Vector Inequalities for a Projection in Hilbert Spaces and Applications

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Abstract In this paper we establish some vector inequalities related to Schwarz and Buzano results. Applications for norm and numerical radius inequalities of two bounded operators are given as well.

Keywords Hilbert space • Schwarz inequality • Buzano inequality • Orthogonal projection • Numerical radius • Norm inequalities

1 Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz's inequality*

$$\|x\| \, \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1}$$

The equality case holds in (1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [23]) established the following refinement of (1):

$$\|x\| \, \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

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and by (2) we get

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \\ \geq 2 |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle| ,$$

which implies the *Buzano's inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

For other Schwarz and Buzano related inequalities in inner product spaces, see [1–10, 12–15, 17, 19–25, 27–36], and the monographs [11, 16] and [18].

Now, let us recall some basic facts on *orthogonal projection* that will be used in the sequel.

If K is a subset of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, the set of *vectors orthogonal* to K is defined by

$$K^{\perp} := \{x \in H : \langle x, k\rangle = 0 \text{ for all } k \in K\} .$$

We observe that $K^{\perp}$ is a *closed subspace* of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^{\perp}$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $K^{\perp}$. Therefore, H is the *internal Hilbert direct sum* of V and $V^{\perp}$, and we denote that as $H = V \oplus V^{\perp}$.

The linear operator $P_V : H \to H$ that maps x to v is called *the orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all *bounded self-adjoint* operators P such that $P^2 = P$. Specifically, the orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of V, which minimizes the distance $\|x - v\|$. This provides the geometrical interpretation of $P_V(x)$: it is *the best approximation* to x by elements of V.

Projections P_U and P_V are called *mutually orthogonal* if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. The sum of the two projections P_U and P_V is a projection only if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$.The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U\cap V}$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2 \le \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j\in J}$ an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \left\langle x, e_j\right\rangle e_j\,,\ x \in H. \tag{4}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y\rangle = \sum_{j\in J} \left\langle x, e_j\right\rangle \left\langle e_j, y\right\rangle,\ x, y \in H \text{ and } \langle P_{\mathcal{E}} x, x\rangle = \sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2,\ x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}\,,\ \|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle\, e,\ x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle\, \langle e, y\rangle\,,\ x, y \in H \tag{5}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \ge |\langle P_e x, y\rangle| \tag{6}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

Motivated by the above results we establish in this paper some vector inequalities for an orthogonal projection P that generalizes amongst others the Buzano's inequality (6). Applications for norm and numerical radius inequalities are provided as well.

2 Vector Inequalities for a Projection

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathscr{B}(H)$ that $0 \le P \le 1_H$.

The following result holds:

Theorem 1. *Let $P : H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$\|x\| \|y\| - \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \ge |\langle x, y\rangle - \langle Px, y\rangle| . \tag{7}$$

and

$$\|x\| \|y\| - \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} \ge |\langle Px, y\rangle| . \tag{8}$$

Proof. Using the properties of projection, we have

$$\begin{aligned}\langle x - Px, y - Py\rangle &= \langle x, y\rangle - \langle Px, y\rangle - \langle x, Py\rangle + \langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, y\rangle + \left\langle P^2x, y\right\rangle \\ &= \langle x, y\rangle - \langle Px, y\rangle\end{aligned} \tag{9}$$

for any $x, y \in H$.

By Schwarz's inequality we have

$$\|x - Px\|^2 \|y - Py\|^2 \ge |\langle x - Px, y - Py\rangle|^2 \tag{10}$$

for any $x, y \in H$.

Since, by (7), we have

$$\|x - Px\|^2 = \|x\|^2 - \langle Px, x\rangle , \quad \|y - Py\|^2 = \|y\|^2 - \langle Py, y\rangle ,$$

then by (10) we have

$$\left(\|x\|^2 - \langle Px, x\rangle\right)\left(\|y\|^2 - \langle Py, y\rangle\right) \ge |\langle x, y\rangle - \langle Px, y\rangle|^2 \tag{11}$$

for any $x, y \in H$.

Using the elementary inequality that holds for any real numbers a, b, c, d

$$(ac - bd)^2 \ge \left(a^2 - b^2\right)\left(c^2 - d^2\right),$$

we have

$$\left(\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2}\right)^2 \geq \left(\|x\|^2 - \langle Px, x\rangle\right)\left(\|y\|^2 - \langle Py, y\rangle\right) \quad (12)$$

for any $x, y \in H$.

Since

$$\|x\| \geq \langle Px, x\rangle^{1/2}, \ \ \|y\| \geq \langle Py, y\rangle^{1/2},$$

then

$$\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} \geq 0,$$

for any $x, y \in H$.

By (11) and (12) we get

$$\left(\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2}\right)^2 \geq |\langle x, y\rangle - \langle Px, y\rangle|^2$$

for any $x, y \in H$, which, by taking the square root, is equivalent to the desired inequality (7).

Observe that, if P is an orthogonal projection, then $Q := 1_H - P$ is also a projection. Indeed we have

$$Q^2 = (1_H - P)^2 = 1_H - 2P + P^2 = 1_H - P = Q.$$

Now, if we write the inequality (7) for the projection Q we get the desired inequality (8).

Corollary 1. *With the assumptions of Theorem 1, we have the following refinements of Schwarz inequality:*

$$\begin{aligned}\|x\|\,\|y\| &\geq \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \\ &\geq |\langle Px, y\rangle| + |\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle|\end{aligned} \quad (13)$$

and

$$\begin{aligned}\|x\|\,\|y\| &\geq \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + |\langle Px, y\rangle| \\ &\geq |\langle x, y\rangle - \langle Px, y\rangle| + |\langle Px, y\rangle| \geq |\langle x, y\rangle|\end{aligned} \quad (14)$$

for any $x, y \in H$.

Remark 1. Since

$$|\langle x,y\rangle - \langle Px,y\rangle| \geq |\langle x,y\rangle| - |\langle Px,y\rangle|$$

then by the first inequality in (13) we have

$$\|x\|\,\|y\| \geq \langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} + |\langle x,y\rangle| - |\langle Px,y\rangle|$$

that produces the inequality

$$\|x\|\,\|y\| - |\langle x,y\rangle| \geq \langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} - |\langle Px,y\rangle| \geq 0 \tag{15}$$

for any $x,y \in H$.

We notice that the second inequality follows by Schwarz's inequality for the nonnegative self-adjoint operator P.

Since

$$|\langle x,y\rangle - \langle Px,y\rangle| \geq |\langle Px,y\rangle| - |\langle x,y\rangle|$$

then by (13) we have

$$\begin{aligned}\|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq \langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} + |\langle Px,y\rangle| - |\langle x,y\rangle|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x\|\,\|y\| + |\langle x,y\rangle| &\geq \langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} + |\langle Px,y\rangle| \\ &\geq 2\,|\langle Px,y\rangle|\end{aligned}$$

and is equivalent to

$$\begin{aligned}\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] &\geq \frac{1}{2}\left[\langle Px,x\rangle^{1/2}\,\langle Py,y\rangle^{1/2} + |\langle Px,y\rangle|\right] \\ &\geq |\langle Px,y\rangle|\end{aligned} \tag{16}$$

for any $x,y \in H$.

The inequality between the first and last term in (16), namely

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq |\langle Px,y\rangle| \tag{17}$$

for any $x,y \in H$ is a generalization of Buzano's inequality (3).

From the inequality (14) we can state that

$$\|x\|\,\|y\| - |\langle Px, y\rangle| \geq \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} \geq |\langle x, y\rangle - \langle Px, y\rangle| \tag{18}$$

for any $x, y \in H$.

From the inequality (14) we also have

$$\begin{aligned}\|x\|\,\|y\| &\geq \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + |\langle Px, y\rangle| \\ &\geq |\langle x, y\rangle - \langle Px, y\rangle| + |\langle Px, y\rangle| \geq |\langle Px, y\rangle| - |\langle x, y\rangle| + |\langle Px, y\rangle| \\ &= 2\,|\langle Px, y\rangle| - |\langle x, y\rangle|,\end{aligned}$$

which implies that

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq \frac{1}{2}\left[\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}\right] + \frac{1}{2}\left[|\langle Px, y\rangle| + |\langle x, y\rangle|\right] \geq |\langle Px, y\rangle| \tag{19}$$

for any $x, y \in H$.

The case of orthonormal families which is related to Bessel's inequality is of interest.

Let $\mathscr{E} = \{e_j\}_{j\in J}$ be an othonormal family in H. Then for any $x, y \in H$ we have from (13) and (14) the inequalities

$$\begin{aligned}\|x\|\,\|y\| &\geq \left(\sum_{j\in J} \left|\langle x, e_j\rangle\right|^2\right)^{1/2} \left(\sum_{j\in J} \left|\langle y, e_j\rangle\right|^2\right)^{1/2} \\ &\quad + \left|\langle x, y\rangle - \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| \\ &\geq \left|\sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| + \left|\langle x, y\rangle - \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| \geq |\langle x, y\rangle|\end{aligned} \tag{20}$$

and

$$\|x\|\,\|y\| \geq \left(\|x\|^2 - \sum_{j\in J} \left|\langle x, e_j\rangle\right|^2\right)^{1/2} \left(\|y\|^2 - \sum_{j\in J} \left|\langle y, e_j\rangle\right|^2\right)^{1/2} \tag{21}$$

$$+\left|\left\langle \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right|$$

$$\geq \left|\langle x,y\rangle-\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|+\left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|\geq |\langle x,y\rangle|\,.$$

By (15) and (16) we have

$$\begin{aligned}&\|x\|\,\|y\|-|\langle x,y\rangle| \\ &\geq \left(\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\left(\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2}-\left|\left\langle\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right|\geq 0\end{aligned}\tag{22}$$

and

$$\begin{aligned}\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right] &\geq \frac{1}{2}\left(\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\left(\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2} \\ &\quad+\frac{1}{2}\left|\left\langle\left\langle\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right\rangle\right| \\ &\geq \left|\left\langle\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right|\end{aligned}\tag{23}$$

for any $x, y \in H$.

The inequality between the first and last term in (23) provides a generalization of Buzano's inequality for orthonormal families $\mathscr{E}=\{e_j\}_{j\in J}$.

The following result holds:

Theorem 2. *Let $P: H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$|\langle x,y\rangle-2\langle Px,y\rangle|\leq \|x\|\,\|y\|\,,\tag{24}$$

$$\begin{aligned}&|\langle x,y\rangle-\langle Px,y\rangle| \\ &\leq \min\left\{\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2},\|y\|\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\right\}\end{aligned}\tag{25}$$

$$\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2}+\|y\|\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-\langle Py,y\rangle-\langle Px,x\rangle\right)^{1/2}$$

and

$$|\langle Px,y\rangle| \leq \min\left\{\|x\|\langle Py,y\rangle^{1/2},\|y\|\langle Px,x\rangle^{1/2}\right\} \tag{26}$$
$$\leq \frac{1}{2}\left[\|x\|\langle Py,y\rangle^{1/2}+\|y\|\langle Px,x\rangle^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\langle Px,x\rangle+\langle Py,y\rangle\right)^{1/2}.$$

Proof. Observe that

$$\begin{aligned}\|x-2Px\|^2 &= \|x\|^2-4\operatorname{Re}\langle x,Px\rangle+4\langle Px,Px\rangle\\ &= \|x\|^2-4\langle x,Px\rangle+4\left\langle P^2x,x\right\rangle\\ &= \|x\|^2-4\langle x,Px\rangle+4\langle Px,x\rangle=\|x\|^2\end{aligned}$$

for any $x \in H$.

Using Schwarz's inequality we have

$$\|x\|\|y\| = \|x-2Px\|\|y\| \geq |\langle x-2Px,y\rangle| = |\langle x,y\rangle-2\langle Px,y\rangle|$$

for any $x, y \in H$ and the inequality (24) is proved.

By Schwarz's inequality we also have

$$\|x-Px\|\|y\| \geq |\langle x-Px,y\rangle| = |\langle x,y\rangle-\langle Px,y\rangle|$$

and

$$\|x\|\|y-Py\| \geq |\langle x,y-Py\rangle| = |\langle x,y\rangle-\langle x,Py\rangle| = |\langle x,y\rangle-\langle Px,y\rangle|$$

for any $x, y \in H$, which implies the first inequality in (25).

The second and the third inequalities are obvious by the elementary inequalities

$$\min\{a,b\} \leq \frac{1}{2}(a+b),\ a,b \in \mathbb{R}_+$$

and

$$ac+bd \leq \left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2},\ a,b,c,d \in \mathbb{R}_+.$$

The inequality (26) follows from (25) by replacing P with $1_H - P$.

Remark 2. By the triangle inequality we have

$$\|x\|\,\|y\| + |\langle x, y\rangle| \geq |\langle x, y\rangle - 2\langle Px, y\rangle| + |\langle x, y\rangle| \geq 2\,|\langle Px, y\rangle|\,,$$

which implies that [see also (16) and (19)]

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq |\langle Px, y\rangle| \tag{27}$$

for any $x, y \in H$.

From (25) we also have

$$\begin{aligned} &|\langle Px, y\rangle| \\ &\leq |\langle x, y\rangle| + \min\left\{\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right\} \end{aligned} \tag{28}$$

and

$$\begin{aligned} &|\langle x, y\rangle| \\ &\leq |\langle Px, y\rangle| + \min\left\{\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right\} \end{aligned} \tag{29}$$

for any $x, y \in H$.

Now, if $\mathscr{E} = \{e_j\}_{j\in J}$ is an orthonormal family, then by the inequalities (24) and (25) we have

$$\left|\langle x, y\rangle - 2\sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right| \leq \|x\|\,\|y\|\,, \tag{30}$$

and

$$\begin{aligned} &\left|\langle x, y\rangle - \sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\right| \\ &\leq \min\left\{\|x\|\left(\|y\|^2 - \sum_{j\in J}\left|\langle y, e_j\rangle\right|^2\right)^{1/2}, \|y\|\left(\|x\|^2 - \sum_{j\in J}\left|\langle x, e_j\rangle\right|^2\right)^{1/2}\right\} \end{aligned} \tag{31}$$

$$\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2-\sum_{j\in J}\left|\left\langle y,e_j\right\rangle\right|^2\right)^{1/2}+\|y\|\left(\|x\|^2-\sum_{j\in J}\left|\left\langle x,e_j\right\rangle\right|^2\right)^{1/2}\right]$$

$$\leq \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-\sum_{j\in J}\left|\left\langle y,e_j\right\rangle\right|^2-\sum_{j\in J}\left|\left\langle x,e_j\right\rangle\right|^2\right)^{1/2}$$

for any $x, y \in H$.

From (28) we also have

$$\left|\sum_{j\in J}\left\langle x,e_j\right\rangle\left\langle e_j,y\right\rangle\right|$$
$$\leq |\langle x,y\rangle| + \min\left\{\|x\|\left(\|y\|^2-\sum_{j\in J}\left|\left\langle y,e_j\right\rangle\right|^2\right)^{1/2}, \|y\|\left(\|x\|^2-\sum_{j\in J}\left|\left\langle x,e_j\right\rangle\right|^2\right)^{1/2}\right\} \tag{32}$$

for any $x, y \in H$.

3 Inequalities for Norm and Numerical Radius

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by Gustafson and Rao [26, p. 1]:

$$W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by Gustafson and Rao [26, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true:

$$w(T) \leq \|T\| \leq 2w(T),\ \text{for any } T \in B(H).$$

Utilizing Buzano's inequality (3) we obtained the following inequality for the numerical radius [13] or [15]:

Theorem 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{33}$$

The constant $\frac{1}{2}$ is best possible in (33).

The following general result for the product of two operators holds [26, p. 37]:

Theorem 4. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then $w(AB) \le 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \le 2w(A)w(B)$. The constant 2 is best possible here.*

The following results are also well known [26, p. 38].

Theorem 5. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{34}$$

If A is an isometry and $AB = BA$, then (34) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. The following result holds [26, p. 38].

Theorem 6. *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|. \tag{35}$$

As a consequence of the above, we have [26, p. 39]:

Corollary 2. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)w(B). \tag{36}$$

A related problem with the inequality (35) is to find the best constant c for which the inequality

$$w(AB) \le cw(A)\|B\|$$

holds for any two commuting operators $A, B \in B(H)$. It is known that $1.064 < c < 1.169$, see [3, 32] and [33].

In relation to this problem, it has been shown in [24] that

Theorem 7. *For any $A, B \in B(H)$ we have*

$$w\left(\frac{AB+BA}{2}\right) \le \sqrt{2} w(A) \|B\|. \tag{37}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

The following result holds.

Theorem 8. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle BPAx, x\rangle| \le \frac{1}{2}\left[\|Ax\| \|B^* x\| + |\langle BAx, x\rangle|\right] \tag{38}$$

and

$$\|BPAx\| \le \frac{1}{2}\left[\|Ax\| \|B\| + \|BAx\|\right] \tag{39}$$

for any $x \in H$.

Moreover, we have

$$w(BPA) \le \frac{1}{2}\left[\|A\| \|B\| + w(BA)\right] \tag{40}$$

and

$$\|BPA\| \le \frac{1}{2}\left[\|A\| \|B\| + \|BA\|\right]. \tag{41}$$

Proof. From the inequality (17) we have

$$|\langle PAx, B^* y\rangle| \le \frac{1}{2}\left[\|Ax\| \|B^* y\| + |\langle Ax, B^* y\rangle|\right]$$

that is equivalent to

$$|\langle BPAx, y\rangle| \le \frac{1}{2}\left[\|Ax\| \|B^* y\| + |\langle BAx, y\rangle|\right] \tag{42}$$

for any $x, y \in H$.

If we take $y = x$ in (42), then we get (38).

Taking the supremum over $y \in H$ with $\|y\| = 1$ in (42) we have

$$\|BPAx\| = \sup_{\|y\|=1} |\langle BPAx, y\rangle| \le \frac{1}{2} \sup_{\|y\|=1} \left[\|Ax\| \|B^* y\| + |\langle BAx, y\rangle|\right]$$

$$\leq \frac{1}{2}\left[\|Ax\| \sup_{\|y\|=1} \|B^{*}y\| + \sup_{\|y\|=1} |\langle BAx, y\rangle|\right]$$
$$= \frac{1}{2}\left[\|Ax\| \, \|B\| + \|BAx\|\right]$$

for any $x \in H$.

The inequalities (40) and (41) follow from (38) and (39) by taking the supremum over $x \in H$ with $\|x\| = 1$.

Corollary 3. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle APAx, x\rangle| \leq \frac{1}{2}\left[\|Ax\| \, \|A^{*}x\| + \left|\langle A^{2}x, x\rangle\right|\right] \tag{43}$$

and

$$\|APAx\| \leq \frac{1}{2}\left[\|Ax\| \, \|A\| + \left\|A^{2}x\right\|\right] \tag{44}$$

for any $x \in H$.

Moreover, we have

$$w(APA) \leq \frac{1}{2}\left[\|A\|^{2} + w\left(A^{2}\right)\right] \tag{45}$$

and

$$\|APA\| \leq \frac{1}{2}\left[\|A\|^{2} + \left\|A^{2}\right\|\right]. \tag{46}$$

Remark 3. Let $e \in H,\ \|e\| = 1$. If we write the inequalities (38) and (39) for the projector P_e defined by $P_e x = \langle x, e\rangle e,\ x \in H$, we have

$$|\langle Ax, e\rangle| \, |\langle Be, x\rangle| \leq \frac{1}{2}\left[\|Ax\| \, \|B^{*}x\| + |\langle BAx, x\rangle|\right] \tag{47}$$

and

$$|\langle Ax, e\rangle| \, \|Be\| \leq \frac{1}{2}\left[\|Ax\| \, \|B\| + \|BAx\|\right] \tag{48}$$

for any $x \in H$.

Now, if we take the supremum over $x \in H,\ \|x\| = 1$ in (48), then we get

$$\|A^*e\| \, \|Be\| \leq \frac{1}{2} \left[\|A\| \, \|B\| + \|BA\|\right] \tag{49}$$

for any $e \in H,\ \|e\| = 1$.

If in (49) we take $B = A$, we have

$$\|A^*e\| \, \|Ae\| \leq \frac{1}{2} \left[\|A\|^2 + \left\|A^2\right\|\right] \tag{50}$$

for any $e \in H,\ \|e\| = 1$.

If in (47) we take $B = A$, then we get

$$|\langle Ax, e\rangle| \, |\langle e, A^*x\rangle| \leq \frac{1}{2} \left[\|Ax\| \, \|A^*x\| + \left|\langle A^2x, x\rangle\right|\right] \tag{51}$$

for any $x \in H$ and $e \in H,\ \|e\| = 1$, and in particular

$$|\langle Ae, e\rangle|^2 \leq \frac{1}{2} \left[\|Ae\| \, \|A^*e\| + \left|\langle A^2e, e\rangle\right|\right] \tag{52}$$

for any $e \in H,\ \|e\| = 1$.

Taking the supremum over $e \in H,\ \|e\| = 1$ in (52) we recapture the result in Theorem 3.

For a given operator T we consider the modulus of T defined as $|T| := (T^*T)^{1/2}$.

Corollary 4. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If A, B are two bounded linear operators on H, then*

$$w(BPA) \leq \frac{1}{2} w(BA) + \frac{1}{4} \left\| |A|^2 + |B^*|^2 \right\|. \tag{53}$$

In particular, we have

$$w(APA) \leq \frac{1}{2} w\left(A^2\right) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\|. \tag{54}$$

Proof. From the inequality (38) we have

$$\begin{aligned} |\langle BPAx, x\rangle| &\leq \frac{1}{2} \left[\|Ax\| \, \|B^*x\| + |\langle BAx, x\rangle|\right] \\ &\leq \frac{1}{2} |\langle BAx, x\rangle| + \frac{1}{4} \left[\|Ax\|^2 + \|B^*x\|^2\right] \end{aligned} \tag{55}$$

for any $x \in H$, where for the second inequality we used the elementary inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right), \ a, b \in \mathbb{R}. \tag{56}$$

Since

$$\begin{aligned}\|Ax\|^2 + \|B^*x\|^2 &= \langle Ax, Ax\rangle + \langle B^*x, B^*x\rangle = \langle A^*Ax, x\rangle + \langle BB^*x, x\rangle \\ &= \left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle\end{aligned}$$

for any $x \in H$, then from (55) we have

$$|\langle BPAx, x\rangle| \leq \frac{1}{2}|\langle BAx, x\rangle| + \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle \tag{57}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (57) we get the desired result (53).

Remark 4. We observe that by (52) we have

$$\begin{aligned}|\langle Ae, e\rangle|^2 &\leq \frac{1}{2}\left[\|Ae\|\,\|A^*e\| + \left|\left\langle A^2e, e\right\rangle\right|\right] \\ &\leq \frac{1}{2}\left|\left\langle A^2e, e\right\rangle\right| + \frac{1}{4}\left[\|Ae\|^2 + \|A^*e\|^2\right] \\ &= \frac{1}{2}\left|\left\langle A^2e, e\right\rangle\right| + \frac{1}{4}\left\langle\left(|A|^2 + |A^*|^2\right)e, e\right\rangle\end{aligned} \tag{58}$$

for any $e \in H$, $\|e\| = 1$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (58) we get

$$w^2(A) \leq \frac{1}{2}w\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|, \tag{59}$$

for any bounded linear operator A.

Since

$$\left\||A|^2 + |A^*|^2\right\| \leq \left\||A|^2\right\| + \left\||A^*|^2\right\| = 2\|A\|^2,$$

then the inequality (59) is better than the inequality in Theorem 3.

The following result also holds:

Theorem 9. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w\left(B\left(\frac{1}{2}1_H - P\right)A\right) \le \frac{1}{4}\left\| |A|^2 + |B^*|^2 \right\|. \tag{60}$$

In particular, we have

$$w\left(A\left(\frac{1}{2}1_H - P\right)A\right) \le \frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\|. \tag{61}$$

Proof. From the inequality (24) we have

$$|\langle (1_H - 2P) Ax, B^*x\rangle| \le \|Ax\| \|B^*x\|,$$

that is equivalent to

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \le \frac{1}{2}\|Ax\| \|B^*x\| \tag{62}$$

for any $x \in H$.

Using the elementary inequality (56) we have

$$\frac{1}{2}\|Ax\| \|B^*x\| \le \frac{1}{4}\left(\|Ax\|^2 + \|B^*x\|^2\right) = \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle$$

and by (62) we get

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \le \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle \tag{63}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (63) we get the desired result (60).

Remark 5. If we take in (60) $P = 1_H$, then we get [18, p. 6]

$$w(BA) \le \frac{1}{2}\left\| |A|^2 + |B^*|^2 \right\| \tag{64}$$

for any A, B bounded linear operators on H.

References

1. Aldaz, J.M.: Strengthened Cauchy-Schwarz and Hölder inequalities. J. Inequal. Pure Appl. Math. **10**(4), Article 116, 6pp. (2009)
2. Buzano, M.L.: Generalizzazione della diseguaglianza di Cauchy-Schwarz (Italian). Rend. Sem. Mat. Univ. e Politech. Torino **31**, 405–409 (1974) (1971/73)

3. Davidson, K.R., Holbrook, J.A.R.: Numerical radii of zero-one matrices. Mich. Math. J. **35**, 261–267 (1988)
4. De Rossi, A.: A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets. Math. Inequal. Appl. **2**(2), 263–282 (1999)
5. Dragomir, S.S.: Some refinements of Schwartz inequality, Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania, 1–2 Noiembrie 1985, pp. 13–16
6. Dragomir, S.S.: A generalization of Grüss' inequality in inner product spaces and applications. J. Math. Anal. Appl. **237**, 74–82 (1999)
7. Dragomir, S.S.: Some Grüss type inequalities in inner product spaces. J. Inequal. Pure Appl. Math. **4**(2), Art. 42 (2003)
8. Dragomir, S.S.: New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. Aust. J. Math. Anal. Appl. **1**(1), Art. 1, 18pp. (2004)
9. Dragomir, S.S.: Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces. J. Inequal. Pure Appl. Math. **5**(3), Article 76, 18pp. (2004)
10. Dragomir, S.S.: A potpourri of Schwarz related inequalities in inner product spaces. I. J. Inequal. Pure Appl. Math. **6**(3), Article 59, 15pp. (2005)
11. Dragomir, S.S.: Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces. Nova Science Publishers, Hauppauge, NY (2005). viii+249 pp. ISBN: 1-59454-202-3
12. Dragomir, S.S.: A potpourri of Schwarz related inequalities in inner product spaces. II. J. Inequal. Pure Appl. Math. **7**(1), Article 14, 11pp. (2006)
13. Dragomir, S.S.: Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. Demonstratio Math. **40**(2), 411–417 (2007)
14. Dragomir, S.S.: New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces. Linear Algebra Appl. **428**, 2750–2760 (2008)
15. Dragomir, S.S.: Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces. Tamkang J. Math. **39**(1), 1–7 (2008)
16. Dragomir, S.S.: Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces. Nova Science Publishers, Hauppauge, NY (2007)
17. Dragomir, S.S.: Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality. Integral Transforms Spec. Funct. **20**(9–10), 757–767 (2009)
18. Dragomir, S.S.: Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces. Springer Briefs in Mathematics. Springer, Berlin (2013). x+120 pp. ISBN: 978-3-319-01447-0; 978-3-319-01448-7
19. Dragomir, S.S.: Some inequalities in inner product spaces related to Buzano's and Grüss' results. Preprint. RGMIA Res. Rep. Coll. **18**, Art. 16 (2015). [Online http://rgmia.org/papers/v18/v18a16.pdf]
20. Dragomir, S.S.: Some inequalities in inner product spaces. Preprint. RGMIA Res. Rep. Coll. **18**, Art. 19 (2015). [Online http://rgmia.org/papers/v18/v18a19.pdf]
21. Dragomir, S.S., Goşa, A.C.: Quasilinearity of some composite functionals associated to Schwarz's inequality for inner products. Period. Math. Hung. **64**(1), 11–24 (2012)
22. Dragomir, S.S., Mond, B.: Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. Soochow J. Math. **21**(4), 413–426 (1995)
23. Dragomir, S.S., Sándor, I.: Some inequalities in pre-Hilbertian spaces. Studia Univ. Babeş-Bolyai Math. **32**(1), 71–78 (1987)
24. Fong, C.K., Holbrook, J.A.R.: Unitarily invariant operators norms. Can. J. Math. **35**, 274–299 (1983)
25. Gunawan, H.: On *n*-inner products, *n*-norms, and the Cauchy-Schwarz inequality. Sci. Math. Jpn. **55**(1), 53–60 (2002)
26. Gustafson, K.E., Rao, D.K.M.: Numerical Range. Springer, New York (1997)
27. Lorch, E.R.: The Cauchy-Schwarz inequality and self-adjoint spaces. Ann. Math. (2) **46**, 468–473 (1945)

28. Lupu, C., Schwarz, D.: Another look at some new Cauchy-Schwarz type inner product inequalities. Appl. Math. Comput. **231**, 463–477 (2014)
29. Marcus, M.: The Cauchy-Schwarz inequality in the exterior algebra. Q. J. Math. Oxf. Ser. (2) **17**, 61–63 (1966)
30. Mercer, P.R.: A refined Cauchy-Schwarz inequality. Int. J. Math. Educ. Sci. Tech. **38**(6), 839–842 (2007)
31. Metcalf, F.T.: A Bessel-Schwarz inequality for Gramians and related bounds for determinants. Ann. Mat. Pura Appl. (4) **68**, 201–232 (1965)
32. Müller, V.: The numerical radius of a commuting product. Mich. Math. J. **39**, 255–260 (1988)
33. Okubo, K., Ando, T.: Operator radii of commuting products. Proc. Am. Math. Soc. **56**, 203–210 (1976)
34. Precupanu T.: On a generalization of Cauchy-Buniakowski-Schwarz inequality. An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **22**(2), 173–175 (1976)
35. Trenčevski, K., Malčeski, R.: On a generalized *n*-inner product and the corresponding Cauchy-Schwarz inequality. J. Inequal. Pure Appl. Math. **7**(2), Article 53, 10pp. (2006)
36. Wang, G.-B., Ma, J.-P.: Some results on reverses of Cauchy-Schwarz inequality in inner product spaces. Northeast. Math. J. **21**(2), 207–211 (2005)