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Narendra Kumar Govil
Ram Mohapatra
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Gerhard Schmeisser *Editors*

Progress in Approximation Theory and Applicable Complex Analysis

In Memory of Q.I. Rahman

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Aims and Scope
Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics, and other sciences.

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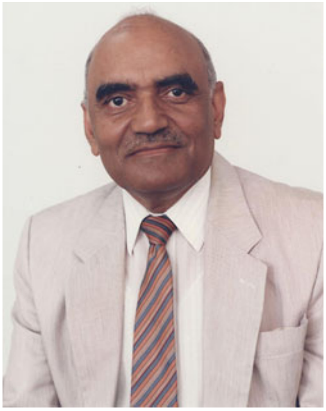
Foreword



Qazi Ibadur Rahman (QIR)

Q.I. Rahman was with Dick Askey one of the world's greatest experts on estimates in two important areas of complex analysis: functions of exponential type and polynomials.

The number and variety of papers in this book show how great his influence has been in this and other areas. His book with Gerhard Schmeisser *The Analytic Theory of Polynomials* remains the fundamental textbook for anyone studying the latter area. His distinction has been recognized by his being only the 21st person and the second mathematician to be awarded an honorary doctorate from the University Maria Curie-Skłodowska in Lublin, Poland. The first awardee was Irène Curie.

Let me mention some of his most memorable results.

Theorem 1 (with B. D. Bojanov and J. Szynal in *Math Z.* **190** (1985), 281–285).

Let $f(z)$ *be a polynomial of degree* n, *all of whose zeros* z_ν *lie in the closed unit disk. Then each disk*

$$|z - z_\nu| \leq 2^{1/n} \tag{1}$$

contains at least one zero of $f'(z)$. This is still the strongest result obtained for general n toward Sendov's conjecture that the right-hand side can be replaced by 1 in (1).

Two further striking results are contained in his paper in *Trans. Amer. Math. Soc.* **163** (1972), 447–455.

Theorem 2. *Suppose that* $p_n(z)$ *is a polynomial of degree* n. *Then if* $|p_n(x)| \leq (1-x^2)^{1/2}$ *for* $-1 < x < 1$, *then*

$$|p'(x)| \leq 2(n-1) \qquad \textit{for} -1 \leq x \leq 1\,.$$

If $|p_n(x)| \leq |x|$, $-1 \leq x \leq 1$, *then*

$$|p'(x)| \leq (n-1)^2 + 1 \qquad \textit{for} -1 \leq x \leq 1\,.$$

Both these results are sharp.

Qazi Rahman was with us at Imperial College from 1959 to 1961 and was a most valuable member of our group. Other members were Jim Clunie, Noel Baker, and Thomas Kovari.

Later in 1965 when he became professor at the Université de Montréal, he invited several of us there to visit him on a number of occasions. He was a devout Muslim and so were his wife Fatima and his sons Amer and Mujtaba. It has been rightly said that the family who prays together, stays together.

I particularly remember his kindness to fellow Muslims, my wife Waficka and myself. He even offered to move out of his home so we could stay in it. I am deeply grateful for his friendship and love throughout his life and that of Fatima, Amer, and Mujtaba.

This book, which follows, is dedicated to Q.I. Rahman, who will long be remembered for his mathematics, his enthusiasm and dedication for doing research in mathematics, and for the overwhelming kindness and understanding he showered on all who came in contact with him.

London, UK W.K. Hayman, F.R.S.

Preface

This volume is a collection of chapters dedicated to Professor Q. I. Rahman who passed away on July 21, 2013, in Montreal, Canada. Professor Rahman was a leading mathematician whose research spanned several areas of approximation theory and classical analysis, including complex analysis. Professor Rahman was viewed as a world expert in the analytic theory of polynomials and entire functions of exponential type by his collaborators and many other colleagues.

We invited outstanding mathematicians, friends, and collaborators of Professor Rahman to submit chapters to be included in this volume. This collection contains original research articles and comprehensive survey contributions by 36 mathematicians from 18 countries. All the chapters were refereed. We hope that the chapters will interest graduate students and researchers in analysis and approximation theory.

Professor Walter K. Hayman, F.R.S., who was one of Professor Rahman's teachers, has prepared the monograph's Foreword. We are extremely grateful to him for the time and effort that he has devoted to the writing of this piece.

The chapters of the monograph are grouped by four themes which reflect some of Professor Rahman's areas of research. The first theme is *Polynomials*. It includes inequalities for polynomials and rational functions, orthogonal polynomials and location of zeros, and comprises the chapters entitled "On the L_2 Markov Inequality with Laguerre Weight", "Markov-Type Inequalities for Products of Müntz Polynomials Revisited", "On Bernstein-Type Inequalities for the Polar Derivative of a Polynomial", "On Two Inequalities for Polynomials in the Unit Disk", "Inequalities for Integral Norms of Polynomials via Multipliers", "Some Rational Inequalities Inspired by Rahman's Research", "On an Asymptotic Equality for Reproducing Kernels and Sums of Squares of Orthonormal Polynomials" and "Two Walsh-Type Theorems for the Solutions of Multi-Affine Symmetric Polynomials". The second theme is *Inequalities and Extremal Problems*, where functions other than polynomials are considered. This theme consists of chapters entitled "Vector Inequalities for a Projection in Hilbert Spaces and Applications", "A Half-Discrete Hardy-Hilbert-Type Inequality with a Best Possible Constant Factor Related to the Hurwitz Zeta Function", "Quantum Integral Inequalities for Generalized Convex Functions", "Quantum Integral Inequalities for Generalized Preinvex Functions" and “On the Bohr Inequality”. The third theme is *Approximation of Functions*, the approximants being polynomials, rational functions and other types of functions; see Chapters entitled “Bernstein-Type Polynomials on Several Intervals”, “Best Approximation by Logarithmically Concave Classes of Functions”, “Local Approximation Using Hermite Functions”, “Approximating the Riemann Zeta and Related Functions”, “Overconvergence of Rational Approximants of Meromorphic Functions” and “Approximation by Bernstein-Faber-Walsh and Szász-Mirakjan-Faber-Walsh Operators in Multiply Connected Compact Sets of $\mathbb{C}$”. The last theme is *Quadrature, Cubature and Applications*. It comprises three chapters, including a posthumous article of Professor Rahman co-authored by one of the editors of this book. This theme includes chapters entitled “Summation Formulas of Euler-Maclaurin and Abel-Plana: Old and New Results and Applications”, “A New Approach to Positivity and Monotonicity for the Trapezoidal Method and Related Quadrature Methods” and “A Unified and General Framework for Enriching Finite Element Approximations”.

In the first chapter, the authors Nikolov and Shadrin have considered L_2 Markov inequality with Laguerre weight over a semi-infinite interval of the real line. They have also obtained an asymptotic value of the constant in their inequality.

In the chapter by Erdélyi, new Markov-type inequalities for products of Müntz polynomials have been proved. These results extend some of the earlier contributions of the author and answer some questions posed by Thomas Bloom.

Govil and Kumar in their survey article mention in a chronological manner Bernstein-type inequalities for polar derivatives of a polynomial. This chapter provides a comprehensive account of results on polar derivatives.

Fournier and Ruscheweyh consider two very different generalizations and refinements of Bernstein’s inequality for polynomials that have been obtained more than 30 years ago. Here, the authors show that one of these inequalities implies the other. They also study the cases of equality.

Pritsker considers a wide range of polynomial inequalities for norms defined by contour and area integrals over the unit disk in the complex plane. He has also proved inequalities using the Schur-Szegő composition.

The chapter by Li, Mohapatra, and Ranasinghe is concerned with some rational inequalities inspired by Rahman’s research. The results include Bernstein-type inequalities for rational functions with prescribed poles and prescribed zeros.

Ignjatovic and Lubinsky investigate an asymptotic equality for reproducing kernels and sums of squares of orthonormal polynomials. These results are motivated by the recent work of Ignjatovic on orthonormal polynomials associated with a symmetric measure with unbounded support and satisfying a recurrence relation. The authors have studied the case of even exponential weights and weights on a finite interval.

In the chapter by B. Sendov and H. Sendov, the authors have considered two Walsh-type theorems for the solution of multi-affine symmetric polynomials. These results can be considered as extensions of the Grace-Walsh-Szegő coincidence theorem.

In the chapter by Dragomir, some vector inequalities related to those of Schwarz and Buzano are established. Also, inequalities involving the numerical range and the numerical radius for two bounded operators are obtained.

Rassias and Yang have used methods of weight functions to obtain a half-discrete Hardy-Hilbert-type inequality with a best constant related to the Hurwitz-Zeta function. Equivalent forms, normed operator expressions, their reverses, and some special cases are also considered.

In the chapter by M. Noor, K. Noor, and Awan, the authors have considered and generalized convex functions involving two arbitrary functions and established some new quantum integral inequalities for the generalized convex functions. Besides, several special cases of interest have been mentioned as corollaries.

M. Noor, Rassias, K. Noor, and Awan have considered quantum integral inequalities involving generalized preinvex functions. They give an account of quantum integral inequalities and in a certain limiting case use these inequalities to obtain many well-known results as special cases. The contents of this chapter are related to that of the previous chapter.

In the chapter by Abu Muhanna, Ali, and Ponnusamy, the Bohr inequality is considered. This survey article considers recent advances and generalizations of the Bohr inequality in the unit disk of the complex plane. Among other things, they have discussed the Bohr radius for harmonic and starlike logharmonic mappings in the unit disk.

In the chapter by Szabados, Bernstein-type polynomials for a set J_s of s finitely many intervals have been considered. On such sets, approximating operators resembling Bernstein polynomials have been defined, and their interpolation properties and rate of convergence are obtained.

The chapter by Dryanov contains results on best approximation by a class of logarithmically concave functions. Exact values of best approximations are found for two specific cases.

The chapter by Mhaskar considers local approximation using Hermite functions. He develops a wavelet-like representation in $L_p(R)$ where the local behavior of the terms characterizes the local smoothness of the target function. He gives new proofs for the localization of certain kernels as well as for the Markov-Bernstein inequality.

Stenger considers a function G which has the same zeros as the well-known Riemann zeta function in the critical strip. For studying its behavior for intermediate values of z, he uses Fourier series and derives an asymptotic approximation for large values of z.

The chapter by Blatt deals with overconvergence of rational approximants of meromorphic functions. It contains results on the degree of convergence and distribution of zeros of the rational approximants. In addition, well-known results on polynomial approximation of holomorphic functions are generalized.

The chapter by Gal is concerned with approximation by Bernstein-Faber-Walsh and Szász-Mirakjan-Faber-Walsh operators in multiply connected compact sets of the complex plane. These results are generalizations of earlier results of the author on q-Bernstein-Faber polynomials and Szász-Faber-type operators in simply connected compact sets in C. This study leads to a conjecture concerning the use of truncated classical Szász-Mirakjan operators in weighted approximation.

Milovanović discusses old and new results on summation formulas of Euler-Maclaurin and Abel-Plana. He has shown connections between Euler-Maclaurin formula and basic quadrature rules of Newton-Cotes-type as well as the composite Gauss-Legendre rule and its Lobato modifications. Summation formulas such as the midpoint summation formula, the Binet formula, and the Lindelöf formula are also extended and analyzed.

The chapter by Rahman and Schmeisser provides a new approach to positivity and monotonicity for quadrature methods. In all of the known results, sign conditions on some derivatives of the given function are required. The authors propose a new approach based on Fourier analysis and the theory of positive definite functions. This method makes it possible to describe much wider classes of functions for which positivity and monotonicity occur. Their results include the trapezoidal method on a compact interval and also on the whole real line.

The chapter by Guessab and Zaim is devoted to a unified and general framework for enriching finite element approximations through the use of additional enrichment functions. They prove a general theorem that characterizes the existence of an enriched finite element approximation. They also show that their method can be used to obtain a new class of enriched nonconforming finite elements in any dimension. For concrete constructions, the authors employ new families of multivariate trapezoidal, midpoint, and Simpson-type cubature formulas.

It is a pleasure to express our gratitude to all the authors and referees without whose contributions this volume would not have been possible. We would like to thank Ziqin Feng, Dmitry Glotov, and Eze Nwaze for their help in compiling and formatting some of the chapters in this volume. Finally, our thanks are due to the publisher for support and careful handling of this volume.

Auburn, AL, USA N.K. Govil
Orlando, FL, USA Ram Mohapatra
Tuskegee, AL, USA M.A. Qazi
Erlangen, Germany G. Schmeisser

Contents

Qazi Ibadur Rahman, 1934–2013[1]

Qazi Ibadur Rahman was born in 1934 in Deoria in the State of Uttar Pradesh (India). The English notions of *christian name* or *first name* and *last name* in their literal sense did not really suit for an Indian Moslem and so he became known in the Western world under the name Rahman, which was really his given name, while Qazi was his family name. When Rahman was a school boy, he was considered as a wunderkind in mathematics in his district. Already before having reached the age of 15, he was admitted to the University of Allahabad for studying mathematics, physics and chemistry. He graduated with a bachelor's degree in 1951 and with a masters' degree in 1953. The same year, at the age of 19, Rahman became Lecturer at the Aligarh Muslim University where he stayed till 1961. In 1956 he was awarded his first Ph.D. in mathematics, his adviser being S.M. Shah. In 1957/58, he took leave for 15 months and went to the Northwestern University in Evanston (Illinois) as a Research Associate to study with R.P. Boas. From 1959 to 1961 Rahman again took leave and went to the Imperial College in London to acquire a second Ph.D. in mathematics, his first adviser being Jim Clunie and his second Walter Hayman. Incidentally, he was Clunie's first Ph.D. student.

From 1961 to 1965 Rahman was Head of the Mathematics Department at the Regional Engineering College in Srinagar (Kashmir), except for the entire year of 1963 which he spent again at the Northwestern University in Evanston. In 1965 he joined the University of Montreal and remained there till the end of his life. He never retired.

Rahman's central field of research comprised polynomials and entire functions of exponential type. After R.P. Boas had passed away in 1992, he became the most prominent expert for the latter class of functions. His particular fields of interest were extremal problems. By utilizing tools from other areas such as variational principles, optimization techniques, subordination principles, duality principles and subharmonic functions, he obtained numerous sharp results that gave a final answer to certain interesting and important questions. Rahman's research portfolio is truly impressive. During a career that spanned 58 years, he published 203 papers and three books.

Q.I. Rahman was very inspiring for students since he slowly and patiently explained his ideas. At the University of Montreal, he had 14 Ph.D. students. At least eight of them became professors at a university or a college. A 15th student had completed his thesis but could not defend it while Rahman was alive.

Rahman's scientific activities were supported by grants of the National Research Council of Canada. For some periods, his level of funding was the highest among all Canadian mathematicians. He was very careful in using such support efficiently, enabling him to invite many famous people in his field to learn from them. One of the

[1]Except for the last three paragraphs, this biography has been essentially reproduced from *J. Approx. Theory* **179** (2014), 94–111 with the permission of Elsevier (license number 3914960806137).

editors (GS) had the pleasure to meet mathematicians such as R.P. Boas, P. Erdős, A. Gončar, W. Hayman, P. Turán and A. Zymund at Rahman's place and to profit from their lectures. For a period of about 20 years, Paul Erdős was a visitor nearly every year. Rahman also supported students and gave a chance to young promising scientists by inviting them to do research with him. In particular, at a time when the world was divided into two parts—East and West—he did not hesitate to invite young people from communist countries such as Poland, Bulgaria and Hungary. It is remarkable that Rahman held a research grant of the National Research Council of Canada for 48 years consecutively till his death, without any interruption.

In 1984 the University Marie Curie-Skłodowska (UMCS) in Lublin (Poland) awarded an honorary doctoral degree to Q.I. Rahman not only for his scientific work but also for the contacts between UMCS and the University of Montreal that he had established. He was the second mathematician to receive this honor from UMCS.

In 1965, Rahman married Imtiyaz Fatima from Lucknow. They had two sons Amer and Mujtaba, the elder son Amer being himself a professor of mathematics. A stroke of fate met Rahman, when his beloved wife died all of a sudden in 2001. Again he was very sad when he learnt that his esteemed teacher Jim Clunie passed away in 2013. In June 2013, he went with his son to a conference in Shantou (China). The day they arrived, Rahman had a fall and was brought to hospital suffering from severe brain trauma which led to a deep coma. After 5 weeks in China, he was brought by an ambulance aeroplane to Montreal where he died on July 21, 2013, exactly 43 days after his accident.

Gerhard Schmeisser (GS) had the privilege to work with Q.I. Rahman for a period of about three decades starting in 1972 when he came to Montreal as a Postdoctoral Assistant. He visited Montreal 26 times. The collaboration resulted in 44 joint papers, two joint books and a deep friendship. At most of his visits, he lived in Rahman's house and was treated like a family member. In particular, he enjoyed the delicious Indian dishes prepared by Mrs. Rahman. Of course, Rahman also visited GS in Erlangen. A few days before Rahman left for the tragic journey to China, he phoned GS and spoke to him for an hour. In retrospect it was as if he knew about his fate and wanted to say Good Bye forever. GS is very sad to have lost one of his best friends.

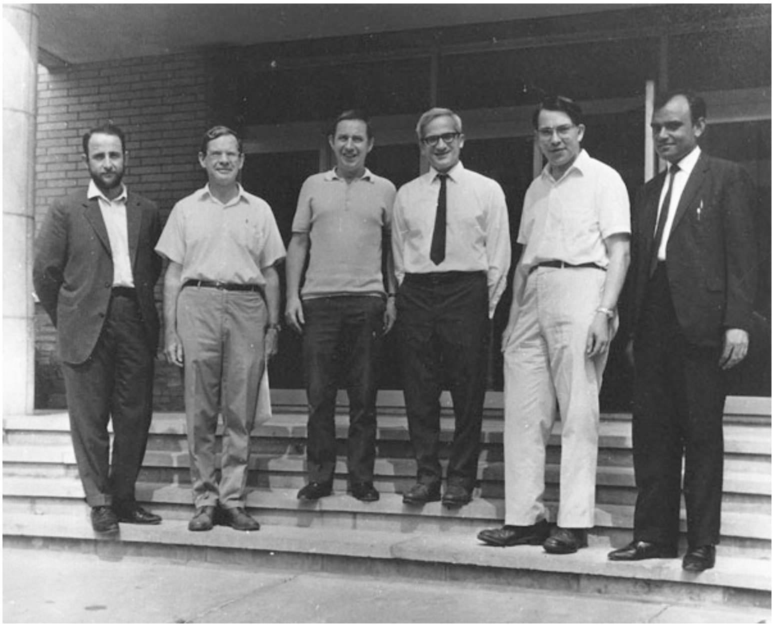
Narendra Kumar Govil (NKG) had already been a Ph.D. student for one year at the University of Montreal when Q.I. Rahman joined the University in 1965. Rahman asked the then Chair of the department, Professor Maurice Labbée, to suggest two students to work with him for doctoral studies, and Professor Labbée proposed the names of Gilbert Labelle and of NKG. NKG started working full-time on his Ph.D. dissertation in April 1966 under Rahman and, in just about one year, his dissertation was complete which NKG attributes mainly to the precious guidance and encouragement he received from Rahman. During that year, Rahman and NKG worked together for several hours a day in research. While working on problems, many times the idea of a solution came from Rahman, but he never took credit for such ideas. He gave them to NKG, perhaps to encourage him. Rahman was surely a most generous and honest person in terms of research and also in other matters. He was an extremely well-read and knowledgeable person in matters outside of mathematics as well and NKG learned so many great and useful things from him. Rahman was truly a noble soul whom NKG loved and respected deeply. NKG is very sad to have lost his teacher, mentor and a trusted friend, whom he misses greatly.

Ram N. Mohapatra (RNM) met Q.I. Rahman in 1975 at the American University of Beirut where he delivered a lecture in the Mathematics Colloquium at the invitation of Professor Walid Al-Salam who was visiting from the University of Alberta. RNM was impressed with his warmth and simplicity. In March 1995, Rahman visited Orlando to speak at the Sectional meeting of AMS at RNM's request. After the conference he asked RNM if he and his friend and colleague Xin Li could meet him on the following day to discuss what they were doing. Since that day RNM has remained in touch with Rahman and always regarded him and addressed him as an elder brother. In September 1995, RNM visited him and met Professor Frappier and Professor Bojanov in his house in Montreal. Rahman was very kind and encouraging. His research inspired RNM and some of his colleagues to a great extent. In him RNM had found a very loving mentor.

M. Amer Qazi (MAQ) and his brother Mujtaba suffered irreparable losses when their father and mother passed away. Both parents were role models for them. With their father often immersed in research at home, MAQ and his brother were fortunate to grow up in an academic environment in which they also had the distinct privilege of meeting countless visiting scientists whom their father invited over to his house. MAQ remembers how during such visits his mother would make everyone feel at home. Aside from work, MAQ recalls his father's passion for the news and many informed discussions on various topics of interest. He also liked to relax by taking the family for leisurely activities with other families in the community such as picnics and general get-togethers. MAQ and his brother feel truly blessed to have experienced a rich and diverse upbringing by their parents. They miss their parents immensely.

Professor Q. I. Rahman Memorial Volume
A Few Photographs

QIR in 1963

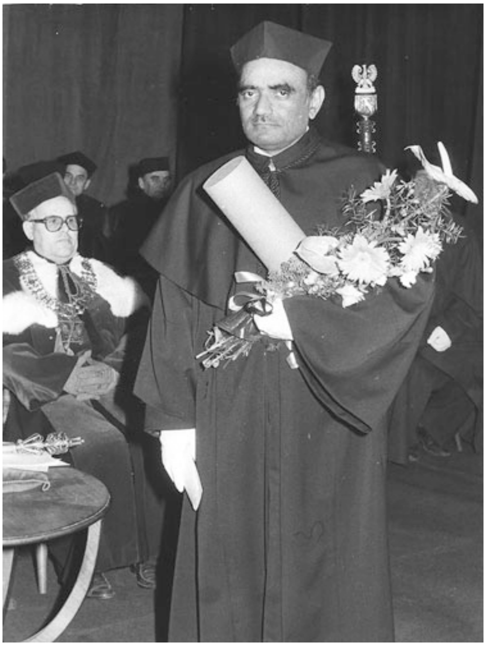


QIR, W. Hayman (2. right) and other participants of 1967 Summer Math. Seminar, Montreal

QIR with P. Turán around 1970

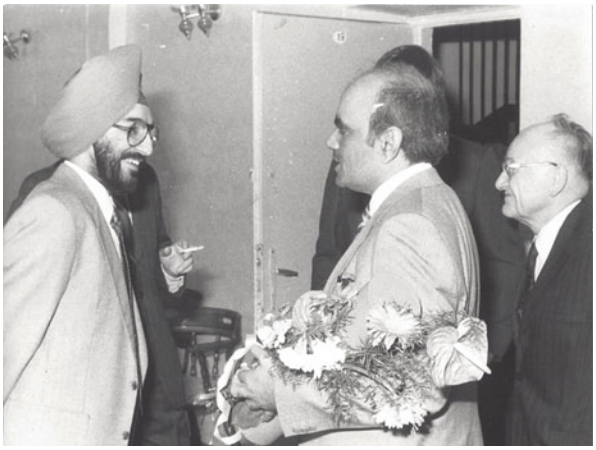
QIR with J. Krzyż at ICM 1978 in Helsinki (Finland)

QIR being awarded the Doctorate Honoris Causa by UMCS in Lublin (Poland), March 1984



QIR as Doctorate Honoris Causa in 1984

QIR with J. Krzyż during the ceremony in 1984



QIR with a Delegate of the Indian Embassy during the ceremony in 1984

QIR with his family and G. Schmeisser (2. left) in Erlangen (Germany), July 1984

QIR with G. Schmeisser (right) and D. Dryanov in Montreal, April 1987

QIR around 2004

Publications of Qazi Ibadur Rahman

Books

1. Q. I. RAHMAN and G. SCHMEISSER. **2002.** *Analytic Theory of Polynomials,* Clarendon Press, Oxford, 742 pp.
2. Q. I. RAHMAN et G. SCHMEISSER. **1983.** *Les inégalités de Markoff et de Bernstein.* Séminaire de Mathématiques Supérieures, No. **86** (Été, 1981). Les Presses de l' Université de Montréal, Qué. 173 pp.
3. Q. I. RAHMAN. **1968.** *Applications of Functional Analysis to Extremal Problems for Polynomials.* Séminaire de Mathématiques Supérieures, No. **29** (Été, 1967). Les Presses de l'Université de Montréal, Montréal, Qué., 65 pp.

Research Papers

1. Q. I. RAHMAN and G. SCHMEISSER. A new approach to positivity and monotonicity for the trapezoidal method and related quadrature methods. This volume.
2. M. A. QAZI and Q. I. RAHMAN. **2014.** Distribution of the zeros of a polynomial with prescribed lower and upper bounds for its modulus on a compact set. *Complex Variables and Elliptic Equations* **59 (9):** 1223–1235.
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On the L_2 Markov Inequality with Laguerre Weight

Geno Nikolov and Alexei Shadrin

Abstract Let $w_\alpha(t) = t^\alpha\, e^{-t}$, $\alpha > -1$, be the Laguerre weight function, and $\|\cdot\|_{w_\alpha}$ denote the associated L_2-norm, i.e.,

$$\|f\|_{w_\alpha} := \Big(\int_0^\infty w_\alpha(t)|f(t)|^2\,dt\Big)^{1/2}.$$

Denote by $\mathscr{P}_n$ the set of algebraic polynomials of degree not exceeding n. We study the best constant $c_n(\alpha)$ in the Markov inequality in this norm,

$$\|p'\|_{w_\alpha} \le c_n(\alpha)\,\|p\|_{w_\alpha}\,, \quad p \in \mathscr{P}_n\,,$$

namely the constant

$$c_n(\alpha) = \sup_{\substack{p\in\mathscr{P}_n\\ p\neq 0}} \frac{\|p'\|_{w_\alpha}}{\|p\|_{w_\alpha}}\,,$$

and we are also interested in its asymptotic value

$$c(\alpha) = \lim_{n\to\infty} \frac{c_n(\alpha)}{n}\,.$$

In this paper we obtain lower and upper bounds for both $c_n(\alpha)$ and $c(\alpha)$.

Note that according to a result of P. Dörfler from 2002, $c(\alpha) = [j_{(\alpha-1)/2,1}]^{-1}$, with $j_{\nu,1}$ being the first positive zero of the Bessel function $J_\nu(z)$, hence our bounds for $c(\alpha)$ imply bounds for $j_{(\alpha-1)/2,1}$ as well.

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1 Introduction and Statement of the Results

The Markov inequality (or, to be more precise, the inequality of the brothers Markov) has proven to be one of the most important polynomial inequalities, with numerous applications in approximation theory, numerical analysis, and many other branches of mathematics. In its classical variant it reads as follows:

The Inequality of the Brothers Markov *If $p \in \mathscr{P}_n$, then for $k = 1, \dots, n$,*

$$\|p^{(k)}\| \le T_n^{(k)}(1)\, \|p\|\,.$$

The equality is attained if and only if $p = c\,T_n$, where T_n is the n-th Chebyshev polynomial of the first kind, $T_n(x) = \cos n \arccos x$, $x \in [-1, 1]$.

Here, $\mathscr{P}_n$ is the set of algebraic polynomials of degree not exceeding n and $\|\cdot\|$ is the uniform norm in $[-1, 1]$, $\|f\| := \sup\{|f(x)| \,:\, x \in [-1, 1]\}$.

Proved for $k = 1$ in 1889 by Andrey Markov [14], and for $k \ge 1$, in 1892, by his kid brother, Vladimir Markov [15], throughout the years Markov inequality has got many alternative proofs and various generalizations. For the intriguing story of Markov's inequality in the uniform norm, and twelve of its proofs, we refer the reader to the survey paper [27]. Another survey on the subject is [2]. For some recent developments, see [3, 17, 18, 20–24, 26].

Generally, Markov-type inequalities provide upper bounds for a certain norm of a derivative of an algebraic polynomial $p \in \mathscr{P}_n$ in terms of some (usually the same) norm of p. Our subject here is Markov-type inequalities in L_2-norms for the first derivative of an algebraic polynomial. For a weight function w on the finite or infinite interval (a, b) with all moments finite, let $\|\cdot\|_w$ be the associated L_2-norm,

$$\|f\|_w := \Big(\int_a^b w(t) |f(t)|^2 \, dt \Big)^{1/2},$$

and let $c_n(w)$ be the best (i.e., the smallest) constant in the L_2 Markov inequality

$$\|p'\|_w \le c_n(w)\, \|p\|_w, \qquad p \in \mathscr{P}_n\,.$$

This constant possesses a simple characterization: it is the largest singular value of a certain matrix, see, e.g., [7] or [16]; however, the exact values of the best Markov constants are generally unknown even in the cases of the classical weight functions of Laguerre and Jacobi, and, in particular, of Gegenbauer.

The Hermite Weight $w_H(t) = e^{-t^2}$, $t \in \mathbb{R}$ This is the only case where both the sharp Markov constant and the extremal polynomial are known. Namely, in this case the sharp Markov constant is $c_n(w_H) = \sqrt{2n}$, and the unique (up to a constant factor) extremal polynomial is the n-th Hermite polynomial $H_n(t) = (-1)^n e^{t^2}\left(\frac{d}{dt}\right)^n e^{-t^2}$. The extremality of H_n persists in the L_2 Markov inequalities for higher order derivatives,

$$\|p^{(k)}\|_{w_H} \le c_n^{(k)}(w_H)\, \|p\|_{w_H}\,, \quad k = 1, \dots, n\,,$$

with the sharp Markov constants given by $c_n^{(k)}(w_H) = \left(2^k \frac{n!}{(n-k)!}\right)^{1/2}$. The reason for this case to be trivial comes from the fact that the derivatives of Hermite's polynomials are Hermite's polynomials of lower degrees [28, Chap. 5], and as a result, the sharp Markov constant is simply the largest entry in a diagonal matrix.

The Gegenbauer Weight $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$, $\lambda > -1/2$, $t \in [-1, 1]$ Neither the sharp Markov constant nor the extremal polynomial is known explicitly in that case. For $\lambda = 1/2$ (a constant weight function) Schmidt [25] found tight estimates for the Markov constant, which in a slightly weaker form look like

$$\frac{1}{\pi}(n+3/2)^2 < c_n(w_{1/2}) < \frac{1}{\pi}(n+2)^2\,, \qquad n > 5.$$

Recently, Kroó [13] turned back to this case, identifying $c_n(w_{1/2})$ as the largest positive root of a polynomial of degree n. This polynomial was found explicitly (to some extent) by Kroó.

Nikolov [19] studied two further special cases $\lambda = 0$ and $\lambda = 1$; in particular, he obtained the following two-sided estimates for the corresponding best Markov constants:

$$0.472135\, n^2 \le c_n(w_0) \le 0.478849\,(n+2)^2\,,$$
$$0.248549\, n^2 \le c_n(w_1) \le 0.256861\,(n+\tfrac{5}{2})^2\,.$$

In [1] we obtained an upper bound for $c_n(w_\lambda)$, which is valid for all $\lambda > -1/2$:

$$c_n(w_\lambda) < \frac{(n+1)(n+2\lambda+1)}{2\sqrt{2\lambda+1}}\,,$$

However, it seems that the correct order with respect to λ should be $O(1/\lambda)$. Also, it has been shown in [1] that the extremal polynomial in the L_2 Markov inequality associated with w_λ is even or odd when n is even or odd, accordingly (for $\lambda \ge 0$ this result was established, by a different argument, in [19]).

The Laguerre Weight $w_\alpha(t) = t^\alpha e^{-t}$, $t \in (0, \infty)$, $\alpha > -1$ In the present paper we study the best constant in the Markov inequality for the first derivative of an algebraic polynomial in the L_2-norm, induced by the Laguerre weight function. We denote this norm by $\|\cdot\|_{w_\alpha}$,

$$\|f\|_{w_\alpha} := \Big(\int_0^\infty t^\alpha e^{-t}|f(t)|^2\,dt\Big)^{1/2}. \tag{1}$$

Further, we denote by $c_n(\alpha)$ the best constant in the Markov inequality in this norm,

$$c_n(\alpha) = \sup_{\substack{p\in\mathscr{P}_n\\ p\neq 0}} \frac{\|p'\|_{w_\alpha}}{\|p\|_{w_\alpha}}. \tag{2}$$

Before formulating our results, let us give a brief account on the known results on the Markov inequality in the L_2 norm induced by the Laguerre weight function. Turán [29] found the sharp Markov constant in the case $\alpha = 0$, namely

$$c_n(0) = \Big(2\sin\frac{\pi}{4n+2}\Big)^{-1}. \tag{3}$$

In 1991, Dörfler [8] proved the inequalities

$$\frac{n^2}{(\alpha+1)(\alpha+3)} \le \big[c_n(\alpha)\big]^2 \le \frac{n(n+1)}{2(\alpha+1)}, \tag{4}$$

(the first one in a somewhat stronger form), and in 2002 he found [9] the sharp asymptotic of $c_n(\alpha)$, namely

$$c(\alpha) := \lim_{n\to\infty}\frac{c_n(\alpha)}{n} = \frac{1}{j_{(\alpha-1)/2,1}}, \tag{5}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function $J_\nu(z)$.

In a series of recent papers [4–6] A. Böttcher and P. Dörfler studied the asymptotic values of the best constants in L_2 Markov-type inequalities of a rather general form, namely (1) they include estimates for higher order derivatives and (2) different L_2-norms of Laguerre or Jacobi type are applied to the polynomial and its derivatives (i.e., at the two sides of their Markov inequalities).

Precisely, they proved that those asymptotic values are equal to the norms of certain Volterra operators. It seems, however, that finding the norms of these related Volterra operators explicitly is equally difficult task. They provide also some upper and lower bounds for the asymptotic values, but they do not match (they are similar to those given in (4)).

Our main goal is upper and lower bounds for the Markov constant $c_n(\alpha)$ which are valid for all n and α.

In this paper we prove the following:

Theorem 1. *For all* $\alpha > -1$ *and* $n \in \mathbb{N}$, $n \geq 3$, *the best constant* $c_n(\alpha)$ *in the Markov inequality*

$$\|p'\|_{w_\alpha} \leq c_n(\alpha)\,\|p\|_{w_\alpha}\,, \qquad p \in \mathscr{P}_n$$

admits the estimates

$$\frac{2\big(n+\frac{2\alpha}{3}\big)\big(n-\frac{\alpha+1}{6}\big)}{(\alpha+1)(\alpha+5)} < \big[c_n(\alpha)\big]^2 < \frac{\big(n+1\big)\big(n+\frac{2(\alpha+1)}{5}\big)}{(\alpha+1)\big((\alpha+3)(\alpha+5)\big)^{\frac{1}{3}}}\,,$$

where for the left-hand inequality it is additionally assumed that $n > (\alpha+1)/6$.

For $n = 1,\,2$, the exact values are readily computable:

$$\big[c_1(\alpha)\big]^2 = \frac{1}{1+\alpha}\,, \qquad \big[c_2(\alpha)\big]^2 = \frac{3(\alpha+2)+\sqrt{(\alpha+2)(\alpha+10)}}{2(\alpha+1)(\alpha+2)}\,.$$

Compared to Dörfler's result (4), we improve the lower bound for $c_n(\alpha)$ by the factor of $\sqrt{2}$, and obtain for the upper bound the order $O(n/\alpha^{5/6})$ instead of $O(n/\alpha^{1/2})$.

As an immediate consequence of Theorem 1 we obtain the following:

Corollary 1. *The asymptotic Markov constant* $c(\alpha) = \lim_{n\to\infty}\{n^{-1}\,c_n(\alpha)\}$ *satisfies the inequalities*

$$\underline{c}(\alpha) := \frac{\sqrt{2}}{\sqrt{(\alpha+1)(\alpha+5)}} \leq c(\alpha) \leq \frac{1}{\sqrt{\alpha+1}\,\sqrt[6]{(\alpha+3)(\alpha+5)}} =: \overline{c}(\alpha)\,. \qquad (6)$$

Let us comment now on the bounds for $c(\alpha)$ given by Corollary 1. First of all,

$$\lim_{\substack{\alpha\to-1\\ \alpha>-1}} \frac{\overline{c}(\alpha)}{\underline{c}(\alpha)} = 1\,,$$

which indicates that for small α our bounds are pretty tight. In particular, in the case $\alpha = 0$, when we have $c(0) = 2/\pi$ (see (3)), the relative errors satisfy

$$\frac{c(0)}{\underline{c}(0)} = \frac{\sqrt{10}}{\pi} < 1.006585\,, \qquad \frac{\overline{c}(0)}{c(0)} = \frac{\pi}{2\sqrt[6]{15}} < 1.000242\,.$$

Second, Corollary 1 gives rise to the question: what is the right order of α in $c(\alpha)$ as $\alpha \to \infty$? The answer follows below:

Theorem 2. *For the asymptotic Markov constant* $c(\alpha)$ *we have* $c(\alpha) = O(\alpha^{-1})$ *as* $\alpha \to \infty$. *More precisely,* $c(\alpha)$ *satisfies the inequalities*

$$\frac{\sqrt{2}}{\sqrt{(\alpha+1)(\alpha+5)}} < c(\alpha) < \frac{2}{\alpha+2\pi-2}\,, \quad \alpha > 1\,. \qquad (7)$$

Proof. The lower bound for $c(\alpha)$ is simply $\underline{c}(\alpha)$ (in fact, the left-hand inequality in (7) holds for all $\alpha > -1$). For the right-hand inequality in (7), we recall that, by Dörfler's result (5), $c(\alpha) = \left[j_{(\alpha-1)/2,1}\right]^{-1}$, with $j_{\nu,1}$ being the first positive zero of the Bessel function $J_\nu(z)$. On using some lower bounds for the zeros of Bessel functions, obtained by Ifantis and Siafarikas [11] (see [10, Eq. (1.6)]), we get

$$\frac{1}{j_{(\alpha-1)/2,1}} < \frac{2}{\alpha + 2\pi - 2}, \quad \alpha > 1.$$

The inequalities in (7) imply that $c(\alpha) = O(\alpha^{-1})$ as $\alpha \to \infty$. □

Notice that the lower bound $\underline{c}(\alpha)$ has the right order with respect to α as $\alpha \to \infty$. Moreover, from (7) it follows that, roughly, this lower bound can only be improved by a factor of at most $\sqrt{2}$.

The upper bound $\bar{c}(\alpha)$ does not exhibit the right asymptotic of $c(\alpha)$ as $\alpha \to \infty$. Nevertheless, $\bar{c}(\alpha)$ is less than the upper bound in (7) for $\alpha \in [2.045, 47.762]$. Moreover, the ratio $r(\alpha) = \bar{c}(\alpha)/\underline{c}(\alpha)$ tends to infinity as $\alpha \to \infty$ rather slowly; for instance, $r(\alpha)$ is less than two for $-1 < \alpha < 500$ (see Fig. 1).

Finally, in view of (5), Corollary 1 provides bounds for $j_{\nu,1}$, the first positive zero of the Bessel function J_ν, which, for some particular values of ν, are better than some of the bounds known in the literature (e.g., the lower bound below is better than the one given in [10, Eq. (1.6)] for $\nu \in [0.53, 23.38]$).

Corollary 2. *The first positive zero $j_{\nu,1}$ of the Bessel function J_ν, $\nu > -1$, satisfies the inequalities*

$$2^{\frac{5}{6}} \sqrt{\nu+1} \sqrt[6]{(\nu+2)(\nu+3)} < j_{\nu,1} < \sqrt{2(\nu+1)(\nu+3)}.$$

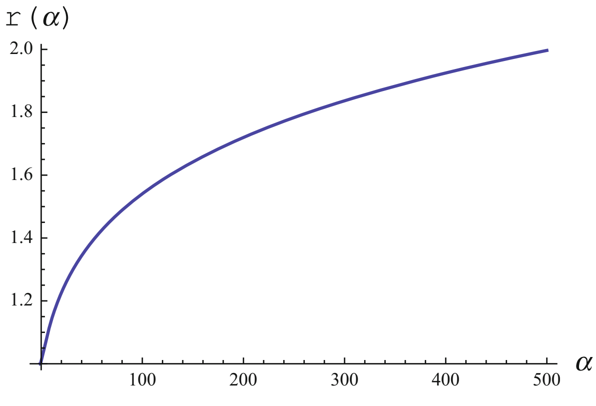

Fig. 1 The graph of the ratio $r(\alpha) = \frac{\bar{c}(\alpha)}{\underline{c}(\alpha)}$

The rest of the paper is organized as follows. In Sect. 2 we present some preliminary facts, which are needed for the proof of Theorem 1. In Sect. 2.1 we quote a known relation between the best Markov constant $c_n(\alpha)$ and the smallest (positive) zero of a polynomial $Q_n(x) = Q_n(x,\alpha)$ of degree n, defined by a three-term recurrent relation. By this definition, Q_n is identified as an orthogonal polynomial with respect to a measure supported on $\mathbb{R}_+$. In Sect. 2.2 we give lower and upper bounds for the largest zero of a polynomial, which has only positive zeros, in terms of a few of its highest degree coefficients. In Sect. 3 we prove formulae for the four lowest degree coefficients of the polynomial Q_n. The proof of our main result, Theorem 1, is given in Sect. 4. As the proof involves some lengthy tough straightforward calculations, for performing part of them we have used the assistance of a computer algebra system. Section 5 contains some final remarks.

2 Preliminaries

In this section we quote some known facts, and prove some results which will be needed for the proof of Theorem 1.

2.1 A Relation Between $c_n(\alpha)$ and an Orthogonal Polynomial

As was already said in the introduction, the best constant in an L_2 Markov inequality for polynomials of degree not exceeding n is equal to the largest singular value of a certain $n \times n$ matrix, say $\mathbf{A}_n$. The latter is equal to a square root of the largest eigenvalue of $\mathbf{A}_n^\top \mathbf{A}_n$ (or $\|\mathbf{A}_n\|_2$, the second matrix norm of $\mathbf{A}_n$). However, finding explicitly $\|\mathbf{A}_n\|_2$ (and for all $n \in \mathbb{N}$) is a fairly difficult task, and this explains the lack of many results on the sharp constants in the L_2 Markov inequalities. To avoid this difficulty, some authors simply try to estimate $\|\mathbf{A}_n\|_2$, or use other matrix norms, e.g., $\|\mathbf{A}_n\|_\infty$, the Frobenius norm, etc.

Our approach to the proof of Theorem 1 makes use of the following theorem:

Theorem 3 ([9, p. 85]). *The quantity* $1/[c_n(\alpha)]^2$ *is equal to the smallest zero of the polynomial* $Q_n(x) = Q_n(x,\alpha)$*, which is defined recursively by*

$$Q_{n+1}(x) = (x - d_n)Q_n(x) - \lambda_n^2 Q_{n-1}(x), \quad n \geq 0\,;$$

$$Q_{-1}(x) := 0, \;\; Q_0(x) := 1\,;$$

$$d_0 := 1 + \alpha, \;\; d_n := 2 + \frac{\alpha}{n+1}\,, \quad n \geq 1\,;$$

$$\lambda_0 > 0 \;\text{ arbitrary}, \;\; \lambda_n^2 := 1 + \frac{\alpha}{n} \quad n \geq 1\,.$$

By Favard's theorem, for any $\alpha > -1$, $\{Q_n(x,\alpha)\}_{n=0}^{\infty}$ form a system of monic orthogonal polynomials, and, in addition, we know that the support of their orthogonality measure is in $\mathbb{R}_+$. Theorem 3 transforms the problem of finding or estimating $c_n(\alpha)$ to a problem for finding or estimating the extreme zeros of orthogonal polynomials, or, equivalently, the extreme eigenvalues of certain tri-diagonal (Jacobi) matrices. For the latter problem one can apply numerous powerful methods such as the Gershgorin circles and the ovals of Cassini. For more details on this kind of methods we refer the reader to the excellent paper of van Doorn [30].

However, we choose here a different approach for estimating the smallest positive zero of $Q_n(x,\alpha)$, which seems to be efficient, too.

2.2 *Bounds for the Largest Zero of a Polynomial Having Only Positive Roots*

In view of Theorem 3, we need to estimate the smallest (positive) zero of the polynomial $Q_n(x,\alpha)$. On using the three-term recurrence relation for $\{Q_m\}_{m=0}^{\infty}$, we can evaluate (at least theoretically) as many coefficients of $Q_n(x)$ as we wish (and thus coefficients of the reciprocal polynomial $x^n Q_n(x^{-1})$, too). Our proof of Theorem 1 makes use of the following statement:

Proposition 1. *Let* $P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} - \cdots + (-1)^{n-1} b_{n-1} x + (-1)^n b_n$ *be a polynomial with positive roots* $x_1 \le x_2 \le \cdots \le x_n$. *Then the largest zero* x_n *of* P *satisfies the inequalities:*

(i) $\dfrac{b_1}{n} \le x_n < b_1\,;$

(ii) $b_1 - 2\dfrac{b_2}{b_1} \le x_n < (b_1^2 - 2b_2)^{\frac{1}{2}}\,;$

(iii) $\dfrac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} \le x_n < (b_1^3 - 3b_1 b_2 + b_3)^{\frac{1}{3}}\,.$

Proof. Part (i) follows trivially from

$$\frac{b_1}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n} \le x_n < x_1 + x_2 + \cdots + x_n = b_1\,.$$

For the proof of parts (ii) and (iii) we make use of Newton's identities to obtain

$$x_1^2 + x_2^2 + \cdots + x_n^2 = b_1^2 - 2b_2, \qquad x_1^3 + x_2^3 + \cdots + x_n^3 = b_1^3 - 3b_1 b_2 + 3b_3\,.$$

Now (ii) follows from

$$\frac{b_1^2 - 2b_2}{b_1} = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} \le x_n < (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} = (b_1^2 - 2b_2)^{\frac{1}{2}}$$

and (iii) follows from

$$\frac{b_1^3 - 3b_1 b_2 + 3b_3}{b_1^2 - 2b_2} = \frac{x_1^3 + \cdots + x_n^3}{x_1^2 + \cdots + x_n^2} \le x_n < (x_1^3 + \cdots + x_n^3)^{\frac{1}{3}} = (b_1^3 - 3b_1 b_2 + 3b_3)^{\frac{1}{3}} .$$

It is clear from the proof that the lower bounds for x_n are attained only when $x_1 = x_2 = \cdots = x_n$. □

3 The Lowest Degree Coefficients of the Polynomial $Q_{n,\alpha}$

Let us denote by $a_{k,n} = a_{k,n}(\alpha)$, $k = 0, \ldots, n$, the coefficients of the monic polynomial $Q_n(x) = Q_n(x, \alpha)$, introduced in Theorem 3, i.e.,

$$Q_n(x) = Q_n(x, \alpha) = x^n + a_{n-1,n} x^{n-2} + \cdots + a_{3,n} x^3 + a_{2,n} x^2 + a_{1,n} x + a_{0,n} .$$

For the sake of convenience, we set $a_{m,m} = 1$, $m \ge 0$, and

$$a_{k,m} = 0 , \text{ if } k < 0 \text{ or } k > m .$$

From the recursive definition of Q_n we have $Q_0(x) = 1$, $Q_1(x) = x - \alpha - 1$, thus

$$a_{0,1} = -\alpha - 1 ,$$

and for $n \in \mathbb{N}$ we obtain a recurrence relations for the coefficients of Q_{n-1}, Q_n, and Q_{n+1} :

$$a_{k,n+1} = a_{k-1,n} - \Big(2 + \frac{\alpha}{n+1}\Big) a_{k,n} - \Big(1 + \frac{\alpha}{n}\Big) a_{k,n-1} , \quad k = 0, \ldots, n . \tag{8}$$

Now recurrence relation (8) will be used to prove consecutively formulae for the coefficients $a_{k,n}$, $0 \le k \le 3$.

Proposition 2. *For all $n \in \mathbb{N}_0$, the coefficient $a_{0,n}$ of the polynomial Q_n is given by*

$$a_{0,n} = (-1)^n \prod_{k=1}^{n} \Big(1 + \frac{\alpha}{k}\Big) .$$

Proof. We apply induction with respect to n. Since $a_{0,0} = 1$ and $a_{0,1} = -(1 + \alpha)$, Proposition 2 is true for $n = 0$ and $n = 1$. For $k = 0$ the recurrence relation (8) becomes

$$a_{0,n+1} = -\Big(2 + \frac{\alpha}{n+1}\Big) a_{0,n} - \Big(1 + \frac{\alpha}{n}\Big) a_{0,n-1} , \quad n \in \mathbb{N} .$$

Assuming Proposition 2 is true for $m \le n$, for $m = n + 1$ we obtain

$$
\begin{aligned}
a_{0,n+1} &= -\Big(2 + \frac{\alpha}{n+1}\Big)\,(-1)^n \prod_{k=1}^{n}\Big(1 + \frac{\alpha}{k}\Big) - \Big(1 + \frac{\alpha}{n}\Big)\,(-1)^{n-1}\prod_{k=1}^{n-1}\Big(1 + \frac{\alpha}{k}\Big) \\
&= (-1)^{n+1}\prod_{k=1}^{n+1}\Big(1 + \frac{\alpha}{k}\Big),
\end{aligned}
$$

hence the induction step is done, and Proposition 2 is proved. □

Before proceeding with the proof of the formulae for $a_{k,n}$, $1 \le k \le 3$, let us point out to the relation

$$
a_{0,m+1} = -\Big(1 + \frac{\alpha}{m+1}\Big)\,a_{0,m}\,, \quad m \in \mathbb{N}_0\,, \tag{9}
$$

which follows from Proposition 2, and will be used in the proof of the next propositions.

Proposition 3. *For all $n \in \mathbb{N}_0$, the coefficient $a_{1,n}$ of the polynomial Q_n is given by*

$$
a_{1,n} = -\frac{n(n+1)}{2(\alpha+1)}\,a_{0,n}\,.
$$

Proof. Again, we apply induction on n. Proposition 3 is true for $n = 0$ and $n = 1$. Indeed, by our convention, $a_{1,0} = 0$, and $a_{1,1} = 1$ also obeys the desired representation, as $a_{0,1} = -(1+\alpha)$. Assume that Proposition 3 is true for $m \le n$, $m \in \mathbb{N}$. From the recurrence relation (8) (with $k = 1$), the induction hypothesis and (9) we obtain

$$
\begin{aligned}
a_{1,n+1} &= a_{0,n} - \Big(2 + \frac{\alpha}{n+1}\Big)\,a_{1,n} - \Big(1 + \frac{\alpha}{n}\Big)\,a_{1,n-1} \\
&= a_{0,n} + \Big(2 + \frac{\alpha}{n+1}\Big)\frac{n(n+1)}{2(\alpha+1)}\,a_{0,n} + \Big(1 + \frac{\alpha}{n}\Big)\frac{(n-1)n}{2(\alpha+1)}\,a_{0,n-1} \\
&= a_{0,n}\Big[1 + \Big(2 + \frac{\alpha}{n+1}\Big)\frac{n(n+1)}{2(\alpha+1)} - \frac{(n-1)n}{2(\alpha+1)}\Big] \\
&= \frac{a_{0,n}}{2(\alpha+1)}\big[n^2 + (\alpha+3)n + 2(\alpha+1)\big] = a_{0,n}\,\frac{(n+2)(n+\alpha+1)}{2(\alpha+1)} \\
&= \frac{(n+1)(n+2)}{2(\alpha+1)}\Big(1 + \frac{\alpha}{n+1}\Big)\,a_{0,n} = -\frac{(n+1)(n+2)}{2(\alpha+1)}\,a_{0,n+1}\,.
\end{aligned}
$$

Hence, the induction step is done, and the proof of Proposition 3 is complete. □

Proposition 4. *For all $n \in \mathbb{N}_0$, the coefficient $a_{2,n}$ of the polynomial Q_n is given by*

$$a_{2,n} = \frac{(n-1)n(n+1)}{24(\alpha+1)(\alpha+2)(\alpha+3)} \big[3(\alpha+2)n + 2(\alpha+6)\big]\, a_{0,n}\,.$$

Proof. The claim is true for $n = 0, 1$ (according to our convention), and also for $n = 2$, as in this case, taking into account that $a_{0,2} = 1/\big((1+\alpha)(1+\alpha/2)\big)$, the above formula produces $a_{2,2} = 1$. Assume now that the proposition is true for $m \leq n$, where $n \in \mathbb{N}$, $n \geq 2$. We shall prove that it is true for $m = n+1$, too, thus proving Proposition 4 by induction. On using the recurrence relation (8) (with $k = 2$), the inductional hypothesis, Proposition 3, and (9) we obtain

$$\begin{aligned}
a_{2,n+1} &= a_{1,n} - \Big(2 + \frac{\alpha}{n+1}\Big)\, a_{2,n} - \Big(1 + \frac{\alpha}{n}\Big)\, a_{2,n-1} \\
&= -\frac{n(n+1)}{2(\alpha+1)}\, a_{0,n} - \Big(2 + \frac{\alpha}{n+1}\Big) \frac{(n-1)n(n+1)\big[3(\alpha+2)n+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\, a_{0,n} \\
&\quad + \frac{(n-2)(n-1)n\big[3(\alpha+2)(n-1)+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\, a_{0,n} \\
&= \frac{n(n+1)}{n+\alpha+1}\Bigg[\frac{n+1}{2(\alpha+1)} + \Big(2 + \frac{\alpha}{n+1}\Big) \frac{(n^2-1)\big[3(\alpha+2)n+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)} \\
&\qquad - \frac{(n-2)(n-1)\big[3(\alpha+2)(n-1)+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\Bigg]\, a_{0,n+1}\,.
\end{aligned}$$

After some calculations the expression in the big brackets simplifies to

$$\frac{(n+2)(n+\alpha+1)\big[(3(\alpha+2)(n+1)+2(\alpha+6)\big]}{24(\alpha+1)(\alpha+2)(\alpha+3)}\,.$$

and substitution of this expression yields the desired formula for $a_{2,n+1}$. The induction proof of Proposition 4 is complete. □

Proposition 5. *For all $n \in \mathbb{N}_0$, the coefficient $a_{3,n}$ of the polynomial Q_n is given by the expression*

$$-\frac{(n-2)(n-1)n(n+1)\big[5(\alpha+2)(\alpha+4)n(n+1)+8(7\alpha+20)n+12(\alpha+20)\big]}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\, a_{0,n}.$$

Proof. Again, induction is applied with respect to n. The formula for $a_{3,n}$ is easily verified to be true for $0 \leq n \leq 3$. Then, assuming that this formula is true for $m \leq n$, where $n \in \mathbb{N}$, $n \geq 3$, we prove that it is true also for $m = n+1$, too. The induction step is performed along the same lines as the one in the proof of Proposition 4. First, we make use of the recurrence relation (8) with $k = 3$ to express

$a_{3,n+1}$ as a linear combination of $a_{2,n}$, $a_{3,n}$, and $a_{3,n-1}$. Next, we apply the inductional hypothesis and (9) to represent $a_{3,n+1}$ in the form

$$a_{3,n+1} = \frac{-(n-1)n(n+1)}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)} \, \frac{r(n)}{n+\alpha+1} \, a_{0,n+1} \,,$$

where $r(n) = r(n,\alpha)$ is a polynomial of 4-th degree. With some lengthy though straightforward calculation (we used a computer algebra program for verification) we obtain that

$$r(n) = (n+2)(n+\alpha+1)\big[5(\alpha+2)(\alpha+4)(n+1)(n+2) + 8(7\alpha+20)(n+1) + 12(\alpha+20)\big]$$

and this expression substituted in the above formula implies the desired representation of $a_{3,n+1}$. To keep the paper condensed, we omit the details. □

4 Proof of Theorem 1

For the proof of Theorem 1 we prefer to work with the (constant multiplier of) reciprocal polynomial of Q_n

$$P_n(x) = P_n(x,\alpha) = (-1)^n \left(a_{0,n}\right)^{-1} x^n \, Q_n\!\left(x^{-1}\right).$$

Clearly, P_n is a monic polynomial of degree n,

$$P_n(x) = x^n - b_1 \, x^{n-1} + b_2 \, x^{n-2} - b_3 \, x^{n-3} + \cdots$$

and, in view of Propositions 2–5, its coefficients b_1, b_2, and b_3 are

$$b_1 = \frac{n(n+1)}{2(\alpha+1)}, \quad b_2 = \frac{(n-1)n(n+1)}{24(\alpha+1)(\alpha+2)(\alpha+3)} \big[3(\alpha+2)n + 2(\alpha+6)\big],$$

$$b_3 = \frac{(n-2)(n-1)n(n+1)\big[5(\alpha+2)(\alpha+4)n(n+1) + 8(7\alpha+20)n + 12(\alpha+20)\big]}{240(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}.$$

As was indicated in Sect. 2.1, $Q_n(x,\alpha)$ is identified an orthogonal polynomial with positive and distinct zeros. Therefore, the same can be said for the zeros of P_n (as reciprocal of Q_n). If x_n is the largest zero of P_n, then, according to Theorem 3, we have $\big[c_n(\alpha)\big]^2 = x_n$.

Now Proposition 1 (iii) applied to $P = P_n$ yields immediately the following:

Proposition 6. *For all* $n \in \mathbb{N}$, $n \geq 3$, *the best Markov constant* $c_n(\alpha)$ *satisfies*

$$\frac{b_1^3 - 3b_1 \, b_2 + 3b_3}{b_1^2 - 2b_2} < \big[c_n(\alpha)\big]^2 < (b_1^3 - 3b_1 \, b_2 + 3b_3)^{\frac{1}{3}}$$

with b_1, b_2, *and* b_3 *as given above.*

The estimates for $c_n(\alpha)\big]$ in Theorem 1 are a consequence of Proposition 6. For the proof of the lower bound, we obtain that

$$b_1^3 - 3b_1\,b_2 + 3b_3 - \frac{2}{(\alpha+3)(\alpha+5)}\Big(n+\frac{2\alpha}{3}\Big)\Big(n-\frac{\alpha+1}{6}\Big)(b_1^2-2b_2)$$
$$= \frac{1}{(\alpha+1)^3(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\sum_{j=1}^{5}\kappa_j(\alpha)\,n^j\,,$$

with

$$\kappa_1(\alpha) = \frac{1}{270}\,(1+\alpha)^2(10\,\alpha^3+100\,\alpha^2+321\,\alpha+1620)\,,$$
$$\kappa_2(\alpha) = \frac{1}{36}\,(1+\alpha)(4\,\alpha^4+35\,\alpha^3+166\,\alpha^2+417\,\alpha+660)\,,$$
$$\kappa_3(\alpha) = \frac{1}{54}\,(4\,\alpha^5+36\,\alpha^4+192\,\alpha^3+625\,\alpha^2+1527\,\alpha+1332)\,,$$
$$\kappa_4(\alpha) = \frac{1}{36}\,(\alpha^4-\alpha^3+157\,\alpha^2+579\,\alpha+780)\,,$$
$$\kappa_5(\alpha) = \frac{1}{30}\,(\alpha^3+7\,\alpha^2+136\,\alpha+280)\,.$$

Obviously, $\kappa_j(\alpha) > 0$ for $\alpha > -1$, $1 \le j \le 5$, and hence the lower bound holds:

$$\big[c_n(\alpha)\big]^2 > \frac{b_1^3-3b_1\,b_2+3b_3}{b_1^2-2\,b_2} > \frac{2}{(\alpha+3)(\alpha+5)}\Big(n+\frac{2\alpha}{3}\Big)\Big(n-\frac{\alpha+1}{6}\Big)\,.$$

For the proof of the upper bound for $c_n(\alpha)$ in Theorem 1, we find that

$$\frac{1}{(\alpha+1)^3(\alpha+3)(\alpha+5)}\,(n+1)^3\Big(n+\frac{2(\alpha+1)}{5}\Big)^3 - \big(b_1^3-3b_1\,b_2+3b_3\big)$$
$$= \frac{1}{(\alpha+1)^2(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}\sum_{j=0}^{5}\nu_j(\alpha)\,n^j\,,$$

where

$$\nu_0(\alpha) = \frac{8}{125}\,(1+\alpha)^2(2+\alpha)(4+\alpha)\,;$$
$$\nu_1(\alpha) = \frac{3}{250}\,(1+\alpha)(16\,\alpha^3+152\,\alpha^2+439\,\alpha-52)\,,$$
$$\nu_2(\alpha) = \frac{1}{500}\,(96\,\alpha^4+1363\,\alpha^3+5656\,\alpha^2+9167\,\alpha+2828)\,,$$

$$\nu_3(\alpha) = \frac{1}{250}\,(16\,\alpha^4 + 363\,\alpha^3 + 2506\,\alpha^2 + 7167\,\alpha + 4708)\,,$$
$$\nu_4(\alpha) = \frac{1}{100}\,(23\,\alpha^3 + 446\,\alpha^2 + 1657\,\alpha + 2164)\,,$$
$$\nu_5(\alpha) = \frac{3}{5}\,(5\alpha + 16)\,.$$

We shall show now that

$$\sum_{j=0}^{5} \nu_j(\alpha)\, n^j \ge 0\,, \qquad n \ge 2\,,\ \alpha > -1\,. \tag{10}$$

Notice that, unlike the case with the coefficients $\{\kappa_j(\alpha)\}_{j=1}^5$, which are all positive for all admissible values of α, i.e., $\alpha > -1$, here the coefficients $\nu_j(\alpha)$, $1 \le j \le 3$, assume negative values for some $\alpha \in (-1,0)$ ($\nu_1(\alpha)$ is negative also for some $\alpha > 0$).

Since $\nu_4(\alpha)$ and $\nu_5(\alpha)$ are positive for $\alpha > -1$, for $n \ge 2$ we have

$$\sum_{j=3}^{5} \nu_j(\alpha)\, n^j \ge \big(4\,\nu_5(\alpha) + 2\,\nu_4(\alpha) + \nu_3(\alpha)\big) n^3 =: \tilde{\nu}_3(\alpha)\, n^3\,,$$

where

$$\tilde{\nu}_3(\alpha) = \frac{1}{125}\,(8\,\alpha^4 + 239\,\alpha^3 + 2368\,\alpha^2 + 9226\,\alpha + 12564)\,.$$

Since $\tilde{\nu}_3(\alpha) > 0$ for $\alpha > -1$, we have

$$\sum_{j=2}^{5} \nu_j(\alpha)\, n^j \ge \big(2\tilde{\nu}_3(\alpha) + \nu_2(\alpha)\big)\, n^2 =: \tilde{\nu}_2(\alpha)\, n^2\,, \qquad n \ge 2\,,$$

where

$$\tilde{\nu}_2(\alpha) = \frac{1}{100}\,(32\,\alpha^4 + 655\,\alpha^3 + 4920\,\alpha^2 + 16595\,\alpha + 20668)\,.$$

Now, from $\tilde{\nu}_2(\alpha) > 0$ for $\alpha > -1$, we obtain

$$\sum_{j=1}^{5} \nu_j(\alpha)\, n^j \ge \big(2\tilde{\nu}_2(\alpha) + \nu_1(\alpha)\big)\, n =: \tilde{\nu}_1(\alpha)\, n\,, \qquad n \ge 2\,,$$

with

$$\tilde{\nu}_1(\alpha) = \frac{1}{250}\,(160\,\alpha^4 + 3323\,\alpha^3 + 25056\,\alpha^2 + 84292\,\alpha + 103184) > 0\,, \quad \alpha > -1\,.$$

Hence, $\sum_{j=0}^{5} \nu_j(\alpha)\, n^j \ \geq\ \tilde{\nu}_1(\alpha)\, n + \nu_0(\alpha)\ >\ 0$, and (10) is proved. From (10) we conclude that

$$\frac{1}{(\alpha+1)^3(\alpha+3)(\alpha+5)}\,(n+1)^3\Big(n+\frac{2(\alpha+1)}{5}\Big)^3 > b_1^3 - 3b_1\, b_2 + 3b_3\,,$$

In view of Proposition 6, the latter inequality proves the upper bound for $c_n(\alpha)$ in Theorem 1.

5 Concluding Remarks

1. Our main concern here is the major terms in the bounds for the best Markov constant $c_n(\alpha)$, obtained through Proposition 1. We did not care much about the lower degree terms, where perhaps some improvement is possible.

2. Obviously, Dörfler's upper bound for $c_n(\alpha)$ in (4) is a consequence of Proposition 1 (i). Dörfler's lower bound for $c_n(\alpha)$ in [8], which is slightly better than the one given in (4), is obtained from Proposition 1 (ii). Both our lower and upper bounds for the asymptotic constant $c(\alpha)$, given in Corollary 1, are superior for all $\alpha > -1$ to Dörfler's bounds obtained from (4) .

3. The upper bounds for the largest zero x_n of a polynomial having only real and positive zeros in Proposition 1 (ii) and (ii) admit some improvement. For instance, in Proposition 1 (ii) one can apply the quadratic mean–arithmetic mean inequality to obtain

$$b_1^2 - 2b_2 = x_n^2 + \sum_{i=1}^{n-1} x_i^2 \geq x_n^2 + \frac{\Big(\sum_{i=1}^{n-1}|x_i|\Big)^2}{n-1} \geq x_n^2 + \frac{(b_1-x_n)^2}{n-1}\,,$$

which yields a (slightly stronger) quadratic inequality for x_n (actually, for any of the zeros of the polynomial P),

$$n\, x_n^2 - 2b_1\, x_n + 2(n-1)b_2 - (n-2)b_1^2 \leq 0\,.$$

The solution of the latter inequality,

$$\frac{1}{n}\Big[b_1 - \sqrt{(n-1)^2 b_1^2 - 2(n-1)n\, b_2}\,\Big] \leq x_n \leq \frac{1}{n}\Big[b_1 + \sqrt{(n-1)^2 b_1^2 - 2(n-1)n\, b_2}\,\Big],$$

provides lower and upper bounds for the zeros of an arbitrary real-root monic polynomial of degree n in terms of its two leading coefficients b_1 and b_2. This result, due to Laguerre, is also known as Laguerre–Samuelson inequality (for more details, see, e.g., [12] and the references therein).

In a similar way one can obtain a slight improvement for the upper bound in Proposition 1 (iii). However, in our case this improvement is negligible (it affects only the lower degree terms in the upper bound for $c_n(\alpha)$).

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Markov-Type Inequalities for Products of Müntz Polynomials Revisited

Tamás Erdélyi

Abstract Professor Rahman was a great expert of Markov- and Bernstein-type inequalities for various classes of functions, in particular for polynomials under various constraints on their zeros, coefficients, and so on. His books are great sources of such inequalities and related matters. Here we do not even try to survey Rahman's contributions to Markov- and Bernstein-type inequalities and related results. We focus on Markov-type inequalities for products of Müntz polynomials. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. We denote the linear span of $x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}$ over $\mathbb{R}$ by $M(\Lambda_n) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$. Elements of $M(\Lambda_n)$ are called Müntz polynomials. The principal result of this paper is a Markov-type inequality for products of Müntz polynomials on intervals $[a, b] \subset (0, \infty)$ which extends a less general result proved in an earlier publication. It allows us to answer some questions asked by Thomas Bloom recently in e-mail communications. The author believes that the new results in this paper are sufficiently interesting and original to serve as a tribute to the memory of Professor Rahman in this volume.

Keywords Markov-type inequality • Markov–Nikolskii-type inequality • Bernstein-type inequality • Müntz polynomials • Lacunary polynomials • Dirichlet sums • Exponential sums • Products

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1 Introduction and Notation

Let $\mathcal{P}_n$ denote the family of all algebraic polynomials of degree at most n with real coefficients. We use the notation

$$\|f\|_A := \|f\|_{L_\infty(A)} := \|f\|_{L_\infty A} := \sup_{t\in A} |f(t)|$$

and

$$\|f\|_{L_q A} := \|f\|_{L_q(A)} := \left(\int_a^b |f(t)|^q \, dt \right)^{1/q}, \qquad q > 0\,,$$

for measurable functions f defined on a nonempty set $A \subset \mathbb{R}$. Two classical inequalities for polynomials are the following:

Markov Inequality. *We have*

$$\|f'\|_{[a,b]} \leq \frac{2n^2}{b-a} \|f\|_{[a,b]}$$

for every $f \in \mathcal{P}_n$ and for every subinterval $[a,b]$ of the real line.

Bernstein Inequality. *We have*

$$|f'(y)| \leq \frac{n}{\sqrt{(b-y)(y-a)}} \|f\|_{[a,b]}\,, \qquad y \in (a,b)\,,$$

for every $f \in \mathcal{P}_n$ and for every $[a,b] \subset \mathbb{R}$.

For proofs see [4] or [16], for example. Professor Rahman was a great expert of Markov- and Bernstein-type inequalities for various classes of functions, in particular for polynomials under various constraints on their zeros, coefficients, and so on. His books [33] and [34] are great sources of such inequalities. See also [4, 19, 22], for instance. Here we do not even try to survey Rahman's contributions to Markov- and Bernstein-type inequalities and related results. We focus only on Markov- and Bernstein-type inequalities for products of Müntz polynomials. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. We denote the linear span of $x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}$ over $\mathbb{R}$ by

$$M(\Lambda_n) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}\,.$$

Elements of $M(\Lambda_n)$ are called Müntz polynomials. We denote the linear span of $e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ over $\mathbb{R}$ by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\}\,.$$

Elements of $E(\Lambda_n)$ are called exponential sums. Observe that the substitution $x = e^t$ transforms exponential sums into Müntz polynomials and the interval $(-\infty, 0]$ onto $(0, 1]$.

Newman [31] established an essentially sharp Markov-type inequality for $M(\Lambda_n)$.

Theorem 1.1 (Newman's Inequality). *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers. We have*

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \leq \sup_{0\not\equiv f\in M(\Lambda_n)} \frac{|f'(1)|}{\|f\|_{[0,1]}} \leq \sup_{0\not\equiv f\in M(\Lambda_n)} \frac{\|xf'(x)\|_{[0,1]}}{\|f\|_{[0,1]}} \leq 11\sum_{j=0}^{n}\lambda_j\,.$$

Frappier [28] showed that the constant 11 in Newman's inequality can be replaced by 8.29. By modifying and simplifying Newman's arguments, Borwein and Erdélyi [9] showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the "right" L_q version $(1 \leq q \leq \infty)$ of Newman's inequality [9] (an L_2 version of which was proved earlier by Borwein et al. [13]). Note that Newman's inequality can be rewritten as

$$\frac{2}{3}\sum_{j=0}^{n}\lambda_j \leq \sup_{0\not\equiv g\in E(\Lambda_n)} \frac{|g'(0)|}{\|g\|_{(-\infty,0]}} \leq \sup_{0\not\equiv g\in E(\Lambda_n)} \frac{\|g'\|_{(-\infty,0]}}{\|g\|_{(-\infty,0]}} \leq 11\sum_{j=0}^{n}\lambda_j\,,$$

whenever $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is a set of nonnegative real numbers.

It is non-trivial and proved by Borwein and Erdélyi [4] that under a growth condition, $\|xf'(x)\|_{[0,1]}$ in Newman's inequality can be replaced by $\|f'\|_{[0,1]}$. More precisely, the following result holds:

Theorem 1.2 (Newman's Inequality Without the Factor x). *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of nonnegative real numbers with $\lambda_0 = 0$ and $\lambda_j \geq j$ for each j. We have

$$\|f'\|_{[0,1]} \leq 18\left(\sum_{j=0}^{n}\lambda_j\right)\|f\|_{[0,1]}$$

for every $f \in M(\Lambda_n)$.

It can be shown that the growth condition in Theorem 1.2 is essential. This observation is based on an example given by Len Bos (non-published communication). The statement below is proved in [18].

Example 1.3. For every $\delta \in (0, 1)$ there exists a sequence $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with $\lambda_0 := 0$, $\lambda_1 \geq 1$, and

$$\lambda_{j+1} - \lambda_j \geq \delta\,, \qquad j = 0, 1, 2, \ldots,$$

such that with $\Lambda_\mu := \{\lambda_0 < \lambda_1 < \cdots < \lambda_\mu\}$ we have

$$\lim_{\mu\to\infty} \sup_{0\not\equiv f\in M(\Lambda_\mu)} \frac{|f'(0)|}{\left(\sum_{j=0}^{\mu} \lambda_j\right) \|f\|_{[0,1]}} = \infty\,.$$

Note that the interval [0, 1] plays a special role in the study of Müntz polynomials. A linear transformation $y = \alpha x + \beta$ does not preserve membership in $M(\Lambda_n)$ in general (unless $\beta = 0$), that is, $f \in M(\Lambda_n)$ does not necessarily imply that $g(x) := f(\alpha x + \beta) \in M(\Lambda_n)$. Analogs of the above results on $[a, b]$, $a > 0$, cannot be obtained by a simple transformation. However, Borwein and Erdélyi [8] proved the following result:

Theorem 1.4 (Newman's Inequality on $[a, b] \subset (0, \infty)$**).** *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$$

be a set of real numbers. Suppose there exists a $\varrho > 0$ *such that* $\lambda_j \ge \varrho j$ *for each j. Suppose* $a, b \in \mathbb{R}$ *and* $0 < a < b$. *There exists a constant* $c(a, b, \varrho)$ *depending only on a, b, and* ϱ *such that*

$$\|f'\|_{[a,b]} \le c(a, b, \varrho) \left(\sum_{j=0}^{n} \lambda_j\right) \|f\|_{[a,b]}$$

for every $f \in M(\Lambda_n)$.

The above theorem is essentially sharp, as one can easily deduce it from the first inequality of Theorem 1.1 by a linear scaling. The novelty of Theorem 1.5 proved in [2] later is the fact that $\Lambda_n := \{\lambda_0 < \lambda_1 < \ldots < \lambda_n\}$ is an arbitrary set of $n + 1$ distinct real numbers, not even the non-negativity of the exponents λ_j is needed.

Theorem 1.5. *Let* $n \ge 1$ *be an integer. Let* $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ *be a set of* $n + 1$ *distinct real numbers. Suppose* $a, b \in \mathbb{R}$ *and* $0 < a < b$. *We have*

$$\frac{1}{3}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4\log(b/a)}(n-1)^2 \le \sup_{0\not\equiv f\in M(\Lambda_n)} \frac{\|xf'(x)\|_{[a,b]}}{\|f\|_{[a,b]}} \le 11\sum_{j=0}^{n} |\lambda_j| + \frac{128}{\log(b/a)}(n+1)^2\,.$$

Remark 1.6. Of course, we can have $f'(x)$ instead of $xf'(x)$ in the above estimate, as an obvious corollary of the above theorem is

$$\frac{1}{3b}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4b\log(b/a)}(n-1)^2 \le \sup_{0\not\equiv f\in M(\Lambda_n)} \frac{\|f'\|_{[a,b]}}{\|f\|_{[a,b]}} \le \frac{11}{a}\sum_{j=0}^{n} |\lambda_j| + \frac{128}{a\log(b/a)}(n+1)^2$$

for every $a, b \in \mathbb{R}$ such that $0 < a < b$. Observe also that Theorem 1.1 can be obtained from Theorem 1.5 (with the constant $1/3$ in the lower bound) as a limiting case by letting $a > 0$ tend to 0.

The following $L_q[a,b]$ version of Theorem 1.5 is also proved in [2] for $q \geq 1$.

Theorem 1.7. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of $n+1$ distinct real numbers. Suppose $a, b \in \mathbb{R}$, $0 < a < b$, and $1 \leq q < \infty$. There is a positive constant $c_1(a,b)$ depending only on a and b such that*

$$\sup_{0 \not\equiv f \in M(\Lambda_n)} \frac{\|f'\|_{L_q[a,b]}}{\|f\|_{L_q[a,b]}} \leq c_1(a,b) \left(n^2 + \sum_{j=0}^{n} |\lambda_j| \right) .$$

Theorem 1.7 was proved earlier under the additional assumptions that $\lambda_0 := 0$ and $\lambda_j \geq \delta j$ for each j with a constant $\delta > 0$ and with $c_1(a,b)$ replaced by $c_1(a,b,\delta)$, see [17]. The novelty of Theorem 1.7 is the fact again that $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ is an arbitrary set of $n+1$ distinct real numbers, not even the non-negativity of the exponents λ_j is needed.

In [21] the following Markov–Nikolskii-type inequality has been proved for $E(\Lambda_n)$ on $(-\infty, 0]$.

Theorem 1.8. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of nonnegative real numbers and $0 < q \leq p \leq \infty$. Let μ be a nonnegative integer. There are constants $c_2 = c_2(p,q,\mu) > 0$ and $c_3 = c_3(p,q,\mu)$ depending only on p, q, and μ such that*

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}} \leq \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{\|f^{(\mu)}\|_{L_p(-\infty,0]}}{\|f\|_{L_q(-\infty,0]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q} - \frac{1}{p}} ,$$

where the lower bound holds for all $0 < q \leq p \leq \infty$ and $\mu \geq 0$, while the upper bound holds when $\mu = 0$ and $0 < q \leq p \leq \infty$, and when $\mu \geq 1$, $p \geq 1$, and $0 < q \leq p \leq \infty$. Also, there are constants $c_2 = c_2(q,\mu) > 0$ and $c_3 = c_3(q,\mu)$ depending only on q and μ such that

$$c_2 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}} \leq \sup_{0 \not\equiv f \in E(\Lambda_n)} \frac{|f^{(\mu)}(y)|}{\|f\|_{L_q(-\infty,y]}} \leq c_3 \left(\sum_{j=0}^{n} \lambda_j \right)^{\mu + \frac{1}{q}}$$

for all $0 < q \leq \infty$, $\mu \geq 1$, and $y \in \mathbb{R}$.

Motivated by a question of Michel Weber, the following Markov-Nikolskii-type inequalities have been proved in [25] for $E(\Lambda_n)$ on $[a,b] \subset (-\infty, \infty)$.

Theorem 1.9. *Suppose $0 < q \leq p \leq \infty$, $a, b \in \mathbb{R}$, and $a < b$. There are constants $c_4 = c_4(p,q,a,b) > 0$ and $c_5 = c_5(p,q,a,b)$ depending only on p, q, a, and b such* *that*

$$c_4\left(n^2+\sum_{j=0}^{n}|\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}} \leq \sup_{0\not\equiv f\in E(\Lambda_n)} \frac{\|f\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \leq c_5\left(n^2+\sum_{j=0}^{n}|\lambda_j|\right)^{\frac{1}{q}-\frac{1}{p}}.$$

Theorem 1.10. *Suppose* $0 < q \leq p \leq \infty$*,* $a, b \in \mathbb{R}$*, and* $a < b$*. There are constants* $c_6 = c_6(p, q, a, b) > 0$ *and* $c_7 = c_7(p, q, a, b)$ *depending only on* p*,* q*,* a*, and* b *such that*

$$c_6\left(n^2+\sum_{j=0}^{n}|\lambda_j|\right)^{1+\frac{1}{q}-\frac{1}{p}} \leq \sup_{0\not\equiv f\in E(\Lambda_n)} \frac{\|f'\|_{L_p[a,b]}}{\|f\|_{L_q[a,b]}} \leq c_7\left(n^2+\sum_{j=0}^{n}|\lambda_j|\right)^{1+\frac{1}{q}-\frac{1}{p}},$$

where the lower bound holds for all $0 < q \leq p \leq \infty$*, while the upper bound holds when* $p \geq 1$ *and* $0 < q \leq p \leq \infty$*.*

We note that even more general Nikolskii-type inequalities are proved in [12] for shift invariant function spaces.

Müntz's classical theorem characterizes the sequences $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space

$$M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$$

is dense in $C[0, 1]$. Here $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ denotes the collection of all *finite* linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \ldots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0, \infty)$ equipped with the supremum norm. If $A := [a, b]$ is a finite closed interval, then the notation $C[a, b] := C([a, b])$ is used. Müntz's Theorem states the following:

Müntz's Theorem. *Suppose* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$*. The space* $M(\Lambda)$ *is dense in* $C[0, 1]$ *if and only if* $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$*.*

Proofs are available in [4, 14, 16], for example. The original Müntz Theorem proved by Müntz [30] and Szász [38] and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. There are many generalizations and variations of Müntz's Theorem. See [4–6, 10, 15, 16, 20, 27, 29, 35, 39] among others. There are also many problems still open today.

Somorjai [37] in 1976 and Bak and Newman [1] in 1978 proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0,1]$ whenever $\Lambda := (\lambda_j)_{j=0}^{\infty}$ contains infinitely many distinct real numbers. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0,1]$, whenever the underlying sequence Λ contains infinitely many distinct real numbers. In the light of this result, Newman [32] (p. 50) raises "the very sane, if very prosaic question". Are the functions

$$\prod_{j=1}^{k}\left(\sum_{i=0}^{n_j} a_{i,j}x^{i^2}\right), \qquad a_{i,j}\in\mathbb{R}, \quad n_j\in\mathbb{N},$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak–Newman–Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \ldots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied to resolve the denseness or non-denseness of H_4 in $C[0,1]$.

Borwein and Erdélyi [4, 5, 10] deal with products of Müntz spaces and, in particular, the question of Newman is answered in the negative. In fact, in [10] we presented a number of inequalities each of which implies the answer to Newman's question. One of them is the following bounded Bernstein-type inequality for products of Müntz polynomials from non-dense Müntz spaces. For

$$\Lambda^{(j)} := (\lambda_{i,j})_{i=0}^{\infty}, \qquad 0=\lambda_{0,j}<\lambda_{1,j}<\lambda_{2,j}<\cdots, \qquad j=1,2,\ldots,$$

we define the sets

$$M(\Lambda^{(1)},\Lambda^{(2)},\ldots,\Lambda^{(k)}) := \left\{ f=\prod_{j=1}^{k} f_j : f_j\in M(\Lambda^{(j)}) \right\}.$$

Theorem 1.11. *Suppose*

$$\Lambda^{(j)} := (\lambda_{i,j})_{i=0}^{\infty}, \qquad 0=\lambda_{0,j}<\lambda_{1,j}<\lambda_{2,j}<\cdots, \qquad j=1,2,\ldots,k,$$

and

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_{i,j}}<\infty \qquad \text{and} \qquad \lambda_{1,j}\geq 1, \qquad j=1,2,\ldots,k.$$

Let $s > 0$. There exits a constant c depending only on $\Lambda^{(1)},\Lambda^{(2)},\ldots,\Lambda^{(k)}$, s, and k (and not on ϱ or A) such that

$$\|f'\|_{[0,\varrho]} \leq c\,\|f\|_A$$

for every $f \in M(\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(k)})$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

In [18] the right Markov-type inequalities for products of Müntz polynomials are established when the factors come from arbitrary (not necessarily non-dense) Müntz spaces. More precisely, associated with the sets

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

of real numbers we examined the magnitude of

$$K(M(\Lambda_n), M(\Gamma_m)) := \sup\left\{\frac{\|x(pq)'(x)\|_{[0,1]}}{\|pq\|_{[0,1]}} : 0 \not\equiv p \in M(\Lambda_n)\,, 0 \not\equiv q \in M(\Gamma_m)\right\}, \tag{1}$$

$$\widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b) := \sup\left\{\frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : 0 \not\equiv p \in M(\Lambda_n)\,, 0 \not\equiv q \in M(\Gamma_m)\right\}, \tag{2}$$

where $[a, b] \subset [0, \infty)$, and

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) := \sup\left\{\frac{\|(pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : 0 \not\equiv p \in E(\Lambda_n)\,, 0 \not\equiv q \in E(\Gamma_m)\right\}, \tag{3}$$

where $[a, b] \subset (-\infty, \infty)$.

The result below proved in [18] is an essentially sharp Newman-type inequality for products of Müntz polynomials.

Theorem 1.12. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \textit{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,.$$

Let $K(M(\Lambda_n), M(\Gamma_m))$ *be defined by (1). We have*

$$\frac{1}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \le K(M(\Lambda_n), M(\Gamma_m)) \le 18\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \le K(M(\Lambda_n), M(\Lambda_n)) \le 36\,(2n+1)\lambda_n\,.$$

The factor x from $\|x(pq)'(x)\|_{[0,1]}$ in Theorem 1.12 can be dropped in the expense of a growth condition. The result below proved in [18] establishes an essentially sharp Markov-type inequality on $[0, 1]$.

Theorem 1.13. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

with $\lambda_j \geq j$ *and* $\gamma_j \geq j$ *for each* j. *Let* $\widetilde{K}(M(\Lambda_n), M(\Gamma_m), 0, 1)$ *be defined by (2). We have*

$$\frac{1}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M(\Lambda_n), M(\Gamma_m), 0, 1) \leq 36\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M(\Lambda_n), M(\Lambda_n), 0, 1) \leq 72\,(2n+1)\lambda_n\,.$$

Under a growth condition again, Theorem 1.13 can be extended to the interval $[0, 1]$ replaced by $[a, b] \subset (0, \infty)$. The essentially sharp Markov-type inequality below is also proved in [18].

Theorem 1.14. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}\,.$$

Suppose there exists a $\varrho > 0$ *such that* $\lambda_j \geq \varrho j$ *and* $\gamma_j \geq \varrho j$ *for each* j. *Suppose* $a, b \in \mathbb{R}$ *and* $0 < a < b$. *Let* $\widetilde{K}(M_n(\Lambda), M_m(\Gamma), a, b)$ *be defined by (2). There is a constant* $c(a, b, \varrho)$ *depending only on* a, b, *and* ϱ *such that*

$$\frac{b}{3}\left((m+1)\lambda_n + (n+1)\gamma_m\right) \leq \widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b) \leq c(a, b, \varrho)\,(n+m+1)(\lambda_n + \gamma_m)\,.$$

In particular,

$$\frac{2b}{3}\,(n+1)\lambda_n \leq \widetilde{K}(M(\Lambda_n), M(\Lambda_n), a, b) \leq 2\,c(a, b, \varrho)\,(2n+1)\lambda_n\,.$$

Remark 1.15. Analogs of the above three theorems dealing with products of several Müntz polynomials can also be proved by straightforward modifications.

Remark 1.16. Let $\lambda_j = \gamma_j := j^2, j = 0, 1, \ldots, n$. If we multiply pq out, where $p, q \in M(\Lambda_n)$, and we apply Newman's inequality, we get

$$K(M_n(\Lambda), M_n(\Lambda)) \leq cn^4$$

with an absolute constant c. However, if we apply Theorem 1.12, we obtain

$$K(M_n(\Lambda), M_n(\Lambda)) \leq 36\,(2n+1)n^2\,.$$

It is quite remarkable that $K(M_n(\Lambda), M_n(\Lambda))$ is of the same order of magnitude as the Markov factor $11\left(\sum_{j=0}^{n} j^2\right)$ in Newman's inequality for $M_n(\Lambda)$. When the exponents λ_j grow sufficiently slowly, similar improvements can be observed in each of our Theorems 1.12–1.14 compared with the "natural first idea" of "multiply out and use Newman's inequality".

The essentially sharp Bernstein-type inequality below for

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\} = \bigcup E(\Lambda_n)$$

is proved in [7] (the union above is taken for all $\Lambda_n = \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ for which $0 \in \Lambda_n$).

Theorem 1.17. *We have*

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \le \sup_{0 \not\equiv f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a, b-y\}}\,, \qquad y \in (a,b)\,.$$

We note that pointwise Remez- and Nikolskii-type inequalities for E_n are also proved in [11].

2 New Results

The results of this section were motivated by e-mail communications with Thomas Bloom who was interested in Corollaries 2.3–2.6 in particular.

We examine what happens when in Theorem 1.14 we drop the growth condition "there exists a $\varrho > 0$ such that $\lambda_j \ge \varrho j$ and $\gamma_j \ge \varrho j$ for each j".

Modifying the proof of Theorem 1.14 we can prove the result below.

Theorem 2.1. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be sets of real numbers such that $\lambda_0 \le 0 \le \lambda_n$ and $\gamma_0 \le 0 \le \gamma_m$. Suppose $a, b \in \mathbb{R}$ and $0 < a < b$. Let $\widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b)$ be defined by (2). We have*

$$\widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b) \le 22(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{512}{\log(b/a)}(n+m+1)^2\,.$$

If, in addition, $\lambda_0 = \gamma_0 = 0$, then

$$\frac{1}{6}\left((m+1)\lambda_n + (n+1)\gamma_m\right) + \frac{1}{16\log(b/a)}(n+m-2)^2 \le \widetilde{K}(M(\Lambda_n), M(\Gamma_m), a, b)\,.$$

Corollary 2.2. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$. Suppose $a, b \in \mathbb{R}$ and $0 < a < b$. Let*

$$\widetilde{K}(M(\Lambda_n), M(\Lambda_n), a, b)$$

be defined by (2). We have

$$\widetilde{K}(M(\Lambda_n), M(\Lambda_n), a, b) \leq 44(2n+1)(\lambda_n - \lambda_0) + \frac{512}{\log(b/a)}(2n+1)^2 .$$

If, in addition, $\lambda_0 = 0$, then

$$\frac{1}{3}(n+1)\lambda_n + \frac{1}{4}\log(b/a)(n-1)^2 \leq \widetilde{K}(M(\Lambda_n), M(\Lambda_n), a, b) .$$

By using the substitution $x = e^t$ it is easy to see that the theorem below is equivalent to Theorem 2.1.

Theorem 2.1*. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_m := \{\gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be sets of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$, $\gamma_0 \leq 0 \leq \gamma_m$. Suppose $a, b \in \mathbb{R}$ and $a < b$. Let*

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b)$$

be defined by (3). We have

$$\widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) \leq 22(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{512}{b-a}(n+m+1)^2 .$$

If, in addition, $\lambda_0 = \gamma_0 = 0$, then

$$\frac{1}{6}((m+1)\lambda_n + (n+1)\gamma_m) + \frac{1}{16(b-a)}(n+m-2)^2 \leq \widetilde{K}(E(\Lambda_n), E(\Gamma_m), a, b) .$$

Corollary 2.2*. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \leq 0 \leq \lambda_n$. Suppose $a, b \in \mathbb{R}$ and $a < b$. Let*

$$\widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b)$$

be defined by (3). We have

$$\widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b) \leq 22(2n+1)\lambda_n + \frac{512}{b-a}(2n+1)^2 .$$

If, in addition, $\lambda_0 = 0$, then

$$\frac{1}{3}(n+1)\lambda_n + \frac{1}{4}(b-a)(n-1)^2 \leq \widetilde{K}(E(\Lambda_n), E(\Lambda_n), a, b) .$$

Theorem 2.1 gives the size of

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) := \sup \left\{ \frac{\left\| \frac{d}{dx}(p(x^\alpha)q(x^\beta)) \right\|_{[a,b]}}{\|p(x^\alpha)q(x^\beta)\|_{[a,b]}} : \;\; p \in \mathcal{P}_n\,, \;\; q \in \mathcal{P}_m \right\} \tag{4}$$

immediately for real numbers $0 < a < b$, $\alpha > 0$, and $\beta > 0$.

Corollary 2.3. *Suppose $a, b, \alpha, \beta \in \mathbb{R}$, $0 < a < b$, $\alpha > 0$, and $\beta > 0$. Let*

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta)$$

be defined by (4). We have

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) \le 22(n+m+1)(n\alpha + m\beta) + \frac{512}{b-a}(n+m+1)^2$$

and

$$\frac{1}{6}\,((m+1)n\alpha + (n+1)m\beta) + \frac{1}{16(b-a)}(n+m-2)^2 \le \widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta)\,.$$

Corollary 2.4. *Suppose $a, b, \alpha, \beta \in \mathbb{R}$, $0 < a < b$, $\alpha > 0$, and $\beta > 0$. Let*

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta)$$

be defined by (4). We have

$$\widetilde{K}(\mathcal{P}_n, \mathcal{P}_m, a, b, \alpha, \beta) \sim (n+m)^2\,,$$

where $x \sim y$ means that $c_1 \le x/y \le c_2$ with some constants $c_1 > 0$ and $c_2 > 0$ depending only on a, b, α, and β.

Finding the size of

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) := \sup \left\{ \frac{\|pq)'\|_{[a,b]}}{\|pq\|_{[a,b]}} : \;\; p \in E(\Lambda_n)\,, \;\; q \in \mathcal{P}_m \right\}$$

can also be viewed as a special case of Theorem 2.1*.

Corollary 2.5. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers such that $\lambda_0 \le 0 \le \lambda_n$. Suppose $a, b \in \mathbb{R}$ and $a < b$. We have*

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) \le 22(n+m+1)(\lambda_n - \lambda_0) + \frac{512}{b-a}(n+m+1)^2\,.$$

If, in addition, $\lambda_0 = 0$, then

$$\frac{1}{6}\,(m+1)\lambda_n + \frac{1}{16(b-a)}(n+m-2)^2 \le \widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b)\,.$$

As a special case of Corollary 2.5 we record the following:

Corollary 2.6. *Suppose $a, b \in \mathbb{R}$ and $a < b$. Let $\Lambda_n := \{0, 1, \ldots, n\}$. We have*

$$\widetilde{K}(E(\Lambda_n), \mathcal{P}_m, a, b) \sim (n+m)^2\,,$$

where $x \sim y$ means that $c_1 \le x/y \le c_2$ with some constants $c_1 > 0$ and $c_2 > 0$ depending only on a and b.

Let $\Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$ be a set of nonnegative real numbers. We denote the collection of all linear combinations of

$$1,\ \cosh(\gamma_1 t),\ \cosh(\gamma_2 t),\ \ldots,\ \cosh(\gamma_m t)$$

over $\mathbb{R}$ by

$$G(\Gamma_m) := \operatorname{span}\{1\,,\ \cosh(\gamma_1 t)\,,\ \cosh(\gamma_2 t)\,,\ \ldots\,,\ \cosh(\gamma_m t)\}\,.$$

Our next result is a Bernstein-type inequality for product of exponential sums. It would be desirable to replace $G(\Gamma_m)$ with $E(\Gamma_m)$ in the theorem below but our method of proof does not seem to allow us to do so.

Theorem 2.7. *Let*

$$\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\}$$

be sets of real numbers. We have

$$|f'(0)| \le (2n + 2m + 1)\|f\|_{[-1,1]}$$

for all f of the form

$$f = pq, \qquad p \in E(\Lambda_n),\ \ q \in G(\Gamma_m)\,.$$

3 Lemmas for Theorem 2.1*

Our first four lemmas have been stated as Lemmas 3.1–3.4 in [22], where their proofs are also presented. Our first lemma can be proved by a simple compactness argument and may be viewed as a simple exercise.

Lemma 3.1. *Let $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ be a set of real numbers. Let $a, b, c \in \mathbb{R}$ and $a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. There exists a $0 \not\equiv T \in E(\Delta_n)$ such that*

$$\frac{|T(c)|}{\|Tw\|_{L_q[a,b]}} = \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} :\ 0 \not\equiv P \in E(\Delta_n)\right\}\,,$$

and there exists a $0 \not\equiv S \in E(\Delta_n)$ *such that*

$$\frac{|S'(c)|}{\|Sw\|_{L_q[a,b]}} = \sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Delta_n)\right\} .$$

Our next lemma is an essential tool in proving our key lemmas, Lemmas 3.3 and 3.4.

Lemma 3.2. *Let* $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ *be a set of real numbers. Let* $a, b, c \in \mathbb{R}$ *and* $a < b < c$. *Let* $q \in (0, \infty]$. *Let* T *and* S *be the same as in Lemma 3.1. The function* T *has exactly* n *zeros in* $[a, b]$ *by counting multiplicities. Under the additional assumption* $\delta_n \geq 0$, *the function* S *also has exactly* n *zeros in* $[a, b]$ *by counting multiplicities.*

The heart of the proof of our theorems is the following pair of comparison lemmas. Lemmas 3.3 and 3.4 below are proved in [24] based on Descartes' Rule of Sign and a technique used earlier by Pinkus and Smith [36].

Lemma 3.3. *Let* $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ *and* $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ *be sets of real numbers satisfying* $\delta_j \leq \gamma_j$ *for each* $j = 0, 1, \ldots, n$. *Let* $a, b, c \in \mathbb{R}$ *and* $a < b \leq c$. *Let* $0 \not\equiv w$ *be a continuous function defined on* $[a, b]$. *Let* $q \in (0, \infty]$. *We have*

$$\sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Delta_n)\right\} \leq \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Gamma_n)\right\} .$$

Under the additional assumption $\delta_n \geq 0$ *we also have*

$$\sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Delta_n)\right\} \leq \sup\left\{\frac{|P'(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Gamma_n)\right\} .$$

Lemma 3.4. *Let* $\Delta_n := \{\delta_0 < \delta_1 < \cdots < \delta_n\}$ *and* $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ *be sets of real numbers satisfying* $\delta_j \leq \gamma_j$ *for each* $j = 0, 1, \ldots, n$. *Let* $a, b, c \in \mathbb{R}$ *and* $c \leq a < b$. *Let* $0 \not\equiv w$ *be a continuous function defined on* $[a, b]$. *Let* $q \in (0, \infty]$. *We have*

$$\sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Delta_n)\right\} \geq \sup\left\{\frac{|P(c)|}{\|Pw\|_{L_q[a,b]}} : \ 0 \not\equiv P \in E(\Gamma_n)\right\} .$$

Under the additional assumption $\gamma_0 \leq 0$ *we also have*

$$\sup\left\{\frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} : \ 0 \not\equiv Q \in E(\Delta_n)\right\} \geq \sup\left\{\frac{|Q'(c)|}{\|Qw\|_{L_q[a,b]}} : \ 0 \not\equiv Q \in E(\Gamma_n)\right\} .$$

We will also need the following result which may be obtained from Theorem 1.5 by a substitution $x = e^t$.

Lemma 3.5. *Let $n \geq 1$ be an integer. Let $\Lambda_n := \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be a set of $n+1$ distinct real numbers. Let $a, b \in \mathbb{R}$ and $0 < a < b$. We have*

$$\frac{1}{3}\sum_{j=0}^{n} |\lambda_j| + \frac{1}{4(b-a)}(n-1)^2 \leq \sup_{0 \not\equiv P \in E(\Lambda_n)} \frac{\|P'\|_{[a,b]}}{\|P\|_{[a,b]}} \leq 11 \sum_{j=0}^{n} |\lambda_j| + \frac{128}{b-a}(n+1)^2 .$$

4 Lemmas for Theorem 2.7

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of distinct positive real numbers. We denote the collection of all linear combinations of

$$\sinh(\lambda_0 t),\ \sinh(\lambda_1 t),\ \dots,\ \sinh(\lambda_n t)$$

over $\mathbb{R}$ by

$$H(\Lambda_n) := \text{span}\{\sinh(\lambda_0 t),\ \sinh(\lambda_1 t),\ \dots,\ \sinh(\lambda_n t)\} .$$

The first lemma is stated and proved in Sect. 4 of [23].

Lemma 4.1. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of positive real numbers satisfying $\lambda_j \leq \gamma_j$ for each $j = 0, 1, \dots, n$. Let $a, b \in \mathbb{R}$ and $0 \leq a < b$. Let $0 \not\equiv w$ be a continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. We have*

$$\sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} :\ P \in H(\Gamma_n)\right\} \leq \sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} :\ P \in H(\Lambda_n)\right\} .$$

As before, associated with $\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\}$, we denote the collection of all linear combinations of

$$1\,,\ \cosh(\lambda_1 t)\,,\ \cosh(\lambda_2 t)\,,\ \dots\,,\ \cosh(\lambda_n t)$$

over $\mathbb{R}$ by

$$G(\Lambda_n) := \text{span}\{1\,,\ \cosh(\lambda_1 t)\,,\ \cosh(\lambda_2 t)\,,\ \dots\,,\ \cosh(\lambda_n t)\} .$$

The next lemma is stated and proved in Sect. 3 of [26].

Lemma 4.2. *Let*

$$\Lambda_n := \{0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\} \qquad \text{and} \qquad \Gamma_n := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n\}$$

be sets of nonnegative real numbers satisfying $\lambda_j \leq \gamma_j$ *for each* $j = 0, 1, \dots, n$. *Let* $a, b \in \mathbb{R}$ *and* $0 \leq a < b$. *Let* $0 \not\equiv w$ *be a continuous function defined on* $[a, b]$. *Let* $q \in (0, \infty]$. *We have*

$$\sup\left\{\frac{|P(0)|}{\|Pw\|_{L_q[a,b]}} : \;\; P \in G(\Gamma_n)\right\} \leq \sup\left\{\frac{|P(0)|}{\|Pw\|_{L_q[a,b]}} : \;\; P \in G(\Lambda_n)\right\} .$$

5 Proofs

*Proof of Theorem 2.1**. First we prove the lower bound of the theorem. The lower bound of Lemma 3.5 guarantees a

$$0 \not\equiv f \in \operatorname{span}\{e^{(\lambda_n+\gamma_0)t}, e^{(\lambda_n+\gamma_1)t}, \dots, e^{(\lambda_n+\gamma_m)t}\}$$

such that

$$\begin{aligned}\|f'\|_{[a,b]} &\geq \left(\frac{1}{3}\sum_{j=0}^{m}(\lambda_n + \gamma_j) + \frac{1}{4(b-a)}(m-1)^2\right)\|f\|_{[a,b]} \\ &\geq \left(\frac{1}{3}\,(m+1)\lambda_n + \frac{1}{4(b-a)}(m-1)^2\right)\|f\|_{[a,b]} \,.\end{aligned}$$

Observe that $f = pq$ with $p \in E(\Lambda_n)$ defined by $p(x) := e^{\lambda_n t}$ and with some $q \in E(\Gamma_m)$.

Similarly, the lower bound of Lemma 3.5 guarantees a

$$0 \not\equiv f \in \operatorname{span}\{e^{(\gamma_m+\lambda_0)t}, e^{(\gamma_m+\lambda_1)t}, \dots, e^{(\gamma_m+\lambda_n)t}\}$$

such that

$$\begin{aligned}\|f'\|_{[a,b]} &\geq \left(\frac{1}{3}\sum_{j=0}^{n}(\gamma_m + \lambda_j) + \frac{1}{4(b-a)}(n-1)^2\right)\|f\|_{[a,b]} \\ &\geq \left(\frac{1}{3}\,(n+1)\gamma_m + \frac{1}{4(b-a)}(n-1)^2\right)\|f\|_{[a,b]} \,.\end{aligned}$$

Observe that $f = pq$ with some $p \in E(\Lambda_n)$ and with $q \in E(\Gamma_m)$ defined by $q(x) := e^{\gamma_m t}$. Hence the lower bound of the theorem is proved.

We now prove the upper bound of the theorem. We want to prove that

$$|(p'q)(y)| \leq 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{256}{b-a}(n+m+1)^2 \, \|pq\|_{[a,b]} \tag{5}$$

for every $p \in E(\Lambda_n)$, $q \in E(\Gamma_m)$, and $y \in [a, b]$. The rest follows from the product rule of differentiation (the role of Λ_n and Γ_m can be interchanged). For $\alpha < \beta$ let

$$M(n, m, \alpha, \beta) := 11(n + m + 1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{128}{\beta - \alpha}(n + m + 1)^2 .$$

Let $d := (a + b)/2 \in (a, b)$.

First let $y \in [d, b]$. We show that

$$|(p'q)(y)| \leq M(n, m, a, y) \, \|pq\|_{[a,y]} \tag{6}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$. To show (8), it is sufficient to prove that

$$|p'q)(y)| \leq (1 + \eta)M(n, m, a, y) \, \|pq\|_{[a,y-\delta]} \tag{7}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$, where η denotes a quantity that tends to 0 as $\delta \in (0, y - a)$ tends to 0. The rest follows by taking the limit when $\delta \in (0, y - a)$ tends to 0.

To see (7), by Lemmas 3.3 and 3.4 we may assume that

$$\lambda_j := \lambda_n - (n - j)\varepsilon , \qquad j = 0, 1, \ldots, n ,$$
$$\gamma_j := \gamma_m - (m - j)\varepsilon , \qquad j = 0, 1, \ldots, m,$$

for some $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(a, y - \delta)$ and q has m zeros in $(a, y - \delta)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) > 0$ and $q'(y) > 0$. Therefore

$$|(p'q)(b)| \leq |(pq)'(b)| .$$

Now observe that $f := pq \in E(\Omega_k)$, where $k := n + m$ and $\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$ with

$$\omega_j := \lambda_n + \gamma_m - (n + m - j)\varepsilon , \qquad j = 0, 1, \ldots, k .$$

Hence Lemma 3.5 implies

$$|(p'q)(y)| \leq |(pq)'(y)| = |f'(y)| \leq M(n, m, a, y) \, \|f\|_{[a,y]} = M(n, m, a, y) \, \|pq\|_{[d,b]} .$$

By this (7), and hence (6), is proved. Combining (6) with

$$M(n,m,a,y) = 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{128}{y-a}(n+m+1)^2$$
$$= 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{256}{b-a}(n+m+1)^2\,,$$

we conclude (7) for all $y \in [d,b]$.

Now let $y \in [a,d]$. We show that

$$|(p'q)(y)| \leq K(n,m,y,b)\|pq\|_{[y,b]} \tag{8}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$. To show (8), it is sufficient to prove that

$$|(p'q)(y)| \leq (1+\eta)M(n,m,y,b)\,\|pq\|_{[y+\delta,b]} \tag{9}$$

for every $p \in E(\Lambda_n)$ and $q \in E(\Gamma_m)$, where η denotes a quantity that tends to 0 as $\delta \in (0, b-y)$ tends to 0 tends to 0. The rest follows by taking the limit when $\delta \in (0, b-\delta)$ tends to 0.

To see (9), by Lemmas 3.3 and 3.4 we may assume that

$$\lambda_j := \lambda_0 + \varepsilon j\,, \qquad j = 0,1,\dots,n\,,$$
$$\gamma_j := \gamma_0 + \varepsilon j\,, \qquad j = 0,1,\dots,m\,,$$

with a sufficiently small $\varepsilon > 0$. By Lemma 3.2 we may also assume that p has n zeros in $(y+\delta, b)$ and q has m zeros in $(y+\delta, b)$. We normalize p and q so that $p(y) > 0$ and $q(y) > 0$. Then, using the information on the zeros of p and q, we can easily see that $p'(y) < 0$ and $q'(y) < 0$. Therefore

$$|(p'q)(y)| \leq |(pq)'(y)|\,.$$

Now observe that $f := pq \in E(\Omega_k)$, where $\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$ with $k := n+m$ and

$$\omega_j := \lambda_0 + \gamma_0 + j\varepsilon\,, \qquad j = 0,1,\dots,k\,.$$

Hence Lemma 3.5 implies

$$|(p'q)(y)| \leq |(pq)'(y)| = |f'(y)| \leq M(n,m,y,b)\,\|f\|_{[y,b]} = M(n,m,y,b)\,\|pq\|_{[y,b]}\,.$$

By this (9), and hence (8), is proved. Combining (8) with

$$\begin{aligned} M(n,m,y,b) &= 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{128}{b-y}(n+m+1)^2 \\ &\le 11(n+m+1)(\lambda_n - \lambda_0 + \gamma_m - \gamma_0) + \frac{256}{b-a}(n+m+1)^2 , \end{aligned}$$

we conclude (5) for all $y \in [a,d]$. The proof of the theorem is now complete. □

Corollaries 2.3 and 2.4 follow from Theorem 2.1 immediately.

Proof of Corollary 2.5. Observe that

$$t = \lim_{\varepsilon \to 0+} \frac{e^{\varepsilon t} - 1}{\varepsilon} ,$$

hence every $q \in \mathcal{P}_m$ and $\eta > 0$ there is a sufficiently small $\varepsilon > 0$ and a

$$q_\varepsilon \in E(\Gamma_{m,\varepsilon}) := \text{span}\{0, \varepsilon, 2\varepsilon, \dots, m\varepsilon\}$$

such that

$$\|q_\varepsilon - q\|_{[a,b]} < \eta \qquad \text{and} \qquad \|q'_\varepsilon - q'\|_{[a,b]} < \eta .$$

Therefore the corollary follows from Theorem 2.1* as a limit case. □

Corollary 2.6 follows from Corollary 2.5 immediately.

Proof of Theorem 2.7. Let

$$f = pq, \qquad p \in E(\Lambda_n), \;\; q \in G(\Gamma_m) .$$

Observe that $q \in G(\Gamma_m)$ is even, hence $q(t) = q(-t)$ for all t, and $q'(0) = 0$. Hence, replacing p with $\tilde{p}$ defined by $\tilde{p}(t) := (p(t) - p(-t))/2$ we have $(\tilde{p}q)'(0) = (pq)'(0)$ and

$$\|\tilde{p}q\|_{[-1,1]} \le \|pq\|_{[-1,1]} ,$$

without loss of generality we may assume that

$$\Lambda_{n+1} = \{\lambda_0 < \lambda_1 < \cdots < \lambda_{n+1}\} \subset (0,\infty)$$

and $p \in H(\Lambda_{n+1})$. So let $f = pq$ with $p \in H(\Lambda_{n+1})$ and $q \in G(\Gamma_m)$, where

$$\Lambda_{n+1} := \{\lambda_0 < \lambda_1 < \cdots < \lambda_{n+1}\} \subset (0,\infty) \qquad \text{and} \qquad \Gamma_m := \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m\} .$$

As $\|pq\|_{[-1,1]} = \|pq\|_{[0,1]}$, we want to prove that

$$|(pq)'(0)| = |p'q)(0)| \le (2n + 2m + 1)\|pq\|_{[0,1]} \tag{10}$$

for all $p \in H(\Lambda_{n+1})$ and $q \in G(\Gamma_m)$. To prove (10), by Lemmas 4.1 and 4.2 we may assume that

$$\lambda_j := j\varepsilon\,, \qquad j = 0, 1, \dots, n+1\,,$$

$$\gamma_j := j\varepsilon\,, \qquad j = 0, 1, \dots, m\,,$$

for some $\varepsilon > 0$. Now observe that $f := pq \in H(\Omega_k)$, where $k := n + m + 1$ and

$$\Omega_k := \{\omega_1 < \omega_2 < \cdots < \omega_k\}$$

with

$$\omega_j := j\varepsilon\,, \qquad j = 0, 1, \dots, k\,.$$

Hence Theorem 1.17 implies (10). □

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On Bernstein-Type Inequalities for the Polar Derivative of a Polynomial

N. K. Govil and P. Kumar

Abstract If $P(z)$ is a polynomial of degree n, and α a complex number, then polar derivative of $P(z)$ with respect to the point α, denoted by $D_\alpha P(z)$, is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Clearly, $D_\alpha P(z)$ is a polynomial of degree $n-1$, and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[\frac{D_\alpha P(z)}{\alpha} \right] = P'(z).$$

It is well known that if $P(z)$ is a polynomial of degree n, then $\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|$. This inequality is known as Bernstein's inequality (Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle. Gauthier-Villars, Paris, 1926), although this inequality was also proved by Riesz (Jahresber Dtsch Math-Verein 23:354–368, 1914) about 12 years before it was proved by Bernstein. The subject of inequalities for polynomials and related classes of functions plays an important and crucial role in obtaining inverse theorems in Approximation Theory. Frequently, the further progress in inverse theorems has depended upon first obtaining the corresponding analogue or generalization of Markov's and Bernstein's inequalities. These inequalities have been the starting point of a considerable literature in Mathematics, and was one of the areas in which Professor Q. I. Rahman worked for more than 50 years, and made some of the most important and significant contributions. Over a period, this Bernstein's inequality and corresponding inequality concerning the growth of polynomials have been generalized in different domains, in different norms, and for

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different classes of functions, and in the literature one can find hundreds of papers on this topic. Here we study some of the research centered around Bernstein-type inequalities for polar derivatives of polynomials. The chapter is purely expository in nature and an attempt has been made here to provide results starting from the beginning to some of the most recent ones.

Keywords Bernstein-type inequality • Polar derivative • Polynomial • Zero

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1 Introduction

Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree at most n. Then it is well known that

$$\max_{|z|\leq 1} |P'(z)| \leq n \max_{|z|\leq 1} |P(z)|. \tag{1}$$

The result is best possible and equality holds in (1) for $P(z) = \lambda z^n$, λ being a complex number.

The above inequality, which is known as Bernstein's inequality, was proved by Bernstein [17], although this inequality was also proved by Riesz [78] about 12 years before Bernstein.

By the maximum modulus principle, $\max_{|z|\leq 1} |P(z)| = \max_{|z|=1} |P(z)|$ and so if we let $\|P\| = \max_{|z|=1} |P(z)|$, then the Inequality (1) can be written as

$$\|P'\| \leq n\|P\|. \tag{2}$$

For $R \geq 1$, it is well known (see [65], [66, Problem 269] or [79, volume 1, p. 137]) that

$$\max_{|z|=R} |P(z)| \leq R^n \|P\|, \tag{3}$$

with equality holding for $P(z) = \lambda z^n$, where λ is a complex number.

An excellent introduction to this topic is given in the well-known books due to Marden [56], Milovanović et al. [57], and Rahman and Schmeisser [68].

Let us begin with the motivation for the genesis of the concept of polar derivative of a polynomial. Let $P(z)$ be a polynomial of degree $n \geq 1$. Let

$$\psi(z) = \frac{\alpha z + \beta}{z + \delta}, \quad \beta \neq \alpha\delta, \tag{4}$$

and

$$\phi(z) = (z + \delta)^n P(\psi(z)).$$

Then $\phi(z) = P(\alpha)z^n + \cdots$ is a polynomial of degree at most n, and is of degree n if and only if $P(\alpha) \neq 0$.

In order to obtain the critical points of ϕ, we differentiate $\phi(z)$ with respect to z and rearrange the terms, to obtain

$$\phi'(z) = (z + \delta)^{n-1} \left(nP(\psi(z)) + (\alpha - \psi(z))P'(\psi(z)) \right). \tag{5}$$

From this one can observe that, if ξ is a critical point of $\phi(z)$, then either $\psi(\xi)$ is a zero of

$$F(\alpha, z) = nP(z) + (\alpha - z)P'(z) = \left[n + (\alpha - z)\frac{d}{dz} \right] P(z), \tag{6}$$

or $\xi = -\delta$ and $F(\alpha, z)$ is of degree less than $n - 1$. Conversely, if ω is a zero of $F(\alpha, z)$, then $\psi^{-1}(\omega)$ is a critical point of ϕ unless $\psi^{-1}(\omega) = \infty$, in which case $\omega = \alpha$, and then again $P(\alpha) = 0$.

These observations lead us to believe that the first order differential operator

$$D_\alpha := n + (\alpha - z)\frac{d}{dz}, \tag{7}$$

appearing in the Eq. (6) is an interesting one and hence the study of its behavior on the class of polynomials in general and more pointedly on the class of polynomials with restricted zeros is of significant importance. The problem concerning estimation of the bound of this operator on polynomials in terms of moduli of polynomials has been evolved subsequently over the last many years.

The fact that the operator D_α in (7) is a "generalization" of the concept of "ordinary derivative" of a polynomial $P(z)$ is evident, and convincing from the fact that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z),$$

uniformly with respect to z for $|z| \leq R,\ R > 0$.

Definition 1. If $P(z)$ is a polynomial of degree n, and α is any complex number, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \tag{8}$$

is called the *polar derivative* of $P(z)$.

As mentioned earlier, $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, and of degree $n-1$ if and only if $P^{(n-1)}(\alpha) \neq 0$. It is also interesting to see that if α is a zero of $P(z)$ so it is of $D_\alpha P(z)$.

The function $D_\alpha P(z)$ has been called by Laguerre [48] *the "emanant" of* $P(z)$ and by Pólya-Szegö [66] *"the derivative of* $P(z)$ *with respect to the point* α." Marden [56] called it *"the polar derivative of* $P(z)$ *with respect to the pole* α*"* or simply, *"the polar derivative of* $P(z)$."

One of the important results at the dawn of evolution of theory on polar derivatives is the relationship between the zeros of a polynomial and that of its polar derivative in so-called *circular regions.*

The class of circular regions includes the open or closed interior or exterior of circle, and open or closed half plane. It is clear that a circular domain is invariant under a bilinear transformation, given in (4).

The following result due to Laguerre [48, Theorem 13,1] which is analogous to Gauss–Lucas Theorem for polynomials in convex domains is for polar derivatives of polynomials on circular regions, and some interesting proofs of this can be found in the monograph by Marden [56].

Theorem 1. *If the zeros of the n^{th} degree polynomial $P(z)$ lie in a circular region C and if w is any zero of $D_\alpha P(z)$, then not both points w and α can lie outside of C. Furthermore, if $P(w) \neq 0$, any circle S through w and α either passes through all the zeros of $P(z)$ or separates these zeros.*

The following result which is equivalent to the first part of the above Laguerre's Theorem is due to Szegö [83].

Theorem 2. *If $P(z)$ is a polynomial of degree n having no zeros in the circular region C and if $\alpha \in C$, then the polar derivative $D_\alpha P(z)$ has no zeros in C.*

During the course of development of inequalities involving polar derivative of a polynomial, some interesting fundamental relations emerged. A couple of them, given below were established by Aziz [4].

If $P(z)$ is a polynomial of degree n, and α any real or complex number, then clearly on $|z| = 1$

$$|D_\alpha Q(z)| = |n\bar{\alpha}P(z) + (1 - \bar{\alpha}z)P'(z)|, \tag{9}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, is the inverse polynomial of $P(z)$.

As is easy to see, the above identity (9) leads us to the following identity, which has played an important role in proving many interesting inequalities in the literature. Simply dividing (9) by $|\overline{\alpha}|$ on both the sides, and using the definition (8), we get

$$|D_\alpha Q(z)| = \left|\alpha D_{\frac{1}{\bar{\alpha}}} P(z)\right| \text{ on } |z| = 1. \tag{10}$$

At this stage, it is quite natural to think if the concept of higher order ordinary derivative of a polynomial can be extended to polar derivative of a polynomial, and this is done as follows.

Definition 2 ([56]). Given a polynomial $P(z)$ of degree n, we can construct the sequence of polar derivatives or so-called higher order derivatives with respect to finitely many poles as given below

$$P_k(z) = (n-k+1)P_{k-1}(z) + (\alpha_k - z)P'_{k-1}(z), \quad k = 1,\ 2,\ 3, \cdots, n$$

where $P_0(z) = P(z)$, and "α_k" may be identical or distinct.

For easier understanding of the concept, one can interpret the same as follows:

$$\begin{aligned}
D_{\alpha_1}P(z) &= nP(z) + (\alpha_1 - z)P'(z) \\
D_{\alpha_2}D_{\alpha_1}P(z) &= (n-1)D_{\alpha_1}P(z) + (\alpha_2 - z)\left(D_{\alpha_1}P(z)\right)' \\
\ldots \quad & \quad \ldots \\
D_{\alpha_k}\ldots D_{\alpha_1}P(z) &= (n-k+1)D_{\alpha_{k-1}}\ldots D_{\alpha_1}P(z) + (\alpha_k - z)\left(D_{\alpha_{k-1}}\ldots D_{\alpha_1}P(z)\right)',
\end{aligned}$$

for $2 \leq k \leq n$.

It is clear that the k^{th} polar derivative $D_{\alpha_k}\ldots D_{\alpha_1}P(z) = P_k(z)$ of $P(z)$ is a polynomial of degree at most $n-k$, just like the k^{th} ordinary derivative $P^{(k)}(z)$ of the polynomial $P(z)$. Also, it is interesting to observe some clear parallels between the location of zeros of the ordinary derivative and that of polar derivative of a polynomial. One can recall that the location of zeros of $P^{(k)}(z)$ is determined by the repeated application of Gauss–Lucas Theorem. Similarly, the location of zeros of polar derivative $P_k(z)$ can be obtained by the repeated application of Laguerre's Theorem.

Let us present the Laguerre's Theorem ([56], p. 52) for higher order polar derivatives as follows.

Theorem 3. *Let $P(z)$ be a polynomial of degree n. If all the zeros of $P(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ lies in the region C, then each of the polar derivatives $D_{\alpha_k}D_{\alpha_{k-1}}\ldots D_{\alpha_1}P(z) = P_k(z)$, $(k = 1, 2, 3, \ldots,$ $n-1)$ has all of its zeros in C.*

Inverse polynomials are widely seen in the literature on polynomial inequalities, so in the inequalities involving polar derivatives as well.

Definition 3. Given a polynomial $P(z)$ of degree n, if $P(z) = uQ(z)$ where $|u| = 1$, and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then $P(z)$ is said to be self-inversive.

Note that if $P(z)$ has nonzero constant term, that is $P(0) \neq 0$, and a is a zero of $P(z)$ then $\dfrac{1}{\bar{a}}$ is a zero of $Q(z)$. Also, a polynomial $P(z)$ is self-inversive, if and only if $|P(z)| = |Q(z)|$. This is evident from the maximum-modulus principle, and this feature of self-inversive polynomials is more employable friendly in studying the bounds or the bounds of associated polynomials in different norms.

An interesting equality relation between the polar derivative of a self-inversive polynomial and that of its inverse polynomial (See [4], p. 190) is given below.

For any self-inversive polynomial $P(z)$ and its inverse polynomial $Q(z)$, and for any complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, we have

$$D_{\alpha_m} \ldots D_{\alpha_1} P(z) = D_{\alpha_m} \ldots D_{\alpha_1} Q(z).$$

The present chapter consists of three sections with some subsections, including Sect. 1 which is introductory in nature and contains preliminary notions, basic definitions, and some of the known results that are relevant and related to our study, and would be needed for the development of the subject in the subsequent sections. Section 2 begins with a brief introduction to the problems discussed in this section, and is followed by three subsections. In Sect. 2.1 we discuss the inequalities on uniform norm for the polar derivative of a polynomial with no restriction on zeros, in Sect. 2.2 the inequalities on uniform norm for polar derivative of a polynomial having no zeros in a circle, and in Sect. 2.3 the inequalities on uniform norm for polar derivative of a polynomial having all their zeros in a circle.

Section 3 is on L^p inequalities and has three subsections. Section 3.1 is devoted to the L^P inequalities for polar derivative of a polynomial with no restriction on zeros, Sect. 3.2 unravels the inequalities of L^P type for polar derivative of a polynomial with no zeros inside a circle, and lastly, in Sect. 3.3 we discuss the L^P inequalities for polar derivative of a polynomial that has all the zeros in a circle.

2 Bounds on the Uniform Norm of Polar Derivative of a Polynomial

We begin with a classical result due to Bernstein [17], mentioned in Sect. 1, which states that, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{11}$$

For $R \geq 1$, it is well known (see [79] or Problem 269 of [66], p. 346) that

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{12}$$

The inequalities (11) and (12) are sharp and equality holds, if $P(z)$ has all its zeros at the origin.

It is well known that the inequalities of Bernstein type are fundamental for the proof of many subsequent theorems in the area of Approximation Theory and Polynomial Approximations. There are many results concerning Bernstein's inequality and their generalizations in different forms, an area in which Professor Q. I. Rahman worked more than 50 years. There are many books, survey articles, and research monographs written on this subject, and we refer to the readers the books due to Borwein and Erdélyi [22], Marden [56], Milovanović, Mitrinović, and Rassias [57], and Rahman and Schmeisser [68].

The results presented in this section, as well as in the coming sections on polar derivatives are some of the extensions and generalizations of the Bernstein-type inequalities appeared in the literature for the ordinary derivatives of any complex polynomials. It may be noted that many times interesting inequalities hold when some conditions on the location of zeros of polynomials are satisfied. This could be seen in the enunciation of most of the results presented in Sects. 3.2 and 3.3. But still, there are many results with no condition on the zeros of the polynomials under consideration or when the underlying conditions are presented independently of the location of zeros, even if they are connected to.

We begin with the following section that deals with some of the results for polar derivative of a polynomial with no restriction on its zeros.

2.1 Inequalities for Polynomials with no Restriction on Their Zeros

Various versions of Bernstein's inequality (11) are derived in which the underlying disks, choice of norms, and the polynomials are replaced by more general sets, extended norms, and some special classes of polynomials. These inequalities have their own intrinsic beauty, relevance, and significance.

The Bernstein-type inequalities for the class of polynomials with "derivative" replaced by "polar derivative" have attracted number of mathematicians. In this direction, the first result is probably due to Aziz [4, p. 189], (see also [13]) who proved the following result which is an analogue of the Inequality (11) for the polar derivatives.

If $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$, we have

$$|D_\alpha P(z)| \leq n|\alpha z^{n-1}| \max_{|z|=1} |P(z)|, \text{ for } |z| \geq 1. \tag{13}$$

The result is best possible and equality holds in (13) for $P(z) = \lambda z^n,\ \lambda \neq 0$.

If we apply the above result recursively to the polynomial $P(z)$, we get the more generalized form [4] as follows.

Theorem 4. *If $P(z)$ is a polynomial of degree n, then for $|z| \geq 1$,*

$$|D_{\alpha_k} D_{\alpha_{k-1}} \dots D_{\alpha_1} P(z)| \leq n(n-1)\dots(n-k+1)|\alpha_1\alpha_2\dots\alpha_k z^{n-k}| \max_{|z|=1} |P(z)|, \tag{14}$$

where $|\alpha_i| \geq 1$ for all $i = 1, 2, \dots, k$ and $1 \leq k \leq n-1$.

The result is best possible and equality holds for the polynomial $P(z) = \lambda z^n,\ |\lambda| = 1$.

If we divide both the sides of (13) by $|\alpha|$, and letting $|\alpha| \to \infty$, with the fact that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha P(z)}{\alpha}\right] = P'(z),$$

we easily get

$$|P'(z)| \le n|z^{n-1}| \max_{|z|=1}|P(z)|, \textit{ for } |z| \ge 1, \tag{15}$$

and this in particular, for $|z| = 1$ gives (11).

Next, if in (13), we take $z = \alpha$, then $\{D_\alpha P(z)\}_{z=\alpha} = nP(\alpha)$, and (13) gives

$$|P(\alpha)| \le |\alpha|^n \max_{|z|=1}|P(z)|,$$

for every α with $|\alpha| \ge 1$, which clearly is equivalent to (12).

We do not know the corresponding inequality under the conditions of Theorem 4, when the condition on $\alpha_1, \alpha_2, \ldots, \alpha_k,\ (1 \le k \le n-1)$, *is replaced by* $|\alpha_i| < 1,\ \ 1 \le i \le k$.

Aziz also established two more generalizations in the same paper [4], which are stated below.

Theorem 5. *If P(z) is a polynomial of degree n such that* $\max_{|z|=1}|P(z)| = 1$, *and* $\alpha_1, \alpha_2, \ldots, \alpha_k,\ (1 \le k \le n-1)$, *are complex numbers with* $|\alpha_i| \ge 1$, *for* $i = 1, 2, \ldots, k$, *then for* $|z| \ge 1$,

$$\begin{aligned}&|D_{\alpha_k}\ldots D_{\alpha_1}P(z)| + |D_{\alpha_k}\ldots D_{\alpha_1}Q(z)|\\ &\qquad \le n(n-1)\ldots(n-k+1)\{|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k} + 1\},\end{aligned} \tag{16}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar z})}$.

Theorem 6. *If P(z) is a polynomial of degree n such that* $\max_{|z|=1}|P(z)| = 1$, *and* $\alpha_1, \alpha_2, \ldots, \alpha_k,\ (1 \le k \le n-1)$, *are complex numbers with* $|\alpha_i| \le 1$, *for* $i = 1, 2, \ldots, k$, *then for* $|z| \le 1$,

$$\begin{aligned}&|D_{\alpha_k}\ldots D_{\alpha_1}P(z)| + |D_{\alpha_k}\ldots D_{\alpha_1}Q(z)|\\ &\qquad \le n(n-1)\ldots(n-k+1)\{|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k} + 1\},\end{aligned} \tag{17}$$

where $Q(z)$ *is same as given in Theorem 5.*

Again, we do not know how the inequalities in Theorems 5, and 6 will look like if the conditions on $\alpha_1, \alpha_2, \ldots, \alpha_k,\ (1 \le k \le n-1)$, *are replaced by* $|\alpha_i| < 1,\ \ (1 \le i \le k)$, *and* $|\alpha_i| > 1,\ \ (1 \le i \le k)$, *respectively.*

As can be easily seen, the following two theorems (see [4]) are special cases of Theorems 5, and 6 respectively.

Theorem 7. *If P(z) is a polynomial of degree n and α is any real or complex number with* $|\alpha| \geq 1$, *then for* $|z| \geq 1$,

$$|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n \left(|\alpha z^{n-1}| + 1\right) \max_{|z|=1} |P(z)|, \tag{18}$$

where $Q(z)$ *is same as given in Theorem 5.*

Theorem 8. *If P(z) is a polynomial of degree n and α is any real or complex number with* $|\alpha| \leq 1$, *then for* $|z| \leq 1$,

$$|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n \left(|\alpha z^{n-1}| + 1\right) \max_{|z|=1} |P(z)|, \tag{19}$$

where $Q(z)$ *is same as given in Theorem 5.*

We state below an interesting consequence (given in [4]) of the Theorem 8, which generalizes a well-known result due to Visser [85].

If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree n and* $Q(z)$ *is its reciprocal polynomial, then by Theorem 8, we have*

$$|D_\alpha P(z)|_{z=0} + |D_\alpha Q(z)|_{z=0} \leq n \max_{|z|=1} |P(z)|,$$

for any α *with* $|\alpha| \leq 1$.

As can be easily seen, the above inequality is clearly equivalent to

$$|na_0 + \alpha a_1| + |na_n + \overline{\alpha} a_{n-1}| \leq n \max_{|z|=1} |P(z)|,$$

for any α with $|\alpha| \leq 1$. The case $\alpha = 0$ of the above result is the result of Visser [85].

On combining Theorems 7 and 8 (again see [4]), one would immediately get

Theorem 9. *If P(z) is a polynomial of degree n and α is any real or complex number, then on* $|z| = 1$,

$$|D_\alpha P(z)| + |D_\alpha Q(z)| \leq n \left(|\alpha| + 1\right) \max_{|z|=1} |P(z)|, \tag{20}$$

where $Q(z)$ *is same as given in Theorem 5.*

If we divide both the sides of (20) by $|\alpha|$, and make $|\alpha| \to \infty$, we get the following result due to Govil and Rahman [43, Inequality (3.2)].

Theorem 10. *If* $P(z)$ *is a polynomial of degree n, then on* $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|, \tag{21}$$

where $Q(z)$ *is same as given in Theorem 5.*

Recently, Liman et al. [51, p.1204], generalized the Inequality (13) as follows.

Theorem 11. *If $P(z)$ is a polynomial of degree n, then for all complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1,\ \ |\beta| \leq 1$,*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \leq n\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| \max_{|z|=1}|P(z)|,$$

on $|z| = 1$.

Liman et al. [51, p.1205] also generalized the Theorems 9 and 10, as follows.

Theorem 12. *If $P(z)$ is a polynomial of degree n, then for all complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1,\ \ |\beta| \leq 1$,*

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| + \left|zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|$$

$$\leq n\left\{\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|z + \beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\} \max_{|z|=1}|P(z)|,$$

on $|z| = 1$, and $Q(z)$ is same as given in Theorem 5.

We next turn towards the inequalities for the so-called self-inversive polynomials, and in this direction we begin with the following two results due to Aziz [4, p. 186].

Theorem 13. *If $P(z)$ is a self-inversive polynomial of degree n and $\alpha_1, \alpha_2, \ldots, \alpha_k$, $1 \leq k \leq n-1$, are all real or complex numbers, then*

$$|D_{\alpha_k} D_{\alpha_{k-1}} \ldots D_{\alpha_1} P(z)| \leq \frac{n(n-1)\cdots(n-k+1)}{2}\left(|\alpha_1 \cdots \alpha_k z^{n-k}| + 1\right) \max_{|z|=1}|P(z)|, \tag{22}$$

for $|z| \geq 1$, and $|\alpha_i| \geq 1,\ \ i = 1, 2, \cdots k$. The result is best possible and equality holds for $P(z) = \dfrac{(z^n + 1)}{2}$.

Remark 1. Note that the Inequality (22) also holds if the conditions in the above theorem are replaced by $|z| \leq 1$, and $|\alpha_i| \leq 1,\ i = 1, 2, \cdots k$.

As a particular case of the above theorem we have the following [4].

Theorem 14. *If $P(z)$ is a self-inversive polynomial of degree n and α is any complex number such that $|\alpha| \geq 1$, then for $|z| \geq 1$,*

$$|D_\alpha P(z)| \leq \frac{n}{2}\left(|\alpha z^{n-1}| + 1\right) \max_{|z|=1}|P(z)|. \tag{23}$$

Remark 2. The Inequality (23) is true for $|z| \leq 1$, and for all complex numbers α such that $|\alpha| \leq 1$.

We conclude this section with an inequality (given in [69]) that relates polar derivatives with ordinary derivatives, and has proved to be very useful in establishing inequalities on polar derivatives.

Let $P(z)$ be a polynomial of degree n and $Q(z)$ be its inverse polynomial. Then for any complex numbers α and γ, with $0 \le \theta < 2\pi$, we have

$$\left|D_\alpha P(e^{i\theta}) + e^{i\gamma} D_\alpha Q(e^{i\theta})\right| \le (|\alpha| + 1)\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|. \tag{24}$$

To prove this, note that for any complex numbers α and γ, with $0 \le \theta < 2\pi$, we have

$$\begin{aligned}
&\left|D_\alpha P(e^{i\theta}) + e^{i\gamma} D_\alpha Q(e^{i\theta})\right| \\
&\qquad = \left|nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + e^{i\gamma}\left(nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\right)\right| \\
&\qquad = \left|e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} + e^{i\gamma}e^{i(n-1)\theta}\overline{P'(e^{i\theta})} + \alpha\left(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\right)\right| \\
&\qquad \le (|\alpha| + 1)\left|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\right|,
\end{aligned}$$

from which (24) follows.

This inequality has its own significance in the literature because the relation established over here is between the moduli of terms involving polar derivatives and that of ordinary derivatives. It is greatly advantageous, as the bounds established for the right-hand side of the above inequality could be used to estimate the bound for polar derivatives.

2.2 Inequalities for Polynomials Having no Zeros in a Circle

As mentioned in the previous section, Bernstein-type inequalities are known on various regions of the complex plane for different norms and for different class of polynomials with various constraints, like restrictions on the zeros of polynomials. This is quite natural because one will always like to see what happens if the polynomials are restricted in certain ways. This section starts discussing in this direction, that is, inequalities for constrained polynomials. Since the equality in (11) holds for polynomials having all their zeros at the origin, it should be possible to improve upon the bound in (11), if we restrict to the class of polynomials having no zeros in the unit circle. It was Erdös [34], who in this direction conjectured the following result, which was later proved by Lax [49].

Theorem 15. *If $P(z)$ is a polynomial of degree at most n having no zeros in $|z| < 1$, then*

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{25}$$

The result is best possible and equality holds for any polynomial which has all its zeros on $|z| = 1$.

In the special case, when $P(z)$ has all its zeros on $|z| = 1$, Theorem 15 was proved independently by Pólya and by Szegö [66]. Simple proofs of this result were later given by de Bruijn [23], and Aziz and Mohammed [8].

Although (25) seems to be the right inequality for the class of polynomials having no zeros in the unit disc, the inequality becomes equality only when all the zeros of $P(z)$ are on $|z| = 1$. Now naturally a question arises as to what happens if the polynomial has all the zeros outside the closed unit disc, and in this direction Aziz and Dawood [7] proved the following which sharpens Theorem 15.

Theorem 16. *If* $P(z)$ *is a polynomial of degree at most* n *having no zeros in* $|z| < 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}. \tag{26}$$

The result is best possible and equality holds for $P(z) = \lambda z^n + \mu$, *where* $|\lambda| = |\mu|$.

Also, for a polynomial of degree n having no zeros in $|z| < 1$, Ankeny and Rivlin [2] proved that

$$\max_{|z|=R} |P(z)| \leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)|. \tag{27}$$

The Inequality (27) becomes equality for $P(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

Aziz and Dawood [7] also sharpened the Inequality (27) by proving,

Theorem 17. *If* $P(z)$ *is a polynomial of degree* n *having no zeros in* $|z| < 1$, *then for* $R \geq 1$,

$$\max_{|z|=R} |P(z)| \leq \left(\frac{R^n + 1}{2} \right) \max_{|z|=1} |P(z)| - \left(\frac{R^n - 1}{2} \right) \min_{|z|=1} |P(z)|. \tag{28}$$

The above result is best possible and again the equality holds for the polynomial $P(z) = \lambda z^n + \mu$, *where* $|\lambda| = |\mu|$.

Malik [53] (for related result, see Govil [37]) considered the class of polynomials having no zeros in the circle of radius K, and proved the following generalization of Theorem 15.

Theorem 18. *If* $P(z)$ *is a polynomial of degree at most* n *having no zeros in* $|z| < K$, $K \geq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \left(\frac{n}{1 + K} \right) \max_{|z|=1} |P(z)|. \tag{29}$$

The result is best possible and equality holds for $P(z) = (z + K)^n$.

Govil and Rahman [43] generalized the above result of Malik by proving the Inequality (29) for s-th order derivatives. Later Govil [39] obtained the following generalizations of Theorem 18, and so of Theorem 16.

Theorem 19. *If* $P(z)$ *is a polynomial of degree at most n having no zeros in* $|z| < K$, $K \geq 1$, *then*

$$\max_{|z|=1} |P^{(s)}(z)| \leq \left(\frac{n(n-1)\ldots(n-s+1)}{1+K^s}\right)\left(\max_{|z|=1}|P(z)| - \min_{|z|=K}|P(z)|\right). \quad (30)$$

The above inequality is sharp in the case $s = 1$, and in this case, equality is attained for $P(z) = (z + K)^n$. For $s = 1$, it sharpens the Inequality (29) of Malik, and for $s = 1$, $K = 1$, it reduces to the Inequality (26) of Aziz and Dawood.

Aziz [4, p. 184] proved the following result for polar derivatives which extends Theorem 15, due to Lax [49], to polar derivatives.

Theorem 20. *If P(z) is a polynomial of degree n and* $P(z)$ *has no zeros in* $|z| < 1$, *then for* $|z| \geq 1$,

$$|D_{\alpha_k}\ldots D_{\alpha_1}P(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{2}\left(|\alpha_1\alpha_2\ldots\alpha_k z^{n-k}| + 1\right)\max_{|z|=1}|P(z)|, \quad (31)$$

where $|\alpha_i| \geq 1$, *for all* $i = 1, 2, 3, \ldots, k$. *The result is best possible and equality in (31) holds for the polynomial* $P(z) = \left(\frac{z^n+1}{2}\right)$.

The following result of Aziz [4], which follows from Theorem 20, generalizes the Inequality (25) due to Lax [49], and the Inequality (27) due to Ankeny and Rivlin [2].

Theorem 21. *If P(z) is a polynomial of degree n such that* $P(z)$ *has no zeros in* $|z| < 1$, *then for every real or complex number* $|\alpha| \geq 1$, *we have for* $|z| \geq 1$,

$$|D_\alpha P(z)| \leq \frac{n}{2}\left(|\alpha z^{n-1}| + 1\right)\max_{|z|=1}|P(z)|. \quad (32)$$

The result is best possible and equality in (32) holds for the polynomial $P(z) = \lambda z^n + \mu$ *where* $|\lambda| = |\mu| = \frac{1}{2}$ *and* $\alpha \geq 1$.

To obtain Inequality (25), divide both the sides of (32) by $|\alpha|$ and make $|\alpha| \to \infty$. In order to obtain Inequality (27) from (32), take $z = \alpha$, and use the fact that $\{D_\alpha P(z)\}_{z=\alpha} = nP(\alpha)$.

Theorem 20 was later sharpened by Aziz and Shah [14, p. 265], who proved that

Theorem 22. *If P(z) is a polynomial of degree n which does not vanish in $|z| < 1$, then for $|z| \geq 1$,*

$$|D_{\alpha_k} \ldots D_{\alpha_1} P(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{2} \tag{33}$$
$$\times\{(|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k}+1)\max_{|z|=1}|P(z)| - (|\alpha_1,\alpha_2\ldots\alpha_k||z|^{n-k}-1)\min_{|z|=1}|P(z)|\},$$

where $|\alpha_i| \geq 1$, for all $i = 1, 2, 3, \ldots, k$. The result is best possible and equality in (33) holds for the polynomial $P(z) = \frac{z^n+1}{2}$.

From the above Theorem 22, as its special case one can obtain the following result due to Aziz and Shah [14].

Theorem 23. *If P(z) is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real and complex number α with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \leq \frac{n}{2}\left((|\alpha||z|^{n-1}+1)\max_{|z|=1}|P(z)| - (|\alpha||z|^{n-1}-1)\min_{|z|=1}|P(z)|\right), \tag{34}$$

for $|z| \geq 1$. The result is best possible and equality in (34) holds for the polynomial $P(z) = \lambda z^n + \mu$, where $|\lambda| = |\mu| = \frac{1}{2}$, and $\alpha \geq 1$.

The above Theorem 23 includes as special cases Theorems 16 and 17 due to Aziz and Dawood [7]. Again, to obtain Theorem 16 from Theorem 23, divide both the sides of (34) by $|\alpha|$, and make $|\alpha| \to \infty$. To obtain Theorem 17 from Theorem 23, simply take $z = \alpha$ in (34), and use the property that $\{D_\alpha P(z)\}_{z=\alpha} = nP(\alpha)$.

Aziz [4, p. 187] also obtained the following result which extends Theorem 18, due to Malik [53], to the polar derivative of a polynomial.

Theorem 24. *If P(z) is a polynomial of degree n having no zeros in $|z| < K$, where $K \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$, we have*

$$\max_{|z|=1}|D_\alpha P(z)| \leq n\left(\frac{|\alpha|+K}{1+K}\right)\max_{|z|=1}|P(z)|. \tag{35}$$

The result is best possible and equality in (35) holds for the polynomial $P(z) = \frac{(z+K)^n}{(1+K)^n}$, with any real number $\alpha \geq 1$ and $K \geq 1$.

Govil and Labelle [40] extended Theorem 18 to the class of polynomials having no zeros in $|z| < 1$, and obtained a bound that depends on the location of all the zeros, rather then depending on the zero of smallest modulus. In this regard, they proved

Theorem 25. *If $P(z) = a_n\prod_{\nu=1}^{n}(z-z_\nu)$, $a_n \neq 0$, is a polynomial of degree n such that $|z_\nu| \geq K_\nu \geq 1$, $1 \leq \nu \leq n$, then*

$$\max_{|z|=1}|P'(z)| \leq n\frac{\sum_{\nu=1}^{n}\frac{1}{K_\nu-1}}{\sum_{\nu=1}^{n}\frac{K_\nu+1}{K_\nu-1}}\max_{|z|=1}|P(z)|. \tag{36}$$

If $K_\nu = K \geq 1$ for some $1 \leq \nu \leq n$, then the above inequality reduces to inequality (29). The result is sharp and equality holds for $P(z) = (z+K)^n, \quad K \geq 1$.

The above Theorems 24 and 25 have recently been sharpened and generalized by Rather et al. [76] as follows.

Theorem 26. *If $P(z) = a_n \sum_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree n such that $|z_\nu| \geq K_\nu \geq 1$, for $1 \leq \nu \leq n$, then for any complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + t_0}{1 + t_0} \right) \max_{|z|=1} |P(z)|, \tag{37}$$

where

$$t_0 = 1 + \frac{n}{\sum_{\nu=1}^{n} \frac{1}{K_\nu - 1}}$$

if $K_\nu > 1$ for all $\nu, \ 1 \leq \nu \leq n$, and $t_0 = 1$ if $K_\nu = 1$ for some $\nu, \ 1 \leq \nu \leq n$.

Dividing both the sides of Inequality (37) by $|\alpha|$, and making $|\alpha| \to \infty$, we get Theorem 25.

If $K = \min\{K_1, K_2, \cdots K_n\}$, then $K_\nu \geq K \geq 1$, for $1 \leq \nu \leq n$. But then $t_0 \geq K$, and hence for any real or complex number α with $|\alpha| \geq 1$, we have $\dfrac{|\alpha| + t_0}{1 + t_0} \leq \dfrac{|\alpha| + K}{1 + K}$. This establishes the improvement of the bound in Theorem 26, compared to the bound in Theorem 24. A related result can be seen in [29].

Aziz and Shah [13, p. 164] sharpened Theorem 24, by proving

Theorem 27. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K$, where $K \geq 1$, then for every real or complex number α, with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{1+K} \left((|\alpha| + K) \max_{|z|=1} |P(z)| - (|\alpha| - 1) \min_{|z|=K} |P(z)| \right). \tag{38}$$

The result is best possible and equality in (38) holds for the polynomial $P(z) = (z + K)^n$ with a real number $\alpha \geq 1$ and $K \geq 1$.

Again, dividing both the sides of (38) by $|\alpha|$ and making $|\alpha| \to \infty$, we get

Theorem 28. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K$, where $K \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+K} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=K} |P(z)| \right\}. \tag{39}$$

The result is best possible and equality in (39) holds for the polynomial $P(z) = (z + K)^n$ with any real number $\alpha \geq 1$ and $K \geq 1$.

As is easy to see, Theorem 28 is a special case of Theorem 19 due to Govil [39], and for $K = 1$, it reduces to Theorem 16 due to Aziz and Dawood [7].

As an improvement to Theorem 23 for the case $|z| = 1$, Liman et al. [51] proved the following theorem with an additional parameter β, and is as follows.

Theorem 29. *If $P(z)$ is a polynomial of degree n that does not vanish in $|z| < 1$, then for every complex number $\alpha,\ \beta$ with $|\alpha| \geq 1,\ |\beta| \leq 1$, and $|z| = 1$,*

$$\left|zD_\alpha P(z) + n\beta\left(\tfrac{|\alpha|-1}{2}\right)P(z)\right|$$

$$\leq \frac{n}{2}\left(\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|z + \beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\max_{|z|=1}|P(z)|$$

$$-\frac{n}{2}\left(\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| - \left|z + \beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\min_{|z|=1}|P(z)|.$$

Recently, Singh et al. [82] (see also [60]) generalized the above result Theorem 29, as follows.

Theorem 30. *If $P(z)$ is a polynomial of degree n that does not vanish in $|z| < K,\ K \leq 1$, then for all complex numbers $\alpha_i,\ 1 \leq i \leq t,\ 1 \leq t \leq n-1$, with $|\alpha_i| \geq K,\ K \leq 1$, and for any real or complex number β with $|\beta| \leq 1$, and for $|z| = 1$,*

$$\left|z^t D_{\alpha_t}\ldots\ldots D_{\alpha_2}D_{\alpha_1}P(z) + \beta n(n-1)(n-2)\ldots(n-t+1)L_t P(z)\right|$$

$$\leq \frac{n(n-1)(n-2)\ldots(n-t+1)}{2}X,$$

where

$$X = \left(\frac{1}{K^n}\left|\alpha_1\alpha_2\cdots\alpha_t + \beta L_t\right| + \left|z^t + \beta L_t\right|\right)\max_{|z|=1}|P(z)|$$

$$-\left(\frac{1}{K^n}\left|\alpha_1\alpha_2\cdots\alpha_t + \beta L_t\right| - \left|z^t + \beta L_t\right|\right)\min_{|z|=K}|P(z)|,$$

and

$$L_t = \left(\frac{(|\alpha_1|-K)(|\alpha_2|-K)\cdots(|\alpha_t|-K)}{(1+K)^t}\right).$$

Note that, as is easy to see, for $K = t = 1$, Theorem 30 reduces to Theorem 29.

The most recent extension of Theorem 29, to our knowledge, is due to Zireh and Bidkham [90] which is for the class of polynomials having no zeros in $|z| < 1$, except s-fold zeros at the origin. They have, in fact, proved

Theorem 31. *Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < 1$, except s-fold zeros at the origin. Then for any complex numbers $\alpha, \ \beta$ with $|\alpha| \geq 1, \ |\beta| \leq 1$, and $|z| = 1$, we have*

$$\left|zD_\alpha P(z) + (n+s)\beta\left(\tfrac{|\alpha|-1}{2}\right)P(z)\right|$$

$$\leq \frac{1}{2}\left(\left|n\alpha + (n+s)\beta\left(\frac{|\alpha|-1}{2}\right)\right| + \left|(n-s)z + s\alpha + (n+s)\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\max_{|z|=1}|P(z)|$$

$$-\frac{1}{2}\left(\left|n\alpha + (n+s)\beta\left(\frac{|\alpha|-1}{2}\right)\right| - \left|(n-s)z + s\alpha + (n+s)\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\min_{|z|=1}|P(z)|.$$

It can be easily seen that when $s = 0$, Theorem 31 reduces to Theorem 29.

Some more results related to inequalities for polar derivative of a polynomial whose zeros are outside an open circle can be found in [12, 26, 27] and also in [88].

In all the above theorems it has been assumed that $|\alpha_i| \geq 1 (\geq K)$, or $|\alpha| \geq 1$, and we do not know if the corresponding results under the condition $|\alpha_i| < 1 (< K)$, or $|\alpha| < 1$, are known or are still to be discovered.

2.3 Inequalities for Polynomials Having all Their Zeros in a Circle

Since the equality in the Bernstein's inequality (11) holds for polynomials which have all their zeros at the origin, improvement in (11) is not possible if we consider polynomials having all their zeros inside the unit circle. For this reason, in this case, it may be interesting to obtain inequality in the reverse direction, and in this connection, Turán [84] proved that if a polynomial $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \left(\frac{n}{2}\right)\max_{|z|=1}|P(z)|. \tag{40}$$

The result is best possible and equality holds in (40) for any polynomial which has all its zeros on $|z| = 1$.

For polynomials $P(z)$ of degree n having all their zeros in $|z| \leq K$, where $K \leq 1$, Malik [53] proved that

$$\max_{|z|=1}|P'(z)| \geq \left(\frac{n}{1+K}\right)\max_{|z|=1}|P(z)|. \tag{41}$$

The Inequality (41) is sharp and equality holds for $P(z) = (z+K)^n$.

A simple and direct proof of (41) was given by Govil [36], who also settled the problem for polynomials having all their zeros in $|z| \leq K$, where $K \geq 1$, by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, where $K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{n}{1+K^n}\right) \max_{|z|=1} |P(z)|. \tag{42}$$

The above result is also best possible and equality holds in (42) for $P(z) = z^n + K^n$.

Govil [39] sharpened inequalities (41) and (42) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, then

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{n}{1+K}\right) \max_{|z|=1} |P(z)| + \left(\frac{n}{K^{n-1}(1+K)}\right) \min_{|z|=K} |P(z)| \tag{43}$$

if $K \leq 1$, and

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{n}{1+K^n}\right) \max_{|z|=1} |P(z)| + \left(\frac{n}{1+K^n}\right) \min_{|z|=K} |P(z)|, \tag{44}$$

if $K \geq 1$. Both these inequalities are best possible. In (43), equality is attained for $P(z) = (z+K)^n$, and in (44), for $P(z) = z^n + K^n$.

Now we turn towards the extension of these fundamental ineqaulities to the polar derivative of a complex polynomial. Shah [80] extended the Inequality (40) of Turán [84] to the polar derivative, by proving,

Theorem 32. *If P(z) is a polynomial of degree n having all its zeros in* $|z| \leq 1$*, then for every real or complex number* α *with* $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2}(|\alpha| - 1) \max_{|z|=1} |P(z)|. \tag{45}$$

The result is best possible and equality in (45) holds for $P(z) = \dfrac{(z-1)^n}{2}$ *for any real number* $\alpha \geq 1$.

The above Theorem 32 has been sharpened by Aziz and Rather [9], who proved

Theorem 33. *If P(z) is a polynomial of degree n having all its zeros in* $|z| \leq 1$*, then for every real or complex number* α *with* $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2} \left\{ (|\alpha| - 1) \max_{|z|=1} |P(z)| + |\alpha| + 1) \min_{|z|=1} |P(z)| \right\}. \tag{46}$$

The result is best possible and equality in (46) holds for $P(z) = (z-1)^n$ *for any real number* $\alpha \geq 1$.

Also, recently Jain [47] obtained a generalization of Theorem 33 to higher order polar derivatives, and proved

Theorem 34. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex numbers $\alpha_1, \alpha_2 \cdots, \alpha_t$ such that $t < n$, and $|\alpha_i| \geq 1, \quad 1 \leq i \leq t$, we have*

$$\max_{|z|=1} |D_{\alpha_t} \cdots D_{\alpha_2} D_{\alpha_1} P(z)|$$

$$\geq \frac{n(n-1)\cdots(n-t+1)}{2^t} \left[L_{\alpha_t} \max_{|z|=1} |P(z)| + \left\{ 2^t |\alpha_1 \alpha_2 \cdots \alpha_t - L_{\alpha_t}| \right\} \min_{|z|=1} |P(z)| \right], \tag{47}$$

where $L_{\alpha_t} = (|\alpha_1| - 1) \cdots (|\alpha_t| - 1)$. This result is best possible and equality holds in (47) for $P(z) = (z-1)^n$, for each real $\alpha_i \geq 1, \quad 1 \leq i \leq t$.

Aziz and Rather [9] extended the Inequality (41) of Malik [53] to the polar derivative of a polynomial. They proved

Theorem 35. *If P(z) is a polynomial of degree n having all its zeros in $|z| \leq K$, where $K \leq 1$, then for every real or complex number α with $|\alpha| \geq K$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K}{1 + K} \right) \max_{|z|=1} |P(z)|. \tag{48}$$

The result is sharp and equality in (48) holds for $P(z) = (z-K)^n$, with $\alpha \geq 1$.

Aziz and Rather [9] also extended the Inequality (42) of Govil [36] to the polar derivative, in which they proved

Theorem 36. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, where $K \geq 1$, then for every real or complex number α with $|\alpha| \geq K$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K}{1 + K^n} \right) \max_{|z|=1} |P(z)|. \tag{49}$$

Govil and McTume [41] sharpened Theorem 35 of Aziz and Rather, by proving

Theorem 37. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K$, where $K \leq 1$, then for every real or complex number α with $|\alpha| \geq K$,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n(|\alpha| - K)}{1 + K} \max_{|z|=1} |P(z)| + \frac{n(|\alpha| + 1)}{K^{n-1}(1 + K)} \min_{|z|=K} |P(z)|. \tag{50}$$

The inequality is sharp and equality holds for $P(z) = (z-K)^n, \;\; \alpha \geq K$.

Clearly, the above Theorem 37 sharpens Theorem 35 of Aziz and Rather [9], and includes as a special case, the Inequality (43). To obtain the above Inequality (43) of Govil [39] from the above theorem, simply divide both sides of (50) by $|\alpha|$, and make $|\alpha| \to \infty$.

Theorem 37 can be sharpened further by using the information about the coefficients, $|a_{n-1}|$ and $|a_n|$, as follows, and this has been done by Govil and McTume [41].

Theorem 38. *Let $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having all its zeros in $|z| \le K$, where $K \le 1$. If $L = \dfrac{nK^2|a_n| + |a_{n-1}|}{|a_{n-1}| + n|a_n|}$, then for every real or complex number α with $|\alpha| \ge K$,*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n}{1+K}\left\{(|\alpha| - L)\max_{|z|=1}|P(z)| + \frac{(|\alpha|K + L)}{K^n}\min_{|z|=K}|P(z)|\right\}. \tag{51}$$

As a generalization of the Inequality (51), Dewan et al. [30] proved that

Theorem 39. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n, having all their zeros in $|z| \le K$, where $K \le 1$, then for any complex number α with $|\alpha| \ge K^\mu$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n}{1+K^\mu}\left\{(|\alpha| - A_\mu)\max_{|z|=1}|P(z)| + \frac{(|\alpha|K^\mu + A_\mu)}{K^n}\min_{|z|=K}|P(z)|\right\}, \tag{52}$$

where

$$A_\mu = \frac{n(|a_n| - \frac{m}{K^n})K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n(|a_n| - \frac{m}{K^n})K^{\mu-1} + \mu|a_{n-\mu}|}.$$

Zireh [89] further refined the above theorem for which he proved the following.

Theorem 40. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ is a polynomial of degree n having all its zeros in $|z| \le K$, $K \le 1$, then for any complex number α with $|\alpha| \ge A_\mu$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n}{1+A_\mu}\left\{(|\alpha| - A_\mu)\max_{|z|=1}|P(z)| + \frac{(|\alpha| + 1)A_\mu}{K^n}\min_{|z|=K}|P(z)|\right\}, \tag{53}$$

where A_μ is as given in Theorem 39.

To see that the bound in Theorem 40 is sharper than the bound in Theorem 39, we need to show that

$$\frac{n}{1+K^\mu}\left\{(|\alpha| - A_\mu)\max_{|z|=1}|P(z)| + \frac{(|\alpha|K^\mu + A_\mu)}{K^n}\min_{|z|=K}|P(z)|\right\}$$

$$< \frac{n}{1+A_\mu}\left\{(|\alpha|-A_\mu)\max_{|z|=1}|P(z)| + \frac{(|\alpha|+1)A_\mu}{K^n}\min_{|z|=K}|P(z)|\right\}. \tag{54}$$

To prove this, note that $|\alpha| \geq A_\mu$, and $K^\mu \geq A_\mu$ implies $\frac{\min_{|z|=K}|P(z)|}{K^n} < \max\limits_{|z|=1}|P(z)|$ (see [89, Lemma 2.7]). This gives the inequality

$$\frac{\min_{|z|=K}|P(z)|}{K^n}\left(\frac{|\alpha|K^\mu + A_\mu}{1+K^\mu} - \frac{(|\alpha|+1)A_\mu}{1+A_\mu}\right) < \frac{(|\alpha|-A_\mu)(K^\mu - A_\mu)}{(1+K^\mu)(1+A_\mu)}\max_{|z|=1}|P(z)|,$$

from which our claim (54) follows.

For more inequalities of these types we refer to Gulzar and Rather [44], Mir and Dar [64], and Liman et al. [52].

Dewan and Upadhye [33] considered the problem of obtaining a bound for the polar derivative of a polynomial that depends on the location of every zero of a polynomial and maximum of the moduli rather than the zero of greatest modulus, and in this regard they proved

Theorem 41. *Let* $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z - z_\nu),\ a_n \neq 0$, *be a polynomial of degree* $n \geq 2,\ |z_\nu| \leq K_\nu, 1 \leq \nu \leq n$, *and let* $K = \max\{K_1,\ K_2, \cdots K_n\} \geq 1$. *Then for any complex number* α *with* $|\alpha| \geq K$,

$$\max_{|z|=1}|D_\alpha P(z)| \geq (|\alpha|-K)\sum_{\nu=1}\frac{K}{K+K_\nu}$$

$$\left[\frac{2}{1+K^n}\max_{|z|=1}|P(z)| + \frac{1}{K^n}\left(\frac{K^n-1}{K^n+1}\right)\min_{|z|=K}|P(z)| + \frac{2|a_{n-1}|}{K(1+K^n)}\left(\frac{K^n-1}{K} - \frac{K^{n-2}-1}{n-2}\right)\right]$$

$$+\left(1-\frac{1}{K^2}\right)|na_0 + \alpha a_1|, \quad for \ \ n > 2,$$

and

$$\max_{|z|=1}|D_\alpha P(z)| \geq (|\alpha|-K)\sum_{\nu=1}\frac{K}{K+K_\nu}$$

$$\left[\frac{2}{1+K^n}\max_{|z|=1}|P(z)| + \frac{1}{K^n}\left(\frac{K^n-1}{K^n+1}\right)\min_{|z|=K}|P(z)| + |a_1|\frac{(K-1)^n}{K(1+K^n)}\right]$$

$$+\left(1-\tfrac{1}{K}\right)|na_0 + \alpha a_1|, \quad for \ \ n = 2.$$

If $P(z)$ has a zero on $|z| = K$, then dividing both sides of above inequalities by $|\alpha|$, and letting $|\alpha| \to \infty$, we obtain the results due to Govil [38].

According to a well-known result due to Bernstein [18] on polynomials, if $P(z)$ and $Q(z)$ are two polynomials with degree of $P(z)$ not exceeding that of $Q(z)$, where $Q(z)$ has all its zeros in $|z| \leq 1$, and $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then

$$|P'(z)| \leq |Q'(z)|, \quad for \ \ |z| = 1. \tag{55}$$

Malik and Vong [55] improved the Inequality (55) by proving

$$|zP'(z) + n\frac{\beta}{2}P(z)| \leq |zQ'(z) + n\frac{\beta}{2}Q(z)|, \tag{56}$$

for every β with $|\beta| \leq 1$, and n is the degree of $Q(z)$.

As is easy to see for $\beta = 0$, the Inequality (56) reduces to (55). A result similar to this can also be found in a paper due to Bidkham et al. [21].

Liman et al. [51] extended the above Inequality (56) to polar derivative of a polynomial by proving the following.

Theorem 42. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, and $P(z)$ a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1,\ \ |\beta| \leq 1$,*

$$\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \leq \left| zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|, \tag{57}$$

for $|z| \geq 1$.

The Inequality (56) due to Malik and Vong [55] follows from the above Theorem 42, if we divide the two sides of (57) by $|\alpha|$, and make $|\alpha| \to \infty$.

The case $\beta = 0$ of the above Inequality (57) is also interesting and useful, and we state this below as a theorem.

Theorem 43. *Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, and $P(z)$ a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for any complex number α, with $|\alpha| \geq 1$,*

$$|D_\alpha P(z)| \leq |D_\alpha Q(z)| \tag{58}$$

for $|z| \geq 1$.

Dividing the two sides of Inequality (58) by $|\alpha|$, and letting $|\alpha| \to \infty$ we get the Inequality (55).

It may be remarked here that Zargar [87], and Bidkham and Mezerji [19] also obtained related results for k^{th} polar derivative of a polynomial.

Li [50] has given a new perspective to the results mentioned in Theorems 42 and 43. He showed how the inequalities (57) and (58) can be obtained in a more natural way from a generalized inequality for rational functions with prescribed poles. Before proceeding towards his result, let us introduce the set of rational functions involved.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n given points in $|z| > 1$, and $P(z)$ a polynomial of degree at most n. Consider the following space of rational functions with prescribed poles:

$$R_n := R_n(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left\{ \frac{P(z)}{W(z)} \right\},$$

where $W(z) := (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$.

Denote

$$B(z) = \frac{z^n \overline{W(1/\overline{z})}}{W(z)} = \prod_{k=1}^{n} \frac{1 - \overline{\alpha_k} z}{z - \alpha_k}.$$

Then it is easy to observe that $|B(z)| = 1$ on $|z| = 1$.

Now we are in a position to state the following result due to Li [50] on rational functions, which is of interest because of its yielding several results on polar derivatives of polynomials.

Theorem 44. *Let $R(z),\ S(z) \in R_n$ and assume $S(z)$ has all its n zeros in $|z| \leq 1$, and $|R(z)| \leq |S(z)|$ for $|z| = 1$. Then, for any ρ with $|\rho| \leq 1/2$,*

$$|R'(z) + \rho B'(z)R(z)| \leq |S'(z) + \rho B'(z)S(z)| \text{ for } |z| = 1. \tag{59}$$

The result is sharp and equality holds if $R(z) \equiv S(z)$.

The above theorem besides including several results on polar derivatives also includes Theorem 42 (so also Theorem 43) due to Liman et al. [51]. For this, take $|\alpha| > 1$ and apply Theorem 44 to the rational function, $R(z) = \frac{P(z)}{(z-\alpha)^n}$ and $S(z) = \frac{Q(z)}{(z-\alpha)^n}$ with poles all at one point $z = \alpha$, which gives us

$$\left| \left(\frac{P(z)}{(z-\alpha)^n} \right)' \right| \leq \left| \left(\frac{Q(z)}{(z-\alpha)^n} \right)' \right| \tag{60}$$

for all $|z| = 1$. Here $Q(z)$ is a polynomial of degree n and $P(z)$ is any polynomial of degree atmost n.

Now, using the fact that

$$\left(\frac{P(z)}{(z-\alpha)^n} \right)' = -\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}$$

in (60), we will easily get the Inequality (58), when $|\alpha| > 1$. Taking the limit as $|\alpha| \to 1$, we get the inequality when $|\alpha| \geq 1$. In the same way, Theorem 44 implies that, for $|z| = 1$,

$$\left| -\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}} + \rho \frac{n(|\alpha|^2 - 1)}{|z-\alpha|^2} \frac{P(z)}{(z-\alpha)^n} \right| \leq \left| -\frac{D_\alpha Q(z)}{(z-\alpha)^{n+1}} + \rho \frac{n(|\alpha|^2 - 1)}{|z-\alpha|^2} \frac{Q(z)}{(z-\alpha)^n} \right|,$$

and if we take $\beta \left(\frac{|\alpha| - 1}{2} \right) = \rho \frac{|\alpha|^2 - 1}{\overline{z} - \overline{\alpha}}$, then we will get the Inequality (57) due to Liman et al. [51].

It may be remarked that the proof of Theorem 44 presented in Li [50] may be useful in proving new inequalities for polynomials and rational functions.

We conclude this section by stating a result due to Rather et al. [76], which can be proved using Theorem 26.

Theorem 45. *If $P(z) = a_n \sum_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree n such that $P'(0) = 0$, and $|z_\nu| \leq K_\nu \leq 1$, for $1 \leq \nu \leq n$, then for any real or complex number α with $|\alpha| \leq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{1 + |\alpha| s_0}{1 + s_0} \right) \max_{|z|=1} |P(z)|, \tag{61}$$

where

$$s_0 = 1 + \frac{n}{\sum_{\nu=1}^{n} \frac{K_\nu}{1-K_\nu}},$$

if $K_\nu < 1$ for all $\nu,\ 1 \leq \nu \leq n$, and $s_0 = 1$ if $K_\nu = 1$ for some $\nu,\ 1 \leq \nu \leq n$.

If $\max\{K_\nu,\ 1 \leq \nu \leq n\} = K$, then $K \leq 1$, and therefore $s_0 \geq \frac{1}{K}$. Thus one can easily get

$$\left(\frac{1 + |\alpha| s_0}{1 + s_0} \right) \leq \left(\frac{|\alpha| + K}{1 + K} \right),$$

for $|\alpha| \leq 1$.

This will lead us to a more natural result (see [76]) that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| < K$, where $K \leq 1$, then for any complex number α with $|\alpha| \leq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + K}{1 + K} \right) \max_{|z|=1} |P(z)|. \tag{62}$$

There are many papers on this and related subjects, and here we refer the reader to some of them [1, 10, 20, 24, 28, 32, 45, 58, 59, 61–63, 73, 86, 91].

3 Bounds on the Integral Mean Values of Polar Derivative of a Polynomial

As mentioned in Sect. 1, the classical Bernstein's inequality asserts that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \tag{63}$$

for any polynomial $P(z)$ of degree n.

We have seen various generalizations of (63) in earlier sections, and by looking at these results one would think of the generalizations of Bernstein inequality (63) in

L^p norm, and in this section we will discuss some of the L^p inequalities for the polar derivatives of polynomials. It may be remarked that Bernstein-type inequalities in L^P norm play an important role in approximation theory and related topics. This section has three subsections, Sect. 3.1 dealing with inequalities for polynomials with no restriction on their zeros, Sect. 3.2 for inequalities having no zeros in a circle, and finally Sect. 3.3 is on polynomials having all their zeros in a circle.

3.1 Inequalities for Polynomials with no Restriction on Their Zeros

The sharp L^P inequality analogous to Bernstein's Inequality (63) was first established by Zygmund [93], who proved that for any polynomial $P(z)$ of degree n and for any $p \geq 1$, we have

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}. \tag{64}$$

The Inequality (64) is best possible and equality holds for polynomial $P(z)$ having all its zeros at the origin.

Arestov [3] extended the above inequality of Zygmund (64) to the case $0 < p < 1$ as well.

It is quite natural to seek an extension of Zygmund's result to polar derivative of a polynomial. In view of the uniform norm extension (13) and L^P extension (64) of Bernstein's inequality (63), one may attempt to obtain the L^P version for corresponding inequality for polar derivative as

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n|\alpha| \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \tag{65}$$

for any polynomial $P(z)$ of degree n and for any real or complex number α with $|\alpha| \geq 1$, and any $p > 0$. But that is not true, and in this regard Aziz and Rather [11] constructed a counterexample by considering the polynomial $P(z) = (1 - iz)^n$, $p = 2$, and $\alpha = i\delta$, where δ is any positive real number such that $1 \leq \delta < \frac{n+\sqrt{2n(2n-1)}}{3n-2}$. Then it is easy to verify that for this polynomial the Inequality (65) does not hold.

However, Aziz and Rather [11] in this direction proved the following

Theorem 46. *If $P(z)$ is a polynomial of degree n then for every complex number α and $p \geq 1$,*

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n(1 + |\alpha|) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}. \tag{66}$$

Later, in a subsequent paper, Rather [70] showed that, the Inequality (66) is true for $0 < p < 1$ also. As is easy to see, if we divide the two sides of (66) by $|\alpha|$, and make $|\alpha| \to \infty$, we get Zygmund Inequality (64) for each $p > 0$.

Govil et al. [46] (See also [42]) obtained a best possible bound for polar derivative of a self reciprocal polynomial for $1 \leq p < \infty$, which was later extended by Rather [70], who proved it for all values of $p > 0$. His result is as follows.

Theorem 47. *If $P(z)$ is a self- inversive polynomial of degree n, then for every complex number α and $p > 0$,*

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq n\,(|\alpha| + 1)\, C_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{67}$$

where $C_p = \left\{\frac{2\pi}{\int_0^{2\pi} |1 + e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$. In the limiting case, when $p \to \infty$, the above inequality is sharp and equality holds for the polynomial $P(z) = a + bz^n$ where $|a| = |b|$.

The above inequality extends the L^P inequality on self-reciprocal polynomials due to Dewan and Govil [25]. We can observe this by dividing both the sides of (67) by $|\alpha|$ and letting $|\alpha| \to \infty$. By doing so we get the following result due to Dewan and Govil [25].

Theorem 48. *If $P(z)$ is a self- inversive polynomial of degree n, then $p > 0$,*

$$\left\{\int_0^{2\pi} |P'(z)|^p d\theta\right\}^{\frac{1}{p}} \leq nC_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{68}$$

where $C_p = \left\{\frac{2\pi}{\int_0^{2\pi} |1 + e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$. In the limiting case, when $p \to \infty$, the above inequality is sharp and equality holds for the polynomial $P(z) = a + bz^n$ where $|a| = |b|$.

3.2 Inequalities for Polynomials Having no Zeros in a Circle

L^P extensions of Erdös–Lax inequality (25) for the polar derivative of a polynomial having no zeros in a circle have been studied in different ways. We begin this section with L^P analogue of (25) for ordinary derivative of a polynomial. The first and foremost result in this direction is due to de Bruijn [23]. In fact, he extended Zygmund's inequality (64) to the class of polynomials having no zeros in the disc

$|z| < 1$ by proving that, if $P(z)$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then for any $p \geq 1$,

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq nC_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{69}$$

where $C_p = \left\{\dfrac{2\pi}{\int_0^{2\pi} |1 + e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$.
Equality holds in (69) for $P(z) = (1 + z^n)$.

Rahman and Schmeisser [67] showed that de Bruijn's result is in fact true for all $p > 0$. Gardner and Govil [35] extended de Bruijn's result for the class of polynomials having no zeros in the disc $|z| < K, \quad K \geq 1$.

Similar results on polar derivative of a complex polynomial were studied significantly. Govil et al. [46] (see also Aziz and Rather [11]) have proved the analogue of de Bruijn's inequality (69) for polar derivatives. They proved that, if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for any $p \geq 1$, and for every real or complex number α, with $|\alpha| \geq 1$,

$$\left\{\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \leq n(|\alpha| + 1)C_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{70}$$

where $C_p = \left\{\dfrac{2\pi}{\int_0^{2\pi} |1 + e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$. Equality holds in (70) for $P(z) = (1 + z^n)$ in the limiting case when $p \to \infty$.

The above Inequality (70) has been presented in a different form with an additional parameter, in a paper due to Singh and Shah [81].

Rather [69] extended the above result (70) to the class of polynomials having no zeros in the disk $|z| < K, \ K \geq 1$, and for all $p > 0$, as follows.

Theorem 49. *If $P(z)$ is a polynomial of degree n and $P(z)$ does not vanish in $|z| < K$ where $K \geq 1$, then for every complex number α with $|\alpha| \geq 1$, and $p > 0$,*

$$\left\{\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \leq n(|\alpha| + K)F_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{71}$$

where $F_p = \left\{\dfrac{2\pi}{\int_0^{2\pi} |K + e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$. In the limiting case, when $p \to \infty$, the above inequality is sharp and equality holds for $P(z) = (z + K)^n$ where α is any real number with $\alpha \geq 1$.

The validity of the Inequality (70) for all $p > 0$ (also see [70]) immediately follows from the above Theorem 49 by taking $K = 1$.

If we take $p \to \infty$, in (71), then Theorem 49 reduces to Theorem 24. Dividing the two sides of Inequality (71) by $|\alpha|$, and then letting $|\alpha| \to \infty$, we get the corresponding result for ordinary derivative on uniform norm given by Gardner and Govil [35].

We do not know if there exist in literature the corresponding inequalities, for the case $|\alpha| < 1$.

3.3 Inequalities for Polynomials Having all Their Zeros in a Circle

We begin with the following result, which is an immediate consequence of Inequality (70), and was first introduced in the paper of Govil et al. [46].

If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then for every complex number α with $|\alpha| \le 1$, and $p \ge 1$,

$$\left\{\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \le n(|\alpha|+1)C_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{72}$$

where $C_p = \left\{\dfrac{2\pi}{\int_0^{2\pi} |1+e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$.

Rather [70] (see also [15]) validated the above result for $0 < p < 1$. He [69] further extended this result for class of polynomials having all their zeros in the closed disc $|z| \le K$, where $K \le 1$, and for all $p > 0$. His result is as given below.

Theorem 50. *If $P(z)$ is a polynomial of degree n and $P(z)$ has all its zeros in $|z| \le K$, where $K \le 1$, and $P(0) \ne 0$, then for every complex number α with $|\alpha| \le 1$, and $p > 0$,*

$$\left\{\int_0^{2\pi} |D_\alpha\{P(e^{i\theta})\}|^p d\theta\right\}^{\frac{1}{p}} \le n(|\alpha|+K)F_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}, \tag{73}$$

where $F_p = \left\{\dfrac{2\pi}{\int_0^{2\pi} |K+e^{i\theta}|^p d\theta}\right\}^{\frac{1}{p}}$. *In the limiting case, when $p \to \infty$, the above inequality is sharp and equality holds for $P(z) = (z+K)^n$, where α is any non-negative real number with $\alpha \le 1$.*

Again, we do not know if there is a sharp inequality available under the conditions of Theorem 50 when $|\alpha| > 1$.

On the other hand, Malik [54] obtained a generalization of (40) due to Turán [84] in L^P norm by proving that, if $P(z)$ has all its zeros in $|z| \leq 1$, then for each $p > 0$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} |1 + e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1} |P'(z)|. \tag{74}$$

The above result of (74) was generalized by Aziz [5] who proved that, if $P(z)$ has all its zeros in $|z| \leq K \leq 1$, then for each $p > 0$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} |1 + Ke^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1} |P'(z)|. \tag{75}$$

As a generalization of (75), Aziz and Ahemad [6] proved that, if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, then for each $p > 0,\ r > 1,\ q > 1$, with $r^{-1} + q^{-1} = 1$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} |1 + Ke^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} \left(|P'(e^{i\theta})|\right)^{qp} d\theta\right\}^{\frac{1}{qp}}. \tag{76}$$

Dewan et al. [31] generalized the inequalities (74) and (75) for polar derivatives. They in fact proved that, if $P(z)$ has all its zeros in $|z| \leq K \leq 1$, then for every complex number α with $|\alpha| \geq K$, and for each $p > 0$,

$$n(|\alpha| - K)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} |1 + Ke^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1} |D_\alpha P(z)|. \tag{77}$$

Recently, Rather et al. [74] (see also [75]) further extended the Inequality (76) to lacunary polynomials, and their result is as follows.

Theorem 51. *If* $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ *is a polynomial of degree n having all its zeros in* $|z| \leq K,\ K \leq 1$, *then for any complex numbers* α, β *with* $|\alpha| \geq K^\mu,\ |\beta| \leq 1$, *and for every* $p > 0,\ r > 1,\ q > 1$, *with* $r^{-1} + q^{-1} = 1$, *we have*

$$n(|\alpha| - K^\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) + \frac{\beta m}{K^{n-\mu}}|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_0^{2\pi} |1 + K^\mu e^{i\theta}|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} \left(|D_\alpha P(e^{i\theta})| - \frac{mn}{K^{n-\mu}}\right)^{qp} d\theta\right\}^{\frac{1}{qp}}, \tag{78}$$

where $m = \min_{|z|=K} |P(z)|$.

Making $q \to \infty$, so that $r \to 1$, we obtain the following result due to Rather et al. [74], which is a generalization of the Inequality (77).

Theorem 52. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$ is a polynomial of degree n having all its zeros in $|z| \leq K,\ K \leq 1$, then for any $\alpha, \beta \in C$ with $|\alpha| \geq K^\mu,\ |\beta| \leq 1$, and for every $p > 0$, we have*

$$n(|\alpha| - K^\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) + \frac{\beta m}{K^{n-\mu}}|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_0^{2\pi} |1 + K^\mu e^{i\theta}|^p d\theta\right\}^{\frac{1}{p}} \left\{\int_0^{2\pi} \left(|D_\alpha P(e^{i\theta})| - \frac{mn}{K^{n-\mu}}\right)^p d\theta\right\}^{\frac{1}{p}}, \tag{79}$$

where $m = \min_{|z|=K} |P(z)|$.

If in the above inequality we make $p \to \infty$ in (79), and choosing β accordingly, we get the corresponding inequality on uniform norm proved by Rather and Mir [73].

Zireh et al. [92] obtained a similar result but took into account the moduli of certain coefficients of the underlying polynomial, and their result is as given below.

Theorem 53. *If $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, and $m = \min_{|z|=K} P(z)$, then for any complex numbers $\lambda,\ \alpha$, with $|\lambda| \leq 1,\ |\alpha| \geq s_\mu$, and $p > 0,\ r > 1,\ q > 1$, with $r^{-1} + q^{-1} = 1$, we have*

$$n(|\alpha| - s_\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq \left\{\int_0^{2\pi} |1 + s_\mu e^{i\theta})|^{pr} d\theta\right\}^{\frac{1}{pr}} \left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^{qp} d\theta\right\}^{\frac{1}{qp}}, \tag{80}$$

where

$$s_\mu = \frac{n|a_n|K^{2\mu} + \mu|a_{n-\mu}|K^{\mu-1}}{n|a_n|K^{\mu-1} + \mu|a_{n-\mu}|}.$$

In the limiting case, when $r \to \infty$, the above Inequality (80) is sharp and equality holds for the polynomial $P(z) = (z - K)^n$, with $\alpha \geq K$.

As an immediate consequence of Theorem 53 we have the following result (see again [92]), which is of some interest, because it refines and generalizes the Inequality (77). To obtain this, make $q \to \infty$, in the Inequality (80), so that $r \to 1$.

Theorem 54. *$P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq K \leq 1$, and $m = \min_{|z|=K} |P(z)|$, then for any complex numbers $\lambda,\ \alpha$, with $|\lambda| \leq 1,\ |\alpha| \geq s_\mu$, and $p > 0$, we have*

$$n(|\alpha| - s_\mu)\left\{\int_0^{2\pi} |P(e^{i\theta}) + \lambda m|^p d\theta\right\}^{\frac{1}{p}} \leq \left\{\int_0^{2\pi} |1 + s_\mu e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \max_{|z|=1} |D_\alpha P(z)|, \tag{81}$$

where s_μ is defined as above.

Inequalities somewhat closer in spirit to the above ones have been obtained among others by Baba and Mir [16], Rather and Gulzar (see [71, 72]) and Rather et al. [77].

We close this chapter with the remark that although a considerable amount of research has been done in the direction of the Bernstein type inequalities for the polar derivatives of polynomials but still there are many questions that are left unanswered in this subject.

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On Two Inequalities for Polynomials in the Unit Disk

Richard Fournier and Stephan Ruscheweyh

This work is dedicated to the late Professor Q.I. Rahman

Abstract We show that a certain interpolation type inequality for polynomials in the unit disk, generalizing Bernstein's inequality, is actually contained in an older, even more general one. We also discuss the cases of equality.

Keywords Bernstein inequality • Inequalities for complex polynomials • Interpolation formulas • Convolution

2000 Mathematics Subject Classification: 41A17.

1 Introduction

Let $\mathcal{P}_n$ denote the class of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ with complex coefficients. We write $\mathbb{D}$ for the unit disk in the complex plane $\mathbb{C}$ and $|p|_{\mathbb{D}} := \max_{|z|=1} |p(z)|$. The famous S. Bernstein theorem says

$$|p'|_{\mathbb{D}} \leq n|p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n, \tag{1}$$

and equality holds here only for the monomials $p(z) = cz^n$, where c is an arbitrary complex constant. Inequality (1) has found numerous generalizations and

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refinements. A rather complete reference to many of these can be found in the very encyclopedic book of Rahman and Schmeisser [8]. In the present chapter we are dealing with two of those generalizations and how they are related to each other.

For a polynomial $Q \in \mathcal{P}_n$ we define $\widetilde{Q}(z) := z^n\ \overline{Q(1/\overline{z})}$. Note that, by writing $Q \in \mathcal{P}_n$, we do not automatically assume that the n-th coefficient q_n in the standard representation of Q is different from zero. In particular, we shall always apply the convention that $\widetilde{Q}$ belongs to $\mathcal{P}_n$ if Q is assumed to be. Note that this convention implies that

$$\widetilde{(\widetilde{Q})} = Q, \tag{2}$$

which cannot be guaranteed otherwise (f.i. if $Q(z) = z^2$ is considered an element of $\mathcal{P}_3$, say, then

$$\widetilde{Q}(z) = z^3\overline{(1/\overline{z^2})} = z$$

can be considered to be in $\mathcal{P}_2$ which would imply $\widetilde{(\widetilde{Q})}(z) = z \neq Q(z)$).

We use the following notation: let $P_{1/2}$ be the class of analytic functions f satisfying $f(0) = 1$ and $\mathrm{Re} f(z) > \frac{1}{2}$ in $\mathbb{D}$; the Hadamard product of two functions $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ analytic in $\mathbb{D}$ is the function $(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n$, also analytic in $\mathbb{D}$.

Theorem 1.1 ([9], Corollary 4.3). *For $p, Q \in \mathcal{P}_n$ with $Q(0) = 0$ and $\widetilde{Q} \in P_{1/2}$ we have*

$$|(p * Q)(z)| + |(p * \widetilde{Q})(z)| \leq |p|_{\mathbb{D}}, \quad z \in \mathbb{D}. \tag{3}$$

Theorem 1.2 ([5], Theorem 8). *For $p, Q \in \mathcal{P}_n$ as in Theorem 1.1 we have*

$$|(p * Q)(z)|_{\mathbb{D}} \leq \max_{1 \leq j \leq 2n} |p(w_j)|, \tag{4}$$

where $w_j := \exp(i\pi j/n), j \in \{1, \ldots, 2n\}$.

Note that both of these inequalities are refinements of (1). Indeed, writing

$$Q(z) := \frac{1}{n} \sum_{j=1}^{n} j z^j$$

we see that $Q \in \mathcal{P}_n$ with $Q(0) = 0$ and

$$\widetilde{Q}(z) = F_{n-1}(z) := \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) z^j$$

is the classical Fejér kernel which is known to belong to $P_{1/2}$. Obviously we have $(Q*p)(z) = \frac{1}{n}zp'(z)$, and therefore both, (3) and (4), contain and refine (1). It is also easy to see that (3) contains a result of Malik [7] according to which, for $p \in \mathcal{P}_n$

$$|p'(z)| + |\widetilde{p}'(z)| \leq n|p|_{\mathbb{D}}, \quad z \in \mathbb{D}.$$

Besides this, Theorems 1.1 and 1.2 contain other classical inequalities. For instance, Theorem 1.1 with $Q(z) = z^n$ contains Visser's inequality (see Visser [11]): for $p(z) = \sum_{k=0}^{n} a_k z^k$ we have

$$|a_0| + |a_n| \leq |p|_{\mathbb{D}},$$

and Theorem 1.2 can be considered to be an equivalent for the unit disk of the Duffin and Schaeffer [3] refinement of the Markov inequality for polynomials on the unit interval $[-1, 1]$. We recall that the Duffin and Schaeffer result states that for all $p \in \mathcal{P}_n$ and $x \in [-1, 1]$,

$$|p'(x)| \leq n^2 \max_{0 \leq j \leq n} |p(\cos(j\pi/n))|.$$

The original proof of Theorem 1.2 as given in [5] is not related to Theorem 1.1 in [9]. It is our aim in this note to show that Theorem 1.2 indeed follows from Theorem 1.1, using an interpolation formula argument. This will be done in Sect. 3. Section 4 discusses cases of equality in Theorems 1.1 and 1.2.

2 Background Material

A polynomial q in $\mathcal{P}_n$ is called bound-preserving over $\mathcal{P}_n$ if

$$|q * p|_{\mathbb{D}} \leq |p|_{\mathbb{D}} \quad \text{for all } p \in \mathcal{P}_n.$$

It is known (see, for example, [9, Chap. 4] or [10] for a multivariable version) that a necessary and sufficient condition for $q \in \mathcal{P}_n$ to be bound-preserving is that

$$q(z) + \mathrm{o}(z^n) = \int_{\partial\mathbb{D}} \frac{1}{1 - xz} d\mu(x), \quad z \in \mathbb{D}, \tag{5}$$

where μ is a complex Borel measure over the boundary $\partial\mathbb{D}$ of the unit disk with total variation $\int_{\partial\mathbb{D}} |d\mu(x)| \leq 1$ and $\mathrm{o}(z^n)$ is a function analytic in $\mathbb{D}$ with an $(n+1)$-fold zero at the origin. Moreover, if $q(0) = 1$, we obtain

$$1 = \int_{\partial\mathbb{D}} d\mu(x) \leq \int_{\partial\mathbb{D}} |d\mu(x)| \leq 1$$

so that μ is in fact a probability measure over $\partial\mathbb{D}$. In this context the representation formula (5) is called the Herglotz formula and the function $q(z) + \mathrm{o}(z^n)$ appearing there belongs to $P_{1/2}$. It is also well known that if $q(z) = 1 + \cdots + a_n z^n \in P_{1/2}$ for some $n \in \mathbb{N}$, we have $|a_n| \leq 1$ with equality if and only if

$$q(z) = \sum_{j=1}^{n} \frac{\ell_j}{1 - w_j a_n^{1/n} z} + \mathrm{o}(z^n),$$

where $\ell_j \geq 0$, $\sum_{j=1}^{n} \ell_j = 1$ and $\{w_j\}_{j=1}^{n}$ is the set of distinct n^{th} roots of unity.

3 Theorem 1.1 Implies Theorem 1.2

First we recall the proof of Theorem 1.1 as given in [9]. Let Q be in $\mathcal{P}_n$ with $Q(0) = 0$, $\widetilde{Q} \in P_{1/2}$ and $\zeta \in \mathbb{D}$. The rational function

$$F_\zeta(z) := \frac{\widetilde{Q}(z) + \zeta Q(z) - \zeta z^n}{1 - \zeta z^n}$$

is analytic in $\overline{\mathbb{D}}$, and since $Q(0) = 0$ and $\widetilde{Q}(0) = 1$, we have

$$\begin{aligned} F_\zeta(z) &= \widetilde{Q}(z) + \zeta\, Q(z) + \frac{\zeta z^n (\widetilde{Q}(z) - 1 + \zeta\, Q(z))}{1 - \zeta\, z^n} \\ &= \widetilde{Q}(z) + \zeta\, Q(z) + \mathrm{o}(z^n), \quad z \to 0. \end{aligned}$$

Moreover, for $|z| = 1$,

$$\begin{aligned} \operatorname{Re} F_\zeta(z) &= \operatorname{Re} \frac{\widetilde{Q}(z) + \zeta Q(z) - \zeta z^n}{1 - \zeta z^n} \\ &= \operatorname{Re} \frac{\widetilde{Q}(z) + \zeta \widetilde{\widetilde{Q}}(z) - \zeta z^n}{1 - \zeta z^n} \\ &= \operatorname{Re} \frac{\widetilde{Q}(z) + \zeta z^n \overline{\widetilde{Q}(z)} - \zeta z^n}{1 - \zeta z^n} \\ &= \operatorname{Re} \frac{\widetilde{Q}(z) + \zeta z^n \overline{\widetilde{Q}(z)} - \zeta z^n}{1 - \zeta z^n} \\ &= \operatorname{Re} \frac{\widetilde{Q}(z) + \overline{\widetilde{Q}(z)} + (\zeta z^n - 1)\overline{\widetilde{Q}(z)} + 1 - \zeta z^n}{1 - \zeta z^n} \end{aligned}$$

$$= \operatorname{Re}\left(\frac{2\operatorname{Re}\widetilde{Q}(z)-1}{1-\zeta z^n} + 1 - \overline{\widetilde{Q}(z)}\right)$$

$$= \left(2\operatorname{Re}\widetilde{Q}(z)-1\right)\operatorname{Re}\left(\frac{1}{1-\zeta z^n}\right) + \operatorname{Re}\left(1-\widetilde{Q}(z)\right)$$

$$\geq \frac{2\operatorname{Re}\widetilde{Q}(z)-1}{2} + \operatorname{Re}\left(1-\widetilde{Q}(z)\right)$$

$$= \frac{1}{2}.$$

This, combined with $F_\zeta(0) = 1$ and the minimum principle for harmonic functions, implies $F_\zeta \in P_{1/2}$. Therefore, for every $\zeta \in \mathbb{D}$, there exists a probability measure μ_ζ on $\partial\mathbb{D}$ such that

$$\widetilde{Q}(z) + \zeta Q(z) + \mathrm{o}(z^n) = \int_{\partial\mathbb{D}} \frac{1}{1-xz} d\mu_\zeta(x), \quad z \in \mathbb{D}, \tag{6}$$

and this implies that for any $p \in \mathcal{P}_n$, we have

$$(p * \widetilde{Q})(z) + \zeta(p * Q)(z) = \int_{\partial\mathbb{D}} p(xz) d\mu_\zeta(x), \quad z \in \mathbb{D}. \tag{7}$$

Using proper choices of ζ this can be used to prove (3).

Starting out from (7) we now prove Theorem 1.2. In the case $|\zeta| = 1$ the measure μ in (6) can be discretized. Indeed, we then have $F := F_\zeta \in P_{1/2}$ with

$$\left|\frac{F^{(n)}(0)}{n!}\right| = 1 = \max_{G \in P_{1/2}} \left|\frac{G^{(n)}(0)}{n!}\right|,$$

so that F is a support point of $P_{1/2}$ (for details see [6, Chap. 4]) and therefore F has a representation

$$F(z) = \widetilde{Q}(z) + \zeta Q(z) + \mathrm{o}(z^n) = \sum_{j=1}^{n} \frac{l_j}{1 - w_j \zeta^{1/n} z},$$

where $l_j \geq 0$, $\sum_{j=1}^n l_j = 1$ and the w_j are distinct n-roots of unity. This is equivalent to

$$(p * \widetilde{Q})(z) + \zeta(p * Q)(z) = \sum_{j=1}^{n} l_j(\zeta) p(w_j \zeta^{1/n} z), \tag{8}$$

for arbitrary $z \in \mathbb{C}$ and $p \in \mathcal{P}_n$.

The particular choice

$$p_j(z) := \frac{1 - (\overline{w_j}(\overline{\zeta})^{1/n} z)^n}{1 - \overline{w_j}(\overline{\zeta})^{1/n} z}$$

leads to the explicit representation

$$l_j = l_j(\zeta) = \frac{1}{n}(2\,\mathrm{Re}\,\widetilde{Q}(\overline{w_j}(\overline{\zeta})^{1/n}) - 1), \quad 1 \le j \le n. \tag{9}$$

Replacing ζ by $-\zeta$ in (8) we get

$$(p * \widetilde{Q})(z) - \zeta(p * Q)(z) = \sum_{j=1}^{n} l_j(-\zeta) p(w_j(-\zeta)^{1/n} z), \tag{10}$$

and adding (10) to (8) yields

$$(p * \widetilde{Q})(z) = \frac{1}{2} \sum_{j=1}^{n} \big(l_j(\zeta) p(w_j \zeta^{1/n} z) + l_j(-\zeta) p(w_j(-\zeta)^{1/n} z) \big) \tag{11}$$

$$= \sum_{j=1}^{2n} \Lambda_j(\zeta) p(v_j(\zeta)^{1/n} z), \tag{12}$$

where $\{v_j\}_{j=1}^{2n}$ stands for the set of distinct $2n$-th roots of unity, and

$$\Lambda_j(\zeta) := \frac{1}{2n}(2\,\mathrm{Re}\,\widetilde{Q}(\overline{v_j}(\overline{\zeta})^{1/n}) - 1) \ge 0, \quad 1 \le j \le 2n, \tag{13}$$

since $\widetilde{Q} \in P_{1/2}$. Furthermore, using the special case $p \equiv 1$ in (11) yields $\sum_{j=1}^{2n} \Lambda_j(\zeta) \equiv 1$ for $|\zeta| = 1$. Therefore, noting that (11) holds for $\widetilde{p}$ as well as for $p \in \mathcal{P}_n$ and that generally

$$\widetilde{\widetilde{p} * \widetilde{Q}} \equiv p * Q$$

we finally obtain

$$\zeta(p * Q(z)) = \sum_{j=1}^{2n} \Lambda_j(\zeta)(-1)^j p(v_j(\zeta)^{1/n} z), \quad z, \zeta \in \overline{\mathbb{D}}. \tag{14}$$

This, using the properties of the Λ_j just discovered, implies (4) □

4 Cases of Equality

The cases of equality in Theorem 1.2 have not been identified in [5] but, in a special case only 20 years later, in [2]. Now we can prove the following general Result:

Theorem 4.1. *For Q as in Theorem 1.2 we have equality in (4) if and only if $p(z) = cz^n$ with some complex constant c.*

Proof. Note that according to (13) the numbers $\Lambda_j(\zeta)$, for ζ fixed, are values of a trigonometric polynomial of degree at most $n-1$, which means that (unless the polynomial is identically zero, which is impossible in our case) only $n-1$ of the $\Lambda_j(\zeta)$ can be zero. Hence at least $n+1$ of those must be different from zero and so, according to (14) we can have equality in (4) only if $\widetilde{p}$ assumes a fixed value Me^{ix} in at least $n+1$ points on the unit circle. This implies that $\widetilde{p}$, being a polynomial of degree n, must be a constant function which implies the conclusion for p in our assertion. □

We wish to point out that a rather trivial and merely formal refinement of inequality (4) is

$$|(p * Q)|_{\mathbb{D}} \leq \min_{|x|=1} \max_{1 \leq j \leq 2n} |p(xw_j)|. \tag{15}$$

This is because the left-hand side of (15) does not change by replacing $p(z)$ by $p(xz)$ when $|x| = 1$. It is also clear that the cases of equality with respect to p do not change under this transformation. On the other hand, it is interesting that the right-hand side of (15) does not depend on the choice of Q. So it would be interesting to find out for which choice of Q, depending on p, the left-hand side of (15) becomes maximal. This is an open question.

The cases of equality in Theorem 1.1 are rather numerous, and we have only partial results which we briefly report. The question is clearly equivalent to the study of solutions $p \in \mathcal{P}_n$ of the equation

$$|p * \widetilde{Q}(z) + \zeta p * Q(z)| = |p|_{\mathbb{D}}, \tag{16}$$

for a given $Q \in \mathcal{P}_n$ with $Q(0) = 0$, $\widetilde{Q} \in P_{1/2}$ and suitable $z, \zeta \in \partial\mathbb{D}$. It is easily verified that any polynomial p of the form $A + Bz^n$ is a solution with z, ζ chosen properly.

Also, if we choose

$$\widetilde{Q}(z) = \sum_{k=0}^{n-1} (1 - \frac{k}{n})(\zeta^{1/n} z)^k, \quad \zeta \in \partial\mathbb{D},$$

so that

$$\widetilde{Q}(z) + \zeta Q(z) = \frac{1}{1 - \zeta^{1/n} z} + \mathrm{o}(z^n),$$

then equality holds in (16) for any $p \in \mathcal{P}_n$ and z suitably chosen.

More generally, using a result of Brickman et al. [1], given a subset K of $\{1, 2, \ldots, n-1\}$ and $\zeta \in \partial\mathbb{D}$, we find $Q \in \mathcal{P}_n$ with $Q(0) = 0$ and $\widetilde{Q} \in P_{1/2}$ such that

$$\operatorname{Re} \widetilde{Q}(\overline{w_j}\, (\bar{\zeta})^{1/n}) = \frac{1}{2} \quad \Leftrightarrow \quad j \in K,$$

and therefore (16) will hold for a polynomial $p \in \mathcal{P}_n$ and $z \in \partial\mathbb{D}$ if and only if $p(w_j \zeta^{1/n} z) = |p|_{\mathbb{D}} e^{ix}$ for all $j \in \{1, 2, \ldots, n\} \setminus K$ and fixed x. There does not seem to exist much information about such polynomials (however, compare [4]).

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Inequalities for Integral Norms of Polynomials via Multipliers

Igor E. Pritsker

Dedicated to the memory of Professor Q.I. Rahman

Abstract We consider a wide range of polynomial inequalities for norms defined by the contour or the area integrals over the unit disk. Special attention is devoted to the inequalities obtained by using the Schur-Szegő composition.

Keywords Polynomial inequalities • Hardy spaces • Bergman spaces • Mahler measure

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1 The Schur-Szegő Composition and Polynomial Inequalities

We survey and develop a large variety of polynomial inequalities for the integral norms on the unit disk. An especially important tool in this study is the Schur-Szegő composition (or convolution) of polynomials, which is defined via certain coefficient multipliers. In particular, it played prominent role in the development of polynomial inequalities in Hardy spaces. Let $\mathbb{C}_n[z]$ be the set of all polynomials of degree at most n with complex coefficients. Define the standard Hardy space H^p norm for $P_n \in \mathbb{C}_n[z]$ by

$$\|P_n\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

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It is well known that the supremum norm of the space H^∞ satisfies

$$\|P_n\|_{H^\infty} = \max_{|z|=1} |P_n(z)| = \lim_{p\to\infty} \|P_n\|_{H^p}.$$

We note the other limiting case [12, p. 139] of the so-called H^0 norm:

$$\|P_n\|_{H^0} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(e^{i\theta})|\,d\theta\right) = \lim_{p\to 0+} \|P_n\|_{H^p}.$$

It is also known as the *contour geometric mean* or the *Mahler measure* of a polynomial $P_n \in \mathbb{C}_n[z]$. An application of Jensen's inequality for $P_n(z) = a_n \prod_{j=1}^n (z - z_j) \in \mathbb{C}_n[z]$ immediately gives that

$$\|P_n\|_{H^0} = |a_n| \prod_{j=1}^{n} \max(|z_j|, 1).$$

The above explicit expression is very convenient, and it is frequently used in our paper and other literature. This direct connection with the roots of P_n explains why the Mahler measure and its close counterpart the Weil height play an important role in number theory, see a survey of Smyth [20].

For a polynomial $\Lambda_n(z) = \sum_{k=0}^n \lambda_k \binom{n}{k} z^k \in \mathbb{C}_n[z]$, we define the *Schur-Szegő composition* with another polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$ by

$$\Lambda P_n(z) := \sum_{k=0}^{n} \lambda_k a_k z^k. \tag{1}$$

If Λ_n is a fixed polynomial, then ΛP_n is a multiplier (or convolution) operator acting on a space of polynomials P_n. More information on the history and applications of this composition may be found in [1, 2, 6] and [18]. De Bruijn and Springer [6] proved a remarkable inequality stated below.

Theorem 1. *Suppose that $\Lambda_n \in \mathbb{C}_n[z]$ and $P_n \in \mathbb{C}_n[z]$. If $\Lambda P_n \in \mathbb{C}_n[z]$ is defined by (1), then*

$$\|\Lambda P_n\|_{H^0} \le \|\Lambda_n\|_{H^0} \|P_n\|_{H^0}. \tag{2}$$

If $\Lambda_n(z) = (1+z)^n$, then $\Lambda P_n(z) \equiv P_n(z)$ and $\|\Lambda_n\|_{H^0} = 1$, so that (2) turns into equality, showing sharpness of Theorem 1. This result was not sufficiently recognized for a long time. In fact, Mahler [14] proved the following special case of (2) nearly 15 years later by using a more complicated argument.

Corollary 1. $\|P_n'\|_{H^0} \leq n\|P_n\|_{H^0}$

We add that equality holds in Corollary 1 if and only if the polynomial P_n has all zeros in the closed unit disk, and present a proof of this fact in Sect. 3. To see how Theorem 1 implies the above estimate for the derivative, just note that if $\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}z^k$, then $\Lambda P_n(z) = zP_n'(z)$ and $\|\Lambda_n\|_{H^0} = n$. Furthermore, (2) immediately answers the question about a lower bound for the Mahler measure of derivative raised in [9, pp. 12 and 194]. Following Storozhenko [21], we consider $P_n'(z) = \sum_{k=0}^{n-1} a_k z^k$ and write

$$\frac{1}{z}\left(P_n(z) - P_n(0)\right) = \sum_{k=0}^{n-1} \frac{a_k}{k+1} z^k = \Lambda P_n'(z),$$

where

$$\Lambda_{n-1}(z) = \sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n-1}{k} z^k = \frac{(1+z)^n - 1}{nz}.$$

The result of de Bruijn and Springer (2) gives

Corollary 2 ([21]). *For any $P_n \in \mathbb{C}_n[z]$, we have*

$$\|P_n(z) - P_n(0)\|_{H^0} \leq c_n \|P_n'\|_{H^0},$$

where

$$c_n := \frac{1}{n}\|(z+1)^n - 1\|_{H^0} = \frac{1}{n} \prod_{n/6<k<5n/6} 2\sin\frac{k\pi}{n}.$$

It is easy to see that $c_n \approx (1.4)^n$ as $n \to \infty$. Moreover, equality holds in Corollary 2 for $P_n(z) = (z+1)^n - 1$.

Another interesting consequence of (2) is the well-known estimate for coefficients (usually attributed to Mahler).

Corollary 3. *If $P_n(z) = \sum_{k=0}^{n} a_k z^k$, then*

$$|a_k| \leq \binom{n}{k} \|P_n\|_{H^0}, \quad k = 0, \ldots, n.$$

The above inequality follows at once from (2) by letting $\Lambda_n(z) = \binom{n}{k}z^k$, $k = 1, \ldots, n$, and taking into account that $\|\Lambda P_n\|_{H^0} = \|a_k z^k\|_{H^0} = |a_k|$ and $\|\Lambda_n\|_{H^0} = \binom{n}{k}$.

Many other inequalities may be obtained from Theorem 1, including the one below, found in [16].

Corollary 4. *Let* $P_n(z) = \sum_{k=0}^{n} a_k z^k$ *and* $m = 0, \dots, n$. *We have*

$$\left\| \sum_{k \neq m} a_k z^k \right\|_{H^0} \leq \left\| (1+z)^n - \binom{n}{m} z^m \right\|_{H^0} \|P_n\|_{H^0}.$$

In particular, if $m = 0$, *then* $\|(1+z)^n - 1\|_{H^0} = \prod_{n/6<k<5n/6} 2\sin\frac{k\pi}{n} \approx (1.4)^n$ *as* $n \to \infty$.

Again, the proof is a simple application of (2) with $\Lambda_n(z) = (1+z)^n - \binom{n}{m} z^m$, so that $\lambda_m = 0$ and $\lambda_k = 1,\ k \neq m$.

An important generalization of Theorem 1 for the H^p norms was obtained by Arestov [1].

Theorem 2. *Suppose that* $\Lambda_n \in \mathbb{C}_n[z]$ *and* $P_n \in \mathbb{C}_n[z]$. *If* $\Lambda P_n \in \mathbb{C}_n[z]$ *is defined by* (1), *then*

$$\|\Lambda P_n\|_{H^p} \leq \|\Lambda_n\|_{H^0} \|P_n\|_{H^p}, \quad 0 \leq p \leq \infty. \tag{3}$$

In fact, Arestov obtained an even more general inequality, and also described the set of extremal polynomials for it, see [1] for details. One of the main motivations for such a result was the Bernstein inequality for derivative of a polynomial in $H^p,\ p \in (0,1)$.

Corollary 5. *For any* $P_n \in \mathbb{C}_n[z]$ *we have*

$$\|P_n'\|_{H^p} \leq n \|P_n\|_{H^p},\ 0 \leq p \leq \infty. \tag{4}$$

If $p > 0$, *then equality holds in* (4) *only for polynomials of the form* $P_n(z) = cz^n,\ c \in \mathbb{C}$.

This inequality was originally proved by Bernstein for $p = \infty$ [5, 15, 18], and generalized to $p \geq 1$ by Zygmund, see [24]. For $p = 0$, (4) reduces to the result of de Bruijn–Springer–Mahler stated in Corollary 1. The case $p \in (0,1)$ remained open for a long time, and was finally settled by Arestov [1]. A more complete history of this result can be found in the book [18] and the recent survey [3].

Lower bounds for the derivative are also of interest. While Theorem 2 immediately gives the analogue of Corollary 2 for H^p (in the same manner as before), the resulting constant c_n of Corollary 2 is no longer sharp. In fact, one can prove much better estimates.

Theorem 3. *If* $P_n \in \mathbb{C}_n[z]$, *then*

$$\|P_n - P_n(0)\|_{H^\infty} \leq \pi \, \|P_n'\|_{H^1} \tag{5}$$

and

$$\|P_n - P_n(0)\|_{H^\infty} \le \pi\, n^{1/p-1} \|P_n'\|_{H^p}, \quad 0 < p < 1. \tag{6}$$

The constant π *in* (5) *cannot be replaced by a smaller number.*

A different application of Theorem 2 gives the solution of the Chebyshev minimization problem in H^p.

Corollary 6. *Any monic polynomial* $P_n(z) = z^n + \ldots \in \mathbb{C}_n[z]$ *satisfies*

$$\|P_n\|_{H^p} \ge 1, \quad 0 \le p \le \infty. \tag{7}$$

If $p > 0$*, then equality holds in* (7) *only for the monomial* $P_n(z) = z^n$*.*

The case of $p = \infty$ in (7) reduces to the classical Chebyshev problem for the unit disk. It is readily seen that for $p = 0$ equality holds in Corollary 6 if and only if P_n has all zeros in the closed unit disk.

Yet another useful application of (3) is the following sharp estimate of the growth for the circular means of polynomials.

Corollary 7. *For any* $P_n \in \mathbb{C}_n[z]$ *and any* $R > 1$*, we have*

$$\|P_n(Rz)\|_{H^p} \le R^n \|P_n\|_{H^p}, \quad 0 \le p \le \infty. \tag{8}$$

If $p > 0$*, then equality holds in* (8) *only for polynomials of the form* $P_n(z) = cz^n$*,* $c \in \mathbb{C}$*.*

The above estimate is a special case of the classical Bernstein–Walsh Lemma on the growth of polynomials outside the set, when $p = \infty$.

If we use Theorem 2 to estimate the coefficients of a polynomial as in Corollary 3, then the result is certainly valid, but is not best possible. Given any polynomial $P_n(z) = \sum_{k=0}^{n} a_k z^k$, we obtain that

$$|a_k| \le \binom{n}{k} \|P_n\|_{H^p}, \quad k = 0, \ldots, n, \quad 0 \le p \le \infty.$$

Apart from the cases $k = 0$ and $k = n$, this is far from being precise. In particular, recall the well-known elementary (and sharp) estimate:

$$|a_k| \le \|P_n\|_{H^1}, \quad k = 0, \ldots, n.$$

Many more interesting estimates for the coefficients of a polynomial may be found in Chap. 16 of [18].

It is useful to have a bound for the regular convolution (or the Hadamard product) of two polynomials, in addition to the Schur-Szegő convolution we mainly consider here. In fact, one version of such an estimate follows directly from Theorem 2, as observed by Tovstolis [22].

Theorem 4. *If* $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$ *and* $Q_n(z) = \sum_{k=0}^n b_k z^k \in \mathbb{C}_n[z]$*, then we have for* $P_n * Q_n(z) = \sum_{k=0}^n a_k b_k z^k$ *that*

$$\|P_n * Q_n\|_{H^p} \le \|\Theta_n\|_{H^0} \|P_n\|_{H^0} \|Q_n\|_{H^p}, \quad 0 \le p \le \infty,$$

where

$$\Theta_n(z) = \sum_{k=0}^n \binom{n}{k}^2 z^k \quad \text{and} \quad \lim_{n\to\infty} \|\Theta_n\|_{H^0}^{1/n} \approx 3.20991230072\ldots.$$

We conclude this section with a bound for the derivative of a polynomial without zeros in the unit disk that was originally proved by Lax for $p = \infty$, then by de Bruijn for $p \ge 1$, and finally by Rahman and Schmeisser for all $p \ge 0$. See [18, p. 553] for a detailed account.

Theorem 5. *If* $P_n \in \mathbb{C}_n[z]$ *has no zeros in the unit disk, then*

$$\|P_n'\|_{H^p} \le \frac{n}{\|z+1\|_{H^p}} \|P_n\|_{H^p}, \ 0 \le p \le \infty,$$

where

$$\|z+1\|_{H^p} = 2\left(\frac{\Gamma(p/2+1/2)}{\sqrt{\pi}\,\Gamma(p/2+1)}\right)^{1/p}, \quad 0 < p < \infty,$$
$$\|z+1\|_{H^0} = 1 \quad \text{and} \quad \|z+1\|_{H^\infty} = 2.$$

Note that Theorem 5 is sharp as equality holds for polynomials of the form $P_n(z) = az^n + b$ with $|a| = |b| \ne 0$. Since $\|z+1\|_{H^p} > 1$ for $p > 0$, this result is an improvement over the standard Bernstein inequality stated in Corollary 5. Arestov [2] considered generalizations of Theorem 5 in the spirit of Theorem 2.

2 Polynomial Inequalities in Bergman Spaces

Polynomial inequalities for Bergman spaces (with norms defined by the area measure) are not developed as well as those for Hardy spaces considered in the previous section. Given a non-negative radial function $w(z) = w(|z|),\ z \in \mathbb{D}$, with $b_w = \iint_{\mathbb{D}} w\, dA > 0$, we define the weighted Bergman space A_w^p norm by setting

$$\|P_n\|_{A_w^p} := \left(\frac{1}{b_w}\iint_{\mathbb{D}} |P_n(z)|^p\, w(z) dA(z)\right)^{1/p}, \quad 0 < p < \infty,$$

where dA is the Lebesgue area measure. If $w \equiv 1$, then we use the standard notation A^p for the regular Bergman space, with $b_w = \pi$. Detailed information on Bergman spaces can be found in the books [8] and [13]. We also define the A^0_w norm by

$$\|P_n\|_{A^0_w} := \exp\left(\frac{1}{b_w}\iint_{\mathbb{D}} \log|P_n(z)|\, w(z)dA(z)\right).$$

This norm was studied in [16] and [17], and it has the same relation to Bergman spaces as H^0 norm to Hardy spaces:

$$\|P_n\|_{A^0_w} = \lim_{p\to 0+} \|P_n\|_{A^p_w},$$

see [12, p. 139]. If $w \equiv 1$, then the following explicit form for $\|P_n\|_{A^0}$ is found in [16, 17].

Theorem 6. *Let $P_n(z) = a_n \prod_{j=1}^n (z - z_j) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$. If P_n has no roots in $\mathbb{D}$, then $\|P_n\|_{A^0} = |a_0|$. Otherwise,*

$$\|P_n\|_{A^0} = \|P_n\|_{H^0} \exp\left(\frac{1}{2}\sum_{|z_j|<1}(|z_j|^2 - 1)\right). \tag{9}$$

We immediately obtain the following comparison result from Theorem 6.

Corollary 8. *For any $P_n \in \mathbb{C}_n[z]$, we have*

$$e^{-n/2}\|P_n\|_{H^0} \le \|P_n\|_{A^0} \le \|P_n\|_{H^0}.$$

Equality holds in the lower estimate if and only if $P_n(z) = cz^n$, $c \in \mathbb{C}$. The upper estimate turns into equality if and only if P_n has no zeros in $\mathbb{D}$.

We state the following generalization of Theorem 2 for the weighted Bergman space.

Theorem 7. *Suppose that $\Lambda_n \in \mathbb{C}_n[z]$ and $P_n \in \mathbb{C}_n[z]$. If $\Lambda P_n \in \mathbb{C}_n[z]$ is defined by (1), then*

$$\|\Lambda P_n\|_{A^p_w} \le \|\Lambda_n\|_{H^0}\|P_n\|_{A^p_w}, \quad 0 \le p \le \infty. \tag{10}$$

Note that equality holds in (10) for any polynomial $P_n \in \mathbb{C}_n[z]$ when $\Lambda_n(z) = (1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k$, because $\Lambda P_n \equiv P_n$ and $\|(1+z)^n\|_{H^0} = 1$. This result allows to treat many problems in a unified way, and it has numerous interesting consequences.

We start with the following version of the Bernstein inequality for derivative of a polynomial in Bergman spaces.

Theorem 8. *For any $P_n \in \mathbb{C}_n[z]$, we have that*

$$\|zP_n'\|_{A_w^p} \le n\|P_n\|_{A_w^p}, \qquad 0 \le p < \infty.$$

If $p > 0$, then equality holds here only for polynomials on the form $P_n(z) = cz^n$, $c \in \mathbb{C}$. The same is true for $p = 0$ provided $0 \in \operatorname{supp} w$.

By writing $0 \in \operatorname{supp} w$ we mean that $\iint_{|z|<\varepsilon} w\, dA > 0$ for all $\varepsilon > 0$, which is the same as $\int_0^\varepsilon w(r)\, dr > 0 \ \forall\, \varepsilon > 0$. While the set of extremal polynomials remains the same, note the difference in the left-hand side comparing to the classical H^p case. It is clear that the norms of H^∞ and A^∞ coincide, and that Theorem 8 reduces to Corollary 5 in this case.

Continuing in the parallel pattern to the results for H^p spaces, we turn to the lower bounds of the derivative for polynomials in Bergman norms. The approach used in Corollary 2 can be applied to produce a similar inequality for A_w^p (with the same constant c_n). But that inequality is not sharp even for $p = 0$ now, in contrast with Corollary 2. Instead, we follow different approaches to obtain the following estimates for A^p.

Theorem 9. *Any $P_n \in \mathbb{C}_n[z]$ satisfies*

$$\|P_n - P_n(0)\|_{A^\infty} \le \frac{n^{2/p-1}}{(2-p)^{1/p}}\, \|P_n'\|_{A^p}, \quad 1 \le p < 2, \tag{11}$$

$$\|P_n - P_n(0)\|_{A^\infty} \le \left(\sum_{k=1}^n \frac{1}{k}\right)^{1/2} \|P_n'\|_{A^2} \le \sqrt{1+\log n}\, \|P_n'\|_{A^2}, \quad n \in \mathbb{N}, \tag{12}$$

and

$$\|P_n - P_n(0)\|_{A^\infty} \le \frac{p}{p-2}\, \|P_n'\|_{A^p}, \quad p > 2. \tag{13}$$

Note that the first inequality in (12) turns into equality for $Q_n(z) = \sum_{k=1}^n z^k/k$, as

$$\|Q_n - Q_n(0)\|_{A^\infty} = \sum_{k=1}^n \frac{1}{k} \quad \text{and} \quad \|Q_n'\|_{A^2} = \left(\sum_{k=1}^n \frac{1}{k}\right)^{1/2}.$$

We also show in the proof of Theorem 9 that the exponent of n in (11) is optimal.

Theorem 7 implies, among many other results, that z^n has the smallest Bergman space norm among all monic polynomials.

Corollary 9. *If $P_n \in \mathbb{C}_n[z]$ is a monic polynomial, then*

$$\|P_n\|_{A_w^p} \ge \|z^n\|_{A_w^p}, \quad 0 \le p < \infty. \tag{14}$$

If $p > 0$, then equality holds above only for $P_n(z) = z^n$. This is also true for $p = 0$ provided $0 \in \operatorname{supp} w$.

For $w \equiv 1$ we have

$$\|z^n\|_{A^p} = \begin{cases} e^{-n/2}, & p = 0, \\ \left(\dfrac{2}{pn+2}\right)^{1/p}, & 0 < p < \infty. \end{cases}$$

Since $\|P_n\|_{A^\infty} = \|P_n\|_{H^\infty}$, the inequality $\|P_n\|_{A^\infty} \geq \|z^n\|_\infty = 1$ is well known, see Corollary 6 and [5, 18].

Another useful estimate compares norms on the concentric disks $D_R := \{z : |z| < R\}$ to that on the unit disk.

Corollary 10. *If $P_n \in \mathbb{C}_n[z]$ and $R \geq 1$, then*

$$\left(\frac{1}{\pi R^2}\iint_{D_R} |P_n(z)|^p \, dA(z)\right)^{1/p} \leq R^n \, \|P_n\|_{A^p}, \qquad p \in (0, \infty),$$

and

$$\exp\left(\frac{1}{\pi R^2}\iint_{D_R} \log|P_n(z)| \, dA(z)\right) \leq R^n \, \|P_n\|_{A^0},$$

where equality holds for $P_n(z) = z^n$.

Again, in the case $p = \infty$, it is already known that $\max_{z\in D_R} |P_n(z)| \leq R^n \|P_n\|_\infty$ (cf. Corollary 7 and [18]).

Another consequence relates $\|P_n\|_p$ to the coefficients of P_n.

Corollary 11. *If $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$, then*

$$|a_k| \leq \left(\|z^k\|_{A^p_w}\right)^{-1} \binom{n}{k} \|P_n\|_{A^p_w}, \quad k = 0, \ldots, n, \quad 0 \leq p < \infty.$$

If $w \equiv 1$, then we have

$$|a_k| \leq \left(\frac{pk+2}{2}\right)^{1/p} \binom{n}{k} \|P_n\|_{A^p}, \quad k = 0, \ldots, n, \quad 0 < p < \infty,$$

and

$$|a_k| \leq e^{k/2} \binom{n}{k} \|P_n\|_{A^0}, \quad k = 0, \ldots, n.$$

If $k = 0$ or $k = n$, then the estimates of Corollary 11 are sharp for the corresponding monomials, but this is not generally so because binomial coefficients grow very fast with n. One can often improve the estimates of Corollary 11 by using the coefficient estimates for general functions from Bergman spaces. For example, the result of Horowitz (cf. [8, p. 81]) states that for any $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A^p$ we have

$$\left(\sum_{k=0}^{\infty} \frac{|a_k|^q}{(k+1)^{q-1}}\right)^{1/q} \le \|f\|_{A^p}, \quad 1 < p \le 2,\ q = p/(p-1). \tag{15}$$

It is certainly possible to extend the list of corollaries by choosing appropriate coefficient multipliers through the polynomials Λ_n.

A somewhat different kind of inequalities are related to comparing the norms of polynomials in Hardy and Bergman spaces. It is well known [8, 13] that for any function $f \in H^p$ we have

$$\|f\|_{A^p} \le \|f\|_{H^p}, \qquad 0 \le p \le \infty.$$

Clearly, we have equality for $p = \infty$. One can prove inequalities for polynomials in the opposite direction, of the form

$$\|P_n\|_{H^p} \le C(n,p)\, \|P_n\|_{A^p}.$$

For example, we have by Corollary 8 that

$$\|P_n\|_{H^0} \le e^{n/2}\, \|P_n\|_{A^0},$$

where equality holds for $P_n(z) = z^n$.

The case $p = 2$ is easy to handle, because

$$\|P_n\|_{H^2}^2 = \sum_{k=0}^{n} |a_k|^2 \le (n+1) \sum_{k=0}^{n} \frac{|a_k|^2}{k+1} = (n+1)\, \|P_n\|_{A^2}^2,$$

where $P_n(z) = \sum_{k=0}^{n} a_k z^k$. Hence we obtain that

$$\|P_n\|_{H^2} \le \sqrt{n+1}\, \|P_n\|_{A^2}, \quad P_n \in \mathbb{C}_n[z],$$

with equality for $P_n(z) = z^n$. It is plausible that more generally

$$\|P_n\|_{H^p} \le (pn/2+1)^{1/p}\, \|P_n\|_{A^p}, \quad 0 < p < \infty,$$

with equality for $P_n(z) = z^n$.

Estimates for the Bergman space norms of zero-free polynomials in the unit disk are not available to the best of our knowledge. We give a bound for the derivative of a polynomial without zeros in the unit disk that generalizes Theorem 5.

Theorem 10. *If $P_n \in \mathbb{C}_n[z]$ has no zeros in the unit disk, then*

$$\left(\frac{1}{b_w}\int_0^{2\pi}\int_0^1 |P_n'(re^{i\theta})|^p \,\|rz+1\|_{H^p}^p\, w(r) r\, dr d\theta\right)^{1/p} \le n\,\|P_n\|_{A_w^p},\ 0<p<\infty,$$

where $b_w = \iint_{\mathbb{D}} w\, dA$. In particular, we have

$$\|P_n'\|_{A_w^p} \le n\,\|P_n\|_{A_w^p}, \quad 0 \le p \le \infty.$$

It is a peculiar fact that the original form of the Bernstein inequality holds for the zero-free polynomials in this case. However, the above estimates are not sharp, see the proof of Theorem 10.

3 Proofs

We prove all new results and also selected known results where reasonably concise proofs can be given. In particular, the proofs of Theorems 1 and 2 are not included, and may be, respectively, found in the original papers of de Bruijn and Springer [6] and of Arestov [1]. An alternative exposition of methods that include a proof of Theorem 2 is contained in Sect. 13.2 of [18]. Proofs of Corollaries 1–4 are already outlined in Sect. 1. We start with characterization of all extremal polynomials in Corollary 1 by the location of their zeros in the closed unit disk. We are not aware of this observation made previously in the literature.

Proof of Corollary 1. We present an alternative proof of this corollary, independent of Theorem 1, that gives a description of all polynomials achieving equality. Consider any $P_n(z) = a_n \prod_{k=1}^n (z-z_k) \in \mathbb{C}_n[z],\ a_n \ne 0$, and note that the inequality of Corollary 1 is equivalent to the following

$$\log \frac{\|P_n'\|_{H^0}}{\|P_n\|_{H^0}} = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{|P_n'(e^{i\theta})|}{|P_n(e^{i\theta})|}\, d\theta \le \log n.$$

On the other hand, we have that

$$\frac{1}{2\pi}\int_0^{2\pi} \log \left|\frac{P_n'(e^{i\theta})}{P_n(e^{i\theta})}\right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log \left|\sum_{k=1}^n \frac{1}{e^{i\theta}-z_k}\right| d\theta.$$

Denote the above expression by $u(z_1,\dots,z_n)$, and observe that it is a continuous function of the roots $z_k \in \mathbb{C}$. Moreover, u is subharmonic in each $z_k \in \mathbb{D}$,

$k = 1, \dots, n$, by Theorem 2.4.8 of [19, p. 38]. It is also subharmonic in each variable in $\Omega := \{z \in \mathbb{C} : |z| > 1\}$. Applying the maximum principle for u with respect to every variable z_k in the domains $\mathbb{D}$ and Ω, we obtain that the largest value of u is attained for a polynomial $Q_n(z) = b_n z^n + \dots$ with all roots $w_k, k = 1, \dots, n$, located on the unit circumference. But we can explicitly find that $\|Q_n\|_{H^0} = |b_n|$ for such an extremal polynomial. Since all zeros of Q_n' are contained in the closed unit disk by the Gauss-Lukas Theorem, we also find that $\|Q_n'\|_{H^0} = n|b_n|$. Thus the largest value of u is $\log n$ for all n-tuples of points $\{z_k\}_{k=1}^n$, i.e., for all polynomials P_n. Furthermore, the same argument gives that $\|P_n\|_{H^0} = |a_n|$ and $\|P_n'\|_{H^0} = n|a_n|$ for any polynomial P_n with all zeros in the closed unit disk, so that equality holds in Corollary 1 as claimed. If P_n has a zero in Ω, then we have a strict inequality. Indeed, assume to the contrary that $z_n \in \Omega$ and $u(z_1 \dots, z_n) = \log n$. Since u is subharmonic and achieves maximum in Ω, it must be constant with respect to $z_n \in \Omega$. Letting $z_n \to \infty$ (and keeping other roots fixed), we now have that

$$\begin{aligned} \log n = \lim_{z_n \to \infty} u(z_1, \dots, z_n) &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{k=1}^{n-1} \frac{1}{e^{i\theta} - z_k} \right| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R_{n-1}'(e^{i\theta})}{R_{n-1}(e^{i\theta})} \right| d\theta, \end{aligned}$$

where $R_{n-1}(z) = \prod_{k=1}^{n-1}(z - z_k)$ is of degree $n - 1$. This is in contradiction with the already proved inequality

$$\log \frac{\|R_{n-1}'\|_{H^0}}{\|R_{n-1}\|_{H^0}} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{R_{n-1}'(e^{i\theta})}{R_{n-1}(e^{i\theta})} \right| d\theta \le \log(n-1).$$

□

Proof of Corollary 5. Inequality (4) is obtained from (3) by using the polynomial

$$\Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k \binom{n}{k} z^k.$$

Indeed, given any polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$, we obtain from the definition of the Schur-Szegő composition in (1) that

$$\Lambda P_n(z) = \sum_{k=0}^n k a_k z^k = z P_n'(z).$$

Furthermore, it is immediate that $\|\Lambda_n\|_{H^0} = n$, so that (4) follows.

The uniqueness part for $0 < p < \infty$ is a consequence of Theorem 5 from [1], because the coefficients of $\Lambda_n(z) = \sum_{k=0}^n \lambda_k z^k$ satisfy $\lambda_n = n > \lambda_0 = 0$, and the function $\phi(u) = u^p$ clearly satisfies that $u\phi'(u)$ is strictly increasing. The case of $p = \infty$ is classical. Uniqueness is also explicitly discussed in Theorem 6 of [10].

□

Proof of Theorem 3. Let f be analytic in $\mathbb{D}$, with the derivative f' in the Hardy space H^1, and apply the Fejér-Riesz inequality, see [7, p. 46]. For any $r \in [0, 1]$ and $\theta \in [0, 2\pi)$, we obtain that

$$|f(re^{i\theta}) - f(0)| = \left| \int_0^r f'(te^{i\theta}) e^{i\theta}\, dt \right| \le \int_0^r |f'(te^{i\theta})|\, dt$$
$$\le \int_{-1}^1 |f'(te^{i\theta})|\, dt \le \frac{1}{2} \int_0^{2\pi} |f'(e^{i(\theta+\phi)})|\, d\phi = \pi \|f'\|_{H^1}.$$

It follows that

$$\|f - f(0)\|_{H^\infty} \le \pi \|f'\|_{H^1},$$

which contains (5) as $P_n' \in H^1$ for any polynomial $P_n \in \mathbb{C}_n[z]$.

We now prove that the constant π in the above inequality and in (5) is sharp. Consider the conformal mapping ψ of the unit disk $\mathbb{D}$ onto the rectangle $R := (-\varepsilon, 1) \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, that satisfies $\psi(0) = 0$ and $\psi'(0) > 0$. It is easy to see that $\|\psi - \psi(0)\|_{H^\infty} = \|\psi\|_{H^\infty} = \sqrt{1 + \varepsilon^2}$. Moreover, the perimeter of R is expressed as

$$2 + 6\varepsilon = \int_0^{2\pi} |\psi'(e^{i\theta})|\, d\theta = 2\pi \|\psi'\|_{H^1}.$$

Hence we have that

$$\lim_{\varepsilon \to 0} \frac{\|\psi - \psi(0)\|_{H^\infty}}{\pi \|\psi'\|_{H^1}} = \lim_{\varepsilon \to 0} \frac{\sqrt{1 + \varepsilon^2}}{1 + 3\varepsilon} = 1,$$

which shows asymptotic sharpness for $f = \psi$ as $\varepsilon \to 0$. On the other hand, polynomials are dense in H^1, and there is a sequence of polynomials $Q_n \in \mathbb{C}_n[z]$ such that $\|\psi' - Q_n'\|_{H^1} \to 0$ as $n \to \infty$. The Fejér-Riesz inequality again gives

$$\|\psi - Q_n - (\psi(0) - Q_n(0))\|_{H^\infty} \le \pi \|\psi' - Q_n'\|_{H^1} \to 0 \quad \text{as } n \to \infty.$$

Thus $\lim_{n\to\infty} \|Q_n'\|_{H^1} = \|\psi'\|_{H^1}$ and $\lim_{n\to\infty} \|Q_n - Q_n(0)\|_{H^\infty} = \|\psi - \psi(0)\|_{H^\infty}$, so that (5) is asymptotically sharp for Q_n as $n \to \infty$.

We obtain (6) from (5) with the help of the Nikolskii-type inequality [18, p. 463]:

$$\|P_n'\|_{H^1} \le ((n-1)\lceil p/2 \rceil + 1)^{1/p-1} \|P_n'\|_{H^p} = n^{1/p-1} \|P_n'\|_{H^p}, \quad 0 < p < 1,$$

where $\lceil \cdot \rceil$ is the standard ceiling function. □

Proof of Corollary 6. Let $\Lambda_n(z) = z^n$, so that for any monic polynomial $P_n \in \mathbb{C}_n[z]$ we have the Schur-Szegő composition $\Lambda P_n(z) = z^n$. Since $\|\Lambda_n\|_{H^0} = 1$, (7) follows from (3).

The uniqueness part for $0 < p < \infty$ follows from Theorem 5 of [1], because the coefficients of Λ_n satisfy $\lambda_n = 1 > \lambda_0 = 0$, and the function $\phi(u) = u^p$ satisfies that $u\phi'(u)$ is strictly increasing. If $p = \infty$, then uniqueness of the extremal polynomial in (7) is the content of Tonelli's theorem [23, p. 72]. □

Proof of Corollary 7. For $\Lambda_n(z) = (Rz + 1)^n = \sum_{k=0}^n \binom{k}{n} R^k z^k$ and $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}_n[z]$, we have that

$$\Lambda P_n(z) = \sum_{k=0}^{n} a_k R^k z^k = P_n(Rz).$$

Note that $\|\Lambda_n\|_{H^0} = R^n$, because the only root of Λ_n is in $\mathbb{D}$. Thus (8) follows from (3). The case of equality is again a consequence of Theorem 5 of [1], as $\lambda_n = R^n > \lambda_0 = 1$. □

Proof of Theorem 4. We apply Theorem 2 and the definition of the Schur-Szegő composition to obtain that

$$\|P_n * Q_n\|_{H^p} = \left\| \sum_{k=0}^{n} a_k b_k z^k \right\|_{H^p} \le \left\| \sum_{k=0}^{n} \binom{n}{k} a_k z^k \right\|_{H^0} \|Q_n\|_{H^p}, \quad 0 \le p \le \infty.$$

Using Theorem 1 for the first factor on the right (or Theorem 2 again), we have

$$\left\| \sum_{k=0}^{n} \binom{n}{k} a_k z^k \right\|_{H^0} \le \left\| \sum_{k=0}^{n} \binom{n}{k}^2 z^k \right\|_{H^0} \left\| \sum_{k=0}^{n} a_k z^k \right\|_{H^0} = \|\Theta_n\|_{H^0} \|P_n\|_{H^0}.$$

The asymptotic value

$$\lim_{n\to\infty} \|\Theta_n\|_{H^0}^{1/n} \approx 3.20991230072\ldots$$

is found from the product of zeros of Θ_n outside the unit disk, see [22] for details and more precise asymptotic results. □

A proof of Theorem 5 may be found in [18, pp. 554–555].

Proof of Theorem 6. If P_n does not vanish in $\mathbb{D}$, then $\log |P_n(z)|$ is harmonic in $\mathbb{D}$. Hence $\|P_n\|_{H^0} = |a_0|$ and $\|P_n\|_{A^0} = |a_0|$ follow from the contour and the area mean value theorems, respectively. Assume now that P_n has zeros in $\mathbb{D}$. Applying Jensen's formula, we obtain that

$$\log \|P_n\|_{H^0} = \frac{1}{2\pi} \int_0^{2\pi} \log |P_n(e^{i\theta})| \, d\theta = \log |a_n| + \sum_{|z_j| \ge 1} \log |z_j|.$$

Furthermore,

$$\begin{aligned}\log \|P_n\|_{A^0} &= \frac{1}{\pi}\int_0^1 \int_0^{2\pi} \log |P_n(re^{i\theta})|\, rdrd\theta \\ &= 2\int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \log |P_n(re^{i\theta})|\, d\theta\right) rdr \\ &= 2\int_0^1 \left(\log |a_n| + \sum_{|z_j|\geq r} \log |z_j| + \sum_{|z_j|<r} \log r\right) rdr \\ &= \log |a_n| + \sum_{|z_j|\geq 1} \log |z_j| + \frac{1}{2}\sum_{|z_j|<1} (|z_j|^2 - 1).\end{aligned}$$

Hence

$$\|P_n\|_{A^0} = \|P_n\|_{H^0} \exp\left(\frac{1}{2}\sum_{|z_j|<1} (|z_j|^2 - 1)\right).$$

□

Proof of Corollary 8. The lower bound for $\|P_n\|_{A^0}$ follows from (9) because the smallest value of the sum

$$\sum_{|z_j|<1} (|z_j|^2 - 1)$$

is equal to $-n$, which is achieved if and only if all $z_j = 0$. The largest value of this sum is clearly 0 iff all $|z_j| \geq 1$, giving us the upper bound. □

Proof of Theorem 7. Applying (2) to the polynomial $P_n(rz)$, $r \in [0, 1]$, instead of $P_n(z)$, we obtain that

$$\int_0^{2\pi} \log |\Lambda P_n(re^{i\theta})|\, d\theta \leq 2\pi \log \|\Lambda_n\|_{H^0} + \int_0^{2\pi} \log |P_n(re^{i\theta})|\, d\theta.$$

Next we integrate the above inequality with respect to $w(r)r\,dr$ from 0 to 1:

$$\begin{aligned}\int_0^{2\pi}\int_0^1 \log |\Lambda P_n(re^{i\theta})|\, w(r)r\,drd\theta &\leq \log \|\Lambda_n\|_{H^0}\, 2\pi \int_0^1 w(r)r\,dr \\ &\quad + \int_0^{2\pi}\int_0^1 \log |P_n(re^{i\theta})|\, w(r)r\,drd\theta.\end{aligned}$$

Dividing by $b_w = 2\pi \int_0^1 w(r)r\,dr$ and taking exponential, we prove (10) for $p = 0$. Similarly, we obtain from (3) that

$$\int_0^{2\pi} |\Lambda P_n(re^{i\theta})|^p \, d\theta \leq \|\Lambda_n\|_{H^0}^p \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta, \quad 0 < p < \infty,$$

which implies that

$$\int_0^{2\pi} \int_0^1 |\Lambda P_n(re^{i\theta})|^p \, w(r) r \, dr d\theta \leq \|\Lambda_n\|_{H^0}^p \int_0^{2\pi} \int_0^1 |P_n(re^{i\theta})|^p \; w(r) r \, dr d\theta.$$

Dividing by b_w and taking the power $1/p$, we now have (10) for $p > 0$. If $p = \infty$, then (10) follows from (3) again:

$$\begin{aligned}\|\Lambda P_n\|_{A_w^\infty} &= \sup_{\substack{0\le\theta<2\pi \\ 0\le r<1}} |\Lambda P_n(re^{i\theta})| w(r) \leq \sup_{\substack{0\le\theta<2\pi \\ 0\le r<1}} \|\Lambda_n\|_{H^0} \max_{0\le\theta<2\pi} |P_n(re^{i\theta})| \, w(r) \\ &= \|\Lambda_n\|_{H^0} \|P_n\|_{A_w^\infty}.\end{aligned}$$

□

Proof of Theorem 8. Observe that the derivative of P_n can be expressed in the form of the Schur-Szegő convolution as in the proof of Corollary 5:

$$zP_n'(z) = \Lambda P_n(z) \quad \text{with} \quad \Lambda_n(z) = nz(1+z)^{n-1} = \sum_{k=0}^n k \binom{n}{k} z^k.$$

Since $\|\Lambda_n\|_{H^0} = n$, the inequality of Theorem 8 follows from (10).

Turning to the case of equality in Theorem 8, we first let $p > 0$. We assume that

$$\int_0^{2\pi} \int_0^1 |rP_n'(re^{i\theta})|^p \, w(r) r \, dr d\theta = n^p \int_0^{2\pi} \int_0^1 |P_n(re^{i\theta})|^p \; w(r) r \, dr d\theta$$

holds for a polynomial P_n. Note that Corollary 5 applied to the polynomial $P_n(rz), r > 0$, gives that

$$\int_0^{2\pi} |rP_n'(re^{i\theta})|^p \, d\theta \leq n^p \int_0^{2\pi} |P_n(re^{i\theta})|^p \, d\theta.$$

Since we have equality for the area integrals over $\mathbb{D}$, we must also have equality in the latter inequality for almost every $r \in \operatorname{supp} w$. But this is only possible when $P_n(z) = cz^n, \; c \in \mathbb{C}$, by Corollary 5.

For $p = 0$, we argue in a similar fashion to show that

$$\int_0^{2\pi} \log |rP_n'(re^{i\theta})| \, d\theta = 2\pi \log n + \int_0^{2\pi} \log |P_n(re^{i\theta})| \, d\theta$$

holds for almost every $r \in \operatorname{supp} w$, provided we have equality in Theorem 8. It follows that the family of polynomials $Q_n(z) = P_n(rz)$ is extremal in Corollary 1 for all such r. Hence $P_n(rz)$ has all zeros in the closure of $\mathbb{D}$, while $P_n(z)$ has all zeros in $\{z \in \mathbb{C} : |z| \leq r\}$. Since this holds for a sequence of radii $r \to 0$ such that $r \in \operatorname{supp} w$, we conclude that all zeros of P_n are at the origin. □

Proof of Theorem 9. We start with the case $1 < p \leq 2$. Let $P_n(z) = \sum_{k=0}^n a_k z^k$, so that $P_n'(z) = \sum_{k=1}^n k a_k z^{k-1}$. Applying Theorem 2 of [8, p. 81] [also see (15)], we obtain that

$$\|P_n'\|_{A^p} \geq \left(\sum_{k=1}^n \frac{k^q |a_k|^q}{k^{q-1}}\right)^{1/q} = \left(\sum_{k=1}^n k|a_k|^q\right)^{1/q}, \quad 1 < p \leq 2,\ q = p/(p-1).$$

Using this inequality together with Hölder's inequality, we estimate

$$\begin{aligned}\|P_n - P_n(0)\|_{A^\infty} &\leq \sum_{k=1}^n k^{1/q}|a_k| k^{-1/q} \leq \left(\sum_{k=1}^n k|a_k|^q\right)^{1/q} \left(\sum_{k=1}^n k^{-p/q}\right)^{1/p} \\ &\leq \|P_n'\|_{A^p} \left(\sum_{k=1}^n k^{1-p}\right)^{1/p} \leq \|P_n'\|_{A^p} \left(1 + \int_1^n x^{1-p}\right)^{1/p}.\end{aligned}$$

Evaluating the latter integral, we arrive at (11) and (12). The case $p = 1$ in (11) is obtained by letting $p \to 1+$.

We now show that the exponent of n in (11) is sharp. Consider the polynomial

$$Q_{2n-1}(z) = \int_0^z \left(\sum_{k=1}^n k t^{k-1}\right)^2 dt, \quad \deg(Q_{2n-1}) = 2n-1.$$

The second part of Theorem 2 in [8, p. 81] states a reverse inequality to (15) for $p \geq 2$. Although $p \in (1, 2]$ in our case, we use this fact for $2p \in (2, 4]$ and $r = 2p/(2p-1)$ to estimate that

$$\begin{aligned}\|Q_{2n-1}'\|_{A^p}^p &= \left\| \left(\sum_{k=0}^n k z^{k-1}\right)^2 \right\|_{A^p}^p = \left\| \sum_{k=0}^n k z^{k-1} \right\|_{A^{2p}}^{2p} \leq \left(\sum_{k=1}^n \frac{k^r}{k^{r-1}}\right)^{2p/r} \\ &= \left(\sum_{k=1}^n k\right)^{2p-1} = \left(\frac{n(n+1)}{2}\right)^{2p-1}.\end{aligned}$$

Hence the right-hand side of (11) for Q_{2n-1} is of the order $O(n^3)$. Note that both Q_{2n-1} and its derivative have positive coefficients. This immediately implies that

$$\|Q_{2n-1}\|_{A^\infty} = Q_{2n-1}(1) = \int_0^1 \left(\sum_{k=1}^n k t^{k-1}\right)^2 dt.$$

Given any polynomial $P_m(z) = \sum_{k=0}^m a_k z^k$ of degree m with positive coefficients, we have that

$$\int_0^1 P_m(x)\,dx = \sum_{k=0}^m \frac{a_k}{k+1} \ge \frac{P_m(1)}{m+1}.$$

The latter inequality applied to Q'_{2n-1} gives that

$$\|Q_{2n-1}\|_{A^\infty} = \int_0^1 \left(\sum_{k=1}^n k t^{k-1}\right)^2 dt \ge \frac{1}{2n-1}\left(\sum_{k=1}^n k\right)^2 = \frac{1}{2n-1}\left(\sum_{k=1}^n \frac{n(n+1)}{2}\right)^2.$$

Hence the left-hand side of (11) for Q_{2n-1} grows like n^3 as $n \to \infty$, matching the right-hand side.

Turning to the case $p > 2$, we apply the area submean inequality for a subharmonic function $|P'_n(z)|^p$ on the disk $\{t \in \mathbb{C} : |t-z| < 1-|z|\}$ contained in $\mathbb{D}$ for any $z \in \mathbb{D}$:

$$|P'_n(z)|^p \le \frac{1}{\pi(1-|z|)^2}\int_{\{|t-z|<1-|z|\}} |P'_n(t)|^p\,dA(t) \le \frac{\|P'_n\|_{A^p}^p}{(1-|z|)^2}, \quad z \in \mathbb{D}.$$

Hence (13) follows from

$$|P_n(e^{i\theta}) - P_n(0)| \le \int_0^1 |P'_n(re^{i\theta})|\,dr \le \|P'_n\|_{A^p}\int_0^1 (1-r)^{-2/p}\,dr = \frac{p}{p-2}\|P'_n\|_{A^p}.$$

□

Proof of Corollary 9. Consider any monic polynomial $P_n(z) = z^n + \ldots$ and the multiplier polynomial $\Lambda_n(z) = z^n$. Then the Schur-Szegő convolution is given by $\Lambda P_n(z) = z^n$. Hence (14) follows from (10). Equality holds trivially for $P_n(z) = z^n$, and we now show that this is the only extremal polynomial. Assume first that $p > 0$. Equality in (14) is equivalent to

$$\int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p\,w(r) r\,drd\theta = \int_0^{2\pi}\int_0^1 r^{np}\,w(r) r\,drd\theta$$

holding for a monic polynomial P_n. Corollary 6 gives by a scaling change of variable that

$$\frac{1}{2\pi}\int_0^{2\pi} |P_n(re^{i\theta})|^p\,d\theta \ge r^{np}, \quad 0 \le r \le 1,$$

for any monic polynomial $P_n \in \mathbb{C}_n[z]$, with equality only for $P_n(z) = z^n$. Hence equality in (14) implies that equality must hold in the above inequality for a.e. $r \in$

supp w, which means that $P_n(z) = z^n$ by Corollary 6. The case $p = 0$ is handled similarly. It is immediate to see that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta \ge \log r^n, \quad 0 \le r \le 1,$$

for any monic polynomial $P_n \in \mathbb{C}_n[z]$, with equality only if all zeros of $P_n(rz)$ are in the closed unit disk. Equality in (14) for $p = 0$ can be written as

$$\int_0^{2\pi}\int_0^1 \log|P_n(re^{i\theta})|\,w(r)r\,drd\theta = \int_0^{2\pi}\int_0^1 \log r^n\,w(r)r\,drd\theta$$

for a monic polynomial P_n, which implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\,d\theta = \log r^n$$

for almost every $r \in \operatorname{supp} w$. Thus $P_n(rz)$ has all zeros in the closure of $\mathbb{D}$, and $P_n(z)$ has all zeros in $\{z \in \mathbb{C} : |z| \le r\}$ for a sequence of radii $r \to 0$ such that $r \in \operatorname{supp} w$. It follows that $P_n(z) = z^n$.

The values of $\|z^n\|_{A^p}$ given in this corollary are found by a routine computation. □

Proof of Corollary 10. Let $\Lambda_n(z) = (1 + Rz)^n = \sum_{k=0}^n \binom{n}{k} R^k z^k$. Then $\Lambda P_n(z) = P_n(Rz)$ and $\|\Lambda_n\|_{H^0} = R^n$. Hence (10) gives that

$$\|P_n(Rz)\|_{A^p} \le R^n \|P_n\|_{A^p}, \quad 0 \le p < \infty,$$

for any $R \ge 1$. Changing variable and passing to the integral over D_R, we obtain that

$$\|P_n(Rz)\|_{A^p} = \left(\frac{1}{\pi R^2}\iint_{D_R} |P_n(z)|^p\,dA(z)\right)^{1/p}, \quad 0 \le p < \infty,$$

and

$$\|P_n(Rz)\|_{A^0} = \exp\left(\frac{1}{\pi R^2}\iint_{D_R} \log|P_n(z)|\,dA(z)\right), \quad p = 0.$$

The case of equality for $P_n(z) = z^n$ is verified by the same substitution. □

Proof of Corollary 11. Let $\Lambda_n(z) = \binom{n}{k} z^k,\ 0 \le k \le n$. Then $\Lambda P_n(z) = a_k z^k$ and $\|\Lambda_n\|_{H^0} = \binom{n}{k}$. It follows from (10) that

$$|a_k|\|z^k\|_{A^p_w} = \|a_k z^k\|_{A^p_w} \le \binom{n}{k} \|P_n\|_{A^p_w}, \quad 0 \le p \le \infty.$$

If $w \equiv 1$, then we can use explicit values of $\|z^k\|_{A^p_w}$ as given in Corollary 9 to obtain the last two inequalities of Corollary 11. $\square$

Proof of Theorem 10. We recall the following estimate for a polynomial P_n without zeros in the disk $\{z \in \mathbb{C} : |z| < R\},\ R \geq 1$:

$$\|P_n'\|_{H^p} \leq \frac{n}{\|z+R\|_{H^p}} \|P_n\|_{H^p},\ 0 \leq p \leq \infty.$$

This extension of Theorem 5 was originally proved by Govil and Rahman [11] for $p \geq 1$, and later by Aziz and Shah [4] for any $p > 0$. While equality may hold in Theorem 5 as explained after its statement, the above inequality cannot turn into equality for any P_n without zeros in the disk $\{z \in \mathbb{C} : |z| < R\},\ R > 1$. The cases $p = 0$ and $p = \infty$ follow immediately by taking limits as $p \to 0$ and $p \to \infty$. We apply the stated result to the family of polynomials $P_n(rz),\ r \in (0, 1]$. It is clear that if P_n is zero-free in $\mathbb{D}$, then $P_n(rz)$ has no zeros in the disk $\{z \in \mathbb{C} : |z| < 1/r\},\ r \in (0, 1]$. Hence

$$\int_0^{2\pi} |rP_n'(re^{i\theta})|^p\, d\theta \leq \frac{n^p}{\|z+1/r\|^p_{H^p}} \int_0^{2\pi} |P_n(re^{i\theta})|^p\, d\theta,\ 0 < p < \infty.$$

Simplifying, we obtain that

$$\|rz+1\|^p_{H^p} \int_0^{2\pi} |P_n'(re^{i\theta})|^p\, d\theta \leq n^p \int_0^{2\pi} |P_n(re^{i\theta})|^p\, d\theta,\ 0 < p < \infty.$$

We now integrate the above inequality with respect to $w(r)r\,dr$ from 0 to 1:

$$\int_0^{2\pi}\int_0^1 |P_n'(re^{i\theta})|^p\, \|rz+1\|^p_{H^p}\, w(r)r\,drd\theta \leq n^p \int_0^{2\pi}\int_0^1 |P_n(re^{i\theta})|^p\, w(r)r\,drd\theta.$$

Thus the first inequality follows for $p \in (0, \infty)$. It remains to observe that $\|rz + 1\|^p_{H^p} \geq 1$ by the submean inequality for the subharmonic function $|rz+1|^p$, so that the second inequality is a consequence of the first one for $p \in (0, \infty)$. The endpoints are handled by the standard limits as $p \to 0$ and $p \to \infty$. $\square$

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Some Rational Inequalities Inspired by Rahman's Research

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Abstract This paper describes three instances of our research activity in rational inequalities inspired by Professor Rahman's research. The results include Bernstein-type inequalities for rational functions with prescribed poles, comparison inequalities for rational functions, and integral inequalities with prescribed poles and prescribed zeros.

Keywords Bernstein-type inequality • Polar derivative • Zeros of a polynomial/Rational functions

2000 Mathematics Subject Classification: 30A10, 30C10, 30E10, 30C15

1 General Introduction

Professor Q.I. Rahman developed many novel techniques to solve extremal problems arising in polynomials, rational functions, and entire functions of exponential type. He used tools from variational principles, optimization methods, duality theory, subordination technique, etc. Over the years many researchers have used the methods developed by Rahman and his collaborators (See references [1] through [32]). In this paper, we describe how his research inspired us to obtain rational inequalities.

Let $\mathscr{P}_n$ denote the complex algebraic polynomials of degree at most n (with complex coefficients). Let $a_j \in \mathbb{C}, j = 1, 2, \ldots, n$, be n fixed numbers with moduli larger than 1: $|a_j| > 1$ ($j = 1, 2, \ldots, n$). Here $\mathbb{C}$ denotes the set of all complex numbers. Define $w_n(z) = \prod_{j=1}^{n}(z - a_j)$ and

$$\mathscr{R}_n = \left\{ \frac{p(z)}{w_n(z)} \mid p \in \mathscr{P}_n \right\}.$$

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Then $\mathscr{R}_n$ is the set of rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at infinity. We shall also write

$$B_n(z) = \prod_{j=1}^{n} \frac{1 - \overline{a_j} z}{z - a_j}.$$

Let $\mathbb{T} := \{z : |z| = 1\}$ be the unit circle in $\mathbb{C}$ and for a function f defined on $\mathbb{T}$, we set $\|f\| = \sup_{|z|=1} |f(z)|$. Also let $D^- = \{z : |z| < 1\}$ and $D^+ = \{z : |z| > 1\}$, respectively, be the inside and outside of the unit circle. The well-known Bernstein inequality is the following:

Theorem 1. *If $p \in \mathscr{P}_n$, then*

$$\|p'\| \leq n\|p\|. \tag{1}$$

For real algebraic polynomials of degree n on the interval $[-1, 1]$ Markoff inequality can be written as:

Theorem 2. *If $p(x)$ is a real polynomial of degree at most n,*

$$\max_{-1 \leq x \leq 1} |p'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |p(x)|. \tag{2}$$

Duffin and Schafer [9] showed that, in (2), $\max_{-1 \leq x \leq 1} |p(x)|$ can be replaced by the maximum of $|p(x)|$ in the extrema $\{\cos(k\pi/n)\}_{k=0}^{n}$ of the nth Chebyshev polynomial of the first kind. Motivated by the above, Frappier et al. [10, Theorem 8, p. 89] proved the following improvement of Theorem 1.

Theorem 3. *If $p \in \mathscr{P}_n$, then*

$$\|p'\| \leq n \max_{1 \leq k \leq 2n} |p(e^{k\pi i/n})|. \tag{3}$$

That is, in (1), $\|p\|$ may be replaced by the maximum of $|p(z)|$ in the $2n$th roots of unity. On the other hand, the maximum in the $(n+m)$th roots of unity with $m < n$ does not suffice.

It is natural to ask if there exist any polynomial $p \in \mathscr{P}_n$ for which $\|p\| > \max_{1 \leq k \leq 2n} |p(e^{k\pi i/n})|$. Rahman et al. [10, p. 95] provided the counterexample $p(z) = 1 + iz^n$.

Motivated by Theorem 3, Mohapatra et al. [25, Theorem B, p. 630] proved the following improvement of it.

Theorem 4. *Let $z_1, z_2, \ldots, z_{2n}$ be any $2n$ equally spaced points on $\mathbb{T}$ in order, say $z_k = ue^{k\pi i/n}$, where $|u| = 1$ and $k = 1, 2, \ldots 2n$. If $p \in \mathscr{P}_n$, then*

$$\|p'\| \leq \frac{n}{2}\left\{\max_{k\ odd} |p(z_k)| + \max_{k\ even} |p(z_k)|\right\}.$$

The rational analogue of Theorem 4 was proved by Li et al. in [20], which we will discuss in the next section. It is needless to say that the rational analogue of Theorem 4 had its motivation in Theorem 3 of Frappier et al.

2 Bernstein-Type Inequalities for Rational Functions with Prescribed Poles

In this section, we will state the rational analogue of Theorems 3 and 4, and also other related inequalities for rational functions. For reader's convenience, all stated results for rational functions will be proved in Sect. 7. Our first result is an identity which is the rational analogue of the result in Mohapatra et al. [25, p. 630] and which allows us to introduce the "sampling points" used in the discrete norms.

Theorem 5 ([20, Theorem 1]). *Suppose that $\lambda \in \mathbb{T}$. Then the following hold:*

(i) $B_n(z) = \lambda$ *has exactly n simple roots, say* $t_1, t_2, \ldots, t_n$, *and all lie on the unit circle,* $\mathbb{T}$;
(ii) *if* $r \in \mathscr{R}_n$ *and* $z \in \mathbb{T}$, *then*

$$B_n{}'(z)r(z) - r'(z)[B_n(z) - \lambda] = \frac{B_n(z)}{z} \sum_{k=1}^{n} c_k r(t_k) \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2, \tag{4}$$

where $c_k = c_k(\lambda)$ *is defined by*

$$c_k^{-1} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2}, \quad for \ \ k = 1, 2, \ldots, n. \tag{5}$$

The following interpolating formula for the derivative of rational functions is useful and follows easily from Theorem 5.

Theorem 6 ([20, Corollary 1]). *Let* c_k *and* t_k *(for* $k = 1, 2, \ldots, n$*) be as in Theorem 5 and let the n roots of* $B_n(z) = -\lambda$ *be* $s_1, s_2, \ldots, s_n$. *If* d_k *is defined as* c_k *in Eq. (5) with* t_k *replaced by* s_k, *for* $k = 1, 2, \ldots, n$, *then*

$$\frac{2\lambda z r'(z)}{B_n(z)} = \sum_{k=1}^{n} c_k r(t_k) \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2 - \sum_{k=1}^{n} d_k r(s_k) \left| \frac{B_n(z) + \lambda}{z - s_k} \right|^2. \tag{6}$$

Now the rational analogue of Theorems 3 and 4 can be stated as follows:

Theorem 7 ([20, Corollary 2]). *Let* t_k *and* s_k *be as in Theorem 6. Then*

$$|r'(z)| \leq \frac{1}{2}|B_n'(z)| \left[\max_{1 \leq k \leq n} |r(t_k)| + \max_{1 \leq k \leq n} |r(s_k)| \right]. \tag{7}$$

The inequality in (7) reduces to an equality when $r(z) = uB_n(z)$ *with* $u \in \mathbb{T}$.

By considering the class of polynomials having no zero in $|z| < 1$, P. Erdös conjectured and P.D. Lax proved the following Erdös–Lax inequality:

Theorem 8 ([17]). *If $p \in \mathscr{P}_n$ with $|p(z)| \leq 1$ on $z \in \mathbb{T}$ and $p(z)$ has no zero in D^-, then*

$$|p'(z)| \leq \frac{n}{2} \text{ for } z \in \mathbb{T}.$$

The result is best possible and the extremal polynomial is $p(z) = \alpha + \beta z^n/2$, where $|\alpha| = |\beta| = 1$.

It follows from inequality (7) the rational Bernstein inequality, (see [20]), for any $r \in \mathscr{R}_n$,

$$|r'(z)| \leq |B_n{}'(z)|\|r\| \text{ for } z \in \mathbb{T}, \tag{8}$$

where the equality holds for $r(z) = uB_n(z)$ with $u \in \mathbb{T}$. The rational version of the Erdös–Lax inequality was established in [20] as well.

Theorem 9 ([20, Theorem 3, p. 526]). *If $r \in \mathscr{R}_n$ has all its zeros in $\mathbb{T} \cup D^+$, the inequality (8) can be strengthened to*

$$|r'(z)| \leq \frac{1}{2}|B_n'(z)|\|r\| \text{ for } z \in \mathbb{T}. \tag{9}$$

Equality holds for $r(z) = \alpha B_n(z) + \beta$ with $|\alpha| = |\beta| = 1$.

Aziz and Shah [3] considered the rational functions not vanishing in D^- but using the discrete norm.

Theorem 10 ([3, Theorem 1]). *Let $r \in \mathscr{R}_n$ and all the zeros of r lie in $\mathbb{T} \cup D^+$. If t_k and s_k are as before, then for $z \in \mathbb{T}$,*

$$|r'(z)| \leq \frac{1}{2}|B_n'(z)| \left[\left(\max_{1\leq k\leq n} |r(t_k)| \right)^2 + \left(\max_{1\leq k\leq n} |r(s_k)| \right)^2 \right]^{1/2}.$$

Remark. From Theorem 10, Erdös–Lax result for polynomials cannot be deduced. Still we haven't seen either a polynomial or a rational analogue of Erdös–Lax inequality using the discrete norm. For reader's convenience, we will give a proof of Theorem 10 in Sect. 7 and state the following open problem.

Open Problem. Formulate and prove a sharp Erdos–Lax inequality using the discrete norm to replace $\|p\|$.

3 Comparison Inequalities

If p is a complex polynomial of degree at most n, then (1) can be written as

$$\|p'\| \leq \left|\frac{d(\|p\|z^n)}{dz}\right| \quad \text{at} \quad |z| = 1. \tag{10}$$

In fact, Bernstein in 1930 (see [31, p. 510, Theorem 14.1.2]) proved the following result.

Theorem 11. *If $F(z) = \sum_{k=0}^{n} A_k z^k$ is a polynomial of degree n with all its zeros in the closed unit disc. If in addition, $f(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$, then*

$$|f'(z)| \leq |F'(z)| \;\; \textit{for} \;\; 1 \leq |z| < \infty. \tag{11}$$

Equality in (11) holds at some point outside the closed unit disc if and only if $f(z) = e^{i\nu} F(z)$ for some real number ν.

In 1969, Rahman used such a comparison inequality to obtain derivatives of entire functions of exponential type (see Lemma 2 and Sect. 3.2 [29]). In Sect. 5 of that paper, Rahman introduced an operator preserving inequality between polynomials which has led to considerable research for inequalities between polar derivatives.

Although our interest in this paper is to write about the impact of this work on rational functions, it is worthwhile to quote one result of Rahman from [29] and mention a few more results of interest since they together motivate the rational version of the comparison inequalities to be discussed in the next section

Let

$$p(z) = \sum_{k=0}^{n} a_k z^k \tag{12}$$

and consider an operator B that carries $p(z)$ into $B[p(z)]$ given by

$$B[p(z)] = \lambda_0 p(z) + \lambda_1 \left(\frac{nz}{2}\right) p'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{p''(z)}{2!}, \tag{13}$$

where λ_0, λ_1, and λ_2 such that all the zeros of

$$u(z) = \lambda_0 + \binom{n}{1} \lambda_1 z + \binom{n}{2} \lambda_2 z^2 \tag{14}$$

lie in the half plane $|z| \leq |z - n/2|$.

Theorem 12 ([29, Theorem 4, p. 304]). *If $p \in \mathscr{P}_n$, then*

$$|p(z)| \leq M \ \ for \ \ z \in \mathbb{T}$$

implies

$$|B[p(z)]| \leq M|B[z^n]| \ \ for \ \ z \in \mathbb{T} \cup D^+. \tag{15}$$

In 1985, Malik and Vong [23] presented the following related result:

Theorem 13. *Assume that $p \in \mathscr{P}_m$ and $q \in \mathscr{P}_n$ are polynomials with $m \leq n$. If $q(z)$ has all its zeros in $|z| \leq 1$ and $|p(z)| \leq |q(z)|$ for $|z| = 1$, then*

$$\left|\frac{zp'(z)}{n} + \beta\frac{p(z)}{2}\right| \leq \left|\frac{zq'(z)}{n} + \beta\frac{q(z)}{2}\right|, \tag{16}$$

for all z such that $|z| = 1$ and $|\beta| \leq 1$.

Remark. If $\beta = 0$, then Theorem 13 reduces to Theorem 11.

Open Problem. Prove or disprove: *Assume that $p \in \mathscr{P}_m$ and $q \in \mathscr{P}_n$ are polynomials with $m \leq n$. If $q(z)$ has all its n zeros in $|z| \leq 1$ and $|p(z)| \leq |q(z)|$ for $|z| = 1$, then*

$$|B[p(z)]| \leq |B[q(z)]| \ \ for \ \ |z| > 1.$$

For a complex number α and for any $p \in \mathscr{P}_n$, the polar derivative of p with respect to α, $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

In 1988, Aziz [2] showed that for $p \in \mathscr{P}_n$, $p(z) \neq 0$ for $|z| < 1$,

$$|D_\alpha(p(z))| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1) \ \text{ for } \ |z| \geq 1.$$

In 1998, Aziz and Shah [4] extended the Bernstein inequality by proving

$$|D_\alpha(p(z))| \leq n|\alpha z^{n-1}|\|p\| \ \text{ for } \ |z| \geq 1.$$

Recently, for polar derivative, Liman et al. [21, Theorem 1] proved the following:

Theorem 14. *Let p and q as in Theorem 13. Then*

$$\left|zD_\alpha p(z) + n\beta\left(\frac{|\alpha| - 1}{2}\right)p(z)\right| \leq \left|zD_\alpha q(z) + n\beta\left(\frac{|\alpha| - 1}{2}\right)q(z)\right|,$$

for all z such that $|z| = 1$, $|\alpha| \geq 1$, $|\beta| \geq 1$.

Li [19] established some comparison inequalities for rational functions and deduced Theorems 13 and 14 as special cases which is discussed in the next section.

4 Comparison Inequality for Rational Functions

The following result gives a rational extension of Theorem 11

Theorem 15 ([19, Theorem 3.1]). *Let $r, s \in \mathscr{R}_n$ and assume that s has all its n zeros in $|z| \leq 1$. Further, let*

$$|r(z)| \leq |s(z)| \ \ for \ \ |z| = 1.$$

Then

$$|r'(z)| \leq |s'(z)| \ \ for \ \ |z| = 1. \tag{17}$$

Remark. Using $s(z) = B_n(z)$ in Theorem 15 will yield Bernstein inequality.

Theorem 15 is a special case of the following more general result.

Theorem 16 ([19, Theorem 3.2]). *Suppose $r, s \in \mathscr{R}_n$ and assume s has all its n zeros in $|z| \leq 1$ and*

$$|r(z)| \leq |s(z)| \ \ for \ \ |z| = 1.$$

Then, for any ρ satisfying $|\rho| \leq 1/2$,

$$|r'(z) + \rho {B_n}'(z) r(z)| \leq |s'(z) + \rho {B_n}'(z) s(z)| \ \ for \ \ |z| = 1. \tag{18}$$

Equality holds if $r(z) \equiv s(z)$.

Remark. Inequality (18) should be compared with the inequality (16) of Malik and Vong [23]. Here we mention some results which can be deduced from this theorem. We will provide a proof of Theorem 16 in Sect. 8.

Corollary 1. *If $p, q \in \mathscr{P}_n$ where q has all its zeros in $|z| \leq 1$ and*

$$|p(z)| \leq |q(z)| \ \ for \ \ |z| = 1,$$

then, for $|a| \geq 1$,

$$|D_a p(z)| \leq |D_a q(z)| \ \ for \ \ |z| = 1. \tag{19}$$

Proof. In Theorem 15, take $a_i = a$ for $i = 1, 2, \ldots, n$, for $|a| > 1$, with the rational functions $r(z) = p(z)/(z - a)^n$ and $s(z) = q(z)/(z - a)^n$. Both $r(z)$ and $s(z)$ have a pole of order n at $z = a$. By Theorem 15, we have

$$\left(\frac{p(z)}{(z-a)^n}\right)' \leq \left(\frac{q(z)}{(z-a)^n}\right)' \tag{20}$$

for all z such that $|z| = 1$. Note that

$$\left(\frac{p(z)}{(z-a)^n}\right)' = \frac{p'(z)}{(z-a)^n} - \frac{np(z)}{(z-a)^{n+1}} = \frac{-D_a p(z)}{(z-a)^{n+1}}. \tag{21}$$

Hence the inequality (19) can be deduced from the inequality (20) using (21) when $|a| > 1$. Letting $|a| \to 1$ gives the inequality for $|a| \geq 1$. □

Let $r, s \in \mathscr{R}_n$ be given by

$$r(z) = \frac{p(z)}{(z-a)^n} \text{ and } s(z) = \frac{p^*(z)}{(z-a)^n},$$

where $p^*(z) = z^n\overline{p(1/\overline{z})}$ with $p \in \mathscr{P}_n$ and $|a| \geq 1$. Now we can deduce:

Corollary 2 ([1, p. 190]). *For $p \in \mathscr{P}_n$ with its zeros in $|z| \geq 1$ and for any a with $|a| \geq 1$, we have*

$$|D_a p(z)| \leq |D_a p^*(z)| \;\; for \;\; |z| = 1. \tag{22}$$

Now Theorem 14 can be obtained from Theorem 16 by taking the poles $a_i = a, \;\; i = 1, 2, \ldots, n, \;\; |a| > 1$. In fact, we will get

$$\left|-\frac{D_a p(z)}{(z-a)^{n+1}} + \rho \frac{n(|a|^2-1)p(z)}{|z-a|^2\;(z-a)^n}\right| \leq \left|-\frac{D_a q(z)}{(z-a)^{n+1}} + \rho \frac{n(|a|^2-1)q(z)}{|z-a|^2\;(z-a)^n}\right|. \tag{23}$$

Next, by defining β such that

$$\beta\left(\frac{|a|-1}{2}\right) = \rho\,\frac{|a^2|-1}{(\overline{z}-\overline{a})}. \tag{24}$$

we see that Theorem 14 follows from Theorem 16.

Corollary 3 ([19, Corollary 3.5]). *Let $r, s \in \mathscr{R}_n$ and assume that s has n zeros lying in $|z| \leq 1$ and*

$$|r(z)| \leq |s(z)| \;\; for \;\; |z| = 1.$$

Then, for any $c, \;\; 0 < c < 1/3$,

$$|r'(z)| + c|(r^*)'(z)| \leq |s'(z)| + c|(s^*)'(z)| \;\; for \;\; |z| = 1. \tag{25}$$

Proof. By direct calculation (or see (22) in [20]) we get

$$|(r^*)'(z)| = |B_n{}'(z)r(z) - r'(z)B_n(z)| \ \text{ for } \ |z| = 1.$$

Let λ satisfy $|\lambda| \geq 3$. Let us take $\rho = -1/(B_n(z)+\lambda)$. Then $|\rho| = 1/|B_n(z) + \lambda| \leq 1/2$. Hence, Theorem 16 will yield

$$|\lambda r'(z) + (B_n(z)r'(z) - B_n{}'(z)r(z))| \leq |\lambda s'(z)| + (B_n(z)r'(z) - B_n{}'(z)r(z))|.$$

Now, choosing the argument of λ such that

$$|\lambda r'(z) + (B_n(z)r'(z) - B_n{}'(z)r(z))| = |\lambda r'(z)| + |B_n(z)r'(z) - B_n{}'(z)r(z)|$$

This will yield for $|z| = 1$,

$$\begin{aligned} |\lambda||r'(z)| + |(r^*)'(z)| &\leq |\lambda s'(z) + B_n(z)s'(z) - B_n{}'(z)s(z)| \\ &\leq |\lambda||s'(z)| + |(s^*)'(z)|. \end{aligned}$$

Taking $c = 1/|\lambda|$, we get the desired result for $|z| = 1$. □

Corollary 4 ([19, Corollary 3.6]). *Let $r \in \mathscr{R}_n$. Then the following hold:*

(i) *If r has n zeros all lying in $|z| \leq 1$, then for any ρ such that $|\rho| \leq 1/2$, then*

$$|(r^*)'(z) + \rho B_n{}'(z)r^*(z)| \leq |r'(z) + \rho B_n{}'(z)r(z)| \ \textit{ for } \ |z| = 1. \tag{26}$$

(ii) *If $r \in \mathscr{R}_n$ has all its zeros in $|z| \geq 1$, then*

$$|r'(z) + \rho B_n{}'(z)r(z)| \leq |(r^*)'(z) + \rho B_n{}'(z)r^*(z)| \ \textit{ for } \ |z| = 1. \tag{27}$$

Proof. (i) Since $|r^*(z)| = |r(z)|$ for $|z| = 1$ and zeros or r are in $|z| \leq 1$, by applying Theorem 16 with replacing $r(z)$ and $s(z)$, respectively, by $r^*(z)$ and $r(z)$ yield the desired result. (ii) Apply Theorem 16 by replacing $r(z)$ and $s(z)$, respectively, by $r(z)$ and $r^*(z)$ yield the desired result. □

Remarks. (1) It is easy to show that (i) and (ii) above are equivalent. (2) When $\rho = 0$, (ii) yields $|r'(z)| \leq |(r^*)'(z)|$ for $|z| = 1$ which is a known result from Li et al. [20].

This observation leads to:

Corollary 5 ([19, Corollary 3.7]). *Let $r \in \mathscr{R}_n$. Then the following hold:*

(i) *If r has n zeros lying in $|z| \leq 1$, then*

$$|(r^*)'(z)| \leq |r'(z)| \ \textit{ for } \ |z| = 1.$$

(ii) *If r has n zeros lying in $|z| \geq 1$, then*

$$|r'(z)| \leq |(r^*)'(z)| \;\; for \;\; |z| = 1.$$

5 An Inequality of De Bruijn

Rahman established many inequalities for entire functions of exponential type τ. Recall that a function $f(z)$ is an entire function of exponential type τ if for every $\epsilon > 0$, there exists a positive number $M_\epsilon > 0$ such that

$$|f(z)| \leq M_\epsilon e^{(\tau+\epsilon)|z|}, \quad z \in \mathbb{C},$$

as $|z| \to \infty$.

A model example of such functions is given by $p(e^{iz})$ for $p \in \mathscr{P}_n$ with $n \leq \tau$. In [29], Rahman gave a unified method for arriving at many familiar inequalities for entire functions of exponential type τ and their generalizations. He then used the same idea to give an alternative proof for an inequality of De Bruijn which improved Zygmund's version of Bernstein inequality using L^p-norm ($p > 1$).

Theorem 17 ([8, Theorem 13]). *If the polynomial $p \in \mathscr{P}_n$ has no zeros for $|z| < 1$, then for $\delta \geq 1$,*

$$\int_0^{2\pi} |p'(e^{i\theta})|^\delta \, d\theta \leq C_\delta n^\delta \int_0^{2\pi} |p_n(e^{i\theta})|^\delta \, d\theta$$

where $C_\delta = 2\pi / \int_0^{2\pi} |1 + e^{i\theta}|^\delta \, d\theta$.

Rahman's proof employed the reverse polynomials and basically established several important polynomial inequalities:

(1) The following generalized version of Bernstein's polynomial inequality on the unit circle: If $|p(z)| \leq M$ for $|z| = 1$, then, for $|z| = 1$,

$$|p'(z)| + |(p^*)'(z)| \leq Mn, \tag{28}$$

recalling that $p^*(z)$ is the reverse polynomial of $p(z)$ defined by $p^*(z) = z^n \overline{p(1/\overline{z})}$.

(2) If $p(z)$ has no zero in $|z| < 1$, then, for $|z| = 1$,

$$|p'(z)| \leq |(p^*)'(z)|. \tag{29}$$

The essence of Rahman's proof consists of the recognition of DeBruijn's use of an arbitrary constant of modulus larger than 1: Consider $R_n(z) = p(z) - \lambda M$ for $|\lambda| > 1$ then Rahman observed that $R_n(z)$ has no zero in $|z| < 1$ and obtained (by applying a result of De Bruijn), for $0 \leq \theta < 2\pi$,

$$|dp(e^{i\theta})/d\theta| \le |dp^*(e^{i\theta})/d\theta - ine^{in\theta}\overline{\lambda}M|.$$

Then by choosing $\arg(\lambda)$ suitably, we get

$$|dp(e^{i\theta})/d\theta| \le n|\lambda|M - |dp^*(e^{i\theta})/d\theta|,$$

which implies (28).

Inequality (28) was rediscovered in ([2, 15, 22]). But more importantly, the proof of Rahman has been used many times ([2, 12, 19, 22, 28]). One might call this *the method of Rahman* according to the criterion of Polya and Szego ("An idea which can be used only once is a trick. If one can use it more than once it becomes a method." [27, p. VIII, line 6]). Now, both inequalities in (28) and (29) have been extended to rational functions $r_n \in \mathscr{R}_n$ in [20, Theorems 2 and 3]. Indeed, Theorems 15 and 16 post another example of the successful application of the method of Rahman. (See the proof of Theorem 16 in Sect. 8.) In an effort to obtain an L^p-norm Bernstein inequality for rational functions that is an analog of De Bruijn's theorem, [18] established an integral representation for the derivatives of rational functions and used this integral representation to prove an inequality that allows poles on both sides of the unit circle. For comparison purpose, we state the special case when all poles are restricted to the outside of the unit circle: If $\psi(u)$ is a non-decreasing, non-negative, and convex function for $u \ge 0$, then the inequality

$$\int_0^{2\pi} \psi\left(\left|\frac{r'(z)}{B_n'(z)}\right|\right)|B_n'(z)||dz| \le \int_0^{2\pi} \psi\left(|r(z)|\right)|B_n'(z)||dz| \tag{30}$$

holds for any $r \in \mathscr{R}_n$. This inequality is sharp the sense that the equality holds if r is a multiple of B_n.

We now show that, in a special case (when all poles and all zeros are outside of the unit circle, and in L^p norm), Rahman's idea [29] above can be used to improve inequality (30) and obtain a new result for rational functions.

Theorem 18. *If $p \ge 1$ and if $r \in \mathscr{R}_n$ has all its zeros outside the unit circle, then the inequality*

$$\int_0^{2\pi} \left|\frac{r'(z)}{B_n'(z)}\right|^p |B_n'(z)||dz| \le C_p \int_0^{2\pi} |r(z)|^p\,|B_n'(z)||dz| \tag{31}$$

holds for any $r \in \mathscr{R}_n$, where $C_p = 2\pi / \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha$.

We will give the proof of this theorem in Sect. 9 after our discussion in Sect. 6 on the integral representation which is again inspired an integral formula given for the polynomial case in one of papers of Rahman (more precisely, [13]).

Remark. When all the poles are at ∞, we are reduced to the polynomial case and $B_n(z) = z^n$ and so $|B_n'(z)| = n$ for $|z| = 1$. Thus, the above inequality (31) becomes the inequality as in De Bruijn's theorem.

6 An Integral Formula

In this section we will give one more example of how useful the ideas in Rahman's work on polynomial inequalities can be used as guidelines when working with rational functions. The paper in mind is by Giroux and Rahman [13]. In this paper, they proved many polynomial inequalities that improve the Bernstein inequality when a zero of the polynomials is fixed. A model result in [13] is as follows.

Theorem 19. *Let $p \in \mathscr{P}_n$. If $\|p\| = 1$ and $p(1) = 0$, then*

$$\|p'\| < n - \frac{C}{n}, \tag{32}$$

and the inequality is sharp in the sense that there exists a $\tilde{p} \in \mathscr{P}_n$ such that $\|\tilde{p}\| = 1$, $\tilde{p}_n(1) = 0$, and

$$\|\tilde{p}'\| > n - \frac{c}{n},$$

where $c > 0$ and $C > 0$ are constants independent of n.

This paper motivated further research on improving Bernstein inequality for polynomials with a prescribed zero [11, 26]. We want to discuss the rational version of the above theorem in [16] and, in particular, we point out the influence of Giroux and Rahman [13] as well as another paper by Rahman and Stenger [30] in the key steps of the proofs in [16].

The rational version of the above theorem of Giroux and Rahman is the following results. Let $a \in [0, 1]$ and define a subset of $\mathscr{R}_n$ similarly as in [13] by

$$\mathscr{R}_{n,a} := \{r \in \mathscr{R}_n \mid \min_{|z|=1} |r(z)| \leq a\}.$$

Denote $m := \min_{|z|=1} |B_n'(z)|$ and $M := \max_{|z|=1} |B_n'(z)|$.

Theorem 20 ([16, Theorems 3.1 and 3.3]). *Assume $r \in \mathscr{R}_{n,a}$ and $\|r\| = 1$. Then*

$$|r'(z)| \leq |B_n'(z)| - \frac{1-a}{4\pi M}\left\{\frac{m}{M}(1-a) - \sin\frac{m}{M}(1-a)\right\}, \quad |z| = 1. \tag{33}$$

Furthermore, the above inequality is sharp in the sense that there exists a constant $c > 0$ such that

$$\max_{r \in \mathscr{R}_{n,a}, \|r\|=1} \max_{|z|=1}(|r'(z)| - |B_n'(z)|) \geq -\frac{c}{m}(1-a). \tag{34}$$

When $a = 0$ and all the poles at ∞, we see that (33) reduces to (32) for the polynomial case. One of the key ingredients in the proof of the Theorem of Giroux and Rahman is the following integral formula for the derivative of a polynomial implied in their proof (see [13, p. 88, line 5]).

Theorem 21. *For* $|z| = 1$ *and* $p_n \in \mathscr{P}_n$,

$$p'(z) = \frac{1}{2\pi} \int_0^{2\pi} p(\zeta)\overline{\zeta} \left\{ \frac{1 - (z/\zeta)^n}{1 - z/\zeta} \right\}^2 d\theta, \quad \zeta = e^{i\theta}.$$

The proof of this integral formula in [13] is also a gem: Note that

$$\left(\sum_{k=1}^{n} z^{k-1} \right)^2 = \sum_{k=1}^{n} k z^{k-1} + \text{higher power terms },$$

so we can write

$$e^{i\theta} p'(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} p(e^{i(\theta+t)}) e^{-it} \{1 + e^{-it} + e^{-i2t} + \cdots + e^{-i(n-1)t}\}^2 dt,$$

which implies the formula.

The rational version of this formula plays a critical role in the proof of the rational extension.

Theorem 22 ([16, Lemma 4.3]). *For* $|z| = 1$ *and* $r \in \mathscr{R}_n$,

$$r'(z) = \frac{1}{2\pi} \int_0^{2\pi} r_n(\zeta)\overline{\zeta} \left\{ \frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\overline{\zeta}} \right\}^2 d\theta, \quad \zeta = e^{i\theta}. \tag{35}$$

To mimic the above proof of Giroux and Rahman, [16] uses the Malmquist-Walsh system $\{\varphi_k(z)\}$ of orthogonal rational functions in $\mathscr{R}_n$ to replace the system $\{z^k\}_{k=0}^n$ in the polynomial case, where $\varphi_k(z)$ is defined by

$$\varphi_0(z) := 1, \ \ \varphi_k(z) := \frac{h_k z}{1 - \overline{a_k} z} B_k(z), \ k = 1, 2, \ldots, n,$$

with $h_k > 0$ chosen to ensure that $\frac{1}{2\pi} \int_{|z|=1} |\varphi_k(z)|^2 |dz| = 1$. So, the sum $\sum_{k=1}^n z^{k-1}$ is substituted by the sum

$$S_n(z, \zeta) := \sum_{k=0}^{n} \varphi_k(z) \overline{\varphi_k(\zeta)},$$

which is the reproducing kernel of $\mathscr{R}_n$ in the following sense:

$$r(z) = \frac{1}{2\pi} \int_0^{2\pi} r(\zeta) S_n(z, \zeta) d\theta \ \text{ for every } r \in \mathscr{R}_n. \tag{36}$$

Luckily, by the Darboux–Christoffel formula for orthogonal rational functions, this reproducing kernel can be written in a closed form:

$$\sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\zeta)} = \frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\bar{\zeta}} + B_n(z)\overline{B_n(\zeta)}.$$

Next, the proof of (35) can be carried out by first differentiating the reproducing identity (36):

$$r'(z) = \frac{1}{2\pi}\int_0^{2\pi} r(\zeta)\frac{\partial}{\partial z}S_n(z,\zeta)d\theta,$$

and now we can follow the proof of Giroux and Rahman (although one line in their proof becomes more than one page's calculation (see [16, pp. 482–483])).

Based on this integral formula, the main ideas of Giroux and Rahman [13] can be carried out in obtaining (33) for the rational case. In the proof of (34) for the sharpness, one can easily trace the construction in the papers of Rahman in [13, Proof of Theorem 2] and [30, pp. 87–88]. Many steps require some work for the rational case and in particular, one technique of Rahman and Stenger in [30] plays an important role in obtaining the following key inequality in establishing the right estimates:

$$\begin{aligned}
&1 - \left(\frac{\sin(\gamma(\theta+\theta_j) - \gamma(\theta_j))/2}{M\sin(\theta/2)}\right)^2 \\
&\geq \left(1 - \frac{1}{M}\right)\left\{1 - \left(\frac{\sin(\gamma(\theta+\theta_j) - \gamma(\theta_j))/2}{(\gamma(\theta+\theta_j) - \gamma(\theta_j))/2}\right)^2\right\},
\end{aligned}$$

where

$$\gamma(\theta) := \int_{\theta_0}^{\theta} |B_n'(e^{i\theta})|d\theta$$

satisfying $|\gamma(\theta+\theta_j) - \gamma(\theta_j)| \leq M|\theta|$.

Later, the integral formula in (35) was extended to rational functions having poles on both sides of the unit circle [18, Lemma 3] where the proof is based on applying the residue theorem. We remark that the integral formula has been proved yet again by a different method in a recent paper by Baranov and Zarouf in [5] using their theory of model spaces where they used the integral formula to obtain many Bernstein-type inequalities various integral norms. Indeed, the integral formula is not hard to verify but getting the right formulation[1] in the first place is critical for carrying out the whole proof of Giroux and Rahman to the rational case. This is where the influence of Giroux and Rahman [13] lies: it allowed us to make intelligent guess on the formulation for the rational case as well as the general guideline for its proof, and the rest is just verification. The paper of Giroux and Rahman contains many other interesting results that have not been extended to the rational case. We hope that our story on the impact of the paper on one of our papers would motivate interested readers to work out some of these extensions.

7 Proofs of Theorems 5–7 and 10 in Sect. 2

The proofs of Theorems 5–7 can be found in [20] but for reader's convenience, we include their proofs here.

Before begin the proofs, we need the following lemma.

Lemma 1 ([20, Lemma 1]).

(i) *If $z \in \mathbb{T}$, then the following hold:*

$$\frac{zB_n'(z)}{B_n(z)} = \sum_{k=1}^{n} \frac{|a_k|^2 - 1}{|z - a_k|^2} \tag{37}$$

and

$$\frac{zB_n'(z)}{B_n(z)} = |B_n'(z)|. \tag{38}$$

(ii) *If $\lambda \in \mathbb{T}$, then $B_n(z) = \lambda$ has exactly n simple roots, say $t_1, t_2, \ldots, t_n$, and all of them lie on $\mathbb{T}$. Moreover,*

$$\frac{t_k B_n'(t_k)}{\lambda} = \sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|t_k - a_j|^2}, \quad \text{for } k = 1, 2, \ldots, n. \tag{39}$$

Proof of Theorem 5. Lemma 1(ii) shows that the equation $B(z) - \lambda = 0$ has exactly n simple roots, say $t_1, t_2, \ldots, t_n$, with each t_i, $i = 1, 2, \ldots, n$ satisfying $|t_i| = 1$. Hence, it remains to prove the identity. Let

$$w_n^*(z) := z^n \overline{w_n\left(\frac{1}{\bar{z}}\right)} \tag{40}$$

[1] By "right formulation" we mean in the sense that when taking modulus inside the integration, the equality could be attained for some special cases - not every integral formula allows such sharp estimation.

and let $q(z) = w_n^*(z) - \lambda w_n(z)$. Then $q \in \mathscr{P}_n$ and $q(z) = w_n(z)\,[B_n(z) - \lambda] = \eta \prod_{k=1}^{n}(z-t_k)$ for some $\eta \neq 0$. Now, $r(z) = p(z)/w_n(z) \in \mathscr{R}_n$. Let $p(z) = \tau z^n + \dots$. Then

$$p(z) - \left(\frac{\tau}{\eta}\right) q(z) \in \mathscr{P}_{n-1}. \tag{41}$$

Since the roots $t_1, t_2, \dots, t_n$ are distinct, we have, by Lagrange interpolation formula,

$$p(z) - \frac{\tau}{\eta} q(z) = \sum_{k=1}^{n} \frac{p(t_k)q(z)}{q'(t_k)(z - t_k)}. \tag{42}$$

Dividing both sides of (42) by $q(z)$, we get

$$\frac{p(z)}{q(z)} - \frac{\tau}{\eta} = \sum_{k=1}^{n} \frac{p(t_k)}{q'(t_k)(z - t_k)}. \tag{43}$$

Differentiating both sides of (43) with respect to z, we have

$$\left(\frac{p(z)}{q(z)}\right)' = -\sum_{k=1}^{n} \frac{p(t_k)}{q'(t_k)(z - t_k)^2}. \tag{44}$$

Now, using $q(z) = w_n(z)\,[B_n(z) - \lambda]$ and $p(z) = w_n(z)r(z)$, we have $p(t_k) = w_n(t_k)r(t_k))$ $q'(t_k) = w_n(t_k)B_n'(t_k)$. Substituting these values in (44), we get

$$\left[\frac{r(z)}{B_n(z) - \lambda}\right]' = -\sum_{k=1}^{n} \frac{[p(t_k)/w_n(t_k)]}{q'(t_k)(z - t_k)^2} = -\sum_{k=1}^{n} \frac{r(t_k)}{q'(t_k)(z - t_k)^2}.$$

Hence,

$$\frac{r'(z)[B_n(z) - \lambda] - r(z)B_n'(z)}{[B_n(z) - \lambda]^2} = -\sum_{k=1}^{n} \frac{r(t_k)}{q'(t_k)(z - t_k)^2}. \tag{45}$$

Multiplying both sides of (45) by $|B_n(z) - \lambda|^2$, and using $q'(t_k) = w_n(t_k)B'(t_k)$, we obtain

$$B_n'(z)r(z) - r'(z)[B_n(z) - \lambda] = \sum_{k=1}^{n} \frac{r(t_k)[B_n(z) - \lambda]^2}{B_n{}'(t_k)(z - t_k)^2}. \tag{46}$$

Observe that if $|u| = |v| = 1$, then $(u - v)^2 = -uv|u - v|^2$. So

$$(z - t_k)^2 = -zt_k|z - t_k|^2, \tag{47}$$

and

$$[B_n(z) - \lambda]^2 = -\lambda B_n(z)|B_n(z) - \lambda|^2. \tag{48}$$

Using (47) and (48), (46) yields

$$B_n'(z)r(z) - r'(z)[B_n(z) - \lambda] = \frac{B_n(z)}{z} \sum_{k=1}^{n} \frac{\lambda r(t_k)}{t_k B_n{}'(t_k)} \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2. \tag{49}$$

Now, using (39) from Lemma 1 and the definition of c_k from Lemma 1, we obtain the identity (4). This completes the proof of Theorem 1. □

We next prove Theorem 6 as an application of Theorem 5.

Proof of Theorem 6. Using λ and $-\lambda$ in Theorem 1, we get

$$B_n'(z)r(z) - r'(z)[B_n(z) - \lambda] = \frac{B_n(z)}{\lambda} \sum_{k=1}^{n} c_k r(t_k) \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2. \tag{50}$$

and

$$B_n'(z)r(z) - r'(z)[B_n(z) + \lambda] = \frac{B_n(z)}{\lambda} \sum_{k=1}^{n} d_k r(s_k) \left| \frac{B_n(z) + \lambda}{z - s_k} \right|^2. \tag{51}$$

Subtracting (51) from (50) yields

$$2\lambda r'(z) = \frac{B_n(z)}{z} \left[\sum_{k=1}^{n} c_k r(t_k) \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2 - \sum_{k=1}^{n} d_k r(s_k) \left| \frac{B_n(z) + \lambda}{z - s_k} \right|^2 \right]. \tag{52}$$

Multiplying both sides by $z/B_n(z)$, we get the desired result. □

Proof of Theorem 7. From Theorem 6 we have

$$\left| \frac{2\lambda z r'(z)}{B_n(z)} \right| \le \sum_{k=1}^{n} |c_k||r(t_k)| \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2 + \sum_{k=1}^{n} |d_k||r(s_k)| \left| \frac{B_n(z) + \lambda}{z - s_k} \right|^2. \tag{53}$$

For $z \in \mathbb{T}$ with $|B_n(z)| = 1$ we get

$$2|r'(z)| \le \max_{1 \le k \le n} |r(t_k)| \sum_{k=1}^{n} |c_k| \left| \frac{B_n(z) - \lambda}{z - t_k} \right|^2 + \max_{1 \le k \le n} |r(s_k)| \sum_{k=1}^{n} |d_k| \left| \frac{B_n(z) + \lambda}{z - s_k} \right|^2. \tag{54}$$

Since c_k and d_k are positive and for $z \in \mathbb{T}$, we have from Theorem 5,

$$z\frac{B_n{}'(z)}{B_n(z)} = \sum_{k=1}^{n} c_k \left| \frac{B_n(z)-\lambda}{z-t_k} \right|^2, \tag{55}$$

$$z\frac{B_n{}'(z)}{B_n(z)} = \sum_{k=1}^{n} d_k \left| \frac{B_n(z)+\lambda}{z-s_k} \right|^2, \tag{56}$$

and

$$\frac{zB_n{}'(z)}{B_n(z)} = |B_n{}'(z)|,$$

we conclude from (54) that

$$|r'(z)| \leq \frac{1}{2}|B_n{}'(z)| \left\{ \max_{1\leq k\leq n} |r(t_k)| + \max_{1\leq k\leq n} |r(s_k)| \right\}.$$

This completes the proof of Theorem 7. □

The following proof is essentially reproduced from [3].

Proof of Theorem 10. Adding (49) and (50) to get

$$2\left(B_n{}'(z)r(z) - r'(z)B_n(z)\right) = \frac{B_n(z)}{\lambda}\left[\sum_{k=1}^{n} c_k r(t_k) \left| \frac{B_n(z)-\lambda}{z-t_k} \right|^2 + \sum_{k=1}^{n} d_k r(s_k) \left| \frac{B_n(z)+\lambda}{z-s_k} \right|^2 \right]. \tag{57}$$

Writing the first sum as T and the second sum as S, we can rewrite (51) and (57) as

$$2\lambda r'(z) = \frac{B_n(z)}{\lambda}[T-S] \tag{58}$$

$$2\left(B_n{}'(z)r(z) - r'(z)B_n(z)\right) = \frac{B_n(z)}{\lambda}[T+S]. \tag{59}$$

Taking the derivative of r^* (or see (22) in [20]), $|B_n{}'(z)r(z) - r'(z)B_n(z)| = |(r^*)'(z)|$, for $|z| = 1$. So by taking the modulus for $|z| = 1$ on both sides in (58) and (59), we arrive at $2|r'(z)| = |T-S|$ and $2|(r^*)'(z)| = |T+S|$. Then, as in [3], by using the parallelogram theorem,

$$\begin{aligned}(2|r'(z)|)^2 + (2|(r^*)'(z)|)^2 &= |T-S|^2 + |T+S|^2 \\ &= 2\left(|T|^2 + |S|^2\right).\end{aligned}$$

So

$$2\left(|r'(z)|^2 + |(r^*)'(z)|^2\right) = |T|^2 + |S|^2. \tag{60}$$

Finally, using comparison inequality in Corollary 5, we have

$$|r'(z)| \le |(r^*)'(z)|, \quad |z| = 1.$$

Thus, (60) implies

$$4|r'(z)|^2 \le |T|^2 + |S|^2,$$

which gives

$$|r'(z)| \le \frac{1}{2}\sqrt{|T|^2 + |S|^2}. \tag{61}$$

Finally, notice that

$$|T| \le \sum_{k=1}^{n} c_k |r(t_k)| \le \max_{1\le k\le n} |r(t_k)|,$$

and

$$|S| \le \sum_{k=1}^{n} d_k |r(s_k)| \le \max_{1\le k\le n} |r(s_k)|.$$

Using these two inequalities in (61) gives the desired result. □

8 Proof of Theorem 16 in Sect. 4

The proofs (see [19]) used the techniques different from those used to prove analogous results for polynomials. The main reason is that proofs for polynomials use Laguerre's theorem or Graces' theorem (see [24]) which are not readily available for rational functions. In fact, Bonsal and Marden [6] showed that the counting of critical points of $r \in \mathscr{R}_n$ depends on the number of distinct poles of r in addition to the zeros. In view of this, there is no extension of Laguerre's theorem and Grace's theorem for rational functions. We will need the following lemmas for the proof of our theorems.

Lemma 2 ([20, Theorem 4]). *If $r \in \mathscr{R}_n$ has exactly n zeros and all the zeros lie in $|z| \le 1$, then*

$$|r'(z)| \ge \frac{1}{2}|B_n{}'(z)||r(z)| \text{ for } |z| = 1.$$

The following lemma in essence summarizes the simple trick used by De Bruijn [8] and referred to as the method of Rahman in Sect. 5.

Lemma 3 ([19, Lemma 4.2]). *Let A, B be any two complex numbers. Then the following hold:*

(i) *Let $|A| \geq |B|$ and $B \neq 0$. Then $A = \delta B$ for some δ satisfying $|\delta| < 1$.*
(ii) *Conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.*

Proof. (i) Let $|A| \geq |B|$ and $B \neq 0$. If $A = \delta B$ for some δ with $\delta < 1$, then $|A| = |\delta||B| < |B|$, which is a contradiction.
(ii) Let $A \neq \delta B$ for any δ with $|\delta| < 1$. If $|A| < |B|$, then $B \neq 0$. Suppose $\delta = A/B$, then

$$A = \delta B \text{ and } |\delta| = \frac{|A|}{|B|} < 1,$$

contradicting the assumption. □

Now, we are ready to prove Theorem 16.

Proof of Theorem 16. Assume at the outset that no zeros of $s(z)$ are on the unit circle $|z| = 1$. Hence, all the zeros of $s(z)$ lie in $|z| < 1$. Let α be an arbitrary complex number such that $|\alpha| < 1$. Let us consider the function $\alpha r(z) + s(z)$. This is a rational function with no poles in $|z| < 1$. Since $|r(z)| \leq |s(z)|$ for $|z| = 1$, by Rouche's theorem $\alpha r(z) + s(z)$ and $s(z)$ have the same number of zeros in $|z| < 1$. Thus, $\alpha r(z) + s(z)$ also has n zeros in $|z| < 1$. By Lemma 2,

$$|\alpha r'(z) + s'(z)| \geq \frac{1}{2}|B_n{}'(z)||\alpha r(z) + s(z)| \text{ for } |z| = 1. \tag{62}$$

Now, taking the logarithmic derivative of the Blaschke product yields

$$\frac{zB_n{}'(z)}{B_n(z)} = \sum_{k=1}^{n} \frac{|a_k|^2 - 1}{|z - a_k|^2},$$

which implies that $B_n{}'(z) \neq 0$. Hence the expression on the right-hand side of (62) is not zero. Now, using Lemma 3, we have for all β satisfying $|\beta| < 1$,

$$\alpha r'(z) + s'(z) \neq \frac{1}{2}B_n{}'(z)[\alpha r(z) + s(z)] \text{ for } |\alpha| < 1 \text{ and } |\beta| < 1.$$

By using (ii) of Lemma 3, we conclude that

$$\left|s'(z) - \frac{\beta}{2}B_n{}'(z)s(z)\right| \geq \left|r'(z) - \frac{\beta}{2}B_n{}'(z)r(z)\right| \text{ for } |z| = 1 \text{ and } |\beta| < 1.$$

Now, taking $\rho := \beta/2$, we get for $|\rho| \le 1/2$,

$$\left|r'(z) - \frac{\beta}{2}{B_n}'(z)r(z)\right| \le \left|s'(z) - \frac{\beta}{2}{B_n}'(z)s(z)\right| \quad \text{for} \quad |z| = 1.$$

This completes the proof of Theorem 16.

9 Proof of Theorem 18 in Sect. 5

We provide a proof of Theorem 18 by following the ideas of Rahman in [29] as discussed in Sect. 5.

Recall that for $r \in \mathscr{R}_n$, the reverse rational function r^* is defined as

$$r^*(z) = B_n(z)\overline{r(1/\bar{z})}.$$

We see that if $r(z) = p(z)/w_n(z)$ for some $p \in \mathscr{P}_n$, then $r^*(z) = p^*(z)/w_n(z)$.

First, we establish a lemma.

Lemma 3. *We have, for* $|z| = 1$,

$$\int_0^{2\pi} r^*(\zeta)\bar{\zeta}\left[\frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right]^2 d\theta = 0, \quad \zeta = e^{i\theta}.$$

Proof. Note that, as a function of ζ, the integrand is analytic outside the unit circle, even at ∞. So, the lemma follows from Cauchy's theorem. □

Proof of Theorem 18. Using (35) and Lemma 1 above, we have, for $|z| = 1$,

$$\begin{aligned} r'(z) + e^{i\alpha}(r^*)'(z) &= \frac{1}{2\pi i}\int_0^{2\pi} [r(\zeta) + e^{i\alpha}r^*(\zeta)]\bar{\zeta}\left[\frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right]^2 d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} r(\zeta)\bar{\zeta}\left[\frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right]^2 d\theta. \end{aligned}$$

Therefore,

$$|r'(z) + e^{i\alpha}(r^*)'(z)| \le \frac{1}{2\pi}\int_0^{2\pi} |r(\zeta)|\left|\frac{1 - B_n(z)\overline{B_n(\zeta)}}{1 - z\bar{\zeta}}\right|^2 d\theta,$$

and so, by Jensen's inequality, for $p \ge 1$,

$$\left|\frac{r'(z) + e^{i\alpha}(r^*)'(z)}{B_n'(z)}\right|^p \le \int_0^{2\pi} |r(\zeta)|^p \rho(\theta; z) d\theta,$$

where

$$\rho(\theta; z) = \left| \frac{1 - B_n(z)\overline{B_n(e^{i\theta})}}{2\pi |B_n'(z)|(1 - ze^{-i\theta})} \right|^2$$

with (by Li [18, Lemma 4]) $\int_0^{2\pi} \rho(\theta; z) d\theta = 1$.

Now, integrating in z to obtain

$$\int_{|z|=1} \left| \frac{r'(z) + e^{i\alpha}(r^*)'(z)}{B_n'(z)} \right|^p |dz| \le \int_{|z|=1} \int_0^{2\pi} |r(\zeta)|^p \rho(\theta; z) d\theta |dz|.$$

By interchanging the order of integration on the right side, we get

$$\int_{|z|=1} \left| \frac{r'(z) + e^{i\alpha}(r^*)'(z)}{B_n'(z)} \right|^p |dz| \le \int_0^{2\pi} |r(\zeta)|^p |B_n'(\zeta)| d\theta.$$

Now, by the comparison inequality stated in Sect. 5, we have $|r'(z)| \le |(r^*)'(z)|$ for $|z| = 1$ since $|r(z)| = |r^*(z)|$ and $r^*(z)$ has all its zeros inside the unit circle. The proof can now be completed in the same way as in Rahman [29, Proof of Theorem 2]. □

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On an Asymptotic Equality for Reproducing Kernels and Sums of Squares of Orthonormal Polynomials

A. Ignjatovic and D.S. Lubinsky

This article is dedicated to the memory of Q.I. Rahman

Abstract In a recent paper, the first author considered orthonormal polynomials $\{p_n\}$ associated with a symmetric measure with unbounded support and with recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + A_{n-1} p_{n-1}(x), \quad n \geq 0.$$

Under appropriate restrictions on $\{A_n\}$, the first author established the identity

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}},$$

uniformly for x in compact subsets of the real line. Here, we establish and evaluate this limit for a class of even exponential weights, and also investigate analogues for weights on a finite interval, and for some non-even weights.

Keywords Orthogonal polynomials • Christoffel functions • Recurrence coefficients

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1 Introduction

Let μ be a symmetric positive measure on the real line, with all finite power moments. Then we may define orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$, $\gamma_n > 0$, $n \geq 0$, satisfying

$$\int p_n p_m d\mu = \delta_{mn}.$$

Because of the symmetry, the three term recurrence relation takes a simple form:

$$xp_n(x) = A_n p_{n+1}(x) + A_{n-1} p_{n-1}(x), \quad n \geq 1,$$

where

$$A_n = \frac{\gamma_n}{\gamma_{n+1}}, \quad n \geq 1.$$

The asymptotic behavior of p_n as $n \to \infty$ has been intensively investigated for over a century, and has a myriad of applications. In a recent paper [4], the first author presented a novel approach, and placed the following hypotheses on the recurrence coefficients:

(C1) $\lim_{n\to\infty} A_n = \infty$;
(C2) $\lim_{n\to\infty} (A_{n+1} - A_n) = 0$;
(C3) There exist m_0, n_0 such that $A_{m+n} > A_n$ for all $n \geq n_0$ and $m \geq m_0$;
(C4)

$$\sum_{j=0}^{\infty} \frac{1}{A_j} = \infty;$$

(C5) There exists $k > 1$ such that

$$\sum_{j=0}^{\infty} \frac{1}{A_j^k} < \infty;$$

(C6)

$$\sum_{j=0}^{\infty} \frac{|A_{j+1} - A_j|}{A_j^2} < \infty;$$

(C7)

$$\sum_{j=0}^{\infty} \frac{\left|A_{j+2} - 2A_{j+1} + A_j\right|}{A_j} < \infty.$$

He proved [4]:

Theorem A. *Under the hypotheses* (C1)–(C7)*, the following limits exist, are finite and positive, and satisfy*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}}, \tag{1}$$

uniformly for x in compact subsets of the real line.

It was shown in [4], that if $0 < p < 1$, $c > 0$, and

$$A_n = c\,(n+1)^p\,, \quad n \geq 0,$$

then (C1)–(C7) hold. This rate of growth of recurrence coefficients is typically associated with an exponential weight such as $\exp\left(-|x|^{-1/p}\right)$, $0 < p < 1$. Indeed the asymptotics for recurrence coefficients given in [2, p. 50, Theorem 1.3] show that (C1)–(C7) are valid for these specific exponential weights.

Here, we shall evaluate the limit in (1), showing that it equals $(2\pi\mu'(x))^{-1}$, for a large class of exponential weights. We do this by using asymptotics for orthonormal polynomials and Christoffel functions that were established in [5].

This chapter is organized as follows: in Sect. 2, we briefly discuss the case of weights on $[-1, 1]$. This simple case illustrates some of the ideas of proof. Our main results, for even exponential weights, are stated and proved in Sect. 3. In Sect. 4, we discuss some limited extensions to non-even weights. In the sequel $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, polynomials of degree $\leq n$, and possibly other parameters. We use $\sim$ in the following sense: given sequences of non-zero real numbers $\{x_n\}$ and $\{y_n\}$, we write $x_n \sim y_n$ if there exists a constant $C > 1$ such that

$$C^{-1} \leq x_n/y_n \leq C \quad \text{for } n \geq 1.$$

Similar notation is used for functions and sequences of functions.

2 Weights on $[-1, 1]$

The result of this section is:

Theorem 2.1. *Let* μ *be a positive measure supported on* $[-1, 1]$ *that satisfies Szegő's condition*

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty. \tag{2}$$

Assume that for some $\eta \in \left(0, \frac{1}{2}\right)$*,* μ *is absolutely continuous in* $[-\eta, \eta]$*, that* μ' *is positive and continuous in* $[-\eta, \eta]$*, and satisfies for some* $C > 0$*,* $\rho > 1$*, and* $x, y \in [-\eta, \eta]$,

$$\left|\mu'(x) - \mu'(y)\right| \leq C \left|\log|x-y|\right|^{-\rho}. \tag{3}$$

Let $\{x_n\}$ *be a sequence of real numbers with limit* 0 *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} p_k^2(x_n)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x_n) + p_{2n+1}^2(x_n)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi}. \tag{4}$$

Of course this result is quite restricted as we need $x_n \to 0$.

We turn to

The Proof of Theorem 2.1. Under the assumptions of Theorem 2.1, there is the asymptotic as $m \to \infty$,

$$p_m(x)\, \mu'(x)^{1/2} \left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \cos\left(m\theta + \gamma(\theta)\right) + o(1), \tag{5}$$

uniformly for $x = \cos\theta$ in a compact subset of $(-\eta, \eta)$, where

$$\gamma(\theta) = \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \left[\log f(t) - \log f(\theta)\right] \cot\frac{\theta - t}{2} dt.$$

Here PV denotes principal value, and

$$f'(\theta) = \mu'(\cos\theta)\left|\sin\theta\right|.$$

This follows from Theorem 2 in [1, p. 41]. We note that other criteria for asymptotics are given in, for example, [3, p. 246, Table II(a)], or Theorem 5 in [6, p. 77].

Now let $\{x_n\}$ be a sequence with limit 0, and for $n \geq 1$, write $x_n = \cos\theta_n$, where $\theta_n \in (0, \pi)$. We see that $\theta_n \to \frac{\pi}{2}$ as $n \to \infty$. The asymptotic (5) gives

$$
\begin{aligned}
&\left[p_{2n}^{2}(x_n)+p_{2n+1}^{2}(x_n)\right]\mu'(x_n)^{1/2}\left(1-x_n^2\right)^{1/2}\\
&=\frac{2}{\pi}\left[\cos^2(2n\theta_n+\gamma(\theta_n))+\cos^2((2n+1)\theta_n+\gamma(\theta_n))\right]+o(1)\\
&=\frac{2}{\pi}\left[\cos^2(2n\theta_n+\gamma(\theta_n))+\cos^2\left(2n\theta_n+\gamma(\theta_n)+\frac{\pi}{2}\right)\right]+o(1)\\
&=\frac{2}{\pi}\left[\cos^2(2n\theta_n+\gamma(\theta_n))+\sin^2(2n\theta_n+\gamma(\theta_n))\right]+o(1)=\frac{2}{\pi}+o(1).
\end{aligned}
$$

Next, the limit

$$\lim_{n\to\infty}A_n=\lim_{n\to\infty}\frac{\gamma_n}{\gamma_{n+1}}=\frac{1}{2} \tag{6}$$

is an immediate consequence of the fact that μ satisfies Szegő's condition [11, p. 309]. Thus the second part in (4) satisfies

$$\lim_{n\to\infty}\frac{p_{2n}^{2}(x_n)+p_{2n+1}^{2}(x_n)}{A_{2n}^{-1}+A_{2n+1}^{-1}}=\frac{1}{2\pi}. \tag{7}$$

Next, asymptotics for Christoffel functions and the continuity of μ' in $[-\eta,\eta]$ yield [7], [10, Theorem 3.11.9, p. 220] that uniformly for $x\in[-\eta,\eta]$,

$$\lim_{n\to\infty}\frac{1}{n+1}\left(\sum_{k=0}^{n}p_k^2(x)\right)\mu'(x)=\frac{1}{\pi\sqrt{1-x^2}}.$$

In particular, then, as μ' is continuous at 0,

$$\lim_{n\to\infty}\frac{1}{n+1}\left(\sum_{k=0}^{n}p_k^2(x_n)\right)=\frac{1}{\pi}. \tag{8}$$

Finally (6) gives

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}A_k^{-1}=2.$$

Combining the last two limits, we have

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}p_k^2(x_n)}{\sum_{k=0}^{n}A_k^{-1}}=\frac{1}{2\pi},$$

so the result follows using (7). ■

3 Even Weights on $(-\infty, \infty)$

Following is the class of even weights we shall consider. It is a subclass of that in [5, Definition 1.1, p. 7]:

Definition 3.1. Let $\mu'(x) = e^{-2Q(x)}$, $x \in \mathbb{R}$, where Q is even,

(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$;
(b) Q'' exists and is positive in $\mathbb{R}\setminus\{0\}$;
(c)

$$\lim_{t\to\infty} Q(t) = \infty;$$

(d) The function $T(t) = \frac{tQ'(t)}{Q(t)}$, $t \in (0,\infty)$ is quasi-increasing in $(0,\infty)$, in the sense that for some constant C and $0 \leq x < y \Rightarrow$

$$T(x) \leq CT(y);$$

In addition we assume that T is bounded below in $\mathbb{R}\setminus\{0\}$ by a constant Λ larger than 1.

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{Q'(x)}{Q(x)} \quad \text{, a.e. } x \in (0\,\infty).$$

Then we write $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2, even\right)$.

Examples of Q satisfying the conditions above on $(-\infty,\infty)$ include [5, pp. 8–9]

$$Q(x) = |x|^\alpha,$$

where $\alpha > 1$, and

$$Q(x) = \exp_\ell\left(|x|^\alpha\right) - \exp_\ell(0),$$

where $\alpha > 1$, $\ell \geq 0$, and $\exp_k(x) = \exp(\exp(\cdots(\exp(x)))$ is the kth iterated exponential. We could actually allow a more general (but more technical) class of weights, namely the even weights of class $\mathcal{F}\left(lip\frac{1}{2}\right)$ from [5]. We shall prove

Theorem 3.2. *Let $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2, even\right)$. Then uniformly for x in compact subsets of the real line,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n p_k^2(x)}{\sum_{k=0}^n A_k^{-1}} = \lim_{n\to\infty} \frac{p_{2n}^2(x) + p_{2n+1}^2(x)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi e^{-2Q(x)}}. \tag{9}$$

In considering orthogonal polynomials associated with the measure $d\mu(t) = e^{-2Q(t)}dt$, a crucial role is played by the Mhaskar-Rakhmanov-Saff numbers $a_t, t > 0$. These are defined by equations [5, p. 13], [8, 9]

$$t = \frac{1}{\pi}\int_0^1 \frac{a_t x Q'(a_t x)}{\sqrt{1-x^2}}dx. \tag{10}$$

We note that a_t increases with t and $a_t \to \infty$ as $t \to \infty$. As an example of Mhaskar-Rakhmanov-Saff numbers, let $Q(x) = |x|^\alpha$, $x \in \mathbb{R}$, $\alpha > 0$. It is known that then [8, 9]

$$a_t = \left\{ \frac{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2}{\Gamma(\alpha)} \right\}^{1/\alpha} t^{1/\alpha}, \quad t > 0.$$

Another important quantity associated with Q is the nth equilibrium density [5, p. 16]

$$\sigma_n(x) = \frac{1}{\pi^2}\sqrt{a_n^2 - x^2}\int_0^{a_n} \frac{sQ'(s) - xQ'(x)}{s^2 - x^2}\frac{ds}{\sqrt{a_n^2 - s^2}}, \quad x \in [-a_n, a_n]. \tag{11}$$

It has total mass n

$$\int_{-a_n}^{a_n} \sigma_n = n,$$

and satisfies the equilibrium equation

$$\int_{-a_n}^{a_n} \log\left|\frac{1}{x-s}\right| \sigma_n(s)ds + Q(x) = c_n, \quad x \in [-a_n, a_n].$$

Here c_n is a constant.

In many contexts, it is convenient to map σ_n onto a density function that is supported on $[-1, 1]$. Let

$$\sigma_n^*(t) = \frac{a_n}{n}\sigma_n(a_n t), \quad t \in [-1, 1]. \tag{12}$$

It satisfies

$$\int_{-1}^{1} \sigma_n^* = 1.$$

Proof of Theorem 3.2.. In [5, p. 403, Theorem 15.3, (15.11)], it is shown that uniformly for $x = \cos\theta$ in a closed subinterval of $(-1, 1)$ and $m = n-1, n$,

$$a_n^{1/2} p_m(a_n x) W(a_n x) \left(1 - x^2\right)^{1/4}$$
$$= \sqrt{\frac{2}{\pi}} \cos\left(\left(m - n + \frac{1}{2}\right)\theta + n\pi \int_x^1 \sigma_n^* - \frac{\pi}{4}\right) + o(1). \tag{13}$$

Note that the linear transformation L_n there reduces to $L_n(x) = x/a_n$ and $L_n^{[-1]}(t) = a_n t$. Setting

$$\Delta_n(x) = \frac{1}{2}\theta + n\pi \int_x^1 \sigma_n^* - \frac{\pi}{4},$$

we see that uniformly for x in a closed subset of $(-1, 1)$,

$$a_n W(a_n x)^2 \left(1 - x^2\right)^{1/2} \left\{p_n^2(a_n x) + p_{n-1}^2(a_n x)\right\}$$
$$= \frac{2}{\pi}\left\{(\cos \Delta_n(x))^2 + (\cos(\Delta_n(x) - \theta))^2\right\} + o(1).$$

In particular, setting $a_n x = y$ where y lies in a compact set, so that $x = y/a_n = \cos\theta$ has $\theta = \arccos(y/a_n) = \arccos(o(1)) = \frac{\pi}{2} + o(1)$, we obtain

$$a_n W(y)^2 \left(1 - \left(\frac{y}{a_n}\right)^2\right)^{1/2} \left\{p_n^2(y) + p_{n-1}^2(y)\right\}$$
$$= \frac{2}{\pi}\left\{\left(\cos \Delta_n\left(\frac{y}{a_n}\right)\right)^2 + \left(\cos\left(\Delta_n\left(\frac{y}{a_n}\right) - \frac{\pi}{2}\right)\right)^2\right\} + o(1)$$
$$= \frac{2}{\pi}\left\{\left(\cos \Delta_n\left(\frac{y}{a_n}\right)\right)^2 + \left(\sin \Delta_n\left(\frac{y}{a_n}\right)\right)^2\right\} + o(1) = \frac{2}{\pi} + o(1).$$

Replacing y by x, and n by $2n+1$, we have that uniformly for x in a compact subset of $\mathbb{R}$,

$$a_{2n+1} W(x)^2 \left\{p_{2n+1}^2(x) + p_{2n}^2(x)\right\} = \frac{2}{\pi} + o(1). \tag{14}$$

Next, (1.124) of Theorem 1.23 in [5, p. 26] gives

$$\lim_{n\to\infty} \frac{A_n}{a_n} = \frac{1}{2}. \tag{15}$$

In addition, [5, p. 81, Eq. (3.50)]

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Then the second part of (9) can be calculated as

$$\lim_{n\to\infty} \frac{p_{2n+1}^2(x) + p_{2n}^2(x)}{A_{2n+1}^{-1} + A_{2n}^{-1}} = \lim_{n\to\infty} \frac{a_{2n+1}}{4} \left\{p_{2n+1}^2(x) + p_{2n}^2(x)\right\} = \frac{1}{2\pi W^2(x)}, \tag{16}$$

uniformly for x in a compact subset of $\mathbb{R}$.

It is more difficult to deal with the left-hand side in (9). We first note that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} p_k^2(x) W^2(x) / \sigma_n(x) = 1, \tag{17}$$

uniformly for x in a range that certainly includes compact subsets of the real line [5, Theorem 1.25, p. 26]. Our task is to compare this to

$$\sum_{k=0}^{n-1} A_k^{-1} = 2 \sum_{k=0}^{n-1} a_k^{-1} (1 + o(1)).$$

We shall show that this last right-hand side behaves like

$$2 \int_0^n \frac{1}{a_t} dt (1 + o(1)),$$

and using an alternative representation for σ_n, due to Rakhmanov, that this in turn is close to $\sigma_n(x)$ when x is bounded. Let us now make this rigorous. First note that a_t is a differentiable increasing function of $t \in (0, \infty)$, with $a_t \to 0$ as $t \to 0+$ and $a_t \to \infty$ as $t \to \infty$. Define the inverse $b : (0, \infty) \to (0, \infty)$ of a by

$$b(a_t) = t, \quad t > 0.$$

Rakhmanov's representation for σ_n for even weights asserts that [5, p. 46, Eq. (2.35)]

$$\sigma_n(x) = \frac{1}{\pi} \int_{|b(x)|}^n \frac{1}{\sqrt{a_s^2 - x^2}} ds, \quad x \in (-a_n, a_n). \tag{18}$$

In particular,

$$\sigma_n(0) = \frac{1}{\pi} \int_0^n \frac{1}{a_s} ds. \tag{19}$$

(the convergence of the integral is established in Chap. 2 of [5].) Now let $m = m(n) = \left[\sqrt{n}\right]$, where $[x]$ denotes the greatest integer $\leq x$. Assume that $x \in (0, r]$. For n large enough (the threshold depends on r), we have

$$\begin{aligned}
|\sigma_n(x) - \sigma_n(0)| &= \frac{1}{\pi}\left|\int_{b(x)}^{n}\left\{\frac{1}{\sqrt{a_s^2 - x^2}} - \frac{1}{a_s}\right\} ds - \int_0^{b(x)} \frac{1}{a_s} ds\right| \\
&\leq \frac{1}{\pi}\int_{b(x)}^{n} \frac{a_s - \sqrt{a_s^2 - x^2}}{\sqrt{a_s^2 - x^2}a_s} ds + \int_0^{b(r)} \frac{1}{a_s} ds \\
&= \frac{x^2}{\pi}\int_{b(x)}^{n} \frac{1}{\sqrt{a_s^2 - x^2}a_s\left(a_s + \sqrt{a_s^2 - x^2}\right)} ds + \int_0^{b(r)} \frac{1}{a_s} ds.
\end{aligned} \tag{20}$$

Here as $a_s \geq a_{b(x)} = x$ in the first integral, we see that

$$\begin{aligned}
&\frac{x^2}{\pi}\int_{b(x)}^{n} \frac{1}{\sqrt{a_s^2 - x^2}a_s\left(a_s + \sqrt{a_s^2 - x^2}\right)} ds \\
&\leq x^2\left\{\frac{1}{\pi x^2}\int_{b(x)}^{m} \frac{ds}{\sqrt{a_s^2 - x^2}} + \frac{1}{a_m^2}\int_m^n \frac{1}{\sqrt{a_s^2 - x^2}} ds\right\} \\
&\leq \sigma_m(x) + \left(\frac{x}{a_m}\right)^2 \sigma_n(x).
\end{aligned}$$

Combining this and (20) gives

$$\left|1 - \frac{\sigma_n(0)}{\sigma_n(x)}\right| \leq \frac{\sigma_m(x)}{\sigma_n(x)} + \left(\frac{x}{a_m}\right)^2 + \frac{C}{\sigma_n(x)}.$$

Here as $x \in [0, r]$ and $m \to \infty$, we have $\left(\frac{x}{a_m}\right)^2 \to 0$ uniformly for $x \in [0, r]$. In addition, it follows from Theorem 5.2(b) in [5, p. 110] and then Lemma 3.5(c) in [5, p. 72] that uniformly for $x \in [-r, r]$,

$$\frac{\sigma_m(x)}{\sigma_n(x)} \leq C\frac{(m/a_m)}{(n/a_n)} \leq C\left(\frac{m}{n}\right)^{1-1/\Lambda} \leq C\left(\frac{1}{\sqrt{n}}\right)^{1-1/\Lambda}. \tag{21}$$

Here $\Lambda > 1$ is a lower bound for T in $\mathbb{R}$. Thus, using also evenness of σ_n, we have

$$\lim_{n\to\infty} \sup_{x\in[-r,r]} \left|1 - \frac{\sigma_n(0)}{\sigma_n(x)}\right| = 0.$$

This, (17), and (19) give, uniformly for $x \in [-r, r]$, as $n \to \infty$,

$$\sum_{k=0}^{n-1} p_k^2(x)\, W^2(x) = \frac{1}{\pi} \int_0^n \frac{1}{a_s} ds \,(1 + o(1)) \,. \tag{22}$$

Finally, [5, p. 81, Eq. (3.50)] gives, for $k \geq 1$,

$$\begin{aligned}\left| \int_{k-1}^{k} \frac{1}{a_s} ds - \frac{1}{a_k} \right| &\leq \int_{k-1}^{k} \frac{1}{a_s} \left| 1 - \frac{a_s}{a_k} \right| ds \\ &\leq C \int_{k-1}^{k} \frac{1}{a_s} \left| 1 - \frac{s}{k} \right| ds \leq \frac{C}{k} \int_{k-1}^{k} \frac{1}{a_s} ds,\end{aligned}$$

so from (15), and using monotonicity of a_t in t,

$$\begin{aligned}\left| \sum_{k=0}^{n-1} A_k^{-1} - 2 \int_0^n \frac{1}{a_s} ds \right| &= \left| 2 \sum_{k=2}^{n-1} \left(a_k^{-1} - \int_{k-1}^{k} \frac{1}{a_s} ds \right) + o\left(\sum_{k=2}^{n-1} a_k^{-1} \right) + O(1) \right| \\ &\leq C \sum_{k=2}^{n-1} \frac{1}{k} \int_{k-1}^{k} \frac{1}{a_s} ds + o\left(\int_1^n \frac{1}{a_s} ds \right) + O(1) \\ &\leq C \int_1^m \frac{1}{a_s} ds + \frac{C}{m} \int_m^n \frac{1}{a_s} ds + o\left(\int_1^n \frac{1}{a_s} ds \right) \\ &\leq C\sigma_m(0) + \frac{C}{m} \sigma_n(0) + o(\sigma_n(0)) \\ &\leq C\sigma_n(0) \left\{ \left(\frac{1}{\sqrt{n}} \right)^{1-1/\Lambda} + \frac{1}{\sqrt{n}} + o(1) \right\} \\ &= o\left(\frac{1}{\pi} \int_0^n \frac{1}{a_s} ds \right),\end{aligned}$$

by (21). Thus

$$\left(\sum_{k=0}^{n-1} A_k^{-1} \right) \Big/ \left(2 \int_0^n \frac{1}{a_s} ds \right) = 1 + o(1) \,.$$

This and (22) give uniformly for x in $[-r, r]$,

$$\frac{\sum_{k=0}^{n-1} p_k^2(x)\, W^2(x)}{\left(\sum_{k=0}^{n-1} A_k^{-1} \right)} = \frac{1}{2\pi} (1 + o(1)) \,.$$

Combining this with (16) gives the result. ■

4 The Non-Even, Not Necessarily Unbounded Case

In this section, we briefly explore the extent to which we can extend Theorem 3.2 to non-even exponential weights, possibly not on an infinite interval. To this end, we first define the full class $\mathcal{F}\left(C^2\right)$ from [5, p. 7]:

Definition 4.1. Let $\mathcal{I} = (c, d)$ be a bounded or unbounded interval containing 0. Let $\mu'(x) = e^{-2Q(x)}$, $x \in \mathcal{I}$, where

(a) Q' is continuous in $\mathcal{I}$ and $Q(0) = 0$;
(b) Q'' exists and is positive in $\mathcal{I} \setminus \{0\}$;
(c)

$$\lim_{t \to c+} Q(t) = \infty = \lim_{t \to d-} Q(t);$$

(d) The function $T(t) = \frac{tQ'(t)}{Q(t)}$, $t \in \mathcal{I} \setminus \{0\}$is quasi-increasing in $(0, d)$, in the sense that for some constant C and $0 \le x < y < d \Rightarrow$

$$T(x) \le CT(y);$$

T is also assumed quasi-decreasing in $(c, 0)$. In addition we assume that T is bounded below in $\mathcal{I} \setminus \{0\}$ by a constant larger than 1.
(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{Q'(x)}{Q(x)}, \qquad \text{a.e. } x \in \mathcal{I} \setminus \{0\}.$$

Then we write $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2\right)$.

Examples of Q satisfying the conditions above on $(-\infty, \infty)$ include [5, pp. 8–9]

$$Q(x) = \begin{cases} x^\alpha, & x \in [0, \infty) \\ |x|^\beta, & x \in (-\infty, 0), \end{cases}$$

where $\alpha, \beta > 1$. A more general example is

$$Q(x) = \begin{cases} \exp_\ell(x^\alpha) - \exp_\ell(0), & x \in [0, \infty) \\ \exp_k\left(|x|^\beta\right) - \exp_k(0) & x \in (-\infty, 0), \end{cases}$$

where $\alpha, \beta > t, k, \ell \ge 0$, and $\exp_k(x) = \exp(\exp(\cdots(\exp(x)))$ is the kth iterated exponential. An example on $(-1, 1)$ is

$$Q(x) = \begin{cases} \exp_\ell\left(\left(1-x^2\right)^{-\alpha}\right) - \exp_\ell(1), & x \in [0,1) \\ \exp_k\left(\left(1-x^2\right)^{-\beta}\right) - \exp_k(1), & x \in (-1,0), \end{cases}$$

where $\alpha, \beta > 0$ and $k, \ell \geq 0$.

Instead of just one Mhaskar-Rakhmanov-Saff number, there are now two: a_{-n}, a_n are defined by equations [5, p. 13]

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx;$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx.$$

The nth equilibrium density now takes the form [5, p. 16]

$$\sigma_n(x) = \frac{1}{\pi^2}\sqrt{(x-a_{-n})(a_n-x)}\int_{a_{-n}}^{a_n} \frac{Q'(s)-Q'(x)}{s-x}$$

$$\times \frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}, \quad x \in [a_{-n}, a_n].$$

The contraction of σ_n to $[-1,1]$ is more complicated than for the even case: let

$$\beta_n := \frac{1}{2}(a_n + a_{-n}); \quad \delta_n = \frac{1}{2}(a_n + |a_{-n}|).$$

We can then define the linear map of $[a_{-n}, a_n]$ onto $[-1,1]$ by

$$L_n(x) = (x-\beta_n)/\delta_n, \quad x \in [a_{-n}, a_n] \Leftrightarrow x = L_n^{[-1]}(t) = \beta_n + \delta_n t, \quad t \in [-1,1].$$

The transformed (and renormalized) density is

$$\sigma_n^*(t) = \frac{\delta_n}{n}\sigma_n \circ L_n^{[-1]}(t), \quad t \in [-1,1].$$

Instead of the asymptotic (13), we have uniformly for $x = \cos\theta \in [-1+\varepsilon, 1-\varepsilon]$ [5, p. 403]

$$a_n^{1/2} p_m\left(L_n^{[-1]}(x)\right) W\left(L_n^{[-1]}(x)\right)\left(1-x^2\right)^{1/4}$$
$$= \sqrt{\frac{2}{\pi}}\cos\left(\left(m-n+\frac{1}{2}\right)\theta + n\pi\int_x^1 \sigma_n^* - \frac{\pi}{4}\right) + o(1). \tag{23}$$

Instead of the asymptotic (15) for the recurrence coefficients, we have [5, p. 26]

$$\lim_{n\to\infty}\frac{A_n}{\delta_n}=\frac{1}{2}. \tag{24}$$

By proceeding as in Sect. 3, it is straightforward to see that

$$\lim_{n\to\infty}\frac{p_{2n}^2\left(L_{2n+1}^{[-1]}(x_n)\right)+p_{2n+1}^2\left(L_{2n+1}^{[-1]}(x_n)\right)}{A_{2n}^{-1}+A_{2n+1}^{-1}}W^2\left(L_{2n+1}^{[-1]}(x_n)\right)=\frac{1}{2\pi}, \tag{25}$$

for any sequence $\{x_n\}$ with limit 0. Setting $y_n=L_{2n+1}^{[-1]}(x_n)$, we see that this becomes

$$\lim_{n\to\infty}\frac{p_{2n}^2(y_n)+p_{2n+1}^2(y_n)}{A_{2n}^{-1}+A_{2n+1}^{-1}}W^2(y_n)=\frac{1}{2\pi}, \tag{26}$$

for any sequence $\{y_n\}$ with

$$\lim_{n\to\infty}L_{2n+1}(y_n)=\lim_{n\to\infty}\frac{y_n-\beta_{2n+1}}{\delta_{2n+1}}=0. \tag{27}$$

Unfortunately, it is more problematic to establish an analogue of (22). The asymptotic (16) holds uniformly for $x\in[a_{-\alpha n},a_{\alpha n}]$, for any fixed $\alpha\in(0,1)$. For this to be compatible with (27), we need $\beta_n\in[a_{-\alpha n},a_{\alpha n}]$, for some $\alpha>0$, so in particular, we cannot have a very asymmetric weight. Rakhmanov's representation for σ_n now takes the form

$$\sigma_n(x)=\frac{1}{\pi}\int_{|b(x)|}^{n}\frac{1}{\sqrt{(x-a_{-s})(a_s-x)}}ds=\frac{1}{\pi}\int_{|b(x)|}^{n}\frac{1}{\sqrt{\delta_s^2-(x-\beta_s)^2}}ds.$$

Provided both for some $r>0$,

$$|y_n-\beta_n|\le r,\quad n\ge 1, \tag{28}$$

and

$$|a_{-n}|/a_n\to 1 \text{ as } n\to\infty, \tag{29}$$

we can prove, much as in the last section, that

$$\lim_{n\to\infty}\left|1-\frac{\sigma_n(\beta_n)}{\sigma_n(y_n)}\right|=0.$$

Of course (29) is a very severe asymptotic symmetry requirement. With the aid of (29), we can prove much as in the previous section, that

$$\sigma_n(\beta_n) = \frac{1}{\pi}\int_0^n \frac{1}{\delta_s} ds\,(1+o(1)).$$

$$\left(\sum_{k=0}^{n-1} A_k^{-1}\right) / \left(2\int_0^n \frac{1}{\delta_s} ds\right) = 1 + o(1).$$

Thus for $W \in \mathcal{F}(C^2)$ satisfying the additional condition (29), and sequences $\{y_n\}$ satisfying (27), we obtain

$$\frac{\sum_{k=0}^{n-1} p_k^2(y_n) W^2(y_n)}{\left(\sum_{k=0}^{n-1} A_k^{-1}\right)} = \frac{1}{2\pi}(1+o(1)),$$

and hence

$$\lim_{n\to\infty} \frac{W^2(y_n)\sum_{k=0}^{n} p_k^2(y_n)}{\sum_{k=0}^{n} A_k^{-1}} = \lim_{n\to\infty} \frac{W^2(y_n)\left(p_{2n}^2(y_n) + p_{2n+1}^2(y_n)\right)}{A_{2n}^{-1} + A_{2n+1}^{-1}} = \frac{1}{2\pi}. \tag{30}$$

It would be interesting to see if the severe symmetry condition (28) can be weakened.

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Two Walsh-Type Theorems for the Solutions of Multi-Affine Symmetric Polynomials

Blagovest Sendov and Hristo Sendov

Abstract The spirit of the classical Grace-Walsh-Szegő coincidence theorem states that if there is a solution of a multi-affine symmetric polynomial in a domain with certain properties, then in it there exists another solution with other properties. We present two results in the same spirit, which may be viewed as extensions of the Grace-Walsh-Szegő result.

Keywords Grace-Walsh-Szegő coincidence theorem • Zeros and critical points of polynomials • Apolarity • Locus of a polynomial • Locus holder

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1 Introduction

The classical Grace-Walsh-Szegő coincidence theorem states the following:

Theorem 1.1 (Grace-Walsh-Szegő Coincidence). *Let $P(z_1, \dots, z_n)$ be a multi-affine symmetric polynomial. If the degree of P is n, then every circular domain containing the points $z_1, \dots, z_n$ contains at least one point z such that $P(z_1, \dots, z_n) = P(z, \dots, z)$. If the degree of P is less than n, then the same conclusion holds, provided the circular domain is convex.*

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In spirit, the theorem states that if there is a solution of a multi-affine symmetric polynomial in a domain with some properties, then in it there exists another solution with other properties. The main goal of this chapter is to present two new results that are in the spirit of the Grace-Walsh-Szegő coincidence theorem. The first, Theorem 4.2, states that if a multi-affine symmetric polynomial has a (extended) solution in a *zero-free* sector, then it has a (extended) solution on a ray in the sector originating from the vertex of the sector. In other words, up to a translation, all the components of the solution have the same argument. The second result, Theorem 4.4, states that if a multi-affine symmetric polynomial has a solution in an annulus, then it has a solution there, that is either (a) located on a circle concentric to the annulus, or (b) has at most two distinct components.

The broader goal of this chapter is to explain how the Grace-Walsh-Szegő coincidence theorem is related to the notion of a locus of complex polynomials. This is objective of Sect. 3, where we give a brief overview of the known facts about loci, and conclude with the new findings that every bounded extended locus is a locus, see Proposition 3.1.

2 Solutions of the Polarization of a Complex Polynomial

Denote by $\mathcal{C}$ the complex plane and let $\mathcal{C}^* := \mathcal{C} \cup \{\infty\}$. By $\mathcal{P}_n$ denote the set of all complex polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n(z - z_1)(z - z_2)\cdots(z - z_n) \tag{1}$$

of degree n, where $a_0, \ldots, a_n \in \mathcal{C}$ are constants and $a_n \neq 0$. Let

$$\overline{\mathcal{P}}_n = \bigcup_{s=0}^{n} \mathcal{P}_s. \tag{2}$$

For every polynomial $p(z) \in \overline{\mathcal{P}}_n$, we consider its *polarization* or *symmetrization* with n variables. That is, we consider the multi-affine symmetric polynomial in n complex variables $z_1, z_2, \ldots, z_n \in \mathcal{C}$:

$$P(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1, z_2, \ldots, z_n), \tag{3}$$

where

$$S_k(z_1, z_2, \ldots, z_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} z_{i_1} z_{i_2} \cdots z_{i_k}$$

are the elementary symmetric polynomials of degree $k = 1, 2, \ldots, n$, with

$$S_0(z_1, z_2, \ldots, z_n) := 1.$$

Clearly, one has $p(z) = P(z, z, \ldots, z)$. We say that an n-tuple $\{z_1, z_2, \ldots, z_n\}$ is a *solution* of P (or $p(z)$) if $P(z_1, z_2, \ldots, z_n) = 0$.

Definition 2.1. A polynomial $q(z) \in \overline{\mathcal{P}}_n$, given by

$$q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

is called *apolar with* $p(z) \in \overline{\mathcal{P}}_n$ if

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = 0. \tag{4}$$

This definition of apolarity extends the one in [1, Definition 3.3.1, p. 102] in that it depends on n. In particular, it allows the leading coefficients of $p(z)$ or $q(z)$ to be zero. The following lemma is easy to verify so we state it without a proof.

Lemma 2.2. *The n-tuple* $\{z_1, z_2, \ldots, z_n\} \subset \mathcal{C}$ *is a solution of* $p(z)$ *if and only if the polynomial* $q(z) = (z - z_1) \cdots (z - z_n)$ *is apolar with* $p(z)$.

With Lemma 2.2 in mind we extend the definition of a solution of $p(z)$.

Definition 2.3. Let $1 \leq m \leq n$. An m-tuple $\{z_1, z_2, \ldots, z_m\}$ is an *extended solution* of $p(z) \in \overline{\mathcal{P}}_n$ if the polynomial $q(z) = (z - z_1) \cdots (z - z_m)$ is apolar with $p(z)$.

In other words, an m-tuple $\{z_1, z_2, \ldots, z_m\}$ is an extended solution of $p(z)$ if

$$\sum_{k=0}^{m} \frac{a_{k+(n-m)}}{\binom{n}{k+(n-m)}} S_k(z_1, \ldots, z_m) = 0. \tag{5}$$

Clearly, when $m = n$ an extended solution is a solution.

It will be convenient to formally complete each extended solution $\{z_1, z_2, \ldots, z_m\}$ with $m - n$ infinities. That is, we say that

$$\{z_1, z_2, \ldots, z_m, \underbrace{\infty, \ldots, \infty}_{n-m}\} \tag{6}$$

is an extended solution of $p(z)$ if the polynomial $q(z) = (z - z_1) \cdots (z - z_m)$ is apolar with $p(z)$.

Definition 2.4. We say that a sequence of n-tuples in $\mathcal{C}^*$ *converges* to the n-tuple $\{z_1, z_2, \ldots, z_n\}$ if it is possible to order the elements $\{z_{1,m}, z_{2,m}, \ldots, z_{n,m}\}$ of the n-tuples in the sequence, so that $\lim_{m \to \infty} z_{k,m} = z_k$ for all $k = 1, \ldots, n$.

Lemma 2.5. *Let $\{Z_m\}$ be a sequence of extended solutions of $p(z) \in \mathcal{P}_n$. If $\{Z_m\}$ converges to Z, then Z is an extended solution of $p(z)$. In addition, at least one component of the extended solution Z is finite.*

Proof. It is clear that for all m large, the solutions Z_m have the form

$$Z_m = \{z_{1,m}, \dots, z_{n-s,m}, z_{n-s+1,m}, \dots, z_{n-s+\ell,m}, \underbrace{\infty, \dots, \infty}_{s-\ell}\},$$

where $z_{k,m} \in \mathcal{C}$ for $k = 1, \dots n - s + \ell$ with $\lim_{m\to\infty} z_{k,m} = z_k \in \mathcal{C}$ for all $k = 1, \dots, n-s$ and $\lim_{m\to\infty} z_{k,m} = \infty$ for all $k = n-s+1, \dots, n-s+\ell$. That is, the n-tuple Z has the form

$$Z = \{z_1, \dots, z_{n-s}, \underbrace{\infty, \dots, \infty}_{s}\}.$$

For all large m, the polynomial

$$q(z) = (z - z_{1,m}) \cdots (z - z_{n-s,m})(z - z_{n-s+1,m}) \cdots (z - z_{n-s+\ell,m})$$

is apolar to $p(z)$, meaning that

$$\sum_{k=0}^{n-s+\ell} \frac{a_{k+(s-\ell)}}{\binom{n}{k+(s-\ell)}} S_k(z_{1,m}, \dots, z_{n-s,m}, z_{n-s+1}, \dots, z_{n-s+\ell,m}) = 0.$$

Dividing this equality by $z_{n-s+1,m} \cdots z_{n-s+\ell,m}$ and taking the limit as m goes to infinity shows

$$0 = \sum_{k=\ell}^{n-s+\ell} \frac{a_{k+(s-\ell)}}{\binom{n}{k+(s-\ell)}} S_{k-\ell}(z_1, \dots, z_{n-s}) = \sum_{k=0}^{n-s} \frac{a_{k+s}}{\binom{n}{k+s}} S_k(z_1, \dots, z_{n-s}). \tag{7}$$

A comparison with (5) shows that Z is a solution of $p(z)$.

If all components of solution Z are infinity, that is $n = s$, then (7) becomes $0 = a_n$, contradicting the assumption that $p(z) \in \mathcal{P}_n$. □

Extended solutions of $p(z)$ are also solutions of its derivative of appropriate order, as the next lemma shows.

Lemma 2.6. *If $p(z) \in \overline{\mathcal{P}}_n$ has an extended solution $\{z_1, \dots, z_{n-s}\}$, then it is an extended solution of $p^{(s)}(z) \in \overline{\mathcal{P}}_{n-s}$, as well.*

Proof. The symmetrization of $p^{(s)}(z)$ with $n-s$ variables is

$$\sum_{m=s}^{n} m(m-1)\cdots(m-s+1)\frac{a_m}{\binom{n-s}{m-s}}S_{m-s}(z_1,\ldots,z_{n-s})$$

$$= n(n-1)\cdots(n-s+1)\sum_{m=s}^{n}\frac{a_m}{\binom{n}{m}}S_{m-s}(z_1,\ldots,z_{n-s})$$

$$= n(n-1)\cdots(n-s+1)\sum_{k=0}^{n-s}\frac{a_{k+s}}{\binom{n}{k+s}}S_k(z_1,\ldots,z_{n-s})$$

$$= 0,$$

where the last equality follows from the fact that $\{z_1,\ldots,z_{n-s}\}$ is a solution of $p(z)$, see (5). □

3 Loci and Extended Loci of Complex Polynomials

In this section, we present a connection between the extended solutions and the notion of a locus introduced in [2] and [3]. We introduce the notion of extended locus and after a review of the known facts about the loci, we conclude the section with Proposition 3.1, showing that, when bounded loci are considered, these two notions are the same. For proofs of the claims made in this section see [2] and [3].

Definition 3.1. Let Ω be a closed subset of $\mathcal{C}^*$. We say that Ω is an *(extended) locus holder* of $p(z) \in \mathcal{P}_n$ if Ω contains at least one point from every (extended) solution of $p(z)$. A minimal by inclusion locus holder Ω is called a *(extended) locus* of $p(z)$.

In [2] and [3] extended solutions of $p(z)$ were not considered, that is, these works considered only solutions with $m = n$ in (6). Thus, a priori, one should distinguish between the notion of a locus and the notion of extended locus given in Definition 3.1. For convenience we need the following definition:

Definition 3.2. If $p(z) \in \overline{\mathcal{P}}_n$ has degree $m \leq n$, then we say that ∞ is a zero of $p(z)$ of multiplicity $n-m$.

The notion of a locus is related to the classical Grace theorem, [1, p. 107], which gives a relationship between the zeros of two apolar polynomials. We present it in a slightly extended version. A circular domain, open or closed, is the interior or exterior of a circle, or a half-plane determined by a line in the complex plane.

Theorem 3.3 (Extended Grace Theorem). *If $p(z), q(z) \in \overline{\mathcal{P}}_n$ are apolar, then every circular domain, containing all the zeros of $p(z)$ contains at least one zero of $q(z)$ and vice versa.*

Thus, every circular domain containing the zeros of $p(z)$, in the sense of Definition 3.2, is a locus holder for $p(z)$. Since every locus holder contains a locus (similarly, every extended locus holder contains an extended locus) every $p(z) \in \mathcal{P}_n$ has a bounded locus. Alternatively, if $p(z) \in \overline{\mathcal{P}}_n$ has degree $m < n$, then every locus of $p(z)$ is unbounded. If $p(z) \in \mathcal{P}_n$ has a single root α with multiplicity n, then it has a unique locus $\{\alpha\}$. Henceforth, we exclude this trivial case and the case of a constant polynomial from our discussion. In general, a polynomial has infinitely many loci, but each one of them:

- contains all the zeros of $p(z)$;
- has no isolated points;
- is the closure of its interior.

Since every solution of $p(z)$ is an extended solution, every extended locus is a locus holder. Hence, every extended locus contains a locus. Conversely, starting with a locus, one can easily enlarge it to an extended locus holder, which on its part contains an extended locus. The question, whether the enlargement and extended locus can be chosen in such a way so that the latter contains the given locus, is still open.

Given a non-degenerate Möbius transformation

$$T(z) = (az + b)/(cz + d) \quad \text{with} \quad ad - bc \neq 0 \tag{8}$$

and a polynomial $p(z) \in \overline{\mathcal{P}}_n$, we define

$$T[p](z) := (cz + d)^n p(T(z)). \tag{9}$$

An argument similar to [1, Remark 3.3.4, p. 103] shows that if $p(z), q(z) \in \overline{\mathcal{P}}_n$ are apolar, then so are $T[p](z)$ and $T[q](z)$. Möbius transformations preserve the property of a set being a locus of a polynomial: the set Ω is a locus of $p \in \overline{\mathcal{P}}_n$ if and only if $T^{-1}(\Omega)$ is a locus of $T[p](z)$. An application of this fact and the Grace-Walsh-Szegő coincidence theorem leads to

Theorem 3.4. *Suppose $p(z) \in \overline{\mathcal{P}}_n$ has at least two distinct zeros. If all zeros of $p(z)$ are on the boundary of a closed circular domain D, then D is a locus of $p(z)$.*

If $p(z) \in \overline{\mathcal{P}}_n$, then

- the intersection of all loci of $p(z)$ is equal to the set of all zeros of $p(z)$;
- the intersection of all bounded loci of $p(z)$ is equal to the set of all zeros of $p^{(\ell)}(z)$, for $\ell = 0, 1, \ldots, n-1$.

For any polynomial $p(z)$ of degree n, the linear operator

$$\mathcal{D}_u(p; z) := np(z) - (z - u)p'(z)$$

is called the *polar derivative* of p with pole u. Clearly

$$\lim_{u\to\infty} \frac{1}{u}\mathcal{D}_u(p;z) = p'(z),$$

allowing us to extend the notation to $\mathcal{D}_\infty(p;z) := p'(z)$. The polar derivative of order ℓ is defined recursively:

$$\mathcal{D}_{u_1,\dots,u_{\ell-1},u_\ell}(p;z) := \mathcal{D}_{u_\ell}\Big(\mathcal{D}_{u_1,\dots,u_{\ell-1}}(p;z)\Big). \tag{10}$$

Note that the degree of $\mathcal{D}_{u_1,\dots,u_\ell}(p;z)$ as a polynomial in z is not bigger than $n-\ell$, and sometimes can be strictly smaller. The fundamental theorem for polar derivatives, see [1, p. 98], is the following:

Theorem 3.5 (Laguerre). *Let p be a polynomial of degree $n \geq 2$ and let $u \in \mathcal{C}$. A circular domain containing the zeros of p, but not the point u, contains all zeros of the polar derivative $\mathcal{D}_u(p;z)$.*

The notion of a locus allows us to strengthen Laguerre's theorem.

Theorem 3.6. *Let Ω be any bounded locus of $p \in \mathcal{P}_n$. If $u_1,\dots,u_\ell \in \mathcal{C}^* \setminus \Omega$, where $\ell \in \{1,\dots,n-1\}$, then all zeros of the polar derivative $D_{u_1,\dots,u_\ell}(p;z)$ are in Ω. Moreover, Ω is a minimal, by inclusion, closed set with this property.*

In addition, under the conditions of the last theorem, the polar derivative (10) has maximal degree, that is $n-\ell$. A corollary of Theorem 3.6 has bearing for the main result of this section.

Corollary 3.1. *Let $p(z) \in \mathcal{P}_n$ and let $s \in \{1,\dots,n-1\}$. If Ω is bounded locus of $p(z)$, then it is a locus holder for $D_{u_1,\dots,u_s}(p;z)$, whenever $u_1,\dots,u_s \in \mathcal{C}^* \setminus \Omega$.*

In particular, if Ω is a bounded locus of $p(z) \in \mathcal{P}_n$, then it is a locus holder for all its derivatives $p^{(\ell)}(z)$, $\ell = 1,2,\dots,n-1$. A bit more could be said in connection with the last corollary. If Ω is a locus of $p(z) \in \mathcal{P}_n$, then the zeros of $D_{u_1,\dots,u_s}(p;z)$, when the poles $u_1,\dots,u_s$ go over $\mathcal{C} \setminus \Omega$ and s goes over the set $\{1,\dots,n-1\}$, form a set that is dense in Ω. Corollary 3.1 is needed for the main result of this section.

Proposition 3.1. *A bounded subset of $\mathcal{C}$ is an extended locus of $p(z) \in \mathcal{P}_n$ if and only if it is a locus of $p(z)$.*

Proof. Let Ω_{ext} be an extended locus. Since the set of all extended solutions of $p(z)$ contains the set of all solutions, Ω_{ext} is a locus holder for $p(z)$, hence it contains a locus, call it Ω, of $p(z)$. We show next that Ω is an extended locus holder of $p(z)$, which, together with the minimality property of the extended loci, implies that $\Omega_{\text{ext}} = \Omega$. Let $Z = \{z_1, z_2, \dots, z_m\}$ be an extended solution of $p(z)$, $m \leq n$. By Lemma 2.6, Z is a extended solution of $p^{(n-m)}(z) \in \mathcal{P}_m$, in fact a solution. Since Ω_{ext}

is a bounded set, so is Ω. Then, by Corollary 3.1, Ω is a locus holder of $p^{(n-m)}(z)$, that is, it contains at least one component of Z. This shows that Ω is an extended locus holder of $p(z)$. □

3.1 Examples of Loci

Polynomials of Degree 2. Let

$$p(z) = (z - \alpha_1)(z - \alpha_2) = z^2 + a_1 z + a_0$$

be a polynomial with $\alpha_1 \neq \alpha_2$. The solutions of $p(z)$ are $\{z, T(z)\}$, where $z \in \mathcal{C}$ and $T(z)$ is the symmetric Möbius transformation

$$T(z) = -\frac{a_1 z + 2a_0}{2z + a_1}. \tag{11}$$

The elementary properties of symmetric Möbius transformations imply that every closed circular domain, D, having α_1 and α_2 on its boundary, is a locus of $p(z)$. More loci can be constructed in the following way. Let C be a domain in D, having a simple (not self intersecting) Jordan curve for a boundary. Then, $(D \setminus C) \cup T(C)$ is a locus of $p(z)$.

Polynomials of Degree 3. Let

$$p(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) = z^3 + a_2 z^2 + a_1 z + a_0.$$

The zeros of a polynomial of degree three, with distinct zeros, define a closed circular domain. Hence, by Theorem 3.4, it (as well as the closure of its complement) is a locus. The drawback of such loci is that if the zeros are close to being co-linear, then this locus is close to being a half-plane, in other words, too big. On the other hand, the commentary after Theorem 3.3 shows the existence of a locus in the smallest disk containing the zeros.

When the smallest disc containing the zeros, call it S, is the same as one of the circular domains, call it D, having the zeros on its boundary, there is nothing to do. So, suppose $S \neq D$. The latter condition implies that (a) the three zeros are distinct, and (b) one of the zeros, say α_3, is in the interior of S and the segment $[\alpha_1, \alpha_2]$ is a diameter of S. Let Γ be the semi-circle of the boundary of S, that is not on the same side with α_3 with respect to the segment $[\alpha_1, \alpha_2]$. Finally, define the rational-quadratic map, obtained by solving $P(z_1, z, z) = 0$ for z_1:

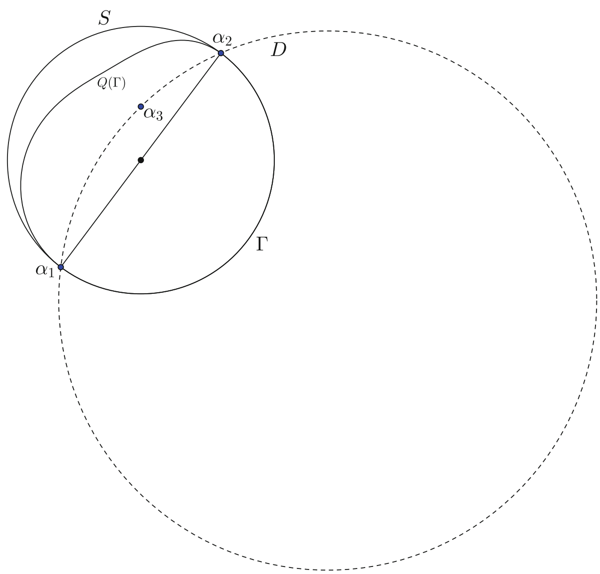

Fig. 1 Theorem 3.7 for $p(z) = (z + 2i)(z - (3 + 2i))(z - (1.5 + i))$

$$Q(z) := -\frac{a_2 z^2 + 2a_1 z + 3a_0}{3z^2 + 2a_2 z + a_1}.$$

We have the following result. Refer to Fig. 1 for an illustration.

Theorem 3.7. *The closed, bounded domain with boundary* $\Gamma \cup Q(\Gamma)$ *is a locus of* $p(z)$ *in the smallest disk containing the zeros of* $p(z)$.

Finally, every polynomial of degree three with distinct roots has a locus consisting of two discs that we now proceed to describe. Denote by $e_1 := e^{-i\pi/3}$ and $e_2 := e^{i\pi/3}$ two of the cube roots of -1, and let W be the symmetric Möbius transformation uniquely defined by $W(e_k) = \alpha_k$ for $k = 1, 2, 3$. Let B_1 be the image under W of the lower half-plane determined by the line through e_1 and 0. Let B_2 be the image under W of the upper half-plane determined by the line through 0 and e_2.

Theorem 3.8. *The set* $B_1 \cup B_2$ *is a locus for* $p(z)$. *Moreover, it is*

1. *bounded if and only if* $[\alpha_1, \alpha_2]$ *is the strictly longest segment between the zeros;*
2. *contained in the closed disc with diameter* $[\alpha_1, \alpha_2]$ *if and only if* $\alpha_3 \in (\alpha_1, \alpha_2)$.

The theorem is illustrated in Fig. 2.

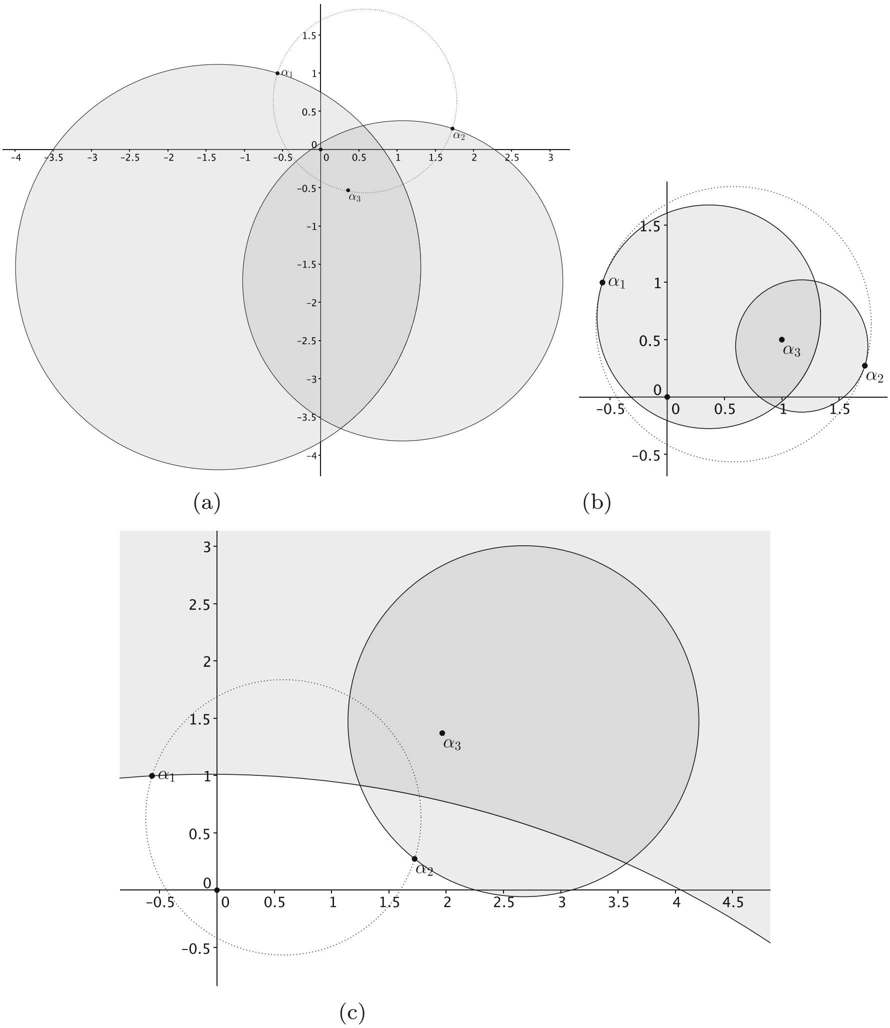

Fig. 2 Illustrating Theorem 3.8. (**a**) Bounded locus. (**b**) Locus in the smallest disc. (**c**) Unbounded locus

4 Argument Coincidence Theorem

This section presents two results that follow the spirit of Grace-Walsh-Szegő coincidence theorem. In what follows, an extended solution will be called simply a solution, for brevity.

For $\alpha, \beta \in [-\pi, \pi]$, with $\alpha \leq \beta$, and $u \in \mathcal{C}$, define the sector

$$S_u(\alpha, \beta) := \{u + re^{i\varphi} : r \geq 0,\ \alpha \leq \varphi \leq \beta\}.$$

Definition 4.1. Sector $S_u(\alpha, \beta)$ is called a *zero-free sector* for the polynomial $p(z) \in \mathcal{P}_n$ if it does not contain a zero of $p^{(k)}(z)$ for all $k \in \{0, 1, \ldots, n-1\}$.

In other words, a sector is zero-free if it does not contain a zero of $p(z), p'(z), \ldots$, and of $p^{(n-1)}(z)$.

Theorem 4.2 (Argument Coincidence). *Let $S_u(\alpha, \beta)$ be a zero-free sector for $p(z) \in \mathcal{P}_n$ with $\beta - \alpha < \pi$. Suppose there is a solution of $p(z)$ in $S_u(\alpha, \beta)$. Then, there exists a solution of $p(z)$ of the form*

$$\{u + s_1 e^{i\psi}, u + s_2 e^{i\psi}, \ldots, u + s_\ell e^{i\psi}\} \tag{12}$$

for some $\psi \in [\alpha, \beta]$, where $s_k \geq 0$ for all $k = 1, 2, \ldots, \ell$, $\ell \geq 2$, and at least one of $\{s_1, s_2, \ldots, s_\ell\}$ is strictly positive.

Proof. No solution $\{z_1, z_2, \ldots, z_\ell\}$ of $p(z)$ inside of $S_u(\alpha, \beta)$, where the displayed points are all finite, can have equal entries. Indeed, if $z_1 = \cdots = z_\ell =: a$, then by Lemma 2.6, a is a zero of $p^{(n-\ell)}(z)$, contradicting the fact that $S_u(\alpha, \beta)$ is a zero-free sector. This shows that if a solution of the form (12) exists, then at least one of $\{s_1, s_2, \ldots, s_\ell\}$ is strictly positive. This argument also implies that it is not possible to have $\ell = 1$.

Given a solution Z of $p(z)$, define the angles $\alpha(Z), \beta(Z) \in [-\pi, \pi]$, with $\alpha(Z) \leq \beta(Z)$, so that $S_u(\alpha(Z), \beta(Z))$ is the smallest sector containing the points in Z.

Let $\{Z_m\}$ be a sequence of solutions in $S_u(\alpha, \beta)$ such that

$$\lim_{m\to\infty} \big(\beta(Z_m) - \alpha(Z_m)\big) = \inf\big\{\beta(Z) - \alpha(Z) : \ Z \text{ is a solution of } p(z) \text{ in } S_u(\alpha, \beta)\big\}.$$

Without loss of generality, or else choose a subsequence, the following limits exist:

$$\alpha' := \lim_{m\to\infty} \alpha(Z_m) \text{ and } \beta' := \lim_{m\to\infty} \beta(Z_m).$$

Without loss of generality, or else choose a subsequence, we may assume that the sequence of solutions $\{Z_m\}$ has a limit

$$Z = \{z_1, \ldots, z_\ell, \underbrace{\infty, \ldots, \infty}_{n-\ell}\},$$

which is a solution of $p(z)$ in $S_u(\alpha', \beta')$, by Lemma 2.5, with at least one finite point. By the first paragraph of this proof, Z has at least two distinct finite points.

If $\alpha' = \beta'$, we are done. Suppose that $\alpha' < \beta'$ and suppose Z is a solution of $p(z)$ contained in $S_u(\alpha', \beta')$ with minimal number of (finite) points on the boundary of $S_u(\alpha', \beta')$. Since those points are at least two, we let $z_1 = u + r_1 e^{i\alpha'}$, $z_2 = u + r_2 e^{i\beta'}$, where $z_1 \neq z_2$ and they can be chosen to be distinct from u as well.

By Lemma 2.6, Z is a solution of $p^{(n-\ell)}(z)$. Let $P^{(n-\ell)}(z_1,\ldots,z_\ell)$ be the symmetrization of $p^{(n-\ell)}(z) \in \mathcal{P}_\ell$. Consider the Möbius transformation $w = T(v)$, defined by $P^{(n-\ell)}(v, w, z_3,\ldots,z_\ell) = 0$ after fixing the points $z_3,\ldots,z_\ell$. Since Z is a solution of $p^{(n-\ell)}(z)$, we have $z_2 = T(z_1)$.

If the Möbius transformation $w = T(v)$ is degenerate, then, $T(v) = z_2$ for all $v \in \mathcal{C}$. Hence, $P^{(n-\ell)}(v, z_2,\ldots,z_\ell) = 0$ for all $v \in \mathcal{C}$ and by moving v to the interior of $S_u(\alpha', \beta')$ we reach a contradiction with the assumption that solution Z has minimal number of points on the border of $S_u(\alpha', \beta')$.

Assume now that the Möbius transformation $w = T(v)$ is non-degenerate (in particular, it is a conformal map) with fixed points ζ_1, ζ_2.

Let C be the unique circle, called the *joint circle* of the pair z_1, z_2, that passes through the points $z_1, z_2, \zeta_1, \zeta_2$. Such a circle exists, since T is an involution:

$$T(T(v)) = v \text{ for all } v \in \mathcal{C}.$$

The joint circle has the property that it is invariant under T. That is, when v moves over C, $w = T(v)$ moves over C in the opposite direction and they meet over the fixed points ζ_1, ζ_2. In particular, v and w are on different arcs of C defined by ζ_1, ζ_2. Finally, if v is inside C, $w = T(v)$ is outside of C, and vice versa.

If one of the fixed points, say ζ_1, is inside of $S_u(\alpha', \beta')$, then $\{\zeta_1, \zeta_1, z_3,\ldots,z_\ell\}$ is a solution with smaller number of points on the border of the sector, a contradiction. So, the two fixed points ζ_1 and ζ_2 are not inside the sector $S_u(\alpha', \beta')$. This implies that the vertex u of the sector $S_u(\alpha', \beta')$ is also not inside the joint circle C, or otherwise one of the fixed points will be on the arc between z_1 and z_2 inside the sector $S_u(\alpha', \beta')$.

Having that in mind, the joint circle has to cross the border of $S_u(\alpha', \beta')$ in two more points: $u_1 = \rho_1 e^{i\alpha'}$ and $u_2 = \rho_2 e^{i\beta'}$. As the fixed points are not inside $S_u(\alpha', \beta')$, we have either

(1) $\rho_1 \geq r_1$ and $\rho_2 \leq r_2$, or
(2) $\rho_1 \leq r_1$ and $\rho_2 \geq r_2$.

By symmetry, it is sufficient to consider only case (1). Denote by $\nu \in [0, \pi/2]$ the angle at which the circle C crosses the border of $S_u(\alpha', \beta')$ at the point z_1 and by $\mu \in [0, \pi/2]$ the angle at which the circle C crosses the border of $S_u(\alpha', \beta')$ at the point z_2, see Fig. 3. If $\nu < \mu$, we move z_2 along the border of $S_u(\alpha', \beta')$ and outside the circle C, see Fig. 3. Then, z_1 will move along a circular arc inside the circle C. As the Möbius transformation preserves the angle, z_1 will move inside the sector $S_u(\alpha', \beta')$. This produces a solution $\{z_1', z_2', z_3,\ldots,z_\ell\}$ within the sector $S_u(\alpha', \beta')$, but with fewer points on its boundary, a contradiction.

If $\nu > \mu$, we move z_1 along the border of $S_u(\alpha', \beta')$ and outside the circle C, see Fig. 4. Then, z_2 will move along a circular arc inside the circle C. As the Möbius transformation preserves the angle, z_2 will move inside the sector $S_u(\alpha', \beta')$. This produces a solution $\{z_1', z_2', z_3,\ldots,z_\ell\}$ within the sector $S_u(\alpha', \beta')$, but with fewer points on its boundary, a contradiction.

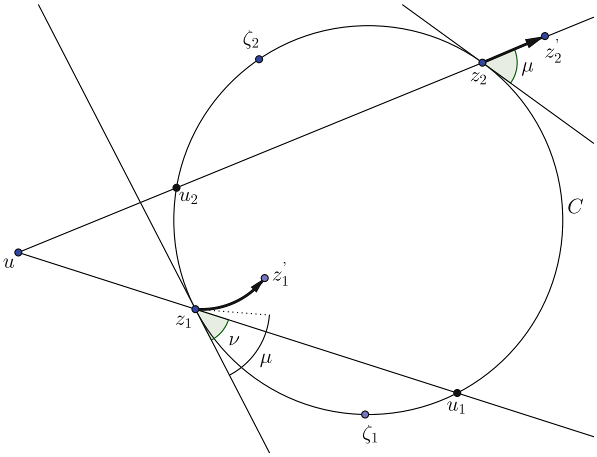

Fig. 3 Illustrating the proof of Theorem 4.2, case $\nu < \mu$

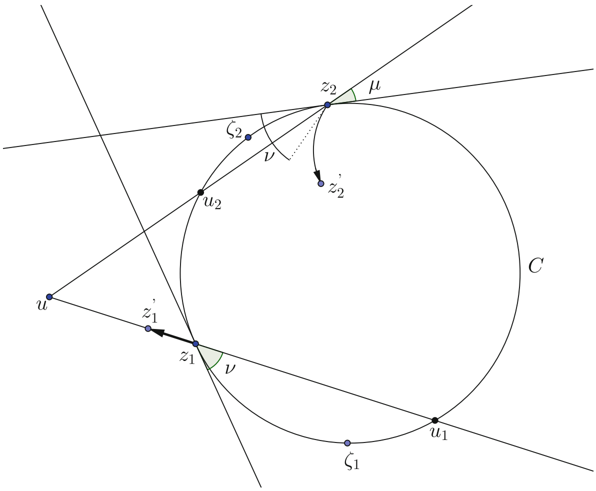

Fig. 4 Illustrating the proof of Theorem 4.2, case $\nu > \mu$

The final case $\nu = \mu$ is similar to the previous ones, but has a twist. We move z_1 along the border of $S_u(\alpha', \beta')$ and outside the circle C, see Fig. 5. We know that z_2 has to enter C and has to move along a circular arc. Now, the two circles, denoted C_1 and C_2, intersecting C at an angle μ do not both enter the sector $S_u(\alpha', \beta')$. If z_2 moves along C_2 it enters the sector, while if it moves along C_1, it does not. We now show that z_2 moves along C_2. Consider again the Möbius transformation $T(z)$ defined by z_1 and z_2 as above, and recall that when z is on the joint circle C, then so is $T(z)$. The circle along which z_2 moves is the image under T of the line through u, z_1 and u_1. Starting at $z = z_1$ and moving along C, the image $T(z)$ starts at z_2 and

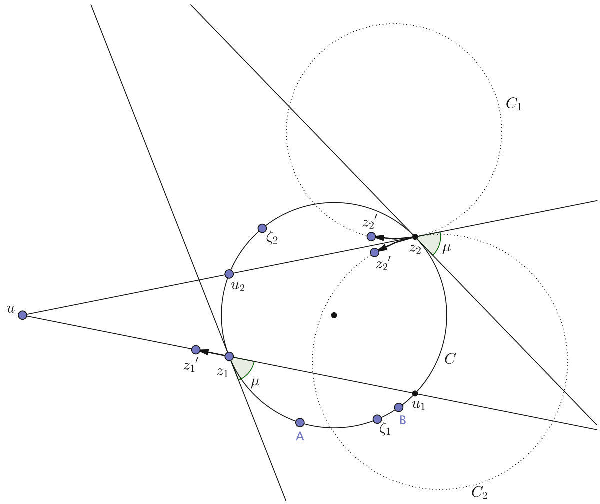

Fig. 5 Illustrating the proof of Theorem 4.2, case $\nu = \mu$

moves along C in the opposite direction, until they meet at the fixed point ζ_1. After that, while z goes over the arc ζ_1, B, u_1, its image $T(z)$ stays on the arc ζ_1, A, z_1. Thus, $T(u_1)$ is on the arc ζ_1, A, z_1. But u_1 is at the intersection of C and the line through u, z_1, u_1. Hence, $T(u_1)$ is at the intersection of C and the circle traversed by z_2. This shows that z_2 goes over C_2 since C_1 does not intersect C in the arc ζ_1, A, z_1. (Note that this argument uses the fact that the angle z_1, u, u_2 of the sector $S_u(\alpha', \beta')$ is less than π.) □

It is important to see that the premises of Theorem 4.2 are not vacuous. That is, there is a zero-free sector that contains a solution of $p(z)$. This is the goal of the next example.

Example 4.3. Consider the polynomial $p(z) = (z-1)(z-i)(z-(1+i))$. Denote its zeros by $\alpha_1, \alpha_2, \alpha_3$ as shown in Fig. 6. The zeros of its derivative are denoted by β_1, β_2 and are equal to

$$\frac{2}{3} + \frac{\sqrt{2}}{6} + \left(\frac{2}{3} - \frac{\sqrt{2}}{6}\right)i \quad \text{and} \quad \frac{2}{3} - \frac{\sqrt{2}}{6} + \left(\frac{2}{3} + \frac{\sqrt{2}}{6}\right)i.$$

Finally, the zero of its second derivative is $\gamma := \frac{2}{3} + \frac{2}{3}i$. Figure 6 shows a solution of the polarization

$$P(z_1, z_2, z_3) = z_1 z_2 z_3 - \frac{2+2i}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) + i(z_1 + z_2 + z_3) + (1-i)$$

that is in a zero-free sector with vertex at the origin. The solution is approximately

$$(z_1, z_2, z_3) = (0.55 + 0.45i, 2 + 1.5i, 0.4481 + 0.2623i).$$

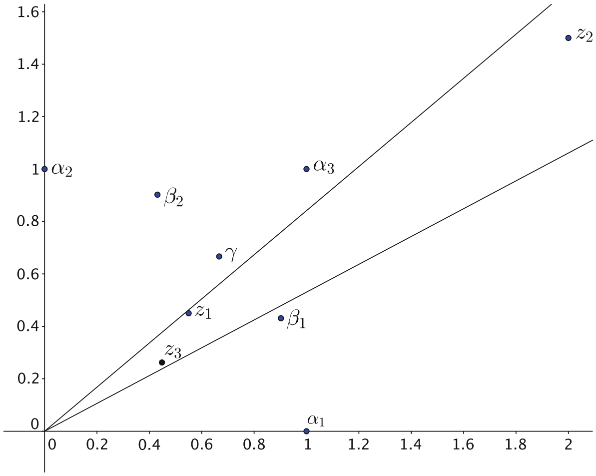

Fig. 6 Illustrating the premises of Theorem 4.2

Using ideas similar to those used in the proof of Theorem 4.2, it is possible to give another extension in the spirit of the Grace-Walsh-Szegő coincidence theorem.

The closed set enclosed between two concentric circles is referred to as *annulus* or *circular ring*. By the *width* of an annulus, we understand the distance between the two concentric circles. A solution of $p \in \mathcal{P}_n$ is called a *bi-solution* if the set $\{z_1, \ldots, z_n\}$ contains at most two distinct points. In the next theorem, by a solution we understand a solution consisting of n points.

Theorem 4.4 (Bi-solution Coincidence). *If $p(z) \in \mathcal{P}_n$ has a solution contained in an annulus, then either*

(a) $p(z)$ has a solution on a circle concentric to the annulus and inside of it; or
(b) $p(z)$ has a bi-solution in the annulus.

Proof. Let A be the annulus containing the solution of $p(z)$. Since the limit of a sequence of solutions of $p(z)$ is also a solution, let A^* be an annulus, concentric to A, contained in A, with the smallest width, containing a solution of $p(z)$. (Note that since the annulus is a closed and bounded set, a limit of a sequence of solutions in it with n points is a solution in the annulus with n points as well.) If the width of A^* is zero, then A^* is a circle and we arrive at conclusion (a). □

Assume now that the width of A^* is strictly positive. Clearly, A^* will have an element of the solution on its inner boundary and an element on its outer boundary. Among all solutions of $p(z)$ in A^* let $Z = \{z_1, \ldots, z_n\}$ be one with fewest elements on the boundary of A^*.

If the degree of $p(z)$ is one or two, then any solution Z is trivially a bi-solution. So, assume the degree of $p(z)$ is at least three.

We want to show that all elements of the solution Z are on the boundary of A^*. Indeed, let z_1 be on the boundary of A^* and suppose there is a z_2 in the interior of A^*. Consider the Möbius transformation $w = T(v)$, defined by $P(v, w, z_3, \ldots, z_n) = 0$ after fixing the points $z_3, \ldots, z_n$. (Since Z is a solution of $p(z)$, we have $z_2 = T(z_1)$.) If T is degenerate, then we can freely move z_1 in the interior of A^* and arrive at a solution with fewer elements on the boundary of A^*. If T is non-degenerate, then it is an open map, and we can perturb z_1 to the interior of A^*, so that $z_2 = T(z_1)$ remains in the interior of A^*. We arrive at a new solution with fewer elements on the boundary. This contradiction shows that all elements of Z are on the boundary of A^*.

Thus, *every* solution Z of $p(z)$ in A^* has the form

$$Z = \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_n\},$$

where $z_1, \ldots, z_k$ are on the outer boundary circle of A^* and $z_{k+1}, \ldots, z_n$ are on the inner boundary circle. Next, we specialize the solution Z further. Among all solutions of $p(z)$ with points on the boundary of A^*, let Z be the one that minimizes the sum

$$\operatorname{diam}\{z_1, \ldots, z_k\} + \operatorname{diam}\{z_{k+1}, \ldots, z_n\},$$

where diam stands for the diameter of a set.

Next, we show that the joint circle of any two points $z_r, z_s \in \{z_1, \ldots, z_k\}$ coincides with the outer boundary circle of A^* and similarly the joint circle of any two points $z_r, z_s \in \{z_{k+1}, \ldots, z_n\}$ coincides with the inner boundary circle of A^*. Since the arguments are analogous we deal with the outer circle only. Suppose that the joint circle of z_1 and z_2 is not the outer boundary circle of A^*. Consider the Möbius transformation $w = T(v)$, defined by $P(v, w, z_3, \ldots, z_n) = 0$ after fixing the points $z_3, \ldots, z_n$. (Since Z is a solution of $p(z)$, we have $z_2 = T(z_1)$.) If T is degenerate, then we can freely move z_1 in the interior of A^*. Otherwise, T has two distinct fixed points ζ_1, ζ_2. One of the fixed points, say ζ_1, of T has to be in the unbounded component of the exterior of A^*, see Fig. 7. Then, moving z_1 towards ζ_2 along the joint circle forces $z_2 = T(z_1)$ to move also towards ζ_2 along the joint circle in the opposite direction. In this way both z_1 and z_2 enter the interior of A^* creating a solution of $p(z)$ in A^* that does not have all of its elements on the boundary of A^*. This is a contradiction.

Suppose now that $\operatorname{diam}\{z_1, \ldots, z_k\} > 0$. Let z_1 and z_k be such that one of the arcs between them contains the points $z_2, \ldots, z_{k-1}$, and the segment between that arc and the cord $[z_1, z_k]$ has diameter equal to $\operatorname{diam}\{z_1, \ldots, z_k\}$. Call the arc ℓ. One of the fixed points, say ζ_1 of the Möbius transformation T, determined by z_1 and z_k, must be on the arc ℓ, so we can move z_1 towards ζ_1 forcing $z_k = T(z_1)$ to move towards ζ_1 as well. This may not automatically reduce the diameter of the set $\{z_1, \ldots, z_k\}$ since both z_1 and z_k may have been multiple points. But it is clear that we can repeat this argument until we obtain a solution of $p(z)$ with a smaller diameter of the set of

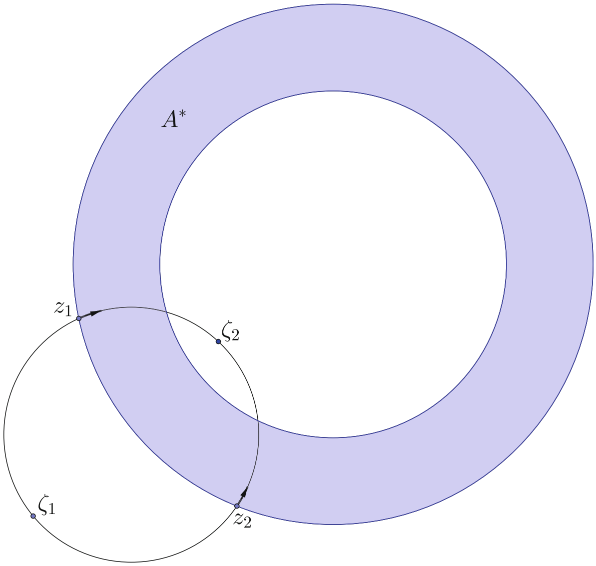

Fig. 7 Illustrating part of the proof of Theorem 4.4

points on the outer circle of A^*. This contradiction shows that there is a bi-solution with one multiple point on the outer boundary circle of A^* and one multiple point on the inner. □

We conclude with a result that complements Theorem 4.4. For $u, v, w \in \mathcal{C}$, define the infinite strip

$$S(u, v, w) := \{u + t(v - u) + sw : t \in [0, 1], s \in \mathbb{R}\}.$$

An extended solution $\{z_1, \dots, z_\ell\}$ of $p \in \mathcal{P}_n$ is called a *bi-solution* if the set $\{z_1, \dots, z_\ell\}$ contains at most two distinct points.

Theorem 4.5 (Bi-solution in a Strip). *If $p(z) \in \mathcal{P}_n$ has an extended solution contained in a strip, then either*

(a) $p(z)$ has an extended solution on a line parallel to the strip and inside of it; or
(b) $p(z)$ has an extended bi-solution in the strip.

Proof. Let S be a strip containing a solution of $p(z)$. By Lemma 2.5, the limit of a sequence of extended solutions of $p(z)$ is also an extended solution with at least one finite component. Using a limiting argument, we can find a strip S^* parallel to S, contained in S, with the smallest width, containing a solution of $p(z)$. If the width of S^* is zero, then S^* is a line in the strip parallel to it, and we arrive at conclusion (a).

If the width of S^* is strictly positive, then it will have an element of an extended solution on each of its boundary lines. The next step is to show that *every* extended solution Z of $p(z)$ in S^* has the form

$$Z = \{z_1, \dots, z_k, z_{k+1}, \dots, z_\ell\},$$

where $z_1, \dots, z_k$ are on one of the boundary lines of S^* and $z_{k+1}, \dots, z_\ell$ are on the other boundary line. The rest of the argument is analogous to that in the proof of Theorem 4.4 and is omitted. □

Of course, Theorem 4.5 is trivial if the strip contains a zero of $p(z)$ or of one of its derivatives. Thus, the interesting case is when the strip is a zero-free strip.

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Vector Inequalities for a Projection in Hilbert Spaces and Applications

Silvestru Sever Dragomir

Abstract In this paper we establish some vector inequalities related to Schwarz and Buzano results. Applications for norm and numerical radius inequalities of two bounded operators are given as well.

Keywords Hilbert space • Schwarz inequality • Buzano inequality • Orthogonal projection • Numerical radius • Norm inequalities

1 Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz's inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1}$$

The equality case holds in (1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [23]) established the following refinement of (1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

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and by (2) we get

$$\begin{aligned}\|x\| \|y\| &\geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \\ &\geq 2 |\langle x, e\rangle \langle e, y\rangle| - |\langle x, y\rangle|,\end{aligned}$$

which implies the *Buzano's inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle x, e\rangle \langle e, y\rangle| \tag{3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

For other Schwarz and Buzano related inequalities in inner product spaces, see [1–10, 12–15, 17, 19–25, 27–36], and the monographs [11, 16] and [18].

Now, let us recall some basic facts on *orthogonal projection* that will be used in the sequel.

If K is a subset of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, the set of *vectors orthogonal* to K is defined by

$$K^{\perp} := \{x \in H : \langle x, k\rangle = 0 \text{ for all } k \in K\}.$$

We observe that $K^{\perp}$ is a *closed subspace* of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^{\perp}$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $K^{\perp}$. Therefore, H is the *internal Hilbert direct sum* of V and $V^{\perp}$, and we denote that as $H = V \oplus V^{\perp}$.

The linear operator $P_V : H \to H$ that maps x to v is called *the orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all *bounded self-adjoint* operators P such that $P^2 = P$. Specifically, the orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of V, which minimizes the distance $\|x - v\|$. This provides the geometrical interpretation of $P_V(x)$: it is *the best approximation* to x by elements of V.

Projections P_U and P_V are called *mutually orthogonal* if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. The sum of the two projections P_U and P_V is a projection only if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$.The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U \cap V}$.

A family $\{e_j\}_{j \in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j, k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J} \left|\langle x, e_j\rangle\right|^2 \le \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j\in J} \left|\langle x, e_j\rangle\right|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j\in J}$ an othonormal basis in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j\,,\ x \in H. \tag{4}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\,,\ x, y \in H \text{ and } \langle P_{\mathcal{E}} x, x\rangle = \sum_{j\in J} \left|\langle x, e_j\rangle\right|^2,\ x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}\,,\ \|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle\, e,\ x \in H,$

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle\,,\ x, y \in H \tag{5}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right] \ge |\langle P_e x, y\rangle| \tag{6}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

Motivated by the above results we establish in this paper some vector inequalities for an orthogonal projection P that generalizes amongst others the Buzano's inequality (6). Applications for norm and numerical radius inequalities are provided as well.

2 Vector Inequalities for a Projection

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathscr{B}(H)$ that $0 \le P \le 1_H$.

The following result holds:

Theorem 1. *Let $P : H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$\|x\| \, \|y\| - \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \ge |\langle x, y\rangle - \langle Px, y\rangle| . \tag{7}$$

and

$$\|x\| \, \|y\| - \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} \ge |\langle Px, y\rangle| . \tag{8}$$

Proof. Using the properties of projection, we have

$$\begin{aligned} \langle x - Px, y - Py\rangle &= \langle x, y\rangle - \langle Px, y\rangle - \langle x, Py\rangle + \langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, y\rangle + \left\langle P^2 x, y\right\rangle \\ &= \langle x, y\rangle - \langle Px, y\rangle \end{aligned} \tag{9}$$

for any $x, y \in H$.

By Schwarz's inequality we have

$$\|x - Px\|^2 \, \|y - Py\|^2 \ge |\langle x - Px, y - Py\rangle|^2 \tag{10}$$

for any $x, y \in H$.

Since, by (7), we have

$$\|x - Px\|^2 = \|x\|^2 - \langle Px, x\rangle , \quad \|y - Py\|^2 = \|y\|^2 - \langle Py, y\rangle ,$$

then by (10) we have

$$\left(\|x\|^2 - \langle Px, x\rangle\right) \left(\|y\|^2 - \langle Py, y\rangle\right) \ge |\langle x, y\rangle - \langle Px, y\rangle|^2 \tag{11}$$

for any $x, y \in H$.

Using the elementary inequality that holds for any real numbers a, b, c, d

$$(ac - bd)^2 \ge \left(a^2 - b^2\right)\left(c^2 - d^2\right) ,$$

we have

$$\left(\|x\|\,\|y\| - \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}\right)^2 \geq \left(\|x\|^2 - \langle Px,x\rangle\right)\left(\|y\|^2 - \langle Py,y\rangle\right) \tag{12}$$

for any $x, y \in H$.

Since

$$\|x\| \geq \langle Px,x\rangle^{1/2},\ \ \|y\| \geq \langle Py,y\rangle^{1/2},$$

then

$$\|x\|\,\|y\| - \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} \geq 0,$$

for any $x, y \in H$.

By (11) and (12) we get

$$\left(\|x\|\,\|y\| - \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}\right)^2 \geq |\langle x,y\rangle - \langle Px,y\rangle|^2$$

for any $x, y \in H$, which, by taking the square root, is equivalent to the desired inequality (7).

Observe that, if P is an orthogonal projection, then $Q := 1_H - P$ is also a projection. Indeed we have

$$Q^2 = (1_H - P)^2 = 1_H - 2P + P^2 = 1_H - P = Q.$$

Now, if we write the inequality (7) for the projection Q we get the desired inequality (8).

Corollary 1. *With the assumptions of Theorem 1, we have the following refinements of Schwarz inequality:*

$$\begin{aligned}\|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq |\langle Px,y\rangle| + |\langle x,y\rangle - \langle Px,y\rangle| \geq |\langle x,y\rangle|\end{aligned} \tag{13}$$

and

$$\begin{aligned}\|x\|\,\|y\| &\geq \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\left(\|y\|^2 - \langle Py,y\rangle\right)^{1/2} + |\langle Px,y\rangle| \\ &\geq |\langle x,y\rangle - \langle Px,y\rangle| + |\langle Px,y\rangle| \geq |\langle x,y\rangle|\end{aligned} \tag{14}$$

for any $x, y \in H$.

Remark 1. Since

$$|\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle| - |\langle Px, y\rangle|$$

then by the first inequality in (13) we have

$$\|x\| \|y\| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle| - |\langle Px, y\rangle|$$

that produces the inequality

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} - |\langle Px, y\rangle| \geq 0 \tag{15}$$

for any $x, y \in H$.

We notice that the second inequality follows by Schwarz's inequality for the nonnegative self-adjoint operator P.

Since

$$|\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle Px, y\rangle| - |\langle x, y\rangle|$$

then by (13) we have

$$\begin{aligned}\|x\| \|y\| &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \\ &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle Px, y\rangle| - |\langle x, y\rangle|,\end{aligned}$$

which implies that

$$\begin{aligned}\|x\| \|y\| + |\langle x, y\rangle| &\geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle Px, y\rangle| \\ &\geq 2|\langle Px, y\rangle|\end{aligned}$$

and is equivalent to

$$\begin{aligned}\frac{1}{2}[\|x\| \|y\| + |\langle x, y\rangle|] &\geq \frac{1}{2}\left[\langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle Px, y\rangle|\right] \\ &\geq |\langle Px, y\rangle|\end{aligned} \tag{16}$$

for any $x, y \in H$.

The inequality between the first and last term in (16), namely

$$\frac{1}{2}[\|x\| \|y\| + |\langle x, y\rangle|] \geq |\langle Px, y\rangle| \tag{17}$$

for any $x, y \in H$ is a generalization of Buzano's inequality (3).

From the inequality (14) we can state that

$$\|x\| \|y\| - |\langle Px, y\rangle| \ge \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} \ge |\langle x, y\rangle - \langle Px, y\rangle| \tag{18}$$

for any $x, y \in H$.

From the inequality (14) we also have

$$\begin{aligned}\|x\| \|y\| &\ge \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + |\langle Px, y\rangle| \\ &\ge |\langle x, y\rangle - \langle Px, y\rangle| + |\langle Px, y\rangle| \ge |\langle Px, y\rangle| - |\langle x, y\rangle| + |\langle Px, y\rangle| \\ &= 2\,|\langle Px, y\rangle| - |\langle x, y\rangle|,\end{aligned}$$

which implies that

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x, y\rangle|\right] \ge \frac{1}{2}\left[\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}\right] + \frac{1}{2}\left[|\langle Px, y\rangle| + |\langle x, y\rangle|\right] \ge |\langle Px, y\rangle| \tag{19}$$

for any $x, y \in H$.

The case of orthonormal families which is related to Bessel's inequality is of interest.

Let $\mathscr{E} = \{e_j\}_{j\in J}$ be an othonormal family in H. Then for any $x, y \in H$ we have from (13) and (14) the inequalities

$$\begin{aligned}\|x\| \|y\| &\ge \left(\sum_{j\in J} \left|\langle x, e_j\rangle\right|^2\right)^{1/2} \left(\sum_{j\in J} \left|\langle y, e_j\rangle\right|^2\right)^{1/2} \\ &\quad + \left|\langle x, y\rangle - \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| \\ &\ge \left|\sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| + \left|\langle x, y\rangle - \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right| \ge |\langle x, y\rangle|\end{aligned} \tag{20}$$

and

$$\|x\| \|y\| \ge \left(\|x\|^2 - \sum_{j\in J} \left|\langle x, e_j\rangle\right|^2\right)^{1/2} \left(\|y\|^2 - \sum_{j\in J} \left|\langle y, e_j\rangle\right|^2\right)^{1/2} \tag{21}$$

$$+\left|\left\langle \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right|$$

$$\geq \left|\langle x,y\rangle - \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right| + \left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right| \geq |\langle x,y\rangle| .$$

By (15) and (16) we have

$$\begin{aligned} &\|x\|\,\|y\| - |\langle x,y\rangle| \\ &\geq \left(\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\left(\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2} - \left|\left\langle \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right| \geq 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} \frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] &\geq \frac{1}{2}\left(\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\left(\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2} \\ &\quad + \frac{1}{2}\left|\left\langle\left\langle \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right\rangle\right| \\ &\geq \left|\left\langle \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right\rangle\right| \end{aligned} \tag{23}$$

for any $x, y \in H$.

The inequality between the first and last term in (23) provides a generalization of Buzano's inequality for orthonormal families $\mathscr{E} = \{e_j\}_{j\in J}$.

The following result holds:

Theorem 2. *Let $P : H \to H$ is an orthogonal projection on H. Then for any $x, y \in H$ we have the inequalities*

$$|\langle x,y\rangle - 2\langle Px,y\rangle| \leq \|x\|\,\|y\| , \tag{24}$$

$$\begin{aligned} &|\langle x,y\rangle - \langle Px,y\rangle| \\ &\leq \min\left\{\|x\|\left(\|y\|^2 - \langle Py,y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\right\} \end{aligned} \tag{25}$$

$$\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2} + \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\|x\|^2 + \|y\|^2 - \langle Py, y\rangle - \langle Px, x\rangle\right)^{1/2}$$

and

$$|\langle Px, y\rangle| \leq \min\left\{\|x\| \langle Py, y\rangle^{1/2}, \|y\| \langle Px, x\rangle^{1/2}\right\} \tag{26}$$
$$\leq \frac{1}{2}\left[\|x\| \langle Py, y\rangle^{1/2} + \|y\| \langle Px, x\rangle^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\langle Px, x\rangle + \langle Py, y\rangle\right)^{1/2}.$$

Proof. Observe that

$$\|x - 2Px\|^2 = \|x\|^2 - 4\operatorname{Re}\langle x, Px\rangle + 4\langle Px, Px\rangle$$
$$= \|x\|^2 - 4\langle x, Px\rangle + 4\left\langle P^2x, x\right\rangle$$
$$= \|x\|^2 - 4\langle x, Px\rangle + 4\langle Px, x\rangle = \|x\|^2$$

for any $x \in H$.

Using Schwarz's inequality we have

$$\|x\| \|y\| = \|x - 2Px\| \|y\| \geq |\langle x - 2Px, y\rangle| = |\langle x, y\rangle - 2\langle Px, y\rangle|$$

for any $x, y \in H$ and the inequality (24) is proved.

By Schwarz's inequality we also have

$$\|x - Px\| \|y\| \geq |\langle x - Px, y\rangle| = |\langle x, y\rangle - \langle Px, y\rangle|$$

and

$$\|x\| \|y - Py\| \geq |\langle x, y - Py\rangle| = |\langle x, y\rangle - \langle x, Py\rangle| = |\langle x, y\rangle - \langle Px, y\rangle|$$

for any $x, y \in H$, which implies the first inequality in (25).

The second and the third inequalities are obvious by the elementary inequalities

$$\min\{a, b\} \leq \frac{1}{2}(a + b), \ a, b \in \mathbb{R}_+$$

and

$$ac + bd \leq \left(a^2 + b^2\right)^{1/2}\left(c^2 + d^2\right)^{1/2}, \ a, b, c, d \in \mathbb{R}_+.$$

The inequality (26) follows from (25) by replacing P with $1_H - P$.

Remark 2. By the triangle inequality we have

$$\|x\|\,\|y\| + |\langle x, y\rangle| \geq |\langle x, y\rangle - 2\langle Px, y\rangle| + |\langle x, y\rangle| \geq 2\,|\langle Px, y\rangle|\,,$$

which implies that [see also (16) and (19)]

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq |\langle Px, y\rangle| \tag{27}$$

for any $x, y \in H$.

From (25) we also have

$$\begin{aligned} &|\langle Px, y\rangle| \\ &\leq |\langle x, y\rangle| + \min\left\{\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right\} \end{aligned} \tag{28}$$

and

$$\begin{aligned} &|\langle x, y\rangle| \\ &\leq |\langle Px, y\rangle| + \min\left\{\|x\|\left(\|y\|^2 - \langle Py, y\rangle\right)^{1/2}, \|y\|\left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2}\right\} \end{aligned} \tag{29}$$

for any $x, y \in H$.

Now, if $\mathscr{E} = \left\{e_j\right\}_{j\in J}$ is an orthonormal family, then by the inequalities (24) and (25) we have

$$\left|\langle x, y\rangle - 2\sum_{j\in J}\left\langle x, e_j\right\rangle\left\langle e_j, y\right\rangle\right| \leq \|x\|\,\|y\|\,, \tag{30}$$

and

$$\begin{aligned} &\left|\langle x, y\rangle - \sum_{j\in J}\left\langle x, e_j\right\rangle\left\langle e_j, y\right\rangle\right| \\ &\leq \min\left\{\|x\|\left(\|y\|^2 - \sum_{j\in J}\left|\left\langle y, e_j\right\rangle\right|^2\right)^{1/2}, \|y\|\left(\|x\|^2 - \sum_{j\in J}\left|\left\langle x, e_j\right\rangle\right|^2\right)^{1/2}\right\} \end{aligned} \tag{31}$$

$$\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2-\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2}+\|y\|\left(\|x\|^2-\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\right]$$

$$\leq \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2-\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}$$

for any $x, y \in H$.

From (28) we also have

$$\left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|$$

$$\leq |\langle x,y\rangle| + \min\left\{\|x\|\left(\|y\|^2-\sum_{j\in J}\left|\langle y,e_j\rangle\right|^2\right)^{1/2}, \|y\|\left(\|x\|^2-\sum_{j\in J}\left|\langle x,e_j\rangle\right|^2\right)^{1/2}\right\} \tag{32}$$

for any $x, y \in H$.

3 Inequalities for Norm and Numerical Radius

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by Gustafson and Rao [26, p. 1]:

$$W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by Gustafson and Rao [26, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true:

$$w(T) \leq \|T\| \leq 2w(T),\ \text{for any } T \in B(H).$$

Utilizing Buzano's inequality (3) we obtained the following inequality for the numerical radius [13] or [15]:

Theorem 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{33}$$

The constant $\frac{1}{2}$ is best possible in (33).

The following general result for the product of two operators holds [26, p. 37]:

Theorem 4. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then $w(AB) \le 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \le 2w(A)w(B)$. The constant 2 is best possible here.*

The following results are also well known [26, p. 38].

Theorem 5. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{34}$$

If A is an isometry and $AB = BA$, then (34) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. The following result holds [26, p. 38].

Theorem 6. *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|. \tag{35}$$

As a consequence of the above, we have [26, p. 39]:

Corollary 2. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)w(B). \tag{36}$$

A related problem with the inequality (35) is to find the best constant c for which the inequality

$$w(AB) \le cw(A)\|B\|$$

holds for any two commuting operators $A, B \in B(H)$. It is known that $1.064 < c < 1.169$, see [3, 32] and [33].

In relation to this problem, it has been shown in [24] that

Theorem 7. *For any $A, B \in B(H)$ we have*

$$w\left(\frac{AB+BA}{2}\right) \le \sqrt{2}w(A)\|B\|. \tag{37}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

The following result holds.

Theorem 8. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle BPAx, x\rangle| \le \frac{1}{2}\left[\|Ax\|\|B^*x\| + |\langle BAx, x\rangle|\right] \tag{38}$$

and

$$\|BPAx\| \le \frac{1}{2}\left[\|Ax\|\|B\| + \|BAx\|\right] \tag{39}$$

for any $x \in H$.

Moreover, we have

$$w(BPA) \le \frac{1}{2}\left[\|A\|\|B\| + w(BA)\right] \tag{40}$$

and

$$\|BPA\| \le \frac{1}{2}\left[\|A\|\|B\| + \|BA\|\right]. \tag{41}$$

Proof. From the inequality (17) we have

$$|\langle PAx, B^*y\rangle| \le \frac{1}{2}\left[\|Ax\|\|B^*y\| + |\langle Ax, B^*y\rangle|\right]$$

that is equivalent to

$$|\langle BPAx, y\rangle| \le \frac{1}{2}\left[\|Ax\|\|B^*y\| + |\langle BAx, y\rangle|\right] \tag{42}$$

for any $x, y \in H$.

If we take $y = x$ in (42), then we get (38).

Taking the supremum over $y \in H$ with $\|y\| = 1$ in (42) we have

$$\|BPAx\| = \sup_{\|y\|=1}|\langle BPAx, y\rangle| \le \frac{1}{2}\sup_{\|y\|=1}\left[\|Ax\|\|B^*y\| + |\langle BAx, y\rangle|\right]$$

$$\leq \frac{1}{2}\left[\|Ax\| \sup_{\|y\|=1}\|B^* y\| + \sup_{\|y\|=1}|\langle BAx, y\rangle|\right]$$
$$= \frac{1}{2}\left[\|Ax\| \, \|B\| + \|BAx\|\right]$$

for any $x \in H$.

The inequalities (40) and (41) follow from (38) and (39) by taking the supremum over $x \in H$ with $\|x\| = 1$.

Corollary 3. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle APAx, x\rangle| \leq \frac{1}{2}\left[\|Ax\| \, \|A^* x\| + \left|\left\langle A^2 x, x\right\rangle\right|\right] \tag{43}$$

and

$$\|APAx\| \leq \frac{1}{2}\left[\|Ax\| \, \|A\| + \left\|A^2 x\right\|\right] \tag{44}$$

for any $x \in H$.

Moreover, we have

$$w(APA) \leq \frac{1}{2}\left[\|A\|^2 + w\left(A^2\right)\right] \tag{45}$$

and

$$\|APA\| \leq \frac{1}{2}\left[\|A\|^2 + \left\|A^2\right\|\right]. \tag{46}$$

Remark 3. Let $e \in H$, $\|e\| = 1$. If we write the inequalities (38) and (39) for the projector P_e defined by $P_e x = \langle x, e\rangle e$, $x \in H$, we have

$$|\langle Ax, e\rangle| \, |\langle Be, x\rangle| \leq \frac{1}{2}\left[\|Ax\| \, \|B^* x\| + |\langle BAx, x\rangle|\right] \tag{47}$$

and

$$|\langle Ax, e\rangle| \, \|Be\| \leq \frac{1}{2}\left[\|Ax\| \, \|B\| + \|BAx\|\right] \tag{48}$$

for any $x \in H$.

Now, if we take the supremum over $x \in H,\ \|x\| = 1$ in (48), then we get

$$\|A^*e\| \, \|Be\| \le \frac{1}{2} \left[\|A\| \, \|B\| + \|BA\| \right] \tag{49}$$

for any $e \in H,\ \|e\| = 1.$

If in (49) we take $B = A$, we have

$$\|A^*e\| \, \|Ae\| \le \frac{1}{2} \left[\|A\|^2 + \left\|A^2\right\| \right] \tag{50}$$

for any $e \in H,\ \|e\| = 1.$

If in (47) we take $B = A$, then we get

$$|\langle Ax, e\rangle| \, |\langle e, A^*x\rangle| \le \frac{1}{2} \left[\|Ax\| \, \|A^*x\| + \left|\left\langle A^2x, x\right\rangle\right| \right] \tag{51}$$

for any $x \in H$ and $e \in H,\ \|e\| = 1$, and in particular

$$|\langle Ae, e\rangle|^2 \le \frac{1}{2} \left[\|Ae\| \, \|A^*e\| + \left|\left\langle A^2e, e\right\rangle\right| \right] \tag{52}$$

for any $e \in H,\ \|e\| = 1.$

Taking the supremum over $e \in H,\ \|e\| = 1$ in (52) we recapture the result in Theorem 3.

For a given operator T we consider the modulus of T defined as $|T| := (T^*T)^{1/2}$.

Corollary 4. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w(BPA) \le \frac{1}{2} w(BA) + \frac{1}{4} \left\| |A|^2 + |B^*|^2 \right\|. \tag{53}$$

In particular, we have

$$w(APA) \le \frac{1}{2} w\left(A^2\right) + \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\|. \tag{54}$$

Proof. From the inequality (38) we have

$$\begin{aligned} |\langle BPAx, x\rangle| &\le \frac{1}{2} \left[\|Ax\| \, \|B^*x\| + |\langle BAx, x\rangle| \right] \\ &\le \frac{1}{2} |\langle BAx, x\rangle| + \frac{1}{4} \left[\|Ax\|^2 + \|B^*x\|^2 \right] \end{aligned} \tag{55}$$

for any $x \in H$, where for the second inequality we used the elementary inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right), \ a, b \in \mathbb{R}. \tag{56}$$

Since

$$\begin{aligned}\|Ax\|^2 + \|B^*x\|^2 &= \langle Ax, Ax\rangle + \langle B^*x, B^*x\rangle = \langle A^*Ax, x\rangle + \langle BB^*x, x\rangle \\ &= \left\langle \left(|A|^2 + |B^*|^2\right) x, x\right\rangle\end{aligned}$$

for any $x \in H$, then from (55) we have

$$|\langle BPAx, x\rangle| \leq \frac{1}{2}|\langle BAx, x\rangle| + \frac{1}{4}\left\langle \left(|A|^2 + |B^*|^2\right) x, x\right\rangle \tag{57}$$

for any $x \in H$.

Taking the supremum over $x \in H, \ \|x\| = 1$ in (57) we get the desired result (53).

Remark 4. We observe that by (52) we have

$$\begin{aligned}|\langle Ae, e\rangle|^2 &\leq \frac{1}{2}\left[\|Ae\| \|A^*e\| + \left|\langle A^2e, e\rangle\right|\right] \\ &\leq \frac{1}{2}\left|\langle A^2e, e\rangle\right| + \frac{1}{4}\left[\|Ae\|^2 + \|A^*e\|^2\right] \\ &= \frac{1}{2}\left|\langle A^2e, e\rangle\right| + \frac{1}{4}\left\langle \left(|A|^2 + |A^*|^2\right) e, e\right\rangle\end{aligned} \tag{58}$$

for any $e \in H, \ \|e\| = 1$.

Taking the supremum over $e \in H, \ \|e\| = 1$ in (58) we get

$$w^2(A) \leq \frac{1}{2}w\left(A^2\right) + \frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\|, \tag{59}$$

for any bounded linear operator A.

Since

$$\left\| |A|^2 + |A^*|^2 \right\| \leq \left\| |A|^2 \right\| + \left\| |A^*|^2 \right\| = 2\|A\|^2,$$

then the inequality (59) is better than the inequality in Theorem 3.

The following result also holds:

Theorem 9. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w\left(B\left(\frac{1}{2}1_H - P\right)A\right) \le \frac{1}{4}\left\||A|^2 + |B^*|^2\right\|. \tag{60}$$

In particular, we have

$$w\left(A\left(\frac{1}{2}1_H - P\right)A\right) \le \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|. \tag{61}$$

Proof. From the inequality (24) we have

$$|\langle(1_H - 2P)Ax, B^*x\rangle| \le \|Ax\|\,\|B^*x\|,$$

that is equivalent to

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \le \frac{1}{2}\|Ax\|\,\|B^*x\| \tag{62}$$

for any $x \in H$.

Using the elementary inequality (56) we have

$$\frac{1}{2}\|Ax\|\,\|B^*x\| \le \frac{1}{4}\left(\|Ax\|^2 + \|B^*x\|^2\right) = \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle$$

and by (62) we get

$$\left|\left\langle B\left(\frac{1}{2}1_H - P\right)Ax, x\right\rangle\right| \le \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x, x\right\rangle \tag{63}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (63) we get the desired result (60).

Remark 5. If we take in (60) $P = 1_H$, then we get [18, p. 6]

$$w(BA) \le \frac{1}{2}\left\||A|^2 + |B^*|^2\right\| \tag{64}$$

for any A, B bounded linear operators on H.

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A Half-Discrete Hardy-Hilbert-Type Inequality with a Best Possible Constant Factor Related to the Hurwitz Zeta Function

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Abstract Using methods of weight functions, techniques of real analysis as well as the Hermite-Hadamard inequality, a half-discrete Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor related to the Hurwitz zeta function and the Riemann zeta function is obtained. Equivalent forms, normed operator expressions, their reverses and some particular cases are also considered.

Keywords Hardy-Hilbert-type inequality • Hurwitz zeta function • Riemann zeta function • weight function • operator

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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = (\int_0^\infty f^p(x)dx)^{\frac{1}{p}} > 0,$$

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$||g||_q > 0$, then we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that

$$a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p,\ b = \{b_n\}_{n=1}^\infty \in l^q,$$

$$||a||_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0,\ ||b||_q > 0,$$

we have the following Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [3]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in Analysis and its applications (cf. [3, 11, 19, 20, 22]).

If $\mu_i, \upsilon_j > 0 (i,j \in \mathbf{N} = \{1, 2, \cdots\})$,

$$U_m := \sum_{i=1}^m \mu_i, V_n := \sum_{j=1}^n \upsilon_j (m, n \in \mathbf{N}), \tag{3}$$

then we have the following inequality (cf. [3], Theorem 321) :

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\mu_m^{1/q} \upsilon_n^{1/p} a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{4}$$

Replacing $\mu_m^{1/q} a_m$ and $\upsilon_n^{1/p} b_n$ by a_m and b_n in (4), respectively, we obtain the following equivalent form of (4):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^\infty \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i = \upsilon_j = 1 (i,j \in \mathbf{N})$, both (4) and (5) reduce to (2). We call (4) and (5) as Hardy-Hilbert-type inequalities.

Note. The authors did not prove that (4) is valid with the best possible constant factor in [3].

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [17] gave an extension of (1) with the kernel $1/(x+y)^{\lambda}$ for $p = q = 2$. Optimizing the method used in [17], Yang [20] provided some extensions of (1) and (2) as follows:

If $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_{\lambda}(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^{\infty} k_{\lambda}(t, 1)t^{\lambda_1 - 1}dt \in \mathbf{R}_+,$$

$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(x) = x^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := \{ \int_0^{\infty} \phi(x)|f(x)|^p dx \}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^{\infty} \int_0^{\infty} k_{\lambda}(x, y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is the best possible.

Moreover, if $k_{\lambda}(x, y)$ remains finite and $k_{\lambda}(x, y)x^{\lambda_1 - 1}(k_{\lambda}(x, y)y^{\lambda_2 - 1})$ is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := (\sum_{n=1}^{\infty} \phi(n)|a_n|^p)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi},\ ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{\lambda}(m, n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{7}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for

$$\lambda = 1, k_1(x, y) = \frac{1}{x+y},\ \lambda_1 = \frac{1}{q},\ \lambda_2 = \frac{1}{p},$$

inequality (6) reduces to (1), while (7) reduces to (2). For

$$0 < \lambda_1, \lambda_2 \leq 1,\ \lambda_1 + \lambda_2 = \lambda,$$

we set

$$k_{\lambda}(x,y) = \frac{1}{(x+y)^{\lambda}} \ ((x,y) \in \mathbf{R}_{+}^{2}).$$

Then by (7), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)||a||_{p,\phi}||b||_{q,\psi}, \tag{8}$$

where the constant $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B\,(u,v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt (u, v > 0)$$

is the beta function.

In 2015, subject to further conditions, Yang [26] proved an extension of (8) and (5) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(U_m+V_n)^{\lambda}} \tag{9}$$

$$< B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\nu_n^{q-1}}\right)^{\frac{1}{q}}, \tag{10}$$

where the constant $B(\lambda_1, \lambda_2)$ is still the best possible.

Further results including some multidimensional Hilbert-type inequalities can be found in [18, 21, 23–25, 27, 33].

On the topic of half-discrete Hilbert-type inequalities with non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But, they did not prove that the constant factors are the best possible. However, Yang [18] presented a result with the kernel $1/(1+nx)^{\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [21] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B\,(\lambda_1, \lambda_2)$:

$$\int_0^{\infty} f\,(x)\left[\sum_{n=1}^{\infty}\frac{a_n}{(x+n)^{\lambda}}\right]dx < B\,(\lambda_1,\lambda_2)\,||f||_{p,\phi}||a||_{q,\psi}, \tag{11}$$

where $\lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$. Zhong et al. [36, 37, 39–41] investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using methods of weight functions and techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_\lambda(x,n) a_n dx < k(\lambda_1) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{12}$$

which is an extension of (11) (cf. Yang and Chen [28]). Additionally, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [24]. The reader is referred to the three books of Yang [23, 25] and Yang and Debnath [29], where half-discrete Hilbert-type inequalities and their operator expressions are extensively treated. The interested reader will find a vast literature on both old and new results on half-discrete Hardy-Hilbert-type inequality with emphasis to the study of best constants in references [1–42].

In this chapter, using methods of weight functions, techniques of real analysis as well as the Hermite-Hadamard inequality, a half-discrete Hardy-Hilbert-type inequality with multi-parameters and a best possible constant factor related to the Hurwitz zeta function and the Riemann zeta function is studied, which is an extension of (12) for $\lambda = 0$ in a particular kernel. Equivalent forms, normed operator expressions, their reverses and some particular cases are also considered.

2 An Example and Some Lemmas

In the following, we assume that $\mu_i, \nu_j > 0$ $(i, j \in \mathbf{N})$, U_m and V_n are defined by (3),

$$\tilde{V}_n := V_n - \tilde{\nu}_n (\tilde{\nu}_n \in [0, \frac{\nu_n}{2}])(n \in \mathbf{N}),$$

$\mu(t)$ is a positive continuous function in $\mathbf{R}_+ = (0, \infty)$,

$$U(x) := \int_0^x \mu(t) dt < \infty (x \in [0, \infty)),$$

$\nu(t) := \nu_n, t \in (n - \frac{1}{2}, n + \frac{1}{2}](n \in \mathbf{N})$, and

$$V(y) := \int_{\frac{1}{2}}^{y} \nu(t) dt (y \in [\frac{1}{2}, \infty)),$$

$p \neq 0, 1,\ \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, f(x), a_n \geq 0 (x \in \mathbf{R}_+, n \in \mathbf{N}),$

$$||f||_{p,\Phi_\delta} = (\int_0^{\infty} \Phi_\delta(x) f^p(x) dx)^{\frac{1}{p}},$$

$||a||_{q,\tilde{\Psi}} = (\sum_{n=1}^{\infty} \tilde{\Psi}(n) b_n^q)^{\frac{1}{q}}$, where,

$$\Phi_\delta(x) := \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}, \tilde{\Psi}(n) := \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} (x \in \mathbf{R}_+, n \in \mathbf{N}).$$

Example 1. For $0 < \gamma < \sigma, 0 \le \alpha \le \rho$ $(\rho > 0)$,

$$\csc h(u) := \frac{2}{e^u - e^{-u}} \ (u > 0)$$

is the hyperbolic cosecant function (cf. [34]). We set

$$h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} \ (t \in \mathbf{R}_+).$$

(i) Setting $u = \rho t^\gamma$, we find

$$\begin{aligned} k(\sigma) &:= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{\csc h(u)}{e^{\frac{\alpha}{\rho} u}} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-\frac{\alpha}{\rho} u} u^{\frac{\sigma}{\gamma}-1}}{e^u - e^{-u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-(\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1}}{1 - e^{-2u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du. \end{aligned}$$

By the Lebesgue term by term integration theorem (cf. [34]), setting $v = \left(2k + \frac{\alpha}{\rho} + 1\right) u$, we have

$$\begin{aligned} k(\sigma) &= \int_0^\infty \frac{\csc h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \int_0^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{1}{(2k+\frac{\alpha}{\rho}+1)^{\sigma/\gamma}} \int_0^\infty e^{-v} v^{\frac{\sigma}{\gamma}-1} dv \end{aligned}$$

$$= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{1}{(k+\frac{\alpha+\rho}{2\rho})^{\sigma/\gamma}}$$

$$= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) \in \mathbf{R}_+, \tag{13}$$

where

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \ (s > 1; 0 < a \leq 1)$$

is the Hurwitz zeta function, $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function, and

$$\Gamma(y) := \int_0^{\infty} e^{-v} v^{y-1} dv \ (y > 0)$$

is the Gamma function (cf. [16]).

In particular, (1) for $\alpha = \rho > 0$,we have $h(t) = \frac{\csc h(\rho t^{\gamma})}{e^{\rho t^{\gamma}}}$ and $k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}$. In this case, for $\gamma = \frac{\sigma}{2}$, we have $h(t) = \frac{\csc h(\rho\sqrt{t^{\sigma}})}{e^{\rho\sqrt{t^{\sigma}}}}$ and $k(\sigma) = \frac{\pi^2}{6\sigma\rho^2}$; (2) for $\alpha = 0$, we have $h(t) = \csc h(\rho t^{\gamma})$ and $\frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}\zeta(\frac{\sigma}{\gamma}, \frac{1}{2})$. In this case, for $\gamma = \frac{\sigma}{2}$, we find $h(t) = \csc h(\rho\sqrt{t^{\sigma}})$ and $k(\sigma) = \frac{\pi^2}{2\sigma\rho^2}$.

(ii) We obtain for $u > 0$, $\frac{1}{e^u - e^{-u}} > 0$,

$$\frac{d}{du}(\frac{1}{e^u - e^{-u}}) = -\frac{e^u + e^{-u}}{(e^u - e^{-u})^2} < 0,$$

$$\frac{d^2}{du^2}(\frac{1}{e^u - e^{-u}}) = \frac{2(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u - e^{-u})^3} > 0.$$

If $g(u) > 0,\ g'(u) < 0,\ g''(u) > 0$, then for $0 < \gamma \leq 1$,

$$g(\rho t^{\gamma}) > 0, \frac{d}{dt} g(\rho t^{\gamma}) = \rho\gamma t^{\gamma-1} g'(\rho t^{\gamma}) < 0,$$

$$\frac{d^2}{dt^2} g(\rho t^{\gamma}) = \rho\gamma(\gamma-1)t^{\gamma-2} g'(\rho t^{\gamma}) + \rho^2\gamma^2 t^{2\gamma-2} g''(\rho t^{\gamma}) > 0;$$

for $y \in (n - \frac{1}{2}, n + \frac{1}{2}), g(V(y)) > 0$,

$$\frac{d}{dy} g(V(y)) = g'(V(y))\nu_n < 0,$$

$$\frac{d^2}{dy^2} g(V(y)) = g''(V(y))\nu_n^2 > 0 (n \in \mathbf{N}).$$

If $g_i(u) > 0, g_i'(u) < 0, g_i''(u) > 0 (i = 1, 2)$, then

$$\begin{aligned} g_1(u)g_2(u) &> 0, \\ (g_1(u)g_2(u))' &= g_1'(u)g_2(u) + g_1(u)g_2'(u) < 0, \\ (g_1(u)g_2(u))'' &= g_1''(u)g_2(u) + 2g_1'(u)g_2'(u) + g_1(u)g_2''(u) > 0 (u > 0). \end{aligned}$$

(iii) Therefore, for $0 < \gamma < \sigma \le 1, 0 \le \alpha \le \rho (\rho > 0)$, we have $k(\sigma) \in \mathbf{R}_+$, with $h(t) > 0, h'(t) < 0, h''(t) > 0$, and then for $c > 0, y \in (n - \frac{1}{2}, n + \frac{1}{2}) (n \in \mathbf{N})$, it follows that

$$\begin{aligned} & h(cV(y))V^{\sigma-1}(y) > 0, \\ & \frac{d}{dy} h(cV(y))V^{\sigma-1}(y) < 0, \\ & \frac{d^2}{dy^2} h(cV(y))V^{\sigma-1}(y) > 0. \end{aligned}$$

Lemma 1. *If $g(t) (> 0)$ is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty)$ where $n_0 \in \mathbf{N}$, satisfying $\int_0^\infty g(t)dt \in \mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)dt. \tag{14}$$

Proof. Since we have

$$\begin{aligned} & \int_n^{n+1} g(t)dt \le g(n) \le \int_{n-1}^n g(t)dt (n = 1, \cdots, n_0), \\ & \int_{n_0+1}^{n_0+2} g(t)dt < g(n_0 + 1) < \int_{n_0}^{n_0+1} g(t)dt, \end{aligned}$$

then it follows that

$$0 < \int_1^{n_0+2} g(t)dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1} \int_{n-1}^n g(t)dt = \int_0^{n_0+1} g(t)dt < \infty.$$

Similarly, we still have

$$0 < \int_{n_0+2}^\infty g(t)dt \le \sum_{n=n_0+2}^\infty g(n) \le \int_{n_0+1}^\infty g(t)dt < \infty.$$

Hence, (14) follows and therefore the lemma is proved.

Lemma 2. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1$, *define the following weight coefficients:*

$$\omega_\delta(\sigma, x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{\delta\sigma}(x)\nu_n}{\tilde{V}_n^{1-\sigma}}, x \in \mathbf{R}_+, \tag{15}$$

$$\varpi_\delta(\sigma, n) := \int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^\sigma \mu(x)}{U^{1-\delta\sigma}(x)} dx, n \in \mathbf{N}. \tag{16}$$

Then, we have the following inequalities:

$$\omega_\delta(\sigma, x) < k(\sigma)(x \in \mathbf{R}_+), \tag{17}$$

$$\varpi_\delta(\sigma, n) \leq k(\sigma)(n \in \mathbf{N}), \tag{18}$$

where $k(\sigma)$ *is given by (13).*

Proof. Since we find

$$\tilde{V}_n = V_n - \tilde{\nu}_n \geq V_n - \frac{\nu_n}{2}$$
$$= \int_{\frac{1}{2}}^{n+\frac{1}{2}} \nu(t)dt - \int_n^{n+\frac{1}{2}} \nu(t)dt = \int_{\frac{1}{2}}^{n} \nu(t)dt = V(n),$$

and for $t \in (n - \frac{1}{2}, n + \frac{1}{2}], V'(t) = \nu_n$, hence by Example 1(iii) and Hermite-Hadamard's inequality (cf. [8]), we have

$$\frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\nu_n}{\tilde{V}_n^{1-\sigma}}$$
$$\leq \frac{\csc h(\rho(U^\delta(x)V(n))^\gamma)}{e^{\alpha(U^\delta(x)V(n))^\gamma}} \frac{\nu_n}{V^{1-\sigma}(n)}$$
$$< \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{V'(t)}{V^{1-\sigma}(t)} dt,$$

$$\omega_\delta(\sigma, x) < \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt$$
$$= \int_{\frac{1}{2}}^{\infty} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt.$$

Setting $u = U^{\delta}(x)V(t)$, by (13), we find

$$\omega_{\delta}(\sigma, x) < \int_0^{U^{\delta}(x)V(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{U^{\delta\sigma}(x)U^{-\delta}(x)}{(uU^{-\delta}(x))^{1-\sigma}} du$$
$$\leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du = k(\sigma).$$

Hence, (17) follows.

Setting $u = \tilde{V}_n U^{\delta}(x)$ in (16), we find $du = \delta \tilde{V}_n U^{\delta-1}(x)\mu(x)dx$ and

$$\varpi_{\delta}(\sigma, n) = \frac{1}{\delta} \int_{\tilde{V}_n U^{\delta}(0)}^{\tilde{V}_n U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} \frac{\tilde{V}_n^{\sigma} \tilde{V}_n^{-1} (\tilde{V}_n^{-1} u)^{\frac{1}{\delta}-1}}{(\tilde{V}_n^{-1} u)^{\frac{1}{\delta}-\sigma}} du$$
$$= \frac{1}{\delta} \int_{\tilde{V}_n U^{\delta}(0)}^{\tilde{V}_n U^{\delta}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

If $\delta = 1$, then

$$\varpi_1(\sigma, n) = \int_0^{\tilde{V}_n U(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du$$
$$\leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

If $\delta = -1$, then

$$\varpi_{-1}(\sigma, n) = -\int_{\infty}^{\tilde{V}_n U^{-1}(\infty)} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du$$
$$\leq \int_0^{\infty} \frac{\csc h(\rho u^{\gamma})}{e^{\alpha u^{\gamma}}} u^{\sigma-1} du.$$

Then by (13), we have (18). The lemma is proved.

Remark 1. We do not need the constraint $\sigma \leq 1$ to obtain (18). If $U(\infty) = \infty$, then we have

$$\varpi_{\delta}(\sigma, n) = k(\sigma)(n \in \mathbf{N}). \tag{19}$$

For example, if we set $\mu(t) = \frac{1}{(1+t)^{\beta}}(t > 0; 0 \leq \beta \leq 1)$, then for $x \geq 0$, we find

$$U(x) = \int_0^x \frac{1}{(1+t)^{\beta}} dt$$
$$= \begin{cases} \frac{(1+x)^{1-\beta}-1}{1-\beta}, 0 \leq \beta < 1 \\ \ln(1+x), \beta = 1 \end{cases} < \infty,$$

and

$$U(\infty) = \int_0^\infty \frac{1}{(1+t)^\beta} dt = \infty.$$

Lemma 3. *If* $0 \leq \alpha \leq \rho\ (\rho > 0),\ 0 < \gamma < \sigma \leq 1$, *there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $V(\infty) = \infty$, *then*

(i) for $x \in \mathbf{R}_+$, *we have*

$$k(\sigma)(1 - \theta_\delta(\sigma, x)) < \omega_\delta(\sigma, x), \tag{20}$$

where, $\theta_\delta(\sigma, x) = O((U(x))^{\delta(\sigma-\gamma)}) \in (0, 1)$;

(ii) for any $b > 0$, *we have*

$$\sum_{n=1}^\infty \frac{v_n}{\tilde{V}_n^{1+b}} = \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + bO(1)\right). \tag{21}$$

Proof. Since $v_n \geq v_{n+1} (n \geq n_0)$, and

$$\tilde{V}_n = V_n - \tilde{v}_n \leq V_n = \int_{\frac{1}{2}}^{n+\frac{1}{2}} v(t)dt = V(n + \frac{1}{2}),$$

by Example 1(iii), we have

$$\begin{aligned}
\omega_\delta(\sigma, x) &= \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{\delta\sigma}(x)v_n}{\tilde{V}_n^{1-\sigma}} \\
&\geq \sum_{n=n_0}^\infty \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\csc h(\rho(U^\delta(x)V(n+\frac{1}{2}))^\gamma)}{e^{\alpha(U^\delta(x)V(n+\frac{1}{2}))^\gamma}} \frac{U^{\delta\sigma}(x)v_{n+1}dt}{(V(n+\frac{1}{2}))^{1-\sigma}} \\
&> \sum_{n=n_0}^\infty \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t))^{1-\sigma}} dt \\
&= \int_{n_0+\frac{1}{2}}^\infty \frac{\csc h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t))^{1-\sigma}} dt.
\end{aligned}$$

Setting $u = U^\delta(x)V(t)$, in view of $V(\infty) = \infty$, by (13), we find

$$\begin{aligned}
\omega_\delta(\sigma, x) &> \int_{U^\delta(x)V_{n_0}}^\infty \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\
&= k(\sigma) - \int_0^{U^\delta(x)V_{n_0}} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du
\end{aligned}$$

$$= k(\sigma)(1 - \theta_\delta(\sigma, x)),$$

$$\theta_\delta(\sigma, x) \ : \ = \frac{1}{k(\sigma)} \int_0^{U^\delta(x)V_{n_0}} \frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \in (0, 1).$$

Since $F(u) = \frac{u^\gamma \csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}}$ is continuous in $(0, \infty)$, satisfying

$$F(u) \to \frac{1}{\rho}(u \to 0^+), F(u) \to 0(u \to \infty),$$

there exists a constant $L > 0$, such that $F(u) \leq L$, namely,

$$\frac{\csc h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \leq L u^{-\gamma} (u \in (0, \infty)).$$

Hence we find

$$0 < \theta_\delta(\sigma, x) \leq \frac{L}{k(\sigma)} \int_0^{U^\delta(x)V_{n_0}} u^{\sigma-\gamma-1} du$$
$$= \frac{L(U^\delta(x)V_{n_0})^{\sigma-\gamma}}{k(\sigma)(\sigma - \gamma)},$$

and then (20) follows.

For $b > 0$, we find

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1+b}} \leq \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \sum_{n=n_0+1}^{\infty} \frac{\nu_n}{V^{1+b}(n)}$$
$$< \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \sum_{n=n_0+1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{V'(x)}{V^{1+b}(x)} dx$$
$$= \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \int_{n_0+\frac{1}{2}}^{\infty} \frac{dV(x)}{V^{1+b}(x)}$$
$$= \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}} + \frac{1}{bV^b(n_0 + \frac{1}{2})}$$
$$= \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + b \sum_{n=1}^{n_0} \frac{\nu_n}{\tilde{V}_n^{1+b}}\right),$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1+b}} \geq \sum_{n=n_0}^{\infty} \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{\nu_{n+1}}{V^{1+b}(n + \frac{1}{2})} dx$$

$$> \sum_{n=n_0}^{\infty} \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{V'(x)}{V^{1+b}(x)} dx = \int_{n_0+\frac{1}{2}}^{\infty} \frac{dV(x)}{V^{1+b}(x)}$$

$$= \frac{1}{bV^b(n_0 + \frac{1}{2})} = \frac{1}{bV_{n_0}^b}.$$

Hence we have (21). The lemma is proved.

Note. For example, $\nu_n = \frac{1}{(n-\tau)^\beta} (n \in \mathbf{N}; 0 \le \beta \le 1, 0 \le \tau < 1)$ satisfies the conditions of Lemma 3 (for $n_0 \ge 1$).

3 Equivalent Inequalities and Operator Expressions

Theorem 1. *If* $0 \le \alpha \le \rho (\rho > 0), 0 < \gamma < \sigma \le 1, k(\sigma)$ *is given by (13), then for* $p > 1,\ 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities:*

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx < k(\sigma)||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \quad (22)$$

$$J_1 := \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p$$

$$< k(\sigma)||f||_{p,\Phi_\delta}, \quad (23)$$

$$J_2 := \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$< k(\sigma)||a||_{q,\tilde{\Psi}}. \quad (24)$$

Proof. By Hölder's inequality with weight (cf. [8]), we have

$$\left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p$$

$$= \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \left(\frac{U^{\frac{1-\delta\sigma}{q}}(x) f(x)}{\tilde{V}_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)} \right) \left(\frac{\tilde{V}_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)} \right) dx \right]^p$$

$$\le \int_0^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \left(\frac{U^{\frac{p(1-\delta\sigma)}{q}}(x) f^p(x)}{\tilde{V}_n^{1-\sigma} \mu^{\frac{p}{q}}(x)} \right) dx$$

$$\times\left[\int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(p-1)}\mu(x)}{U^{1-\delta\sigma}(x)}dx\right]^{p-1}$$

$$= \frac{(\varpi_\delta(\sigma,n))^{p-1}}{\tilde{V}_n^{p\sigma-1}\nu_n}\int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)}f^p(x)dx. \quad (25)$$

In view of (18) and the Lebesgue term by term integration theorem (cf. [9]), we find

$$J_1 \le (k(\sigma))^{\frac{1}{q}}\left[\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)}f^p(x)dx\right]^{\frac{1}{p}}$$

$$= (k(\sigma))^{\frac{1}{q}}\left[\int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)}f^p(x)dx\right]^{\frac{1}{p}}$$

$$= (k(\sigma))^{\frac{1}{q}}\left[\int_0^\infty \omega_\delta(\sigma,x)\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}f^p(x)dx\right]^{\frac{1}{p}}. \quad (26)$$

Then by (17), we have (23).

By Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^\infty\left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}}\int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}f(x)dx\right]\left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma}a_n}{\nu_n^{\frac{1}{p}}}\right)$$
$$\le J_1||a||_{q,\tilde{\Psi}}. \quad (27)$$

Then by (23), we have (22).

On the other hand, assuming that (22) is valid, we set

$$a_n := \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}\left[\int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}f(x)dx\right]^{p-1}, n \in \mathbf{N}.$$

Then, we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = 0$, then (23) is trivially valid.

If $J_1 = \infty$, then (23) keeps impossible.

Suppose that $0 < J_1 < \infty$. By (22), it follows that

$$||a||_{q,\tilde{\Psi}}^q = J_1^p = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 < k(\sigma)||f||_{p,\Phi_\delta},$$

and then (23) follows, which is equivalent to (22).

By Hölder's inequality with weight (cf. [8]), we obtain

$$\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}a_n\right]^q$$

$$= \left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\left(\frac{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1-\sigma}{p}}}\right)\left(\frac{\tilde{V}_n^{\frac{1-\sigma}{p}}a_n}{U^{\frac{1-\delta\sigma}{q}}(x)\nu_n^{\frac{1}{p}}}\right)\right]^q$$

$$\leq \left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}}\right]^{q-1}$$

$$\times\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\frac{\tilde{V}_n^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q$$

$$= \frac{(\omega_{\delta}(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)}\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q. \tag{28}$$

Then by (17) and Lebesgue term by term integration theorem (cf. [9]), it follows that

$$J_2 < (k(\sigma))^{\frac{1}{p}}\left\{\int_0^{\infty}\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q dx\right\}^{\frac{1}{q}}$$

$$= (k(\sigma))^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}\frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}}a_n^q dx\right\}^{\frac{1}{q}}$$

$$= (k(\sigma))^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varpi_{\delta}(\sigma,n)\frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}}a_n^q\right\}^{\frac{1}{q}}. \tag{29}$$

Then by (18), we have (24).

By Hölder's inequality (cf. [8]), we have

$$I = \int_0^{\infty}\left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)}f(x)\right)\left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)}\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}a_n\right]dx$$

$$\leq ||f||_{p,\Phi_{\delta}}J_2. \tag{30}$$

Then by (24), we have (22).

On the other hand, assuming that (24) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\operatorname{csc}h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}}a_n\right]^{q-1},\ x\in\mathbf{R}_+.$$

Then we find $J_2^q = ||f||_{p,\Phi_\delta}^p$.

If $J_2 = 0$, then (24) is trivially valid.

If $J_2 = \infty$, then (24) keeps impossible.

Suppose that $0 < J_2 < \infty$. By (22), it follows that

$$||f||_{p,\Phi_\delta}^p = J_2^q = I < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||f||_{p,\Phi_\delta}^{p-1} = J_2 < k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (24) follows, which is equivalent to (22).

Therefore, (22), (23) and (24) are equivalent. The theorem is proved.

Theorem 2. *With the assumptions of Theorem 1, if there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k(\sigma)$ *in (22), (23) and (24) is the best possible.*

Proof. For $\varepsilon \in (0, q(\sigma-\gamma))$, we set $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q} (\in (\gamma, 1))$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+, \tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), 0 < x^\delta \leq 1 \\ 0, x^\delta > 0 \end{cases}, \tag{31}$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-1} v_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1} v_n, n \in \mathbf{N}. \tag{32}$$

Then for $\delta = \pm 1$, since $U(\infty) = \infty$, we find

$$\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx = \frac{1}{\varepsilon} U^{\delta\varepsilon}(1). \tag{33}$$

By (21), (33) and (20), we obtain

$$||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}} = \left(\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)}\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{v_n}{\tilde{V}_n^{1+\varepsilon}}\right)^{\frac{1}{q}}$$
$$= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}, \tag{34}$$

$$\tilde{I} := \int_0^\infty \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \tilde{a}_n \tilde{f}(x) dx$$
$$= \int_{\{x>0;0<x^\delta\leq 1\}} \sum_{n=1}^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{\tilde{\sigma}-1} v_n \mu(x)}{U^{1-\delta(\tilde{\sigma}+\varepsilon)}(x)} dx$$

$$
\begin{aligned}
&= \int_{\{x>0;0<x^\delta\le 1\}} \omega_\delta(\tilde{\sigma},x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&\ge k(\tilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}} (1-\theta_\delta(\tilde{\sigma},x))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&= k(\tilde{\sigma})\int_{\{x>0;0<x^\delta\le 1\}} (1-O((U(x))^{\delta(\sigma-\frac{\varepsilon}{q}-\gamma)}))\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\
&= k(\tilde{\sigma})\left[\int_{\{x>0;0<x^\delta\le 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \right.\\
&\quad \left. -\int_{\{x>0;0<x^\delta\le 1\}} O(\frac{\mu(x)}{U^{1-\delta(\sigma-\gamma+\frac{\varepsilon}{p})}(x)})dx\right] \\
&= \frac{1}{\varepsilon}k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O(1)).
\end{aligned}
$$

If there exists a positive constant $K \le k(\sigma)$, such that (22) is valid when replacing $k(\sigma)$ to K, then in particular, by Lebesgue term by term integration theorem, we have $\varepsilon\tilde{I} < \varepsilon K||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$
k(\sigma-\frac{\varepsilon}{q})(U^{\delta\varepsilon}(1)-\varepsilon O(1)) < K\cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}.
$$

It follows that $k(\sigma) \le K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (22).

The constant factor $k(\sigma)$ in (23) (respectively, (24)) is still the best possible. Otherwise, we would reach a contradiction by (27) (respectively, (30)) that the constant factor in (22) is not the best possible. The theorem is proved.

For $p > 1$, we find

$$
\tilde{\Psi}^{1-p}(n) = \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}(n\in \mathbf{N}), \Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}(x\in\mathbf{R}_+),
$$

and define the following real normed spaces:

$$
\begin{aligned}
L_{p,\Phi_\delta}(\mathbf{R}_+) &= \{f; f = f(x), x\in\mathbf{R}_+, ||f||_{p,\Phi_\delta} < \infty\},\\
l_{q,\tilde{\Psi}} &= \{a; a=\{a_n\}_{n=1}^{\infty}, ||a||_{q,\tilde{\Psi}} < \infty\},\\
L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+) &= \{h; h = h(x), x\in\mathbf{R}_+, ||h||_{q,\Phi_\delta^{1-q}} < \infty\},\\
l_{p,\tilde{\Psi}^{1-p}} &= \{c; c=\{c_n\}_{n=1}^{\infty}, ||c||_{p,\tilde{\Psi}^{1-p}} < \infty\}.
\end{aligned}
$$

Assuming that $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx, n \in \mathbf{N},$$

we can rewrite (23) as follows:

$$||c||_{p,\tilde{\Psi}^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta} < \infty,$$

namely, $c \in l_{p,\tilde{\Psi}^{1-p}}$.

Definition 1. Define a half-discrete Hardy-Hilbert-type operator

$$T_1 : L_{p,\Phi_\delta}(\mathbf{R}_+) \to l_{p,\tilde{\Psi}^{1-p}}$$

as follows:

For any $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = c \in l_{p,\tilde{\Psi}^{1-p}}$. Define the formal inner product of $T_1 f$ and $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\tilde{\Psi}}$ as follows:

$$(T_1 f, a) := \sum_{n=1}^{\infty} \left[\int_0^\infty \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x)dx \right] a_n. \tag{35}$$

Then we can rewrite (22) and (23) as:

$$(T_1 f, a) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{36}$$

$$||T_1 f||_{p,\tilde{\Psi}^{1-p}} < k(\sigma)||f||_{p,\Phi_\delta}. \tag{37}$$

Define the norm of operator T_1 as follows:

$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\Phi_\delta}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\tilde{\Psi}^{1-p}}}{||f||_{p,\Phi_\delta}}.$$

Then by (37), it is evident that $||T_1|| \leq k(\sigma)$. Since by Theorem 2, the constant factor in (37) is the best possible, we have

$$||T_1|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}). \tag{38}$$

Assuming that $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\tilde{\Psi}}$, setting

$$h(x) := \sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n, x \in \mathbf{R}_+,$$

we can rewrite (24) as follows:

$$||h||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\tilde{\Psi}} < \infty,$$

namely, $h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$.

Definition 2. Define a half-discrete Hardy-Hilbert-type operator

$$T_2 : l_{q,\tilde{\Psi}} \to L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$$

as follows:

For any $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\tilde{\Psi}}$, there exists a unique representation $T_2 a = h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$. Define the formal inner product of $T_2 a$ and $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$ by:

$$(T_2 a, f) := \int_0^\infty \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right] f(x)dx. \tag{39}$$

Then we can rewrite (22) and (24) as follows:

$$(T_2 a, f) < k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{40}$$

$$||T_2 a||_{q,\Phi_\delta^{1-q}} < k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{41}$$

Define the norm of operator T_2 by:

$$||T_2|| := \sup_{a(\neq\theta)\in l_{q,\tilde{\Psi}}} \frac{||T_2 a||_{q,\Phi_\delta^{1-q}}}{||a||_{q,\tilde{\Psi}}}.$$

Then by (41), we find $||T_2|| \leq k(\sigma)$. Since by Theorem 2, the constant factor in (41) is the best possible, we have

$$||T_2|| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}\zeta(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}) = ||T_1||. \tag{42}$$

4 Some Equivalent Reverses

In the following, we also set

$$\tilde{\Phi}_\delta(x) := (1 - \theta_\delta(\sigma, x))\frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}(x \in \mathbf{R}_+).$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $||f||_{p,\Phi_\delta}$, $||f||_{p,\tilde{\Phi}_\delta}$ and $||a||_{q,\tilde{\Psi}}$.

Theorem 3. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then for* $p < 0$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $k(\sigma)$*:*

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{43}$$

$$J_1 = \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p > k(\sigma)||f||_{p,\Phi_\delta}, \tag{44}$$

$$J_2 = \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{45}$$

Proof. By the reverse Hölder's inequality with weight (cf. [8]), since $p < 0$, similarly to the way we obtained (25) and (26), we have

$$\left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p \leq \frac{\tilde{V}_n^{1-p\sigma}}{(\varpi_\delta(\sigma,n))^{1-p}\nu_n} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x) dx,$$

and then by (19) and Lebesgue term by term integration theorem, it follows that

$$J_1 \geq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}$$
$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_\delta(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}.$$

Then by (17), we have (44).

By the reverse Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^{\infty} \left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right] \left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma} a_n}{\nu_n^{\frac{1}{p}}} \right) \geq J_1 ||a||_{q,\tilde{\Psi}}. \tag{46}$$

Then by (44), we have (43).

On the other hand, assuming that (43) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = \infty$, then (44) is trivially valid.

If $J_1 = 0$, then (44) is impossible.

Suppose that $0 < J_1 < \infty$. By (43), it follows that

$$||a||_{q,\tilde{\Psi}}^q = J_1^p = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$

$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 > k(\sigma)||f||_{p,\Phi_\delta},$$

and then (44) follows, which is equivalent to (43).

By the reverse of Hölder's inequality with weight (cf. [8]), since $0 < q < 1$, similarly to the way we obtained (28) and (29), we have

$$\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right]^q$$
$$\geq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q,$$

and then by (17) and Lebesgue term by term integration theorem, it follows that

$$J_2 > (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right]^{\frac{1}{q}}$$
$$= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right]^{\frac{1}{q}}.$$

Then by (19), we obtain (45).

By the reverse Hölder's inequality (cf. [8]), we get

$$I = \int_0^\infty \left(\frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right) \left[\frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right] dx$$
$$\geq ||f||_{p,\Phi_\delta} J_2. \tag{47}$$

Then by (45), we derive (43).

On the other hand, assuming that (45) is valid, we set $f(x)$ as in Theorem 1. Then we find $J_2^q = ||f||_{p,\Phi_\delta}^p$.

If $J_2 = \infty$, then (45) is trivially valid.

If $J_2 = 0$, then (45) keeps impossible.

Suppose that $0 < J_2 < \infty$. By (43), it follows that

$$||f||_{p,\Phi_\delta}^p = J_2^q = I > k(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}},$$
$$||f||_{p,\Phi_\delta}^{p-1} = J_2 > k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (45) follows, which is equivalent to (43).

Therefore, inequalities (43), (44) and (45) are equivalent.

For $\varepsilon \in (0, q(\sigma-\gamma))$, we set $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}(\in (\gamma, 1))$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+$, $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), 0 < x^\delta \leq 1 \\ 0, x^\delta > 0 \end{cases},$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-1}\nu_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, n \in \mathbf{N}.$$

By (21), (33) and (17), we obtain

$$||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}} = \frac{1}{\varepsilon}U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}, \tag{48}$$

$$\begin{aligned}\tilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\tilde{a}_n\tilde{f}(x)dx \\ &= \int_{\{x>0;0<x^\delta\leq 1\}} \omega_\delta(\tilde{\sigma}, x)\frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx \\ &\leq k(\tilde{\sigma})\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)}dx = \frac{1}{\varepsilon}k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1).\end{aligned}$$

If there exists a positive constant $K \geq k(\sigma)$, such that (43) is valid when replacing $k(\sigma)$ by K, then in particular, we have $\varepsilon\tilde{I} > \varepsilon K||\tilde{f}||_{p,\Phi_\delta}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$k(\sigma - \frac{\varepsilon}{q})U^{\delta\varepsilon}(1) > K \cdot U^{\frac{\delta\varepsilon}{p}}(1)\left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \geq K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (43).

The constant factor $k(\sigma)$ in (44) (respectively, (45)) is still the best possible. Otherwise, we would reach a contradiction by (46) (respectively, (47)) that the constant factor in (43) is not the best possible. The theorem is proved.

Theorem 4. *With the assumptions of Theorem 3, if*

$$0 < p < 1, \ 0 < ||f||_{p,\Phi_\delta}, \ ||a||_{q,\tilde{\Psi}} < \infty,$$

then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}||a||_{q,\tilde{\Psi}}, \tag{49}$$

$$J_1 = \sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}, \tag{50}$$

$$J := \left\{ \int_0^{\infty} \frac{(1-\theta_{\delta}(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\tilde{\Psi}}. \tag{51}$$

Proof. By the reverse Hölder's inequality with weight (cf. [8]), since $0 < p < 1$, similarly to the way we obtained (25) and (26), we have

$$\left[\int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right]^p$$
$$\geq \frac{(\varpi_{\delta}(\sigma,n))^{p-1}}{\tilde{V}_n^{p\sigma-1}\nu_n} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx,$$

and then in view of (19) and Lebesgue term by term integration theorem, we find

$$J_1 \geq (k(\sigma))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} \frac{U^{(1-\delta\sigma)(p-1)}(x)\nu_n}{\tilde{V}_n^{1-\sigma}\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}$$
$$= (k(\sigma))^{\frac{1}{q}} \left[\int_0^{\infty} \omega_{\delta}(\sigma,x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x)dx \right]^{\frac{1}{p}}.$$

Then by (20), we have (50).

By the reverse Hölder's inequality (cf. [8]), we have

$$I = \sum_{n=1}^{\infty} \left[\frac{\nu_n^{\frac{1}{p}}}{\tilde{V}_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)\tilde{V}_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)\tilde{V}_n)^{\gamma}}} f(x)dx \right] \left(\frac{\tilde{V}_n^{\frac{1}{p}-\sigma} a_n}{\nu_n^{\frac{1}{p}}} \right)$$
$$\geq J_1||a||_{q,\tilde{\Psi}}. \tag{52}$$

Then by (50), we have (49).

On the other hand, assuming that (49) is valid, we set a_n as in Theorem 1. Then we find $J_1^p = ||a||_{q,\tilde{\Psi}}^q$.

If $J_1 = \infty$, then (50) is trivially valid.
If $J_1 = 0$, then (50) remains impossible.
Suppose that $0 < J_1 < \infty$. By (49), it follows that

$$||a||_{q,\tilde{\Psi}}^q = J_1^p = I > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}},$$

$$||a||_{q,\tilde{\Psi}}^{q-1} = J_1 > k(\sigma)||f||_{p,\tilde{\Phi}_\delta},$$

and then (50) follows, which is equivalent to (49).
By the reverse Hölder's inequality with weight (cf. [8]), since $q < 0$, we have

$$\left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right]^q$$

$$\leq \frac{(\omega_\delta(\sigma,x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q,$$

and then by (20) and Lebesgue term by term integration theorem, it follows that

$$J > (k(\sigma))^{\frac{1}{p}} \left[\int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)\nu_n^{q-1}} a_n^q dx\right]^{\frac{1}{q}}$$

$$= (k(\sigma))^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \varpi_\delta(\sigma,n) \frac{\tilde{V}_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} a_n^q\right]^{\frac{1}{q}}.$$

Then by (19), we have (51).
By the reverse Hölder's inequality (cf. [8]), we have

$$\begin{aligned} I = &\int_0^\infty \left[(1-\theta_\delta(\sigma,x))^{\frac{1}{p}} \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x)\right] \\ &\times \left[\frac{(1-\theta_\delta(\sigma,x))^{\frac{-1}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} a_n\right] dx \\ &\geq ||f||_{p,\tilde{\Phi}_\delta} J. \end{aligned} \tag{53}$$

Then by (51), we have (49).
On the other hand, assuming that (49) is valid, we set $f(x)$ as in Theorem 1. Then we find $J^q = ||f||_{p,\tilde{\Phi}_\delta}^p$.
If $J = \infty$, then (51) is trivially valid.
If $J = 0$, then (51) remains impossible.
Suppose that $0 < J < \infty$. By (49), it follows that

$$||f||_{p,\tilde{\Phi}_\delta}^p = J^q = I > k(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}},$$

$$||f||_{p,\tilde{\Phi}_\delta}^{p-1} = J > k(\sigma)||a||_{q,\tilde{\Psi}},$$

and then (51) follows, which is equivalent to (49).

Therefore, inequalities (49), (50) and (51) are equivalent.

For $\varepsilon \in (0, p(\sigma - \gamma))$, we set $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p}(> \gamma)$, and $\tilde{f} = \tilde{f}(x), x \in \mathbf{R}_+, \tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$,

$$\tilde{f}(x) = \begin{cases} U^{\delta\tilde{\sigma}-1}(x)\mu(x), 0 < x^\delta \leq 1 \\ 0, x^\delta > 0 \end{cases},$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-\varepsilon-1}\nu_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1}\nu_n, n \in \mathbf{N}.$$

By (20), (21) and (33), we obtain

$$\begin{aligned}
&||\tilde{f}||_{p,\tilde{\Phi}_\delta}||\tilde{a}||_{q,\tilde{\Psi}} \\
&= \left[\int_{\{x>0;0<x^\delta\leq 1\}} (1 - O((U(x))^{\delta(\sigma-\gamma)}))\frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)}\right]^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(U^{\delta\varepsilon}(1) - \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\tilde{O}(1)\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}}\tilde{a}_n\tilde{f}(x)dx \\
&= \sum_{n=1}^\infty \left[\int_{\{x>0;0<x^\delta\leq 1\}} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{\tilde{\sigma}}\mu(x)}{U^{1-\delta\tilde{\sigma}}(x)}dx\right]\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&\leq \sum_{n=1}^\infty \left[\int_0^\infty \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\alpha(U^\delta(x)\tilde{V}_n)^\gamma}} \frac{\tilde{V}_n^{\tilde{\sigma}}\mu(x)}{U^{1-\delta\tilde{\sigma}}(x)}dx\right]\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&= \sum_{n=1}^\infty \varpi_\delta(\tilde{\sigma}, n)\frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} = k(\tilde{\sigma})\sum_{n=1}^\infty \frac{\nu_n}{\tilde{V}_n^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}k(\sigma + \frac{\varepsilon}{p})\left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\tilde{O}(1)\right).
\end{aligned}$$

If there exists a positive constant $K \geq k(\sigma)$, such that (43) is valid when replacing $k(\sigma)$ by K, then in particular, we have $\varepsilon\tilde{I} > \varepsilon K||\tilde{f}||_{p,\tilde{\Phi}_\delta}||\tilde{a}||_{q,\tilde{\Psi}}$, namely,

$$k(\sigma + \frac{\varepsilon}{p}) \left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon \tilde{O}(1) \right)$$

$$> K \left(U^{\delta\varepsilon}(1) - \varepsilon O(1) \right)^{\frac{1}{p}} \left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon \tilde{O}(1) \right)^{\frac{1}{q}}.$$

It follows that $k(\sigma) \geq K(\varepsilon \to 0^+)$. Hence, $K = k(\sigma)$ is the best possible constant factor of (49).

The constant factor $k(\sigma)$ in (50) (respectively, (51)) is still the best possible. Otherwise, we would reach a contradiction by (52) (respectively, (53)) that the constant factor in (49) is not the best possible. The theorem is proved.

5 Some Particular Inequalities

For $\tilde{\nu}_n = 0$, $\tilde{V}_n = V_n$, we set

$$\Psi(n) := \frac{V_n^{q(1-\sigma)-1}}{\nu_n^{q-1}} \ (n \in \mathbf{N}).$$

In view of Theorems 2–4, we have

Corollary 1. *If* $0 \leq \alpha \leq \rho(\rho > 0), 0 < \gamma < \sigma \leq 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $\upsilon_n \geq \upsilon_{n+1}$ $(n \in \{n_0, n_0 + 1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n f(x) dx < k(\sigma) ||f||_{p,\Phi_\delta} ||a||_{q,\Psi}, \quad (54)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} f(x) dx \right]^p < k(\sigma) ||f||_{p,\Phi_\delta}, \quad (55)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) ||a||_{q,\Psi}; \quad (56)$$

(ii) for $p < 0$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\Phi_{\delta}}||a||_{q,\Psi}, \tag{57}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_{\delta}}, \tag{58}$$

$$\left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{59}$$

(iii) for $0 < p < 1,\ 0 < ||f||_{p,\Phi_{\delta}}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}||a||_{q,\Psi}, \tag{60}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} f(x)dx\right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_{\delta}}, \tag{61}$$

$$\left\{\int_0^{\infty} \frac{(1-\theta_{\delta}(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^{\delta}(x)V_n)^{\gamma})}{e^{\alpha(U^{\delta}(x)V_n)^{\gamma}}} a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}. \tag{62}$$

The above inequalities have the best possible constant factor $k(\sigma)$.

In particular, for $\delta = 1$, we have the following inequalities with the non-homogeneous kernel:

Corollary 2. *If* $0 \le \alpha \le \rho (\rho > 0), 0 < \gamma < \sigma \le 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \ge v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1,\ 0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x)dx < k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi}, \tag{63}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x)dx\right]^p < k(\sigma)||f||_{p,\Phi_1}, \tag{64}$$

$$\left\{\int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n\right]^q dx\right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi}; \tag{65}$$

(ii) for $p < 0,\ 0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_nf(x)dx > k(\sigma)||f||_{p,\Phi_1}||a||_{q,\Psi}, \tag{66}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\Phi_1}, \tag{67}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{68}$$

(iii) for $0 < p < 1,\ 0 < ||f||_{p,\Phi_1}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_nf(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_1}||a||_{q,\Psi}, \tag{69}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{V_n^{1-p\sigma}}\left[\int_0^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}f(x)dx\right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_1}, \tag{70}$$

$$\left\{\int_0^{\infty}\frac{(1-\theta_1(\sigma,x))^{1-q}\mu(x)}{U^{1-q\sigma}(x)}\left[\sum_{n=1}^{\infty}\frac{\csc h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}}a_n\right]^q dx\right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}. \tag{71}$$

The above inequalities involve the best possible constant factor $k(\sigma)$.

For $\delta = -1$, we have the following inequalities with the homogeneous kernel of degree 0:

Corollary 3. *If* $0 \le \alpha \le \rho(\rho > 0), 0 < \gamma < \sigma \le 1, k(\sigma)$ *is given by (13), there exists* $n_0 \in \mathbf{N}$, *such that* $v_n \ge v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1,\ 0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, *we have the following equivalent inequalities:*

$$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\csc h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}}a_nf(x)dx < k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi}, \tag{72}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} f(x)dx \right]^p < k(\sigma)||f||_{p,\Phi_{-1}}, \tag{73}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma)||a||_{q,\Psi}; \tag{74}$$

(ii) for $p < 0,\ 0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\Phi_{-1}}||a||_{q,\Psi}, \tag{75}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} f(x)dx \right]^p > k(\sigma)||f||_{p,\Phi_{-1}}, \tag{76}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma)||a||_{q,\Psi}; \tag{77}$$

(iii) for $0 < p < 1,\ 0 < ||f||_{p,\Phi_{-1}}, ||a||_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n f(x)dx > k(\sigma)||f||_{p,\tilde{\Phi}_{-1}}||a||_{q,\Psi}, \tag{78}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{V_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} f(x)dx \right]^p > k(\sigma)||f||_{p,\tilde{\Phi}_{-1}}, \tag{79}$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_{-1}(\sigma,x))^{1-q}\mu(x)}{U^{1+q\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\operatorname{csc} h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$
$$> k(\sigma)||a||_{q,\Psi}. \tag{80}$$

The above inequalities involve the best possible constant factor $k(\sigma)$.
For $\alpha = \rho$ in Theorems 2–4, we have

Corollary 4. *If $\rho > 0, 0 < \gamma < \sigma \leq 1$,*

$$K(\sigma) := \frac{2\Gamma(\frac{\sigma}{\gamma})\zeta(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}}, \tag{81}$$

there exists $n_0 \in \mathbf{N}$, *such that* $v_n \geq v_{n+1}$ $(n \in \{n_0, n_0+1, \cdots\})$, *and* $U(\infty) = V(\infty) = \infty$, *then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx < K(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{82}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p < K(\sigma)||f||_{p,\Phi_\delta}, \tag{83}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < K(\sigma)||a||_{q,\tilde{\Psi}}. \tag{84}$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx > K(\sigma)||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{85}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p > K(\sigma)||f||_{p,\Phi_\delta}, \tag{86}$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > K(\sigma)||a||_{q,\tilde{\Psi}}; \tag{87}$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$*:*

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n f(x) dx > K(\sigma)||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}}, \tag{88}$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} f(x) dx \right]^p > K(\sigma)||f||_{p,\tilde{\Phi}_\delta}, \tag{89}$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^\gamma)}{e^{\rho(U^\delta(x)\tilde{V}_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > K(\sigma)||a||_{q,\tilde{\Psi}}. \tag{90}$$

In particular, for $\gamma = \frac{\sigma}{2}, \theta_\delta(\sigma, x) = O((U(x))^{\frac{\delta\sigma}{2}})$,

(i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x)dx < \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \quad (91)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x)dx \right]^p < \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta}, \quad (92)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\pi^2}{6\sigma\rho^2} ||a||_{q,\tilde{\Psi}}; \quad (93)$$

(ii) for $p < 0$,we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x)dx > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta} ||a||_{q,\tilde{\Psi}}, \quad (94)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x)dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\Phi_\delta}, \quad (95)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma\rho^2} ||a||_{q,\tilde{\Psi}}; \quad (96)$$

(iii) for $0 < p < 1$,we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma\rho^2}$:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n f(x)dx > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\tilde{\Phi}_\delta} ||a||_{q,\tilde{\Psi}}, \quad (97)$$

$$\sum_{n=1}^{\infty} \frac{\nu_n}{\tilde{V}_n^{1-p\sigma}} \left[\int_0^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} f(x)dx \right]^p > \frac{\pi^2}{6\sigma\rho^2} ||f||_{p,\tilde{\Phi}_\delta}, \quad (98)$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[\sum_{n=1}^{\infty} \frac{\csc h(\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2})}{e^{\rho(U^\delta(x)\tilde{V}_n)^{\sigma/2}}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$> \frac{\pi^2}{6\sigma\rho^2}||a||_{q,\tilde{\Psi}}. \tag{99}$$

For $\alpha = 0, \gamma = \frac{\sigma}{2}, \theta_\delta(\sigma, x) = O((U(x))^{\frac{\delta\sigma}{2}})$ in Theorems 2–4, we have

Corollary 5. *If $\rho > 0, 0 < \sigma \leq 1$, there exists $n_0 \in \mathbf{N}$, such that $\upsilon_n \geq \upsilon_{n+1}$ ($n \in \{n_0, n_0+1, \cdots\}$), and $U(\infty) = V(\infty) = \infty$, then*

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty}\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n f(x)dx < \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{100}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}\left[\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})f(x)dx\right]^p < \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}, \tag{101}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n\right]^q dx\right\}^{\frac{1}{q}} < \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}; \tag{102}$$

(ii) for $p < 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty}\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n f(x)dx > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}||a||_{q,\tilde{\Psi}}, \tag{103}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}\left[\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})f(x)dx\right]^p > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\Phi_\delta}, \tag{104}$$

$$\left\{\int_0^{\infty}\frac{\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n\right]^q dx\right\}^{\frac{1}{q}} > \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}; \tag{105}$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\tilde{\Psi}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{2\sigma\rho^2}$:

$$\sum_{n=1}^{\infty}\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n f(x)dx > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\tilde{\Phi}_\delta}||a||_{q,\tilde{\Psi}}, \tag{106}$$

$$\sum_{n=1}^{\infty}\frac{\nu_n}{\tilde{V}_n^{1-p\sigma}}\left[\int_0^{\infty} \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})f(x)dx\right]^p > \frac{\pi^2}{2\sigma\rho^2}||f||_{p,\tilde{\Phi}_\delta}, \tag{107}$$

$$\left\{\int_0^\infty \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)}\left[\sum_{n=1}^\infty \csc h(\rho(U^\delta(x)\tilde{V}_n)^{\frac{\sigma}{2}})a_n\right]^q dx\right\}^{\frac{1}{q}}$$

$$> \frac{\pi^2}{2\sigma\rho^2}||a||_{q,\tilde{\Psi}}. \tag{108}$$

Remark 2. (i) For $\mu(x) = \nu_n = 1$ in (54), we have the following inequality with the best possible constant factor $k(\sigma)$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(x^\delta n)^\gamma)}{e^{\alpha(x^\delta n)^\gamma}} a_n f(x)dx \tag{109}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{110}$$

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(xn)^\gamma)}{e^{\alpha(xn)^\gamma}} a_n f(x)dx \tag{111}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}; \tag{112}$$

for $\delta = -1$, we have the following inequality with the homogeneous kernel:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}} a_n f(x)dx \tag{113}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1+\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{114}$$

(ii) For $\mu(x) = \nu_n = 1$, $\tilde{\nu}_n = \tau \in (0, \frac{1}{2}]$ in (22), we have the following more accurate inequality than (82) with the best possible constant factor $k(\sigma)$:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho[x^\delta(n-\tau)]^\gamma)}{e^{\alpha(x^\delta(n-\tau)]^\gamma)^\gamma}} a_n f(x)dx \tag{115}$$

$$< k(\sigma)\left[\int_0^\infty x^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\tau)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{116}$$

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h([x(n-\tau)]^{\gamma})}{e^{\alpha\{[x(n-\tau)]\}^{\gamma}}} a_n f(x) dx \tag{117}$$

$$< k(\sigma) \left[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\tau)^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}; \tag{118}$$

for $\delta = -1$, we have the following inequality with the homogeneous kernel:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(\rho(\frac{n-\tau}{x})^{\gamma})}{e^{\alpha(\frac{n-\tau}{x})^{\gamma}}} a_n f(x) dx \tag{119}$$

$$< k(\sigma) \left[\int_0^{\infty} x^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\tau)^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \tag{120}$$

We can still obtain a large number of other inequalities by using some special parameters in the above theorems and corollaries.

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Quantum Integral Inequalities for Generalized Convex Functions

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Abstract In this chapter, we consider generalized convex functions involving two arbitrary functions. We establish some new quantum integral inequalities for the generalized convex functions. Several spacial cases are also discussed which can be obtained from our main results. We expect that the techniques and ideas developed here would be useful in future research. Exploring the applications of general convex functions and quantum integral inequalities is an interesting and fascinating problem.

Keywords Generalized convex functions • Quantum estimates • Hermite–Hadamard inequalities • Convex functions • Convex sets

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1 Introduction

Theory of convexity plays an important role in different fields of pure and applied sciences. Theory of convex functions has been extended in different directions using innovative and novel techniques. For the various generalizations and extensions, for example, see [1, 2, 9, 11–17, 19, 20, 26, 31]. Noor [15] and Jian [9] introduced the notion of generalized convexity involving two arbitrary functions. This type of convexity is quite general and flexible. One can obtain a wide class of convex functions and its variant forms by selecting appropriate choice of the arbitrary functions. These generalized convex functions and convex sets include the g-convex

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sets introduced by Youness [31]. Noor [15] has shown that the minimum of the differentiable generalized convex functions involving two arbitrary functions on the generalized convex set can be characterized by a class of variational inequalities, which is called extended general variational inequalities. This result inspired a deal of research activities in variational inequalities and optimization.

It is well known that a function is a convex function if and only if it satisfies the inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a, b \in I = [a, b],$$

which is called the Hermite–Hadamard inequality. In recent years, several types of integral inequalities have been derived for various classes of convex functions. See [2–4, 7, 11, 16–20, 24–26].

Quantum calculus is the study of calculus without limits. Euler (17-7-1783) introduced the q in tracks of Newton's infinite series. F. H. Jackson started the study of quantum calculus. In quantum calculus, we obtain the q-analogues of mathematical objects, which can be obtained as $q \longrightarrow 1$. It has been noticed that quantum calculus is a subfield of time scale calculus. The time scale provides us a unified and flexible framework for studying dynamic equations on both the discrete and continuous domains. The quantum calculus can be treated as bridge between Mathematics and Physics. In recent years, quantum calculus has emerged as a fascinating and interesting field. Several researchers have utilized the concepts of quantum calculus to obtain integral inequalities via different classes of convex functions, see [6, 21–23, 28, 30]. We would like to emphasize that the analysis of these problems requires a blend of techniques from convex analysis, functional analysis, numerical analysis and other optimization theory.

In this chapter, we consider the class of generalized convexity involving two arbitrary functions. We establish some new quantum estimates for Hermite–Hadamard type inequalities via generalized convexity. We also discuss some special cases which can be deduced from the main results. Our results continue to hold for these new and known special cases. It is an interesting problem to find the derivative estimates of the polynomials on the unit interval using the quantum calculus. For an excellent exposition of the theory of polynomials, see Rahman and Schmeisser [27]. The readers are encouraged to find applications of the generalized convexity and quantum integral inequalities in various fields of pure and applied sciences.

2 Basic Results from Quantum Calculus

In this section, we discuss some basic concepts and results pertaining to quantum calculus. For more details interested readers may consult [5, 10].

Let us start with q-analogue of differentiation. Then, consider

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{\mathrm{d}f}{\mathrm{d}x},$$

the above expression gives the derivative of a function $f(x)$ at $x = x_0$.

If we take $x = qx_0$ where $0 < q < 1$ is a fixed number and do not take limits, then we enter in the world of Quantum calculus. The q-derivative of x^n is $[n]x^{n-1}$, where

$$[n] = \frac{q^n - 1}{q - 1},$$

is the q-analogue of n in the sense that n is the limit of $[n]$ as $q \to 1$.

We now give the formal definition of q-derivative of a function f.

Definition 1. The q-derivative is defined as

$$\mathrm{D}_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}. \tag{1}$$

Note that when $q \to 1$, then we have ordinary derivative.

Definition 2. The function $F(x)$ is a q-antiderivative of $f(x)$, if $D_q F(x) = f(x)$. It is denoted by

$$\int f(x)\mathrm{d}_q x. \tag{2}$$

Our next definition is due to Jackson.

Definition 3. The Jackson integral of $f(x)$ is defined as

$$\int f(x)\mathrm{d}_q x = (1 - q)x \sum_{j=0}^{\infty} q^j f(q^j x). \tag{3}$$

It is evident from the above definition that

$$\begin{aligned} \int f(x) D_q g(x) \mathrm{d}_q x &= (1 - q)x \sum_{j=0}^{\infty} q^j f(q^j x) D_q g(q^j x) \\ &= (1 - q)x \sum_{j=0}^{\infty} q^j f(q^j x) \frac{g(q^j x) - g(q^{j+1} x)}{(1 - q)q^j x}. \end{aligned}$$

Definite q-integrals are defined as:

Definition 4 ([8]). Let $0 < a < b$. Then definite q-integral is defined as

$$\int_0^b f(x)\mathrm{d}_q x = (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b), \tag{4}$$

provided the sum converges absolutely.

A more general formula for definite integrals is given as

$$\int_0^b f(x)\mathrm{d}_q x = \sum_{j=0}^{\infty} f(q^j b)(g(q^j b) - g(q^{j+1}b)).$$

Remark 1. From the above definition of definite q-integral, we have

$$\int_a^b f(x)\mathrm{d}_q x = \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x.$$

We now recall some basic concepts of quantum calculus on finite intervals. These results are mainly due to Tariboon and Ntouyas [29, 30].

Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. Then q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ is defined as follows:

Definition 5. Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as

$$\mathscr{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{5}$$

A function f is q-differentiable on J if $\mathscr{D}_q f(x)$ exists for all $x \in J$. We illustrate this fact by an example.

Example 1. Let $x \in [a, b]$ and $0 < q < 1$. Then, for $x \neq a$, we have

$$\begin{aligned}
\mathscr{D}_q x^2 &= \frac{x^2 - (qx + (1-q)a)^2}{(1-q)(x-a)} \\
&= \frac{(1+q)x^2 - 2qax - (1-q)x^2}{x-a} \\
&= (1+q)x + (1-q)a.
\end{aligned}$$

Note that, when $x = a$, we have $\lim_{x \to a}(\mathscr{D}_q x^2) = 2a$.

Definition 6. Let $f : J \to \mathbb{R}$ is a continuous function. Then, a second-order q-derivative on J, which is denoted as $\mathscr{D}_q^2 f$, provided $\mathscr{D}_q f$ is q-differentiable on J is defined as $\mathscr{D}_q^2 f = \mathscr{D}_q(\mathscr{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivatives on J are defined by $\mathscr{D}_q^n f =: J \to \mathbb{R}$.

Lemma 1. *Let* $\alpha \in \mathbb{R}$. *Then*

$$\mathscr{D}_q (x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}.$$

Tariboon and Ntouyas [29, 30] defined the q-integral as:

Definition 7. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as

$$\int_a^x f(t)\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{6}$$

for $x \in J$.

These integrals can be viewed as Riemann-type q-integral. If $a = 0$ in (6), then we have the classical q-integral, that is

$$\int_0^x f(t)\mathrm{d}_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in [0,\infty).$$

Moreover, if $c \in (a, x)$, then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\mathrm{d}_q t &= \int_a^x f(t)\mathrm{d}_q t - \int_a^c f(t)\mathrm{d}_q t \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &- (1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Theorem 1. *Let* $f : I \to \mathbb{R}$ *be a continuous function. Then for* $x \in J$

1. $\mathscr{D}_q \int_a^x f(t)\,\mathrm{d}_q t = f(x)$
2. $\int_c^x \mathscr{D}_q f(t)\mathrm{d}_q t = f(x) - f(c)$ *for* $x \in (c, x)$.

Theorem 2. *Let $f, g : I \to \mathbb{R}$ be a continuous functions, $\alpha \in \mathbb{R}$. Then for $x \in J$*

1. $\int_a^x [f(t) + g(t)]\, \mathrm{d}_q t = \int_a^x f(t)\, \mathrm{d}_q t + \int_a^x g(t)\, \mathrm{d}_q t$
2. $\int_a^x (\alpha f(t))(t)\, \mathrm{d}_q t = \alpha \int_a^x f(t)\, \mathrm{d}_q t$
3. $\int_a^x f(t)\, {}_a\mathscr{D}_q g(t)\, \mathrm{d}_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\mathscr{D}_q f(t) \mathrm{d}_q t$ *for* $c \in (a, x)$.

Lemma 2. *Let $\alpha \in \mathbb{R} \setminus \{-1\}$. Then*

$$\int_a^x (t-a)^\alpha \mathrm{d}_q t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}.$$

3 Generalized Convexity

In this section, we recall the concepts of generalized convex sets and generalized convex functions involving two arbitrary functions $h, g : \mathbb{R} \to \mathbb{R}$, which are mainly due to Noor [15].

Definition 8 ([15]). Let $K \subset \mathbb{R}$ be any set. Then, the set K is said to be generalized convex set with respect to arbitrary functions $h, g : \mathbb{R} \to \mathbb{R}$ such that

$$(1-t)h(u) + tg(v) \in K, \quad \forall u, v \in \mathbb{R} : h(u), g(v) \in K, t \in [0, 1].$$

Remark 2. If $h = I = g$, the identity functions, then the definition of generalized convex set reduces to the definition of classical convex sets. If $h = I$, then we have the definition of Noor-type convex sets [14]. If $g = I$, then we have the definition of h-convex sets [16]. If $h = g$, then the definition of generalized convex set coincides with the definition of convex sets (E-convex sets), see [31]. This shows that the generalized convex sets include several kinds of convex sets as special cases. Clearly, this shows that the convex sets provide us with a unified framework.

Definition 9 ([15]). A function $f : K \to \mathbb{R}$ is said to be generalized convex, if there exist two arbitrary functions $h, g : \mathbb{R} \to \mathbb{R}$, such that

$$f((1-t)h(u) + tg(v)) \le (1-t)f(h(u)) + tf(g(v)), \quad \forall u, v \in \mathbb{R} : h(u), g(v) \in K, t \in [0, 1]. \tag{7}$$

Definition 10 ([16]). The function $f : K \to \mathbb{R}$ is said to be a generalized quasi-convex, if there exist two arbitrary functions $h, g : \mathbb{R} \to \mathbb{R}$ such that

$$f((1-t)h(u) + tg(v)) \leq \max\{f(h(u)), f(g(v))\},$$
$$\forall u, v \in \mathbb{R} : h(u), g(v) \in K, t \in [0, 1]. \tag{8}$$

Definition 11. Let $I \subseteq \mathbb{R}$ be the interval and $h, g : \mathbb{R} \to \mathbb{R}$ be any two arbitrary functions. Then f is generalized convex function, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ h(a) & x & g(b) \\ f(h(a)) & f(x) & f(g(b)) \end{vmatrix} \geq 0; \quad h(a) \leq x \leq g(b),$$

where $x = (1-t)h(a) + tg(b)$.

From the above definition, one can obtain the following equivalent forms:

1. f is generalized convex function.
2. $f(x) \leq f(h(a)) + \frac{f(g(b))-f(h(a))}{g(b)-h(a)}(x - h(a))$.
3. $\frac{f(x)-f(h(a))}{x-h(a)} \leq \frac{f(g(b))-f(h(a))}{g(b)-h(a)} \leq \frac{f(g(b))-f(x)}{g(b)-x}$.
4. $\frac{f(h(a))}{(x-h(a))(g(b)-h(a))} + \frac{f(x)}{(g(b)-x)(x-h(a))} + \frac{f(g(b))}{(g(b)-h(a))(g(b)-x)} \geq 0$.
5. $(g(b) - x)f(h(a)) + (g(b) - h(a))f(x) + (x - h(a))f(g(b)) \geq 0$.

Definition 12 ([26]). Two functions f and g are said to be similarly ordered (f is g-monotone) on $I \subseteq \mathbb{R}$, if

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq 0, \quad \forall x, y \in I.$$

Theorem 3. *The product of two similarly generalized convex functions involving two arbitrary functions is a generalized convex function.*

Proof. Let f and w be two generalized convex functions. Then

$$\begin{aligned}
&f((1-t)h(a) + tg(b))w((1-t)h(a) + tg(b)) \\
&\leq [(1-t)f(h(a)) + tf(g(b))][(1-t)w(h(a)) + tw(g(b))] \\
&= [1-t]^2 f(h(a))w(h(a)) + t(1-t)f(h(a))w(g(b)) + t(1-t)f(g(b))w(h(a)) \\
&\quad + [t]^2 f(g(b))w(g(b)) \\
&= (1-t)f(h(a))w(h(a)) + tf(g(b))w(g(b)) - (1-t)f(h(a))w(h(a)) \\
&\quad - tf(g(b))w(g(b)) + [1-t]^2 f(h(a))w(h(a)) + t(1-t)f(h(a))w(g(b)) \\
&\quad + t(1-t)f(g(b))w(h(a)) + [t]^2 f(g(b))w(g(b)) \\
&= (1-t)f(h(a))w(h(a)) + tf(g(b))w(g(b))
\end{aligned}$$

$$-t(1-t)[f(h(a))w(h(a)) + f(g(b))w(g(b))$$
$$-f(g(b))w(h(a)) - f(h(a))w(g(b))]$$
$$\leq (1-t)f(h(a))w(h(a)) + tf(g(b))w(g(b)).$$

This completes the proof. □

4 Quantum Estimates

In this section, we establish some quantum estimates of Hermite–Hadamard type inequalities via generalized convexity.

Theorem 4. *Let $f : I \to \mathbb{R}$ be generalized convex and continuous function on J with respect to arbitrary functions $h, g : \mathbb{R} \to \mathbb{R}$. Then, for $0 < q < 1$, we have*

$$f\left(\frac{h(a)+g(b)}{2}\right) \leq \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(t)\mathrm{d}_q t \leq \frac{qf(h(a)) + f(g(b))}{1+q}. \tag{9}$$

Proof. Let f be a generalized convex function, then q-integrating with respect to t on $[0, 1]$. Then

$$\begin{aligned}
f\left(\frac{h(a)+g(b)}{2}\right) &= \int_0^1 f\left(\frac{(1-t)h(a)+tg(b)+th(a)+(1-t)g(b)}{2}\right) \mathrm{d}_q t \\
&\leq \frac{1}{2}\left[\int_0^1 f((1-t)h(a)+tg(b))\mathrm{d}_q t + \int_0^1 f(th(a)+(1-t)g(b))\mathrm{d}_q t\right] \\
&= \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(t)\mathrm{d}_q t \\
&= \int_0^1 f((1-t)h(a)+tg(b))\mathrm{d}_q t \\
&\leq f(h(a)) \int_0^1 (1-t)\mathrm{d}_q t + f(g(b)) \int_0^1 t\,\mathrm{d}_q t \\
&= \frac{qf(h(a)) + f(g(b))}{1+q}.
\end{aligned}$$

This completes the proof. □

Note that, if $q \to 1$, then Theorem 4 reduces to:

Theorem 5. *Let $f : [h(a), g(b)] \to \mathbb{R}$ be a generalized convex function. Then*

$$f\left(\frac{h(a)+g(b)}{2}\right) \leq \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)\mathrm{d}x \leq \frac{f(h(a))+f(g(b))}{2}.$$

Theorem 6. *Let $f, w : I \to \mathbb{R}$ be generalized convex functions involving two arbitrary functions. Then*

$$\begin{aligned}
&\frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)w(x)\mathrm{d}_q x \\
&\leq \theta_1 f(h(a))w(h(a)) + \theta_2 N(h(a), g(b)) + \theta_3 f(g(b))w(g(b)),
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &= \frac{q(1+q^2)}{(1+q)(1+q+q^2)}; \\
\theta_2 &= \frac{q^2}{(1+q)(1+q+q^2)}; \\
\theta_3 &= \frac{1}{1+q+q^2},
\end{aligned}$$

and $N(a, b; h; g)$ is given by

$$N(a, b; h; g) = f(h(a))w(g(b)) + f(g(b))w(h(a)). \tag{10}$$

Proof. Since f and w are generalized convex functions involving two arbitrary functions, so we have

$$\begin{aligned}
f((1-t)h(a) + tg(b)) &\leq tf(h(a)) + (1-t)f(g(b)), \\
w((1-t)h(a) + tg(b)) &\leq tw(h(a)) + (1-t)w(g(b)).
\end{aligned}$$

Multiplying the above inequalities, we have

$$\begin{aligned}
&f((1-t)h(a) + tg(b))w((1-t)h(a) + tg(b)) \\
&\leq (1-t)^2 f(h(a))w(h(a)) + t(1-t)\{f(h(a))w(g(b)) + f(g(b))w(h(a))\} \\
&+ t^2 f(g(b))w(g(b)).
\end{aligned}$$

Taking q-integral of the above inequality with respect to t on [0, 1], we have

$$\int_0^1 f((1-t)h(a)+tg(b))w((1-t)h(a)+tg(b))\mathrm{d}_q t$$

$$\leq f(h(a))w(h(a))\int_0^1 (1-t)^2\mathrm{d}_q t + \{f(h(a))w(g(b))+f(g(b))w(h(a))\}\int_0^1 t(1-t)\mathrm{d}_q t$$

$$+f(g(b))w(g(b))\int_0^1 t^2\mathrm{d}_q t.$$

This implies that

$$\frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)w(x)\mathrm{d}_q x$$

$$\leq \left[\frac{q(1+q^2)}{(1+q)(1+q+q^2)}\right] f(h(a))w(h(a))$$

$$+\left[\frac{q^2}{(1+q)(1+q+q^2)}\right]\{f(h(a))w(g(b))+f(g(b))w(h(a))\}$$

$$+\left[\frac{1}{1+q+q^2}\right] f(g(b))w(g(b)).$$

This completes the proof. □

Theorem 7. *Let f and w be two generalized convex functions involving two arbitrary functions. Then*

$$2f\left(\frac{h(a)+g(b)}{2}\right)w\left(\frac{h(a)+g(b)}{2}\right)-\frac{2q^2M(a,b;h;g)+(1+2q+q^3)N(a,b;h;g)}{2(1+q)(1+q+q^2)}$$

$$\leq \frac{1}{(g(b)-h(a))}\int_{h(a)}^{g(b)} f(x)w(x)\mathrm{d}_q x,$$

where

$$M(a,b;h;g)=f(h(a))w(h(a))+f(g(b))w(g(b)), \tag{11}$$

and $N(a,b;h;g)$ is given by (8).

Proof. Since f and w are generalized convex functions, so we have

$$\begin{aligned}
& f\left(\frac{h(a)+g(b)}{2}\right) w\left(\frac{h(a)+g(b)}{2}\right) \\
& \leq \frac{1}{4}\left[f((1-t)h(a)+tg(b))+f(th(a)+(1-t)g(b))\right. \\
& \left.+w((1-t)h(a)+tg(b))+w(th(a)+(1-t)g(b))\right] \\
& \leq \frac{1}{4}\Bigg[f((1-t)h(a)+tg(b))w((1-t)h(a)+tg(b)) \\
& \quad +f(th(a)+(1-t)g(b))w(th(a)+(1-t)g(b)) \\
& \quad +[f(h(a))w(h(a))+f(g(b))w(g(b))]\{2t(1-t)\} \\
& \quad +[f(h(a))w(g(b))+f(g(b))w(h(a))]\{t^2+(1-t)^2\}\Bigg].
\end{aligned}$$

By q-integrating the above inequality with respect to t on $[0, 1]$, we have

$$\begin{aligned}
& f\left(\frac{h(a)+g(b)}{2}\right) w\left(\frac{h(a)+g(b)}{2}\right) \\
& \leq \frac{1}{4}\Bigg[\int_0^1 [f((1-t)h(a)+tg(b))w((1-t)h(a)+tg(b)) \\
& \quad +f(th(a)+(1-t)g(b))w(th(a)+(1-t)g(b))]\,\mathrm{d}_q t \\
& \quad +[f(h(a))w(h(a))+f(g(b))w(g(b))]\int_0^1 \{2t(1-t)\}\mathrm{d}_q t \\
& \quad +[f(h(a))w(g(b))+f(g(b))w(h(a))]\int_0^1 \{t^2+(1-t)^2\}\mathrm{d}_q t\Bigg] \\
& = \frac{1}{2(g(b)-h(a))}\int_{h(a)}^{g(b)} f(x)w(x)\mathrm{d}_q x \\
& \quad +\frac{1}{4}\Bigg[\frac{2q^2\{f(h(a))w(h(a))+f(g(b))w(g(b))\}}{(1+q)(1+q+q^2)} \\
& \quad +\frac{(1+2q+q^3)[f(h(a))w(g(b))+f(g(b))w(h(a))]}{(1+q)(1+q+q^2)}\Bigg].
\end{aligned}$$

This completes the proof. □

We now prove an auxiliary result, which will be useful in proving our coming results.

Lemma 3. *Let* $f : I = [h(a), g(b)] \subset \mathbb{R} \to \mathbb{R}$ *be a q-differentiable function on the interior* I° *of* I *and let* $\mathscr{D}_q$ *be continuous and integrable on* I, *where* $0 < q < 1$. *Then*

$$\frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x - \frac{qf(h(a))+f(g(b))}{1+q}$$

$$= \frac{q(g(b)-h(a))}{1+q}\int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)h(a)+t(g(b)))\,\mathrm{d}_q t.$$

Proof. The proof is left for interested readers. □

Remark 3. We would like to emphasize that if $h = I = g$ in Lemma 3, then we have Lemma 3.1 [22]. If, along with $h = I = g$, $q \to 1$, we have Lemma 2.1 [3].

Theorem 8. *Let* $f : I \to \mathbb{R}$ *be a q-differentiable function on the interior* I° *of* I *and let* $\mathscr{D}_q$ *be continuous and integrable on* I, *where* $0 < q < 1$. *Let* $|\mathscr{D}_q f|^r$, $r \geq 1$ *be a generalized convex function. Then*

$$\left|\frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x - \frac{qf(h(a))+f(g(b))}{1+q}\right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(h(a))|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(g(b))|^r\right]^{\frac{1}{r}}.$$

Proof. Since $|\mathscr{D}_q f|^r$ is generalized convex function, using Lemma 3 and power mean inequality, we have

$$\left|\frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x - \frac{qf(h(a))+f(g(b))}{1+q}\right|$$

$$= \left|\frac{q(g(b)-h(a))}{1+q}\int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)h(a)+t(g(b)))\,\mathrm{d}_q t\right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q}\left(\int_0^1 |1-(1+q)t|\,\mathrm{d}_q t\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)t||\mathscr{D}_q f((1-t)h(a)+t(g(b)))|^r \, \mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$\leq \frac{q(g(b)-h(a))}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)t|[(1-t)|\mathscr{D}_q f(h(a))|^r + t|\mathscr{D}_q f(g(b))|^r] \, \mathrm{d}_q t\right)^{\frac{1}{r}}$$

$$= \frac{q(g(b)-h(a))}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(h(a))|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(g(b))|^r\right]^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 9. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a generalized convex function, where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$. Then*

$$\left|\frac{qf(h(a))+f(g(b))}{1+q} - \frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x\right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(h(a))|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|\mathscr{D}_q f(g(b))|^r\right]^{\frac{1}{r}}.$$

Proof. Since $|\mathscr{D}_q f|^r$ is generalized convex function, so using Lemma 3 and Holder's inequality, we have

$$\left|\frac{qf(h(a))+f(g(b))}{1+q} - \frac{1}{g(b)-h(a)}\int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x\right|$$

$$= \left|\frac{q(g(b)-h(a))}{1+q}\int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)h(a)+t(g(b)))\mathrm{d}_q t\right|$$

$$\leq \left| \frac{q(g(b)-h(a))}{1+q} \int_0^1 (1-(1+q)t)^{1-\frac{1}{r}} (1-(1+q)t)^{\frac{1}{r}} \mathscr{D}_q f((1-t)h(a)+t(g(b)))\mathrm{d}_q t \right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q} \left(\int_0^1 |1-(1+q)t| \mathrm{d}_q t \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 |1-(1+q)t| |\mathscr{D}_q f((1-t)h(a)+t(g(b)))|^r \mathrm{d}_q t \right)^{\frac{1}{r}}$$

$$= \frac{q(g(b)-h(a))}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{\frac{1}{p}}$$

$$\times \left[\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^2} |\mathscr{D}_q f(h(a))|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3} |\mathscr{D}_q f(g(b))|^r \right]^{\frac{1}{r}}.$$

This completes the proof. □

Now we prove some quantum analogues of Iyengar type inequalities via generalized convex functions involving two arbitrary functions.

Theorem 10. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a generalized quasi-convex function such that $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$. Then*

$$\left| \frac{qf(h(a)) + f(g(b))}{1+q} - \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x \right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{\frac{1}{p}}$$

$$\times \left(\frac{q(2+q+q^3)}{(1+q)^3} \left[\max\{|\mathscr{D}_q f(h(a))|, |\mathscr{D}_q f(g(b))|\} \right] \right)^{\frac{1}{r}}.$$

Proof. Using Lemma 3, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is generalized quasi-convex function, we have

$$\left| \frac{qf(h(a)) + f(g(b))}{1+q} - \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x \right|$$

$$= \left| \frac{q(g(b)-h(a))}{1+q} \int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)h(a)+t(g(b)))\mathrm{d}_q t \right|$$

$$\leq \left| \frac{q(g(b)-h(a))}{1+q} \int_0^1 (1-(1+q)t)^{1-\frac{1}{r}}(1-(1+q)t)^{\frac{1}{r}} \mathscr{D}_q f((1-t)h(a)+t(g(b)))\mathrm{d}_q t \right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q} \left(\int_0^1 |1-(1+q)t|\mathrm{d}_q t \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 |1-(1+q)t||\mathscr{D}_q f((1-t)h(a)+t(g(b)))|^r \mathrm{d}_q t \right)^{\frac{1}{r}}$$

$$= \frac{q(g(b)-h(a))}{1+q} \left(\frac{2q}{(1+q)^2} \right)^{\frac{1}{p}}$$

$$\times \left(\frac{q(2+q+q^3)}{(1+q)^3} \Big[\max\{|\mathscr{D}_q f(h(a))|, |\mathscr{D}_q f(g(b))|\} \Big] \right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 11. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a generalized quasi-convex function, $r > 1$. Then*

$$\left| \frac{qf(h(a))+f(g(b))}{1+q} - \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x \right|$$

$$\leq \frac{q^2(g(b)-h(a))(2)}{(1+q)^3} \left(\max\{|\mathscr{D}_q f(h(a))|, |\mathscr{D}_q f(g(b))|\} \right)^{\frac{1}{r}}.$$

Proof. Using Lemma 3, power mean inequality and the fact that $|\mathscr{D}_q f|^r$ is generalized quasi-convex function, we have

$$\left| \frac{qf(h(a))+f(g(b))}{1+q} - \frac{1}{g(b)-h(a)} \int_{h(a)}^{g(b)} f(x)\mathrm{d}_q x \right|$$

$$= \left| \frac{q(g(b)-h(a))}{1+q} \int_0^1 (1-(1+q)t)\mathscr{D}_q f((1-t)h(a)+t(g(b)))\mathrm{d}_q t \right|$$

$$\leq \frac{q(g(b)-h(a))}{1+q}\left(\int_0^1 |1-(1+q)t|\mathrm{d}_qt\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)t||\mathscr{D}_qf((1-t)h(a)+t(g(b)))|^r\mathrm{d}_qt\right)^{\frac{1}{r}}$$

$$= \frac{q^2(g(b)-h(a))(2)}{(1+q)^3}\left(\max\{|\mathscr{D}_qf(h(a))|,|\mathscr{D}_qf(g(b))|\}\right)^{\frac{1}{r}}.$$

This completes the proof. □

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Quantum Integral Inequalities for Generalized Preinvex Functions

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Abstract We consider the generalized preinvex functions, which unify the preinvex and φ-convex functions. We give an account of the quantum integral inequalities via the generalized preinvex functions. Results obtained in this chapter represent significant and important refinements of the known results. These inequalities involve Riemann-type quantum integrals. We would like to emphasize that these results reduce to classical results, when $q \to 1$. It is expected that ideas and techniques given here would inspire further research.

Keywords Preinvex functions • Integral inequalities • Quantum estimates • Convex functions • Invex sets

2000 Mathematics Subject Classification: 26A33, 26D15, 49J40, 90C33

1 Introduction

Theory of convexity has a great impact in our daily life through its numerous applications in various fields of pure and applied sciences, see [3, 4, 6, 11, 15, 36–40]. Many classical and famous inequalities have been obtained via convex functions and

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their various generalizations, see [5, 7–9, 16, 20, 24–28, 30–34, 41, 43]. One of the most extensively studied inequalities is Hermite–Hadamard type inequality, which provides a necessary and sufficient condition for a function to be convex.

Theorem 1. *Let* $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

The left side of Hermite–Hadamard's inequality is estimated by Ostrowski's inequality, which reads as:

Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| < M$, where M is a constant, then we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

The right side of Hermite–Hadamard inequality can be estimated by the inequality of Iyengar, which reads as:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{M(b-a)}{4} - \frac{1}{4M(b-a)}(f(b)-f(a))^2,$$

where by M, we denote the Lipschitz constant, that is, $M = \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|; x \neq y\right\}$.

In recent years, convex sets and convex functions have been generalized in several directions using novel and innovative ideas. In this chapter, we mainly consider two different classes of functions, which are investigated recently. The origin of one of these classes can be traced back to Hanson [14], which was introduced in mathematical programming. This class of functions is called invex functions. Ben Israel and Mond [2] introduced the concept of the invex set and defined the class of nonconvex functions on the invex sets. This class of nonconvex functions is known as preinvex functions. Under certain conditions, one can show that these two classes of nonconvex functions are equivalent. Noor [21] had shown that the minimum of the differentiable preinvex functions on the invex sets can be characterized by a class of variational inequalities, which is called variational-like inequalities. For the numerical methods and other aspects of the variational-like inequalities, see [21, 22] and the references therein. Gordji et al. [13] introduced the notion of φ-convex functions with respect to the bifunction $\varphi(.,.)$. They have shown that this class of convex functions is quite different than the classical convex functions and preinvex functions. Gordi et al. [13] also introduced another class of nonconvex functions, which is called the generalized preinvex functions. This class of nonconvex functions includes the preinvex functions and φ-convex functions as special cases.

In recent years, some authors have utilized the concepts of quantum calculus to obtain the integral inequalities via different classes of convex functions, see [12, 26–29, 31, 41, 43]. Noor et al. [28, 31] obtained some q-analogues of certain integral inequalities involving preinvex functions and φ-convex functions, respectively.

In Sect. 2, we recall the basic concepts and results. We also give a brief account of the Hermite–Hadamard inequalities for the φ-convex functions. Preinvex functions and their characterizations are also given. Section 3 is devoted to the fundamentals of the quantum calculus. In Sect. 4, we establish the quantum inequalities for φ-convex functions and preinvex functions. These results are due to Noor et al. [28, 31]. In Sect. 5, we introduce the class of the generalized preinvex functions. We derive several new quantum integral inequalities for the generalized preinvex functions. Results obtained here continue to hold for the previously known and new classes of convex functions. It is an interesting problem to explore the applications of the quantum calculus and the generalized preinvex functions in the theory of polynomials. For an excellent exposition of the theory of polynomials, see Rahman and Schmeisser [35].

2 φ-Convex Functions and Preinvex Functions

In this section, we recall some definitions and basic results of φ-convex functions and preinvex functions. We include all the necessary details to convey the main ideas involved.

Definition 1 ([13]). A function $f : K \to \mathbb{R}$ is said to be φ-convex function, if there exists a bifunction $\varphi(.,.)$, such that

$$f((1-\mu)u+\mu v) \leq f(u)+\mu\varphi(f(v),f(u)), \quad \forall u,v \in K, \mu \in [0,1]. \tag{1}$$

Also f is said to be φ-affine, if

$$f((1-\mu)u+\mu v) = f(u)+\mu\varphi(f(v),f(u)), \quad \forall u,v \in K. \tag{2}$$

If $\varphi(f(v),f(u)) = f(v)-f(u)$ in (1) and (2), then we have classical convex and classical affine functions.

If $u = v$ in (1), then we have $\varphi(f(v),f(u)) \geq 0$. If $\mu = 1$, then

$$f(v)-f(u) \leq \varphi(f(v),f(u)).$$

We show that the φ-convex functions are not convex functions. We give some examples of φ-convex functions, which are due to Gordi et al. [13].

Examples. 1. For a convex function f, we may find another function $\varphi(u,v) = u - v$ such that f is φ-convex. Let $f(x) = x^2$ and $\varphi(u,v) = 2u + v$. Then

$$\begin{aligned} f(\mu u + (1-\mu)v) &= (\mu u + (1-\mu)v)^2 \\ &\leq v^2 + \mu u^2 + \mu(1-\mu)2uv \\ &\leq v^2 + \mu u^2 + \mu(1-\mu)(u^2+v^2) \\ &\leq v^2 + \mu(2u^2+v^2) \\ &= f(v) + \mu\varphi(f(u), f(v)). \end{aligned}$$

Also the facts $u^2 \leq v^2 + (2u^2 + v^2)$ and $v^2 \leq v^2$, for $u, v \in \mathbb{R}$, show the correctness of inequality for $\mu = 0$, respectively. This means that f is φ-convex. We would like to remark here that function $f(u) = u^2$ is a φ-convex function with respect to $\varphi(u,v) = au + bv$ with $a \geq 1, b \geq -1$ and $u, v \in \mathbb{R}$.

2. There is a φ-convex function f which is not convex. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} -u, \textit{ if} & u \geq 0, \\ u, \quad \textit{if} & u < 0, \end{cases} \tag{3}$$

and $\varphi : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R}$ be defined as

$$\varphi(u,v) = \begin{cases} u, & \textit{if} \quad v = 0, \\ -v, & \textit{if} \quad u = 0, \\ -u - v, & \textit{if} \quad u < 0, v < 0. \end{cases} \tag{4}$$

Then it is clear that f is φ-convex and is not convex in the classical sense.

Definition 2. A function $f : K \to \mathbb{R}$ is said to be φ-quasiconvex, if

$$f(\mu u + (1-\mu)v) \leq \max\{f(u), f(u) + \varphi(f(y), f(u))\}, \quad \forall u, v \in K, \mu \in [0,1].$$

Theorem 2. *A function $f : I \to \mathbb{R}$ is said to a φ-convex if and only if*

$$\begin{vmatrix} 1 & x_1 & \varphi(f(x_1), f(x_3)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{vmatrix} \geq 0; \quad x_1 < x_2 < x_3,$$

and

$$f(x_2) \leq f(x_3) + \varphi(f(x_1), f(x_3)).$$

Our next result is Hermite–Hadamard type inequality for φ-convex functions.

Theorem 3 ([13]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a φ-convex function and φ be bounded from above on $f(I) \times f(I)$. Then, for any $a, b \in I$ with $a < b$, we have*

$$2f\left(\frac{a+b}{2}\right) - M_\varphi \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq f(b) + \frac{\varphi(f(a), f(b))}{2},$$

where M_φ is an upper bound of φ on $f([a,b]) \times f([a,b])$.

Proof. Let f be a φ-convex function. Then

$$f(\mu a + (1-\mu)b) \leq f(b) + \mu\varphi(f(a), f(b)) \leq f(b) + M_\varphi.$$

This shows that f has an upper bound. To find the lower bound of f, consider

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{4} + \frac{t}{2} + \frac{a+b}{4} - \frac{t}{2}\right) \\ &= f\left(\frac{1}{2}\left(\frac{a+b}{2} + t\right) + \frac{1}{2}\left(\frac{a+b}{2} - t\right)\right) \\ &\leq f\left(\frac{a+b}{2} - t\right) + \frac{1}{2}\varphi\left(f\left(\frac{a+b}{2} + t\right), f\left(\frac{a+b}{2} - t\right)\right) \\ &\leq f\left(\frac{a+b}{2} - t\right) + \frac{M_\varphi}{2}. \end{aligned}$$

Now suppose $m = f\left(\frac{a+b}{2}\right) - \frac{M_\varphi}{2}$. For the right-hand side of the inequality, let $x = \mu a + (1-\mu)b$. Then this implies that $f(x) \leq f(b) + \mu\varphi(f(a), f(b))$, where $\mu = \frac{x-b}{a-b}$. It follows that

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &\leq \frac{1}{b-a}\left(f(b)(b-a) + \frac{\varphi(f(a), f(b))}{b-a} \cdot \frac{(b-a)^2}{2}\right) \\ &= f(b) + \frac{\varphi(f(a), f(b))}{2}. \end{aligned}$$

Since f is φ-convex, hence

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{4} - \frac{t(b-a)}{4} + \frac{a+b}{4} + \frac{t(b-a)}{4}\right) \\ &= f\left(\frac{1}{2}\left(\frac{a+b-t(b-a)}{2}\right) + \frac{1}{2}\left(\frac{a+b+t(b-a)}{2}\right)\right) \\ &\leq f\left(\frac{a+b+t(b-a)}{2}\right) + \frac{1}{2}\varphi\left(f\left(\frac{a+b-t(b-a)}{2}\right),\right. \end{aligned}$$

$$f\left(\frac{a+b+t(b-a)}{2}\right)\Bigg)$$
$$\leq f\left(\frac{a+b+t(b-a)}{2}\right)+\frac{1}{2}M_{\varphi}, \quad \forall t\in[0,1].$$

Also

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x+\int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x\right]\\ &= \int_0^1\left[f\left(\frac{a+b-t(b-a)}{2}\right)+f\left(\frac{a+b+t(b-a)}{2}\right)\right]\mathrm{d}t\\ &\geq \int_0^1\left[f\left(\frac{a+b-t(b-a)}{2}\right)+f\left(\frac{a+b}{2}\right)-\frac{1}{2}M_{\varphi}\right]\mathrm{d}t\\ &\geq m+f\left(\frac{a+b}{2}\right)-\frac{1}{2}M_{\varphi}\\ &= 2f\left(\frac{a+b}{2}\right)-M_{\varphi}.\end{aligned}$$

This completes the proof. □

We now recall the concepts of invex sets and preinvex functions, which are mainly due to Ben-Isreal and Mond [2]. See also [44].

Definition 3 ([44]). A set $K\subset\mathbb{R}$ is said to be invex set with respect to an arbitrary bifunction $\eta(.,.)$, if

$$u+\mu\eta(v,u)\in K, \quad \forall u,v\in K, t\in[0,1]. \tag{5}$$

The invex set K is also called η-connected set.

Remark 1 ([1]). We note that every convex set is also an invex set with respect to $\eta(v,u)=v-u$, but the converse is not necessarily true. See [45] and the references therein.

Definition 4 ([44]). A function f is said to be preinvex with respect to an arbitrary bifunction $\eta(.,.)$, if

$$f(u+t\eta(v,u))\leq(1-t)f(u)+tf(v), \quad \forall u,v\in K, t\in[0,1]. \tag{6}$$

For $\eta(v, u) = v - u$ in (6), the preinvex functions reduce to convex functions in the classical sense. In general, preinvex functions are not convex functions.

We also need the well-known Condition C, which was introduced by Mohan and Neogy in [19]. This condition is automatically satisfied for the convex functions.

Condition C. Let $K \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(.,.)$. Then, $\forall x, y \in K$ and $t \in [0, 1]$,

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y),$$
$$\eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).$$

Note that for every $x, y \in K$, $t_1, t_2 \in [0, 1]$ and from Condition C, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

It is worth mentioning that Condition C plays a crucial and significant role in the development of the variational-like inequalities, see [21, 22] and the references therein.

Noor [23] has shown that a function f is a preinvex function, if and only if, it satisfies the inequality

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a) + f(b)}{2},$$

which is called the Hermite–Hadamard–Noor inequality.

3 Quantum Calculus

In this section, we recall some basic concepts and results of quantum calculus. For further details see [10, 18]. We start with the q-analogue of differentiation. For this purpose, we consider

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{\mathrm{d}f}{\mathrm{d}x}.$$

This implies the derivative of a function $f(x)$ at $x = x_0$.

If we take $x = qx_0$, where $0 < q < 1$ is a fixed number and do not take limits, then we enter in the world of Quantum calculus. The q-derivative of x^n is $[n]x^{n-1}$, where

$$[n] = \frac{q^n - 1}{q - 1},$$

is the q-analogue of n in the sense that n is the limit of $[n]$ as $q \to 1$.

Formal definition of q-derivative of a function f is given as:

Definition 5. The q-derivative is defined as

$$\mathrm{D}_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{7}$$

Note that when $q \to 1$, then we have ordinary derivative.

Usually, q-analogue of antiderivatives of function f is defined as follows:

Definition 6. The function $F(x)$ is a q-antiderivative of $f(x)$, if $D_q F(x) = f(x)$. It is denoted by $\int f(x)\mathrm{d}_q x$.

Jackson [17] defined q-integral as follows:

Definition 7. The Jackson integral of $f(x)$ is defined:

$$\int f(x)\mathrm{d}_q x = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x). \tag{8}$$

It is evident from the above definition that

$$\begin{aligned} \int f(x) D_q g(x) \mathrm{d}_q x &= (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) D_q g(q^j x) \\ &= (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) \frac{g(q^j x) - g(q^{j+1} x)}{(1-q) q^j x}. \end{aligned}$$

Definite q-integrals are defined as follows:

Definition 8 ([17]). Let $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(x)\mathrm{d}_q x = (1-q)b \sum_{j=0}^{\infty} q^j f(q^j b), \tag{9}$$

provided the sum converges absolutely.

A more general formula for definite integrals is given as follows:

$$\int_0^b f(x)\mathrm{d}_q x = \sum_{j=0}^{\infty} f(q^j b)(g(q^j b) - g(q^{j+1} b)).$$

Remark 2. Definite q-integral in a generic interval $[a, b]$ is given by

$$\int_a^b f(x)\mathrm{d}_q x = \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x.$$

Tariboon et al. [42, 43] discussed the concepts of quantum calculus on finite intervals. To be more precise, let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ is defined as follows.

Definition 9. Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as

$$\mathscr{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{10}$$

A function f is q-differentiable on J, if $\mathscr{D}_q f(x)$ exists for all $x \in J$. We elaborate this definition with the help of an example.

Example 1. Let $x \in [a, b]$ and $0 < q < 1$. Then, for $x \neq a$, we have

$$\begin{aligned} \mathscr{D}_q x^2 &= \frac{x^2 - (qx + (1-q)a)^2}{(1-q)(x-a)} \\ &= \frac{(1+q)x^2 - 2qax - (1-q)x^2}{x-a} \\ &= (1+q)x + (1-q)a. \end{aligned}$$

Note that when $x = a$, we have $\lim_{x \to a}(\mathscr{D}_q x^2) = 2a$.

Definition 10. Let $f : J \to \mathbb{R}$ be a continuous function. Then a second-order q-derivative on J, which is denoted as $\mathscr{D}_q^2 f$, provided $\mathscr{D}_q f$ is q-differentiable on J is defined as $\mathscr{D}_q^2 f = \mathscr{D}_q(\mathscr{D}_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by $\mathscr{D}_q^n f =: J \to \mathbb{R}$.

Lemma 1. *Let $\alpha \in \mathbb{R}$. Then*

$$\mathscr{D}_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}.$$

Tariboon et al. [42, 43] defined the q-integral as follows:

Definition 11. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as

$$\int_a^x f(t)\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \qquad \forall x \in J. \tag{11}$$

These integrals can be viewed as Riemann-type q-integral. If $a = 0$ in (11), then we have the classical q-integral, that is

$$\int_0^x f(t)\mathrm{d}_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in [0,\infty).$$

Moreover, if $c \in (a,x)$, then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)\mathrm{d}_q t &= \int_a^x f(t)\mathrm{d}_q t - \int_a^c f(t)\mathrm{d}_q t \\ &= (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad -(1-q)(c-a)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Theorem 4. *Let $f : I \to \mathbb{R}$ be a continuous function. Then*

1. $\mathscr{D}_q \int_a^x f(t)\,\mathrm{d}_q t = f(x)$
2. $\int_c^x \mathscr{D}_q f(t)\mathrm{d}_q t = f(x) - f(c), \qquad$ *for* $x \in (c,x)$.

Theorem 5. *Let $f, g : I \to \mathbb{R}$ be continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

1. $\int_a^x [f(t) + g(t)]\,\mathrm{d}_q t = \int_a^x f(t)\,\mathrm{d}_q t + \int_a^x g(t)\,\mathrm{d}_q t$
2. $\int_a^x (\alpha f(t))(t)\,\mathrm{d}_q t = \alpha \int_a^x f(t)\,\mathrm{d}_q t$
3. $\int_a^x f(t)\,{}_a\mathscr{D}_q g(t)\,\mathrm{d}_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\mathscr{D}_q f(t)\mathrm{d}_q t, \quad$ *for* $c \in (a,x)$.

Tariboon et al. [43] obtained the following q-analogue of Hermite–Hadamard's inequality for convex functions as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

4 Quantum Integral Inequalities Via φ-Convex and Preinvex Functions

In this section, we recall the recent results, which are mainly due to Noor et al. [31]. They have derived the quantum Hermite–Hadamard inequalities for φ-convex functions and preinvex functions.

Theorem 6. *Let $f, g : I \to \mathbb{R}$ be two φ-convex functions. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}_q x \leq f(a)g(a) + \frac{1}{1+q}T(a,b;\varphi;f;g) + \frac{1}{1+q+q^2}R(a,b;\varphi;f;g),$$

where

$$T(a,b;\varphi;f;g) = f(a)\varphi(g(b),g(a)) + g(a)\varphi(f(b),f(a)),$$

and

$$R(a,b;\varphi;f;g) = \varphi(g(b),g(a))\varphi(f(b),f(a)).$$

Proof. Let f and g be two φ-convex functions. Then

$$f((1-\mu)a+\mu b) \leq f(a) + \mu\varphi(f(b),f(a))$$
$$g((1-\mu)a+\mu b) \leq g(a) + \mu\varphi(g(b),g(a)).$$

Multiplying the above inequalities, we have

$$f((1-\mu)a+\mu b)g((1-\mu)a+\mu b) \leq f(a)g(a) + \mu f(a)\varphi(g(b),g(a)) + \mu g(a)\varphi(f(b),f(a)) + \mu^2\varphi(g(b),g(a))\varphi(f(b),f(a)).$$

Now by q-integrating the above inequality with respect to μ on [0, 1], we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}_q x\\&\leq f(a)g(a)+\frac{1}{1+q}\left[f(a)\varphi(g(b),g(a))+g(a)\varphi(f(b),f(a))\right]\\&\quad+\frac{1}{1+q+q^2}\varphi(g(b),g(a))\varphi(f(b),f(a)).\end{aligned}$$

□

The following two auxiliary results are due to Noor et al. [27]. These results play an important role in the development of our results.

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Then*

$$\begin{aligned}H_f(a,b;q)&=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}_q x-\frac{qf(a)+f(b)}{1+q}\\&=\frac{q(b-a)}{1+q}\int_0^1(1-(1+q)\mu)\mathrm{D}_q f((1-\mu)a+\mu b)\,\mathrm{d}_q\mu.\end{aligned}$$

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Then*

$$\begin{aligned}K_f(a,b;q)&=f(x)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\\&=\frac{q(x-a)^2}{b-a}\int_0^1\mu\mathscr{D}_q f(\mu x+(1-\mu)a)\mathrm{d}_q\mu\\&\quad+\frac{q(b-x)^2}{b-a}\int_0^1\mu\mathscr{D}_q f(\mu x+(1-\mu)b)\mathrm{d}_q\mu\end{aligned}$$

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|$ be a φ-convex function. Then*

$$\begin{aligned}&\left|H_f(a,b;q)\right|\\&\le \Omega(b-a)\\&\times\left[2(1+q)(1+q+q^2)|\mathscr{D}_q f(a)|+(1+4q+q^2)\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)\right],\end{aligned}$$

where

$$\Omega=\frac{q^2}{(1+q+q^2)(1+q)^4}.$$

Proof. Using Lemma 2 and the fact that $|\mathscr{D}_q f|$ is a φ-convex function, we have

$$\begin{aligned}&\left|H_f(a,b;q)\right|\\&=\left|\frac{q(b-a)}{1+q}\int_0^1(1-(1+q)\mu)\mathscr{D}_q f((1-\mu)a+\mu b)\,\mathrm{d}_q\mu\right|\\&\le\frac{q(b-a)}{1+q}\int_0^1|1-(1+q)\mu||\mathscr{D}_q f((1-\mu)a+\mu b)|\,\mathrm{d}_q\mu\\&\le\frac{q(b-a)}{1+q}\int_0^1|1-(1+q)\mu|[|\mathscr{D}_q f(a)|+\mu\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)]\\&=\frac{q(b-a)}{1+q}\left[\int_0^1|1-(1+q)\mu||\mathscr{D}_q f(a)|\mathrm{d}_q\mu\right.\\&\qquad\left.+\int_0^1\mu|1-(1+q)\mu|\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)\mathrm{d}_q\mu\right]\\&=\frac{q(b-a)}{1+q}\left[\frac{2q}{(1+q)^2}|\mathscr{D}_q f(a)|+\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)\right]\\&=\frac{q^2(b-a)}{(1+q+q^2)(1+q)^4}\\&\times\left[2(1+q)(1+q+q^2)|\mathscr{D}_q f(a)|+(1+4q+q^2)\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)\right].\end{aligned}$$

□

Theorem 8. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a φ-convex function, where $r > 1$. Then*

$$\begin{aligned}&\left|H_f(a,b;q)\right|\\&\leq \frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left[\theta_1|\mathscr{D}_q f(a)|^r + \theta_2\varphi(|\mathscr{D}_q f(b)|^r, |\mathscr{D}_q f(a)|^r)\right]^{\frac{1}{r}},\end{aligned}$$

where

$$\theta_1 = \frac{2q}{(1+q)^2}, \quad \text{and} \quad \theta_2 = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}.$$

Proof. Since $|\mathscr{D}_q f|^r$ is a φ-convex function, hence, from Lemma 2 and using Holder's inequality, we have

$$\begin{aligned}&\left|H_f(a,b;q)\right|\\&= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)\mu)\mathscr{D}_q f((1-\mu)a+\mu b)\mathrm{d}_q\mu\right|\\&\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)\mu|\mathrm{d}_q\mu\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f((1-\mu)a+\mu b)|^r\mathrm{d}_q\mu\right)^{\frac{1}{r}}\\&= \frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\\&\quad\times\left[\frac{2q}{(1+q)^2}|\mathscr{D}_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\varphi(|\mathscr{D}_q f(b)|^r, |\mathscr{D}_q f(a)|^r)\right]^{\frac{1}{r}}.\end{aligned}$$

This completes the proof. □

We now derive some quantum analogues for Iyengar type inequalities via φ-quasiconvex functions.

Theorem 9. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a φ-quasiconvex function, where $r > 1$. Then*

$$\left|H_f(a,b;q)\right| \le \frac{q^2(b-a)2}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\varphi(f(a),f(b))|^r\}\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 2, power mean inequality and the fact that $|\mathscr{D}_q f|^r$ is a φ-quasiconvex function, we have

$$\begin{aligned}\left|H_f(a,b;q)\right| &= \left|\frac{q(b-a)}{1+q}\int_0^1 (1-(1+q)\mu)_a D_q f((1-\mu)a+\mu b)\mathrm{d}_q\mu\right| \\ &\le \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)\mu|\mathrm{d}_q\mu\right)^{1-\frac{1}{r}} \\ &\quad\times\left(\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f((1-\mu)a+\mu b)|^r \mathrm{d}_q\mu\right)^{\frac{1}{r}} \\ &= \frac{q^2(b-a)2}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\varphi(f(a),f(b))|^r\}\right)^{\frac{1}{r}}.\end{aligned}$$

This completes the proof. □

Our next results are quantum analogues of Ostrowski type inequalities via φ-convex functions.

Theorem 10. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. Let $|\mathscr{D}_q f|$ be a φ-convex function. Then*

$$\begin{aligned}\left|K_f(a,b;q)\right| \le{}& \frac{q(x-a)^2}{b-a}\left[\frac{1}{1+q}|\mathscr{D}_q f(a)| + \frac{1}{1+q+q^2}\varphi(|\mathscr{D}_q f(x)|, |\mathscr{D}_q f(a)|)\right] \\ &+ \frac{q(b-x)^2}{b-a}\left[\frac{1}{1+q}|\mathscr{D}_q f(b)| + \frac{1}{1+q+q^2}\varphi(|\mathscr{D}_q f(x)|, |\mathscr{D}_q f(b)|)\right].\end{aligned}$$

Proof. If we use Lemma 3 and the fact that $|\mathscr{D}_q f|$ is a φ-convex function, we have

$$\begin{aligned}\left|K_f(a,b;q)\right| = {}&\left|\frac{q(x-a)^2}{b-a}\int_0^1 \mu\mathscr{D}_q f(\mu x+(1-\mu)a)\mathrm{d}_q\mu\right. \\ &\left.+\frac{q(b-x)^2}{b-a}\int_0^1 \mu\mathscr{D}_q f(\mu x+(1-\mu)b)\mathrm{d}_q\mu\right|\end{aligned}$$

$$\leq \frac{q(x-a)^2}{b-a}\int_0^1 \mu|\mathscr{D}_q f(\mu x+(1-\mu)a)|\mathrm{d}_q\mu$$

$$+\frac{q(b-x)^2}{b-a}\int_0^1 \mu|\mathscr{D}_q f(\mu x+(1-\mu)b)|\mathrm{d}_q\mu$$

$$\leq \frac{q(x-a)^2}{b-a}\int_0^1 \mu[|\mathscr{D}_q f(a)|+\mu\varphi(|\mathscr{D}_q f(x)|,|\mathscr{D}_q f(a)|)]\,\mathrm{d}_q\mu$$

$$+\frac{q(b-x)^2}{b-a}\int_0^1 \mu[|\mathscr{D}_q f(b)|+\mu\varphi(|\mathscr{D}_q f(x)|,|\mathscr{D}_q f(b)|)]\,\mathrm{d}_q\mu$$

$$= \frac{q(x-a)^2}{b-a}\left[\frac{1}{1+q}|\mathscr{D}_q f(a)|+\frac{1}{1+q+q^2}\varphi(|\mathscr{D}_q f(x)|,|\mathscr{D}_q f(a)|)\right]$$

$$+\frac{q(b-x)^2}{b-a}\left[\frac{1}{1+q}|\mathscr{D}_q f(b)|+\frac{1}{1+q+q^2}\varphi(|\mathscr{D}_q f(x)|,|\mathscr{D}_q f(b)|)\right].$$

This completes the proof. □

Theorem 11. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0<q<1$. Let $|\mathscr{D}_q f|^r$ be a φ-convex function. Then, for $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, we have*

$$\left|K_f(a,b;q)\right|$$
$$\leq \frac{q(x-a)^2}{b-a}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(|\mathscr{D}_q f(a)|^r+\frac{1}{1+q+q^2}\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(a)|^r)\right)^{\frac{1}{r}}$$
$$+\frac{q(b-x)^2}{b-a}\left(\frac{1-q}{1-q^{p+1}}\right)^{\frac{1}{p}}\left(|\mathscr{D}_q f(b)|^r+\frac{1}{1+q}\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(b)|^r)\right)^{\frac{1}{r}}.$$

Proof. Using Lemma 3, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is a φ-convex function, we have

$$\left|K_f(a,b;q)\right| = \left|\frac{q(x-a)^2}{b-a}\int_0^1 \mu\mathscr{D}_q f(\mu x+(1-\mu)a)\mathrm{d}_q\mu\right.$$
$$\left.+\frac{q(b-x)^2}{b-a}\int_0^1 \mu\mathscr{D}_q f(\mu x+(1-\mu)b)\mathrm{d}_q\mu\right|$$

$$\leq \frac{q(x-a)^2}{b-a}\Big(\int_0^1 \mu^p \mathrm{d}_q\mu\Big)^{\frac{1}{p}}\Big(\int_0^1 |\mathscr{D}_q f(\mu x+(1-\mu)a)|^r \mathrm{d}_q\mu\Big)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-a}\Big(\int_0^1 \mu^p \mathrm{d}_q\mu\Big)^{\frac{1}{p}}\Big(\int_0^1 |\mathscr{D}_q f(\mu x+(1-\mu)b)|^r \mathrm{d}_q\mu\Big)^{\frac{1}{r}}$$

$$\leq \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [|\mathscr{D}_q f(a)|^r+\mu\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(a)|^r)]\,\mathrm{d}_q\mu\Big)^{\frac{1}{r}}$$

$$+\frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(\int_0^1 [|\mathscr{D}_q f(b)|^r+\mu\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(b)|^r)]\,\mathrm{d}_q\mu\Big)^{\frac{1}{r}}$$

$$\leq \frac{q(x-a)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|\mathscr{D}_q f(a)|^r+\frac{1}{1+q}\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(a)|^r)\Big)^{\frac{1}{r}}$$
$$+\frac{q(b-x)^2}{b-a}\Big(\frac{1-q}{1-q^{p+1}}\Big)^{\frac{1}{p}}\Big(|\mathscr{D}_q f(b)|^r+\frac{1}{1+q}\varphi(|\mathscr{D}_q f(x)|^r,|\mathscr{D}_q f(b)|^r)\Big)^{\frac{1}{r}}.$$

This completes the proof. □

We now establish some quantum estimates of certain integral inequalities via preinvex functions. For more details, see [28]. For simplicity, we denote the interval by $I_\eta=[a,a+\eta(b,a)]\subset\mathbb{R}$ and by I_η^0 the interior of I_η.

Theorem 12. *Let $f : I_\eta \to \mathbb{R}$ be a preinvex function with $\eta(b,a) > 0$. If the bifunction $\eta(.,.)$ satisfies the Condition C, then, we have*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x \leq \frac{qf(a)+f(b)}{2}.$$

Proof. Let f be a preinvex function with respect to the bifunction $\eta(.,.)$. Then, using Condition C, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq \frac{1}{2}\left[f(a+\mu\eta(b,a))+f(a+(1-\mu)\eta(b,a))\right].$$

By q-integrating the above inequality with respect to t on [0, 1], we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\ \leq \frac{1}{2}\left[\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x\right]$$

$$= \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x. \tag{12}$$

Since f is a preinvex function, hence

$$f(a+\mu\eta(b,a)) \leq (1-\mu)f(a) + \mu f(b), \quad \mu \in [0,1]..$$

Again q-integrating the above inequality with respect to μ on $[0,1]$, we have

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x \leq \frac{qf(a)+f(b)}{2}. \tag{13}$$

Combining (12) and (13) completes the proof. □

Remark 3. If $q \to 1$, then, Theorem 12 reduces to (2). If $\eta(b,a) = b-a$, then, Theorem 12 reduces to (3). If $\eta(b,a) = b-a$ and $q \to 1$, then Theorem 12 reduces to (1).

Theorem 13. *Let $f,g : I_\eta \to \mathbb{R}$ be integrable and preinvex functions with $\eta(b,a) > 0$ and let Condition C hold. Then, for $0 < q < 1$, we have*

$$2f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - [\theta_1 M(a,b) + \theta_2 N(a,b)]$$
$$\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}_q x,$$

where

$$\theta_1 = \frac{q^2}{(1+q)(1+q+q^3)},$$
$$\theta_2 = \frac{1+2q+q^2}{2(1+q)(1+q+q^2)},$$
$$M(a,b) = f(a)g(a) + f(b)g(b),$$

and

$$N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Let f and g be preinvex functions with respect to bifunction $\eta(.,.)$. Then, using Condition C, we have

$$
\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\
&= f\left(\frac{a+\mu\eta(b,a)+a+(1-\mu)\eta(b,a)}{2}\right) \\
&\quad \times g\left(\frac{a+\mu\eta(b,a)+a+1-\mu\eta(b,a)}{2}\right) \\
&\leq \frac{1}{4}\left[\{f(a+\mu\eta b,a)+f(a+(1-\mu)\eta b,a)\}\right. \\
&\quad \left.\{g(a+\mu\eta b,a)+g(a+(1-\mu)\eta b,a)\}\right] \\
&\leq \frac{1}{4}\left[\{f(a+\mu\eta b,a)f(a+\mu\eta b,a)+f(a+(1-\mu)\eta b,a)f(a+(1-\mu)\eta b,a)\}\right. \\
&\quad \left. + \left\{2\mu(1-\mu)M(a,b)+(\mu^2+(1-\mu)^2)N(a,b)\right\}\right].
\end{aligned}
$$

By q-integrating both sides of the above inequality with respect to μ on $[0,1]$, we have

$$
\begin{aligned}
& 2f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{2q^2M(a,b)+(1+2q+q^2)N(a,b)}{2(1+q)(1+q+q^2)} \\
&\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}_qx.
\end{aligned}
$$

This completes the proof. □

Theorem 14. *Let $f,g : I_\eta \to \mathbb{R}$ be integrable and preinvex functions with $\eta(b,a) > 0$. Then, for $0 < q < 1$, we have*

$$
\begin{aligned}
& \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}_qx \\
&\leq \phi_1 f(a)g(a) + \phi_2\left[q(1+q^2)f(b)g(b)+q^2N(a,b)\right],
\end{aligned}
$$

where

$$
\phi_1 = \frac{1}{1+q+q^2},
$$

$$
\phi_2 = \frac{1}{(1+q)(1+q+q^2)},
$$

and

$$N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Let f and g be two preinvex functions. Then

$$f(a + \mu\eta(b,a)) \leq (1-\mu)f(a) + \mu f(b), \tag{14}$$

and

$$g(a + \mu\eta(b,a)) \leq (1-\mu)g(a) + \mu g(b). \tag{15}$$

Multiplying (14) and (15), we have

$$\begin{aligned} &f(a + \mu\eta(b,a))g(a + \mu\eta(b,a)) \\ &\leq (1-\mu)^2 f(a)g(a) + \mu(1-\mu)f(a)g(b) + \mu(1-\mu)f(b)g(a) + \mu^2 f(b)g(b). \end{aligned}$$

By q-integrating both sides of the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned} &\int_0^1 f(a + \mu\eta(b,a))g(a + \mu\eta(b,a))\, \mathrm{d}_q\mu \\ &\leq f(a)g(a)\int_0^1 (1-\mu)^2\, \mathrm{d}_q\mu + f(a)g(b)\int_0^1 \mu(1-\mu)\, \mathrm{d}_q\mu \\ &\quad + f(b)g(a)\int_0^1 \mu(1-\mu)\, \mathrm{d}_q\mu + f(b)g(b)\int_0^1 \mu^2\, \mathrm{d}_q\mu. \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\, \mathrm{d}_q x \\ &\leq \Omega_1 f(a)g(a) + \Omega_2\left[q(1+q^2)f(b)g(b) + q^2 N(a,b)\right]. \end{aligned}$$

This completes the proof. □

Theorem 15. *Let $f, g : I_\eta \to \mathbb{R}$ be integrable and preinvex functions. Then*

$$\frac{1+q+q^2}{\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right)$$
$$\times g\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right) \mathrm{d}_q\mu \, \mathrm{d}_q y$$
$$\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x) \, \mathrm{d}_q x + \frac{q(1+q^2)}{4(1+q)}[M(a,b)+N(a,b)]$$
$$+\frac{q^2}{2(1+q)^2}\left[2(qf(a)g(a)+f(b)g(b)) + (1+q)N(a,b)\right].$$

Proof. Let f and g be preinvex functions. Then

$$f\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right)$$
$$\leq (1-\mu)f\left(\frac{2a+\eta(b,a)}{2}\right) + \mu f(y), \tag{16}$$

and

$$g\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right) \leq (1-\mu)g\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$+\mu g(y). \tag{17}$$

Multiplying (16) and (17), we have

$$f\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right)$$
$$\times g\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right)$$
$$\leq (1-\mu)^2 f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$+\mu(1-\mu)\left[f\left(\frac{2a+\eta(b,a)}{2}\right)g(y) + f(y)g\left(\frac{2a+\eta(b,a)}{2}\right)\right] + \mu^2 f(y)g(y),$$

By q-integrating both sides of the above inequality with respect to μ on $[0, 1]$, we have

$$\int_0^1 \left[f\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right) \right. \\ \left. \times g\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right)\right] \mathrm{d}_q\mu \\ \leq \frac{q(1+q^2)}{(1+q)(1+q+q^2)} f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\ + \frac{q^2}{(1+q)(1+q+q^2)}\left[f\left(\frac{2a+\eta(b,a)}{2}\right)g(y) + f(y)g\left(\frac{2a+\eta(b,a)}{2}\right)\right] \\ + \frac{1}{1+q+q^2} f(y)g(y).$$

Again q-integrating both sides of the above inequality with respect to y on $[a, a+\eta(b,a)]$ and using Theorem 12, we have

$$\int_a^{a+\eta(b,a)} \int_0^1 f\left(\frac{2a+\eta(b,a)}{2} + t\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right) \\ \times g\left(\frac{2a+\eta(b,a)}{2} + \mu\eta\left(y, \frac{2a+\eta(b,a)}{2}\right)\right) \mathrm{d}_q\mu \, \mathrm{d}_q y \\ \leq \frac{q(1+q^2)}{(1+q)(1+q+q^2)} \int_a^{a+\eta(b,a)} f(y)g(y) \, \mathrm{d}_q y \\ + \frac{q^2\eta(b,a)}{2(1+q)(1+q+q^2)} [2\{(qf(a)g(a) + f(b)g(b))\} + (1+q)N(a,b)] \\ + \frac{q(1+q^2)\eta(b,a)}{4(1+q)(1+q+q^2)} [M(a,b) + N(a,b)].$$

Multiplying both sides of the above inequality by $\frac{(1+q)(1+q+q^2)}{\eta^2(b,a)}$ completes the proof. □

Theorem 16. *Let $f, g : I_\eta \to \mathbb{R}$ be integrable and preinvex functions. Then*

$$\frac{(1+q)(1+q+q^2)}{\eta^2(b,a)} \int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f(x+\mu\eta(y,x))g(x+\mu\eta(y,x)) \, \mathrm{d}_q\mu \, \mathrm{d}_q x \, \mathrm{d}_q y$$

$$\leq \frac{1+2q+q^2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}_q x + \frac{2q^2}{(1+q)^2}[q^2 f(a)g(a) + f(b)g(b) + qN(a,b)].$$

Proof. Let f and g be preinvex functions. Then

$$\begin{aligned} &f(x+\mu\eta(y,x))g(x+\mu\eta(y,x)) \\ &\leq (1-\mu)^2 f(x)g(x) + \mu(1-\mu)[f(x)g(y) + f(y)g(x)] + \mu^2 f(y)g(y), \end{aligned}$$

By q-integrating the above inequality with respect to μ on $[0,1]$, we have

$$\begin{aligned} &\int_0^1 f(x+\mu\eta(y,x))g(x+\mu\eta(y,x))\,\mathrm{d}_q\mu \\ &\leq \frac{f(x)g(x)q(1+q^2)}{(1+q)(1+q+q^2)} + \frac{q^2[f(x)g(y) + f(y)g(x)]}{(1+q)(1+q^2+q^3)} + \frac{f(y)g(y)}{1+q+q^2}. \end{aligned}$$

Again q-integrating both sides of the above inequality, with respect to x, y on $[a + a + \eta(b,a)]$ and using Theorem 12, we have

$$\int_a^{a+\eta(b,a)} \int_a^{a+\eta(b,a)} \int_0^1 f(x+\mu\eta(y,x))g(x+\mu\eta(y,x))\,\mathrm{d}_q\mu\,\mathrm{d}_q x\,\mathrm{d}_q y$$

$$\begin{aligned} &\leq \left(\frac{1}{1+q^2+q^3} + \frac{q(1+q^2)}{(1+q)(1+q+q^2)}\right)\eta(b,a) \int_a^{a+\eta(b,a)} f(x)g(x)\,\mathrm{d}_q x \\ &\quad + \frac{2q^2\eta^2(b,a)}{(1+q)^3(1+q+q^2)}[q^2 f(a)g(a) + f(b)g(b) + qN(a,b)]. \end{aligned}$$

Multiplying both sides of the above inequality by $\frac{(1+q)(1+q+q^2)}{\eta^2(b,a)}$ completes the proof. □

The next Lemma will be helpful in obtaining our next results.

Lemma 4 ([28]). *Let* $f : I_\eta \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* $\mathscr{D}_q f$ *is an integrable function on* I_η^0, *then*

$$\begin{aligned} &\Xi_f(a, a+\eta(b,a); q; \eta) \\ &= \frac{q\eta(b,a)}{1+q} \int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu, \end{aligned}$$

where

$$\Xi_f(a, a+\eta(b,a); q; \eta) = \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}_q x - \frac{qf(a) + f(a+\eta(b,a))}{1+q}.$$

Theorem 17. *Let $f : I_\eta \to \mathbb{R}$ be a q-differentiable function on I_η° and let $\mathscr{D}_q$ be continuous and integrable on I_η, where $0 < q < 1$. If $|\mathscr{D}_q f|$ is a preinvex function, then*

$$\begin{aligned}&\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right| \\ &\le \psi_\eta(a,b;q)\left[q(1+3q^2+2q^3)|\mathscr{D}_q f(a)| + (1+4q+q^2)|\mathscr{D}_q f(b)|\right],\end{aligned}$$

where

$$\psi_\eta(a,b;q) = \frac{q^2\eta(b,a)}{(1+q)^4(1+q+q^2)}.$$

Proof. Using Lemma 4, property of modulus and the fact that $|\mathscr{D}_q f|$ is a preinvex function, we have

$$\begin{aligned}&\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right| \\ &= \left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu\right| \\ &\le \frac{q\eta(b,a)}{1+q}\Bigg[|\mathscr{D}_q f(a)|\int_0^1 |1-(1+q)\mu|(1-\mu)\,\mathrm{d}_q\mu \\ &\qquad + |\mathscr{D}_q f(b)|\int_0^1 |1-(1+q)\mu|\mu\,\mathrm{d}_q\mu\Bigg] \\ &= \frac{q^2\eta(b,a)}{(1+q)^4(1+q+q^2)}\left[(1+3q^2+2q^3)|\mathscr{D}_q f(a)| + (1+4q+q^2)|\mathscr{D}_q f(b)|\right].\end{aligned}$$

This completes the proof. □

Theorem 18. *Let $f : I_\eta \to \mathbb{R}$ be a q-differentiable function on the interior I_η° and let $\mathscr{D}_q$ be continuous and integrable on I_η, where $0 < q < 1$. Let $|\mathscr{D}_q f|^r$ be a preinvex function, where $r \ge 1$. Then*

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right|$$
$$\leq \theta_\eta(a,b;q)\left[\frac{q(1+3q^2+2q^3)|_a\mathrm{D}_q f(a)|^r + (1+4q+q^2)|_a\mathrm{D}_q f(b)|^r}{(1+q+q^2)2q}\right]^{\frac{1}{r}},$$

where

$$\theta_\eta(a,b;q) = \frac{q^2 2\eta(b,a)}{(1+q)^3}.$$

Proof. Using Lemma 4, property of modulus, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is a preinvex function, we have

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right|$$
$$= \left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu\right|$$
$$\leq \left(\int_0^1 |1-(1+q)\mu|\,\mathrm{d}_q\mu\right)^{1-\frac{1}{r}}$$
$$\times\left(\int_0^1 |1-(1+q)\mu|[(1-\mu)|\mathscr{D}_q f(a)|^r + \mu\mathscr{D}_q f(b)|^r]\,\mathrm{d}_q\mu\right)^{\frac{1}{r}}$$
$$= \left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$
$$\times\left(\frac{q}{(1+q)^3(1+q+q^2)}\left[(1+3q^2+2q^3)|\mathscr{D}_q f(a)|^r + (1+4q+q^2)|\mathscr{D}_q f(b)|^r\right]\right)^{\frac{1}{r}}.$$

This completes the proof. □

We now prove some new quantum Iyengar type inequalities for quasi preinvex functions.

Theorem 19. *Let $f : I_\eta \to \mathbb{R}$ be a q-differentiable function on I_η° and let $\mathscr{D}_q$ be continuous and integrable on I_η, where $0 < q < 1$. If $|\mathscr{D}_q f|^r$ is a quasi preinvex function, where $r \geq 1$, then*

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right| \leq \frac{q^2\eta(b,a)2}{(1+q)^3}\left(\max\{|\mathscr{D}_q f(a)|^r, |\mathscr{D}_q f(b)|^r\}\right)^{\frac{1}{r}}.$$

Proof. If we use Lemma 4, property of modulus, Holder's inequality and the fact that $|\mathscr{D}_q f|^r$ is a quasi preinvex function, we have

$$\begin{aligned}
&\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right| \\
&= \left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\mathscr{D}_q f(a+\mu\eta(b,a))\mathrm{d}_q\mu\right| \\
&\leq \frac{q\eta(b,a)}{1+q}\left(\int_0^1 |1-(1+q)\mu|\mathrm{d}_q\mu\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f(a+\mu\eta(b,a))|^r\mathrm{d}_q\mu\right)^{\frac{1}{r}} \\
&= \frac{q^2\eta(b,a)2}{(1+q)^3}\left(\max\{|\mathrm{D}_q f(a)|^r, |\mathrm{D}_q f(b)|^r\}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 20. *Under the conditions of Theorem 19, ifor $r=1$, we have*

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right| \leq \frac{q^2\eta(b,a)2}{(1+q)^3}\left(\max\{|{}_a\mathrm{D}_q f(a)|, |{}_a\mathrm{D}_q f(b)|\}\right).$$

5 Quantum Inequalities for Generalized Preinvex Functions

In this section, we derive some q-analogues of certain integral inequalities for generalized preinvex functions. These classes of φ-convex functions and preinvex functions are distinctly different extensions of the classical functions. Gordji et al. [13] introduced the generalized preinvex functions only. For integral inequalities and their various variant forms, see Noor [24].

Definition 12 ([13]). Let $K \subset \mathbb{R}$ be an invex set with respect to the bifunction $\eta(.,.)$. Then a function $f: K \to \mathbb{R}$ is said to be generalized preinvex with respect to $\eta(.,.)$ and $\varphi(.,.)$, if

$$f(u+\mu\eta(v,u)) \leq f(u)+\mu\varphi(f(v), f(u)), \quad \forall u, v \in K, \mu \in [0,1].$$

For different values of μ, $\eta(.,.)$, and $\varphi(.,.)$, one can easily show that the generalized preinvex functions include φ-convex functions and preinvex functions as special cases. We now introduce the following concept of generalized quasi preinvex functions.

Definition 13. Let $K \subset \mathbb{R}$ be an invex set with respect to the bifunction $\eta(.,.)$. A function $f : K \to \mathbb{R}$ is said to be generalized quasi preinvex with respect to $\eta(.,.)$ and $\varphi(.,.)$, if

$$f(u + \mu\eta(v,u)) \le \max\{f(u), f(u) + \varphi(f(v), f(u))\}, \quad \forall u, v \in K, \mu \in [0,1].$$

We now establish some quantum integral inequalities for generalized preinvex functions.

Theorem 21. *Let $f, g : I \to \mathbb{R}$ be two generalized preinvex functions. Then*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x$$
$$\le f(a)g(a) + \frac{1}{1+q}K_1(a,b;\varphi;f;g) + \frac{1}{1+q+q^2}K_2(a,b;\varphi;f;g),$$

where

$$K_1(a,b;\varphi;f;g) = f(a)\varphi(g(b),g(a)) + g(a)\varphi(f(b),f(a)),$$

and

$$K_2(a,b;\varphi;f;g) = \varphi(g(b),g(a))\varphi(f(b),f(a)).$$

Proof. Let f and g be two generalized preinvex functions. Then

$$f(a + \mu\eta(b,a)) \le f(a) + \mu\varphi(f(b), f(a))$$
$$g(a + \mu\eta(b,a)) \le g(a) + \mu\varphi(g(b), g(a)).$$

Multiplying the above inequalities, we have

$$f(a + \mu\eta(b,a))g(a + \mu\eta(b,a))$$
$$\le f(a)g(a) + \mu f(a)\varphi(g(b),g(a)) + \mu g(a)\varphi(f(b),f(a))$$
$$+\mu^2\varphi(g(b),g(a))\varphi(f(b),f(a)).$$

Now by q-integrating the above inequality with respect to μ on [0, 1], we have

$$\begin{aligned}&\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q x\\ &\leq f(a)g(a)+\frac{1}{1+q}\left[f(a)\varphi(g(b),g(a))+g(a)\varphi(f(b),f(a))\right]\\ &\quad+\frac{1}{1+q+q^2}\varphi(g(b),g(a))\varphi(f(b),f(a)).\end{aligned}$$

This completes the proof. □

Theorem 22. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I with $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. If $|\mathscr{D}_q f|$ is a generalized preinvex function, then*

$$\begin{aligned}&\left|\Xi_f(a,a+\eta(b,a);q;\eta)\right|\\ &\leq \omega(b-a)\\ &\times\left[2(1+q)(1+q+q^2)|\mathscr{D}_q f(a)|+(1+4q+q^2)\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)\right],\end{aligned}$$

where

$$\omega=\frac{q^2}{(1+q+q^2)(1+q)^3}.$$

Proof. Using Lemma 4 and the fact that $|\mathscr{D}_q f|$ is a generalized preinvex function, we have

$$\begin{aligned}&\left|\Xi_f(a,a+\eta(b,a);q;\eta)\right|\\ &=\left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu\right|\\ &\leq \frac{q\eta(b,a)}{1+q}\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f(a+\mu\eta(b,a))|\,\mathrm{d}_q\mu\\ &\leq \frac{q\eta(b,a)}{1+q}\int_0^1 |1-(1+q)\mu|[|\mathscr{D}_q f(a)|+\mu\varphi(|\mathscr{D}_q f(b)|,|\mathscr{D}_q f(a)|)]\end{aligned}$$

$$= \frac{q\eta(b,a)}{1+q}\Bigg[\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f(a)|\mathrm{d}_q\mu$$

$$+ \int_0^1 \mu|1-(1+q)\mu|\varphi(|\mathscr{D}_q f(b)|, |\mathscr{D}_q f(a)|)\mathrm{d}_q\mu\Bigg]$$

$$= \frac{q\eta(b,a)}{1+q}\left[\frac{2q}{(1+q)^2}|\mathscr{D}_q f(a)| + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\varphi(|\mathscr{D}_q f(b)|, |\mathscr{D}_q f(a)|)\right]$$

$$= \frac{q^2\eta(b,a)}{(1+q+q^2)(1+q)^4}$$

$$\times\left[2(1+q)(1+q+q^2)|\mathscr{D}_q f(a)| + (1+4q+q^2)\varphi(|\mathscr{D}_q f(b)|, |\mathscr{D}_q f(a)|)\right].$$

□

Theorem 23. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. If $|\mathscr{D}_q f|^r$ is a generalized preinvex function where $r > 1$, then*

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta)\right|$$

$$\le \frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left[\vartheta_1|\mathscr{D}_q f(a)|^r + \vartheta_2\varphi(|\mathscr{D}_q f(b)|^r, |\mathscr{D}_q f(a)|^r)\right]^{\frac{1}{r}},$$

where

$$\vartheta_1 = \frac{2q}{(1+q)^2}, \quad \textit{and} \quad \vartheta_2 = \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}.$$

Proof. Let $|\mathscr{D}_q f|^r$ be a generalized preinvex function. Then, from Lemma 4 and using Holder's inequality, we have

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta\right|$$

$$= \left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu\right|$$

$$\le \frac{q\eta(b,a)}{1+q}\left(\int_0^1 |1-(1+q)\mu|\mathrm{d}_q\mu\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f(a+\mu\eta(b,a))|^r \mathrm{d}_q\mu\right)^{\frac{1}{r}}$$

$$= \frac{q\eta(b,a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}$$

$$\times\left[\frac{2q}{(1+q)^2}|\mathscr{D}_q f(a)|^r + \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\varphi(|\mathscr{D}_q f(b)|^r, |\mathscr{D}_q f(a)|^r)\right]^{\frac{1}{r}}.$$

This completes the proof. □

We now derive some quantum Iyengar type inequalities via generalized quasi preinvex functions.

Theorem 24. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on the interior I° of I and let $\mathscr{D}_q$ be continuous and integrable on I, where $0 < q < 1$. If $|\mathscr{D}_q f|^r$ is a generalized quasi preinvex function, where $r > 1$, then*

$$\left|\Xi_f(a, a+\eta(b,a); q; \eta\right|$$
$$\leq \frac{q^2(b-a)2}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\varphi(f(a), f(b))|^r\}\right)^{\frac{1}{r}}.$$

Proof. If we use Lemma 4, power mean inequality and the fact that $|\mathscr{D}_q f|^r$ is a generalized quasi preinvex function, we have

$$\left|H_f(a,b;q)\right|$$

$$= \left|\frac{q\eta(b,a)}{1+q}\int_0^1 (1-(1+q)\mu)\,\mathscr{D}_q f(a+\mu\eta(b,a))\,\mathrm{d}_q\mu\right|$$

$$\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)\mu|\mathrm{d}_q\mu\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1 |1-(1+q)\mu||\mathscr{D}_q f(a+\mu\eta(b,a))|^r \mathrm{d}_q\mu\right)^{\frac{1}{r}}$$

$$= \frac{q^2(b-a)2}{(1+q)^3}\left(\max\{|f(a)|^r, |f(a)+\varphi(f(a), f(b))|^r\}\right)^{\frac{1}{r}}.$$

□

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On the Bohr Inequality

Yusuf Abu Muhanna, Rosihan M. Ali, and Saminathan Ponnusamy

Abstract The Bohr inequality, first introduced by Harald Bohr in 1914, deals with finding the largest radius r, $0 < r < 1$, such that $\sum_{n=0}^{\infty} |a_n| r^n \leq 1$ holds whenever $|\sum_{n=0}^{\infty} a_n z^n| \leq 1$ in the unit disk $\mathbb{D}$ of the complex plane. The exact value of this largest radius, known as the *Bohr radius*, has been established to be $1/3$. This paper surveys recent advances and generalizations on the Bohr inequality. It discusses the Bohr radius for certain power series in $\mathbb{D}$, as well as for analytic functions from $\mathbb{D}$ into particular domains. These domains include the punctured unit disk, the exterior of the closed unit disk, and concave wedge-domains. The analogous Bohr radius is also studied for harmonic and starlike logharmonic mappings in $\mathbb{D}$. The Bohr phenomenon which is described in terms of the Euclidean distance is further investigated using the spherical chordal metric and the hyperbolic metric. The exposition concludes with a discussion on the n-dimensional Bohr radius.

Keywords Bohr inequality • Bohr radius • Bohr phenomenon • Analytic functions • Harmonic mappings • Subordination • Majorant series • Alternating series • Dirichlet series • Multidimensional Bohr radius

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1 Harald August Bohr (1887–1951)

Harald August Bohr was born on the 22nd day of April, 1887 in Copenhagen, Denmark, to Christian and Ellen Adler Bohr. His father was a distinguished professor of physiology at the University of Copenhagen and his elder brother Niels was to become a famous theoretical physicist.

Harald and Niels were prolific football (soccer) players. Harald made his playing debut as a 16-year-old in 1903 with *Akademisk Boldklub*. He represented the Danish national football team in the 1908 Summer Olympics, where football was first introduced as an official event. Denmark faced hosts Great Britain in the final, but eventually lost 2-0, and Bohr and the Danish team came home as silver medalists.

Bohr enrolled at the University of Copenhagen in 1904 to study mathematics. It was reported that during his doctoral dissertation examination, there were more football fans in attendance than there were mathematicians!

Bohr became a professor in mathematics at the Copenhagen's Polytechnic Institute in 1915. He was later appointed as professor at the University of Copenhagen in 1930, where he remained in that position until his demise on January 22, 1951.

Bohr was an extremely capable teacher. Indeed to his honour, the annual award for outstanding teacher at the University of Copenhagen is called *The Harald*. With Johannes Mollerup, Bohr wrote an influential four-volume textbook entitled *Lærebog i Matematisk Analyse* (Textbook in mathematical analysis).

Bohr worked on Dirichlet series, and applied analysis to the theory of numbers. During this period, Edmund Landau was at Göttingen, studying the Riemann zeta function $\zeta(s)$, and whom was also renowned for his unsolved problem on Landau's constant (see, for example, [64]). Bohr collaborated with Landau, and in 1914, they proved the Bohr–Landau theorem on the distribution of zeros for the zeta function. All but an infinitesimal proportion of these zeros lie in a small neighbourhood of the line $s = 1/2$. Although Niels Bohr was an accomplished physicist and Nobel Laureate, Harald and Niels only had one joint publication.

Bohr's interest in functions which could be represented by a Dirichlet series led to the development of almost periodic functions. These are functions which, after a period, take values within e of the values in the previous period. Bohr pioneered this theory and presented it in three major works during the years 1923 and 1926 in *Acta Mathematica*. It is with these works that his name is now most closely associated.

Titchmarsh [68] made the following citation on Bohr's work on almost periodic functions: "*The general theory was developed for the case of a real variable, and then, in the light of it, was developed the most beautiful theory of almost periodic functions of a complex variable. The creation of the theory of almost periodic functions of a real variable was a performance of extraordinary power, but was not based on the most up-to-date methods, and the main results were soon simplified and improved. However, the theory of almost periodic functions of a complex variable remains up to now in the same perfect form in which it was given by Bohr*".

Bohr devoted his life to mathematics and to the theory of almost periodic functions. Four months before his death, Bohr was still actively engaged with the mathematical community at the International Congress of Mathematicians in Cambridge, Massachusetts, in September, 1950; he died soon after the New Year. Besicovitch wrote: "*For most of his life Bohr was a sick man. He used to suffer from bad headaches and had to avoid all mental effort. Bohr the man was not less remarkable than Bohr the mathematician. He was a man of refined intellect, harmoniously developed in many directions. He was also a most humane person. His help to his pupils, to his colleagues and friends, and to refugees belonging to the academic world was generous indeed. Once he had decided to help he stopped at nothing and he seldom failed. He was very sensitive to literature. His favourite author was Dickens; he had a deep admiration of Dickens' love of the human being and deep appreciation of his humour*".

Harald Bohr was elected an Honorary Member of the London Mathematical Society in 1939. Additional biographical account on Bohr may be obtained from http://en.wikipedia.org/wiki/Harald_Bohr

2 The Classical Bohr Inequality

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and let $\mathscr{A}$ denote the space of functions analytic in the unit disk $\mathbb{D}$. Then $\mathscr{A}$ is a locally convex linear topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. We may assume that $f \in \mathscr{A}$ has continuous boundary values and $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. It is evident that each $f \in \mathscr{A}$ has a power series expansion about the origin. What can be deduced from the sum of the moduli of the terms in the series?

In 1914, Harald Bohr [27] studied this property and made the observation: "In particular, the solution of what is called the "absolute convergence problem" for Dirichlet series of the type $\sum a_n n^{-s}$ must be based upon a study of the relations between the absolute value of a power-series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms on the other. It was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only, which we discussed last year, and which seems to be of some interest in itself." More precisely, Bohr obtained the following remarkable result.

Theorem A (Bohr Inequality (1914)). *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic in* $\mathbb{D}$*, and* $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty$ *for all* $z \in \mathbb{D}$*. Then*

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq \|f\|_\infty \tag{1}$$

for $0 \leq r \leq 1/3$.

Here M_f is the associated majorant series for f. Bohr actually obtained the inequality (1) only for $r \leq 1/6$. M. Riesz, I. Schur, and N. Wiener independently proved its validity for $r \leq 1/3$ and showed that the bound $1/3$ was sharp. The best constant r in (1), which is $1/3$, is called the Bohr radius for the class of all analytic self-maps of the unit disk $\mathbb{D}$. Other proofs can also be found, for example, by Sidon [67], Tomic [69], and Paulsen et al. in [61] and [62, 63]. Similar problems were considered for Hardy spaces or for more abstract spaces, for instance, by Boas and Khavinson in [25]. More recently, Aizenberg [12, 14] extended the inequality in the context of several complex variables which we shall discuss with some details in Sect. 3.

2.1 Bohr Phenomenon for the Space of Subordinate Mappings

In recent years, two types of spaces are widely considered in the study of Bohr inequality. They are the space of subordinations and the space of complex-valued bounded harmonic mappings. One way of generalizing the notion of the Bohr phenomenon, initially defined for mappings from $\mathbb{D}$ to itself, is to rewrite Bohr inequality in the equivalent form

$$\sum_{k=1}^{\infty} |a_k| r^k \leq 1 - |a_0| = 1 - |f(0)|.$$

The distance to the boundary is an important geometric quantity. Observe that the number $1 - |f(0)|$ is the distance from the point $f(0)$ to the boundary $\partial\mathbb{D}$ of the unit disk $\mathbb{D}$. Using this "distance form" formulation of Bohr inequality, the notion of Bohr radius can be generalized to the class of functions f analytic in $\mathbb{D}$ which take values in a given domain Ω. For our formulation, we first introduce the notion of subordination.

If f and g are analytic in $\mathbb{D}$, then g is *subordinate* to f, written $g \prec f$ or $g(z) \prec f(z)$, if there exists a function w analytic in $\mathbb{D}$ satisfying $w(0) = 0$, $|w(z)| < 1$ and $g(z) = f(w(z))$ for $z \in \mathbb{D}$. If f is univalent in $\mathbb{D}$, then $g \prec f$ if and only if $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$ (see [44, p. 190 and p. 253]). By the Schwarz lemma, it follows that

$$|g'(0)| = |f'(w(0))w'(0)| \leq |f'(0)|.$$

Now for a given f, let $S(f) = \{g : g \prec f\}$ and $\Omega = f(\mathbb{D})$. The family $S(f)$ is said to satisfy a Bohr phenomenon if there exists an r_f, $0 < r_f \leq 1$ such that whenever $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$, then

$$\sum_{n=1}^{\infty} |b_n| r^n = M_g(r) - |b_0| \leq \operatorname{dist}(f(0), \partial f(\mathbb{D})) \tag{2}$$

for $|z| = r \leq r_f$. We observe that if $f(z) = (a_0 - z)/(1 - \overline{a_0}z)$ with $|a_0| < 1$, and $\Omega = \mathbb{D}$, then $\mathrm{dist}(f(0), \partial\Omega) = 1 - |a_0| = 1 - |b_0|$ so that (2) holds with $r_f = 1/3$, according to Theorem A. We say that the family $S(f)$ satisfies the *classical* Bohr phenomenon if (2) holds for $|z| = r < r_0$ with $1-|f(0)|$ in place of $\mathrm{dist}(f(0), \partial f(\mathbb{D}))$. Hence the distance form allows us to extend Bohr's theorem to a variety of distances provided the Bohr phenomenon exists. The following theorem was obtained in [4, Theorem 1]:

Theorem 1. *If f, g are analytic in $\mathbb{D}$ such that f is univalent in $\mathbb{D}$ and $g \in S(f)$, then inequality* (2) *holds with $r_f = 3-2\sqrt{2} \approx 0.17157$. The sharpness of r_f is shown by the Koebe function $f(z) = z/(1-z)^2$.*

Proof. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \prec f(z)$, where f is a univalent mapping of $\mathbb{D}$ onto a simply connected domain $\Omega = f(\mathbb{D})$. Then it is well known that (see, for instance, [44, p. 196] and [33])

$$\frac{1}{4}|f'(0)| \leq \mathrm{dist}(f(0), \partial\Omega) \leq |f'(0)| \text{ and } |b_n| \leq n|f'(0)|. \tag{3}$$

It follows that $|b_n| \leq 4n\mathrm{dist}(f(0), \partial\Omega)$, and thus

$$\sum_{n=1}^{\infty} |b_n| r^n \leq 4\mathrm{dist}(f(0), \partial\Omega) \sum_{n=1}^{\infty} nr^n = 4\mathrm{dist}(f(0), \partial\Omega) \frac{r}{(1-r)^2} \leq \mathrm{dist}(f(0), \partial\Omega)$$

provided $4r \leq (1-r)^2$, that is, for $r \leq 3-2\sqrt{2}$. When $f(z) = z/(1-z)^2$, we obtain $\mathrm{dist}(f(0), \partial\Omega) = 1/4$ and a simple calculation gives sharpness. □

In [4], it was also pointed out that for $f(z) = z/(1-z)^2$, $S(f)$ does not have the classical Bohr phenomenon. Moreover, from the proof of Theorem 1, it is easy to see that $r_f = 3-2\sqrt{2}$ could be replaced by $1/3$ if f is univalent in $\mathbb{D}$ with convex image. In this case, instead of (3), one uses the following (see [44, p. 195, Theorem 6.4]):

$$\frac{1}{2}|f'(0)| \leq \mathrm{dist}(f(0), \partial\Omega) \leq |f'(0)| \text{ and } |b_n| \leq |f'(0)|.$$

Hence the deduced result clearly contains the classical Bohr inequality (1). In the next section, we shall present various other generalizations and improved results.

2.2 Bohr Radius for Alternating Series and Symmetric Mappings

The majorant series of f defined by (1) belongs to a very important class of series, namely, series with non-negative terms. As pointed out by the authors in [11], there is yet another class of interesting series—the class of alternating series. Thus, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, its associated *alternating series* is given by

$$A_f(r) = \sum_{k=0}^{\infty} (-1)^k |a_k|\, r^k. \tag{4}$$

In [11], the authors obtained several results on Bohr radius, which include the following counterpart of Theorem A.

Theorem 2. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic and bounded in* $\mathbb{D}$*, then* $\left|A_f(r)\right| \le \|f\|_\infty$ *for* $0 \le r \le 1/\sqrt{3}$*. The radius* $r = 1/\sqrt{3}$ *is best possible.*

In [11], the authors proved the following.

Theorem 3. *Let* $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ *be an odd analytic function in* $\mathbb{D}$ *such that* $|f(z)| \le 1$ *in* $\mathbb{D}$*. Then* $M_f(r) \le 1$ *for* $0 \le r \le r_*$*, where* r_* *is a solution of the equation*

$$5r^4 + 4r^3 - 2r^2 - 4r + 1 = 0,$$

which is unique in the interval $1/\sqrt{3} < r < 1$*. The value of* r_* *can be calculated in terms of radicals as*

$$r_* = -\frac{1}{5} + \frac{1}{10}\sqrt{\frac{A+32}{3}} + \frac{1}{10}\sqrt{\frac{64}{3} - \frac{A}{3} + 144\sqrt{\frac{3}{A+32}}} = 0.7313\ldots,$$

where

$$A = 10 \cdot 2^{\frac{2}{3}} \left((47 - 3\sqrt{93})^{\frac{1}{3}} + (47 + 3\sqrt{93})^{\frac{1}{3}} \right).$$

In [11], an example was given to conclude that the Bohr radius for the class of odd functions satisfies the inequalities $r_* \le r \le r^* \approx 0.7899$, where

$$r^* = \frac{1}{4}\sqrt{\frac{B-2}{6}} + \frac{1}{2}\sqrt{3\sqrt{\frac{6}{B-2}} - \frac{B}{24} - \frac{1}{6}},$$

with

$$B = (3601 - 192\sqrt{327})^{\frac{1}{3}} + (3601 + 192\sqrt{327})^{\frac{1}{3}}.$$

This raises the following open problem.

Problem 1 ([11]). *Find the Bohr radius for the class of odd functions f satisfying $|f(z)| \le 1$ for all $z \in \mathbb{D}$.*

Apart from the majorant and alternating series defined by (1) and (4), one can consider a more general type of series associated with f given by

$$S_f^n(r) = \sum_{k=0}^{\infty} e^{\frac{2\pi ik}{n}} |a_k| r^k, \tag{5}$$

where n is a positive integer. Note that

$$M_f(r) = S_f^1(r) \text{ and } A_f(r) = S_f^2(r).$$

The arguments of coefficients of series (5) are equally spaced over the interval $[0, 2\pi)$, and thus $S_f^n(r)$ can be thought of as an *argument symmetric series* associated with f. This raised the next problem.

Problem 2 ([11]). Given a positive integer $n \ge 2$, and $|f(z)| \le 1$ in $\mathbb{D}$, find the largest radius r_n such that $|S_f^n(r)| \le 1$ for all $r \le r_n$.

We recall that an analytic function in $\mathbb{D}$ is called n-symmetric, where $n \ge 1$ is an integer, if $f(e^{2\pi i/n}z) = f(z)$ for all $z \in \mathbb{D}$. It is a simple exercise to see that $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is n-symmetric if and only if its Taylor expansion has the n-symmetric form

$$f(z) = a_0 + \sum_{k=1}^{\infty} a_{nk} z^{nk}.$$

In [11], the authors generalized Theorem A as follows.

Theorem 4. *If $f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}$ is analytic in $\mathbb{D}$, and $|f(z)| \le 1$ in $\mathbb{D}$, then $M_f(r) \le 1$ for $0 \le r \le 1/\sqrt[n]{3}$. The radius $r = 1/\sqrt[n]{3}$ is best possible.*

Proof. Put $\zeta = z^n$ and consider a function $g(\zeta) = \sum_{k=0}^{\infty} a_{nk} \zeta^k$. Clearly, g is analytic in $\mathbb{D}$ and $|g(\zeta)| = |f(z)| \le 1$ for all $|\zeta| < 1$. Thus, $|a_{nk}| \le 1 - |a_0|^2$ for all $k \ge 1$ and this well-known inequality is easily established (see [27] and [60, Exercise 8, p.172]). For $r^n \le 1/3$,

$$\begin{aligned} M_f(r) &\le |a_0| + (1 - |a_0|^2) \sum_{k=1}^{\infty} r^{nk} = |a_0| + (1 - |a_0|^2) \frac{r^n}{1 - r^n} \\ &\le |a_0| + (1 - |a_0|^2) \frac{1/3}{1 - (1/3)} := \frac{1}{2} h(|a_0|), \end{aligned}$$

where $h(x) = 1 + 2x - x^2$, $0 \le x < 1$. Since $h(x) \le h(1)$, it follows that

$$M_f(r) = \sum_{k=0}^{\infty} |a_{nk}| r^{nk} \le 1 \quad \text{for } r \le 1/\sqrt[n]{3}.$$

To show that the radius $1/\sqrt[n]{3}$ is best possible, consider

$$\varphi_\alpha(\zeta) = \frac{\alpha - \zeta}{1 - \overline{\alpha}\zeta}$$

with $\zeta = z^n$. Then, for each fixed $\alpha \in \mathbb{D}$, φ_α is analytic in $\mathbb{D}$, $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$ and $\varphi_\alpha(\partial\mathbb{D}) = \partial\mathbb{D}$. It suffices to restrict α such that $0 < \alpha < 1$. Moreover, for $|\zeta| < 1/\alpha$,

$$\varphi_\alpha(\zeta) = (\alpha - \zeta) \sum_{k=0}^{\infty} \alpha^k \zeta^k = \alpha - (1 - \alpha^2) \sum_{k=1}^{\infty} \alpha^{k-1} \zeta^k,$$

and with $|\zeta| = \rho$,

$$M_{\varphi_\alpha}(\rho) = \alpha + (1 - \alpha^2) \sum_{k=1}^{\infty} \alpha^{k-1} \rho^k = 2\alpha - \varphi_\alpha(\rho).$$

It follows that $M_{\varphi_\alpha}(\rho) > 1$ if and only if $(1 - \alpha)((1 + 2\alpha)\rho - 1) > 0$, which gives $\rho > 1/(1 + 2\alpha)$. Since α can be chosen arbitrarily close to 1, this means that the radius $r = 1/\sqrt[n]{3}$ in Theorem 4 is best possible. □

In Theorem 4, it would be interesting to find the smallest constant for Bohr inequality to hold when $r > 1/\sqrt[n]{3}$. From the proof, it is clear that

$$M(r) = \sup\left\{ t + (1 - t^2)\frac{r^n}{1 - r^n} \,:\, 0 \le t = |a_0| \le 1 \right\}$$
$$= \begin{cases} 1 & \text{for } 0 \le r \le 1/\sqrt[n]{3} \\ \dfrac{4r^{2n} + (1 - r^n)^2}{4r^n(1 - r^n)} & \text{for } 1/\sqrt[n]{3} < r < 1. \end{cases}$$

On the other hand, it follows from the argument of Landau, which is an immediate consequence of the Cauchy-Bunyakovskii inequality, that

$$M_f(r) \le \left(\sum_{k=0}^{\infty} |a_{nk}|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} r^{2nk} \right)^{1/2} = \frac{\|f\|_2}{\sqrt{1 - r^{2n}}} \le \frac{\|f\|_\infty}{\sqrt{1 - r^{2n}}},$$

where $\|f\|_2$ stands for the norm of the Hardy space $H^2(\mathbb{D})$. Thus the following result is obtained.

Corollary 1. *If* $f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}$ *is bounded and analytic in* $\mathbb{D}$, *then*

$$M_f(r) \le A_n(r) \|f\|_\infty,$$

where $A_n(r) = \inf\{M(r), 1/\sqrt{1-r^{2n}}\}$.

We remark that for r close to 1, $M(r)\sqrt{1-r^{2n}} > 1$ which is reversed for r close to $1/\sqrt[n]{3}$. So a natural question is to look for the best such constant $A(r)$. In [28], Bombieri determined the exact value of this constant for the case $n = 1$ and for r in the range $1/3 \le r \le 1/\sqrt{2}$. This constant is

$$A(r) = \frac{3 - \sqrt{8(1-r^2)}}{r}.$$

Later Bombieri and Bourgain in [29] considered the function

$$m(r) = \sup \left\{ \frac{M_f(r)}{\|f\|_\infty} \right\}$$

for the case $n = 1$, and studied the behaviour of $m(r)$ as $r \to 1$ (see also [43]). More precisely, the authors in [29, Theorem 1] proved the following result which validated a question raised in [61, Remark 1] in the affirmative.

Theorem 5. *If* $r > 1/\sqrt{2}$, *then* $m(r) < 1/\sqrt{1-r^2}$. *With* $\alpha = 1/\sqrt{2}$, *the function* $\varphi_\alpha(z) = (\alpha - z)/(1 - \alpha z)$ *is extremal giving* $m(1/\sqrt{2}) = \sqrt{2}$.

A lower estimate for $m(r)$ as $r \to 1$ is also obtained in [29, Theorem 2]. Given $\epsilon > 0$, there exists a positive constant $C(\epsilon) > 0$ such that

$$\frac{1}{1-r^2} - C(\epsilon)\left(\log \frac{1}{1-r}\right)^{\frac{3}{2}+\epsilon} \le m(r)$$

as $r \to 1$. A multidimensional generalization of the work in [29] along with several other issues, including on the Rogosinski phenomena, is discussed in a recent article by Aizerberg [13]. More precisely, the following problems were treated in [13] (see also Sect. 3):

1. Asymptotics of the majorant function in the Reinhardt domains in $\mathbb{C}^n$.
2. The Bohr and Rogosinski radii for Hardy classes of functions holomorphic in the disk.
3. Neither Bohr nor Rogosinski radius exists for functions holomorphic in an annulus with natural basis.
4. The Bohr and Rogosinski radii for the mappings of the Reinhardt domains into Reinhardt domains.

If $a_0 = 0$, it follows from the proof of Theorem 4 that the number $r = 1/\sqrt[n]{3}$ in Theorem 4 can be evidently replaced by $r = 1/\sqrt[n]{2}$.

Corollary 2. *If* $f(z) = \sum_{k=1}^{\infty} a_{nk} z^{nk}$ *is analytic in* $\mathbb{D}$*, and* $|f(z)| \leq 1$ *in* $\mathbb{D}$*, then* $M_f(r) \leq 1$ *for* $0 \leq r \leq 1/\sqrt[n]{2}$*. The radius* $r = 1/\sqrt[n]{2}$ *is best possible as demonstrated by the function*

$$\varphi_\alpha(z) = z^n \left(\frac{\alpha - z^n}{1 - \alpha z^n} \right)$$

with $\alpha = 1/\sqrt[n]{2}$.

We now state another simple extension of Theorem 4 which again contains the classical Bohr inequality for the special case $n = 1$.

Theorem 6. *If* $f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}$ *is analytic in* $\mathbb{D}$ *satisfying* $\mathrm{Re}\, f(z) \leq 1$ *in* $\mathbb{D}$ *and* $f(0) = a_0$ *is positive, then* $M_f(r) \leq 1$ *for* $0 \leq r \leq 1/\sqrt[n]{3}$.

Proof. The proof requires the well-known coefficient inequality for functions with positive real part. If $p(z) = \sum_{k=0}^{\infty} p_k z^k$ is analytic in $\mathbb{D}$ such that $\mathrm{Re}\, p(z) > 0$ in $\mathbb{D}$, then $|p_k| \leq 2\mathrm{Re}\, p_0$ for all $k \geq 1$. Applying this result to $p(z) = 1 - f(z)$ leads to $|a_{nk}| \leq 2(1 - a_0)$ for all $k \geq 1$. Thus

$$M_f(r) \leq a_0 + 2(1 - a_0) \sum_{k=1}^{\infty} r^{nk} = a_0 + 2(1 - a_0) \frac{r^n}{1 - r^n},$$

which is clearly less than or equal to 1 if $r^n \leq 1/3$. □

A minor change in the proof of Theorem 4 gives the following result, which for $a_0 = 0$ provides a vast improvement on the Bohr radius.

Corollary 3. *If* $f(z) = \sum_{k=0}^{\infty} a_{nk} z^{nk}$ *is analytic in* $\mathbb{D}$*, and* $|f(z)| \leq 1$ *in* $\mathbb{D}$*, then*

$$|a_0|^2 + \sum_{k=1}^{\infty} |a_{nk}| r^{nk} \leq 1$$

for $0 \leq r \leq 1/\sqrt[n]{2}$*. The radius* $r = 1/\sqrt[n]{2}$ *is best possible.*

Proof. As in the proof of Theorem 4, it follows easily that for $r^n \leq 1/2$,

$$|a_0|^2 + \sum_{k=1}^{\infty} |a_{nk}| r^{nk} \leq |a_0|^2 + (1 - |a_0|^2) \frac{r^n}{1 - r^n} \leq |a_0|^2 + (1 - |a_0|^2) = 1.$$

Also, it is easy to see that this inequality fails to hold for larger r. □

The case $n = 1$ of Theorem 6 and Corollary 3 appeared in [61].

2.3 Bohr Phenomenon for Harmonic Mappings

Suppose that $f = u + iv$ is a complex-valued harmonic function defined on a simply connected domain D. Then f has the canonical form $f = h + \overline{g}$, where h and g are analytic in D. A generalization of Bohr inequality for harmonic functions from $\mathbb{D}$ into $\mathbb{D}$ was initiated by Abu Muhanna in [4].

Theorem 7. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function in $\mathbb{D}$. If $|f(z)| < 1$ in $\mathbb{D}$, then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \frac{2}{\pi} \approx 0.63662$$

and

$$\sum_{n=1}^{\infty} |\, e^{i\mu} a_n + e^{-i\mu} b_n | r^n + |\mathrm{Re}\, e^{i\mu} a_0| \leq 1 \tag{6}$$

for $r \leq 1/3$ and any real μ. Equality in (6) is attained by the Möbius transformation

$$\varphi(z) = \frac{z - a}{1 - az}, \quad 0 < a < 1, \quad \textit{as } a \to 1.$$

From the proof of Theorem 1, it suffices to have sharp upper estimates for $|a_n| + |b_n|$ and $|\, e^{i\mu} a_n + e^{-i\mu} b_n|$. With the help of these estimates (see [4, Lemma 4] and [30, 31]), the proof of Theorem 7 is readily established. In [4], an example of a harmonic function was given to show that the inequality (6) fails when $|\mathrm{Re}\, e^{i\mu} a_0|$ is replaced by $|a_0|$.

Theorem 7 was extended to bounded domains in [7]. If D is a bounded set, denote by $\overline{D}$ the closure of D, and $\overline{D}_{\min}$ the smallest closed disk containing the closure of D.

Theorem 8 ([7, Theorem 4.4]). *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ be a complex-valued harmonic function in $\mathbb{D}$. If $f : \mathbb{D} \to D$ for some bounded domain D, then, for $r \leq 1/3$,*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \frac{2}{\pi} \rho$$

and

$$\sum_{n=1}^{\infty} |e^{i\mu} a_n + e^{-i\mu} b_n|\, r^n + |\mathrm{Re}\, e^{i\mu} (a_0 - w_0)| \leq \rho,$$

where ρ and w_0 are, respectively, the radius and centre of $\overline{D}_{\min}$.

The bound $1/3$ *is sharp as demonstrated by an analytic univalent mapping* f *from* $\mathbb{D}$ *onto* D*. In particular, if* D *is an open disk with radius* $\rho > 0$ *centred at* ρw_0*, then sharpness is shown by the Möbius transformation*

$$\varphi(z) = e^{i\mu_0}\rho\left(\frac{z+a}{1+az} + |w_0|\right)$$

for some $0 < a < 1$ *and* μ_0 *satisfying* $w_0 = |w_0|e^{i\mu_0}$*.*

2.4 Bohr Inequality in Hyperbolic Metric

In [5], Abu Muhanna and Ali expressed the Bohr inequality in terms of the spherical chordal distance

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad z_1, z_2 \in \mathbb{C}.$$

Thus the Euclidean distance in inequality (2) is replaced by the chordal distance χ.

Let $c\overline{\mathbb{D}}$ denote the complement of $\mathbb{D} \cup \partial\mathbb{D}$ and $\mathscr{H}(\mathbb{D}, \Omega)$ be the class consisting of all analytic functions mapping $\mathbb{D}$ into Ω. Denote by $\mathscr{H}(\mathbb{D}) := \mathscr{H}(\mathbb{D}, \mathbb{D})$. The following theorem generalizes Bohr's theorem for the class $\mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$.

Theorem 9. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$*, then*

$$\chi\left(\sum_{n=0}^{\infty} |a_n z^n|, |a_0|\right) \leq \chi(a_0, \partial c\overline{\mathbb{D}}) \tag{7}$$

for $|z| \leq 1/3$*. Moreover, the bound* $1/3$ *is sharp.*

Interestingly, if $f \in \mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$, then $f \prec \exp \circ W$ for some univalent function W mapping $\mathbb{D}$ onto the right-half plane. Thus $\exp \circ W \in \mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$ is a universal covering map. The proof of Theorem 9 in [5] used the following key result.

Lemma 1 (see [9]). *If* F *is a univalent function mapping* $\mathbb{D}$ *onto* Ω*, where the complement of* Ω *is convex, and* $F(z) \neq 0$*, then any analytic function* $f \in S(F^n)$ *for a fixed* $n = 1, 2, \ldots$*, can be expressed as*

$$f(z) = \int_{|x|=1} F^n(xz)\, d\mu(x)$$

for some probability measure μ *on the unit circle* $|x| = 1$. *(Here* $S(F^n)$ *is defined as in Sect. 2.1.) Consequently,*

$$f(z) = \int_{|x|=1} \exp(F^n(xz))\, d\mu(x),$$

for every $f \in S(\exp \circ F)$.

In the same paper [5], a result under a more general setting than Theorem 9 was also obtained.

Theorem 10. *Let* Δ *be a compact convex body with* $0 \in \Delta$, $1 \in \partial\Delta$. *Suppose that there is a universal covering map from* $\mathbb{D}$ *into* $c\Delta$ *with a univalent logarithmic branch that maps* $\mathbb{D}$ *into the complement of a convex set. If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D}, c\Delta)$ *satisfies* $a_0 > 1$, *then inequality (7) holds for* $|z| < 3 - 2\sqrt{2} \approx 0.17157$.

Another paper by Abu Muhanna and Ali [6] considered the hyperbolic metric. Recall that [21] the hyperbolic metric for $\mathbb{D}$ is defined by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2},$$

the hyperbolic length by

$$L_{\mathbb{D}}(\gamma) = \int_{\gamma} \lambda_{\mathbb{D}}(z)\, |dz|,$$

and the hyperbolic distance by

$$d_{\mathbb{D}}(z, w) = \inf_{\gamma} L_{\mathbb{D}}(\gamma) = \log \frac{1 + \left|\frac{z-w}{1-z\overline{w}}\right|}{1 - \left|\frac{z-w}{1-z\overline{w}}\right|}.$$

Here the infimum is taken over all smooth curves γ joining z to w in $\mathbb{D}$. The function $\lambda_{\mathbb{D}}(z) = 2/(1 - |z|^2)$ is known as the density of the hyperbolic metric on $\mathbb{D}$. For any simply connected domain Ω, λ_{Ω} can be computed via the formula

$$\lambda_{\Omega}(w) = \frac{\lambda_{\mathbb{D}}(f^{-1}(w))}{|f'(f^{-1}(w))|}, \quad w \in \Omega,$$

where f maps $\mathbb{D}$ conformally onto Ω. Note that the metric λ_Ω is independent of the choice of the conformal map f used. The metrics d_Ω and $d_\mathbb{D}$ satisfy the following relation:

Theorem 11. *Let $f : \mathbb{D} \to \Omega$ be analytic, where Ω is a simply connected subdomain of $\mathbb{C}$. Then*

$$d_\Omega(f(z), f(w)) \le d_\mathbb{D}(z, w).$$

Equality is possible only when f maps $\mathbb{D}$ conformally onto Ω.

The authors in [6] incorporated the hyperbolic metric into the Bohr inequality for three classes of functions. Let $\mathbb{H} = \{z = x + iy : \operatorname{Re} z > 0\}$ be the right half-plane. Then

$$\lambda_\mathbb{H}|dz| = \frac{|dz|}{\operatorname{Re} z}$$

as shown in [21, Example 7.2].

Theorem 12 ([6, Theorem 2.1]). *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D}, \mathbb{H})$. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \le \frac{1}{\lambda_\mathbb{H}(a_0)} = \operatorname{dist}(a_0, \partial\mathbb{H})$$

for $|z| \le 1/3$. The bound is sharp.

The next result is on the class $\mathscr{H}(\mathbb{D}, \mathbb{P})$, where $\mathbb{P} = \{z : |\arg z| < \pi\}$ is a slit domain. Its hyperbolic metric [21, Example 7.7] is given by

$$\lambda_\mathbb{P}|dz| = \frac{|dz|}{2|\sqrt{z}|\operatorname{Re}\sqrt{z}} = \frac{|dz|}{2|z|\cos[(\arg z)/2]} \ge \frac{|dz|}{2|z|}.$$

Theorem 13 ([6, Theorem 2.3]). *Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathscr{H}(\mathbb{D}, \mathbb{P})$. Then*

$$\sum_{n=1}^{\infty} |b_n z^n| \le \frac{1}{2\lambda_\mathbb{P}(|b_0|)} = \operatorname{dist}(|b_0|, \partial\mathbb{P})$$

for $|z| \le 3 - 2\sqrt{2} \approx 0.17157$. The bound is sharp.

The final class treated was the class $\mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$. By the formula given earlier, it can be deduced that

$$\lambda_{c\overline{\mathbb{D}}}|dz| = \frac{|dz|}{|z|\log|z|}.$$

Theorem 14 ([6, Theorem 2.5]). *Let* $c_0 > 1$ *and* $h(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathscr{H}(\mathbb{D}, c\overline{\mathbb{D}})$. *If* $|z| \leq 1/3$, *then*

(a) $\log\left(\sum_{n=0}^{\infty} |c_n z^n|\right) - \log c_0 \leq \dfrac{1}{\lambda_{\mathbb{H}}(\log c_0)} = \operatorname{dist}(\log c_0, \partial\mathbb{H})$,

(b) $\sum_{n=1}^{\infty} |c_n z^n| \leq \dfrac{2}{\lambda_{c\overline{\mathbb{D}}}(c_0)}$, *provided* $c_0 \leq 2$.

2.5 Bohr Radius for Concave-Wedge Domain

In [7], another class $\mathscr{H}(\mathbb{D}, W_\alpha)$ was considered, where

$$W_\alpha := \left\{w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2}\right\}, \quad 1 \leq \alpha \leq 2,$$

is a concave-wedge domain. In this instance, the conformal map of $\mathbb{D}$ onto W_α is given by

$$F_{\alpha,t}(z) = t\left(\frac{1+z}{1-z}\right)^{\alpha} = t\left(1 + \sum_{n=1}^{\infty} A_n z^n\right), \quad t > 0. \tag{8}$$

When $\alpha = 1$, the domain reduces to a convex half-plane, while the case $\alpha = 2$ yields a slit domain. The results are as follows:

Theorem 15. *Let* $\alpha \in [1, 2]$. *If* $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D}, W_\alpha)$ *with* $a_0 > 0$, *then*

$$\sum_{n=1}^{\infty} |a_n z^n| \leq \operatorname{dist}(a_0, \partial W_\alpha)$$

for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$. *The function* $f = F_{\alpha,a_0}$ *in* (8) *shows that the Bohr radius* r_α *is sharp.*

Theorem 16. *Let* $\alpha \in [1, 2]$. *If* $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D}, W_\alpha)$, *then*

$$\sum_{n=0}^{\infty} |a_n z^n| - |a_0|^* \leq \operatorname{dist}(|a_0|^*, \partial W_\alpha)$$

for $|z| \leq r_\alpha = (2^{1/\alpha} - 1)/(2^{1/\alpha} + 1)$, *where* $|a_0|^* = F_{\alpha,1}\left(\left|F_{\alpha,1}^{-1}(a_0)\right|\right)$ *and* $F_{\alpha,1}$ *is given by* (8). *The function* $f = F_{\alpha,|a_0|^*}$ *shows that the Bohr radius* r_α *is sharp.*

Theorem 15 is in the standard distance form for the Bohr theorem but under the condition $a_0 > 0$. Theorem 16, however, has no extra condition, but the inequality is only nearly Bohr-like. In a recent paper [11], Theorem 15 is shown to hold true without the assumption $a_0 > 0$.

2.6 Bohr Radius for a Special Subordination Class

The link between Bohr and differential subordination was also established in [7]. For $\alpha \geq \gamma \geq 0$, and for a given convex function $h \in \mathscr{H}(\mathbb{D})$, let

$$R(\alpha, \gamma, h) := \{f \in \mathscr{H}(\mathbb{D}) : f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z), \quad z \in \mathbb{D}\}.$$

An easy exercise shows that

$$f(z) \prec q(z) \prec h(z), \quad \text{for all } f \in R(\alpha, \gamma, h),$$

where

$$q(z) = \int_0^1 \int_0^1 h(zt^{\mu} s^{\nu})\, dt\, ds \in R(\alpha, \gamma, h). \tag{9}$$

Thus $R(\alpha, \gamma, h) \subset S(h)$ which is a subordination class.

Theorem 17. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha, \gamma, h)$, and h be convex. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \leq \operatorname{dist}(h(0), \partial h(\mathbb{D}))$$

for all $|z| \leq r_{CV}(\alpha, \gamma)$, where $r_{CV}(\alpha, \gamma)$ is the smallest positive root of the equation

$$\sum_{n=1}^{\infty} \frac{1}{(1 + \mu n)(1 + \nu n)} r^n = \frac{1}{2}.$$

Further, this bound is sharp. An extremal case occurs when $f(z) = q(z)$ as defined in (9) and $h(z) = z/(1 - z)$.

Analogously, if $h \in \mathscr{H}(\mathbb{D})$ is starlike, that is, h is univalent in $\mathbb{D}$ and the domain $h(\mathbb{D})$ is starlike with respect to the origin, then the function q in (9) is also starlike. A similar result for subordination to a starlike function is readily obtained.

Theorem 18. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in R(\alpha, \gamma, h)$*, and* h *be starlike. Then*

$$\sum_{n=1}^{\infty} |a_n z^n| \leq \operatorname{dist}(h(0), \partial h(\mathbb{D}))$$

for all $|z| \leq r_{ST}(\alpha, \gamma)$*, where* $r_{ST}(\alpha, \gamma)$ *is the smallest positive root of the equation*

$$\sum_{n=1}^{\infty} \frac{n}{(1+\mu n)(1+\nu n)} r^n = \frac{1}{4}.$$

This bound is sharp. An extremal case occurs when $f(z) = q(z)$ *as defined in* (9) *and* $h(z) = z/(1-z)^2$.

2.7 *Bohr's Theorem for Starlike Logharmonic Mappings*

We end Sect. 2 by discussing a recent extension of Bohr's theorem to the class of starlike logharmonic mappings. These mappings have been widely studied, for example, the works in [10, 57, 59] and the references therein.

Let $\mathscr{B}(\mathbb{D})$ denote the set of all functions a analytic in $\mathbb{D}$ satisfying $|a(z)| < 1$ in $\mathbb{D}$. A logharmonic mapping defined in $\mathbb{D}$ is a solution of the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = a \frac{f_z}{f},$$

where the second dilatation function a lies in $\mathscr{B}(\mathbb{D})$. Thus the Jacobian

$$J_f = |f_z|^2 \left(1 - |a|^2\right)$$

is positive and all non-constant logharmonic mappings are therefore sense-preserving and open in $\mathbb{D}$.

If f is a non-constant logharmonic mapping of $\mathbb{D}$ which vanishes only at $z = 0$, then f admits the representation [1]

$$f(z) = z^m |z|^{2\beta m} h(z) \overline{g(z)}, \tag{10}$$

where m is a nonnegative integer, $\operatorname{Re} \beta > -1/2$, and h and g are analytic functions in $\mathbb{D}$ satisfying $g(0) = 1$ and $h(0) \neq 0$. The exponent β in (10) depends only on $a(0)$ and can be expressed by

$$\beta = \overline{a(0)}\frac{1 + a(0)}{1 - |a(0)|^2}.$$

Note that $f(0) \neq 0$ if and only if $m = 0$, and that a univalent logharmonic mapping in $\mathbb{D}$ vanishes at the origin if and only if $m = 1$, that is, f has the form

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \quad z \in \mathbb{D},$$

where $\operatorname{Re}\beta > -1/2$, $0 \notin (hg)(\mathbb{D})$ and $g(0) = 1$. In this case, it follows that $F(\zeta) = \log f(e^{\zeta})$ is a univalent harmonic mapping of the half-plane $\{\zeta : \operatorname{Re}(\zeta) < 0\}$.

Denote by S_{Lh} the class consisting of univalent logharmonic maps f of the form

$$f(z) = zh(z)\overline{g(z)}$$

with the normalization $h(0) = g(0) = 1$. Also denote by ST^0_{Lh} the class consisting of functions $f \in S_{Lh}$ which maps $\mathbb{D}$ onto a starlike domain (with respect to the origin). Further let S^* be the usual class of normalized analytic functions f satisfying $f(\mathbb{D})$ is a starlike domain.

Lemma 2 ([2]). *Let $f(z) = zh(z)\overline{g(z)}$ be logharmonic in $\mathbb{D}$. Then $f \in ST^0_{Lh}$ if and only if $\varphi(z) = zh(z)/g(z) \in S^*$.*

This lemma shows the connection between starlike logharmonic functions and starlike analytic functions. The authors made use of Lemma 2 to obtain necessary and sufficient conditions on h and g so that the function $f(z) = zh(z)\overline{g(z)}$ belongs to ST^0_{Lh} (see [10, Theorem 1]), from which resulted in a sharp distortion theorem.

Theorem 19. *Let $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. Then*

$$\frac{1}{1+|z|}\exp\left(\frac{-2|z|}{1+|z|}\right) \le |h(z)| \le \frac{1}{1-|z|}\exp\left(\frac{2|z|}{1-|z|}\right),$$
$$(1+|z|)\exp\left(\frac{-2|z|}{1+|z|}\right) \le |g(z)| \le (1-|z|)\exp\left(\frac{2|z|}{1-|z|}\right),$$

and

$$|z|\exp\left(\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le |z|\exp\left(\frac{4|z|}{1-|z|}\right).$$

Equalities occur if and only if h, g, and f are, respectively, appropriate rotations of h_0, g_0, and f_0, where

$$h_0(z) = \frac{1}{1-z}\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2+\frac{1}{n}\right)z^n\right),$$

$$g_0(z) = (1-z)\exp\left(\frac{2z}{1-z}\right) = \exp\left(\sum_{n=1}^{\infty}\left(2-\frac{1}{n}\right)z^n\right),$$

and

$$f_0(z) = zh_0(z)\overline{g_0(z)} = \frac{z(1-\bar{z})}{1-z}\exp\left(Re\left(\frac{4z}{1-z}\right)\right).$$

The function f_0 is the logharmonic Koebe function. Theorem 19 then gives

Corollary 4. *Let* $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. *Also, let* $H(z) = zh(z)$ *and* $G(z) = zg(z)$. *Then*

$$\frac{1}{2e} \le \mathrm{dist}(0, \partial H(\mathbb{D})) \le 1, \quad \frac{2}{e} \le \mathrm{dist}(0, \partial G(\mathbb{D})) \le 1,$$

and

$$\frac{1}{e^2} \le \mathrm{dist}(0, \partial f(\mathbb{D})) \le 1.$$

Equalities occur if and only if h, g, *and* f *are, respectively, suitable rotations of* h_0, g_0, *and* f_0.

Finally, with the help of Corollary 4 and the sharp coefficient bounds from [3, Theorem 3.3], the Bohr theorems are obtained.

Theorem 20. *Let* $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$, $H(z) = zh(z)$ *and* $G(z) = zg(z)$. *Then*

(a) $|z|\exp\left(\sum_{n=1}^{\infty}|a_n||z|^n\right) \le \mathrm{dist}(0, \partial H(\mathbb{D}))$ *for* $|z| \le r_H \approx 0.1222$, *where* r_H *is the unique root in* $(0, 1)$ *of*

$$\frac{r}{1-r}\exp\left(\frac{2r}{1-r}\right) = \frac{1}{2e},$$

(b) $|z|\exp\left(\sum_{n=1}^{\infty}|b_n||z|^n\right) \le \mathrm{dist}(0, \partial G(\mathbb{D}))$ *for* $|z| \le r_G \approx 0.3659$, *where* r_G *is the unique root in* $(0, 1)$ *of*

$$r(1-r)\exp\left(\frac{2r}{1-r}\right) = \frac{2}{e}.$$

Both radii are sharp and are attained, respectively, by appropriate rotations of $H_0(z) = zh_0(z)$ *and* $G_0(z) = zg_0(z)$. *Here* h_0 *and* g_0 *are given in Theorem 19.*

Theorem 21. *Let* $f(z) = zh(z)\overline{g(z)} \in ST^0_{Lh}$. *Then, for any real* t,

$$|z| \exp\left(\sum_{n=1}^{\infty} \left|a_n + e^{it} b_n\right| |z|^n\right) \leq \mathrm{dist}(0, \partial f(\mathbb{D}))$$

for $|z| \leq r_0 \approx 0.09078$, *where* r_0 *is the unique root in* $(0, 1)$ *of*

$$r \exp\left(\frac{4r}{1-r}\right) = \frac{1}{e^2}.$$

The bound is sharp and is attained by a suitable rotation of the logharmonic Koebe function f_0.

3 Dirichlet Series and *n*-Dimensional Bohr Radius

3.1 Bohr and the Dirichlet Series

In [19], Balasubramanian et al. extended the Bohr inequality to the setting of Dirichlet series. This paper brings the Bohr phenomenon back to its origins since Bohr radius for power series on the disk originated from studying problems on absolute convergence [27] in the theory of Dirichlet series.

For $1 \leq p < \infty$, let D^p be the space of ordinary Dirichlet series consisting of $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathbb{H} = \{s = \sigma + it : \sigma > 0\}$ corresponding to the Hardy space of order p. The space D^p is the completion of the space of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ in the norm

$$\|P\| = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt\right)^{1/p},$$

which is equivalent to requiring $\sum |a_n|^2 < \infty$ when $p = 2$. The space D^∞ consists of the space of Dirichlet series as above with $\|f\|_\infty := \sup\{|f(s)| : \sigma = \mathrm{Re}\, s > 0\} < \infty$. Then the Bohnenblust–Hille theorem [26] takes the form

Theorem 22. *The infimum of* ρ *such that* $\sum_{n=1}^{\infty} |a_n| n^{-\rho} < \infty$ *for every* $\sum_{n=1}^{\infty} a_n n^{-s}$ *in* D^∞ *equals* $1/2$.

For $k \geq 1$, let D_k^∞ denote the subspace of D^∞ consisting of $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ such that $a_n = 0$ whenever the number of prime divisors of n exceeds k. If $(E, \|\cdot\|)$ is a Banach space of Dirichlet series, the isometric Bohr abscissa and the isomorphic Bohr abscissa are, respectively, defined as

$$\rho_1(E) = \min\left\{\sigma \geq 0 : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \leq \|f\| \text{ for all } f \in E\right\},$$

and

$$\rho(E) = \inf\left\{\sigma \geq 0 : \exists\, C_\sigma \in (0,\infty) \text{ such that } \sum_{n=1}^{\infty} |a_n| \|n^{-\sigma} \leq C_\sigma \|f\| \text{ for all } f \in E\right\}.$$

By using a number of recent developments in this topic (some of them related to the hypercontractivity properties of the Poisson kernel), the authors in [19] obtained among others the following results:

(1) If $1 \leq p < \infty$, then $\rho(D^p) = 1/2$, but this value is not attained. For $p = \infty$, this is equivalent to determining the maximum possible width of the strip of uniform, but not absolute, convergence of Dirichlet series (see Theorem 22).
(2) $\rho(D^\infty) = 1/2$, and this value is attained. So $\sum_{n=1}^{\infty} |a_n| n^{-1/2} \leq C\|f\|_\infty$ for some absolute constant C (see Theorem 22).
(3) Let $p \in [0, 1]$. Every $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in D^\infty$ satisfies $\sum_{n=1}^{\infty} |a_n n^{-\sigma}|^p < \infty$ whenever $\sigma \geq \sigma_0 := 1/p - 1/2$. If $\sigma < \sigma_0$, there is $f \in D^\infty$ such that the last sum is infinite.
(4) $\rho(D_k^\infty) = 1/2 - 1/(2k)$, and it is attained.
(5) $\rho_1(D_1^\infty) = 0$.
(6) $1.5903 < \rho_1(D_2^\infty) < 1.5904$.
(7) $1.585 < \log 3/\log 2 \leq \rho_1(D^\infty) \leq 1.8154$. In particular, $\sum_{n=1}^{\infty} |a_n| n^{-2} \leq \|f\|_\infty$.

3.2 The n-Dimensional Bohr Radius

Mathematicians have studied various generalizations of Bohr theorem, for example, in the works of [12, 14, 24, 25, 40, 41, 48, 49] and the references therein. One generalization uses power series representation of holomorphic functions defined on a complete Reinhardt domain, that is, a bounded complete n-circular domain in $\mathbb{C}^n$. In order to present and summarize certain multidimensional analogs of the Bohr and related inequalities, consider an n-variable power series $\sum_\alpha c_\alpha z^\alpha$ in the standard multi-index notation, where α denotes an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, $|\alpha|$ denotes the sum $\alpha_1 + \cdots + \alpha_n$ of its components, $\alpha!$ denotes the product $\alpha_1! \cdots \alpha_n!$ of the factorials of its components, z denotes the n-tuple $(z_1, \ldots, z_n)$ of complex numbers, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Let D be a complete Reinhardt domain. Denote by $R(D)$ the largest nonnegative number r such that whenever the power series $\sum_\alpha c_\alpha z^\alpha$ converges in D with $\left|\sum_\alpha c_\alpha z^\alpha\right| < 1$, then $\sum_\alpha |c_\alpha z^\alpha| < 1$ in the homothety rD. The number $R(D)$ is called the Bohr radius. In the case of n-dimensional unit polydisk $\mathbb{D}^n = \{(z_1, \ldots, z_n) :$

$\max_{1\leq j\leq n}|z_j| < 1\}$, Boas and Khavinson in [25, Theorem 2] showed the following estimate for the Bohr radius $R(D)$ (also known as the *first Bohr radius*, and denoted by K_n):

$$\frac{1}{3\sqrt{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

Note that the radius decreases to zero as the dimension of the domain increases. The result of Boas and Khavinson stimulated a lot of interest in Bohr type questions and has brought Bohr theorem to prominence even though the generalization of Bohr radius to the unit polydisk in $\mathbb{C}^n$ was first studied in [40]. By using the fact that the Bohnenblust–Hille inequality is hypercontractive, Defant et al. [35] obtained the optimal asymptotic estimate for this radius to be

$$K_n = b(n)\sqrt{\frac{\log n}{n}},$$

where $1/\sqrt{2} + o(1) \leq b(n) \leq 2$. Bayart et al. [20] proved that K_n behaves asymptotically as $\sqrt{(\log n)/n}$ and further improved the bounds for K_n to

$$K_n = c(n)\sqrt{\frac{\log n}{n}}, \tag{11}$$

where $1 + o(1) \leq c(n) \leq 2$. The article of Bohnenblust–Hille remains a seminal contribution, and the hypercontractive polynomial Bohnenblust–Hille inequality is the best one can hope for. It has several interesting consequences, and leads to precise asymptotic results regarding certain Sidon sets, Bohr radii for polydisks, and the moduli of the coefficients of functions in H^∞.

A few years after the first appearance of K_n in [25], Boas [24] extended the Bohr theorem to the complex Banach space ℓ_p^n whose norm is defined by

$$\|z\|_{\ell_p^n} := \left(\sum_{j=1}^{n}|z_j|^p\right)^{1/p}.$$

When $p = \infty$, the unit ball is to be interpreted as the unit polydisk in $\mathbb{C}^n$. Denote by $K(B_{\ell_p^n})$ the corresponding Bohr radius. For $1 \leq p \leq \infty$, it was shown in [24] that

$$\frac{1}{c}\left(\frac{1}{n}\right)^{1-\frac{1}{\min\{p,2\}}} \leq K(B_{\ell_p^n}) \leq c\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min\{p,2\}}},$$

where $c > 0$ is a constant independent of p, n. The lower bound was then improved to the value $\sqrt{(\log n/\log\log n)/n}$ in [34]. On the other hand, Aizenberg [12] proved that

$$\frac{1}{3e^{1/3}} \le K(B_{\ell_1^n}) \le 1/3,$$

where $1/3$ is the best upper bound. Also note that this estimate does not depend on n. In the same paper, Aizenberg defined the *second Bohr radius* $B_n(G)$, which is the largest radius r such that whenever a multidimensional power series $\sum_\gamma a_\gamma z^\gamma$ is bounded by 1 in the complete Reinhardt domain G, then $\sum_\gamma \sup_{rG} |a_\gamma z^\gamma| \le 1$. General lower and upper estimates for the first and the second Bohr radii of bounded complete Reinhardt domains are given in [37]. Results from both papers [34] and [37] were proved using certain theorems from [36], which was the first paper linking multidimensional Bohr study to local Banach space theory. The estimates for K_n obtained in [36] were in terms of unconditional basis constants and Banach-Mazur distances. We refer to [39] for a survey on these studies.

3.3 Bohr Radius in the Study of Banach Spaces

A new Bohr-type radius can be found in [38] which relates to the study of Banach spaces. Let $v : X \to Y$ be a bounded operator between complex Banach spaces, $n \in \mathbb{N}$, and $\lambda \ge \|v\|$. The λ-*Bohr radius of* v, denoted by $K_n(v, \lambda)$, is the supremum of all $r \ge 0$ such that for all holomorphic functions $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha$ on the n-dimensional unit polydisk $\mathbb{D}^n$,

$$\sup_{z \in r\mathbb{D}^n} \sum_{\alpha \in \mathbb{N}_0^n} \|v(c_\alpha) z^\alpha\|_Y \le \lambda \sup_{z \in \mathbb{D}^n} \Big\| \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha \Big\|_X.$$

If $X = \mathbb{C}$, v is the identity on X and $\lambda = 1$, then $K_n(v, \lambda) = K_n$ is exactly the n-dimensional Bohr radius previously defined. Thus, the main goal in [38] was to study the Bohr radii of the n-dimensional unit polydisk for holomorphic functions defined on $\mathbb{D}^n$ with values in Banach spaces, for example, by obtaining upper and lower estimates for Bohr radii $K_n(v, \lambda)$ of specific operators v between Banach spaces. Interestingly, Dixon [42] paid attention to applications of Bohr phenomena to operator theory showing that Bohr theorem is useful in the characterization of Banach algebras that satisfy von Neuman's inequality. Later Paulsen et al. in [61] continued the work in this line of investigation.

It was shown in [22] that the analogous Bohr theorem fails in the Hardy spaces H^q, $0 < q < \infty$, equipped with the corresponding Hardy norm. The authors also showed how renorming a space affected the Bohr radius. In [8], Abu-Muhanna and Gunatillake found the Bohr radius for the weighted Hardy Hilbert spaces, and again showed that no Bohr radius exists for the classical Hardy space H^2. The Bohr research on multidimensional weighted Hardy-Hilbert was earlier done in [61]. Thus, in view of the supremum norm, we can say that for the standard basis $(z^n)_{n=0}^\infty$, there exists a compact set $\{z \in \mathbb{C} : |z| \leq 1/3\}$ which lies in the open set $U \subset \mathbb{C}$ such that the norm version Bohr inequality occurred. Hence $(z^n)_{n=0}^{\infty}$ is said to have the Bohr property.

In [15] and [16], Aizenberg et al. considered the general bases for the space of holomorphic functions $\mathscr{H}(M)$ on a complex manifold M. A basis $(\phi_n)_{n=0}^{\infty}$ in $\mathscr{H}(M)$ is said to have the *Bohr Property* (**BP**) if there exist an open set $U \subset M$ and a compact set $K \subset M$ satisfying

$$\sum |c_n| \sup_U |\phi_n(z)| \leq \sup_K |f(z)|$$

for all $f = \sum c_n\phi_n \in \mathscr{H}(M)$. It was shown in [16] that when $\phi_0 = 1$ and $\phi_n(z_0)$, $n \geq 1$, vanishes for some $z_0 \in M$, then $(\phi_n)_n^{\infty}$ has the **BP**. A generalization of this result can be found in [15, Theorem 4]. Aytuna and Djakov in [18] introduced the term Global Bohr Property: a basis $(\phi_n)_n^{\infty}$ in the space of entire functions $\mathscr{H}(\mathbb{C}^n)$ has the *Global Bohr Property* (**GBP**) if for every compact $K \subset \mathbb{C}^n$ there is a compact $K_1 \supset K$ such that

$$\sum |c_n| \sup_K |\phi_n(z)| \leq \sup_{K_1} |f(z)|$$

for all $f = \sum c_n\phi_n \in \mathscr{H}(\mathbb{C}^n)$. They showed that a basis $(\phi_n)_n^{\infty}$ has the **GBP** if and only if one of the functions ϕ_n is a constant. As pointed out in [18], the inequality of **GBP** was first stated in a paper by Lassère and Mazzilli [55]. They studied the power series expressed in Faber polynomial basis which is associated with a compact continuum in $\mathbb{C}$. In fact, the relation between **BP** and Faber polynomials was first discovered by Kaptanoğlu and Sadık [52]. The radius obtained in [52] was not sharp. It was later solved in [56] by using better elliptic Carathéodory's inequalities.

The connection between Hadamard real part theorem and Bohr theorem can be seen in [53], in which Kresin and Maz'ya introduced the Bohr type real part estimates and proved the following theorem by applying the ℓ_p norm on the remainder of the power series expansion:

Theorem 23. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{A}$ *with*

$$\sup_{|\zeta|<1} \operatorname{Re}\,(e^{-i\arg f(0)} f(\zeta)) < \infty,$$

where $\arg f(0)$ *is replaced by zero if* $f(0) = 0$*. Then for any* $q \in (0, \infty]$*, integer* $m \geq 1$*, and* $|z| \leq r_{m,q}$*, the inequality*

$$\left(\sum_{n=m}^{\infty} |a_n z^n|^q \right)^{1/q} \leq \sup_{|\zeta|<1} \operatorname{Re}\,(e^{-i\arg f(0)} f(\zeta)) - |f(0)|$$

holds, where $r_{m,q} \in (0, 1)$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *if* $0 < q < \infty$*, and* $r_{m,\infty} := 2^{-1/m}$*. The radius* $r_{m,q}$ *is best possible.*

With $(q, m) = (1, 1)$, the result reduces to the sharp inequality obtained by Sidon [67] and which contains the classical Bohr inequality. For a discussion on the Bohr-type real part estimates, we refer to [54, Chap. 6].

There are still many possible directions of extending the Bohr theorem. For example, in [52], the authors considered the domain of functions bounded by ellipse instead of the unit disk $\mathbb{D}$. However, the Bohr radius does not exist for the space of holomorphic functions in an annulus equipped with the natural basis [13]. The Bohr radius for the class of analytic functions defined on

$$\{z : |z + \gamma/(1-\gamma)| < 1/(1-\gamma)\}, \quad 0 \le \gamma < 1,$$

was given in [46]. Liu and Wang [58] proved another kind of extension of the classical Bohr inequality involving bounded symmetric domains.

Theorem 24. *Let Ω denote one of the four classical domains in the sense of Hua [50] or the unit polydisk in $\mathbb{C}^n$. Denote by $\|\cdot\|_\Omega$ the Minkowski norm associated with Ω. Let $f : \Omega \to \Omega$ be a holomorphic map with*

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

as its Taylor expansion in k-homogeneous polynomials f_k. Let $\phi \in \operatorname{Aut}\Omega$ such that $\phi(f(0)) = 0$. Then

$$\sum_{k=0}^{\infty} \frac{\|D\phi(f(0)) \cdot f_k(z)\|_\Omega}{\|D\phi(f(0))\|_\Omega} < 1$$

for all $z \in \Omega$ satisfying $\|z\|_\Omega < 1/3$.

Using a different approach, Roos [66] extended the theorem to any bounded circled symmetric domain. Earlier results on the generalization of Bohr theorem using homogeneous expansions can also be found in [12, Theorem 8] and [14]. Meanwhile, the generalization of both the results [58] and [12, Theorem 8] was obtained by Hamada et al. [49].

On the other hand, Guadarrama [47] considered the polynomial Bohr radius defined by

$$R_n = \sup_{p \in \mathscr{P}_n} \left\{ r \in (0,1) : \sum_{k=0}^{n} |a_k| r^k \le \|p\|_\infty, \quad p(z) = \sum_{k=0}^{n} a_k z^k \right\},$$

where $\mathscr{P}_n$ consists of all the complex polynomials of degree at most n. The author showed that

$$C_1 \frac{1}{3^{n/2}} \le R_n - 1/3 \le C_2 \frac{\log n}{n}$$

for some positive constants C_1 and C_2. Subsequently, Fournier [45] computed and obtained an explicit formula for R_n by applying the notion of bounded-preserving operators. The following result concerning the asymptotic behaviour of R_n was proved only recently in [32]:

$$\lim_{n\to\infty} n^2\left(R_n - \frac{1}{3}\right) = \frac{\pi^2}{3}.$$

As remarked earlier, the authors in [61] square the constant term in the expansion of f and obtained the sharp Bohr radius $1/2$. A similar idea was adopted by Blasco in [23]. The author introduced and studied the radius

$$R_{p,q}(X) = \inf\{R_{p,q}(f, X) : \sup_{|z|<1} \|f\|_X \le 1\},$$

where $X = L_p(\mu)$ or $X = \ell_p$ spaces and

$$R_{p,q}(f, X) = \sup\left\{ r \ge 0 : \|a_0\|_X^p + \left(\sum_{n=1}^{\infty} \|a_n\|_X r^n\right)^q \le 1,\ f(z) = \sum_{n=0}^{\infty} a_n z^n \right\}.$$

Popescu et al. [61, 63, 65] established the operator-theoretic Bohr radius. In [51], Kaptanoğlu studied the Bohr phenomenon for elliptic equations by considering the case of harmonic functions for the Laplace-Beltrami operator. The Bohr radii for classes of harmonic, separately harmonic and pluriharmonic functions were evaluated in [17]. Extension of Bohr theorem to uniform algebra can also be found in [62].

3.4 Concluding Remarks on Multidimensional Bohr Radius

To conclude, we discuss some results with regard to the multidimensional Bohr radius. For functions f holomorphic in $\mathbb{D}^n$ of the form

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha z^\alpha, \tag{12}$$

its associated majorant series is given by

$$M_f(z) = \sum_{\alpha} |c_\alpha z^\alpha| = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} |c_\alpha z^\alpha|.$$

Also for an integer $m \geq 1$, we extend the definition of an m-symmetric analytic function of single variable to n-variable: a function f holomorphic in $\mathbb{D}^n$ is called m-symmetric if $f(e^{2\pi i/m}z) = f(z)$ for all $z \in \mathbb{D}^n$. Also note that a holomorphic function f of the form (12) is m-symmetric if and only if its Taylor expansion has the m-symmetric form

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_{m\alpha} z^{m\alpha},$$

where $m\alpha = (m\alpha_1, \ldots, m\alpha_n)$. Therefore if $f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_{m\alpha} z^{m\alpha}$ is holomorphic in $\mathbb{D}^n$, then by letting $\zeta = z^m$, it follows that the function $g(\zeta) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_{m\alpha} \zeta^{\alpha}$ is also holomorphic in $\mathbb{D}^n$. Hence the following result is obtained as a consequence of the n-dimensional Bohr theorem (11).

Theorem 25. *If* $f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_{m\alpha} z^{m\alpha}$ *is holomorphic in* $\mathbb{D}^n$ *for some integer* $m \geq 1$, *and* $|f(z)| < 1$ *in* $\mathbb{D}^n$, *then* $M_f(z) < 1$ *holds in* $K_{n,m} \cdot \mathbb{D}^n$ *with*

$$K_{n,m} = \sqrt[m]{c(n)\sqrt{\frac{\log n}{n}}},$$

and $1 + o(1) \leq c(n) \leq 2$.

Similar to the case of single variable, for a function f holomorphic in $\mathbb{D}^n$ of the form (12), its alternating series can be defined as

$$A_f(z) = \sum_{k=0}^{\infty} (-1)^k \sum_{|\alpha|=k} |c_\alpha z^\alpha|.$$

Adopting the idea from [12], denote by $B_{A,n}$ the largest number r such that

$$M_r A_f(z) = \sum_{k=0}^{\infty} (-1)^k \sum_{|\alpha|=k} \sup_{\mathbb{D}_r^n} |c_\alpha z^\alpha| < 1,$$

where $r > 0$ and $\mathbb{D}_r^n = r \cdot \mathbb{D}^n$ is the homothetic transformation of $\mathbb{D}^n$. The following majorant-type series

$$M_{r,f}(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \sup_{\mathbb{D}_r^n} |c_\alpha z^\alpha|$$

will be required in the sequel.

Theorem 26. *If* $f(z) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} c_\alpha z^\alpha$ *is holomorphic in* $\mathbb{D}^n$ *and* $|f(z)| < 1$ *in* $\mathbb{D}^n$, *then* $|M_r A_f(z)| < 1$ *holds in* $B_{A,n} \cdot \mathbb{D}^n$, *where* $1 - \sqrt[n]{2/5} \leq B_{A,n}$.

Proof. If $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, then $-z = (-z_1, \ldots, -z_n)$ and

$$(-z)^\alpha = (-z_1)^{\alpha_1} \cdots (-z_n)^{\alpha_n} = (-1)^{|\alpha|} z^\alpha.$$

Define the even and odd parts of f to, respectively, be

$$f_e(z) = \frac{1}{2}(f(z) + f(-z)) = \sum_{k=0}^{\infty} \sum_{|\alpha|=2k} c_\alpha z^\alpha,$$

and

$$f_o(z) = \frac{1}{2}(f(z) - f(-z)) = \sum_{k=0}^{\infty} \sum_{|\alpha|=2k+1} c_\alpha z^\alpha.$$

As $\mathbb{D}$ is convex, it follows that $|f_e(z)| < 1$ and $|f_o(z)| < 1$ in $\mathbb{D}^n$.

Now, Wiener method (see the proof of [25, Theorem 2]) and the multidimensional Cauchy estimate yield

$$|c_\alpha| \le 1 - |c_0|^2 \ \text{ for } \ |\alpha| \ge 1.$$

The inequality then gives

$$\begin{aligned} M_r A_f(z) &= M_{r,f_e}(z) - M_{r,f_o}(z) \le M_{r,f_e}(z) \\ &= \sum_{k=0}^{\infty} \sum_{|\alpha|=2k} \sup_{r\cdot\mathbb{D}^n} |c_\alpha z^\alpha| = \sum_{k=0}^{\infty} \sum_{|\alpha|=2k} |c_\alpha| \sup_{r\cdot\mathbb{D}^n} |z^\alpha| \\ &\le |c_0| + (1 - |c_0|^2) \sum_{k=1}^{\infty} r^{2k} \sum_{\alpha_1+\cdots+\alpha_n=2k} 1 \\ &= |c_0| + (1 - |c_0|^2) \sum_{k=1}^{\infty} \binom{2k+n-1}{2k} r^{2k}. \end{aligned}$$

Since $(1-r)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k$ and $|c_0| < 1$, it follows that

$$M_r A_f(z) < |c_0| + (1 - |c_0|) \left(\frac{1}{(1-r)^n} + \frac{1}{(1+r)^n} - 2 \right).$$

Thus $M_r A_f(z) < 1$ when

$$\frac{1}{(1-r)^n} + \frac{1}{(1+r)^n} \le 3. \tag{13}$$

On the other hand,

$$\begin{aligned} M_r A_f(z) &= M_{r f_e}(z) - M_{r f_o}(z) \geq -M_{r f_o}(z) \\ &= -\sum_{k=0}^{\infty} \sum_{|\alpha|=2k+1} \sup_{r\cdot\mathbb{D}^n} |c_\alpha z^\alpha| = -\sum_{k=0}^{\infty} \sum_{|\alpha|=2k+1} |c_\alpha| \sup_{r\cdot\mathbb{D}^n} |z^\alpha| \\ &\geq -(1-|c_0|^2) \sum_{k=0}^{\infty} r^{2k+1} \sum_{\alpha_1+\cdots+\alpha_n=2k+1} 1 \\ &> -\sum_{k=0}^{\infty} \binom{n+2k}{2k+1} r^{2k+1} = \frac{1}{2(1+r)^n} - \frac{1}{2(1-r)^n}. \end{aligned}$$

Thus $M_r A_f(z) > -1$ when

$$\frac{1}{(1-r)^n} - \frac{1}{(1+r)^n} \leq 2. \tag{14}$$

Adding (13) and (14) gives $r \leq 1 - \sqrt[n]{2/5}$. □

Note: After the present survey was completed, Problem 1 on the Bohr radius for odd analytic functions was solved by I. Kayumov and S. Ponnusamy in a more general setting: https://arxiv.org/pdf/1701.03884.pdf

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Bernstein-Type Polynomials on Several Intervals

J. Szabados

Dedicated to the memory of Professor Qazi Ibadur Rahman

Abstract We construct the analogues of Bernstein polynomials on the set J_s of s finitely many intervals. Two cases are considered: first when there are no restrictions on J_s, and then when J_s has a so-called T-polynomial. On such sets we define approximating operators resembling the classic Bernstein polynomials. Reproducing and interpolation properties as well as estimates for the rate of convergence are given.

Keywords Bernstein polynomial • T-polynomial • Set of intervals • Rate of convergence

2000 Mathematics Subject Classification: 41A10, 41A25

1 Introduction

For any $s \geq 1$ let

$$0 = a_1 < b_1 < \cdots < a_s < b_s = 1$$

be a finite partition of the interval [0, 1], let $I_j := [a_j, b_j],\ j = 1, \ldots, s$, and let

$$J_s := \bigcup_{j=1}^{s} I_j, \quad I_j := [a_j, b_j]$$

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be the corresponding set of s pairwise disjoint intervals, and denote by Π_n the set of polynomials of degree at most n. Denote by $C(J_s)$ the set of continuous functions on J_s.

Some basic problems of approximation theory like Markov–Bernstein inequalities or Lagrange interpolation have been considered for functions $f \in C(J_s)$; see, for example, [4] or [2]. It is equally important to establish analogues of classic operators on this set. In this context we will be concerned with generalizations of Bernstein polynomial.

2 The General Case

The simplest construction for a general set J_s is the following. If $f \in C(J_s)$ is not differentiable, then extend it to $[0, 1] \setminus J_s$ by defining it as a linear function in each $[b_j, a_{j+1}]$, $j = 1, \dots, s-1$, continuously connecting the different parts of the original function. If $f' \in C(J_s)$, then in each $[b_j, a_{j+1}]$, $j = 1, \dots, s-1$, define f as the third degree Hermite polynomial satisfying

$$H_j(b_j) = f(b_j),\ H_j'(b_j) = f'(b_j),\ H_j(a_{j+1}) = f(a_{j+1}),\ H_j'(a_{j+1}) = f'(a_{j+1}),$$

$1 \le j \le s-1$. Then evidently the modulus of continuity $\omega(\tilde{f}, t) \le c\omega(f, t)$ in the first case, and $\omega(\tilde{f}', t) \le c\omega(f', t)$ in the second case. Thus considering the ordinary Bernstein polynomials for the extended function $\tilde{f}$ on $[0, 1]$, we obtain the usual error estimates $c\omega(f, 1/\sqrt{n})$ and $\omega(f', 1/\sqrt{n})/\sqrt{n}$ in the corresponding cases, respectively. (In the latter case besides function values, derivative values at the endpoints I_j of the function also appear in the construction.)

This method has two disadvantages. First, in the classic case the difference between the number of data used in the construction and the degree of the Bernstein polynomials is 1, while here it is cn. Second, the classic Bernstein polynomials interpolate at the endpoints of the interval, while here not.

3 The Case of T-Polynomials

It is well known that given arbitrary positive integer m, a set J_s always has a so-called Chebyshev polynomial of degree m, i.e., a polynomial which attains its maximum and minimum on J_s at consecutive $m+1$ points in J_s. If, in addition, this polynomial attains its maximum or minimum on J_s at the endpoints of I_j, $j = 1, \dots, s$, then we say that this is a T-polynomial. Of course, this additional property does not hold for all sets J_s. It was F. Peherstorfer who introduced this notion, and thoroughly investigated its existence and properties (see [5, 6]).

It will be convenient to assume that a T-polynomial has a minimum 0 and maximum 1 on J_s.

Assume that J_s has a T-polynomial $p(x) \in \Pi_m,\ m \geq s$ normalized such that $p(0) = 0$. For $n \in \mathbb{N}$, let $x_{k1} < \cdots < x_{km_k}$ be defined by

$$p(x_{ki}) = \frac{k}{n}, \qquad i = 1, \dots, m_k; \quad k = 0, \dots, n$$

where

$$m_k = \begin{cases} m + s - \left[\frac{m+s}{2}\right], & \text{if} \quad k = 0, \\ m, & \text{if} \quad k = 1, \dots, n-1, \\ \left[\frac{m+s}{2}\right], & \text{if} \quad k = n. \end{cases}$$

The existence of such x_{ki}'s follows from the properties of T-polynomials (they are monotone between two adjacent extremal values).

For an arbitrary $f(x) \in C(J_s)$, let

$$L_k(f, x) = \sum_{i=1}^{m_k} f(x_{ki}) \ell_{ki}(x) \in \Pi_{m_k - 1}, \qquad k = 0, 1, \dots, n\,,$$

be the Lagrange interpolation polynomial with respect to the nodes x_{ki}. Here $\ell_{ki}(x) \in \Pi_{m_k-1}$ are the fundamental polynomials with the property

$$\ell_{ki}(x_{kj}) = \delta_{ij}, \qquad i, j = 1, \dots, m_k, \quad k = 0, 1, \dots, n\,. \tag{1}$$

Consider the discrete linear operator

$$B_n(f, x) := \sum_{k=0}^{n} L_k(f, x) b_{nk}(p(x)), \qquad x \in J_s\,, \tag{2}$$

where

$$b_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0, \dots, n\,,$$

are the fundamental functions of the Bernstein polynomials. Evidently, $B_n(f, x) \in \Pi_{mn+m-1}$, and there are $mn + s$ function values used in the construction of the operator. This means that the difference between the number of function values and the degree of the operator is $m - s + 1$, i.e., independent of n, just like in case of the classic Bernstein polynomials. In this respect they are better than the polynomials defined in the previous section.

Although this is not a positive operator, it still has a bounded norm; this will follow from Theorem 1 below.

We mention two properties of this operator. The first one is about reproducing polynomials:

$$B_n(q,x) \equiv q(x) \qquad \text{if} \quad q \in \Pi_{m_n-1} \quad \text{or} \quad q = p\,. \tag{3}$$

Namely, if $q \in \Pi_{m_n-1}$, then $L_k(q,x) \equiv q(x),\ k = 0,1,\dots,n$, by the reproducing property of Lagrange interpolation, and thus

$$B_n(q,x) = q(x) \sum_{k=0}^{n} b_{nk}(p(x)) = q(x)\,.$$

And if $q = p$, then $L_k(p,x) \equiv \frac{k}{n},\ k = 0,1,\dots,n$, whence

$$B_n(p,x) = \sum_{k=0}^{n} \frac{k}{n} b_{nk}(p(x)) = p(x)\,,$$

since the classic Bernstein polynomials reproduce linear functions. Again, this is better than the corresponding reproducing property of the classic Bernstein polynomials (which reproduce only linear functions).

The second property is about interpolation:

$$B_n(f,x_{ki}) = f(x_{ki}), \qquad i = 1,\dots,m_k; \quad k = 0 \text{ or } n\,. \tag{4}$$

Namely, $p(x_{0i}) = 0,\ i = 1,\dots,m_0$, and thus

$$B_n(f,x_{0i}) = L_0(f,x_{0i}) = f(x_{0i}), \qquad i = 1,\dots,m_0\,.$$

Similarly for the x_{ni}'s. Notice that the polynomials defined in the previous section interpolate only at 0 and 1.

Now we state a pointwise convergence estimate. Let

$$\varphi(x) = \sqrt{(x-a_j)(b_j-x)} \qquad \text{if} \quad x \in I_j,\ j = 1,\dots,s,$$

and define the Ditzian–Totik modulus of continuity (cf. [1, Chap. 1]) as

$$\omega_\varphi(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi(x)} f(x)\|_{J_s}$$

where the difference is meant to be zero if any of the arguments is outside J_s, and we assume that t is so small that both $x \pm \varphi(x)$ fall into the same interval I_j. Further let

$$V(f) = \sup_{x,y\in J_s} |f(x) - f(y)| \,.$$

Theorem 1. *For an arbitrary $f \in C(J_s)$ we have*

$$\|f(x) - B_n(f,x)\|_{J_s} \le c\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right) + c\frac{V(f)}{\sqrt{n}} \,.$$

(Here and in what follows, c will always denote a positive constant depending on J_s and m, but independent of n, not necessarily the same at each occurrence.) Note that the second term in the error estimate cannot be dropped, since for functions which are constant on each interval I_j, the modulus of continuity is zero, while the operator does not reproduce such functions.

For the proof we need the following:

Lemma 1. *Let* (α, β) *be one of the pairs of numbers* $(0,1), (1/2,1), (1,1), (1,2), (3/2,2)$. *Then*

$$\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{\alpha} \left|x - \frac{k}{n}\right|^{\beta} b_{nk}(x) \le \begin{cases} c\left(\frac{x}{n}\right)^{\frac{\beta-\alpha}{2}}, & \text{if} \quad 0 \le x \le 1/n, \\ c\frac{x^{\beta/2-\alpha}}{n^{\beta/2}}, & \text{if} \quad 1/n \le x \le 1 \,. \end{cases}$$

Proof. By Cauchy–Schwarz inequality we get for $0 \le x \le 1/n$,

$$\begin{aligned} T := \sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{\alpha} \left|x - \frac{k}{n}\right|^{\beta} b_{nk}(x) &\le \left(\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{2\alpha} \left(x - \frac{k}{n}\right)^{2\beta} b_{nk}(x)\right)^{1/2} \\ &\le \left(\sum_{k=1}^{[n/2]} \left(x - \frac{k}{n}\right)^{2\beta-2\alpha} b_{nk}(x)\right)^{1/2} . \end{aligned}$$

Now we use the inequalities

$$\sum_{k=0}^{n} \left|x - \frac{k}{n}\right|^{2\gamma} b_{nk}(x) \le c\left(\frac{x}{n}\right)^{\gamma} , \tag{5}$$

where $\gamma = 1/2$ or 1 and $0 \le x \le 1$, or $\gamma = 2$ and $1/n \le x \le 1$. These follow from well-known relations (cf. Lorentz [3, p. 14]). Hence we obtain the first statement of the lemma.

Next let $1/n \le x \le 1$. We obtain, using again (5),

$$T \le \left(\sum_{k=1}^{[n/2]} \left(\frac{n}{k}\right)^{2\alpha} \left(x - \frac{k}{n}\right)^{2\beta} b_{nk}(x)\right)^{1/2} \le \frac{1}{x^{\alpha}} \left(\sum_{k=1}^{[n/2]} \left(x - \frac{k}{n}\right)^{2\beta} b_{n+2\alpha,k}(x)\right)^{1/2}$$

$$\le \frac{c}{x^\alpha}\left(\sum_{k=1}^{[n/2]}\left(x-\frac{k}{n+2\alpha}\right)^{2\beta} b_{n+2\alpha,k}(x)\right)^{1/2} + c\frac{1}{n^{\beta-\alpha}} \le c\frac{x^{\beta/2-\alpha}}{n^{\beta/2}}. \qquad \square$$

Proof of Theorem 1. Let $x \in (\xi,\eta) \subset I_j = [a_j,b_j]$ be such that $p(\xi)=0,\ p(\eta)=1$ and $p'(x)>0$ (the case $p'(x)<0$ can be handled similarly). Let $p(x_{kt})=\frac{k}{n}$ where $x_{kt}\in[\xi,\eta]$. Let

$$S_k = \begin{cases} \{t\}, & \text{if} \quad \xi = a_j, \\ \{t-1,t\}, & \text{if} \quad \xi > a_j. \end{cases}$$

Since both operators L_k and the classic Bernstein polynomials reproduce constants, we get

$$|f(x)-B_n(f,x)| \le \sum_{k=0}^{n}\sum_{i=1}^{m_k}|f(x)-f(x_{ki})|\cdot|\ell_{ki}(x)|b_{nk}(p(x))$$

$$\le c\sum_{k=0}^{n}\left\{\omega_\varphi\left(f,\frac{1}{\sqrt{n}}\right)\sum_{i\in S_k}\left[\frac{\sqrt{n}}{\varphi(x)}(x-x_{ki})+1\right]|\ell_{ki}(x)|\right.$$

$$\left. +V(f)\sum_{i\notin S_k}|\ell_{ki}(x)|\right\} b_{nk}(p(x)).$$

Here we used the inequality $\varphi\left(\frac{x+x_{ki}}{2}\right) \ge c\varphi(x),\ i\in S_k$.

We estimate the right-hand side sum for $0\le k\le n/2$; the other part can be handled similarly. Then it is sufficient to consider the case $0\le p(x)\le 4/5$, since for $4/5\le p(x)\le 1$ we get

$$\sum_{k=0}^{[n/2]}\left(\frac{n}{k}\right)^\alpha b_{nk}(p(x)) \le n^\alpha(1-p(x))^{n/2}\sum_{k=0}^{n}\binom{n}{k}p(x)^k$$

$$\le n^\alpha 2^n (1/5)^{n/2} = n^\alpha(4/5)^{n/2}.$$

This geometric convergence to zero implies that all similar sums to be handled subsequently are minor compared to the estimates for $0\le p(x)\le 4/5$. Also, it is sufficient to consider the case when $\eta<b_j$, since $\eta=b_j$ corresponds to the case $\xi=a_j$.

Let first $k=0$. Then we have $|\ell_{0i}(x)|\le c,\ i=0,1,\dots,m_0$, and

$$\frac{(x-x_{0i})|\ell_{0i}(x)|}{\varphi(x)} \le c\frac{x-x_{0t}}{\varphi(x)} \le c\sqrt{p(x)}, \qquad i=1,\dots,m_0.$$

Hence

$$\left(\frac{\sqrt{n}}{\varphi(x)}(x-x_{0i})+1\right)|\ell_{0i}(x)|b_{n0}(p(x)) \le c(\sqrt{np(x)}(1-p(x))^n + c \le c\,.$$

Now let $1 \le k \le n/2$. In order to complete the proof of Theorem 1 we need to prove the following estimates:

$$A := \sum_{k=1}^{[n/2]}\sum_{i\in S_k}|\ell_{ki}(x)|b_{nk}(p(x)) \le c\,,$$

$$B := \sum_{k=1}^{[n/2]}\sum_{i\in S_k}|x-x_{ki}|\cdot|\ell_{ki}(x)|b_{nk}(p(x)) \le \frac{\varphi(x)}{\sqrt{n}}\,,$$

$$C := \sum_{k=1}^{[n/2]}\sum_{i\notin S_k}|\ell_{ki}(x)|b_{nk}(p(x)) \le \frac{c}{\sqrt{n}}\,.$$

We distinguish two cases.

Case 1: $S_k = \{t\}$. Then $|\ell_{kt}(x)| \le c$, whence $A \le c\sum_{k=1}^{[n/2]} b_{nk}(p(x)) \le c$. On the other hand, $|x-x_{kt}| \le c\left|p(x)-\frac{k}{n}\right|$, therefore by Lemma 1 applied with $\alpha = 0, \beta = 1$ yields

$$B \le c\sum_{k=1}^{[n/2]}|x-x_{kt}|b_{nk}(p(x)) \le c\sum_{k=1}^{[n/2]}\left|p(x)-\frac{k}{n}\right|b_{nk}(p(x)) \le c\sqrt{\frac{p(x)}{n}} \le c\frac{\varphi(x)}{\sqrt{n}}\,.$$

Finally, using $|\ell_{ki}(x)| \le c\sqrt{\frac{n}{k}}|x-x_{kt}|,\ i\notin S_k$, and Lemma 1 with $\alpha = \frac{1}{2},\ \beta = 1$,

$$C \le c\sum_{k=1}^{[n/2]}\sqrt{\frac{n}{k}}|x-x_{kt}|b_{nk}(p(x)) \le c\sum_{k=1}^{[n/2]}\sqrt{\frac{n}{k}}\left|p(x)-\frac{k}{n}\right|b_{nk}(p(x)) \le \frac{c}{\sqrt{n}}\,.$$

Case 2: $S_k = \{t-1, t\}$. Then

$$|\ell_{kt}(x)| \le c\frac{x-x_{k,t-1}}{x_{kt}-x_{k,t-1}} \le 1 + c\sqrt{\frac{n}{k}}|x-x_{kt}|\,,$$

$$|\ell_{k,t-1}(x)| \le c\frac{|x-x_{kt}|}{x_{kt}-x_{k,t-1}} \le c\sqrt{\frac{n}{k}}|x-x_{kt}|\,.$$

$$|\ell_{ki}(x)| \le c\frac{|x-x_{kt}|(x-x_{k,t-1})}{x_{ki}-x_{k,i\pm 1}} \le c|x-x_{kt}| + \sqrt{\frac{n}{k}}(x-x_{kt})^2, \qquad i\notin S_k\,.$$

We also need the inequalities

$$|x-x_{kt}| \le \begin{cases} c\sqrt{\frac{n}{k}}\left(p(x)-\frac{k}{n}\right), & \text{if} \quad 1 \le k \le n\min(p(x),1/2), \\ c\min\left(\sqrt{\frac{k}{n}}, \frac{1}{\sqrt{p(x)}}\left(\frac{k}{n}-p(x)\right)\right) & \text{if} \quad n\min(p(x),1/2) < k \le n/2, \end{cases}$$

and

$$0 < x - x_{k,t-1} \le c|x-x_{kt}| + c\sqrt{\frac{k}{n}}.$$

These relations can be easily seen by using the mean value theorem and considering that $p'(x) \sim \sqrt{p(x)}$.

Using Lemma 1 with $\alpha = \beta = 1$ we have

$$A \le c\sum_{k=1}^{[n/2]}\left(1+\sqrt{\frac{n}{k}}|x-x_{kt}|\right)b_{nk}(p(x))$$

$$\le c + c\sum_{k=1}^{[n\min(p(x),1/2)]}\frac{n}{k}\left(\frac{k}{n}-p(x)\right)b_{nk}(p(x)) + \sum_{k=[n\min(p(x),1/2)]}^{[n/2]} b_{nk}(p(x)) \le c,$$

and since now $\varphi(x) \ge c$, Lemma 1 for $(\alpha,\beta) = (1/2,1), (3/2,2), (0,1)$ gives

$$B \le c\sum_{k=1}^{[n/2]}|x-x_{kt}|\left(1+\sqrt{\frac{n}{k}}|x-x_{kt}|\right)b_{nk}(p(x))$$

$$\le c\sum_{k=1}^{[n\min(p(x),1/2)]}\left\{\sqrt{\frac{n}{k}}\left(p(x)-\frac{k}{n}\right)+\left(\frac{n}{k}\right)^{3/2}\left(p(x)-\frac{k}{n}\right)^2\right\}b_{nk}(p(x))$$

$$+\sum_{k=[n\min(p(x),1/2)]}^{[n/2]}\left\{\frac{1}{\sqrt{p(x)}}\left(\frac{k}{n}-p(x)\right)\right\}b_{nk}(p(x)) \le \frac{c}{\sqrt{n}} \le c\frac{\varphi(x)}{\sqrt{n}}.$$

Finally, using

$$|\ell_{ki}(x)| \le c\left|\frac{(x-x_{kt})(x-x_{k,t-1})}{x_{ki}-x_{k,i\pm1}}\right| \le c|x-x_{kt}| + c\sqrt{\frac{n}{k}}(x-x_{kt})^2, \qquad i \notin S.$$

we get

$$C \le c\sum_{k=1}^{[n/2]}\left(|x-x_{kt}|+\sqrt{\frac{n}{k}}(x-x_{kt})^2\right)b_{nk}(p(x)).$$

But this leads to the same estimate as for B above. □

Next, we state an equivalence result.

Theorem 2. *We have*

$$\|f - B_n(f)\|_{J_s} = O(n^{-\alpha/2}) \iff \omega_\varphi(f,t) = O(t^\alpha), \qquad 0 < \alpha < 1\,.$$

For the proof we need two lemmas.

Lemma 2. *We have*

$$\|\varphi B_n'(f)\|_{J_s} \le c\sqrt{n}V(f)$$

for all $f \in C(J_s)$.

Proof of Lemma 2. We make the same assumptions on x and on the summation for k as in the proof of Theorem 1. Differentiating the identity

$$\sum_{k=0}^{n}\sum_{i=1}^{m_k} \ell_{ki}(x) b_{nk}(p(x)) = 1$$

we obtain

$$\sum_{k=0}^{n}\sum_{i=1}^{m_k} f(x)\ell_{ki}(x)b_{nk}'(p(x))p'(x) + \sum_{k=0}^{n}\sum_{i=1}^{m_k} f(x)\ell_{ki}'(x)b_{nk}(p(x)) = 0\,.$$

Also, differentiating the operator $B_n(f,x)$ and subtracting the pervious relation we obtain

$$\begin{aligned}\varphi(x)|B_n'(f,x)| &\le \varphi(x)V(f)\sum_{k=0}^{n}\sum_{i=1}^{m_k}\left\{|\ell_{ki}(x)|b_{nk}'(p(x))|p'(x)| + |\ell_{ki}'(x)|b_{nk}(p(x))\right\} \\ &\le cV(f)\sum_{k=0}^{n}\sum_{i=1}^{m_k}\left\{\sqrt{p(x)}|\ell_{ki}(x)|b_{nk}'(p(x)) + |\ell_{ki}'(x)|b_{nk}(p(x))\right\}\,,\end{aligned}$$

since

$$\varphi(x)|p'(x)| \le c\sqrt{p(x)}\,. \tag{6}$$

Thus we have to prove:

$$A := \sqrt{p(x)}\sum_{k=0}^{[n/2]}\left(\sum_{i=1}^{m_k}|\ell_{ki}(x)|\right)b_{nk}'(p(x)) \le c\sqrt{n}$$

and

$$B := \sum_{k=0}^{[n/2]} \left(\sum_{i=1}^{m_k} |\ell'_{ki}(x)| \right) b_{nk}(p(x)) \le c\sqrt{n}\,.$$

For $k = 0$ we have $|\ell_{ki}(x)| \le c$, and by the Markov inequality $|\ell'_{ki}(x)| \le c$ for all $i = 1, \dots, m_0$, thus the contribution to A and B will be

$$n\sqrt{p(x)}(1 - p(x))^{n-1} \le c\sqrt{n} \quad \text{and} \quad c(1 - p(x))^n \le c\,,$$

respectively.

So let $1 \le k \le [n/2]$. Since

$$|b'_{nk}(p(x))| \le \frac{cn}{p(x)} \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x))\,, \tag{7}$$

we have

$$A \le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \sum_{k=1}^{[n/2]} \left(\sum_{i=1}^{m_k} |\ell_{ki}(x)| \right) \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x))\,.$$

From the proof of Theorem 1 we can see that

$$|\ell_{ki}(x)| \le c + c\sqrt{\frac{n}{k}}|x - x_{kt}|$$

$$\le c + \begin{cases} c\frac{n}{k}\left(p(x) - \frac{k}{n}\right), & \text{if} \quad 1 \le k \le n\min(p(x), 1/2), \\ 0, & \text{if} \quad n\min(p(x), 1/2) < k \le [n/2]\,, \end{cases}$$

$i = 1, \dots, m_k;\ k = 1, \dots, [n/2]$, where again x_{kt} is the nearest node to x on the right. Thus we obtain, using Lemma 1 with $\alpha = 1, \beta = 2$,

$$A \le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \sum_{k=1}^{n\min(p(x),1/2)} \frac{n}{k} \left(p(x) - \frac{k}{n} \right)^2 b_{nk}(p(x))$$
$$\le c\sqrt{n} + \frac{cn}{\sqrt{p(x)}} \left(\sqrt{\frac{p(x)}{n}} + \frac{1}{n} \right) \le c\sqrt{n}\,,$$

since the sum appears only if $p(x) \ge 1/n$.

Finally, since $|\ell_{ki}(x)| \le c\sqrt{n}$, consequently $|\ell'_{ki}(x)| \le c\sqrt{n}$, we obtain

$$B \le c\sqrt{n} \sum_{k=0}^{n} b_{nk}(p(x)) \le c\sqrt{n}\,. \qquad \square$$

Lemma 3. *We have*

$$\|\varphi B_n'(f)\|_{J_s} \le c(\|\varphi f'\|_{J_s} + V(f))$$

for all $f' \in C(J_s)$.

Remark. The example of a piecewise constant function shows that the second term on the right-hand side is necessary.

Proof of Lemma 3. Similarly as in the proof of Lemma 2, by (6) and (7) we obtain

$$\begin{aligned}\varphi(x)|B_n'(f,x)| \le & c\varphi(x)|p'(x)| \sum_{k=0}^{n} \left(\sum_{i=1}^{m_k} |f(x) - f(x_{ki})| \cdot |\ell_{ki}(x)| \right) |b'_{nk}(p(x))| \\ & + c\varphi(x) \sum_{k=0}^{n} \left(\sum_{i=1}^{m_k} |f(x) - f(x_{ki})| \cdot |\ell'_{ki}(x)| \right) b_{nk}(p(x))\,. \end{aligned} \tag{8}$$

Since $\varphi(x) \ge \sqrt{x - a_j}$, we have

$$\begin{aligned} \varphi(x)|f(x) - f(x_{ki})| &= \varphi(x) \left| \int_{x_{ki}}^{x} f'(t)\,dt \right| \\ \le c\varphi(x)\|\varphi f'\|_{J_s} \int_{x_{ki}}^{x} \frac{dt}{\varphi(t)} &\le c|x - x_{ki}| \cdot \|\varphi f'\|_{J_s}, \qquad i \in S_k\,. \end{aligned} \tag{9}$$

Hence (8) will take the form

$$\begin{aligned} \varphi(x)|B_n'(f,x)| \le c\|\varphi f'\|_{J_s} \sum_{k=0}^{n} \Bigg\{ & \left(\sum_{i \in S_k} |(x - x_{ki})\ell_{ki}(x)| \right) p'(x)|b'_{nk}(p(x))| \\ & + \left(\sum_{i \in S_k} |(x - x_{ki})\ell'_{ki}(x)| \right) b_{nk}(p(x)) \Bigg\} \\ + cV(f)\varphi(x) \sum_{k=0}^{n} \Bigg\{ & \left(\sum_{i \notin S_k} |\ell_{ki}(x)| \right) p'(x)|b'_{nk}(p(x))| + \left(\sum_{i \notin S_k} |\ell'_{ki}(x)| \right) b_{nk}(p(x)) \Bigg\}\,. \end{aligned}$$

Let

$$A_k := p'(x) \sum_{i \in S_k} |(x - x_{ki})\ell_{ki}(x)|, \quad B_k := \sum_{i \in S_k} |(x - x_{ki})\ell'_{ki}(x)|,$$

$$C_k := p'(x)\varphi(x) \sum_{i \notin S_k} |\ell_{ki}(x)|, \quad D_k := \varphi(x) \sum_{i \notin S_k} |\ell'_{ki}(x)|.$$

Let first $k = 0$. Then $A_0 \sim C_0 \leq c(x - x_{0t})$, and both contribute

$$A_0|b'_{n0}(p(x))| \sim C_0|b'_{n0}(p(x))| \leq$$
$$\leq cnp'(x)(x - x_{0t})(1 - p(x))^{n-1} \leq cnp(x)(1 - p(x))^{n-1} \leq c.$$

Similarly, $B_0 \sim D_0 \leq c$, and their contribution is also majorized by constant.

Now let $1 \leq k \leq n/2$. Assume first that $a_j = \xi < x < \eta$ (i.e. x is at the left "edge" of the interval I_j). Then

$$A_k \sim B_k \leq c\left|p(x) - \frac{k}{n}\right|, \quad C_k \leq c\sqrt{\frac{np(x)}{k}}\left|p(x) - \frac{k}{n}\right|, \quad D_k \sim \sqrt{\frac{np(x)}{k}},$$

Thus summing up for $1 \leq k \leq n$ using these estimates and the relation $|b'_{nk}(x)| \leq c\frac{n}{x}\left|x - \frac{k}{n}\right| b_{nk}(x)$ we get

$$\sum_{k=1}^{n} A_k b'_{nk}(p(x)) \leq c\frac{n}{p(x)} \sum_{k=1}^{n} \left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x)) \leq c$$

(cf. (5) with $\gamma = 2$),

$$\sum_{k=1}^{n} B_k b_{nk}(p(x)) \leq c \sum_{k=1}^{n} \left|p(x) - \frac{k}{n}\right| b_{nk}(p(x)) \leq c\sqrt{\frac{p(x)}{n}}$$

(cf. (5) with $\gamma = 1$),

$$\sum_{k=1}^{n} C_k |b'_{nk}(p(x))| \leq c\frac{n}{p(x)} \sum_{k=1}^{n} \sqrt{\frac{np(x)}{k}} \left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x))$$

$$\leq \frac{cn}{\sqrt{p(x)}} \left(\sum_{k=1}^{n} \frac{k}{n}\left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x)) \sum_{k=1}^{n} \left(p(x) - \frac{k}{n}\right)^2 b_{nk}(p(x))\right)^{1/2}$$

$$\leq \frac{cn}{\sqrt{p(x)}} \left(\frac{1}{n} \cdot \frac{p(x)}{n}\right)^{1/2} \leq c$$

(cf. Lemma 1 with $\alpha = 1,\ \beta = 2$), and

$$\sum_{k=1}^{n} D_k b_{nk}(p(x)) \le c \sum_{k=1}^{n} \sqrt{\frac{np(x)}{k}} b_{nk}(p(x)) \le \sqrt{p(x)} \sum_{k=1}^{n} \left(\frac{n}{k} b_{nk}(p(x))\right)^{1/2} \le c$$

(cf. Lemma 1 with $\alpha = 1,\ \beta = 0$).

Finally, let $a_j < \xi < x < \eta$, i.e., x is "inside" the interval I_j. Since now $p'(x) \le c\sqrt{p(x)}$, we have

$$A_k \sim C_k \le c\sqrt{p(x)} \left(\sqrt{\frac{n}{k}} (x - x_{kt})^2 + x - x_{kt} \right), \quad B_k \sim D_k \le c\sqrt{\frac{n}{k}} (x - x_{kt}) + 1 .$$

Using previous estimates for $|x - x_{kt}|$ we get

$$\sum_{k=1}^{n} A_k |b'_{nk}(p(x))| \le c \frac{n}{\sqrt{p(x)}} \sum_{k=1}^{n} \left(\sqrt{\frac{n}{k}} (x - x_{kt})^2 + x - x_{kt} \right) \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x))$$

$$\le \frac{cn}{\sqrt{p(x)}} \sum_{1 \le k \le np(x)} \left[\left(\frac{n}{k}\right)^{3/2} \left| p(x) - \frac{k}{n} \right|^3 + \sqrt{\frac{n}{k}} \left(p(x) - \frac{k}{n} \right)^2 \right] b_{nk}(p(x))$$

$$+ \frac{cn}{p(x)} \sum_{np(x) \le k \le n} \left(p(x) - \frac{k}{n} \right)^2 b_{nk}(p(x)) \le cn \sum_{k=1}^{n} \left(\frac{n}{k}\right)^2 \left| p(x) - \frac{k}{n} \right|^3 b_{nk}(p(x))$$

$$+ \frac{n}{\sqrt{p(x)}} \left(\sum_{k=1}^{n} \frac{n}{k} \left(p(x) - \frac{k}{n} \right)^2 b_{nk}(p(x)) \right)^{1/2} + c$$

$$\le cn \left(\frac{1}{n} + \frac{1}{\sqrt{x} n^{3/2}} \right) + \frac{1}{\sqrt{np(x)}} + c \le c .$$

And finally,

$$\sum_{k=1}^{n} \left[\sqrt{\frac{n}{k}} + 1 \right] b_{nk}(p(x)) \le c \sum_{1 \le k \le np(x)} \frac{n}{k} \left| p(x) - \frac{k}{n} \right| b_{nk}(p(x))$$

$$+ \sum_{np(x) \le k \le n} b_{nk}(p(x)) + c \le c .$$

□

Proof of Theorem 2. The implication "$\Longleftarrow$" follows from Theorem 1. The proof of the other direction is modelled after the proof of (9.3.3) in [1] for the classic Bernstein-type operators. Define the K-functional by

$$K_\varphi(f,t) := \inf(\|f-g\|_{J_s} + t\|\varphi f'\|_{J_s}),$$

where the infimum is taken over all functions g which are absolutely continuous on each interval I_j. By applying Lemmas 2 and 3 for *fig* and g, respectively, we get

$$\begin{aligned}K_\varphi\left(f,\frac{1}{\sqrt{n}}\right) &\le \|f-B_k(f)\|_{J_s} + \frac{1}{\sqrt{n}}\|\varphi B_k'(f)\|_{J_s}\\ &\le \|f-B_k(f)\|_{J_s} + \frac{c}{\sqrt{n}}\left(\sqrt{k}\|f-g\|_{J_s} + \|\varphi g'\|_{J_s} + c\right)\\ &\le \|f-B_k(f)\|_{J_s} + \frac{c}{\sqrt{k}} + c\sqrt{\frac{k}{n}}K_\varphi\left(f,\frac{1}{\sqrt{k}}\right)\\ &\le \frac{c}{k^{\alpha/2}} + c\sqrt{\frac{k}{n}}K_\varphi\left(f,\frac{1}{\sqrt{k}}\right), \qquad 0<\alpha<1.\end{aligned}$$

Applying the Berens–Lorentz lemma (cf. Lemma 9.3.4 from [1]) with $r=1/2$ we get

$$K_\varphi\left(f,\frac{1}{\sqrt{n}}\right) \le \frac{c}{n^{\alpha/2}}, \quad \text{whence} \quad K_\varphi(f,t)\le ct^\alpha, \qquad 0<\alpha<1.$$

This easily implies $\omega_\varphi(f,t)\le t^\alpha$ (cf. the proof in Sect. 2.4 in [1]). □

We do not know if the operator B_n is saturated with $O(n^{-1/2})$. The following example seems to support this conjecture.

Example. Let $m-s\ge 1$. Then there are $m-s$ extremal points of $p(x)$ inside J_s. Assume that there exists an extremal point such that $p(x)=1$, and let

$$f_0(x) = \prod_{i=1}^{m_n}(x-x_{ni}) \in \Pi_{m_n}, \qquad m_n = \left[\frac{m+s}{2}\right].$$

Then

$$\|f_0(x) - B_n(f_0,x)\|_{J_s} \sim \frac{1}{\sqrt{n}}.$$

Here the upper estimate follows from Theorem 1. To prove the lower estimate, note that

$$L_k(f_0, x) = \begin{cases} f_0(x), & \text{if} \quad k = 0, \dots, n-1, \\ 0, & \text{if} \quad k = n\,. \end{cases}$$

Thus

$$f_0(x) - B_n(f_0, x) = f_0(x) p(x)^n\,.$$

Now let $x_{nr} \in \text{int} J_s$ be such that $p(x_{nr}) = 1$ and let $x = x_{nr} + \frac{1}{\sqrt{n}}$. Then

$$|f_0(x) - B_n(f_0, x)| \geq c(x - x_{nr})(1 - |p(x_{nr}) - p(x)|)^n$$
$$\geq c(x - x_{nr})(1 - c(x - x_{nr})^2)^n \geq \frac{c}{\sqrt{n}}\,.$$

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Best Approximation by Logarithmically Concave Classes of Functions

Dimiter Dryanov

Abstract The paper contains results on best approximation by logarithmically concave classes of functions. For example, we prove the following: Let $\mathscr{P}_c$ denote the class of real polynomials, having -1 and 1 as consecutive zeros, and whose zeros $z_k = x_k + \mathbf{i}y_k$, $\mathbf{i}^2 = -1$, satisfy the inequality $|y_k| \le |x_k| - 1$. Let $i(x) = 1,\ x \in [-1, 1]$ be the unit function on the interval $[-1, 1]$ and $1 \le p < \infty$. Then, there exists a unique constant c_p such that

$$\inf_{q\in\mathscr{P}_c} \int_{-1}^{1} |i(x) - q(x)|^p dx = \int_{-1}^{1} |i(x) - c_p(1 - x^2)|^p dx.$$

The exact values of the best approximation are found in the particular cases $p = 1$ and $p = 2$.

Keywords Uniqueness of best approximation • Logarithmically concave classes of functions • Polynomials with only real zeros • Laguerre–Pólya class of entire functions

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In a lecture given at the Université de Montreal by J.G. Clunie, the following proposition has been formulated as crucial in the proof of a necessary and sufficient condition for an entire function to have only real zeros.

Proposition 1. *Let $\mathscr{P}_r$ denote the class of polynomials with only real zeros, having -1 and 1 as consecutive zeros. Then for the best $L_1[-1, 1]$ approximation of the unit function $i(x) = 1,\ x \in [-1, 1]$ by polynomials $q \in \mathscr{P}_r$ the following estimate from below holds:*

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$$\inf_{q\in\mathscr{P}_r}\int_{-1}^{1}|i(x)-q(x)|dx > 0.1.$$

Proposition A says that the best $L_1[-1,1]$ approximation of the unit function i by polynomials from the class $\mathscr{P}_r$ is bounded from below by a positive constant.

Question Asked by Q.I. Rahman Inspired by the result formulated in Proposition 1, Q.I. Rahman asked the following question: Let $0 < p < \infty$. Find the best $L_p[-1,1]$ approximant to the unit function i by polynomials q from the class $\mathscr{P}_r$. In other words, given $p \in (0,\infty)$, find $q^* \in \mathscr{P}_r$ such that

$$\inf_{q\in\mathscr{P}_r}\int_{-1}^{1}|i(x)-q(x)|^p dx = \int_{-1}^{1}|i(x)-q^*(x)|^p dx\,. \tag{1}$$

Note that the class $\mathscr{P}_r$ is not a linear space and the usual theorems and techniques when approximating by linear spaces of functions are not applicable. Next, if q is negative on $(-1,1)$, then obviously

$$\int_{-1}^{1}|i(x)-[-q(x)]|^p dx < \int_{-1}^{1}|i(x)-q(x)|^p dx, \quad p>0$$

and in view of this, if $\mathscr{P}_{r,+}$ denote the polynomials from $\mathscr{P}_r$ that are positive on $(-1,1)$, then

$$\inf_{q\in\mathscr{P}_r}\int_{-1}^{1}|i(x)-q(x)|^p dx = \inf_{q\in\mathscr{P}_{r,+}}\int_{-1}^{1}|i(x)-q(x)|^p dx.$$

Generalization of the Problem We consider a generalization of the above question by using a characteristic property of the approximation polynomial class $\mathscr{P}_r$: The polynomials with only real zeros are logarithmically concave on any interval bounded by two consecutive zeros. Namely, if $q \in \mathscr{P}_r$ with zeros $x_1, x_2, \ldots, x_n$, then

$$\frac{d^2 \ln|q(x)|}{dx^2} = \left[\frac{q'(x)}{q(x)}\right]' = -\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} < 0$$

at each point x where it is defined. Taking this property of polynomials belonging to $\mathscr{P}_r$ as a characteristic one, we generalize the approximation problem (1) by considering as an approximation tool the class $\mathscr{RD}_{lc,+}$ of all functions of the form $f(x) = (1-x^2)\psi(x)$, where $\psi(x)$ is logarithmically concave on $(-1,1)$. Without any restriction we can suppose that $\psi(x) > 0$ on $(-1,1)$.

Note also that the polynomial class $\mathscr{P}_{r,+}$ is a subclass of the function class $\mathscr{RD}_{lc,+}$.

Logarithmically Concave Polynomials with Complex Zeros Let $\mathscr{P}_c$ denote the class of real polynomials (with real coefficients), having -1 and 1 as consecutive zeros, and whose zeros $z_k = x_k + \mathbf{i}y_k$, $\mathbf{i}^2 = -1$ satisfy the inequality $|y_k| \le |x_k| - 1$. Then, each polynomial $q \in \mathscr{P}_c$ has the form

$$q(x) = c(1-x^2)\prod_{j=1}^{m_1}(x-x_j)\prod_{k=1}^{m_2}(x-z_k)(x-\bar{z}_k) = (1-x^2)\psi(x).$$

Analogously to the above considerations we have

$$\inf_{q\in\mathscr{P}_c}\int_{-1}^{1}|i(x)-q(x)|^p dx = \inf_{q\in\mathscr{P}_{c,+}}\int_{-1}^{1}|i(x)-q(x)|^p dx,$$

where $\mathscr{P}_{c,+}$ denote the polynomials from $\mathscr{P}_c$ that are positive on $(-1,1)$. In other words $q \in \mathscr{P}_{c,+}$ if $\psi(x) > 0$ for $|x| < 1$. Then

$$\frac{d^2 \ln|\psi(x)|}{dx^2} = \left[\frac{\psi'(x)}{\psi(x)}\right]' = -\sum_{j=1}^{m_1}\frac{1}{(x-x_j)^2}$$

$$-2\sum_{k=1}^{m_2}\frac{(x-x_k)^2 - y_k^2}{(x^2-2xx_k+x_k^2+y_k^2)^2} < 0 \text{ for } |x| < 1.$$

In view of this, each polynomial from $\mathscr{P}_c$ is logarithmically concave in $(-1,1)$. Hence, $\mathscr{P}_{c,+}$ is also a subclass of $\mathscr{RD}_{lc,+}$.

Laguerre–Pólya Class of Entire Functions A real entire function is said to belong to Laguerre–Pólya class, $\mathscr{LP}$ for short, if it is a local uniform limit in the complex plane of a sequence of polynomials with only real zeros; see [4, 5] and the references given there for additional facts about $\mathscr{LP}$ class of entire functions. Let us denote by $\mathscr{LP}_{1,+}$ the set of functions in $\mathscr{LP}$ having -1 and 1 as consecutive zeros and positive on $(-1,1)$. Each function $f(x)$ from $\mathscr{LP}_{1,+}$ can be written (see [4, 5]) as $f(x) = (1-x^2)\psi(x)$, where $\psi(x) > 0$ on $(-1,1)$ and $\psi(x)$ has the following representation:

$$\psi(x) = ce^{-ax^2+bx}\prod_{k=1}^{\infty}(1-t_kx)e^{t_kx} \quad (-1 \le t_k \le 1, c > 0, a \ge 0, b \text{ real})$$

such that $\sum_{k=1}^{\infty} t_k^2 < \infty$. Note that ψ is logarithmically concave between two consecutive zeros hence, $\mathscr{LP}_{1,+}$ is a subclass of $\mathscr{RD}_{lc,+}$. Note also that $f \in \mathscr{LP}_{1,+}$ if and only if f is a local inform limit in the complex plane of a sequence of polynomials in $\mathscr{P}_{r,+}$.

Let $\widetilde{\mathscr{LP}}_{1,+}$ denote the subclass of $\mathscr{LP}_{1,+}$ consisting of entire functions of the form $c(1-x^2)e^{\rho x}$, where $c > 0$ and ρ is real.

Theorem 1. *Let $0 < p < \infty$. Then the following holds:*

$$\inf_{f \in \mathscr{DR}_{lc,+}} \int_{-1}^{1} |i(x) - f(x)|^p dx = \inf_{e \in \mathscr{LP}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx$$
$$= \inf_{e \in \widetilde{\mathscr{LP}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx.$$

Proof. Let $p \in (0, \infty)$. Note that $\widetilde{\mathscr{LP}}_{1,+}$ is a subclass of $\mathscr{DR}_{lc,+}$ being a subclass of $\mathscr{LP}_{1,+}$. Let $f \in \mathscr{DR}_{lc,+}$ hence, $f(x) = (1 - x^2)\psi(x)$ but f is not in the class $\widetilde{\mathscr{LP}}_{1,+}$; in other words, $\psi(x)$ is not in the form $c\, e^{\rho x}$.

Obviously, $1 - f(x) = 0$ if and only if $\ln \psi(x) = -\ln(1 - x^2)$ and since the left side of this equation is concave and the right side is strictly convex in $(-1, 1)$, this equation can have at most two solutions in $(-1, 1)$. This means that the curve $y = f(x)$ intersects the line $y = 1$ either twice, or touches it, or remains below it.

Case 1. Suppose that the equation $\ln \psi(x) = -\ln(1 - x^2)$ has two solutions $x_1 < x_2$ in $(-1, 1)$. Consider the line $y = \rho x + b$ interpolating the points $(x_1, -\ln(1 - x_1^2))$ and $(x_2, -\ln(1 - x_2^2))$

$$y = \rho x + b = -\ln(1 - x_1^2) + \frac{x - x_1}{x_2 - x_1} \ln \frac{1 - x_1^2}{1 - x_2^2}.$$

Replacing $\ln \psi(x)$ by $\rho x + b$ in $|1 - (1 - x^2)e^{\ln \psi(x)}|$ we obtain a better pointwise approximation to the unit function $i(x)$ on the interval $[-1, 1]$. The function $-\ln(1 - x^2)$ is strictly convex and $\ln \psi$ is concave hence on the intervals $[-1, x_1)$ and $(x_2, 1]$ we have

$$\ln \psi(x) \le \rho x + b \le -\ln(1 - x^2) \quad \Leftrightarrow \quad (1 - x^2)\psi(x) \le (1 - x^2)e^{\rho x + b} < 1$$

and because $f \notin \widetilde{\mathscr{LP}}_{1,+}$, the first inequality is strict on some non-degenerate subinterval $[\alpha, \beta]$ contained in $[-1, x_1) \cup (x_2, 1]$. On the other hand, on (x_1, x_2) we have

$$\ln \psi(x) \ge \rho x + b > -\ln(1 - x^2) \quad \Leftrightarrow \quad (1 - x^2)\psi(x) \ge (1 - x^2)e^{\rho x + b} > 1.$$

Hence,

$$\left|i(x) - (1 - x^2)\psi(x)\right|^p \ge \left|i(x) - (1 - x^2)e^{\rho x + b}\right|^p \quad \text{for all} \quad x \in [-1, 1]$$

with strict inequality on (α, β).

Case 2. The equation $\ln\psi(x) = -\ln(1-x^2)$ has a unique solutions x_0 in $(-1,1)$. In this case we consider the line $y = \rho x + b$ that is tangent to the curve $y = -\ln(1-x^2)$ at the point $(x_0, -\ln(1-x_0^2))$. Similar to the Case 1 we conclude that the inequality

$$\left|i(x) - (1-x^2)\psi(x)\right|^p \geq \left|i(x) - (1-x^2)e^{\rho x+b}\right|^p \quad \text{for all} \quad x \in [-1,1]$$

holds and this inequality is strict on some non-degenerate subinterval of $(-1,1)$.

Case 3. The equation $\ln\psi(x) = -\ln(1-x^2)$ does not have a solution in $(-1,1)$ that is equivalent to

$$(1-x^2)\psi(x) < 1 \quad \text{for all} \quad x \in [-1,1].$$

In this case the graphs of the functions $y = -\ln(1-x^2)$ and $y = \ln\psi(x)$ do not have common points. In addition, the function $y = -\ln(1-x^2)$ is convex but the function $y = \ln\psi(x)$ is concave. Hence, there exists a line $y = \rho x + b$ that separates the graphs of both functions, that is

$$\ln\psi(x) < \rho x + b < -\ln(1-x^2) \quad \text{for all} \quad x \in [-1,1].$$

Hence, taking into account that $p > 0$ we have the strict inequality

$$\left|i(x) - (1-x^2)\psi(x)\right|^p > \left|i(x) - (1-x^2)e^{\rho x+b}\right|^p \quad \text{for all} \quad x \in [-1,1]\,.$$

In view of the above three cases we conclude the following:
If $f(x) = (1-x^2)\psi(x) \in \mathscr{D}\mathscr{R}_{lc,+}$ but f is not in the class $\widetilde{\mathscr{L}\mathscr{P}}_{1,+}$, then substituting $e^b = c > 0$ we obtain

$$\int_{-1}^{1} |i(x) \ -f(x)|^p \ dx > \int_{-1}^{1} \left|i(x) - c(1-x^2)e^{\rho x}\right|^p dx.$$

We end the proof of Theorem 1 by observing that

$$\widetilde{\mathscr{L}\mathscr{P}}_{1,+} \subset \mathscr{L}\mathscr{P}_{1,+} \subset \mathscr{D}\mathscr{R}_{lc,+}\,.$$

Remark. Substituting $v = -x$ we obtain

$$\int_{-1}^{1} |i(x) - c(1-x^2)e^{\rho x}|^p dx = \int_{-1}^{1} |i(v) - c(1-v^2)e^{-\rho v}|^p dv$$

and in view of this

$$\inf_{e \in \widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx = \inf_{c \geq 0, \rho \geq 0} \int_{-1}^{1} |i(x) - c(1-x^2)e^{\rho x}|^p dx\,.$$

How to Proceed Following Theorem 1, we attempt to find the best $L_p[-1,1]$ approximants to the unit function i from $\mathscr{RD}_{lc,+}$ by studding the best $L_p[-1,1]$ approximation of the unit function i from the $(c,\ \rho)$-parametric class $\widetilde{\mathscr{LP}}_{1,+}$. Note that $\widetilde{\mathscr{LP}}_{1,+}$ is a subclass of $\mathscr{RD}_{lc,+}$.

Next theorem gives existence and uniqueness of the best $L_p[-1,1]$ approximant, $0 < p < \infty$, to i from the linear space $\{c(1-x^2),\ -\infty < c < \infty\}$.

Theorem 2. *Let $0 < p < \infty$. Then, there exists a unique constant $c_p \in (-\infty, \infty)$ such that*

$$\inf_{-\infty<c<\infty} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx = \int_{-1}^{1} |i(x) - c_p(1-x^2)|^p dx.$$

In addition, $c_p > 1$.

Proof. Let $p \in (0,\infty)$ be a fixed number. Obviously

$$\inf_{-\infty<c<\infty} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx = \inf_{c\geq 0} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx.$$

We have to show that there exists a unique $c_p \geq 0$ such that

$$\inf_{c\geq 0} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx = \int_{-1}^{1} |i(x) - c_p(1-x^2)|^p dx.$$

Consider the p-parametric function of the variable c

$$I_p(c) = \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx.$$

For $0 < p < 1$, $L_p[-1,1]$ is a metric space and in view of this

$$|I_p(c_1) - I_p(c_2)| \leq |c_1 - c_2|^p \int_{-1}^{1} (1-x^2)^p dx\,.$$

For $p \geq 1$, $L_p[-1,1]$ is a normed space hence,

$$|[I_p(c_1)]^{1/p} - [I_p(c_2)]^{1/p}| \leq |c_1 - c_2| \left(\int_{-1}^{1} (1-x^2)^p dx\right)^{1/p}.$$

By the above two inequalities we conclude that the function $I_p(c)$ is continuous in $c \in (0,\infty)$.

In addition, it is obvious that $I_p(c) \to \infty$ as $c \to \infty$ and from here we conclude that there is c_p^* such that

$$\inf_{c\geq 0} I_p(c) = \inf_{0\leq c\leq c_p^*} I_p(c) = \min_{0\leq c\leq c_p^*} I_p(c) = \min_{1\leq c\leq c_p^*} I_p(c)$$

where c_p^* is a positive constant which must be greater than or equal to 1 ($c_p \geq 1$), taking into account that $I_p(c)$ is a decreasing function for $0 \leq c \leq 1$. Now, $I_p(c)$ is continuous in $c \in (0, \infty)$ and $[1, c_p^*]$ is a compact set hence, by the Extreme Value Theorem [6] we conclude that there exists $c_p \in [1, c_p^*]$ such that

$$\inf_{c\geq 0} I_p(c) = \min_{1\leq c\leq c_p^*} I_p(c) = I_p(c_p).$$

Hence, the infimum of $I_p(c)$ over the open interval $(0, \infty)$ is attained. Then, the point of minimum c_p must be a critical number, i.e., to satisfy the normal equation

$$\left.\frac{dI_p(c)}{d\,c}\right|_{c=c_p} = 0$$

that is equivalent to

$$\left.\int_{-1}^{1} \left|1 - c(1-x^2)\right|^{p-1} \operatorname{sign}\left[1 - c(1-x^2)\right]\,(1-x^2)dx\right|_{c=c_p} = 0.$$

The above equality can hold only if the integrand changes sign in $(-1, 1)$ and in view of this, c_p must be greater than 1. Therefore, for $c > 1$, the even curve $y = c(1-x^2)$ intersects the even line $y = 1$ exactly twice, say at $x = -\xi$ and $x = \xi$ for some $\xi \in (0, 1)$. Then, $c = 1/(1-\xi^2)$ and it will be sufficient to show that

$$\min_{\xi\in(0,1)} \omega(\xi) = \min_{\xi\in(0,1)} \int_{-1}^{1} \left|\frac{x^2-\xi^2}{1-\xi^2}\right|^p dx = \min_{\xi\in(0,1)} 2\int_{0}^{1} \left|\frac{x^2-\xi^2}{1-\xi^2}\right|^p dx$$

is attained for only one value $\xi_p \in (0, 1)$, where $c_p = 1/(1-\xi_p^2)$. Substituting $\eta = \xi^2$ and $u = x^2$ in the above equation we obtain an equivalent minimum problem in terms of $\eta \in (0, 1)$:

$$\min_{\xi\in(0,1)} \omega(\xi) = \min_{\eta\in(0,1)} \Omega(\eta) = \min_{\eta\in(0,1)} \int_{0}^{1} u^{-1/2} \left|\frac{u-\eta}{1-\eta}\right|^p du.$$

Hence, in order to solve the initial c minimum problem we may show that the above minimum of $\Omega(\eta)$ over $\eta \in (0, 1)$ is attained for only one value $\eta_p \in (0, 1)$.

The minimum of $\Omega(\eta)$ can occur only at the roots of the normal equation $d\Omega/d\eta = 0$ that is

$$\int_0^1 u^{-1/2} \left|\frac{u-\eta}{1-\eta}\right|^{p-1} \operatorname{sign}\left[\frac{u-\eta}{1-\eta}\right] \frac{u-1}{(1-\eta)^2} du = 0$$

and since $0 < \eta < 1$, the above equation is equivalent to

$$\int_0^1 (1-u)u^{-1/2} \, |u-\eta|^{p-1} \operatorname{sign}[u-\eta] \, du = 0 \, .$$

Now, replacing $\eta \in (0,1)$ by $(1-\beta)/2, \ -1 < \beta < 1$ and making the substitution $u = (1-v)/2$ in the above integral, we obtain the following equivalent equation:

$$\int_{-1}^1 (1+v)(1-v)^{-1/2}|v-\beta|^{p-1}\operatorname{sign}[v-\beta] \, dv = 0, \quad \beta \in (-1,1)$$

and we have to show that it is satisfied for only one value $\beta_p \in (-1,1)$. Substituting $\beta = \cos\theta_1, -\pi < \theta_1 < \pi$ and $v = \cos\theta$, it is easily seen that the above equation is equivalent to the following equation, involving trigonometric integral (taking into account that the integrand is even trigonometric function)

$$\int_{-\pi}^{\pi} \lambda(\theta) \, \cos\frac{\theta}{2} \, d\theta = 0, \tag{2}$$

where

$$\lambda(\theta) = \cos^2\frac{\theta}{2} \left|\sin\frac{\theta+\theta_1}{2} \, \sin\frac{\theta-\theta_1}{2}\right|^{p-1} \operatorname{sign}\left[\sin\frac{\theta+\theta_1}{2} \, \sin\frac{\theta-\theta_1}{2}\right].$$

Observing that the function $\lambda(\theta)$ is even and $\sin(\theta/2)$ is odd, the next equality is satisfied for all $\theta_1 \in [-\pi, \pi]$.

$$\int_{-\pi}^{\pi} \lambda(\theta) \, \sin\frac{\theta}{2} \, d\theta = 0, \tag{3}$$

Now we shall use the following result contained in [2]: Let m be positive integer. Let $\alpha > 0$ and $\alpha_\mu > 0$, $\mu = 0, 1, \ldots, m$. Then, there exists a unique

$$(t_1^*, \ldots, t_m^*) \in \{\bar{t} = (t_1, \ldots, t_m) \, : \, -\pi \le t_1 \le \cdots \le t_m \le \pi\}$$

that satisfies the following system of equation with respect to $t_1, \ldots, t_m$:

$$\int_{-\pi}^{\pi} \left|\cos\frac{\theta}{2}\right|^{\alpha(\alpha_0-1)} \prod_{\mu=1}^{m} \left|\sin\frac{\theta - t_\mu}{2}\right|^{\alpha(\alpha_\mu-1)}$$

$$\times \operatorname{sign}\left[\prod_{\mu=1}^{m} \sin\frac{\theta - t_\mu}{2}\right] e^{\mathbf{i}\frac{(m-1)-2k}{2}}\, d\theta, \quad k = 0, \ldots, m-1,\ \mathbf{i}^2 = -1.$$

In addition, the unique solution $(t_1^*, \ldots, t_m^*)$ satisfies $-\pi < t_1^* < \cdots < t_m^* < \pi$.

The particular case $m = 2$, $\alpha = 1$, $\alpha_0 = 3$, $\alpha_1 = \alpha_2 = p > 0$ of the above result (see [2] for details) gives that the system of non-linear equations

$$\int_{-\pi}^{\pi} \cos^2\frac{\theta}{2} \left|\sin\frac{\theta - t_1}{2} \sin\frac{\theta - t_2}{2}\right|^{p-1} \operatorname{sign}\left[\sin\frac{\theta - t_1}{2} \sin\frac{\theta - t_2}{2}\right] e^{\pm\mathbf{i}\frac{\theta}{2}}\, d\theta \qquad (4)$$

has a unique solution $-\pi < t_1 < t_2 < \pi$.

Substituting θ by $-\theta$ in (4) we observe that $-\pi < -t_2 < -t_1 < \pi$ is also a solution of the same system. However, the solution of the system is unique and from here we conclude that t_2^* must be equal to $-t_1^*$. Hence, if we substitute $t_2 = \theta_1$, then the unique solution has the form $-\pi < -\theta_1 < \theta_1 < \pi$. In view of this, we can now claim that the system of Eqs. (2), (3) has a unique solution. However, Eq. (3) is satisfied for all values of $\theta_1 \in [-\pi, \pi]$. Hence, Eq. (2) must have a unique solution.

The Case p = 2 Next theorem gives in particular that the unique best $L_2[-1, 1]$ approximant to i from $\widetilde{\mathscr{L}\mathscr{P}}_{1,+}$ must have $\rho = 0$, i.e., it must be of the form $c(1-x^2)$.

Theorem 3. *Let i denote the unit function on the interval* $[-1, 1]$. *Then, there exists a unique constant* $c_2 \in (-\infty, \infty)$ *such that*

$$\inf_{f \in \mathscr{D}\mathscr{R}_{lc,+}} \int_{-1}^{1} [i(x) - f(x)]^2 dx = \inf_{e \in \mathscr{L}\mathscr{P}_{1,+}} \int_{-1}^{1} [i(x) - e(x)]^2 dx$$

$$= \inf_{e \in \widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} [i(x) - e(x)]^2 dx = \inf_{q \in \mathscr{P}_{c,+}} \int_{-1}^{1} [i(x) - q(x)]^2 dx$$

$$= \inf_{q \in \mathscr{P}_{r,+}} \int_{-1}^{1} [i(x) - q(x)]^2 dx = \inf_{-\infty < c < \infty} \int_{-1}^{1} [i(x) - c(1 - x^2)]^2 dx$$

$$= \int_{-1}^{1} [i(x) - c_2(1 - x^2)]^2 dx.$$

In addition, $c_2 = 5/4$ *hence,* $(5/4)(1 - x^2)$ *is the unique best approximant and the exact value of the best approximation is*

$$\int_{-1}^{1} \left[i(x) - \frac{5}{4}(1 - x^2)\right]^2 dx = \frac{1}{3}.$$

Proof. Denote

$$I(c,\rho) = \int_{-1}^{1} (1 - c(1-x^2)\, e^{\rho x})^2\, dx.$$

First obviously, we have

$$\inf_{-\infty<c<\infty,\, -\infty<\rho<\infty} I(c,\rho) = \inf_{c\geq 0,\, \rho\geq 0} I(c,\rho).$$

Let $\rho \in (-\infty,\infty)$ be fixed. Observing that $I(c,\rho)$ tends to infinity as $c \to \infty$ and $I(c,\rho)$ is a decreasing function of c in a *small* interval of the form $[0,\delta_\rho]$, $\delta_\rho > 0$ we conclude that, for a fixed ρ, there exists a positive number c_ρ^* such that

$$\inf_{c\geq 0} I(c,\rho) = \min_{\delta_\rho \leq c \leq c_\rho^*} I(c,\rho).$$

The minimum points on the right side of the above equality must be critical numbers for $I(c,\rho)$ hence, they must be roots of the normal equation

$$\frac{\partial I}{\partial c} = -2\int_{-1}^{1} [1 - c(1-x^2)e^{\rho x}](1-x^2)e^{\rho x}\, dx = 0$$

and because the above equation is linear in c, it can be solved for c to obtain that the unique minimum point $c(\rho) \in (-\infty,\infty)$, located in $[\delta_\rho, c_\rho^*]$ is determined by

$$c(\rho) = \frac{\int_{-1}^{1}(1-x^2)e^{\rho x}\, dx}{\int_{-1}^{1}(1-x^2)^2 e^{2\rho x}\, dx}.$$

Then,

$$\begin{aligned}\min_{\delta_\rho \leq c \leq c_\rho^*} I(c,\rho) = I(c(\rho),\rho) &= \int_{-1}^{1} \left[1 - 2c(\rho)(1-x^2)e^{\rho x} + c^2(\rho)(1-x^2)^2 e^{2\rho x}\right] dx \\ &= 2 - \frac{\left[\int_{-1}^{1}(1-x^2)e^{\rho x}\, dx\right]^2}{\int_{-1}^{1}(1-x^2)^2 e^{2\rho x}\, dx}\end{aligned}$$

Consider

$$I(c(\rho),\rho) = 2 - \frac{\left[\int_{-1}^{1}(1-x^2)e^{\rho x}\, dx\right]^2}{\int_{-1}^{1}(1-x^2)^2 e^{2\rho x}\, dx}.$$

Next, we shall use Maclaurin series representations of entire functions in order to prove the following: If $\rho \neq 0$ is a real number, then

$$I(c(\rho), \rho) > I(c(0), 0)$$

and from here we conclude that

$$\min_{-\infty<\rho<\infty} I(c(\rho), \rho) = I(c(0), 0)$$

and the unique point of minimum of $I(c(\rho), \rho)$ is $\rho = 0$. Suppose that $\rho \neq 0$. Then, straight computations show that

$$\left[\int_{-1}^{1} (1-x^2)e^{\rho x}\,dx\right]^2 = \frac{4}{\rho^4}\left[e^{2\rho}\left(1-\frac{2}{\rho}+\frac{1}{\rho^2}\right)+e^{-2\rho}\left(1+\frac{2}{\rho}+\frac{1}{\rho^2}\right)+2-\frac{2}{\rho^2}\right]$$

and

$$\int_{-1}^{1} (1-x^2)^2 e^{2\rho x}\,dx = \frac{1}{\rho^3}\left[e^{2\rho}\left(1-\frac{3}{2\rho}+\frac{3}{4\rho^2}\right)-e^{-2\rho}\left(1-\frac{3}{2\rho}+\frac{3}{4\rho^2}\right)\right].$$

Since, $c(0) = 5/3$ and $I(c(0), 0) = 1/3$ we need to show that

$$2-\frac{\frac{4}{\rho^4}\left[e^{2\rho}\left(1-\frac{2}{\rho}+\frac{1}{\rho^2}\right)+e^{-2\rho}\left(1+\frac{2}{\rho}+\frac{1}{\rho^2}\right)+2-\frac{2}{\rho^2}\right]}{\frac{1}{\rho^3}\left[e^{2\rho}\left(1-\frac{3}{2\rho}+\frac{3}{4\rho^2}\right)-e^{-2\rho}\left(1-\frac{3}{2\rho}+\frac{3}{4\rho^2}\right)\right]} > \frac{1}{3} \quad \text{for} \quad \rho \neq 0.$$

Taking into account that $\int_{-1}^{1}(1-x^2)^2 e^{2\rho x}\,dx > 0$, the above inequality is equivalent to the inequality

$$G(\rho) > 0, \quad \text{where} \quad G(\rho) = e^{2\rho}\left(5\rho^3 - \frac{39}{2}\rho^2 + \frac{111}{4}\rho - 12\right)$$
$$-e^{-2\rho}\left(5\rho^3 + \frac{39}{2}\rho^2 + \frac{111}{4}\rho + 12\right) + 24(1-\rho^2).$$

Denote,

$$h(\rho) = e^{2\rho}\left(5\rho^3 - \frac{39}{2}\rho^2 + \frac{111}{2}\rho - 12\right).$$

Then

$$G(\rho) = h(\rho) + h(-\rho) + 24(1-\rho^2), \quad G(0) = 0$$

and we have to show that $G(\rho) > 0$ for each $\rho \neq 0$. Obviously, G is an even entire function. In view of this, its Maclaurin expansion has the form

$$G(\rho) = \sum_{k=1}^{\infty} \frac{G^{(2k)}(0)}{(2k)!}\rho^{2k}.$$

Calculating the even order derivatives of h we conclude that

$$h^{(2)}(\rho) = e^{2\rho}(20\rho^3 - 18\rho^2 - 15\rho + 24), \quad g^{(2)}(0) = 2h^{(2)}(0) - 48 \Rightarrow g^{(2)}(0) = 0;$$

$$h^{(4)}(\rho) = 4e^{2\rho}(20\rho^3 + 42\rho^2 - 21\rho), \quad g^{(4)}(0) = 2h^{(4)}(0) = 0;$$

$$h^{(6)}(\rho) = 16e^{2\rho}(20\rho^3 + 102\rho^2 + 93\rho), \quad g^{(6)}(0) = 2h^{(6)}(0) = 0;$$

Observing that $G^{(2k)}(0) = 2h^{(2k)}(0)$ for $k \geq 2$ and taking into account that the coefficients of ρ^3, ρ^2, and ρ in the expression of $h^{(6)}(\rho)$ are positive we make an important conclusion that the coefficients of ρ^3, ρ^2, ρ, and the free term in the expression $h^{(2k)}(\rho)$ for all $k \geq 4$ are all positive.

Hence, for $k \geq 3$

$$h^{(2k)}(\rho) = e^{2\rho}(a_k\rho^3 + b_k\rho^2 + c_k\rho + d_k),\ a_k > 0, b_k > 0, c_k > 0, d_k > 0,$$

where the coefficients $a_k,\ b_k,\ c_k,\ d_k$ are all positive and in view of this

$$G^{(2k)}(0) = 2h^{(2k)}(0) = 2d_k > 0 \quad \text{for} \quad k \geq 4.$$

Thus

$$G(\rho) = \sum_{k=1}^{\infty} \frac{2d_k}{(2k)!}\rho^{2k} > 0 \quad \text{for} \quad \rho \neq 0.$$

The Case $\mathbf{p = 1}$ Next theorem gives in particular that the unique best $L_1[-1,1]$ approximant to i from $\widetilde{\mathscr{L}\mathscr{P}}_{1,+}$ must have $\rho = 0$, i.e., it is of the form $c(1 - x^2)$.

Theorem 4. *Let i denote the unit function on the interval $[-1,1]$. Then, there exists a unique constant $c_1 \in (-\infty, \infty)$ such that*

$$\inf_{f \in \mathscr{D}\mathscr{R}_{lc,+}} \int_{-1}^{1} |i(x) - f(x)|dx = \inf_{e \in \mathscr{L}\mathscr{P}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|dx$$

$$= \inf_{e \in \widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|dx = \inf_{q \in \mathscr{P}_{c,+}} \int_{-1}^{1} |i(x) - q(x)|dx$$

$$= \inf_{q \in \mathscr{P}_{r,+}} \int_{-1}^{1} |i(x) - q(x)|dx = \inf_{-\infty < c < \infty} \int_{-1}^{1} |i(x) - c(1 - x^2)|dx$$

$$= \int_{-1}^{1} |i(x) - c_1(1 - x^2)|dx.$$

In addition, $c_1 = \left[1 - 4\sin^2\left(\frac{\pi}{18}\right)\right]^{-1}$ *hence, the unique best approximant is* $\left[1 - 4\sin^2\left(\frac{\pi}{18}\right)\right]^{-1}(1-x^2)$ *and the exact value of the best approximation is*

$$\int_{-1}^{1} |i(x) - c_1(1-x^2)|dx = 2 - 8\sin\left(\frac{\pi}{18}\right).$$

Proof. Taking into account Theorems 1 and 2 it is sufficient to prove that

$$\inf_{e\in\widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|dx = \inf_{-\infty<c<\infty} \int_{-1}^{1} |i(x) - c(1-x^2)|dx.$$

We have $i(x) = 1$ for $x \in [-1, 1]$, and for simplicity we denote

$$I(c,\rho) = \int_{-1}^{1} \left|i(x) - c(1-x^2)e^{\rho x}\right| dx = \int_{-1}^{1} \left|1 - c(1-x^2)e^{\rho x}\right| dx.$$

It is obvious that minimizing $I(c, \rho)$, without any restriction, we can assume $\rho \geq 0$ and $c \geq 0$.

Let us fix ρ. Then, $I(c, \rho)$ tends to ∞ as c goes to ∞. In addition, $I(c, \rho) < I(0, \rho)$ for all small positive values of c. Hence, the minimum of $I(c, \rho)$ over $c \in [0,\infty)$ is attained at some finite points in the open interval $c \in (0,\infty)$. Hence, a point of minimum (being a critical number) must be a solution of the normal equation

$$-\frac{\partial I}{\partial c} = \int_{-1}^{1} \operatorname{sign}\left[1 - c(1-x^2)e^{\rho x}\right](1-x^2)e^{\rho x}dx = 0. \tag{5}$$

The above equation can be satisfied for some c if the function $\left[1 - c(1-x^2)e^{\rho x}\right]$ changes sign at least once in $(-1, 1)$. However, it must do so exactly twice since it has only one critical number in $(-1, 1)$. We claim that (5) is satisfied for only one value of c. For a fixed c, denote by $\xi_{1,c} < \xi_{2,c}$ the points, where the function $\left[1 - c(1-x^2)e^{\rho x}\right]$ changes sign in $(-1, 1)$. Taking into account that the optimal c is strictly positive, the above normal equation can be written in the following equivalent form:

$$\int_{-1}^{1} \operatorname{sign}\left[\frac{1}{c} - (1-x^2)e^{\rho x}\right](1-x^2)e^{\rho x}dx = 0.$$

Next, consider the planar region bounded by the curve $y = (1-x^2)e^{\rho x}$, the lines $x = -1$, $x = 1$ and the x-axis. Divide this region into three parts by the lines $x = \xi_{1,c}$, $x = \xi_{2,c}$. Denote by $A_1(c)$ the area of the part to the left of $x = \xi_{1,c}$; by

$A_2(c)$ the area of the one lying to the right of $x = \xi_{2,c}$; and by $A_3(c)$ the area of the part in the middle. Then (5) can be written as

$$A_1(c) + A_2(c) - A_3(c) = 0.$$

If $1/c$ is small, then $A_1(c) + A_2(c)$ is small and $A_3(c)$ is large; but as $1/c$ increases, then $A_1(c) + A_2(c)$ increases while $A_3(c)$ decreases. Hence, by the Intermediate Value Theorem, taking into account that $A(c)$ is strictly increasing with $1/c$, there is one and only one value of c for which $A(c) = 0$. Obviously for each $\rho \geq 0$, the unique solution $c(\rho)$ must be greater than 1.

The next steps of the proof are based on a technique similar to what has been developed in [3]. Denote $\xi_1(\rho) = \xi_{1,c(\rho)}$, $\xi_2(\rho) = \xi_{2,c(\rho)}$. Then, we obtain

$$\begin{aligned} \min_{c\geq 0} \int_{-1}^{1} \left|1 - c(1-x^2)e^{\rho x}\right| dx &= \int_{-1}^{1} \left|1 - c(\rho)(1-x^2)e^{\rho x}\right| dx \\ &= \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right]\left[1 - c(\rho)(1-x^2)e^{\rho x}\right] dx \qquad (6) \\ &= \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] dx \\ &= \int_{-1}^{\xi_1(\rho)} dx - \int_{\xi_1(\rho)}^{\xi_2(\rho)} dx + \int_{\xi_2(\rho)}^{1} dx = 2 - 2\left[\xi_2(\rho) - \xi_1(\rho)\right] . \end{aligned}$$

by using the fact that (5) is equivalent to

$$\int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](1-x^2)e^{\rho x} dx = 0. \qquad (7)$$

Next, we show that $\xi_2(\rho) - \xi_1(\rho)$ is a decreasing function of ρ for all $\rho > 0$. For this we use (7) and the two relations

$$c(\rho)\left[1 - \xi_1^2(\rho)\right] e^{\rho \xi_1(\rho)} = 1 \qquad (8)$$

$$c(\rho)\left[1 - \xi_2^2(\rho)\right] e^{\rho \xi_2(\rho)} = 1 \qquad (9)$$

Observe that $\xi_2(\rho) > 0$ for $\rho \geq 0$. Assume to the contrary: $-1 < \xi_1(\rho) < \xi_2(\rho) \leq 0$. Then

$$\begin{aligned} 0 &= \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](1-x^2)e^{\rho x} dx \\ &= \int_{-1}^{0} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](1-x^2)e^{\rho x} dx + \int_{0}^{1} (1-x^2)e^{\rho x} dx \end{aligned}$$

$$> -\int_{-1}^{0}(1-x^2)e^{\rho x} + \int_{0}^{1}(1-x^2)e^{\rho x}dx \geq 0$$

and we get a contradiction. In addition, for $\rho > 0$, by (8) and (9), $(1-\xi_1^2(\rho))/(1-\xi_2^2(\rho)) = e^{\rho[\xi_2(\rho)-\xi_1(\rho)]} > 1$ and in view of this $\xi_2(\rho) > |\xi_1(\rho)|$. Hence,

$$\xi_2(\rho) + \xi_1(\rho) > 0 \quad \text{for} \quad \rho > 0 \quad \text{and evidently,} \quad \xi_2(0) + \xi_1(0) = 0. \tag{10}$$

By the definition of $I(c,\rho)$ we clearly have that $I(c,-\rho) = I(c,\rho)$. Then $c(\rho)$, $\xi_1(\rho)$, and $\xi_2(\rho)$ can be extended to negative values of ρ. We have $c(-\rho) = c(\rho)$, $\xi_1(-\rho) = -\xi_2(\rho)$, and $\xi_2(-\rho) = -\xi_1(\rho)$. In particular, $\xi_2(\rho) - \xi_1(\rho)$ is an even function of ρ.

Using (7)–(9) we compute the Jacobian of the non-linear system

$$F_1(\xi_1,\xi_2,c,\rho) = \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](1-x^2)e^{\rho x}dx = 0;$$

$$F_2(\xi_1,\xi_2,c,\rho) = \ln\left[c(1-\xi_1^2)e^{\rho\xi_1}\right] = 0; F_3(\xi_1,\xi_2,c,\rho) = \ln\left[c(1-\xi_2^2)e^{\rho\xi_2}\right] = 0$$

to obtain

$$\frac{\partial(F_1,F_2,F_3)}{\partial(\xi_1,\xi_2,c)} = \frac{4}{c^2}\left[\frac{\xi_2}{1-\xi_2^2} - \frac{\xi_1}{1-\xi_1^2}\right] > 0 \; (\neq 0)$$

taking into account that $\xi/(1-\xi^2)$ is an increasing function of $\xi \in (-1,1)$. Hence, by Implicit Function Theorem [6, Theorem 9.18] we conclude that $c(\rho)$, $\xi_1(\rho)$, $\xi_2(\rho)$ are continuously differentiable functions of ρ.

Next, differentiating (7) with respect to ρ we obtain

$$\xi_2'(\rho) - \xi_1'(\rho) = \frac{c(\rho)}{2}\int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](x-x^3)e^{\rho x}dx \tag{11}$$

Simple computations, by using (7)–(9) give for $\rho > 0$

$$\int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right]xe^{\rho x}dx$$
$$= (1-\xi_2^2(\rho))e^{\rho\xi_2(\rho)} - (1-\xi_1^2(\rho))e^{\rho\xi_1(\rho)} = 0. \tag{12}$$

By continuity, the above equality holds for $\rho = 0$. Hence, (11) can be simplified to

$$\xi_2'(\rho) - \xi_1'(\rho) = -\frac{c(\rho)}{2}\int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right]x^3e^{\rho x}dx \tag{13}$$

We need the sign of the left-hand side in (13) and it will be determined by the following equality that holds for $\rho > 0$:

$$\left[\frac{4}{\rho} + \xi_2(\rho) + \xi_1(\rho)\right] \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x^3 e^{\rho x} dx$$
$$= \int_{-1}^{1} |(x-\xi_1(\rho))(x-\xi_2(\rho)|\,(1-x^2) e^{\rho x} dx$$
$$+ \frac{2}{\rho c(\rho)} \left[\xi_2^2(\rho) - \xi_1^2(\rho)\right] . \tag{14}$$

Now we prove the above equality. Obviously,

$$(x-\xi_1(\rho))(x-\xi_2(\rho))(1-x^2) = 1 - x^4 + [\xi_1(\rho) + \xi_2(\rho)]\, x^3$$
$$- [1 - \xi_1(\rho)\xi_2(\rho)]\,(1-x^2) - [\xi_1(\rho) + \xi_2(\rho)]\, x.$$

Multiplying the above equality by $\operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] e^{\rho x}$ and integrating, we obtain by using (7) and (12)

$$\int_{-1}^{1} |(x-\xi_1(\rho))(x-\xi_2(\rho)|\,(1-x^2) e^{\rho x} dx$$
$$= \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] (1-x^4) e^{\rho x} dx$$
$$+ [\xi_1(\rho) + \xi_2(\rho)] \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x^3 e^{\rho x} dx\,. \tag{15}$$

The first integral on the right-hand side of (15) is equal to

$$-\frac{2}{\rho} \left[(1-\xi_2^4(\rho)) e^{\rho \xi_2(\rho)} - (1-\xi_1^4(\rho)) e^{\rho \xi_1(\rho)}\right]$$
$$+ \frac{4}{\rho} \int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x^3 e^{\rho x}\, dx\,.$$

From (8) and (9) we obtain

$$(1-\xi_2^4(\rho)) e^{\rho \xi_2(\rho)} - (1-\xi_1^4(\rho)) e^{\rho \xi_1(\rho)} = \frac{1}{c(\rho)} \left[\xi_2^2(\rho) - \xi_1^2(\rho)\right] .$$

In view of this, Eq. (15) is equivalent to Eq. (14). Taking into account (10), we conclude that the right-hand side of (14) and the multiplier on the left-hand side of (14) before the integral are positive hence,

$$\int_{-1}^{1} \operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x^3 e^{\rho x} dx > 0 \quad \text{for} \quad \rho > 0.$$

By (13) we conclude that

$$[\xi_2(\rho)-\xi_1(\rho)]' = \xi_2'(\rho)-\xi_1'(\rho) < 0 \quad \text{for} \quad \rho > 0\,.$$

From here, the function $[\xi_2(\rho)-\xi_1(\rho)]$ is strictly decreasing for $\rho \in (0,\infty)$ and using (6) and by Theorem 2 we conclude that there exists a unique $c_1 \in (-\infty,\infty)$ such that

$$\min_{(c,\ \rho)\in(-\infty,\infty)\times(-\infty,\infty)} \int_{-1}^{1} \left|1-c(1-x^2)e^{\rho x}\right| dx = \min_{c\in(-\infty,\infty)} \int_{-1}^{1} \left|1-c(1-x^2)\right| dx$$
$$= \int_{-1}^{1} \left|1-c_1(1-x^2)\right| dx.$$

Next, we compute the extremal constant c_1 and the minimum value of the integral. The extremal constant c_1 is the unique solution of the equation

$$\int_{-1}^{1} \operatorname{sign}\left[1-c(1-x^2)\right](1-x^2)dx = 0. \tag{16}$$

In addition, $c_1 > 1$ and by (10) $-1 < \xi_1(0) < 0 < \xi_2(0) < 1$ and $\xi_1(0) = -\xi_2(0) = -\xi$, $0 < \xi < 1$. Hence, $-\xi, \xi$, $0 < \xi < 1$ are the two points in $(-1,1)$, where the graph of the function $y = c_1(1-x^2)$ intersects the line $y = 1$. Then (16) is equivalent to

$$\int_{-1}^{1} \operatorname{sign}\left[(x+\xi)(x-\xi)\right](1-x^2)dx = 0 \quad \Rightarrow \quad \int_{-1}^{1} \operatorname{sign}\left[x^2-\xi^2\right](1-x^2)dx = 0$$
$$\Rightarrow \quad \int_{0}^{1} \operatorname{sign}\left[x-\xi\right](1-x^2)dx = 0 \quad \Rightarrow \quad \xi^3-3\xi+1 = 0, \quad 0 < \xi < 1$$
$$\Rightarrow \quad 4\omega^3-3\omega+\frac{1}{2}, \quad \xi = 2\omega, \quad 0 < \omega < \frac{1}{2}.$$

Substituting $\omega = \cos(\gamma)$ and using the trigonometric identity $\cos(3\gamma) = 4\cos^3(\gamma)-3\cos(\gamma)$ we obtain an equivalent equation in terms of γ:

$$\cos(3\gamma) = -1/2, \quad \frac{\pi}{3} < \gamma < \frac{\pi}{2}.$$

The above equation has three solutions in $[0, \pi]$: $\gamma_1 = 2\pi/9$, $\gamma_2 = 4\pi/9$, $\gamma_3 = 8\pi/9$ but only γ_2 is in $(\pi/3, \pi/2)$. Hence,

$$c_1 = \frac{1}{1-\xi_1^2} = \frac{1}{1-4\sin^2\left(\frac{\pi}{18}\right)}, \quad \xi_1 = 2\cos\left(\frac{4\pi}{9}\right) = 2\sin\left(\frac{\pi}{18}\right) \in (0,1).$$

By using (6) we obtain the exact value of the best $L_1[-1,1]$ approximation to i from $\mathscr{D}\mathscr{R}_{lc,+}$:

$$\int_{-1}^{1} |1 - c_1(1-x^2)|dx = 2 - 4\xi_1 = 2 - 8\sin\left(\frac{\pi}{18}\right), \quad \xi_1 = 2\sin\left(\frac{\pi}{18}\right).$$

The Case $1 < p < \infty$ Next theorem gives in particular that the unique best $L_p[-1,1]$ approximant to i from $\widetilde{\mathscr{L}\mathscr{P}}_{1,+}$ must have $\rho = 0$, i.e., it is the form $c(1-x^2)$.

Theorem 5. *Let i denote the unit function on the interval $[-1,1]$ and let $1 < p < \infty$. Then, there exists a unique constant $c_p \in (-\infty, \infty)$ such that*

$$\begin{aligned}
\inf_{f\in\mathscr{D}\mathscr{R}_{lc,+}} \int_{-1}^{1} |i(x) - f(x)|^p dx &= \inf_{e\in\mathscr{L}\mathscr{P}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx \\
= \inf_{e\in\widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx &= \inf_{q\in\mathscr{P}_{c,+}} \int_{-1}^{1} |i(x) - q(x)|^p dx \\
= \inf_{q\in\mathscr{P}_{r,+}} \int_{-1}^{1} |i(x) - q(x)|^p dx &= \inf_{-\infty<c<\infty} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx \\
&= \int_{-1}^{1} |i(x) - c_p(1-x^2)|^p dx.
\end{aligned}$$

Proof. Taking into account Theorems 1 and 2, it is sufficient to prove that

$$\inf_{e\in\widetilde{\mathscr{L}\mathscr{P}}_{1,+}} \int_{-1}^{1} |i(x) - e(x)|^p dx = \inf_{-\infty<c<\infty} \int_{-1}^{1} |i(x) - c(1-x^2)|^p dx\,.$$

For simplicity we denote

$$I(c,\rho) = \int_{-1}^{1} \left|i(x) - c(1-x^2)e^{\rho x}\right|^p\, dx = \int_{-1}^{1} \left|1 - c(1-x^2)e^{\rho x}\right|^p\, dx.$$

Next, our considerations follow the technique that has been developed in [3, pp. 32–36]. Here are the details. Without any restriction, when minimizing $I(c,\rho)$, we can assume $\rho \geq 0$ and $c \geq 0$. Let us fix ρ and minimize $I(c,\rho)$ with respect to c. Then, $I(c,\rho)$ tends to infinity as c goes to infinity. In addition, $I(c,\rho) < I(0,\rho)$ for small positive values of c. Hence, the minimum is attained at some finite point $c(\rho)$ in $(0, \infty)$ that is a critical number of $I(c, \rho)$ so, we must have $dI/dc = 0$ at $c = c(\rho)$ that is equivalent to

$$\int_{-1}^{1} \left| \frac{1}{c} - (1-x^2)e^{\rho x} \right|^{p-1} \operatorname{sign}\left[1 - c(1-x^2)e^{\rho x}\right] (1-x^2)e^{\rho x}\, dx = 0. \tag{17}$$

By (17), the function $[1 - c(\rho)(1-x^2)e^{\rho x}]$ must change sign at exactly two points $-1 < \xi_1(\rho) < \xi_2(\rho) < 1$ in $(-1, 1)$ and also, $c(\rho) > 1$. Similar to the case $p = 1$, we conclude that (17) is satisfied for only one value $c(\rho)$ of c. Hence, the function $I(c(\rho), \rho)$ is well defined. Note that $I(c(\rho), \rho)$ is the minimum of $I(c, \rho)$ for a fixed ρ, with respect to $c \in (-\infty, \infty)$.

Our goal is to show that the function $I(c(\rho), \rho)$ is increasing for $\rho \geq 0$. We extend the function $I(c(\rho), \rho)$ for all $\rho \in (-\infty, \infty)$ taking into account that $c(-\rho) = c(\rho)$, $\xi_1(-\rho) = -\xi_2(\rho)$, and $\xi_2(-\rho) = -\xi_1(\rho)$.

Next, we observe that the Jacobian

$$\frac{\partial(G_1, G_2, G_3)}{\partial(\xi_1, \xi_2, c)} = -(p-1)\left(-\frac{2\xi_1}{1-\xi_1^2} + \rho\right)\left(-\frac{2\xi_2}{1-\xi_2^2} + \rho\right) \times \int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p-2}(1-x^2)^2 e^{2\rho x}\, dx$$

of the non-linear system of equations (the first one of the equations below is equivalent to (17))

$$G_1(\xi_1, \xi_2, c, \rho) := \int_{-1}^{1} \left| \frac{1}{c} - (1-x^2)e^{\rho x} \right|^{p-1} \operatorname{sign}\left[(x-\xi_1)(x-\xi_2)\right] (1-x^2)e^{\rho x} dx = 0$$

$$G_2(\xi_1, \xi_2, c, \rho) := c(1-\xi_1^2)e^{\rho\xi_1} - 1 = 0$$

$$G_3(\xi_1, \xi_2, c, \rho) := c(1-\xi_2^2)e^{\rho\xi_2} - 1 = 0$$

is distinct from zero since the function $\left[-2x/(1-x^2) + \rho\right]$ vanishes in $(-1, 1)$ only at the point of maximum of $(1-x^2)e^{\rho x}$, i.e., not at ξ_1 and ξ_2. Hence, by Implicit Function Theorem [6] we conclude that $c(\rho)$, $\xi_1(\rho)$, $\xi_2(\rho)$ are continuously differentiable functions of ρ.

Next by (17), for $c = c(\rho)$, $\xi_1 = \xi_1(\rho)$, and $\xi_2 = \xi_2(\rho)$ $(\rho \neq 0)$, taking into account that

$$c(\rho)\left[1 - \xi_1^2(\rho)\right] e^{\rho\xi_1(\rho)} = 1, \quad c(\rho)\left[1 - \xi_2^2(\rho)\right] e^{\rho\xi_2(\rho)} = 1,$$

the following relation holds:

$$
\begin{aligned}
0 &= \int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1)(x-\xi_2)\right](1-x^2)e^{\rho x}dx \\
&= \frac{2}{\rho}\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\left[(x-\xi_1)(x-\xi_2)\right] xe^{\rho x}dx \\
&-\frac{1}{c\rho}\left[\int_{-1}^{\xi_1}+\int_{\xi_2}^{1}\right]\left(1-c(1-x^2)e^{\rho x}\right)^{p-1} d\left[1-c(1-x^2)e^{\rho x}\right] \\
&+\frac{1}{c\rho}\int_{\xi_1}^{\xi_2}\left(c(1-x^2)e^{\rho x}-1\right)^{p-1} d\left[c(1-x^2)e^{\rho x}-1\right].
\end{aligned} \tag{18}
$$

In view of (18) and by continuity we have for $\rho \geq 0$

$$
\int_{-1}^{1} |1-c(\rho)(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x\, e^{\rho x}\, dx = 0. \tag{19}
$$

We prove that $I(c(\rho),\rho)$ is an increasing function of $\rho \geq 0$ by studying the sign of its derivative with respect to ρ, $\rho > 0$. At $c = c(\rho)$ we have $\partial I/\partial c = 0$ and taking into account (17) and (19), we are to show that

$$
\frac{d\,I(c(\rho),\rho)}{d\,\rho} = \frac{\partial I}{\partial c}\frac{d\,c}{d\,\rho} + \frac{\partial I}{\partial \rho} = \frac{\partial I}{\partial \rho} \tag{20}
$$

$$
\begin{aligned}
&= pc(\rho)\int_{-1}^{1} |1-c(\rho)(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right](x^3-x)e^{\rho x}dx \\
&= pc(\rho)\int_{-1}^{1} |1-c(\rho)(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1(\rho))(x-\xi_2(\rho))\right] x^3\, e^{\rho x}dx > 0\,.
\end{aligned}
$$

Next, with $c = c(\rho)$, $\xi_1 = \xi_1(\rho)$, $\xi_2 = \xi_2(\rho)$ the following holds:

$$
\begin{aligned}
0 &< \int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1)(x-\xi_2)\right]\left[(x-\xi_1)(x-\xi_2)\right](1-x^2)e^{\rho x}dx \\
&= \int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1)(x-\xi_2)\right](1-x^4)e^{\rho x}dx \\
&+ (\xi_1+\xi_2)\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\operatorname{sign}\left[(x-\xi_1)(x-\xi_2)\right] x^3 e^{\rho x}dx
\end{aligned} \tag{21}
$$

By (17)–(19), and representing $(1-x^4)$ in the form

$$
1-x^4 = 1-x^2 + \frac{1}{\rho}\left[-2x + \rho(1-x^2) + 2x\right]x^2
$$

the relation (21) can be written in the following equivalent form:

$$0 < \frac{1}{\rho}\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]\left[-2x+\rho(1-x^2)\right]x^2 e^{\rho x}dx$$

$$+\left[\xi_1+\xi_2+\frac{2}{\rho}\right]\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]\; x^3\, e^{\rho x}$$

$$= \frac{2}{p\,c\,\rho}\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p}\,x\,dx + \left[\xi_1+\xi_2+\frac{2}{\rho}\right]\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}$$

$$\times\,\text{sign}\,[(x-\xi_1)(x-\xi_2)]\,x^3\,e^{\rho x}\,dx \tag{22}$$

$$= \frac{2}{p\,c\,\rho}\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\,\text{sign}\,[(x-\xi_1)(x-\xi_2)]\;x\,dx$$

$$+\left[\xi_1+\xi_2+\frac{2}{\rho}+\frac{2}{p\rho}\right]\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]\,x^3\,e^{\rho x}\,dx$$

taking into account (19) and the following identity:

$$\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p}xdx = \int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]$$

$$\times\left[1-c(1-x^2)e^{\rho x}\right]xdx\,. \tag{23}$$

Analogously to the case $p = 1$ (see (10)), $\xi_1(\rho) + \xi_2(\rho) > 0$ for $\rho > 0$ hence, (20) will follow by (22) if

$$\int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]\,x\,dx \le 0\,. \tag{24}$$

In order to prove (24) we consider 2 cases.

First, let $\xi_1\xi_2 \le 0$, i.e., $-1 < \xi_1 \le 0 < \xi_2 < 1$. It is easily seen that the auxiliary function

$$h(x) := \frac{1}{\xi_2-\xi_1}\begin{vmatrix} 1 & \xi_1 & \xi_1 e^{\rho\xi_1} \\ 1 & \xi_2 & \xi_2 e^{\rho\xi_2} \\ 1 & x & xe^{\rho x}\end{vmatrix} = xe^{\rho x} - \beta x - \alpha$$

has two simple zeros at ξ_1 and ξ_2 and it is positive on $(-1, \xi_1)$ and $(\xi_2, 1)$ and negative on (ξ_1, ξ_2). Then by (17), (19), and (23) ($\xi_2 > |\xi_1|$)

$$0 < \int_{-1}^{1} |1-c(1-x^2)e^{\rho x}|^{p-1}\text{sign}\,[(x-\xi_1)(x-\xi_2)]\,(\xi_2-\xi_1)h(x)dx$$

$$= \xi_1\xi_2(e^{\rho\xi_2} - e^{\rho\xi_1}) \int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p-1} \text{sign}\,[(x-\xi_1)(x-\xi_2)]\,dx$$

$$-(\xi_2 e^{\rho\xi_2} - \xi_1 e^{\rho\xi_1}) \int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p-1} \text{sign}\,[(x-\xi_1)(x-\xi_2)]\; x\,dx$$

$$= \xi_1\xi_2(e^{\rho\xi_2} - e^{\rho\xi_1}) \int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p} dx$$

$$-(\xi_2 e^{\rho\xi_2} - \xi_1 e^{\rho\xi_1}) \int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p-1} \text{sign}\,[(x-\xi_1)(x-\xi_2)]\; x\,dx$$

and in view of this (24) follows taking into account that $\xi_1\xi_2(e^{\rho\xi_2} - e^{\rho\xi_1}) \leq 0$ and $\xi_2 e^{\rho\xi_2} - \xi_1 e^{\rho\xi_1} > 0$.

Now let $\xi_1\xi_2 > 0$ hence, $0 < \xi_1 < \xi_2 < 1$. Since $\rho > 0$, $c = c(\rho) > 0$, and $p > 1$ we have

$$|1 - c(1-x^2)e^{\rho x}|^{p-1} < |1 - c(1-x^2)e^{-\rho x}|^{p-1} \quad \text{for} \quad 0 < x < 1$$

and then, trivially

$$\int_{-1}^{1} |1 - c(1-x^2)e^{\rho x}|^{p-1} \text{sign}\,[(x-\xi_1)(x-\xi_2)]\, x\,dx$$

$$= \left[\int_{-1}^{0} + \int_{0}^{\xi_1} - \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^{1} \right] |1 - c(1-x^2)e^{\rho x}|^{p-1}\, x\,dx$$

$$< -\int_{\xi_1}^{\xi_2} |1 - c(1-x^2)e^{\rho x}|^{p-1}\, x\,dx - \int_{\xi_1}^{\xi_2} |1 - c(1-x^2)e^{-\rho x}|^{p-1}\, x\,dx < 0$$

hence, (24) holds. Then

$$\min_{\rho \geq 0} I(c(\rho), \rho) = I(c(0), 0) = \min_{c \in (-\infty,\infty)} \int_{-1}^{1} |1 - c(1-x^2)|^p\, dx.$$

This completes the proof.

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Local Approximation Using Hermite Functions

H.N. Mhaskar

Abstract We develop a wavelet-like representation of functions in $L^p(\mathbb{R})$ based on their Fourier–Hermite coefficients; i.e., we describe an expansion of such functions where the local behavior of the terms characterize completely the local smoothness of the target function. In the case of continuous functions, a similar expansion is given based on the values of the functions at arbitrary points on the real line. In the process, we give new proofs for the localization of certain kernels, as well as for some very classical estimates such as the Markov–Bernstein inequality.

Keywords Approximation with Hermite polynomials • Localized kernels • Quadrature formulas • Wavelet-like representation

1 Introduction

The subject of weighted polynomial approximation is by now fairly well studied in approximation theory, with several books (e.g., [12, 14, 27]) devoted to various aspects of this subject. One of the first papers in the modern theory was by Freud et al. [11]. The purpose of this paper is to revisit this theory in the context of approximation by Hermite functions.

To describe our motivation, we consider the case of uniform approximation of periodic functions by trigonometric polynomials. In view of the direct and converse theorems of approximation, both the functions

$$f_1(x) = \sqrt{|\cos x|}, \qquad f_2(x) = \sum_{k=0}^{\infty} \frac{\cos(4^k x)}{2^k}, \qquad x \in \mathbb{R},$$

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are in the same Hölder class Lip$(1/2)$, with the uniform degree of approximation by trigonometric polynomials of order $< n$ to both of these being $\mathscr{O}(n^{-1/2})$. However, f_1 has an analytic extension except at $x = (2k+1)\pi/2$, $k \in \mathbb{Z}$, while f_2 is nowhere differentiable. Also, the Fourier coefficients of neither of the two functions reveal this fact. One of the reasons for developing the very popular wavelet analysis is to be able to detect the fact that f_1 is only locally in Lip$(1/2)$ at $x = (2k+1)\pi/2$, $k \in \mathbb{Z}$, and infinitely smooth at other points by means of local behavior of the wavelet coefficients of f_1 rather than its Fourier coefficients [4, Chap. 9]. Motivated by this theory, we have developed in a series of papers (e.g., [2, 3, 7, 9, 17–19, 21–25]) a theory of wavelet-like representations of functions on the torus, compact interval, sphere, manifolds, and graphs using the expansion coefficients of classical orthogonal systems on these domains, for example, Jacobi polynomials on the interval. In this paper, we develop such a theory for the whole real line using Hermite functions as the underlying orthogonal system.

Naturally, the basic ideas and ingredients involved in this development are the same as in our previous work. However, there are several technical difficulties. The infinite–finite range inequalities (see Proposition 6.1) help us, as expected, to deal with the fact that the domain of approximation here is obviously not compact. An additional technical difficulty is the following "product problem." The product of two polynomials P_1, P_2 of degree $< n$ is also a polynomial of degree $< 2n$. In contrast, the product of two "weighted polynomials" $\exp(-x^2/2)P_1(x)$, $\exp(-x^2/2)P_2(x)$ is not another weighted polynomial. A straightforward attempt to approximate $\exp(-x^2/2)$ by its Taylor polynomial or even the more sophisticated approach described in [14, Chap. 7] is not adequate to obtain the correct rates of approximation of such a product with weighted polynomials. The other important components in our theory are the availability of localized kernels and quadrature formulas based on arbitrary points on $\mathbb{R}$. While the localization estimates on certain kernels as in Theorem 3 are given in [6, 8], we give a more elementary proof based on the Mehler identity and a new Tauberian theorem proved in [13]. As a consequence, we also give a new proof of certain classical inequalities such as the estimates on the Christoffel functions and Markov–Bernstein inequalities.

The paper is organized as follows. We define the basic notations and definitions and summarize some preliminary facts in Sect. 2. In Sect. 3, we develop the machinery to help us surmount the product problem by reviewing and interpreting certain equivalence theorems from the theory of weighted polynomial approximation. Localized kernels will be described next in Sect. 4 (Theorem 3). These will be used in Sect. 5 to develop certain localized, uniformly bounded summability operators (Lemma 2, Theorem 6). In turn, these will be used to give a new proof of the Markov–Bernstein inequality in Corollary 5.1. The summability operators are analogues of the shifted average operators in [14, Sect. 3.4]. When defined in terms of the Lebesgue measure, they reproduce weighted polynomials. This may not hold when they are defined with other measures. For this purpose, we will prove in Sect. 6 the existence of measures supported on an arbitrary set of real numbers which integrate products of weighted polynomials exactly. Finally, the wavelet-like representation is given in Sect. 7.

2 Basic Notation and Definitions

In this section, we collect together different notations and definitions, as well as some preliminary facts which we will use often in this paper.

If $x \in \mathbb{R}$ and $r \geq 0$, we will write $\mathbb{B}(x,r) = [x-r, x+r]$.

Let $\{\psi_j\}$ denote the sequence of orthonormalized Hermite functions; i.e., [28, Formulas (5.5.3), (5.5.1)]

$$\psi_j(x) = \frac{(-1)^j}{\pi^{1/4}2^{j/2}\sqrt{j!}} \exp(x^2/2) \left(\frac{d}{dx}\right)^j (\exp(-x^2)), \qquad x \in \mathbb{R},\ j = 0, 1, \cdots. \tag{1}$$

We note that

$$\int_{\mathbb{R}} \psi_j(z)\psi_\ell(z)dz = \delta_{j,\ell}, \qquad j, \ell = 0, 1, \cdots. \tag{2}$$

We denote $w(x) = \exp(-x^2/2)$. For $t > 0$, let $\mathbb{P}_t$ be the class of all algebraic polynomials of degree $< t$. The space Π_t is defined by

$$\Pi_t = \mathsf{span}\{\psi_j : \sqrt{j} < t\} = \{wP : P \in \mathbb{P}_{t^2}\}, \qquad t > 0. \tag{3}$$

In this paper, the term measure will denote a signed, complex valued Borel measure (or a positive, sigma–finite Borel measure). We recall that if μ is an extended complex valued Borel measure on $\mathbb{R}$, then its total variation measure is defined for a Borel set B by

$$|\mu|(B) = \sup \sum |\mu(B_k)|,$$

where the sum is over a partition $\{B_k\}$ of B comprising Borel sets, and the supremum is over all such partitions.

Definition 1. If $t > 0$, a Borel measure ν will be called t-**regular** if there exists a constant $A > 0$ such that

$$|\nu|(\mathbb{B}(x,r)) \leq A(r + 1/t), \qquad x \in \mathbb{R},\ r > 0. \tag{4}$$

We will define the regularity norm of ν by

$$|\!|\!|\nu|\!|\!|_t = \sup_{r>0,\ x\in\mathbb{R}} \frac{|\nu|(\mathbb{B}(x,r))}{r + 1/t}. \tag{5}$$

The set of all Borel measures for which $|\!|\!|\nu|\!|\!|_t < \infty$ is a vector space, denoted by $\mathscr{R}_t$. □

It is easy to verify that $|||\cdot|||_t$ is a norm on $\mathscr{R}_t$. It is not difficult to deduce from the definition that

$$|||\nu|||_t \le \max(1, t/u)|||\nu|||_u, \qquad t, u > 0.$$

In particular, when $t < u$, $\mathscr{R}_u \subseteq \mathscr{R}_t$, and for any constant $c > 0$, the spaces of measures $\mathscr{R}_t$ and $\mathscr{R}_{ct}$ are the same, with the constants involved in the norm equivalence depending upon c.

For example, the Lebesgue measure on $\mathbb{R}$ is in $\mathscr{R}_\infty$, and its regularity norm is obviously 1. If $\mathscr{C} \subset \mathbb{R}$, the **density content** of $\mathscr{C}$ is defined by

$$\delta(\mathscr{C}) = \sup_{y,z\in\mathscr{C}} |y - z|. \tag{6}$$

If $\mathscr{C}$ is a finite set, and ν is a measure that associates the mass 1 with each of these points, then ν is clearly $1/\delta(\mathscr{C})$-regular.

Definition 2. Let $n > 0$. A Borel measure ν on $\mathbb{R}$ is called **quadrature measure** of order n if

$$\int_{\mathbb{R}} P(y)Q(y)dy = \int_{\mathbb{R}} P(y)Q(y)d\nu(y), \qquad P, Q \in \Pi_n. \tag{7}$$

The set of all quadrature measures of order n which are in $\mathscr{R}(n)$ is denoted by $MZ(n)$. □

We note that the formula (7) is required for **products of weighted polynomials**. Clearly, the Lebesgue measure itself is in $MZ(n)$ for all $n > 0$. In Theorem 7, we will prove the existence of measures in $MZ(n)$ supported on a sufficiently dense set of points in $\mathbb{R}$.

If ν is any Borel measure on $\mathbb{R}$, for $1 \le p \le \infty$, and ν-measurable set $B \subseteq \mathbb{R}$ and ν-measurable function $f : B \to \mathbb{R}$

$$\|f\|_{\nu;p,B} := \begin{cases} \left\{\displaystyle\int_B |f(x)|^p d|\nu|(x)\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ |\nu| - \text{ess sup}_{x\in B} |f(x)|, & \text{if } p = \infty. \end{cases}$$

The class of all functions f for which $\|f\|_{\nu;p,B} < \infty$ is denoted by $L^p(\nu; B)$, with the usual convention that functions that are equal $|\nu|$-almost everywhere are considered to be equal. If ν is the Lebesgue measure, its mention will be omitted from the notation, and if $B = \mathbb{R}$, its mention will also be omitted from the notation. The set X^p will denote L^p if $1 \le p < \infty$, and the set of all continuous functions on $\mathbb{R}$ which vanish at infinity if $p = \infty$.

Constant Convention

The symbols $c, c_1, \cdots$ will denote generic positive constants depending only on the fixed parameters in the discussion, such as the norms and smoothness parameters. Their value may be different at different occurrences, even within a single formula. The notation $A \sim B$ means that $c_1 A \le B \le c_2 A$. □

3 Weighted Approximation

In this section, we review some results from [16] for the sake of making this paper more self-contained. The main purpose is to point out Corollary 3.1, which will help us later in Sect. 7 to get around the difficulty that the product of $P, Q \in \Pi_n$ is not in any Π_{cn}.

Let $1 \le p \le \infty$, $t > 0$. If $f \in L^p$, we define

$$E_{t,p}(f) = \inf_{Q \in \Pi_t} \|f - Q\|_p. \tag{8}$$

For $t > 0$ and integer $k \ge 0$, the forward difference of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Delta_t^k f(x) := \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x + \ell t).$$

With

$$Q_\delta(x) := \min\left(\delta^{-1}, (1 + x^2)^{1/2}\right), \qquad \delta > 0,\ x \in \mathbb{R},$$

we define a modulus of smoothness for $f \in L^p$, $\delta > 0$ by the formula

$$\omega_r(p; f, \delta) := \sum_{k=0}^{r} \delta^{r-k} \sup_{|t| \le \delta} \|Q_\delta^{r-k} \Delta_t^k f\|_p. \tag{9}$$

The results in [16] lead to the following theorem:

Theorem 1. *Let $1 \le p \le \infty$, $f \in X^p$, $r, n \ge 1$ be integers. Then*

$$E_{n,p}(f) \le c\,\omega_r(p; f, 1/n), \tag{10}$$

and

$$\omega_r(p; f, 1/n) \le \frac{c}{n^r} \left\{ \|f\|_p + \sum_{k=0}^{n} (k+1)^{r-1} E_{k,p}(f) \right\}. \tag{11}$$

For the present paper, we need the following equivalence theorem, Theorem 2, which is obtained from Theorem 1 using standard methods of approximation theory as in [5].

For a sequence $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$, $0 < \rho \le \infty$, $\gamma \in (0, \infty)$, we define the sequence (quasi-)norm

$$[\![\mathbf{a}]\!]_{\rho,\gamma} = \begin{cases} \left(\sum_{n=0}^{\infty} (2^{\gamma n}|a_n|)^{\rho}\right)^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \sup_{n\ge 0} 2^{n\gamma}|a_n|, & \text{if } \rho = \infty. \end{cases} \tag{12}$$

The space of all sequences $\mathbf{a}$ with $[\![\mathbf{a}]\!]_{\rho,\gamma} < \infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$.

Definition 3. Let $1 \le p \le \infty$, $0 < \rho \le \infty$, $0 < \gamma < \infty$. The **Besov space** $B_{p,\rho,\gamma}$ is the space of all $f \in X^p$ for which $\|f\|_p + [\![\{E_{2^n,p}(f)\}_{n=0}^{\infty}]\!]_{\rho,\gamma} < \infty$. □

Theorem 2. *Let $0 < \rho \le \infty$, $0 < \gamma < \infty$, $1 \le p \le \infty$, $f \in X^p$, and $r > \gamma$ be an integer. Then $f \in B_{p,\rho,\gamma}$ if and only if $[\![\{\omega_r(p; f, 1/2^n)\}_{n=0}^{\infty}]\!]_{\rho,\gamma} < \infty$.*

A consequence of this theorem is the following. Let $w(x) = \exp(-x^2/2)$. Let $1 \le p \le \infty$, $t > 0$. If $f \in L^p$, we define

$$\tilde{E}_{t,p}(f) = \inf_{R\in\mathbb{P}_{t^2}} \|f - Rw^2\|_p. \tag{13}$$

With $\tilde{f}(x) = f(x/\sqrt{2})$, it is elementary to see that $\tilde{E}_{n,p}(f) \sim E_{n,p}(\tilde{f})$. Since $\omega_r(p; \tilde{f}, \delta) \sim \omega_r(p; f, \delta)$ for $\delta > 0$, we obtain as a corollary to Theorem 2 from the following:

Corollary 3.1. *Let $0 < \rho \le \infty$, $0 < \gamma < \infty$, $1 \le p \le \infty$, $f \in X^p$. Then $f \in B_{p,\rho,\gamma}$ if and only if $[\![\{\tilde{E}_{2^n,p}(f)\}_{n=0}^{\infty}]\!]_{\rho,\gamma} < \infty$.*

4 Localized Kernels

If $H : [0, \infty) \to \mathbb{R}$ is a compactly supported function, we write

$$\Phi_n(H; x, y) = \sum_{j=0}^{\infty} H\left(\frac{\sqrt{j}}{n}\right) \psi_j(x)\psi_j(y), \qquad n > 0,\ x, y \in \mathbb{R}. \tag{14}$$

Theorem 3. *Let $H : \mathbb{R} \to \mathbb{R}$ be a compactly supported, infinitely differentiable, even function. For $x, y \in \mathbb{R}$, $n \ge 1$, $S \ge 3$, we have*

$$|\Phi_n(H; x, y)| \le c\frac{n}{\max(1, (n|x-y|)^S)}, \qquad \left|\frac{\partial}{\partial x}\Phi_n(H; x, y)\right| \le c\frac{n^2}{\max(1, (n|x-y|)^S)}, \tag{15}$$

where the constants c may depend upon S.

The proof of this theorem requires some preparation. First, we recall some terminology.

A measure μ on $\mathbb{R}$ is called an even measure if $\mu((-u,u)) = 2\mu([0,u))$ for all $u > 0$, and $\mu(\{0\}) = 0$. If μ is an extended complex valued measure on $[0,\infty)$, and $\mu(\{0\}) = 0$, we define a measure μ_e on $\mathbb{R}$ by

$$\mu_e(B) = \mu\left(\{|x| : x \in B\}\right),$$

and observe that μ_e is an even measure such that $\mu_e(B) = \mu(B)$ for $B \subset [0,\infty)$. In the sequel, we will assume that all measures on $[0,\infty)$ which do not associate a nonzero mass with the point 0 are extended in this way, and will abuse the notation μ also to denote the measure μ_e. In the sequel, the phrase "measure on $\mathbb{R}$" will refer to an extended complex valued Borel measure having bounded total variation on compact intervals in $\mathbb{R}$, and similarly for measures on $[0,\infty)$.

The proof of Theorem 3 uses two Tauberian theorems. The first of these [13, Theorem 2.1] is the following:

Theorem 4. *Let μ be an extended complex valued measure on $[0,\infty)$, and $\mu(\{0\}) = 0$. We assume that there exist $Q, r > 0$, such that each of the following conditions are satisfied:*

1.

$$\sup_{u\in[0,\infty)} \frac{|\mu|([0,u))}{(u+2)^Q} < \infty, \tag{16}$$

2. There are constants $c, C > 0$, such that

$$\left|\int_{\mathbb{R}} \exp(-u^2 t)d\mu(u)\right| \le c_1 t^{-C} \exp(-r^2/t) \sup_{u\in[0,\infty)} \frac{|\mu|([0,u))}{(u+2)^Q}, \qquad 0 < t \le 1. \tag{17}$$

Let $H : [0,\infty) \to \mathbb{R}$, $S > Q+1$ be an integer, and suppose that there exists a measure $H^{[S]}$ such that

$$H(u) = \int_0^\infty (y^2 - u^2)_+^S dH^{[S]}(y), \qquad u \in \mathbb{R}, \tag{18}$$

and

$$V_{Q,S}(H) = \max\left(\int_0^\infty (y+2)^Q y^{2S} d|H^{[S]}|(y), \int_0^\infty (y+2)^Q y^S d|H^{[S]}|(y)\right) < \infty. \tag{19}$$

Then for $n \ge 1$,

$$\left|\int_0^\infty H(u/n)d\mu(u)\right| \le c\frac{n^Q}{\max(1,(nr)^S)} V_{Q,S}(H) \sup_{u\in[0,\infty)} \frac{|\mu|([0,u))}{(u+2)^Q}. \tag{20}$$

The second theorem we need is the following [20, Lemma 5.2]:

Theorem 5. *Let $C > 0$, $\{\ell_j\}$ be a non-increasing sequence of nonnegative numbers such that $\ell_0 = 0$ and $\lim_{j\to\infty} \ell_j = \infty$. Let $\{a_j\}$ be a sequence of nonnegative numbers such that $\sum_{j=0}^{\infty} \exp(-\ell_j^2 t)a_j$ converges for $t \in (0,1]$. Then*

$$c_1 L^C \le \sum_{\ell_j \le L} a_j \le c_2 L^C, \qquad L > 0, \tag{21}$$

if and only if

$$c_3 t^{-C/2} \le \sum_{j=0}^{\infty} \exp(-\ell_j^2 t)a_j \le c_4 t^{-C/2}, \qquad t \in (0,1]. \tag{22}$$

We are now in a position to prove Theorem 3. We note that the estimates (27) and (33) below were obtained in [14, Theorem 3.3.4] assuming the Markov–Bernstein inequality using more complicated machinery. In the present paper, the Markov–Bernstein inequality will be deduced as a consequence of Theorem 3.

Proof of Theorem 3. The starting point of the proof is the Mehler formula [1, Formula (6.1.13)]: For $x, y \in \mathbb{R}$, $|r| < 1$,

$$\begin{aligned}\sum_{j=0}^{\infty} \psi_j(x)\psi_j(y)r^j &= \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{2xyr - (x^2+y^2)r^2}{1-r^2}\right) \exp(-(x^2+y^2)/2)\\ &= \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(-\frac{r}{1-r^2}(x-y)^2 - \frac{1-r}{1+r}\frac{x^2+y^2}{2}\right).\end{aligned} \tag{23}$$

Writing $r = e^{-t}$, $t > 0$, we get the explicit expression for the "heat kernel":

$$\begin{aligned}&\sum_{j=0}^{\infty} e^{-jt}\psi_j(x)\psi_j(y)\\ &= \frac{e^{t/2}}{\sqrt{2\pi \sinh t}} \exp\left(-\frac{2}{\sinh t}(x-y)^2\right) \exp(-(1/2)\tanh(t/2)(x^2+y^2)).\end{aligned} \tag{24}$$

Hence,

$$\left|\sum_{j=0}^{\infty} e^{-jt}\psi_j(x)\psi_j(y)\right| \le \frac{c_1}{\sqrt{t}} \exp\left(-\frac{c(x-y)^2}{t}\right), \qquad 0 < t \le 1. \tag{25}$$

Taking $x = y$ above, we see that

$$\sum_{j=0}^{\infty} e^{-jt}\psi_j(x)^2 \le ct^{-1/2}. \tag{26}$$

Consequently, Theorem 5 used with $\ell_j = \sqrt{j}$ and $a_j = \psi_j(x)^2$ yields

$$\sum_{0\le\sqrt{j}<u} \psi_j(x)^2 \le cu, \qquad u \ge 1. \tag{27}$$

We now define a family of measures $\mu_{x,y}$ by

$$\mu_{x,y}(u) = \sum_{0\le\sqrt{j}<u} \psi_j(x)\psi_j(y), \qquad u, x, y \in \mathbb{R}.$$

Using Schwarz inequality and (27), we conclude that

$$\sup_{u>0} \frac{|\mu_{x,y}|(u)}{u+2} \le c, \qquad x, y \in \mathbb{R}. \tag{28}$$

In view of (25), the estimate (17) is satisfied by each of the measures $\mu_{x,y}$ with $r = |x - y|$. Moreover, it is clear that H satisfies the conditions required in Theorem 4. Since

$$\Phi_n(H; x, y) = \int_0^{\infty} H(u/n)d\mu_{x,y}(u),$$

we may use Theorem 4 with $Q = 1$ to arrive at the first inequality in (15).

In order to prove the second estimate in (15), we define a family of measures $\mu^{(1)}_{x,y}$ by

$$\mu^{(1)}_{x,y}(u) = \sum_{0\le\sqrt{j}<u} \psi_j'(x)\psi_j(y), \qquad u, x, y \in \mathbb{R},$$

and observe that

$$\frac{\partial}{\partial x}\Phi_n(H; x, y) = \int_0^{\infty} H(u/n)d\mu^{(1)}_{x,y}(u), \qquad x, y \in \mathbb{R}.$$

We will verify that (17) is satisfied by each of the measures $\mu^{(1)}_{x,y}$ with $r = |x - y|$, and

$$\sup_{u>0} \frac{|\mu^{(1)}_{x,y}|(u)}{(u+2)^2} \le c, \qquad x, y \in \mathbb{R}. \tag{29}$$

An application of Theorem 4 with $Q = 2$ then implies the desired second inequality in (15) as before.

Since $\psi_n'(x) = \sqrt{2n}\psi_{n-1}(x) - x\psi_n(x)$ (cf. [28, Eqs. (5.5.1), (5.5.10)]), it follows from (27) that $\|\psi_n'\|_\infty \le cn^2$. Therefore, we may differentiate the left-hand side of (24) term by term to obtain for $t > 0$

$$\sum_{j=0}^{\infty} e^{-jt}\psi_j'(x)\psi_j(y) = \frac{e^{t/2}}{\sqrt{2\pi \sinh t}}\left\{\frac{4(y-x)}{\sinh t} - x\tanh(t/2)\right\} \times \exp\left(-\frac{2}{\sinh t}(x-y)^2 - (1/2)\tanh(t/2)(x^2+y^2)\right), \tag{30}$$

and

$$\sum_{j=0}^{\infty} e^{-jt}\psi_j'(x)\psi_j'(y) = \frac{e^{t/2}}{\sqrt{2\pi \sinh t}}\left\{\frac{4}{\sinh t} + \left(\frac{4(y-x)}{\sinh t} - x\tanh(t/2)\right)\left(\frac{4(x-y)}{\sinh t} - y\tanh(t/2)\right)\right\} \times \exp\left(-\frac{2}{\sinh t}(x-y)^2 - (1/2)\tanh(t/2)(x^2+y^2)\right). \tag{31}$$

Since $\max_{x\in\mathbb{R}} |x|^m \exp(-ax^2) = (2a/(em))^{-m/2}$, $m = 1, 2, \cdots$, we deduce from (30) and (31) that for $0 < t \le 1$,

$$\left|\sum_{j=0}^{\infty} e^{-jt}\psi_j'(x)\psi_j(y)\right| \le \frac{c_1}{t}\exp\left(-\frac{c(x-y)^2}{t}\right), \quad \sum_{j=0}^{\infty} e^{-jt}\psi_j'(x)^2 \le ct^{-3/2}. \tag{32}$$

Thus, each of the measures $\mu_{x,y}^{(1)}$ satisfies (17) with $r = |x-y|$. Using Theorem 5 with $\psi_j'(x)^2$ in place of a_j, (32) leads to

$$\sum_{0\le\sqrt{j}<u} \psi_j'(x)^2 \le cu^3, \qquad u \ge 1. \tag{33}$$

Therefore, using Schwarz inequality and (27), we conclude that for $u \ge 1$,

$$|\mu_{x,y}^{(1)}|(u) \le \sum_{0\le\sqrt{j}<u} |\psi_j'(x)\psi_j(y)| \le cu^2.$$

This leads to (29) and completes the proof of the second inequality in (15) as explained before. □

5 Summability Operators

Definition 4. A function $h : \mathbb{R} \to [0, 1]$ is called a **low pass filter** if each of the following conditions is satisfied:

1. h is an even, infinitely differentiable function on $\mathbb{R}$,
2. $h(u) = 1$ for $|u| \le 1/2$,
3. h is non-increasing on $[1/2, 1]$,
4. $h(u) = 0$ if $|u| \ge 1$. □

In the sequel we will fix an infinitely differentiable low pass filter h, and will omit its mention from the notations, unless necessary to avoid confusion. In particular, the constants may depend upon h.

Let $n > 0$, ν be a Borel measure on $\mathbb{R}$, $f \in L^1(\nu) + L^\infty$, and $x \in \mathbb{R}$. We define

$$\hat{f}(\nu; j) = \int_{\mathbb{R}} f(y)\psi_j(y)d\nu(y), \qquad j = 0, 1, \cdots, \tag{34}$$

and with $\Phi_n(x, y) = \Phi_n(h; x, y)$ as defined in (14),

$$\sigma_n(\nu; f, x) = \sigma_n(h; \nu; f, x) = \int_{\mathbb{R}} \Phi_n(x, y) f(y) d\nu(y) = \sum_{j=0}^{\infty} h(\sqrt{j}/n)\hat{f}(\nu; j)\psi_j(x). \tag{35}$$

As usual, we will omit the mention of ν if ν is the Lebesgue measure on $\mathbb{R}$, e.g.,

$$\hat{f}(j) = \int_{\mathbb{R}} f(y)\psi_j(y)dy, \qquad j = 0, 1, \cdots . \tag{36}$$

In this section, we will also find it useful to introduce the notation

$$\sigma_n^{(1)}(f, x) = \frac{d}{dx}\sigma_n(f, x), \qquad x \in \mathbb{R},\ f \in L^1 + L^\infty. \tag{37}$$

The main theorem of this section is the following:

Theorem 6. *Let $n > 0$, $\nu \in MZ(n)$. If $P \in \Pi_{n/2}$, then $\sigma_n(\nu; P) = P$. If $1 \le p \le \infty$ and $f \in L^p$, then*

$$E_{n,p}(f) \le \|\sigma_n(\nu; f) - f\|_p \le cE_{n/2,p}(f). \tag{38}$$

In preparation for the proof of this theorem, we first prove two lemmas.

Lemma 1. *If $t > 0$, $\nu \in \mathscr{R}_t$, $r > 0$, $S \ge 2$, and $x \in \mathbb{R}$, then*

$$\int_{\mathbb{R}\setminus\mathbb{B}(x,r)} |y - x|^{-S} d|\nu|(y) \le \frac{2^S}{2^S - 2} |||\nu|||_t r^{-S+1}(2 + 1/(rt)). \tag{39}$$

In particular, if $n > 0$*, and* $\nu \in \mathscr{R}_n$*, then*

$$\int_{\mathbb{R}} |\Phi_n(x,y)| d|\nu|(y) \le c |||\nu|||_n, \qquad \int_{\mathbb{R}} \left|\frac{\partial}{\partial x}\Phi_n(x,y)\right| d|\nu|(y) \le cn |||\nu|||_n. \tag{40}$$

Proof. By re-normalization if necessary, we may assume in this proof that $|||\nu|||_t = 1$. Then (5) can be used to deduce that

$$\begin{aligned}
\int_{\mathbb{R}\setminus\mathbb{B}(x,r)} |y-x|^{-S} d|\nu|(y) &= \sum_{j=0}^{\infty} \int_{\mathbb{B}(x,2^{j+1}r)\setminus\mathbb{B}(x,2^j r)} |y-x|^{-S} d|\nu|(y) \\
&\le \sum_{j=0}^{\infty} (2^j r)^{-S} |\nu|(\mathbb{B}(x, 2^{j+1} r)) \\
&\le \sum_{j=0}^{\infty} (2^j r)^{-S} (2^{j+1} r + 1/t) = \frac{2^S r^{-S+1}}{2^{S-1}-1} + \frac{2^S r^{-S}}{(2^S-1)t} \\
&\le \frac{2^S r^{-S+1}}{2^S - 2}(2 + 1/(rt)).
\end{aligned}$$

Using the first estimate in (15) with $S \ge 2$, we deduce from (39) (with n in place of t) that

$$\begin{aligned}
\int_{\mathbb{R}} |\Phi_n(x,y)| d|\nu|(y) &= \int_{\mathbb{B}(x,1/n)} |\Phi_n(x,y)| d|\nu|(y) + \int_{\mathbb{R}\setminus\mathbb{B}(x,1/n)} |\Phi_n(x,y)| d|\nu|(y) \\
&\le cn\left\{|\nu|(\mathbb{B}(x,1/n)) + n^{-S} n^{S-1}\right\} \le c.
\end{aligned}$$

The second estimate in (40) is proved in the same way using the second estimate in (15). □

As a consequence of this lemma, we obtain the following:

Lemma 2. *Let* $n > 0$*,* $\mu, \nu \in \mathscr{R}_n$*, and* $1 \le p \le \infty$*. Then*

$$\|\sigma_n(\nu; f)\|_{\mu;p} \le c\|f\|_{\nu;p}, \qquad f \in L^p(\nu), \tag{41}$$

$$\|\sigma_n^{(1)}(f)\|_p \le cn\|f\|_p, \qquad f \in L^p(\nu) \tag{42}$$

Proof. In view of (40), for all $x \in \mathbb{R}$, and $f \in L^\infty(\nu)$,

$$|\sigma_n(\nu; f, x)| \le \int_{\mathbb{R}} |\Phi_n(x,y)| |f(y)| d|\nu|(y) \le c\|f\|_{\nu;\infty},$$

and similarly, using Tonelli's theorem, if $f \in L^1(\nu)$,

$$\int_{\mathbb{R}} |\sigma_n(\nu; f, x)| d|\mu|(x) \le \int_{\mathbb{R}} \int_{\mathbb{R}} |\Phi_n(x, y)||f(y)| d|\nu|(y) d|\mu|(x)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |\Phi_n(y, x)||f(y)| d|\mu|(x) d|\nu|(y) \le c\|f\|_{\nu;1}.$$

The estimate (41) follows from these and the Riesz-Thorin interpolation theorem. The proof of (42) is similar. □

We are now in a position to prove Theorem 6.

Proof of Theorem 6. We recall that $h(u) = 1$ if $|u| \le 1/2$. If $P \in \Pi_{n/2}$, then for $x \in \mathbb{R}$,

$$P(x) = \sum_{0 \le \sqrt{j} < n/2} \hat{P}(j)\psi_j(x) = \sum_{j=0}^{\infty} h(\sqrt{j}/n)\hat{P}(j)\psi_j(x) = \sigma_n(P, x) = \int_{\mathbb{R}} P(y)\Phi_n(x, y) dy.$$

Since $\nu \in MZ(n)$, the definition (7) now shows that

$$P(x) = \int_{\mathbb{R}} P(y)\Phi_n(x, y) d\nu(y) = \sigma_n(\nu; P, x).$$

The first inequality in (38) is obvious. In view of Lemma 2, we obtain for any $P \in \Pi_{n/2}$,

$$\|\sigma_n(\nu; f) - f\|_p = \|\sigma_n(\nu; f - P) - (f - P)\|_p \le c\|f - P\|_p.$$

This leads to the second inequality in (38). □

We end this section by pointing out that the estimate (42) leads immediately to the following Markov–Bernstein inequality. This deduction is the same in spirit as that given in [14], but we consider it to be a new proof, since the proof of (42) is significantly different from that in [14].

Corollary 5.1. *For* $1 \le p \le \infty$,

$$\|P'\|_p \le cn\|P\|_p, \qquad n > 0, \; P \in \Pi_n. \tag{43}$$

Proof. If $P \in \Pi_n$, Theorem 6 shows that $\sigma_{2n}(P) = P$, so that $P' = \sigma_{2n}^{(1)}(P)$. The inequality (43) follows from this and (42). □

6 Quadrature Formula

In this section, we wish to demonstrate the existence of measures in $MZ(n)$, supported on sufficiently dense finite point sets in $\mathbb{R}$, in the sense made precise below. We recall that if $\mathscr{C} \subset \mathbb{R}$, the density content of $\mathscr{C}$ is defined by

$$\delta(\mathscr{C}) = \sup_{y,z \in \mathscr{C}} |y - z|. \tag{44}$$

Theorem 7. *There exists $C, \alpha > 0$ with the following property: With $A_n = (n\sqrt{2})(1 + Cn^{-4/3})$, if $\mathscr{C} = \{y_1 < \cdots < y_{M+1}\} \subset \mathbb{R}$, $[-A_n, A_n] \subseteq [y_1, y_{M+1}]$, and $\delta(\mathscr{C}) \le c$, then there exist real numbers $w_1, \cdots, w_M$ such that with $n = \alpha\delta(\mathscr{C})^{-1}$,*

$$\int_{\mathbb{R}} P(y)Q(y)dy = \sum_{k=1}^{M} w_k P(y_k)Q(y_k), \qquad P, Q \in \Pi_n, \tag{45}$$

and

$$|w_k| \le c|y_{k+1} - y_k|, \qquad k = 1, \cdots, M. \tag{46}$$

In particular, the measure ν that associates the mass w_k with each of the points y_k is in $MZ(n)$. Further, if $[y_1, y_{M+1}] \subset [-cn^{\beta}, cn^{\beta}]$ for some $\beta > 0$, then

$$\sum_{k=1}^{M} |w_k| \le cn^{\beta}. \tag{47}$$

This theorem will be deduced by making some changes in variable in the following theorem:

Theorem 8. *There exists $C, \alpha_1 > 0$ with the following property: With $A'_n = 2n(1 + Cn^{-4/3})$, if $\mathscr{C}' = \{x_1 < \cdots < x_{M+1}\} \subset \mathbb{R}$, $[-A'_n, A'_n] \subseteq [x_1, x_{M+1}]$, and $\delta(\mathscr{C}') \le c$, then there exist real numbers $\tilde{w}_1, \cdots, \tilde{w}_M$ such that with $n = \alpha_1\delta(\mathscr{C}')^{-1}$,*

$$\int_{\mathbb{R}} P(x)dx = \sum_{k=1}^{M} \tilde{w}_k P(x_k), \qquad P \in \Pi_{n\sqrt{2}}, \tag{48}$$

and

$$|\tilde{w}_k| \le c|x_{k+1} - x_k|, \qquad k = 1, \cdots, M. \tag{49}$$

The proof of Theorem 8 follows the now standard methods (e.g., [10, 15, 19, 26]). We first use the Markov–Bernstein inequality (43) with $p = 1$ to prove the so-called Marcinkiewicz–Zygmund inequalities (Lemma 3 below), and then use the Hahn–Banach theorem.

Before starting this program, we recall some finite–infinite range inequalities.

Proposition 6.1. *Let $n > 0$, $1 \le p, r \le \infty$, $P \in \Pi_n$. Then*

$$\|P\|_{p,\mathbb{R}\setminus[-2n,2n]} \le c\exp(-c_1 n)\|P\|_{r,[-2n,2n]}. \tag{50}$$

Moreover, there exists $D > 0$ such that with $B_n = (n\sqrt{2})(1 + Dn^{-4/3})$, we have for $n \ge c$,

$$\int_{\mathbb{R}\setminus[-B_n,B_n]} |P(x)|dx \le (1/8)\int_{-B_n}^{B_n} |P(x)|dx. \tag{51}$$

Proof. The estimate (50) is proved in [14, Proposition 6.2.8] (and its proof). The estimate (51) is proved in [15, Corollary 2.1]. (To reconcile the notation in [15], we use $\alpha = 2$ and $2n^2$ in place of n which yields the interval denoted there by $\Delta_{n,\alpha}$ to be of the form $[-B_n, B_n]$ with a suitable value of D.) □

Lemma 3. *We assume the set up in Theorem 8. Then*

$$(3/4)\int_{\mathbb{R}} |P(x)|dx \le \sum_{k=1}^{M}(x_{k+1} - x_k)|P(x_k)| \le (5/4)\int_{\mathbb{R}} |P(x)|dx, \qquad P \in \Pi_{n\sqrt{2}}. \tag{52}$$

Proof. Let $P \in \Pi_{n\sqrt{2}}$, and $C = 2^{-2/3}D$, where D is defined in Proposition 6.1. Since $[-A'_n, A'_n] \subseteq [x_1, x_{M+1}]$, we obtain from (51) that for $n \ge c$

$$\int_{\mathbb{R}\setminus[x_1,x_{M+1}]} |P(x)|dx \le (1/8)\int_{x_1}^{x_{M+1}} |P(x)|dx. \tag{53}$$

For $k = 1, \cdots, M$, we have

$$\begin{aligned}\left|\int_{x_k}^{x_{k+1}} |P(x)|dx - (x_{k+1} - x_k)|P(x_k)|\right| &\le \int_{x_k}^{x_{k+1}} \left||P(x)| - |P(x_k)|\right| dx\\ \le \int_{x_k}^{x_{k+1}} |P(x) - P(x_k)|dx &\le \int_{x_k}^{x_{k+1}}\int_{x_k}^{y} |P'(u)|dudx\\ \le (x_{k+1} - x_k)\int_{x_k}^{x_{k+1}} |P'(u)|du.\end{aligned}$$

Consequently, we deduce from (53) and (43) that

$$\begin{aligned}&\left|\int_{\mathbb{R}} |P(x)|dx - \sum_{k=1}^{M}(x_{k+1} - x_k)|P(x_k)|\right|\\ &\le \int_{\mathbb{R}\setminus[x_1,x_{M+1}]} |P(x)|dx + \left|\int_{x_1}^{x_{M+1}} |P(x)|dx - \sum_{k=1}^{M}(x_{k+1} - x_k)|P(x_k)|\right|\end{aligned}$$

$$\leq (1/8)\int_{\mathbb{R}} |P(x)|dx + \sum_{k=1}^{M} \left| \int_{x_k}^{x_{k+1}} |P(x)|dx - (x_{k+1}-x_k)|P(x_k)| \right|$$

$$\leq (1/8)\int_{\mathbb{R}} |P(x)|dx + \sum_{k=1}^{M}(x_{k+1}-x_k)\int_{x_k}^{x_{k+1}} |P'(u)|du$$

$$\leq (1/8)\int_{\mathbb{R}} |P(x)|dx + c\delta(\mathscr{C}')\int_{\mathbb{R}} |P'(u)|du$$

$$\leq (1/8)\int_{\mathbb{R}} |P(x)|dx + cn\delta(\mathscr{C}')\int_{\mathbb{R}} |P(x)|dx.$$

Therefore, choosing α_1 sufficiently small, we obtain for $n = \alpha_1\delta(\mathscr{C}')^{-1}$,

$$\left| \int_{\mathbb{R}} |P(x)|dx - \sum_{k=1}^{M}(x_{k+1}-x_k)|P(x_k)| \right| \leq (1/4)\int_{\mathbb{R}} |P(x)|dx.$$

This completes the proof. □

We are now in a position to complete the proof of Theorem 8.

Proof of Theorem 8. In this proof only, we define a norm on $\mathbb{R}^M$ by

$$|||(z_1,\cdots,z_M)||| = \sum_{k=1}^{M}(x_{k+1}-x_k)|z_k|,$$

the sampling operator $\mathscr{U} : \Pi_{n\sqrt{2}} \to \mathbb{R}^M$ by $\mathscr{U}P = (P(x_1),\cdots,P(x_M))$, and denote the range of $\mathscr{U}$ by V. Then (52) shows that the operator $\mathscr{U}$ is invertible on V, and we may define a linear functional on V by

$$x^*(\mathscr{U}P) = \int_{\mathbb{R}} P(x)dx.$$

The dual norm of this functional can be estimated easily using (52):

$$|x^*(\mathscr{U}P)| \leq \int_{\mathbb{R}} |P(x)|dx \leq (4/3)|||\mathscr{U}P|||,$$

so that the norm is $\leq 4/3$. In view of the Hahn–Banach theorem, this functional can be extended from V to $\mathbb{R}^M$, where the extended functional has the same norm as x^*; i.e., $\leq 4/3$. This extended functional can be identified with $(\tilde{w}_1,\cdots,\tilde{w}_M) \in \mathbb{R}^M$. Then for $P \in \Pi_{n\sqrt{2}}$,

$$\sum_{k=1}^{M} \tilde{w}_k P(x_k) = x^*(\mathscr{U} P) = \int_{\mathbb{R}} P(x)dx,$$

proving (48). The norm of the extended functional is

$$\max_{1\le k\le M} \frac{|\tilde{w}_k|}{x_{k+1}-x_k} \le (4/3).$$

This proves (49). □

Having proved Theorem 8, the proof of Theorem 7 is only a change of variables.

Proof of Theorem 7. Let $x_k = y_k\sqrt{2}$, $k = 1, \cdots, M+1$, and $\mathscr{C}' = \{x_1, \cdots, x_{M+1}\}$. Then with A'_n, α_1 as defined in Theorem 8, $\delta(\mathscr{C}') = \sqrt{2}\delta(\mathscr{C})$, and $[-A'_n, A'_n] \supset [x_1, x_{M+1}]$. Further, with $\alpha = \alpha_1/\sqrt{2}$, $n = \alpha\delta(\mathscr{C})^{-1} = \alpha_1\delta(\mathscr{C}')^{-1}$. Therefore, Theorem 8 yields $\tilde{w}_k$ satisfying (48) and (49).

If $P(y) = R_1(y)\exp(-y^2/2)$, $Q(y) = R_2(y)\exp(-y^2/2)$, $R_1, R_2 \in \mathbb{P}_{n^2}$, then $x \mapsto R_1(x/\sqrt{2})R_2(x/\sqrt{2})\exp(-x^2/2) \in \Pi_{n\sqrt{2}}$. Hence, with $w_k = \tilde{w}_k/\sqrt{2}$, (48) implies that

$$\begin{aligned}
\int_{\mathbb{R}} P(y)Q(y)dy &= \int_{\mathbb{R}} R_1(y)R_2(y)\exp(-y^2)dy \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} R_1(x/\sqrt{2})R_2(x/\sqrt{2})\exp(-x^2/2)dx \\
&= \sum_{k=1}^{M} w_k R_1(y_k)R_2(y_k)\exp(-y_k^2) \\
&= \sum_{k=1}^{M} w_k P(y_k)Q(y_k),
\end{aligned}$$

which is (45). Also, (49) implies that

$$|w_k| = \frac{1}{\sqrt{2}}|\tilde{w}_k| \le \frac{c}{\sqrt{2}}|x_{k+1}-x_k| = c|y_{k+1}-y_k|,$$

which is (46). □

7 Wavelet-Like Representation

We recall Definition 3 of Besov spaces $B_{p,\rho,\gamma}$. Our first theorem is a characterization of these spaces in terms of an expansion of a function in L^p based either on the Fourier–Hermite coefficients or values of the target function at arbitrary points on $\mathbb{R}$.

Let $\aleph = \{\nu_n\}$ be a sequence of measures. We define the **frame operators** by

$$\tau_n(\aleph;f) = \begin{cases} \sigma_1(\nu_0;f), & \text{if } n = 0, \\ \sigma_{2^n}(\nu_n;f) - \sigma_{2^{n-1}}(\nu_{n-1};f), & \text{if } n = 1, 2, \cdots, \end{cases} \tag{54}$$

for all f for which the operators involved are well defined. If each of the measures ν_n is the Lebesgue measure, we will omit the mention of the sequence in the notations. In this case, the operators are defined for $f \in L^1 + L^\infty$. If each ν_n is a finitely supported measure, then the operators are defined for $f \in X^\infty$.

The following theorem is easy to deduce from Theorem 6 and [2, Theorem 3.1]:

Theorem 9. *Let $1 \le p \le \infty$, $\aleph = \{\nu_n\}$ be a sequence of measures such that each $\nu_n \in MZ(2^{n+1})$. Let $f \in X^p$.*

(a) *We have*

$$f = \sum_{n=0}^{\infty} \tau_n(\aleph;f). \tag{55}$$

(b) *If $0 < \rho \le \infty$, $0 < \gamma < \infty$, then $f \in B_{p,\rho,\gamma}$ if and only if $\{\|f - \sigma_{2^n}(f)\|_p\} \in \mathsf{b}_{\rho,\gamma}$. In turn, $f \in B_{p,\rho,\gamma}$ if and only if $\{\|\tau_n(f)\|_p\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$.*

(c) *Let $\aleph = \{\nu_n\}$ be a sequence of measures such that each $\nu_n \in MZ(2^{n+1})$, $f \in X^\infty$, $0 < \rho \le \infty$, and $0 < \gamma < \infty$. Then $f \in B_{\infty,\rho,\gamma}$ if and only if $\{\|f - \sigma_{2^n}(\nu_n;f)\|_\infty\} \in \mathsf{b}_{\rho,\gamma}$. In turn, $f \in B_{\infty,\rho,\gamma}$ if and only if $\{\|\tau_n(\aleph;f)\|_\infty\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$.*

(d) *If $f \in L^2$, then*

$$\|f\|_2^2 \sim \sum_{n=0}^{\infty} \|\tau_n(f)\|_2^2. \tag{56}$$

The main purpose of this section is to show that (55) is a wavelet-like representation; i.e., the local behavior of the sequence $\{\tau_n(f)\}_{n=0}^\infty$ characterizes the membership of f in local Besov spaces, defined below.

Definition 5. If $x_0 \in \mathbb{R}$, the **local Besov space** $B_{p,\rho,\gamma}(x_0)$ is the space of all $f \in X^p$ with the following property : There exists a $\delta > 0$ such that for every infinitely differentiable function ϕ supported on $\mathbb{B}(x_0, \delta)$, $\phi f \in B_{p,\rho,\gamma}$. □

The wavelet-like representation property is described in the following theorem:

Theorem 10. *Let $1 \le p \le \infty$, $f \in X^p$, $x_0 \in \mathbb{R}$, $0 < \rho \le \infty$, and $0 < \gamma < \infty$. The following statements are equivalent:*

(a) $f \in B_{p,\rho,\gamma}(x_0)$.
(b) *There exists a* $\delta > 0$ *such that* $\{\|f - \sigma_{2^n}(f)\|_{p,\mathbb{B}(x_0,\delta)}\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$.
(c) *There exists a* $\delta > 0$ *such that* $\{\|\tau_n(f)\|_{p,\mathbb{B}(x_0,\delta)}\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$.

In the case of functions in X^∞, one can obtain a similar theorem also based on the samples of the target function at arbitrary points.

Theorem 11. *Let* $f \in X^\infty$, $x_0 \in \mathbb{R}$, $0 < \rho \le \infty$, *and* $0 < \gamma < \infty$. *Let* $\aleph = \{\nu_n\}$ *be a sequence of measures such that each* $\nu_n \in MZ(2^{n+1})$. *The following statements are equivalent:*

(a) $f \in B_{\infty,\rho,\gamma}(x_0)$.
(b) *There exists a* $\delta > 0$ *such that* $\{\|f - \sigma_{2^n}(\nu_n;f)\|_{\infty,\mathbb{B}(x_0,\delta)}\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$.
(c) *There exists a* $\delta > 0$ *such that* $\{\|\tau_n(\aleph;f)\|_{\infty,\mathbb{B}(x_0,\delta)}\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$.

We will prove Theorem 11 in some detail, and then indicate the changes required to prove Theorem 10.

Proof of Theorem 11. In this proof, we will choose and fix an integer $S > \gamma + 3$. All constants may depend upon x_0, δ, and S.

Let (a) hold, and $\delta > 0$ be such that for every infinitely differentiable function ϕ supported on $\mathbb{B}(x_0,\delta)$, $\{E_{2^n,\infty}(\phi f)\}_{n=0}^{\infty} \in \mathsf{b}_{\rho,\gamma}$. In this part of the proof, let ϕ be an infinitely differentiable function supported on $\mathbb{B}(x_0,\delta)$ and equal to 1 on $\mathbb{B}(x_0,3\delta/4)$. We use the first estimate in (15) and (39) (with $S+1$ in place of S) to conclude that for $x \in I = \mathbb{B}(x_0,\delta/2)$,

$$
\begin{aligned}
|\sigma_{2^n}(\nu_n;(1-\phi)f,x)| &= \left|\int_{\mathbb{R}\setminus\mathbb{B}(x_0,3\delta/4)} (1-\phi(y))f(y)\Phi_n(x,y)d\nu_n(y)\right| \\
&\le c\|f\|_\infty \int_{\mathbb{R}\setminus\mathbb{B}(x_0,3\delta/4)} |\Phi_n(x,y)|d|\nu_n|(y) \\
&\le c\|f\|_\infty \int_{\mathbb{R}\setminus\mathbb{B}(x,\delta/4)} |\Phi_n(x,y)|d|\nu_n|(y) \le c2^{-nS}\|f\|_\infty. \qquad (57)
\end{aligned}
$$

Therefore, (38) leads to

$$
\begin{aligned}
\|f - \sigma_{2^n}(\nu_n;f)\|_{\infty,I} &= \|\phi f - \sigma_{2^n}(\nu_n;f)\|_{\infty,I} \\
&\le \|\phi f - \sigma_{2^n}(\nu_n;\phi f)\|_{\infty,I} + \|\sigma_{2^n}((1-\phi)f\|_{\infty,I} \\
&\le c\left\{E_{2^{n-1},\infty}(\phi f) + 2^{-nS}\|f\|_\infty\right\}. \qquad (58)
\end{aligned}
$$

Since $S > \gamma + 3$, each of the sequences $\{E_{2^{n-1},\infty}(\phi f)\}_{n=0}^{\infty}$ and $\{2^{-nS}\|f\|_\infty\}_{n=0}^{\infty}$ belongs to $\mathsf{b}_{\rho,\gamma}$. Therefore, (58) implies the statement in part (b).

Conversely, let part (b) hold, and ϕ be any infinitely differentiable function supported on $I = \mathbb{B}(x_0, \delta)$. Since ϕ is in particular $2S$ times continuously differentiable, the direct theorem of approximation [14, Theorem 4.2.1] shows that for $n \ge c$, there exists $R_n \in \Pi_{2^n}$ such that $\|R_n\|_\infty \le c$, and

$$\|\phi - R_n\|_\infty \le c2^{-nS}. \tag{59}$$

Therefore, using the notation introduced in (13),

$$\begin{aligned}
\tilde{E}_{2^{n+1},\infty}(\phi f) &\le \|\phi f - R_n \sigma_{2^n}(\nu_n; f)\|_\infty \\
&\le \|\phi(f - \sigma_{2^n}(\nu_n; f))\|_\infty + \|(\phi - R_n)\sigma_{2^n}(\nu_n; f)\|_\infty \\
&\le c\left\{\|(f - \sigma_{2^n}(\nu_n; f)\|_{\infty,I} + \|\phi - R_n\|_\infty \|\sigma_{2^n}(\nu_n; f)\|_\infty\right\} \\
&\le c\left\{\|(f - \sigma_{2^n}(\nu_n; f)\|_{\infty,I} + c2^{-nS}\|f\|_\infty\right\}.
\end{aligned}$$

As before, the statement in part (b) now leads to $\{\tilde{E}_{2^n,\infty}(\phi f)\}_{n=0}^\infty \in \mathsf{b}_{\rho,\gamma}$. In view of Corollary 3.1, this implies the statement in part (a).

The equivalence of parts (b) and (c) follows from (55), and an application of the discrete Hardy inequalities [5, p. 27]. □

Proof of Theorem 10. The proof is almost verbatim the same as that of Theorem 11, except for one difference, which we now point out. We continue the notation as in the proof of (a)⇒ (b). All the constants in this proof will depend upon x_0 and δ. As shown in (57) (with the Lebesgue measure in place of ν_n),

$$\|\sigma_{2^n}((1 - \phi)f\|_{\infty,I} \le c2^{-nS}\|f\|_\infty, \qquad f \in L^\infty. \tag{60}$$

If $f \in L^1$, then (15) (with $S + 1$ in place of S) implies that

$$\begin{aligned}
&\int_I |\sigma_{2^n}((1 - \phi)f, x)|dx \\
&\le \int_I \int_{\mathbb{R}\setminus\mathbb{B}(x_0, 3\delta/4)} |(1 - \phi(y))f(y)||\Phi_n(x, y)|dydx \\
&\le c\|f\|_1 \sup_{y\in\mathbb{R}\setminus\mathbb{B}(x_0, 3\delta/4)} \int_I |\Phi_n(x, y)|dx \le c2^{-nS}\|f\|_1.
\end{aligned} \tag{61}$$

The Riesz–Thorin interpolation theorem applied with the operator $f \mapsto \sigma_{2^n}((1 - \phi)f)$, together with (60) and (61), now implies that for $1 \le p \le \infty$,

$$\|\sigma_{2^n}((1 - \phi)f\|_{p,I} \le c2^{-nS}\|f\|_p, \qquad f \in L^p.$$

The remainder of the proof is almost verbatim the same as that of Theorem 11. □

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Approximating the Riemann Zeta and Related Functions

Frank Stenger

This chapter is written in memory of Q.I. Rahman

Abstract In this chapter we study the well-known function G, as well as some other functions that have the same zeros as the Riemann zeta function $\zeta(z)$ in the critical strip. To this end, we first derive a Fourier series expansion of G. Next, we use asymptotic methods to derive another function which also has the same zeros in the critical strip as $\zeta(z)$, but which lacks the extreme oscillatory behavior and extreme amplitude values that $\zeta(z)$ possesses, and which is therefore more suitable for computational purposes.

Keywords Riemann-zeta function • Sinc approximation • Asymptotic approximation

AMS Subject Classification: 42A15, 42A05, 41A55, 41A20

1 Introduction and Summary

Q.I. Rahman posed a mathematical problem to me many years ago when I was a visitor at the Centre de Récherches Mathématiques at the Université de Montréal. I was lucky to have been able to solve that problem for him, and I soon forgot about it. However, about 2 years later I received a reprint [11], authored by him and me, which surprised me, albeit mildly, inasmuch as I knew Rahman to be a very kind person. Rahman and Schmeisser [10] improved the result of [11], and Lachance et al. [9] later obtained the exact solution to that problem.

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P. Erdős frequently visited the University of Montreal, and that is how Rahman learned about this problem. Erdős' interest in this subject stemmed from work with P. Turán, who worked on problems associated with Riemann's hypothesis (see, e.g., [5]).

It thus seems appropriate to write this chapter about the Riemann zeta function, and the Riemann Hypothesis. Section 2 thus begins with a derivation of the well-known function, G , which has the same zeros as the zeta function in the critical strip, $\mathcal{D} = \{z \in C : 0 < \Re z < 1\}$. Section 3 then describes some explicit Fourier series and Fourier polynomials that are obtained by approximating G using Sinc quadrature [§1.5.7 of 15]. These formulas may be used to approximate $G(z)$ for values of z in $\mathcal{D}$ that are not unduly large.[1] Section 4 presents some novel asymptotic approximations in order to facilitate the approximation of G for very large[2] values of z. Section 5 presents two examples of applications of the results.

The literature is replete with excellent papers about computing the Riemann zeta function (see, e.g., [2, 8, 13], with the latter describing a procedure based on Sinc approximation). There also exist several asymptotics papers, but those are based on the Euler–MacLauren formula; they require partial sums of the series representation of the zeta function, they yield results based on fixed numerical values of z, and I believe that the formulas derived here are more suitable for studying $\zeta(z)$ for arbitrarily large values of z.

2 Function Related Riemann Zeta

The usual series expansion of the Riemann Zeta function can be obtained via term-wise integration of the expression

$$\begin{aligned}\zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1}}{e^x - 1}\, dx \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} e^{-x}\, x^{z-1} \left(1 + e^{-x} + e^{-2x} + \ldots\right) dx\,,\end{aligned} \tag{1}$$

where $\Gamma(z)$ denotes the usual Gamma function. Unfortunately, the above infinite series converges only in the region $\Re z > 1$, and it is thus not convenient for studying

[1]This formula may be used to approximate G on an interval where the usual FFT formula is applicable.

[2]The formulas of Sects. 3 and 4 are applicable on overlapping regions.

the zeros of the Zeta function in the critical strip, $\mathcal{D} = \{z \in C : 0 < \Re z < 1\}$. However, the above series for $\zeta(z)$ immediately yields

$$\frac{2}{2^z}\,\zeta(z) = \sum_{n=1}^{\infty} \frac{2}{(2\,n)^z}\,, \tag{2}$$

and by subtracting the series (2) from the series in (1), and then proceeding as in (1) above, we immediately get

$$\left(1 - \frac{2}{2^z}\right)\,\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x + 1}\,dx\,. \tag{3}$$

Theorem 2.1. *Set*

$$G(z) = \int_0^\infty \frac{x^{z-1}}{e^x + 1}\,dx\,. \tag{4}$$

Then G is analytic on the right half plane, and moreover, the zeros of G coincide with the zeros of the Riemann zeta function in the critical strip, $\mathcal{D} = \{z \in C : 0 < \Re z < 1\}$.

Proof. The analyticity of G on the right half plane follows by inspection of the integral (4). That G and $\zeta(z)$ have the same zeros in $\mathcal{D}$ follows from (3) as a consequence of the fact that neither of the functions $\left(1 - \frac{2}{2^z}\right)$ and $1/\,\Gamma(z)$ vanishes in $\mathcal{D}$. ■

Corollary 2.2. *The Riemann Hypothesis is equivalent to having all zeros of G in the critical strip* $\mathcal{D}$ *be on the line* $\{z \in C : \sigma = \Re z = 1/2\}$.

3 Fourier Series Approximation of G

Let us briefly recall the conditions for convergence of Sinc quadrature from [Stp, §4.2] or [15, §1.5.7], or [15].

Theorem 3.1. *Assume that the function f in the integral*

$$I = \int_0^\infty f(x)\,dx \tag{5}$$

satisfies the following assumptions:

1. *f is analytic in the sector, $S_\varepsilon \equiv \{x \in C : |\arg(x)| < \pi/2 - \varepsilon\}$, with $\varepsilon \in (0\,,1)$ an arbitrary positive constant; and*
2. *$|f(x)| = \mathcal{O}\left(x^{\sigma-1}\exp(-|x|)\right)$, $x \in S_\varepsilon$, where x is a positive number. Set $\beta = 2\,\pi\,(\pi/2 - \varepsilon)$.*

(a) *Then, [Stp, Theorem 4.2.2 (b)] we have for positive numbers h and C, with C depending only on f, such that*

$$\left| I - h \sum_{k=-\infty}^{\infty} e^{k\,h} f\left(e^{kh}\right) \right| < C \, \exp\left(-\frac{\beta}{h}\right) . \tag{6}$$

(b) *By taking* $h = (\beta/N)^{1/2}$ *and* $M = [N/\sigma]$, *there exists a constant* C' *depending only on g, such that*

$$\left| I - h \sum_{k=-M}^{N} e^{k\,h} f\left(e^{kh}\right) \right| < C' \, \exp\left(-(\beta\, N)^{1/2}\right) . \tag{7}$$

Now, by taking $f(x) = x^{z-1}/(\exp(x)+1)$, we get a function f which not only depends on x but also on a complex parameter z. Furthermore, if $z \in D_\varepsilon = \{z \in C : \varepsilon < \Re z < 1 - \varepsilon\}$ (where we assume that $0 < \varepsilon < 1/2$), then this function f does, in fact, satisfy all of the above conditions, enabling us to arrive at the following result.

Theorem 3.2. *Let G be defined by the integral (4), let* $z = \sigma + i\,t \in D_\varepsilon$, *define M and N and h as in Theorem 3.1, set*

$$a(\sigma, x) = \frac{\exp(\sigma\, x)}{1 + \exp\left(e^x\right)} , \tag{8}$$

and set

$$\begin{aligned} G_h(\sigma + i\,t) &= \begin{cases} h \displaystyle\sum_{k=-\infty}^{\infty} a(\sigma, kh)\, e^{i\,k\,h\,t} \text{ if } |t| < \pi/h\,, \\ 0 \text{ if } |t| > \pi/h\,. \end{cases} \\ G_{h,M,N}(\sigma + i\,t) &= \begin{cases} h \displaystyle\sum_{k=-M}^{N} a(\sigma\,, k\,h)\, e^{i\,k\,h\,t} \text{ if } |t| < \pi/h \\ 0 \text{ if } |t| > \pi/h\,. \end{cases} \end{aligned} \tag{9}$$

Then, in the notation of Theorem 3.1, we have for all $z = \sigma + i\,t \in \mathcal{D}_\varepsilon$, *and for constants* C_ε *and* C'_ε *depending only on ε and a, that*

$$\begin{aligned} |G(\sigma + i\,t) - G_h(\sigma + i\,t)| &< C_\varepsilon \, \exp\left(-\tfrac{\beta}{h}\right) , \\ |G(\sigma + i\,t) - G_{M,N,h}(\sigma + i\,t)| &< C'_\varepsilon \exp\left(-(\beta\, N)^{1/2}\right) . \end{aligned} \tag{10}$$

Remark 3.3. The function G of (8) can also be expressed as a Fourier transform. For, in the notation of (8) above,

$$G(\sigma + i\,y) = \int_R a(\sigma\,, t)\, e^{i\,t\,y}\, dt\,,$$

$$a(\sigma, t) = \frac{1}{2\,\pi} \int_R G(\sigma + iy)\, e^{-i\,t\,y}\, dt, \tag{11}$$

$$G(1/2 + i\,y) = \int_R a(1/2\,, t)\, e^{i\,t\,y}\, dy\,.$$

It is easy to deduce that G_h and $G_{M,N,h}$ do, in fact, converge to G uniformly on the strip D_ε, as $h \to 0$. It follows, therefore, that these functions converge to G at all points of the critical strip $\mathcal{D}$. Clearly, $a(\sigma\,, \cdot) \in \mathbf{L}^1(0\,, \infty)$ for all $\sigma \in (0\,, 1)$.

4 Asymptotic Approximations

There are a number of papers that enable asymptotic estimates of the Riemann zeta function; see, e.g., [6, 13] and the references in these papers. In this section we present an approach, which follows the one used to get the asymptotic expansion of the Gamma function [3]. The excellent thesis [7] and the papers [13] and [6] start with a partial sum of the series for $\zeta(z)$ as given in (1) and then to express the remainder with an integral which must somehow be evaluated, an evaluation that is usually carried out via use of the Euler–MacLauren formula, with this formula requiring the evaluation of an asymptotic series that has limited accuracy.[3]

Similarly, the usual asymptotic expression for the Gamma function $\Gamma(z)$ as obtained in [3] is the leading term i.e., Stirling's formula-times an integral factor, which is a non-convergent power series in powers of $1/z$, and which also only has limited accuracy. In this section we present an approach which is similar to that used to get the asymptotic expansion of the Gamma function.

The procedures of Erdély and Wyman [4] as well as that in Copson [3] will thus be used to get asymptotic estimates of the integral for $G(z)$ in (4) to derive a novel function which also has the same zeros in the critical strip as the Riemann Zeta function, but for which the values of amplitude and oscillations are less extreme. As for the case of the asymptotic expansion of the Gamma function, we shall assume at the outset that z is real.

[3] W. Galway shows, there is no difference between the asymptotic expansion and use of the trapezoidal rule for the large values of z of his computations.

Let us first sketch the procedure of Erdély and Wyman [4]. We are interested only in dominant behavior of the asymptotic expansions, and we shall thus omit the details required of getting a complete asymptotic expansion, which can be found in [4]. Our aim is to soon write another paper based on the novel formulas obtained here.

Starting with the function G as defined in (4), we replace x with $z\,x$ in this integral, to get

$$G(z) = z^z \int_0^\infty \frac{x^{-1}\, \exp(-z\,(x - \log(x)))}{1 + \exp(-z\,x)}\, dx\,. \tag{12}$$

Next, we locate the *critical point* by solving the equation $(d/dx)(x - \log(x)) = 0$; this yields $x = 1$. The critical points are the points whose neighborhood contains the complete asymptotic expansion.

Then, replacing x with $1 + x$ (to replace the critical point $x = 1$ with $x = 0$) yields

$$\begin{aligned} G(z) &= \left(\frac{z}{e}\right)^z \int_{-1}^\infty h(z,x)\, dx \\ h(z,x) &:= \frac{\exp(-z\,(x - \log(1+x)))}{(1+x)\,(1 + \exp(-z\,x - z))}\,. \end{aligned} \tag{13}$$

Expanding[4]

$$-\,z\,(x - \log(1+x)) = -z\,x^2/2 + z\,x^3/3 - \ldots\,, \tag{14}$$

we select a neighborhood of the origin, such as $N_\alpha = \{x \in R : |x| < z^{-\alpha}\}$, for some $\alpha > 0$, with α a positive number to be determined, such that one of the terms of this expansion *dominates* in N_α. In our case, this neighborhood has the property that all but the dominant term approach zero in this neighborhood, whereas when $x = z^{-\alpha}$, the dominant term approaches infinity. If the first term of the expansion on the right-hand side of (14), i.e., if $-z\,x^2/2$ is to dominate, then we must have $z\,x^{-2\,\alpha} = z^{1-2\,\alpha} \to \infty$ as $z \to \infty$, requiring $1 - 2\,\alpha > 0$, i.e., $\alpha < 1/2$. For all other terms, we require $z\,x^n \to 0$ in N_α for $n = 3\,, 4\,, \ldots$, i.e., $n\,\alpha > 1$ for $n \geq 3$. This yields $\alpha > 1/3$. Combining with $\alpha < 1/2$, this means that any α in the range $1/3 < \alpha < 1/2$ will suffice. One such α is, e.g., $\alpha = 5/12$.

[4]This present analysis is for $z \to \infty$, with z real. By taking $x = 0$, the function $(1 + \exp(-zx - z))^{-1} = 1 - \exp(-z))^{-1} := 1 + 0$, for large positive values of z, i.e., the power series of $(1 + \exp(-z(1 + x))$ only has one term that contributes to the asymptotic expansion, the remainder being of negative exponential order, and hence asymptotically zero. We have, however, included this function, since *it does* have an important contribution for large values of z in the critical strip.

Thus, setting

$$k(z\,,x) := \exp(-z(x - \log(1+x)) + z\,x^2/2) \tag{15}$$

and expanding in powers of x, we get

$$k(z,x) = \exp(z\,(x^3/3 + \mathcal{O}(x^4)))\,, \quad x \to 0\,, \tag{16}$$

where, with $\alpha = 5/12$,

$$\begin{aligned}
&\int_{-1}^{\infty} h(z,x)\,dx \\
&= \int_{N_\alpha} \exp(-z\,x^2/2)\frac{k(z\,,x)}{(1+x)\,(1+\exp(-z-z\,x))}\,dx + E \\
&= (2z)^{-1/2}\int_{|y|<z^{-5/12}} \frac{k(z\,(2/z)^{1/2}\,y)}{(1+(2/z)^{1/2}\,y)(1+\exp(-z-(2/z)^{1/2}\,y)}\,e^{-y^2}\,dy \\
&\quad + E \\
&= (2/z)^{1/2}\int_{|y|<z^{-5/12}} e^{-y^2}\,\frac{1}{(1+x)\,(1+\exp(-z-z\,x))}dy \\
&\quad +\mathcal{O}(1/z) + \ E \\
&= (2/z)^{1/2}\int_{R} e^{-y^2}\,dy + E' \\
&:= \left(\frac{2\,\pi}{z}\right)^{1/2} + \ E'\,.
\end{aligned} \tag{17}$$

In (17) the terms E and E' can be shown using Watson's lemma (see [4]) to be of order $\mathcal{O}(\exp(-\gamma z))$ for some constant $\gamma > 0$; such terms decrease exponentially and hence they do not contribute to an asymptotic expansion involving powers of $1/z$, i.e., they are *asymptotically zero* with respect to such an asymptotic sequence.

Recall now, the Stirling formula result, that

$$\Gamma(z) = \left(\frac{2\,\pi}{z}\right)^{1/2}\left(\frac{z}{e}\right)^z\,(1+\mathcal{O}(1/z))\;,\; z \to \infty, \tag{18}$$

where, as shown in [3], this asymptotic relation holds for all $z \in \ C_\varepsilon \equiv \{C : |\arg(z)| < \pi - \varepsilon\}$ for any number ε in $(0\,,\pi/2)$.

Note also that the factor $(z/e)^z$ is non-vanishing in $\mathcal{D}$. Combining this with the above asymptotic derivation yields

Theorem 4.1. *Let $h(z,x)$ be defined as in (13), and let K be defined by the equation*

$$\begin{aligned} K(z) &= \int_{-1}^{\infty} h(z,x)\, dx \\ &= \int_{-1}^{\infty} \frac{\exp(-z(x - \log(1+x))}{(1+x)(1+\exp(-z - z\,x))}\, dx \end{aligned} \tag{19}$$

Then K has the same zeros in $\mathcal{D}$ as the Riemann zeta function $\zeta(z)$.

Now let K be defined as in Theorem 4.1. Then

$$G(z) = \left(\frac{z}{e}\right)^z K(z)\,. \tag{20}$$

Let us next examine the $L(z) := G(z)/K(z)$, i.e., the factor preceding $K(z)$ in (20), for $z = \sigma + i\,t$, with $\sigma > 0$ fixed, and t large. The purpose of this examination is just to get an insight into the behavior of $G(\sigma + i\,t)$ for large t. Thus,

$$L(z) = \left(\frac{z}{e}\right)^z = |L(z)|\, e^{i\,\theta}\,. \tag{21}$$

As is shown in [3] that for $z = \sigma + i\,t$ with σ fixed in R,

$$|\Gamma(z)| = \left|\left(\frac{2\,\pi}{z}\right)^{1/2} \left(\frac{z}{e}\right)^z\right| = (\pi/(2\,\pi/|t|))^{1/2}|t|^{\sigma}\ \exp(-\pi\,|t|/2)\,(1 + O(1/|t|))\,, \tag{22}$$

as $t \to \infty$.

Note, also, if $z = \sigma + i\,t$ with $\sigma > 0$ fixed, then

$$\begin{aligned} (1+\exp(-z))^{-1} &= \frac{1}{2}\,\frac{\exp((\sigma + i\,t)/2)}{\cosh((\sigma + i\,t)/2)}\,; \\ |1+\exp(-z)|^{-1} &= \frac{|t|^{\sigma}\ \exp(-\pi\,|t|/2 + \sigma/2)}{(\sinh^2(\sigma/2) + \cos^2(t/2))^{1/2}}\,(1 + \mathcal{O}(1/|t|))\,. \end{aligned} \tag{23}$$

so that this term remains bounded, e.g., if $\sigma \ge 1/2$.

Furthermore, we have

$$\begin{aligned} \theta &= \Im \log(L(z)) \\ &= \sigma\ \arctan(t/\sigma) + t\ \log(|\sigma + i\,t|) - t - \arg(1 + e^{-(\sigma + i\,t)}) \\ &= \sigma\,(\pi/2 - \arctan(\sigma/t)) \\ &\quad + (t/2)\ \log(\sigma^2 + t^2) - t - \arctan\left(\frac{e^{-\sigma}\ \sin(t)}{1 + e^{-\sigma}\ \cos(t)}\right)\,. \end{aligned} \tag{24}$$

The following lemma summarizes the above derived result thus illustrates the extreme variation of the magnitude and the wild oscillatory behavior of the function L. The removal of L from G yields the function $K(z)$ which is thus affected less by such oscillatory phenomena, and should therefore be more suitable for studying the behavior of the Riemann zeta function for large values of z in the critical strip.

Lemma 4.2. *Let $L(z) = L(\sigma + i\,t)$ and θ be defined as in (21). Then, with $\sigma + i\,t \in \mathcal{D}$, with $\sigma \in (1/2 + \varepsilon, 1)$ fixed, and $t \to \pm\infty$,*

$$|L(\sigma + i\,t)| = (\pi/2)^{1/2} \frac{|t|^{\sigma}\, \exp(-\pi\,|t|/2 + \sigma/2)}{\left(\sinh^2(\sigma/2) + \cos^2(t/2)\right)^{1/2}} \left(1 + \mathcal{O}(1/|t|\right), \tag{25}$$

$$\theta = \sigma\,\pi/2 + t\,(\log(|t|) - 1) + \arctan\left(\frac{e^{-\sigma}\ \sin(t)}{1 + e^{-\sigma}\ \cos(t)}\right).$$

5 Examples

We present here two examples illustrating the application of our results.

Example 5.1. Let us use formula (9)-(b) to compute $G(z)$ via (b) using $M = 240$, $N = 160$ and $h = \pi/\sqrt{M}$. The printout is given in Fig. 1 for the interval $(13.5\,,14.5)$, which is known to contain the first zero of the Riemann zeta function. Note the small magnitude of the function G on this interval, as predicted in Sect. 4.

Example 5.2. An Asymptotic Estimate of $K(z)$

It follows from the above, that

$$K(z) = \int_{-1}^{\infty} \exp(-z\,x^2/2)\,p(z,x)\,dx\,; \tag{26}$$

$$p(z,x) = \frac{\exp(z\,x^2/2)\ \exp(-z(x - \log(1 + x))}{(1 + x)(1 + \exp(-z - z\,x))}.$$

This expression of course readily yields an asymptotic expansion for $K(z)$ with z real and $x \to \infty$. For example, taking only the x^0 term of the expansion of $p(z,x)$ in powers of x and integrating over $(-1\,,\infty)$, we get

$$K(z) \sim \left(\frac{2\,\pi}{z}\right)^{1/2} (1 + \exp(-z))^{-1}\,, \quad z \to \infty\,. \tag{27}$$

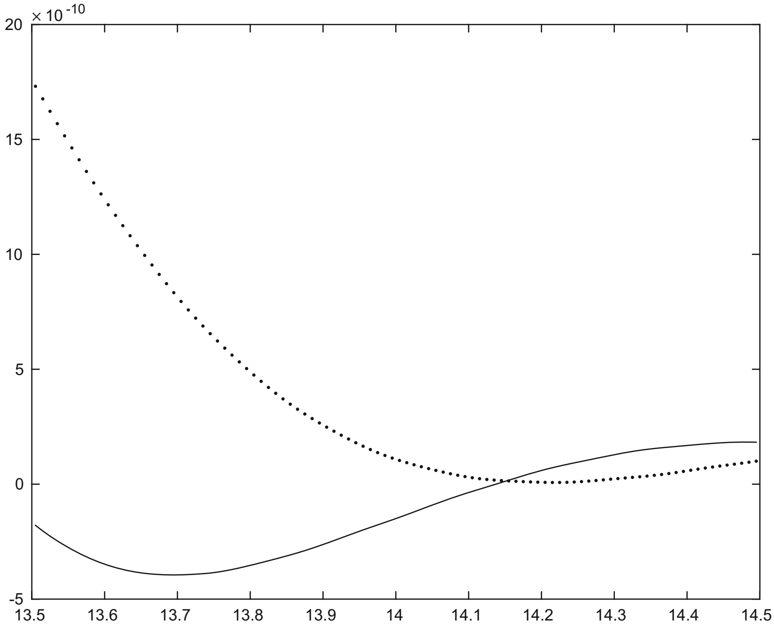

Fig. 1 Real (*solid line*) and imaginary (*dotted line*) parts of G

Note at this point, in view of our above estimates of L, that if $\sigma > 1/2$, then the function

$$\kappa(z) := \left(\frac{2\,\pi}{z}\right)^{-1/2} (1 + \exp(-z))K(z) \tag{28}$$

has the same zeros in the region $D_{1/2} := \{z \in C : 1/2 < \Re z < 1\}$ as the Riemann zeta function, $\zeta(z)$.

This leads to the question regarding the validity of the result (27), e.g., as $z = \sigma + i\,t \to \infty$ with σ fixed in $(1/2 + \varepsilon, 1)$, and with $\varepsilon \in (0\,, 1/2)$ as $t \to \infty$.

We shall again examine this approach via use of more careful analysis in a future paper, using, e.g., the results of [1, 12, 14] and §1.5.8 of [15]. We would like to be able to prove, for example, that given such an $\varepsilon > 0$, there exists a positive number $T = T(\varepsilon)$, such that whenever $t > T$, then $\zeta(z)$ has no zeros in the region $z \in D_{1/2} \cap (T\,, \infty)$. I believe that such a result could be an important step towards proving the Riemann Hypothesis.

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Overconvergence of Rational Approximants of Meromorphic Functions

Hans-Peter Blatt

Dedicated to the memory of Q. I. Rahman

Abstract Let E be a compact set in $\mathbb{C}$ with regular connected complement Ω, and let f be meromorphic on E with maximal Green domain of meromorphy $E_{\rho(f)}$, $\rho(f) < \infty$. We investigate rational approximants r_{n,m_n} of f with numerator degree $\le n$ and denominator degree $\le m_n$ and deduce overconvergence properties from geometric convergence rates of $f - r_{n,m_n}$ near the boundary of E if $n \to \infty$ and $m_n = o(n)$ (resp. $m_n = o(n/\log n)$) as $n \to \infty$. Moreover, results about the limiting distribution of the zeros of r_{n,m_n}, as well as for the distribution of the interpolation points of multipoint Padé approximation can be derived. Hereby, well-known results for polynomial approximation of holomorphic functions are generalized for rational approximation of meromorphic functions.

Keywords Rational approximation • Convergence in m_1-measure and capacity • Distribution of zeros • Harmonic majorant • Padé approximation

MSC: 41A20, 41A21, 41A25, 30E10, 30D35

1 Introduction: Polynomial Approximation

For $B \subset \mathbb{C}$, we denote by B° the set of interior points of E, by $\overline{B}$ its closure and by ∂B the boundary of B, and we use $\|\cdot\|_B$ for the supremum norm on B. A compact set B in $\mathbb{C}$, resp. $\overline{\mathbb{C}}$, is called a continuum if B is connected and consists of more than a single point.

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Let K be a compact set in the complex plane and let $\mathscr{B}(K)$ denote the collection of all probability measures with support in K. The logarithmic energy $I(\mu)$ of $\mu \in \mathscr{B}(K)$ is defined by

$$I(\mu) := \int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t)$$

and the energy V of K by

$$V := \inf\{I(\mu) : \mu \in \mathscr{B}(K)\}.$$

V is either finite or $+\infty$. The quantity

$$\operatorname{cap} K := e^{-V}$$

is called the capacity of K. The capacity of any set $B \subset \mathbb{C}$ is defined by

$$\operatorname{cap} B := \sup\{\operatorname{cap} K : K \subset B, K \text{ compact}\}.$$

If K is compact, then there exists a measure $\mu_K \in \mathscr{B}(K)$ such that $I(\mu_K) = -\log \operatorname{cap} K = V$. μ_K is called an equilibrium measure of K. If $\operatorname{cap} K > 0$, then μ_K is unique.

In the following, E is a compact set of $\mathbb{C}$ with regular connected complement $\Omega = \overline{\mathbb{C}} \setminus E$ in the extended complex plane $\overline{\mathbb{C}}$. The set Ω is called *regular* if there exists a Green function $G(z) = G(z, \infty)$ on Ω with pole at ∞ satisfying $G(z) \to 0$ as $z \in \Omega$ tends to the boundary $\partial\Omega$ of Ω. Then $\operatorname{cap} E > 0$ and

$$\lim_{z \to \infty} (G(z) - \log |z|) = -\log \operatorname{cap} E.$$

If E is a continuum, then Ω is regular (cf. [17, Theorem. I. 11, p. 7]).

For $\rho > 1$ we define the *Green domains* E_ρ by

$$E_\rho := \{z \in \Omega : G(z) < \log \rho\} \cup E$$

with boundaries $\Gamma_\rho := \partial E_\rho$. If Ω is regular, then the equilibrium measure μ_E of E exists and is unique, as well as the equilibrium measure $\mu_{\overline{E}_\rho}$ of $\overline{E}_\rho$ for all $\rho > 1$.

For $B \subset \mathbb{C}$, we denote by $C(B)$ the class of continuous functions on B, and $\mathscr{A}(B)$ (resp. $\mathscr{M}(B)$) represents the class of functions f that are holomorphic (resp. meromorphic) in some open neighborhood of B. Moreover, we will denote by $\mathscr{M}_m(B)$ the subset of functions f of $\mathscr{M}(B)$ such that for some neighborhood U of B, the function f has at most m poles in U, each pole counted with its multiplicity.

Let $f \in C(E) \cap \mathscr{A}(E^\circ)$, then by Mergelyan's theorem

$$\lim_{n \to \infty} \inf\{\|f - p\|_E : p \in \mathscr{P}_n\} = 0,$$

where $\mathscr{P}_n$ denotes the collection of all algebraic polynomials having degree at most n. If $\{p_n\}_{n\in\mathbb{N}}$, $p_n \in \mathscr{P}_n$, is a sequence of polynomials with $\lim_{n\to\infty} \|f - p_n\|_E = 0$, then the Bernstein–Walsh lemma [21, p. 77] implies that for any compact set S in $\Omega = \overline{\mathbb{C}} \setminus E$,

$$\lim_{n\to\infty} \max_{z\in S} \left(\frac{1}{n} \log |p_n(z)| - G(z) \right) \le 0,$$

i.e., the Green function $G(z)$ is a *harmonic majorant* for the sequence of *subharmonic functions* $(1/n) \log |p_n(z)|$, $n \in \mathbb{N}$, in Ω (cf. [20]).

If $f \in \mathscr{A}(E)$ is not an entire function, then there exists a maximal $\rho > 1$ such that f has a holomorphic continuation to E_ρ. Then a sequence $p_n \in \mathscr{P}_n$, $n \in \mathbb{N}$, is said to *converge maximally to f on E* if

$$\limsup_{n\to\infty} \|f - p_n\|_E^{1/n} = \frac{1}{\rho}. \tag{1}$$

For example, the polynomials p_n^* of best uniform approximation to f on E are maximally convergent. Moreover, Walsh ([19], Corollary on p. 81, Sect. 4.7) proved that for such maximally convergent polynomials

$$\limsup_{n\to\infty} \|f - p_n\|_{E_\sigma}^{1/n} = \frac{\sigma}{\rho}, \quad 1 < \sigma < \rho. \tag{2}$$

Then again the Bernstein–Walsh lemma implies that for the sequence $(1/n) \log |p_n(z)|$, $n \in \mathbb{N}$, the Green function

$$G_\sigma(z) = G(z) - \log \sigma$$

is a harmonic majorant in $\overline{\mathbb{C}} \setminus \overline{E}_\sigma$ for any σ, $1 < \sigma < \rho$. Hence, $G_\rho(z) = G(z) - \log \rho$ is a harmonic majorant for $\{(1/n) \log |p_n(z)|\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{C}} \setminus \overline{E}_\rho$, or

$$\limsup_{n\to\infty} \max_{z\in S} \left(\frac{1}{n} \log |p_n(z)| - G_\rho(f) \right) \le 0 \tag{3}$$

for any compact set $S \subset \overline{\mathbb{C}} \setminus \overline{E}_\rho$. Moreover, Walsh [19] proved that in (3) always the equality sign holds if $\{p_n\}_{n\in\mathbb{N}}$ is maximally converging to f on E and if S is a continuum. This observation leads Walsh to the following terminology [20]:

The sequence $\{(1/n) \log |p_n(z)|\}_{n\in\mathbb{N}}$ has the Green function $G_\rho(z)$ as an *exact harmonic majorant* if in (3) the equality sign holds for any continuum $S \in \overline{\mathbb{C}} \setminus \overline{E}_\rho$, i.e.,

$$\limsup_{n\to\infty} \max_{z\in S} \left(\frac{1}{n} \log |p_n(z)| - G_\rho(z) \right) = 0. \tag{4}$$

For (4) to be true for *any* continuum S it is sufficient that (4) holds for some compact set $S \subset \overline{\mathbb{C}} \setminus \overline{E}_\rho$ and S can be a single point [20].

As a consequence of (4), the normalized counting measures ν_n of the zeros of p_n converge weakly to the equilibrium measure $\mu_{\overline{E}_\rho}$ of $\overline{E}_\rho$ ([11], cf. [1]), where ν_n is defined for $B \subset \mathbb{C}$ by

$$\nu_n(B) := \frac{\#\text{ number of zeros of } p_n \text{ in } B}{\text{degree } (p_n)}$$

and the zeros are counted with respect to their multiplicity.

Another type of maximally convergent polynomials are Hermite–Lagrange interpolating polynomials p_n to f at points

$$z_{0,n}, z_{1,n}, \ldots, z_{n,n} \in E, \ n \in \mathbb{N}. \tag{5}$$

Let us introduce the probability measure τ_n such that

$$\tau_n(\{z_{\nu,n}\}) = \frac{c_{\nu,n}}{n+1} \tag{6}$$

where $c_{\nu,n}$ is the multiplicity of $z_{\nu,n}$ in (5). Then p_n, $n \in \mathbb{N}$, converge maximally to f on E if $\tau_n \underset{n\to\infty}{\overset{*}{\to}} \mu_E$ in the weak*-sense [21].

Conversely, let $f \in \mathscr{A}(E)$ with maximal parameter ρ of holomorphy, $1 < \rho < \infty$. If there exists Hermite–Lagrange interpolating polynomials p_n at the points (5), which are maximally convergent to f on E, then for the sequence of measures τ_n in (6) there exists a subsequence $\Lambda \subset \mathbb{N}$ such that the associated balayage measures $\widehat{\tau}_n$ to the boundary of E satisfy

$$\widehat{\tau}_n \overset{*}{\to} \mu_E \text{ as } n \in \Lambda, \ n \to \infty.$$

proved by Grothmann [12] if E_ρ is a domain.

2 Rational Approximation, Overconvergence in m_1-Measure

Let E be compact in $\mathbb{C}$ with regular connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, $G(z)$ denotes the Green function on Ω with pole at ∞.

Given $n, m \in \mathbb{N} \cup \{0\}$, let $\mathscr{R}_{n,m}$ denote the collection of all rational functions

$$\mathscr{R}_{n,m} := \{r = p/q : p \in \mathscr{P}_n, q \in \mathscr{P}_m, \ q \not\equiv 0\}.$$

For fixed $f \in \mathscr{M}(E)$ we define $\rho = \rho(f)$ as the maximal parameter $\rho > 1$ such that $f \in \mathscr{M}(E_\rho)$. $\rho(f) = \infty$ if and only if f is meromorphic on $\mathbb{C}$.

Our starting point is a sequence of rational approximants $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ to $f \in \mathscr{M}(E) \setminus \mathscr{M}(\mathbb{C})$ with

$$\lim_{n\to\infty} m_n = \infty \text{ and } m_n = o(n/\log n) \text{ as } n \to \infty, \tag{7}$$

and

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leq \frac{1}{\rho(f)}. \tag{8}$$

Since f is meromorphic on E, but not on $\mathbb{C}$, the parameter $\rho(f)$ satisfies $1 < \rho(f) < \infty$. By Walsh's theorem, we know that such a sequence $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ with (7) and (8) always exists.

In this section, we consider the overconvergence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ to f in m_1-measure (cf. [10]):

Let B be a subset of $\mathbb{C}$ and set

$$m_1(B) := \inf \left\{ \sum_{\nu} |U_\nu| \right\},$$

where the infimum is taken over all denumerable coverings $\{U_\nu\}$ of B by disks U_ν and $|U_\nu|$ is the radius of U_ν. Then the overconvergence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ outside E (resp. ∂E) is described by

Theorem 2.1 ([4], Theorem 2.1). *Let E be compact with regular connected complement. Under the conditions (7) and (8), for any $\varepsilon > 0$ there exists a subset $\Omega(\varepsilon) \subset \mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\overline{E}_\sigma \setminus \Omega(\varepsilon)}^{1/n} \leq \frac{\sigma}{\rho(f)} \tag{9}$$

for any σ, $1 < \sigma < \rho(f)$.

If E is connected in Theorem 2.1, we obtain from (9) that for any σ, $1 < \sigma < \rho(f)$,

$$\liminf_{\sigma^*\to\sigma} \limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\Gamma_{\sigma^*}}^{1/n} \leq \frac{\sigma}{\rho(f)} \tag{10}$$

(cf. [6], Remark 4).

The property (9) was called in [6] *maximal m_1-convergence.* Furthermore, we remark that the exceptional set $\Omega(\varepsilon)$ can be defined explicitly: Let us denote by

$$\eta_1, \eta_2, \ldots, \eta_s$$

the finite number of poles of f in E, notified according to their multiplicities. Since f is meromorphic in $E_{\rho(f)}$, the total number of poles of f in $E_{\rho(f)}$ is denumerable. Hence we may arrange the poles η_i of f outside E, $i > s$, such that

$$G(\eta_i) \leq G(\eta_{i+1}) \text{ for } i \geq s.$$

If the number of poles of f on $E_{\rho(f)}$ is finite, say $\widetilde{s}$, then we set $\eta_{\widetilde{s}+j} = \eta$, $j \geq 1$, where η is a fixed point on $E_{\rho(f)}$. Therefore, in any case we have defined an infinite sequence

$$\{\eta_i\}_{i=1}^{\infty} \text{ with } \lim_{i\to\infty} G(\eta_i) = \rho(f),$$

where all poles of f in $E_{\rho(f)}$ are listed. Next, let $\xi_{n,i}$ denote the poles of r_{n,m_n}, according to their multiplicities again. For $0 < \varepsilon < 1$, we define the open sets

$$\Omega_n(\varepsilon) := \bigcup_{\xi_{n,i}} \left\{ z \in \mathbb{C} : |z - \xi_{n,i}| < \frac{\varepsilon}{2n^3} \right\} \cup \left\{ z \in \mathbb{C} : |z - \eta_n| < \frac{\varepsilon}{2n^3} \right\}$$

and

$$\Omega(\varepsilon) := \bigcup_{n=1}^{\infty} \Omega_n(\varepsilon). \tag{11}$$

Then $m_1(\Omega(\varepsilon)) < \varepsilon$ and in [4] it was shown that $\Omega(\varepsilon)$ satisfies Theorem 2.1.

For obtaining results about the distribution of r_{n,m_n} we have to consider functions f which guarantee that the inequality (9) is an equality, and consequently, that the growth of r_{n,m_n} outside $E_{\rho(f)}$ can be characterized by the Green function of $\overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$.

To this end we use a result of Gonchar [9] connecting the rate of rational approximation with the property of single-valuedness of an analytic function in the neighborhood of an isolated singular point of the boundary of $E_{\rho(f)}$:

Let $z_0 \in \Gamma_{\rho(f)} = \partial E_{\rho(f)}$ and U a neighborhood of z_0 such that f can be continued to any point of $U \setminus \{z_0\}$ to a function which is locally holomorphic. Moreover, let $K \subset U$ be a continuum such that $z_0 \in K$ and $(K \setminus \{z_0\}) \subset E_{\rho(f)}$.

Define

$$K_\varepsilon := \{z \in K : |z - z_0| \geq \varepsilon\},$$
$$\rho_n(f, K_\varepsilon) := \inf_{r \in \mathscr{R}_{n,n}} \|f - r\|_{K_\varepsilon}^{1/n},$$
$$\rho(f, K_\varepsilon) := \liminf_{n\to\infty} \rho_n(f, K_\varepsilon)$$

and

$$\rho(f, K) := \sup_{\varepsilon > 0} \rho(f, K_\varepsilon). \tag{12}$$

Since f is holomorphic on K_ε for any $\varepsilon > 0$, we have $0 \le \rho(f, K_\varepsilon) < 1$ and

$$\rho(f, K) = \lim_{\varepsilon \to 0} \rho(f, K_\varepsilon) \le 1.$$

Theorem 2.2 (Gonchar [9], Theorem 1). *Under the above conditions, let $\rho(f, K) < 1$. Then f is single-valued in $U \setminus \{z_0\}$.*

Hence, this characterization leads to the following definition.

Definition. Let $f \in \mathscr{M}(E)$ with $\rho(f) < \infty$. A point z_0 of the boundary of $E_\rho(f)$ is called *a singularity of multivalued character of f* if there exists a neighborhood U of z_0 such that f can be continued to any point of $U \setminus \{z_0\}$ and f is locally holomorphic, but not single-valued.

For example, a branch point is a special type of a singularity of multivalued character.

Theorem 2.3. *Let E be a compact, connected set in $\mathbb{C}$ with regular connected complement and let (7) and (8) hold. If there exists a singularity of multivalued character on the boundary of $E_{\rho(f)}$, then*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} = \frac{1}{\rho(f)} \tag{13}$$

and for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon)$ in $\mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{E_\sigma \setminus \Omega(\varepsilon)}^{1/n} = \frac{\sigma}{\rho(f)} \tag{14}$$

for any σ, $1 < \sigma < \rho(f)$. Moreover, $\Omega(\varepsilon)$ can be defined by (11).

Proof. We note that (13) is the first part of Theorem 2 in [6].

Concerning (14), we know from Theorem 2.1 that for the sets $\Omega(\varepsilon)$, defined by (11), the inequality (9) holds. Hence, if we assume that (14) is not true, then for some σ, $1 < \sigma < \rho(f)$, we have

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{E_\sigma \setminus \Omega(\varepsilon)}^{1/n} < \frac{\sigma}{\rho(f)},$$

where $\varepsilon > 0$ and $\Omega(\varepsilon)$ is defined by (11). Then there exists a parameter σ^*, $1 < \sigma^* < \rho(f)$, such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\Gamma_{\sigma^*}}^{1/n} \le \frac{1}{\tau} < \frac{\sigma^*}{\rho(f)}$$

(compare Remark 4 in [6]). Following the proof of Theorem 2 in [6], we can construct a continuum K such that $(K \setminus \{z_0\}) \subset E_{\rho(f)}$ and

$$\rho(f, K) < 1,$$

where $\rho(f, K)$ is defined in (12). This is a contradiction to the above Theorem 2.2 of Gonchar, since z_0 is a singularity of multivalued character. □

The property (14) is essential for obtaining estimates of the growth of r_{n,m_n} in $\Omega = \overline{\mathbb{C}} \setminus E_{\rho(f)}$. Hence, let us introduce the following class of approximants.

Definition. Let $f \in \mathcal{M}(E)$ and $\rho(f) < \infty$. A sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, is called *exactly* m_1*-maximally convergent* to f on $E_{\rho(f)}$ if for any $\varepsilon > 0$ there exists a subset $\Omega(\varepsilon)$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n}_{\overline{E}_\sigma \setminus \Omega(\varepsilon)} \le \frac{\sigma}{\rho(f)} \tag{15}$$

and

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \|f - r_n\|^{1/n}_{\overline{E}_\sigma \setminus \Omega(\varepsilon)} = \frac{\sigma}{\rho(f)} \tag{16}$$

for any σ, $1 < \sigma \le \rho(f)$.

Remark. In [6], a sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, which is m_1-maximally convergent to f on $E_\rho(f)$, was called *exact* if the convergence rate (8) holds and

$$\liminf_{\sigma^*\to\sigma} \limsup_{n\to\infty} \|f - r_n\|^{1/n}_{\Gamma_{\sigma^*}} = \frac{\sigma}{\rho(f)}$$

for any σ, $1 < \sigma < \rho(f)$, i.e., this condition replaces (16).

In any case, the condition (8) is not necessary for getting estimates of the growth of $r_n = r_{n,m_n}$ outside $E_{\rho(f)}$ if $\{m_n\}_{n\in\mathbb{N}}$ satisfies (7) and (15) holds. Moreover, if E is connected both definitions are equivalent up to the condition (8). This observation immediately leads to

Theorem 2.4. *Let E be a compact, connected set in $\mathbb{C}$ with regular connected complement and let (7) hold. Moreover, let $f \in \mathcal{M}(E)$ with $\rho(f) < \infty$ and let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, be an exact m_1-maximally convergent sequence to f on $E_{\rho(f)}$. Then the Green function*

$$G_{\rho(f)}(z) = G(z) - \log \rho(f) \;\; \textit{of} \;\; \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$$

is an exact harmonic majorant for the sequence

$$\left\{\frac{1}{n}\log|p_n(z)|\right\}_{n\in\mathbb{N}} \quad in\ \overline{\mathbb{C}}\setminus\overline{E}_{\rho(f)},$$

where $r_{n,m_n} = p_n/q_{m_n}$ *is normalized by* $\|q_{m_n}\|_E = 1$.

Proof. Theorem 2.4 was proved in [6], Theorem 3, for the case that $r_{n,m_n} = p_n^*/q_{m_n}^*$ was normalized with respect to E_τ, where $0 \in E_\tau$ and $\tau > \rho(f)$, i.e.

$$q_{m_n}^*(z) = q_{m_n,1}^*(z)q_{m_n,2}^*(z) \tag{17}$$

with

$$q_{m_n,1}^*(z) = \prod_{\xi_{n,i}\in E_\tau}(z-\xi_{n,i}),\ \ q_{m_n,2}^*(z) = \prod_{\xi_{n,i}\notin E_\tau}\left(1-\frac{z}{\xi_{n,i}}\right) \tag{18}$$

and $\xi_{n,i}$ denotes the zero of $q_{m_n}^*$. Then

$$\left\{\frac{1}{n}\log|p_n^*(z)|\right\}\ n\in\mathbb{N}$$

has the Green function $G_{\rho(f)}(z)$ as an exact harmonic majorant in $\overline{\mathbb{C}}\setminus\overline{E}_{\rho(f)}$.

Let $r_{n,m_n} = p_n/q_{m_n}$ with $\|q_{m_n}\|_E = 1$ as in Theorem 2.4. We define $p_n^* \in \mathscr{P}_n$, $q_{m_n}^* \in \mathscr{P}_{m_n}$ such that

$$\deg(p_n^*) = \deg(p_n) \text{ and } \deg(q_{m_n}^*) = \deg(q_{m_n}),$$

and moreover, we can define a constant $\gamma_n \neq 0$ such that

$$p_n = \gamma_n p_n^* \text{ and } q_{m_n} = \gamma_n q_{m_n}^*$$

and $q_{m_n}^*$ is normalized according to (17), (18). Let $\ell_{n,1} = \deg(q_{m_n,1}^*)$ and $\ell_{n,2} = \deg(q_{m_n,2}^*)$. Then

$$\begin{aligned}\|q_{m_n}^*\|_E &\geq \|q_{m_n,1}^*\|_E \min_{z\in E}|q_{m_n,2}^*(z)|\\ &\geq (\text{cap } E)^{\ell_{n,1}}\alpha_1^{\ell_{n,2}},\end{aligned}$$

where

$$\alpha_1 = \min_{z\in E}\ \min_{\xi\in\overline{\mathbb{C}}\setminus E_\tau}\left|1-\frac{z}{\xi}\right| > 0.$$

Define

$$\alpha := \min(\alpha_1, \text{cap } E, 1),$$

then

$$\|q^*_{m_n}\|_E \geq \alpha^{m_n}. \tag{19}$$

On the other hand,

$$\|q^*_{m_n}\|_E \leq \|q^*_{m_n,1}\|_E \; \|q^*_{m_n,2}\|_E \leq \beta^{m_n}, \tag{20}$$

where

$$\beta = \max(1, \beta_1, \beta_2)$$

and

$$\beta_1 := \max_{z\in E} \max_{\xi\in \overline{E}_\tau} |z-\xi| \text{ and } \beta_2 := \max_{z\in E} \max_{\xi\in\overline{\mathbb{C}}\setminus E_\tau} \left|1 - \frac{z}{\xi}\right|$$

Using (19) and (20), we get

$$|\gamma_n|\alpha^{m_n} \leq 1 = \|q_{m_n}\|_E = |\gamma_n|\|q^*_{m_n}\|_E \leq |\gamma_n|\beta^{m_n}$$

or

$$\beta^{-m_n} \leq |\gamma_n| \leq \alpha^{-m_n}.$$

Consequently,

$$-\frac{m_n}{n}\log\beta \leq \frac{1}{n}\log|\gamma_n| \leq -\frac{m_n}{n}\log\alpha$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n}\log|\gamma_n| = 0.$$

Since

$$\frac{1}{n}\log|p_n(z)| = \frac{1}{n}\log|\gamma_n| + \frac{1}{n}\log|p^*_n(z)|$$

we have proved that $G_{\rho(f)}$ is also an exact harmonic majorant of $\{(1/n)\log|p_n(z)|\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{C}}\setminus E_{\rho(f)}$. □

Let us denote by ν_n the normalized zero counting of r_{n,m_n}, zeros counted with their multiplicities, i.e.,

$$\nu_n(B) := \frac{\#\text{ number of zeros of } r_{n,m_n} \text{ in } B}{\text{total number of zeros of } r_{n,m_n} \text{ in } \mathbb{C}}. \tag{21}$$

Then we obtain the following distribution result for the zeros.

Theorem 2.5 ([6], Theorem 4). *Let $E, f, \{m_n\}_{n\in\mathbb{N}}$ and $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ be as in Theorem 2.4. Then the normalized zero counting measures ν_n of r_{n,m_n} converge weakly to the equilibrium measure $\mu_{E_{\rho(f)}}$ of the maximal Green domain $E_{\rho(f)}$ of meromorphy, at least for a subsequence $\Lambda \subset \mathbb{N}$ as $n \in \Lambda$, $n \to \infty$.*

We remark that all theorems remain valid with some minor modifications if in (7) the sequence $\{m_n\}_{n\in\mathbb{N}}$ is bounded, i.e., if

$$\limsup_{n\to\infty} m_n = m < \infty.$$

Let us denote by $\mathscr{M}_m(E_\rho)$, $\rho > 1$, the class of meromorphic functions on E_ρ with at most m poles in E_ρ. Define for $f \in \mathscr{M}(E)$

$$\rho_m(f) := \{\sup \rho : \rho > 1, f \in \mathscr{M}_m(E_\rho)\},$$

then all theorems above hold if $\rho_m(f) < \infty$ and if we replace $\rho(f)$ by $\rho_m(f)$. Moreover, m_1-maximal convergence and exact m_1-maximal convergence are equivalent in this case. Hence, there is a complete analogue between polynomial approximation and rational approximation with fixed degree m of the denominator (cf. [15], [10]). Recently, new results for generalized multipoint Padé-approximation were obtained by M. Bello Hernández, B. de la Calle Ysern, and J. Mínguez Ceniceros (cf. [2, 8]).

3 Rational Approximation: Overconvergence in Capacity

As in the previous section, let $E \subset \mathbb{C}$ be compact with regular connected complement $\Omega = \overline{\mathbb{C}} \setminus E$.

We weaken the conditions (7) and (8) of Sect. 2. First, let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathscr{R}_{n,m_n}$, be a sequence of rational approximants of $f \in \mathscr{M}(E)$ with $\rho(f) < \infty$, and let

$$\lim_{n\to\infty} m_n = \infty, \; m_n = o(n) \text{ as } n \to \infty. \tag{22}$$

Second, define $\Gamma_1 := \partial E$ and let $\{\sigma_n\}_{n=1}^{\infty}$, $\sigma_n \geq 1$, be a sequence with $\lim_{n\to\infty} \sigma_n = 1$. Then we consider a sequence $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\Gamma_{\sigma_n}}^{1/n} \leq \frac{1}{\rho(f)}. \tag{23}$$

Hence, (8) of Sect. 2 is a special case of (23) of Sect. 3 if all $\sigma_n = 1$ and $\Gamma_{\sigma_n} = \partial E$. (23) of this Section has the advantage that it is more appropriate for applications, for example for multipoint Padé approximation in Sect. 4 (cf. proof of Theorem 4.2).

To clarify the different notions of convergence, let D be an open set in $\mathbb{C}$, φ a function in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to *converge in capacity inside D to* φ if for any compact set $K \subset D$ and any $\varepsilon > 0$

$$\text{cap}\,\{z \in K : |(\varphi - \varphi_n)(z)| \geq 0\} \to 0 \text{ as } n \to \infty.$$

Moreover, $\{\varphi_n\}_{n\in\mathbb{N}}$ *converges uniformly in capacity inside D to* φ if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that cap $K_\varepsilon < \varepsilon$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ in $K \setminus K_\varepsilon$ (cf. [10]).

Theorem 3.1 (cf. [3], Theorem 2.2). *Let E be compact in* $\mathbb{C}$ *with regular connected complement,* $f \in \mathscr{M}(E)$ *with* $\rho(f) < \infty$, *let* $\{m_n\}_{n\in\mathbb{N}}$ *be a sequence satisfying (22), i.e.,*

$$\lim_{n\to\infty} m_n = \infty, \; m_n = o(n) \text{ as } n \to \infty.$$

Moreover, let $\{\sigma_n\}_{n=1}^{\infty}$, $\sigma_n \geq 1$, *be a sequence with* $\lim_{n\to\infty} \sigma_n = 1$ *and let* $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathscr{R}_{n,m_n}$, *be such that (23) holds, i.e.*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\Gamma_{\sigma_n}}^{1/n} \leq \frac{1}{\rho(f)}.$$

If σ, $1 < \sigma < \rho(f)$, *and* $1 < \theta < \rho(f)/\sigma$, *then there exists a number* $n_0 = n_0(\sigma,\theta) \in \mathbb{N}$ *and compact sets* $\Omega_n(\sigma,\theta) \subset \overline{E}_\sigma$ *such that for all* $n \geq n_0(\sigma,\theta)$

$$\text{cap}\;\; \Omega_n(\sigma,\theta) \leq d^{1/2}\left(1 - \frac{\theta - 1}{1 + 3\theta}\right)^{\frac{n}{2m_n}} \tag{24}$$

and

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma \setminus \Omega_n(\sigma,\theta)} \leq \left(\frac{\theta\sigma}{\rho(f)}\right)^n, \tag{25}$$

where d is the diameter of $E_{\rho(f)}$.

Proof. For abbreviation, we write $\rho = \rho(f)$. Let $\varepsilon := (\theta - 1)/4$; then we get

$$\varepsilon = \frac{\theta - 1}{4} < \frac{\rho/\sigma - 1}{4} = \frac{1}{4}\frac{\rho - \sigma}{\sigma} < \rho - \sigma.$$

We choose τ such that $\rho - \varepsilon < \tau < \rho$, and we denote by h^τ the monic polynomial whose zeros are the poles of f in $\overline{E}_\tau$, counted with their multiplicities. Then

$$(fh^\tau)(z) = f(z)h^\tau(z)$$

is holomorphic in $\overline{E}_\tau$. Let us denote by $p_n^\tau \in \mathscr{P}_n$ the best uniform approximation of fh^τ on E. Then there exists $n_1 = n_1(\sigma, \varepsilon)$ such that for $n \geq n_1(\sigma, \varepsilon)$

$$\|fh^\tau - r_{n,m_n}h^\tau\|_{\Gamma_{\sigma_n}} \leq \frac{1}{2}\left(\frac{1}{\rho - \varepsilon}\right)^n, \tag{26}$$

$$\|fh^\tau - p_n^\tau\|_{\overline{E}_{\sigma_n}} \leq \frac{1}{2}\left(\frac{1}{\rho - \varepsilon}\right)^n, \tag{27}$$

$$\|fh^\tau - p_n^\tau\|_{\overline{E}_\sigma} \leq \frac{1}{2}\left(\frac{\sigma}{\rho - \varepsilon}\right)^n, \tag{28}$$

$$\text{degree }(h^\tau) \leq m_n. \tag{29}$$

For (26) we have used (23), the theorem of Bernstein–Walsh for (27) and (28), (29) follows from (22).

Equation (26) together with (27) yields

$$\|r_{n,m_n}h^\tau - p_n^\tau\|_{\Gamma_{\sigma_n}} \leq \left(\frac{1}{\rho - \varepsilon}\right)^n, \; n \geq n_1(\sigma, \varepsilon). \tag{30}$$

Let $r_{n,m_n}(z) = p_n(z)/q_{m_n}(z)$ be normalized by

$$q_{m_n}(z) = q_{m_n}^*(z) \prod_{\xi_{n,i} \notin E_\rho}\left(1 - \frac{z}{\xi_{n,i}}\right) \text{ and } q_{m_n}^*(z) = \prod_{\xi_{n,i} \in E_\rho}(z - \xi_{n,i}).$$

Then for any compact set $K \subset \mathbb{C}$

$$\limsup_{n\to\infty} \|q_{m_n}\|_K^{1/n} \leq 1.$$

Because of (9) and the normalization of q_{m_n}, there exists a constant $c > 0$ such that for $z \in E_{\sigma_n}$

$$|(p_n h^\tau - p_n^\tau q_{m_n})(z)| \leq c^{m_n}\left(\frac{1}{\rho - \varepsilon}\right)^n.$$

We apply the lemma of Bernstein–Walsh to the polynomial

$$w(z) = (p_n h^\tau - p_n^\tau q_{m_n})(z) \in \mathscr{P}_{n+m_n}$$

and obtain

$$|w(z)| \le (c\sigma)^{m_n} \left(\frac{\sigma}{\rho-\varepsilon}\right)^n \text{ for } z \in \overline{E}_\sigma .$$

Consequently, for $z \in \overline{E}_\sigma$, where $q_{m_n}(z) \neq 0$, we get

$$|(r_{n,m_n}h^\tau - p_n^\tau)(z)| \le (c\sigma)^{m_n} \left(\frac{\sigma}{\rho-\varepsilon}\right)^n \frac{1}{|q_{m_n}(z)|}.$$

Hence, there exists $n_2 = n_2(\sigma,\varepsilon)$, $n_2 \ge n_1$, such that

$$|(r_{n,m_n}h^\tau - p_n^\tau)(z)| \le \frac{1}{2}\left(\frac{(1+\varepsilon)\sigma}{\rho-\varepsilon}\right)^n \frac{1}{|q^*_{m_n}(z)|}$$

for all $z \in \overline{E}_\sigma$ with $q^*_{m_n}(z) \neq 0$ and $n \ge n_2$.

Let us consider the set

$$S_n(\sigma,\varepsilon) := \left\{ z \in \overline{E}_\sigma : |(r_{n,m_n}h^\tau - p_n^\tau)(z)| \ge \frac{1}{2}\left(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\right)^n \right\},$$

then

$$S_n(\sigma,\varepsilon) \subset e_n = e_n(\sigma,\varepsilon) := \left\{ z \in \overline{E}_\sigma : |q^*_{m_n}(z)| \le \left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^n \right\}.$$

Since $q^*_{m_n}$ is monic and degree $(q^*_{m_n}) \le m_n$, we obtain

$$\operatorname{cap} e_n \le \left(\frac{1+\varepsilon}{1+2\varepsilon}\right)^{\frac{n}{m_n}}. \tag{31}$$

Therefore, we have for $z \in \overline{E}_\sigma \setminus e_n$ and $n \ge n_2$

$$|(r_{n,m_n}h^\tau - p_n^\tau)(z)| \le \frac{1}{2}\left(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\right)^n. \tag{32}$$

By (28) and (32) we get for $z \in \overline{E}_\sigma \setminus e_n$ and $n \ge n_2$

$$|(fh^\tau - r_{n,m_n}h^\tau)(z)| \le \left(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\right)^n$$

or

$$|(f - r_{n,m_n})(z)| \le \left(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\right)^n \frac{1}{|h^\tau(z)|},$$

where $h^\tau(z) \neq 0$. Let us consider

$$\widetilde{S}_n(\sigma,\varepsilon) := \left\{ z \in \overline{E}_\sigma : |(f - r_{n,m_n})(z)| \geq \left(\frac{(1+3\varepsilon)\sigma}{\rho - \varepsilon} \right)^n \right\}$$

then

$$\widetilde{S}_n(\sigma,\varepsilon) \subset \widetilde{e}_n = \widetilde{e}_n(\sigma,\varepsilon) := \left\{ z \in \overline{E}_\sigma : |h^\tau(z)| \leq \left(\frac{1+2\varepsilon}{1+3\varepsilon} \right)^n \right\}$$

and with (29),

$$\text{cap } \widetilde{e}_n \leq \left(\frac{1+2\varepsilon}{1+3\varepsilon} \right)^{\frac{n}{m_n}}. \tag{33}$$

Summarizing, we have obtained for $z \in \overline{E}_\sigma \setminus (e_n \cup \widetilde{e}_n)$ and $n \geq n_2$

$$|f(z) - r_{n,m_n}(z)| \leq \left(\frac{(1+3\varepsilon)\sigma}{\rho - \varepsilon} \right)^n. \tag{34}$$

Because of the subadditivity property of the capacity, due to Nevanlinna (cf. [17])

$$1/\log \frac{d}{\text{cap }(e_n \cup \widetilde{e}_n)} \leq 1/\log \frac{d}{\text{cap } e_n} + 1/\log \frac{d}{\text{cap } \widetilde{e}_n},$$

where d is greater than or equal to the diameter of $e_n \cup \widetilde{e}_n$. Hence, we can choose d as the diameter of $E_{\rho(f)}$.Then we fix $n_3 = n_3(\varepsilon)$ such that

$$d \left(\frac{1+3\varepsilon}{1+2\varepsilon} \right)^{\frac{n}{2m_n}} > 1 \text{ for all } n \geq n_3(\varepsilon)$$

and we obtain for $n \geq n_3(\varepsilon)$

$$d \left(\frac{1+2\varepsilon}{1+\varepsilon} \right)^{\frac{n}{2m_n}} > d \left(\frac{1+3\varepsilon}{1+2\varepsilon} \right)^{\frac{n}{2m_n}} > 1.$$

Define $n_0(\sigma,\theta) := \max(n_2(\sigma,\varepsilon), n_3(\varepsilon))$, then we get for $n \geq n_0(\sigma,\theta)$ by using (31) and (33)

$$\begin{aligned} 1/\log \frac{d}{\text{cap }(e_n \cup \widetilde{e}_n)} &\leq 1/\log \left[d \left(\frac{1+2\varepsilon}{1+\varepsilon} \right)^{\frac{n}{m_n}} \right] + 1/\log \left[d \left(\frac{1+3\varepsilon}{1+2\varepsilon} \right)^{\frac{n}{m_n}} \right] \\ &\leq 2/\log \left[d \left(\frac{1+3\varepsilon}{1+2\varepsilon} \right)^{\frac{n}{m_n}} \right] \end{aligned}$$

or

$$\log \frac{d}{\text{cap }(e_n \cup \widetilde{e}_n)} \geq \frac{1}{2} \log \left[d \left(\frac{1+3\varepsilon}{1+2\varepsilon} \right)^{\frac{n}{m_n}} \right]$$

and finally

$$\text{cap }(e_n \cup \widetilde{e}_n) \leq d^{1/2} \left(\frac{1+2\varepsilon}{1+3\varepsilon} \right)^{\frac{n}{2m_n}}. \tag{35}$$

Since $\varepsilon = (\theta - 1)/4$, we get

$$\frac{1+2\varepsilon}{1+3\varepsilon} = \frac{2+2\theta}{1+3\theta} = 1 - \frac{\theta - 1}{1+3\theta} < 1$$

and some calculations show that

$$\frac{1+3\varepsilon}{\rho - \varepsilon} < \frac{\theta}{\rho}.$$

Inserting the last inequalities into (34) and (35) and define the compact sets

$$\Omega_n(\sigma, \theta) := e_n(\sigma, \varepsilon) \cup \widetilde{e}_n(\sigma, \varepsilon),$$

where $\varepsilon = (\theta - 1)/4$, then we have proved the inequalities (24) and (25) for $n \geq n_0(\sigma, \theta)$. □

Hence, (24) and (25) imply that $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ converges in capacity to f inside $E_{\rho(f)}$. Concerning uniform convergence in capacity, the following theorem holds.

Theorem 3.2 ([4], Theorem 3.2). *Let E be compact in $\mathbb{C}$ with regular connected complement, $f \in \mathscr{M}(E)$ with $\rho(f) < \infty$, $\{m_n\}$ as in (22) and let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathscr{R}_{n,m_n}$, satisfy (23). Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the property: For any σ, $1 < \sigma < \rho(f)$, and any $\varepsilon > 0$ there exists a denumerable union of closed sets $\Omega(\sigma, \varepsilon) \subset \overline{E}_\sigma$ (a so-called F_σ-set) with* cap $\Omega(\sigma, \varepsilon) < \varepsilon$ *such that*

$$\limsup_{n\in\Lambda, n\to\infty} \|f - r_{n,m_n}\|^{1/n}_{\overline{E}_\sigma \setminus \Omega(\sigma,\varepsilon)} \leq \frac{\sigma}{\rho(f)}. \tag{36}$$

In the above theorem and in the following, we will use the standard notion of an F_σ-set as the denumerable union of closed sets; there is no connection with the parameter σ where $1 < \sigma < \rho(f)$. Though we have weakened the condition (7) by (22), we could obtain overconvergence in capacity. At first glance, this appears somewhat contradicting because convergence in capacity seems to be stronger than m_1-convergence due to an inequality of H. Cartan, namely that there exists a constant $c > 0$ such that $m_1(B) \leq c$ cap B for any Borel set $B \subset \mathbb{C}$ ([13, p. 203]). But by

Theorem 3.2 we obtain maximal convergence in capacity *only for a subsequence* $\Lambda \subset \mathbb{N}$, in contrast to Theorem 2.3 where we have maximal m_1-convergence for the *whole* sequence.

For abbreviation, we will call the property (36) of $\{r_{n,m_n}\}_{n\in\Lambda}$ *maximal convergence in capacity to* f *on* $E_{\rho(f)}$ (cf. [4], Theorem 3.2 and Corollary 3.4), used also if $r_{n,m_n} \in \mathcal{R}_{n,m_n}$ is replaced by $r_n \in \mathcal{R}_{n,n}$.

In polynomial approximation, we get by (2) trivially

$$\limsup_{n\to\infty} \|f - p_n\|_{\Gamma_\sigma}^{1/n} \le \frac{\sigma}{\rho}, 1 < \sigma < \rho,$$

if $\{p_n\}_{n\in\mathbb{N}}$ is maximally convergent to f on the maximal Green domain E_ρ of holomorphy of E. Apparently, this cannot hold for rational approximation since the poles of the rational approximants can destroy such an inequality. Nevertheless, a modified version of such a result holds, analogous to the inequality (10).

Theorem 3.3 ([4], Theorem 3.5). *In addition to the conditions of Theorem 3.2, let E be connected. Then for any* σ, $1 < \sigma < \rho(f)$, *there exists a sequence*

$$\{\sigma_n\}_{n=1}^\infty, \sigma_n \le \sigma, \text{ with } \lim_{n\to\infty} \sigma_n = \sigma$$

such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\Gamma_{\sigma_n}}^{1/n} \le \frac{\sigma}{\rho(f)}.$$

As a consequence, we can prove the analogue of Theorem 2.4 under the weaker condition $m_n = o(n)$ as $n \to \infty$.

Theorem 3.4 ([4], Theorem 3.7). *Under the conditions of Theorem 3.3 the Green function*

$$G_{\rho(f)}(z) = G(z) - \log \rho(f) \text{ of } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$$

is a harmonic majorant for the sequence

$$\left\{\frac{1}{n} \log |p_n(z)|\right\}_{n\in\mathbb{N}} \text{ in } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)},$$

where $r_{n,m_n} = p_n/q_{m_n}$ *is normalized by* $\|q_{m_n}\|_E = 1$.

Next, we want to obtain results about the zeros of r_{n,m_n}, but now for the case that $m_n = o(n)$ as $n \to \infty$, instead of $m_n = o(n/\log n)$ as in Sect. 2.

Henceforth, let $\{r_n\}_{n\in\Lambda}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence that converges maximally in capacity to f on $E_{\rho(f)}$. We define for fixed $\varepsilon > 0$ and σ, $1 < \sigma < \rho(f)$, the set

$$F(\sigma, \varepsilon) := \{\Omega(\sigma, \varepsilon) : \Omega(\sigma, \varepsilon) \text{ is a } F_\sigma\text{-set in } \overline{E}_\sigma \text{ with cap } \Omega(\sigma, \varepsilon) < \varepsilon\}$$

and

$$m(\Lambda,\sigma,\varepsilon) := \inf\left\{ \limsup_{n\in\Lambda, n\to\infty} \|f - r_n\|^{1/n}_{E_\sigma\setminus\Omega(\sigma,\varepsilon)} : \Omega(\sigma,\varepsilon) \in F(\sigma,\varepsilon) \right\}.$$

Then

$$m(\Lambda,\sigma,\varepsilon) \geq m(\Lambda,\sigma,\varepsilon') \text{ for } \varepsilon < \varepsilon'$$

and

$$m(\Lambda,\sigma) := \liminf_{\varepsilon\to 0} m(\Lambda,\sigma,\varepsilon) = \lim_{\varepsilon\to 0} m(\Lambda,\sigma,\varepsilon).$$

By definition, the maximal convergence in capacity of the sequence $\{r_n\}_{n\in\Lambda}$ implies

$$m(\Lambda,\sigma) \leq \frac{\sigma}{\rho(f)} \text{ for all } \sigma, 1 < \sigma < \rho(f).$$

Therefore,

$$\liminf_{\sigma\to\rho(f)} \left(m(\Lambda,\sigma)\frac{\rho(f)}{\sigma} \right) \leq \limsup_{\sigma\to\rho(f)} \left(m(\Lambda,\sigma)\frac{\rho(f)}{\sigma} \right) \leq 1.$$

This observation leads to the following definition.

Definition. Let $\{r_n\}_{n\in\Lambda}$ be maximally convergent in capacity to f on $E_{\rho(f)}$. Then the sequence $\{r_n\}_{n\in\Lambda}$ is called *exactly maximally convergent* if

$$\liminf_{\sigma\to\rho(f)} \left(m(\Lambda,\sigma)\frac{\rho(f)}{\sigma} \right) = 1,$$

or, equivalently, if

$$\liminf_{\sigma\to\rho(f)} m(\Lambda,\sigma) = 1.$$

Comparing the definition of exactness for maximal convergence in m_1-measure and in capacity, we can note the following: If $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathscr{R}_{n,n}$, is exactly m_1-maximally convergent to f on $E_{\rho(f)}$ and if the subsets $\Omega(\varepsilon)$ in (15) and (16) are F_σ-sets with $\operatorname{cap}\Omega(\varepsilon) < \varepsilon$, then the sequence $\{r_n\}_{n\in\mathbb{N}}$ is also exactly maximally convergent in capacity. On the other hand, if $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathscr{R}_{n,n}$, is exactly maximally convergent in capacity to f on $E_{\rho(f)}$, and if $m(\mathbb{N},\sigma) = \sigma/\rho(f)$ for all σ, $1 < \sigma < \rho(f)$, then this sequence is exactly m_1-maximally convergent.

Again with the characterization of Gonchar (Theorem 2.2) we can obtain such an exact maximal convergent sequence if f has again a singularity of multivalued character on $\Gamma_{\rho(f)}$, namely

Theorem 3.5 ([5], Theorem 1). *Let E be a compact and connected set in $\mathbb{C}$ with regular connected complement, $\Lambda \subset \mathbb{N}$, and let $\{r_n\}_{n\in\Lambda}$ be maximally convergent in capacity to f on $E_{\rho(f)}$, $\rho(f) < \infty$. If f has a singularity of multivalued character on the boundary of $E_{\rho(f)}$, then $\{r_n\}_{n\in\Lambda}$ is exactly maximally convergent to f on $E_{\rho(f)}$.*

As a consequence of the exact maximal convergence we obtain

Theorem 3.6 ([5], Theorem 2). *Let E be a compact, connected set in $\mathbb{C}$ with regular connected complement, and let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ with (22) and (23) be maximally convergent to f on $E_{\rho(f)}$. If there exists a subset $\Lambda \subset \mathbb{N}$ such that $\{r_{n,m_n}\}_{n\in\Lambda}$ is exactly maximally convergent to f on $E_{\rho(f)}$, then the Green function*

$$G_{\rho(f)}(z) = G(z) - \log \rho(f) \text{ of } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)}$$

is an exact harmonic majorant for the sequence

$$\left\{ \frac{1}{n} \log |p_n(z)| \right\}_{n\in\mathbb{N}} \text{ on } \overline{\mathbb{C}} \setminus \overline{E}_{\rho(f)},$$

where $r_{n,m} = p_n/q_{m_n}$ is normalized by $\|q_{m_n}\|_E = 1$.

Finally, we are in position to get the analogous result as in Theorem 2.5 for the distribution of the zero of r_{n,m_n}, but now for $m_n = o(n)$ as $n \to \infty$.

Theorem 3.7 ([5], Theorem 4). *Let E be a compact, connected set in $\mathbb{C}$, let $\{m_n\}$ be a sequence satisfying (22), and let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathscr{R}_{n,m_n}$, be a sequence with the property (23). If there exists a subsequence $\Lambda \subset \mathbb{N}$ such that $\{r_{n,m_n}\}_{n\in\Lambda}$ is exactly maximally convergent to f on $E_{\rho(f)}$, then there exists a subsequence $\Lambda_1 \subset \mathbb{N}$ such that*

$$\nu_n \overset{*}{\to} \mu_{E_{\rho(f)}} \text{ as } n \to \infty,\ n \in \Lambda_1,$$

where ν_n is the normalized zero counting measure of ν_n, defined in (21).

We point out that the proof in [5] does not imply that Λ_1 has to be a subset of Λ.

4 Multipoint Padé Approximation

Let α be an infinite triangular table

$$\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \text{ where } \alpha_n = \{\alpha_{n,k}\}_{k=1}^{n},\ \alpha_{n,k} \in E,$$

and let us define

$$w_n(z) = \prod_{k=1}^{n}(z - \alpha_{n,k}).$$

Let $f \in \mathscr{M}(E)$ and let (n, m) be a fixed pair of nonnegative integers. We assume moreover that f is holomorphic at all points of the table α. Then there exists a unique rational function

$$\pi_{n,m}^{\alpha} = \frac{p}{q} \in \mathscr{R}_{m,n}$$

such that the function $(fq - p)/w_{n+m+1} \in \mathscr{M}(E)$ is holomorphic at all points of the table α. The rational function $\pi_{n,m}^{\alpha}$ is called *multipoint Padé approximant* to f with respect to the table α. If $0 \in E$ and if all $\alpha_{n,k} = 0$ in the table α, then we obtain the classical Padé approximants, denoted by $\pi_{n,m}$. In this classical case, the convergence of $\pi_{n,m}$ to f was first studied by Montesssus de Ballore and generalized for multipoint Padé by Saff [16] and Wallin [18] if the degree m of the denominators of $\pi_{n,m}$ are fixed and $n \to \infty$.

Convergence results for $\pi_{n,m}$ to f are based on analytic properties of f and the degree m of the denominators. For example, Pommerenke [14] showed that $\pi_{n,n}$ converges in capacity to f if f is meromorphic on $\mathbb{C} \setminus e$, where the exceptional set e has capacity 0. Moreover, Wallin [18] has given an example of an entire function f such that the poles of $\pi_{n,n}$ are dense everywhere in $\mathbb{C}$, $n \to \infty$ (except at 0).

A natural method for controlling the behavior of the poles of $\pi_{n,m}^{\alpha}$ is the limitation of the growth of the degrees m of the denominators. In [10], Gonchar considered two cases of sequences $\{\pi_{n,m_n}^{\alpha}\}_{n\in\mathbb{N}}$, where $r_{n,m_n} \in \mathscr{R}_{n,m_n}$ and

$$m_n = o(n/\log n) \text{ or } m_n = o(n) \text{ as } n \to \infty.$$

A main role for the convergence of Padé approximants π_{n,m_n}^{α}, $n \to \infty$, is played by the normalized counting measures τ_n^{α} of the table α, defined by

$$\tau_n^{\alpha}(B) := \frac{\#\{\alpha_{n,k} \in B : 1 \le k \le n\}}{n}, \; B \subset \mathbb{C}.$$

τ_n^{α} is a probability measure on E with logarithmic potential

$$U^{\tau_n^{\alpha}}(z) = \frac{1}{n} \log \frac{1}{|w_n(z)|}.$$

Let us denote by $\widehat{\tau}_n^{\alpha}$ the balayage measure of τ_n^{α} to the boundary of E.

Theorem 4.1. *Let E be compact in $\mathbb{C}$ with regular connected complement and let $f \in \mathscr{M}(E)$, $\rho(f) < \infty$, be holomorphic at all points of the triangular table α.*

Moreover, let $\widehat{\tau}_n^\alpha$ *converge weakly to the equilibrium measure* μ_E *of* E. *If* $\{m_n\}_{n=\mathbb{N}}^\infty$ *satisfies (7) (i.e.,* $m_n = o(n/\log n)$*), then the Padé approximants* π_{n,m_n}^α *converge* m_1*-maximally to* f *on* $E_{\rho(f)}$ *as* $n \to \infty$.

For the proof we refer to a result of Gonchar [10] (cf. Theorem A in [7]).

Theorem 4.2. *In addition to the conditions of Theorem 4.1, let* E *be connected. If* $\{m_n\}_{n=\mathbb{N}}^\infty$ *satisfies (22) (i.e.,* $m_n = o(n)$ *as* $n \to \infty$*), then there exists a subsequence* $\Lambda \subset \mathbb{N}$ *such that* $\{\pi_{n,m_n}^\alpha\}_{n\in\Lambda}$ *converges maximally in capacity to* f *on* $E_{\rho(f)}$*, as* $n \in \Lambda$*,* $n \to \infty$.

Proof. Let Φ be the conformal mapping from Ω to the exterior of the unit disk, normalized by $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. Then, according to the transfer lemma for the logarithmic capacity (Lemma 5.1 in [4]), there exists a constant $c(\sigma) > 0$ such that

$$\operatorname{cap} \Phi(B) \le c(\sigma) \operatorname{cap} B \tag{37}$$

for any compact set $B \subset \mathbb{C} \setminus E_\sigma$.

Now, let us fix σ with $1 < \sigma < \rho(f)$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be strictly decreasing with $\lim_{k\to\infty} \varepsilon_k = 0$ and $\varepsilon_1 < (\rho(f) - \sigma)/2$.

Define for $k \in \mathbb{N}$

$$\sigma_k = \sigma + \varepsilon_k \text{ and } \rho_k = \rho(f) - \varepsilon_k. \tag{38}$$

Let $\pi_{n,m_n}^\alpha = p_n/q_{m_n}$ be the multipoint Padé approximant, normalized by $q_{m_n} = q_{m_n,1}q_{m_n,2}$ with

$$q_{m_n,1}(z) = \prod_{\xi_{n,i}\in E_\tau} (z - \xi_{n,i}), \;\; q_{m_n,2}(z) = \prod_{\xi_{n,i}\notin E_\tau} \left(1 - \frac{z}{\xi_{n,i}}\right),$$

where $\xi_{n,i}$ denote the poles of π_{n,m_n}^α and τ is chosen in such a way that $0 \in E_\tau$ and $\tau > \rho(f)$. For fixed $k \in \mathbb{N}$, we numerize by

$$\eta_{k,1}, \eta_{k,2}, \ldots, \eta_{k,s_k}$$

all poles of f in $\overline{E}_{\rho_k}$, counted with their multiplicities, and set

$$Q_k(z) := \prod_{i=1}^{s_k} (z - \eta_{k,i}).$$

The Lagrange–Hermite formula (cf. [21]) yields for $z \in E_{\rho_k}$ and for n with $m_n \geq s_k$

$$(fq_m Q_k - p_n Q_k)(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\rho_k}} \frac{w_{n+m_n+1}(z)}{w_{n+m_n+1}(t)} \frac{(fq_m Q_k)(t)}{t-z} dt. \tag{39}$$

Using the weak*-convergence of $\widehat{\tau}_n^{\alpha}$ to μ_E and $m_n = o(n)$ as $n \to \infty$, we obtain by (39) (cf. [10]), that there exists a number n_k such that for $n \geq n_k$ we have $m_n \geq s_k$ and

$$|(fq_{m_n} Q_k - p_n Q_k)(z)| \leq \left(\frac{(1+\varepsilon_k)\sigma_k}{\rho(f)} \right)^n \text{ for } z \in \overline{E}_{\sigma_k}. \tag{40}$$

Since we may define the numbers n_k inductively, we can assume that $n_{k+1} > n_k (k \in \mathbb{N})$ and

$$c(\sigma) \left(\frac{1+\varepsilon_k}{1+2\varepsilon_k} \right)^{\frac{n}{2m_n}} \leq \frac{\varepsilon_k}{8} \text{ for all } n \geq n_k. \tag{41}$$

Let

$$c_1 = \min \left(1, \min_{z \in \overline{E}_{\rho(f)}} \min_{\xi \notin E_\tau} \left| 1 - \frac{z}{\xi} \right| \right)$$

then $0 < c_1 \leq 1$, and

$$|q_{m_n,2}(z)| \geq c_1^{m_n} \text{ for } z \in \overline{E}_{\sigma_k}.$$

We obtain with (40)

$$|(f - \pi_{n,m_n}^{\alpha})(z)| \leq \frac{1}{c_1^{m_n}} \left(\frac{(1+\varepsilon_k)\sigma_k}{\rho(f)} \right)^n \frac{1}{|(q_{m_n,1} Q_k)(z)|}$$

for $z \in \overline{E}_{\sigma_k}$ with $(q_{m_n,1} Q_k)(z) \neq 0$. Let us define

$$S_n(\sigma_k, \varepsilon_k) := \left\{ z \in \overline{E}_{\sigma_k} : |(q_{m_n,1} Q_k)(z)| \leq \left(\frac{1+\varepsilon_k}{1+2\varepsilon_k} \right)^n \right\},$$

then

$$|(f - \pi_{n,m_n}^{\alpha}(z)| \leq \frac{1}{c_1^{m_n}} \left((1+2\varepsilon_k) \frac{\sigma_k}{\rho(f)} \right)^n \tag{42}$$

for $z \in \overline{E}_{\sigma_k} \setminus S_n(\sigma_k, \varepsilon_k)$ and

$$\operatorname{cap} S_n(\sigma_n, \varepsilon_k) \le \left(\frac{1+\varepsilon_k}{1+2\varepsilon_k}\right)^{\frac{n}{2m_n}} \tag{43}$$

for all $n \ge n_k$.

Next, we want to define a sequence $\{\sigma_n^*\}_{n=1}^{\infty}$, $\sigma \le \sigma_n^* < \rho(f)$, such that $\lim_{n\to\infty} \sigma_n^* = \sigma$ and

$$\limsup_{n\to\infty} \|f - \pi_{n,m_n}^{\alpha}\|_{\Gamma_{\sigma_n^*}}^{1/n} \le \frac{\sigma}{\rho(f)}. \tag{44}$$

To construct this sequence, we consider the projection $p_1 : \mathbb{C} \to \mathbb{R}_+$ defined by $p_1(z) = |z|$. The contraction principle of the logarithmic capacity yields

$$\operatorname{cap} p_1(B) \le \operatorname{cap} B \text{ for any compact set } B \subset \mathbb{C}.$$

Then the contraction principle together with the transfer lemma, applied to the compact sets

$$\widetilde{S}_n(\sigma_k, \varepsilon_k) := S_n(\sigma_k, \varepsilon_k) \cap (\mathbb{C} \setminus E_\sigma),$$

implies

$$\begin{aligned} \operatorname{cap} p_1(\Phi(\widetilde{S}_n(\sigma_k, \varepsilon_k)) &\le \operatorname{cap} \Phi(\widetilde{S}_n(\sigma_k, \varepsilon_k)) \\ &\le c(\sigma) \operatorname{cap} \widetilde{S}_n(\sigma_k, \varepsilon_k) \le c(\sigma) \left(\frac{1+\varepsilon_k}{1+2\varepsilon_k}\right)^{\frac{n}{2m_n}} \\ &\le \frac{\varepsilon_k}{8} \text{ for } n \ge n_k, \end{aligned} \tag{45}$$

where we have used (41) and (43).

On the other hand, consider the annulus

$$R_k := \{z \in \mathbb{C} : \sigma \le |z| \le \sigma + \varepsilon_k\},$$

then again by the contraction principle,

$$\operatorname{cap} p_1(R_k) = \operatorname{cap} ([\sigma, \sigma + \varepsilon_k]) = \frac{\varepsilon_k}{4}. \tag{46}$$

Combining (45) and (46), we conclude that there exists for each $n \ge n_k$ a parameter $\sigma_{n,k}$ with $\sigma \le \sigma_{n,k} \le \sigma + \varepsilon_k$ such that the level line $\Gamma_{\sigma_{n,k}}$ of Green's function $G(z)$ satisfies

$$\Gamma_{\sigma_{n,k}} \subset \overline{E}_{\sigma_k} \setminus S_n(\sigma_k, \varepsilon_k),$$

and therefore for $n \geq n_k$, using (42) and (43),

$$\|f - \pi^{\alpha}_{n,m_n}\|^{1/n}_{\Gamma_{\sigma_{n,k}}} \leq \left(\frac{1}{c_1}\right)^{\frac{m_n}{n}} (1 + 2\varepsilon_k)\frac{\sigma_k}{\rho(f)}. \tag{47}$$

Now, we define

$$\sigma^*_n := \sigma_{n,1} \text{ for } 1 \leq n < n_2$$

and

$$\sigma^*_n := \sigma_{n,k} \text{ for } n_k \leq n < n_{k+1},\ k \geq 2.$$

Then we obtain for $\{\sigma^*_n\}_{n=n_1}^{\infty}$, using (47),

$$\limsup_{n\to\infty} \|f - \pi^{\alpha}_{n,m_n}\|^{1/n}_{\Gamma_{\sigma^*_n}} \leq \frac{\sigma}{\rho(f)},$$

and (44) is proved.

Consider a sequence $\{\tau_k\}_{k\in\mathbb{N}}$, $\tau_k > 1$, with $\lim_{k\to\infty} \tau_k = 1$. Replacing σ in (44) by τ_k, we can find parameters $\tau_{n,k} \geq \tau_k$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \tau_{n,k} = \tau_k$ and a number $n_k \in \mathbb{N}$ such that

$$\|f - \pi^{\alpha}_{n,m_n}\|^{1/n}_{\Gamma_{\tau_{n,k}}} \leq \frac{(\tau_k)^2}{\rho(f)} \text{ for } n \geq n_k.$$

Moreover, we may assume that $n_k < n_{k+1}$, $k \in \mathbb{N}$. Then we define the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ by

$$\sigma_n := \tau_{n,1} \quad \text{for } 1 \leq n < n_2$$

and

$$\sigma_n := \tau_{n,k} \quad \text{for } n_k \leq n < n_{k+1}, k \geq 2.$$

Hence, $\lim_{n\to\infty} \sigma_n = 1$ and

$$\limsup_{n\to\infty} \|f - \pi^{\alpha}_{n,m_n}\|^{1/n}_{\Gamma_{\sigma_n}} \leq \frac{1}{\rho(f)},$$

and Theorem 3.2 implies the maximal convergence of $\{\pi^{\alpha}_{n,m_n}\}_{n\in\Lambda}$ to f in $E_{\rho(f)}$, at least for a subsequence $\Lambda \subset \mathbb{N}$. □

Finally, we can apply the theorems in Sects. 2 and 3 to obtain insights to the distribution of the zeros of π^{α}_{n,m_n} if f has a singularity of multivalued character on the boundary of $E_{\rho(f)}$, or more generally, if $\{\pi^{\alpha}_{n,m_n}\}_{n\in\mathbb{N}}$ is exactly m_1-maximally convergent (resp. exactly maximally convergent in capacity) to f in $E_{\rho(f)}$.

In the polynomial case, we mentioned already a converse result of Grothmann [12]. If $m_n = o(n/\log n)$, an analogue for multipoint Padé approximation was proved via m_1-maximal convergence in [7], namely

Theorem 4.3 ([7]). *Let $E \subset \mathbb{C}$ be compact and connected with regular, connected complement, let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ satisfying (7), and let α be a triangular table with points in E.*

Moreover, let $f \in \mathcal{M}(E)$, $\rho(f) < \infty$, be holomorphic at all points of the table α. If $\{\pi^{\alpha}_{n,m_n}\}_{n\in\mathbb{N}}$ is a m_1-maximally convergent sequence of Padé approximants to f in $E_{\rho(f)}$ such that for some σ^*, $1 < \sigma^* < \rho(f)$,

$$\liminf_{\sigma\to\sigma^*} \limsup_{n\to\infty} \|f - \pi^{\alpha}_{n,m_n}\|^{1/n}_{\Gamma_\sigma} = \frac{\sigma^*}{\rho(f)},$$

then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that the balayage measures $\widehat{\tau}^{\alpha}_n$ of the normalized counting measures τ^{α}_n of the interpolation point sets α_n satisfy

$$\widehat{\tau}^{\alpha}_n \underset{n\in\Lambda, n\to\infty}{\overset{*}{\longrightarrow}} \mu_E.$$

Final Remark The exact convergence (in m_1-measure or in capacity) is proved via the result of Gonchar [9] only under the additional condition that E is connected. Moreover, this condition was needed in the proofs if the transfer lemma (Lemma 5.1 in [4]) for the logarithmic capacity was applied. Hence, Theorems 2.3–2.5, Theorems 3.3–3.6, and Theorems 4.2–4.3 were proved under the condition that E is connected. It would be interesting to get rid of this condition.

Furthermore, an analogue of Theorem 4.3 for the distribution of the interpolation points under the weaker condition (22), i.e., if $m_n = o(n)$ as $n \to \infty$ is open.

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Approximation by Bernstein–Faber–Walsh and Szász–Mirakjan–Faber–Walsh Operators in Multiply Connected Compact Sets of $\mathbb{C}$

Sorin G. Gal

Abstract By considering a multiply connected compact set $G \subset \mathbb{C}$ and an analytic function on G, we attach the q-Bernstein–Faber–Walsh polynomials with $q \geq 1$, for which Voronovskaja-type results with quantitative upper estimates are given and the exact orders of approximation in G for these polynomials, namely $\frac{1}{n}$ if $q = 1$ and $\frac{1}{q^n}$ if $q > 1$, are obtained. Also, given a sequence with the property $\lambda_n \searrow 0$ as fast as we want, a type of Szász–Mirakjan–Faber–Walsh operator is attached to G, for which the approximation order $O(\lambda_n)$ is proved. The results are generalizations of those previously obtained by the author for the q-Bernstein–Faber polynomials and Szász–Faber type operators attached to simply connected compact sets of the complex plane. The proof of existence for the Faber–Walsh polynomials used in our constructions is strongly based on some results on the location of critical points obtained in the book of Walsh (The location of critical points of analytic and harmonic functions, vol 34. American Mathematical Society, New York, 1950), which is also used in the major book of Rahman–Schmeisser (Analytic theory of polynomials, vol 26. Oxford University Press Inc, New York, 2002). At the end of the chapter, we present and motivate a conjecture and an open question concerning the use of truncated classical Szász–Mirakjan operators in weighted approximation and in solving a generalization of the Szegó's problem concerning the zeroes distribution for the partial sums of the exponential function, respectively. Concerning the open question, the extensions of Eneström–Kakeya Theorem in Govil–Rahman (Tôhoku Math J 20(2):126–136, 1968) and other results on the location of the zeroes of polynomials in the Rahman–Schmeisser's book (Analytic theory of polynomials, vol 26. Oxford University Press Inc, New York, 2002) are of interest.

Keywords Conformal mapping • Compact sets of several components • Faber–Walsh polynomials • q-Bernstein–Faber–Walsh polynomials with $q \geq 1$ • Voronovskaja type formula • Exact order of approximation • Szász–Mirakjan

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–Faber–Walsh type operators • Truncated Szász–Mirakjan operators • Weighted approximation • Szegö domain • Szegö curve • Zeroes distribution

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1 Introduction

The history of the overconvergence phenomenon in complex approximation by Bernstein-type operators goes back to the work of Wright [46], Kantorovich [22], Bernstein [1–3], Lorentz [25, Chap. 4], and Tonne [40], who in the case of complex Bernstein operators defined by

$$B_n(f)(z) = \sum_{k=0}^{n} p_{n,k}(z)f(\frac{k}{n}), \quad p_{n,k}(z) = \binom{n}{k} z^k(1-z)^{n-k}, z \in \mathbb{C},$$

have given interesting qualitative results, but without giving quantitative estimates. Also, qualitative results without any quantitative estimates were obtained for the complex Favard–Szász–Mirakjan operators by Dressel–Gergen and Purcell [4] and for the complex Jakimovski–Leviatan operators by Wood [45]. We notice that the qualitative results are theoretically based on the “bridge” made by the classical result of Vitali, between the (well-established) approximation results for these Bernstein-type operators of real variable and those for the Bernstein-type operators of complex variable.

In Chap. 1 of each of the recent books of Gal [9, 13], the exact orders in approximation for several important classes of complex Bernstein-type operators attached to an analytic function f in closed disks were obtained, as for example for the operators of Bernstein, Bernstein–Butzer, q-Bernstein with $0 < q \leq 1$, Bernstein–Stancu, Bernstein–Kantorovich, Beta of first kind, Bernstein–Durrmeyer, Lorentz, q-Stancu, and so on. It is worth noting here that in this topic, many other studies were made for other kinds of complex approximation operators, for a selective bibliography see, e.g., [10–12, 14–21, 26–28, 31, 32, 36].

A progress was realized when by using the Faber polynomials too, the Bernstein kind operators were constructed attached to arbitrary simply connected compact subsets in $\mathbb{C}$ (not necessarily disks centered at origin). Thus, for example, in the recent book [13], in Sects. 1.9 and 1.11, we introduced the so-called q-Bernstein–Faber polynomials, $B_{n,q}(f;G)(z)$, $q \geq 1$, attached to a simply connected compact subset $G \subset \mathbb{C}$ and an analytic function f on G, for which the upper approximation estimate of order $\frac{1}{\tilde{q}^n}$ and a Voronovskaja-type results were obtained. Here $\tilde{q}^n = q^n$ if $q > 1$ and $\tilde{q}^n = n$ if $q = 1$.

Looking closer at the proofs of the results obtained in Sect. 1.11 of [13], it is worth noting that instead to define $B_{n,q}(f;G)(z)$ by Definition 1.11.1, p. 86 in [13] which makes use of $F(f)(w) = \frac{1}{2\pi i} \cdot \int_{|u|=1} \frac{f(\Psi(u))}{u-w} du$, we could define $B_{n,q}$ for $f(z) = \sum_{k=0}^{\infty} a_k(f) \cdot F_k(z)$ (the development of f in Faber series, $F_k(z)$-Faber polynomial of degree k), directly by

$$B_{n,q}(f;G)(z) = \sum_{k=0}^{\infty} a_k(f) \cdot \sum_{p=0}^{n} D_{n,p,k}^{(q)} \cdot F_p(z), \tag{1}$$

where $D_{n,p,k}^{(q)} = \binom{n}{p}_q \cdot [\Delta^p_{1/[n]_q} e_k(0)]_q$ are positive constants given by the explicit formula (1.11.6), p. 91 in [13], satisfying $\sum_{p=0}^{n} D_{n,p,k}^{(q)} = 1$. This new definition presents the advantage that avoids the additional hypothesis on $F(f)$ in Definition 1.11.1 in [13].

Indeed, firstly since f is analytic in G, there exists $R > 1$ such that f is analytic in G_R, i.e. $f(z) = \sum_{k=0}^{\infty} a_k(f) \cdot F_k(z)$, $z \in G_R$, where G_R denotes the interior of the closed level curve $\Gamma_R = \{\Psi(w); |w| = R\}$, and Ψ is the conformal mapping of $\tilde{\mathbb{C}} \setminus \overline{\mathbb{D}_1}$ onto $\tilde{\mathbb{C}} \setminus G$. Let $1 < r < \beta < R$. Since by relationships (1.11.2) and (1.11.3), pp. 88–89 in [13], we have $|F_p(z)| \le C(r) r^p$, for all $z \in \overline{G_r}$, $p \ge 0$ and $|a_k(f)| \le \frac{C(\beta,f)}{\beta^k}$, for all $k \ge 0$, it immediately follows

$$|B_{n,q}(f;G)(z)| \le C(r) \cdot C(\beta,f) \cdot \sum_{k=0}^{\infty} \left(\frac{r}{\beta}\right)^k = \frac{1}{1 - r/\beta} < +\infty,$$

that is $B_{n,q}(f;G)(z)$ is well defined at each $z \in \overline{G_r}$, $n \in \mathbb{N}$.

Also, note that $B_{n,q}(f;G)(z)$ given by (1) is a polynomial of degree n. Indeed, we can write $B_{n,q}(f;G)(z) = \sum_{p=0}^{n} F_p(z) \left[\sum_{k=0}^{\infty} a_k(f) \cdot D_{n,p,k}^{(q)}\right]$, where denoting $S_{n,p}^{(q)} = \sum_{k=0}^{\infty} a_k(f) \cdot D_{n,p,k}^{(q)}$, for any $0 \le p \le n$ we have

$$|S_{n,p}^{(q}| \le \sum_{k=0}^{\infty} |a_k(f)| \cdot D_{n,p,k}^{(q)} \le C_1 \sum_{k=0}^{\infty} \frac{1}{\beta^k} \cdot [0, [1]_q/[n]_q, \ldots, [p]_q/[n]_q; e_k]$$
$$\le C_2 \sum_{k=p+1}^{\infty} \frac{1}{\beta^k} \cdot \frac{k^p}{p!} < +\infty.$$

It is worth noting that similar considerations with those made for formula (1), can be made for the Szász–Mirakjan–Faber operators too, see [14].

Based on the generalization of Faber polynomials attached to compact sets consisting of several components introduced in Walsh [43] and called Faber–Walsh polynomials, here we show that all the above-mentioned approximation results in the case of q-Bernstein–Faber polynomials and Szász–Mirakjan–Faber type operators attached to simply connected compact sets can be extended to the corresponding q-Bernstein–Faber–Walsh polynomials and Szász–Mirakjan–Faber

–Walsh type operators, attached to compact sets of several components. By using in this case a formula of definition similar to that in (1), the proofs will considerably be simplified.

The plan of this chapter goes as follows. Section 2 contains some preliminaries on Faber–Walsh polynomials, on q-Bernstein–Faber–Walsh polynomials and on Szász–Mirakjan–Faber–Walsh type operators. In Sect. 3, firstly we prove a Voronovskaja-type results with quantitative estimates and then, results concerning the exact orders of approximation for the q-Bernstein–Faber–Walsh polynomials, $q \geq 1$, on a multiply connected compact domain G. Section 4 firstly deals with an upper estimate in approximation by Szász–Mirakjan–Faber–Walsh type operators. The chapter ends with the conjecture that the classical truncated Szász–Mirakjan operators could represent a constructive solution to the weighted approximation in $\{|z| < 1/2\}$ intersected with the Szegö's domain. Also, a generalization of the Szegö's result concerning the zeroes distribution for the partial sums of the exponential function is raised as an open question.

It is worth mentioning that the proof of existence of the Faber–Walsh polynomials used in our constructions and the results concerning the zeroes distributions in the original and in the generalized Szegö's problem are strongly based on some results on the location of critical points and of zeroes of polynomials, two topics in which Professor Rahman has made very important contributions, summarized in, e.g., the major book Rahman–Schmeisser [30].

2 Definitions and Preliminaries

The Faber polynomials were introduced by Faber in [5] as associated with a simply connected compact set. They allow the expansion of functions analytic on that set into a series with similar properties to the classical power series.

In Walsh [43], were introduced polynomials that generalize the Faber polynomials, to compact sets consisting of several components (i.e., whose complement is a multiply connected domain). These generalized Faber polynomials are called Faber–Walsh polynomials and also allow the expansion of an analytic function into a series with properties again similar to the power series.

In what follows, let us briefly recall some basic concepts on Faber–Walsh polynomials and Faber–Walsh expansions we need in this chapter.

Everywhere in this chapter $G \subset \mathbb{C}$ will be considered a compact set consisting of several components, that is $\tilde{\mathbb{C}} \setminus G$ is multiply connected.

Definition 1 (See, e.g., Walsh [43]). A lemniscatic domain is a domain of the form $\{w \in \tilde{C}; |U(w)| > \mu\}$, where $\mu > 0$ is some constant and $U(w) = \Pi_{j=1}^{\nu}(w - \alpha_j)^{m_j}$ for some points $\alpha_1, \ldots, \alpha_\nu \in \mathbb{C}$ and real exponents $m_1, \ldots, m_\nu > 0$ with $\sum_{j=1}^{\nu} m_j = 1$.

In all what follows, we will consider that the points $\alpha_1, \ldots, \alpha_\nu$ have the property that from them can be chosen a sequence $(\alpha_j)_{j\in\mathbb{N}}$ such that for any closed set C not containing any of the points $\alpha_1, \ldots, \alpha_\nu$, there exist constants $A_1(C), A_2(C) > 0$ with

$$A_1(C) < \frac{|u_n(w)|}{|U(w)|^n} < A_2(C),\ n = 0, 1, 2, \ldots,\ w \in C, \tag{2}$$

where $u_n(w) = \Pi_{j=1}^n (w - \alpha_j)$.

Let $D_1, \ldots, D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set). According to Theorem 3 in Walsh [42], there exists a lemniscatic domain

$$K_1 = \{w \in \tilde{\mathbb{C}}; |U(w)| > \mu\}$$

and a conformal bijection

$$\Phi : \tilde{\mathbb{C}} \setminus G \to \{w \in \tilde{\mathbb{C}}; |U(w)| > \mu\},\ \text{with } \Phi(\infty) = \infty,\ \text{and } \Phi'(\infty) = 1.$$

Here μ is the logarithmic capacity of G. Further, the inverse conformal bijection satisfies

$$\Psi = \Phi^{-1} = \{w \in \tilde{\mathbb{C}}; |U(w) > \mu\} \to \tilde{\mathbb{C}} \setminus G,\ \text{with } \Psi(\infty) = 1 \text{ and } \Psi'(\infty) = 1.$$

Consider the Green's functions $H_1(w) = \log(|U(w)|) - \log(\mu)$, $H = H_1 \circ \Phi$ and for $r > 1$ their level curves

$$\Lambda_r = \{w \in \mathbb{C}; H_1(w) = \log(r)\} = \{w \in \mathbb{C}; |U(w)| = r\mu\},$$
$$\Gamma(r) = \{z \in \mathbb{C}; H(z) = \log(r)\}.$$

We have $\Gamma_r = \Psi(\Lambda_r)$. Denote by G_r the interior of Γ_r and by D_r^∞ the exterior of Λ_r (including ∞).

Notice that for $1 < r < \beta < R$ we have $G \subset G_r \subset G_\beta \subset G_R$.

According to Theorem 3 in Walsh [43], for $z \in \Gamma_r$ and $w \in D_r^\infty$ we have

$$\frac{\Psi'(w)}{\Psi(w) - z} = \sum_{n=0}^{\infty} \frac{B_n(z)}{u_{n+1}(w)},\ \text{with } B_n(z) = \frac{1}{2\pi i} \int_{\Lambda_\lambda} u_n(t) \cdot \frac{\Psi'(t)}{\Psi(t) - z} dt, \lambda > r.$$

The polynomial $B_n(z)$ is called the n-th Faber–Walsh polynomial attached to G and $(\alpha_j)_{j\in\mathbb{N}}$ and according to Lemma 2.5 in Sète [33], the Faber–Walsh polynomials are independent of the lemniscatic domain and the exterior mapping function Ψ.

Remark 1. The proof of existence of the above conformal mapping Ψ (and implicitly of the existence of Faber–Walsh polynomials) was obtained in Walsh [42] and it is based on some results on critical points of polynomials obtained in the book of Walsh [41]. It is worth mentioning that this book of Walsh [41] was also used in the important book of Rahman–Schmeisser [30], where many other results of the authors in this topic are presented.

Remark 2. A nice property of the Faber–Walsh polynomials obtained in Walsh [43, p. 31], relation (34), is that

$$\lim\sup_{k\to\infty}[\|B_k\|_G]^{1/k}=\mu,$$

property which is similar to that for Chebyshev polynomials attached to the multiply connected compact set G and also holds for many sets of polynomials defined by extremal properties (see Fekete–Walsh [6, 7]). Here $\|\cdot\|_G$ denotes the uniform norm on G.

Similar to the Faber polynomials, according to Theorem 3 in Walsh [43], the Faber–Walsh polynomials allow the series expansion of functions analytic in compact sets. Namely, if f is analytic on the compact set G (with multiply connected complement), there exists $R>1$ such that f is analytic in G_R and inside G_R admits (locally uniformly) the series expansion $f(z)=\sum_{k=0}^{\infty}a_k(f)B_k(z)$, with

$$a_k(f)=\frac{1}{2\pi i}\int_{\Lambda_\beta}\frac{f(\Psi(t))}{u_{k+1}(t)}dt,\ 1<\beta<R. \tag{3}$$

Remark 3. If G is simply connected, then the Faber–Walsh polynomials become the Faber polynomials.

Remark 4. In our reasonings, we will also need the following estimate, see, e.g., Walsh [43], p. 29, relation (26)

$$|B_k(z)|\le A_1(r\mu)^k,\ \text{ for all } z\in\Gamma_r,\ 1<r<R,k\ge 0, \tag{4}$$

where A_1 depends on r only.

Also, by the relationship $\lim\sup_{n\to\infty}|a_k|^{1/k}\le\frac{1}{\beta\mu}$ in Walsh [43], page 30, we immediately get the estimate

$$|a_k(f)|\le\frac{C(\beta,\mu,f)}{(\beta\mu)^k},\ \text{ for all } k=0,1,\ldots, \tag{5}$$

where $C(\beta,\mu,f)>0$ is independent of k. Note that here and in all the next reasonings we will choose $1<r<\beta<R$.

For further properties of Faber–Walsh polynomials, see, e.g., Chap. 13 in Suetin [37].

Now, denoting $[n]_q=\frac{q^n-1}{q-1}$ for $q>1$, $[n]_q=n$ for $q=1$,

$$[n]_q!=[1]_q\cdot[2]_q\cdot\ldots\cdot[n]_q,\ \binom{n}{k}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$$

and using the Faber–Walsh polynomials $B_p(z)$, attached to the multiple connected compact set G, for f analytic on G, suggested by the comments on Bernstein–Faber polynomials in Introduction, we can introduce the following.

Definition 2. For $q, \mu \geq 1$, $z \in G$, $n \in \mathbb{N}$, the q-Bernstein–Faber–Walsh polynomials attached to G and f are defined by the formulas

$$W_{n,q}(f;G)(z) = \sum_{k=0}^{\infty} a_k(f) \cdot [\sum_{p=0}^{n} D_{n,p,k}^{(q)} \cdot B_p(z)] = \sum_{p=0}^{n} B_p(z) [\sum_{k=0}^{\infty} a_k(f) \cdot D_{n,p,k}^{(q)}], \quad (6)$$

where $B_p(z)$ is the Faber–Walsh polynomial of degree p attached to G, $a_k(f)$ are the coefficients in the Faber–Walsh expansion of f and $D_{n,p,k}^{(q)}$ are the constants mentioned in Introduction, i.e. more precisely

$$D_{n,p,k}^{(q)} = \binom{n}{p}_q [\Delta^p_{1/[n]_q} e_k(0)]_q$$

$$= (1 - [1]_q/[n]_q) \cdot \ldots \cdot (1 - [p-1]_q/[n]_q) \cdot [0, [1]_q/[n]_q, \ldots, [p]_q/[n]_q; e_k].$$

Here $[0, [1]_q/[n]_q, \ldots, [p]_q/[n]_q; e_k]$ denotes the divided difference of $e_k(z) = z^k$ and we suppose $\mu \geq 1$ due to the fact that in the classical case when G is a compact disk with radius r, we have considered that $r \geq 1$ and therefore in that case $\mu = r \geq 1$.

Remark 5. The expression $W_{n,q}(f;G)(z)$ in Definition 2 is indeed a polynomial (of degree n), because denoting $S_{n,p} = \sum_{k=0}^{\infty} a_k(f) D_{n,p,k}^{(q)}$, applying the mean value theorem to the divided difference and taking into account (5) too, for any $0 \leq p \leq n$ we obtain

$$|S_{n,p}^{(q)}| \leq \sum_{k=0}^{\infty} |a_k(f)| \cdot D_{n,p,k}^{(q)} \leq C_1 \sum_{k=0}^{\infty} \frac{1}{(\beta\mu)^k} \cdot [0, [1]_q/[n]_q, \ldots, [p]_q/[n]_q; e_k]$$

$$\leq C_2 \sum_{k=p+1}^{\infty} \frac{1}{(\beta\mu)^k} \cdot \frac{k^p}{p!} < +\infty.$$

Remark 6. In the past, while the Faber polynomials were studied and used in many previously published papers, the Faber–Walsh polynomials have rarely been studied, excepting the Suetin's book [37], which contains a short section about them. The main reason for neglecting the Faber–Walsh polynomials was the fact that no explicit examples of Walsh's lemniscatic conformal maps were known. But very recently, by the papers [33–35], Sète, Oral communication, the Faber–Walsh polynomials were brought again into attention. Thus, the first example of Walsh's lemniscatic conformal maps seems to be mentioned in the very recent paper of Sète–Liesen [35]. Also, the first explicit formulas for the Faber–Walsh polynomials were obtained for the case when G consists in two disjoint compact intervals in Sète–Liesen [34]. The results in the present work are also new contributions to the topic of Faber–Walsh polynomials.

3 Approximation by q-Bernstein–Faber–Walsh Polynomials

In this section we present the approximation results by q-Bernstein–Faber–Walsh polynomials, $q \geq 1$. The notations used are those in Sect. 2.

For the proofs of results, we will use the methods in the case of q-Bernstein–Faber polynomials in [9], pp. 19–20 and [13], pp. 88–98), adapted to the case of Faber–Walsh polynomials.

We can summarize all the results by the following theorem.

Theorem 1. *Let $D_1, \ldots, D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set) and suppose that μ, the logarithmic capacity of G, satisfies $\mu \geq 1$. Let f be analytic in G, that is there exists $R > 1$ such that f is analytic in G_R. Also, denote $S_k^{(q)} = [1]_q + \ldots + [k-1]_q$, $k \geq 2$. Above, we recall that G_R denotes the interior of the closed level curve Γ_R, both defined as in Sect. 2.*

(i) Then for any $1 < r < R$ the following upper estimate

$$|W_{n,1}(f;G)(z) - f(z)| \leq \frac{C}{n},\ z \in \overline{G_r},\ n \in \mathbb{N},$$

holds, where $C > 0$ depends on f, r, and G_r but it is independent of n.

(ii) Let $1 < q < R$. Then for any $1 < r < \frac{R}{q}$, the following upper estimate

$$|W_{n,q}(f;G)(z) - f(z)| \leq \frac{C}{q^n},\ \text{for all}\ z \in \overline{G}_r,\ n \in \mathbb{N},$$

holds, where $C > 0$ depends on f, r, G_r and q but is independent of n and z.

(iii) Assume that $1 \leq q < R$ and denote $S_k^{(q)} = [1]_q + \ldots + [k-1]_q$, $k \geq 2$.

(iii_a) If $q = 1$, then for any $1 < r < R$, $z \in \overline{G_r}$ and $n \in \mathbb{N}$, the following upper estimate

$$\left| W_{n,1}(f;G)(z) - f(z) - \sum_{k=2}^{\infty} a_k(f) \cdot \frac{S_k^{(1)}}{n}[B_{k-1}(z) - B_k(z)] \right| \leq \frac{C}{n^2}$$

holds, where $C > 0$ depends on f, r, G_r but it is independent of n.

(iii_b) If $q > 1$, $1 < r < \frac{R}{q^2}$, $z \in \overline{G_r}$ and $n \in \mathbb{N}$, then the following upper estimate

$$\left| W_{n,q}(f;G)(z) - f(z) - \sum_{k=2}^{\infty} a_k(f) \cdot \frac{S_k^{(q)}}{[n]_q}[B_{k-1}(z) - B_k(z)] \right| \leq \frac{C}{q^{2n}}$$

holds, where $C > 0$ depends on f, r, G_r, and q but is independent of n.

(iv) If $q \geq 1$, then for any $1 < r < \frac{R}{q}$ we have

$$\lim_{n\to\infty}[n]_q(W_{n,q}(f;G)(z) - f(z)) = A_q(f)(z), \text{ uniformly in } \overline{G_r},$$

where $A_q(f)(z) = \sum_{k=2}^{\infty} a_k(f) \cdot S_k^{(q)} \cdot [B_{k-1}(z) - B_k(z)]$.

(v) Let $1 \leq q < R$. If $1 < r < \frac{R}{q}$ and f is not a polynomial of degree ≤ 1 in G, then

$$\|W_{n,q}(f;G) - f\|_{\overline{G_r}} \sim \frac{1}{\tilde{q}^n}, n \in \mathbb{N},$$

where $\|f\|_{\overline{G_r}} = \sup\{|f(z)|; z \in \overline{G_r}\}$ and the constants in the equivalence depend on f, r, G_r, and q, but are independent of n.

Proof.

(i) By the considerations in Sect. 2, for any fixed $1 < \beta < R$ we have $f(z) = \sum_{k=0}^{\infty} a_k(f) B_k(z)$ uniformly in $\overline{G_\beta}$, where $a_k(f)$ are the Faber–Walsh coefficients given by (3).

Consequently, by (6) we obtain

$$\begin{aligned}|W_{n,1}(f;G)(z) - f(z)| &\leq \sum_{k=0}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)| \\ &= \sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)| \\ &\quad + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)|.\end{aligned}$$

Since for the classical complex Bernstein polynomials attached to a disk of center at origin we can write $B_n(e_k)(z) = \sum_{p=0}^{n} D_{n,p,k}^{(1)} z^p$, since each e_k is convex of any order and $B_n(e_k)(1) = e_k(1) = 1$ for all k, it follows that all $D_{n,p,k}^{(1)} \geq 0$ and $\sum_{p=0}^{n} D_{n,p,k}^{(1)} = 1$, for all k and n. Also, note that $D_{n,k,k}^{(1)} = \frac{n(n-1)\ldots(n-k+1)}{n^k}$.

In the estimation of $|a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)|$ we distinguish two cases: (1) $0 \leq k \leq n$; (2) $k > n$.

Case 1. We have

$$|\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)| \leq |B_k(z)| \cdot |1 - D_{n,k,k}^{(1)}| + \sum_{p=0}^{k-1} D_{n,p,k}^{(1)} \cdot |B_p(z)|.$$

Fix now $1 < r < \beta$. By using the estimate in (4), we immediately get

$$|\sum_{p=0}^{n} D_{n,p,k}^{(1)} B_p(z) - B_k(z)| \leq 2A_1(r)[1 - D_{n,k,k}^{(1)}](r\mu)^k \leq A_1(r)\frac{k(k-1)}{n}(r\mu)^k,$$

for all $z \in \overline{G}_r$. Here we used the inequality $1 - \Pi_{i=1}^{k} x_i \leq \sum_{i=1}^{k}(1 - x_i)$ (valid if all $x_i \in [0, 1]$) which implies the inequality

$$1 - D^{(1)}_{n,k,k} = 1 - \frac{n(n-1)\dots(n-k+1)}{n^k} = 1 - \Pi_{i=1}^{k-1} \frac{n-i}{n}$$
$$\leq \sum_{i=1}^{k-1}(1 - (n-i)/n) = \frac{1}{n}\sum_{i=0}^{k-1} i = \frac{k(k-1)}{2n}.$$

By the estimate for $|a_k(f)|$ in (5), for all $z \in \overline{G}_r$ and $k = 0, 1, 2, \dots n$, it follows

$$|a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C(r, \beta, \mu, f)}{n} k(k-1) \left[\frac{r}{\beta}\right]^k,$$

(above μ was simplified), that is

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C(r, \beta, \mu, f)}{n} \sum_{k=2}^{n} k(k-1) d^k, \text{ for all } z \in \overline{G}_r,$$

where $0 < d = r/\beta < 1$. Also, clearly we have $\sum_{k=2}^{n} k(k-1)d^k \leq \sum_{k=2}^{\infty} k(k-1)d^k < \infty$, which finally implies that

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C^*(r, \beta, \mu, f)}{n}.$$

Case 2. We have

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - B_k(z)| \leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z)| + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |B_k(z)|.$$

By the estimates mentioned in the Case (1), we immediately get

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |B_k(z)| \leq C(r, \beta, \mu, f) \sum_{k=n+1}^{\infty} d^k, \text{ for all } z \in \overline{G}_r,$$

with $d = r/\beta$.

Also,

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z)| = \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \left| \sum_{p=0}^{n} D^{(1)}_{n,p,k} \cdot B_p(z) \right|$$

$$\leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \sum_{p=0}^{n} D^{(1)}_{n,p,k} \cdot |B_p(z)|.$$

But for $p \leq n < k$ and taking into account the estimates obtained in the Case (1) we get

$$|a_k(f)| \cdot |B_p(z)| \leq C(r,\beta,\mu,f) \frac{(r\mu)^p}{(\beta\mu)^k} \leq C(r,\beta,\mu,f) \frac{r^k}{\beta^k}, \text{ for all } z \in \overline{G}_r,$$

which implies

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - B_k(z)| \leq C(r,\beta,\mu,f) \sum_{k=n+1}^{\infty} \sum_{p=0}^{n} D^{(1)}_{n,p,k} \left[\frac{r}{\beta}\right]^k$$

$$= C(r,\beta,\mu,f) \sum_{k=n+1}^{\infty} \left[\frac{r}{\beta}\right]^k$$

$$= C(r,\beta,\mu,f) \frac{d^{n+1}}{1-d},$$

with $d = r/\beta$.

In conclusion, collecting the estimates in the Cases (1) and (2) we obtain

$$|\sum_{p=0}^{n} D^{(1)}_{n,p,k} B_p(z) - f(z)| \leq \frac{C_1}{n} + C_2 d^{n+1} \leq \frac{C}{n}, \ z \in \overline{G}_r, \ n \in \mathbb{N}.$$

(ii) By using (6), we obtain

$$|W_{n,q}(f;G)(z) - f(z)| \leq \sum_{k=0}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| =$$

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)|.$$

Since each e_k is convex of any order, we get that all $D^{(q)}_{n,p,k} \geq 0$ and $\sum_{p=0}^{n} D^{(q)}_{n,p,k} = 1$, for all k and n.

Also, note that for all $k \geq 1$

$$D^{(q)}_{n,k,k} = \left(1 - \frac{[1]_q}{[n]_q}\right) \dots \left(1 - \frac{[k-1]_q}{[n]_q}\right)$$

and that $D^{(q)}_{n,0,0} = 1$.

In the estimation of $|a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)|$ we distinguish two cases: (1) $0 \leq k \leq n$; (2) $k > n$.

Case 1. Since $D^{(q)}_{n,0,0} = 1$, we may suppose that $1 \leq k \leq n$. We have

$$|\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| \leq |B_k(z)| \cdot |1 - D^{(q)}_{n,k,k}| + \sum_{p=0}^{k-1} D^{(q)}_{n,p,k} \cdot |B_p(z)|.$$

Fix now $1 < r < \frac{\beta}{q}$. By the proof of Theorem 3, p. 3766 in Mahmudov [26] (taking there $\alpha = \gamma = 0$), taking into account (4) and by similar reasonings with those in the case $q = 1$, we immediately get

$$|\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| \leq 2A_1(r)[1 - D^{(q)}_{n,k,k}](r\mu)^k \leq c(r) \frac{k[k-1]_q}{[n]_q} (r\mu)^k$$

$$\leq c(r) \frac{kq^k}{(q-1)([n]_q)} (r\mu)^k = c(r) \frac{k(qr\mu)^k}{(q-1)([n]_q)} \leq c(r,q) \frac{k(qr\mu)^k}{[n]_q},$$

for all $z \in \overline{G}_r$.

Then, by the estimate in (5) of $|a_k(f)|$, for all $z \in \overline{G}_r$ and $k = 0, 1, 2, \dots n$, it follows

$$|a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C(r,\beta,\mu,f,q)}{[n]_q} k \left[\frac{qr}{\beta}\right]^k,$$

that is

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C(r,\beta,\mu,f,q)}{[n]_q} \sum_{k=1}^{n} k d_1^k, \text{ for all } z \in \overline{G}_r,$$

where $d_1 = \frac{rq}{\beta} < 1$

Also, clearly we have $\sum_{k=1}^{n} k d_1^k \leq \sum_{k=1}^{\infty} k d_1^k < \infty$, which finally implies that

$$\sum_{k=0}^{n} |a_k(f)| \cdot |\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)| \leq \frac{C^*(r,\beta,\mu,f,q)}{[n]_q} \leq \frac{qC^*(r,\beta,\mu,f,q)}{q^n}.$$

Here we used the inequality $\frac{1}{[n]_q} \leq \frac{q}{q^n}$.

Case 2. We have

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| \leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z)| + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |B_k(z)|.$$

By the estimates mentioned in the Case (1), we immediately get

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |B_k(z)| \leq C(r,\beta,f) \sum_{k=n+1}^{\infty} d^k, \text{ for all } z \in \overline{G}_r,$$

with $d = r/\beta$.

Also,

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z)| \leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \sum_{p=0}^{n} D_{n,p,k}^{(q)} \cdot |B_p(z)|.$$

But for $p \leq n < k$ and taking into account the estimates obtained in the Case (1) we get

$$|a_k(f)| \cdot |B_p(z)| \leq C(r,\beta,\mu,f) \frac{(r\mu)^p}{(\beta\mu)^k} \leq C(r,\beta,\mu,f) \frac{r^k}{\beta^k}, \text{ for all } z \in \overline{G}_r,$$

which therefore implies

$$\begin{aligned} &\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| \\ &\leq C(r,\beta,\mu,f) \sum_{k=n+1}^{\infty} \sum_{p=0}^{n} D_{n,p,k}^{(q)} \left[\frac{r}{\beta}\right]^k + C(r,\beta,\mu,f) \sum_{k=n+1}^{\infty} \left[\frac{r}{\beta}\right]^k \\ &\leq C(r,\beta,\mu,f) \sum_{k=n+1}^{\infty} \left[\frac{r}{\beta}\right]^k = C(r,\beta,\mu,f) \frac{d^{n+1}}{1-d} \\ &= \frac{rC(r,\beta,\mu,f)}{\beta - r} \cdot d^n \leq \frac{rC(r,\beta,\mu,f)}{\beta - r} \cdot \frac{1}{q^n}, \end{aligned}$$

with $d = \frac{r}{\beta} < \frac{1}{q} < 1$.

In conclusion, collecting the estimates in the Cases (1) and (2) we obtain

$$|W_{n,q}(f;G)(z) - f(z)| \leq \frac{c_1}{q^n} + \frac{c_2}{q^n} \leq \frac{C}{q^n}, \; z \in \overline{G}_r, \; n \in \mathbb{N},$$

with the constants $c_1, c_2, C > 0$ depending on r, β, μ, f, q, but independent of n and z.

(iii) Let $1 < r < \frac{R}{q}$. Simple calculation shows that

$$S_k^{(q)} = \frac{k(k-1)}{2}, \text{ for } q = 1 \text{ and } S_k^{(q)} = \frac{q^k - k(q-1) - 1}{(q-1)^2}, \text{ for } q > 1. \quad (7)$$

Note that by Lemma 3, p. 245 in [28], we have

$$D_{n,k-1,k}^{(q)} = \frac{S_k^{(q)}}{[n]_q} \cdot \Pi_{i=1}^{k-2}\left(1 - \frac{[i]_q}{[n]_q}\right), \; k \le n. \quad (8)$$

In what follows, first we prove that $A_q(f)(z)$ given by

$$A_q(f)(z) = \sum_{k=2}^{\infty} a_k(f) \cdot S_k^{(q)} \cdot [B_{k-1}(z) - B_k(z)],$$

is analytic in $\overline{G_r}$, for $1 < r < \frac{R}{q}$.

Indeed, by the inequality

$$|A_q(f)(z)| \le \sum_{k=0}^{\infty} |a_k(f)| \cdot S_k^{(q)} \cdot [|B_{k-1}(z)| + |B_k(z)|],$$

since by (7) we get $S_k^{(q)} \le \frac{q^k}{(q-1)^2}$ for $q > 1$, by (4) and (5) it immediately follows

$$|A_q(f)(z)| \le \frac{2C(r) \cdot C(\beta, \mu, f)}{(q-1)^2} \cdot \sum_{k=0}^{\infty} d^k = \frac{2C(r) \cdot C(\beta, \mu, f)}{(1-d)(q-1)^2}, \text{ if } q > 1,$$

and

$$|A_q(f)(z)| \le C(r) \cdot C(\beta, \mu, f) \sum_{k=0}^{\infty} k(k-1)d^k, \text{ if } q = 1,$$

with $d = \frac{rq}{\beta} < 1$, for all $z \in \overline{G_r}$. But by the ratio test the above series is uniformly convergent, which immediately shows that for $q \ge 1$, the function $A_q(f)$ is well defined and analytic in $\overline{G_r}$.

Now, by (6) we obtain

$$\left| W_{n,q}(f; G)(z) - f(z) - \sum_{k=0}^{\infty} a_k(f) \cdot \frac{S_k^{(q)}}{[n]_q} [B_{k-1}(z) - B_k(z)] \right|$$
$$\le \sum_{k=0}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(q)}(G)(z)|,$$

where

$$E^{(q)}_{k,n}(G)(z) = \sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z) - \frac{S^{(q)}_k}{[n]_q}[B_{k-1}(z) - B_k(z)].$$

Because simple calculations imply that

$$E^{(q)}_{0,n}(G)(z) = E^{(q)}_{1,n}(G)(z) = E^{(q)}_{2,n}(G)(z) = 0,$$

in fact we have to estimate the expression

$$\sum_{k=3}^{\infty} |a_k(f)| \cdot |E^{(q)}_{k,n}(G)(z)|$$

$$= \sum_{k=3}^{n} |a_k(f)| \cdot |E^{(q)}_{k,n}(G)(z)| + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |E^{(q)}_{k,n}(G)(z)|.$$

To estimate $|E^{(q)}_{k,n}(G)(z)|$, we distinguish two cases : (1) $3 \le k \le n$; (2) $k \ge n+1$.

Case 1. By using (4), we obtain

$$[n]_q|E^{(q)}_{k,n}(G)(z)| = |[n]_q(\sum_{p=0}^{n} D^{(q)}_{n,p,k} B_p(z) - B_k(z)) - S^{(q)}_k \cdot (B_{k-1}(z) - B_k(z))|$$

$$\le C(r)(r\mu)^k [n]_q \sum_{i=1}^{k-2} D^{(q)}_{n,i,k} + C(r)(r\mu)^k |[n]_q D^{(q)}_{n,k-1,k} - S^{(q)}_k|$$

$$+ C(r)(r\mu)^k |[n]_q (1 - D^{(q)}_{n,k,k}) - S^{(q)}_k|.$$

Taking now into account (8) and following exactly the reasonings in the proof of Lemma 3, p. 747 in [44], we arrive at

$$|E^{(q)}_{k,n}(G)(z)| \le \frac{4C(r)(k-1)^2[k-1]_q^2}{[n]_q^2} \cdot (r\mu)^k, \text{ for all } z \in \overline{G_r}. \tag{9}$$

Let β satisfy $qr < \beta < R$. By (5) and (9) it follows

$$\sum_{k=3}^{n} |a_k(f)| \cdot |E^{(q)}_{k,n}(G)(z)| \le \frac{4C(r) \cdot C(\beta, \mu, f)}{[n]_q^2} \cdot \sum_{k=3}^{n} (k-1)^2 [k-1]_q^2 \rho^k, \tag{10}$$

for all $z \in \overline{G_r}$ and $n \in \mathbb{N}$, where $\rho = \frac{r}{\beta} < \frac{1}{q}$.

Case 2. We get

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(q)}(G)(z)|$$

$$\leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z)| + \sum_{k=n+1}^{\infty} |a_k(f)| \cdot |B_k(z)|$$

$$+\frac{1}{[n]_q} \sum_{k=n+1}^{\infty} |a_k(f)| \cdot S_k^{(q)} \cdot |B_{k-1}(z)| + \frac{1}{[n]_q} \sum_{k=n+1}^{\infty} |a_k(f)| \cdot S_k^{(q)} \cdot |B_k(z)|$$

$$=: L_{1,q}(z) + L_{2,q}(z) + L_{3,q}(z) + L_{4,q}(z). \tag{11}$$

Wa have two subcases : (iii_a) $q = 1$; (iii_b) $q > 1$.

Subcase (iii_a). By (4) and (5), for $1 < r < \beta < R$, denoting $\rho = \frac{r}{\beta}$, it immediately follows

$$L_{1,1}(z) \leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \sum_{p=0}^{n} D_{n,p,k}^{(1)} |B_p(z)| \leq \frac{C(r,\beta,\mu,f)}{n^2} \sum_{k=n+1}^{\infty} (k-1)^2 \rho^k,$$

and similarly

$$L_{2,1}(z) \leq \frac{C(r,\beta,\mu,f)}{n^2} \sum_{k=n+1}^{\infty} (k-1)^2 \rho^k.$$

for all $z \in \overline{G_r}$.

Next, by similar reasonings as above and by (7), we obtain

$$L_{3,1}(z) \leq \frac{C(r,\beta,\mu,f)}{n} \sum_{k=n+1}^{\infty} \frac{k(k-1)}{2} \rho^k \leq \frac{C(r,\beta,\mu,f)}{n^2} \sum_{k=n+1}^{\infty} (k-1)^3 \rho^k,$$

and

$$L_{4,1}(z) \leq \frac{C(r,\beta,\mu,f)}{n^2} \sum_{k=n+1}^{\infty} (k-1)^3 \rho^k,$$

which by (11) implies that there exists a constant $K(r,\beta,\mu,f) > 0$ independent of n, such that

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(1)}(G)(z)| \leq \frac{K(r,\beta,\mu,f)}{n^2} \sum_{k=n+1}^{\infty} (k-1)^3 \rho^k,$$

for all $z \in \overline{G_r}$ and $n \in \mathbb{N}$.

But the sequence $\{a_n = \sum_{k=n+1}^{\infty}(k-1)^3\rho^k, n \in \mathbb{N}\}$ is convergent to zero (therefore bounded by a constant $M > 0$ independent of n), as the remainder of the convergent series $\sum_{k=0}^{\infty}(k-1)^3\rho^k$ (applying, for example, the ratio test), which will imply

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(1)}(G)(z)| \leq \frac{M \cdot K(r, \beta, \mu, f)}{n^2},$$

for all $z \in \overline{G_r}$ and $n \in \mathbb{N}$.

Now, taking $q = 1$ in (10) and taking into account that by the ratio test the series $\sum_{k=3}^{\infty}(k-1)^4\rho^k$ is convergent, we get

$$\sum_{k=3}^{n} |a_k(f)| \cdot |E_{k,n}^{(1)}(G)(z)| \leq \frac{4C(r) \cdot C(\beta, f)}{n^2} \cdot \sum_{k=3}^{n}(k-1)^4\rho^k$$
$$\leq \frac{4C(r) \cdot C(\beta, \mu, f)}{n^2} \cdot \sum_{k=3}^{\infty}(k-1)^4\rho^k = \frac{K'(r, \beta, \mu, f)}{n^2},$$

which combined with the previous estimate immediately implies

$$\sum_{k=0}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(1)}(G)(z)| \leq \frac{C}{n^2},$$

for all $z \in \overline{G_r}$ and $n \in \mathbb{N}$, where $C > 0$ is a constant independent of n.

This proves (iii_a) in the statement of (iii).

Subcase (iii_b). By (4) and (5), for $1 < r < \frac{\beta}{q} < \frac{R}{q}$ and denoting $\rho = \frac{r}{\beta} < \frac{1}{q} < 1$, for all $z \in \overline{G_r}$ it follows

$$L_{1,q}(z) \leq \sum_{k=n+1}^{\infty} |a_k(f)| \cdot \sum_{p=0}^{n} D_{n,p,k}^{(q)} |B_p(z)| \leq C(r, \beta, \mu, f) \sum_{k=n+1}^{\infty} \rho^k$$
$$\leq \frac{C(r, \beta, \mu, f)}{[n]_q^2} \sum_{k=n+1}^{\infty} [k-1]_q^2 \rho^k \leq \frac{C(r, \beta, \mu, f, q)}{[n]_q^2} \sum_{k=n+1}^{\infty} (q^2\rho)^k$$

and similarly

$$L_{2,q}(z) \leq C(r, \beta, \mu, f) \sum_{k=n+1}^{\infty} \rho^k \leq \frac{C(r, \beta, \mu, f)}{[n]_q^2} \sum_{k=n+1}^{\infty} [k-1]_q^2 \rho^k$$
$$\leq \frac{C(r, \beta, \mu, f, q)}{[n]_q^2} \sum_{k=n+1}^{\infty} (q^2\rho)^k.$$

Also, since by (7) we get $S_k^{(q)} \leq \frac{q^k}{(q-1)^2}$, by using (4) and (5) too, for all $z \in \overline{G_r}$ it follows

$$L_{3,q}(z) \leq \frac{C(r,\beta,\mu,f,q)}{[n]_q} \sum_{k=n+1}^{\infty} q^k \rho^k = \frac{C(r,\beta,\mu,f,q)}{[n]_q} \sum_{k=n+1}^{\infty} (q\rho)^k$$
$$\leq \frac{C'(r,\beta,\mu,f,q)}{[n]_q^2} \sum_{k=n+1}^{\infty} (q^2\rho)^k,$$

and

$$L_{4,q}(z) \leq \frac{C(r,\beta,\mu,f,q)}{[n]_q} \sum_{k=n+1}^{\infty} q^k \rho^k = \frac{C(\beta,f,q)}{[n]_q} \sum_{k=n+1}^{\infty} (q\rho)^k$$
$$\leq \frac{C'(r,\beta,\mu,f,q)}{[n]_q^2} \sum_{k=n+1}^{\infty} (q^2\rho)^k.$$

By (11), we immediately obtain

$$\sum_{k=n+1}^{\infty} |a_k(f)| \cdot |E_{k,n}^{(q)}(G)(z)| \leq \frac{K(r,\beta,\mu,f,q)}{[n]_q^2},$$

for all $z \in \overline{G_r}$, if $q^2\rho < 1$ (which holds for $1 < r < \frac{\beta}{q^2} < \frac{R}{q^2}$). Also, by (10), since $[k-1]_q^2 \leq [k]_q^2 \leq \frac{q^{2k}}{(q-1)^2}$, for $z \in \overline{G_r}$ with $1 < r < \frac{\beta}{q^2} < \frac{R}{q^2}$, we easily obtain

$$\sum_{k=3}^{n} |a_k(f)| \cdot |E_{k,n}^{(q)}(G)(z)| \leq \frac{K'(r,\beta,\mu,f,q)}{[n]_q^2}.$$

Collecting these results, we immediately obtain the upper estimate in (iii_b) too.

(iv) The case $q = 1$ follows directly by multiplying by n in the estimate in (iii_a) and by passing to limit with $n \to \infty$. In the case of $q > 1$, if $1 < r < \frac{R}{q^2}$ then by multiplying in (iii_b) with $[n]_q$ and passing to limit with $n \to \infty$, we get the desired conclusion.

What remained to be proved is that the limit in (iv) still holds under the more general condition $1 < r < \frac{R}{q}$.

Since $\frac{R}{q^{1+t}} \nearrow \frac{R}{q}$ as $t \searrow 0$, evidently that given $1 < r < \frac{R}{q}$, there exists a $t \in (0,1)$, such that $q^{1+t}r < R$. Because f is analytic in G, choosing β with $q^{1+t}r < \beta < R$, by (4) and (5) this implies that $\sum_{k=2}^{\infty} |a_k(f)| k^4 q^{(1+t)k} r^k \leq \sum_{k=2}^{\infty} k^4 \left(\frac{q^{1+t}r}{\beta}\right)^k < \infty$, for all $z \in \overline{G_r}$. Also, the convergence of the previous series implies that for arbitrary $\varepsilon > 0$, there exists n_0, such that $\sum_{k=n_0+1}^{\infty} |a_k(f)| k^2 q^k r^k < \varepsilon$.

By using (9), for all $z \in \overline{G_r}$ and $n > n_0$ we get

$$\left|[n]_q(\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - f(z)) - A_q(f)(z)\right|$$

$$\leq \sum_{k=2}^{n_0} |a_k(f)| \cdot \left|[n]_q(\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)) - S_k^{(q)}[B_{k-1}(z) - B_k(z)]\right|$$

$$+ \sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot \left([n]_q|\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| + S_k^{(q)}|B_{k-1}(z) - B_k(z)|\right)$$

$$\leq C(r) \sum_{k=2}^{n_0} |a_k(f)| \cdot \frac{4(k-1)^2[k-1]_q^2}{[n]_q} \cdot (r\mu)^k$$

$$+ \sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot \left([n]_q|\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| + S_k^{(q)}|B_{k-1}(z) - B_k(z)|\right).$$

But by the proof of the above points (i), (ii), Case (1), for $k \leq n$ we have

$$|\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| \leq C(r)\frac{k[k-1]_q}{[n]_q} \cdot (r\mu)^k,$$

while for $k > n$ and using (4), we get

$$|\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)|$$

$$\leq |\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z)| + |B_k(z)| \leq \sum_{p=0}^{n} D_{n,p,k}^{(q)} |B_p(z)| + |B_k(z)|$$

$$\leq C(r)(r\mu)^n + C(r)(r\mu)^k \leq C'(r)(r\mu)^k \leq 2C'(r)\frac{k[k-1]_q}{[n]_q} \cdot (r\mu)^k,$$

for all $z \in \overline{G_r}$.

Also, since $S_k^{(q)} \leq (k-1)[k-1]_q$, by using (4) it is immediate that

$$S_k^{(q)} \cdot |B_{k-1}(z) - B_k(z)| \leq S_k^{(q)} \cdot [|B_{k-1}(z)| + |B_k(z)|]$$
$$\leq 2C(r)(k-1)[k-1]_q(r\mu)^k.$$

Therefore, we easily obtain

$$\sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot \left([n]_q | \sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - B_k(z)| + S_k^{(q)} |B_{k-1}(z) - B_k(z)| \right)$$

$$\leq K(r, \mu) \cdot \sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot (k-1)[k-1]_q (r\mu)^k,$$

valid for all $z \in \overline{G_r}$, where $K(r, \mu) > 0$ is a constant depending only on r.

From this we conclude that, for all $z \in \overline{G_r}$ and $n > n_0$ we have

$$\left| [n]_q (\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - f(z)) - A_q(f)(z) \right|$$

$$\leq C(r) \sum_{k=2}^{n_0} |a_k(f)| \cdot \frac{4(k-1)^2 [k-1]_q^2}{[n]_q} \cdot (r\mu)^k$$

$$+ K(r, \mu) \cdot \sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot (k-1)[k-1]_q (r\mu)^k$$

$$\leq \frac{4C(r)}{[n]_q^t} \cdot \sum_{k=2}^{n_0} |a_k(f)| \cdot k^2 [k-1]_q^{1+t} \cdot (r\mu)^k + K(r, \mu) \cdot \sum_{k=n_0+1}^{\infty} |a_k(f)| \cdot k^2 q^k (r\mu)^k$$

$$\leq \frac{4C(r)}{[n]_q^t} \cdot \sum_{k=2}^{\infty} |a_k(f)| \cdot k^4 q^{(1+t)k} \cdot (r\mu)^k + K(r, \mu)\varepsilon.$$

Now, since $\frac{4C(r)}{[n]_q^t} \to 0$ as $n \to \infty$ and $\sum_{k=2}^{\infty} |a_k(f)| \cdot k^4 q^{(1+t)k} \cdot (r\mu)^k < \infty$, for the given $\varepsilon > 0$, there exists an index n_1, such that

$$\frac{4C(r)}{[n]_q^t} \cdot \sum_{k=2}^{\infty} |a_k(f)| \cdot k^4 q^{(1+t)k} \cdot (r\mu)^k < \varepsilon,$$

for all $n > n_1$.

As a final conclusion, for all $n > \max\{n_0, n_1\}$ and $z \in \overline{G_r}$, we get

$$\left| [n]_q (\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - f(z)) - A_q(f)(z) \right| \leq (1 + K(r))\varepsilon,$$

which shows that

$$\lim_{n\to\infty} [n]_q (\sum_{p=0}^{n} D_{n,p,k}^{(q)} B_p(z) - f(z)) = A_q(f)(z), \text{ uniformly in } \overline{G_r}.$$

(v) Suppose that we would have $\|W_{n,1}(f;G) - f\|_{\overline{G_r}} = o([n]_q^{-1})$. Then, combining the above points (i), (ii), and (iv) would immediately imply that $A_q(f) = 0$ for all $z \in \overline{G_r}$, where $A_q(f)(z)$.

But $A_q(f)(z) = 0$ for all $z \in \overline{G_r}$ by simple calculation implies

$$2a_2(f)S_2^{(q)}B_1(z) + \sum_{k=2}^{\infty}[S_{k+1}^{(q)}a_{k+1}(f) - S_k^{(q)}a_k(f)]B_k(z) = 0,\ z \in \overline{G_r}.$$

By the uniqueness of Walsh polynomial series (see [33]), since by (7) it is clear that $S_k^{(q)} > 0$ for all $k \geq 2$, we would get that $a_2(f) = 0$ and

$$S_{k+1}^{(q)}a_{k+1}(f) - S_k^{(q)}a_k(f) = 0,\ \text{for all } k = 2, 3, \ldots, .$$

For $k = 2$ we easily get $a_3(f) = 0$ and taking above step by step $k = 3, 4, \ldots,$ we easily would obtain that $a_k(f) = 0$ for all $k \geq 2$.

Therefore we would get $f(z) = a_0(f)B_0(z) + a_1(f)B_1(z)$ for all $z \in \overline{G_r}$. But because $B_k(z)$ is a polynomial of exact degree k, would imply that f is a polynomial of degree ≤ 1 in $\overline{G_r}$, a contradiction with the hypothesis.

In conclusion, if f is not a polynomial of degree ≤ 1, then the approximation order is exactly $\frac{1}{[n]_q}$, which ends the proof of the theorem.

Remark 7. In the case when $G = [-\beta, -\alpha]\bigcup[\alpha, \beta]$, with $0 < \alpha < \beta$, the conformal mapping Ψ and the Faber–Walsh polynomials are calculated in [35]. This would allow to calculate the coefficients in the Faber–Walsh expansion and therefore the q-Bersnstein–Faber–Walsh polynomials too.

4 Approximation by Szász-Mirakjan–Faber–Walsh Operators

Everywhere in this section, $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of real positive numbers with the property that $\lambda_n \searrow 0$ as fast as we want. Without the loss of generality, we may suppose that $\lambda_n \leq \frac{1}{2}$, for all $n \in \mathbb{N}$.

Definition 3. Let $D_1, \ldots, D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G := \bigcup_{j=1}^{\nu} D_j$ in the extended plane is a ν-times connected region (open and connected set) and suppose that f is analytic in G, that is there exists $R > 1$ such that f is analytic in G_R, i.e. $f(z) = \sum_{k=0}^{\infty} a_k(f)B_k(z)$ for all $z \in G_R$, where $B_k(z)$ denotes the Faber–Walsh polynomials attached to G and G_R denotes the interior of the closed level curve Γ_R, all defined as in Sect. 2.

The Szász–Mirakjan–Faber–Walsh operators attached to G and f will be formally defined by

$$M_n(f;\lambda_n,G;z)=\sum_{k=0}^{\infty}a_k(f)\cdot\sum_{j=0}^{k}[0,\lambda_n,\ldots,j\lambda_n;e_k]\cdot B_j(z). \tag{12}$$

We are in a position to state the main result.

Theorem 2. *Let $D_1,\ldots,D_\nu$ be mutually exterior compact sets (none a single point) of the complex plane such that the complement of $G:=\bigcup_{j=1}^{\nu}D_j$ in the extended plane is a ν-times connected region (open and connected set) and suppose that f is analytic in G, that is there exists $R>1$ such that f is analytic in G_R, i.e. $f(z)=\sum_{k=0}^{\infty}a_k(f)B_k(z)$ for all $z\in G_R$. Also, suppose that there exist $M>0$ and $A\in\left(\frac{1}{R\mu},\frac{1}{\mu}\right)$, with $|a_k(f)|\le M\frac{A^k}{k!}$, for all $k=0,1,\ldots$, (which implies $|f(z)|\le C(r)Me^{\mu Ar}$ for all $z\in G_r$, $1<r<R$). Here μ, G_R, and G_r are those defined in Sect. 2.*

Let $1<r<\frac{1}{A\mu}$ be arbitrary fixed. Then, there exist $n_0\in\mathbb{N}$ and $C(r,f,\mu)>0$ depending on r and f only, such that for all $z\in\overline{G_r}$ and $n\ge n_0$ we have

$$|M_n(f;\lambda_n,G;z)-f(z)|\le C(r,f,\mu)\cdot\lambda_n.$$

Proof. Firstly, we prove that $M_n(f;\lambda_n,G;z)$ given by formula (12) is well defined. Thus, from Lemma 3.2 in [18], we easily get

$$\sum_{j=0}^{k}[0,\lambda_n,2\lambda_n,\ldots,j\lambda_n;e_k]\le(k+1)!,\ \text{for all } k\ge 0 \text{ and } n\in\mathbb{N},$$

which for $1<r<1/(\mu A)$ and by (4) implies

$$|M_n(f;\lambda_n,G;z)|\le C(r)\sum_{k=0}^{\infty}|a_k(f)|\cdot(k+1)!(r\mu)^k\le MC(r)\sum_{k=0}^{\infty}(k+1)(\mu Ar)^k<\infty,$$

for all $z\in\overline{G_r}$ and $n\in\mathbb{N}$.

Let us denote $T_{n,k}(z)=\sum_{j=0}^{k}[0,\lambda_n,2\lambda_n,\ldots,j\lambda_n;e_k]\cdot B_j(z)$, $z\in G$, $n\in\mathbb{N}, k\ge 0$, $m(n)=[1/\lambda_n]$. For $1<r<R$ and $z\in\overline{G_r}$, by the formula in Definition 3, we obtain

$$|M_n(f;\lambda_n,G;z)-f(z)|\le\sum_{k=0}^{\infty}|a_k(f)|\cdot|T_{n,k}(z)-B_k(z)|$$

$$=\sum_{k=0}^{m(n)}|a_k(f)|\cdot|T_{n,k}(z)-B_k(z)|+\sum_{k=m(n)+1}^{\infty}|a_k(f)|\cdot|T_{n,k}(z)-B_k(z)|$$

$$\leq \sum_{k=0}^{m(n)} |a_k(f)| \cdot |T_{n,k}(z) - B_k(z)| + \sum_{k=m(n)+1}^{\infty} |a_k(f)| \cdot |T_{n,k}(z)| + \sum_{k=m(n)+1}^{\infty} |a_k(f)| \cdot |B_k(z)|$$

$$:= S_1 + S_2 + S_3.$$

In the case of S_1, by using Lemma 3.1 in [14] and the estimate (4) in Sect. 2, $|B_j(z)| \leq C(r) \cdot (r\mu)^j$, $z \in \overline{G_r}$, $j \geq 0$, it follows

$$S_1 \leq \sum_{k=0}^{m(n)} |a_k(f)| \cdot \sum_{j=0}^{k-1} [0, \lambda_n, 2\lambda_n, \ldots, j\lambda_n; e_k] \cdot |B_j(z)|$$

$$\leq C(r) \cdot \lambda_n \cdot \sum_{k=0}^{m(n)} |a_k(f)| \cdot \frac{(k+1)!}{2} \cdot (\mu r)^k.$$

By $|a_k(f)| \leq M \cdot \frac{A^k}{k!}$, for all $k = 0, 1, \ldots,$, where $A \in \left(\frac{1}{R\mu}, \frac{1}{r\mu}\right)$, we get

$$S_1 \leq \frac{M \cdot C(r)}{2} \cdot \lambda_n \sum_{k=0}^{n} (k+1) \cdot (\mu A r)^k \leq \frac{M \cdot C(r)}{2} \cdot \lambda_n \cdot \sum_{k=0}^{\infty} (k+1) \cdot (\mu A r)^k,$$

where $\sum_{k=0}^{\infty} (k+1) \cdot (\mu A r)^k < +\infty$.

To estimate S_2, firstly from the relationships $\mu A r < 1$, $a - 1 \leq [a] \leq a + 1$ and $m(n) \nearrow +\infty$ as $n \to \infty$, it is easy to see that there exists n_0 depending on r and A, such that

$$(m(n) + 2) \cdot (\mu A r)^{m(n)+1} \leq (1/\lambda_n + 2) \cdot (\mu A r)^{a_n/b_n} \leq \lambda_n, \text{ for all } n \geq n_0.$$

Now, by using Lemma 3.2 in [18], for all $n \geq n_0$ we have

$$S_2 \leq \sum_{k=m(n)+1}^{\infty} |a_k(f)| \cdot |T_{n,k}(z)| \leq \sum_{k=m(n)+1}^{\infty} |a_k(f)| \cdot \sum_{j=0}^{k} [0, \lambda_n, 2\lambda_n, \ldots, j\lambda_n; e_k] \cdot |B_j(z)|$$

$$\leq C(r) \cdot \sum_{k=m(n)+1}^{\infty} |a_k(f)| \cdot (k+1)! \cdot (\mu r)^k$$

$$\leq (m(n) + 2) \cdot (\mu A r)^{m(n)+1} \cdot M \cdot C(r) \sum_{k=0}^{\infty} (k+1)(\mu A r)^k \leq M C(r) \lambda_n \cdot \sum_{k=0}^{\infty} (k+1)(\mu A r)^k.$$

Finally, to estimate S_3, since $\frac{k}{m(n)+1} \geq 1$, we obtain

$$S_3 \leq \frac{C(r) \cdot M}{m(n) + 1} \cdot \sum_{k=m(n)+1}^{\infty} k \cdot \frac{A^k}{k!} \cdot (\mu r)^k \leq \mu \cdot A \cdot r \cdot C(r) \cdot M \lambda_n \cdot \sum_{k=m(n)+1}^{\infty} \frac{(\mu A r)^{k-1}}{(k-1)!}$$

$$\leq e^{\mu A r} \cdot \mu \cdot A \cdot r \cdot C(r) \cdot M \cdot \lambda_n.$$

Collecting all the above estimates for S_1, S_2, and S_3, we immediately get the estimate in the statement.

Remark 8. Since $\sum_{k=0}^{\infty}(k+1)P_m(k)(\mu Ar)^k < +\infty$ and $\sum_{k=0}^{\infty} P_m(k+1)\cdot\frac{(\mu Ar)^k}{k!} < +\infty$ for any algebraic polynomial P_m of degree $\le m$ satisfying $P_m(k) > 0$ for all $k \ge 0$, it is immediate from the proof that Theorem 2 holds under the more general hypothesis $|a_k(f)| \le P_m(k)\cdot\frac{A^k}{k!}$, for all $k \ge 0$.

Remark 9. In the case when the set G is simply connected compact set, Theorem 2 was proved in [14].

Remark 10. It is worth noting that in fact the condition $\mu \ge 1$ in Theorem 1 can be dropped and the conditions on A and r in the statement of Theorem 2 can be written as $A \in (1/R, 1)$, $1 < r < 1/A$ (i.e., independent of the logarithmic capacity μ), simply by suitably normalizing the lemniscatic domain to be $K_1 = \{w : |U(w)| > 1\}$ and choosing $\Phi'(\infty) = 1/\mu > 0$. Indeed, in this case, the attached Faber–Walsh polynomials $\tilde{B}_k(z)$ and the Faber–Walsh coefficients $\tilde{a}_k(f)$ in the expansion $f(z) = \sum_{k=0}^{\infty}\tilde{a}_k(f)\cdot\tilde{B}_k(z)$, satisfy (4) and (5) without the appearance of μ^k in these estimates (see Sète, Oral communication).

At the end of this chapter, firstly we conjecture that the (classical) truncated Szász–Mirakjan complex operators $T_n(f)(z) = e^{-nz}s_n(f)(z) := e^{-nz}\sum_{k=0}^{n}\frac{(nz)^k}{k!}f(k/n)$, attached to the unit disk, may represent a constructive solution to the so-called weighted approximation problem, obtained in the Szegö domain :

Theorem 3 ([29], Theorems 3.2 and 3.4). *Denote by $G = \{z; |z| < 1, |ze^{1-z}| < 1\}$, the Szegö domain interior to the Szegö curve $S = \{z; |ze^{1-z}| = 1, |z| \le 1\}$ (see [39]). If f is analytic in G and continuous on compact subsets of $\overline{G}\setminus\{1\}$, then for any compact $E \subset \overline{G}\setminus\{1\}$, there exists a sequence of polynomials $(P_n(f)(z))_{n\in\mathbb{N}}$, with degree $P_n \le n$, such that denoting $T_n(f)(z) = e^{-nz}P_n(f)(z)$, we have $\lim_{n\to\infty}T_n(f)(z) = f(z)$, uniformly on E.*

Furthermore, if f is analytic in G and continuous on $\overline{G}$ with $f(1) = 0$, then a sequence of polynomials $(P_n(f)(z))_{n\in\mathbb{N}}$ exists, such that $\lim_{n\to\infty}T_n(f)(z) = f(z)$, uniformly on $\overline{G}$.

Above, the rate of convergence is geometric.

In order to motivate our open question, firstly let us make some considerations on the truncated Szász–Mirakjan operators of real variable introduced in [24],

$$\tilde{S}_n(f)(x) = e^{-nx}\sum_{k=0}^{n}\frac{(nx)^k}{k!}f(k/n),\ n\in\mathbb{N},\ x\in[0,1],$$

attached to $f \in C[0,1] = \{f : [0,1] \to \mathbb{R}; f \text{ is continuous on } [0,1]\}$. Defining $f^* : [0,+\infty) \to \mathbb{R}$ by $f^*(x) = f(x)$ if $x \in [0,1]$ and $f^*(x) = f(1)$ if $x \ge 1$, we obviously can write, for all $x \in [0,1]$ and $n \in \mathbb{N}$,

$$\tilde{S}_n(f)(x) = S_n(f^*)(x) - f(1)\left[1 - \frac{\sum_{k=0}^{n}(nx)^k/k!}{e^{nx}}\right]$$
$$= S_n(f^*)(x) - f(1)[e^{-nx}\sum_{k>n}(nx)^k/k!],$$

where $S_n(f^*)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f^*(k/n)$ is the classical Szász–Mirakjan operator attached to f^*.

Reasoning exactly as in the proof of Theorem 5 in [24], we easily get that $\lim_{n\to\infty}\tilde{S}_n(f)(x) = f(x)$, uniformly on every compact subinterval of $x \in [0, 1)$.

Now, for all $n \in \mathbb{N}$, $x \in [0, 1]$, let us denote $R_n(x) = \sum_{k>n}^{\infty}\frac{(nx)^k}{k!}$ and

$$L_n(x) = e^{-nx}\left[\sum_{k=0}^{n}\frac{(nx)^k}{k!}\right] = [1 - e^{-nx}R_n(x)].$$

By the above formula, we can write

$$\tilde{S}_n(f)(x) = S_n(f^*)(x) - f(1)[e^{-nx}\sum_{k>n}(nx)^k/k!] = S_n(f^*)(x) - f(1)[1 - L_n(x)], \tag{13}$$

for all $n \in \mathbb{N}$ and $x \in [0, 1]$.

But, according to the book [8, p. 229] (see also the papers [23, 38]), we have

$$\lim_{n\to\infty} L_n(x) = 1,\ \text{if } x \in [0, 1),\ \lim_{n\to\infty} L_n(1) = \frac{1}{2},\ \lim_{n\to\infty} L_n(x) = 0,\ \text{if } x > 1.$$

Therefore, we obtain $\lim_{n\to\infty}\tilde{S}_n(f)(x) = \lim_{n\to\infty} S_n(f^*)(x)$, for all $x \in [0, 1)$ and since f^* is obviously of exponential growth, according to the Szász's result, we have $\lim_{n\to\infty} S_n(f^*)(x) = f(x)$, uniformly on $[0, 1]$. On the other hand, passing to limit with $n \to \infty$ in (13), we obtain

$$\lim_{n\to\infty}\tilde{S}_n(f)(1) = f(1) - f(1)[1 - 1/2] = \frac{1}{2}f(1),$$

where obviously that $\frac{1}{2}f(1) = f(1)$ only if $f(1) = 0$.

From here it also follows that for the partial sums of the Szász–Mirakjan operators, in order to have uniform convergence on the closed interval $[0, 1]$ (and consequently on the whole closure of Szegö's set in the complex case), we necessarily need to have $f(1) = 0$. Contrariwise, for $f(1) \neq 0$, the partial sums of the Szász–Mirakjan operators do not converge at $x = 1$ to $f(1)$.

In particular, if $f(x) = e_j(x) = x^j$, with $j = 0, 1, \ldots$, arbitrary, then we get

$$\lim_{n\to\infty}\left|e^{-nx}\sum_{k=0}^{n}\frac{(nx)^k}{k!}\left(\frac{k}{n}\right)^j - x^j\right| = 0,$$

uniformly for all compact subintervals in $[0, 1)$.

Then, it is natural to look for the convergence of the truncated Szász–Mirakjan complex operators

$$\tilde{S}_n(f)(z) = e^{-nz} \sum_{k=0}^{n} \frac{(nz)^k}{k!} f(k/n),\ n \in \mathbb{N},$$

attached to an analytic function $f(z)$ in $\mathbb{D}_1 = \{z; |z| < 1\}$, for example, in the open unit disk $|z| < 1/2$, i.e., to $f(z) = \sum_{j=0}^{\infty} c_j e_j(z)$, $|z| < 1/2$. Here we consider $|z| < 1/2$, because according to Theorem 3.8 in [29], the weighted approximation by polynomials is possible, uniformly on compacts subsets in any open disk of radius $1/2$.

Since we immediately can write

$$|\tilde{S}_n(f)(z) - f(z)| \leq \sum_{j=0}^{\infty} |c_j| \cdot \left| e^{-nz} \sum_{k=0}^{n} \frac{(nz)^k}{k!} \cdot \left(\frac{k}{n}\right)^j - z^j \right|,$$

it is natural to consider the following.

Conjecture. If $f : \mathbb{D}_1 \to \mathbb{C}$ is analytic in $\mathbb{D}_1$, then we believe that a constructive solution of the weighted approximation in $\mathbb{D}_{1/2} \cap G$ is represented by $P_n(f)(z) = \sum_{k=0}^{n} \frac{(nz)^k}{k!} f(k/n)$, $n \in \mathbb{N}$, that is for any compact $K \subset \mathbb{D}_{1/2} \cap G$, uniformly in K we have $\lim_{n\to\infty} e^{-nz} P_n(f)(z) = f(z)$.

Remark 11. If in the above conjecture f is supposed analytic in only $\{|z| < 1/2\}$, then the conjecture is that $\lim_{n\to\infty} e^{-nz} \sum_{k=0}^{[n/2]} \frac{(nz)^k}{k!} f(k/n) = f(z)$, uniformly in any compact set $K \subset \{|z| < 1/2\}$. The choice for this partial sum is motivated by the fact that similar reasonings with those in the proof of Theorem 5 in [24] leads to the conclusion that $\lim_{n\to\infty} e^{-nx} \sum_{k=0}^{[n/2]} \frac{(nx)^k}{k!} f(k/n) = f(x)$, uniformly in any compact subinterval of $[0, 1/2)$.

For a quantitative estimate in the above conjecture, it is natural to consider the following generalization of the Szegö's problem:

Open Problem. *Let $j = 0, 1, \ldots,$ be fixed and let us denote $s_{n,j}(z) = \sum_{k=0}^{n} \frac{z^k}{k!} \cdot \left(\frac{k}{n}\right)^j$. For each $j = 0, 1, \ldots,$, find a quantitative (asymptotic) estimate for*

$$\left| e^{-nz} \sum_{k=0}^{n} \frac{(nz)^k}{k!} \cdot \left(\frac{k}{n}\right)^j - z^j \right| = |e^{-nz} s_{n,j}(nz) - z^j|$$

and the distribution of the zeros for $s_{n,j}(nz)$.

Note that in the particular case $j = 0$, we recapture the classical Szegö's result in [39], which says that each accumulation point of all the zeroes of $s_{n,0}(z)$ lies on the Szegö's curve. Also, it is known, for example, the following estimate (see [29]),

$$|e^{-nz} s_{n,0}(nz) - 1| \leq \frac{4}{\sqrt{2\pi n}|z-1|},\ z \in \overline{G} \setminus \{1\}, n \geq 1.$$

Remark 12. In solving the above open problem, we mention that the extensions of Eneström–Kakeya Theorem in Govil–Rahman [21] and other results on the location of the zeroes of polynomials in the Rahman–Schmeisser's book [30] could be useful.

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Summation Formulas of Euler–Maclaurin and Abel–Plana: Old and New Results and Applications

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Abstract Summation formulas of the Euler–Maclaurin and Abel–Plana and their connections with several kinds of quadrature rules are studied. Besides the history of these formulas, several of their modifications and generalizations are considered. Connections between the Euler–Maclaurin formula and basic quadrature rules of Newton–Cotes type, as well as the composite Gauss–Legendre rule and its Lobatto modification are presented. Besides the basic Plana summation formula a few integral modifications (the midpoint summation formula, the Binet formula, Lindelöf formula) are introduced and analysed. Starting from the moments of their weight functions and applying the recent MATHEMATICA package `OrthogonalPolynomials`, recursive coefficients in the three-term recurrence relation for the corresponding orthogonal polynomials are constructed, as well as the parameters (nodes and Christoffel numbers) of the corresponding Gaussian quadrature formula.

Keywords Summation • Euler–Maclaurin formula • Abel–Plana formula • Gaussian quadrature formula • Orthogonal polynomial • Three-term recurrence relation

Mathematics Subject Classification (2010): 33C45, 33C47, 41A55, 65B15, 65D30, 65D32

1 Introduction and Preliminaries

A summation formula was discovered independently by Leonhard Euler [18, 19] and Colin Maclaurin [35] plays an important role in the broad area of numerical analysis, analytic number theory, approximation theory, as well as in many applications in

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other fields. This formula, today known as the Euler–Maclaurin summation formula,

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,\mathrm{d}x + \frac{1}{2}(f(0) + f(n)) + \sum_{\nu=1}^{r} \frac{B_{2\nu}}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r(f), \tag{1}$$

was published first time by Euler in 1732 (without proof) in connection with the problem of determining the sum of the reciprocal squares,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots, \tag{2}$$

which is known as the *Basel problem*. The brothers Johann and Jakob Bernoulli, Leibnitz, Stirling, etc. also dealt intensively by such a kind of problems. In modern terminology, the sum (2) represents the zeta function of 2, where more generally

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad (s > 1).$$

Although at that time the theory of infinite series was not exactly based, it was observed a very slow convergence of this series, e.g. in order to compute directly the sum with an accuracy of six decimal places it requires taking into account at least a million first terms, because

$$\frac{1}{n+1} < \sum_{k=n+1}^{+\infty} \frac{1}{k^2} < \frac{1}{n}.$$

Euler discovered the remarkable formula with much faster convergence

$$\zeta(2) = \log^2 2 + \sum_{k=1}^{+\infty} \frac{1}{2^{k-1}k^2},$$

and obtained the value $\zeta(2) = 1.644944\ldots$ (with seven decimal digits). But the discovery of a general summation procedure (1) enabled Euler to calculate $\zeta(2)$ to 20 decimal places. For details see Gautschi [25, 26] and Varadarajan [61].

Using a generalized Newton identity for polynomials (when their degree tends to infinity), Euler [19] proved the exact result $\zeta(2) = \pi^2/6$. Using the same method he determined $\zeta(s)$ for even $s = 2m$ up to 12,

$$\zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450},\ \zeta(10) = \frac{\pi^{10}}{93555},\ \zeta(12) = \frac{691\pi^{12}}{638512875}.$$

Sometime later, using his own partial fraction expansion of the cotangent function, Euler obtained the general formula

$$\zeta(2\nu) = (-1)^{\nu-1}\frac{2^{2\nu-1}B_{2\nu}}{(2\nu)!}\pi^{2\nu},$$

where $B_{2\nu}$ are the Bernoulli numbers, which appear in the general Euler–Maclaurin summation formula (1). Detailed information about Euler's complete works can be found in *The Euler Archive* (http://eulerarchive.maa.org).

We return now to the general Euler–Maclaurin summation formula (1) which holds for any $n, r \in \mathbb{N}$ and $f \in C^{2r}[0, n]$. As we mentioned before this formula was found independently by Maclaurin. While in Euler's case the formula (1) was applied for computing slowly converging infinite series, in the second one Maclaurin used it to calculate integrals. A history of this formula was given by Barnes [5], and some details can be found in [3, 8, 25, 26, 38, 61].

Bernoulli numbers B_k ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$) can be expressed as values at zero of the corresponding Bernoulli polynomials, which are defined by the generating function

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{k=0}^{+\infty} B_k(x)\frac{t^k}{k!}.$$

Similarly, Euler polynomials can be introduced by

$$\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t + 1} = \sum_{k=0}^{+\infty} E_k(x)\frac{t^k}{k!}.$$

Bernoulli and Euler polynomials play a similar role in numerical analysis and approximation theory like orthogonal polynomials. First few Bernoulli polynomials are

$$B_0(x) = 1,\ \ B_1(x) = x - \frac{1}{2},\ \ B_2(x) = x^2 - x + \frac{1}{6},\ \ B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2},$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},\ \ B_5(x) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6},\ \ \text{etc.}$$

Some interesting properties of these polynomials are

$$B_n'(x) = nB_{n-1}(x),\ \ B_n(1-x) = (-1)^n B_n(x),\ \ \int_0^1 B_n(x)\,\mathrm{d}x = 0\ \ (n \in \mathbb{N}).$$

The error term $E_r(f)$ in (1) can be expressed in the form (cf. [8])

$$E_r(f) = (-1)^r \sum_{k=1}^{+\infty} \int_0^n \frac{\mathrm{e}^{\mathrm{i}2\pi kt} + \mathrm{e}^{-\mathrm{i}2\pi kt}}{(2\pi k)^{2r}} f^{(2r)}(x)\,\mathrm{d}x,$$

or in the form

$$E_r(f) = -\int_0^n \frac{B_{2r}(x - \lfloor x \rfloor)}{(2r)!} f^{(2r)}(x)\,\mathrm{d}x, \tag{3}$$

where $\lfloor x \rfloor$ denotes the largest integer that is not greater than x. Supposing $f \in C^{2r+1}[0,n]$, after an integration by parts in (3) and recalling that the odd Bernoulli numbers are zero, we get (cf. [28, p. 455])

$$E_r(f) = \int_0^n \frac{B_{2r+1}(x - \lfloor x \rfloor)}{(2r+1)!} f^{(2r+1)}(x)\,\mathrm{d}x. \tag{4}$$

If $f \in C^{2r+2}[0,n]$, using Darboux's formula one can obtain (1), with

$$E_r(f) = \frac{1}{(2r+2)!} \int_0^1 [B_{2r+2} - B_{2r+2}(x)] \left(\sum_{k=0}^{n-1} f^{(2r+2)}(k+x) \right) \mathrm{d}x \tag{5}$$

(cf. Whittaker and Watson [65, p. 128]). This expression for $E_r(f)$ can be also derived from (4), writting it in the form

$$\begin{aligned} E_r(f) &= \int_0^1 \frac{B_{2r+1}(x)}{(2r+1)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+x) \right) \mathrm{d}x \\ &= \int_0^1 \frac{B'_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+x) \right) \mathrm{d}x, \end{aligned}$$

and then by an integration by parts,

$$\begin{aligned} E_r(f) = &\left[\frac{B_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+x) \right) \right]_0^1 \\ &- \int_0^1 \frac{B_{2r+2}(x)}{(2r+2)!} \left(\sum_{k=0}^{n-1} f^{(2r+2)}(k+x) \right) \mathrm{d}x. \end{aligned}$$

Because of $B_{2r+2}(1) = B_{2r+2}(0) = B_{2r+2}$, the last expression can be represented in the form (5).

Since

$$(-1)^r [B_{2r+2} - B_{2r+2}(x)] \geq 0, \quad x \in [0,1],$$

and

$$\int_0^1 [B_{2r+2} - B_{2r+2}(x)]\,\mathrm{d}t = B_{2r+2},$$

according to the *Second Mean Value Theorem for Integrals*, there exists $\eta \in (0, 1)$ such that

$$E_r(f) = \frac{B_{2r+2}}{(2r+2)!}\left(\sum_{k=0}^{n-1} f^{(2r+2)}(k+\eta)\right) = \frac{nB_{2r+2}}{(2r+2)!} f^{(2r+2)}(\xi), \quad 0 < \xi < n. \tag{6}$$

The Euler–Maclaurin summation formula can be considered on an arbitrary interval (a, b) instead of $(0, n)$. Namely, taking $h = (b-a)/n$, $t = a + xh$, and $f(x) = f((t-a)/h) = \varphi(t)$, formula (1) reduces to

$$h\sum_{k=0}^{n} \varphi(a+kh) = \int_a^b \varphi(t)\,\mathrm{d}t + \frac{h}{2}\left[\varphi(a)+\varphi(b)\right] + \sum_{\nu=1}^{r} \frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\varphi^{(2\nu-1)}(b) - \varphi^{(2\nu-1)}(a)\right] + E_r(\varphi), \tag{7}$$

where, according to (6),

$$E_r(\varphi) = (b-a)\frac{B_{2r+2}h^{2r+2}}{(2r+2)!}\varphi^{(2r+2)}(\xi), \quad a < \xi < b. \tag{8}$$

Remark 1. An approach in the estimate of the remainder term of the Euler–Maclaurin formula was given by Ostrowski [47].

Remark 2. The Euler–Maclaurin summation formula is implemented in MATHEMATICA as the function `NSum` with option `Method -> Integrate`.

2 Connections Between Euler–Maclaurin Summation Formula and Some Basic Quadrature Rules of Newton–Cotes Type

In this section we first show a direct connection between the Euler–Maclaurin summation formula (1) and the well-known composite trapezoidal rule,

$$T_n f := \sum_{k=0}^{n}{}'' f(k) = \frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(k) + \frac{1}{2}f(n), \tag{9}$$

for calculating the integral

$$I_n f := \int_0^n f(x)\,\mathrm{d}x. \tag{10}$$

This rule for integrals over an arbitrary interval $[a, b]$ can be presented in the form

$$h \sum_{k=0}^{n}{}'' \varphi(a + kh) = \int_a^b \varphi(t)\,\mathrm{d}t + E^T(\varphi), \tag{11}$$

where, as before, the sign $\sum''$ denotes summation with the first and last terms halved, $h = (b - a)/n$, and $E^T(\varphi)$ is the remainder term.

Remark 3. In general, the sequence of the composite trapezoidal sums converges very slowly with respect to step refinement, because of $|E^T(\varphi)| = O(h^2)$. However, the trapezoidal rule is very attractive in numerical integration of analytic and periodic functions, for which $\varphi(t + b - a) = \varphi(t)$. In that case, the sequence of trapezoidal sums

$$T_n(\varphi; h) := h \sum_{k=0}^{n}{}'' \varphi(a + kh) = h \sum_{k=1}^{n} \varphi(a + kh) \tag{12}$$

converges geometrically when applied to analytic functions on periodic intervals or the real line. A nice survey on this subject, including history of this phenomenon, has been recently given by Trefethen and Weideman [59] (see also [64]). For example, when φ is a $(b - a)$-periodic and analytic function, such that $|\varphi(z)| \le M$ in the half-plane $\mathrm{Im}\, z > -c$ for some $c > 0$, then for each $n \ge 1$, the following estimate

$$|E^T(\varphi)| = \left| T_n(\varphi; h) - \int_a^b \varphi(t)\,\mathrm{d}t \right| \le \frac{(b - a)M}{\mathrm{e}^{2\pi cn/(b-a)} - 1}$$

holds. A similar result holds for integrals over $\mathbb{R}$.

It is well known that there are certain types of integrals which can be transformed (by changing the variable of integration) to a form suitable for the trapezoidal rule. Such transformations are known as *Exponential* and *Double Exponential Quadrature Rules* (cf. [44–46, 57, 58]). However, the use of these transformations could introduce new singularities in the integrand and the analyticity strip may be lost. A nice discussion concerning the error theory of the trapezoidal rule, including several examples, has been recently given by Waldvogel [63].

Remark 4. In 1990 Rahman and Schmeisser [51] gave a specification of spaces of functions for which the trapezoidal rule converges at a prescribed rate as $n \to +\infty$, where a correspondence is established between the speed of convergence and regularity properties of integrands. Some examples for these spaces were provided in [64].

In a general case, according to (1), it is clear that

$$T_n f - I_n f = \sum_{\nu=1}^{r} \frac{B_{2\nu}}{(2\nu)!} \left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0) \right] + E_r^T(f), \tag{13}$$

where $T_n f$ and $I_n f$ are given by (9) and (10), respectively, and the remainder term $E_r^T(f)$ is given by (6) for functions $f \in C^{2r+2}[0, n]$.

Similarly, because of (7), the corresponding formula on the interval $[a, b]$ is

$$h\sum_{k=0}^{n}{}'' \varphi(a+kh) - \int_a^b \varphi(t)\,\mathrm{d}t = \sum_{\nu=1}^{r} \frac{B_{2\nu}h^{2\nu}}{(2\nu)!}\left[\varphi^{(2\nu-1)}(b) - \varphi^{(2\nu-1)}(a)\right] + E_r^T(\varphi),$$

where $E_r^T(\varphi)$ is the corresponding remainder given by (8). Comparing this with (11) we see that $E^T(\varphi) = E_0^T(\varphi)$.

Notice that if $\varphi^{(2r+2)}(x)$ does not change its sign on (a, b), then $E_r^T(\varphi)$ has the same sign as the first neglected term. Otherwise, when $\varphi^{(2r+2)}(x)$ is not of constant sign on (a, b), then it can be proved that (cf. [14, p. 299])

$$|E_r^T(\varphi)| \le h^{2r+2}\frac{|2B_{2r+2}|}{(2r+2)!}\int_a^b |\varphi^{(2r+2)}(t)|\,\mathrm{d}t \approx 2\left(\frac{h}{2\pi}\right)^{2r+2}\int_a^b |\varphi^{(2r+2)}(t)|\,\mathrm{d}t,$$

i.e., $|E_r^T(\varphi)| = O(h^{2r+2})$. Supposing that $\int_a^{+\infty} |\varphi^{(2r+2)}(x)|\,\mathrm{d}x < +\infty$, this holds also in the limit case as $b \to +\infty$. This limit case enables applications of the Euler–Maclaurin formula in summation of infinite series, as well as for obtaining asymptotic formulas for a large b.

A standard application of the Euler–Maclaurin formula is in numerical integration. Namely, for a small constant h, the trapezoidal sum can be dramatically improved by subtracting appropriate terms with the values of derivatives at the endpoints a and b. In such a way, the corresponding approximations of the integral can be improved to $O(h^4)$, $O(h^6)$, etc.

Remark 5. Rahman and Schmeisser [52] obtained generalizations of the trapezoidal rule and the Euler–Maclaurin formula and used them for constructing quadrature formulas for functions of exponential type over infinite intervals using holomorphic functions of exponential type in the right half-plane, or in a vertical strip, or in the whole plane. They also determined conditions which equate the existence of the improper integral to the convergence of its approximating series.

Remark 6. In this connection an interesting question can be asked. Namely, what happens if the function $\varphi \in C^\infty(\mathbb{R})$ and its derivatives are $(b-a)$-periodic, i.e., $\varphi^{(2\nu-1)}(a) = \varphi^{(2\nu-1)}(b)$, $\nu = 1, 2, \ldots$? The conclusion that $T_n(\varphi; h)$, in this case, must be exactly equal to $\int_a^b \varphi(t)\,\mathrm{d}t$ is wrong, but the correct conclusion is that $E^T(\varphi)$ decreases faster than any finite power of h as n tends to infinity.

Remark 7. Also, the Euler–Maclaurin formula was used in getting an extrapolating method well-known as Romberg's integration (cf. [14, pp. 302–308 and 546–523] and [39, pp. 158–164]).

In the sequel, we consider a quadrature sum with values of the function f at the points $x = k + \frac{1}{2}$, $k = 0, 1, \ldots, n-1$, i.e., the so-called midpoint rule

$$M_n f := \sum_{k=0}^{n-1} f\left(k + \frac{1}{2}\right).$$

Also, for this rule there exists the so-called second Euler–Maclaurin summation formula

$$M_nf - I_nf = \sum_{\nu=1}^{r} \frac{(2^{1-2\nu}-1)B_{2\nu}}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^M(f), \tag{14}$$

for which

$$E_r^M(f) = n\,\frac{(2^{-1-2r}-1)B_{2r+2}}{(2r+2)!}f^{(2r+2)}(\xi), \qquad 0 < \xi < n,$$

when $f \in C^{2r+2}[0,n]$ (cf. [39, p. 157]). As before, I_nf is given by (10).

The both formulas, (13) and (14), can be unified as

$$Q_nf - I_nf = \sum_{\nu=1}^{r} \frac{B_{2\nu}(\tau)}{(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^Q(f),$$

where $\tau = 0$ for $Q_n \equiv T_n$ and $\tau = 1/2$ for $Q_n \equiv M_n$. It is true, because of the fact that [50, p. 765] (see also [10])

$$B_\nu(0) = B_\nu \quad \text{and} \quad B_\nu\left(\frac{1}{2}\right) = (2^{1-\nu}-1)B_\nu.$$

If we take a combination of T_nf and M_nf as

$$Q_nf = S_nf = \frac{1}{3}(T_nf + 2M_nf),$$

which is, in fact, the well-known classical composite Simpson rule,

$$S_nf := \frac{1}{3}\left[\frac{1}{2}f(0) + \sum_{k=1}^{n-1} f(k) + 2\sum_{k=0}^{n-1} f\left(k+\frac{1}{2}\right) + \frac{1}{2}f(n)\right],$$

we obtain

$$S_nf - I_nf = \sum_{\nu=2}^{r} \frac{(4^{1-\nu}-1)B_{2\nu}}{3(2\nu)!}\left[f^{(2\nu-1)}(n) - f^{(2\nu-1)}(0)\right] + E_r^S(f). \tag{15}$$

Notice that the summation on the right-hand side in the previous equality starts with $\nu = 2$, because the term for $\nu = 1$ vanishes. For $f \in C^{2r+2}[0,n]$ it can be proved that there exists $\xi \in (0,n)$, such that

$$E_r^S(f) = n\,\frac{(4^{-r}-1)B_{2r+2}}{3(2r+2)!}f^{(2r+2)}(\xi).$$

For some modification and generalizations of the Euler–Maclaurin formula, see [2, 7, 20–22, 37, 55, 60]. In 1965 Kalinin [29] derived an analogue of the Euler–Maclaurin formula for C^∞ functions, for which there is Taylor series at each positive integer $x = \nu$,

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=0}^{+\infty} \frac{\theta^{k+1} - (\theta - 1)^{k+1}}{(k+1)!} h^{k+1} \sum_{\nu=1}^{n} f^{(k)}(a + (\nu - \theta)h),$$

where $h = (b - a)/n$, and used it to find some new expansions for the gamma function, the ψ function, as well as the Riemann zeta function.

Using Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$, in 1960 Keda [30] established a quadrature formula similar to the Euler–Maclaurin,

$$\int_0^1 f(x)\,\mathrm{d}x = T_n + \sum_{k=0}^{n-1} A_k \left[f^{(2k+2)}(0) + f^{(2k+2)}(1)\right] + R_n,$$

where

$$T_n = \frac{1}{n} {\sum_{k=0}^{n}}'' f\Big(\frac{k}{n}\Big), \quad A_k = \sum_{\nu=1}^{2k+2} \frac{B_\nu E_{2k+3-\nu}}{\nu!(2k+3-\nu)!n^\nu} \quad (k = 0, 1, \ldots, n-1),$$

and

$$R_n = f^{(2n+2)}(\xi) \sum_{m=1}^{n+1} \frac{2B_{2m}E_{2n-2m+3}}{(2m)!(2n-2m+3)!n^{2m}} \quad (0 \le \xi \le 1)$$

for $f \in C^{2n+2}[0, 1]$, where $B_n = B_n(0)$ and $E_n = E_n(0)$. The convergence of Euler–Maclaurin quadrature formulas on a class of smooth functions was considered by Vaskevič [62].

Some periodic analogues of the Euler–Maclaurin formula with applications to number theory have been developed by Berndt and Schoenfeld [6]. In the last section of [6], they showed how the composite Newton–Cotes quadrature formulas (Simpson's parabolic and Simpson's three-eighths rules), as well as various other quadratures (e.g., Weddle's composite rule), can be derived from special cases of their periodic Euler–Maclaurin formula, including explicit formulas for the remainder term.

3 Euler–Maclaurin Formula Based on the Composite Gauss–Legendre Rule and Its Lobatto Modification

In the papers [15, 48, 56], the authors considered generalizations of the Euler–Maclaurin formula for some particular Newton–Cotes rules, as well as for 2- and 3-point Gauss–Legendre and Lobatto formulas (see also [4, 17, 33, 34]).

Recently, we have done [40] the extensions of Euler–Maclaurin formulas by replacing the quadrature sum Q_n by the composite Gauss–Legendre shifted formula, as well as by its Lobatto modification. In these cases, several special rules have been obtained by using the MATHEMATICA package `OrthogonalPolynomials` (cf. [9, 43]). Some details on construction of orthogonal polynomials and quadratures of Gaussian type will be given in Sect. 5.

We denote the space of all algebraic polynomials defined on $\mathbb{R}$ (or some its subset) by $\mathcal{P}$, and by $\mathcal{P}_m \subset \mathcal{P}$ the space of polynomials of degree at most m ($m \in \mathbb{N}$).

Let $w_\nu = w_\nu^G$ and $\tau_\nu = \tau_\nu^G$, $\nu = 1, \ldots, m$, be weights (Christoffel numbers) and nodes of the Gauss–Legendre quadrature formula on $[0, 1]$,

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{\nu=1}^{m} w_\nu^G f(\tau_\nu^G) + R_m^G(f). \tag{16}$$

Note that the nodes τ_ν are zeros of the shifted (monic) Legendre polynomial

$$\pi_m(x) = \binom{2m}{m}^{-1} P_m(2x-1).$$

Degree of its algebraic precision is $d = 2m - 1$, i.e., $R_m^G(f) = 0$ for each $f \in \mathcal{P}_{2m-1}$. The quadrature sum in (16) we denote by $Q_m^G f$, i.e.,

$$Q_m^G f = \sum_{\nu=1}^{m} w_\nu^G f(\tau_\nu^G).$$

The corresponding composite Gauss–Legendre sum for approximating the integral $I_n f$, given by (10), can be expressed in the form

$$G_m^{(n)} f = \sum_{k=0}^{n-1} Q_m^G f(k + \cdot) = \sum_{\nu=1}^{m} w_\nu^G \sum_{k=0}^{n-1} f(k + \tau_\nu^G). \tag{17}$$

In the sequel we use the following expansion of a function $f \in C^s[0, 1]$ in Bernoulli polynomials for any $x \in [0, 1]$ (see Krylov [31, p. 15])

$$f(x) = \int_0^1 f(t)\,\mathrm{d}t + \sum_{j=1}^{s-1} \frac{B_j(x)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] - \frac{1}{s!}\int_0^1 f^{(s)}(t) L_s(x, t)\,\mathrm{d}t, \tag{18}$$

where $L_s(x,t) = B_s^*(x-t) - B_s^*(x)$ and $B_s^*(x)$ is a function of period one, defined by

$$B_s^*(x) = B_s(x), \quad 0 \le x < 1, \quad B_s^*(x+1) = B_s^*(x). \tag{19}$$

Notice that $B_0^*(x) = 1$, $B_1^*(x)$ is a discontinuous function with a jump of -1 at each integer, and $B_s^*(x)$, $s > 1$, is a continuous function.

Suppose now that $f \in C^{2r}[0,n]$, where $r \ge m$. Since the all nodes $\tau_\nu = \tau_\nu^G$, $\nu = 1,\dots,m$, of the Gaussian rule (16) belong to $(0,1)$, using the expansion (18), with $x = \tau_\nu$ and $s = 2r+1$, we have

$$f(\tau_\nu) = I_1 f + \sum_{j=1}^{2r} \frac{B_j(\tau_\nu)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] \\ - \frac{1}{(2r+1)!}\int_0^1 f^{(2r+1)}(t) L_{2r+1}(\tau_\nu, t)\,\mathrm{d}t,$$

where $I_1 f = \int_0^1 f(t)\,\mathrm{d}t$.

Now, if we multiply it by $w_\nu = w_\nu^G$ and then sum in ν from 1 to m, we obtain

$$\sum_{\nu=1}^{m} w_\nu f(\tau_\nu) = \left(\sum_{\nu=1}^{m} w_\nu\right) I_1 f + \sum_{j=1}^{2r} \frac{1}{j!}\left(\sum_{\nu=1}^{m} w_\nu B_j(\tau_\nu)\right)\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] \\ - \frac{1}{(2r+1)!}\int_0^1 f^{(2r+1)}(t)\left(\sum_{\nu=1}^{m} w_\nu L_{2r+1}(\tau_\nu, t)\right)\mathrm{d}t,$$

i.e.,

$$Q_m^G f = Q_m^G(1)\int_0^1 f(t)\,\mathrm{d}t + \sum_{j=1}^{2r} \frac{Q_m^G(B_j)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] + E_{m,r}^G(f),$$

where

$$E_{m,r}^G(f) = -\frac{1}{(2r+1)!}\int_0^1 f^{(2r+1)}(t)\, Q_m^G\left(L_{2r+1}(\cdot, t)\right)\,\mathrm{d}t.$$

Since

$$\int_0^1 B_j(x)\,\mathrm{d}x = \begin{cases} 1, j = 0, \\ 0, j \ge 1, \end{cases}$$

and

$$Q_m^G(B_j) = \sum_{\nu=1}^{m} w_\nu B_j(\tau_\nu) = \begin{cases} 1, j = 0, \\ 0,\ 1 \le j \le 2m-1, \end{cases}$$

because the Gauss–Legendre formula is exact for all algebraic polynomials of degree at most $2m-1$, the previous formula becomes

$$Q_m^G f - \int_0^1 f(t)\,\mathrm{d}t = \sum_{j=2m}^{2r} \frac{Q_m^G(B_j)}{j!}\left[f^{(j-1)}(1) - f^{(j-1)}(0)\right] + E_{m,r}^G(f). \tag{20}$$

Notice that for Gauss–Legendre nodes and the corresponding weights the following equalities

$$\tau_\nu + \tau_{m-\nu+1} = 1, \quad w_\nu = w_{m-\nu+1} > 0, \quad \nu = 1, \dots, m,$$

hold, as well as

$$w_\nu B_j(\tau_\nu) + w_{m-\nu+1} B_j(\tau_{m-\nu+1}) = w_\nu B_j(\tau_\nu)(1 + (-1)^j),$$

which is equal to zero for odd j. Also, if m is odd, then $\tau_{(m+1)/2} = 1/2$ and $B_j(1/2) = 0$ for each odd j. Thus, the quadrature sum

$$Q_m^G(B_j) = \sum_{\nu=1}^{m} w_\nu B_j(\tau_\nu) = 0$$

for odd j, so that (20) becomes

$$Q_m^G f - \int_0^1 f(t)\,\mathrm{d}t = \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right] + E_{m,r}^G(f). \tag{21}$$

Consider now the error of the (shifted) composite Gauss–Legendre formula (17). It is easy to see that

$$\begin{aligned} G_m^{(n)} f - I_n f &= \sum_{k=0}^{n-1}\left[Q_m^G f(k+\cdot) - \int_k^{k+1} f(t)\,\mathrm{d}t\right] \\ &= \sum_{k=0}^{n-1}\left[Q_m^G f(k+\cdot) - \int_0^1 f(k+x)\,\mathrm{d}x\right]. \end{aligned}$$

Then, using (21) we obtain

$$\begin{aligned} G_m^{(n)} f - I_n f &= \sum_{k=0}^{n-1}\left\{\sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(k+1) - f^{(2j-1)}(k)\right] + E_{m,r}^G(f(k+\cdot))\right\} \\ &= \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!}\left[f^{(2j-1)}(n) - f^{(2j-1)}(0)\right] + E_{n,m,r}^G(f), \end{aligned}$$

where $E^G_{n,m,r}(f)$ is given by

$$E^G_{n,m,r}(f) = -\frac{1}{(2r+1)!}\int_0^1 \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+t)\right) Q^G_m\left(L_{2r+1}(\cdot,t)\right)\, \mathrm{d}t. \tag{22}$$

Since $L_{2r+1}(x,t) = B^*_{2r+1}(x-t) - B^*_{2r+1}(x)$ and

$$B^*_{2r+1}(\tau_\nu) = B_{2r+1}(\tau_\nu), \quad B^*_{2r+1}(\tau_\nu - t) = -\frac{1}{2r+2}\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\tau_\nu - t),$$

we have

$$\begin{aligned} Q^G_m\left(L_{2r+1}(\cdot,t)\right) &= Q^G_m\left(B^*_{2r+1}(\cdot - t)\right) - Q^G_m\left(B^*_{2r+1}(\cdot)\right) \\ &= -\frac{1}{2r+2}\, Q^G_m\left(\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\cdot - t)\right), \end{aligned}$$

because $Q^G_m\left(B_{2r+1}(\cdot)\right) = 0$. Then for (22) we get

$$(2r+2)!E^G_{n,m,r}(f) = \int_0^1 \left(\sum_{k=0}^{n-1} f^{(2r+1)}(k+t)\right) Q^G_m\left(\frac{\mathrm{d}}{\mathrm{d}t}B^*_{2r+2}(\cdot - t)\right)\, \mathrm{d}t.$$

By using an integration by parts, it reduces to

$$(2r+2)!E^G_{n,m,r}(f) = F(t)Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\Big|_0^1 - \int_0^1 Q^G_m\left(B^*_{2r+2}(\cdot - t)\right) F'(t)\, \mathrm{d}t,$$

where $F(t)$ is introduced in the following way

$$F(t) = \sum_{k=0}^{n-1} f^{(2r+1)}(k+t).$$

Since $B^*_{2r+2}(\tau_\nu - 1) = B^*_{2r+2}(\tau_\nu) = B_{2r+2}(\tau_\nu)$, we have

$$\begin{aligned} F(t)Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\Big|_0^1 &= \big(F(1) - F(0)\big)Q^G_m\left(B^*_{2r+2}(\cdot)\right) \\ &= Q^G_m\left(B_{2r+2}(\cdot)\right)\int_0^1 F'(t)\, \mathrm{d}t, \end{aligned}$$

so that

$$(2r+2)!E^G_{n,m,r}(f) = \int_0^1 \left[Q^G_m\left(B_{2r+2}(\cdot)\right) - Q^G_m\left(B^*_{2r+2}(\cdot - t)\right)\right] F'(t)\, \mathrm{d}t.$$

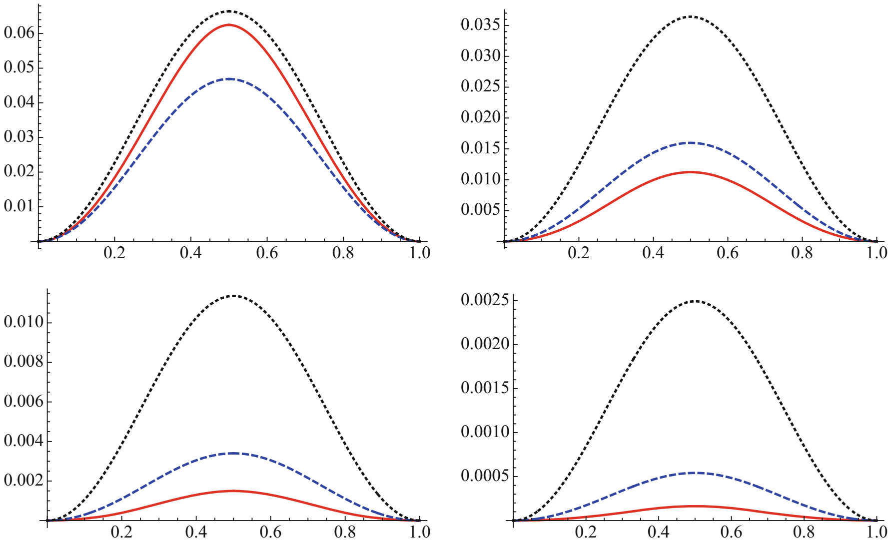

Fig. 1 Graphs of $t \mapsto g^G_{m,r}(t)$, $r = m$ (*solid line*), $r = m+1$ (*dashed line*), and $r = m+2$ (*dotted line*), when $m = 1$, $m = 2$ (*top*), and $m = 3$, $m = 4$ (*bottom*)

Since

$$g^G_{m,r}(t) := (-1)^{r-m} Q^G_m \left[B_{2r+2}(\,\cdot\,) - B^*_{2r+2}(\,\cdot - t)\right] > 0, \qquad 0 < t < 1, \tag{23}$$

there exists an $\eta \in (0, 1)$ such that

$$(2r+2)! E^G_{n,m,r}(f) = F'(\eta) \int_0^1 Q^G_m \left[B_{2r+2}(\,\cdot\,) - B^*_{2r+2}(\,\cdot - t)\right] \mathrm{d}t.$$

Typical graphs of functions $t \mapsto g^G_{m,r}(t)$ for some selected values of $r \geq m \geq 1$ are presented in Fig. 1.

Because of continuity of $f^{(2r+2)}$ on $[0, n]$ we conclude that there exists also $\xi \in (0, n)$ such that $F'(\eta) = n f^{(2r+2)}(\xi)$.

Finally, because of $\int_0^1 Q^G_m \left[B^*_{2r+2}(\,\cdot - t)\right] \mathrm{d}t = 0$, we obtain that

$$(2r+2)! E^G_{n,m,r}(f) = n f^{(2r+2)}(\xi) \int_0^1 Q^G_m \left[B_{2r+2}(\,\cdot\,)\right] \mathrm{d}t.$$

In this way, we have just proved the Euler–Maclaurin formula for the composite Gauss–Legendre rule (17) for approximating the integral $I_n f$, given by (10) (see [40]):

Theorem 1. *For $n, m, r \in \mathbb{N}$ $(m \le r)$ and $f \in C^{2r}[0, n]$ we have*

$$G_m^{(n)} f - I_n f = \sum_{j=m}^{r} \frac{Q_m^G(B_{2j})}{(2j)!} \left[f^{(2j-1)}(n) - f^{(2j-1)}(0) \right] + E_{n,m,r}^G(f), \tag{24}$$

where $G_m^{(n)} f$ is given by (17), and $Q_m^G B_{2j}$ denotes the basic Gauss–Legendre quadrature sum applied to the Bernoulli polynomial $x \mapsto B_{2j}(x)$, i.e.,

$$Q_m^G(B_{2j}) = \sum_{\nu=1}^{m} w_\nu^G B_{2j}(\tau_\nu^G) = -R_m^G(B_{2j}), \tag{25}$$

where $R_m^G(f)$ is the remainder term in (16).

If $f \in C^{2r+2}[0, n]$, then there exists $\xi \in (0, n)$, such that the error term in (24) can be expressed in the form

$$E_{n,m,r}^G(f) = n \, \frac{Q_m^G(B_{2r+2})}{(2r+2)!} f^{(2r+2)}(\xi). \tag{26}$$

We consider now special cases of the formula (24) for some typical values of m. For a given m, by $G^{(m)}$ we denote the sequence of coefficients which appear in the sum on the right-hand side in (24), i.e.,

$$G^{(m)} = \left\{ Q_m^G(B_{2j}) \right\}_{j=m}^{\infty} = \left\{ Q_m^G(B_{2m}), Q_m^G(B_{2m+2}), Q_m^G(B_{2m+4}), \ldots \right\}.$$

These Gaussian sums we can calculate very easily by using MATHEMATICA Package `OrthogonalPolynomials` (cf. [9, 43]). In the sequel we mention cases when $1 \le m \le 6$.

Case $m = 1$. Here $\tau_1^G = 1/2$ and $w_1^G = 1$, so that, according to (25),

$$Q_1^G(B_{2j}) = B_{2j}(1/2) = (2^{1-2j} - 1) B_{2j},$$

and (24) reduces to (14). Thus,

$$G^{(1)} = \left\{ -\frac{1}{12}, \frac{7}{240}, -\frac{31}{1344}, \frac{127}{3840}, -\frac{2555}{33792}, \frac{1414477}{5591040}, -\frac{57337}{49152}, \frac{118518239}{16711680}, \ldots \right\}.$$

Case $m = 2$. Here we have

$$\tau_1^G = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad \tau_2^G = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \quad \text{and} \quad w_1^G = w_2^G = \frac{1}{2},$$

so that $Q_2^G(B_{2j}) = \frac{1}{2}\left(B_{2j}(\tau_1^G) + B_{2j}(\tau_2^G)\right) = B_{2j}(\tau_1^G)$. In this case, the sequence of coefficients is

$$G^{(2)} = \left\{ -\frac{1}{180}, \frac{1}{189}, -\frac{17}{2160}, \frac{97}{5346}, -\frac{1291411}{21228480}, \frac{16367}{58320}, -\frac{243615707}{142767360}, \ldots \right\}.$$

Case $m = 3$. In this case

$$\tau_1^G = \frac{1}{10}\left(5 - \sqrt{15}\right), \quad \tau_2^G = \frac{1}{2}, \quad \tau_3^G = \frac{1}{10}\left(5 + \sqrt{15}\right)$$

and

$$w_1^G = \frac{5}{18}, \quad w_2^G = \frac{4}{9}, \quad w_3^G = \frac{5}{18},$$

so that

$$Q_3^G(B_{2j}) = \frac{5}{9}B_{2j}(\tau_1^G) + \frac{4}{9}B_{2j}(\tau_2^G)$$

and

$$G^{(3)} = \left\{-\frac{1}{2800}, \frac{49}{72000}, -\frac{8771}{5280000}, \frac{4935557}{873600000}, -\frac{15066667}{576000000}, \frac{3463953717}{21760000000}, \dots\right\}.$$

Cases $m = 4, 5, 6$. The corresponding sequences of coefficients are

$$G^{(4)} = \left\{-\frac{1}{44100}, \frac{41}{565950}, -\frac{3076}{11704875}, \frac{93553}{75631500}, -\frac{453586781}{60000990000}, \frac{6885642443}{117354877500}, \dots\right\},$$

$$G^{(5)} = \left\{-\frac{1}{698544}, \frac{205}{29719872}, -\frac{100297}{2880541440}, \frac{76404959}{352578272256}, -\frac{839025422533}{496513166929920}, \dots\right\},$$

$$G^{(6)} = \left\{-\frac{1}{11099088}, \frac{43}{70436520}, -\frac{86221}{21074606784}, \frac{147502043}{4534139665440}, -\frac{1323863797}{4200045163776}, \dots\right\}.$$

The Euler–Maclaurin formula based on the composite Lobatto formula can be considered in a similar way. The corresponding Gauss-Lobatto quadrature formula

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{\nu=0}^{m+1} w_\nu^L f(\tau_\nu^L) + R_m^L(f), \tag{27}$$

with the endnodes $\tau_0 = \tau_0^L = 0$, $\tau_{m+1} = \tau_{m+1}^L = 1$, has internal nodes $\tau_\nu = \tau_\nu^L$, $\nu = 1, \dots, m$, which are zeros of the shifted (monic) Jacobi polynomial,

$$\pi_m(x) = \binom{2m+2}{m}^{-1} P_m^{(1,1)}(2x - 1),$$

orthogonal on the interval $(0, 1)$ with respect to the weight function $x \mapsto x(1 - x)$. The algebraic degree of precision of this formula is $d = 2m + 1$, i.e., $R_m^L(f) = 0$ for each $f \in \mathcal{P}_{2m+1}$.

For constructing the Gauss-Lobatto formula

$$Q_m^L(f) = \sum_{\nu=0}^{m+1} w_\nu^L f(\tau_\nu^L), \tag{28}$$

we use parameters of the corresponding Gaussian formula with respect to the weight function $x \mapsto x(1-x)$, i.e.,

$$\int_0^1 g(x)x(1-x)\,\mathrm{d}x = \sum_{\nu=1}^{m} \widehat{w}_\nu^G g(\widehat{\tau}_\nu^G) + \widehat{R}_m^G(g).$$

The nodes and weights of the Gauss-Lobatto quadrature formula (27) are (cf. [36, pp. 330–331])

$$\tau_0^L = 0, \quad \tau_\nu^L = \widehat{\tau}_\nu^G \quad (\nu = 1, \ldots, m), \quad \tau_{m+1}^L = 1,$$

and

$$w_0^L = \frac{1}{2} - \sum_{\nu=1}^{m} \frac{\widehat{w}_\nu^G}{\widehat{\tau}_\nu^G}, \quad w_\nu^L = \frac{\widehat{w}_\nu^G}{\widehat{\tau}_\nu^G(1-\widehat{\tau}_\nu^G)} \; (\nu = 1, \ldots, m), \quad w_{m+1}^L = \frac{1}{2} - \sum_{\nu=1}^{m} \frac{\widehat{w}_\nu^G}{1-\widehat{\tau}_\nu^G},$$

respectively. The corresponding composite rule is

$$\begin{aligned} L_m^{(n)} f &= \sum_{k=0}^{n-1} Q_m^L f(k+\cdot) = \sum_{\nu=0}^{m+1} w_\nu^L \sum_{k=0}^{n-1} f(k+\tau_\nu^L), \\ &= (w_0^L + w_{m+1}^L) \sum_{k=0}^{n}{}'' f(k) + \sum_{\nu=1}^{m} w_\nu^L \sum_{k=0}^{n-1} f(k+\tau_\nu^L). \end{aligned} \tag{29}$$

As in the Gauss–Legendre case, there exists a symmetry of nodes and weights, i.e.,

$$\tau_\nu^L + \tau_{m+1-\nu}^L = 1, \; w_\nu^L = w_{m+1-\nu}^L > 0 \quad \nu = 0, 1, \ldots, m+1,$$

so that the Gauss-Lobatto quadrature sum

$$Q_m^L(B_j) = \sum_{\nu=0}^{m+1} w_\nu^L B_j(\tau_\nu^L) = 0$$

for each odd j.

By the similar arguments as before, we can state and prove the following result.

Theorem 2. *For $n, m, r \in \mathbb{N}$ $(m \le r)$ and $f \in C^{2r}[0, n]$ we have*

$$L_m^{(n)} f - I_n f = \sum_{j=m+1}^{r} \frac{Q_m^L(B_{2j})}{(2j)!} \left[f^{(2j-1)}(n) - f^{(2j-1)}(0) \right] + E_{n,m,r}^L(f), \tag{30}$$

where $L_m^{(n)} f$ is given by (29), and $Q_m^L B_{2j}$ denotes the basic Gauss-Lobatto quadrature sum (28) applied to the Bernoulli polynomial $x \mapsto B_{2j}(x)$, i.e.,

$$Q_m^L(B_{2j}) = \sum_{\nu=0}^{m+1} w_\nu^L B_{2j}(\tau_\nu^L) = -R_m^L(B_{2j}),$$

where $R_m^L(f)$ is the remainder term in (27).

If $f \in C^{2r+2}[0, n]$, then there exists $\xi \in (0, n)$, such that the error term in (30) can be expressed in the form

$$E_{n,m,r}^L(f) = n \frac{Q_m^L(B_{2r+2})}{(2r+2)!} f^{(2r+2)}(\xi).$$

In the sequel we give the sequence of coefficients $L^{(m)}$ which appear in the sum on the right-hand side in (30), i.e.,

$$L^{(m)} = \left\{ Q_m^L(B_{2j}) \right\}_{j=m+1}^{\infty} = \left\{ Q_m^L(B_{2m+2}), Q_m^L(B_{2m+4}), Q_m^L(B_{2m+6}), \ldots \right\},$$

obtained by the Package `OrthogonalPolynomials`, for some values of m.

Case $m = 0$. This is a case of the standard Euler–Maclaurin formula (1), for which $\tau_0^L = 0$ and $\tau_1^L = 1$, with $w_0^L = w_1^L = 1/2$. The sequence of coefficients is

$$L^{(0)} = \left\{ \frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, -\frac{3617}{510}, \frac{43867}{798}, -\frac{174611}{330}, \frac{854513}{138}, \ldots \right\},$$

which is, in fact, the sequence of Bernoulli numbers $\{B_{2j}\}_{j=1}^{\infty}$.

Case $m = 1$. In this case $\tau_0^L = 0$, $\tau_1^L = 1/2$, and $\tau_2 = 1$, with the corresponding weights $w_0^L = 1/6$, $w_1^L = 2/3$, and $w_2^L = 1/6$, which is, in fact, the Simpson formula (15). The sequence of coefficients is

$$L^{(1)} = \left\{ \frac{1}{120}, -\frac{5}{672}, \frac{7}{640}, -\frac{425}{16896}, \frac{235631}{2795520}, -\frac{3185}{8192}, \frac{19752437}{8355840}, -\frac{958274615}{52297728}, \ldots \right\}.$$

Case $m = 2$. Here we have

$$\tau_0^L = 0, \quad \tau_1^L = \frac{1}{10}(5 - \sqrt{5}), \quad \tau_2^L = \frac{1}{10}(5 + \sqrt{5}), \quad \tau_3^L = 1$$

and $w_0^L = w_3^L = 1/12$, $w_1^L = w_2^L = 5/12$, and the sequence of coefficients is

$$L^{(2)} = \left\{ \frac{1}{2100}, -\frac{1}{1125}, \frac{89}{41250}, -\frac{25003}{3412500}, \frac{3179}{93750}, -\frac{2466467}{11953125}, \frac{997365619}{623437500}, \ldots \right\}.$$

Case $m = 3$. Here the nodes and the weight coefficients are

$$\tau_0^L = 0, \quad \tau_1^L = \frac{1}{14}(7 - \sqrt{31}), \quad \tau_2^L = \frac{1}{2}, \quad \tau_3^L = \frac{1}{14}(7 + \sqrt{31}), \quad \tau_4^L = 1$$

and

$$w_0^L = \frac{1}{20}, \quad w_1^L = \frac{49}{180}, \quad w_2^L = \frac{16}{45}, \quad w_3^L = \frac{49}{180}, \quad w_4^L = \frac{1}{20},$$

respectively, and the sequence of coefficients is

$$L^{(3)} = \left\{ \frac{1}{35280}, -\frac{65}{724416}, \frac{38903}{119857920}, -\frac{236449}{154893312}, \frac{1146165227}{122882027520}, \ldots \right\}.$$

Cases $m = 4, 5$. The corresponding sequences of coefficients are

$$L^{(4)} = \left\{ \frac{1}{582120}, -\frac{17}{2063880}, \frac{173}{4167450}, -\frac{43909}{170031960}, \frac{160705183}{79815002400}, -\frac{76876739}{3960744480}, \ldots \right\},$$

$$L^{(5)} = \left\{ \frac{1}{9513504}, -\frac{49}{68999040}, \frac{5453}{1146917376}, -\frac{671463061}{17766424811520}, \frac{1291291631}{3526568534016}, \ldots \right\}.$$

Remark 8. Recently Dubeau [16] has shown that an Euler–Maclaurin like formula can be associated with any interpolatory quadrature rule.

4 Abel–Plana Summation Formula and Some Modifications

Another important summation formula is the so-called Abel–Plana formula, but it is not so well known like the Euler–Maclaurin formula. In 1820 Giovanni (Antonio Amedea) Plana [49] obtained the summation formula

$$\sum_{k=0}^{+\infty} f(k) - \int_0^{+\infty} f(x)\,\mathrm{d}x = \frac{1}{2} f(0) + \mathrm{i} \int_0^{+\infty} \frac{f(\mathrm{i}y) - f(-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\,\mathrm{d}y, \qquad (31)$$

which holds for analytic functions f in $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ which satisfy the conditions:

1° $\lim_{|y| \to +\infty} \mathrm{e}^{-|2\pi y|} |f(x \pm \mathrm{i}y)| = 0$ uniformly in x on every finite interval,

2° $\int_0^{+\infty} |f(x + \mathrm{i}y) - f(x - \mathrm{i}y)| \mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$ exists for every $x \geq 0$ and tends to zero when $x \to +\infty$.

This formula was also proved in 1823 by Niels Henrik Abel [1]. In addition, Abel also proved an interesting *"alternating series version"*, under the same conditions,

$$\sum_{k=0}^{+\infty}(-1)^k f(k) = \frac{1}{2}f(0) + \mathrm{i}\int_0^{+\infty}\frac{f(\mathrm{i}y)-f(-\mathrm{i}y)}{2\sinh \pi y}\,\mathrm{d}y. \tag{32}$$

Otherwise, this formula can be obtained only from (31). Note that, by subtracting (31) from the same formula written for the function $z \mapsto 2f(2z)$, we get (32).

For the finite sum $S_{n,m}f = \sum_{k=m}^{n}(-1)^k f(k)$, (32) the *Abel summation formula* becomes

$$S_{n,m}f = \frac{1}{2}\big[(-1)^m f(m) + (-1)^n f(n+1)\big] \\ - \int_{-\infty}^{+\infty}\big[(-1)^m\psi_m(y) + (-1)^n\psi_{n+1}(y)\big]w^A(y)\,\mathrm{d}y, \tag{33}$$

where the *Abel weight* on $\mathbb{R}$ and the function $\phi_m(y)$ are given by

$$w^A(x) = \frac{x}{2\sinh \pi x} \quad \text{and} \quad \phi_m(y) = \frac{f(m+\mathrm{i}y)-f(m-\mathrm{i}y)}{2\mathrm{i}y}. \tag{34}$$

The moments for the Abel weight can be expressed in terms of Bernoulli numbers as

$$\mu_k = \begin{cases} 0, & k \text{ odd}, \\ \left(2^{k+2}-1\right)\dfrac{(-1)^{k/2}B_{k+2}}{k+2}, & k \text{ even}. \end{cases} \tag{35}$$

A general Abel–Plana formula can be obtained by a contour integration in the complex plane. Let $m, n \in \mathbb{N}$, $m < n$, and $C(\varepsilon)$ be a closed rectangular contour with vertices at $m \pm \mathrm{i}b$, $n \pm \mathrm{i}b$, $b > 0$ (see Fig. 2), and with semicircular indentations of radius ε round m and n. Let f be an analytic function in the strip $\Omega_{m,n} = \{z \in \mathbb{C} : m \le \operatorname{Re} z \le n\}$ and suppose that for every $m \le x \le n$,

$$\lim_{|y|\to+\infty} \mathrm{e}^{-|2\pi y|}|f(x \pm \mathrm{i}y)| = 0 \quad \text{uniformly in } x,$$

and that

$$\int_0^{+\infty} |f(x+\mathrm{i}y) - f(x-\mathrm{i}y)|\mathrm{e}^{-|2\pi y|}\,\mathrm{d}y$$

exists.

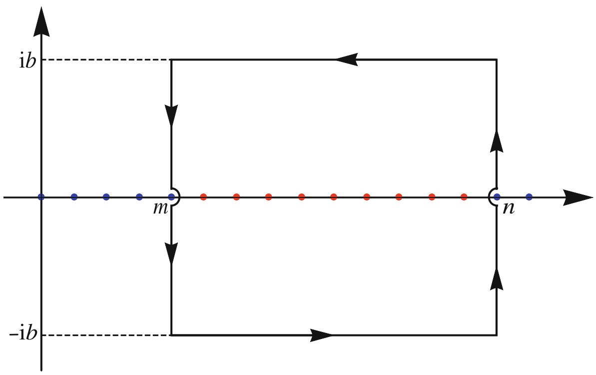

Fig. 2 Rectangular contour $C(\varepsilon)$

The integration

$$\int_{C(\varepsilon)} \frac{f(z)}{\mathrm{e}^{-\mathrm{i}2\pi z}-1}\,\mathrm{d}z,$$

with $\varepsilon \to 0$ and $b \to +\infty$, leads to the *Plana formula* in the following form (cf. [42])

$$T_{m,n}f - \int_m^n f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big(\phi_n(y)-\phi_m(y)\big) w^P(y)\,\mathrm{d}y, \tag{36}$$

where

$$\phi_m(y) = \frac{f(m+\mathrm{i}y)-f(m-\mathrm{i}y)}{2\mathrm{i}y} \quad \text{and} \quad w^P(y) = \frac{|y|}{\mathrm{e}^{|2\pi y|}-1}. \tag{37}$$

Practically, the Plana formula (36) gives the error of the composite trapezoidal formula (like the Euler–Maclaurin formula). As we can see the formula (36) is similar to the Euler–Maclaurin formula, with the difference that the sum of terms

$$\frac{B_{2j}}{(2j)!}\big(f^{(2j-1)}(n)-f^{(2j-1)}(m)\big)$$

replaced by an integral. Therefore, in applications this integral must be calculated by some quadrature rule. It is natural to construct the Gaussian formula with respect to the *Plana* weight function $x \mapsto w^P(x)$ on $\mathbb{R}$ (see the next section for such a construction).

In order to find the moments of this weight function, we note first that if k is odd, the moments are zero, i.e.,

$$\mu_k(w^P) = \int_{\mathbb{R}} x^k w^P(x)\,\mathrm{d}x = \int_{\mathbb{R}} x^k \frac{|x|}{\mathrm{e}^{|2\pi x|}-1}\,\mathrm{d}x = 0.$$

For even k, we have

$$\mu_k(w^P) = 2\int_0^{+\infty} \frac{x^{k+1}}{e^{2\pi x}-1}\,dx = \frac{2}{(2\pi)^{k+2}}\int_0^{+\infty} \frac{t^{k+1}}{e^t-1}\,dt,$$

which can be exactly expressed in terms of the Riemann zeta function $\zeta(s)$,

$$\mu_k(w^P) = \frac{2(k+1)!\zeta(k+2)}{(2\pi)^{k+2}} = (-1)^{k/2}\frac{B_{k+2}}{k+2},$$

because the number $k+2$ is even. Thus, in terms of Bernoulli numbers, the moments are

$$\mu_k(w^P) = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2}\dfrac{B_{k+2}}{k+2}, & k \text{ is even}. \end{cases} \tag{38}$$

Remark 9. By the Taylor expansion for $\phi_m(y)$ (and $\phi_n(y)$) on the right-hand side in (36),

$$\phi_m(y) = \frac{f(m+iy)-f(m-iy)}{2iy} = \sum_{j=1}^{+\infty}\frac{(-1)^{j-1}y^{2j-2}}{(2j-1)!}f^{(2j-1)}(m),$$

and using the moments (38), the Plana formula (36) reduces to the Euler–Maclaurin formula,

$$\begin{aligned} T_{m,n}f - \int_m^n f(x)\,dx &= \sum_{j=1}^{+\infty}\frac{(-1)^{j-1}}{(2j-1)!}\mu_{2j-2}(w^P)\left(f^{(2j-1)}(n)-f^{(2j-1)}(m)\right) \\ &= \sum_{j=1}^{+\infty}\frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(n)-f^{(2j-1)}(m)\right), \end{aligned}$$

because of $\mu_{2j-2}(w^P) = (-1)^{j-1}B_{2j}/(2j)$. Note that $T_{m,n}f$ is the notation for the composite trapezoidal sum

$$T_{m,n}f := \sum_{k=m}^{n}{}'' f(k) = \frac{1}{2}f(m) + \sum_{k=m+1}^{n-1} f(k) + \frac{1}{2}f(n). \tag{39}$$

For more details see Rahman and Schmeisser [53, 54], Dahlquist [11–13], as well as a recent paper by Butzer, Ferreira, Schmeisser, and Stens [8].

A similar summation formula is the so-called *midpoint summation formula*. It can be obtained by combining two Plana formulas for the functions $z \mapsto f(z-1/2)$

and $z \mapsto f((z+m-1)/2)$. Namely,

$$T_{m,2n-m+2}f\Big(\frac{z+m-1}{2}\Big) - T_{m,n+1}f\Big(z-\frac{1}{2}\Big) = \sum_{k=m}^{n} f(k),$$

i.e.,

$$\sum_{k=m}^{n} f(k) - \int_{m-1/2}^{n+1/2} f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big[\phi_{m-1/2}(y) - \phi_{n+1/2}(y)\big] w^M(y)\,\mathrm{d}y, \qquad (40)$$

where the *midpoint weight function* is given by

$$w^M(x) = w^P(x) - w^P(2x) = \frac{|x|}{\mathrm{e}^{|2\pi x|}+1}, \qquad (41)$$

and $\phi_{m-1/2}$ and $\phi_{n+1/2}$ are defined in (37), taking $m := m-1/2$ and $m := n+1/2$, respectively. The moments for the midpoint weight function can be expressed also in terms of Bernoulli numbers as

$$\mu_k(w^M) = \int_{\mathbb{R}} x^k \frac{|x|}{\mathrm{e}^{|2\pi x|}+1}\,\mathrm{d}x = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2}(1-2^{-(k+1)})\dfrac{B_{k+2}}{k+2}, & k \text{ is even}. \end{cases} \qquad (42)$$

An interesting weight function and the corresponding summation formula can be obtained from the Plana formula, if we integrate by parts the right side in (36) (cf. [13]). Introducing the so-called *Binet* weight function $y \mapsto w^B(y)$ and the function $y \mapsto \psi_m(y)$ by

$$w^B(y) = -\frac{1}{2\pi}\log\big(1-\mathrm{e}^{-2\pi|y|}\big) \quad \text{and} \quad \psi_m(y) = \frac{f'(m+\mathrm{i}y)+f'(m-\mathrm{i}y)}{2}, \qquad (43)$$

respectively, we see that $\mathrm{d}w^B(y)/\,\mathrm{d}y = -w^P(y)/y$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\Big\{\big[\phi_n(y)-\phi_m(y)\big]y\Big\} &= \frac{1}{2\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}y}\Big\{\big[f(n+\mathrm{i}y)-f(n-\mathrm{i}y)\big]-\big[f(m+\mathrm{i}y)-f(m-\mathrm{i}y)\big]\Big\}\\ &= \psi_n(y)-\psi_m(y),\end{aligned}$$

so that

$$\begin{aligned}\int_{-\infty}^{+\infty}\big[\phi_n(y)-\phi_m(y)\big]w^P(y)\,\mathrm{d}y &= \int_{-\infty}^{+\infty}\big[\phi_n(y)-\phi_m(y)\big](-y)\,\mathrm{d}w^B(y)\\ &= \int_{-\infty}^{+\infty}\big[\psi_n(y)-\psi_m(y)\big]w^B(y)\,\mathrm{d}y,\end{aligned}$$

because $w^B(y) = O(\mathrm{e}^{-2\pi|y|})$ as $|y| \to +\infty$. Thus, the *Binet summation formula* becomes

$$T_{m,n}f - \int_m^n f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \big[\psi_n(y) - \psi_m(y)\big] w^B(y)\,\mathrm{d}y. \tag{44}$$

Such a formula can be useful when $f'(z)$ is easier to compute than $f(z)$.

The moments for the Binet weight can be obtained from ones for w^P. Since

$$\mu_k(w^P) = \int_{\mathbb{R}} y^k w^P(y)\,\mathrm{d}y = \int_{\mathbb{R}} y^k(-y)\,\mathrm{d}w^B(y) = (k+1)\mu_k(w^B),$$

according to (38),

$$\mu_k(w^B) = \begin{cases} 0, & k \text{ is odd}, \\ (-1)^{k/2} \dfrac{B_{k+2}}{(k+1)(k+2)}, & k \text{ is even}. \end{cases} \tag{45}$$

There are also several other summation formulas. For example, the *Lindelöf formula* [32] for alternating series is

$$\sum_{k=m}^{+\infty} (-1)^k f(k) = (-1)^m \int_{-\infty}^{+\infty} f(m - 1/2 + \mathrm{i}y) \frac{\mathrm{d}y}{2\cosh \pi y}, \tag{46}$$

where the *Lindelöf* weight function is given by

$$w^L(x) = \frac{1}{2\cosh \pi y} = \frac{1}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}}. \tag{47}$$

Here, the moments

$$\mu_k(w^L) = \int_{\mathbb{R}} \frac{x^k}{\mathrm{e}^{\pi x} + \mathrm{e}^{-\pi x}}\,\mathrm{d}x$$

can be expressed in terms of the generalized Riemann zeta function $z \mapsto \zeta(z, a)$, defined by

$$\zeta(z, a) = \sum_{\nu=0}^{+\infty} (\nu + a)^{-z}.$$

Namely,

$$\mu_k(w^L) = \begin{cases} 0, & k \text{ odd}, \\ 2(4\pi)^{-k-1} k! \big[\zeta\left(k+1, \tfrac{1}{4}\right) - \zeta\left(k+1, \tfrac{3}{4}\right)\big], & k \text{ even}. \end{cases} \tag{48}$$

5 Construction of Orthogonal Polynomials and Gaussian Quadratures for Weights of Abel–Plana Type

The weight functions w ($\in \{w^P, w^M, w^B, w^A, w^L\}$) which appear in the summation formulas considered in the previous section are even functions on $\mathbb{R}$. In this section we consider the construction of (monic) orthogonal polynomials π_k ($\equiv \pi_k(w;\,\cdot)$) and corresponding Gaussian formulas

$$\int_{\mathbb{R}} f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{n} A_\nu f(x_\nu) + R_n(w;f), \tag{49}$$

with respect to the inner product $(p,q) = \int_{\mathbb{R}} p(x)q(x)w(x)\,\mathrm{d}x$ $(p,q \in \mathcal{P})$. We note that $R_n(w;f) \equiv 0$ for each $f \in \mathcal{P}_{2n-1}$.

Such orthogonal polynomials $\{\pi_k\}_{k\in\mathbb{N}_0}$ and Gaussian quadratures (49) exist uniquely, because all the moments for these weights μ_k ($\equiv \mu_k(w)$), $k = 0, 1, \ldots,$ exist, are finite, and $\mu_0 > 0$.

Because of the property $(xp, q) = (p, xq)$, these (monic) orthogonal polynomials π_k satisfy the fundamental *three–term recurrence relation*

$$\pi_{k+1}(x) = x\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0, 1, \ldots, \tag{50}$$

with $\pi_0(x) = 1$ and $\pi_{-1}(x) = 0$, where $\{\beta_k\}_{k\in\mathbb{N}_0}$ ($= \{\beta_k(w)\}_{k\in\mathbb{N}_0}$) is a sequence of recursion coefficients which depend on the weight w. The coefficient β_0 may be arbitrary, but it is conveniently defined by $\beta_0 = \mu_0 = \int_{\mathbb{R}} w(x)\,\mathrm{d}x$. Note that the coefficients α_k in (50) are equal to zero, because the weight function w is an even function! Therefore, the nodes in (49) are symmetrically distributed with respect to the origin, and the weights for symmetrical nodes are equal. For odd n one node is at zero.

A characterization of the Gaussian quadrature (49) can be done via an eigenvalue problem for the symmetric tridiagonal Jacobi matrix (cf. [36, p. 326]),

$$J_n = J_n(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

constructed with the coefficients from the three-term recurrence relation (50) (in our case, $\alpha_k = 0$, $k = 0, 1, \ldots, n-1$).

The nodes x_ν are the eigenvalues of J_n and the weights A_ν are given by $A_\nu = \beta_0 v_{\nu,1}^2$, $\nu = 1, \ldots, n$, where β_0 is the moment $\mu_0 = \int_{\mathbb{R}} w(x)\,\mathrm{d}x$, and $v_{\nu,1}$ is the first component of the normalized eigenvector $\mathbf{v} = [v_{\nu,1} \;\cdots\; v_{\nu,n}]^T$ (with $\mathbf{v}_\nu^T\mathbf{v}_\nu = 1$)

corresponding to the eigenvalue x_ν,

$$J_n \mathbf{v}_\nu = x_\nu \mathbf{v}_\nu, \quad \nu = 1, \ldots, n.$$

An efficient procedure for constructing the Gaussian quadrature rules was given by Golub and Welsch [27], by simplifying the well-known QR algorithm, so that only the first components of the eigenvectors are computed.

The problems are very sensitive with respect to small perturbations in the data.

Unfortunately, the recursion coefficients are known explicitly only for some narrow classes of orthogonal polynomials, as e.g. for the classical orthogonal polynomials (Jacobi, the generalized Laguerre, and Hermite polynomials). However, for a large class of the so-called *strongly non-classical polynomials* these coefficients can be constructed numerically, but procedures are very sensitive with respect to small perturbations in the data. Basic procedures for generating these coefficients were developed by Walter Gautschi in the eighties of the last century (cf. [23, 24, 36, 41]).

However, because of progress in symbolic computations and variable-precision arithmetic, recursion coefficients can be today directly generated by using the original Chebyshev method of moments (cf. [36, pp. 159–166]) in symbolic form or numerically in sufficiently high precision. In this way, instability problems can be eliminated. Respectively symbolic/variable-precision software for orthogonal polynomials and Gaussian and similar type quadratures is available. In this regard, the MATHEMATICA package `OrthogonalPolynomials` (see [9] and [43]) is downloadable from the web site http://www.mi.sanu.ac.rs/~gvm/. Also, there is Gautschi's software in MATLAB (packages `OPQ` and `SOPQ`). Thus, all that is required is a procedure for the symbolic calculation of moments or their calculation in variable-precision arithmetic.

In our case we calculate the first $2N$ moments in a symbolic form (list `mom`), using corresponding formulas (for example, (38) in the case of the Plana weight w^P), so that we can construct the Gaussian formula (49) for each $n \leq N$. Now, in order to get the first N recurrence coefficients `{al,be}` in a symbolic form, we apply the implemented function `aChebyshevAlgorithm` from the Package `OrthogonalPolynomials`, which performs construction of these coefficients using Chebyshev algorithm, with the option `Algorithm->Symbolic`. Thus, it can be implemented in the MATHEMATICA package `OrthogonalPolynomials` in a very simple way as

```
<<orthogonalPolynomials`
 mom=Table[<expression for moments>,{k,0,199}];
 {al,be}=aChebyshevAlgorithm[mom,Algorithm->Symbolic]
 pq[n_]:=aGaussianNodesWeights[n,al,be,
           WorkingPrecision->65,Precision -> 60]
 xA = Table[pq[n],{n,5,40,5}];
```

where we put $N = 100$ and the `WorkingPrecision->65` in order to obtain very precisely quadrature parameters (nodes and weights) with `Precision->60`. These parameters are calculated for $n = 5(5)40$, so that `xA[[k]][[1]]` and `xA[[k]][[2]]` give lists of nodes and weights for five-point formula when `k=1`, for ten-point formula when `k=2`, etc. Otherwise, here we can calculate the n-point Gaussian quadrature formula for each $n \le N = 100$.

All computations were performed in MATHEMATICA, Ver. 10.3.0, on MacBook Pro (Retina, Mid 2012) OS X 10.11.2. The calculations are very fast. The running time is evaluated by the function `Timing` in MATHEMATICA and it includes only CPU time spent in the MATHEMATICA kernel. Such a way may give different results on different occasions within a session, because of the use of internal system caches. In order to generate worst-case timing results independent of previous computations, we used also the command `ClearSystemCache[]`, and in that case the running time for the Plana weight function w^P has been 4.2 ms (calculation of moments), 0.75 s (calculation of recursive coefficients), and 8 s (calculation quadrature parameters for $n = 5(5)40$).

In the sequel we mention results for different weight functions, whose graphs are presented in Fig. 3.

1. Abel and Lindelöf Weight Functions $\mathbf{w^A}$ and $\mathbf{w^L}$ These weight functions are given by (34) and (47), and their moments by (35) and (48), respectively. It is interesting that their corresponding coefficients in the three-term recurrence relation (50) are known explicitly (see [36, p. 159])

$$\beta_0^A = \mu_0^A = \frac{1}{4}, \quad \beta_k^A = \frac{k(k+1)}{4}, \quad k = 1, 2, \dots ,$$

and

$$\beta_0^L = \mu_0^L = \frac{1}{2}, \quad \beta_k^L = \frac{k^2}{4}, \quad k = 1, 2, \dots .$$

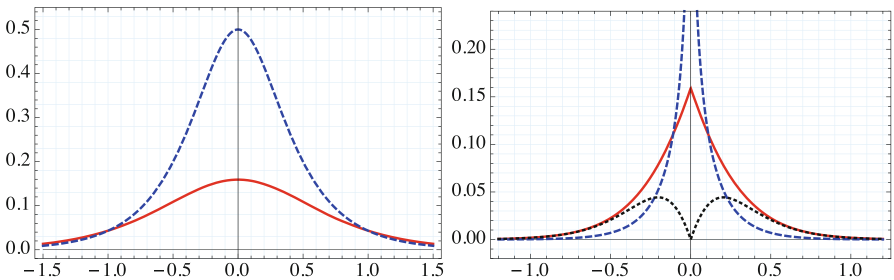

Fig. 3 Graphs of the weight functions: (*left*) w^A (*solid line*) and w^L (*dashed line*); (*right*) w^P (*solid line*), w^B (*dashed line*) and w^M (*dotted line*)

Thus, for these two weight functions we have recursive coefficients in the explicit form, so that we go directly to construction quadrature parameters.

2. Plana Weight Function $\mathbf{w^P}$ This weight function is given by (37), and the corresponding moments by (38). Using the Package `OrthogonalPolynomials` we obtain the sequence of recurrence coefficients $\{\beta_k^P\}_{k\geq 0}$ in the rational form:

$$\beta_0^P = \frac{1}{12},\ \beta_1^P = \frac{1}{10},\ \beta_2^P = \frac{79}{210},\ \beta_3^P = \frac{1205}{1659},\ \beta_4^P = \frac{262445}{209429},\ \beta_5^P = \frac{33461119209}{18089284070},$$

$$\beta_6^P = \frac{361969913862291}{137627660760070},\ \beta_7^P = \frac{85170013927511392430}{24523312685049374477},$$

$$\beta_8^P = \frac{10643272151859884438142889951130}{2361552627563909211512391211153},$$

$$\beta_9^P = \frac{286789982254764757195675003870137955697117}{51246435664921031688705695412342990647850},$$

$$\beta_{10}^P = \frac{15227625889136643989610717434803027240375634452808081047}{22121475212911039111935495289204379122003759800113006650},$$

$$\beta_{11}^P = \frac{587943441754746283972138649821948554273878447469233852697401814148410885}{71529318090286333175985287358122471724664434392542372273400541405857921},$$

etc.

As we can see, the fractions are becoming more complicated, so that already β_{11}^P has the "form of complexity" $\{72/71\}$, i.e., it has 72 decimal digits in the numerator and 71 digits in the denominator. Further terms of this sequence have the "form of complexity" $\{88/87\}$, $\{106/05\}$, $\{129/128\}$, $\{152/151\}, \ldots, \{13451/13448\}$.

Thus, the last term β_{99}^P has more than 13 thousand digits in its numerator and denominator. Otherwise, its value, e.g. rounded to 60 decimal digits, is

$$\beta_{99}^P = 618.668116294139071216871819412846078447729830182674784697227.$$

3. Midpoint Weight Function $\mathbf{w^M}$ This weight function is given by (41), and the corresponding moments by (42). As in the previous case, we obtain the sequence of recurrence coefficients $\{\beta_k^M\}_{k\geq 0}$ in the rational form:

$$\beta_0^M = \frac{1}{24},\ \beta_1^M = \frac{7}{40},\ \beta_2^M = \frac{2071}{5880},\ \beta_3^M = \frac{999245}{1217748},\ \beta_4^M = \frac{21959166635}{18211040276},$$

$$\beta_5^M = \frac{108481778600414331}{55169934195679160},\ \beta_6^M = \frac{208385239691564817344 1543}{8137828947445883350085 20},$$

$$\beta_7^M = \frac{25698543837390957571411809266308135}{7116536885169433586426285918882662},$$

$$\beta_8^M = \frac{2022217398360507246593127286050156180973495555485}{457883445996331837976313744446948175389676295 98},$$

$$\beta_9^M = \frac{140775644932548533751440756528783842684097847772368692345390683 57}{244608717049998332714170591533096152188800133593490040277740 2200},$$

etc. In this case, the last term β_{99}^{M} has slightly complicated the "form of complexity" {16401/16398} than one in the previous case, precisely. Otherwise, its value (rounded to 60 decimal digits) is

$$\beta_{99}^{M} = 619.562819405146668677971154899553589896235540274133472854031.$$

4. Binet Weight Function $\mathbf{w^B}$ The moments for this weight function are given in (38), and our Package `OrthogonalPolynomials` gives the sequence of recurrence coefficients $\{\beta_k^B\}_{k\geq 0}$ in the rational form:

$$\beta_0^B = \frac{1}{12}, \quad \beta_1^B = \frac{1}{30}, \quad \beta_2^B = \frac{53}{210}, \quad \beta_3^B = \frac{195}{371}, \quad \beta_4^B = \frac{22999}{22737}, \quad \beta_5^B = \frac{29944523}{19733142},$$

$$\beta_6^B = \frac{109535241009}{48264275462}, \quad \beta_7^B = \frac{29404527905795295658}{9769214287853155785},$$

$$\beta_8^B = \frac{455377030420113432210116914702}{113084128923675014537885725485},$$

$$\beta_9^B = \frac{26370812569397719001931992945645578779849}{5271244267917980801966553649147604697542},$$

$$\beta_{10}^B = \frac{152537496709054809881638897472985990866753853122697839}{24274291553105128438297398108902195365373879212227726},$$

$$\beta_{11}^B = \frac{100043420063777451042472529806266909090824649341814868347109676190691}{13346384670164266280033479022693768890138348905413621178450736182873},$$

etc. Numerical values of coefficients β_k^B for $k = 12, \ldots, 39$, rounded to 60 decimal digits, are presented in Table 1.

For this case we give also quadrature parameters x_ν^B and A_ν^B, $\nu = 1, \ldots, n$, for $n = 10$ (rounded to 30 digits in order to save space). Numbers in parenthesis indicate the decimal exponents (Table 2).

Table 1 Numerical values of the coefficients β_k^B, $k = 12, \ldots, 39$

k	β_k^B
12	9.0406602343677266995311393602604817493362196353707222267 5357
13	10.489303654509482277188371304592629522097237989383404999 3209
14	12.297193610386205863989437140091917659736550900451645361 0177
15	13.982876953992430188259760651278730085908033315470050643 1789
16	16.053551416704935469715616365006260178351576497091771136 1702
17	17.976607399870277592569472307671554399314783855650011718 7847
18	20.309762027441653743805414720494896893701648534519688152 6453
19	22.470471639933132495517941571507922108995386290182352089 3038
20	25.065846548945972029163400322506305368238517635457020708 4270
21	27.464451825029133609175558982646222673228647385791386492 1713
22	30.321821231673047126882599306405786994487378731380997742 6698
23	32.958533929972987219994066451412088206960100099972479634 9878
24	36.077698931299242645153320900855452336776003311554346830 1504
25	38.952706682311555734544390410481046299159323380561658839 7077
26	42.333490043576957211381853948856097339914786141195344671 7663
27	45.446960850061621014424175737541451082848436831140766578 2656
28	49.089203129012597708164883350275087292449199889806803667 7541
29	52.441288751415337312569856046996108427147860745593015552 9787
30	56.344845345341843538420365947476113542133304662352360702 5848
31	59.935683907165858207852583492752112110134546409037694062 1335
32	64.100422755920354527906611892237917752909220210767957094 3670
33	67.930140788018221186367702745198535816522551006935119301 3587
34	72.355940555211701969680052963236217910751758534556246288 0100
35	76.424654626829689752585090422287526403570045911230834815 3069
36	81.111403237247965484814230985683460974502694224629639582 4649
37	85.419221276410972614585638717348682726988822368168470459 9999
38	90.366814723864108595513574581683377780787091193972158162 5005
39	94.913837100009887953076231291986927458767824186893694016 5561

Table 2 Gaussian quadrature parameters x_ν^B and A_ν^B, $\nu = 1, \ldots, n$, for ten-point rule

ν	$x_{\nu+5}^B$ $(= -x_{6-\nu}^B)$	$A_{\nu+5}^B$ $(= A_{6-\nu}^B)$
1	1.1902613441086993104129971729 6(−1)	3.9510754133470557773378844004 5(−2)
2	5.9858925774221969335795616210 7(−1)	2.1095688322136396724373959659 4(−3)
3	1.2505802881902493465303354222 2	4.6079950342739755966914606588 6(−5)
4	2.1202092556917260535590485324 7	2.6357427235200110647978103032 9(−7)
5	3.3492781964583583334922310650 4	1.7636737746377703230858748653 1(−10)

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A New Approach to Positivity and Monotonicity for the Trapezoidal Method and Related Quadrature Methods

Q. I. Rahman and G. Schmeisser

Abstract For positive integers n let $R_n[f]$ be the remainders of a quadrature method applied to a function f. It is of practical importance to know sufficient conditions on f which guarantee that the remainders are non-negative and converge monotonically to zero as $n \to \infty$. For most of the familiar quadrature methods such conditions are known as sign conditions on certain derivatives of f. However, conditions of this type specify only a small subset of the desired functions. In particular, they exclude oscillating functions. In the case of the trapezoidal method, we propose a new approach based on Fourier analysis and the theory of positive definite functions. It allows us to describe much wider classes of functions for which positivity and monotonicity occur. Our considerations include not only the trapezoidal method on a compact interval but also that for integration over the whole real line as well as some related methods.

Keywords Quadrature • Trapezoidal method • Positive remainders • Monotonically decreasing remainders • Positive definite functions

Mathematics Subject Classification (2010): 41A55, 41A80, 42A82, 65D30, 65D32

Preliminary Comments

The second named author had the privilege to work with Q.I. Rahman for a period of three decades starting in 1972. The collaboration resulted in 44 joint papers, two books, and a deep friendship. From 1982 to 1994 the authors studied various themes

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on quadrature such as generalizations, refinements, and interconnections concerning some classical summation formulae interpreted as quadrature formulae and characterization of rates of convergence by function spaces. They communicated their results in 13 publications. Another aim was to find new and less restrictive conditions which guarantee that the remainders of a quadrature method are non-negative and converge monotonically to zero. They drafted several versions of two manuscripts: one for quadrature on a compact interval and one for the whole real line. The second named author presented the results for $\mathbb{R}$ at the Winter Meeting of the Canadian Mathematical Society in December 1989 in Montreal, the results for a compact interval at an Oberwolfach meeting in November 1992 and the results for both types of intervals in a colloquium at the University of Hildesheim in January 1995. He also lectured on this theme at local workshops of his university. Since the authors became very busy with a book project on polynomials, they postponed the work for final versions of their manuscripts and somehow they never came back to this subject.

With great pleasure, the second named author now finalized and united these manuscripts. Although around 25 years have passed since the beginning of this theme, he thinks that the attribute "new" in the title is still justified. He is very delighted commemorating his dear friend with a belated joint paper.

1 Introduction

A sequence of quadrature formulae

$$\int_a^b f(x)\,dx = \sum_{\kappa=0}^{m_n} A_{n,\kappa} f(x_{n,\kappa}) + R_n[f] \quad (n \in \mathbb{N}), \qquad (1)$$
$$(a \le x_{n,0} < x_{n,1} < \cdots < x_{n,m_n} \le b)$$

will be called a *quadrature method*. Here $R_n[f]$ denotes the *remainder* or *error* and

$$Q_n[f] := \sum_{\kappa=0}^{m_n} A_{n,\kappa} f(x_{n,\kappa})$$

is the approximation of the integral. Clearly, when $[\alpha, \beta]$ is a compact interval different from $[a, b]$, we can easily derive from (1) a quadrature method for integration over $[\alpha, \beta]$ by an affine transformation. Without loss of generality, we may therefore restrict ourselves to integration over $[0, 1]$.

Often a quadrature method is generated from one special quadrature formula by decomposing the interval $[a, b]$ into n congruent subintervals, applying the chosen formula after an appropriate transformation to each of them and summing up the results. The sequence of quadrature formulae obtained this way is called a *compound method*. As an example, we consider the trapezoidal rule on $[0, 1]$, described by

$$\int_0^1 f(x)\,dx = \frac{1}{2}\big[f(0)+f(1)\big] + R^{\mathrm{tr}}[f],$$

which generates the *trapezoidal method*

$$\int_0^1 f(x)\,dx = \frac{1}{2n}\left[f(0) + 2\sum_{\kappa=1}^{n-1} f\left(\frac{\kappa}{n}\right) + f(1)\right] + R_n^{\mathrm{tr}}[f] \quad (n \in \mathbb{N}). \tag{2}$$

It is not only of theoretical interest but also of practical importance for error estimates and stop criteria in computations to know sufficient conditions on f which guarantee that the quadrature method yields a one-sided approximation of the integral and that the remainders converge monotonically to zero as $n \to \infty$. This leads us to the following questions:

Positivity: How to guarantee that $R_n[f] \geq 0$ for all $n \in \mathbb{N}$?
Monotonicity: How to guarantee that $0 \leq R_{n+1}[f] \leq R_n[f]$ for all $n \in \mathbb{N}$?

In many cases, an answer to the question of positivity can be obtained by considering Peano's representation of the remainder $R_n[f]$; see, e.g., [4, Sect. II.1]. If the corresponding Peano kernel does not change sign on $[a, b]$, then positivity can be guaranteed by a sign condition on a certain derivative of f.

Finding sufficient conditions for monotonicity is a more difficult problem that has attracted some renowned mathematicians. Contributions have been obtained for nearly all of the familiar quadrature methods; see [23] for sequences of Riemann sums and a discussion of unpublished work of Fejér for the rectangle method, Stenger [20] for Gaussian methods, Albrecht [2] and Ström [22] for Romberg's method, Rivlin [19] for the midpoint method, Newman [17] for compound Newton–Côtes methods and a communication of unpublished results by Newman and Rivlin for the trapezoidal method and by Molluzo for Simpson's method, Brass [5] and Locher [12] for Gaussian and Newton–Côtes methods, Kütz [11] for interpolatory methods, Förster [9] for Gregory's method, and Nikolov [18] for certain compound methods. In all these cases monotonicity is guaranteed by sign conditions on successive derivatives.

For the trapezoidal method (2), the known results concerning positivity and monotonicity are as follows.

Theorem A. *Let* $f \in C^2[0,1]$ *and suppose that* $f''(x) \leq 0$ *for* $x \in [0,1]$. *Then* $R_n^{\mathrm{tr}}[f] \geq 0$ *for all* $n \in \mathbb{N}$.

Theorem B (Newman and Rivlin). *Let* $f \in C^3[0,1]$ *and suppose that* $f''(x) \leq 0$, $f'''(x) \neq 0$ *for* $x \in [0,1]$. *Then*

$$R_{n_1}^{\mathrm{tr}}[f] \geq R_{n_2}^{\mathrm{tr}}[f] \geq 0 \quad \text{for } n_2 > n_1 \ \ (n_1, n_2 \in \mathbb{N}).$$

We believe that the conditions given in the literature describe only a very small subset of the collection of all functions for which positivity or monotonicity occurs. Numerical experiments show positivity and monotonicity even for some highly oscillating functions which violate sign conditions on derivatives drastically. Let us consider some examples in case of the trapezoidal method.

Example 1. The function $-e^x$ would satisfy the hypotheses of Theorem A, but now we consider

$$f(x) := -e^x \cos(30\pi x)$$

By Rolle's theorem, the second derivative of f has at least 28 sign changes on the interval $[0, 1]$. Hence the hypotheses of Theorem A are heavily violated. Nevertheless the remainders of the trapezoidal method are positive as the first column of Table 1 shows.

Example 2. Let

$$f(x) := \begin{cases} -\dfrac{\sin(33\pi x)}{33\sin(\pi x)} & \text{if } x \in (0, 1), \\ -1 & \text{if } x \in \{0, 1\}. \end{cases}$$

By Rolle's theorem, the second derivative of f has at least 30 and the third at least 29 sign changes on the interval $[0, 1]$. Hence the hypotheses of Theorems A and B are heavily violated. Nevertheless the remainders of the trapezoidal method are nonnegative and nonincreasing as the second column of Table 1 shows.

Example 3. The second derivative of the Bernoulli polynomial $B_4(x)$ has two sign changes on $[0, 1]$ and the third derivative also changes sign. Hence Theorems A and B do not apply to $f(x) := B_4(x)$. But the remainders of the trapezoidal method are positive and converge monotonically to zero as the third column of Table 1 shows.

Our aim is to describe wide classes of functions which guarantee positivity or monotonicity and contain the functions specified in Theorem A or Theorem B, respectively. For this purpose, the concept of positive definite functions or—in the language of probability—the characteristic functions of a distribution provides an efficient approach, at least for the trapezoidal method. In particular, it explains the behavior of Examples 1–3. However, we still do not get the maximal classes of functions for which positivity or monotonicity occurs; see Remark 3. They seem to be of a complicated structure. For an attempt to describe the maximal class that guarantees positivity for smooth functions, see Proposition 2 below.

The paper is organized as follows. In Sect. 2 we recall some fundamental properties of positive definite function useful for our consideration. In Sect. 3 we study positivity and monotonicity for the trapezoidal method on $[0, 1]$ and discuss interconnections with some related quadrature methods. Finally, in Sect. 4, we establish analogous results for the trapezoidal method on $\mathbb{R}$.

Table 1 Examples for remainders of the trapezoidal method

Example 1		Example 2		Example 3	
n	$R_n^{tr}[f]$	n	$R_n^{tr}[f]$	n	$R_n^{tr}[f]$
1	1.85894749	1	0.96969697	1	0.03333333
2	0.10501640	2	0.48484848	2	0.00208333
3	1.73396904	3	0.30303030	3	0.00041152
4	0.05241149	4	0.24242424	4	0.00013021
5	1.72381220	5	0.18181818	5	0.00005333
6	0.01171155	6	0.12121212	6	0.00002572
7	0.04529855	7	0.12121212	7	0.00001388
8	0.04456332	8	0.12121212	8	0.00000814
9	0.00686319	9	0.06060606	9	0.00000508
10	0.00409871	10	0.06060606	10	0.00000333
11	0.00409238	11	0.06060606	11	0.00000228
12	0.00575906	12	0.06060606	12	0.00000161
13	0.01150753	13	0.06060606	13	0.00000117
14	0.04299383	14	0.06060606	14	0.00000087
15	1.71872476	15	0.06060606	15	0.00000066
16	0.04281955	16	0.06060606	16	0.00000051
17	0.01112852	17	0.00000000	17	0.00000040
18	0.00509632	18	0.00000000	18	0.00000032

2 Positive Definite Functions

Pioneering work on positive definite functions is due to Mathias [15] and Bochner [6]. We recall some fundamental knowledge extracted from a survey article of Stewart [21].

Definition 1. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be *positive definite* if for every choice of $x_1, \ldots, x_n \in \mathbb{R}$ and $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$, we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} f(x_j - x_k)\zeta_j \overline{\zeta}_k \ \geq \ 0.$$

Some authors require in addition that f is continuous; see, e.g., [14]. The following properties, observed already by Mathias [15] are of interest.

Properties.

(i) If f is positive definite, then $|f(x)| \leq f(0)$ and $f(x) = \overline{f(-x)}$ for all $x \in \mathbb{R}$.
(ii) If f and g are positive definite, then fg is positive definite.
(iii) If $f_1, \ldots, f_n$ are positive definite and $c_1, \ldots, c_n \in [0, \infty)$, then $c_1 f_1 + \cdots + c_n f_n$ is positive definite.
(iv) If $(f_n)_{n\in\mathbb{N}}$ is a pointwise convergent sequence of positive definite functions, then $f := \lim_{n\to\infty} f_n$ is positive definite.

Properties (ii)–(iv) show that the positive definite functions form a closed, multiplicative cone.

Some examples of positive definite functions are:

$$e^{ix}, \quad \cos x, \quad (1+x^2)^{-1}, \quad e^{-|x|}, \quad e^{-x^2}.$$

A celebrated representation theorem for positive definite functions is due to Bochner [6].

Bochner's Theorem. *A continuous function $f : \mathbb{R} \to \mathbb{C}$ is positive definite if and only if there exists a bounded, nondecreasing function V on $\mathbb{R}$ such that*

$$f(x) = \int_{-\infty}^{\infty} e^{i\alpha x}\, dV(\alpha) \quad (x \in \mathbb{R}).$$

The following corollaries are of interest.

Corollary 1. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous, p-periodic function. Then f is positive definite if and only if*

$$\frac{1}{p}\int_0^p f(x)e^{-i2\pi nx/p}\, dx \geq 0$$

for all $n \in \mathbb{Z}$.

Corollary 2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function belonging to $L^1(\mathbb{R})$ and suppose that its Fourier transform*

$$\hat{f}(v) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-itv}\, dt$$

also belongs to $L^1(\mathbb{R})$. Then f is positive definite if and only if $\hat{f}(v) \geq 0$ for all $v \in \mathbb{R}$.

As usual $L^1(\mathbb{R})$ denotes the set of functions which are Lebesgue integrable over $\mathbb{R}$. Many sophisticated results on positive functions have been obtained in the framework of probability where these functions play an important role as characteristic functions of distributions. More precisely, f is a characteristic function of a distribution if and only if f is a continuous, positive definite function and $f(0) = 1$.

3 The Trapezoidal Method on a Compact Interval

When we consider a compact interval, it is no loss of generality to restrict ourselves to $[0, 1]$ where the trapezoidal method is given by (2). We shall use a familiar terminology only. By $C[0,1]$, we denote the class of all continuous functions $f : [0,1] \to \mathbb{R}$ and by $C^k[0,1]$ the subclass of functions that are k times continuously differentiable on $(0,1)$ with one-sided derivatives at the endpoints 0 and 1. For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the largest integer not exceeding x.

Since the approximation $Q_n^{\mathrm{tr}}[f]$ of $\int_0^1 f(x)dx$ by the trapezoidal method is a Riemann sum, it is clear that $f \in C[0,1]$ implies that $R_n^{\mathrm{tr}}[f] \to 0$ as $n \to \infty$. In the subsequent statements, we shall therefore not mention the convergence to zero once again explicitly.

3.1 Some Lemmas

We start with a representation of $R_n^{\mathrm{tr}}[f]$ in terms of Fourier coefficients.

Lemma 1. *Let $f \in C[0,1]$ and define*

$$a_n := 2\int_0^1 f(x)\cos(2\pi n x)\,dx \quad (n \in \mathbb{N}_0). \tag{3}$$

Then

$$R_n^{\mathrm{tr}}[f] = -\lim_{N\to\infty} \sum_{j=1}^{\lfloor N/n \rfloor} \left(1 - \frac{jn}{N+1}\right) a_{jn}. \tag{4}$$

Proof. Define

$$h(t) := \frac{1}{2}\big[f(t) + f(1-t)\big]. \tag{5}$$

Then

$$\int_0^1 f(t)\,dt = \int_0^1 h(t)\,dt = \frac{a_0}{2} \quad \text{and} \quad Q_n^{\mathrm{tr}}[f] = Q_n^{\mathrm{tr}}[h] = \frac{1}{n}\sum_{\kappa=1}^{n} h\left(\frac{\kappa}{n}\right). \tag{6}$$

Moreover, the 1-periodic continuation $\tilde{h}$ of h is an even continuous function on $\mathbb{R}$. For the trigonometric series associated with $\tilde{h}$, we find

$$\tilde{h}(x) \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(2\pi j x)$$

with coefficients given by (3). It is known from the theory of Fourier series (see, e.g., [7, p. 173, Problem 4]) that the Cesàro limit of the right-hand side represents $\tilde{h}$, i.e.,

$$\tilde{h}(x) = \frac{a_0}{2} + \lim_{N\to\infty} \sum_{j=1}^{N} \left(1 - \frac{j}{N+1}\right) a_j \cos(2\pi jx).$$

From (6) it now follows that

$$R_n^{\mathrm{tr}}[f] = -\lim_{N\to\infty} \sum_{j=1}^{N} \left(1 - \frac{j}{N+1}\right) a_j \cdot \frac{1}{n} \sum_{\kappa=1}^{n} \cos\left(2\pi \frac{j\kappa}{n}\right). \tag{7}$$

Finally, expressing $\cos x$ as $(e^{ix} + e^{-ix})/2$, we note that

$$\frac{1}{n} \sum_{\kappa=1}^{n} \cos\left(2\pi \frac{j\kappa}{n}\right) = \begin{cases} 1 & \text{if } n \text{ divides } j, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this equation with (7), we arrive at (4). □

Remark 1. From the theory of summability one knows that the right-hand side of (4) may be replaced by $-\sum_{j=1}^{\infty} a_{jn}$ if this series converges.

Lemma 2. *Let $g \in C[0,1]$ and suppose that $g(x) \equiv g(1-x)$. Denote by $\tilde{g}$ the 1-periodic continuation of g. Then the coefficients*

$$\alpha_n := 2\int_0^1 g(x)\cos(2\pi nx)\,dx \quad (n \in \mathbb{N}_0) \tag{8}$$

are non-negative for $n \in \mathbb{N}$ and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is nonincreasing, if and only if the 2-periodic function $x \mapsto \tilde{g}(x)\sin(\pi x)$ has an associated trigonometric series

$$\sum_{n=0}^{\infty} b_n \sin\big((2n+1)\pi x\big) \tag{9}$$

with $b_n \geq 0$ for $n \in \mathbb{N}$.

Proof. Considering the symmetries of $\tilde{g}(x)\sin(\pi x)$, we readily see that the associated trigonometric series is of the form (9). Furthermore,

$$\begin{aligned} b_n &= \int_0^2 \tilde{g}(x)\sin(\pi x)\sin\big((2n+1)\pi x\big)\,dx \\ &= \frac{1}{2}\int_0^2 \tilde{g}(x)\big[\cos(2n\pi x) - \cos\big((2n+2)\pi x\big)\big]\,dx \\ &= \int_0^1 \tilde{g}(x)\cos(2n\pi x)\,dx - \int_0^1 \tilde{g}(x)\cos\big((2n+2)\pi x\big)\,dx \end{aligned}$$

$$= \frac{1}{2}\big(\alpha_n - \alpha_{n+1}\big) \qquad (n \in \mathbb{N}_0).$$

Hence, for $n \in \mathbb{N}$, we have $b_n \geq 0$ if and only if $\alpha_n \geq \alpha_{n+1}$. Since by a standard result $\alpha_n \to 0$ as $n \to \infty$ (see, e.g., [7, p. 51, Problem 8]), we conclude that $\alpha_n \geq 0$ for $n \in \mathbb{N}$. □

The desired properties of the coefficients α_n can also be described in terms of positive definite functions. Since the trapezoidal rule is exact for affine functions, an additive constant c does not change the remainder, that is, $R_n^{\mathrm{tr}}[f] = R_n^{\mathrm{tr}}[f + c]$. For this reason, the following notion will be convenient.

Definition 2. We say that a function f is *essentially positive definite* if there exists a constant c such that $f + c$ is positive definite.

The following lemma has been designed for our needs. For a related result, see [16, Theorem 2.2].

Lemma 3. *Let $g \in C^1[0,1]$ and suppose that $g(x) \equiv g(1-x)$. Denote by $\tilde{g}$ the 1-periodic continuation of g. Then the coefficients (8) are non-negative for $n \in \mathbb{N}$ and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is nonincreasing if and only if the function ϕ given by*

$$\phi(x) := \tilde{g}(x) + \frac{1}{2\pi i}\left(e^{2\pi i x} - 1\right)\tilde{g}\,'(x) \tag{10}$$

is essentially positive definite.

Proof. Define $\phi_c := \phi + c$ and $\tilde{g}_c := \tilde{g} + c$. Then, by the properties and examples listed in Sect. 2, the function ϕ_c is positive definite if and only if

$$\begin{aligned} \phi_c(x)e^{-2\pi i x} &= \tilde{g}_c(x)e^{-2\pi i x} + \frac{1}{2\pi i}\left(1 - e^{-2\pi i x}\right)\tilde{g}_c'(x) \\ &= \frac{1}{2\pi i}\frac{d}{dx}\Big(\tilde{g}_c(x)\left(1 - e^{-2\pi i x}\right)\Big) \end{aligned} \tag{11}$$

is positive definite. By Corollary 1, the latter holds if and only if

$$c_n := \int_0^1 \phi_c(x)e^{-2\pi i x}e^{-2\pi i n x}\,dx \geq 0 \quad (n \in \mathbb{Z}). \tag{12}$$

Using (11), we find by integration by parts that

$$\begin{aligned} c_n &= n\int_0^1 \big(g(x) + c\big)\big(1 - e^{-2\pi i x}\big)e^{-2\pi i n x}\,dx \\ &= \frac{n}{2}\left(\alpha_{|n|} - \alpha_{|n+1|}\right) \qquad \text{for } n \in \mathbb{Z} \setminus \{0, -1\}, \end{aligned}$$

$c_0 = 0$ and $c_{-1} = \frac{1}{2}(\alpha_0 + c - \alpha_1)$. From these relations we conclude that if (12) holds, then $\alpha_n \geq \alpha_{n+1}$ for $n \in \mathbb{N}$. Since $\alpha_n \to 0$ as $n \to \infty$, we also have $\alpha_n \geq 0$ for $n \in \mathbb{N}$. Conversely, if the coefficients α_n have these properties, then $c_n \geq 0$ for $n \in \mathbb{Z} \setminus \{-1\}$ and $c_{-1} \geq 0$ if c is chosen appropriately, namely such that ϕ_c becomes positive definite. This completes the proof. □

For the subsequent applications of Lemmas 1–3, the following observations will be useful.

Remark 2. For $f \in C[0,1]$, define g and $\tilde{g}$ by

$$g(x) := -\frac{1}{2}\big[f(x)+f(1-x)\big],\ x \in [0,1] \quad \text{and} \quad \tilde{g}(x) := g\big(x-\lfloor x\rfloor\big),\ x \in \mathbb{R}. \tag{13}$$

Then $\tilde{g}$ is the 1-periodic continuation of g. In view of (5) and (6), we have $R_n^{\mathrm{tr}}[g] = -R_n^{\mathrm{tr}}[f]$. Furthermore, g satisfies $g(x) \equiv g(1-x)$, as required in Lemmas 2 and 3, and the coefficients a_n in (3) and α_n in (8) are related by $a_n = -\alpha_n$ $(n \in \mathbb{N}_0)$.

3.2 Positivity

The following theorem extends the conclusion of Theorem A to a much wider class of functions.

Theorem 1. *Let $f \in C[0,1]$ and suppose that $\tilde{g}$ defined in (13) is essentially positive definite. Then the remainders $R_n^{\mathrm{tr}}[f]$ are non-negative. Moreover,*

$$R_n^{\mathrm{tr}}[f] \geq R_{2n}^{\mathrm{tr}}[f] \geq \cdots \geq R_{2^k n}^{\mathrm{tr}}[f] \geq \cdots \geq 0 \quad (n \in \mathbb{N}). \tag{14}$$

Proof. Since $\tilde{g}$ is an even, 1-periodic, essentially positive definite function, it follows from Corollary 1 that the coefficients α_n given by (8) are non-negative for $n \in \mathbb{N}$. Hence in view of Remark 2 and Lemma 1, we have

$$R_n^{\mathrm{tr}}[f] = -R_n^{\mathrm{tr}}[g] = \lim_{N\to\infty} \sum_{j=1}^{\lfloor N/n\rfloor} \left(1 - \frac{jn}{N+1}\right)\alpha_{jn}. \tag{15}$$

In the sum on the right-hand side, each term is non-negative, and so $R_n^{\mathrm{tr}}[f] \geq 0$.

Finally, if in (15) we replace n by $2n$, then each term of the resulting new sum also appears in the former one. This shows that (14) holds. □

Theorem A is a special case of Theorem 1 as the following proposition shows.

Proposition 1. *Let $f \in C^2[0,1]$ and suppose that $f''(x) \leq 0$ for $x \in [0,1]$. Define g and $\tilde{g}$ by (13). Then for every $c \geq \int_0^1 f(x)dx$ the function $\tilde{g} + c$ is positive definite.*

Proof. By Corollary 1, it suffices to verify that the integrals

$$I_n := \int_0^1 \big(g(x) + c\big) e^{-2\pi i n x}\, dx$$

are non-negative for all $n \in \mathbb{Z}$. For $n = 0$ we have

$$I_0 = c - \int_0^1 f(x)\, dx,$$

which is non-negative by the restriction on c. For $n \in \mathbb{Z} \setminus \{0\}$ we find by two integration by parts:

$$\begin{aligned} I_n &= -\frac{1}{2}\int_0^1 \big(f(x) + f(1-x)\big) e^{-2\pi i n x}\, dx \\ &= -\int_0^1 f(x)\cos(2\pi n x)\, dx \\ &= -\frac{1}{(2\pi n)^2}\int_0^1 f''(x)\big(1 - \cos(2\pi n x)\big)\, dx \geq 0. \end{aligned}$$

This completes the proof. □

While Theorem A cannot explain why $R_n^{\mathrm{tr}}[f]$ is non-negative for a function $f(x) := -e^x \cos(2m\pi x)$ with $m \in \mathbb{N}$, as considered in Example 1 for $m = 15$, Theorem 1 does apply to f. Indeed, the function (13) becomes

$$g(x) = \frac{e^x + e^{1-x}}{2} \cdot \cos(2m\pi x)\,.$$

As a consequence of Proposition 1, the 1-periodic continuation of the first factor on the right-hand side is positive definite and so is the cosine. Thus, $\tilde{g}$ being the product of two positive definite functions, it is itself positive definite. Therefore the conclusion of Theorem 1 is valid for f.

Remark 3. We want to mention that there are functions with non-negative remainders that are not covered by Theorem 1. Indeed, suppose that

$$\sum_{n=1}^{\infty} n^{\varepsilon}\, |\alpha_n| < \infty \tag{16}$$

for some $\varepsilon > 0$. Then, in view of Remark 2, formula (15) simplifies to

$$R_n^{\mathrm{tr}}[f] = -R_n^{\mathrm{tr}}[g] = \sum_{j=1}^{\infty} \alpha_{jn} \quad (n \in \mathbb{N}). \tag{17}$$

Moreover, the Möbius inversion formula applies; see [13, p. 19, Corollary]. It yields that

$$\alpha_n = \sum_{j=1}^{\infty} \mu(j) R_{jn}^{\mathrm{tr}}[f] \quad (n \in \mathbb{N}), \tag{18}$$

where $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined by

$$\mu(j) := \begin{cases} 1 & \text{if } j = 1, \\ (-1)^n & \text{if } j = p_1 \cdots p_n \text{ with distinct primes } p_1, \dots, p_n, \\ 0 & \text{if } j \text{ is divisible by a square of a prime.} \end{cases}$$

Formulae (17) and (18) show that there is a one-to-one correspondence between the sequence of coefficients and the sequence of remainders. However, while by (17) non-negative coefficients entail non-negative remainders $R_n^{\mathrm{tr}}[f]$, the converse is not true since the Möbius function in (18) may attain negative values. For example, if

$$f(x) = \cos(4\pi x) + \cos(6\pi x) - \cos(12\pi x),$$

then $\alpha_2 = \alpha_3 = -1,\ \alpha_6 = 1$ and $\alpha_n = 0$ for $n \in \mathbb{N} \setminus \{2, 3, 6\}$ while the remainders are all non-negative. More precisely, $R_6^{\mathrm{tr}}[f] = 1$ and $R_n^{\mathrm{tr}}[f] = 0$ for $n \neq 6$.

Under some smoothness on f, we can now give a (rather implicit) necessary and sufficient condition for the remainders to be non-negative. It shows that a simple characterization cannot be expected.

Proposition 2. *Let $f \in C^2[0, 1]$. Then $R_n^{\mathrm{tr}}[f] \geq 0$ for all $n \in \mathbb{N}$ if and only if there exists a $\delta > 0$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of non-negative numbers such that $\sum_{n=1}^{\infty} n^{\delta} c_n < \infty$ and the coefficients (8) can be represented as*

$$\alpha_n = \sum_{j=1}^{\infty} \mu(j) c_{jn} \quad (n \in \mathbb{N}).$$

Proof. Starting with formula (8), we find by two integrations by parts that

$$\alpha_n = \frac{2}{(2\pi n)^2} \int_0^1 g''(x)\big[1 - \cos(2\pi n x)\big]\, dx \quad (n \in \mathbb{N}).$$

Hence

$$|\alpha_n| \leq \frac{1}{\pi^2 n^2} \int_0^1 \big|f''(x)\big|\, dx \quad (n \in \mathbb{N}),$$

and so (16) holds for any $\varepsilon \in (0, 1)$. Thus [13, p. 19, Corollary] is applicable. It implies that $R_n^{\mathrm{tr}}[f] = c_n$ for all $n \in \mathbb{N}$. □

3.3 Monotonicity

Lemmas 2 and 3 allow us to obtain the conclusion of Theorem B for a much wider class of functions.

Theorem 2. *Let* $f \in C[0,1]$ *and let* $\tilde{g}$ *be given by* (13)*. If the trigonometric series* (9) *associated with* $\tilde{g}(x)\sin(\pi x)$ *has coefficients* $b_n \geq 0$ *for* $n \in \mathbb{N}$*, then*

$$R^{\mathrm{tr}}_{n_1}[f] \geq R^{\mathrm{tr}}_{n_2}[f] \geq 0 \quad \textit{for } n_2 > n_1 \;\; (n_1, n_2 \in \mathbb{N}). \tag{19}$$

Proof. In view of Remark 2, Lemmas 1 and 2 yield

$$R^{\mathrm{tr}}_{n_\ell}[f] = \lim_{N\to\infty} \sum_{j=1}^{\lfloor N/n_\ell \rfloor} \left(1 - \frac{jn_\ell}{N+1}\right) \alpha_{jn_\ell} \quad (\ell = 1, 2),$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a nonincreasing sequence of non-negative numbers. Noting that $\lfloor N/n_2 \rfloor \leq \lfloor N/n_1 \rfloor$ and

$$0 \leq \left(1 - \frac{jn_2}{N+1}\right) \alpha_{jn_2} \leq \left(1 - \frac{jn_1}{N+1}\right) \alpha_{jn_1} \quad \text{for } n_2 > n_1,$$

we obtain (19) immediately. □

For the function f of Example 2, we have

$$\tilde{g}(x)\sin(\pi x) = \frac{\sin(33\pi x)}{33}\,.$$

Hence $b_{16} = 1/33$ and $b_n = 0$ for $n \in \mathbb{N} \setminus \{16\}$, and so Theorem 2 guarantees that (19) holds.

For $f(x) := B_4(x)$ of Example 3, we have $g(x) = -B_4(x)$ and

$$\tilde{g}(x) = \frac{3}{\pi^4} \sum_{n=1}^{\infty} \frac{\cos(2n\pi x)}{n^4}\,;$$

see [1, formulae (23.1.8) and (23.1.18)]. Hence

$$\begin{aligned} \tilde{g}(x)\sin(\pi x) &= \frac{3}{2\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4}\Big[\sin\big((2n+1)\pi x\big) - \sin\big((2n-1)\pi x\big)\Big] \\ &= \frac{3}{2\pi^4}\left[-\sin(\pi x) + \sum_{n=1}^{\infty}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right)\sin\big((2n+1)\pi x\big)\right]. \end{aligned}$$

Again we see that Theorem 2 applies and yields that (19) holds.

The proof of the following theorem is analogous to that of Theorem 2 with the role of Lemma 2 taken by Lemma 3.

Theorem 3. *Let $f \in C^1[0,1]$ and let $\tilde{g}$ be given by (13). If the function ϕ, defined by (10), is essentially positive definite, then*

$$R_{n_1}^{\mathrm{tr}}[f] \geq R_{n_2}^{\mathrm{tr}}[f] \geq 0 \quad \textit{for } n_2 > n_1 \ \ (n_1, n_2 \in \mathbb{N}).$$

The following proposition shows that Theorem B is a special case of Theorems 2 and 3.

Proposition 3. *Let $f \in C^3[0,1]$ and suppose that $f''(x) \leq 0$ and $f'''(x) \neq 0$ for $x \in [0.1]$. Then f satisfies the hypotheses of Theorems 2 and 3.*

Proof. In view of Lemmas 2 and 3 in conjunction with Remark 2, it suffices to show that if a_n is defined by (3), then $a_n \leq 0$ for $n \in \mathbb{N}$ and the sequence $(a_n)_{n\in\mathbb{N}}$ is nondecreasing.

We may assume that $f'''(x) \geq 0$ for $x \in [0,1]$; otherwise, the role of $f(x)$ may be taken by $f(1-x)$ in the following considerations.

Starting with (3) for $n \in \mathbb{N}$, we find by two integrations by parts that

$$\begin{aligned} a_n &= -\frac{1}{2\pi n}\int_0^1 f'(x)\sin(2\pi nx)\,dx \\ &= \frac{1}{(2\pi n)^2}\int_0^1 f''(x)\big(1-\cos(2\pi nx)\big)\,dx. \end{aligned}$$

Performing a further integration by parts, we obtain

$$\begin{aligned} a_n &= \frac{1}{(2\pi n)^2}\left[f''(x)\left(x-\frac{\sin(2\pi nx)}{2\pi n}\right)\Bigg|_0^1 - \int_0^1 f'''(x)\left(x-\frac{\sin(2\pi nx)}{2\pi n}\right)dx\right] \\ &= \frac{f''(1)}{(2\pi n)^2} - \int_0^1 x^3 f'''(x)\left[\frac{2\pi nx-\sin(2\pi nx)}{(2\pi nx)^3}\right]dx\,. \end{aligned} \tag{20}$$

The assumptions on f show immediately that $a_n \leq 0$ for $n \in \mathbb{N}$. It remains to prove that $a_{n+1} \geq a_n$ for $n \in \mathbb{N}$. Obviously, the first term in (20) is nondecreasing. Hence it suffices to show that for each fixed $x \in (0,1)$ the term in square brackets is decreasing. Clearly, this is guaranteed if

$$\varphi(t) := \frac{t-\sin t}{t^3}$$

is decreasing on $(0,\infty)$. Thus it is enough to show that

$$\psi(t) := t^4\varphi'(t) = 3\sin t - t\cos t - 2t < 0$$

for $t > 0$. For this we consider two cases:

If $t \geq \pi$, then

$$\psi(t) \leq 3\sin t - t(1+\cos t) - \pi < 3 - \pi < 0.$$

For $t \in [0, \pi]$, we employ Taylor's formula. Noting that $\psi(0) = 0$, $\psi'(0) = 0$ and $\psi''(t) = t\cos t - \sin t < 0$ for $t \in (0, \pi)$, we find that

$$\psi(t) = \frac{t^2}{2}\,\psi''(\theta)$$

for some $\theta \in (0, \pi)$. Hence $\psi(t) < 0$ for $t \in (0, \pi)$ as well. This completes the proof. □

A relatively simple condition guarantees monotonic convergence of the remainders $R_n^{\mathrm{tr}}[f]$ for sufficiently large n.

Theorem 4. *Let $f \in C^3[0,1]$ and suppose that $f'(0) \neq f'(1)$. Then there exists an $n_0 \in \mathbb{N}$ such that the sequence $(R_n^{\mathrm{tr}}[f])_{n \geq n_0}$ converges monotonically to zero.*

Proof. Since f is continuous, we know that $R_n^{\mathrm{tr}}[f] \to 0$ as $n \to \infty$. By Lemma 1 it is therefore enough to show that for the coefficients (3) the difference $a_n - a_{n+1}$ has a fixed sign for all sufficiently large n.

First we note that

$$\begin{aligned} a_n - a_{n+1} &= \int_0^1 f(x)\big[\cos(2\pi n x) - \cos\big(2\pi(n+1)x\big)\big]\,dx \\ &= 2\int_0^1 f(x)\sin(\pi x)\sin\big((2n+1)\pi x\big)\,dx. \end{aligned}$$

Introducing $F(x) := f(x)\sin(\pi x)$, we find by three integrations by part

$$a_n - a_{n+1} = \frac{-1}{((2n+1)\pi)^3}\left[F''(1) + F''(0) + \int_0^1 F'''(x)\cos\big((2n+1)\pi x\big)\,dx\right].$$

Since $F''(0) = 2\pi f'(0)$ and $F''(1) = -2\pi f'(1)$, we finally obtain

$$a_n - a_{n+1} = \frac{1}{((2n+1)\pi)^3}\left[2\pi\big(f'(1) - f'(0)\big) + \int_0^1 F'''(x)\cos\big((2n+1)\pi x\big)\,dx\right].$$

By a standard result, the integral converges to zero as $n \to \infty$. Hence

$$\operatorname{sgn}(a_n - a_{n+1}) = \operatorname{sgn}\big(f'(1) - f'(0)\big)$$

for all sufficiently large n. □

Theorem 4 applies to Example 1. Unfortunately we do not know the value of n_0. Table 1 shows that $n_0 \geq 15$.

3.4 Related Quadrature Methods

The essential key to the previous results was Lemma 1. It establishes a representation of the remainders $R_n^{\mathrm{tr}}[f]$ of the trapezoidal method in terms of Fourier coefficients of f. One cannot expect that a modification of the proof of Lemma 1 will give an extension to an arbitrary quadrature method. However, if the method has some periodic structure, as in the case of a sequence of compound formulae, then there is some hope.

The Midpoint Method

For the interval [0, 1] it takes the form

$$\int_0^1 f(x)\,dx = \frac{1}{n}\sum_{\kappa=1}^{n} f\left(\frac{2\kappa-1}{2n}\right) + R_n^{\mathrm{mi}}[f] \quad (n \in \mathbb{N}).$$

Obviously this formula is equivalent to applying the nth formula of the trapezoidal method to the function $\tilde{h}(x - 1/(2n))$ with $\tilde{h}$ as in the proof of Lemma 1. With this observation, we arrive at the following result.

Lemma 4. *Let $f \in C[0,1]$ and define a_n by (3). Then*

$$R_n^{\mathrm{mi}}[f] = -\lim_{N\to\infty}\sum_{j=1}^{\lfloor N/n\rfloor}\left(1 - \frac{jn}{N+1}\right)(-1)^j a_{jn}. \tag{21}$$

A comparison of (4) and (21) allows us to establish interconnections between $R_n^{\mathrm{tr}}[f]$ and $R_n^{\mathrm{mi}}[f]$.

Proposition 4. *Under the hypotheses of Theorem 1, there holds*

$$\left|R_n^{\mathrm{mi}}[f]\right| \leq R_n^{\mathrm{tr}}[f] \quad (n \in \mathbb{N}). \tag{22}$$

Proof. The hypotheses of Theorem 1 imply that for the coefficients (3) we have $a_n \leq 0$ for $n \in \mathbb{N}$. Now a comparison of (4) and (21) show that $R_n^{\mathrm{tr}}[f] \pm R_n^{\mathrm{mi}}[f] \geq 0$, which gives (22). □

Proposition 5. *Under the hypotheses of Theorem 2 or 3, there hold*

$$-R_n^{\mathrm{tr}}[f] \leq R_n^{\mathrm{mi}}[f] \leq 0 \quad (n \in \mathbb{N}) \tag{23}$$

and

$$\left|R_n^{\mathrm{mi}}[f] - R_{n+1}^{\mathrm{mi}}[f]\right| \;\leq\; R_n^{\mathrm{tr}}[f] - R_{n+1}^{\mathrm{tr}}[f] \quad (n \in \mathbb{N}). \tag{24}$$

Proof. The hypotheses imply that $a_n \leq 0$ for $n \in \mathbb{N}$ and the sequence $(a_n)_{n\in\mathbb{N}}$ is nondecreasing. Now a comparison of (4) and (21) shows that (23) holds and that

$$R_{n+1}^{\mathrm{tr}}[f] \pm R_{n+1}^{\mathrm{mi}}[f] \;\leq\; R_n^{\mathrm{tr}}[f] \pm R_n^{\mathrm{mi}}[f],$$

which gives (24). □

Inequality (23) implies that

$$Q_n^{\mathrm{tr}}[f] \;\leq\; \int_0^1 f(x)\,dx \;\leq\; Q_n^{\mathrm{mi}}[f],$$

and so the approximations obtained by the trapezoidal and the midpoint method yield an inclusion of the exact value of the integral.

The study of the behavior of $R_n^{\mathrm{mi}}[f]$ itself is somewhat complicated for the following reason. Let m be an integer of the form $2\ell(2j+1)$ with $\ell, j \in \mathbb{N}$. Then a_m appears in the representations (21) of $R_{2j+1}^{\mathrm{mi}}[f]$ and $R_{2\ell}^{\mathrm{mi}}[f]$ with opposite signs. Thus if $|a_m|$ is very large these two remainders will have different signs even if the coefficients a_n are of the same sign for all $n \in \mathbb{N}$. We can avoid this phenomenon if we restrict ourselves to remainders with odd indices.

For $f \in C[0,1]$, define

$$u(x) \;:=\; \begin{cases} \frac{1}{2}\left[f(\frac{1}{2}+x) + f(\frac{1}{2}-x)\right] & \text{if } \; 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}\left[f(\frac{3}{2}-x) + f(x-\frac{1}{2})\right] & \text{if } \; \frac{1}{2} < x \leq 1. \end{cases} \tag{25}$$

Then $u \in C[0,1]$, and it can be shown that

$$R_{2k+1}^{\mathrm{mi}}[f] = R_{2k+1}^{\mathrm{tr}}[u] \quad (k \in \mathbb{N}).$$

Theorems 1 and 2 applied to u provide results on positivity, respectively, monotonicity of the sequence $(R_{2k+1}^{\mathrm{mi}}[f])_{k\in\mathbb{N}}$.

Simpson's Method

For the interval [0, 1] it takes the form

$$\int_0^1 f(x)\,dx = \frac{1}{6n}\left[f(0) + 4\sum_{\kappa=1}^{n} f\left(\frac{2\kappa-1}{2n}\right) + 2\sum_{\kappa=1}^{n-1} f\left(\frac{\kappa}{n}\right) + f(1)\right] + R_n^{\mathrm{Si}}[f].$$

We see that

$$R_n^{\mathrm{Si}}[f] = \frac{1}{3}R_n^{\mathrm{tr}}[f] + \frac{2}{3}R_n^{\mathrm{mi}}[f].$$

Thus, defining

$$v(x) := \frac{1}{3}f(x) + \frac{2}{3}u(x)$$

with u given by (25), we obtain $R^{\mathrm{Si}}_{2k+1}[f] = R^{\mathrm{tr}}_{2k+1}[v]$. Theorems 1 and 2 applied to v provide results on positivity and monotonicity of the sequence $(R^{\mathrm{Si}}_{2k+1}[f])_{k\in\mathbb{N}}$.

The Gauss–Chebyshev Method

In general, the ideas of this paper do not apply to Gaussian methods. An exception is the Gaussian formula with a Chebyshev weight:

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2n}\left[f(1) + 2\sum_{\kappa=1}^{n-1} f\left(\cos\frac{\kappa\pi}{n}\right) + f(1)\right] + R^{\mathrm{GC}}_n[f].$$

Introducing $\varphi(x) := f(\cos(\pi x))$, we find that $R^{\mathrm{GC}}_n[f] = \pi R^{\mathrm{tr}}_n[\varphi]$. Theorems 1–3 applied to φ provide results on positivity and monotonicity of the sequence $(R^{\mathrm{GC}}_n[f])_{n\in\mathbb{N}}$.

4 The Trapezoidal Method on the Whole Real Line

It may be introduced as

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sigma}\sum_{\kappa=-\infty}^{\infty} f\left(\frac{\kappa\pi}{\sigma}\right) + \mathcal{R}^{\mathrm{tr}}_{\sigma}[f] \quad (\sigma > 0). \tag{26}$$

Of course we need some assumptions on f for the integral to exist and for the series to converge. In [8, Definition 2], the authors specified the following class of functions.

Definition 3. Denote by $\mathscr{C}$ the class of functions $f : \mathbb{R} \to \mathbb{C}$ satisfying the following conditions:

(i) f is continuous;
(ii) $f(x) \to 0$ as $x \to \pm\infty$;
(iii) $\int_x^{2x} |f(\xi) - f(-\xi)|\,d\xi < \Lambda$ for some $\Lambda > 0$ and all $x > 0$;
(iv) for every (fixed) $\omega > 0$ the sequence of functions

$$F_N(\omega, x) := \omega \sum_{\substack{k=-N\\k\neq 0}}^{N} f\big(\omega(k+x)\big)$$

converges uniformly for $x \in [-\frac{1}{2}, \frac{1}{2}]$ to a function $F(\omega, x)$ as $N \to \infty$;
(v) $\int_{-1/2}^{1/2} |F(\omega, x)|\,dx$ is bounded for $\omega > 0$.

These conditions may look complicated, but they cover a wide class of functions and guarantee that (26) is applicable in some sense. Several more familiar classes of functions are subclasses of $\mathscr{C}$. For a justification of the following statements, see [8, pp. 327–328].

If f is continuous on $\mathbb{R}$ and $f(x) = \mathcal{O}(|x|^{-1}\log^{-2}|x|)$ as $x \to \pm\infty$, then $f \in \mathscr{C}$.

If $f \in L^1(\mathbb{R})$ is uniformly continuous and of bounded variation on $\mathbb{R}$, then $f \in \mathscr{C}$.

The function

$$f(x) := \begin{cases} \dfrac{\sin x}{x} & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 1 & \text{if } x = 0 \end{cases}$$

belongs to $\mathscr{C}$. Here the integral in (26) exists as an improper integral but not as a Lebesgue integral and the series converges but not absolutely.

The function $f(x) = (1 + ix)^{-1}$ belongs to $\mathscr{C}$. In this case the integral and the series in (26) exist as a Cauchy principal values only.

4.1 Some Lemmas

For the following two lemmas, see [8, Lemmas 4.1 and 4.2].

Lemma 5. *For $f \in \mathscr{C}$ the Fourier transform $\hat{f}$ exists as a Cauchy principal value, that is,*

$$\hat{f}(v) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(t)e^{-itv}\,dt,$$

and $\hat{f}$ is bounded on $\mathbb{R}$.

Lemma 6. *For $f \in \mathscr{C}$ the quadrature formula (26) is applicable with the integral and the series existing as a Cauchy principal value and*

$$\mathcal{R}_\sigma^{\rm tr}[f] = -\sqrt{2\pi} \lim_{\delta\to 0+} \sum_{n=1}^{\infty} \left[\hat{f}(2n\sigma) + \hat{f}(-2n\sigma)\right] e^{-2n\sigma\delta}. \tag{27}$$

Lemma 7. *Let $f \in \mathscr{C}$ and $\sigma > 0$. Then $\mathcal{R}_{n\sigma}^{\rm tr}[f] \to 0$ as $n \to \infty$.*

Proof. Let $\sigma > 0$ and $\varepsilon > 0$ be given. Since the integral of f over $\mathbb{R}$ exists as a Cauchy principal value and conditions (ii) and (iv) of Definition 3 are satisfied, we can find (by a little manipulation) an integer $m > 0$ such that for $a := m\pi/\sigma$ we have

$$\left| \int_a^\infty \left[f(x) + f(-x)\right] dx \right| \le \frac{\varepsilon}{3} \tag{28}$$

and simultaneously

$$\left|\frac{\pi}{\sigma}\sum_{\kappa=m}^{\infty}\left[f\left(\frac{\kappa\pi}{\sigma}+x\right)+f\left(-\frac{(\kappa+1)\pi}{\sigma}+x\right)\right]\right| \le \frac{\varepsilon}{3} \tag{29}$$

for all $x \in [0, \pi/\sigma]$. Here $\int_a^\infty$ and $\sum_{\kappa=m}^{\infty}$ have to be understood as $\lim_{A\to\infty}\int_a^A$ and $\lim_{M\to\infty}\sum_{\kappa=m}^{M}$, respectively.

Now we have

$$\begin{aligned}\mathcal{R}_{n\sigma}^{\mathrm{tr}}[f] = {} & \int_{-a}^{a} f(x)\,dx + \int_a^\infty [f(x)+f(-x)]\,dx \\ & - \frac{\pi}{n\sigma}\sum_{\kappa=-nm+1}^{nm} f\left(\frac{\kappa\pi}{n\sigma}\right) \\ & - \frac{\pi}{n\sigma}\sum_{\kappa=nm}^{\infty}\left[f\left(\frac{(\kappa+1)\pi}{n\sigma}\right)+f\left(-\frac{\kappa\pi}{n\sigma}\right)\right].\end{aligned} \tag{30}$$

Considering the series as $\lim_{M\to\infty}\sum_{\kappa=nm}^{M}$ and recalling (29), we find after a short reflection that (30) may be rewritten as

$$-\frac{1}{n}\sum_{j=1}^{n}\left\{\frac{\pi}{\sigma}\sum_{\kappa=m}^{\infty}\left[f\left(\frac{\kappa\pi}{\sigma}+\frac{j\pi}{n\sigma}\right)+f\left(-\frac{(\kappa+1)\pi}{\sigma}+\frac{j\pi}{n\sigma}\right)\right]\right\}.$$

By (29), this expression is of modulus less than $\varepsilon/3$. Taking also (28) into account, we see that

$$\left|\mathcal{R}_{n\sigma}^{\mathrm{tr}}[f]\right| \le \left|\int_{-a}^{a} f(x)\,dx - \frac{\pi}{n\sigma}\sum_{\kappa=-nm+1}^{nm} f\left(\frac{\kappa\pi}{n\sigma}\right)\right| + \frac{2\varepsilon}{3}.$$

The sum on the right-hand side is an approximation of $\int_{-a}^{a} f(x)dx$ by a Riemann sum of step size $\pi/(n\sigma)$. Since $[-a,a]$ is a compact interval on which f is continuous, the Riemann sum converges to the integral as $n \to \infty$. Thus $\left|\mathcal{R}_{n\sigma}^{\mathrm{tr}}[f]\right| < \varepsilon$ for sufficiently large n. This completes the proof. □

In the next two lemmas we assume that the Fourier transform of a function g exists as a Cauchy principal value and give conditions which assure that $\hat{g}(v)$ is non-negative, respectively, non-negative and nonincreasing for growing $|v|$.

Lemma 8. *Let g be a continuous, positive definite function whose Fourier transform exists as a Cauchy principal value. Then $\hat{g}(v) \ge 0$ for $v \in \mathbb{R}$.*

Proof. The conclusion is known to be true if in addition $g \in L^1(\mathbb{R})$; see [14, Theorem 3.2.2]. We shall therefore construct a sequence of positive definite functions $g_n \in L^1(\mathbb{R})$ whose Fourier transforms converge pointwise to $\hat{g}$.

For this, we consider the functions

$$h_n : x \longmapsto \begin{cases} 1 - \dfrac{|x|}{n} & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases} \qquad (n \in \mathbb{N}), \tag{31}$$

which are positive definite since they may be represented as [7, p. 516]

$$h_n(x) = \frac{n}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(nt/2)}{nt/2} \right)^2 e^{-itx}\, dt \qquad (n \in \mathbb{N}).$$

Hence, by Property (ii) of positive definite function, $g_n := h_n g$ is a continuous positive definite function of compact support. As such $g_n \in L^1(\mathbb{R})$, and so in view of the aforementioned result in [14]

$$\hat{g}_n(v) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \left(1 - \frac{|t|}{n}\right) g(t) e^{-itv}\, dt \ge 0 \tag{32}$$

for all $n \in \mathbb{N}$.

Now consider

$$\lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \frac{|t|}{n}\right)^{\alpha} g(t) e^{-itv}\, dt$$

for $\alpha \ge 0$. It is the (C,α)-limit of the Cauchy principal value of $(2\pi)^{-1/2} \int_{-\infty}^{\infty} g(t)e^{-itv} dt$. By hypothesis, it exists for $\alpha = 0$ and gives what we call $\hat{g}(v)$. Hence [10, Sect. 5.15] the (C,α)-limits exist for all $\alpha \ge 0$ and are equal to $\hat{g}(v)$. In particular, $\lim_{n\to\infty} \hat{g}_n(v) = \hat{g}(v)$, and so (32) implies that $\hat{g}(v)$ is non-negative. □

The subsequent lemma is related to a result of Khinchine [14, Theorem 4.5.1] considered in a Fourier transform setting. In the language of probability, the following function g, if normalized to $g(0) = 1$, is the characteristic function of a unimodal distribution.

Lemma 9. *Let γ be a continuous, positive definite function such that $\gamma(t) \to 0$ as $t \to \infty$. Define*

$$g(0) := \gamma(0), \quad g(t) := \frac{1}{t} \int_0^t \gamma(x)\, dx \quad (t \in \mathbb{R} \setminus \{0\}). \tag{33}$$

If the Fourier transform of g exists as a Cauchy principal value, then $\hat{g}$ is non-negative, nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$.

Proof. The hypotheses of the lemma imply that g is itself a continuous, positive definite function; this can be extracted from the proof in [14, pp. 93–94]. Thus, by Lemma 8, we have $\hat{g}(v) \geq 0$ for $v \in \mathbb{R}$.

The remaining part of the assertion needs more efforts. Since the $(C,0)$-limit and the $(C,1)$-limit of the Cauchy principal value of the Fourier transform of g produce the same value for $\hat{g}(v)$, as we have seen in the previous proof, we conclude that

$$\varphi_n(v) := \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \frac{|t|}{n} g(t) e^{-itv}\, dt \tag{34}$$

approaches zero pointwise as $n \to \infty$. The definition of g in (33) implies that g is continuously differentiable on $\mathbb{R} \setminus \{0\}$, $\gamma(t) = g(t) + tg'(t)$ for $t \neq 0$ and $tg'(t) \to 0$ as $t \to 0$; furthermore, $g(t) \to 0$ as $t \to \pm\infty$. Thus, writing (34) as

$$\varphi_n(v) = \frac{1}{\sqrt{2\pi}\, n} \int_0^n \Big(tg(t)e^{-itv} + tg(-t)e^{itv} \Big)\, dt,$$

we may perform integration by parts for $v \neq 0$ and obtain

$$\varphi_n(v) = \frac{i}{\sqrt{2\pi}\, v} \Bigg[g(n)e^{-inv} - g(-n)e^{inv} - \frac{1}{n} \int_0^n \Big(\big(g(t) + tg'(t)\big)e^{-itv} - \big(g(-t) - tg'(-t)\big)e^{itv} \Big)\, dt \Bigg].$$

This shows that

$$|\varphi_n(v)| \leq \frac{1}{\sqrt{2\pi}\, |v|} \left(|g(n)| + |g(-n)| + \frac{1}{n} \int_{-n}^{n} \big| g(t) + tg'(t) \big|\, dt \right). \tag{35}$$

By hypothesis and Property (i), $|g(t)|$ and $|g(t) + tg'(t)|$ approach zero as $t \to \pm\infty$. Hence given $\delta > 0$, there exists a $T > 0$ such that $|g(t) + tg'(t)| < \delta$ for $|t| \geq T$, and so for all $n \geq T$,

$$\frac{1}{n} \int_{-n}^{n} \big| g(t) + tg'(t) \big|\, dt \leq \frac{1}{n} \int_{-T}^{T} \big| g(t) + tg'(t) \big|\, dt + \frac{2(n-T)}{n}\, \delta.$$

This implies that

$$\lim_{n\to\infty} \frac{1}{n} \int_{-n}^{n} \big| g(t) + tg'(t) \big|\, dt = 0.$$

Therefore we can conclude from (35) that, given $v^* > 0$ and $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$|\varphi_n(v)| < \varepsilon \qquad \text{for } n \geq n_0,\ |v| \geq v^*. \tag{36}$$

In the next step, we turn to $\hat{g}_n$ as given by (32) and note that it is continuously differentiable with

$$\hat{g}_n'(v) \;=\; -\frac{i}{\sqrt{2\pi}} \int_{-n}^{n} \left(1-\frac{|t|}{n}\right) t g(t) e^{-itv}\, dt.$$

The integrand is also continuously differentiable, and so we may perform an integration by parts to obtain

$$\begin{aligned} v\hat{g}_n'(v) &= -\frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \left(\left(1-\frac{|t|}{n}\right)\big(g(t)+tg'(t)\big) - \frac{|t|}{n} g(t)\right) e^{-itv}\, dt \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(t)\big(g(t)+tg'(t)\big)e^{-itv}\, dt \;+\; \varphi_n(v), \end{aligned}$$

with h_n given by (31). The last integral is the Fourier transform of a continuous positive definite function of compact support. By [14, Theorem 3.2.2], it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(t)\big(g(t)+tg'(t)\big)e^{-itv}\, dt \;\geq\; 0 \quad (v \in \mathbb{R}).$$

Consequently, taking into account (36), we see that, given $v^* > 0$ and $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$v\hat{g}_n'(v) \;<\; \varepsilon \quad \text{for } n \geq n_0,\ |v| \geq v^*. \tag{37}$$

We are now ready for showing that $\hat{g}$ is nonincreasing on $(0,\infty)$. Assume to the contrary that there exist points $0 < v_1 < v_2$ at which $\hat{g}(v_1) < \hat{g}(v_2)$. Set

$$\Delta := \hat{g}(v_2) - \hat{g}(v_1), \quad v^* := v_1, \quad \varepsilon := \frac{\Delta}{2}\,\frac{v_1}{v_2 - v_1}$$

and let n_0 be such that (37) holds. As we know from the proof of Lemma 8, $\hat{g}_n(v) \to \hat{g}(v)$ as $n \to \infty$, and so there exists an $n_1 \in \mathbb{N}$ such that

$$\hat{g}_n(v_2) > \hat{g}(v_2) - \frac{\Delta}{4}, \quad \hat{g}_n(v_1) < \hat{g}(v_1) + \frac{\Delta}{4} \quad \text{for } n \geq n_1. \tag{38}$$

Thus, if $n \geq \max\{n_0, n_1\}$, we obtain from (38) and the mean value theorem that

$$0 \;<\; \frac{\Delta}{2} \;<\; \hat{g}_n(v_2) - \hat{g}_n(v_1) \;=\; \hat{g}_n'(\tilde{v})\,(v_2 - v_1),$$

where $\tilde{v} \in [v_1, v_2]$. This implies

$$\tilde{v}\,\hat{g}_n'(\tilde{v}) \;>\; \frac{\Delta}{2}\,\frac{\tilde{v}}{v_2 - v_1} \;\geq\; \frac{\Delta}{2}\,\frac{v_1}{v_2 - v_1} = \varepsilon,$$

contradicting (37). Hence $\hat{g}$ is nonincreasing on $(0,\infty)$.

The proof is completed by an analogous consideration on $(-\infty, 0)$ or by replacing g by $\overline{g}$ and noting that $\hat{\overline{g}}(v) = \overline{\hat{g}}(-v)$. □

For the subsequent applications of Lemmas 8 and 9, the following observations will be useful, which correspond to Remark 2.

Remark 4. For real-valued $f \in \mathscr{C}$, define g by

$$g(x) := -\frac{1}{2}\big[f(x) + f(-x)\big], \quad x \in \mathbb{R}. \tag{39}$$

Then $\mathcal{R}_\sigma^{\mathrm{tr}}[f] = -\mathcal{R}_\sigma^{\mathrm{tr}}[g]$. In terms of g, formula (27)
takes the form

$$\mathcal{R}_\sigma^{\mathrm{tr}}[f] = 2\sqrt{2\pi} \lim_{\delta \to 0+} \sum_{n=1}^{\infty} \hat{g}(2n\sigma) e^{-2n\sigma\delta}. \tag{40}$$

4.2 Positivity

The following theorem corresponds to Theorem 1. With Remark 4 in mind, it is easily proved by employing Lemmas 5–8.

Theorem 5. *Let $f \in \mathscr{C}$ be real-valued and suppose that g, defined in* (39)*, is positive definite. Then the remainders $\mathcal{R}_\sigma^{\mathrm{tr}}[f]$ are non-negative. Moreover,*

$$\mathcal{R}_\sigma^{\mathrm{tr}}[f] \geq \mathcal{R}_{2\sigma}^{\mathrm{tr}}[f] \geq \cdots \geq \mathcal{R}_{2^k\sigma}^{\mathrm{tr}}[f] \geq \cdots \geq 0 \tag{41}$$

for each $\sigma \in (0, \infty)$ and $\mathcal{R}_{2^k\sigma}^{\mathrm{tr}}[f] \to 0$ as $k \to \infty$.

A simple sufficient condition for g to be positive definite is due to Pólya [14, Theorem 4.3.1]. In our setting it may be stated as follows.

Proposition 6. *Let $f \in \mathscr{C}$ be real-valued and suppose that g, defined in* (39)*, is convex on* $(0, \infty)$*. Then g is positive definite.*

Examples. Proposition 6 applies to $-e^{-|x|}$ but it does not apply to $f(x) := -e^{-|x|}\cos x$. However, Theorem 5 applies to f since $g = -f$ and g is positive definite as it is a product of two positive definite functions.

4.3 Monotonicity

The following theorem corresponds to Theorem 3.

Theorem 6. *Let $f \in \mathscr{C}$ be real-valued and continuously differentiable on $\mathbb{R} \setminus \{0\}$ with finite one-sided derivatives at* 0*. Suppose that $tf'(t) \to 0$ as $t \to \pm\infty$. Define $\gamma : \mathbb{R} \to \mathbb{R}$ by*

$$\gamma(0) := g(0), \quad \gamma(t) := g(t) + tg'(t) \quad (t \in \mathbb{R} \setminus \{0\})$$

with g given by (39)*. If γ is positive definite, then*

$$\mathcal{R}^{\mathrm{tr}}_{\sigma_1}[f] \geq \mathcal{R}^{\mathrm{tr}}_{\sigma_2}[f] \geq 0 \quad \text{for } \sigma_2 > \sigma_1 > 0 \;\; (\sigma_1, \sigma_2 \in \mathbb{R}) \tag{42}$$

and $\mathcal{R}^{\mathrm{tr}}_{\sigma}[f] \to 0$ as $\sigma \to \infty$.

Proof. Noting that g may be represented as in (33), we find in view of Lemma 5 that the hypotheses of Lemma 9 are satisfied. The conclusion of this lemma, used in connection with Lemma 6 and formula (40), easily shows that (42) holds. The convergence to 0 follows from (42) and the convergence to 0 of the sequence $(\mathcal{R}^{\mathrm{tr}}_{n\sigma}[f])_{n\in\mathbb{N}}$ known from Lemma 7. ☐

A simple sufficient condition for f to satisfy the hypotheses of Theorem 6 can be obtained with the help of a result of Askey [3, Theorem 1].

Proposition 7. *Let $f \in \mathscr{C}$ be real-valued and continuously differentiable on $\mathbb{R} \setminus \{0\}$ with finite one-sided derivatives at* 0*. Suppose that $f(0) < 0$ and $f'(x) - f'(-x)$ is convex for $x > 0$. Then f satisfies the hypotheses of Theorem 6.*

Examples. Let $f(x) := -e^{-|x|}$. Then Theorem 6 and Proposition 7 are applicable and yield that (42) holds. The proposition is a little bit more convenient.

Next let

$$f(0) := -1, \quad f(x) := -\frac{\sin x}{x}\, e^{-|x|} \quad (x \in \mathbb{R} \setminus \{0\}).$$

Here Proposition 7 certainly fails since f oscillates an infinite number of times. But Theorem 6 applies. We find that

$$\gamma(t) := e^{-|t|}\big(\cos t - \sin |t|\big).$$

By expressing γ with the help of Euler's formulae as a sum of exponential functions, we can calculate its Fourier transform and obtain

$$\hat{\gamma}(v) = \frac{1}{\sqrt{2\pi}}\, \frac{4v^2}{4 + v^4}\,.$$

This is a continuous, non-negative function that belongs to $L^1(\mathbb{R})$. Therefore the Fourier inversion formula applies and shows in view of Bochner's theorem that γ is positive definite. Hence (42) holds.

4.4 Interconnections with the Midpoint Method

On $\mathbb{R}$, the midpoint method may be introduced as

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sigma}\sum_{\kappa=-\infty}^{\infty} f\left(\frac{(2\kappa+1)\pi}{2\sigma}\right) + \mathcal{R}_\sigma^{\mathrm{mi}}[f] \quad (\sigma>0).$$

Since the integral is invariant under a translation of the argument of f, we see that the midpoint method is obtained by applying the trapezoidal method to the function $f(\cdot + \pi/(2\sigma))$. Its Fourier transform is $\hat{f}(v)e^{iv\pi/(2\sigma)}$. Therefore (27) yields

$$\begin{aligned}\mathcal{R}_\sigma^{\mathrm{mi}}[f] &= -\sqrt{2\pi}\lim_{\delta\to 0+}\sum_{n=1}^{\infty}(-1)^n\left[\hat{f}(2n\sigma)+\hat{f}(-2n\sigma)\right]e^{-2n\sigma\delta}\\ &= 2\sqrt{2\pi}\lim_{\delta\to 0+}\sum_{n=1}^{\infty}(-1)^n\,\hat{g}(2n\sigma)\,e^{-2n\sigma\delta}\end{aligned}$$

with g given by (39).

Analogously to Propositions 4 and 5, the following statements are easily verified.

Proposition 8. *Under the hypotheses of Theorem 5, there holds*

$$\left|\mathcal{R}_\sigma^{\mathrm{mi}}[f]\right| \le \mathcal{R}_\sigma^{\mathrm{tr}}[f] \quad (\sigma>0).$$

Proposition 9. *Under the hypotheses of Theorem 6, there hold*

$$-\mathcal{R}_\sigma^{\mathrm{tr}}[f] \le \mathcal{R}_\sigma^{\mathrm{mi}}[f] \le 0 \quad (\sigma>0) \tag{43}$$

and

$$\left|\mathcal{R}_{\sigma_1}^{\mathrm{mi}}[f]-\mathcal{R}_{\sigma_2}^{\mathrm{mi}}[f]\right| \le \mathcal{R}_{\sigma_1}^{\mathrm{tr}}[f]-\mathcal{R}_{\sigma_2}^{\mathrm{tr}}[f] \quad (0<\sigma_1<\sigma_2).$$

Inequalities (43) imply that

$$\mathcal{Q}_\sigma^{\mathrm{tr}}[f] \le \int_{-\infty}^{\infty} f(x)\,dx \le \mathcal{Q}_\sigma^{\mathrm{mi}}[f],$$

where the quantities on the left-hand side and the right-hand side denote the approximations obtained by the trapezoidal and the midpoint method, respectively.

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A Unified and General Framework for Enriching Finite Element Approximations

Allal Guessab and Yassine Zaim

Abstract In this chapter our goal is to develop a unified and general framework for enriching finite element approximations via the use of additional enrichment functions. A crucial point in such an approach is to determine conditions on enrichment functions which guarantee that they generate a well-defined finite element. We start by giving under some conditions an abstract general theorem characterizing the existence of any enriched finite element approximation. After proving four key lemmas, we then establish under a unisolvence condition a more practical characterization result. We show that this proposed method easily allows us to establish a new class of enriched non-conforming finite elements in any dimension. This new family is inspired by the Han rectangular element and the nonconforming rotated element of Rannacher and Turek. They are all obtained as applications of a new family of multivariate trapezoidal, midpoint, and Simpson type cubature formulas, which employ integrals over facets. In addition, we provide analogously a general class of perturbed trapezoidal and midpoint cubature formulas, and use them to build a new enriched nonconforming finite element of Wilson-type.

Keywords Enriched finite element approximations • Nonconforming finite element • Trapezoidal and midpoint cubature formulas • Wilson-type nonconforming finite elements

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1 Introduction and Motivation

Let us start by recalling some conventions and definitions. A finite element is a triplet $(K, \Sigma_K^{\text{org}}, \mathscr{F}_K^{\text{org}})$, where

1. K is a polytope in $\mathbb{R}^d$ (e.g., a polygon in $\mathbb{R}^2$ or a polyhedron in $\mathbb{R}^3$);
2. $\mathscr{F}_K^{\text{org}}$ is a finite-dimensional space of real-valued functions defined over the set K, we let n denote the dimension of $\mathscr{F}_K^{\text{org}}$;
3. $\Sigma_K^{\text{org}} = \{L_i^{\text{org}}, i = 1, \ldots, n\}$ a unisolvent, linearly independent set of linear functionals on $\mathscr{F}_K^{\text{org}}$.

This means that for each vector $\mathbf{a} := (a_1, \ldots, a_n) \in \mathbb{R}^n$, there exists one and only one element in $\mathscr{F}_K^{\text{org}}$ such that a_i is the image of the i-th functional $L_i^{\text{org}}, i = 1, \ldots, n$. Functionals L_i^{org} are known as degrees of freedom of the finite element, while the basis functions are the functions $f_j \in \mathscr{F}_K^{\text{org}}, j = 1, \ldots, n$, such that

$$L_i^{\text{org}}(f_j) = \delta_{ij}, i = 1, \ldots, n \; , \tag{1}$$

where δ_{ij} stands for the Kronecker delta symbol. In engineering literature basis functions are called shape functions. It can easily be verified that the basis functions $f_i, i = 1, \ldots, n$ spans the finite element space $\mathscr{F}_K^{\text{org}}$, that is

$$\mathscr{F}_K^{\text{org}} = \text{span}\,\{f_i, i = 1, \ldots, n\} \, . \tag{2}$$

Also note that the triplet $(K, \Sigma_K^{\text{org}}, \mathscr{F}_K^{\text{org}})$ is a finite element if and only if there exists a basis $\{q_i^{\text{org}}, i = 1, \ldots, n\}$ for $\mathscr{F}_K^{\text{org}}$ such that the $n \times n$ matrix with entries $L_j^{\text{org}}(q_i^{\text{org}})$ is nonsingular. Recall that in the context of standard local finite element the space $\mathscr{F}_K^{\text{org}}$ usually consists of polynomial functions. However, it already observed that the approximations used of classical finite element method are not effective for treating problems with singular or oscillatory solutions. In order to overcome this issue, various approaches have been proposed in literature. A natural way to improve its effectiveness is to enrich the approximation space $\mathscr{F}_K^{\text{org}}$ by adding more appropriate additional functions (not necessary polynomials), we refer, e.g., to [8, 9, 15, 16]. Therefore, here we would like to enrich the space $\mathscr{F}_K^{\text{org}}$ with a given set of new linearly independent functions $e_1^{\text{enr}}, \ldots, e_{n^{\text{enr}}}^{\text{enr}}$ belonging to a subset of $C(K)$, as follows:

$$\mathscr{F}_K^{\text{enr}} := \mathscr{F}_K^{\text{org}} \oplus \left\{e_1^{\text{enr}}, \ldots, e_{n^{\text{enr}}}^{\text{enr}}\right\} . \tag{3}$$

Here $C(K)$ denotes the space of continuous real-valued functions on K. We assume throughout that the dimension of the enriched approximation space $\mathscr{F}_K^{\text{enr}}$ is $n + n^{\text{enr}}$.

In this chapter these latter are referred to as “enrichment functions”, since their incorporation in the space $\mathscr{F}_K^{\text{org}}$ can provide better approximation than standard polynomial basis functions, used in the classical finite element. The enrichment functions that appear in (3) are usually chosen through the partition-of-unity concept, see, e.g., [1–3, 5–7, 13, 19, 21]. In general, the choice of these functions depends on the geometry, the boundary conditions, and the equation being solved.

Let us assume that we are given a set of enrichment functions

$$\left\{e_1^{\mathrm{enr}}, \ldots, e_{n^{\mathrm{enr}}}^{\mathrm{enr}}\right\}, \tag{4}$$

and a set of distinct degrees of freedom

$$\Sigma_K^{\mathrm{enr}} := \{L_i, i = 1, \ldots, n + n^{\mathrm{enr}}\}, \tag{5}$$

such that the original approximation space $\mathscr{F}_K^{\mathrm{org}}$ is Σ_K^{enr}-unisolvent, then one of the main challenges in such enriched approximation method is:

> How to suitably choose them, in such a way that the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ also generates a well-defined finite element?

Note that in our consideration the enriched degrees of freedom Σ_K^{enr} could not include the original ones Σ_K^{org}. Our main focus here is the development of a concept for local enrichment of *any* conforming or nonconforming finite element. This chapter is a continuation of the recent work initiated in an earlier paper [2], where such enriched finite element approximations were considered for the first time (for standard *linear* elements).

Hence, it is the purpose of this chapter to answer the above question. As we will see below, the results obtained by this method provide a systematic way of an enrichment strategy of *any* given original space $\mathscr{F}_K^{\mathrm{org}}$. The organization of the chapter is as follows: Sect. 2 establishes an abstract general theorem given necessary and sufficient conditions to guarantee that the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ is a well-defined finite element. Furthermore, after proving some key properties, we then give under a unisolvence condition a more practical characterization result. For the purpose of illustration, in Sect. 3, we apply our proposed method, in a practical situation, to obtain new enriched nonconforming finite elements in any dimension. This new family is inspired by the Han rectangular element [17] and the nonconforming rotated element of Rannacher and Turek [23]. Our significant results, derived in Theorems 3, 5 and 7, show that the approximation errors of a new family of multivariate trapezoidal, midpoint, and Simpson cubature formulas play a central role in the existence of our new elements. These latter natural generalize the one-dimensional midpoint, trapezoidal, and Simpson's rules, and employ integrals over facets. In addition, we provide analogously a general class of perturbed trapezoidal and midpoint cubature formulas, and use them to build a new enriched nonconforming finite element of Wilson-type, see, e.g., [12] and [25]. Finally, in Sect. 4 we summarize our main results and state some open problems.

Finding suitable enrichment functions is crucial for the success of the enriched finite-element approximation, see, e.g., [18, Sect. 2.3]. All our new practical elements use enrichment functions which are based on a single univariate function that depends on additional free parameters. Hence, the proposed general class may offer more appropriate and adapted ones to solve particular problems.

2 Existence and Characterization of Enriched Finite Element Approximations

We start by establishing an abstract general theorem given necessary and sufficient conditions to ensure that the enriched triplet $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ is actually a well-defined finite element. Here, $\mathscr{F}_K^{\text{enr}}$ is the enriched approximation space (3) and Σ_K^{enr} is the set of enriched degrees of freedom (5). Our results extend those of Achchab et al. [2] obtained in the case when the original space $\mathscr{F}_K^{\text{org}}$ is the space of affine functions. In particular, many of our proofs in this section are adapted from [2]. However, let us mention that we do not use here the concept of center of gravity with respect to a functional. This notion played an important role in this earlier paper.

Theorem 1. *Given a linearly independent set of vectors*

$$\left\{\mathbf{c}^i := (c_{i,1}, \ldots, c_{i,n+n^{\text{enr}}})^\top \in \mathbb{R}^{n+n^{\text{enr}}}, \qquad i = 1, \ldots, n^{\text{enr}}\right\}. \tag{6}$$

Define the set of distinct linear functionals on $\mathscr{F}_K^{\text{enr}}$

$$\Xi_K^{\text{enr}} := \left\{\mathscr{L}_i := \sum_{j=1}^{n+n^{\text{enr}}} c_{ij} L_j, i = 1, \ldots, n^{\text{enr}}\right\}, \tag{7}$$

and assume that for each i, $\mathscr{L}_i$ *vanishes on* $\mathscr{F}_K^{org}$. *Let* $\mathscr{F}_K^{\text{enr}}$, Σ_K^{enr} *be, respectively, defined as in (3) and (5), such that* $\mathscr{F}_K^{org}$ *is* Σ_K^{enr}*-unisolvent. Then, the following statements are logically equivalent:*

(i) $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ *is a finite element,*
(ii) $(K, \Xi_K^{\text{enr}}, \mathscr{E}_K^{\text{enr}})$ *is a finite element,*

with

$$\mathscr{E}_K^{\text{enr}} = \text{span}\left\{e_1^{\text{enr}}, \ldots, e_{n^{\text{enr}}}^{\text{enr}}\right\}.$$

Proof. Recall first that in our consideration we have tacitly assumed:

$$\dim(\mathscr{F}_K^{\text{enr}}) = \mathbf{card}(\Sigma_K^{\text{enr}}) = n + n^{\text{enr}},$$

$$\dim(\mathscr{E}_K^{\text{enr}}) = \mathbf{card}(\Xi_K^{\text{enr}}) = n^{\text{enr}},$$

where the cardinality of a set A, written as $\mathbf{card}(A)$, is the number of elements in A. In order to prove the sufficient condition, let us assume that we have a function $f \in \mathscr{F}_K^{\text{enr}}$ such that

$$L_j(f) = 0 \qquad (j = 1, \ldots, n + n^{\text{enr}}). \tag{8}$$

Since each $f \in \mathscr{F}_K^{\text{enr}}$ may be decomposed into the standard part $p \in \mathscr{F}_K^{\text{org}}$ and into an enriched part $e \in \mathscr{E}_K^{\text{enr}}$, as:

$$f = p + e,$$

and the $\mathscr{L}_i$'s belong to $\text{span}(\Sigma_K^{\text{enr}})$, it follows from (8) that for each $i = 1 \ldots, n^{\text{enr}}$ we have

$$\begin{aligned} 0 &= \mathscr{L}_i(f) \\ &= \mathscr{L}_i(p) + \mathscr{L}_i(e) \\ &= \mathscr{L}_i(e). \end{aligned}$$

To obtain the last inequality, we have used the fact that functional $\mathscr{L}_i$ vanishes on $\mathscr{F}_K^{\text{org}}$. Therefore, since $(K, \Xi_K^{\text{enr}}, \mathscr{E}_K^{\text{enr}})$ is a finite element, we have that $e = 0$. Hence $f = p$ is a function in $\mathscr{F}_K^{\text{org}}$. But then the unisolvence of $\mathscr{F}_K^{\text{org}}$ with respect to Σ_K^{enr} implies that $f = 0$. Hence we have shown that $\mathscr{F}_K^{\text{enr}}$ is Σ_K^{enr} unisolvent, and so $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ is a finite element.

In order to establish the necessary condition, let us assume the contrary, that is, the triplet $(K, \Xi_K^{\text{enr}}, \mathscr{E}_K^{\text{enr}})$ is not a finite element. This means that there exists a function $e \in \mathscr{E}_K^{\text{enr}}$, such that e is not identically zero on K and

$$\mathscr{L}_i(e) = 0, i = 1, \ldots, n^{\text{enr}}.$$

Since $e = \sum_{i=1}^{n^{\text{enr}}} \lambda_i e_i^{\text{enr}}$, where $\lambda_1, \ldots, \lambda_{n^{\text{enr}}} \in \mathbb{R}$, then the above equations can be represented in the matrix form as:

$$\begin{pmatrix} \mathscr{L}_1(e_1^{\text{enr}}) & \ldots & \mathscr{L}_1(e_{n^{\text{enr}}}^{\text{enr}}) \\ \vdots & & \vdots \\ \mathscr{L}_{n^{\text{enr}}}(e_1^{\text{enr}}) & \ldots & \mathscr{L}_{n^{\text{enr}}}(e_{n^{\text{enr}}}^{\text{enr}}) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{n^{\text{enr}}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{9}$$

in which the matrix on the left-hand side has zero determinant. This consequently implies that there exist numbers $\lambda_1, \ldots, \lambda_{n^{\text{enr}}} \in \mathbb{R}$, not all zero, for which

$$\mathscr{L} := \sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathscr{L}_i$$

vanishes at all enrichment functions $e_1^{\text{enr}}, \ldots, e_{n^{\text{enr}}}^{\text{enr}}$. Since $\mathscr{L}$ is linear then it vanishes on $\mathscr{E}_K^{\text{enr}}$. Now, using the fact that for each i, $\mathscr{L}_i$ vanishes also on $\mathscr{F}_K^{\text{org}}$, we can deduce that $\mathscr{L}$ vanishes on the whole space $\mathscr{F}_K^{\text{enr}}$. Expressing $\mathscr{L}_i$ in terms of $L_1, \ldots, L_{n+n^{\text{enr}}}$ and interchanging the order of summation, we then have for all $f \in \mathscr{F}_K^{\text{enr}}$

$$\begin{aligned}\mathscr{L}(f) &= \sum_{i=1}^{n^{\text{enr}}} \lambda_i \sum_{j=1}^{n+n^{\text{enr}}} c_{i,j} L_j(f) \\ &= \sum_{j=1}^{n+n^{\text{enr}}} \left(\sum_{i=1}^{n^{\text{enr}}} c_{i,j} \lambda_i \right) L_j(f) \\ &= 0.\end{aligned}$$

Now, since $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ is a finite element then the linear independence of $L_1, \ldots, L_{n+n^{\text{enr}}}$ implies that

$$\sum_{i=1}^{n^{\text{enr}}} c_{i,j} \lambda_i = 0, \qquad (j = 1, \ldots, n + n^{\text{enr}}), \tag{10}$$

or equivalently:

$$\sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathbf{c}^i = 0. \tag{11}$$

But, taking into account that $\mathbf{c}^i, i = 1, \ldots, n^{\text{enr}}$, are assumed linearly independent vectors in $\mathbb{R}^{n+n^{\text{enr}}}$, then the coefficients $\lambda_1, \ldots, \lambda_{n^{\text{enr}}}$ must all be zero. This is a contradiction to our assumption. This completes the proof of the characterization Theorem. □

It is noteworthy that this proof is clearly not constructive, since it gives no indication of a way to choose the needed vectors $\mathbf{c}^i$. Hence, the numerical success of the above Theorem 1 will depend on the answers to the following two questions:

1. Under what conditions on Σ_K^{enr} does the required vectors of the form given in (6) exist?
2. How to find explicit expressions of such vectors?

Before answering these questions, we first need to introduce some additional notation and terminology. Since we have assumed that $\mathscr{F}_K^{\text{org}}$ has dimension n, then it has a basis

$$\left\{q_i^{\mathrm{org}}, i = 1, \ldots, n\right\},$$

whose elements belong to $\mathscr{F}_K^{\mathrm{org}}$. Define the set of vectors

$$\mathbf{w}^i := (L_i(q_1^{\mathrm{org}}), \ldots, L_i(q_n^{\mathrm{org}}))^\top \in \mathbb{R}^n,\ (i = 1, \ldots, n + n^{\mathrm{enr}}), \tag{12}$$

where $L_i, i = 1, \ldots, n + n^{\mathrm{enr}}$, is any linear functional from Σ_K^{enr}. Let us denote by $\mathbf{A}$ the matrix defined by

$$\mathbf{A} := \left(\mathbf{w}^1, \ldots, \mathbf{w}^{n+n^{\mathrm{enr}}}\right) \in \mathbb{R}^{n \times (n+n^{\mathrm{enr}})}, \tag{13}$$

whose columns are the vectors defined in (12). This matrix will play an important role in determining the required vectors $\mathbf{c}^i$ in the characterization Theorem 1. Throughout the chapter, the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ with appropriate size.

In order to state and prove a more practical characterization result, we need to establish under certain conditions four key lemmas on the enriched degrees of freedom Σ_K^{enr}.

The following important but simple step helps us to characterize all linear functionals

$$\mathscr{L} \in \mathrm{span}\left\{L_i,\ i = 1, \ldots, n + n^{\mathrm{enr}}\right\},$$

which vanish on the original space $\mathscr{F}_K^{\mathrm{org}}$. This result is extremely useful, since it will be used to find explicitly the required vectors $\mathbf{c}^i$ in the characterization Theorem 1.

Lemma 1. *Given a vector* $\mathbf{c} = (c_1, \ldots, c_{n+n^{\mathrm{enr}}})^\top \in \mathbb{R}^{n+n^{\mathrm{enr}}}$, *and define the functional* $\mathscr{L}$ *by*

$$\mathscr{L} := \sum_{i=1}^{n+n^{\mathrm{enr}}} c_i L_i. \tag{14}$$

Then the following statements are equivalent:

(i) $\mathscr{L}$ *vanishes on* $\mathscr{F}_K^{org}$.
(ii) *The vector* $\mathbf{c}$ *belongs to* $\ker(\mathbf{A})$, *where* $\mathbf{A}$ *is the* $n \times (n + n^{\mathrm{enr}})$ *matrix defined by* (13) *and*

$$\ker(\mathbf{A}) := \left\{\mathbf{v} \in \mathbb{R}^{n+n^{\mathrm{enr}}},\ \mathbf{A}\mathbf{v} = \mathbf{0}\right\}.$$

Proof. We begin by showing that (i) implies (ii). Assume that $\mathscr{L}(q) = 0$ for all $q \in \mathscr{F}_K^{\mathrm{org}}$. For each $\mathbf{v} \in \mathbb{R}^n$, let us define the function $q : \mathbb{R}^d \to \mathbb{R}$ by

$$q(\mathbf{x}) := \langle \mathbf{v}, \boldsymbol{\phi}(\mathbf{x}) \rangle, \tag{15}$$

with $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{R}^n$ having as component functions q_k^{org}, $k = 1, \ldots, n$. It is not difficult to observe that for every $\mathbf{v} \in \mathbb{R}^n$, q belongs to $\mathscr{F}_K^{\text{org}}$. Since, functional $\mathscr{L}$ is linear, then it follows that

$$\begin{aligned} \mathscr{L}(q) &= \langle \mathbf{v}, \mathscr{L}(\boldsymbol{\phi}) \rangle \\ &= \left\langle \mathbf{v}, \sum_{i=1}^{n+n^{\text{enr}}} c_i L_i(\boldsymbol{\phi}) \right\rangle \\ &= \left\langle \mathbf{v}, \sum_{i=1}^{n+n^{\text{enr}}} c_i \mathbf{w}^i \right\rangle \\ &= \langle \mathbf{v}, \mathbf{A}\mathbf{c} \rangle . \end{aligned} \tag{16}$$

Hence we get for each $\mathbf{v} \in \mathbb{R}^n$

$$\langle \mathbf{v}, \mathbf{A}\mathbf{c} \rangle = 0.$$

According to the fact that the above equality holds for all $\mathbf{v} \in \mathbb{R}^n$, we conclude that $\mathbf{A}\mathbf{c} = \mathbf{0}$. Hence, $\mathbf{c}$ must belong to the kernel of $\mathbf{A}$. This verifies (ii).

We show that (ii) implies (i). Let us assume now that the vector $\mathbf{c}$ belongs to $\ker(\mathbf{A})$. But, if q belongs to $\mathscr{F}_K^{\text{org}}$, then by (16), $\mathscr{L}(q)$ may be represented as

$$\mathscr{L}(q) = \langle \mathbf{v}, \mathbf{A}\mathbf{c} \rangle ,$$

with some vector $\mathbf{v} \in \mathbb{R}^n$. As the vector $\mathbf{c}$ belongs to $\ker(\mathbf{A})$, this implies that $\mathscr{L}$ vanishes on $\mathscr{F}_K^{\text{org}}$ and completes the proof. □

The second Lemma characterizes the linear independence of a given set of linear functionals:

$$\mathscr{L}_i \in \text{span}\left\{L_j,\ j = 1, \ldots, n + n^{\text{enr}}\right\}, \qquad (i = 1, \ldots, n^{\text{enr}}).$$

Lemma 2. *Given n^{enr} vectors*

$$\mathbf{c}^i = (c_{i1}, \ldots, c_{i(n+n^{\text{enr}})})^\top \in \mathbb{R}^{n+n^{\text{enr}}}, \qquad (i = 1, \ldots, n^{\text{enr}})$$

and define the n^{enr} functionals

$$\mathscr{L}_i := \sum_{j=1}^{n+n^{\text{enr}}} c_{ij} L_j \qquad (i = 1, \ldots, n^{\text{enr}}). \tag{17}$$

Assume that $L_1, \ldots, L_{n+n^{\text{enr}}}$ are linearly independent on some subspace of $C(K)$. Then the following statements are equivalent:

(i) *The* $\mathbf{c}^i$*'s are linearly independent.*
(ii) *The* $\mathscr{L}_i$*'s are linearly independent.*

Proof. Let the vectors $\mathbf{c}^i, i = 1, \ldots, n^{\text{enr}}$ be linearly independent and let us assume that $\sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathscr{L}_i = 0$ for some real numbers $\lambda_1, \ldots, \lambda_{n^{\text{enr}}}$. Expressing $\mathscr{L}_i$ in terms of $L_1, \ldots, L_{n+n^{\text{enr}}}$ and interchanging the order of summation, we then have

$$\sum_{i=1}^{n^{\text{enr}}} \lambda_i \sum_{j=1}^{n+n^{\text{enr}}} c_{i,j} L_j = \sum_{j=1}^{n+n^{\text{enr}}} \left(\sum_{i=1}^{n^{\text{enr}}} c_{i,j} \lambda_i \right) L_j = 0\,.$$

Since $L_1, \ldots, L_{n+n^{\text{enr}}}$ are assumed to be linear independent, then it follows

$$\sum_{i=1}^{n^{\text{enr}}} c_{i,j} \lambda_i = 0, \qquad (j = 1, \ldots, n + n^{\text{enr}}). \tag{18}$$

Equivalently we therefore have:

$$\sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathbf{c}^i = 0. \tag{19}$$

But the $\mathbf{c}^i$'s are linearly independent in $\mathbb{R}^{n+n^{\text{enr}}}$, then the λ_i's must all be zero. This shows that the $\mathscr{L}_i$'s must be linearly independent. This verifies that (i) implies (ii).

Now, let the functionals $\mathscr{L}_i, i = 1, \ldots, n^{\text{enr}}$ be linearly independent and let us assume that $\sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathbf{c}^i = 0$ for some real numbers $\lambda_1, \ldots, \lambda_{n^{\text{enr}}}$. This obviously implies that

$$\sum_{i=1}^{n^{\text{enr}}} \lambda_i c_{i,j} = 0 \qquad (j = 1, \ldots, n + n^{\text{enr}}).$$

Thus multiplying by L_j and summing over j and interchanging the order of summation, we then have

$$\begin{aligned}
\sum_{j=1}^{n+n^{\text{enr}}} \left(\sum_{i=1}^{n^{\text{enr}}} c_{i,j} \lambda_i \right) L_j &= \sum_{i=1}^{n^{\text{enr}}} \lambda_i \sum_{j=1}^{n+n^{\text{enr}}} c_{i,j} L_j \\
&= \sum_{i=1}^{n^{\text{enr}}} \lambda_i \mathscr{L}_i \\
&= 0.
\end{aligned}$$

Hence, the linear independence of the $\mathscr{L}_i$'s implies that the λ_i's must all be zero. This verifies that (ii) implies (i) and completes the proof of the Lemma. □

Remark 1. We note in passing that from the above proof, it is clear that (ii) implies (i) continue to hold without the assumption that the enriched degrees of freedom $L_1, \ldots, L_{n+n^{\rm enr}}$ are assumed to be linearly independent.

Lemma 1 tells us that we should seek the required vectors $\mathbf{c}^i$ in the characterization Theorem 1, among those belonging to the kernel of the matrix $\mathbf{A}$.

For the third step, we prove the following Lemma, which will be the key to the characterization of the required $\mathscr{F}_K^{\rm org}$−unisolvence property needed in Theorem 1.

Lemma 3. *The following statements are equivalent:*

(i) *The set of linear functionals* $\Sigma_K^{\rm enr}$ *is* $\mathscr{F}_K^{org}$*-unisolvent.*
(ii) *The vectors* $\mathbf{w}^i, i = 1, \ldots, n + n^{\rm enr}$, *satisfy*

$$\text{span}\left\{\mathbf{w}^1, \ldots, \mathbf{w}^{n+n^{\rm enr}}\right\} = \mathbb{R}^n. \tag{20}$$

Proof. Let us denote $U := \text{span}\left\{\mathbf{w}^1, \ldots, \mathbf{w}^{n+n^{\rm enr}}\right\}$ and assume the contrary that $\dim U < n$ then the orthogonal complement of U in $\mathbb{R}^n$ has a positive dimension. This implies the existence of a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\langle \mathbf{v}, \mathbf{w}^j \rangle = 0, j = 1, \ldots, n + n^{\rm enr}$. Let us now define the function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f(\mathbf{x}) := \langle \mathbf{v}, \boldsymbol{\phi}(\mathbf{x}) \rangle, \tag{21}$$

with $\boldsymbol{\phi} : \mathbb{R}^d \to \mathbb{R}^n$ the vector valued function having as component functions $q_i^{\rm org}$, $i = 1, \ldots, n$. It is readily verified that function f defined as above satisfies the following three properties:

(a) it belongs to $\mathscr{F}_K^{\rm org}$;
(b) it is different from zero;
(c) it satisfies for $j = 1, \ldots, n + n^{\rm enr}$, $L_j(f) = \langle \mathbf{v}, \mathbf{w}^j \rangle = 0$.

Now a simple inspection shows that these hold together if and only if $\mathscr{F}_K^{\rm org}$ is not $\Sigma_K^{\rm enr}$-unisolvent. A contradiction, which completes the proof. □

The key step is the following Lemma, which shows under the $\mathscr{F}_K^{\rm org}$ unisolvence with respect to $\Sigma_K^{\rm enr}$, the existence of a linearly independent set of $n^{\rm enr}$ vectors belonging to $\ker(\mathbf{A})$, where $\mathbf{A}$ is the matrix defined in (13).

Lemma 4. *Assume that* $\Sigma_K^{\rm enr}$ *is* $\mathscr{F}_K^{org}$*-unisolvent. Then there exists a set of vectors*

$$\boldsymbol{c}^i = (c_{i,1}, \ldots, c_{i,n+n^{\rm enr}})^\top \in \mathbb{R}^{n+n^{\rm enr}} \qquad (i = 1, \ldots, n^{\rm enr}), \tag{22}$$

such that:

(i) *they are linearly independent;*
(ii) *they belong to* $\ker(\mathbf{A})$, *where* $\mathbf{A}$ *is the matrix defined in (13).*

Proof. Lemma 3 immediately tells us that the rank of the matrix A is n. Hence, since its rows are of length $n + n^{\rm enr}$, then they span an n-dimensional subspace

U of $\mathbb{R}^{n+n^{\text{enr}}}$. Thus the orthogonal complement of U with respect to $\mathbb{R}^{n+n^{\text{enr}}}$ is of dimension n^{enr}. We conclude that there exists a set of linearly independent vectors $\mathbf{c}^i, i = 1, \ldots, n^{\text{enr}}$, such that each vector $\mathbf{c}^i$ is orthogonal to all row vectors of A. This shows that statements (i) and (ii) hold. □

We now come to our practical characterization Theorem.

Theorem 2. *Assume that* Σ_K^{enr} *is* $\mathscr{F}_K^{org}$*-unisolvent. Let* $\mathbf{c}^i, i = 1, \ldots, n^{\text{enr}}$ *be the vectors* (22) *for which statements* (i) *and* (ii) *of Lemma* 4 *hold. Define the vectors*

$$\mathbf{e}^i := (L_1(e_i^{\text{enr}}), \ldots, L_{n+n^{\text{enr}}}(e_i^{\text{enr}}))^\top \in \mathbb{R}^{n+n^{\text{enr}}},\ (i = 1, \ldots, n^{\text{enr}}). \tag{23}$$

Then the triplet $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ *constitutes a finite element if and only if*

$$\det \begin{pmatrix} \langle \mathbf{c}^1, \mathbf{e}^1 \rangle & \ldots & \langle \mathbf{c}^1, \mathbf{e}^{n^{\text{enr}}} \rangle \\ \vdots & & \vdots \\ \langle \mathbf{c}^{n^{\text{enr}}}, \mathbf{e}^1 \rangle & \ldots & \langle \mathbf{c}^{n^{\text{enr}}}, \mathbf{e}^{n^{\text{enr}}} \rangle \end{pmatrix} \neq 0\,. \tag{24}$$

Proof. Since the $\mathbf{c}^i$'s are linearly independent and belonging to $\ker(\mathbf{A})$, then by Lemmas 1 the $\mathscr{L}_i$'s vanish on $\mathscr{F}_K^{\text{org}}$. Note that here $\mathscr{L}_i(e_j^{\text{enr}})$ can be expressed simply as

$$\mathscr{L}_i(e_j^{\text{enr}}) = \langle \mathbf{c}^i, \mathbf{e}^j \rangle\,. \tag{25}$$

Hence, Theorem 1 applies and gives the desired result. □

At this stage, we still have the following problem: If the vectors $\mathbf{c}^i$ given by (22) have been determined in such a way that statements (i) and (ii) of Lemma 4 are satisfied, the question now is whether we can choose them in order that the determinant defined in (24) is not zero. We may also note in passing that the value of the latter does not depend on the choice of the vectors $\mathbf{c}^i$. From a practical point of view, as we shall do later, we will choose those that generate a sparse matrix.

The construction and the choice of such vectors will be explained carefully below in the context of the enrichment of the Han rectangular element [17], the nonconforming rotated element of Rannacher and Turek [23], and the Wilson type nonconforming element, see, e.g., [12] and [25].

3 Applications: Enrichment of Nonconforming Finite Elements

For illustration purposes we discuss in detail how the proposed method can be applied to enrich three well-known nonconforming elements in any dimension: the Han rectangular element, the rotated parallelogram element of Rannacher and

Turek, and the Wilson type nonconforming element. As we will see, a key point in the derivation of these elements is a new family of multivariate trapezoidal and Simpson type cubature formulas. The principal results of this section are given in Theorems 3–6 below, where our approach is applied to a general class of enrichment functions.

3.1 A New Enriched Nonconforming Finite Element of Han-Type

The formulation of the rectangular element proposed by Han in [17] is obtained by introducing the following local space

$$\mathscr{Q}_H := \left\{1, x, y, x^2 - \frac{5}{3}x^4, y^2 - \frac{5}{3}y^4\right\}$$

and the $\mathscr{Q}_H$-unisolvent set of linearly independent linear forms read

$$L_i^{\mathrm{tr}}(f) = \frac{1}{|l_i|}\int_{l_i} f\, d\sigma,\ i = 1, \dots, 4,$$

$$L_5^{\mathrm{tr}}(f) = \frac{1}{|K|}\int_K f(\mathbf{x})\, d\mathbf{x},$$

with $l_i, i = 1, \dots, 4$, the four edges of the rectangle K. In our context, this element can be seen as an enriched element. Here, the set of affine functions is the original space enriched by adding two enrichment functions $x^2 - \frac{5}{3}x^4, y^2 - \frac{5}{3}y^4$.

To precisely define our new elements, we first introduce some additional notation. The original space $\mathscr{F}_K^{\mathrm{org}}$ and the triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ of these elements are defined by:

- K is the hyper-rectangle in $\mathbb{R}^d$ defined by

$$K := \{\mathbf{x}, \beta_{i1} \le x_i \le \beta_{i2}, i = 1, \dots, d\},$$

-

$$\mathscr{F}_K^{\mathrm{org}} = \mathscr{P}_1 := \operatorname{span}\{1, x_1, \dots, x_d\}. \tag{26}$$

-

$$\mathscr{F}_K^{\mathrm{enr}} := \mathscr{F}_K^{\mathrm{org}} \oplus \left\{e_1^{\mathrm{enr}}, \dots, e_d^{\mathrm{enr}}\right\}. \tag{27}$$

-

$$\Sigma_K^{\mathrm{enr}} := \left\{L_i^{\mathrm{tr}}, i = 1, \dots, 2d+1\right\}, \tag{28}$$

where

$$L_i^{\mathrm{tr}}(f) = \frac{1}{|F_i|}\int_{F_i} f\,d\sigma,\ i = 1,\ldots,2d,$$
$$L_{2d+1}^{\mathrm{tr}}(f) = \frac{1}{|K|}\int_K f(\mathbf{x})\,d\mathbf{x}.$$

Here $F_1,\ldots,F_{2d}$ are $2d$ facets of K. We choose the special enumeration of all facets as follows: for each $j = 1,\ldots,d$, F_j and F_{j+d} are subsets of the hyperplanes $x_j = \beta_{j1}$ and $x_j = \beta_{j2}$, respectively.

We now introduce a new family of a multivariate version of the well-known trapezoidal rule, which is useful in subsequent manipulation. For each $i = 1,\ldots,d$, we want to call the integration formula

$$L_{2d+1}^{\mathrm{tr}}(f) = \frac{1}{2}\left(L_i^{\mathrm{tr}}(f) + L_{i+d}^{\mathrm{tr}}(f)\right) + T_i^{\mathrm{tr}}(f) \tag{29}$$

the trapezoidal cubature formula, since for the one-dimensional case, $d = 1$, it coincides with the trapezoidal rule.

One property of the trapezoidal cubature formula (29), which is fundamental to our work is the following simple but key observation:

Lemma 5. *For each $i = 1,\ldots,d$, the approximation error T_i^{tr} of the trapezoidal cubature formula vanishes for all affine functions.*

Proof. Indeed, it is obvious that T_i^{tr} necessarily vanishes for constant functions. Moreover, the following basic properties of $L_i^{\mathrm{tr}}, i = 1,\ldots,2d+1$ can be shown to hold

$$L_i^{\mathrm{tr}}(x_j) = \frac{\beta_{j1} + \beta_{j2} + \left(\beta_{j1} - \beta_{j2}\right)\delta_{ij}}{2},\ \ i,j = 1,\ldots,d,$$
$$L_{i+d}^{\mathrm{tr}}(x_j) = \frac{\beta_{j1} + \beta_{j2} + \left(\beta_{j2} - \beta_{j1}\right)\delta_{ij}}{2},\ \ i,j = 1,\ldots,d,$$
$$L_{2d+1}^{\mathrm{tr}}(x_j) = \frac{\beta_{j1} + \beta_{j2}}{2},\ \ i,j = 1,\ldots,d.$$

Hence, we can deduce that

$$L_{2d+1}^{\mathrm{tr}}(x_j) = \frac{L_i^{\mathrm{tr}}(x_j) + L_{i+d}^{\mathrm{tr}}(x_j)}{2}\ \ i,j = 1,\ldots,d. \tag{30}$$

This shows that the approximation error T_i^{tr} vanishes on $\mathscr{P}_1$. □

For later use we need the following result:

Lemma 6. *Let p be an affine function. Then, the following statements are equivalent:*

(i) $L_j^{tr}(p) = 0$, *for* $j = 1, \ldots, 2d$,
(ii) $p = 0$.

Proof. To prove (i) implies (ii), assume that p is an affine function such that $L_j^{\mathrm{tr}}(p) = 0$, for $j = 1, \ldots, 2d$. By Lemma 5, we can deduce that $L_{2d+1}^{\mathrm{tr}}(p) = 0$. Denote by $\boldsymbol{\nu}^j$ the outer normal unit vector on F_j and note that $\partial p/\partial \boldsymbol{\nu}^j$ and $\|\nabla p\|$ are constants. Then, as can easily be verified, by Green's formula, we get

$$\begin{aligned}
\|\nabla p\|^2 \, |K| &= \int_K \|\nabla p\|^2 \mathrm{d}\mathbf{x} \\
&= -\int_K p \, \Delta p \, \mathrm{d}\mathbf{x} + \sum_{j=1}^{2d} \int_{F_j} p \frac{\partial p}{\partial \boldsymbol{\nu}^j} \, \mathrm{d}\sigma \\
&= \sum_{j=1}^{2d} \frac{\partial p}{\partial \boldsymbol{\nu}^j} \int_{F_j} p \, \mathrm{d}\sigma \\
&= \sum_{j=1}^{2d} \frac{\partial p}{\partial \boldsymbol{\nu}^j} \left|F_j\right| L_j^{\mathrm{tr}}(p) \\
&= 0.
\end{aligned}$$

Consequently,

$$\|\nabla p\|^2 = 0,$$

and so p is a constant. But then $L_{2d+1}^{\mathrm{tr}}(p) = 0$ implies that p is identically zero. Since the converse implication is easy to verify, the proof is complete. □

Now we can give the basic result on which our enrichment approach is based. Indeed, with notation of this section, the following result shows that for each $i = 1, \ldots, d$, the functional $\mathscr{L}_i$ figuring in (25) can be taken as the approximation error of the trapezoidal cubature formula T_i^{tr}.

Theorem 3. *The triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ constitutes a finite element if and only if*

$$\det \begin{pmatrix} T_1^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & T_1^{\mathrm{tr}}(e_d^{\mathrm{enr}}) \\ \vdots & & \vdots \\ T_d^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & T_d^{\mathrm{tr}}(e_d^{\mathrm{enr}}) \end{pmatrix} \neq 0 \,. \tag{31}$$

Proof. First we suppose that K is the hypercube $K_0 := [-1, 1]^d$. The crucial point in application of Theorem 2 is now to determine the explicit expressions for the vectors $\mathbf{c}^i$ given by (22), when the degrees of freedom are defined by (28). The first step of this proof is to find the matrix $\mathbf{A}$ defined in (13). In the present situation, taking into account that

$$-L_j^{\mathrm{tr}}(x_i) = L_{j+d}^{\mathrm{tr}}(x_i) = \delta_{ij},\ (j = 1, \ldots, d, i = 1, \ldots d),$$
$$L_{2d+1}^{\mathrm{tr}}(x_i) = 0,\ (i = 1, \ldots d),$$

we get that the matrix $\mathbf{A}$ has the simple form:

$$\mathbf{A} = \begin{bmatrix} & \mathbf{1}_{2d+1} & \\ \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{0}_d \end{bmatrix} \in \mathbb{R}^{(d+1)\times(2d+1)}, \tag{32}$$

where $\mathbf{1}_{2d+1}$ is a $(2d + 1)$-dimensional vector, whose all components are equal to 1, $\mathbf{A}_2 = -\mathbf{A}_1 = \mathbf{I}_{d\times d}$ is the $d \times d$ identity matrix, $\mathbf{0}_d \in \mathbb{R}^d$ is the zero vector. Thanks to Lemma 6, we know that Σ_K^{enr} is $\mathscr{F}_K^{\mathrm{org}}$–unisolvent then Lemma 4 applies. It tells us that there is a linearly independent set of d vectors

$$\mathbf{c}^i = (c_{i,1}, \ldots, c_{i,2d+1})^\top \in \mathbb{R}^{2d+1} \qquad (i = 1, \ldots, d),$$

belonging to $\ker(\mathbf{A})$. Hence for each $i, i = 1, \ldots, d$, $\mathbf{c}^i$ must be the solution of the following linear system:

$$\mathbf{A}\mathbf{c}^i = \mathbf{0}, \tag{33}$$

where $\mathbf{A}$ is the matrix given by (32). Then, the above system is equivalent to:

$$\sum_{j=1}^{2d+1} c_{i,j} = 0, \tag{34}$$

$$-c_{i,j} + c_{i,j+d} = 0,\ (j = 1, \ldots, d). \tag{35}$$

Because there are more unknowns than linear equations, then this system has an infinite number of solutions. Let $\mathbf{C}$ denote the matrix in which the ith row is the transpose of the vector $\mathbf{c}^i$. Now in order that the row vectors of $\mathbf{C}$ satisfy the criterion for linear independence, we may choose $\mathbf{C}$, for instance, the matrix

$$\mathbf{C} = \left[-\tfrac{1}{2}\mathbf{I}_{d\times d}\ -\tfrac{1}{2}\mathbf{I}_{d\times d}\ \mathbf{1}_d\right] \in \mathbb{R}^{d\times(2d+1)}, \tag{36}$$

where $\mathbf{1}_d$ is a d-dimensional vector, whose all components are equal to 1. Here, our conditions (34) and (35) are obviously satisfied. Moreover, the submatrix constituted by the first d columns of the matrix in (36) is $-\frac{1}{2}$ times the identity matrix. Hence its determinant is different from zero, which ensures the linear independence of the row vectors. This shows that Theorem 2 applies, and consequently the $\mathscr{L}_i := T_i^{\mathrm{tr}}$'s form a set of admissible functionals.

The case of general K is reduced to K_0 by a coordinate transformation in the calculation of the integrals. □

We give here a direct proof of Theorem 3, using only Lemmas 1 and 5. Indeed, for each $i = 1, \dots, d$, the approximation error of the trapezoidal cubature formula T_i^{tr} enjoys the following properties:

(i) $T_i^{\mathrm{tr}} \in \operatorname{span}\left\{L_j^{\mathrm{tr}}, j = 1, \dots, 2d+1\right\}$.
(ii) By Lemma 5, we also know that T_i^{tr} vanishes on $\mathscr{F}_K^{\mathrm{org}} := \mathscr{P}_1$.

Then Lemma 1 implies the existence of a system of vectors $\mathbf{c}^i \in \ker(\mathbf{A}), i = 1, \dots, d$, which are associated with the approximation errors T_i^{tr}. Moreover, a simple inspection shows that these vectors are exactly the same as those defined by the matrix $\mathbf{C}$ in (36). Finally, as we have seen before, the row vectors of $\mathbf{C}$ are linear independent, then the $\mathscr{L}_i := T_i^{\mathrm{tr}}$'s form a set of admissible functionals.

Having determined the admissible linear functionals $\mathscr{L}_i, i = 1, \dots, d$, the rest of this section is devoted to seek necessary and sufficient conditions on the enrichment functions $e_i^{\mathrm{enr}}, i = 1, \dots d$, under which the determinant defined in (31) is nonzero.

Our objective here is to obtain a more general set of enrichment functions than those commonly used in the particular case of Han's polynomial functions, see [17, 20, 22]. The main advantage of the new elements is that they have a great freedom in selecting enrichment functions e_i^{enr}. More precisely, we show that these functions can be generated from just one single function $e \in C^0[-1, 1]$ that depends on additional free parameters. It will turn out that the remainder $T^{\mathrm{tr}}(e)$ of the trapezoidal rule

$$\int_{-1}^{1} e(t)\,\mathrm{d}t = e(-1) + e(1) + T^{\mathrm{tr}}(e)$$

will play an important role in the existence of our new elements. Indeed we can show the following elegant characterization:

Theorem 4. *Let $K \subset \mathbb{R}^d$ be a nondegenerate hyper-rectangle and consider the degrees of freedom $L_1^{tr}, \dots, L_{2d+1}^{tr}$, given by (28). Let $T_1^{\mathrm{tr}}, \dots, T_d^{\mathrm{tr}}$ be given by (29). Assume that $F_K = (F_{1K}, \dots, F_{dK})^\top$ is a diagonal affine transformation such that $F_K(\hat{K}) = K$, where $\hat{K}$ the hypercube $\hat{K} := [-1, 1]^d$. For a function $e \in C^0[-1, 1]$ define*

$$e_i^{\mathrm{enr}} : \mathbf{x} \longmapsto \sum_{j=1}^{d} \alpha_{ij} e\left(F_{jK}(\hat{\mathbf{x}})\right) + c_i \qquad (i = 1, \dots, d), \tag{37}$$

where α_{ij} and c_i are some given real numbers, such that

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{d1} \\ \vdots & \ddots & \vdots \\ \alpha_{1d} & \cdots & \alpha_{dd} \end{vmatrix} \neq 0. \tag{38}$$

Then, the following statements are equivalent:

(i) *The determinant*

$$\Delta := \det \begin{pmatrix} T_1^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & T_1^{\mathrm{tr}}(e_d^{\mathrm{enr}}) \\ \vdots & & \vdots \\ T_d^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & T_d^{\mathrm{tr}}(e_d^{\mathrm{enr}}) \end{pmatrix} \neq 0\,. \tag{39}$$

(ii) *The remainder $T^{\mathrm{tr}}(e)$ of the trapezoidal rule is different from zero.*

Proof. To simplify the discussion, let us first assume that K is the hypercube $\hat{K} := [-1,1]^d$. Then, in this case the functions (37) simplify to

$$e_i^{\mathrm{enr}} \,:\, \mathbf{x} \longmapsto \sum_{j=1}^{d} \alpha_{ij} e\left(x_j\right) + c_i,\ \ (i = 1, \ldots, d).$$

Then a simple calculation shows that for each $i = 1, \ldots, d$, we have

$$L_j^{\mathrm{tr}}(e_i^{\mathrm{enr}}) = c_i + \alpha_{ij} e(-1) + \frac{1}{2}\int_{-1}^{1} e(t)\,dt \sum_{\substack{1\le k\le d\\ k\neq j}} \alpha_{ik},\ j = 1, \ldots, d,$$

$$L_{j+d}^{\mathrm{tr}}(e_i^{\mathrm{enr}}) = c_i + \alpha_{ij} e(1) + \frac{1}{2}\int_{-1}^{1} e(t)\,dt \sum_{\substack{1\le k\le d\\ k\neq j}} \alpha_{ik},\ j = 1, \ldots, d,$$

$$L_{2d+1}^{\mathrm{tr}}(e_i^{\mathrm{enr}}) = c_i + \frac{1}{2}\int_{-1}^{1} e(t)\,dt \sum_{1\le k\le d} \alpha_{ik}.$$

Therefore, substituting these values in the general expressions of $T_j^{\mathrm{tr}}(e_i^{\mathrm{enr}})$ gives, after simplification, a simple relation between the errors of the multivariate trapezoidal cubatures $T_j^{\mathrm{tr}}(e_i^{\mathrm{enr}})$ and those, which is produced by the classical trapezoidal rule $T^{\mathrm{tr}}(e)$:

$$\begin{aligned} T_j^{\mathrm{tr}}(e_i^{\mathrm{enr}}) &= \frac{1}{2}\alpha_{ij}\left(\int_{-1}^{1} e(t)\,dt - e(-1) - e(1)\right) \\ &= \frac{1}{2}\,\alpha_{ij} T^{\mathrm{tr}}(e)\,. \end{aligned}$$

Thus, we obviously can deduce

$$\Delta = \frac{(T^{\mathrm{tr}}(e))^d}{2^d} \det \begin{pmatrix} \alpha_{11} & \dots & \alpha_{d1} \\ \vdots & \ddots & \vdots \\ \alpha_{1d} & \dots & \alpha_{dd} \end{pmatrix}. \qquad (40)$$

Hence the desired conclusion holds for $K = \hat{K}$. Now, the case of general K is reduced to the case of $\hat{K}$ by using the affine transformation F_K in the calculation of the functionals (28). This completes the proof of Theorem 4. □

To avoid unnecessary complication, we have stated Theorem 4 for enrichment functions of the form (37), however the latter can be extended as follows:

Remark 2. An inspection of the proof of Theorem 4 shows that if the single-functions e_i are taken such that $T^{\mathrm{tr}}(e_i) \neq 0$, then the above result can be extended to enrichment functions that have the more general forms

$$e_i^{\mathrm{enr}} : \mathbf{x} \longmapsto \sum_{j=1}^{d} \alpha_{ij} e_i \left(F_{jK}(\hat{\mathbf{x}})\right) + c_i \qquad (i = 1, \dots, d). \qquad (41)$$

For $d = 2$ or $d = 3$, let K be a d-dimensional rectangle. Han Nonconforming d-rectangular finite element is useful in practice. It has been extended in many directions with several possible choice of the enriched approximation space $\mathscr{F}_K^{\mathrm{enr}}$. The first choice was proposed by Han in [17] with

$$\mathscr{F}_K^{\mathrm{enr}} = \begin{cases} \{1, x, y\} \oplus \{e(x), e(y)\} & \text{for } d = 2, \\ \{1, x, y, z\} \oplus \{e(x), e(y), e(z)\} & \text{for } d = 3, \end{cases}$$

where $e(t) = t^2 - \frac{5}{3}t^4$, with $t \in [-1, 1]$. Recently, [20] and [22] extended these elements to the case $e(t) = t^2$, with $t \in [-1, 1]$. All these elements can simply be deduced from our general Theorem 4 by choosing the matrix with entries α_{ij} the identity matrix and $c_i = 0$ in enrichment functions (37), and by checking that the remainders $T^{\mathrm{tr}}(e)$ of the trapezoidal rule for all these functions e are different from zero. Hence, in particular, this methodology allowed us to recover the well-known nonconforming Han element [17].

3.2 A New Enriched Nonconforming Finite Element of Rannacher–Turek-Type

Throughout this subsection we assume that: $d \geq 2$. The second choice to illustrate our proposed method is to enrich the well-known nonconforming rotated bilinear finite element proposed by Rannacher and Turek in [23], where the corresponding local finite element space is obtained by rotating the mixed term of the bilinear element, and assuming as local degree of freedom either the average of the function over the edge or its value at the mid side node. In order to extend the latter, here the original space $\mathscr{F}_K^{\mathrm{org}}$ and the enriched triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ of this element are chosen as follows:

- K is the hyper-rectangle in $\mathbb{R}^d$ defined by

$$K := \{\mathbf{x}, \beta_{i1} \le x_i \le \beta_{i2}, i = 1, \ldots, d\},$$

-

$$\mathscr{F}_K^{\mathrm{org}} = \mathscr{P}_1 := \mathrm{span}\,\{1, x_1, \ldots, x_d\}. \tag{42}$$

-

$$\mathscr{F}_K^{\mathrm{enr}} := \mathscr{F}_K^{\mathrm{org}} \oplus \left\{e_1^{\mathrm{enr}}, \ldots, e_{d-1}^{\mathrm{enr}}\right\}. \tag{43}$$

-

$$\Sigma_K^{\mathrm{enr}} := \left\{L_i^{\mathrm{tr}}, i = 1, \ldots, 2d\right\}, \tag{44}$$

  where

$$L_i^{\mathrm{tr}}(f) = \frac{1}{|F_i|} \int_{F_i} f\, d\sigma,\ i = 1, \ldots, 2d, \tag{45}$$

with $F_1, \ldots, F_{2d}$ are $2d$ facets of K. We continue to use the special enumeration of all facets as in the case of the enriched Han element.

Now we present another multivariate version of trapezoid rule. We define the function $\mathscr{L}(f)$ as follows:

$$\mathscr{L}(f) := \frac{1}{2d} \sum_{j=1}^{d} \left(L_j^{\mathrm{tr}}(f) + L_{j+d}^{\mathrm{tr}}(f)\right), \tag{46}$$

and therefore introduce, for each $i = 1, \ldots, d-1$, the trapezoidal cubature formula, which is supposed to approximate:

$$\mathscr{L}(f) = \frac{1}{2} \left(L_i^{\mathrm{tr}}(f) + L_{i+d}^{\mathrm{tr}}(f)\right) + G_i^{\mathrm{tr}}(f). \tag{47}$$

We make the following remarks:

(a) For each $i = 1, \ldots, d-1$, the approximation error of the trapezoidal cubature formula G_i^{tr} vanishes for all affine functions.
(b) For each $i = 1, \ldots, d-1$, $G_i^{\mathrm{tr}} \in \mathrm{span}\left\{L_j^{\mathrm{tr}}, j = 1, \ldots, 2d\right\}$.

(c) In view of property (a), Lemma 1 implies the existence of a system of vectors $\mathbf{c}^i \in \ker(\mathbf{A}), i = 1, \ldots, d-1$, which are associated with the approximation errors G_i^{tr}. Moreover, as can be easily verified, these vectors are the row vectors of the matrix $\mathbf{C} \in \mathbb{R}^{(d-1)\times 2d}$ where

$$\mathbf{C} = \begin{pmatrix} \alpha & \beta & \ldots & \beta & \beta & \alpha & \beta & \ldots & \beta & \beta \\ \beta & \alpha & \ldots & \beta & \beta & \beta & \alpha & \ldots & \beta & \beta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta & \beta & \ldots & \alpha & \beta & \beta & \beta & \ldots & \alpha & \beta \end{pmatrix}, \tag{48}$$

in which

$$\alpha = \frac{1}{2d} - \frac{1}{2},$$
$$\beta = \frac{1}{2d}.$$

Let $\mathbf{D}$ be the square submatrix formed by the first $d-1$ columns of the matrix $\mathbf{C}$. It is easy to see that the determinant of $\mathbf{D}$ is equal to $\frac{1}{(-2d)^{d-1}} \det(\mathbf{E})$, where $\mathbf{E}$ is the $(d-1)\times(d-1)$ matrix

$$\mathbf{E} = \begin{pmatrix} d-1 & -1 & \ldots & -1 \\ -1 & d-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & d-1 \end{pmatrix}. \tag{49}$$

Since $\mathbf{E}$ is strictly diagonally dominant, then its determinant is different from zero. Hence, the rows of $\mathbf{C}$ are linearly independent, consequently, the $\mathscr{L}_i := G_i^{\mathrm{tr}}$'s form a set of admissible functionals.

Finally, since Theorem 2 applies, there follows:

Theorem 5. *The triplet $(K, \Sigma_K^{\mathrm{enr}}, \mathscr{F}_K^{\mathrm{enr}})$ constitutes a finite element if and only if*

$$\det \begin{pmatrix} G_1^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & G_1^{\mathrm{tr}}(e_{d-1}^{\mathrm{enr}}) \\ \vdots & & \vdots \\ G_{d-1}^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & G_{d-1}^{\mathrm{tr}}(e_{d-1}^{\mathrm{enr}}) \end{pmatrix} \neq 0\,. \tag{50}$$

Remark 3. Note that thanks to Lemma 6, the unisolvence condition required in our general Theorem 2 obviously holds.

Proceeding in an identical manner as in the case of the enriched Han element, we have the following result:

Theorem 6. *Let $K \subset \mathbb{R}^d$ be a nondegenerate hyper-rectangle and consider the degrees of freedom $L_1^{tr}, \ldots, L_{2d}^{tr}$, given by (45). Let $G_1^{\mathrm{tr}}, \ldots, G_{d-1}^{\mathrm{tr}}$ be given by (47). Assume that $F_K = (F_{1K}, \ldots, F_{dK})^\top$ is a diagonal affine transformation such that $F_K(\hat{K}) = K$, where $\hat{K}$ the hypercube $\hat{K} := [-1, 1]^d$. For a function $e \in C^0[-1, 1]$ define*

$$e_i^{\mathrm{enr}} : \mathbf{x} \longmapsto \sum_{j=1}^{d} \alpha_{ij} e\left(F_{jK}(\hat{\mathbf{x}})\right) + c_i \qquad (i = 1, \ldots, d-1), \tag{51}$$

where α_{ij} and c_i are some given real numbers, such that

$$\Delta_{\boldsymbol{\alpha}} := \begin{vmatrix} \alpha_{12} - \alpha_{11} & \cdots & \alpha_{(d-1)2} - \alpha_{(d-1)1} \\ \alpha_{13} - \alpha_{11} & \cdots & \alpha_{(d-1)3} - \alpha_{(d-1)1} \\ \vdots & \ddots & \vdots \\ \alpha_{1d} - \alpha_{11} & \cdots & \alpha_{(d-1)d} - \alpha_{(d-1)1} \end{vmatrix} \neq 0. \tag{52}$$

Then, the following statements are equivalent:

(i) *The determinant*

$$\det \begin{pmatrix} G_1^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & G_1^{\mathrm{tr}}(e_{d-1}^{\mathrm{enr}}) \\ \vdots & & \vdots \\ G_{d-1}^{\mathrm{tr}}(e_1^{\mathrm{enr}}) & \ldots & G_{d-1}^{\mathrm{tr}}(e_{d-1}^{\mathrm{enr}}) \end{pmatrix} \neq 0\,. \tag{53}$$

(ii) *The remainder $T^{\mathrm{tr}}(e)$ of the trapezoidal rule is different from zero.*

When $d = 2$ or $d = 3$, we recover the well-known nonconforming rectangular (or cubic) element of Rannacher and Turek, with

$$\mathscr{F}_K^{\mathrm{enr}} = \begin{cases} \{1, x, y\} \oplus \{e(x) - e(y)\} & \text{for } d = 2, \\ \{1, x, y, z\} \oplus \{e(x) - e(y), e(x) - e(z)\} & \text{for } d = 3, \end{cases}$$

and where $e(t) = t^2, t \in [-1, 1]$. Other variants were proposed in [10, 11, 14] by setting

$$e(t) = \begin{cases} t^2 - \frac{5}{3}t^4 & \text{for } d = 2, \\ t^2 - \frac{25}{6}t^4 + \frac{7}{2}t^6 & \text{for } d = 3. \end{cases}$$

The reader can easily verify that all these elements are covered by our general Theorem 6. Moreover, in these two examples the entries α_{ij} are given by:

(a) The case $d = 2$, $\alpha_{11} = 1$ and $\alpha_{12} = -1$, therefore $\alpha_{12} - \alpha_{11} = -2 \neq 0$. Hence (52) is satisfied.

(b) The case $d = 3$, $(\alpha_{11}, \alpha_{12}, \alpha_{13}) = (1, -1, 0)$ and $(\alpha_{21}, \alpha_{22}, \alpha_{23}) = (1, 0, -1)$. Since

$$\Delta_{\boldsymbol{\alpha}} := \begin{vmatrix} \alpha_{12} - \alpha_{11} & \alpha_{22} - \alpha_{21} \\ \alpha_{13} - \alpha_{11} & \alpha_{23} - \alpha_{21} \end{vmatrix} = 3, \tag{54}$$

this determinant is non-zero, then condition (52) is satisfied.

3.3 A New Enriched Nonconforming Finite Element of Wilson-Type

For the third example, we obtain new enriched nonconforming finite elements of Wilson type in any dimension. The proofs are very similar to those in Sect. 3.1, we do not give the full details here. In the previous two examples, the original approximation spaces $\mathscr{F}_K^{\text{org}}$ were taken to be $\mathscr{P}_1$. Here, this latter and the triplet $(K, \Sigma_K^{\text{enr}}, \mathscr{F}_K^{\text{enr}})$ of these elements are chosen to be:

- K is the hyper-rectangle in $\mathbb{R}^d$ defined by

$$K := \{\mathbf{x}, \beta_{i1} \leq x_i \leq \beta_{i2}, i = 1, \ldots, d\},$$

-

$$\mathscr{F}_K^{\text{org}} := \mathscr{P}_1 \oplus \left\{x_1^2, \ldots, x_d^2\right\}. \tag{55}$$

-

$$\mathscr{F}_K^{\text{enr}} := \mathscr{F}_K^{\text{org}} \oplus \left\{e_1^{\text{enr}}, \ldots, e_{2d}^{\text{enr}}\right\}. \tag{56}$$

-

$$\Sigma_K^{\text{enr}} := \left\{L_i^{\text{Wil}}, i = 1, \ldots, 4d + 1\right\}, \tag{57}$$

  where

$$L_i^{\text{Wil}}(f) := \frac{1}{|F_i|} \int_{F_i} f \, d\sigma, \; i = 1, \ldots, 2d,$$
$$L_{2d+1}^{\text{Wil}}(f) := \frac{1}{|K|} \int_K f(\mathbf{x}) \, d\mathbf{x},$$
$$L_{2d+1+i}^{\text{Wil}}(f) := \frac{1}{|S_i|} \int_{S_i} f \, d\sigma, \; i = 1, \ldots, d,$$

$$L^{\mathrm{Wil}}_{3d+1+i}(f) := \frac{1}{2|K|}\int_K \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2}\,d\mathbf{x},\ i=1,\ldots,d.$$

Here S_i is defined by

$$S_i := \left\{ \mathbf{x} = (x_1,\ldots,x_d) \in K : x_i = \frac{\beta_{i,1}+\beta_{i,2}}{2} \right\}.$$

We start with the following remark.

Remark 4. The midpoint rule in one dimension is

$$\frac{1}{b-a}\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right) + M^{\mathrm{mid}}(f). \tag{58}$$

For each $i=1,\ldots,d$, a natural multivariate extension of this rule is

$$L^{\mathrm{Wil}}_{2d+1}(f) = \frac{1}{|S_i|}\int_{S_i} f\,d\sigma + M_i^{\mathrm{mid}}(f).$$

The trapezoidal rule in one dimension is

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{f(a)+f(b)}{2} + T^{\mathrm{tr}}(f). \tag{59}$$

For each $i=1,\ldots,d$, as mentioned before, a natural multivariate extension of this rule is

$$L^{\mathrm{Wil}}_{2d+1}(f) = \frac{1}{2}\left(L_i^{\mathrm{Wil}}(f) + L^{\mathrm{Wil}}_{i+d}(f)\right) + T_i^{\mathrm{Wil}}(f).$$

Let us also observe the following surprising result. We know that the trapezoidal and midpoint rules are each only exact for linear functions. However, it can be easily verified that, for each i, the approximation errors T_i^{Wil} and M_i^{mid} vanish on the space

$$\mathscr{P}_1 \oplus \left\{x_j^2, j=1,\ldots,d, j\neq i\right\}. \tag{60}$$

Moreover, if, for each i, we define A_i^{tr} and A_i^{mid} by

$$A_i^{\mathrm{tr}}(f) = \frac{1}{2}\left(L_i^{\mathrm{Wil}}(f) + L^{\mathrm{Wil}}_{i+d}(f)\right),$$
$$A_i^{\mathrm{mid}}(f) = \frac{1}{|S_i|}\int_{S_i} f\,d\sigma,$$

then, the following identities hold:

$$A_i^{\mathrm{tr}}(x_i^2) = \frac{\beta_{i1}^2 + \beta_{i2}^2}{2}, \tag{61}$$

$$A_i^{\mathrm{mid}}(x_i^2) = \frac{(\beta_{i1} + \beta_{i2})^2}{4}, \tag{62}$$

$$L_{2d+1}^{\mathrm{Wil}}(x_i^2) = \frac{1}{3}A_i^{\mathrm{tr}}(x_i^2) + \frac{2}{3}A_i^{\mathrm{mid}}(x_i^2), \tag{63}$$

$$L_{2d+1}^{\mathrm{Wil}}(x_i^2) - A_i^{\mathrm{mid}}(x_i^2) = \frac{(\beta_{i2} - \beta_{i1})^2}{12}. \tag{64}$$

We now introduce a new family of cubature formulas of the Simpson type. First, for each $i = 1, \ldots, d$, we define the integration formula.

$$\begin{aligned} L_{2d+1}^{\mathrm{Wil}}(f) &= \frac{1}{3}A_i^{\mathrm{tr}}(f) + \frac{2}{3}A_i^{\mathrm{mid}}(f) + S_i^{\mathrm{Sim}}(f) \\ &= \frac{1}{3}\left(\frac{1}{2}\left(L_i^{\mathrm{Wil}}(f) + L_{i+d}^{\mathrm{Wil}}(f)\right)\right) + \frac{2}{3}\left(\frac{1}{|S_i|}\int_{S_i} f\, d\sigma\right) + S_i^{\mathrm{Sim}}(f). \end{aligned} \tag{65}$$

Recall that Simpson's rule can be expressed on the interval $[a, b]$ as:

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{3}\left(\frac{f(a) + f(b)}{2}\right) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + S^{\mathrm{Sim}}(f). \tag{66}$$

Hence, the cubature formula (65) appears as a natural extension to higher dimensions of the classical Simpson's rule.

For the Simpson cubature formula (65), the following result holds.

Lemma 7. *For each $i = 1, \ldots, d$, the approximation error of the Simpson cubature formula S_i^{Sim} vanishes for all functions belonging to $\mathscr{F}_K^{org}$.*

Proof. Remark 4 tells us that, for each $i = 1, \ldots, d$, the trapezoidal and midpoint cubature formulas vanish on the space

$$\mathscr{P}_1 \oplus \left\{x_j^2, j = 1, \ldots, d, j \neq i\right\}. \tag{67}$$

Then, since S_i^{Sim} is a convex combination of these two cubature formulas, it consequently vanishes on the same space. Hence, S_i^{Sim} vanishes identically for any $f \in \mathscr{F}_K^{\mathrm{org}}$ provided that

$$S_i^{\mathrm{Sim}}(x_i^2) = 0.$$

This required equality now follows from identity (63). □

For each $i = 1, \ldots, d$, let us also introduce another class of cubature formulas via

$$L_{2d+1}^{\mathrm{Wil}}(f) = \frac{1}{|S_i|}\int_{S_i} f\,d\sigma + \frac{(\beta_{i2}-\beta_{i1})^2}{24}\left(\frac{1}{|K|}\int_K \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2}d\mathbf{x}\right) + S_{i+d}^{\mathrm{Sim}}(f), \qquad (68)$$

Atkinson [4] defined the corrected or perturbed midpoint rule on the interval $[a, b]$ by

$$\frac{1}{b-a}\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right) + \frac{b-a}{24}\big(f'(b) - f'(a)\big) + S^{\mathrm{per}}(f), \qquad (69)$$

and so the cubature formula (68) is a natural extension of the perturbed midpoint rule in higher dimensions.

It also holds that the cubature formula (68) satisfies the following exactness condition.

Lemma 8. *For each $i = 1, \ldots, d$, the approximation error of the perturbed midpoint cubature formula S_{i+d}^{Sim} vanishes for all functions belonging to $\mathscr{F}_K^{org}$.*

Proof. The proof simply follows from Remark 4. Indeed, since for any $f \in \mathscr{P}_1 \oplus \left\{x_j^2, j = 1, \ldots, d, j \neq i\right\}$, we have

$$\int_K \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2}d\mathbf{x} = 0,$$

then to establish Lemma 8, by Remark 4, (60), it suffices to show that

$$S_{i+d}^{\mathrm{Sim}}(x_i^2) = 0.$$

But the above equality is a simple consequence of identities (64) and (68). □

Next we need the following result.

Lemma 9. *Let p belong to $\mathscr{F}_K^{org}$. Then, the following statements are equivalent:*

(i) $L_j^{Wil}(p) = 0$, *for* $j = 1, \ldots, 4d+1$,
(ii) $p = 0$.

Proof. To prove (i) implies (ii), assume that $p \in \mathscr{F}_K^{\mathrm{org}}$ such that $L_j^{\mathrm{Wil}}(p) = 0$, for $j = 1, \ldots, 4d+1$. Then p can be decomposed as

$$p = p_1 + \sum_{i=1}^{d} c_i x_i^2,$$

where $p_1 \in \mathscr{P}_1$ and the $c_i, (i = 1, \ldots, d)$, are some real coefficients. Now observe that

$$
\begin{aligned}
L^{\text{Wil}}_{2d+1+i}(p) &:= \frac{1}{2|K|}\int_K \frac{\partial^2 p(\mathbf{x})}{\partial x_i^2}\,d\mathbf{x}\\
&= c_i,
\end{aligned}
$$

so that the $c_i, (i = 1,\dots,d)$, are equal to zero. Hence, $p = p_1 \in \mathscr{P}_1$. Since $L^{\text{Wil}}_j(p) = 0$, for $j = 1,\dots,2d+1$, then by Lemma 6, we can deduce that p is identically zero. The converse implication is easy to verify, then the proof is complete. □

With the help of the cubature formulas (65) and (68), we now present a practical characterization result, that can be used to show that the triplet $(K, \Sigma^{\text{enr}}_K, \mathscr{F}^{\text{enr}}_K)$ generates a finite element.

Theorem 7. *The triplet* $(K, \Sigma^{\text{enr}}_K, \mathscr{F}^{\text{enr}}_K)$ *constitutes a finite element if and only if*

$$
\det\begin{pmatrix} S^{\text{sim}}_1(e^{\text{enr}}_1) & \dots & S^{\text{sim}}_1(e^{\text{enr}}_{2d})\\ \vdots & & \vdots\\ S^{\text{sim}}_{2d}(e^{\text{enr}}_1) & \dots & S^{\text{sim}}_{2d}(e^{\text{enr}}_{2d})\end{pmatrix} \neq 0\,. \tag{70}
$$

Proof. Observe first that the needed unisolvence property of Σ^{enr}_K with respect to $\mathscr{F}^{\text{org}}_K$ is guaranteed by Lemma 9. Furthermore, here for each $i = 1,\dots,2d$, the approximation error of the Simpson cubature formula S^{Sim}_i enjoys the following properties:

(i) $S^{\text{Sim}}_i \in \operatorname{span}\left\{L^{\text{Wil}}_j, j = 1,\dots,4d+1\right\}$.

(ii) By Lemmas 7 and 8, we also know that S^{Sim}_i vanishes on $\mathscr{F}^{\text{org}}_K$.

Consequently, by Lemma 1, there exists a system of vectors $\mathbf{c}^i \in \ker(\mathbf{A}), i = 1,\dots,2d$, which are the coefficients in the expressions of the approximation errors $S^{\text{Sim}}_i, i = 1,\dots,2d$. In the present situation, a simple verification shows that the matrix $\mathbf{C}$ has the following block diagonal structure:

$$
\mathbf{C} = \begin{bmatrix} -\frac{1}{6}\mathbf{I}_{d\times d} & -\frac{1}{6}\mathbf{I}_{d\times d} & \mathbf{1}_d & -\frac{2}{3}\mathbf{I}_{d\times d} & \mathbf{0}_{d\times d}\\ \mathbf{0}_{d\times d} & \mathbf{0}_{d\times d} & \mathbf{1}_d & -\mathbf{I}_{d\times d} & \mathbf{A}_{d\times d}\end{bmatrix} \in \mathbb{R}^{2d\times(4d+1)}, \tag{71}
$$

where $\mathbf{A}_{d\times d}$ is the diagonal matrix

$$
\mathbf{A}_{d\times d} = diag\left(-\frac{(\beta_{12}-\beta_{11})^2}{12},\dots,-\frac{(\beta_{d2}-\beta_{d1})^2}{12}\right).
$$

Taking the determinant of the sub-block matrix

$$
X = \begin{bmatrix} -\frac{2}{3}\mathbf{I}_{d\times d} & \mathbf{0}_{d\times d}\\ -\mathbf{I}_{d\times d} & \mathbf{A}_{d\times d}\end{bmatrix} \tag{72}
$$

yields the result

$$\det X = \left(\frac{2}{3}\right)^d \prod_{i=1}^{d} \frac{(\beta_{i2} - \beta_{i1})^2}{12},$$

where we have exploited the fact that the determinant of a block lower triangular matrix is the product of the determinants of its diagonal blocks, see [24]. In view of the fact that X has a non-vanishing determinant, we have shown that the rows of the matrix $\mathbf{C}$ are linearly independent. Hence, the $\mathscr{L}_i := S_i^{\mathrm{Sim}}$'s form a set of admissible functionals. □

Remark 5. For the construction of the matrix $\mathbf{C}$, defined by (36), rather than the perturbed midpoint cubature formula (68) we could have taken the following new cubature formula:

$$L_{2d+1}^{\mathrm{Wil}}(f) = \frac{L_i^{\mathrm{Wil}}(f) + L_{i+d}^{\mathrm{Wil}}(f)}{2} - \frac{(\beta_{i2} - \beta_{i1})^2}{12}\left(\frac{1}{|K|}\int_K \frac{\partial^2 f(\mathbf{x})}{\partial x_i^2}\,d\mathbf{x}\right) + S_{i+d}^{\mathrm{Sim}}(f), \tag{73}$$

which is a natural generalization in higher dimensions of the perturbed trapezoidal rule, see Atkinson [4],

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{f(a)+f(b)}{2} - \frac{b-a}{12}\big(f'(b) - f'(a)\big) + S^{\mathrm{per}}(f). \tag{74}$$

In this case, its corresponding sub-block matrix X is simply a $2d$-by-$2d$ diagonal matrix. We also may take any convex combination of the two cubature formulas (68) and (73).

Remark 6. As in Theorems 4 and 6, we finally observe that the reader can easily reformulate all our results of Theorem 7 for the case where the enrichment functions are based on single univariate functions.

4 Conclusion and Future Work

In this chapter we developed a general method for enriching *any* finite element approximations via the use of additional enrichment functions. Our significant results, derived in Theorems 3, 5 and 7, show that the approximation errors of a new family of multivariate versions of the trapezoidal, midpoint, and Simpson rules and their perturbations play a central role in the existence of our new elements. These latter natural generalize these rules in any dimension, and employ integrals over facets.

From a practical point of view, possibly the principal result of this chapter is that this methodology easily allowed us to recover the well-known nonconforming as the Han rectangular element and the nonconforming rotated element of Rannacher and Turek in any dimension. We note that we have only focused here in describing different practical properties of approximations with appropriate enrichment functions.

From a theoretical point of view, there remain open questions. The following items have not been treated in the present chapter:

(a) What are the associated local basis functions of the enriched elements?
(b) It will also be interesting to study the error estimates for such approximations.

This kind of questions and numerical tests of our new enriched finite elements will be considered in a coming paper.

Note: This chapter is dedicated to the memory of Professor Q.I. Rahman. A. Guessab had the great privilege to work with Q.I. Rahman, when he visited him in Montreal five times. Q.I. Rahman has been a visiting professor at the University of Pau, 1996. After this visit, they became very close family friends. They wrote the following paper together: Quadrature formulas and polynomial inequalities, Journal of Approximation Theory, Volume 90, Issue 2, 1997, 255–282. In this paper they established several inequalities for polynomials and trigonometric polynomials. They are all obtained as applications of certain new quadrature formulas. The topic we have chosen to write in memory of Professor Q.I. Rahman is how a new class of enriched nonconforming finite elements could be deduced from some new cubature formulas?

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