Numerical Simulation of Flows Involving Singularities

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Abstract Many interesting fluid interface problems involve singular events, as breaking-up or merging of the physical domain. In particular, wave propagation and breaking, droplet and bubble break-up, electro-jetting, rain drops, etc. are good examples of such processes. All these mentioned problems can be modeled using the potential flow assumptions, in which an interface needs to be advanced by a velocity determined by the solution of a surface partial differential equation posed on this moving boundary. The standard approach, the Lagrangian-Eulerian formulation together with some sort of front tracking method, is prone to fail when break-up or merging processes appear. The embedded formulation using level sets seamlessly allows topological breakup or merging of the fluid domain. In this work we present the numerical approximation of the embedded model and some computational results regarding electrohydrodynamic applications.

1 The Embedded Model Equations

Let $\Omega_1(t)$ be a fluid domain immersed in an infinite exterior fluid $\Omega_2(t)$, Γ_t be the free boundary separating both domains, and Ω_D be a fixed domain that should contain the free boundary for all $t \in [0, T]$. The level set/extended potential flow model, [3, 4], may be then written as:

$$\mathbf{u} = \nabla\phi \ \text{ in } \ \Omega_1(t) \tag{1}$$

$$\Delta\phi = 0 \ \text{ in } \ \Omega_1(t) \tag{2}$$

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$$\Psi_t + \mathbf{u}_{\text{ext}} \cdot \nabla \Psi = 0 \text{ in } \Omega_D \tag{3}$$

$$G_t + \mathbf{u}_{\text{ext}} \cdot \nabla G = f_{\text{ext}} \text{ in } \Omega_D. \tag{4}$$

Here, ϕ is the velocity potential, $\mathbf{u}$ the velocity field, Ψ the level set function, G the extended potential function, f accounts for the surface forces, and the subscript "ext" refers to the extended quantities off the front into Ω_D. This hydrodynamic problem can be coupled with any other exterior problem on $\Omega_2(t)$. In particular, assuming a uniform electric field $\mathbf{E}$ in $\Omega_2(t)$, acting in the direction of the z axis and $\mathbf{E} = 0$ in $\Omega_1(t)$ (perfect conductor fluid) then:

$$\mathbf{E} = -\nabla U \text{ in } \Omega_2(t) \tag{5}$$

$$\Delta U = 0 \text{ in } \Omega_2(t) \tag{6}$$

$$U = U_0 \text{ on } \Gamma_t \tag{7}$$

$$U = -E_\infty z \text{ at the far field,} \tag{8}$$

where U is the electric potential and E_∞ is the electric field intensity.

2 Numerical Approximation

The semidiscretization in time of the model equations is:

$$\mathbf{u}^n = \nabla \phi^n \text{ in } \Omega_1(t_n) \tag{9}$$

$$\Delta \phi^n(r,z) = 0 \text{ in } \Omega_1(t_n) \tag{10}$$

$$\frac{\Psi^{n+1} - \Psi^n}{\Delta t} = -\mathbf{u}^n_{\text{ext}} \cdot \nabla \Psi^n \text{ in } \Omega_D \tag{11}$$

$$\frac{G^{n+1} - G^n}{\Delta t} = -\mathbf{u}^n_{\text{ext}} \cdot \nabla G^n + f^n_{\text{ext}} \text{ in } \Omega_D, \tag{12}$$

$$\Delta U^n(r,z) = 0 \text{ in } \Omega_2(t_n) \tag{13}$$

where a first order explicit scheme has been applied. For the space discretization of Eqs. (11) and (12) a first order or second order upwind scheme can be used. The approximation of (10) and (13) is crucial in this numerical method, as it provides the velocity to advance the free boundary and also the velocity potential evolution within this front. We have coupled the following solvers for the interior and exterior Laplace equations:

- For 2D and 3D axisymmetric geometries a Galerkin boundary integral solution is established, where the boundary element method with linear elements have been used to approximate the integral equations, see [6, 8].

- For the fully 3D approximation a non conforming Nitsche finite element method has been used together with stabilization techniques of the bilinear forms, as the jump stabilization or the ghost penalty stabilization, see [1, 2, 9].

3 Numerical Results

Several physical scenarios can be simulated using the assumptions and the numerical method presented here. In the case of pure hydrodynamic problems, Eqs. (1)–(4), results for the wave breaking phenomena in a 2D geometry have been presented in [3], where splitting of the fluid domain was not considered. The first simulation involving computations through singular events was presented in [4], where the pinch-off of an infinite fluid jet and subsequent cascade of drop formation was reproduced in a seamless 3D axi-symmetric computation. In Fig. 1 we present the comparison of the satellite break up simulation with laboratory photographs. The interaction of two inviscid fluids of different densities was studied in [5]. The only parameter in the non-dimensional model is the fluid density ratio and simulations of the breaking up transition patterns from air bubbles to water droplets have been

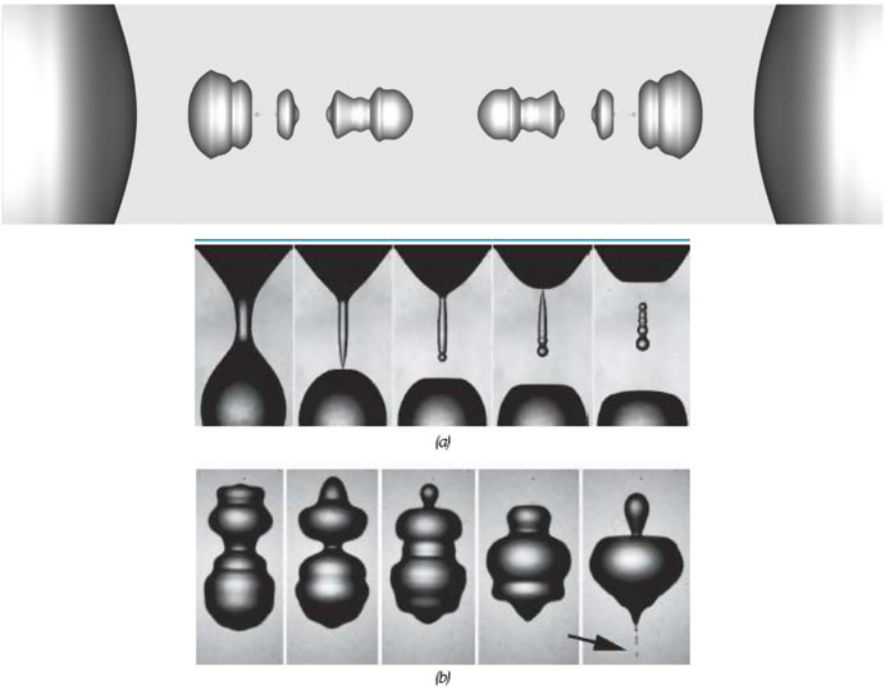

Fig. 1 Satellite drop breaking up, computed profiles (**a**) and Laboratory photographs (**b**), see [10]. Reproduced from [4] with permission from Elsevier

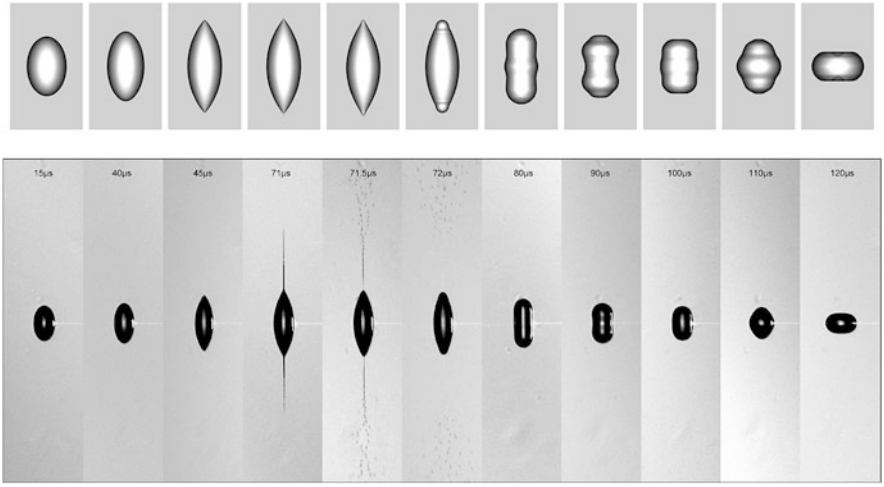

Fig. 2 Laboratory snapshots at indicated times of the evolution of a surface charged super-cooled water droplet, reprinted figure with permission from E. Giglio, D. Duft and T. Leisner, Phys. Rev. E, 77, 036319 (2008). Copyright (2008) by the American Physical Society (*bottom*); and computed profiles at times 80, 101.2, 108.1, 108.5, 109.8, 112.1, 124.2, 133.4, 138, 142, 154.1 μs (*top*)

computed. When electrical forces acting on the free surface are also considered, Eqs. (1)–(8), the flow gets even more interesting: a charged water droplet will elongate until Taylor cones are formed, from which fine filaments will be ejected from both drop tips. As soon as the drop losses enough charge, it will recoil and oscillate back to equilibrium. In Fig. 2 we show also a comparison between computed profiles on top and Laboratory experiments on bottom at corresponding times. See [7].

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