Evolution of Consistent Conjectures in Semi-aggregative Representation of Games, with Applications to Public Good Games and Contests

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1 Introduction

Aggregative games are a special class of games in which the payoff of a player depends on the player's own strategy and on a (common across players) aggregate of players' strategies. An early example of a work on aggregative games is Corchón (1994) but in a more recent series of works, Richard Cornes and Roger Hartley elucidated the usefulness of studying the mathematical structure of these games for establishing equilibrium existence and for finding equilibria in situations going beyond textbook symmetric examples. They applied this methodology to such classic examples of economic analysis as public good games (Cornes and Hartley 2007) and contests (Cornes and Hartley 2003, 2005),[1] as well as studying the general structure of aggregate games further (Cornes and Hartley 2012).

Before turning his attention to aggregative games, Richard also worked on applications of the concept of conjectural variations. This concept was extensively analyzed in the context of industrial organization games (see e.g. Laitner 1980; Bresnahan 1981; Perry 1982); its application in common property exploitation model was considered in Cornes and Sandler (1983) and in public good games in Cornes and Sandler (1984a).

Paper prepared for a volume honoring the memory of Richard Cornes. In his time at the University of Nottingham, Richard was a helpful colleague, ready to give advice in his usual witty and entertaining manner.

[1]Further examples of aggregative games are listed in Cornes and Hartley (2011) and Cornes (2016) discusses the applications of aggregative games in the analysis of environmental problems.

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In this paper, I also focus on a representation of games that is similar to the aggregative one and on conjectural variations. The representation is such that a player's payoff depends on the player's strategy and on a certain aggregate of all player's strategies, personalized for the player. Thus the aggregates do not have to be the same for all players, as in a usual aggregative game. Nevertheless, the aggregates fulfill a similar role of reducing the dimensionality of what a player needs to consider about the other players. In such a representation (which I call *semi-aggregative*), what is relevant for the player is how the aggregate measure of players' strategies possibly changes. This is precisely what the conjectures of the players in such a game are about.

Given their conjectures about possible changes in the respective aggregates, the players in the game behave rationally, that is, maximize their payoff. Their decisions characterize an equilibrium, for the given conjectures. But where do the conjectures come from? I suppose that they represent players' innate beliefs, but those beliefs are subject to evolution. Different conjectures will lead to different equilibrium choices and thus different payoffs. From the point of view of evolution, those conjectures that led to higher payoff are more likely to propagate.[2]

I focus on the setting where a game is not necessarily symmetric, thus players can have different roles (for example, one player can have a larger marginal benefit from a public good than another player, or a lower cost of contributing to it). Since roles are different, evolution is considered as happening within each role separately. Instead of considering an explicit dynamic process, I look for evolutionarily stable conjectures, which are conjectures that no other conjectures can invade by achieving a higher payoff for this player's role, given the conjectures of the other players and the equilibrium that the players play.

I find that the evolutionary stability of conjectures is linked to their consistency. An equilibrium in the model is at the intersection of the reaction functions of the players, which also define the reaction of the aggregates. If a player's strategy changes, for whatever reason, the reaction functions determine how the other players change (optimally) their strategies, and thus how the aggregates change. Conjectures are considered consistent if the belief of a player locally coincides, to the first approximation, to the actual change in the player's personalized aggregate. The main result of the paper is that, in well-behaved games, only consistent conjectures of a player can be evolutionary stable for this player.

The result extends the link between consistent and evolutionarily stable conjectures. Previous works noted this connection in simple duopoly models (Dixon and Somma 2003; Müller and Normann 2005), in two-player games (Possajennikov 2009) and in symmetric aggregative games (Possajennikov 2015). What I add in this paper is that the link between consistent and evolutionarily stable conjectures hold

[2] Another interpretation is that players first choose conjectures and then play the game. The search is then for an equilibrium in the game of choosing conjectures. I nevertheless prefer the evolutionary interpretation, which makes it clearer that the process of forming beliefs and choosing strategies occur at different times. This evolutionary interpretation is an example of the "indirect evolution approach" (Güth and Yaari 1992).

in more general n-player asymmetric situations. Thus it is not only that evolution selects consistent conjectures when other players' conjectures are consistent; for any conjectures of the other players, it is best, from the evolutionary point of view, to have a consistent conjecture.

This result is illustrated on two examples of games that were often the subject of Richard Cornes's work and that are aggregative or naturally semi-aggregative, namely public good games and contests. In these settings, I show that for many parameter values consistent and evolutionarily stable conjectures coincide, thus consistency is not only a necessary but also a sufficient condition for evolutionary stability.

2 Games and Conjectures

2.1 Semi-aggregative Representation of Games

A simultaneous-move game on the real line is $G = (N, \{X_i\}_{i=1}^n, \{u_i\}_{i=1}^n)$, where $N = \{1, \ldots, n\}$ is the set of players, $X_i \subset \mathbb{R}$ is the strategy set of player i, and $u_i : X_1 \times \ldots \times X_n \to \mathbb{R}$ is the payoff function of player i. It is assumed that the game is well-behaved: strategy sets are convex and the payoff functions are differentiable as many times as required.

For any game, the payoff of player i can be written as $u_i(x_i, A_i)$, where $A_i = f_i(x_1, \ldots, x_n)$ for some function $f_i : X_1 \times \ldots \times X_n \to \mathbb{R}$.[3] I call the representation of the payoffs in the form $u_i(x_i, A_i)$ *semi-aggregative*, since A_i can be seen as a personalized aggregate of player i, which summarizes the dependence of the payoff of player i on the strategies of other players. Note that the aggregate A_i can include the strategy x_i of player i. A game is *aggregative* if there exists a semi-aggregative representation with $A = A_i$ for all i, i.e. with the same functions f_i for all players and a common aggregate A.

While in general games the payoff representation discussed above may appear strange, there are classes of games for which a (semi-)aggregative representation is natural. For example, in a differentiated product oligopoly, the price $p_i(q_1, \ldots, q_n)$ for the product of firm i is determined by the inverse demand from the quantities chosen by all firms. This price can then naturally be taken as the personalized aggregate of firm i. The payoff for firm i is the profit $\pi_i(q_i, p_i) = p_i q_i - C_i(q_i)$, where $C_i(q_i)$ is the cost function of firm i.[4]

For another example, consider a (pure) public good game. Each player i contributes a part x_i of the endowment m_i to the public good, leaving $m_i - x_i$

[3]For example, consider the identity $u_i(x_1, \ldots, x_n) = x_i + u_i(x_1, \ldots, x_n) - x_i$. Let $A_i = f_i(x_1, \ldots, x_n) = u_i(x_1, \ldots, x_n) - x_i$ and write $u_i(x_i, A_i) = x_i + A_i$.

[4]In a homogeneous good market with one price, the aggregate (the price) is the same for all firms, and the game is properly aggregative.

for private good consumption. The aggregate production of the public good is $A = \sum_{i=1}^{n} x_i$. Player i's payoff is given by the utility function $u_i(m_i - x_i, A)$, which is already a semi-aggregative representation. In fact, with pure public goods the game is properly aggregative, since the aggregate amount of public good A is the same for all players; it is not needed to have personalized aggregates for each player.

The advantage of the semi-aggregative representation is the reduction in the dimensionality of the problem. In a sense, a player sees his or her opponents as one aggregate opponent and is only concerned about the aggregate effect of such an opponent on payoff. In the next section I discuss how this can be used to formulate in a simple manner players' expectations about the behavior of other players.

2.2 *Conjectures and Conjectural Variation Equilibria*

Suppose that player i has some conjectures r_i about the reaction of other players to a change in the player's own strategy. With the semi-aggregative representation of the game, the conjectures are about the change in the personalized aggregate, $r_i = \left(\frac{dA_i}{dx_i}\right)^e$, where the superindex is meant to convey that it is an *expected*, rather than actual, change. It is assumed that the conjectures are constant, $r_i \in R_i$, where R_i is a convex subset of $\mathbb{R}$, i.e. conjectures do not depend on the current strategies of players. This assumption again reduces the dimensionality of the problem while still allowing consideration of consistent conjectures.

A change in player i's own strategy x_i also can directly affect the aggregate $A_i = f_i(x_1, \ldots, x_n)$. But the conjecture is about the *total* effect of a change in x_i on A_i: it incorporates the direct effect $\frac{\partial A_i}{\partial x_i}$ but also the effect from the expected changes in the other players' strategies. This formulation is slightly more general than the one with the aggregate being a function of the other players' strategies only, as was used in e.g. Perry (1982) for oligopoly and in Cornes and Sandler (1984a) for a public good model. It can still represent the usual Nash behavior: $r_i = \frac{\partial A_i}{\partial x_i}$ means that the strategies of the other players are kept fixed; player i does not expect the other players to react.

Having conjecture r_i, player i maximizes payoff $u_i(x_i, A_i)$. The first-order condition for maximization is

$$F_i(x_i, A_i; r_i) = \frac{\partial u_i}{\partial x_i}(x_i, A_i) + \frac{\partial u_i}{\partial A_i}(x_i, A_i) \cdot r_i = 0. \tag{1}$$

Suppose now that all players have certain conjectures, summarized by vector $\mathbf{r} = (r_1, \ldots, r_n)$. Suppose further that for each player i, the solution of the player's maximization problem is characterized by Eq. (1). A *conjectural variation equilibrium (CVE)* for the given vector $\mathbf{r}$ of conjectures consists of the vector of players' strategies $\mathbf{x}^*(\mathbf{r}) = (x_1^*(\mathbf{r}), \ldots, x_n^*(\mathbf{r}))$ and the vector of personalized aggregates $\mathbf{A}^*(\mathbf{r}) = (A_1^*(\mathbf{r}), \ldots, A_n^*(\mathbf{r}))$ that satisfy the system of equations

$$F_i(x_i, A_i; r_i) = 0,\ i = 1, \ldots, n, \tag{2}$$
$$A_i - f_i(x_i, \ldots, x_n) = 0,\ i = 1, \ldots, n.$$

It is assumed that the solution of this system of equations exists for the values of conjectures in sets R_i. There may be multiple solutions of the system; in the analysis below I consider any particular solution that is locally unique and well-behaved.

Although conjectures are about changes in a player's strategy and reactions to them, the conjectural variation equilibrium is a static concept. However, it can be interpreted as a convenient short-cut summarizing the result of a more explicit dynamic analysis,[5] and this is the interpretation I have in mind by focusing on CVE in this paper.

2.3 Consistent Conjectures

Recall that a conjecture of player i is a belief about the change in the personalized aggregate A_i in response to a change in player i's strategy x_i. To define consistent conjectures, let x_i vary unconstrained and concentrate on optimal responses of the other players. Consider the system of equations

$$F_j(x_j, A_j; r_j) = 0, j = 1, \ldots, n, j \neq i \tag{3}$$
$$A_j - f_j(x_1, \ldots, x_n) = 0, \quad j = 1, \ldots, n,$$

which is like system (2) except that the first-order condition for player i is not there. Thus, the strategy x_i of player i is not constrained to be optimal; it can take any value. The strategies of the other players are still characterized by the first-order conditions; thus the system describes optimal responses of the other players to arbitrary values of x_i. Denote a solution of system (3) as $(x_1^{**}(x_i), \ldots, x_{i-1}^{**}(x_i), x_{i+1}^{**}(x_i), \ldots, x_n^{**}(x_i); A_1^{**}(x_i), \ldots, A_n^{**}(x_i))$.

Consider a vector of conjectures $\mathbf{r}$ and a certain CVE $(\mathbf{x}^*, \mathbf{A}^*) = (x_1^*, \ldots, x_n^*; A_1^*, \ldots, A_n^*)$ for these conjectures. Note that for $x_i = x_i^*$ there exists a solution of system (3) with $x_j^{**}(x_i^*) = x_j^*$ for all $j \neq i$ and $A_j^{**}(x_i^*) = A_j^*$ for all $j = 1, \ldots, n$. Consider such a solution and consider $A_i^{**}(x_i)$. Conjecture r_i of player

[5] In a duopoly context, Dockner (1992) and Cabral (1995) show that a dynamic model indeed can lead to the same outcomes as certain CVEs, and Itaya and Dasgupta (1995) and Itaya and Okamura (2003) do so for a public good game.

i is *consistent* if $r_i = \frac{dA_i^{**}}{dx_i}(\mathbf{x}^*, \mathbf{A}^*)$, i.e. the conjecture about the reaction of the personalized aggregate is, to a first approximation, correct at equilibrium.[6]

Whether a particular conjecture r_i is consistent depends on the vector of conjectures $\mathbf{r}_{-i}$ of the other players. Given a vector $\mathbf{r}$, it is possible that some players hold consistent conjectures and others not. One can define conjectures to be mutually consistent if for all i, r_i is consistent against $\mathbf{r}_{-i}$. However, it will not be important for the analysis of conjectures of player i what conjectures the other players hold thus I do not focus only on mutually consistent conjectures.

3 Evolutionary Stability of Conjectures

Imagine that for each of the n player roles there is a large (infinite) population of players, and players from each population from time to time are called to play the game G against opponents randomly drawn from the other populations. Consider the population for the role of player i. Each player in the population has some conjectures. Suppose that in all other player populations conjectures have stabilized on some values $\mathbf{r}_{-i}$. Thus, if called to play, a player with a certain conjecture r_i from the population of players i will play the game against the other players with conjectures $\mathbf{r}_{-i}$. Suppose that when the game is played, a CVE is played. The question is: for the given conjectures $\mathbf{r}_{-i}$ of the other players, which conjecture of player i is evolutionarily stable?

Different conjectures in the population for the role of player i will lead to different CVEs and thus to different payoffs. Conjecture r_i^{ES} is said to be *evolutionarily stable* (Maynard Smith and Price 1973; Selten 1980) if

$$u_i(x_i^*(r_i^{ES}, \mathbf{r}_{-i}), A_i^*(r_i^{ES}, \mathbf{r}_{-i})) > u_i(x_i^*(r_i, \mathbf{r}_{-i}), A_i^*(r_i, \mathbf{r}_{-i})) \text{ for any } r_i \neq r_i^{ES}.$$

The above inequality means that in the population for the role of player i, a player with conjecture r_i^{ES} will get a higher payoff when called to play than a player with any other value r_i of the conjecture. The evolutionary intuition is that players with any other conjecture r_i in the population for the role of player i would have lower fitness than the players with conjecture r_i^{ES}. Therefore evolution will favor players with conjecture r_i^{ES} to survive and thrive.[7]

With the alternative interpretation that players first choose their conjectures and then play a CVE of the game G, an evolutionarily stable conjecture of player i is a strict best response of player i to the given conjectures of the other players. If a vector of conjectures $\mathbf{r}^{ES} = (r_1^{ES}, \ldots, r_n^{ES})$ is such that for each player i the

[6]This consistency requirement was introduced by Bresnahan (1981) for a duopoly, and also used e.g. in Perry (1982) in an oligopoly and Cornes and Sandler (1984a) in a public good game context.

[7]Note that the definition focuses on player i treating the other players conjectures as fixed; Selten (1980) showed that such an approach is appropriate in asymmetric games.

conjecture r_i^{ES} is evolutionarily stable given $\mathbf{r}_{-i}^{ES}$, then $(r_1^{ES}, \ldots, r_n^{ES})$ is a strict Nash equilibrium in the game where players choose conjectures and their payoffs are determined via conjectural variations equilibria.

Whatever the interpretation, an evolutionary stable conjecture solves

$$\max_{r_i} u_i(x_i^*(r_i, \mathbf{r}_{-i}), A_i^*(r_i, \mathbf{r}_{-i})).$$

The first-order condition for maximization is[8]

$$\frac{\partial u_i}{\partial x_i}\frac{\partial x_i^*}{\partial r_i} + \frac{\partial u_i}{\partial A_i}\frac{dA_i^*}{dr_i} = 0.$$

Therefore (provided that $\frac{\partial u_i}{\partial A_i} \neq 0$ and $\frac{\partial x_i^*}{\partial r_i} \neq 0$), $-\frac{\partial u_i/\partial x_i}{\partial u_i/\partial A_i} = \frac{dA_i^*/dr_i}{\partial x_i^*/\partial r_i}$. Since from Eq. (1) $r_i = -\frac{\partial u_i/\partial x_i}{\partial u_i/\partial A_i}$, an interior evolutionarily stable conjecture satisfies

$$r_i^{ES} = \frac{dA_i^*/dr_i}{\partial x_i^*/\partial r_i}. \tag{4}$$

Speaking somewhat loosely in mathematical terms, if $dr_i = \partial r_i$ is treated as a small change in the independent variable r_i, then it can be canceled from (4). Note also that $dx_i = dx_i^* = \partial x_i^*$ if only r_i changes. Therefore $r_i^{ES} = \frac{dA_i^*}{dx_i}$. Recall that a conjecture is consistent if $r_i = \frac{dA_i^{**}}{dx_i}$. Since at a CVE $A_i^{**}(x_i^*) = A_i^*$, the first-order condition for evolutionary stability and the consistency condition are essentially the same.[9]

For a more formal demonstration of the reasoning, consider system (2). To simplify notation, focus on $i = 1$. Differentiating each line of (2) with respect to r_1,

$$\begin{array}{ccccccccccc}
\frac{\partial F_1}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} & + \ldots + & 0 & + & \frac{\partial F_1}{\partial A_1}\frac{dA_1^*}{dr_1} & + \ldots + & 0 & = & -\frac{\partial F_1}{\partial r_1} \\
\cdots & \cdots & \cdots & & \cdots & \cdots & & & \cdots \\
0 & + \ldots + & \frac{\partial F_n}{\partial x_n}\frac{\partial x_n^*}{\partial r_1} & + & 0 & + \ldots + & \frac{\partial F_n}{\partial A_n}\frac{dA_n^*}{dr_1} & = & 0 \\
-\frac{\partial f_1}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} & + \ldots + & -\frac{\partial f_1}{\partial x_n}\frac{\partial x_n^*}{\partial r_1} & + & \frac{dA_1^*}{dr_1} & + \ldots + & 0 & = & 0 \\
\cdots & \cdots & \cdots & & \cdots & \cdots & & & \cdots \\
-\frac{\partial f_n}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} & + \ldots + & -\frac{\partial f_n}{\partial x_n}\frac{\partial x_n^*}{\partial r_1} & + & 0 & + \ldots + & \frac{dA_n^*}{dr_1} & = & 0
\end{array}$$

[8]To save space, arguments of derivatives are omitted. It is understood that they are evaluated at $\mathbf{r} = (r_i^{ES}, \mathbf{r}_{-i})$ and CVE $(\mathbf{x}^*(\mathbf{r}), \mathbf{A}^*(\mathbf{r}))$.

[9]The relationship between consistent conjectures and the conjectures that maximize the indirect payoff function $u_i(x_i^*(r_i, r_j), A_i^*(r_i, r_j))$ was noted by Itaya and Dasgupta (1995) for a two-player public good game.

Define

$$M = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & 0 & \frac{\partial F_1}{\partial X_1} & \dots & 0 \\ \dots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & \dots & \frac{\partial F_n}{\partial x_n} & 0 & \dots & \frac{\partial F_n}{\partial X_n} \\ -\frac{\partial f_1}{\partial x_1} & \dots & -\frac{\partial f_1}{\partial x_n} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \ddots & \dots \\ -\frac{\partial f_n}{\partial x_1} & \dots & -\frac{\partial f_n}{\partial x_n} & 0 & \dots & 1 \end{pmatrix},$$

$$M_{-11} = \begin{pmatrix} \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X_2} & \dots & 0 \\ \dots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & \dots & \frac{\partial F_n}{\partial x_n} & 0 & \dots & \frac{\partial F_n}{\partial X_n} \\ -\frac{\partial f_2}{\partial x_2} & \dots & -\frac{\partial f_2}{\partial x_n} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \ddots & \dots \\ -\frac{\partial f_n}{\partial x_2} & \dots & -\frac{\partial f_n}{\partial x_n} & 0 & \dots & 1 \end{pmatrix},$$

and

$$M_{-1A} = \begin{pmatrix} 0 & \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X_2} & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & \frac{\partial F_n}{\partial x_n} & 0 & \dots & \frac{\partial F_n}{\partial X_n} \\ -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & \dots & -\frac{\partial f_1}{\partial x_n} & 0 & \dots & 0 \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} & \dots & -\frac{\partial f_2}{\partial x_n} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \ddots & \dots \\ -\frac{\partial f_n}{\partial x_1} & -\frac{\partial f_n}{\partial x_2} & \dots & -\frac{\partial f_n}{\partial x_n} & 0 & \dots & 1 \end{pmatrix}.$$

If $|M| \neq 0$, by Cramer's rule, $\frac{\partial x_1^*}{\partial r_1} = \frac{1}{|M|}\left(-\frac{\partial F_1}{\partial r_1}\right)|M_{-11}|$ and $\frac{\partial A_1^*}{\partial r_1} = \frac{1}{|M|}(-1)^n \frac{\partial F_1}{\partial r_1}|M_{-1A}|$. Therefore, if $|M_{-11}| \neq 0$ (from Eq. (1) $\frac{\partial F_1}{\partial r_1} = \frac{\partial u_i}{\partial A_i}$ thus $\frac{\partial F_1}{\partial r_1} \neq 0$ if $\frac{\partial u_i}{\partial A_i} \neq 0$), Eq. (4) becomes

$$r_1^{ES} = \frac{(-1)^{n-1}|M_{-1A}|}{|M_{-11}|}. \tag{5}$$

To determine $\frac{dA_1^{**}}{dx_1}$, consider system (3). Differentiating each of the equations with respect to x_1,

$$\begin{array}{ccccccccc}
\frac{\partial F_2}{\partial x_2}\frac{\partial x_2^{**}}{\partial x_1} & +\ldots+ & 0 & + & 0 & + \frac{\partial F_2}{\partial A_2}\frac{dA_2^{**}}{dx_1} & +\ldots+ & 0 & = 0 \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & +\ldots+ & \frac{\partial F_n}{\partial x_n}\frac{\partial x_n^{**}}{\partial x_1} & + & 0 & + \quad 0 & +\ldots+ & \frac{\partial F_n}{\partial A_n}\frac{dA_n^{**}}{dx_1} & = 0 \\
-\frac{\partial f_1}{\partial x_2}\frac{\partial x_2^{**}}{\partial x_1} & +\ldots+ & -\frac{\partial f_1}{\partial x_n}\frac{\partial x_n^{**}}{\partial x_1} & + & \frac{dA_1^{**}}{dx_1} & + \quad 0 & +\ldots+ & 0 & = \frac{\partial f_1}{\partial x_1} \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{\partial f_n}{\partial x_2}\frac{\partial x_2^{**}}{\partial x_1} & +\ldots+ & -\frac{\partial f_n}{\partial x_n}\frac{\partial x_n^{**}}{\partial x_1} & + & 0 & + \quad 0 & +\ldots+ & \frac{dA_n^{**}}{dx_1} & = \frac{\partial f_n}{\partial x_1}
\end{array}$$

Define

$$L_{-11} = \begin{pmatrix}
0 & \frac{\partial F_2}{\partial x_2} & \ldots & 0 & \frac{\partial F_2}{\partial A_2} & \ldots & 0 \\
\vdots & \ldots & \ddots & \ldots & \ldots & \ddots & \ldots \\
0 & 0 & \ldots & \frac{\partial F_n}{\partial x_n} & 0 & \ldots & \frac{\partial F_n}{\partial A_n} \\
1 & -\frac{\partial f_1}{\partial x_2} & \ldots & -\frac{\partial f_1}{\partial x_n} & 0 & \ldots & 0 \\
0 & -\frac{\partial f_2}{\partial x_2} & \ldots & -\frac{\partial f_2}{\partial x_n} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ldots & \ddots & \ldots \\
0 & -\frac{\partial f_n}{\partial x_2} & \ldots & -\frac{\partial f_n}{\partial x_n} & 0 & \ldots & 1
\end{pmatrix}$$

and

$$L_{-1A} = \begin{pmatrix}
0 & \frac{\partial F_2}{\partial x_2} & \ldots & 0 & \frac{\partial F_2}{\partial A_2} & \ldots & 0 \\
\vdots & \ldots & \ddots & \ldots & \ldots & \ddots & \ldots \\
0 & 0 & \ldots & \frac{\partial F_n}{\partial x_n} & 0 & \ldots & \frac{\partial F_n}{\partial A_n} \\
\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & \ldots & -\frac{\partial f_1}{\partial x_n} & 0 & \ldots & 0 \\
\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} & \ldots & -\frac{\partial f_2}{\partial x_n} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ldots \\
\frac{\partial f_n}{\partial x_1} & -\frac{\partial f_n}{\partial x_2} & \ldots & -\frac{\partial f_n}{\partial x_n} & 0 & \ldots & 1
\end{pmatrix}$$

Then $|L_{-11}| = (-1)^n|M_{-11}|$ and $|L_{-1A}| = -|M_{-1A}|$. Using Cramer's rule again, $\frac{dA_1^{**}}{dx_1} = \frac{1}{|L_{-11}|}|L_{-1A}| = \frac{-|M_{-1A}|}{(-1)^n|M_{-11}|} = \frac{(-1)^{n-1}|M_{-1A}|}{|M_{-11}|}$, which is the same as the right-hand side of Eq. (5). Thus, the following proposition is proved:

Proposition 1 *Consider a semi-aggregative representation of the game* G *and consider conjecture profile* $\mathbf{r} = (r_1, \ldots, r_n)$. *Suppose that there exists a CVE* $(\mathbf{x}^*(\mathbf{r}), \mathbf{A}^*(\mathbf{r}))$ *for this* $\mathbf{r}$. *If* $\frac{\partial u_i}{\partial A_i} \neq 0$, $|M| \neq 0$ *and* $|M_{-11}| \neq 0$ *at* $\mathbf{r}$ *and*

$(\mathbf{x}^*(\mathbf{r}), \mathbf{A}^*(\mathbf{r}))$*, then if* r_i *is an evolutionarily stable conjecture for player i, then it is a consistent conjecture for player i.*

Since the analysis was based only on the first-order condition for evolutionary stability, it is not necessarily the case that a consistent conjecture is evolutionarily stable. Concavity or quasi-concavity conditions on the indirect function $u_i(x_i(\mathbf{r}), A_i(\mathbf{r}))$ can guarantee this. Instead of stating these conditions in general, evolutionarily stability of consistent conjectures is demonstrated for particular games in Sect. 4.

The practical usefulness of the result is that it is usually easier to find consistent conjectures than to derive the indirect function to search for the evolutionarily stable ones. Since the result shows that in well-behaved games only consistent conjectures can be evolutionarily stable in the interior of the conjecture space, the search for evolutionarily stable conjectures can be reduced to the consistent ones.

The conceptual usefulness of the result is to provide foundations for consistent conjectures. Consistency of conjectures is not always accepted as a plausible criterion for preferring some conjectures over others.[10] The result in this paper shows though, that if players are endowed with conjectures that are subject to evolutionary pressure (or, equivalently, if players could choose conjectures before playing the game), then only consistent conjectures can survive such a process.

Note that the proof of the result concentrated on player i, while taking arbitrary conjectures held by the other players. The conjectures of the other players may or may not be consistent; if one wants all players to have evolutionarily stable conjectures, then only profiles with mutually consistent conjectures can be such. The result shows that it is best for player i to have consistent conjectures whatever the conjectures of the other players are (but which value of the conjecture is consistent, and thus possibly evolutionarily stable, for player i depends on the current conjectures of the other players).[11]

The current result generalizes the previous ones in Possajennikov (2009, 2015) to asymmetric games with more than two players. In principle, the games do not even need to have an obvious aggregative structure: what was used is that the players make conjectures about the appropriate quantity A_i that was relevant for their payoff. In general, the function f_i determining this quantity may be complicated and thus it is not likely that the players would consider conjectures about it; however, in some games, illustrated in the next section, the aggregate quantity A_i arises naturally in the formulation of the problem.

[10] See e.g. Makowski (1987) and Cornes and Sandler (1996, p. 32) say that they do not attach any particular importance to consistent conjectures.

[11] The observation that the consistent conjecture is the best response conjecture of one player to any given conjecture of the other player was made in Dixon and Somma (2003) for a linear-quadratic Cournot duopoly game.

4 Examples

4.1 Semi-public Good Games

Cornes and Sandler (1984a,b) explored the public good model, including the impact of various conjectures and the possibility of impure public goods, where a player's contribution to a public good also provides a private benefit. I will use instead the formulation of semi-public goods from Costrell (1991) that models the same idea—that a player benefits more from his or her own contribution to a public good than the other players do—in a more transparent manner. The formulation also encompasses a pure public good model.

Suppose that each player i has a money endowment m_i that can be spent either on a private good or on a semi-public good. Assuming for simplicity that prices of all goods are equal and normalizing the price to 1, $m_i = y_i + x_i$, where y_i is the amount spent on the private good and x_i the amount spent on the public good. Player i has the utility function $u_i(y_i, G_i)$, where G_i is the quantity of the public good available to player i. The semi-public nature of the public good is modeled by $G_i = x_i + b_i \sum_{j \neq i} x_j$, where $0 < b_i \leq 1$. Player i benefits most from his or her own contribution to the public good, but other players' contributions also spillover to player i's benefit. Quantity G_i naturally plays the role of the personalized aggregate for player i.[12]

To illustrate the result in the previous section, consider the three-player case ($n = 3$) and Cobb-Douglas utility functions for all players

$$u_i(x_i, G_i) = (m_i - x_i)^{\alpha_i} G_i^{1-\alpha_i},$$

with $0 < \alpha_i < 1$. Suppose that each player i has conjecture $r_i \geq 0$. Player i's first-order condition for utility maximization is $-\alpha_i(m_i - x_i)^{\alpha_i - 1} G_i^{1-\alpha_i} + (1 - \alpha_i)(m_i - x_i)^{\alpha_i} G_i^{-\alpha_i} r_i = 0$, or, in the interior where $x_i \neq m_i$ and $G_i \neq 0$, $-\alpha_i G_i + (1 - \alpha_i)(m_i - x_i) r_i = 0$. Therefore

$$\alpha_i G_i + (1 - \alpha_i) x_i r_i = (1 - \alpha_i) m_i r_i$$

characterizes the solution of player i utility maximization problem.[13]

To find consistent conjectures for player 1, consider the system

$$\alpha_2 G_2 + (1 - \alpha_2) x_2 r_2 = (1 - \alpha_2) m_2 r_2$$
$$\alpha_3 G_3 + (1 - \alpha_3) x_3 r_3 = (1 - \alpha_3) m_3 r_3$$

[12]Note that if $b_i = 1$ for all i, then the public good becomes a pure public good and the same aggregate $G = \sum_{i=1}^{n} x_i$ can be used for all players.

[13]The second-order condition $\alpha_i(1-\alpha_i)(m_i - x_i)^{\alpha_i - 2} G_i^{-\alpha_i - 1}(-G_i^2 - 2(m_i - x_i) G_i r_i - (m_i - x_i)^2 r_i^2) < 0$ is satisfied for $r_i \geq 0$ and all interior x_i, G_i.

$$G_1 - x_1 - b_1 x_2 - b_1 x_3 = 0$$
$$G_2 - b_2 x_1 - x_2 - b_2 x_3 = 0$$
$$G_3 - b_3 x_1 - b_3 x_2 - x_3 = 0.$$

Substituting the last three equations into the first two,

$$((1-\alpha_2)r_2 + \alpha_2)x_2 + \alpha_3 b_3 x_3 = (1-\alpha_2)m_2 r_2 - \alpha_2 b_2 x_1$$
$$\alpha_2 b_2 x_2 + ((1-\alpha_3)r_3 + \alpha_3)x_3 = (1-\alpha_3)m_3 r_3 - \alpha_3 b_3 x_1.$$

If $b_j < 1, j = 2, 3$, then the solution of these two equations is guaranteed to exist. It is

$$x_2^{**}(x_1) = \frac{(m_2(1-\alpha_2)r_2 - \alpha_2 b_2 x_1)((1-\alpha_3)r_3 + \alpha_3) - \alpha_2 b_2 (m_2(1-\alpha_3)r_3 - \alpha_3 b_3 x_1)}{((1-\alpha_2)r_2 + \alpha_2)((1-\alpha_3)r_3 + \alpha_3) - \alpha_2 b_2 \alpha_3 b_3}$$
$$x_3^{**}(x_1) = \frac{((1-\alpha_2)r_2 + \alpha_2)(m_3(1-\alpha_3)r_3 - \alpha_3 b_3 x_1) - (m_2(1-\alpha_2)r_2 - \alpha_2 b_2 x_1)\alpha_3 b_3}{((1-\alpha_2)r_2 + \alpha_2)((1-\alpha_3)r_3 + \alpha_3) - \alpha_2 b_2 \alpha_3 b_3}.$$

Since $G_1^{**}(x_1) = x_1 + b_1 x_2^{**}(x_1) + b_1 x_3^{**}(x_1)$, the consistent conjecture is

$$r_1^C = \frac{dG_1^{**}}{dx_1} = 1 - b_1 \frac{\alpha_2 b_2((1-\alpha_3)r_3 + \alpha_3) + ((1-\alpha_2)r_2 + \alpha_2)\alpha_3 b_3 - 2\alpha_2 b_2 \alpha_3 b_3}{((1-\alpha_2)r_2 + \alpha_2)((1-\alpha_3)r_3 + \alpha_3) - \alpha_2 b_2 \alpha_3 b_3}$$

This consistent conjecture is the unique candidate to be evolutionarily stable. Note that the consistent conjecture is less than unity, meaning that player 1 (correctly) expects the other players to partially offset an increase in his or her contribution to the public good. This exacerbates the inefficiency of the private provision of the good.

A CVE of the game is characterized by the equations

$$\alpha_1 G_1 + (1-\alpha_1)x_1 r_1 = (1-\alpha_1)m_1 r_1$$
$$\alpha_2 G_2 + (1-\alpha_2)x_2 r_2 = (1-\alpha_2)m_2 r_2$$
$$\alpha_3 G_3 + (1-\alpha_3)x_3 r_3 = (1-\alpha_3)m_3 r_3$$
$$G_1 - x_1 - b_1 x_2 - b_1 x_3 = 0$$
$$G_2 - b_2 x_1 - x_2 - b_2 x_3 = 0$$
$$G_3 - b_3 x_1 - b_3 x_2 - x_3 = 0.$$

Substituting the last three equations into the first three, the system becomes

$$((1-\alpha_1)r_1 + \alpha_1)x_1 + \alpha_2 b_2 x_2 + \alpha_3 b_3 x_3 = (1-\alpha_1)m_1 r_1$$
$$\alpha_2 b_2 x_1 + ((1-\alpha_2)r_2 + \alpha_2)x_2 + \alpha_3 b_3 x_3 = (1-\alpha_2)m_2 r_2$$
$$\alpha_3 b_3 x_1 + \alpha_2 b_2 x_2 + ((1-\alpha_3)r_3 + \alpha_3)x_3 = (1-\alpha_3)m_3 r_3.$$

Let

$$
\begin{aligned}
|M| = {} & ((1-\alpha_1)r_1+\alpha_1)((1-\alpha_2)r_2+\alpha_2)((1-\alpha_3)r_3+\alpha_3)+2\alpha_1 b_1\alpha_2 b_2\alpha_3 b_3 \\
& -\alpha_1 b_1((1-\alpha_2)r_2+\alpha_2)\alpha_3 b_3-((1-\alpha_1)r_1+\alpha_1)\alpha_2 b_2\alpha_3 b_3 \\
& -\alpha_1 b_1\alpha_2 b_2((1-\alpha_3)r_3+\alpha_3),
\end{aligned}
$$

$$
\begin{aligned}
|M_1| = {} & m_1(1-\alpha_1)r_1((1-\alpha_2)r_2+\alpha_2)((1-\alpha_3)r_3+\alpha_3)+\alpha_1 b_1 m_2(1-\alpha_2)r_2\alpha_3 b_3 \\
& +\alpha_1 b_1 m_2(1-\alpha_2)r_2\alpha_3 b_3-\alpha_1 b_1((1-\alpha_2)r_2+\alpha_2)m_3(1-\alpha_3)r_3 \\
& -m_1(1-\alpha_1)r_1\alpha_2 b_2\alpha_3 b_3-\alpha_1 b_1 m_2(1-\alpha_2)r_2((1-\alpha_3)r_3+\alpha_3),
\end{aligned}
$$

$$
\begin{aligned}
|M_2| = {} & ((1-\alpha_1)r_1+\alpha_1)m_2(1-\alpha_2)r_2((1-\alpha_3)r_3+\alpha_3)+m_1(1-\alpha_1)r_1\alpha_2 b_2\alpha_3 b_3 \\
& +\alpha_1 b_1\alpha_2 b_2 m_3(1-\alpha_3)r_3-\alpha_1 b_1 m_2(1-\alpha_2)r_2\alpha_2 b_2-((1-\alpha_1)r_1 \\
& +\alpha_1)\alpha_2 b_2 m_3(1-\alpha_3)r_3-m_1(1-\alpha_1)r_1\alpha_2 b_2((1-\alpha_3)r_3+\alpha_3),
\end{aligned}
$$

$$
\begin{aligned}
|M_3| = {} & ((1-\alpha_1)r_1+\alpha_1)((1-\alpha_2)r_2+\alpha_2)m_3(1-\alpha_3)r_3+m_1(1-\alpha_1)r_1\alpha_2 b_2\alpha_3 b_3 \\
& +\alpha_1 b_1 m_2(1-\alpha_2)r_2\alpha_3 b_3-m_1(1-\alpha_1)r_1((1-\alpha_2)r_2+\alpha_2)\alpha_3 b_3 \\
& -((1-\alpha_1)r_1+\alpha_1)m_2(1-\alpha_2)r_2\alpha_3 b_3-\alpha_1 b_1\alpha_2 b_2 m_3(1-\alpha_3)r_3.
\end{aligned}
$$

Then $x_1^* = \frac{|M_1|}{|M|}$, $x_2^* = \frac{|M_2|}{|M|}$, $x_3^* = \frac{|M_3|}{|M|}$ and $G_1^* = \frac{|M_1|}{|M|} + b_1\frac{|M_2|+|M_3|}{|M|}$.

Evolutionarily stable conjectures of player 1 are found from the problem

$$\max_{r_1}(m_1 - x_1^*(r_1,\mathbf{r}_{-1}))^{\alpha_i} G_1^*(r_1,\mathbf{r}_{-1})^{1-\alpha_i}$$

The first-order condition for maximization is

$$(m_1 - x_1^*)^{\alpha_1 - 1}(G_1^*)^{-\alpha_1}\left(-\alpha_1 G_1^*\frac{dx_1^*}{dr_1} + (1-\alpha_1)(m_1 - x_1^*)\frac{dG_1^*}{dr_1}\right) = 0.$$

Since in a CVE $-\alpha_i G_i + (1-\alpha_i)(m_i - x_i)r_i = 0$, the condition simplifies to

$$(m_1 - x_1^*)^{\alpha_1}(G_1^*)^{-\alpha_1}(1-\alpha_1)\left(-r_1\frac{dx_1^*}{dr_1} + \frac{dG_1^*}{dr_1}\right) = 0. \tag{6}$$

Consider $\frac{dx_1^*}{dr_1} = \frac{1}{|M|}\left(\frac{\partial |M_1|}{\partial r_1}|M| - |M_1|\frac{\partial |M|}{\partial r_1}\right)$. Since $\frac{\partial |M_1|}{\partial r_1} = m_1(1-\alpha_1)((1-\alpha_2)r_2+\alpha_2)((1-\alpha_3)r_3+\alpha_3) - m_1(1-\alpha_1)\alpha_2 b_2\alpha_3 b_3$ and $\frac{\partial |M|}{\partial r_1} = (1-\alpha_1)((1-\alpha_2)r_2+\alpha_2)((1-\alpha_3)r_3+\alpha_3) - (1-\alpha_1)\alpha_2 b_2\alpha_3 b_3$,

$$\frac{dx_1^*}{dr_1} = \frac{|K_1|}{|M|}(((1-\alpha_2)r_2+\alpha_2)((1-\alpha_2)r_3+\alpha_3) - \alpha_2 b_2\alpha_3 b_3),$$

where $|K_1| = \alpha_1(1-\alpha_1)m_1((1-\alpha_2)r_2(1-\alpha_3)r_3 + (1-\alpha_2)r_2\alpha_3(1-b_1b_3) + \alpha_2(1-\alpha_3)r_3(1-b_1b_2) + \alpha_2\alpha_3(1-b_2b_3-b_1b_3-b_1b_2+2b_1b_2b_3) + b_1(((1-\alpha_2)r_2 + \alpha_2(1-b_2))m_3(1-\alpha_3)r_3 + m_2(1-\alpha_2)r_2((1-\alpha_3)r_3 + \alpha_3(1-b_3)))) > 0$.

Consider now $\frac{dG_1^*}{dr_1} = \frac{dx_1^*}{dr_1} + b_1\left(\frac{dx_2^*}{dr_1} + \frac{dx_3^*}{dr_1}\right)$. Since $\frac{dx_2^*}{dr_1} = \frac{1}{|M|}\left(\frac{\partial |M_2|}{\partial r_1}|M| - |M_2|\frac{\partial |M|}{\partial r_1}\right)$ and $\frac{\partial |M_2|}{\partial r_1} = (1-\alpha_1)m_2(1-\alpha_2)r_2((1-\alpha_3)r_3+\alpha_3)+m_1(1-\alpha_1)\alpha_2b_2\alpha_3b_3-(1-\alpha_1)\alpha_2b_2m_3(1-\alpha_3)r_3 - m_1(1-\alpha_1)\alpha_2b_2((1-\alpha_3)r_3+\alpha_3)$,

$$\frac{dx_2^*}{dr_1} = \frac{|K_1|}{|M|}\alpha_2b_2((1-\alpha_3r_3)+\alpha_3-a_3b_3).$$

Analogously,

$$\frac{dx_3^*}{dr_1} = \frac{|K_1|}{|M|}\alpha_3b_3((1-\alpha_2r_2)+\alpha_2-a_2b_2).$$

Therefore, $-r_1\frac{dx_1^*}{dr_1} + \frac{dG_1^*}{dr_1} = 0$ is equivalent to

$$(1-r_1)(((1-\alpha_2)r_2+\alpha_2)((1-\alpha_2)r_3+\alpha_3)-\alpha_2b_2\alpha_3b_3) +$$
$$b_1(\alpha_2b_2((1-\alpha_3r_3)+\alpha_3-\alpha_3b_3)+a_3b_3((1-\alpha_2r_2)+\alpha_2-a_2b_2)) = 0$$

and thus a candidate evolutionarily stable conjecture is

$$r_1^{ES} = 1 - b_1\frac{(a_2b_2((1-\alpha_3r_3)+\alpha_3-a_3b_3)+a_3b_3((1-\alpha_2r_2)+\alpha_2-a_2b_2))}{(((1-\alpha_2)r_2+\alpha_2)((1-\alpha_2)r_3+\alpha_3)-\alpha_2b_2\alpha_3b_3)},$$

the same as the consistent conjecture.

Now note that the left-hand side of the first order condition (6) is positive for $r_1 < r_1^{ES}$ and negative for $r_1 > r_1^{ES}$. Thus r_1^{ES} is indeed evolutionarily stable.

Proposition 2 *If the parameters of the semi-public good game of this section are such that for given r_j, r_k and consistent*

$$r_i^C = 1 - b_i\frac{(a_jb_j((1-\alpha_kr_k)+\alpha_k-a_kb_k)+a_kb_k((1-\alpha_jr_j)+\alpha_j-a_jb_j))}{(((1-\alpha_j)r_j+\alpha_j)((1-\alpha_j)r_k+\alpha_k)-\alpha_jb_j\alpha_kb_k)}$$

the CVE $(\mathbf{x}^*(\mathbf{r}), \mathbf{A}^*(\mathbf{r}))$ is interior, then conjecture r_i^C is evolutionarily stable for player i.

To illustrate the proposition, consider first the symmetric case $m_1 = m_2 = m_3 = m$, $a_1 = a_2 = a_3 = a$, $b_1 = b_2 = b_3 = b$. It is then natural to expect conjectures to be symmetric too, $r_1 = r_2 = r_3 = r$. The (mutual) consistency condition then reduces to $r = 1 - \frac{2\alpha b^2(\alpha+(1-\alpha)r-\alpha b)}{(\alpha+(1-\alpha)r)^2-\alpha^2b^2} = 1 - \frac{2\alpha b^2}{\alpha+(1-\alpha)r+\alpha b}$. This holds if $(1-\alpha)r^2 + (2\alpha+\alpha b-1)r+\alpha(2b^2-b-1) = 0$. For $r = 0$, the left-hand side is $\alpha(2b^2-b-1) < 0$ for $0 < b < 1$; for $r = 1$, the left-hand side is $2\alpha b^2 > 0$. For positive values of

Table 1 (Mutually) consistent conjectures in the public good game with $m = 1$, $b = 0.5$

α_1	α_2	α_3	$r_1^C = r_1^{ES}$	$r_2^C = r_2^{ES}$	$r_3^C = r_3^{ES}$	x_1^*	x_2^*	x_3^*
0.25	0.25	0.25	0.879	0.879	0.879	0.569	0.569	0.569
0.5	0.5	0.5	0.781	0.781	0.781	0.281	0.281	0.281
0.75	0.75	0.75	0.712	0.712	0.712	0.106	0.106	0.106
0.25	0.5	0.5	0.784	0.821	0.821	0.637	0.217	0.217
0.4	0.5	0.5	0.782	0.797	0.797	0.423	0.255	0.255
0.65	0.5	0.5	0.779	0.758	0.758	0.070	0.320	0.320

conjectures there is thus one consistent $r^C \in (0, 1)$, confirming the result in Costrell (1991) that consistent conjectures correspond to negative reactions, i.e. if a player increases his or her contributions, the other players decrease theirs.[14]

The proposition can be used to find consistent and evolutionarily stable conjectures also for cases that are asymmetric either in parameters or conjectures. For example, consider symmetric values of parameters $m = 1$, $b = 0.5$ and $\alpha = 0.5$. If players 2 and 3 have the (Nash) conjectures $r_2 = r_3 = 1$, then the consistent conjecture for player 1 is $r_1^C = 0.8$ (and it is evolutionarily stable because the CVE for these conjectures is interior). Table 1 shows the numerical calculations to find (mutually) consistent conjectures for some particular values of the parameters, and it also shows that the CVEs for these conjectures are interior. Therefore the consistent conjectures in Table 1 are also evolutionarily stable. Note that as the parameter α increases, less weight is put in the utility function on the public good; mutually consistent conjectures then decrease and so do contributions to the public good. The last line in the table shows that asymmetries between players should not be too large for an interior solution to exist; if the parameter α_1 increased further, x_1^* becomes 0 and the propositions cease to apply.

4.2 Contests

Consider rent-seeking contests introduced in Tullock (1980) and investigated using the techniques for aggregative games in Cornes and Hartley (2003, 2005). Each player i contributes a costly effort $x_i \geq 0$ and can win a prize of value V with probability $\frac{x_i}{x_1+\ldots+x_n}$. Each player i's payoff function is thus given by

$$u_i(x_i, A) = \frac{x_i}{A}V - c_i x_i,$$

[14]Note that if $b = 1$, then $r = 0$ is the solution of the consistency condition (Sugden 1985). However, for $r = 0$ the solution of the players' maximization problem is not interior and the first-order conditions do not characterize it. The propositions do not apply in this case.

where $A = x_1 + \ldots + x_n$ is the aggregate.[15] This aggregate is the same for all players; the game is truly aggregative. The game can still be asymmetric though, represented by possibly different marginal costs c_i of the players.

Consider player i with conjecture $r_i \geq 0$. The first-order conditions for player i's payoff maximization problem is $F_i = \frac{b_i A - b_i x_i r_i}{A^2} V - c_i = 0$, or $\frac{1}{A^2}(b_i V(A - x_i r_i) - c_i A^2) = 0$. Writing $A = x_i + A_{-i}$, the first order condition becomes $\frac{1}{A^2}(-c_i x_i^2 + ((1 - r_i)V - 2c_i A_{-i})x_i + (V - c_i A_{-i})A_{-i} = 0$. The left-hand side is negative as $x_i \to \infty$ and positive at $x_i = 0$ if $V - c_i A_{-i} > 0$. In this case, the equation

$$V(A - x_i r_i) - c_i A^2 = 0 \tag{7}$$

characterizes the choice of player i.

Consider again for illustration the case of three players ($n = 3$). The system describing a CVE is

$$\begin{aligned} V(A - x_1 r_1) - c_1 A^2 &= 0 \\ V(A - x_2 r_2) - c_2 A^2 &= 0 \\ V(A - x_3 r_3) - c_3 A^2 &= 0 \\ A - x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(since there is only one aggregate, there is only one additional accounting equation). To solve the system, from the first three equations $x_i = \frac{A}{r_i V}(V - c_i A)$. Therefore $A - \frac{A}{r_1 V}(V - c_1 A) - \frac{A}{r_2 V}(V - c_2 A) - \frac{A}{r_3 V}(V - c_3 A) = 0$, or $r_1 r_2 r_3 V^3 - r_2 r_3 V^2 (V - c_1 A) - r_1 r_2 V^2 (V - c_2 A) - r_1 r_2 V^2 (V - c_3 A) = 0$. Thus

$$A^* = \frac{r_2 r_3 + r_1 r_3 + r_1 r_2 - r_1 r_2 r_3}{r_2 r_3 c_1 + r_1 r_3 c_2 + r_1 r_2 c_3} V. \tag{8}$$

To find consistent conjectures of player 1, consider the system

$$\begin{aligned} V(A - x_2 r_2) - c_2 A^2 &= 0 \\ V(A - x_3 r_3) - c_3 A^2 &= 0 \\ A - x_1 - x_2 - x_3 &= 0. \end{aligned}$$

Solving for x_2 and x_3 from the first two equations and substituting into the third one gives $A - x_1 - \frac{A}{r_2 V}(V - c_2 A) - \frac{A}{r_3 V}(V - c_3 A) = 0$, or $(r_3 c_2 + r_2 c_3)A^2 + (r_2 r_3 - r_2 - r_3)VA - r_2 r_3 V x_1 = 0$. Using the implicit function theorem,

$$\frac{dA^{**}}{dx_1} = \frac{r_2 r_3 V}{2(r_3 c_2 + r_2 c_3)A^{**} + (r_2 r_3 - r_2 - r_3)V}.$$

[15] To avoid indeterminacies, let $u_i = \frac{1}{n}$ if $A = 0$.

Since $A^{**}(x_1^*) = A^*$, $\frac{dA^{**}}{dx_1}(x_1^*)$ can be found using A^*. Rearranging, the consistent conjecture satisfies

$$r_1 = \frac{r_2r_3(r_2r_3c_1 + r_1r_3c_2 + r_1r_2c_3)}{2(r_3c_2 + r_2c_3)(r_2r_3 + r_1r_3 + r_1r_2 - r_1r_2r_3) + (r_2r_3 - r_2 - r_3)(r_2r_3c_1 + r_1r_3c_2 + r_1r_2c_3)}.$$

Simplifying this expression leads to $(c_2r_1r_3 - c_1r_2r_3 + c_3r_1r_2)(r_2r_3 + r_1r_3 + r_1r_2 - r_1r_2r_3) = 0$. If $r_2r_3 + r_1r_3 + r_1r_2 - r_1r_2r_3 = 0$, then $A^* = 0$ thus $x_1^* = x_2^* = x_3^* = 0$, which is not an interior equilibrium. Thus consider $c_2r_1r_3 - c_1r_2r_3 + c_3r_1r_2 = 0$. The consistent conjecture of player 1 possibly leading to an interior CVE is thus

$$r_1^C = \frac{c_1r_2r_3}{r_3c_2 + r_2c_3}.$$

For evolutionary stability analysis of conjectures of player 1, consider

$$\max_{r_1} \frac{x_1^*(r_1, \mathbf{r}_{-1})}{A^*(r_1, \mathbf{r}_{-1})} V - c_1 x_1^*(r_1, \mathbf{r}_{-1}).$$

The first order condition for maximization is $\frac{1}{(A^*)^2} V \left(\frac{dx_1^*}{dr_1} A^* - x_1^* \frac{dA^*}{dr_1} \right) - c_1 \frac{dx_1^*}{dr_1} = 0$. Using Eq. (7), the condition can be rewritten as

$$\frac{Vx_1^*}{(A^*)^2} \left(r_1 \frac{dx_1^*}{dr_1} - \frac{dA^*}{dr_1} \right) = 0. \tag{9}$$

Equation (7) also implies that $x_1^* = \frac{1}{r_1} A^* - \frac{c_1}{r_1 V} (A^*)^2$. Therefore $\frac{dx_1^*}{dr_1} = \frac{1}{r_1^2} \left(\frac{dA^*}{dr_1} r_1 - A^* \right) - \frac{c_1}{V} \frac{1}{r_1^2} \left(2A^* \frac{dA^*}{dr_1} r_1 - (A^*)^2 \right)$. Equation (9) can then be written as

$$\frac{x_1^*}{(A^*)^2 r_1} \left(c_1 A^* - V - 2c_1 r_1 \frac{dA^*}{dr_1} \right) = 0.$$

Using A^* from Eq. (8), finding $\frac{dA^*}{dr_1} = \frac{-r_2r_3(c_2r_3 + c_3r_2 - c_1r_3 - c_1r_2 + c_1r_2r_3)}{(r_2r_3c_1 + r_1r_3c_2 + r_1r_2c_3)^2} V$, and substituting, the first-order condition (9) becomes

$$\frac{x_1^*}{(A^*)^2} \frac{-(c_3r_1r_2 - c_1r_2r_3 + c_2r_1r_3)(c_2r_3 + c_3r_2 - c_1r_3 - c_1r_2 + c_1r_2r_3)}{(r_2r_3c_1 + r_1r_3c_2 + r_1r_2c_3)^2} V = 0. \tag{10}$$

Therefore, provided that the CVE is interior, the first order condition is satisfied only if $c_3r_1r_2 - c_1r_2r_3 + c_2r_1r_3 = 0$, i.e.

$$r_1^{ES} = \frac{c_1r_2r_3}{c_3r_2 + c_2r_3}.$$

The unique candidate for the evolutionarily stable conjecture of player 1 is the consistent conjecture of the player.

Suppose that $c_2r_3+c_3r_2-c_1r_3-c_1r_2+c_1r_2r_3 = (c_2-c_1)r_3+(c_3-c_1)r_2+c_1r_2r_3 > 0$, which is the case unless all $r_i = 0$ or unless player 1 has marginal cost much higher than those of the other players. Then the left-hand side of (10) is positive if $r_1 < r_1^{ES}$ and negative if $r_1 > r_1^{ES}$. The consistent conjecture is then evolutionarily stable.

Proposition 3 *If the parameters of the contest game of this section are such that for given r_j, r_k and consistent*

$$r_i^C = \frac{c_i r_j r_k}{c_k r_j + c_j r_k}$$

the CVE $(\mathbf{x}^*(\mathbf{r}), \mathbf{A}^*(\mathbf{r}))$ is interior and $(c_j - c_i)r_k + (c_k - c_i)r_j + c_i r_j r_k > 0$, then conjecture r_i^C is evolutionarily stable for player i.

Consider again a few numerical examples to illustrate the proposition. Suppose that the players are symmetric, $c_1 = c_2 = c_3 = c$, and that they hold symmetric conjectures $r_1 = r_2 = r_3 = r$. Then the condition for the (mutually) consistent conjectures becomes $r = \frac{r}{2}$. The consistent conjecture is then $r^C = 0$, i.e. each player expects that a increase in his or her effort is fully offset by the decrease in the effort of the other players, leaving A unchanged. Although $A^* = V$ and $x_i^* = \frac{V}{3}$ is an interior equilibrium with such a conjecture, the proposition does not apply because the condition $(c_j - c_i)r_k + (c_k - c_i)r_j + c_i r_j r_k > 0$ is not satisfied. Indeed, if $r_j = 0$ for one of the players, then Eq. (7) becomes $A(V - A) = 0$, leading to $A = V$ in equilibrium and zero payoff to all players. Any conjecture $r_i \neq 0$ of player i implies a corner solution $x_i^* = 0$ in a CVE, again with zero payoff. Therefore $r^C = 0$ for all players is not evolutionarily stable but can be seen as *neutrally* stable for player *i*: alternative conjectures cannot lead to a higher payoff although they can be equally successful.[16]

Although the proposition does not apply to the symmetric case, it still can be used for asymmetric costs or conjectures. Table 2 shows numerical calculations for finding consistent conjectures of player 1, for given conjectures of players 2 and 3 (conjectures r_2 and r_3 are not consistent; mutually consistent conjectures are always zero for the parameters in the table). Those conjectures of player 1 are also evolutionarily stable because the conditions in Proposition 3 are satisfied. Consistent conjectures of player 1 increase with the given conjectures of the other players and with the cost of player 1 but typically stay below unity, implying that the player correctly expects the aggregate to increase by less than the increase in his or her own effort. However, it is also possible that player 1 correctly anticipates

[16]For $n = 2$, there are non-zero symmetric conjectures that are consistent and evolutionarily stable. The consistency condition for $n = 2$ is $r_i = \frac{c_i}{c_j} r_j$. If $c_i = c_j$, then any r is consistent. It is shown in Possajennikov (2009) that any $0 < r < 2$ is evolutionarily stable then.

Table 2 Consistent conjectures for player 1 in the contest game with $V = 1$

c_1	c_2	c_3	$r_1^C = r_1^{ES}$	r_2	r_3	x_1^*	x_2^*	x_3^*
1	1	1	0.375	0.75	0.75	0.406	0.203	0.203
1	1	1	0.5	1	1	0.375	0.188	0.188
1	1	1	0.625	1.25	0.125	0.344	0.172	0.172
0.75	1	1	0.281	0.75	0.75	0.925	0.027	0.027
0.75	1	1	0.375	1	1	0.764	0.076	0.076
0.75	1	1	0.469	1.25	1.25	0.655	0.010	0.010
1.25	1	1	0.781	1.25	1.25	0.200	0.194	0.194
2	1	1	1.25	1.25	1.25	0.044	0.197	0.197

that the aggregate increases by more than the increase in player's own effort if the cost and the other players' conjectures are high enough (the last line of the table). Equilibrium efforts are inversely related to cost parameters and to conjectures; but it is possible (the penultimate line in the table) that a player with a higher cost makes a higher effort in equilibrium than the other players, due to this player holding lower conjectures (that also happen to be consistent).

5 Conclusion

Richard Cornes has done much work on public good games, on contests, and on games with aggregative structure in general. Some of his work also considered conjectural variations, mostly in public good games. In this paper I also consider conjectures and I use representations of games that share some properties with aggregative games. In such representations, there is a personalized aggregate for each player; I call these representations semi-aggregative.

The idea of a semi-aggregative representation is that a player forms appropriate conjectures about how the aggregate changes and how it affects the player's payoff. In a sense, the game is reduced to just two players: the player him- or herself and the aggregate opponent. Thus the dimensionality of players' conjectures is reduced and such conjectures can be analyzed.

I show that if conjectures are subject to evolution, then only consistent conjectures can be evolutionarily stable. The result provides foundations for the (much discussed) notion of consistent conjectures as the result of evolution. On the other hand, the result can be used to find evolutionarily stable conjectures more easily, through finding first consistent conjectures. While this observation is not new for some classes of games, the result in this paper extends it to any well-behaved game.

The result is illustrated on the examples of (impure) public good games and contests. Although finding the exact value of consistent (and evolutionarily stable) conjectures in specific asymmetric games is still a difficult task (thus only 3-player examples are considered for illustration), the point of the examples is to demonstrate that it can be done, and that often consistent conjectures are indeed evolutionarily stable. The choice of public good games and contests as the examples shows that those games, to which Richard Cornes dedicated much of his work, are still a source of useful insights.

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