3.3].

Theorem 3.8 *With notation as above, if* $\operatorname{char} k = p > 0$*, then the map*

$$\operatorname{sp}_{\bar{K},\bar{k}} \otimes_{\mathbb{Z}} \operatorname{id}_{\mathbb{Z}[1/p]} \colon \operatorname{NS} \mathcal{X}_{\bar{K}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \rightarrow \operatorname{NS} \mathcal{X}_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

is injective and has torsion-free cokernel.

---

[3] This map has a simple description at the level of cycles: given a prime divisor on $\mathcal{X}_L$, take its Zariski closure in $\mathcal{X}_{R'_L}$ and restrict to $\mathcal{X}_{k'}$. This operation respects linear equivalence and can be linearly extended to $\operatorname{Pic} \mathcal{X}_L$.

Proof See [70, Proposition 3.6]. □

Remark 3.9 If Y is a K3 surface over a field then $\mathrm{NS}\,\overline{Y} \simeq \mathrm{Pic}\,\overline{Y}$ (Proposition 2.8), so $\mathrm{sp}_{\bar{K},\bar{k}}$ is the map we already know, and it is already injective before tensoring with $\mathbb{Z}[1/p]$.

The moral of the story so far (Proposition 3.6) is that if X is a smooth projective surface over a number field, then we can use information at a prime of good reduction for X to bound $\rho(\overline{X})$. The key tool is the cycle class map, which we turn to next; this map is the algebraic version of the connecting homomorphism in the long exact sequence in cohomology associated to the exponential sequence.

3.3 The Cycle Class Map

In this section we let X be a smooth projective geometrically integral variety over a finite field $\mathbb{F}_q$ with $q = p^r$ elements (p prime). Write $\overline{\mathbb{F}}_q$ for a fixed algebraic closure of $\mathbb{F}_q$, and let $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the Frobenius automorphism $x \mapsto x^q$. Let $\overline{X}_{\text{ét}}$ denote the (small) étale site of $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and let $\ell \neq p$ be a prime. For an integer $m \geq 1$, the Tate twist $(\mathbb{Z}/\ell^n\mathbb{Z})(m)$ is the sheaf $\mu_{\ell^n}^{\otimes m}$ on $\overline{X}_{\text{ét}}$. For a fixed m there is a natural surjection $(\mathbb{Z}/\ell^{n+1}\mathbb{Z})(m) \to (\mathbb{Z}/\ell^n\mathbb{Z})(m)$; putting these maps together, we define

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m)) := \varprojlim_n \mathrm{H}^2_{\text{ét}}(\overline{X}, (\mathbb{Z}/\ell^n\mathbb{Z})(m)),$$
$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m)) := \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Since $\ell \neq p$, the Kummer sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{[\ell^n]} \mathbb{G}_m \to 0$$

is an exact sequence of sheaves on $\overline{X}_{\text{ét}}$ [75, p. 66], so the long exact sequence in étale cohomology gives a boundary map

$$\delta_n \colon \mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{G}_m) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^n}). \tag{5}$$

Since $\mathrm{H}^1_{\text{ét}}(\overline{X}, \mathbb{G}_m) \simeq \mathrm{Pic}\,\overline{X}$ [75, III.4.9], taking the inverse limit of (5) with respect to the ℓ-th power maps $\{\mu_{\ell^{n+1}} \to \mu_{\ell^n}\}$ we obtain a homomorphism

$$\mathrm{Pic}\,\overline{X} \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)). \tag{6}$$

The kernel of this map is the group $\mathrm{Pic}^\tau\,\overline{X}$ of divisors numerically equivalent to zero [105, pp. 97–98], and since $\mathrm{Pic}\,\overline{X}/\mathrm{Pic}^\tau\,\overline{X} \simeq \mathrm{NS}\,\overline{X}/(\mathrm{NS}\,\overline{X})_{\text{tors}}$, tensoring (6) with $\mathbb{Q}_\ell$ gives an injection

$$c\colon \operatorname{NS} \overline{X} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)). \tag{7}$$

The map c is compatible with the action of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, and moreover, there is an isomorphism of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)) \simeq \underbrace{\left(\varprojlim_n \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\right)}_{=:\, \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)} \otimes_{\mathbb{Z}_\ell} \left(\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n}\right), \tag{8}$$

where $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n}$ according to the usual action of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$. In particular, the Frobenius automorphism σ acts as multiplication by q on $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mu_{\ell^n}$: indeed, we are regarding $\mu_{\ell^n} \subset \overline{\mathbb{F}}_q$ as a $\mathbb{Z}/\ell^n\mathbb{Z}$-module via the multiplication $m \cdot \zeta := \zeta^m$.

Proposition 3.10 *Let X be a smooth proper scheme over a finite field $\mathbb{F}_q$ of cardinality $q = p^r$ with p prime. Write $\sigma \in \operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ for the Frobenius automorphism $x \mapsto x^q$. Let $\ell \neq p$ be a prime and let $\sigma^*(0)$ denote the automorphism of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ induced by σ. Then $\rho(\overline{X})$ is bounded above by the number of eigenvalues of $\sigma^*(0)$, counted with multiplicity, of the form ζ/q, where ζ is a root of unity.*

Proof Write σ^* for the automorphisms of $\operatorname{NS} \overline{X}$ induced by σ. The divisor classes generating $\operatorname{NS} \overline{X}$ are defined over a *finite* extension of k, so some power of σ^* acts as the identity on $\operatorname{NS} \overline{X}$. Hence, all eigenvalues of σ^* are roots of unity. Using the injection (7), we deduce that $\rho(\overline{X})$ is bounded above by the number of eigenvalues of $\sigma^*(1)$ operating on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ that are roots of unity. The isomorphism (8) shows that this number is in turn equal to the number of eigenvalues of $\sigma^*(0)$ operating on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ of the form ζ/q, where ζ is a root of unity. □

Remark 3.11 Let $\mathbb{F} \subseteq \overline{\mathbb{F}}_q$ be a finite extension of $\mathbb{F}_q$. The Tate conjecture [105, p. 98] implies that

$$c(\operatorname{NS} X_{\mathbb{F}} \otimes \mathbb{Q}_\ell) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^{\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F})}.$$

One can deduce that the upper bound in Proposition 3.10 is sharp (exercise!). This conjecture has now been established for K3 surfaces X when q is odd [15, 66, 69, 81, 82], and also for q even if the geometric Picard rank of the surface is ≥ 2 [17].

Proposition 3.10 implies that knowledge of the characteristic polynomial of σ^* acting on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ gives an upper bound for $\rho(\overline{X})$. It turns out that it is easier to calculate the characteristic polynomial of $(\sigma^*)^{-1}$, because we can relate this problem to point counts for X over a finite number of finite extensions of $\mathbb{F}_q$. To this end, we take a moment to understand what $(\sigma^*)^{-1}$ looks like.

3.3.1 Absolute Frobenius

For a scheme Z over a finite field $\mathbb{F}_q$ (with $q = p^r$), we let $F_Z\colon Z \to Z$ be the absolute Frobenius map: this map is the identity on points, and $x \mapsto x^p$ on the structure sheaf; it is *not* a morphism of $\mathbb{F}_q$-schemes. Set $\Phi_Z = F_Z^r$; the map $\Phi_Z \times 1\colon Z \times \overline{\mathbb{F}}_q \to Z \times \overline{\mathbb{F}}_q$ induces a linear transformation $\Phi_Z^*\colon \mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell) \to \mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell)$. The action of F_Z on $Z_{\text{ét}}$ is (naturally equivalent to) the identity [75, VI Lemma 13.2], and since $F_{\overline{Z}}^r = F_Z^r \times F_{\bar{k}}^r = \Phi_Z \times \sigma$, the maps Φ_Z^* and $\sigma^*(0)$ operate as each other's inverses on $\mathrm{H}^2_{\text{ét}}(\overline{Z}, \mathbb{Q}_\ell)$. Using the notation of Proposition 3.10, we conclude that the number of eigenvalues of $\sigma^*(0)$ operating on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ of the form ζ/q is equal to the number of eigenvalues of Φ_X^* operating on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ of the form $q\zeta$, where ζ is a root of unity.

3.4 Upper Bounds I: Putting Everything Together

Theorem 3.12 *Let R be a discrete valuation ring of a number field K, with residue field $k \simeq \mathbb{F}_q$. Fix an algebraic closure $\overline{K}$ of K, and let $\overline{R}$ be the integral closure of R in $\overline{K}$. Choose a nonzero prime $\mathfrak{p} \in \overline{R}$ so that $\bar{k} = \overline{R}/\mathfrak{p}$ is an algebraic closure of k. Let $\ell \neq$ char k be a prime number.*

Let $\mathcal{X} \to R$ be a smooth proper morphism of relative dimension 2, and assume that the surfaces $\mathcal{X}_{\overline{K}}$ and $\mathcal{X}_{\bar{k}}$ are geometrically integral. There are natural injective homomorphisms of $\mathbb{Q}_\ell$-inner product spaces

$$\mathrm{NS}\,\mathcal{X}_{\overline{K}} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{NS}\,\mathcal{X}_{\bar{k}} \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{H}^2_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell(1))$$

and the second map is compatible with $\mathrm{Gal}(\bar{k}/k)$-actions. Consequently, $\rho(\mathcal{X}_{\overline{K}})$ is bounded above by the number of eigenvalues of $\Phi^*_{\mathcal{X}_k}$ operating on $\mathrm{H}^2_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$, counted with multiplicity, of the form $q\zeta$, where ζ is a root of unity. □

Convention 3.13 We will apply Theorem 3.12 to K3 surfaces X over a number field K. In such cases, we will speak of a finite prime $\mathfrak{p} \subseteq \mathcal{O}_K$ of good reduction for X. The model $\mathcal{X} \to \operatorname{Spec} R$ with $R = (\mathcal{O}_K)_\mathfrak{p}$ satisfying the hypotheses of Theorem 3.12 will be implicit, and we will write $\overline{X}$ for the ($\overline{K}$-isomorphic) scheme $\mathcal{X}_{\overline{K}}$, and $\overline{X}_\mathfrak{p}$ for $\mathcal{X}_{\bar{k}}$.

Keep the notation of Theorem 3.12. The number of eigenvalues of $\Phi^*_{\mathcal{X}_k}$ of the form $q\zeta$ can be read off from the characteristic polynomial $\psi_q(x)$ of this linear operator. To compute this characteristic polynomial, we use two ideas. First, the characteristic polynomial of a linear operator on a finite dimensional vector space can be recovered from knowing traces of sufficiently many powers of the linear operator, as follows.

Theorem 3.14 (Newton's identities). *Let T be a linear operator on a vector space V of finite dimension n. Write t_i for the trace of the i-fold composition T^i of T, and define*

$$a_1 := -t_1 \quad \textit{and} \quad a_k := -\frac{1}{k}\left(t_k + \sum_{j=1}^{k-1} a_j t_{k-j}\right) \quad \textit{for } k = 2, \ldots, n.$$

Then the characteristic polynomial of T is equal to

$$\det(x \cdot Id - T) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n.$$

Second, the traces of powers of $\Phi^*_{\mathcal{X}_k}$ operating on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$ can be recovered from the Lefschetz trace formula

$$\mathrm{Tr}\left((\Phi^*_{\mathcal{X}_k})^i\right) = \#\mathcal{X}_k(\mathbb{F}_{q^i}) - 1 - q^{2i};$$

see [68, Sect. 27] for a proof of this formula in the surface case. When $\mathcal{X}_k$ is a K3 surface, we have $n = 22$, so at first glance we have to count points over $\mathbb{F}_{q^i}$ for $i = 1, \ldots, 22$. However, the characteristic polynomial of $\Phi^*_{\mathcal{X}_k}$ happens to satisfy a functional equation, coming from the Weil conjectures (which have all been proved):

$$q^{22}\psi_q(x) = \pm x^{22}\psi_q(q^2/x).$$

If we are lucky, counting points over $\mathbb{F}_{q^i}$ for $i = 1, \ldots, 11$ will be enough to determine the sign of the functional equation, and thus allow us to compute $\psi_q(x)$. If we are unlucky, one can always compute two possible characteristic polynomials, one for each possible sign in the functional equation, and discard the polynomial whose roots provably have absolute value different from q (i.e., absolute value distinct from that predicted by the Weil conjectures). In practice, if we already know a few explicit divisor classes on $\mathcal{X}_{\bar{k}}$, we can cut down the amount of point counting required to determine $\psi_q(x)$. For example, knowing that the hyperplane class is fixed by Galois tells us that $(x - q)$ divides $\psi_q(x)$; this information can be used to get away with point count counts for $i = 1, \ldots, 10$ only. More generally, if one already knows an explicit submodule $M \subseteq \mathrm{NS}\,\mathcal{X}_{\bar{k}}$ as a Galois module, then the characteristic polynomial $\psi_M(x)$ of Frobenius acting on M can be computed, and since $\psi_M(x) \mid \psi_q(x)$, one can compute $\psi_q(x)$ with only a few point counts, depending on the rank of M.

Exercise 3.15 Show that if M has rank r then counting points on $\mathcal{X}_k(\mathbb{F}_{q^i})$ for $i = 1, \ldots, \lceil (22 - r)/2 \rceil$ suffices to determine the two possible polynomials $\psi_q(x)$ (one for each possible sign in the functional equation).

Example 3.16 ([43, Sect. 5.3]). In the polynomial ring $\mathbb{F}_3[x, y, z, w]$, give weights 1, 1, 1 and 3, respectively, to the variables x, y, z and w, and let $\mathbb{P}_{\mathbb{F}_3}(1, 1, 1, 3) = \mathrm{Proj}\,\mathbb{F}_3[x, y, z, w]$ be the corresponding weighted projective plane. We choose a polynomial $p_5(x, y, z) \in \mathbb{F}_3[x, y, z]_5$ so that the hypersurface X given by

$$w^2 = 2y^2(x^2 + 2xy + 2y^2)^2 + (2x + z)p_5(x, y, z) \tag{9}$$

is smooth, hence a K3 surface (of degree 2).

For example, take

$$p_5(x,y,z) = x^5 + x^4y + x^3yz + x^2y^3 + x^2y^2z + 2x^2z^3 + xy^4 + 2xy^3z + xy^2z^2 + y^5 + 2y^4z + 2y^3z^2 + 2z^5.$$

The projection $\pi\colon \mathbb{P}(1,1,1,3) \dashrightarrow \operatorname{Proj} \mathbb{F}_3[x,y,z]$ restricts to a double cover morphism $\pi\colon X \to \mathbb{P}^2_{\mathbb{F}_3}$, branched along the vanishing of the right hand side of (9).

Let $N_i := \#X(\mathbb{F}_{3^i})$; counting points we find

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
7	79	703	6607	60427	532711	4792690	43068511	387466417	3486842479

Applying the procedure described above, this is enough information to determine the characteristic polynomial $\psi_3(x)$. The sign of the functional equation for $\psi_3(x)$ is negative—a positive sign gives rise to roots of absolute value $\neq 3$. Setting $\widetilde{\psi}(x) = 3^{-22}\psi_3(3x)$, we obtain a factorization into irreducible factors as follows:

$$\widetilde{\psi}(x) = \frac{1}{3}(x-1)(x+1)(3x^{20} + 3x^{19} + 5x^{18} + 5x^{17} + 6x^{16} + 2x^{15} + 2x^{14} - 3x^{13} - 4x^{12} - 8x^{11} - 6x^{10} - 8x^9 - 4x^8 - 3x^7 + 2x^6 + 2x^5 + 6x^4 + 5x^3 + 5x^2 + 3x + 3).$$

The roots of the degree 20 factor of $\psi(x)$ are not integral, so they are not roots of unity. We conclude that $\rho(\overline{X}) \leq 2$.

On the other hand, inspecting the right hand side of (9), we see that the line $2x + z = 0$ on $\mathbb{P}^2$ is a tritangent line to the branch curve of the double cover morphism π. The components of the pullback of this line intersect according to the following Gram matrix

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

which has determinant $-5 \neq 0$, and thus they generate a rank 2 sublattice L of $\operatorname{NS}\overline{X}$. We conclude that $\rho(\overline{X}) = 2$. Since the determinant of the lattice L is not divisible by a square, the lattice L must be saturated in $\operatorname{NS}\overline{X}$, so $\operatorname{NS}\overline{X} = L$.

By Theorem 3.12, any K3 surface over $\mathbb{Q}$ whose reduction at $p = 3$ is isomorphic to X has geometric Picard rank at most 2.

3.5 Upper Bounds II

Keep the notation of Theorem 3.12. It is natural to wonder how good the upper bound furnished by Theorem 3.12 really is, at least for K3 surfaces, which are the varieties that concern us. The Weil conjectures tell us that the eigenvalues of $\Phi^*_{\mathcal{X}_k}$ operating on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$ have absolute value[4] q. Since the characteristic polynomial of $\Phi^*_{\mathcal{X}_{\bar{k}}}$ lies in $\mathbb{Q}[x]$, the eigenvalues not of form $q\zeta$ must come in complex conjugate pairs. In particular, the total number of eigenvalues that *are* of the form $q\zeta$ must have the same parity as the ℓ-adic Betti number $b_2 = \dim_{\mathbb{Q}_\ell} \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{k}}, \mathbb{Q}_\ell)$. For a K3 surface, $b_2 = 22$ because the l-adic Betti numbers coincide with the usual Betti numbers (use [75, Theorem 3.12]).

We conclude, for example, that Theorem 3.12 by itself cannot be used to construct a projective K3 surface over a number field of geometric Picard rank 1. This was a distressing state of affairs, since it is a classical fact that outside a countable union of divisors, the points in the coarse moduli space $\mathcal{K}_{2d}$ of complex K3 surfaces of degree $2d$ represent K3 surfaces of geometric Picard rank 1. The complement of these divisors is not empty (by the Baire category theorem!), but since number fields are countable, it was conceivable that there did not exist K3 surfaces over number fields of geometric Picard rank 1. Terasoma and Ellenberg showed that such surfaces do exist [26, 106], and van Luijk constructed the first explicit examples [109].

3.5.1 van Luijk's Method

The idea behind van Luijk's method [109] is beautiful in its simplicity: use information at *two* primes of good reduction. See Convention 3.13 to understand the notation below.

Proposition 3.17 *Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ and $\mathfrak{p}'$ be two finite places of good reduction. Suppose that* $\mathrm{NS}\,\overline{X}_{\mathfrak{p}} \simeq \mathbb{Z}^n$ *and* $\mathrm{NS}\,\overline{X}_{\mathfrak{p}'} \simeq \mathbb{Z}^n$, *and that the discriminants* $\mathrm{Disc}\left(\mathrm{NS}\,\overline{X}_{\mathfrak{p}}\right)$ *and* $\mathrm{Disc}\left(\mathrm{NS}\,\overline{X}_{\mathfrak{p}'}\right)$ *are different in* $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. *Then $\rho(\overline{X}) \leq n - 1$.*

Proof By Theorem 3.12, we know that $\rho(\overline{X}) \leq n$. If $\rho(\overline{X}) = n$, then $\mathrm{NS}\,\overline{X}$ is a full rank sublattice of both $\mathrm{NS}\,\overline{X}_{\mathfrak{p}}$ and $\mathrm{NS}\,\overline{X}_{\mathfrak{p}'}$. This implies that $\mathrm{Disc}\,\mathrm{NS}(\overline{X})$ is equal to *both* $\mathrm{Disc}\left(\mathrm{NS}\,\overline{X}_{\mathfrak{p}}\right)$ and $\mathrm{Disc}\left(\mathrm{NS}\,\overline{X}_{\mathfrak{p}'}\right)$ as elements of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, so the discriminants of the reductions are equal in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. This is a contradiction. □

Example 3.18 ([109, Sect. 3]). The following is van Luijk's original example. Set

$$\begin{aligned} f = x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw \\ + y^3 + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3, \end{aligned}$$

and let X be the quartic surface in $\mathbb{P}^3_{\mathbb{Q}} = \mathrm{Proj}\,\mathbb{Q}[x, y, z, w]$ given by

$$wf + 2z(xy^2 + xyz - xz^2 - yz^2 + z^3) - 3(z^2 + xy + yz)(z^2 + xy) = 0.$$

[4]When we say absolute value here we mean any archimedean absolute value of the field obtained by adjoining to K the eigenvalues of $\Phi^*_{\mathcal{X}_{\bar{k}}}$.

One can check (using the Jacobian criterion), that X is smooth, and that X has good reduction at $p = 2$ and 3. Let $\psi_p(x)$ denote the characteristic polynomial of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell)$, and let $\widetilde{\psi}_p(x) = p^{-22}\psi_p(px)$. Proceeding as in Example 3.16, we use point counts to compute

$$\widetilde{\psi}_2(x) = \frac{1}{2}(x-1)^2(2x^{20} + x^{19} - x^{18} + x^{16} + x^{14} + x^{11} + 2x^{10} + x^9 + x^6 + x^4 - x^2 + x + 2)$$

$$\widetilde{\psi}_3(x) = \frac{1}{3}(x-1)^2(3x^{20} + x^{19} - 3x^{18} + x^{17} + 6x^{16} - 6x^{14} + x^{13} + 6x^{12} - x^{11} - 7x^{10} - x^9 + 6x^8 + x^7 - 6x^6 + 6x^4 + x^3 - 3x^2 + x + 3)$$

The roots of the degree 20 factors of $\widetilde{\psi}_p(x)$ are not integral for $p = 2$ and 3, so they are not roots of unity. We conclude that $\rho(\overline{X}_2)$ and $\rho(\overline{X}_3)$ are both less than or equal to 2.

Next, we compute $\mathrm{Disc}(\mathrm{NS}\,\overline{X}_p)$ for $p = 2$ and 3 by finding explicit generators for $\mathrm{NS}\,\overline{X}_p$. For $p = 2$ note that, besides the hyperplane section H (i.e., the pullback of $\mathscr{O}_{\mathbb{P}^3}(1)$ to $\overline{X}_2$), the surface $\overline{X}_2$ contains the conic

$$C\colon w = z^2 + xy = 0.$$

We have $H^2 = 4$ (it's the degree of $\overline{X}_2$ in $\mathbb{P}^3$), and $C \cdot H = \deg C = 2$. Finally, by the adjunction formula $C^2 = -2$ because C has genus 0 and the canonical class on $\overline{X}_2$ is trivial. All told, we have produced a rank two sublattice of $\mathrm{NS}\,\overline{X}_2$ of discriminant

$$\det\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix} = -12.$$

We conclude that $\mathrm{Disc}(\mathrm{NS}\,\overline{X}_2) = -3 \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

For $p = 3$, the surface $\overline{X}_3$ contains the hyperplane class H and the line $L\colon w = z = 0$, giving a rank two sublattice of $\mathrm{NS}\,\overline{X}_3$ of discriminant

$$\det\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} = -9.$$

Thus $\mathrm{Disc}(\mathrm{NS}\,\overline{X}_3) = -1 \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Proposition 3.17 implies that $\rho(\overline{X}) \leq 1$, and since $\mathrm{NS}\,\overline{X}$ contains the hyperplane class, we conclude that $\rho(\overline{X}) = 1$.

3.6 Further Techniques

In Examples 3.16 and 3.18 above, we computed the discriminant of the Néron-Severi lattice for some K3 surfaces by exhibiting explicit generators. What if we don't have explicit generators? In [60] Kloosterman gets around this problem by using that Artin–Tate conjecture, which states that for a K3 surface X over a finite field $\mathbb{F}_q$ the

Brauer group $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ of X is finite and

$$\lim_{x \to q} \frac{\psi_q(x)}{(x-q)^{\rho(X)}} = q^{21-\rho(X)} \# \operatorname{Br} X |\operatorname{Disc}(\operatorname{NS} X)|, \tag{10}$$

where $\rho(X) = \operatorname{rk}(\operatorname{NS} X)$. The Artin–Tate conjecture follows from the Tate conjecture when $2 \nmid q$ [74], and the Tate conjecture is now known to hold in odd characteristic; see Remark 3.11. Assume then that q is odd. Pass to the finite extension of the ground field so that $\operatorname{NS} X = \operatorname{NS} \overline{X}$. Since the Artin–Tate conjecture holds, so in particular $\operatorname{Br} X$ is finite, a theorem of Lorenzini, Liu and Raynaud states that the quantity $\# \operatorname{Br} X$ is a square [63]. Hence (10) can be used to compute $|\operatorname{Disc}(\operatorname{NS} \overline{X})|$ as an element of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

Elsenhans and Jahnel have made several contributions to the computation of Néron-Severi groups of K3 surfaces. For example, in [28], they explain that one can use the Galois module structures of Néron-Severi groups to refine Proposition 3.17. Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ be a finite place of good reduction for X, with residue field k (see Convention 3.13). The specialization map

$$\operatorname{sp}_{\bar{K},\bar{k}} \otimes \operatorname{id}\colon \operatorname{NS} \overline{X} \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{NS} \overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injective homomorphism. The $\mathbb{Q}$-vector space $\operatorname{NS} \overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\operatorname{Gal}(\bar{k}/k)$-representation, while the $\mathbb{Q}$-vector space $\operatorname{NS} \overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\operatorname{Gal}(\overline{K}/K)$-representation. Let L denote the kernel of the latter representation.

Exercise 3.19 Show that the field extension L/K is finite and unramified at $\mathfrak{p}$.

Exercise 3.19 shows that, after choosing a prime $\mathfrak{q}$ in L lying above $\mathfrak{p}$, there is a unique lift of Frobenius to L, which together with the specialization map, makes $\operatorname{NS} \overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ a $\operatorname{Gal}(\bar{k}/k)$-submodule of $\operatorname{NS} X_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Q}$. By understanding the $\operatorname{Gal}(\bar{k}/k)$-submodules of $\operatorname{NS} X_{\bar{k}} \otimes_{\mathbb{Z}} \mathbb{Q}$ as we vary over several primes of good reduction, we can find restrictions on the structure of $\operatorname{NS} \overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$, and often compute $\rho(\overline{X})$.

The main tool is the characteristic polynomial χ_{Frob} of Frobenius as an endomorphism of $\operatorname{NS} \overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$. If χ_{Frob} has simple roots, then $\operatorname{Gal}(\bar{k}/k)$-submodules of $\operatorname{NS} \overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}$ are in bijection with the monic polynomials dividing χ_{Frob}.

Recall that $\operatorname{NS} \overline{X}_{\mathfrak{p}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is a $\operatorname{Gal}(\bar{k}/k)$-submodule of $\mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ via the cycle class map, so χ_{Frob} divides the characteristic polynomial $\widetilde{\psi}_{\mathfrak{p}}$ of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(1))$, and we have seen that the roots of χ_{Frob} are roots of unity (because some power of Frobenius acts as the identity). Therefore, χ_{Frob} divides the product of the cyclotomic polynomials that divide $\widetilde{\psi}_{\mathfrak{p}}$. The Tate conjecture implies that χ_{Frob} is in fact equal to this product. So let V_{Tate} denote the highest dimensional $\mathbb{Q}_\ell$-subspace of $\mathrm{H}^2_{\text{ét}}(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ on which all the eigenvalues of Frobenius are roots of unity. Let $L \subset \operatorname{NS} \overline{X}_{\mathfrak{p}}$ be a sublattice; typically, L will be generated by the classes of explicit divisors we are aware of on $\overline{X}_{\mathfrak{p}}$. If we are lucky, there are very few possibilities for $\operatorname{Gal}(\bar{k}/k)$-submodules of the quotient $V_{\text{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$, which we compare as we vary over finite places of good reduction. This is best explained through an example.

Example 3.20 ([28, Sect. 5]). The following is an example of a K3 surface X over $\mathbb{Q}$ with good reduction at $p = 3$ and 5, such that $\rho(\overline{X}_3) = 4$ and $\rho(\overline{X}_5) = 14$, for which we can show that $\rho(\overline{X}) = 1$ using only information at these two primes. Let $\mathcal{X}$ be the subscheme of $\mathbb{P}(1, 1, 1, 3) = \operatorname{Proj} \mathbb{Z}_{(15)}[x, y, z, w]$ given by $w^2 = f_6(x, y, z)$, where

$$
\begin{aligned}
f_6(x, y, z) &\equiv 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 \\
&\qquad + xy^4z + xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \bmod 3, \\
f_6(x, y, z) &\equiv y^6 + x^4y^2 + 3x^2y^4 + 2x^5z + 3xz^5 + z^6 \bmod 5.
\end{aligned}
$$

Set $X = \mathcal{X}_{\mathbb{Q}}$. Counting the elements of $\mathcal{X}_{\mathbb{F}_3}(\mathbb{F}_{3^n})$ for $n = 1, \ldots, 10$, we compute the characteristic polynomial of Frobenius on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_3}, \mathbb{Q}_\ell(1))$ (here $\ell \neq 3$ is a prime) and we get

$$
\begin{aligned}
\widetilde{\phi}_3(x) = \frac{1}{3}&(x-1)^2(x^2+x+1) \\
&(3x^{18} + 5x^{17} + 7x^{16} + 10x^{15} + 11x^{14} + 11x^{13} + 11x^{12} + 10x^{11} + 9x^{10} \\
&\quad + 9x^9 + 9x^8 + 10x^7 + 11x^6 + 11x^5 + 11x^4 + 10x^3 + 7x^2 + 5x + 1)
\end{aligned}
$$

Let $L \subset \operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_3}$ be the rank 1 sublattice generated by the pullback of the class of a line for the projection $\mathcal{X}_{\mathbb{F}_3} \to \mathbb{P}^2_{\mathbb{F}_3}$ (i.e., the "hyperplane class"). The characteristic polynomial of Frobenius acting on $V_{\text{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$ is $(x-1)(x^2+x+1)$, which has simple roots. We conclude that, for each dimension 1, 2, 3, and 4, there is at most one $\operatorname{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_3)$-invariant vector subspace of $\operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_3}$ that contains L.

Repeating this procedure[5] at $p = 5$, we find that the characteristic polynomial of Frobenius acting on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell(1))$ is

$$
\begin{aligned}
\widetilde{\phi}_5(x) = \frac{1}{5}&(x-1)^2(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1) \\
&(5x^8 - 5x^7 - 2x^6 + 3x^5 - x^4 + 3x^3 - 2x^2 - 5x + 5)
\end{aligned}
$$

Again, let $L \subset \operatorname{NS} \mathcal{X}_{\overline{\mathbb{F}}_5}$ be the rank 1 sublattice generated by the pullback of the class of a line for the projection $\mathcal{X}_{\mathbb{F}_5} \to \mathbb{P}^2_{\overline{\mathbb{F}}_5}$. The characteristic polynomial of Frobenius acting on $V_{\text{Tate}}/(L \otimes_{\mathbb{Z}} \mathbb{Q}_\ell)$ is

$$
(x-1)(x^4+x^3+x^2+x+1)(x^8-x^7+x^5-x^4+x^3-x+1)
$$

[5] In the interest of transparency, one should add that brute-force point counting of $\mathbb{F}_{5^n}$-points of $\mathcal{X}_{\mathbb{F}_5}$ is usually not feasible for $n \geq 8$. However, the defining equation for $\mathcal{X}_{\mathbb{F}_5}$ contains no monomials involving both y and z. This "decoupling" allows for extra tricks that allow a refined brute-force approach to work. See [27, Algorithm 17]. Alternatively, one can find several divisors on $\mathcal{X}_{\overline{\mathbb{F}}_5}$, given by irreducible components of the pullbacks of lines tritangent to the curve $f_6(x, y, z) = 0$ in $\mathbb{P}^2_{\overline{\mathbb{F}}_5}$, and thus compute a large degree divisor of $\widetilde{\phi}_5(x)$; see the discussion after Theorem 3.14.

which has simple roots. Thus, for each dimension 1, 2, 5, 6, 9, 10, 13, and 14 there is at most one $\mathrm{Gal}(\overline{\mathbb{F}}_5/\mathbb{F}_5)$-invariant vector subspace of $\mathrm{NS}\,\mathcal{X}_{\overline{\mathbb{F}}_5}$ that contains L.

Since $\mathrm{NS}\,\overline{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-invariant subspace of $\mathrm{NS}\,\mathcal{X}_{\overline{\mathbb{F}}_p}$ for $p = 3$ and 5, we already see that $\rho(\overline{X}) = 1$ or 2. If $\rho(\overline{X}) = 2$, then the discriminants of the $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-invariant subspaces of $\mathrm{NS}\,\mathcal{X}_{\overline{\mathbb{F}}_p}$ of rank 2 for $p = 3$ and 5 must be equal in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. These classes *modulo squares* of these discriminants can be calculated using the Artin–Tate formula (10), and they are, respectively -489 and -5. Hence $\rho(\overline{X}) = 1$.

Unless one uses p-adic cohomology methods to count points of a K3 surface over a finite field (e.g. [1, 25]), the slowest step in computing geometric Picard numbers using the above techniques is point counting. One is restricted to using small characteristics, typically 2, 3 and (sometimes) 5, and in practice, it can be difficult to write a model of a surface over a number field with good reduction at these small primes. Remarkably, Elsenhans and Jahnel proved a theorem that requires point counting in only *one* characteristic. Their result is quite general; we explain below how to use it in a concrete situation.

Theorem 3.21 ([29, Theorem 1.4]). *Let R be a discrete valuation ring with quotient field K of characteristic zero and perfect residue field k of characteristic $p > 0$. Write v for the valuation of R, and assume that $v(p) < p - 1$. Let $\pi\colon X \to \operatorname{Spec} R$ be a smooth proper morphism. Then the cokernel of the specialization homomorphism*

$$\mathrm{sp}\colon\ \mathrm{Pic}\, X_{\overline{K}} \to \mathrm{Pic}\, X_{\bar{k}}$$

is torsion-free. □

Recall that for a K3 surface the Picard group and the Néron-Severi group coincide (Proposition 2.8).

Example 3.22 Let $R = \mathbb{Z}_{(3)}$, so that $K = \mathbb{Q}$ and $k = \mathbb{F}_3$. Let X be the K3 surface in $\mathbb{P}(1, 1, 1, 3) = \operatorname{Proj} \mathbb{Z}_{(3)}[x, y, z, w]$ given by

$$w^2 = 2y^2(x^2 + 2xy + 2y^2)^2 + (2x + z)p_5(x, y, z) + 3p_6(x, y, z),$$

where $p_5(x, y, z)$ is the same polynomial as in Example 3.16, and $p_6(x, y, z) \in \mathbb{Z}_{(3)}[x, y, z]_6$ is a polynomial of degree 6 such that X is smooth as a $\mathbb{Z}_{(3)}$-scheme. We saw in Example 3.16 that $\mathrm{NS}\, X_{\overline{\mathbb{F}}_3} = \mathrm{Pic}\, X_{\overline{\mathbb{F}}_3}$ has rank 2 and is generated by the pullbacks C and C' for $X_{\overline{\mathbb{F}}_3} \to \mathbb{P}^2_{\overline{\mathbb{F}}_3}$ of the tritangent line $2x + z = 0$. Theorem 3.21 tell us that if $\mathrm{NS}\, X_{\overline{\mathbb{Q}}}$ has rank 2, then C and C' lift to classes $\widetilde{C}$ and $\widetilde{C}'$, respectively, in $\mathrm{NS}\, X_{\overline{\mathbb{Q}}}$. The Riemann-Roch theorem shows that $\widetilde{C}$ and $\widetilde{C}'$ are effective, and an intersection number computation shows that $\widetilde{C}$ and $\widetilde{C}'$ must be components of the pullback of a line tritangent to the branch curve of the projection $X_{\overline{\mathbb{Q}}} \to \mathbb{P}^2_{\overline{\mathbb{Q}}}$. But now the presence of $p_6(x, y, z)$ could wreck havoc here, and there may not be a line that is tritangent to the branch curve in characteristic zero!

For a particular $p_6(x, y, z)$, how does one look for a line tritangent to the curve

$$2y^2(x^2 + 2xy + 2y^2)^2 + (2x + z)p_5(x, y, z) + 3p_6(x, y, z) = 0$$

in $\mathbb{P}^2_{\overline{\mathbb{Q}}}$? One can use Gröbner bases and [27, Algorithm 8] to carry out this task (on a computer!). Alternatively, one could use a different prime p of good reduction for $X_{\mathbb{Q}}$ and look for tritangent lines to the branch curve of the projection $X_{\overline{\mathbb{F}}_p} \to \mathbb{P}^2_{\overline{\mathbb{F}}_p}$, still using [27, Algorithm 8], hoping of course that there is no such line. No point counting is needed in this second approach, but the Gröbner bases computations over finite fields that take place under the hood are much simpler than the corresponding computations over $\mathbb{Q}$.

Exercise 3.23 Fill in the details in the Example 3.22 to show that $\widetilde{C}$ and $\widetilde{C}'$ must be components of the pullback of a line tritangent to the branch curve of the projection $X_{\overline{\mathbb{Q}}} \to \mathbb{P}^2_{\overline{\mathbb{Q}}}$.

Exercise 3.24 Implement [27, Algorithm 8] in your favorite platform, and use it to write down a specific homogeneous polynomial $p_6(x, y, z)$ of degree 6 for which you can prove that the surface $X_{\mathbb{Q}}$ of Example 3.22 has geometric Picard rank 1.

3.7 More on the Specialization Map

Let X be a K3 surface over a number field K, and let $\mathfrak{p}$ be a finite place of good reduction for X (see Convention 3.13). We have used the injectivity of the specialization map $\mathrm{sp}_{\overline{K},\bar{k}}\colon \mathrm{NS}\,\overline{X} \to \mathrm{NS}\,\overline{X}_{\mathfrak{p}}$ to glean information about the geometric Picard number $\rho(\overline{X})$ of X. On the other hand, we also know that $\rho(\overline{X}_{\mathfrak{p}})$ is even, whereas $\rho(\overline{X})$ can be odd, so the specialization map need not be surjective. In [30], Elsenhans and Jahnel asked if there is always a finite place $\mathfrak{p}$ of good reduction such that $\rho(\overline{X}_{\mathfrak{p}}) - \rho(\overline{X}) \leq 1$.

Using Hodge theory, Charles answers this question in [16]. Although the answer to the original question is "no", Charles' investigation yields sharp bounds for the difference $\rho(\overline{X}_{\mathfrak{p}}) - \rho(\overline{X})$. We introduce some notation to explain his results.

Let $T_{\mathbb{Q}}$ be the orthogonal complement of $\mathrm{NS}\,X_{\mathbb{C}}$ inside the singular cohomology group $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$ with respect to the cup product pairing; $T_{\mathbb{Q}}$ is a sub-Hodge structure of $\mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Q})$. Write E for the endomorphism algebra of $T_{\mathbb{Q}}$. It is known that E is either a totally real field or a CM field[6]; see [117].

Theorem 3.25 ([16, Theorem 1]). *Let X, $T_{\mathbb{Q}}$ and E be as above.*

(1) If E is a CM field or if the dimension of $T_{\mathbb{Q}}$ as an E-vector space is even, then there exist infinitely many places $\mathfrak{p}$ of good reduction for X such that $\rho(\overline{X}_{\mathfrak{p}}) = \rho(\overline{X})$.

[6] Recall a CM field K is a totally imaginary quadratic extension of a totally real number field.

(2) If E is a totally real field and the dimension of $T_{\mathbb{Q}}$ as an E-vector space is odd, and if $\mathfrak{p}$ is a finite place of good reduction for X of residue characteristic ≥ 5, then

$$\rho(\overline{X}_{\mathfrak{p}}) \geq \rho(\overline{X}) + [E : \mathbb{Q}].$$

Equality holds for infinitely many places of good reduction.

Theorem 3.25 gives a theoretical algorithm for computing the geometric Picard number of a K3 surface X defined over a number field, provided the Hodge conjecture for codimension 2 cycles holds for $X \times X$. The idea is to run three processes in parallel; see [16, Sect. 5] for details.

(1) Find divisors on $\overline{X}$ however you can (worst case scenario: start ploughing through Hilbert schemes of curves in the projective space where X is embedded and check whether the curves you see lie on $\overline{X}$). Use the intersection pairing to compute the rank of the span of the divisors you find. This will give a lower bound $\rho'(\overline{X})$ for $\rho(\overline{X})$.
(2) If the Hodge conjecture holds for $X \times X$, then elements of E are induced by codimension 2 cycles. Find codimension 2 cycles on $X \times X$ (again, worst case scenario one can use Hilbert schemes of surfaces on a projective space where $X \times X$ is embedded to look for surfaces that lie on $X \times X$). Use these cycles to compute the degree $[E : \mathbb{Q}]$.
(3) Systematically compute $\rho(\overline{X}_{\mathfrak{p}})$ at places of good reduction.

After a finite amount of computation, Theorem 3.25 guarantees we will have computed $\rho(\overline{X})$: Suppose that after a finite number of steps in the first process we have computed a lower bound $\rho'(\overline{X})$ that is sharp, i.e., $\rho'(\overline{X}) = \rho(\overline{X})$, but say we can't yet justify this equality. If E is a CM field or if the dimension of $T_{\mathbb{Q}}$ as an E-vector space is even, then Theorem 3.25 (1) guarantees that eventually $\rho'(\overline{X}) = \rho(\overline{X}_{\mathfrak{p}})$ for some prime $\mathfrak{p}$ of good reduction. The third process will allow us to conclude $\rho(\overline{X}) = \rho'(\overline{X})$ in this case. If E is a totally real field and the dimension of $T_{\mathbb{Q}}$ as an E-vector space is odd, then the second process allows us to compute $[E : \mathbb{Q}]$, and the third process will eventually give a prime $\mathfrak{p}$ of good reduction such that $\rho(\overline{X}_{\mathfrak{p}}) = \rho'(\overline{X}) + [E : \mathbb{Q}]$, proving that $\rho(\overline{X}) = \rho'(\overline{X})$ in this case as well, using Theorem 3.25 (2). Of course, we should keep running the first process in the meantime in case the lower bound $\rho'(\overline{X})$ is not yet sharp! But eventually it will be, and we will have computed $\rho(\overline{X})$.

This algorithm is not really practical, but it shows that the problem can be solved, in principle. Recent work of Poonen, Testa, and van Luijk shows that there is an *unconditional* algorithm to compute $\mathrm{NS}\,\overline{X}$, as a Galois module, for a K3 surface X defined over a finitely generated field of characteristic $\neq 2$ [86, Sect. 8]. For K3 surfaces of degree 2 over a number field, there is also work by Hassett, Kresch and Tschinkel on this problem [41].

4 Brauer Groups of K3 Surfaces

4.1 Generalities

References: [18, 19, 22, 98, 111]

Through this section, k denotes a number field. Call a smooth, projective geometrically integral variety over k a nice k-variety. Let X be a nice k-variety; is $X(k) \neq \emptyset$? There appears to be no algorithm that could answer this question in this level of generality.[7] On the other hand, the Lang–Nishimura Lemma[8] assures us that if X and Y are nice k-varieties, k-birational to each other, then

$$X(k) \neq \emptyset \iff Y(k) \neq \emptyset.$$

This suggests we narrow down the scope of the original question by fixing some k-birational invariants of X (like dimension). It also suggests we look at birational invariants of X that have some hope of capturing arithmetic. The Brauer group $\operatorname{Br} X := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ is precisely such an invariant [36, Corollaire 7.3].

Let k_v denote the completion of k at a place v of k. Since $k \hookrightarrow k_v$, an obvious necessary condition for $X(k) \neq \emptyset$ is $X(k_v) \neq \emptyset$ for all places v. Detecting if $X(k_v) \neq \emptyset$ is a relatively easy task, thanks to the Weil conjectures and Hensel's lemma (at least for finite places of good reduction and large enough residue field—see Sect. 5 of Viray's Arizona Winter School notes, for example [113]). That these weak necessary conditions are not sufficient has been known for decades [62, 88]; see [18] for a beautiful, historical introduction to this topic.

Let $\mathbb{A}_k$ denote the ring of adeles of k. A nice k-variety such that $X(\mathbb{A}_k) = \prod_v X(k_v) \neq \emptyset$ and $X(k) = \emptyset$ is called a counterexample to the Hasse principle.[9] In 1970 Manin observed that the Brauer group of a variety could be used to explain several of the known counterexamples to the Hasse principle. More precisely, for any subset $\mathcal{S} \subseteq \operatorname{Br} X$, Manin constructed an obstruction set $X(\mathbb{A}_k)^{\mathcal{S}}$ satisfying

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{S}} \subseteq X(\mathbb{A}_k),$$

and he observed that it was possible to have $X(\mathbb{A}_k) \neq \emptyset$, yet $X(\mathbb{A}_k)^{\mathcal{S}} = \emptyset$, and thus $X(k) = \emptyset$. Whenever this happens, we say there is a Brauer–Manin obstruction to the Hasse principle. We will not define the sets $X(\mathbb{A}_k)^{\mathcal{S}}$ here; the focus of these notes is on trying to write down, in a convenient way, the input necessary to compute the sets $X(\mathbb{A}_k)^{\mathcal{S}}$, namely elements of $\operatorname{Br} X$ expressed, for example, as central simple

[7] Hilbert's tenth problem over k asks for such an algorithm. The problem is open even for $k = \mathbb{Q}$, but it is known that no such algorithm exists for large subrings of $\mathbb{Q}$ [85].

[8] See [89, Proposition A.6] for a short proof of this result due to Kollár and Szabó.

[9] The equality $X(\mathbb{A}_k) = \prod_v X(k_v)$ follows from projectivity of X, because $X(\mathcal{O}_k) = X(k)$ in this case; here $\mathcal{O}_k$ denotes the ring of integers of k.

algebras over the function field $\mathbf{k}(X)$. For details on how to define $X(\mathbb{A}_k)^{\mathcal{S}}$, see [98, Sect. 5.2], [111, Sect. 3] and [20, 113].

4.2 Flavors of Brauer Elements

For a map of schemes $X \to Y$, étale cohomology furnishes a map of Brauer groups $\operatorname{Br} Y \to \operatorname{Br} X$; it also recovers Galois cohomology when $X = \operatorname{Spec} K$ for a field K. In fact,

$$\operatorname{Br}\operatorname{Spec}(K) = \mathrm{H}^2_{\text{ét}}(\operatorname{Spec} K, \mathbb{G}_m) \simeq \mathrm{H}^2\left(\operatorname{Gal}(\overline{K}/K), \overline{K}^{\times}\right) = \operatorname{Br} K,$$

where $\overline{K}$ is a separable closure of K, and $\operatorname{Br} K$ is the (cohomological) Brauer group of K.

For a nice k-variety X, write $\overline{X}$ for $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$, where $\bar{k}$ is a separable closure of k. There is a filtration of the Brauer group

$$\operatorname{Br}_0 X \subseteq \operatorname{Br}_1 X \subseteq \operatorname{Br} X,$$

where

$$\begin{aligned} \operatorname{Br}_0 X &:= \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X), && \text{arising from the structure morphism } X \to \operatorname{Spec} k, \text{ and} \\ \operatorname{Br}_1 X &:= \ker\left(\operatorname{Br} X \to \operatorname{Br} \overline{X}\right), && \text{arising from extension of scalars } \overline{X} \to X. \end{aligned}$$

Elements in $\operatorname{Br}_0 X$ are called constant; class field theory shows that if $\mathcal{S} \subseteq \operatorname{Br}_0 X$, then $X(\mathbf{A})^{\mathcal{S}} = X(\mathbf{A})$, so these elements cannot obstruct the Hasse principle. Elements in $\operatorname{Br}_1 X$ are called algebraic; the remaining elements of the Brauer group are transcendental.

The Leray spectral sequence for $X \to \operatorname{Spec} k$ and $\mathbb{G}_m$

$$E_2^{p,q} := \mathrm{H}^p\left(\operatorname{Gal}(\bar{k}/k), \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)\right) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

gives rise to an exact sequence of low-degree terms, which yields an isomorphism

$$\operatorname{Br}_1 X / \operatorname{Br}_0 X \xrightarrow{\sim} \mathrm{H}^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} \overline{X}). \tag{11}$$

Exercise 4.1 Fill in the necessary details to prove the map in (11) is indeed an isomorphism. You will need the vanishing of $\mathrm{H}^3(\operatorname{Gal}(\bar{k}/k), (\bar{k})^{\times})$ for a number field k, due to Tate; see [78, 8.3.11(iv)].

Roughly speaking, the isomorphism (11) tells us that the Galois action on $\operatorname{Pic} \overline{X}$ determines the algebraic part of the Brauer group. There are whole classes of varieties for which $\operatorname{Br} X = \operatorname{Br}_1 X$, e.g., curves [36, Corollaire 5.8] or rational varieties, by the birational invariance of the Brauer group and the following exercise.

Exercise 4.2 Show that $\operatorname{Br} \mathbb{P}^n_{\bar{k}} = 0$. Hint: use the Kummer sequence in étale cohomology to show that $\operatorname{Br} \mathbb{P}^n_{\bar{k}}[\ell] = 0$ for every prime ℓ, and the inclusion $\operatorname{Br} \mathbb{P}^n_{\bar{k}} \hookrightarrow \operatorname{Br} \mathbf{k}(\mathbb{P}^n_{\bar{k}})$ coming from the generic point of $\mathbb{P}^n_{\bar{k}}$ to see that $\operatorname{Br} \mathbb{P}^n_{\bar{k}}$ is torsion (see Sect. 4.3 below).

Exercise 4.3 Let X be a nice k-variety of dimension 2. Show that if the Kodaira dimension of X is negative then $\operatorname{Br} X = \operatorname{Br}_1 X$.

4.3 Computing Algebraic Brauer–Manin Obstructions

On a nice k-variety X with function field $\mathbf{k}(X)$, the inclusion $\operatorname{Spec} \mathbf{k}(X) \to X$ gives rise to a map $\operatorname{Br} X \to \operatorname{Br} \mathbf{k}(X)$ via functoriality of étale cohomology. This map is injective; see [75, Example III.2.22]. When trying to compute the obstruction sets $X(\mathbb{A}_k)^{\mathcal{S}}$, at least when $\mathcal{S} \subseteq \operatorname{Br}_1 X$, one often tries to compute the right hand side of (11); one then tries to invert the map (11) and embed $\operatorname{Br}_1(X)$ into $\operatorname{Br} \mathbf{k}(X)$, thus representing elements of $\operatorname{Br}_1 X$ as central simple algebras over $\mathbf{k}(X)$. This kind of representation is convenient for the computation of the obstruction sets $X(\mathbb{A}_k)^{\mathcal{S}}$. See, for example, [98, p. 145] and [20, 57, 58, 113] for some explicit calculations along these lines, and [57], [110, Sect. 3] and [111, Sect. 3.5] for ideas on how to invert the isomorphism (11).

4.4 Colliot–Thélène's Conjecture

Before moving on to K3 surfaces, we mention a conjecture of Colliot-Thélène [19], whose origins date back to work of Colliot-Thélène and Sansuc in the case of surfaces [21, Question k_1]. Recall a rationally connected variety Y over an algebraically closed field K is a smooth projective integral variety such that any two closed points lie in the image of some morphism $\mathbb{P}^1_K \to Y$. For surfaces, rational connectedness is equivalent to rationality.

Conjecture 4.4 (Colliot-Thélène). *Let X be a nice variety over a number field k. Suppose that X is geometrically rationally connected. Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset \implies X(k) \neq \emptyset$.*

Conjecture 4.4 remains wide open even for geometrically rational surfaces, including, for example, cubic surfaces. See Colliot–Thélène's Arizona Winter School notes [20] for more on this conjecture, including evidence for it and progress towards it.

4.5 Skorobogatov's Conjecture

Based on growing evidence [23, 38, 49, 101], Skorobogatov has put forth [99] the following conjecture.

Conjecture 4.5 (Skorobogatov). *Let X be a projective K3 surface over a number field k. Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset \implies X(k) \neq \emptyset$.*

Remark 4.6 The analogous conjecture for other surfaces of Kodaira dimension 0 is false: Skorobogatov has constructed counter examples of bi-elliptic surfaces for which $X(\mathbb{Q}) = \emptyset$ while $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br} X} \neq \emptyset$. Using [112] as a starting point, Balestrieri, Berg, Manes, Park and Viray constructed an Enriques surface over $\mathbb{Q}$ satisfying the analogous conclusion [6].

4.6 Transcendental Brauer Elements on K3 Surfaces: An Introduction

References: [48, 49, 51, 79, 87, 102, 103, 116]

We have seen that there are no transcendental elements of the Brauer group for curves and surfaces of negative Kodaira dimension. The first place we might see such elements is on surfaces of Kodaira dimension zero. K3 surfaces fit this profile. In fact, if X is an algebraic K3 surface over a number field, the group $\operatorname{Br} \overline{X}$ is quite large: there is an exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{22-\rho} \to \operatorname{Br} \overline{X} \to \bigoplus_{\ell \text{ prime}} \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\text{tors}} \to 0,$$

where $\rho = \rho(\overline{X})$ is the geometric Picard number of X; see [36, (8.7) and (8.9)]. Moreover, since X is a surface, [36, (8.10) and (8.11)] gives, for each prime ℓ, a perfect pairing of finite abelian groups

$$\left(\operatorname{Br} \overline{X}/(\mathbb{Q}/\mathbb{Z})^{22-\rho}\right)\{\ell\} \times \operatorname{NS} \overline{X}\{\ell\} \to \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $A\{\ell\}$ denotes the ℓ-primary torsion of A. Since $\operatorname{NS} \overline{X}$ is torsion-free (by Proposition 2.8 and the fact that $\operatorname{Num} \overline{X}$ is torsion free, essentially by definition), we conclude that $\operatorname{Br} \overline{X} \simeq (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. (Alternatively, one can embed $k \hookrightarrow \mathbb{C}$, and use the vanishing of the singular cohomology group $\mathrm{H}^3(X_{\mathbb{C}}, \mathbb{Z})$ and comparison Theorems [75, III.3.12].)

This result doesn't necessarily imply that $\operatorname{Br} X$ has infinitely many transcendental elements, because it's possible that most elements of $\operatorname{Br} \overline{X}$ might not descend to the ground field. This is indeed the case, as shown by the following remarkable theorem of Skorobogatov and Zarhin.

Theorem 4.7 ([102, Theorem 1.2]). *If X is an algebraic K3 surface over a number field k, then the group* $\operatorname{Br} X/\operatorname{Br}_0 X$ *is finite.* □

It is natural to ask what the possible isomorphism types of $\operatorname{Br} X/\operatorname{Br}_0 X$ are (or for that matter $\operatorname{Br} X/\operatorname{Br}_1 X$), at least at first as abstract abelian groups. A related question is: what prime numbers can divide the order of elements of $\operatorname{Br} X/\operatorname{Br}_0 X$? These kinds of questions have prompted much recent work on Brauer groups of K3 surfaces (e.g., [49, 51, 79, 103]), particularly on surfaces with high geometric Picard rank. Two recent striking results [49, 79] on the transcendental odd-torsion of the Brauer group are the following (for a finite abelian group A, write A_{odd} for its subgroup of odd order elements).

Theorem 4.8 ([49, 50]). *Let $X_{[a,b,c,d]}$ be a smooth quartic in $\mathbb{P}^3_{\mathbb{Q}}$ given by*

$$ax^4 + by^4 = cz^4 + dw^4.$$

Then

$$\left(\operatorname{Br} X_{[a,b,c,d]}/\operatorname{Br}_0 X_{[a,b,c,d]}\right)_{\mathrm{odd}} = (\operatorname{Br} \overline{X}_{[a,b,c,d]})_{\mathrm{odd}}^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \simeq \begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } -3abcd \in \langle -4\rangle \mathbb{Q}^{\times 4}, \\ \mathbb{Z}/5\mathbb{Z} & \text{if } 5^3abcd \in \langle -4\rangle \mathbb{Q}^{\times 4}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, transcendental elements of odd order on $X_{[a,b,c,d]}$ never obstruct the Hasse principle, but they can obstruct weak approximation.

This work builds on earlier work by Bright, Ieronymou, Skorobogatov, and Zarhin [11, 51, 103]. Curiously, transcendental elements of order 5 on surfaces of the form $X_{[a,b,c,d]}$ *always* obstruct weak approximation (density of $X(k)$ in $X(\mathbb{A}_k)$ for the product topology of the v-adic topologies); it is also possible for transcendental elements of order 3 to obstruct weak approximation. The first example of such an obstruction was found by Preu [87] on the surface $X_{[1,3,4,9]}$. See [50, Theorem 2.3] for precise conditions detailing when such obstructions arise.

Newton [79] has found a similar statement for K3 surfaces that are Kummer for the abelian surface $E \times E$, where E is an elliptic curve with complex multiplication.

Theorem 4.9 ([79]). *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by the full ring of integers of an imaginary quadratic field. Let X be the Kummer K3 surface associated to the abelian surface $E \times E$. Suppose that* $(\operatorname{Br} X/\operatorname{Br}_1 X)_{\mathrm{odd}} \neq 0$. *Then* $\operatorname{Br}_1 X = \operatorname{Br} \mathbb{Q}$ *and*

$$\operatorname{Br} X/\operatorname{Br} \mathbb{Q} \simeq \mathbb{Z}/3\mathbb{Z}.$$

Moreover $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br} X} \subsetneq X(\mathbb{A}_{\mathbb{Q}})$; consequently, there is always a Brauer–Manin obstruction to weak approximation on X.

The surfaces of Theorem 4.9 always have rational points by their construction, but it would be interesting to understand the situation for the Hasse principle on torsors for these surfaces; it seems likely that Newton's method will also show that the Hasse principle cannot be obstructed by odd order transcendental Brauer elements for such torsors.

So far, no collection of odd order elements of the Brauer group has been shown to obstruct the Hasse principle on a K3 surface.

Question 4.10 ([49]). Does there exist a K3 surface X over a number field k with $X(\mathbb{A}_k) \neq \emptyset$ such that $X(\mathbb{A}_k)^{(\operatorname{Br} X)_{\mathrm{odd}}} = \emptyset$?

As for transcendental Brauer elements of even order, Hassett and the author showed that they can indeed obstruct the Hasse principle on a K3 surface. We looked at the other end of the Néron-Severi spectrum, i.e., at K3 surfaces of geometric Picard rank one (in fact, we used the technology developed in Sect. 3 to compute Picard numbers!).

Theorem 4.11 ([43]). *Let X be a K3 surface of degree 2 over a number field k, with function field $\mathbf{k}(X)$, given as a sextic in the weighted projective space $\mathbb{P}(1, 1, 1, 3) = \operatorname{Proj} k[x, y, z, w]$ of the form*

$$w^2 = -\frac{1}{2} \cdot \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}, \tag{12}$$

where $A, \dots, F \in k[x, y, z]$ are homogeneous quadratic polynomials. Then the class $\mathscr{A}$ of the quaternion algebra $(B^2 - 4AD, A)$ in $\operatorname{Br}(\mathbf{k}(X))$ extends to an element of $\operatorname{Br}(X)$.

When $k = \mathbb{Q}$, there exist polynomials $A, \dots, F \in \mathbb{Z}[x, y, z]$ such that X has geometric Picard rank 1 and $\mathscr{A}$ gives rise to a transcendental Brauer–Manin obstruction to the Hasse principle on X.

For the second part of Theorem 4.11, one can take

$$\begin{aligned} A &= -7x^2 - 16xy + 16xz - 24y^2 + 8yz - 16z^2, \\ B &= 3x^2 + 2xz + 2y^2 - 4yz + 4z^2, \\ C &= 10x^2 + 4xy + 4xz + 4y^2 - 2yz + z^2, \\ D &= -16x^2 + 8xy - 23y^2 + 8yz - 40z^2, \\ E &= 4x^2 - 4xz + 11y^2 - 4yz + 6z^2, \\ F &= -40x^2 + 32xy - 40y^2 - 8yz - 23z^2. \end{aligned} \tag{13}$$

The reason to look at K3 surfaces with very low Picard rank is that these surfaces have little structure, e.g., they don't have elliptic fibrations or Kummer structures that one can use to construct or control transcendental Brauer elements [31, 37, 48, 49, 79, 87, 101, 116]. Our hope was to give a way to construct Brauer classes that did not depend on extra structure, that could be systematized for large classes of K3 surfaces. So far, we have been able to construct all the possible kinds of 2-torsion elements on K3 surfaces of degree 2 [43, 44, 72]; see Sect. 4.9 below.

Exercise 4.12 Let X be an algebraic K3 surface over $\mathbb{C}$. Prove that if $\rho(X) \geq 5$ then there is a map $\phi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ whose general fiber is a smooth curve of genus 1. Hint: use the Hasse-Minkowski theorem to show there is class $C \in \operatorname{Pic} X$ with $C^2 = 0$. Use the linear system of this class (or a similar class of square zero) to produce the desired fibration.

4.7 Transcendental Brauer Elements on K3 Surfaces: Hodge Theory

References: [13, 43, 44, 53, 72, 77, 100, 108]

The idea behind the construction of transcendental Brauer elements in [43, 44, 72] goes back to work of van Geemen [108], and is most easily explained using sheaf cohomology on complex K3 surfaces; most of this section can be properly rewritten using Kummer sequences for étale cohomology and comparison theorems, e.g., see [91, Proposition 1.3]. The analytic point of view is a little easier to digest.

Let X be a complex K3 surface. Let $\operatorname{Br}' X = \mathrm{H}^2(X, \mathscr{O}_X^\times)_{\mathrm{tors}}$. Since $\mathrm{H}^3(X, \mathbb{Z}) = 0$, the long exact sequence in sheaf cohomology associated to the exponential sequence gives

$$0 \to \mathrm{H}^2(X, \mathbb{Z})/c_1(\mathrm{NS}\, X) \to \mathrm{H}^2(X, \mathscr{O}_X) \to \mathrm{H}^2(X, \mathscr{O}_X^\times) \to 0$$

We apply the functor $\mathrm{Tor}_\bullet^{\mathbb{Z}}(\,\cdot\,, \mathbb{Q}/\mathbb{Z})$ to this short exact sequence of abelian groups. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}^2(X, \mathscr{O}_X), \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^2(X, \mathscr{O}_X)_{\mathrm{tors}} = 0$ and that $\mathrm{H}^2(X, \mathscr{O}_X) \otimes \mathbb{Q}/\mathbb{Z} = 0$ since $\mathbb{Q}/\mathbb{Z}$ is torsion and $\mathrm{H}^2(X, \mathscr{O}_X)$ is divisible. Hence

$$\operatorname{Br}' X \simeq \big(\mathrm{H}^2(X, \mathbb{Z})/c_1(\mathrm{NS}\, X)\big) \otimes \mathbb{Q}/\mathbb{Z}. \tag{14}$$

Let T_X be the orthogonal complement in $\mathrm{H}^2(X, \mathbb{Z})$ of $\mathrm{NS}\, X$ with respect to cup product. We call T_X the transcendental lattice of X. Write $T_X^\vee = \operatorname{Hom}(T_X, \mathbb{Z})$ for the dual of T_X.

Lemma 4.13 *The map*

$$\begin{aligned} \phi\colon \mathrm{H}^2(X, \mathbb{Z})/c_1(\mathrm{NS}\, X) &\to T_X^\vee \\ v + \mathrm{NS}\, X &\mapsto [t \mapsto \langle v, t\rangle] \end{aligned}$$

is an isomorphism of lattices.

Proof First, observe that both $\mathrm{NS}\, X$ and T_X are primitive sublattices of $\mathrm{H}^2(X, \mathbb{Z})$: for the former lattice, note that $\mathrm{H}^2(X, \mathbb{Z})/c_1(\mathrm{NS}\, X)$ injects into $\mathrm{H}^2(X, \mathscr{O}_X)$, which is torsion-free, and that c_1 is an injective map, because $\mathrm{H}^1(X, \mathscr{O}_X) = 0$, by definition of a K3 surface. For the latter, use Exercise 2.26(1).

Since NS X is a primitive sublattice of $\mathrm{H}^2(X, \mathbb{Z})$, we have $T_X^{\perp} = \mathrm{NS}\, X$, by Exercise 2.26(2). Injectivity of the map ϕ follows: if $\phi(v + \mathrm{NS}\, X) = 0$, then $v \in T_X^{\perp} = \mathrm{NS}\, X$, so $v + \mathrm{NS}\, X$ is the trivial class in $\mathrm{H}^2(X, \mathbb{Z})/c_1(\mathrm{NS}\, X)$.

Consider the short exact sequence of abelian groups

$$0 \to T_X \to \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})/T_X \to 0.$$

Apply the functor $\mathrm{Ext}^{\bullet}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z})$. Since $\mathrm{H}^2(X, \mathbb{Z})/T_X$ is torsion free, we have

$$\mathrm{Ext}^1_{\mathbb{Z}}\left(\mathrm{H}^2(X, \mathbb{Z})/T_X, \mathbb{Z}\right) = 0$$

so the natural map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^2(X, \mathbb{Z}), \mathbb{Z}) \to T_X^{\vee}$$

is surjective. Since $\mathrm{H}^2(X, \mathbb{Z})$ is unimodular, and hence self dual, this means that every element of $T_X^{\vee}$ has the form $v \mapsto \langle \lambda, v \rangle$ for some $\lambda \in \mathrm{H}^2(X, \mathbb{Z})$. This gives surjectivity of ϕ. □

Proposition 4.14 *Let X be a complex K3 surface. There are isomorphisms of abelian groups*

$$\mathrm{Br}\, X \simeq T_X^{\vee} \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{Hom}_{\mathbb{Z}}(T_X, \mathbb{Q}/\mathbb{Z}).$$

Proof This follows from (14) and Lemma 4.13. □

Informally, Proposition 4.14 tells us there are bijections

$$\begin{aligned} & \{\text{cyclic subgroups of } \mathrm{Br}'\, X \text{ of order } n\} \\ \overset{1-1}{\longleftrightarrow}\; & \{\text{surjections } T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}\} \\ \overset{1-1}{\longleftrightarrow}\; & \{\text{sublattices } \Gamma \subseteq T_X \text{ of index } n \text{ with cyclic quotient and generator}\} \end{aligned} \tag{15}$$

where the last bijection comes from

($\longrightarrow$) taking the kernel of the surjection $T_X \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$.
($\longleftarrow$) taking the cokernel of the inclusion $\Gamma \subseteq T_X$.

In what follows, we will focus on the case where $n = p$ is a prime number, in which case (15) tells us that subgroups of order p of $\mathrm{Br}'\, X$ are in one-to-one correspondence with sublattices of index p of T_X. Since we are working over a ground field that is already algebraically closed, this discussion asserts that sublattices of T_X contain information about the transcendental classes of K3 surfaces!

4.8 First Examples: Work of van Geemen [108, Sect. 9]

Let's implement the above idea in the simplest possible case. Consider an complex algebraic K3 surface X with $\operatorname{NS} X \simeq \mathbb{Z}h$, $h^2 = 2$. We will study sublattices of index 2 in T_X, up to isometry, corresponding by (15) to elements of order 2 in $\operatorname{Br}' X$.

First, a primitive embedding

$$\operatorname{NS} X = \langle h \rangle \hookrightarrow \Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

exists by Theorem 2.27. Let $\{e, f\}$ be a basis for the first summand of Λ_{K3} equal to the hyperbolic plane U, with intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

A primitive embedding $\langle h \rangle \hookrightarrow \Lambda_{\mathrm{K3}}$ is also unique up to isometry by [80, Theorem 1.14.4], so we may assume that $h = e + f$. Let $v = e - f$; we have $v^2 = -2$, $\langle h, v \rangle = 0$, and

$$T_X \simeq \langle v \rangle \oplus \Lambda', \qquad \text{where } \Lambda' = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}.$$

The lattice Λ' is unimodular (hence equal to its dual lattice), so every $\phi \in \operatorname{Hom}(\Lambda', \mathbb{Z})$ is of the form

$$\phi_\lambda \colon \Lambda' \to \mathbb{Z}, \qquad v \mapsto \langle v, \lambda \rangle.$$

for some $\lambda \in \Lambda'$. In other words, the map

$$\Lambda' \to \operatorname{Hom}(\Lambda', \mathbb{Z}), \qquad \lambda \mapsto \phi_\lambda$$

is an isomorphism. Tensoring with $\mathbb{Z}/2\mathbb{Z}$ we get an isomorphism

$$\Lambda'/2\Lambda' \to \operatorname{Hom}(\Lambda', \mathbb{Z}/2\mathbb{Z}), \qquad \lambda + 2\Lambda' \mapsto \phi_\lambda \otimes \operatorname{id}_{\mathbb{Z}/2\mathbb{Z}}$$

Hence, a surjection $T_X \to \mathbb{Z}/2\mathbb{Z}$ has the form

$$\begin{aligned} \alpha \colon T_X &\to \mathbb{Z}/2\mathbb{Z} \\ nv + \lambda' &\mapsto a_\alpha n + \langle \lambda', \lambda_\alpha \rangle \bmod 2, \end{aligned} \tag{16}$$

for some $\lambda_\alpha \in \Lambda'$, determined only up to an element of $2\Lambda'$, and some $a_\alpha \in \{0, 1\}$. We classify these surjections by studying their kernels (see (15)). These kernels are lattices which, by Theorem 2.25, are determined up to isomorphism by their rank, signature, and discriminant quadratic forms. Recall that the discriminant quadratic form of a lattice $(L, \langle\ ,\ \rangle)$ is

$$q_L : L^\vee/L \to \mathbb{Q}/2\mathbb{Z} \qquad x + L \mapsto \langle x, x\rangle \bmod 2\mathbb{Z}.$$

Proposition 4.15 ([108, Proposition 9.2]). *Let X be a complex algebraic K3 surface with* NS $X \simeq \mathbb{Z}h$*,* $h^2 = 2$*. Let* $\alpha\colon T_X \to \mathbb{Z}/2\mathbb{Z}$ *be a surjective map as above, and put* $\Gamma_\alpha = \ker\alpha$*. Then*

(1) If $a_\alpha = 0$ *then* $\Gamma_\alpha^\vee/\Gamma_\alpha \simeq (\mathbb{Z}/2\mathbb{Z})^3$*. There are* $2^{20} - 1$ *such lattices* Γ_α*, all isomorphic to each other.*
(2) If $a_\alpha = 1$ *then* $\Gamma_\alpha^\vee/\Gamma_\alpha \simeq \mathbb{Z}/8\mathbb{Z}$*. There are* 2^{20} *such lattices* Γ_α*, sorted out into two isomorphism classes by their discriminant forms as follows:*
  (a) The even class*, where* $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle \equiv 0 \bmod 2$*. There are* $2^9(2^{10} + 1)$ *such lattices.*
  (b) The odd class*, where* $\frac{1}{2}\langle\lambda_\alpha, \lambda_\alpha\rangle \equiv 1 \bmod 2$*. There are* $2^9(2^{10} - 1)$ *such lattices.*

Proof In all cases, the order of the discriminant group $\Gamma_\alpha^\vee/\Gamma_\alpha$ is $\operatorname{disc}(\Gamma_\alpha) = 2^2 \operatorname{disc}(T_X) = 8$, because Γ_α has index 2 in T_X. If $a_\alpha = 0$, then Γ_α has an orthogonal direct sum decomposition

$$\Gamma_\alpha = \langle v\rangle \oplus (\Gamma_\alpha \cap \Lambda'),$$

and we obtain a decomposition of the discriminant group

$$\Gamma_\alpha^\vee/\Gamma_\alpha = \langle v\rangle^\vee/\langle v\rangle \oplus (\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda') \simeq \mathbb{Z}/2\mathbb{Z} \oplus (\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda').$$

The discriminant group $(\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda')$ has order 4. Let $\mu \in \Lambda'$ satisfy $\langle\mu, \lambda_\alpha\rangle = 1$. One verifies that $\{\lambda/2, \mu\}$ generates a subgroup of order 4 in $(\Gamma_\alpha \cap \Lambda')^\vee/(\Gamma_\alpha \cap \Lambda')$, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (do this!). The discriminant quadratic form is also determined up to isometry (check this!), so all the lattices Γ_α with $a_\alpha = 0$ are isometric. There are $2^{20} - 1$ choices for λ_α, parametrized by elements in $\Lambda'/2\Lambda'$, except for the zero vector, which would give $\Gamma_\alpha = T_X$.

For the case $a_\alpha = 1$, we check that $w := \frac{1}{4}(-v + 2\lambda_\alpha)$ is in $\Gamma_\alpha^\vee$. The vector $4w$ is not in Γ_α (it is in T_X, but it is not in the kernel of the map α), but $8w \in \Gamma_\alpha$, so w has order 8 in the discriminant group, which is therefore isomorphic to $\mathbb{Z}/8\mathbb{Z}$. The discriminant form q_α of Γ_α is determined by its value on w, which is

$$q(w) = \langle w, w\rangle = \frac{-2 + 4\langle\lambda_\alpha, \lambda_\alpha\rangle}{16} = \frac{-1 + 2\langle\lambda_\alpha, \lambda_\alpha\rangle}{8} \bmod 2\mathbb{Z}$$

Two lattices Γ_α and $\Gamma_{\alpha'}$ of this form, with discriminant groups generated by w and w', respectively, are therefore equivalent if and only if there exists an integer x such that $q_\alpha(xw) = q_{\alpha'}(w')$. In other words, if and only if

$$x^2 \cdot \frac{-1 + 2\langle\lambda_\alpha, \lambda_\alpha\rangle}{8} \equiv \frac{-1 + 2\langle\lambda_{\alpha'}, \lambda_{\alpha'}\rangle}{8} \bmod 2\mathbb{Z}$$

On the other hand, a vector λ_α is determined only up to elements of $2\Lambda'$ and thus can always be modified (check!) to satisfy $\langle \lambda_\alpha, \lambda_\alpha \rangle = 0$ or 2; we assume a normalization like this. If $\langle \lambda_\alpha, \lambda_\alpha \rangle = \langle \lambda_{\alpha'}, \lambda_{\alpha'} \rangle$, then $x = 1$ will show two lattices are isomorphic. If $\langle \lambda_\alpha, \lambda_\alpha \rangle \neq \langle \lambda_{\alpha'}, \lambda_{\alpha'} \rangle$, then we are looking for an integer x such that

$$x^2 \cdot \frac{-1}{8} \equiv \frac{-1+4}{8} \bmod 2\mathbb{Z}$$

i.e., for an integer x such that $x^2 \equiv 13 \bmod 16$. No such integer exists. We conclude there are two isomorphism classes of lattices Γ_α with $a_\alpha = 1$, depending on the parity of $\frac{1}{2}\langle \lambda_\alpha, \lambda_\alpha \rangle$, as claimed. The count of the number of lattices of each type is left as an exercise. □

Exercise 4.16 Formulate and prove the analogue of Proposition 4.15 for complex algebraic K3 surfaces with NS $X \simeq \mathbb{Z}h$, $h^2 = 2d$ (see [72]). Can you do the case when NS $X \simeq U$? Such K3 surfaces are endowed with elliptic fibrations (see Exercise 4.12). What about the case when $\rho(X) = 19$?

4.9 From Lattices to Geometry

Proposition 4.15 is nice, but how are we supposed to extract central simple algebras over the function field of a complex K3 surface from it? The hope here is that the lattices Γ_α of Proposition 4.15 are themselves isomorphic to a piece of the cohomology of a *different algebraic variety*, and that the isomorphism is really a shadow of some geometric correspondence that could shed light on the mysterious transcendental Brauer classes.

For example, in the notation of Sect. 4.8, an obvious sublattice of index 2 of $T_X = \langle v \rangle \oplus \Lambda'$ is $\Gamma := \langle 2v \rangle \oplus \Lambda'$. This lattice is in the even class of Propoposition 4.15(2). Note that $\omega_X \in T_X \otimes \mathbb{C}$, so $\omega_X \in \Gamma \otimes \mathbb{C}$ as well. If we can re-embed Γ *primitively* in Λ_{K3}, say by a map $\iota\colon \Gamma \hookrightarrow \Lambda_{K3}$, then $\iota_{\mathbb{C}}(\omega_X)$ will give a period point in the period domain $\boldsymbol{\Omega}$, and by the surjectivity of the period map (Theorem 2.24) there will exist a K3 surface Y with[10] $\omega_Y = \iota_{\mathbb{C}}(\omega_X)$ and $T_Y \simeq \iota(\Gamma)$. Discriminant and rank considerations imply that NS $Y \simeq \mathbb{Z}h'$, $h'^2 = 8$, i.e., Y is a K3 surface of degree 8, with Picard rank 1.

Exercise 4.17 Show that there is indeed a primitive embedding $\iota\colon \Gamma \hookrightarrow \Lambda_{K3}$. Hint: what would $\iota(\Gamma)^\perp$ have to look like as a lattice (including its discriminant form)? Could you apply Theorem 2.27 and [80, Corollary 1.14.4] to this orthogonal complement instead?

Our discussion suggests there is a correspondence, up to isomorphism, between pairs (X, α) consisting of a K3 surface X of degree 2 and Picard rank 1 together

[10] Note the importance of primitivity of $\iota\colon \Gamma \hookrightarrow \Lambda_{K3}$ here: T_Y must be a primitive sublattice of $\mathrm{H}^2(Y, \mathbb{Z})$; see the proof of Lemma 4.13.

with an even class $\alpha \in \mathrm{Br}' X$, and K3 surfaces of degree 8 and Picard rank 1. This is indeed the case; Mukai had already observed this in [77, Example 0.9]. Starting with a K3 surface Y of degree 8 with $\mathrm{NS}\, Y \simeq \mathbb{Z}h'$, Mukai notes that the moduli space of stable sheaves E (with respect to h') of rank 2, determinant algebraically equivalent to h', and Euler characteristic 4, is birational to a K3 surface X of degree 2. The moduli space is in general not fine, and the obstruction to the existence of a universal sheaf is an element $\alpha \in \mathrm{Br}' X[2]$. See [13, 72] for accounts of this phenomenon. Let $\pi_X \colon X \times Y \to X$ be the projection onto the first factor. In modern lingo, any $\pi_X^{-1}\alpha$-twisted universal sheaf on $X \times Y$ induces a Fourier-Mukai equivalence of bounded derived categories $\mathrm{D^b}(X, \alpha) \simeq \mathrm{D^b}(Y)$.

Before we explain a more geometric approach to the correspondence $(X, \alpha) \longleftrightarrow Y$, we pause to identify the varieties encoded by the remaining isomorphism classes of lattices from Proposition 4.15.

Proposition 4.18 *Let X be a complex algebraic K3 surface with* $\mathrm{NS}\, X \simeq \mathbb{Z}h$, $h^2 = 2$. *Let Γ_α be the kernel of a surjection $\alpha \colon T_X \to \mathbb{Z}/2\mathbb{Z}$. Let $\Gamma_\alpha(-1)$ denote the lattice Γ_α with its bilinear form scaled by -1.*

(1) If $\Gamma_\alpha^\vee/\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^3$, then there is an isometry

$$\Gamma_\alpha(-1) \simeq \langle h_1^2, h_1h_2, h_2^2\rangle^\perp \subseteq \mathrm{H}^4(Y, \mathbb{Z}),$$

where $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ is a double cover branched along a smooth divisor of type $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ and h_i is the pullback of $\mathscr{O}_{\mathbb{P}^2}(1)$ along the projection $\pi_i \colon Y \to \mathbb{P}^2$ for $i = 1, 2$.

(2) If $\Gamma_\alpha^\vee/\Gamma \simeq (\mathbb{Z}/8\mathbb{Z})$, then

(a) if Γ_α belongs to the even class, then there is an isometry

$$\Gamma_\alpha \simeq T_Y \subseteq \mathrm{H}^2(Y, \mathbb{Z}),$$

where T_Y is the transcendental lattice of a K3 surface of degree 8.

(b) if Γ_α belongs to the odd class, then there is an isometry

$$\Gamma_\alpha(-1) \simeq \langle H^2, P\rangle^\perp \subseteq \mathrm{H}^4(Y, \mathbb{Z}),$$

where $Y \subseteq \mathbb{P}^5$ is a cubic fourfold containing a plane P, with hyperplane section H.

Proof We have discussed the case (2)(a). However, all the statements can be deduced from Theorem 2.25 (see also [108, Sects. 9.6–9.8]). For example, let $Y \subseteq \mathbb{P}^5$ be a cubic fourfold, and write H for a hyperplane section of Y. By the Hodge–Riemann relations, the lattice $\mathrm{H}^4(Y, \mathbb{Z})$ has signature $(21, 2)$; it is unimodular by Poincaré duality, and it is odd (i.e. not even), because $\langle H^2, H^2\rangle = 3$. By the analogue of Theorem 2.13 for odd indefinite unimodular lattices [95, Sect. V.2.2], we have

$\mathrm{H}^4(Y,\mathbb{Z}) \simeq \langle +1\rangle^{\oplus 21} \oplus \langle -1\rangle^{\oplus 2}$ If Y contains a plane P, then the Gram matrix for $\langle H^2, P\rangle$ is

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

(see [40, Sect. 4.1] for the calculation of $\langle P, P\rangle$.). One checks that the rank, signature and discriminant form of $\langle H^2, P\rangle^{\perp}$ matches that of Γ_α. Applying Theorem 2.25 finishes the proof in this case. The other cases are left as exercises. □

Exercise 4.19 Let $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ be a double cover branched along a smooth divisor of type (2, 2) in $\mathbb{P}^2 \times \mathbb{P}^2$.

(1) Compute the structure of the lattice $\mathrm{H}^4(Y,\mathbb{Z})$.
(2) For $i = 1, 2$, let h_i be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ along the projection $\pi_i \colon Y \to \mathbb{P}^2$. Compute the Gram matrix of the lattice $\langle h_1^2, h_1h_2, h_2^2\rangle$.
(3) Compute the rank, signature and discriminant quadratic form of $\langle h_1^2, h_1h_2, h_2^2\rangle^{\perp}$. Use this to establish Proposition 4.18(1).

Remark 4.20 The connection between cubic fourfolds containing a plane and K3 surfaces of degree 2 goes back at least to Voisin's proof of the Torelli theorem for cubic fourfolds [114]. See also Hassett's work on this subject [40]. Fans of derived categories should consult [65].

The proof of Proposition 4.18 might make it seem like a numerical coincidence, but the discussion of the case (2)(a) before the Proposition suggests something deeper is going on. Let us describe the geometry that connects a pair (X, α) to the auxiliary variety Y.

Theorem 4.21 *Let Y be either*

(1) a K3 surface of degree 8 *with* NS $Y \simeq \mathbb{Z}$*, or,*
(2) a smooth cubic fourfold containing a plane P such that $\mathrm{H}^4(Y,\mathbb{Z})_{\mathrm{alg}} \simeq \langle H^2, P\rangle$*, where H denotes a hyperplane section, or*
(3) a smooth double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ *branched over a smooth divisor of type* (2, 2) *such that* $\mathrm{H}^4(Y,\mathbb{Z})_{\mathrm{alg}} \simeq \langle h_1^2, h_1h_2, h_2^2\rangle$*, where h_1, h_2 are the respective pullbacks to Y of* $\mathcal{O}_{\mathbb{P}^2}(1)$ *along the two projections* $\pi_1, \pi_2 \colon Y \to \mathbb{P}^2$.

Then there is a quadric fibration $\pi \colon Y' \to \mathbb{P}^2$ *associated to Y such that, for general Y, the discriminant locus* $\Delta \subseteq \mathbb{P}^2$ *of* π *is a smooth curve of degree* 6*, and the Stein factorization for the relative variety of maximal isotropic subspaces* $\mathcal{W} \to \mathbb{P}^2$ *has the form*

$$\mathcal{W} \to X \to \mathbb{P}^2,$$

where X is a double cover of $\mathbb{P}^2$ *branched along* Δ*, and* $\mathcal{W} \to X$ *is a smooth* $\mathbb{P}^n$*-bundle for the analytic topology for some* $n \in \{1, 3\}$.

So there it is! The surface X is a K3 surface of degree 2, and $\mathcal{W} \to X$ is a Severi-Brauer bundle representing a class $\alpha \in \mathrm{Br}' X[2]$. The bundle $\mathcal{W} \to X$ can be turned

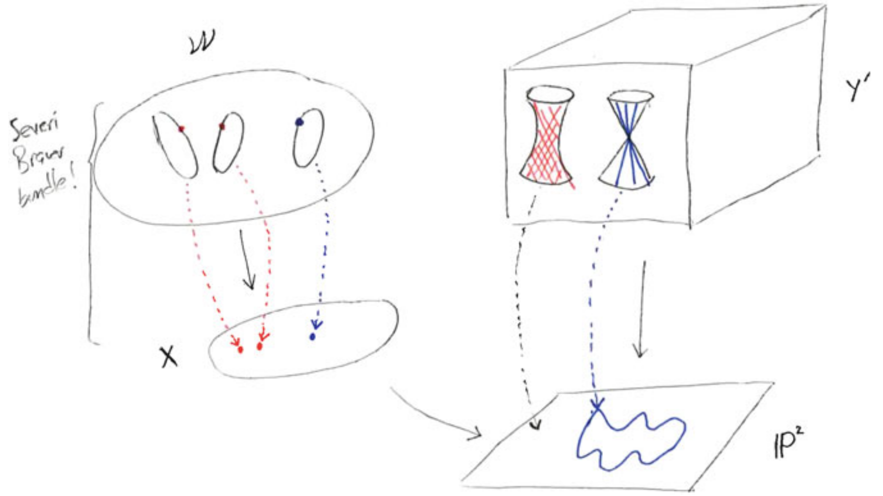

Fig. 1 Pictorial representation of Theorem 4.21. Each point of $\mathcal{W}$ represents a linear subspace of maximal dimension in a fiber of the quadric bundle $Y' \to \mathbb{P}^2$

into a central simple algebra over the function field $\mathbf{k}(X)$ that is suitable for the computation of Brauer–Manin obstructions; see [43, 44, 72] for details. Figure 1 illustrates this idea.

Proof of Theorem 4.21 We explain how to construct the quadric bundles $Y' \to \mathbb{P}^2$. The rest of the theorem can be deduced from [44, Proposition 3.3]; see also [44, Theorem 5.1] in the case of cubic fourfolds, [43, Theorem 3.2] for double covers of $\mathbb{P}^2 \times \mathbb{P}^2$, and [72, Lemmas 13 and 14] for K3 surfaces of degree 8.

If Y is a K3 surface of degree 8 with $\operatorname{NS} Y \simeq \mathbb{Z}$, then it is a complete intersection of three quadrics $V(Q_0, Q_1, Q_2)$ in $\mathbb{P}^5 = \operatorname{Proj} \mathbb{C}[x_0, \dots, x_5]$; see [8, Chap. VIII, Exercise 11] or [52, Proposition 3.8]. There is a net of quadrics

$$Y' = \left\{([x, y, z], [x_0, \dots, x_5]) \in \mathbb{P}^2 \times \mathbb{P}^5 : xQ_0 + yQ_1 + zQ_2 = 0\right\} \subseteq \mathbb{P}^2 \times \mathbb{P}^5,$$

and the projection to the first factor gives the desired bundle of quadrics $Y' \to \mathbb{P}^2$. For a general K3 surface Y, the singular fibers of $Y' \to \mathbb{P}^2$ will have rank 5, and thus the discriminant locus on $\mathbb{P}^2$ will be a smooth sextic curve.

If Y is a smooth cubic fourfold containing a plane P, then blowing up and projecting away from P gives a fibration into quadrics $Y' \to \mathbb{P}^2$. The discriminant locus on $\mathbb{P}^2$ where the fibers of the map drop rank is smooth already because Y does not contain another plane intersecting P along a line [114, §Lemme 2], by hypothesis.

Finally, if $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ is a double cover branched along a type $(2, 2)$-divisor, then the projections $\pi_i \colon Y \to \mathbb{P}^2$ give fibrations into quadrics. Smoothness of the discriminant loci is discussed in [43, Lemma 3.1]. □

Remark 4.22 If Y is defined over a number field, then so is the output data $\mathcal{W} \to \mathbb{P}^2$ of the above construction. This gives a way of writing down transcendental Brauer classes on X defined over a number field(!), provided one uses Y as the starting data. The difficulty here is that one might like to use X as the starting data (over a number field), and compute all the possible Y over number fields that fit into the above recipe.

Remark 4.23 The results developed in [53, 100] contain as special cases extensions of Proposition 4.18 and Theorem 4.21 to K3 surfaces of degree 2 without restrictions on their Néron-Severi groups.

5 Uniform Boundedness and K3 Surfaces: Some Questions

Let X be a K3 surface over a number field k. In this section, we return to the question of possible orders of the finite quotient $|\operatorname{Br} X/\operatorname{Br}_0 X|$, and connect this question to the geometric correspondences we saw in Theorem 4.21. There is a strong analogy between torsion points on elliptic curves over number fields, and nonconstant Brauer classes of K3 surfaces over number fields. We start by exploring this idea: the analogy suggests it is conceivable that if one fixes just the right amount of data, e.g., a geometric lattice polarization, then there are only finitely many possibilities for $|\operatorname{Br} X/\operatorname{Br}_0 X|$.

5.1 Torsion Subgroups of Elliptic Curves

Let E be an elliptic curve over a number field k. By the Mordell-Weil theorem, the group $E(k)$ is finitely generated and abelian. Hence

$$E(k) \cong E(k)_{\text{tors}} \times \mathbb{Z}^r,$$

for some nonnegative integer r. In a 1966 survey paper, Cassels asserts it is a folklore conjecture that there are only finitely many possibilities for $E(k)_{\text{tors}}$ [14, Sect. 22]. Shortly thereafter, Manin showed that for each prime p there is a uniform bound on the p-primary torsion of elliptic curves over k:

Theorem 5.1 ([67]). *Let k be a number field; fix a prime p. There is a constant $c := c(k, p)$ such that $|E(k)_{\text{tors}}| < c(k, p)$ for all elliptic curves E/k.*

Manin proved that the modular curve $X_1(p^r)$, which has high genus for all $r \gg 0$, has only finitely many k-points—before Faltings' theorem was known! Shortly thereafter, Ogg gave a precise conjecture for the possible orders of torsion points on elliptic curves over $\mathbb{Q}$ [83, Conjecture 1]. In a spectacular breakthrough, Mazur proved this conjecture, and classified all possibilities for $E(\mathbb{Q})_{\text{tors}}$.

Theorem 5.2 ([71, Theorem 8]). *Let $E/\mathbb{Q}$ be an elliptic curve. Then $E(\mathbb{Q})_{\mathrm{tors}}$ is isomorphic to one of the following 15 groups:*

$$\mathbb{Z}/n\mathbb{Z} \quad \textit{for } 1 \le n \le 10 \textit{ or } n = 12, \qquad \textit{or} \qquad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \quad \textit{for } 1 \le n \le 4.$$

In fact, Mazur showed that the only rational points of the modular curve $X_1(N)$ are the rational cusps if $N = 11$ or $N \ge 13$. After subsequent work establishing (strong) uniform boundedness of torsion over more classes of number fields [54, 55], Merel showed that in fact $\#E(k)_{\mathrm{tors}}$ could be bounded by a constant depending only on the *degree* of k:

Theorem 5.3 ([73]). *Fix $d \ge 1$. There is a constant $c := c(d)$ such that $|E(k)_{\mathrm{tors}}| < c$ for all elliptic curves E over a number field k for which $[k : \mathbb{Q}] = d$.*

5.2 From Torsion on Elliptic Curves to Brauer Groups of K3 Surfaces

Is there a Mazur/Merel Theorem for K3 surfaces? At first glance, this question makes no sense. K3 surfaces have no group structure: what would torsion subgroup even mean? Perhaps we can reinterpret the group $E(k)_{\mathrm{tors}}$ of an elliptic curve in such a way that it does not depend on the group structure of E, and then look for an analogue on K3 surfaces:

$$\begin{aligned} E(k)_{\mathrm{tors}} &\simeq (\operatorname{Pic}^0 E)_{\mathrm{tors}} && \text{by [96, III.3.4], taking Galois invariants,} \\ &\simeq (\operatorname{Pic} E)_{\mathrm{tors}} && \text{because only degree 0 line bundles are torsion,} \\ &\simeq \mathrm{H}^1(E, \mathscr{O}_E^\times)_{\mathrm{tors}} && \text{[39, Exercise III.4.5],} \\ &\simeq \mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}} && \text{[75, III, Proposition 4.9],} \\ &\simeq \mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) && \text{by Hilbert's Theorem 90.} \end{aligned}$$

The quotient $\mathrm{H}^1_{\text{ét}}(E, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m)$ makes no reference to the group structure of E, and so it is defined for more general varieties. For a K3 surface X/k, we might thus consider the quotient

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\mathrm{tors}}/\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_m) = \operatorname{Br} X/\operatorname{Br}_0 X.$$

Theorem 4.7 guarantees that $\operatorname{Br} X/\operatorname{Br}_0 X$ is finite!

5.3 Moduli Spaces

Understanding the arithmetic of the modular curves $X_0(N)$ and $X_1(N)$ is essential in proving Theorems 5.2 and 5.3. We should expect that defining and understanding moduli spaces for K3 surfaces with level structures coming from the Brauer group will be crucial in investigating uniform boundedness problems for Brauer groups on K3 surfaces. As with modular curves, one can start by studying the geometry of these spaces when defined as complex analytic varieties.

In this context, for example, Proposition 4.15 should have the following interpretation: let $\mathcal{K}_2^o$ denote the locus of the coarse moduli space of complex K3 surfaces of degree 2 whose points correspond to K3 surfaces of Picard rank 1; see [35, Sect. 2.5] for a definition of this space. Then the locus of the (to be defined) moduli space $\mathcal{Y}_0(2,2)$ parametrizing pairs $(X, \langle\alpha\rangle)$, where X is a K3 surface of degree 2 and $0 \neq \alpha \in (\operatorname{Br} X)[2]$, such that $\rho(X) = 1$ has three components. Each component maps dominantly onto $\mathcal{K}_2^o$ via the forget map, with finite degree equal to the number of lattices in the corresponding isomorphism class of Proposition 4.15. Proposition 4.18 identifies each of these three components in turn as moduli spaces of other varieties, and Theorem 4.21 details geometric correspondences realizing the isomorphisms between the moduli spaces of objects in Proposition 4.18 and the components of $\mathcal{Y}_0(2,2)$. Compare this with the discussion in Sect. 4.9.

The lattice-theoretic calculations of [72] show that if $p \nmid 2d$, then the analogous moduli space $\mathcal{Y}_0(2d, p)$ parametrizing pairs $(X, \langle\alpha\rangle)$, where X is a K3 surface of degree $2d$ and $0 \neq \alpha \in (\operatorname{Br} X)[p]$, has three components. One of these components can be identified, á la Mukai, with the moduli space $\mathcal{K}_{2dp^2}$ of K3 surfaces of degree $2dp^2$, and if $d = 1$ and $p \equiv 2 \bmod 3$, then another component is isomorphic to the moduli space $\mathcal{C}_{2p^2}$ of special cubic fourfolds of discriminant $2p^2$. Both $\mathcal{K}_{2dp^2}$ and $\mathcal{C}_{2p^2}$ are varieties of general type for $p \geq 11$ [34, 104]. This leads us to propose the following challenge:

Challenge 5.4 Does there exist a K3 surface $X/\mathbb{Q}$ of degree 2 with $\rho(\overline{X}) = 1$, such that $(\operatorname{Br} X/\operatorname{Br}_0 X)[11] \neq 0$?

The above discussion is admittedly informal, but it should be possible to use ideas of Rizov [90] to make it precise and arithmetic.

5.4 Uniform Boundedness

We conclude by stating optimistic conjectures about Brauer groups of K3 surfaces over number fields suggested by the above discussion.

Conjecture 5.5 (Uniform boundedness). *Fix a number field k and a primitive lattice $L \hookrightarrow \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let X be a K3 surface over k such that* $\operatorname{NS} \overline{X} \simeq L$. *Then there is a constant $c(K, L)$, independent of X, such that*

$$|\operatorname{Br} X/\operatorname{Br}_0 X| < c(k, L).$$

Conjecture 5.6 (Strong uniform boundedness). *Fix a positive integer n and a primitive lattice $L \hookrightarrow \Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let X be a K3 surface over a number* *field k of degree n such that* $\mathrm{NS}\,\overline{X} \simeq L$*. Then there is a constant* $c(n, L)$*, independent of X such that*

$$|\operatorname{Br} X / \operatorname{Br}_0 X| < c(n, L).$$

If, for some lattice L, Conjecture 5.5 is verified with an effectively computable constant $c(k, L)$, then [59, Theorem 1] would imply that the obstruction set $X(\mathbb{A}_k)^{\operatorname{Br} X}$ is effectively computable for the corresponding surfaces. Skorobogatov's Conjecture 4.5 would then imply there is an effective way to determine if $X(k) \neq \emptyset$ for these K3 surfaces.

The relevant moduli spaces with level structures whose rational points would shed light on Conjectures 5.5 and 5.6, have dimension $20 - r$, where $r = \operatorname{rk} L$. These spaces tend to have trivial Albanese varieties (one can use the techniques of [56] to see this); thus, determining the qualitative arithmetic of these spaces is a difficult problem for small values of r. However, special cases of these conjectures may be accessible, e.g., by taking specific L with $r = 19$ or 20, where the moduli spaces to be studied have dimension ≤ 1. This is the subject of upcoming joint work with Bianca Viray. More optimistically, recent work of the author with Dan Abramovich [2, 3] gives "proofs-of-concept" for similar questions on abelian varieties, conditional on Lang's Conjecture and Vojta's Conjecture, respectively. These strong conjectures allow us to control the arithmetic of high-dimensional moduli spaces with level structures. It is our hope that once an arithmetic theory of moduli spaces of K3 surfaces with Brauer level structures is firmly in place, one may obtain similar conditional results strengthening the plausibility of Conjectures 5.5 and 5.6.

6 Epilogue: Results from the Arizona Winter School

We report on the work of three project groups that began at the Arizona Winter School.

6.1 Picard Groups of Degree Two K3 Surfaces

Using the techniques presented in Sect. 3 as a starting point, Bouyer, Costa, Festi, Nicholls, and West [10] have computed not only the geometric Picard rank, but the full Galois module structure for general members of the family of degree 2 K3 surfaces given by

$$X/\mathbb{Q}: \qquad w^2 = ax^6 + by^6 + cz^6 + dx^2y^2z^2.$$

Over $\overline{\mathbb{Q}}$, we can assume that $a = b = c = 1$; for general d, the authors showed that $\rho(\overline{X}) = 19$. Using explicit generators for $\mathrm{NS}(\overline{X})$, the authors are able to compute the Galois cohomology groups $\mathrm{H}^i(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{NS}(\overline{X}))$ for $0 \leq i \leq 2$, and hence compute the algebraic Brauer groups $\mathrm{Br}_1 X/\mathrm{Br}_0 X$ of this family; see Sect. 4.2. The case $d = 0$, where $\rho(\overline{X}) = 20$ is also studied in Nakahara's upcoming Ph. D. thesis.

6.2 Rational Points and Derived Equivalence

Ascher, Dasaratha, Perry, and Zong constructed remarkable further examples of the kind appearing in Theorem 4.11 which showed that, over $\mathbb{Q}$, $\mathbb{Q}_2$ and $\mathbb{R}$, the existence of rational points on K3 surfaces need not be preserved by twisted derived equivalences ([4]). This result stands in sharp contrast with the untwisted derived equivalence over finite fields and p-adic fields; see [46, 61] and [42, Corollary 35].

6.3 Effective Bounds for Brauer Groups of Kummer Surfaces

Let A be a principally polarized abelian surface over a number field k, and let X be the associated Kummer surface. Building on ideas in [102], Cantoral Farfán, Tang, Tanimoto, and Visse ([32]) showed there is an effectively computable constant M, depending on the Faltings' height of A and $\mathrm{NS}(\overline{A})$, such that $|\,\mathrm{Br}\, X/\mathrm{Br}_1 X| < M$. By [59, Theorem 1], it follows that the Brauer–Manin set $X(\mathbf{A})^{\mathrm{Br}\, X}$ for these surfaces is effectively computable. Their work also yields *practical* algorithms for computing the quotient $\mathrm{Br}_1 X/\mathrm{Br}_0 X$ when $\rho(\overline{A}) = 1$ or 2.

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Odd-Dimensional Cohomology with Finite Coefficients and Roots of Unity

Yuri G. Zarhin

Abstract We prove that the triviality of the Galois action on the suitably twisted *odd-dimensional* étale cohomology group with finite coefficients of an absolutely irreducible smooth projective variety implies the existence of certain primitive roots of unity in the field of definition of the variety. This text was inspired by an exercise in Serre's Lectures on the Mordell–Weil theorem.

1 Introduction

We recall some basic facts about *cyclotomic characters*. Let K be a field, $\bar{K}$ its algebraic closure, $\mathrm{G}_K = \mathrm{Aut}(\bar{K}/K)$ the absolute Galois group of K. Let n be a positive integer that is *not* divisible by $\mathrm{char}(K)$. We write $\mu_n \subset \bar{K}$ for the cyclic multiplicative group of nth roots of unity in $\bar{K}$. We write

$$\bar{\chi}_n : \mathrm{G}_K \to \mathrm{Aut}(\mu_n) = (\mathbb{Z}/n\mathbb{Z})^*$$

for the cyclotomic character that defines the Galois action on nth roots of unity. Clearly, $\mu_n \subset K$ if and only if

$$\bar{\chi}_n(g) = 1 \ \forall g \in \mathrm{G}_K.$$

Recall that the order of $(\mathbb{Z}/n\mathbb{Z})^*$ is $\phi(n)$ where ϕ is the *Euler function*. This implies that

$$\bar{\chi}_n^{\phi(n)}(g) = 1 \ \forall g \in \mathrm{G}_K.$$

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Let $K(\mu_n) \subset \bar{K}$ be the nth cyclotomic extension of K. Then the degree $[K(\mu_n) : K]$ of the (abelian) field extension $K(\mu_n)/K$ coincides with the order of the finite commutative Galois group $\mathrm{Gal}(K(\mu_n)/K)$ of this extension. By definition of $\bar{\chi}_n$, its kernel coincides with $\mathrm{G}_K/G_{K(\mu_n)}$ and $\bar{\chi}_n$ is the composition of the surjection

$$\mathrm{G}_K \mapsto \mathrm{G}_K/\mathrm{Gal}(\bar{K}/K(\mu_n)) = \mathrm{Gal}(K(\mu_n)/K)$$

and the embedding

$$\mathrm{Gal}(K(\mu_n)/K) \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^*,$$

which we continue to denote by $\bar{\chi}_n$, slightly abusing notation.

Remark 1.1 Clearly, the *exponent* $\exp(n, K)$ of $\mathrm{Gal}(K(\mu_n)/K)$ divides the order of $\mathrm{Gal}(K(\mu_n)/K)$, which, in turn, divides $\phi(n)$. In addition, if f is an integer then the character $\bar{\chi}_n^f$ is *trivial* if and only if f is divisible by $\exp(n, K)$. In particular, the character $\bar{\chi}_n^{\exp(n,K)}$ is *trivial*. On the other hand, if the degree of the extension $K(\mu_n)/K$ is *even* then so is $\exp(n, K)$; this implies that if f is an *odd* integer then the character $\bar{\chi}_n^f$ is *nontrivial*.

Remark 1.2 If m is (another) positive integer that is relatively prime to n and $\mathrm{char}(K)$, then the map

$$\mu_n \times \mu_m \to \mu_{nm}, \ (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$$

is an isomorphism of groups (and even Galois modules). The natural map

$$\phi_{n,m} : \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \quad c + nm\mathbb{Z} \mapsto (c + n\mathbb{Z}, c + m\mathbb{Z})$$

is a ring homomorphism and the group homomorphism

$$\bar{\chi}_{nm} : \mathrm{G}_K \to (\mathbb{Z}/nm\mathbb{Z})^*$$

coincides with

$$g \mapsto (\bar{\chi}_n(g), \bar{\chi}_m(g)) \in (\mathbb{Z}/n\mathbb{Z})^* \times (\mathbb{Z}/m\mathbb{Z})^* \xrightarrow{\phi_{n,m}^{-1}} (\mathbb{Z}/nm\mathbb{Z})^*.$$

If A is an abelian variety over K then we write $A[n]$ for the kernel of multiplication by n in $A(\bar{K})$. It is well known that $A[n]$ is a finite Galois submodule of $A(\bar{K})$. If we forget about the Galois action then $A[n]$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim(A)$.

The following assertion is stated without proof, as an exercise, in Serre's Lectures on the Mordell–Weil Theorem [7, Sect. 4.6, p. 55].

Theorem 1.3 *If* $\dim(A) > 0$ *and* $A[n] \subset A(K)$ *then* $\mu_n \subset K$.

Proof First, it suffices to check the case when $n = \ell^r$ is a power of a prime $\ell \neq \mathrm{char}(K)$.

Second, if A^t is the dual of A then let us take a K-polarization $\lambda : A \to A^t$ of smallest possible degree. Then λ is not divisible by ℓ, i.e., $\ker(\lambda)$ does not contain the whole $A[\ell]$. Otherwise, divide λ by ℓ to get a K-polarization of lower degree. Thus the image $\lambda(A[\ell^r]) \subset A^t[\ell^r]$ contains a point of exact order ℓ^r, say Q. Otherwise,

$$\lambda(A[\ell^r]) \subset A^t[\ell^{r-1}]$$

and therefore $A[\ell] = \ell^{r-1}A[\ell^r]$ lies in the kernel of λ, which is not the case.

Since $A[\ell^r] \subset A[K]$ and λ is defined over K, the image $\lambda(A[\ell^r])$ lies in $A^t(K)$. In particular, Q is a K-rational point on A^t.

Third, there is a *nondegenerate* Galois-equivariant Weil pairing [5]

$$e_n : A[\ell^r] \times A^t[\ell^r] \to \mu_{\ell^r}.$$

I claim that there is a point $P \in A[\ell^r]$ such that $e_n(P, Q)$ is a primitive ℓ^rth root of unity. Indeed, otherwise

$$e_n(A[\ell^r], Q) \subset \mu_{\ell^{r-1}}$$

so that the nonzero point $\ell^{r-1}Q$ is orthogonal to the whole $A[\ell^r]$ with respect to e_n, which contradicts the nondegeneracy of e_n.

Thus, $\gamma := e_n(P, Q)$ is a primitive ℓ^rth root of unity that lies in K, because both P and Q are K-points. Since μ_{ℓ^r} is generated by γ, $\mu_{\ell^r} \subset K$. □

The aim of this paper is to a prove a variant of Serre's exercise that deals with the Galois action on the twisted odd-dimensional étale cohomology group with finite coefficients of a smooth projective variety (see Theorem 1.6 below). Our proof is based on the Hard Lefschetz Theorem [2] and the unimodularity of Poincaré duality [10].

1.4 If Λ is a commutative ring with 1 and without zero divisors and M is a Λ-module, then we write M_{tors} for its torsion submodule and M/tors for the quotient M/M_{tors}. Usually, we will use this notation when Λ is the ring $\mathbb{Z}_\ell$ of ℓ-adic integers.

If ℓ is a prime different from $\mathrm{char}(K)$ then we write $\mathbb{Z}_\ell(1)$ for the projective limit of the cyclic Galois modules μ_{ℓ^r} with ℓth power as transition map. It is known that $\mathbb{Z}_\ell(1)$ is a free $\mathbb{Z}_\ell$-module of rank 1 with natural continuous action of G_K defined by the *cyclotomic character*

$$\chi_\ell : \mathrm{G}_K \to \mathrm{Aut}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell^*.$$

There are canonical isomorphisms

$$\mathbb{Z}_\ell/\ell^r\mathbb{Z}_\ell = \mathbb{Z}/\ell^r\mathbb{Z},\ \ \mathbb{Z}_\ell(1)/\ell^r\mathbb{Z}_\ell(1) = \mu_{\ell^r};$$

in addition

$$\chi_\ell \bmod \ell^r = \bar{\chi}_{\ell^r}$$

for all positive integers r.

We write $\mathbb{Q}_\ell(1)$ for the one-dimensional $\mathbb{Q}_\ell$-vector space

$$\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

provided with the natural Galois action that is defined by the character χ_ℓ. For each integer a we will need the ath tensor power $\mathbb{Q}_\ell(a) := \mathbb{Q}_\ell(1)^{\otimes a}$, which is a one-dimensional $\mathbb{Q}_\ell$-vector space provided with the Galois action that is defined by the character χ_ℓ^a.

Let X be an absolutely irreducible smooth projective variety over K of positive dimension $d = \dim(X)$. We write $\bar{X}$ for the irreducible smooth projective d-dimensional variety $X \times_K \bar{K}$ over $\bar{K}$. Let ℓ be a prime $\neq \mathrm{char}(K)$ and a an integer. If i is a nonnegative integer then we write $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ for the corresponding (twisted) ith étale ℓ-adic cohomology group. Recall that all the étale cohomology groups $H^i(\bar{X}, \mu_n^{\otimes a})$ are finite $\mathbb{Z}/n\mathbb{Z}$-modules and that the $\mathbb{Z}_\ell$-modules $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ are finitely generated. In particular, each $H^i(\bar{X}, \mathbb{Z}_\ell(a))/\mathrm{tors}$ is a free $\mathbb{Z}_\ell$-module of finite rank. These finiteness results are fundamental finiteness theorems in étale cohomology from **SGA 4, 4$\frac{1}{2}$, 5**, see [3] and [4, pp. 22–24] for precise references. All these groups are provided with the natural linear continuous actions of G_K. We also consider the corresponding finite-dimensional $\mathbb{Q}_\ell$-vector spaces

$$H^i(\bar{X}, \mathbb{Q}_\ell(a)) = H^i(\bar{X}, \mathbb{Z}_\ell(a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Galois action on $H^i(\bar{X}, \mathbb{Z}_\ell(a))$ extends by $\mathbb{Q}_\ell$-linearity to $H^i(\bar{X}, \mathbb{Q}_\ell(a))$. There are natural isomorphisms of G_K-modules

$$H^i(\bar{X}, \mathbb{Q}_\ell(a+b)) = H^i(\bar{X}, \mathbb{Q}_\ell(a)) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(b)$$

for all integers a and b.

Remark 1.5 If a positive integer m is relatively prime to n and $\mathrm{char}(K)$, then the splitting $\mu_{nm} = \mu_n \times \mu_m$ induces the splitting of Galois modules

$$H^i(\bar{X}, {\mu_{nm}}^{\otimes a}) = H^i(\bar{X}, {\mu_n}^{\otimes a}) \oplus H^i(\bar{X}, {\mu_m}^{\otimes a}).$$

The $\mathbb{Q}_\ell$-dimension of $H^i(\bar{X}, \mathbb{Q}_\ell(a))$ is denoted by $\mathbf{b}_i(\bar{X})$ and called the ith *Betti number* of $\bar{X}$: it does not depend on a choice of (a and) ℓ. In characteristic zero it follows from the comparison theorem between classical and étale cohomology [6]. In finite characteristic the independence follows from results of Deligne [1]. It is also known that $\mathbf{b}_i(\bar{X}) = 0$ if $i > 2d$ [3, 4].

Our main result is the following statement.

Theorem 1.6 *Let i be a nonnegative integer.*

(i) Suppose that $i \le d-1$ *and* $\mathbf{b}_{2i+1}(\bar{X}) \neq 0$. *If the Galois action on* $H^{2i+1}(\bar{X}, {\mu_n}^{\otimes i})$ *is trivial then* $\mu_n \subset K$.
(ii) Suppose that $1 \le i \le d$ *and* $\mathbf{b}_{2i-1}(\bar{X}) \neq 0$. *If the Galois action on* $H^{2i-1}(\bar{X}, {\mu_n}^{\otimes i})$ *is trivial then* $\mu_n \subset K$.

Example 1.7 Let us take $i = 1$. Then *Kummer theory* tells us that

$$H^{2i-1}(\bar{X}, {\mu_n}^{\otimes i}) = H^1(\bar{X}, \mu_n) = \mathrm{Pic}(\bar{X})[n]$$

is the kernel of multiplication by n in the Picard group $\mathrm{Pic}(\bar{X})$ of $\bar{X}$. On the other hand if B is an abelian variety over K that is the Picard variety of X [5] then $\dim(B) = \mathbf{b}_1(\bar{X})$ and $B[n]$ is a Galois submodule of $H^1(\bar{X}, \mu_n)$. If we know that the Galois action on $H^1(\bar{X}, \mu_n)$ is trivial then the same is true for its submodule $B[n]$. Now if $\mathbf{b}_1(\bar{X}) \neq 0$ then $B \neq \{0\}$ and Theorem 1.3 applied to B implies that $\mu_n \subset K$.

Theorem 1.6 may be viewed as a special case (when $a = \frac{j \pm 1}{2}$) of the following statement.

Theorem 1.8 *Let* j *be a nonnegative integer and* $\mathbf{b}_j(\bar{X}) \neq 0$. *Let* a *be an integer. Assume that the Galois action on* $H^j(\bar{X}, \mu_n^{\otimes a})$. *is trivial. Then*

$$\bar{\chi} n^{2a-j}(g) = 1 \; \forall g \in G = \mathrm{G}_K.$$

If, in addition, $2a - j$ *is relatively prime to* $\phi(n)$ *then* $\mu_n \subset K$

Corollary 1.9 [Corollary to Theorem 1.8]. *Let* K *be a field,* n *a positive integer prime to* $\mathrm{char}(K)$. *Suppose that* K *does not contain a primitive* n*th root of unity. Suppose that* j *is an odd positive integer. Let* a *be an integer such that* $2a - j$ *is relatively prime to* $\phi(n)$. *Then for each absolutely irreducible smooth projective variety* X *over* K *with* $\mathbf{b}_j(\bar{X}) \neq 0$ *the Galois group* G_K *acts nontrivially on* $H^j(\bar{X}, \mu_n^{\otimes a})$

The next assertion covers (in particular) the case of *quadratic* $\bar{\chi}_n$ (e.g., when K is the maximal real subfield $\mathbb{Q}(\mu_n)^+$ of the nth cyclotomic field $\mathbb{Q}(\mu_n)$ of $\mathbb{Q}$).

Theorem 1.10 *Let* K *be a field,* n *a positive integer prime to* $\mathrm{char}(K)$. *Suppose that the degree* $[K(\mu_n) : K]$ *is even. (E.g.,* $K(\mu_n)/K$ *is a quadratic extension.) Then for each positive odd integer* j*, each integer* a *and every absolutely irreducible smooth projective variety* X *over* K *with* $\mathbf{b}_j(\bar{X}) \neq 0$ *the Galois group* G_K *acts nontrivially on* $H^j(\bar{X}, \mu_n^{\otimes a})$.

Remark 1.11 The special case of Theorem 1.10 when $\bar{\chi}_n$ is a quadratic character follows directly from Theorem 1.6, because in this case the Galois module $H^j(\bar{X}, \mu_n^{\otimes a})$ is isomorphic either to $H^j(\bar{X}, \mu_n^{\otimes [(j+1)/2]})$ or to $H^j(\bar{X}, \mu_n^{\otimes [(j-1)/2]})$.

The paper is organized as follows. Section 2 contains auxiliary results about pairings between finitely generated modules over discrete valuation rings. We use them in Sect. 3, in order to prove Theorems 1.8, 1.6 and 1.10.

2 Linear Algebra

This section contains auxiliary results that will be used in the next section in order to prove main results of the paper.

2.1 Let E be a discrete valuation field, $\Lambda \subset E$ the corresponding discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $\pi \in \mathfrak{m}$ be an uniformizer, i.e., $\mathfrak{m} = \pi\Lambda$.

If U is a finitely generated Λ-module then we write U_E for the corresponding (finite-dimensional) E-vector space $U \otimes_\Lambda E$. The kernel of the homomorphism of Λ-modules

$$\otimes 1 : U \to U \otimes_\Lambda E = U_E,\ x \mapsto x \otimes 1$$

coincides with U_{tors} while the image

$$\tilde{U} := \otimes 1(U) \subset U_E$$

is a Λ-lattice in V_E of (maximal) rank $\dim_E(U_E)$.

Let G be a group and

$$\chi : G \to \Lambda^* \subset E^*$$

is a homomorphism of G to the group Λ^* of invertible elements of Λ. If H is a nonzero finite-dimensional vector space over E and

$$\rho : G \to \mathrm{Aut}_E(H)$$

is a E-linear representation of G in H then H becomes a module over the group algebra $E[G]$ of G over E. Then

$$\rho \otimes \chi : G \to \mathrm{Aut}_E(H),\ \rho \otimes \chi(g) = \chi(g)\rho(g)\ \forall g \in G$$

is also a linear representation of G in H. We denote the corresponding $E[G]$-module by $H(\chi)$ and call it the *twist* of H by χ. Notice that H and $H(\chi)$ coincide as E-vector spaces. It is also clear that if T is a Λ-lattice in H then it is G-stable in $H(\chi)$ if and only if it is G-stable in (the $E[G]$-module) H. On the other hand, let L be a *one-dimensional* E-vector space provided with a structure of G-module defined by

$$gz := \chi(g)z\ \forall g \in G, z \in L.$$

Then the G-modules $H(\chi)$ and $H \otimes_E L$ are isomorphic (noncanonically).

Lemma 2.2 *Suppose that H_1 and H_2 are nonzero finite-dimensional E-vector spaces and*

$$\rho_1 : G \to \mathrm{Aut}_E(H_1),\ \rho_2 : G \to \mathrm{Aut}_E(H_2)$$

are isomorphic E-linear representations of G. Suppose that T_1 is a G-stable Λ-lattice in H_1 of rank $\dim_E(H_1)$ *and T_2 is a G-stable Λ-lattice in H_2 of rank* $\dim_E(H_2)$. *Then* *there is an isomorphism of* $E[G]$*-modules* $u : H_1 \to H_2$ *such that*

$$u(T_1) \subset T_2,\ u(T_1) \not\subset \pi \cdot T_2.$$

Proof Clearly,

$$H_2 = \bigcup_{j=1}^{\infty} \pi^{-j} \cdot T_2,\ \bigcap_{j=1}^{\infty} \pi^{j} \cdot T_2 = \{0\}.$$

Let $u_0 : H_1 \cong H_2$ be an isomorphism of $E[G]$-modules. Since H_1 is a finitely generated Λ-module, there exists an integer j such that $\pi^{-j} \cdot u_0(T_1) \subset T_2$. Let us take the smallest j that enjoys this property and put $u = \pi^{-j} u_0$. □

Theorem 2.3 *Suppose that* U *and* V *are finitely generated* Λ*-modules provided with group homomorphisms*

$$G \to \mathrm{Aut}_\Lambda(U),\ G \to \mathrm{Aut}_\Lambda(V).$$

Let us assume that $U/\mathrm{tors} \neq \{0\}$*, i.e., rank of* U *is positive.*

Suppose that we are given a Λ*-bilinear pairing*

$$e : U \times V \to \Lambda$$

that enjoys the following properties.

(i)

$$e(gx, gy) = \chi(g) \cdot e(x, y)\ \forall g \in G; x \in U, y \in V.$$

(ii) The Λ*-bilinear pairing*

$$U/\mathrm{tors} \times V/\mathrm{tors} \to \Lambda$$

induced by e is perfect *(unimodular).*

(iii) The $E[G]$*-modules* U_E *and* V_E *are isomorphic.*

Let r *be a positive integer such that the induced* G*-action on* $U/\pi^r U$ *is trivial, i.e.,*

$$x - gx \in \pi^r U\ \forall g \in G, x \in U.$$

Then

$$\chi(g) \bmod \pi^r \Lambda = 1 \in \Lambda/\pi^r \Lambda\ \forall g \in G.$$

Proof Clearly,

$$e(U_{\mathrm{tors}}, V) = \{0\} = e(U, V_{\mathrm{tors}}).$$

It is also clear that U_{tors} is a G-submodule of U and V_{tors} is a G-submodule of V. It is also clear that the G-module $[U/\mathrm{tors}]/\pi^r[U/\mathrm{tors}]$ is isomorphic to a quotient of the G-module $U/\pi^r U$. In particular, the G-action on $[U/\text{tors}]/[\pi^r U/\text{tors}]$ is (also) trivial. In the notation of Sect. 2.1, the natural homomorphisms

$$U/\text{tors} = U/U_{\text{tors}} \to \tilde{U},\ x + U_{\text{tors}} \mapsto x \otimes 1,\ V/\text{tors} = V/V_{\text{tors}} \to \tilde{V},\ x + V_{\text{tors}} \mapsto x \otimes 1$$

are G-equivariant isomorphisms of free Λ-modules of finite rank

$$U/\text{tors} \cong \tilde{U},\ \ V/\text{tors} \cong \tilde{V}$$

where $\tilde{U}$ and $\tilde{V}$ are G-stable lattices of maximal rank in U_E and V_E respectively. This implies that the G-action on $\tilde{U}/\pi^r\tilde{U}$ and e induces a Λ-bilinear *perfect* pairing

$$\tilde{e} : \tilde{U} \times \tilde{V} \to \Lambda$$

such that

$$\tilde{e}(gx, gy) = \chi(g) \cdot \tilde{e}(x, y)\ \forall g \in G;\ x \in \tilde{U},\ y \in \tilde{V}.$$

Applying Lemma 2.2 to the isomorphic $E[G]$-modules U_E and V_E, we obtain a "nicer" isomorphism of $E[G]$-modules $u : U_E \cong V_E$ such that

$$u(T_1) \subset T_2,\ \ u(T_1) \not\subset \pi T_2.$$

Let us pick $x_0 \in T_1$ with $y := u(x_0) \notin \pi T_2$. Since $x_0 \bmod \pi^r T_1 \in T_1/\pi^r T_1$ is G-invariant, its image

$$u(x) \bmod \pi^r T_2 = y \bmod \pi^r T_2 \in T_2/\pi^r T_2$$

is also G-invariant. Since y is *not* divisible in T_2, the Λ-submodule $\Lambda \cdot y$ is a direct summand of T_2. Since the pairing $\tilde{e}$ between T_1 and T_2 is perfect, there is $x \in T_1$ with $e(x, y) = 1$. This implies that

$$\chi(g) = \chi(g) \cdot 1 = \chi(g) \cdot \tilde{e}(x, y) = \tilde{e}(gx, gy),$$

i.e.,

$$\chi(g) = \tilde{e}(gx, gy)\ \forall g \in G.$$

On the other hand, since

$$x - gx \in \pi^r T_1,\ \ y - gy \in \pi^r T_2,$$

we have

$$\tilde{e}(gx, gy) - \tilde{e}(x, y) \in \pi^r \Lambda\ \forall g \in G.$$

This means that

$$\chi(g) - 1 = \tilde{e}(gx, gy) - \tilde{e}(x, y) \in \pi^r \Lambda \ \forall g \in G$$

and we are done. □

The next statement is a useful variant of Theorem 2.3 that deals with *twisted* representations.

Theorem 2.4 *Suppose that U and V are finitely generated Λ-modules provided with group homomorphisms*

$$G \to \mathrm{Aut}_\Lambda(U), \ G \to \mathrm{Aut}_\Lambda(V).$$

Assume that $U/\mathrm{tors} \neq \{0\}$, i.e., the rank of U is positive.

Suppose that we have a Λ-bilinear pairing

$$e : U \times V \to \Lambda$$

that enjoys the following properties.

(i)
$$e(gx, gy) = e(x, y) \ \forall g \in G; x \in U, y \in V.$$

(ii) The Λ-bilinear pairing
$$U/\mathrm{tors} \times V/\mathrm{tors} \to \Lambda$$
induced by e is perfect *(unimodular).*

(iii) The $E[G]$-modules U_E and $V_E(\chi)$ are isomorphic.

Let r be a positive integer such that the induced G-action on $U/\pi^r U$ is trivial, i.e.,

$$x - gx \in \pi^r U \ \forall g \in G, x \in U.$$

Then
$$\chi(g) \bmod \pi^r \Lambda = 1 \in \Lambda/\pi^r \Lambda \ \forall g \in G.$$

Proof Let
$$\rho_U : G \to \mathrm{Aut}_\Lambda(U), \ \rho_V : G \to \mathrm{Aut}_\Lambda(V)$$

be the structure homomorphisms that define the actions of G on U and V respectively. In this notation,

$$e(\rho_U(g)x, \rho_V(g)y) = e(x, y) \ \forall g \in G; x \in U, y \in V.$$

Let us twist ρ_V by considering the group homomorphism

$$\rho_{V(\chi)} : G \to \mathrm{Aut}_\Lambda(V), \ g \mapsto \chi(g)\rho(g).$$

We denote the resulting G-module by $V(\chi)$ and call it the *twist* of V by χ. Notice that V coincides with $V(\chi)$ as Λ-module. On the other hand, the $E[G]$-module $V(\chi)_E$ is canonically isomorphic to $V_E(\chi)$. The pairing e defines the Λ-bilinear pairing

$$e_\chi : U \times V(\chi) \to \Lambda,\ e_\chi(x, y) := e(x, y)\ \forall x \in U, y \in V = V(\chi)$$

of G-modules U and $V(\chi)$, which satisfies

$$e_\chi(\rho_U(g)x, \rho_{V(\chi)}(g)y) = e(\rho_U(g)x, \chi(g)\rho_V(g)y) = \chi(g)e(\rho_U(g)x, \rho_V(g)y) =$$

$$\chi(g)e(x, y) = \chi(g)e_\chi(x, y)\ \forall g \in G; x \in U, y \in V(\chi).$$

This implies that

$$e_\chi(\rho_U(g)x, \rho_{V(\chi)}(g)y) = \chi(g)e_\chi(x, y)\ \forall g \in G; x \in U, y \in V(\chi).$$

Now the result follows from Theorem 2.3 applied to U, $V(\chi)$ and e_χ. □

3 Proofs of Main Results

Let ℓ be a prime different from $\mathrm{char}(K)$ and r a positive integer. Let us put

$$E = \mathbb{Q}_\ell, \Lambda = \mathbb{Z}_\ell, \pi = \ell, G = \mathrm{G}_K.$$

We keep the notation and assumptions of Sect. 1.4. Recall that $d = \dim(X) \geq 1$.

Proposition 3.1 *Let j be a nonnegative integer with $j \leq 2d$ and $\mathbf{b}_j(\bar{X}) \neq 0$. Let a be an integer. Assume that the Galois action on $H^j(\bar{X}, \mu_{\ell^r}{}^{\otimes a})$ is trivial. Then*

$$\bar{\chi}_{\ell^r}^{2a-j}(g) = 1\ \forall g \in G = \mathrm{G}_K.$$

Proof Let us put $U := H^j(\bar{X}, \mathbb{Z}_\ell(a))$: it is provided with the natural structure of $G = \mathrm{G}_K$-module. Then the universal coefficients theorem [6, Chap. V, Sect. 1, Lemma 1.11] gives us a canonical G_K-equivariant embedding

$$U/\ell^r U = H^j(\bar{X}, \mathbb{Z}_\ell(a))/\ell^r H^j(\bar{X}, \mathbb{Z}_\ell(a)) \hookrightarrow H^j(\bar{X}, \mu_n{}^{\otimes a}).$$

Since the Galois action on $H^j(\bar{X}, \mu_n{}^{\otimes a})$ is trivial, it is also trivial on $U/\ell^r U$. We have (in the notation of Sect. 2.1)

$$U_E = H^j(\bar{X}, \mathbb{Z}_\ell(a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^j(\bar{X}, \mathbb{Q}_\ell(a)).$$

Let $V := H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a))$: it has the natural structure of $G = \mathrm{G}_K$-module and

$$V_E = H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a)).$$

The cup product pairing gives rise to a $\mathbb{Z}_\ell$-bilinear G_K-invariant pairing known as *Poincaré duality* ([6, Chap. VI, Sect. 11, Cor. 11.2 on p. 276], [4, p. 23], [3, Chap. II, Sect. 1])

$$e : H^j(\bar{X}, \mathbb{Z}_\ell(a)) \times H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a)) \to H^{2d}(\bar{X}, \mathbb{Z}_\ell(d)) \cong \mathbb{Z}_\ell.$$

It is known [10] that the induced pairing of free $\mathbb{Z}_\ell$-modules of finite rank

$$e : H^j(\bar{X}, \mathbb{Z}_\ell(a))/\mathrm{tors} \times H^{2d-j}(\bar{X}, \mathbb{Z}_\ell(d-a))/\mathrm{tors} \to \mathbb{Z}_\ell$$

is *perfect and unimodular*.

Let us choose an invertible very ample sheaf $\mathcal{L}$ on X and let

$$h \in H^2(\bar{X}, \mathbb{Q}_\ell(1))^{\mathrm{G}K} \subset H^2(\bar{X}, \mathbb{Q}_\ell(1))$$

be its first ℓ-adic Chern class. If $j \leq d$ then the Hard Lefschetz Theorem ([2], [3, Chap. IV, Sect. 5, pp. 274–275]) tells us that cup multiplication by $(d-j)$th power of h establishes an isomorphism between $\mathbb{Q}_\ell$-vector spaces $H^j(\bar{X}, \mathbb{Q}_\ell(a))$ and $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$. On the other hand, if $d \geq j$ then cup multiplication by the $(j-d)$th power of h establishes an isomorphism between the $\mathbb{Q}_\ell$-vector spaces $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$ and $H^j(\bar{X}, \mathbb{Q}_\ell(a))$. In both cases the Galois-invariance of h implies that the $\mathbb{Q}_\ell$-vector spaces $U_E = H^j(\bar{X}, \mathbb{Q}_\ell(a))$ and $H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j))$ are isomorphic as G_K-modules. On the other hand, the $\mathrm{G}K$-module

$$H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(a+d-j)) = H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a+2a-j)) =$$

$$H^{2d-j}(\bar{X}, \mathbb{Q}_\ell(d-a)) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(2a-j) \cong V_E(\chi)$$

where

$$\chi := \chi_\ell^{2a-j} : G = \mathrm{G}K \to \mathbb{Z}_\ell^* = \Lambda^*.$$

So, the G-module U_E is isomorphic to $V_E(\chi)$ and Theorem 2.4 tells us that

$$\bar{\chi}_{\ell^r}^{2a-j}(g) = (\chi_\ell(g))^{2a-j} \bmod \ell^r\mathbb{Z}_\ell = \chi(g) \bmod \ell^r\mathbb{Z}_\ell = 1 \ \forall g \in G = \mathrm{G}_K.$$

□

Proof of Theorem 1.8 Since $\mathbf{b}_j(\bar{X}) \neq 0$, we have $j \leq 2d$. Recall that n is a positive integer that is not divisible by $\mathrm{char}(K)$. Let ℓ be a prime dividing n and let $\ell^{r_n(\ell)}$ be the exact power of ℓ that divides n. Applying Proposition 3.1 to all such ℓ with $r = r_n(\ell)$ and using Remarks 1.2 and 1.5, we obtain that the character $\bar{\chi}_n^{2a-j}$ is trivial, which gives us the first assertion of Theorem 1.8. On the other hand, we know that $\bar{\chi}_n^{\phi(n)}$ is trivial. This implies that if $2a - j$ and $\phi(n)$ are relatively prime then $\bar{\chi}_n$ is itself trivial, i.e., $\mu_n \subset K$. This proves the second assertion of Theorem 1.8.

Now we use Theorem 1.8 in order to prove Corollary 1.9 and Theorem 1.10.

Remark 3.2 In the statement of Theorem 1.8 we do not require that j is *odd* and therefore its immediate Corollary 1.9 remains true without this assumption. However, if we drop this assumption in Corollary 1.9 (while keeping all the other ones) and assume instead that j is *even* then $2a - j$ is also even and therefore $\phi(n)$ is odd, because it is relatively prime to $2a - j$. This implies that $n = 2$ and therefore $\mathrm{char}(K) \neq 2$ and K does *not* contain a primitive square root of unity, i.e., K does *not* contain -1, which is absurd.

Remark 3.3 The second assertion Theorem 1.8 (and its proof) remains true (valid) if in its statement we replace $\phi(n)$ by its divisor $\exp(n, K)$.

Proof of Theorem 1.6 Since $a = (j \pm 1)/2$, the integer $2a - j = \pm 1$ is relatively prime to $\phi(n)$. Now the result follows from already proven Theorem 1.8. □

Proof of Theorem 1.10 Suppose that the Galois action on $H^j(\bar{X}, \mu_n{}^{\otimes a})$ is trivial for some absolutely irreducible smooth projective variety X with $\mathbf{b}_j(\bar{X}) \neq 0$. By Theorem 1.8, the character $\bar{\chi}_n^{2a-j}$ is trivial. On the other hand, since $f := 2a - j$ is *odd* and $[K(\mu_n) : K]$ is *even*, Remark 1.1 tells us that $\bar{\chi}_n^{2a-j}$ is *nontrivial*. This gives us a desired contradiction. □

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