A Half-Discrete Hardy-Hilbert-Type Inequality with a Best Possible Constant Factor Related to the Hurwitz Zeta Function

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Abstract Using methods of weight functions, techniques of real analysis as well as the Hermite-Hadamard inequality, a half-discrete Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor related to the Hurwitz zeta function and the Riemann zeta function is obtained. Equivalent forms, normed operator expressions, their reverses and some particular cases are also considered.

Keywords Hardy-Hilbert-type inequality • Hurwitz zeta function • Riemann zeta function • weight function • operator

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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = (\int_0^\infty f^p(x)dx)^{\frac{1}{p}} > 0,$$

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$||g||_q > 0$, then we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that

$$a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p,\ b = \{b_n\}_{n=1}^\infty \in l^q,$$

$$||a||_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0,\ ||b||_q > 0,$$

we have the following Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [3]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [3, 11, 19, 20, 22]).

If $\mu_i, \upsilon_j > 0 (i, j \in \mathbf{N} = \{1, 2, \cdots\})$,

$$U_m := \sum_{i=1}^m \mu_i, V_n := \sum_{j=1}^n \upsilon_j (m, n \in \mathbf{N}), \tag{3}$$

then we have the following inequality (cf. [3], Theorem 321) :

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\mu_m^{1/q} \upsilon_n^{1/p} a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{4}$$

Replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n in (4), respectively, we obtain the following equivalent form of (4):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^\infty \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i = \upsilon_j = 1 (i, j \in \mathbf{N})$, both (4) and (5) reduce to (2). We call (4) and (5) as Hardy-Hilbert-type inequalities.

Note. The authors did not prove that (4) is valid with the best possible constant factor in [3].

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [17] gave an extension of (1) with the kernel $1/(x + y)^{\lambda}$ for $p = q = 2$. Optimizing the method used in [17], Yang [20] provided some extensions of (1) and (2) as follows:

If $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_{\lambda}(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^{\infty} k_{\lambda}(t, 1)t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{\int_0^{\infty} \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^{\infty} \int_0^{\infty} k_{\lambda}(x, y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is the best possible.

Moreover, if $k_{\lambda}(x, y)$ remains finite and $k_{\lambda}(x, y)x^{\lambda_1 - 1}$ $(k_{\lambda}(x, y)y^{\lambda_2 - 1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_{m,} b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := (\sum_{n=1}^{\infty} \phi(n)|a_n|^p)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi},\ ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{\lambda}(m, n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{7}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for

$$\lambda = 1, k_1(x, y) = \frac{1}{x + y},\ \lambda_1 = \frac{1}{q},\ \lambda_2 = \frac{1}{p},$$

inequality (6) reduces to (1), while (7) reduces to (2). For

$$0 < \lambda_1, \lambda_2 \leq 1,\ \lambda_1 + \lambda_2 = \lambda,$$

we set

$$k_\lambda(x,y) = \frac{1}{(x+y)^\lambda} \ ((x,y) \in \mathbf{R}_+^2).$$

Then by (7), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < B(\lambda_1,\lambda_2)||a||_{p,\phi}||b||_{q,\psi}, \tag{8}$$

where the constant $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B(u,v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt (u, v > 0)$$

is the beta function.

In 2015, subject to further conditions, Yang [26] proved an extension of (8) and (5) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(U_m+V_n)^\lambda} \tag{9}$$

$$< B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\nu_n^{q-1}}\right)^{\frac{1}{q}}, \tag{10}$$

where the constant $B(\lambda_1, \lambda_2)$ is still the best possible.

Further results including some multidimensional Hilbert-type inequalities can be found in [18, 21, 23–25, 27, 33].

On the topic of half-discrete Hilbert-type inequalities with non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But, they did not prove that the constant factors are the best possible. However, Yang [18] presented a result with the kernel $1/(1+nx)^\lambda$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [21] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^\infty f(x)\left[\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^\lambda}\right]dx < B(\lambda_1,\lambda_2)\,||f||_{p,\phi}||a||_{q,\psi}, \tag{11}$$

where $\lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$. Zhong et al. [36, 37, 39–41] investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using methods of weight functions and techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x,n) a_n dx < k(\lambda_1)||f||_{p,\phi}||a||_{q,\psi}, \tag{12}$$

which is an extension of (11) (cf. Yang and Chen [28]). Additionally, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [24]. The reader is referred to the three books of Yang [23, 25] and Yang and Debnath [29], where half-discrete Hilbert-type inequalities and their operator expressions are extensively treated. The interested reader will find a vast literature on both old and new results on half-discrete Hardy-Hilbert-type inequality with emphasis to the study of best constants in references [1–42].

In this chapter, using methods of weight functions, techniques of real analysis as well as the Hermite-Hadamard inequality, a half-discrete Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor related to the Hurwitz zeta function and the Riemann zeta function is studied, which is an extension of (12) for $\lambda = 0$ in a particular kernel. Equivalent forms, normed operator expressions, their reverses and some particular cases are also considered.

2 An Example and Some Lemmas

In the following, we assume that $\mu_i, \nu_j > 0$ $(i, j \in \mathbf{N})$, U_m and V_n are defined by (3),

$$\tilde{V}_n := V_n - \tilde{\nu}_n (\tilde{\nu}_n \in [0, \frac{\nu_n}{2}])(n \in \mathbf{N}),$$

$\mu(t)$ is a positive continuous function in $\mathbf{R}_+ = (0, \infty)$,

$$U(x) := \int_0^x \mu(t)dt < \infty (x \in [0, \infty)),$$

$\nu(t) := \nu_n, t \in (n - \frac{1}{2}, n + \frac{1}{2}](n \in \mathbf{N})$, and

$$V(y) := \int_{\frac{1}{2}}^y \nu(t)dt (y \in [\frac{1}{2}, \infty)),$$

$p \neq 0, 1,\ \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, f(x), a_n \geq 0 (x \in \mathbf{R}_+, n \in \mathbf{N})$,

$$||f||_{p,\Phi_\delta} = (\int_0^\infty \Phi_\delta(x) f^p(x)dx)^{\frac{1}{p}},$$

$||a||_{q,\tilde{\Psi}} = (\sum_{n=1}^{\infty} \tilde{\Psi}(n) b_n^q)^{\frac{1}{q}}$, where,

$$\Phi_\delta(x) := \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}, \tilde{\Psi}(n) := \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} (x \in \mathbf{R}_+, n \in \mathbf{N}).$$

Example 1. For $0 < \gamma < \sigma, 0 \leq \alpha \leq \rho$ $(\rho > 0)$,

$$\csc h(u) := \frac{2}{e^u - e^{-u}} \ (u > 0)$$

is the hyperbolic cosecant function (cf. [34]). We set

$$h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} \ (t \in \mathbf{R}_+).$$

(i) Setting $u = \rho t^\gamma$, we find

$$\begin{aligned} k(\sigma) &:= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{\csc h(u)}{e^{\frac{\alpha}{\rho} u}} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-\frac{\alpha}{\rho} u} u^{\frac{\sigma}{\gamma}-1}}{e^u - e^{-u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-(\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1}}{1 - e^{-2u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du. \end{aligned}$$

By the Lebesgue term by term integration theorem (cf. [34]), setting $v = \left(2k + \frac{\alpha}{\rho} + 1\right) u$, we have

$$\begin{aligned} k(\sigma) &= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \int_0^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(2k + \frac{\alpha}{\rho} + 1)^{\sigma/\gamma}} \int_0^\infty e^{-v} v^{\frac{\sigma}{\gamma}-1} dv \end{aligned}$$

$$= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{1}{(k+\frac{\alpha+\rho}{2\rho})^{\sigma/\gamma}}$$

$$= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) \in \mathbf{R}_+, \tag{13}$$

where

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \ (s>1; 0<a\leq 1)$$

is the Hurwitz zeta function, $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function, and

$$\Gamma(y) := \int_0^{\infty} e^{-v} v^{y-1} dv \ (y>0)$$

is the Gamma function (cf. [16]).

In particular, (1) for $\alpha = \rho > 0$,we have $h(t) = \frac{\csc h(\rho t^{\gamma})}{e^{\rho t^{\gamma}}}$ and $k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}$. In this case, for $\gamma = \frac{\sigma}{2}$, we have $h(t) = \frac{\csc h(\rho\sqrt{t^{\sigma}})}{e^{\rho\sqrt{t^{\sigma}}}}$ and $k(\sigma) = \frac{\pi^2}{6\sigma\rho^2}$; (2) for $\alpha = 0$, we have $h(t) = \csc h(\rho t^{\gamma})$ and $\frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}\zeta(\frac{\sigma}{\gamma}, \frac{1}{2})$. In this case, for $\gamma = \frac{\sigma}{2}$, we find $h(t) = \csc h(\rho\sqrt{t^{\sigma}})$ and $k(\sigma) = \frac{\pi^2}{2\sigma\rho^2}$.

(ii) We obtain for $u > 0$, $\frac{1}{e^u - e^{-u}} > 0$,

$$\frac{d}{du}(\frac{1}{e^u - e^{-u}}) = -\frac{e^u + e^{-u}}{(e^u - e^{-u})^2} < 0,$$

$$\frac{d^2}{du^2}(\frac{1}{e^u - e^{-u}}) = \frac{2(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u - e^{-u})^3} > 0.$$

If $g(u) > 0,\ g'(u) < 0,\ g''(u) > 0$, then for $0 < \gamma \leq 1$,

$$g(\rho t^{\gamma}) > 0, \frac{d}{dt} g(\rho t^{\gamma}) = \rho\gamma t^{\gamma-1} g'(\rho t^{\gamma}) < 0,$$

$$\frac{d^2}{dt^2} g(\rho t^{\gamma}) = \rho\gamma(\gamma-1) t^{\gamma-2} g'(\rho t^{\gamma}) + \rho^2\gamma^2 t^{2\gamma-2} g''(\rho t^{\gamma}) > 0;$$

for $y \in (n-\frac{1}{2}, n+\frac{1}{2}), g(V(y)) > 0,$

$$\frac{d}{dy} g(V(y)) = g'(V(y)) \nu_n < 0,$$

$$\frac{d^2}{dy^2} g(V(y)) = g''(V(y)) \nu_n^2 > 0 (n \in \mathbf{N}).$$

If $g_i(u) > 0, g_i'(u) < 0, g_i''(u) > 0 (i = 1, 2)$, then

$$\begin{aligned} &g_1(u)g_2(u) > 0, \\ &(g_1(u)g_2(u))' = g_1'(u)g_2(u) + g_1(u)g_2'(u) < 0, \\ &(g_1(u)g_2(u))'' = g_1''(u)g_2(u) + 2g_1'(u)g_2'(u) + g_1(u)g_2''(u) > 0 (u > 0). \end{aligned}$$

(iii) Therefore, for $0 < \gamma < \sigma \leq 1, 0 \leq \alpha \leq \rho (\rho > 0)$, we have $k(\sigma) \in \mathbf{R}_+$, with $h(t) > 0, h'(t) < 0, h''(t) > 0$, and then for $c > 0, y \in (n - \frac{1}{2}, n + \frac{1}{2}) (n \in \mathbf{N})$, it follows that

$$\begin{aligned} &h(cV(y))V^{\sigma-1}(y) > 0, \\ &\frac{d}{dy} h(cV(y))V^{\sigma-1}(y) < 0, \\ &\frac{d^2}{dy^2} h(cV(y))V^{\sigma-1}(y) > 0. \end{aligned}$$

Lemma 1. *If $g(t) (> 0)$ is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty)$ where $n_0 \in \mathbf{N}$, satisfying $\int_0^\infty g(t)dt \in \mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)dt. \tag{14}$$

Proof. Since we have

$$\begin{aligned} &\int_n^{n+1} g(t)dt \leq g(n) \leq \int_{n-1}^n g(t)dt (n = 1, \cdots, n_0), \\ &\int_{n_0+1}^{n_0+2} g(t)dt < g(n_0 + 1) < \int_{n_0}^{n_0+1} g(t)dt, \end{aligned}$$

then it follows that

$$0 < \int_1^{n_0+2} g(t)dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1} \int_{n-1}^n g(t)dt = \int_0^{n_0+1} g(t)dt < \infty.$$

Similarly, we still have

$$0 < \int_{n_0+2}^\infty g(t)dt \leq \sum_{n=n_0+2}^\infty g(n) \leq \int_{n_0+1}^\infty g(t)dt < \infty.$$

Hence, (14) follows and therefore the lemma is proved.

Lemma 2. *If* $0 \le \alpha \le \rho(\rho > 0), 0 < \gamma < \sigma \le 1$, *define the following weight coefficients:*

$$\omega_\delta(\sigma, x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{\delta\sigma}(x)\nu_n}{\tilde{V}_n^{1-\sigma}}, x \in \mathbf{R}_+, \tag{15}$$

$$\varpi_\delta(\sigma, n) := \int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^\sigma \mu(x)}{U^{1-\delta\sigma}(x)} dx, n \in \mathbf{N}. \tag{16}$$

Then, we have the following inequalities:

$$\omega_\delta(\sigma, x) < k(\sigma)(x \in \mathbf{R}_+), \tag{17}$$

$$\varpi_\delta(\sigma, n) \le k(\sigma)(n \in \mathbf{N}), \tag{18}$$

where $k(\sigma)$ *is given by (13).*

Proof. Since we find

$$\tilde{V}_n = V_n - \tilde{\nu}_n \ge V_n - \frac{\nu_n}{2}$$
$$= \int_{\frac{1}{2}}^{n+\frac{1}{2}} \nu(t)dt - \int_n^{n+\frac{1}{2}} \nu(t)dt = \int_{\frac{1}{2}}^{n} \nu(t)dt = V(n),$$

and for $t \in (n - \frac{1}{2}, n + \frac{1}{2}], V'(t) = \nu_n$, hence by Example 1(iii) and Hermite-Hadamard's inequality (cf. [8]), we have

$$\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\nu_n}{\tilde{V}_n^{1-\sigma}}$$
$$\le \frac{\csc h(\rho(U^\delta(x)V(n))^\gamma)}{e^{\alpha(U^\delta(x)V(n))^\gamma}} \frac{\nu_n}{V^{1-\sigma}(n)}$$
$$< \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{V'(t)}{V^{1-\sigma}(t)} dt,$$

$$\omega_\delta(\sigma, x) < \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt$$
$$= \int_{\frac{1}{2}}^{\infty} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt.$$

Setting $u = U^{\delta}(x)V(t)$, by (13), we find

$$\omega_{\delta}(\sigma, x) < \int_{0}^{U^{\delta}(x)V(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{U^{\delta\sigma}(x)U^{-\delta}(x)}{(uU^{-\delta}(x))^{1-\sigma}} du$$
$$\leq \int_{0}^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du = k(\sigma).$$

Hence, (17) follows.

Setting $u = \tilde{V}_n U^{\delta}(x)$ in (16), we find $du = \delta \tilde{V}_n U^{\delta-1}(x)\mu(x)dx$ and

$$\varpi_{\delta}(\sigma, n) = \frac{1}{\delta} \int_{\tilde{V}_n U^{\delta}(0)}^{\tilde{V}_n U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{\tilde{V}_n^{\sigma} \tilde{V}_n^{-1} (\tilde{V}_n^{-1} u)^{\frac{1}{\delta}-1}}{(\tilde{V}_n^{-1} u)^{\frac{1}{\delta}-\sigma}} du$$
$$= \frac{1}{\delta} \int_{\tilde{V}_n U^{\delta}(0)}^{\tilde{V}_n U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

If $\delta = 1$, then

$$\varpi_{1}(\sigma, n) = \int_{0}^{\tilde{V}_n U(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du$$
$$\leq \int_{0}^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

If $\delta = -1$, then

$$\varpi_{-1}(\sigma, n) = -\int_{\infty}^{\tilde{V}_n U^{-1}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du$$
$$\leq \int_{0}^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

Then by (13), we have (18). The lemma is proved.

Remark 1. We do not need the constraint $\sigma \leq 1$ to obtain (18). If $U(\infty) = \infty$, then we have

$$\varpi_{\delta}(\sigma, n) = k(\sigma)(n \in \mathbf{N}). \tag{19}$$

For example, if we set $\mu(t) = \frac{1}{(1+t)^{\beta}} (t > 0; 0 \leq \beta \leq 1)$, then for $x \geq 0$, we find

$$U(x) = \int_{0}^{x} \frac{1}{(1+t)^{\beta}} dt$$
$$= \begin{cases} \frac{(1+x)^{1-\beta}-1}{1-\beta}, 0 \leq \beta < 1 \\ \ln(1+x), \beta = 1 \end{cases} < \infty,$$

and

$$U(\infty) = \int_0^\infty \frac{1}{(1+t)^\beta} dt = \infty.$$

Lemma 3. *If* $0 \le \alpha \le \rho$ $(\rho > 0)$, $0 < \gamma < \sigma \le 1$, *there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \ge v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $V(\infty) = \infty$, *then*

(i) for $x \in \mathbf{R}_+$, *we have*

$$k(\sigma)(1 - \theta_\delta(\sigma, x)) < \omega_\delta(\sigma, x), \tag{20}$$

where, $\theta_\delta(\sigma, x) = O((U(x))^{\delta(\sigma-\gamma)}) \in (0,1)$;

(ii) for any $b > 0$, *we have*

$$\sum_{n=1}^\infty \frac{v_n}{\tilde{V}_n^{1+b}} = \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + bO(1)\right). \tag{21}$$

Proof. Since $v_n \ge v_{n+1} (n \ge n_0)$, and

$$\tilde{V}_n = V_n - \tilde{v}_n \le V_n = \int_{\frac{1}{2}}^{n+\frac{1}{2}} v(t)dt = V(n + \frac{1}{2}),$$

by Example 1(iii), we have

$$\begin{aligned}
\omega_\delta(\sigma, x) &= \sum_{n=1}^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{\delta\sigma}(x) v_n}{\tilde{V}_n^{1-\sigma}} \\
&\ge \sum_{n=n_0}^\infty \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\operatorname{csc} h(\rho(U^\delta(x)V(n+\frac{1}{2}))^\gamma)}{e^{\alpha(U^\delta(x)V(n+\frac{1}{2}))^\gamma}} \frac{U^{\delta\sigma}(x) v_{n+1} dt}{(V(n+\frac{1}{2}))^{1-\sigma}} \\
&> \sum_{n=n_0}^\infty \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\operatorname{csc} h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t))^{1-\sigma}} dt \\
&= \int_{n_0+\frac{1}{2}}^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x) V'(t)}{(V(t))^{1-\sigma}} dt.
\end{aligned}$$

Setting $u = U^\delta(x)V(t)$, in view of $V(\infty) = \infty$, by (13), we find

$$\begin{aligned}
\omega_\delta(\sigma, x) &> \int_{U^\delta(x)V_{n_0}}^\infty \frac{\operatorname{csc} h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\
&= k(\sigma) - \int_0^{U^\delta(x)V_{n_0}} \frac{\operatorname{csc} h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du
\end{aligned}$$

$$= k(\sigma)(1-\theta_\delta(\sigma,x)),$$

$$\theta_\delta(\sigma,x) \;:\; = \frac{1}{k(\sigma)}\int_0^{U^\delta(x)V_{n_0}} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1}du \in (0,1).$$

Since $F(u) = \frac{u^\gamma \csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}$ is continuous in $(0,\infty)$, satisfying

$$F(u) \to \frac{1}{\rho}(u \to 0^+), F(u) \to 0(u \to \infty),$$

there exists a constant $L > 0$, such that $F(u) \le L$, namely,

$$\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \le Lu^{-\gamma}(u \in (0,\infty)).$$

Hence we find

$$0 < \theta_\delta(\sigma,x) \le \frac{L}{k(\sigma)}\int_0^{U^\delta(x)V_{n_0}} u^{\sigma-\gamma-1}du$$
$$= \frac{L(U^\delta(x)V_{n_0})^{\sigma-\gamma}}{k(\sigma)(\sigma-\gamma)},$$

and then (20) follows.

For $b > 0$, we find

$$\sum_{n=1}^\infty \frac{\nu_n}{\tilde{V}_n^{1+b}} \le \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \sum_{n=n_0+1}^\infty \frac{\nu_n}{V^{1+b}(n)}$$
$$< \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \sum_{n=n_0+1}^\infty \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{V'(x)}{V^{1+b}(x)}dx$$
$$= \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \int_{n_0+\frac{1}{2}}^\infty \frac{dV(x)}{V^{1+b}(x)}$$
$$= \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \frac{1}{bV^b(n_0+\frac{1}{2})}$$
$$= \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + b\sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}}\right),$$

$$\sum_{n=1}^\infty \frac{\nu_n}{\tilde{V}_n^{1+b}} \ge \sum_{n=n_0}^\infty \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\nu_{n+1}}{V^{1+b}(n+\frac{1}{2})}dx$$

$$> \sum_{n=n_0}^{\infty} \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{V'(x)}{V^{1+b}(x)} dx = \int_{n_0+\frac{1}{2}}^{\infty} \frac{dV(x)}{V^{1+b}(x)}$$

$$= \frac{1}{bV^b(n_0 + \frac{1}{2})} = \frac{1}{bV_{n_0}^b}.$$

Hence we have (21). The lemma is proved.

Note. For example, $\nu_n = \frac{1}{(n-\tau)^\beta}(n \in \mathbf{N}; 0 \leq \beta \leq 1, 0 \leq \tau < 1)$ satisfies the conditions of Lemma 3 (for $n_0 \geq 1$).

3 Equivalent Inequalities and Operator Expressions

Theorem 1. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1, k(\sigma)$ *is given by (13), then for* $p > 1,\ 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities:*

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \quad (22)$$

$$J_1 := \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p$$
$$< k(\sigma)||f||_{p,\Phi_\delta}, \quad (23)$$

$$J_2 := \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$< k(\sigma)||a||_{q,\tilde{\Psi}}. \quad (24)$$

Proof. By Hölder's inequality with weight (cf. [8]), we have

$$\left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p$$

$$= \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \left(\frac{U^{\frac{1-\delta\sigma}{q}}(x) f(x)}{\tilde{V}_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)} \right) \left(\frac{\tilde{V}_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)} \right) dx \right]^p$$

$$\leq \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \left(\frac{U^{\frac{p(1-\delta\sigma)}{q}}(x) f^p(x)}{\tilde{V}_n^{1-\sigma} \mu^{\frac{p}{q}}(x)} \right) dx$$

$$\times \left[\int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(p-1)}\mu(x)}{U^{1-\delta\sigma}(x)} dx \right]^{p-1}$$

$$= \frac{(\varpi_\delta(\sigma,n))^{p-1}}{\tilde{V}_n^{p\sigma-1}\nu_n} \int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx. \quad (25)$$

In view of (18) and the Lebesgue term by term integration theorem (cf. [9]), we find

$$J_1 \leq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^\infty \int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}$$

$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \sum_{n=1}^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}$$

$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}. \quad (26)$$

Then by (17), we have (23).

By Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^\infty \left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}} \int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx \right] \left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma} a_n}{\nu_n^{\frac{1}{p}}} \right)$$
$$\leq J_1 ||a||_{q,\tilde{\Psi}}. \quad (27)$$

Then by (23), we have (22).

On the other hand, assuming that (22) is valid, we set

$$a_n := \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx \right]^{p-1}, n \in \mathbf{N}.$$

Then, we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = 0$, then (23) is trivially valid.

If $J_1 = \infty$, then (23) keeps impossible.

Suppose that $0 < J_1 < \infty$. By (22), it follows that

$$||a||_{q,\tilde{\Psi}}^q = J_1^p = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 < k(\sigma)||f||_{p,\Phi_\delta},$$

and then (23) follows, which is equivalent to (22).

By Hölder's inequality with weight (cf. [8]), we obtain

$$\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n\right]^q$$

$$= \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \left(\frac{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1-\sigma}{p}}}\right)\left(\frac{\tilde{V}_n^{\frac{1-\sigma}{p}} a_n}{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{\frac{1}{p}}}\right)\right]^q$$

$$\leq \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}}\right]^{q-1}$$

$$\times \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{\tilde{V}_n^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q$$

$$= \frac{(\omega_{\delta}(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q. \tag{28}$$

Then by (17) and Lebesgue term by term integration theorem (cf. [9]), it follows that

$$J_2 < (k(\sigma))^{\frac{1}{p}} \left\{\int_0^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right\}^{\frac{1}{q}}$$

$$= (k(\sigma))^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right\}^{\frac{1}{q}}$$

$$= (k(\sigma))^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} \varpi_{\delta}(\sigma,n) \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right\}^{\frac{1}{q}}. \tag{29}$$

Then by (18), we have (24).

By Hölder's inequality (cf. [8]), we have

$$I = \int_0^{\infty} \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right)\left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n\right] dx$$

$$\leq ||f||_{p,\Phi_{\delta}} J_2. \tag{30}$$

Then by (24), we have (22).

On the other hand, assuming that (24) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n\right]^{q-1}, \ x \in \mathbf{R}_+.$$

Then we find $J_2^q = ||f||_{p,\Phi_\delta}^p$.

If $J_2 = 0$, then (24) is trivially valid.

If $J_2 = \infty$, then (24) keeps impossible.

Suppose that $0 < J_2 < \infty$. By (22), it follows that

$$||f||_{p,\Phi_\delta}^p = J_2^q = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||f||_{p,\Phi_\delta}^{p-1} = J_2 < k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (24) follows, which is equivalent to (22).

Therefore, (22), (23) and (24) are equivalent. The theorem is proved.

Theorem 2. *With the assumptions of Theorem 1, if there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k(\sigma)$ *in (22), (23) and (24) is the best possible.*

Proof. For $\varepsilon \in (0, q(\sigma - \gamma))$, we set $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q} (\in (\gamma, 1))$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+, \tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), 0 < x^\delta \leq 1 \\ 0, x^\delta > 0 \end{cases}, \tag{31}$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-1} v_n = \tilde{V}_n^{\sigma - \frac{\varepsilon}{q} - 1} v_n, n \in \mathbf{N}. \tag{32}$$

Then for $\delta = \pm 1$, since $U(\infty) = \infty$, we find

$$\int_{\{x>0;0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx = \frac{1}{\varepsilon} U^{\delta\varepsilon}(1). \tag{33}$$

By (21), (33) and (20), we obtain

$$\begin{aligned} ||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}} &= \left(\int_{\{x>0;0<x^\delta \leq 1\}} \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)} \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{v_n}{\tilde{V}_n^{1+\varepsilon}} \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon \tilde{O}(1) \right)^{\frac{1}{q}}, \end{aligned} \tag{34}$$

$$\begin{aligned} \tilde{I} \ :&= \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\ &= \int_{\{x>0;0<x^\delta \leq 1\}} \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{\tilde{\sigma}-1} v_n \mu(x)}{U^{1-\delta(\tilde{\sigma}+\varepsilon)}(x)} dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\{x>0;0<x^{\delta}\leq 1\}} \omega_{\delta}(\tilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&\geq k(\tilde{\sigma})\int_{\{x>0;0<x^{\delta}\leq 1\}} (1-\theta_{\delta}(\tilde{\sigma},x))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&= k(\tilde{\sigma})\int_{\{x>0;0<x^{\delta}\leq 1\}} (1-O((U(x))^{\delta(\sigma-\frac{\varepsilon}{q}-\gamma)}))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&= k(\tilde{\sigma})\left[\int_{\{x>0;0<x^{\delta}\leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \right.\\
&\quad \left. -\int_{\{x>0;0<x^{\delta}\leq 1\}} O(\frac{\mu(x)}{U^{1-\delta(\sigma-\gamma+\frac{\varepsilon}{p})}(x)})dx\right] \\
&= \frac{1}{\varepsilon}k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O(1)).
\end{aligned}
$$

If there exists a positive constant $K \leq k(\sigma)$, such that (22) is valid when replacing $k(\sigma)$ to K, then in particular, by Lebesgue term by term integration theorem, we have $\varepsilon\tilde{I} < \varepsilon K||\tilde{f}||_{p,\Phi_{\delta}}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$
k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O(1)) < K\cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}.
$$

It follows that $k(\sigma) \leq K(\varepsilon \to 0^{+})$. Hence, $K = k(\sigma)$ is the best possible constant factor of (22).

The constant factor $k(\sigma)$ in (23) (respectively, (24)) is still the best possible. Otherwise, we would reach a contradiction by (27) (respectively, (30)) that the constant factor in (22) is not the best possible. The theorem is proved.

For $p > 1$, we find

$$
\tilde{\Psi}^{1-p}(n) = \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}(n\in\mathbf{N}),\ \Phi_{\delta}^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}(x\in\mathbf{R}_{+}),
$$

and define the following real normed spaces:

$$
\begin{aligned}
L_{p,\Phi_{\delta}}(\mathbf{R}_{+}) &= \{f; f = f(x), x\in\mathbf{R}_{+}, ||f||_{p,\Phi_{\delta}} < \infty\}, \\
l_{q,\tilde{\Psi}} &= \{a; a = \{a_n\}_{n=1}^{\infty}, ||a||_{q,\tilde{\Psi}} < \infty\}, \\
L_{q,\Phi_{\delta}^{1-q}}(\mathbf{R}_{+}) &= \{h; h = h(x), x\in\mathbf{R}_{+}, ||h||_{q,\Phi_{\delta}^{1-q}} < \infty\}, \\
l_{p,\tilde{\Psi}^{1-p}} &= \{c; c = \{c_n\}_{n=1}^{\infty}, ||c||_{p,\tilde{\Psi}^{1-p}} < \infty\}.
\end{aligned}
$$

Assuming that $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx, n \in \mathbf{N},$$

we can rewrite (23) as follows:

$$||c||_{p,\tilde{\Psi}^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta} < \infty,$$

namely, $c \in l_{p,\tilde{\Psi}^{1-p}}$.

Definition 1. Define a half-discrete Hardy-Hilbert-type operator

$$T_1 : L_{p,\Phi_\delta}(\mathbf{R}_+) \to l_{p,\tilde{\Psi}^{1-p}}$$

as follows:

For any $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = c \in l_{p,\tilde{\Psi}^{1-p}}$. Define the formal inner product of $T_1 f$ and $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\tilde{\Psi}}$ as follows:

$$(T_1 f, a) := \sum_{n=1}^{\infty} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx \right] a_n. \tag{35}$$

Then we can rewrite (22) and (23) as:

$$(T_1 f, a) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{36}$$

$$||T_1 f||_{p,\tilde{\Psi}^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta}. \tag{37}$$

Define the norm of operator T_1 as follows:

$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\Phi_\delta}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\tilde{\Psi}^{1-p}}}{||f||_{p,\Phi_\delta}}.$$

Then by (37), it is evident that $||T_1|| \le k(\sigma)$. Since by Theorem 2, the constant factor in (37) is the best possible, we have

$$||T_1|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}). \tag{38}$$

Assuming that $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\tilde{\Psi}}$, setting

$$h(x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n, x \in \mathbf{R}_+,$$

we can rewrite (24) as follows:

$$||h||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\tilde{\Psi}} < \infty,$$

namely, $h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$.

Definition 2. Define a half-discrete Hardy-Hilbert-type operator

$$T_2 : l_{q,\tilde{\Psi}} \to L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$$

as follows:

For any $a = \{a_n\}_{n=1}^\infty \in l_{q,\tilde{\Psi}}$, there exists a unique representation $T_2 a = h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$. Define the formal inner product of $T_2 a$ and $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$ by:

$$(T_2 a, f) := \int_0^\infty \left[\sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right] f(x)dx. \tag{39}$$

Then we can rewrite (22) and (24) as follows:

$$(T_2 a, f) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{40}$$

$$||T_2 a||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{41}$$

Define the norm of operator T_2 by:

$$||T_2|| := \sup_{a(\neq\theta)\in l_{q,\tilde{\Psi}}} \frac{||T_2 a||_{q,\Phi_\delta^{1-q}}}{||a||_{q,\tilde{\Psi}}}.$$

Then by (41), we find $||T_2|| \leq k(\sigma)$. Since by Theorem 2, the constant factor in (41) is the best possible, we have

$$||T_2|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}\zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) = ||T_1||. \tag{42}$$

4 Some Equivalent Reverses

In the following, we also set

$$\tilde{\Phi}_\delta(x) := (1-\theta_\delta(\sigma,x))\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}(x \in \mathbf{R}_+).$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $||f||_{p,\Phi_\delta}$, $||f||_{p,\tilde{\Phi}_\delta}$ and $||a||_{q,\tilde{\Psi}}$.

Theorem 3. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $\upsilon_n \geq \upsilon_{n+1}$ $(n \in \{n_0, n_0 + 1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then for* $p < 0,\ 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $k(\sigma)$*:*

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x)dx > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{43}$$

$$J_1 = \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_\delta}, \tag{44}$$

$$J_2 = \left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{45}$$

Proof. By the reverse Hölder's inequality with weight (cf. [8]), since $p < 0$, similarly to the way we obtained (25) and (26), we have

$$\left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx\right]^p \leq \frac{\tilde{V}_n^{1-p\sigma}}{(\varpi_\delta(\sigma,n))^{1-p}\nu_n} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx,$$

and then by (19) and Lebesgue term by term integration theorem, it follows that

$$J_1 \geq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx\right]^{\frac{1}{p}}$$
$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx\right]^{\frac{1}{p}}.$$

Then by (17), we have (44).

By the reverse Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^{\infty} \left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx\right] \left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma} a_n}{\nu_n^{\frac{1}{p}}}\right) \geq J_1||a||_{q,\tilde{\Psi}}. \tag{46}$$

Then by (44), we have (43).

On the other hand, assuming that (43) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = \infty$, then (44) is trivially valid.

If $J_1 = 0$, then (44) is impossible.

Suppose that $0 < J_1 < \infty$. By (43), it follows that

$$||a||_{q,\tilde{\Psi}}^q = J_1^p = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 > k(\sigma)||f||_{p,\Phi_\delta},$$

and then (44) follows, which is equivalent to (43).

By the reverse of Hölder's inequality with weight (cf. [8]), since $0 < q < 1$, similarly to the way we obtained (28) and (29), we have

$$\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right]^q$$
$$\geq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q,$$

and then by (17) and Lebesgue term by term integration theorem, it follows that

$$J_2 > (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right]^{\frac{1}{q}}$$
$$= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right]^{\frac{1}{q}}.$$

Then by (19), we obtain (45).

By the reverse Hölder's inequality (cf. [8]), we get

$$I = \int_0^\infty \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right) \left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right] dx$$
$$\geq ||f||_{p,\Phi_\delta} J_2. \tag{47}$$

Then by (45), we derive (43).

On the other hand, assuming that (45) is valid, we set $f(x)$ as in Theorem 1. Then we find $J_2^q = ||f||_{p,\Phi_\delta}^p$.

If $J_2 = \infty$, then (45) is trivially valid.

If $J_2 = 0$, then (45) keeps impossible.

Suppose that $0 < J_2 < \infty$. By (43), it follows that

$$||f||_{p,\Phi_\delta}^{p} = J_2^q = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$
$$||f||_{p,\Phi_\delta}^{p-1} = J_2 > k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (45) follows, which is equivalent to (43).

Therefore, inequalities (43), (44) and (45) are equivalent.

For $\varepsilon \in (0, q(\sigma - \gamma))$, we set $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}(\in (\gamma, 1))$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+$, $\tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}$,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), 0 < x^\delta \le 1 \\ 0, x^\delta > 0 \end{cases},$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-1}\nu_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, n \in \mathbf{N}.$$

By (21), (33) and (17), we obtain

$$||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}} = \frac{1}{\varepsilon}U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}, \tag{48}$$

$$\begin{aligned}\tilde{I} &= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\tilde{a}_n\tilde{f}(x)dx \\ &= \int_{\{x>0;0<x^\delta\le 1\}}\omega_\delta(\tilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &\le k(\tilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}}\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx = \frac{1}{\varepsilon}k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1).\end{aligned}$$

If there exists a positive constant $K \ge k(\sigma)$, such that (43) is valid when replacing $k(\sigma)$ by K, then in particular, we have $\varepsilon\tilde{I} > \varepsilon K||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1) > K \cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \ge K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (43).

The constant factor $k(\sigma)$ in (44) (respectively, (45)) is still the best possible. Otherwise, we would reach a contradiction by (46) (respectively, (47)) that the constant factor in (43) is not the best possible. The theorem is proved.

Theorem 4. *With the assumptions of Theorem 3, if*

$$0 < p < 1,\ 0 < ||f||_{p,\Phi_\delta},\ ||a||_{q,\tilde{\Psi}} < \infty,$$

then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}||a||_{q,\tilde{\Psi}}, \tag{49}$$

$$J_1 = \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}, \tag{50}$$

$$J := \left\{ \int_0^{\infty} \frac{(1-\theta_{\delta}(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{51}$$

Proof. By the reverse Hölder's inequality with weight (cf. [8]), since $0 < p < 1$, similarly to the way we obtained (25) and (26), we have

$$\left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right]^p$$
$$\geq \frac{(\varpi_{\delta}(\sigma,n))^{p-1}}{\tilde{V}_n^{p\sigma-1}\nu_n} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx,$$

and then in view of (19) and Lebesgue term by term integration theorem, we find

$$J_1 \geq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}$$
$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_{\delta}(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}.$$

Then by (20), we have (50).

By the reverse Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^{\infty} \left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right] \left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma} a_n}{\nu_n^{\frac{1}{p}}} \right)$$
$$\geq J_1||a||_{q,\tilde{\Psi}}. \tag{52}$$

Then by (50), we have (49).

On the other hand, assuming that (49) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = \infty$, then (50) is trivially valid.
If $J_1 = 0$, then (50) remains impossible.
Suppose that $0 < J_1 < \infty$. By (49), it follows that

$$||a||_{q,\tilde{\Psi}}^{q} = J_1^{p} = I > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}},$$
$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 > k(\sigma)||f||_{p,\tilde{\Phi}_\delta},$$

and then (50) follows, which is equivalent to (49).
By the reverse Hölder's inequality with weight (cf. [8]), since $q < 0$, we have

$$\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right]^q$$
$$\leq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q,$$

and then by (20) and Lebesgue term by term integration theorem, it follows that

$$J > (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right]^{\frac{1}{q}}$$
$$= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right]^{\frac{1}{q}}.$$

Then by (19), we have (51).
By the reverse Hölder's inequality (cf. [8]), we have

$$I = \int_0^\infty \left[(1-\theta_\delta(\sigma,x))^{\frac{1}{p}} \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right] \times \left[\frac{(1-\theta_\delta(\sigma,x))^{\frac{-1}{p}}\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right] dx$$
$$\geq ||f||_{p,\tilde{\Phi}_\delta} J. \tag{53}$$

Then by (51), we have (49).
On the other hand, assuming that (49) is valid, we set $f(x)$ as in Theorem 1. Then we find $J^q = ||f||_{p,\tilde{\Phi}_\delta}^p$.
If $J = \infty$, then (51) is trivially valid.
If $J = 0$, then (51) remains impossible.
Suppose that $0 < J < \infty$. By (49), it follows that

$$||f||_{p,\tilde{\Phi}_\delta}^{p} = J^q = I > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}},$$
$$||f||_{p,\tilde{\Phi}_\delta}^{p-1} = J > k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (51) follows, which is equivalent to (49).

Therefore, inequalities (49), (50) and (51) are equivalent.

For $\varepsilon \in (0, p(\sigma-\gamma))$, we set $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p}(> \gamma)$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+, \tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}$,

$$\tilde{f}(x) = \begin{cases} U^{\delta\tilde{\sigma}-1}(x)\mu(x), 0 < x^\delta \le 1 \\ 0, x^\delta > 0 \end{cases},$$
$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-\varepsilon-1}\nu_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, n \in \mathbf{N}.$$

By (20), (21) and (33), we obtain

$$\begin{aligned}
&||\tilde{f}||_{p,\tilde{\Phi}_\delta}||\tilde{a}||_{q,\tilde{\Psi}} \\
&= \left[\int_{\{x>0;0<x^\delta\le 1\}} (1 - O((U(x))^{\delta(\sigma-\gamma)}))\frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)}\right]^{\frac{1}{p}} \left(\sum_{n=1}^{\infty}\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(U^{\delta\varepsilon}(1) - \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\tilde{a}_n\tilde{f}(x)dx \\
&= \sum_{n=1}^{\infty}\left[\int_{\{x>0;0<x^\delta\le 1\}}\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\frac{\tilde{V}_n^{\tilde{\sigma}}\mu(x)}{U^{1-\delta\tilde{\sigma}}(x)}dx\right]\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&\le \sum_{n=1}^{\infty}\left[\int_0^{\infty}\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\frac{\tilde{V}_n^{\tilde{\sigma}}\mu(x)}{U^{1-\delta\tilde{\sigma}}(x)}dx\right]\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&= \sum_{n=1}^{\infty}\varpi_\delta(\tilde{\sigma}, n)\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} = k(\tilde{\sigma})\sum_{n=1}^{\infty}\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}k(\sigma + \frac{\varepsilon}{p})\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right).
\end{aligned}$$

If there exists a positive constant $K \ge k(\sigma)$, such that (43) is valid when replacing $k(\sigma)$ by K, then in particular, we have $\varepsilon\tilde{I} > \varepsilon K||\tilde{f}||_{p,\tilde{\Phi}_\delta}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$k(\sigma + \frac{\varepsilon}{p})\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right)$$

$$> K\left(U^{\delta\varepsilon}(1) - \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \geq K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (49).

The constant factor $k(\sigma)$ in (50) (respectively, (51)) is still the best possible. Otherwise, we would reach a contradiction by (52) (respectively, (53)) that the constant factor in (49) is not the best possible. The theorem is proved.

5 Some Particular Inequalities

For $\tilde{v}_n = 0, \tilde{V}_n = V_n$, we set

$$\Psi(n) := \frac{V_n^{q(1-\sigma)-1}}{v_n^{q-1}} \ (n \in \mathbf{N}).$$

In view of Theorems 2–4, we have

Corollary 1. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0 + 1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_n f(x)dx < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi}, \tag{54}$$

$$\sum_{n=1}^{\infty}\frac{v_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}f(x)dx\right]^p < k(\sigma)||f||_{p,\Phi_\delta}, \tag{55}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi}; \tag{56}$$

(ii) for $p < 0$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_nf(x)dx > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\Psi}, \tag{57}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_\delta}, \tag{58}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{59}$$

(iii) for $0 < p < 1,\ 0 < ||f||_{p,\tilde{\Phi}_\delta}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_nf(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\Psi}, \tag{60}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}, \tag{61}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\Psi}. \tag{62}$$

The above inequalities have the best possible constant factor $k(\sigma)$.

In particular, for $\delta = 1$, we have the following inequalities with the non-homogeneous kernel:

Corollary 2. *If* $0 \le \alpha \le \rho (\rho > 0), 0 < \gamma < \sigma \le 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $\upsilon_n \ge \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1,\ 0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_nf(x)dx < k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi}, \tag{63}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}f(x)dx\right]^p < k(\sigma)||f||_{p,\Phi_1}, \tag{64}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi}; \tag{65}$$

(ii) for $p < 0$, $0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x) dx > k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi}, \tag{66}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x) dx \right]^p > k(\sigma)||f||_{p,\Phi_1}, \tag{67}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{68}$$

(iii) for $0 < p < 1$, $0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x) dx > k(\sigma)||f||_{p,\tilde{\Phi}_1}||a||_{q,\Psi}, \tag{69}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x) dx \right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_1}, \tag{70}$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_1(\sigma,x))^{1-q}\mu(x)}{U^{1-q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\Psi}. \tag{71}$$

The above inequalities involve the best possible constant factor $k(\sigma)$.

For $\delta = -1$, we have the following inequalities with the homogeneous kernel of degree 0:

Corollary 3. *If* $0 \le \alpha \le \rho(\rho > 0), 0 < \gamma < \sigma \le 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \ge v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n f(x) dx < k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi}, \tag{72}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}f(x)dx\right]^p < k(\sigma)||f||_{p,\Phi_{-1}}, \tag{73}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi}; \tag{74}$$

(ii) for $p < 0,\ 0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi}, \tag{75}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_{-1}}, \tag{76}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{77}$$

(iii) for $0 < p < 1,\ 0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n f(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_{-1}}||a||_{q,\Psi}, \tag{78}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_{-1}}, \tag{79}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_{-1}(\sigma,x))^{1-q}\mu(x)}{U^{1+q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}. \tag{80}$$

The above inequalities involve the best possible constant factor $k(\sigma)$.
For $\alpha = \rho$ in Theorems 2–4, we have

Corollary 4. *If* $\rho > 0, 0 < \gamma < \sigma \leq 1$,

$$K(\sigma) := \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}, \tag{81}$$

there exists $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx < K(\sigma)||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \tag{82}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p < K(\sigma)||f||_{p,\Phi_\delta}, \tag{83}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < K(\sigma)||a||_{q,\tilde{\Psi}}. \tag{84}$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx > K(\sigma)||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \tag{85}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p > K(\sigma)||f||_{p,\Phi_\delta}, \tag{86}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > K(\sigma)||a||_{q,\tilde{\Psi}}; \tag{87}$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx > K(\sigma)||f||_{p,\tilde{\Phi}_\delta} ||a||_{q,\tilde{\Psi}}, \tag{88}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p > K(\sigma)||f||_{p,\tilde{\Phi}_\delta}, \tag{89}$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > K(\sigma)||a||_{q,\tilde{\Psi}}. \tag{90}$$

In particular, for $\gamma = \frac{\sigma}{2}, \theta_\delta(\sigma, x) = O((U(x))^{\frac{\delta\sigma}{2}})$,

(i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x) dx < \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \quad (91)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x) dx \right]^p < \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta}, \quad (92)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\pi^2}{6\sigma\rho^2} ||a||_{q,\tilde{\Psi}}; \quad (93)$$

(ii) for $p < 0$,we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x) dx > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \quad (94)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x) dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta}, \quad (95)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma\rho^2} ||a||_{q,\tilde{\Psi}}; \quad (96)$$

(iii) for $0 < p < 1$,we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x) dx > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\tilde{\Phi}_\delta} ||a||_{q,\tilde{\Psi}}, \quad (97)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x) dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\tilde{\Phi}_\delta}, \quad (98)$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$> \frac{\pi^2}{6\sigma\rho^2}||a||_{q,\tilde{\Psi}}. \quad (99)$$

For $\alpha = 0, \gamma = \frac{\sigma}{2}, \theta_\delta(\sigma, x) = O((U(x))^{\frac{\delta\sigma}{2}})$ in Theorems 2–4, we have

Corollary 5. *If $\rho > 0, 0 < \sigma \leq 1$, there exists $n_0 \in \mathbf{N}$, such that $\upsilon_n \geq \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, and $U(\infty) = V(\infty) = \infty$, then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) a_n f(x)dx < \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \quad (100)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) f(x)dx\right]^p < \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}, \quad (101)$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) a_n\right]^q dx\right\}^{\frac{1}{q}} < \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}; \quad (102)$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) a_n f(x)dx > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \quad (103)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) f(x)dx\right]^p > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}, \quad (104)$$

$$\left\{\int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) a_n\right]^q dx\right\}^{\frac{1}{q}} > \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}; \quad (105)$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) a_n f(x)dx > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}}, \quad (106)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^\infty \operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}}) f(x)dx\right]^p > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\tilde{\Phi}_\delta}, \quad (107)$$

$$\left\{\int_0^\infty \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^\infty \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n\right]^q dx\right\}^{\frac{1}{q}}$$

$$> \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}. \tag{108}$$

Remark 2. (i) For $\mu(x) = \nu_n = 1$ in (54), we have the following inequality with the best possible constant factor $k(\sigma)$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(x^\delta n)^\gamma)}{e^{\alpha(x^\delta n)^\gamma}} a_n f(x) dx \tag{109}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{110}$$

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(xn)^\gamma)}{e^{\alpha(xn)^\gamma}} a_n f(x) dx \tag{111}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}; \tag{112}$$

for $\delta = -1$, we have the following inequality with the homogeneous kernel:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}} a_n f(x) dx \tag{113}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1+\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{114}$$

(ii) For $\mu(x) = \nu_n = 1$, $\tilde{\nu}_n = \tau \in (0, \frac{1}{2}]$ in (22), we have the following more accurate inequality than (82) with the best possible constant factor $k(\sigma)$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho[x^\delta(n-\tau)]^\gamma)}{e^{\alpha(x^\delta(n-\tau)]^\gamma)^\gamma}} a_n f(x) dx \tag{115}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{116}$$

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h([x(n-\tau)]^{\gamma})}{e^{\alpha\{[x(n-\tau)]\}^{\gamma}}}a_n f(x)dx \tag{117}$$

$$< k(\sigma)\left[\int_0^{\infty}x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}; \tag{118}$$

for $\delta = -1$, we have the following inequality with the homogeneous kernel:

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{n-\tau}{x})^{\gamma})}{e^{\alpha(\frac{n-\tau}{x})^{\gamma}}}a_n f(x)dx \tag{119}$$

$$< k(\sigma)\left[\int_0^{\infty}x^{p(1+\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{120}$$

We can still obtain a large number of other inequalities by using some special parameters in the above theorems and corollaries.

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