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# Variable Structure Control of Complex Systems

## Analysis and Design

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# Variable Structure Control of Complex Systems

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*This book is dedicated to:*

*Xing-Gang's son Jiaxiao, wife Ru Su and parents with love and gratitude.*
*Sarah's husband Chris and Children Ben, Hannah and Matt with love and thanks for all their support.*

*Chris' Parents.*

# Preface

It is well known that linear dynamical systems cannot adequately describe many phenomena commonly observed in the real world. With the advancement of science and technology, practical systems are becoming more complex in order to complete more advanced tasks. With the increasing requirements for system performance, linear system theory based study cannot satisfy the practical requirements, and the mathematical equations used to model real physical and engineering systems have become more and more complex. In reality, there are many factors which will affect system performance. To describe and explore various natural phenomena, it is necessary to consider these factors and thus to investigate complex systems as a means to model real systems more accurately. This book systemises aspects of the authors' recent achievements in the area of variable structure control alongside with some fundamental knowledge in the area.

This book focuses on the study of complex control systems in which the complexity mainly stems from nonlinearities, uncertainties, time-delay, faults and/or coupling among subsystems. It provides rigorous theoretical solutions to the problem of control of complex systems but has potential application in practical systems. It should be emphasised that many theoretical studies on control systems often assume that all system states are available for control design. This assumption is not valid for real systems in many cases. To implement such control schemes, a pertinent way forward is to construct an appropriate dynamical system which is called an observer, to estimate the state variables. Unfortunately, the traditional separation principle for linear control systems usually does not hold for the nonlinear counterpart, which implies that for nonlinear systems, the properties of a state feedback control law may not be achieved when the control law is implemented with the estimated states. In connection with this, this book focuses on output feedback control design: both static output feedback and dynamical output feedback strategies, including reduced order dynamical output feedback strategies, are proposed to control complex systems such that the closed-loop systems have the desired performance.

Variable structure control techniques have been extensively studied, and widely applied to theoretical research and practical engineering systems due to their high robustness. Specifically, as one special case of variable structure controllers, sliding mode controllers are completely robust to matched uncertainties. Moreover, the sliding motion is determined by reduced order dynamics, which facilitates the reduction of the effects of mismatched uncertainties on the whole systems when compared with other methods. A key development in this book considers variable structure control for complex systems based on only output information, using mainly the Lyapunov direct method and sliding mode techniques, with the objective of enhancing the robustness against uncertainties, reduction of conservatism and enlargement of the admissible systems. Rigorous stability analysis and design methodologies are provided from a theoretical perspective for this theme. Nonlinearities appear in all the considered systems throughout the book. Both the matched and mismatched uncertainties covered in this book are nonlinear and bounded by nonlinear functions. Since the considered systems are complex and all the results are rigorous, the conditions developed for all the main results in this book are sufficient. As there is no general way to obtain the design parameters for an output feedback controller, trying to determine 'easy' test conditions with low conservatism, by separating possible known information from the system and then employing them in the design to reduce the effects of factors such as uncertainties and time-delay on the system, is one of the main targets throughout this book. The book also presents novel contributions to deal with nonlinear uncertainties for time-delay systems by combining the Lyapunov–Razumikhin approach and variable structure techniques for different cases when delay is known and unknown respectively. It is shown that for interconnected systems, decentralised control schemes are available to cancel/reduce the effects of the interconnections on the whole system performance, under certain conditions. One of the characteristics of this monograph is that many examples and case studies with simulations are given to help readers understand the developed theoretical results and the proposed approaches.

The first two chapters present fundamental knowledge used in later developments. Chapter 1 develops some preliminary ideas regarding variable structure control. Specifically, the basic concepts and fundamental methodologies for sliding mode control and decentralised control are provided. Some of them are clarified for the first time based on the authors' understanding as a result of the authors' many years of research work in the areas. Several practical examples are given to show the potential application of complex systems. This helps readers understand the main methods used in the book intuitively from both mathematical and practical points of view. Chapter 2 presents some preliminary mathematical results and some results developed by the authors.

Chapter 3 considers static output feedback control design for both nonlinear systems and interconnected systems. For a class of fully nonlinear systems, a variable structure control based on Lyapunov methods is designed to drive and maintain the system in a 'small' region of the origin. Then, in the region, the nonlinear system is linearised and a sliding mode control is designed to stabilise the system asymptotically. Both controllers combined together stabilise the system globally. For interconnected systems, decentralised control schemes are developed and output variables embedded in the nonlinearity are separated and used in the control design to reduce conservatism. Case studies relating to a mass–spring system, coupled inverted pendulums and a flight control system are provided to illustrate the developed control methodologies.

Chapter 4 considers dynamical output feedback control design for systems with mismatched uncertainties/disturbances such that the corresponding closed-loop systems are asymptotically stable. Compared with Chap. 3, all the uncertainties involved in this chapter are bounded by nonlinear functions of the system state variables instead of the output variables. The bounding functions are assumed to be known and thus it is possible to use them for control design and system analysis to reduce the effects of uncertainties. In Sect. 4.2, a sliding surface is designed which is independent of the designed observer, and then a sliding mode control is synthesised based on the estimated states from the designed observer and the system outputs. The controller design and the observer design are separated. The designed control can be implemented with any appropriate observer but the developed approach requires that the considered system is minimum phase. In Sect. 4.3, a dynamical compensator is designed first. A sliding surface is then designed for the augmented system formed by the considered system and error dynamics. It is not required that the nominal system is minimum phase. Applications to control of the High Incidence Research Model (HIRM) aircraft are given in Sect. 4.4. Both longitudinal and lateral aircraft dynamics based on different trim values of Mach number and height are employed in the simulation study.

Chapter 5 continues to consider dynamical output feedback controller design. It focuses on large-scale interconnected systems and uses reduced order compensators to form the feedback loop which is particularly important for large-scale systems as it may avoid 'the curse of dimensionality'. In Sect. 5.2, sliding mode dynamics are established and the stability is analysed using an equivalent control approach and a local coordinate transformation. A robust decentralised output feedback sliding mode control scheme is synthesised such that the interconnected system can be driven to the predesigned sliding surface. This approach allows both the nominal isolated subsystem and the whole nominal system to be nonminimum phase. In Sect. 5.3, a similar structure is introduced to identify a class of nonlinear large-scale interconnected systems. By exploiting the system structure of similarity, the proposed nonlinear reduced order control schemes allow more general forms of uncertainties. Specifically, based on a constrained Lyapunov equation, the effect of matched uncertainties is canceled completely. The study shows that a similar structure can simplify the analysis and reduce the amount of computation. Numerical simulation examples and a case study on river pollution control are provided to illustrate the results developed.

Chapters 6 and 7 consider complex systems with time-delay. A Lyapunov–Razumikhin approach is employed to deal with time-delay throughout the two chapters. All the developed results are suitable for time-varying delay and there is no limitation to the rate of change of the time-varying delay as with the Lyapunov–Krasovskii approach. Chapter 6 requires that the time-delay is known and thus the time-delay can be used in the design to reduce conservatism. Therefore the controllers are delay dependent. Chapter 7 removes the assumption that the time-delay is known but the results obtained are usually conservative when compared with Chap. 6. In Chap. 6, both static and dynamical output feedback control schemes are presented for complex time-delay systems; decentralised static output feedback sliding mode controllers are designed to stabilise complex interconnected time-delay systems where delay exists in both the interconnections and the isolated subsystems. In Chap. 7, local stabilisation is considered for affine nonlinear control systems with uncertainties involving time-varying delay. It is not assumed that the nominal system is either linearisable or partially linearisable. Section 7.4 focuses on the stabilisation problem for a class of large-scale systems with nonlinear interconnections. A decentralised static output feedback variable structure control is synthesised and a set of conditions is developed to guarantee that the considered large scale interconnected systems are stabilised uniformly asymptotically. Section 7.5 provides some examples to demonstrate the results developed in Sects. 7.2–7.4. Numerical simulation examples and a case study on a mass–spring system are provided to demonstrate the theoretical results.

Chapter 8 considers fault detection and isolation (FDI) for nonlinear systems with uncertainties using particular sliding mode observers for which the design parameters can be obtained using LMI techniques. In Sect. 8.2, a sliding mode observer based approach is presented to estimate system faults using bounds on the uncertainties, and as a special case, a fault reconstruction scheme is available where the reconstructed signal can approximate the fault signal to any accuracy. Section 8.3 considers sensor FDI for nonlinear systems where a nonlinear diffeomorphism is introduced to explore the system structure and a simple filter is presented to ‘transform’ the sensor fault into a pseudo-actuator fault scenario. Both fault estimation and reconstruction are considered. Case studies on a robotic arm system and a mass–spring system demonstrate the effectiveness of the proposed FDI schemes.

Chapter 9 provides a decentralised strategy for the excitation control problem of multimachine power systems which are formed from an interconnected set of lower order subsystems through a network transmission. Both mismatched uncertainties in the interconnections and parametric uncertainties in the direct axis transient short circuit time constants, which affect the subsystem input distribution matrix, are considered. The proposed approach can deal with interconnection terms and parametric disturbances with large magnitude. The results obtained hold in a large region of operation if the control gain is high enough. This allows the operating point of the multimachine power system to vary to satisfy different load demands. Simulations based on a three-machine power system are presented to illustrate the proposed control scheme.

Chapter 10 makes some concluding remarks. Several specific examples are presented to show the complexity of the systems considered in this book. Some comments offer suggestions for future work. Finally, Appendixes A to D provide some results (with rigorous proofs), which are used in the book, and Appendix E presents notation and the parameters of the multimachine power system considered in Chap. 9.

The book aims to disseminate recent results in the area of variable structure control of complex systems. It is suitable for scientists and engineers in academia and industry who are interested in either variable structure techniques or complex systems including nonlinear control, decentralised control, time-delay systems, robust control and fault detection and isolation. It is particularly valuable to have a combined set of references at the end of the book for ease of access to many important theoretical and practical applications. It contains many case studies and numerical examples with simulations to help readers understand and apply the developed theoretical results. The analysis and design methodologies are also useful for both undergraduate and postgraduate students in the field of nonlinear control systems design. We believe mathematicians and control engineers will find this book useful.

Last but not least, we would like to point out that this book only attempts to present part of the authors' recent achievements in the area of complex variable structure control, which is obviously built on many other previous results. Although we have tried to cover most of the recent important ideas and results in the area, the exposition is far from a complete overview of the associated subjects. The bibliography includes only the literature which has been actually used in the book. We sincerely apologise for any serious omissions, large or small.

Canterbury, UK Xing-Gang Yan
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# Contents

# About the Authors

**Dr. Xing-Gang Yan** received B.Sc. and M.Sc. degrees from the Shaanxi Normal University and Qufu Normal University, respectively, and received Ph.D. degree in Control Engineering from the Northeastern University, China. He has been a postdoctoral research fellow or research associate in China, Hong Kong, Singapore and United Kingdom. Currently, he is a Senior Lecturer in the School of Engineering and Digital Arts in the University of Kent, United Kingdom. He is a member of IEEE. He has published more than 140 refereed papers and three books in the area of control theory and applications. His research interests include non-linear control, sliding mode control, decentralised control, time-delay systems, observer design and fault detection and isolation with applications.

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**Prof. Christopher Edwards** is Professor of Control Engineering in the College of Engineering, Mathematics and Physical Sciences at the University of Exeter. Between 1996 and 2012 he was a member of the Control Systems Group in the Engineering department at Leicester University. He is author of over 325 refereed papers, 14 chapter contributions to edited monographs and four books. In 2006, Prof. Edwards was awarded a Royal Academy of Engineering/Leverhulme Senior Research Fellowship to conduct research in the area of 'Sliding mode observers for fault detection, isolation and sensor fault tolerant control in micro/nano-satellites'. He is a subject editor for the International Journal of Nonlinear Control and an associated editor for the International Journal of Systems Science. He is currently the vice-chair of the IEEE Technical Committee on Variable Structure Systems.

# Notation and Symbols

| | |
|---|---|
| $\emptyset$ | The empty set |
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{R}^+$ | The set of nonnegative real numbers |
| $\mathbb{R}^n$ | The $n$ dimensional Euclidean space |
| $\mathbb{R}^{n\times m}$ | The set of $n \times m$ matrices with elements in $\mathbb{R}$ |
| $\lVert \cdot \rVert$ | The Euclidean norm or its induced norm |
| $I_n$ | The unit matrix with dimension $n$ |
| $\mathrm{Im}(A)$ | The range space of the matrix $A$ |
| $A^{(j)}$ | The $j$-th column vector of the matrix $A$ |
| $B_r$ or $\mathscr{B}_r$ | The ball $\{x \mid \lVert x\rVert < r\}$ with radius $r$ where $r \in (0, +\infty)$ |
| $\overline{B}_r$ or $\overline{\mathscr{B}}_r$ | The closure of $\mathcal{B}_r$ |
| $\partial\mathcal{B}_r$ or $\partial\overline{\mathscr{B}}_r$ | The boundary of $B_r$ |
| $A^\tau$ or $A^T$ | The transpose of matrix $A$ |
| $A^{-\tau}$ or $A^{-T}$ | The transpose of matrix $A^{-1}$ |
| $A > 0$ | $A$ is a symmetric positive definite matrix |
| $A < 0$ | $A$ is a symmetric negative definite matrix |
| $\overline{\sigma}(A)$ | The maximum singular value of the matrix $A$ |
| $\underline{\sigma}(A)$ | The minimum singular value of the matrix $A$ |
| $\lambda_{\min}(A)$ | The minimum eigenvalue of the square matrix $A$ |
| $\lambda_{\max}(A)$ | The maximum eigenvalue of the square matrix $A$ |
| $\mathrm{diag}\{A_1, A_2, \cdots, A_N\}$ | A block-diagonal matrix with diagonal elements $A_1, A_2, \cdots, A_N$ |
| $A^{\frac{1}{2}}$ | A symmetric positive definite matrix such that $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$ |
| $f^{-1}(\cdot)$ | The inverse function of the function $f(\cdot)$ |
| $\mathscr{L}_f$ | The Lipschitz constant of the function $f(\cdot)$ |
| $\mathscr{L}_f h$ | Derivative of the mapping $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$, along the vector field $f(x,u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ defined by $L_{f(x,u)}h(x) := \frac{\partial h}{\partial x} f(x,u)$ |
| $< d\lambda(x), f(x) >$ | $L_f\lambda(x)$ where $d\lambda = \frac{\partial\lambda}{\partial x}$ is the differential of $\lambda$ |

| | |
|---|---|
| $J_f(x)$ *or* $\frac{\partial f(x)}{\partial x}$ | The Jacobian matrix of the function $f(x)$ |
| $[f, g]$ | Lie bracket (product) of the vector fields $f(x)$ and $g(x)$, defined by $[f, g](x) = J_g(x)f(x) - J_f(x)g(x)$ |
| $ad_f^k g(x)$ | $[f, ad_f^{k-1} g](x)$ where $ad_f^0 g(x) := g(x)$ |
| $L_\psi(x_1, x_2)$ | Generalised Lipschitz constants about $x_1 \in \mathbb{R}^{n_1}$ uniformly for $x_2 \in \mathbb{R}^{n_2}$ where the function $\psi(x_1, x_2)$ is defined in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ |
| $L_f^r h$ | The $r$th order Lie derivative of the mapping $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$, along the vector field $f(x, u) : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ |
| $f(x, y)$ | $f(x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ where $x = [x_1 x_2 \cdots x_{n_1}]^T \in \mathbb{R}^{n_1}$ and $y = [y_1 y_2 \cdots y_{n_2}]^T \in \mathbb{R}^{n_2}$ |
| $\frac{\partial f(x,y)}{\partial x}$ | Jacobian matrix of function $f(x, y)$ relating the variable $x$ where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ |
| $\mathrm{col}\,(x_1, x_2, \ldots, x_n)$ | The coordinates $[x_1, x_2, \ldots, x_n]^T$ where $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ |
| $\mathrm{col}\,(x_1, x_2)$ | The coordinates $[x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}]^T$ where $x_1 = [x_{11}, x_{12}, \ldots, x_{1n}]^T \in \mathbb{R}^{n_1}$ and $x_2 = [x_{21}, x_{22}, \ldots, x_{2n}]^T \in \mathbb{R}^{n_2}$ |
| $\mathscr{C}_{[a,b]}$ | Represents the set of $\mathbb{R}^n$-valued continuous function on interval $[a, b]$ |
| $x_d$ | $x(t - d)$ where $d$ (may be time varying) represents time-delay |
| $A := B$ | $A$ is defined by $B$ |
| $A \Rightarrow B$ | $A$ implies $B$ |
| $A \Leftrightarrow B$ | $A$ is equivalent to $B$ |

# Chapter 1
# Introduction

Control systems widely exist in the real world. Increasing requirements for system performance and reliability have resulted in increasing complexity in the dynamic systems used to model reality. Control engineers are faced with increasingly complex control systems. The development of computer science and technology coupled with developments in mathematical theory has provided the possibility for study of complex systems from both theoretical and practical viewpoint.

This book systemises some of the authors' recent research works along with fundamental concepts and methodologies in the area of variable structure control for complex systems. The complexity resulting from nonlinearities, uncertainties including modelling error, time-delay, and interconnections between subsystems is considered. For various complex systems, theoretical analysis and control design using static output feedback, observer-based output feedback, and decentralised control ideas is presented based on variable structure techniques. The fault detection and isolation problem is also investigated, using sliding mode observers, where reconstruction and estimation schemes for both system faults and sensor faults will be presented. Numerous numerical examples and case studies with accompanying simulations are provided to help readers understand and apply the developed strategies and approaches.

## 1.1 System Complexity

Linear dynamical systems cannot describe many commonly observed phenomena well. In the real world, nearly all systems exhibit nonlinearity. In order to reveal complex phenomena and study complex systems, it is necessary to investigate nonlinear dynamical systems as a means to model real systems more accurately.

A dynamical control system usually can be expressed by the following differential equation

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$$\dot{x} = f(t, x, u) \tag{1.1}$$

where $x \in \mathbb{R}^n$ denotes the system state, $u \in \mathbb{R}^m$ represents the system input/control and $t \in \mathbb{R}^+$ is time. If a particular system output is of interest, then an algebraic equation

$$y = h(t, x) \tag{1.2}$$

or

$$y = h(t, x, u) \tag{1.3}$$

is used, where $y \in \mathbb{R}^p$ represents the system output. Equation (1.1) is called the state equation while Eq. (1.2) or (1.3) is said to be the output equation.

In this book, only the output Eq. (1.2) is considered which means that the output equation considered in this book does not involve the control variable $u$. The system (1.1)–(1.2) is called a *single-input single-output* (SISO) system if both $u$ and $y$ are scalars. It is called a *multi-input multi-output* (MIMO) system if the dimensions of either $u$ or $y$ are bigger than one.

The complexity of a control system depends on the controlled plant and the environment. Higher requirements on the controlled system's performance usually require more advanced control techniques, which will introduce additional complexity. There are many factors which may affect control system's performance and result in complex phenomena, such as nonlinearities, uncertainties/modelling errors, time-delay and any interconnections existing in the system.

- **Nonlinearity**: Compared with linear systems, the study of nonlinear systems is much more difficult. Analysis and design of nonlinear control systems usually involve more advanced mathematics. Due to the existence of nonlinearities in dynamical systems, phenomena such as finite time escape, multiple isolated equilibria, limit cycles, harmonic oscillation, chaos and multiple modes of behaviour may appear [91]. These rich behaviours which exist in nonlinear dynamical systems greatly increase the complexity of the problem.
- **Uncertainty/modelling error**: Real systems unavoidably experience various uncertainties such as mechanical wear and changes in the external environment. The former may result in parametric uncertainties while the latter may result in unstructured uncertainties. Moreover, it may be impossible to model a system accurately. If these modelling errors and uncertainties or disturbances are not considered, the developed strategies may not work well or may even fail to meet the design objective. Specifically, for a large-scale interconnected system, a perturbation of one subsystem can affect other subsystems and the overall performance of the network. This increases the complexity in the problem.
- **Time-delay**: With the increasing expectations for the closed-loop system's dynamical performance, it is required that the established system model behaves more like the real process. Thus time-delay has to be considered as many processes include after effect phenomena in their inner dynamics: for example, biology, population dynamics, economics, viscoelasticity and engineering science [130, 141]. For a

time-delay system, the future evolution, usually, not only depends on the present state but also on its history. Even a small delay may greatly affect the performance of a system; a stable system may become unstable, or chaotic behaviour may appear due to delay in the system [126].

- **Interconnection**: In order to complete a complex task, systems have to be combined together to provide the desired performance. For example, in a manufacturing process, in order to produce the same engineering components in sufficiently large quantities, many machine tools (isolated subsystems) are interconnected together and monitored to form a large-scale system to complete the task [202]. A complex system may also be formed by interconnections between a collection of simple systems. In this case, although each subsystem may exhibit good performance in isolation, the whole system may not work well due to the interactions between the subsystems. To reduce, minimise or even employ the effects of the interconnections on the whole system is challenging. Moreover, these subsystems are usually distributed geographically in space, which results in problems such as data transfer, the reliability of the network communication channels and economic cost etc. [2, 210].

In this book, the factors mentioned above will be considered. In order to deal with the effects of uncertainties, variable structure control techniques will be employed. The Lyapunov–Razumikhin approach will be used to deal with time-delay. For interconnected systems, decentralised strategies will be developed whenever possible to avoid the reliability problem caused by network links.

## 1.2 Variable Structure Control

Consider the control system (1.1) in the domain $D \in \mathbb{R}^n$. A corresponding *variable structure control* can be expressed as

$$u = \begin{cases} u_1(t,x), & (t,x) \in \mathbb{R}^+ \times D_1 \\ u_2(t,x), & (t,x) \in \mathbb{R}^+ \times D_2 \\ \quad\vdots & \qquad\quad\vdots \\ u_q(t,x), & (t,x) \in \mathbb{R}^+ \times D_q \end{cases} \tag{1.4}$$

where the functions $u_i(t,x)$ are continuous for $i = 1, 2, \ldots, q$. The structures of the functions $u_i(t,x)$ and $u_j(t,x)$ are different for $i \neq j$ and $i, j = 1, 2, \ldots, q$ ($q \geq 2$). The domains $D_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, q$ satisfy

(i) $D_1 \cup D_2 \cup \cdots \cup D_q = D$;
(ii) $D_i \cap D_j = \emptyset$ if $i \neq j$ for $i, j = 1, 2, \ldots, q$.

When the variable structure control in (1.4) is applied to the system (1.1), the corresponding closed-loop system becomes a *variable structure system*. Literally speaking, variable structure control is a control whose structure is changed or keeps changing in order to obtain and maintain the desired system performance during the control process.

For example, in real control design, when the response error/accuracy $e(t)$ is over the threshold, a proportional control is used to increase the response speed; when the response error/accuracy $e(t)$ is within the threshold, an integral control is employed to guarantee that the steady error requirement is satisfied. In this case, the control law may be described by

$$u = \begin{cases} k_p e(t), & \|e(t)\| > k \\ k_i \int e(t)dt, & \|e(t)\| < k. \end{cases}$$

Here the positive constants $k_p$ and $k_i$ are called the proportional gain and integral gain respectively which are tuning parameters, and the positive constant $k$ is called the threshold.

This example shows that sometimes it is desirable to change the control structure in order to get the desired system performance. As pointed out in [12], nonholonomic systems cannot be stabilised by continuously differentiable, time invariant state feedback control laws. However, a discontinuous control law is available to stabilise nonholonomic systems (see, e.g., [1]). This motivates the need for discontinuous control.

When the variable structure controller (1.4) is applied to the system (1.1), it usually produces a discontinuous right-hand side in the corresponding closed-loop dynamical system which consists of a set of ordinary differential equations. This produces an interesting mathematical problem: the traditional definition and existence conditions for the solutions of the closed-loop system are not applicable. It is necessary to extend the classical solution. In this case the solution of the equations is defined in the Filippov sense [46] throughout the book.

In order to reject/reduce the effects of uncertainties and disturbances, different variable structure approaches have been proposed, for example, the approach based on the direct Lyaponov method in [202, 210, 214] and a discontinuous control law for nonholonomic systems in [1]. However, variable structure control which leads to a sliding motion, has underpinned the development of a systematic research methodology, which is the well-known sliding mode control paradigm. Sliding mode control has dominated the literature in the area of variable structure control and thus when people talk about variable structure control, they usually mean sliding mode control. Here, it should be pointed out that not all variable structure control will lead to a sliding motion.

## 1.3 Sliding Mode Control

Sliding mode control, as a particular type of variable structure control, evolved from the pioneering work in Russia of Emel'yanov and Barbashin in the early 1960s. The ideas did not appear outside of Russia until the mid 1970s when a book by Itkis [81] and a survey paper by Utkin were published in English [175]. The ideas underlying the modern analysis and design of sliding mode controllers may be further dated back to publications in the early 1930s. At that time, concerns on relay systems with sliding modes for controlling the course of a ship had been proposed [55] where the terms phase plane, switching line, and even sliding mode appear [172].

Relay systems have been found in many control engineering systems. Relay control systems are a simple nonlinear system which is effective and has low cost. Sometimes they have better dynamical performance than linear systems [171]. Early rigorous studies on relay systems are found in contributions in the 1960s which were presented celebrating Filippov's achievement for differential equations with discontinuous right-hand sides [47]. The study of relay systems stimulated the study of sliding mode control.

In the initial stage (before 1962), nearly all studies focused on second-order linear systems. Later work was extended to higher order systems (i.e., systems with order greater than 2) but most work was still limited to linear systems with single input control. The study of nonlinear systems in state space form commenced in 1970 and multi-input control systems have been widely considered since then. The development of this state space description and multivariable control system theory greatly promoted the development of sliding mode controllers, which also motivated the application of sliding mode techniques in practical systems [172].

In recent decades, various control approaches have been proposed and research on sliding mode control has become very active. Due to its high robustness against uncertainties/disturbances, sliding mode control has been widely combined with other approaches to provide better results in both theoretical research and practical engineering. Many interesting results have been created in adaptive sliding mode control [4, 18, 176], fuzzy sliding mode control [168, 178], backstepping based sliding mode control [162] and decentralised sliding mode control [200, 201] with applications in wide areas such as engineering systems, aircraft control, energy systems, communication networks and biology [7, 77, 82, 129, 153, 172].

## *1.3.1 Sliding Mode Control Methodology*

Sliding mode control changes the system dynamics by employing a discontinuous control signal. This approach has been well developed and extensively used in theoretical research and practical engineering design. It has been successfully employed to solve various control problems in combination with other control approaches.

The sliding mode control method consists of two steps:

- the design of a sliding surface such that the system considered possesses the desired performance when it is restricted to the surface;
- the design of a variable structure control which drives the system trajectory to the sliding surface in finite time and maintains a sliding motion on it thereafter.

A concise description is available in [38, 173]. In view of these two steps, the system motion can be separated into two phases: the *reaching phase* and the *sliding phase*. The former refers to the motion when the system trajectory moves towards the sliding surface and the latter concerns the motion when the system trajectory moves on the sliding surface.

#### 1.3.1.1 Sliding Phase

Consider System (1.1). In order to design a proper switching/sliding function

$$s = s(x)$$

such that the resulting sliding motion has the desired performance, one way is to find the dynamical equations which will govern the sliding motion, and then synthesise the sliding surface based on the characteristics of the sliding mode dynamics or sliding motion. It is assumed that the sliding motion exists. The following two approaches are usually employed to find the sliding mode dynamics and in this way the stability of the sliding motion is transformed to the problem of ensuring stability of an unforced system.

- **Equivalent control**: When the considered system (1.1) is limited to and moving on the sliding surface,

  $$s(x) = 0, \qquad \text{and} \qquad \dot{s}(x) = 0.$$

  The time derivative of $s(x)$ along the system (1.1) is given by

  $$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t, x, u).$$

  In the sliding motion,

  $$\frac{\partial s}{\partial x} f(t, x, u) = 0. \tag{1.5}$$

  Suppose there is a solution for $u$ to the Eq. (1.5) denoted by

  $$u_{eq} = u_{eq}(t, x)$$

  which is the so-called *equivalent control* (see, p. 14 in [174]). Then, the sliding mode dynamics governing the sliding motion may be obtained by

  $$\begin{cases} \quad \dot{x} = f(t, x, u_{eq}(t, x)) \\ s(x) = 0 \end{cases}. \tag{1.6}$$

Now, assume that System (1.1) is in the following affine nonlinear form,

$$\dot{x} = F(t, x) + G(t, x)u. \tag{1.7}$$

Then, for the sliding surface $s(x) = 0$, it follows from $\dot{s}(x) = 0$ that the corresponding equivalent control is given by

$$u_{eq} = -(s(x)G(x, t))^{-1}s(x)F(t, x) \tag{1.8}$$

where $s(x)$ should be chosen such that $s(x)G(x, t)$ is nonsingular for all $x$ in the considered domain and $t \in \mathbb{R}^+$. Substitute $u_{eq}$ from (1.8) into the system (1.1), it follows that the corresponding sliding motion can be described by

$$\begin{cases} \quad \dot{x} = F(t, x) - G(t, x)(s(x)G(x, t))^{-1}s(x)F(t, x) \\ s(x) = 0 \end{cases}.$$

*Remark 1.1* It should be noted that the equivalent control is used only to analyse the sliding motion. It is not the control signal which is actually applied to the system but it may be thought of as the control signal which must be applied "on average" to maintain the sliding motion [38, 174].

- **Regular form**: Another approach to find the sliding mode dynamics relating to the sliding function $s = s(x)$ for System (1.1) is to employ the well-known regular form. Suppose that there exists a coordinate transformation $z = T(x)$ such that in the new coordinate system $z$, the sliding surface $s(x) = 0$ can be described in the form

$$z_2 = \sigma(z_1)$$

where $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^m$, $z := \text{col}(z_1, z_2)$ and System (1.1) can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{1.9}$$

$$\dot{z}_2 = F_2(t, z_1, z_2, u) \tag{1.10}$$

where $u \in \mathbb{R}^m$ is the control. The Jacobian matrix $\frac{\partial F_2(t,z_1,z_2,u)}{\partial u}$ is assumed to be nonsingular in the considered domain. Note that System (1.9) is independent of the control signal and the dimension of $z_2$ is the same as the dimension of the control $u$. System (1.9)–(1.10) is the so-called *regular form*.
Based on the regular form in (1.9)–(1.10), it is clear to see that the corresponding sliding mode dynamics of System (1.1) is described by

$$\dot{z}_1 = F_1(t, z_1, \sigma(z_1)) \tag{1.11}$$

which is a reduced-order system when compared with System (1.1).
Note, if System (1.1) is in the following affine form as given in (1.7), then, the corresponding regular form can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{1.12}$$
$$\dot{z}_2 = F_2(t, z_1, z_2) + G_2(t, z_1, z_2)u \tag{1.13}$$

where the functions $F_1(\cdot)$ and $F_2(\cdot)$, and $G_2(\cdot)$ are dependent on the coordinate transformation $z = T(x)$ and the functions $F(\cdot)$ and $G(\cdot)$ respectively.

#### 1.3.1.2 Reaching Phase

In order to guarantee that the system trajectory can be driven to the sliding surface $s(x) = 0$ in finite time and a sliding motion can be maintained on it thereafter, a proper discontinuous control

$$u = u(t, x)$$

needs to be designed such that the following condition is satisfied [38, 173]

$$s^T(x)\dot{s}(x) \le -\eta\|s(x)\| \tag{1.14}$$

for some constant $\eta > 0$. The inequality (1.14) is the so-called *reachability condition* and $\eta$ is called the *reachability constant*.

From Eq. (1.1), it follows that

$$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t, x, u).$$

Therefore, Inequality (1.14) is equivalent to

$$s^T(x)\frac{\partial s}{\partial x} f(t, x, u) \le -\eta\|s(x)\| \tag{1.15}$$

which explicitly contains the variable $u$. The sliding mode controller guaranteeing reachability can usually be synthesised from (1.15).

The following condition

$$s^T(x)\dot{s}(x) < 0$$

is also called a reachability condition but it cannot guarantee that a sliding motion takes place in finite time and thus a sliding motion may not occur in this case.

It should be emphasised that, when the designed sliding/switching function is time varying, for example,

$$s = s(t, x)$$

it is straightforward to see that the condition (1.15) used to synthesise the sliding mode control law should be updated to

$$s^T(t,x)\left(\frac{\partial s}{\partial t}+\frac{\partial s}{\partial x}f(t,x,u)\right)\le -\eta\|s(t,x)\|.$$

For this case, a design approach has been provided in [27].

### 1.3.2 Sliding Mode Control of a Mass–Spring Damper System

In order to illustrate the sliding mode control methodology, consider the simple mass–spring damper mechanical system in Fig. 1.1 where the mass $M$ slides on a smooth surface. In Fig. 1.1, $X$ denotes the displacement from the reference position, $m$ is the mass of the object $M$ sliding on a horizontal surface, $k$ is the coefficient of spring $K$, $b$ is the coefficient of the damper $B$ and $F$ is an external force which is considered as the control input $u$ ($u = F$).

It is assumed that the mass–spring damper system experiences a hardening spring which produces a restoring force described by (see [91])

$$k(1+a^2X^2)X.$$

The simple viscous damper produces a damping force described by $b\dot{X}$. From Newton's second law, the motion of the object $M$ can be described by

$$m\ddot{X} = -b\dot{X} - k(1+a^2X^2)X + u. \tag{1.16}$$

Let $x = \mathrm{col}(x_1, x_2) = (X, \dot{X})$. Then, $\dot{x}_1 = x_2$ and $\dot{x}_2 = \ddot{X}$. From Eq. (1.16), it follows that

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 + u$$

which can be rewritten by

$$\dot{x}_2 = -\Big(\frac{k}{m}+\frac{k}{m}a^2x_1^2\Big)x_1 - \frac{b}{m}x_2 + u.$$

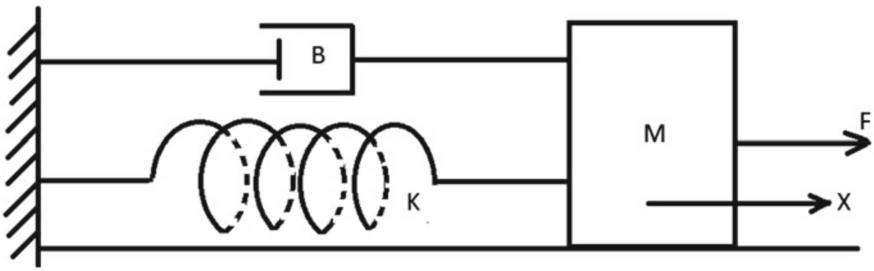


**Fig. 1.1** A mass–spring damper mechanical system

Choose $m = b = k = a = 1$ for simplicity. Then, the system (1.16) can be described in the form of (1.1) by

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -(1 + x_1^2)x_1 - x_2 + u \end{bmatrix}}_{f(x,u)} \tag{1.17}$$

which is a nonlinear system.

The objective is to design a sliding mode control law such that the system (1.17) is asymptotically stable.

(i) **Sliding phase**: Design a linear switching function

$$s(x) = \gamma x_1 + x_2 \tag{1.18}$$

where $\gamma$ is a design parameter. When System (1.17) is limited to the sliding surface, $s(x) = 0$. It follows from (1.18) that

$$x_2 = -\gamma x_1.$$

Considering the structure of System (1.17), it is straightforward to see that the corresponding sliding mode dynamics are

$$\dot{x}_1 = -\gamma x_1. \tag{1.19}$$

Therefore, the sliding motion governed by the sliding mode dynamics (1.19) is asymptotically stable if the parameter $\gamma$ is chosen to satisfy $\gamma > 0$.

(ii) **Reaching phase**: Consider the sliding mode controller

$$u = (1 + x_1^2)x_1 + x_2 - \gamma x_2 - \eta \text{sgn}(\gamma x_1 + x_2) \tag{1.20}$$

where $\eta > 0$ is a constant. Then the closed-loop system obtained by applying the control in (1.20) to System (1.17) is given by

$$\dot{x}_1 = x_2 \tag{1.21}$$

$$\dot{x}_2 = -\gamma x_2 - \eta \text{sgn}(\gamma x_1 + x_2). \tag{1.22}$$

By direct computation, it follows from Eqs. (1.21)–(1.22) that

$$\begin{aligned} s(x)\dot{s}(x) &= -s(x)(\gamma \dot{x}_1 + \dot{x}_2) \\ &= -\eta s(x)\text{sgn}(s(x)) \leq -\eta |s|. \end{aligned}$$

This guarantees that the control (1.20) can drive the trajectories of System (1.17) to the sliding surface $s(x) = 0$ with $s(\cdot)$ defined in (1.18), in finite time and maintain a sliding motion on it thereafter.

From sliding mode control theory, (i) and (ii) above together show that the corresponding closed-loop system is asymptotically stable. For simulation purposes, choose

$$\gamma = 0.5, \qquad \eta = 1$$

and the initial condition $x_0 = \text{col}(2, 1)$.

Figure 1.2 shows the phase plane portrait of the displacement $x_1$ and velocity $x_2$. From Fig. 1.2, the system states $(x_1, x_2)$ are driven to the sliding surface from the initial point $x_0 = (2, 1)$, and then move along the sliding surface to converge to the origin.

The time responses of the displacement and velocity of the object are shown in Fig. 1.3. Figure 1.4 shows the control signal imposed on the system.

It is clear to see that *chattering* appears due to the discontinuity in the control.

Chattering may be undesirable in practice because it may result in unnecessary wear and tear on the actuator components and result in unnecessary energy consumption. One way of overcoming this drawback is to introduce a boundary layer about the discontinuous surfaces (see [13]) which may affect the control accuracy. Another way is to use higher order sliding mode techniques but this requires the considered system to have a certain structure.

In this book, higher order sliding mode techniques will not be discussed. Detailed information about higher order sliding mode control can be found in [5, 45, 100, 153] and the references therein.

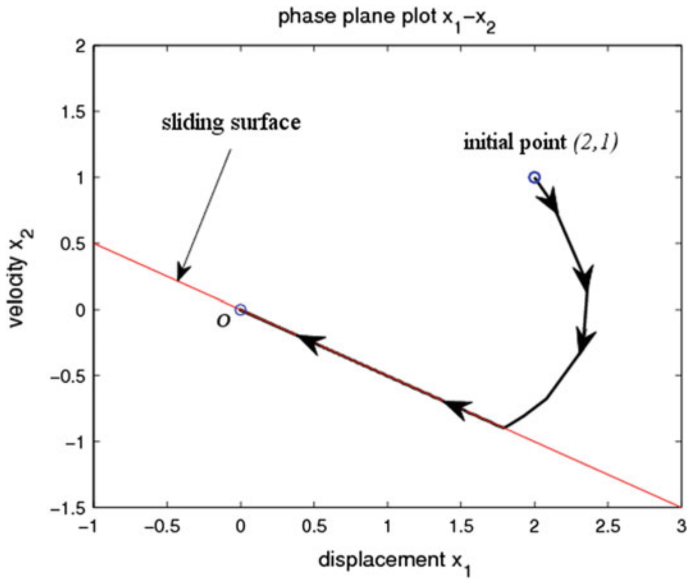


**Fig. 1.2** The phase plane portrait

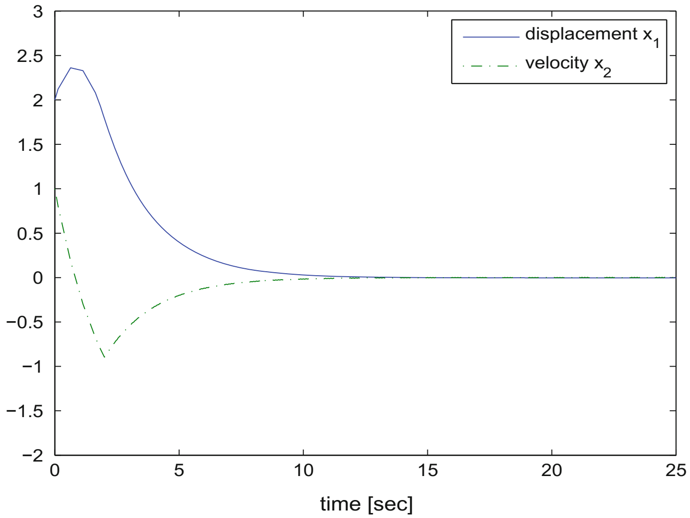


**Fig. 1.3** The time responses of the displacement $x_1$ and velocity $x_2$

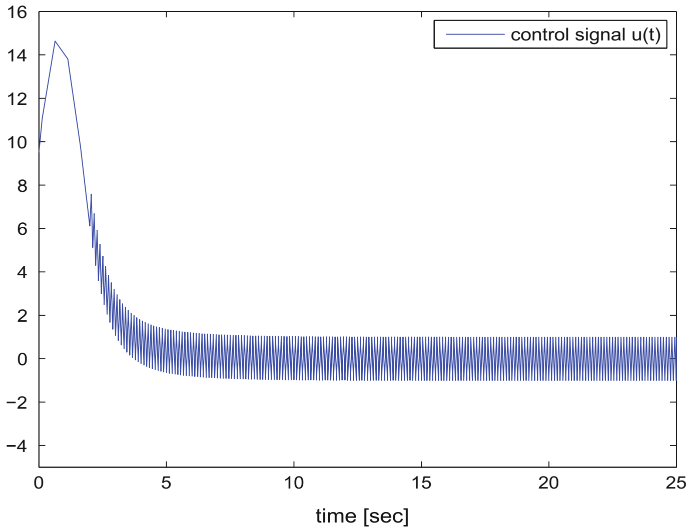


**Fig. 1.4** The time response of the control signal

### *1.3.3 Characteristics of Sliding Mode Control*

It is observed that sliding mode control has the following characteristics:

- The sliding mode dynamics are a reduced-order system when compared with the original system dynamics.

For System (1.1) with sliding surface $s = s(x)$, the corresponding sliding mode dynamics can be described by (1.6) or (1.11). It is clear to see that the order of the system (1.6) or (1.11) is $n - m$ where $n$ is the dimension of the original system and $m$ is the dimension of the control. Therefore, during the sliding motion, the system exhibits reduced-order dynamics when compared with the original system.

- The sliding motion is insensitive to matched uncertainty.
  Suppose System (1.1) experiences an uncertainty/disturbance. If the uncertainty or disturbance acts in the input/control channel or the effects are equivalent to an uncertainty acting in the input channel, it is called *matched uncertainty*. Otherwise it is called *mismatched uncertainty*. For example, assume that the nonlinear affine control system (1.7) experiences uncertainties $\phi(t, x)$ and $\psi(t, x)$ described by

$$\dot{x} = F(t, x) + G(t, x)(u + \phi(t, x)) + \psi(t, x). \tag{1.23}$$

  Then, the term $\phi(t, x)$ is called matched uncertainty. In addition, if the uncertainty $\psi(t, x)$ can be modelled as

$$\psi(t, x) = G(t, x)\chi(t, x)$$

  where $\chi(\cdot)$ represents the uncertainty, it is clear to see that the uncertainty of the term $\psi(\cdot)$ is reflected by the uncertainty $\chi(\cdot)$ which is exactly acting in the input channel. In this case $\psi(t, x)$ is also called matched uncertainty.
  From Eq. (1.6) or (1.11), it is straightforward to see that the dynamics governing the sliding motion are completely independent of the control and thus the system is robust to matched uncertainty.
- Uncertainties/disturbances will affect reachability.
  In order to guarantee that the trajectory of the considered system is driven to the predesigned sliding surface, the reachability condition must be satisfied—which is interpreted as (1.15). It is clear that (1.15) involves all of the right-hand side of Eq. (1.1). Therefore, uncertainties/disturbances may affect the reaching phase no matter whether they are matched or mismatched, but the effects of some uncertainties may be completely rejected by an appropriate control.
- The process of designing the sliding surface and sliding mode control can be 'separated'.
  The main target of sliding surface design is to ensure that the resulting sliding motion has the required performance. The main objective of the control design is that the reachability condition is satisfied so that the system can be driven to the sliding surface. In view of this, sliding surface design and sliding mode control design can be completed separately. This property is called the design 'separation' property in this book.
  The design of a sliding surface is usually not dependent on the process of the sliding mode control design. Once the sliding surface is specified, the study of the stability of the sliding motion and the reachability can be carried out separately. This has advantages when compared with other control approaches. For example, the steady-state response is totally dependent on the sliding mode dynamics

which is independent of the control. Therefore, in order to improve the steady-state response of the control system, it is only necessary to consider the sliding mode dynamics instead of the original system. In the reaching phase, by adjusting the parameters in the sliding mode control law, the reaching time can be reduced which may produce a fast time response, and will also maximise robustness.

## 1.4 Decentralised Control

In the real world, there are a number of important systems which can be modelled as dynamical equations composed of interconnections between a collection of lower-dimensional subsystems. Such classes of systems are called large-scale interconnected systems, which are often widely distributed in space [111, 117, 145]. A fundamental property of an interconnected system is that a perturbation of one subsystem can affect the other subsystems as well as the overall performance of the entire network. Decentralised control has been recognised as an effective method to control such systems.

### *1.4.1 Background*

Large-scale interconnected systems widely exist in society. A typical large-scale interconnected system is the multimachine power system [182, 201]. Other examples of large-scale interconnected systems that present a great challenge to both system analysts and control designers include power networks, ecological systems, biological systems and energy systems [117, 158].

For interconnected systems, the presupposition of centrality fails to hold due to either the lack of centralised information or the lack of centralised computing capability. When the number of subsystems is large, the computation time increases significantly if centralised control is employed. In the extreme case when information transfer among the subsystems is blocked, centralised control schemes simply cannot be applied. Even with engineered systems, issues such as the economic cost and reliability of communication links, particularly when systems are characterised by geographical separation, limit the appetite to develop centralised systems. From the perspective of economics and reliability, decentralised strategies are pertinent for large-scale interconnected systems. This has motivated the application of decentralised control methodologies to interconnected systems [87, 106, 192]. A survey paper [2] has covered several decomposition approaches such as disjoint subsystems, overlapping subsystems, symmetric composite systems, multi-time scale systems and hierarchically structured systems to simplify the analysis and synthesis tasks for large-scale systems to reduce the computational complexity.

Decentralised control for large-scale interconnected systems has been studied extensively. Research on large-scale interconnected systems analysis and synthesis can be traced back to at least the 1970s, and the survey paper [145] clearly shows the development of this topic at that time, when almost all of the work focused on linear cases. With the advancement of technology and increasing requirements for high levels of performance, specifically in recent years, the dynamic systems used to model reality have become more complex involving nonlinearities, uncertainties, time-delay and interconnection. Therefore, the study of complex interconnected systems has become increasingly important. The interest in this subject has been revived by new developments in nonlinear systems and control. The recent survey paper [216] has shown the progress made in the area of decentralised control where some of the work associated with sliding mode control, adaptive control and backstepping control has been covered.

### *1.4.2 Fundamental Concept*

From the mathematical point of view, a nonlinear large-scale interconnected system composed of $N$ $n_i$-th order subsystems can be described by

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x_i)\big(u_i + \Delta g_i\big(t, x_i\big)\big) + \Delta f_i(t, x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \zeta_{ij}(t, x_j) \tag{1.24}$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, N, \tag{1.25}$$

where $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$ ($\Omega_i$ is a neighbourhood of the origin), $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the $i$-th subsystem respectively for $i = 1, 2, \ldots, N$. All the matrix functions $g_i(\cdot) \in \mathbb{R}^{n_i \times m_i}$ and the nonlinear vectors $f_i(\cdot) \in \mathbb{R}^{n_i}$ and $h_i(\cdot) \in \mathbb{R}^{p_i}$ with $h_i(0) = 0$ are known. The terms $\Delta g_i(\cdot)$ and $\Delta f_i(\cdot)$ represent the matched and the mismatched uncertainties respectively. The term

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \zeta_{ij}(t, x_j)$$

represents the interconnection of the $i$-th subsystem with the other subsystems. It is assumed that all the nonlinear functions are smooth enough such that the unforced systems have unique continuous solutions.

**Definition 1.1** Consider System (1.24)–(1.25). The system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x_i)\big(u_i + \Delta g_i\big(t, x_i\big)\big) + \Delta f_i(t, x_i) \tag{1.26}$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, N, \tag{1.27}$$

is called the $i$-th *isolated subsystem* of System (1.24)–(1.25), and the system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x_i)u_i \tag{1.28}$$

$$y_i = h_i(x_i), \qquad i = 1, 2, \ldots, N, \tag{1.29}$$

is called the $i$-th *nominal isolated subsystem* of System (1.24)–(1.25).

It is well known that one of the main problems for interconnected systems is to establish under what conditions the interconnected system (1.24)–(1.25) exhibits the desired performance if all the isolated subsystems (1.26)–(1.27) or all the nominal isolated subsystems (1.28)–(1.29) exhibit the required performance. Therefore, how to deal with interconnections is a key problem of interest in decentralised control.

**Definition 1.2** Consider System (1.24)–(1.25). If the designed controllers $u_i$ for the $i$-th subsystems depend on the time $t$ and states $x_i$ of the $i$-th subsystem only, i.e.,

$$u_i = u_i(t, x_i), \qquad i = 1, 2, \ldots, N \tag{1.30}$$

then (1.30) is called *decentralised state feedback control*. If the controllers in (1.30) have the form

$$u_i = u_i(t, y_i), \qquad i = 1, 2, \ldots, N \tag{1.31}$$

that is, each local controller depends upon the time $t$ and the outputs of the local subsystem only, then they are called *decentralised static output feedback control*. Furthermore, if the designed controllers consist of the dynamical systems

$$\dot{\hat{x}}_i = \phi_i(t, \hat{x}_i, u_i, y_i), \qquad i = 1, 2, \ldots, N \tag{1.32}$$

and controllers

$$u_i = u_i(t, \hat{x}_i, y_i), \qquad i = 1, 2, \ldots, N \tag{1.33}$$

then (1.32)–(1.33) is called *decentralised dynamical output feedback control*. Specifically, if (1.32) is an observer of the system (1.24)–(1.25), then it is called *decentralised observer-based feedback control*.

It is straightforward to see, according to Definition 1.2 above, that it is required that the dynamical systems (1.32) are decoupled in a decentralised dynamical output feedback scheme. It should be mentioned that in some of the existing work, see for example [203, 215], the designed dynamical systems (1.32) are not decoupled (in fact they are interconnected systems). In this case, the developed controllers are sometimes still called a decentralised control. However, in precise terms, such a class of controllers is not decentralised because there exists information transfer between subsystems of the designed dynamical system (see e.g., [203, 215]).

Several decades ago, most work on decentralised control focused on linear interconnected systems due to the limitation of available control paradigms that were able to deal with nonlinearity. However, the dynamics of large-scale natural and engineered interconnected systems are usually highly nonlinear. It is not only the structure of the system and interconnections which produce complexity but also the nonlinearity of the dynamics themselves. It is clear that although linear dynamics may approximate the orbit of a nonlinear system locally, it does not permit the existence of the multiple states observed in real networks and does not accommodate global properties of the system. Such global properties can be crucial because they may become significant when the system is perturbed or a subsystem enters a failure state. Increasing requirements on system performance coupled with the ability to model and simulate reality by means of complex, possibly nonlinear, interconnected systems models has motivated increasing contributions in the study of such systems. This interest has been further stimulated by the simultaneous development of nonlinear systems theory and the emergence of powerful mathematical and computational tools which render the formal and constructive study of nonlinear large-scale systems increasingly possible [210].

In order to help readers to understand the 'decentralised' concept, the following schematic diagram in which the interconnected system has three subsystems, is produced to show that in static decentralised output feedback control scheme, the local controller $u_i$ of the $i$-th subsystem only uses the local output information $y_i$; no output information $y_j$ $(j \neq i)$ is involved in the design of $u_i$. From Fig. 1.5, it is clear that there is no local output information transfer between the local controllers $u_i$ and $u_j$ $(i \neq j)$ for $i, j = 1, 2, 3$.

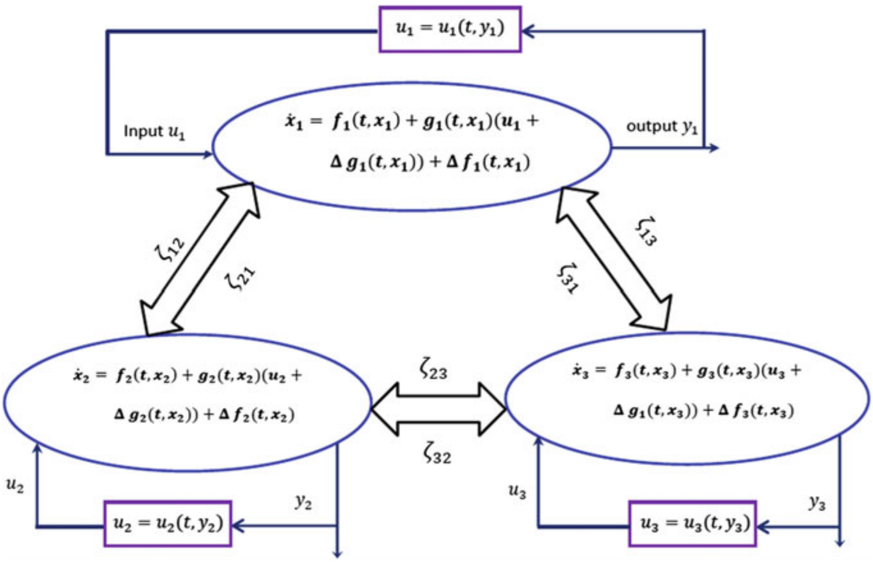


**Fig. 1.5** Decentralised static output feedback schematic diagram

## 1.5 Examples of Complex Systems

In this section, some practical examples will be presented to show that complex systems widely exist in the real world.

### *1.5.1 One-Machine Infinite-Bus System*

Consider a simple power system where a large-turbine generator set connects with an infinite bus. The motion equation of the machine's rotor can be described by (see, for example, [107])

$$H\frac{d^2\delta}{dt^2} = M_m(t) - \frac{E_q V_s}{X_\delta}\sin\delta(t) \tag{1.34}$$

where $\delta(t)$ is the generator's rotor angle, $M_m$ is the mechanical input torque, $H$ is the moment of inertia of the machine, $E_q$ is the transient potential of the $q$-axis of the generator, $V_s$ is the voltage of the infinite bus which is constant, $X_\delta$ is the sum of the transient inductance of the shaft of generator, the inductance of the transformer and the inductance of the transmission line.

For simplicity, assume that $E_q$ is constant. Let

$$x_1 = \delta \qquad \text{and} \qquad x_2 = \dot{\delta}$$

where $x_2$ represents the angular velocity. The letter $M_m$ denotes the control input $u$. Then the system (1.34) modelling the one-machine infinite-bus is described by

$$\dot{x} = \begin{bmatrix} x_2 \\ -a_1 \sin x_1 + a_2 u \end{bmatrix} \tag{1.35}$$

where $x := \text{col}(x_1, x_2)$, and

$$a_1 := \tfrac{E_q V_s}{H X_\delta}$$
$$a_2 := \tfrac{1}{H}.$$

System (1.35) is a nonlinear affine system as it can be described by

$$\dot{x} = \begin{bmatrix} x_2 \\ -a_1 \sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \end{bmatrix} u$$

where the input distribution is a constant matrix.

### 1.5.2 PVTOL Aircraft

The well-known planar vertical take-off and landing (PVTOL) represents a challenging nonlinear control problem. It is motivated by the need to stabilise an aircraft which is able to take-off vertically such as helicopters and some special aircraft.

The mathematical model describing an aircraft that evolves in a vertical plane usually has three degrees of freedom $(X, Y, \phi)$ corresponding to its position $(X, Y)$ and orientation in the plane $\phi$. The PVTOL is composed of two independent thrusters that produce a force and a moment on the aircraft. The dynamical model of the PVTOL aircraft can be obtained using the Lagrangian approach or Newtons laws, which are given in [191], as follows

$$\begin{aligned} m\ddot{X} &= -(\sin\phi)U_1 + \varepsilon_0(\cos\phi)U_2 \\ m\ddot{Y} &= (\cos\phi)U_1 + \varepsilon_0(\cos\phi)U_2 - mg \\ J\ddot{\phi} &= U_2 \end{aligned}$$

where $(X, Y)$ is the centre of mass of the aircraft, $\theta$ is the roll angle, $mg$ is the gravity force imposed at the aircraft's centre of mass and $J$ is the mass moment of inertia around the axis through the aircraft's centre of the mass and along the fuselage, the control $U_1$ is the thrust directed to the bottom of aircraft and the control $U_2$ the moment around the aircraft's centre of the mass, $\varepsilon_0$ is the quantity of lateral force induced by the rolling moment which characterises the coupling between the rolling moment and the lateral acceleration of the aircraft.

Let

$$\bar{x} = -X/g, \quad \bar{y} = -Y/g, \quad u_1 = U_1/mg$$

$$U_2 = U_2/mg, \quad \varepsilon = \varepsilon_0 J/mg.$$

Then the normalised PVTOL aircraft dynamics can be described by [16, 191]

$$\ddot{\bar{x}} = -(\sin\phi)u_1 + \varepsilon(\cos\phi)u_2 \tag{1.36}$$

$$\ddot{\bar{y}} = (\cos\phi)u_1 + \varepsilon(\cos\phi)u_2 - 1 \tag{1.37}$$

$$\ddot{\phi} = u_2. \tag{1.38}$$

The dynamical equations (1.36)–(1.38) can be described in (1.1) as follows

$$\dot{x} = \begin{bmatrix} x_2 \\ -(\sin x_5)u_1 + \varepsilon(\cos x_5)u_2 \\ x_4 \\ (\cos x_5)u_1 + \varepsilon(\cos x_5)u_2 - 1 \\ x_6 \\ u_2 \end{bmatrix} \tag{1.39}$$

where

$$x_1 := \bar{x}, \quad x_2 := \dot{\bar{x}}, \quad x_3 := \bar{y}$$

$$x_4 := \dot{\bar{y}}, \quad x_5 := \phi, \quad x_6 := \dot{\phi}.$$

System (1.39) can be rewritten by

$$\dot{x} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ -1 \\ x_6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\sin x_5 & \varepsilon \cos x_5 \\ 0 & 0 \\ \cos x_5 & \varepsilon \cos x_5 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $x := \mathrm{col}(x_1, x_2, \ldots, x_6)$, and thus it represents an affine nonlinear control system. In general, $\varepsilon$ is unknown but it is very small and can be neglected. In this case, the model can be simplified as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(\sin x_5)u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (\cos x_5)u_1 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2. \end{aligned}$$

This is a nonlinear system.

### *1.5.3 Stirred Tank Reactor*

Consider an industrial jacketed continuous stirred tank reactor (JCSTR) with a delayed recycle stream [116]. The reactions within the JCSTR are assumed unimolecular and irreversible (exothermic). Perfect mixing is assumed and the heat losses are neglected. The reactor accepts a feed of reactant which contains a substance $A$ with initial concentration $C_{A_0}$. Cooling of the tank is achieved by a flow of water around the jacket and the water flow in the jacket $F_J$ is controlled by actuating a valve.

Suppose that a fresh feed of pure substance $A$ is to be mixed with a recycled stream of unreacted substance $A$ with a recycle flow rate

$$1 - c, \qquad (0 \le c \le 1)$$

where $c$ is the coefficient of recirculation.

The change of concentration arises from three terms: the amount of substance $A$ that is added with feed under recycling, the amount of substance $A$ that leaves with the product flow, and the amount of the substance $A$ that is used up in the reaction. The change in the temperature of the fluid comes from the following four factors: the heat that enters with the feed flow under recycling, the heat that leaves with the product flow, the heat created by the reaction and the heat that is transferred to the cooling jacket. There are three terms associated with the changes of the temperature of the fluid in the jacket: one term representing the heat entering the jacket with the cooling fluid flow, another term accounting for the heat leaving the jacket with the outflow of cooling liquid and a third term representing the heat transferred from the fluid in the reaction tank to the fluid in the jacket.

Under conditions of constant hold-up, constant densities and perfect mixing, the energy and material balances can be expressed mathematically as [116]:

$$
\begin{aligned}
\dot{C}_A &= (FV)^{-1}\left(cC_{A_0} - cC_A - cC_A(t-d)\right) - k_1 C_A e^{-\frac{k_2}{T}} \\
\dot{T} &= (FV)^{-1}\left(cT_0 - cT - cT(t-d)\right) - k_1 k_3 C_A e^{-\frac{k_2}{T}} - k_4(T - T_J(t)) \\
\dot{T}_J &= (F_J V_J)^{-1}\left(T_{J_0} - T_J\right) - k_5(T - T_J)
\end{aligned}
\tag{1.40}
$$

where $C_A$ is the concentration of the substance $A$, $T$ is the temperature of the fluid in the tank, $T_J$ is the temperature of the jacket, $V$ is the volume of the tank (gallons), $F$ is the feed entry rate, the initial temperature is $T_0$, and $d$ represents the transport delay in the recycled stream.

It is straightforward to see that System (1.40) is a nonlinear time-delay control system and can be described in the form of (1.1) as

$$
\dot{x} = \begin{pmatrix} (FV)^{-1}\left(cC_{A_0} - cx_1 - cx_1(t-d)\right) - k_1 x_1 e^{-\frac{k_2}{x_2}} \\ (FV)^{-1}\left(cT_0 - cx_2 - cx_2(t-d)\right) - k_1 k_3 x_1 e^{-\frac{k_2}{x_2}} - k_4(x_2 - x_3(t)) \\ (uV_J)^{-1}\left(T_{J_0} - x_3\right) - k_5(x_2 - x_3) \end{pmatrix}
$$

where $x_1 = C_A, x_2 = T, x_3 = T_J, x = \mathrm{col}(x_1, x_2, x_3)$ is the system states and $u = F_J$ is the system input. The letter $d$ represents the time-delay.

## *1.5.4 Coupled Inverted Pendula on Carts*

Consider a coupled inverted pendulum connected by a moving spring mounted on two carts as shown in Fig. 1.6. It is assumed that the pivot position of the moving spring is a function of time which can change along the full length $l$ of the pendula. The input to each pendulum is the torque $u_i$ applied at the pivot which is produced by the external forces $F_1$ and $F_2$ applied to the carts.

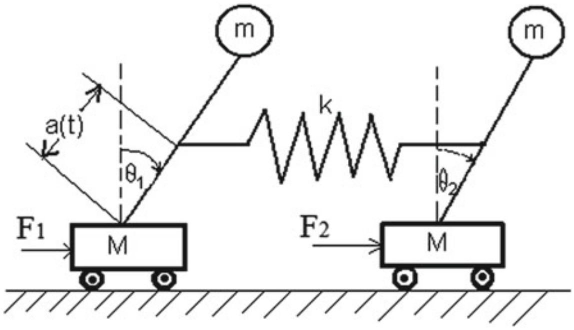


**Fig. 1.6** Two coupled inverted pendula on carts

Let

$$z_1 = \mathrm{col}(\theta_1, \dot{\theta}_1)^T, \qquad \text{and} \qquad z_2 = \mathrm{col}(\theta_2, \dot{\theta}_2)^T.$$

Then the dynamical model for the two coupled inverted pendulum systems is given by (see [149]):

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_2 - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_1)\dot{\theta}_1^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_1 - s_2) \end{bmatrix} \tag{1.41}$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_2 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} x_1 - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_2)\dot{\theta}_2^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_2 - s_1) \end{bmatrix} \tag{1.42}$$

where $s_1$ and $s_2$ are the positions of the two carts,

$$c = M/(M+m)$$

and $k$ and $g$ are the spring and gravity constants, respectively. The system (1.41)–(1.42) is a nonlinear interconnected control system.

### 1.5.5 Multimachine Power Systems

Power systems play an important role in the practical world. The classical model of power systems was given by Bergen [8], and based on this, a multimachine power system consisting of $N$ synchronous generators interconnected through a transmission network is described by the following equations [67]:

- Mechanical equations

$$\dot{\delta}_i = \omega_i, \tag{1.43}$$

$$\dot{\omega}_i = -\frac{D_i}{2H_i}\omega_i + \frac{\omega_0}{2H_i}(P_{mi0} - P_{ei}). \tag{1.44}$$

- Generator electrical dynamics:

$$\dot{E}'_{qi} = \frac{1}{T'_{doi}}(E_{fi} - E_{qi}). \tag{1.45}$$

- Electrical equations

$$E_{qi} = E'_{qi} - (x_{di} - x'_{di})I_{di}, \tag{1.46}$$

$$E_{fi} = K_{ci}u_{fi} \tag{1.47}$$

$$P_{ei} = \sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\sin(\delta_i - \delta_j) \tag{1.48}$$

$$Q_{ei} = -\sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\cos(\delta_i - \delta_j) \tag{1.49}$$

$$I_{qi} = \sum_{j=1}^{N} E'_{qj}B_{ij}\sin(\delta_i - \delta_j) \tag{1.50}$$

$$I_{di} = \sum_{j=1}^{N} E'_{qj}B_{ij}\cos(\delta_i - \delta_j) \tag{1.51}$$

$$E_{qi} = x_{adi}I_{fi} \tag{1.52}$$

$$V_{ti} = \sqrt{(E'_{qi} - x'_{di}I_{di})^2 + (x'_{di}I_{qi})^2} \tag{1.53}$$

where $\delta_i$ is the $i$-th generator power angle [rad], and $\omega_i$ is the relative speed [rad/s], $E'_{qi}$ represents the transient EMF in the quadrature axis [p.u.], and $u_{fi}$ is the input of the amplifier of the $i$-th generator for $i = 1, 2, \ldots, N$. The physical meanings of all the other symbols/notation used above are shown in Appendix E.1.

This model has been used by many authors to study multimachine power systems [67, 108, 182, 193]. The multimachine power system shown above can be expressed in the form of (1.24) (see, for example, Chap. 9).

### 1.5.6 A Biochemical System—Peroxidase–Oxidase Reaction

As a biochemical system, the peroxidase–oxidase (PO) reaction exhibits many complex dynamical behaviours. A great deal of experimental and theoretical work has been devoted to determining the mechanism by which oscillations and chaos arise in the PO reaction.

In addition to oscillatory and chaotic behaviour, the PO reaction exhibits bistability. Due to its suspected kinetic source: the inhibition of the enzyme by molecular oxygen, both autocatalysis and inhibition, i.e., positive and negative feedback are needed in the reaction mechanism for this system. A simple model for the PO reaction is described in [30, 181] as follows

$$\begin{aligned}
\dot{A} &= -k_1 ABX - k_3 ABY + k_7 - k_9 A \\
\dot{B} &= -k_1 ABX - k_3 ABY + k_8 \\
\dot{X} &= k_1 ABX - 2k_2 X^2 + 2k_3 ABY - k_4 X + k_6 \\
\dot{Y} &= -k_3 ABY + 2k_2 X^2 - k_5 Y
\end{aligned}$$

where $A$ is the concentration of dissolved $O_2$, B is the concentration of nicotinamide adenine dinucleotide, and $X$ and $Y$ are concentrations of two critical intermediates, $X$ and $Y$.

Typically all parameters except $k_1$ are constant. The parameter $k_1$ can be considered as a bifurcation parameter. Chaos is found only within a certain range of parameter values. Variations in $k_1$ reproduce the experimental behaviour observed when the enzyme concentration is changed. Thus $k_1$ can be considered as being related to the enzyme catalyst concentration [30, 181].

This section has provided practical examples of complex systems. Some will be used to demonstrate the developed results later in the text and additional examples will be given in the subsequent chapters.

## 1.6 Outline of This Book

This monograph systematically summarises the authors' recent results in the area of variable structure systems. It will focus on the analysis and design of complex systems where sliding mode techniques and the Lyapunov approach are the two main methods used throughout the monograph. Simulation examples and/or case studies are presented in each chapter to help readers understand the obtained theoretical results and utilise the proposed design approaches.

The book is organised as follows. First, the fundamental mathematical knowledge and basic control theory employed in the subsequent analysis and design in this monograph will be presented in Chap. 2. Considering that static output feedback control design is more convenient for real implementation when compared with state feedback control, in Chap. 3, robust static output controllers are designed to globally asymptotically stabilise the system, and then a decentralised static output feedback sliding mode control scheme follows for a class of nonlinear interconnected systems.

As static output feedback control imposes strong limitations on the considered system, dynamical feedback control is investigated in Chap. 4 where both minimum phase and non-minimum phase systems are considered. Chapter 4 studies dynamical output feedback control for nonlinear interconnected systems. Since large-scale interconnected systems have higher dimension, and dynamical output feedback will greatly increase the dimension of the closed-loop system, reduced-order observer-based feedback controllers are considered in Chap. 5.

Time-delay is a factor which increases system complexity. Chapters 6 and 7 concentrate on the study of nonlinear time-delay systems where the Lyapunov–Razumikhin approach is used to deal with the time-delay. Under the assumption that the time-delays are known, control schemes for nonlinear time-delay systems, and a decentralised control strategy for interconnected systems are proposed in Chap. 6. In practice, knowledge of the time-delay is not always available for design. In connection with this, memoryless variable structure controllers are presented in Chap. 7.

Chapter 8 discusses model based fault detection and isolation for nonlinear systems with uncertainties. The reconstruction and/or estimation of both system faults and sensor faults are considered based on a sliding mode observer scheme. LMI techniques are employed to facilitate the design of the parameters. A coordinate transformation is employed to explore the system structure when the considered system is fully nonlinear.

Applications of decentralised sliding mode control schemes to multimachine power systems are presented in Chap. 9. Simulation studies on three machine power systems confirm the theoretical results.

Finally, Chap. 10 concludes the book by providing some comments on the developed methods, some specific examples to show the complexity of control systems, and some suggestions for future developments in the area of variable structure control.

# Chapter 2
# Mathematical Background

This chapter presents some fundamental mathematical knowledge and basic results which facilitate the analysis and design in the subsequent chapters. The motivation is to help readers understand the theoretical work presented in this book.

## 2.1 Lipschitz Function

This section will present the well-known Lipschitz condition and the generalised Lipschitz condition.

### *2.1.1 Lipschitz Condition*

**Definition 2.1** A function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to satisfy the *Lipschitz condition* in the domain $\Omega \subset \mathbb{R}^n$ if there exists a nonnegative constant $L$ such that the inequality

$$\|f(x) - f(\hat{x})\| \leq L\|x - \hat{x}\| \tag{2.1}$$

holds for any $x \in \Omega$ and $\hat{x} \in \Omega$. Then $L$ is called the *Lipschitz constant* and $f(x)$ is called a *Lipschitz function* in $\Omega$. If $\Omega = \mathbb{R}^n$, then $f(x)$ is said to satisfy the *global Lipschitz condition.*

From Definition 2.1, it is clear that a Lipschitz function must be continuous. However, the converse is not true and a typical example is the scalar function

$$f(x) = x^{1/3}$$

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in a neighbourhood of the origin $x = 0$. A Lipschitz function may not be differentiable and a simple example is the scalar function

$$f(x) = |x|$$

at the origin $x = 0$ in $x \in \mathbb{R}$. Moreover, a differentiable function may not be Lipschitz on a compact set, for example the function

$$f(x) = \begin{cases} x^{\alpha} \sin \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases} \tag{2.2}$$

is not Lipschitz in the compact set $x \in [0,\ 1]$ for any constant $\alpha$ satisfying $1 < \alpha < 2$. The reason is that the derivative of the function $f(x)$ defined in (2.2) is not bounded in the interval $[0,\ 1]$.

**Lemma 2.1** ([91]) *Consider a function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ which is differentiable in the domain $\Omega$. If its Jacobian matrix is bounded in $\Omega$, that is, there exists a constant $L$ such that*

$$\|J_f\| \leq L$$

*for any $x \in \Omega$, then $f(x)$ satisfies the Lipschitz condition, and the inequality (2.1) holds.*

### 2.1.2 Generalised Lipschitz Condition

The well-known Lipschitz condition in Sect. 2.1.1 will be extended to a more general case which will be used later in the analysis.

**Definition 2.2** A function $f(x_1, x_2, x_3) : \Omega_1 \times \Omega_2 \times \Omega_3 \mapsto \mathbb{R}^n$ is said to satisfy a generalised Lipschitz condition with respect to (w.r.t.) the variables $x_1 \in \Omega_1 \subset \mathbb{R}^{n_1}$ and $x_2 \in \Omega_2 \subset \mathbb{R}^{n_2}$ uniformly for $x_3$ in $\Omega_3 \subset \mathbb{R}^{n_3}$ if there exist nonnegative continuous functions $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ defined in $\Omega_3$ such that for any $\hat{x}_1, x_1 \in \Omega_1$ and $\hat{x}_2, x_2 \in \Omega_2$, the inequality

$$\left\| f(x_1, x_2, x_3) - f(\hat{x}_1, \hat{x}_2, x_3) \right\| \leq \mathscr{L}_{f1}(x_3) \left\| x_1 - \hat{x}_1 \right\| + \mathscr{L}_{f2}(x_3) \left\| x_2 - \hat{x}_2 \right\|$$

holds for any $x_3 \in \Omega_3$. Then, $f(\cdot)$ is called a generalised Lipschitz function, and $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ are called generalised Lipschitz bounds. Further, if $\Omega_1 = \mathbb{R}^{n_1}$ and $\Omega_2 = \mathbb{R}^{n_2}$, then, it is said that $f(\cdot)$ satisfies a global generalised Lipschitz condition w.r.t. $x_1$ and $x_2$ uniformly for $x_3$ in $\Omega_3$.

*Remark 2.1* The symbols $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ introduced above are usually nonnegative functions instead of constants. This is different from the Lipschitz condition.

Thus, the nonnegative continuous functions $\mathscr{L}_{f1}(x_3)$ and $\mathscr{L}_{f2}(x_3)$ are called generalised Lipschitz bounds which correspond to the Lipschitz constant for the Lipschitz condition.

Clearly, the generalised Lipschitz condition is more relaxed than the Lipschitz condition. For example, the function

$$f(x_1, x_2, x_3) := x_1 x_3^2 + x_2 x_3$$

with $x_1, x_2, x_3 \in \mathbb{R}$ does not satisfy the global Lipschitz condition. However, from the inequality that for any $\mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$ and $\mathrm{col}(\hat{x}_1, \hat{x}_2, x_3) \in \mathbb{R}^3$

$$|f(x_1, x_2, x_3) - f\left(\hat{x}_1, \hat{x}_2, x_3\right)| \le |x_1 - \hat{x}_1| x_3^2 + |x_2 - \hat{x}_2|\, |x_3|$$

it is clear to see that $f(\cdot)$ satisfies the global generalised Lipschitz condition w.r.t. $x_1$ and $x_2$, uniformly for $x_3 \in \mathbb{R}$.

## 2.2 Comparison Functions

This section will present the definitions and properties of the class $\mathscr{K}$ function and related functions.

**Definition 2.3** (*see [91]*) A continuous function $\alpha : [0, a) \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

**Definition 2.4** (*see [91]*) A continuous function $\beta : [0, a) \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for any given $s \in \mathbb{R}^+$, the mapping $\beta(r, s)$ belongs to class $\mathscr{K}$ with respect to the variable $r$, and for any given $r \in [0, a)$, the mapping $\beta(r, s)$ is decreasing with respect to the variable $s$ and $\lim_{s\to\infty} \beta(r, s) = 0$.

**Definition 2.5** If a class $\mathscr{K}$ function is a $C^1$ function, then it is said to belong to class $\mathscr{K}C^1$. A continuous function $\beta : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be a class $\mathscr{K}I$ function if for any given $x \in \mathbb{R}^n$ the function $\beta(x, s)$ is increasing with respect to the variable $s$ in $\mathbb{R}^+$, that is, $\beta(x, s_1) \le \beta(x, s_2)$ for any $0 \le s_1 \le s_2$.

The functions defined in Definitions 2.3 and 2.4 above are directly from [91]. The new concepts of class $\mathscr{K}C^1$ functions and class $\mathscr{K}I$ functions introduced in Definition 2.5 will be used for later analysis.

The following new concept is introduced, which will be termed as a class $\mathscr{W}\mathscr{S}$ function and will be used in Sect. 7.3.

**Definition 2.6** A continuous function $\beta(t, x_1, x_2) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t, 0, 0) = 0$ is said to be weak w.r.t the variable $x_1$ and strong w.r.t. the variable $x_2$ if there exist functions $\chi_1(t, x_1, x_2)$ and $\chi_2(t, x_1, x_2)$ such that

$$\beta(t, x_1, x_2) = \chi_1(t, x_1, x_2)x_1 + \chi_2(t, x_1, x_2)x_2, \tag{2.3}$$

where both $\chi_1(\cdot, \cdot, x_2)$ and $\chi_2(\cdot, \cdot, x_2)$ are continuous and nondecreasing w.r.t. the variable $x_2$. Further, the function $\beta(t, x_1, x_2)$ is said to be a class $\mathscr{W}\mathscr{S}$ function w.r.t. the variables $x_1$ and $x_2$.

*Remark 2.2* It should be noted that if a function $\beta(t, x_1, x_2) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t, 0, 0) = 0$ is smooth enough, then it follows from [3] that there exist continuous functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ such that the expression

$$\beta(t, x_1, x_2) = \beta_1(t, x_1, x_2)x_1 + \beta_2(t, x_1, x_2)x_2$$

holds. Moreover, if $\beta_1(t, x_1, x_2)$ and $\beta_2(t, x_1, x_2)$ are nondecreasing w.r.t. $x_2$, then $\beta(t, x_1, x_2)$ is a class $\mathscr{W}\mathscr{S}$ function w.r.t. $x_1$ and $x_2$.

**Lemma 2.2** (see [91]) *Assume that $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions in* $[0,\ a)$, *$\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class $\mathscr{K}_\infty$ functions, and $\beta(\cdot)$ is a class $\mathscr{K}\mathscr{L}$ function defined in* $[0,\ a) \times \mathbb{R}^+$. *Then, the following results hold:*

- *the inverse function $\alpha_1^{-1}(\cdot)$ is a class $\mathscr{K}$ function defined in* $[0,\ \alpha_1(a))$.
- *the inverse function $\alpha_3^{-1}(\cdot)$ is a class $\mathscr{K}_\infty$ function defined in* $[0,\ \infty)$.
- *the composite function $\alpha_1 \circ \alpha_2$ is a class $\mathscr{K}$ function.*
- *the composite function $\alpha_3 \circ \alpha_4$ is a class $\mathscr{K}_\infty$ function.*
- *the function $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is a class $\mathscr{K}\mathscr{L}$ function.*

**Lemma 2.3** *The following results hold:*

*(i) If $\beta(x, s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function, then $\beta^2(x, s)$ is a class $\mathscr{K}I$ function.*
*(ii) Suppose that a function $\phi_1 :\ [0, a) \mapsto \mathbb{R}^+$ is a $C^1$ function with $\phi_1(0) = 0$. Then there exists a continuous function $\phi_2(\cdot)$ in $[0, a)$ such that*

$$\phi_1(s) = \phi_2(s)s, \qquad s \in [0, a)$$

*Proof* (i) Suppose that $\beta(x, s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function. Then for any $0 \le s_1 \le s_2$ and $x \in \mathbb{R}^n$,

$$\beta(x, s_1) \le \beta(x, s_2)$$

Since $\beta(x, s) \ge 0$ for any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\begin{aligned} &\beta^2(x, s_1) - \beta^2(x, s_2) \\ &= (\beta(x, s_1) + \beta(x, s_2))(\beta(x, s_1) - \beta(x, s_2)) \\ &\le 0 \end{aligned}$$

This shows that $\beta^2(x, s)$ is a class $\mathscr{K}I$ function

(ii) Since the function $\phi_1(\cdot)$ is a $C^1$ function in $[0, a)$, its derivative $\frac{d\phi_1(s)}{ds}$ is continuous in $[0, a)$. For any $s \in [0, a)$, construct a function

$$\phi_2(s) := \begin{cases} \dfrac{\phi_1(s)}{s}, & s \neq 0 \\ \dfrac{d\phi_1(s)}{ds}|_{s=0}, & s = 0 \end{cases} \tag{2.4}$$

From the definition of $\phi_2(\cdot)$, it is clear to see that

(1) if $s \neq 0$, then $\phi_1(s) = \phi_2(s)s$;
(2) if $s = 0$, then from $\phi_1(0) = 0$, $\phi_1(s) = \phi_2(s)s$.

Therefore, the expression

$$\phi_1(s) = \phi_2(s)s$$

holds for $s \in [0, a)$. It remains to prove that the function $\phi_2(\cdot)$ defined in (2.4) is continuous in $[0, a)$.

It is clear that $\phi_2(s)$ is continuous in $(0, a)$. Since $\phi_1$ is a $C^1$ function in $[0, a)$, from the continuity of $\frac{d\phi_1(s)}{ds}$ at $s = 0$,

$$\lim_{s \to 0^+} \phi_2(s) = \lim_{s \to 0^+} \frac{\phi_1(s)}{s} = \frac{d\phi_1(s)}{ds}|_{s=0} = \phi_2(0)$$

which implies that $\phi_2(\cdot)$ is continuous at $s = 0$. Therefore, $\phi_2(\cdot)$ is continuous in $[0, a)$.

Hence the conclusion follows. $\nabla$

## 2.3 Lyapunov Stability Theorems

The results given in this section are available in [91].

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \tag{2.5}$$

where the function $f : \mathbb{R}^+ \times D \mapsto \mathbb{R}^n$ is continuous and $D \subset \mathbb{R}^n$ is a domain which contains the origin $x = 0$. It is assumed that

$$f(t, 0) = 0, \qquad t \in \mathbb{R}^+$$

which implies that the origin is an equilibrium point of the system.

**Definition 2.7** The equilibrium point $x = 0$ of System (2.5) is called exponentially stable if there exist positive constants $c_i$ for $i = 1, 2, 3$ such that for any $x(t_0)$ satisfying $\|x(t_0)\| \leq c_1$,

$$\|x(t)\| \leq c_2 \|x(t_0)\| e^{-c_3(t-t_0)} \tag{2.6}$$

If Inequality (2.6) holds for any $x(t_0) \in \mathbb{R}^n$, then, the equilibrium point $x = 0$ of System (2.5) is called globally exponentially stable.

### 2.3.1 Asymptotic Stability

**Theorem 2.1** *Consider System (2.5). Let* $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ *be a continuously differentiable function such that*

$$\begin{aligned} W_1(x) \leq V(t, x) \leq W_2(x) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \end{aligned}$$

*for any* $t \in \mathbb{R}^+$ *and* $x \in D$*, where* $W_i(x)$ *for* $i = 1, 2, 3$ *are continuous positive definite functions in* $D$*. Then* $x = 0$ *is uniformly asymptotically stable. Further if* $D = \mathbb{R}^n$*, and* $w(x)$ *is radially unbounded, then* $x = 0$ *is globally uniformly asymptotically stable.*

### 2.3.2 Exponential Stability

**Theorem 2.2** *Consider System (2.5). Let* $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ *be a continuously differentiable function such that for* $t \in \mathbb{R}^+$ *and* $x \in D$*,*

$$\begin{aligned} k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^a, \end{aligned}$$

*where* $k_i$ *for* $i = 1, 2, 3$ *and* $a$ *are positive constants. Then* $x = 0$ *is exponentially stable. Further if* $D = \mathbb{R}^n$*, then* $x = 0$ *is global exponentially stable.*

Comparing Theorems 2.1 and 2.2 above, it is straightforward to see that exponential stability implies uniform asymptotic stability.

### 2.3.3 Converse Lyapunov Theorem

The following result is the well-known converse Lyapunov theorem.

**Theorem 2.3** *Consider System (2.5) in domain* $D := \mathscr{B}_r = \{x \in \mathbb{R}^n \mid \|x\| < r\}$*. Let* $\beta(\cdot)$ *be a class* $\mathscr{KL}$ *function and* $r_0$ *be a positive constant such that*

$$\beta(r_0, 0) < r \quad \textit{and} \quad \mathscr{B}_{r_0} := \{x \mid \|x\| < r_0\}$$

*Assume that the Jacobian matrix* $\frac{\partial f}{\partial x}$ *is bounded*[1] *in domain D uniformly for* $t \in \mathbb{R}^+$*, and that the trajectory of System (2.5) satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad x(t_0) \in \mathscr{B}_{r_0}, \quad t \geq t_0 \geq 0$$

*Then, there exists a continuously differentiable function* $V : \mathbb{R}^+ \times \mathscr{B}_{r_0} \mapsto \mathbb{R}^+$ *such that*

$$\begin{aligned} \alpha_1(\|x\|) \leq V(t, x) &\leq \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -\alpha_3(\|x\|) \\ \left\| \frac{\partial V}{\partial x} \right\| &\leq \alpha_4(\|x\|), \end{aligned}$$

*where* $\alpha_i$ *for* $i = 1, 2, 3, 4$ *are class* $\mathscr{K}$ *functions defined on the interval* $[0, r_0]$*. The function* $V(\cdot)$ *can be chosen independent of time* $t$ *if* $f(\cdot)$ *in System (2.5) is independent of the time* $t$*.*

## 2.4 Uniformly Ultimate Boundedness

For a given System (2.5), if asymptotic stability is not possible, uniform ultimate bounded stability can be considered. This is very useful in practical cases.

**Theorem 2.4** *Consider System (2.5). Let* $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ *be a continuously differentiable function such that in* $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}^n$*,*

$$\begin{aligned} &\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \\ &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \textit{for any} \quad \|x\| \geq \mu > 0, \end{aligned}$$

*where* $\alpha_1(\cdot)$ *and* $\alpha_2(\cdot)$ *are class* $\mathscr{K}$ *functions and* $W_3(\cdot)$ *is a continuous positive definite function in domain D. Then* $x = 0$ *is uniformly ultimately bounded.*[2] *Further if* $D = \mathbb{R}^n$*, and* $\alpha_1(\cdot)$ *belongs to class* $\mathscr{K}_\infty$*, then* $x = 0$ *is globally uniformly ultimately bounded.*

*Proof* See the reference [91] (Theorem 4.18, p. 172). #

From Theorem 2.4, the following result is ready to be presented:

---

[1]If the function $f(\cdot)$ in (2.5) is continuously differentiable in the ball $\overline{\mathscr{B}}_r$, then $\frac{\partial f}{\partial x}$ is bounded in the domain $D = \mathscr{B}_r$.

[2]The ultimate bound depends on the parameters $\mu$, which can be estimated using the result given in Theorem 4.18 in [91].

**Lemma 2.4** *Consider the nonlinear system*

$$\dot{x} = \omega(x), \tag{2.7}$$

*where $x \in \mathbb{R}^n$ is the system state, and the function $\omega(\cdot)$ is continuous in $\mathbb{R}^n$. Let $\mathscr{V} : \mathbb{R}^n \mapsto \mathbb{R}^+$ be a continuously differentiable class $\mathscr{K}_\infty$ function of $\|x\|$ such that the inequality*

$$\frac{\partial \mathscr{V}}{\partial x}\omega(x) \leq -\vartheta(\|x\|), \qquad x \in \mathbb{R}^n \setminus \mathscr{B}_\mu \tag{2.8}$$

*holds for some domain $\mathscr{B}_\mu$, where $\vartheta$ is a class $\mathscr{K}$ function, and $\mu$ is a positive constant. Then, the trajectory of System (2.7) enters into the domain $\mathscr{B}_\mu$ in finite time.*

*Proof* From the condition of Lemma 2.4, there exists a class $\mathscr{K}_\infty$ function $\vartheta_1(\cdot)$ such that

$$\mathscr{V}(x) = \vartheta_1(\|x\|). \tag{2.9}$$

Then, from (2.8), (2.9) and using Theorem 2.4, the trajectory of System (2.7) is driven to the domain $\overline{\mathscr{B}}_\mu$ in a finite time, and remains there. That means there exists $t_1$ such that $x \in \overline{\mathscr{B}}_\mu$ for $t \geq t_1$.

The aim now is to prove that the trajectory of System (2.7) enters into $\mathscr{B}_\mu$ in a finite time. Suppose for a contradiction that this is not the case, then there exists some time $t_2$ such that the solution $x(x_0, t)$ of System (2.7) starting from some point $x_0$ satisfies $x(x_0, t) \in \partial\overline{\mathscr{B}}_\mu$ after $t_2$. This is equivalent to

$$\|x(x_0, t)\| = \mu, \qquad t \geq t_2. \tag{2.10}$$

By (2.9) and (2.10), it follows that

$$\mathscr{V}(x(x_0, t)) = \vartheta_1(\|x(x_0, t)\|) = \vartheta_1(\mu), \qquad t \geq t_2, \tag{2.11}$$

where $\mu$ is a positive constant. This shows that $\dot{\mathscr{V}}\mid_{(2.7)} \equiv 0$ after $t_2$, and it contradicts (2.8). Hence, the conclusion follows. #

*Remark 2.3* Lemma 2.4 demonstrates that the solution enters the open set $\mathscr{B}_\mu$ in finite time and remains on $\overline{\mathscr{B}}_\mu$. It does not claim that the solution subsequently remains in $\mathscr{B}_\mu$.

## 2.5 Razumikhin Theorem

Consider a time-delay system

$$\dot{x}(t) = f(t, x(t - d(t)) \tag{2.12}$$

with an initial condition

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0],$$

where the function vector $f : \mathbb{R}^+ \times \mathscr{C}_{[-\overline{d},0]} \mapsto \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}_{[-\overline{d},0]}$) into bounded sets in $\mathbb{R}^n$; $d(t) > 0$ is the time-delay and

$$\overline{d} := \sup_{t \in \mathbb{R}^+} \{d(t)\} < \infty$$

which implies that the time-delay $d(t)$ has a finite upper bound in $t \in \mathbb{R}^+$.

**Theorem 2.5** (Razumikhin Theorem) *If there exist class $\mathscr{K}_\infty$ functions $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, a class $\mathscr{K}$ function $\zeta_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\overline{d}, \infty] \times \mathbb{R}^n \mapsto \mathbb{R}^+$ satisfying*

$$\zeta_1(\|x\|) \le V_1(t, x) \le \zeta_2(\|x\|), \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n$$

*such that the time derivative of $V_1$ along the solution of System (2.12) satisfies*

$$\dot{V}_1(t, x) \le -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t - d, x(t - d)) \le V_1(t, x(t)) \tag{2.13}$$

*for any $d \in [0, \overline{d}]$, then the System (2.12) is uniformly stable. If in addition, $\zeta_3(\tau) > 0$ for $\tau > 0$ and there exists a continuous nondecreasing function $\xi(\tau) > \tau$ for $\tau > 0$ such that (2.13) is strengthened to*

$$\dot{V}_1(t, x) \le -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t - d, x(t - d)) \le \xi(V_1(t, x(t))) \tag{2.14}$$

*for $d \in [0, \overline{d}]$, then the System (2.12) is uniformly asymptotic stable.*

*Proof* See pages 14–15 in [65]. ∇

From the Razumikhin Theorem 2.5, the following conclusion can be obtained directly:

**Lemma 2.5** *Consider the time-delay system (2.12). If there exist constants $\gamma > 0$ and $\zeta > 1$ and a function*

$$V_2(x(t)) = x^T \tilde{P} x$$

*with $\tilde{P} > 0$ such that the time derivative of $V_2(\cdot)$ along the solution of System (2.12) satisfies*

$$\dot{V}_2 \mid_{(2.12)} \le -\gamma \left\| \tilde{P}^{\frac{1}{2}} x(t) \right\|^2 \tag{2.15}$$

*whenever*

$$\|\tilde{P}^{\frac{1}{2}} x(t + \theta)\| \le \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

*for any $\theta \in [-\overline{d}, 0]$, then, System (2.12) is uniformly asymptotic stable.*

*Proof* From the definition of $V_2(\cdot)$ it follows that

$$\lambda_{\min}(\tilde{P})\|x\|^2 \leq V_2(t, x(t)) \leq \lambda_{\max}(\tilde{P})\|x\|^2$$

and from (2.15)

$$\dot{V}_2 \mid_{(2.12)} \leq -\gamma x(t)^T \tilde{P} x(t) \leq -\gamma \lambda_{\max}(\tilde{P})\|x\|^2.$$

It is clear that the inequality

$$V_2(x(t+\theta)) \leq \zeta^2 V_2(x(t))$$

is equivalent to the inequality

$$\|\tilde{P}^{\frac{1}{2}} x(t+\theta)\| \leq \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

Then, from Razumikhin Theorem 2.5 and $\tilde{P} > 0$, the conclusion follows by letting

$$\gamma_1(\tau) = \lambda_{\min}(\tilde{P})\tau^2, \quad \gamma_2(\tau) = \lambda_{\max}(\tilde{P})\tau^2$$
$$\gamma_3(\tau) = \gamma \lambda_{\min}(\tilde{P})\tau^2, \; \gamma_4(\tau) = \zeta^2 \tau$$

in Theorem 2.5. #

## 2.6 Output Sliding Surface Design

In order to form an output feedback sliding mode control scheme, it is usually required that the designed switching function is a function of the system outputs. The corresponding sliding surface is called *an output sliding surface* in this book. The output sliding surface algorithm proposed in [37, 38] is outlined here, and this will be frequently used in the sequel.

Consider initially a linear system

$$\dot{x} = Ax + Bu \tag{2.16}$$
$$y = Cx, \tag{2.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the states, inputs and outputs, respectively, and assume $m \leq p < n$. The triple $(A, B, C)$ comprises constant matrices of appropriate dimensions with $B$ and $C$ both being of full rank.

For System (2.16) and (2.17), it is assumed that

$$\text{rank}(CB) = m$$

Then, from [37] it can be shown that a coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the system triple $(A, B, C)$ with respect to the new coordinate $\widetilde{x}$ has the following structure

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & \breve{T} \end{bmatrix}, \tag{2.18}$$

where $\widetilde{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ and $\breve{T} \in \mathbb{R}^{p\times p}$ is orthogonal. Further, it is assumed that the system $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ with $\widetilde{C}_1$ defined by

$$\widetilde{C}_1 = \begin{bmatrix} 0_{(p-m)\times(n-m)} & I_{p-m} \end{bmatrix} \tag{2.19}$$

is output feedback stabilisable, i.e., there exists a matrix $K \in \mathbb{R}^{m\times(p-m)}$ such that

$$\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$$

is stable. It is shown in [37, 38] that a necessary condition for $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ to be stabilisable is that the invariant zeros of $(A, B, C)$ lie in the open left-half plane. In [37, 38] a sliding surface of the form

$$FCx = 0 \tag{2.20}$$

is proposed, where

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} \breve{T}^{\tau} \tag{2.21}$$

and $F_2 \in \mathbb{R}^{m\times m}$ is any nonsingular matrix.

If a further coordinate change is introduced based on the nonsingular transformation $z = \widehat{T}\widetilde{x}$ with $\widehat{T}$ defined by

$$\widehat{T} = \begin{bmatrix} I_{n-m} & 0 \\ K\widetilde{C}_1 & I_m \end{bmatrix}$$

then in the new coordinates $z$, System (2.16) and (2.17) has the following form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \widehat{C},$$

where $A_{11} = \widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ is stable and $\widehat{C}$ satisfies

$$F\widehat{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$$

with $F_2$ nonsingular. From the analysis above, the following conclusion is obtained directly:

**Lemma 2.6** *Consider System (2.16) and (2.17). Suppose that*

*(i)* $\operatorname{rank}(CB) = m$;

*(ii) the invariant zeros of* $(A, B, C)$ *lie in the open left-half plane;*
*(iii) the matrix triple* $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ *is output feedback stabilisable, where* $(\widetilde{A}_{11}, \widetilde{A}_{12})$ *and* $\widetilde{C}_1$ *are defined, respectively, by (2.18) and (2.19).*

*Then,*

*(i) there exists a transformation* $z = Tx$ *such that the new coordinate* $z$ *system (2.16) and (2.17) has the following form*

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{2.22}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{2.23}$$

*where* $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ *is stable. Both matrices* $B_2 \in \mathbb{R}^{m\times m}$ *and* $C_2 \in \mathbb{R}^{p\times p}$ *are nonsingular;*

*(ii) there exists a matrix* $F$ *such that* $FCx = 0$ *provides a stable sliding motion for System (2.16) and (2.17) and* $F\begin{bmatrix} 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$*, where* $F_2 \in \mathbb{R}^{m\times m}$ *is nonsingular.*

*Proof* All that remains to be shown is that the output distribution matrix has the form given in (2.23) and that $C_2$ is nonsingular. The output distribution matrix in the new coordinates is given by

$$\begin{aligned}
\begin{bmatrix} 0 & \breve{T} \end{bmatrix} \widehat{T}^{-1} &= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -K\widetilde{C}_1 & I_m \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-p} & 0 & 0 \\ 0 & I_{p-m} & 0 \\ 0 & -K & I_m \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ 0 & \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

and so by inspection,

$$C_2 = \breve{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix}$$

which is nonsingular. Hence the result follows. #

From the analysis above, it is clear to see that the coordinate transformation

$$z = Tx,$$

where $T := \widehat{T}\tilde{T}$, transfers the system (2.16) and (2.17) to the regular form (2.22) and (2.23). Choose the sliding surface

$$\mathscr{S} = \{x \mid FCx = 0,\ x \in \mathbb{R}^n\} \tag{2.24}$$

Then, the analysis above shows that the sliding motion of System (2.16) and (2.17) corresponding to the sliding surface (2.24) is asymptotically stable. The sliding surface (2.24) can be described by

$$\mathscr{S} = \{y \mid Fy = 0,\ y \in \mathbb{R}^p\} \tag{2.25}$$

which is a subspace of the *output space*. Therefore, $\mathscr{S}$ in (2.24) or (2.25) denote *output sliding surfaces*.

*Remark 2.4* Lemma 2.6 gives a condition for the existence of the output switching surface (2.20) on which system (2.16) is stable. It should be emphasised that the sliding surface given by Lemma 2.6 can be obtained from a systematic algorithm together with any output feedback pole placement algorithm of choice. Details of appropriate algorithms and how to determine the switching surface (2.20) are described in [37, 38], where the necessary and sufficient condition to guarantee the existence of the matrix $F$ is available in Proposition 5.2 of [38]. If $p = m$ then there is no design freedom and the sliding motion is governed by the invariant zeros of $(A, B, C)$.

## 2.7 Geometric Structure of Nonlinear System

Consider the nonlinear system

$$\dot{x}(t) = F(x(t), u(t)) \tag{2.26}$$

$$y(t) = h(x(t)), \qquad x_0 = x(0), \tag{2.27}$$

where $x \in \Omega \subset \mathscr{R}^n$ (and $\Omega$ is a neighbourhood of $x_0$), $u = \mathrm{col}(u_1, u_2, \ldots, u_m) \in \mathscr{U} \in \mathscr{R}^m$, and $y = \mathrm{col}(y_1, y_2, \ldots, y_p) \in \mathscr{R}^p$ are the state variables, inputs and outputs, respectively, where $\mathscr{U}$ is an admissible control set. $F(x, u)$ is a known smooth vector field in $\Omega \times \mathscr{U}$ and the known function $h : \Omega \mapsto \mathscr{R}^p$ is smooth. For convenience, the system (2.26) and (2.27) is also denoted by the pair $(F(x, u), h(x))$.

**Definition 2.8** (*See, e.g., [58]*) System (2.26) and (2.27) is said to be *observable* at $(x_0, u_0) \in \Omega \times \mathscr{U}$ if there exists a neighbourhood $\mathscr{N}$ of $(x_0, u_0)$ in $\Omega \times \mathscr{U}$ and a set of nonnegative integer numbers $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ such that

(1) for all $(x, u) \in \mathscr{N}$

$$\frac{\partial}{\partial u_j} L^k_{F(x,u)} h_i(x) = 0 \tag{2.28}$$

for indices $i = 1, 2, \ldots, p$, $k = 0, 1, 2, \ldots, r_i - 1$ and $j = 1, 2, \ldots, m$;

(2) the $p \times m$ matrix $M(x, u) := \{\frac{\partial}{\partial u_j} L^{r_i}_{F(x,u)} h_i(x)\}$ has rank $p$ in $(x_0, u_0)$

Then, $\{r_1, r_2, \cdots, r_p\}$ is called the *observability index* of System (2.26) and (2.27) at $(x_0, u_0)$. Further, System (2.26) and (2.27) is said to be *uniformly observable* in $\Omega \times \mathscr{U}$ if for any $(x_0, u_0) \in \Omega \times \mathscr{U}$, the system is observable and the observability indices are fixed.

Assume the pair $(F(x, u), h(x))$ has uniform observability index $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in the domain $\Omega \times \mathscr{U}$. Construct a nonlinear transformation $T : x \mapsto z$ as follows:

$$z_{i1} = h_i(x) \tag{2.29}$$

$$z_{i2} = L_{F(x,u)} h_i(x) \tag{2.30}$$

$$\vdots$$

$$z_{ir_i} = L_{F(x,u)}^{r_i - 1} h_i(x), \tag{2.31}$$

where $z_i := \text{col}(z_{i1}, z_{i2}, \cdots, z_{ir_i})$ for $i = 1, 2, \ldots, p$ and $z := \text{col}(z_1, z_2, \cdots, z_p)$.

It follows from Definition 2.8 that $M(x, u)$ has rank $p$ in $\Omega \times \mathscr{U}$, implying that all $z_i$ are independent of the control $u$, which combined with the restriction $\sum_{i=1}^{p} r_i = n$ means that the corresponding Jacobian matrix of $T(x)$, $\frac{\partial T}{\partial x}$, is nonsingular. Therefore, (2.29) and (2.31) is a diffeomorphism in the domain $\Omega$, and $z = \text{col}(z_1, z_2, \ldots, z_p)$ forms a new coordinate system which can be obtained by direct computation from (2.29) to (2.31).

Since $L_{F(x,u)}^{j} h_i(x)$ is independent of $u$ for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i - 1$, it follows by direct computation that for $i = 1, 2, \ldots, p$

$$\begin{aligned}
\dot{z}_{i1} &= \tfrac{\partial h_i}{\partial x} F(x, u) = L_{F(x,u)} h_i(x) = z_{i2} \\
\dot{z}_{i2} &= \frac{\partial \left( L_{F(x,u)} h_i(x) \right)}{\partial x} F(x, u) = L_{F(x,u)}^{2} h_i(x) = z_{i3} \\
&\vdots \\
\dot{z}_{ir_{i-1}} &= L_{F(x,u)}^{r_i - 1} h_i(x) = z_{ir_i} \\
\dot{z}_{ir_i} &= L_{F(x,u)}^{r_i} h_i(x)
\end{aligned}$$

Therefore, in the new coordinates $z$ defined by (2.29) and (2.31), System (2.26) and (2.27) has the following form:

$$\begin{aligned}
\dot{z} &= Az + B\Phi(z, u) \\
y &= Cz,
\end{aligned}$$

where

$$A = \text{diag}\{A_1, \ldots, A_p\}, \quad B = \text{diag}\{B_1, \ldots, B_p\} \quad \text{and} \quad C = \text{diag}\{C_1, \ldots, C_p\},$$

where $A_i \in \mathscr{R}^{r_i \times r_i}$, $B_i \in \mathscr{R}^{r_i \times 1}$ and $C_i \in \mathscr{R}^{1 \times r_i}$ for $i = 1, 2, \ldots, p$ are defined by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.32}$$

and

$$\Phi(z,u) := \begin{bmatrix} \phi_1(z,u) \\ \phi_2(z,u) \\ \vdots \\ \phi_p(z,u) \end{bmatrix} := \begin{bmatrix} L^{r_1}_{F(x,u)} h_1(x) \\ L^{r_2}_{F(x,u)} h_2(x) \\ \vdots \\ L^{r_p}_{F(x,u)} h_p(x) \end{bmatrix}_{x=T^{-1}(z)}, \tag{2.33}$$

where $\phi_i : T(\Omega) \times \mathscr{U} \mapsto \mathscr{R}$ for $i = 1, 2, \ldots, p$.

## 2.8 Summary

This chapter has presented the fundamental concepts and results which underpin the theoretical analysis in this book. Some of the results are taken from the existing literature and others are developed by the authors, but with rigorous proofs provided. The content covers Lipschitz conditions, comparison functions, stability of nonlinear systems, the converse Lyapunov theorem and uniform ultimate boundedness. The well-known Razumikhin theorem has been presented, for the readers' convenience, and will be employed to deal with time-delay systems throughout the book. Section 2.5 summarises the output sliding surface design approach proposed in [38] which will be frequently used in the sequel. Finally, the geometric structure of nonlinear systems with uniform observability index has been provided.

# Chapter 3
# Static Output Feedback Variable Structure Control

In this chapter, static output feedback variable structure controllers are designed to stabilise a class of nonlinear systems globally. Then a decentralised strategy is proposed for a class of nonlinear large-scale interconnected systems. Finally, case studies are provided to show the application of the proposed design methods.

## 3.1 Introduction

It is well known that knowledge of all the system state variables is often not available in practice. Some state variables may be difficult or costly to measure whilst some may have no physical meaning and thus cannot be measured. Sometimes it may be possible to use an observer to estimate unknown states, but this approach not only requires more hardware resources, but also increases the system dimension. This may result in further difficulties, especially for large-scale interconnected systems. Therefore, it is pertinent to apply static output feedback control whenever possible.

Static output feedback control has advantages in that it is convenient for real implementation when compared with state feedback, as usually only partial state information is available for design, and schemes do not need extra resources for practical implementation when compared with observer-based feedback. However, as static output feedback can only use a subset of the system state information and cannot employ extra dynamical information, strong theoretical limitations are required on the considered systems.

Compared with state feedback, static output feedback control is much more complex. The well-known static output feedback problem is to design a static feedback control law only using output information such that the corresponding closed-loop system has the desired performance, or to show that such a feedback does not exist [165]. For a linear system described by a triple $(A, B, C)$, the static output feedback problem is to find a matrix $K$ such that the matrix $A + BKC$ is Hurwitz stable, or

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to show that such a matrix $K$ does not exist. This is still an open problem even for SISO linear systems as the fundamental problem of the existence of static output controllers is not solved for SISO systems [102, 165].

It should be noted that many problems associated with system analysis and control design can be formulated as a convex linear matrix inequality (LMI) problem which can be tackled using standard convex optimisation techniques [11]. However, this is not the case for the static output feedback problem even for linear control systems since the most general characterisation of the static output feedback design involves bilinear matrix inequalities for which complete/systemised and efficient methods/algorithms are currently not available [102].

Formulating the design problem to establish stability with respect to a quadratic Lyapunov function (so-called quadratic stabilisability), a problem occurs which was termed by Galimidi and Barmish as the constrained Lyapunov problem (CLP) [57]. This problem commonly occurs in uncertain linear systems, where the so-called matching condition is assumed to be satisfied and when full state availability does not exist. In the static output feedback framework, the CLP can be stated as follows:

- For a given triple $(A, B, C)$, find a static output feedback control gain K such that

  $$(A + BKC)^T P + P(A + BKC) < 0,$$

  where $P > 0$ is a symmetric positive definite matrix and subject to the linear constraint

  $$B^T P = FC.$$

  In this problem, the matrix parameters $K$, $P$ and $F$ are treated as variables which must be selected appropriately.

This CLP problem has appeared in different guises in the control systems literature over several decades [24, 29, 57, 214]. The solvability of constrained Lyapunov equations is therefore an interesting problem of practical significance. Many authors have considered this problem. The CLP was posed and solved in [57] for both square and nonsquare systems in the sense that necessary and sufficient algebraic conditions were given to enable its solution. The conditions in [57] are given in algebraic terms, and there is no suggestion as to when they are solvable in system theoretic terms. For square systems, Kim and Park [92] drew parallels between the CLP and the robust output feedback work of Gu [64]. Later the results in [92] are extended to the nonsquare case in [41].

In this chapter, both nonlinear systems and nonlinear interconnected systems are considered. The problem to be addressed is that, under the assumption that the nominal systems are stabilisable with predesigned static output feedback controllers, how can updated static output controllers be designed for nonlinear systems and decentralised static output feedback controllers for interconnected systems such that the corresponding closed-loop systems are asymptotically stable in the presence of uncertainties. The problem of stabilisation of the nominal systems, which is the well-known static output feedback problem as discussed above, is not considered in this book. Applications of the developed results are presented in Sect. 3.4.

## 3.2 Robust Global Stabilisation of Nonlinear Systems

Stabilisation, particularly robust global stabilisation, has always been a challenging problem for nonlinear systems. With the development of nonlinear theory and computer science, a period of significant progress has been experienced in the nonlinear control area over the last two decades. This section focuses on global stabilisation control combining Lyapunov methods and sliding mode techniques.

### *3.2.1 Background*

Lyapunov analysis often plays an important role in the study of nonlinear systems. What is more, many control techniques and methods such as adaptive control, the geometric method, passivity tools, small gain theory and sliding mode control have been widely and successfully used in nonlinear control. Many valuable results have been achieved in the global stabilisation of nonlinear systems (see [14, 88] and the references therein). Notably, most work focuses on special classes of systems such as partially linear composite systems, globally linearisable systems or globally minimum phase systems, and it is assumed that all system state information is available in the majority of results.

In practical engineering systems, however, state variables are not always accessible and only a subset of them is available. In consideration of this, dynamical output feedback control schemes have been considered (see e.g., [95, 157]). Some authors also focus on establishing an observer to measure or estimate the system state: associated work can be found in [40, 43, 203]. Notably the separation principle is no longer true for nonlinear systems. This means that different or even converse results may occur even if the same controller is used for the same nonlinear system using estimated states and true states [203]. Therefore, static output feedback control should be considered if possible.

Work using static output feedback control has been presented in [196, 214, 220]. In the approach proposed by Zak and Hui [220], geometric conditions are presented for the existence of a sliding mode and an associated design algorithm is also derived. However, as pointed out by Kwan [95] and Shyu et al. [157], there are two major assumptions which restrict the application of the corresponding results. In order to overcome these shortcomings, a class of SISO system is considered in [95], and later, Shyu et al. [157] proposed an approach which is applicable to MIMO systems with mismatched uncertainty based on the work of Kwan [95]. This work is based on dynamical output feedback although the results report that the two major limitations in [220] are overcome. Static output feedback control strategies based on Lyapunov techniques proposed in [196, 214] can also avoid the two major limitations in [220]. However, the results place strong limitations on the bounds of the uncertainty, and in most cases are only valid in a small region of the origin. For example, when the bounding function of the uncertainty takes the form

$$\varpi(x) = \begin{cases} \dfrac{1}{\varepsilon}, & \|x\| \le \varepsilon \\ \dfrac{1}{\|x\|}, & \|x\| > \varepsilon \end{cases}$$

with very small positive constant $\varepsilon$, then, the conclusion in [196, 214] may not hold or may only be satisfied in a very small domain which is contained in $\{x \mid \|x\| < \varepsilon\}$.

In this section, a global robust stabilisation problem for a class of nonlinear systems whose nominal system is linearisable locally is considered. The design framework may be described as follows. First, using a Lyapunov technique and knowledge of the desired performance of the nominal subsystem, a nonlinear robust control is designed so that the system is driven to a domain of the origin and remains there even in the presence of matched and mismatched uncertainties. Then, the sliding mode technique is applied to the system such that its trajectories are driven to the predesigned sliding surface on which the system is asymptotically stable. Finally, a variable structure control is synthesised to stabilise the system globally. It should be noted that only static output feedback is used here, and the two major assumptions in Zak and Hui [220] are alleviated. This is in comparison with other reported work in [95, 157] where dynamic output feedback is employed, and in [196, 203, 214] where the results are often valid only in a small region of the origin. Theoretically, the goal is to develop an approach to study the global stabilisation problem for a wide class of nonlinear systems.

### 3.2.2 System Description and Assumptions

Consider a nonlinear system

$$\dot{x} = f(x) + g(x)[u + \Delta g(x)] + \Delta f(x) \tag{3.1}$$

$$y = h(x), \tag{3.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ $(p \ge m)$ are the system state variables, input and output, respectively; $f(x)$ and $g(x)$ with $f(0) = 0$ are both known functions; $\Delta g(x)$ and $\Delta f(x)$ are respectively matched and mismatched uncertainties, and they are all continuous in their arguments. It is assumed that the existence and the uniqueness of the system solution are guaranteed globally.

For System (3.1)–(3.2), its nominal System is described by

$$\dot{x} = f(x) + g(x)u \tag{3.3}$$

$$y = h(x). \tag{3.4}$$

One fundamental problem of robust control is to design a controller so that the controlled signals have some desired properties even in the presence of uncertainties. In this section, it is assumed that the controllers for the nominal system (3.3)–(3.4) have been well designed and the controlled nominal system has desired performance. Then, a variable structure control scheme is to be synthesised to stabilise the system (3.1)–(3.2) globally.

**Assumption 3.1** There exist known continuous functions $\phi_1$, $\phi_2$ and $\psi$ such that

(i) $\|\Delta g(x)\| \leq \phi_1(y)\phi_2(x)$;
(ii) $\|\Delta f(x)\| \leq \psi(y)$,

where $\phi_1(0) = 0$ and $\psi$ is sufficiently smooth with $\psi(0) = 0$.

*Remark 3.1* Assumption 3.1 implies that the origin is an equilibrium point of system (3.1)–(3.2) in the presence of uncertainties. Condition (ii) of Assumption 3.1, that the mismatched uncertainty is bounded by a function of the output, is not essential. It can be extended to a function of the system state variable. This can be seen in the subsequent analysis.

**Assumption 3.2** There exists a continuous function $u_1^I(y)$ with $u_1^I(0) = 0$, a $C^1$ function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ which is a class $\mathscr{K}_\infty$ function of $\|x\|$, and class $\mathscr{K}$ functions $\alpha_i$ for $i = 1, 2$ such that for $x \in \mathbb{R}^n \setminus \mathscr{B}_r$

$$\frac{\partial V}{\partial x}\big(f(x) + g(x)u_1^I(y)\big) \leq -\alpha_1(\|x\|) \tag{3.5}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_2(\|x\|), \tag{3.6}$$

where $r$ is a positive constant, and $\frac{\partial V}{\partial x} \equiv: \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right)$.

*Remark 3.2* Assumption 3.2 requires that $V(x)$ is a class $\mathscr{K}_\infty$ function of $\|x\|$. It should be noted that if there exists a quadratic Lyapunov function $x^T P x$ ($P > 0$) for a system $\dot{x} = \mathscr{F}(x)$, then there exists a Lyapunov function $z^T z$ which is a class $\mathscr{K}_\infty$ function of $\|z\|$ for system

$$\dot{z} = P^{\frac{1}{2}}\mathscr{F}(P^{-\frac{1}{2}}z).$$

Therefore, such a limitation is trivial for any system which has a quadratic Lyapunov function. Furthermore, the conditions (3.5) and (3.6) of Assumption 3.2 are satisfied automatically if the nominal system is exponentially stabilisable using output feedback, or the resulting closed-loop system has a quadratic Lyapunov function. These conditions have been used by many authors in papers on nonlinear analysis [91, 196].

**Assumption 3.3** The function $V$ defined in Assumption 3.2 satisfies

$$\frac{\partial V}{\partial x} g(x) = R(y),$$

where $R(y) \in \mathbb{R}^m$ is continuous in its domain of definition.

*Remark 3.3* Assumption 3.3 is similar to the structural condition

$$B^T P = FC$$

for the linear system $(A, B, C)$ possessing a quadratic Lyapunov function $x^T Px$ (see e.g., [24, 214]). Yan et al. [196] proposed a condition for the nonlinear case which is a direct extension of the linear one. However, the current consideration renders all previous settings as special cases in this regard.

In the following, Lyapunov approach based controllers are initially designed such that the considered system can be driven to the specific domain of the origin, in which the nominal system can be linearised. Then, a sliding mode controller is designed such that the corresponding closed-loop system is asymptotically stable. Finally the globally stabilising controllers are synthesised.

### 3.2.3 Lyapunov Approach Based Control Design

Consider an output feedback control

$$u^I(y) = u_1^I(y) + u_2^I(y), \tag{3.7}$$

where $u_1^I$ is given by Assumption 3.2, and $u_2^I$ is defined as

$$u_2^I(y) = \begin{cases} -\dfrac{\phi_1^2(y) R^T(y)}{2\varepsilon \|R(y)\|}, & R(y) \neq 0 \\ 0, & R(y) = 0, \end{cases} \tag{3.8}$$

where $\varepsilon$ is an adjustable positive constant, $R(\cdot)$ is given in Assumption 3.3 and the function $\phi_1(\cdot)$ is determined by Assumption 3.1.

**Theorem 3.1** *Consider the closed-loop system (3.1)–(3.2) and (3.7). Under Assumptions 3.1–3.3, the trajectories of System (3.1)–(3.2) enter into $\mathscr{B}_r$ in finite time if there exists a class $\mathscr{K}$ function $\alpha_3(\cdot)$ such that*

$$\alpha_1(\|x\|) - \frac{\varepsilon}{2} \|R(y)\| \phi_2^2(x) - \alpha_2(\|x\|)\psi(y) \geq \alpha_3(\|x\|), \qquad x \in \overline{\mathscr{B}_r} \tag{3.9}$$

*for some positive constant $\varepsilon$.*

*Proof* Consider the system (3.1)–(3.2). The closed-loop system obtained by applying the control (3.7) to System (3.1)–(3.2) is described by

$$\dot{x} = f(x) + g(x)\left[u_1^I(y) + u_2^I(y) + \Delta g(x)\right] + \Delta f(x) \tag{3.10}$$

$$y = h(x). \tag{3.11}$$

For System (3.10)–(3.11), consider the Lyapunov function candidate $V(x)$ given by Assumption 3.2. Then, the time derivative of $V(x)$ along the trajectories of System (3.10)–(3.11) is given by

$$\begin{aligned}\dot{V}\mid_{(3.10)-(3.11)} &= \frac{\partial V}{\partial x}\left[f(x) + g(x)u_1^I(y)\right] + \frac{\partial V}{\partial x}g(x)\left[u_2^I(y) + \Delta g(x)\right] \\ &\quad + \frac{\partial V}{\partial x}\Delta f(x).\end{aligned} \tag{3.12}$$

From the structure of $u_2^I$ in (3.8) and Assumptions 3.1 and 3.3, it follows that

(i) if $R(y) = 0$, then

$$\frac{\partial V}{\partial x}g(x)\left[u_2^I(y) + \Delta g(x)\right] = \frac{\partial V}{\partial x}g(x)\Delta g(x) = R(y)\Delta g(x) = 0;$$

(ii) if $R(y) \neq 0$, then

$$\begin{aligned}\frac{\partial V}{\partial x}g(x)\left[u_2^I(y) + \Delta g(x)\right] &= -R(y)\frac{\phi_1^2(y)R^T(y)}{2\varepsilon\|R(y)\|} + R(y)\Delta g(x) \\ &\leq -\|R(y)\|\phi_2^2(y)/(2\varepsilon) + \|R(y)\|\phi_1(y)\phi_2(x) \\ &\leq -\|R(y)\|\left(\frac{1}{2\varepsilon}\phi_1^2(y) - \frac{\varepsilon}{2}\phi_2^2(x) - \frac{1}{2\varepsilon}\phi_1^2(y)\right) \\ &= \frac{\varepsilon}{2}\phi_2^2(x)\|R(y)\|,\end{aligned}$$

where the special case of Young's inequality

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

for $\varepsilon > 0$ is used in the implication above.

Consequently,

$$\frac{\partial V}{\partial x}g(x)\left[u_2^I(y) + \Delta g(x)\right] \leq \frac{\varepsilon}{2}\phi_2^2(x)\|R(y)\|. \tag{3.13}$$

From Assumption 3.1 and (3.6), it also follows that

$$\frac{\partial V}{\partial x}\Delta f(x) \le \left\| \frac{\partial V}{\partial x} \right\| \|\Delta f(x)\| \le \alpha_2(\|x\|)\psi(y). \tag{3.14}$$

Substituting (3.5), (3.13) and (3.14) into (3.12), it can be shown that for $x \in \overline{\mathscr{B}_r}$,

$$\dot{V}\mid_{(3.10)-(3.11)} \le -\alpha_1(\|x\|) + \frac{\varepsilon}{2}\phi_2^2(x)\|R(y)\| + \alpha_2(\|x\|)\psi(y) \le -\alpha_3(\|x\|),$$

where the condition (3.9) is used in the last implication. Since $V(x)$ is a class $K_\infty$ function of $\|x\|$ by Assumption 3.2, the conclusion follows directly from Lemma 2.4 in Sect. 2.4. #

Theorem 3.1 shows that under some conditions, the state of System (3.1)–(3.2) can be driven into the domain $\mathscr{B}_r$ by the control (3.7) in a finite time. The focus is now to analyse the characteristics of the system (3.1)–(3.2) in the domain $\mathscr{B}_r$.

### *3.2.4 Local Output Sliding Surface Design*

It is assumed that System (3.1)–(3.2) is limited to the domain $\mathscr{B}_r$. In order to design a control law such that the system is asymptotically stabilised, some assumptions are imposed.

**Assumption 3.4** In $\overline{\mathscr{B}_r}$, there exists a diffeomorphism $z = T(x)$ with $T(0) = 0$, and an output feedback

$$u^{II} = \alpha(y) + \beta(y)v, \tag{3.15}$$

where $v$ is a new input, $\alpha(\cdot)$ with $\alpha(0) = 0$ and $\beta(\cdot)$ are smooth such that the closed-loop system (3.3)–(3.4) and (3.15) has the following form in the new coordinates $z$

$$\dot{z} = Az + Bv \tag{3.16}$$

$$y = Cz, \tag{3.17}$$

where the triple $(A, B, C)$ is observable and controllable with $rank(CB) = m$.

Under Assumption 3.4, it is observed that in the new coordinates $z$, System (3.1)–(3.2) in the domain

$$T(\mathscr{B}_r) \equiv: \{z \mid z = T(x), x \in \mathscr{B}_r\}$$

is described by

$$\dot{z} = Az + B\left(v + \Delta g(T^{-1}(z))\right) + \left[\frac{\partial T}{\partial x}\right]_{x=T^{-1}(z)} \Delta f(T^{-1}(z)) \tag{3.18}$$

$$y = Cz, \tag{3.19}$$

where the triple $(A, B, C)$ is controllable and observable with $rank(CB) = m$.

*Remark 3.4* The analysis above shows that under Assumption 3.4, system (3.1)–(3.2) can be exactly linearised to be (3.18)–(3.19). However, sometimes Assumption 3.4 may not be satisfied, and in this case some other approaches such as the approximate linearisation technique [62] can also be employed to transfer system (3.1)–(3.2) to (3.18)–(3.19).

*Remark 3.5* Notably, in Assumption 3.4, it is required that the system (3.1)–(3.2) is output feedback linearisable only in the domain $\overline{\mathscr{B}_r}$ instead of the entire state space. Furthermore, Assumption 3.4 does not imply that the nominal system (3.3)–(3.4) is output feedback stabilisable and this is in comparison with the work in [24, 196, 214].

**Assumption 3.5** Consider System (3.18)–(3.19). For a given group of distinct negative real values $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$, there exist full rank matrices $W \in \mathbb{R}^{n\times(n-m)}$, and $W^g \in \mathbb{R}^{(n-m)\times n}$ such that
(i) $W^g W = I_{n-m}$, $W^g B = 0$ and $W^g AW = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$;
(ii) $\text{rank}(CW) = p - m$.

Notably, Assumption 3.5 combined with the controllability and observability of the system $(A, B, C)$, yields from [220] that there exists a matrix $S \in \mathbb{R}^{m\times n}$ such that the sliding motion associated with $\{z \mid Sz = 0\}$ is governed by $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$. Furthermore, matrix $S$ has a decomposition $S = FC$. In the following analysis, the same sliding surface can be used

$$\sigma(z) = Sz = 0 \tag{3.20}$$

with $S = FC$ for System (3.18)–(3.19). The choice of matrices $S$ and $F$ is contained in the reference [220]. Once the matrix $S$ or $F$ is obtained, the sliding surface (3.20) is designed, which, due to the constraint $S = FC$, can be described by

$$\{y \mid Fy = 0, \quad y \in \mathbb{R}^p\}$$

and the designed sliding surface (3.20) is an output sliding surface which helps to form an output feedback control scheme.

### 3.2.5 *Local Stability of Sliding Motion*

It is assumed that System (3.1)–(3.2) is limited to the domain $\mathscr{B}_r$. In order to study the stability of the sliding mode dynamics of the system (3.18)–(3.19), the following linear transformation is introduced to derive the sliding mode dynamics associated with the designed sliding surface (3.20)

$$\eta \equiv: Dz \equiv: \begin{bmatrix} W^g \\ B^g \end{bmatrix} z, \tag{3.21}$$

where $W^g$ is defined by Assumption 3.5 and $B^g$ is a generalised inverse of $B$. It is obvious by Assumption 3.5 and [220] that (3.21) is a nonsingular transformation and the matrix $SB$ is also nonsingular.

By direct computation, it is observed that in the new coordinates $\eta$, the system (3.18)–(3.19) in $T(\mathscr{B}_r)$, that is, system (3.1)–(3.2) in the domain $\mathscr{B}_r$ has the following form

$$\dot{\eta}_1 = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}\eta_1 + W^g AB\eta_2 + W^g \Delta\Theta(\eta), \tag{3.22}$$

$$\dot{\eta}_2 = B^g AW\eta_1 + B^g AB\eta_2 + v + \Delta\Upsilon(\eta) + B^g \Delta\Theta(\eta), \tag{3.23}$$

$$y = CD^{-1}\eta, \tag{3.24}$$

where $\eta_1 \in \mathbb{R}^{n-m}$, $\eta_2 \in \mathbb{R}^m$, $\eta = \text{col}(\eta_1, \eta_2)$ and

$$\Delta\Theta(\eta) \equiv: \left[\frac{\partial T}{\partial x}\right]_{x=T^{-1}\circ D^{-1}(\eta)} \Delta f\left(T^{-1} \circ D^{-1}(\eta)\right) \tag{3.25}$$

$$\Delta\Upsilon(\eta) = \Delta g\left(T^{-1} \circ D^{-1}(\eta)\right). \tag{3.26}$$

Notably, in terms of the new coordinate $\eta$, the sliding surface (3.20) is described by $\eta_2 = 0$ due to $SW = 0$ and the nonsingularity of $SB$. The sliding mode dynamics are prescribed by

$$\dot{\eta}_1 = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}\eta_1 + W^g \Delta\Theta(\eta_1, 0). \tag{3.27}$$

with $\Delta\Theta$ defined by (3.25). Since $z = T(x)$ (with $T(0) = 0$) is a diffeomorphism defined in $\overline{\mathscr{B}}_r$ and $\eta = Dz$ is linear, it follows from Assumption 3.1 and (3.25) that

$$\|\Delta\Theta(\eta_1, 0)\| \le \kappa(\eta_1)\|\eta_1\|, \tag{3.28}$$

where $\kappa(\eta_1)$ is continuous in its domain of definition and it can be obtained using Assumption 3.1.

**Theorem 3.2** *Consider System (3.18)–(3.19). Under Assumptions 3.1 and 3.5, the sliding mode dynamics (3.27) are asymptotically stable if*

$$\|W^g\|\kappa(\eta_1) < \gamma, \tag{3.29}$$

*for any* $\eta_1 \in D \circ T(\mathscr{B}_r) \bigcap \mathbb{R}^{n-m}$, *where*

$$\gamma =: min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_{n-m}|\}.$$

*$W$ and $W^g$ are determined by Assumption 3.5, and $\kappa$ satisfies (3.28).*

*Proof* For the sliding mode dynamics (3.27), consider a Lyapunov function candidate as

$$\widetilde{V}(\eta_1) = \eta_1^\tau \eta_1.$$

Then, the time derivative of $\widetilde{V}$ along the trajectories of the dynamic System (3.27) is given by

$$\begin{aligned}\dot{\widetilde{V}}\,|_{(3.27)} &= 2\eta_1^\tau \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}\eta_1 + 2\eta_1^\tau W^g \Delta\Theta(\eta_1, 0)\\ &\le -2\gamma\|\eta_1\|^2 + 2\|\eta_1\|\,\|W^g\|\kappa(\eta_1)\|\eta_1\|\\ &= -2\Big[\gamma - \|W^g\|\kappa(\eta_1)\Big]\|\eta_1\|^2.\end{aligned}$$

Then, $\widetilde{V}(\eta_1)$ is negative definite if (3.29) is satisfied.

Hence, the conclusion follows. #

*Remark 3.6* Theorem 3.2 shows that the values of $\lambda_1, \lambda_2, \ldots, \lambda_{n-m}$ in Assumption 3.5 directly govern the sliding motion if the bound on the mismatched uncertainty satisfies (3.29). It should be noted that $W^g$ (see Assumption 3.5) is closely related to the value of $\lambda_i$ and thus the value of $\gamma$ in (3.29). From Inequality (3.29), to find an appropriate matrix $W^g$ to maximise the value $\frac{\gamma}{\|W^g\|}$ will reduce the conservatism of the result in Theorem 3.2. It shows that mismatched uncertainty affects the sliding motion.

### 3.2.6 Local Sliding Mode Controller

The objective now is to design a sliding mode control such that the corresponding closed-loop system is driven to the sliding surface (3.20) in a finite time and remains on it.

Consider the system (3.18)–(3.19). Since the transformation $T$ is a diffeomorphism in the domain $\overline{\mathscr{B}}_r$, $T^{-1}$ and $\frac{\partial T}{\partial x}$ are both bounded in their domain of definition. Therefore, it is straightforward to see from Assumption 3.1 that there exist continuous functions $\widetilde{\phi}_2$ and $\widetilde{\psi}$ defined in $T(\mathscr{B}_r)$ such that for any $z \in T(\mathscr{B}_r)$

$$\left\|\Delta g\left(T^{-1}(z)\right)\right\| \le \phi_1(y)\widetilde{\phi}_2(z) \tag{3.30}$$

$$\left\|\left[\frac{\partial T}{\partial x}\right]_{x=T^{-1}(z)} \Delta f(T^{-1}(z))\right\| \le \widetilde{\psi}(y) \tag{3.31}$$

with $\widetilde{\psi}(0) = 0$. It should be noted that the functions $\widetilde{\phi}_2$ and $\widetilde{\psi}$ can be obtained by estimation using Assumption 3.1.

An output feedback control for the system (3.18)–(3.19) is:

$$v = -(SB)^{-1}\frac{Fy}{\|Fy\|}\left(\frac{1}{2}\|SB\|\phi_1^2(y) + \|S\|\widetilde{\psi}(y) + K(y)\right), \tag{3.32}$$

where $K(y)$ is a control gain to be defined later and $S$ is defined by (3.20).

Now, applying the control (3.32) to System (3.18)–(3.19), the following closed-loop system is provided

$$\begin{aligned}\dot{z} &= Az - B(SB)^{-1}\frac{Fy}{\|Fy\|}\left(\tfrac{1}{2}\|SB\|\phi_1^2(y) + \|S\|\widetilde{\psi}(y) + K(y)\right)\\ &\quad + B\Delta g(T^{-1}(z)) + \left[\tfrac{\partial T}{\partial x}\right]_{x=T^{-1}(z)}\Delta f(T^{-1}(z)) \qquad (3.33)\\ y &= Cz. \qquad (3.34)\end{aligned}$$

The following result is now ready to be presented:

**Theorem 3.3** *Under Assumptions 3.1 and 3.5, System (3.18)–(3.19) driven by the control (3.32) converges to the sliding surface (3.20) in finite time and remains on it if the control gain $K(y)$ satisfies*

$$K(y) \ge \|SAz\| + \frac{1}{2}\|SB\|\widetilde{\phi}_2^2(z) + \rho \tag{3.35}$$

*for some positive constant $\rho$, where $\widetilde{\phi}_2$ is determined by (3.30) and $F$ satisfies $S = FC$.*

*Proof* From Assumption 3.1, it is observed that (3.30) and (3.31) are satisfied. It follows from (3.20) that

$$\begin{aligned}\dot{\sigma}(z) &= SAz - \frac{Fy}{\|Fy\|}\left(\tfrac{1}{2}\|SB\|\phi_1^2(y) + \|S\|\widetilde{\psi}(y) + K(y)\right)\\ &\quad + SB\Delta g(T^{-1}(z)) + S\left[\tfrac{\partial T}{\partial x}\right]_{x=T^{-1}(z)}\Delta f(T^{-1}(z)). \qquad (3.36)\end{aligned}$$

Then, since $S = FC$

$$\begin{aligned}\sigma^T(z)\dot{\sigma}(z) &= (Fy)^T SAz - \|Fy\|\left(\tfrac{1}{2}\|SB\|\phi_1^2(y) + \|S\|\widetilde{\psi}(y) + K(y)\right)\\ &\quad + (Fy)^T SB\Delta g(T^{-1}(z))\\ &\quad + (Fy)^T S\left[\frac{\partial T}{\partial x}\right]_{x=T^{-1}(z)}\Delta f(T^{-1}(z)). \qquad (3.37)\end{aligned}$$

From the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, it follows from (3.30) that

$$\begin{aligned}(Fy)^T SB\Delta g(T^{-1}(z)) &\le \|Fy\|\,\|SB\|\phi_1(y)\widetilde{\phi}_2(z)\\ &\le \frac{1}{2}\|Fy\|\,\|SB\|(\phi_1^2(y) + \widetilde{\phi}_2^2(z)). \qquad (3.38)\end{aligned}$$

From (3.31),

$$(Fy)^T S \left[\frac{\partial T}{\partial x}\right]_{x=T^{-1}(z)} \Delta f(T^{-1}(z)) \leq \|Fy\| \, \|S\| \widetilde{\psi}(y). \tag{3.39}$$

Substituting (3.38) and (3.39) into (3.37), it follows that

$$\sigma^T(z)\dot{\sigma}(z) \leq \left(-K(y) + \|SAz\| + \frac{1}{2}\|SB\|\widetilde{\phi}_2^2(z)\right) \|Fy\|.$$

Then, by condition (3.35),

$$\sigma^T(z)\dot{\sigma}(z) \leq -\rho\|\sigma(z)\|.$$

This shows that the reachability condition is satisfied, and thus the conclusion follows. #

*Remark 3.7* Generally speaking, Inequality (3.35) cannot be satisfied in the entire state space since the left-hand side is a function of the output $y$ but the right-hand side is a function of the state variables. However, it is always satisfied in $T(\overline{\mathscr{B}_r})$ because the right-hand side of (3.35) is continuous in $T(\overline{\mathscr{B}_r})$ which is a compact set. In fact, a conservative choice is

$$K(y) = \max_{z \in T(\overline{\mathscr{B}_r})} \left\{ \|SAz\| + \frac{1}{2}\|SB\|\widetilde{\phi}_2^2(z) \right\} + \rho.$$

*Remark 3.8* The proof of Theorems 3.1 and 3.3 shows that the effect of uncertainty may be cancelled completely by designing an appropriate control if the uncertainty is bounded by a function of the output. It also demonstrates that the mismatched uncertainty can be dealt with in the same way as for matched uncertainty even if it is bounded by a function of the state variables. However, in this case, the effect of the mismatched uncertainty may not be cancelled completely if only static output feedback control is available.

### *3.2.7 Global Variable Structure Control Synthesis*

In this section, based on the conclusions provided in Sects. 3.2.3–3.2.6, the synthesis of a control which makes the corresponding closed-loop system globally asymptotically stable is sought.

Consider the control

$$u(y) = \begin{cases} u^I(y), & x \in \mathbb{R}^n \setminus \mathscr{B}_r \\ u^{II}(y), & \text{otherwise} \end{cases} \tag{3.40}$$

where $y = h(x)$ defined by (3.2), is the system output, $u^I$ is defined by (3.7) and $u^{II}$ is defined as

$$u^{II} = \alpha(y) + \beta(y)v(y) \tag{3.41}$$

with $\alpha$, $\beta$ defined by Assumption 3.4 and $v$ defined by (3.32). Then, from the results given in Sects. 3.2.3–3.2.6, the following conclusion is immediate.

**Theorem 3.4** *It is assumed that the conditions of Theorems 3.1–3.3 are satisfied. Then under Assumption 3.4, System (3.1)–(3.2) is asymptotically stable globally if* $T(\mathscr{B}_r)$ *is an invariant set with respect to the system (3.33)–(3.34).*

*Proof* Consider the structure of the control (3.40). From Theorem 3.1, it follows that for any $x_0 \in \mathbb{R}^n$, the control $u^I$ can drive the trajectories of the system (3.1)–(3.2) to the domain $\mathscr{B}_r$ in a finite time. Since $T(\mathscr{B}_r)$ is invariant with respect to the system (3.33)–(3.34), once the trajectories of System (3.33)–(3.34) enter into $T(\mathscr{B}_r)$ it is seen that they will stay there and in this case the control $u^{II}$ will drive the system to the sliding surface in finite time by Theorem 3.3. Based on the theory of sliding mode control, the sliding motion takes place once the system state reaches the sliding surface. Theorem 3.2 has shown that the sliding mode is asymptotically stable.

Note that $T$ is a diffeomorphism defined on $\overline{\mathscr{B}}_r$, and the closed-loop system obtained by applying (3.41) to System (3.1)–(3.2) in the $z$ coordinates has the form (3.33)–(3.34). Hence, the conclusion follows from the relationship between systems (3.1)–(3.2) and (3.18)–(3.19). #

A sufficient condition under which the system (3.1)–(3.2) can be stabilised globally even in the presence of uncertainties, is presented in Theorem 3.4, where the condition that $T(\mathscr{B}_r)$ is invariant with respect to the system (3.33)–(3.34) is necessary.

## 3.3 Decentralised Sliding Mode Control for Large-Scale Interconnected Systems

Large-scale interconnected systems are often modelled as dynamic equations composed of interconnections between lower dimensional subsystems. One of the characteristics of these systems is that they are often widely distributed geographically, and thus the information transfer among subsystems may be very difficult due to high cost, or even impossible due to practical limitations. The lack of centralised information or the lack of centralised computing capacity often makes centralised control difficult to implement. As a valid control method for interconnected systems, decentralised control can avoid such disadvantages, and it has been widely used in practical engineering problems such as interconnected inverted pendula system [59], flight control systems [177], electric power systems and chemical process plants [119].

This section focuses on decentralised static output feedback control design using sliding mode techniques.

### 3.3.1 Introduction

Decentralised output feedback control has received much attention and many interesting results have been obtained. Many of these methods are based on a Lyapunov approach or involve adaptive control. In [196, 214, 215], Lyapunov analysis methods are used to form the control scheme and strict structural conditions are imposed on the system together with some strong limitations on the admissible interconnections. Adaptive control techniques are employed by [83, 86] for the control of interconnected systems, but only parametric uncertainty is considered. The corresponding results can only be applied to certain systems with special structure.

Sliding mode control has been used successfully by many authors [38, 39, 160, 173], and indeed sliding mode control schemes for large-scale systems have been proposed (see for example [69, 97, 124, 170]). In these sliding mode control schemes for interconnected systems, it is required that the uncertainties or the interconnections are matched, or else have linear or polynomials bounds. In addition, unfortunately, most of them focus on state feedback control. Much less attention has been paid to the output feedback case. Lee proposed a decentralised output feedback control scheme using sliding mode techniques in [97] but only the linear case is dealt with. Also all uncertainties and interconnections are required to be matched.

In this section, a class of nonlinear large-scale interconnected systems with both matched and mismatched uncertainties are considered. No statistical information about the uncertainties is imposed. Furthermore, the bounding functions on the uncertainties and interconnections take a more general structure than in the other literature in this area. Not only are nonlinear interconnections considered, but nonlinear nominal subsystems are also treated in this work. Based on the sliding surface design method proposed by Edwards and Spurgeon in [37, 38], a composite sliding surface is synthesised for the interconnected system so that the closed-loop system is asymptotically stable when restricted to the surface. A sliding mode control strategy which can eliminate the major limitations of [219] (pointed out by [96]) is proposed, to drive the system to the sliding surface. Under certain conditions, a global result can be derived. Compared with the previous results [97, 196, 203, 214], it is only necessary to solve an $n_i - m_i$ order instead of a $n_i$-order Lyapunov equation, which is especially useful for large-scale interconnected systems. The robustness is enhanced and the conservatism is reduced because the system output information and bounds on the uncertainties are used fully. The restriction on the interconnections is also relaxed.

### 3.3.2 System Description and Problem Formulation

Consider a nonlinear large-scale system formed by $N$ interconnected subsystems as follows

$$\dot{x}_i = A_i x_i + f_i(x_i) + B_i(u_i + \Delta g_i(x_i)) + H_i(x), \tag{3.42}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N, \tag{3.43}$$

where $x = \mathrm{col}(x_1, x_2, \ldots, x_N)$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the $i$-th subsystem respectively and $m_i \le p_i < n_i$. The triples $(A_i, B_i, C_i)$ represent constant matrices of appropriate dimensions with $B_i$ and $C_i$ of full rank. The function $f_i(x_i)$ represents known nonlinearities in the $i$-th subsystem. The matched uncertainty of the $i$-th isolated subsystem is denoted by $\Delta g_i(x_i)$ and $H_i(x)$ represents system interconnections including all mismatched uncertainties. The functions are all assumed to be continuous in their arguments.

It is clear that the nominal subsystems of System (3.42)–(3.43) are described by

$$\dot{x}_i = A_i x_i + f_i(x_i) + B_i u_i \tag{3.44}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N, \tag{3.45}$$

and the isolated subsystems of System (3.42)–(3.43) are given by

$$\dot{x}_i = A_i x_i + f_i(x_i) + B_i(u_i + \Delta g_i(x_i)), \tag{3.46}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N. \tag{3.47}$$

It should be pointed out that the sliding motion of System (3.44)–(3.45) will be the same as the sliding motion of (3.46)–(3.47).

The object is to find some limitations on the nonlinearities and uncertainties associated with the interconnected system, under which a decentralised sliding mode output feedback control scheme can be established for (3.42)–(3.43) such that the corresponding closed-loop system is asymptotically stable.

### 3.3.3 Basic Assumptions

In order to solve the problems proposed in Sect. 3.3.2, it is necessary to impose the following basic assumptions on System (3.42)–(3.43).

**Assumption 3.6** The equations $\mathrm{rank}(C_i B_i) = m_i$ hold for $i = 1, 2, \ldots, N$.

Under Assumption 3.6, as reviewed in Sect. 2.6, there exists a nonsingular linear coordinate transformation such that the triple $(A_i, B_i, C_i)$ with respect to the new coordinates has the structure

$$\begin{bmatrix} \widetilde{A}_{i11} & \widetilde{A}_{i12} \\ \widetilde{A}_{i21} & \widetilde{A}_{i22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \widetilde{B}_{i2} \end{bmatrix}, \quad \begin{bmatrix} 0 & \widetilde{C}_{i2} \end{bmatrix}, \tag{3.48}$$

where $\widetilde{A}_{i11} \in \mathbb{R}^{(n_i-m_i)\times(n_i-m_i)}$, $\widetilde{B}_{i2} \in \mathbb{R}^{m_i\times m_i}$ and $\widetilde{C}_{i2} \in \mathbb{R}^{p_i\times p_i}$ for $i = 1, 2, \ldots, N$.

**Assumption 3.7** The invariant zeros of $(A_i, B_i, C_i)$ lie in the open left-half plane, and the triple $(\widetilde{A}_{i11}, \widetilde{A}_{i12}, \widetilde{C}_{i2})$ given by (3.48) is output feedback stabilisable for $i = 1, 2, \ldots, N$.

*Remark 3.9* Assumptions 3.6 and 3.7 are based on the linear part of the nominal system (3.44)–(3.45). They guarantee the existence of the output sliding surface (see Sect. 2.6). Notably, Assumption 3.7 requires $(\widetilde{A}_{i11}, \widetilde{A}_{i12}, \widetilde{C}_{i2})$, instead of $(A_i, B_i, C_i)$ to be output feedback stabilisable. It should be emphasised that all the matrices in (3.48) can be obtained directly from $(A_i, B_i, C_i)$ using the algorithm given in [37, 38].

From Assumption 3.7, there exist matrices $K_i$ such that

$$\widetilde{A}_{i11} - \widetilde{A}_{i12}K_i\widetilde{C}_{i2} =: A_{i11}$$

are stable for $i = 1 \ldots N$.

**Assumption 3.8** Suppose that $f_i(x_i)$ has the decomposition

$$f_i(x_i) = \Gamma_i(y_i)x_i, \qquad i = 1, 2, \ldots, N, \tag{3.49}$$

where $\Gamma_i \in \mathbb{R}^{n_i\times n_i}$ is a continuous function matrix for $i = 1, 2, \ldots, N$.

**Assumption 3.9** There exist known continuous functions $\rho_i(\cdot)$ and $\eta_i(\cdot)$ such that

(i) $\|\Delta g_i(x_i)\| \le \rho_i(y_i)$,
(ii) $\|H_i(x)\| \le \eta_i(x)$

for $i = 1, 2, \ldots, N$, where $\eta_i(\cdot)$ satisfies

$$\eta_i(x) \le \beta_i(x)\|x\|$$

for some continuous function $\beta_i$.

*Remark 3.10* Assumption 3.9 ensures that all uncertainties in (3.42)–(3.43) are bounded by known functions, and the matched uncertainty is bounded by a function of the output. It should be emphasised that the approach proposed in this work allows more general bounds for $\|\Delta g_i(x_i)\|$. In this section, the assumption is only used to show that the effect of the matched uncertainty can be eliminated completely if it is bounded by a function of the output.

Clearly, there are no special requirements imposed on the structure of the uncertainties and interconnections. As such the assumption above follows the work described in [196, 203, 214, 215]. The bounds on the uncertainties are allowed to take more general forms compared with other work in this area, where it is required that the interconnections are the functions of the output or have linear or polynomial bounds (see for example [69, 97, 143, 170]).

It has been seen that the sliding mode control technique consists of two steps: (i) the design of the sliding surface such that the system possesses the required performance when it is restricted to the surface; (ii) the design of a variable structure control law which drives the system trajectory to, and maintains motion on, the sliding surface. Based on the preliminaries above, a control strategy for System (3.42)–(3.43) will now be developed.

### 3.3.4 Stability Analysis of Sliding Mode Dynamics

In this section, a sliding surface will first be chosen, and then the stability of system (3.42)–(3.43) when constrained to the chosen surface will be studied.

Under Assumptions 3.6 and 3.7, it is observed from Lemma 2.6 in Sect. 2.6 that there exist matrices such that the system

$$\dot{x}_i = A_i x_i + B_i u_i$$

when restricted to the surface $F_i C_i x_i = 0$ is stable for $i = 1, 2, \ldots, N$.

The composite sliding surface for the interconnected system (3.42)–(3.43) is chosen as

$$\sigma(x) = 0, \tag{3.50}$$

where $\sigma(x) \equiv: \text{col}(\sigma_1(x_1), \sigma_2(x_2), \ldots \sigma_N(x_N))$ and

$$\sigma_i(x_i) = F_i C_i x_i = F_i y_i, \tag{3.51}$$

where the $F_i$ are obtained from the algorithm given in [37, 38].

Under Assumptions 3.6–3.8, and using Lemma 2.6 in Sect. 2.6, there exists a nonsingular coordinate transformation $z_i = T_i x_i$ such that in the new coordinates $z = \text{col}(z_1, z_2, \ldots, z_N)$, System (3.42)–(3.43) has the following form

$$\begin{aligned} \dot{z}_i = & \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix} z_i + T_i \Gamma_i(y_i) T_i^{-1} z_i + \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} \left(u_i + \Delta g_i(T_i^{-1} z_i)\right) \\ & + T_i H_i(T^{-1} z) \end{aligned} \tag{3.52}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, N, \tag{3.53}$$

where $A_{i11}$ is stable and $B_{i2} \in \mathbb{R}^{m_i \times m_i}$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ are both nonsingular with

$$C_{i2} = \breve{T}_i \begin{bmatrix} I_{p_i - m_i} & 0 \\ -K_i & I_{m_i} \end{bmatrix}, \qquad \breve{T}_i \in \mathbb{R}^{p_i \times p_i}. \tag{3.54}$$

Furthermore

$$F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} = \begin{bmatrix} 0 & F_{i2} \end{bmatrix}, \tag{3.55}$$

where $F_{i2} \in \mathbb{R}^{m_i \times m_i}$ is nonsingular. The overall coordinate transformation is given by $T^{-1} \equiv$: diag $\{T_1^{-1}, T_2^{-1}, \ldots, T_N^{-1}\}$. It should be noted that the regular form (3.52)–(3.53) can easily be obtained from the algorithm given in [37, 38].

Since $A_{i11}$ is stable for $i = 1, \ldots N$, for any $Q_i > 0$, the following Lyapunov equation has a unique solution $P_i > 0$ such that

$$A_{i11}^{\tau} P_i + P_i A_{i11} = -Q_i, \qquad i = 1, 2, \ldots, N. \tag{3.56}$$

For convenience, partition

$$T_i \Gamma_i(y_i) T_i^{-1} \equiv: \begin{bmatrix} \Gamma_{i11}(y_i) & \Gamma_{i12}(y_i) \\ \Gamma_{i21}(y_i) & \Gamma_{i22}(y_i) \end{bmatrix}, \quad T_i \equiv: \begin{bmatrix} T_{i1} \\ T_{i2} \end{bmatrix}, \quad T_i^{-1} \equiv: \begin{bmatrix} W_{i1} & W_{i2} \end{bmatrix}, \tag{3.57}$$

where $\Gamma_{i11} \in \mathbb{R}^{(n_i - m_i) \times (n_i - m_i)}$, $T_{i1} \in \mathbb{R}^{(n_i - m_i) \times n_i}$ and $W_{i1} \in \mathbb{R}^{n_i \times (n_i - m_i)}$.

Then, System (3.52)–(3.53) can be rewritten as

$$\dot{z}_i^I = A_{i11} z_i^I + \Gamma_{i11}(y_i) z_i^I + \Big[A_{i12} + \Gamma_{i12}(y_i)\Big] z_i^{II} + T_{i1} H_i(T^{-1} z), \tag{3.58}$$

$$\dot{z}_i^{II} = \Big[A_{i21} + \Gamma_{i21}(y_i)\Big] z_i^I + A_{i22} z_i^{II} + \Gamma_{i22}(y_i) z_i^{II} + B_{i2} \left[u_i + \Delta g_i(T_i^{-1} z_i)\right] + T_{i2} H_i(T^{-1} z) \tag{3.59}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, N, \tag{3.60}$$

where $z_i^I \in \mathbb{R}^{n_i - m_i}$, $z_i^{II} \in \mathbb{R}^{m_i}$ and $z_i = \mathrm{col}(z_i^I, z_i^{II})$.

Now, consider the sliding surface (3.51) in the new coordinate system. From (3.55) it follows

$$F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i = \begin{bmatrix} 0 & F_{i2} \end{bmatrix} \begin{bmatrix} z_i^I \\ z_i^{II} \end{bmatrix} = F_{i2} z_i^{II} \qquad i = 1, 2 \ldots, N$$

and from the nonsingularity of $F_{i2}$, it follows that the sliding surface (3.51) becomes

$$z_i^{II} = 0, \qquad i = 1, 2, \ldots, N. \tag{3.61}$$

Partition $C_{i2}$ and $z_i^I$ as

$$C_{i2} = \begin{bmatrix} C_{i21} & C_{i22} \end{bmatrix}, \qquad z_i^I = \mathrm{col}(z_i^{I_1},\ z_i^{I_2}),$$

where $C_{i21} \in \mathbb{R}^{p_i \times (p_i - m_i)}$, $C_{i22} \in \mathbb{R}^{p_i \times m_i}$, $z_i^{I_1} \in \mathbb{R}^{p_i - m_i}$. Then it is straightforward to see that

$$y_i = C_{i21} z_i^{I_2} + C_{i22} z_i^{II}.$$

When System (3.58)–(3.60) is restricted to the sliding surface (3.61), the sliding mode has the following form

$$\dot{z}_i^I = A_{i11} z_i^I + \Gamma_{i11}(C_{i21} z_i^{I_2}) z_i^I + T_{i1} H_i(W z^I), \qquad i = 1, 2, \ldots, N, \quad (3.62)$$

where

$$z^I \equiv: \mathrm{col}(z_1^I, z_2^I, \ldots, z_N^I), \qquad \text{and} \qquad W \equiv: \mathrm{diag}\{W_{11}, W_{12}, \ldots, W_{1N}\}.$$

From (3.62), it can be observed that all the mismatched uncertainties and mismatched nonlinearities affect the dynamics of the sliding mode and may destroy its stability. It is thus necessary to impose some constraints so that the stability of the sliding mode dynamics is guaranteed.

**Theorem 3.5** *Consider the nonlinear interconnected system (3.42)–(3.43). Under Assumptions 3.6–3.9, the sliding mode is asymptotically stable if there exists a domain* $\Omega$ *of the origin* ($\Omega \subset \mathbb{R}^{\sum_{i=1}^{N}(n_i - m_i)}$) *such that*

$$M^\tau(\cdot) + M(\cdot) > 0$$

*in* $\Omega \setminus \{0\}$, *where* $M(\cdot) = (m_{ij}(\cdot))_{N\times N}$ *and the functions* $m_{ij}$ *with* $i, j = 1, 2, \ldots, N$ *are defined by*

$$m_{ij} = \begin{cases} \lambda_{\min}(Q_i) - \|R_i(C_{i21} z_i^{I_2})\| - 2\|P_i T_{i1}\|\, \|W_{i1}\| \beta_i(W z^I), & i = j \\ -2\|P_i T_{i1}\|\, \|W_{j1}\| \beta_i(W z^I), & i \neq j \end{cases},$$

*where* $P_i$ *and* $Q_i$ *are defined in (3.56),* $\Gamma_{i11}(\cdot)$ *and* $T_{i1}$ *are defined by (3.57) and*

$$R_i(\cdot) := P_i \Gamma_{i11}(\cdot) + \Gamma_{i11}^\tau(\cdot) P_i \qquad i = 1, \ldots N$$

*and* $\beta_i(\cdot)$ *is determined by Assumption 3.9.*

*Proof* From the analysis above, all that needs to be proved is that System (3.62) is asymptotically stable. For System (3.62), consider the Lyapunov function candidate

$$V(z_1^I, z_2^I, \ldots, z_N^I) = \sum_{i=1}^{N} \left(z_i^I\right)^\tau P_i z_i^I. \quad (3.63)$$

Then, the time derivative of $V(z_1^I, z_2^I, \ldots, z_N^I)$ along the trajectories of System (3.62) is given by

$$\dot{V}\mid_{(3.62)} = \sum_{i=1}^{N}\Big\{-\left(z_i^I\right)^{\tau} Q_i z_i^I + \left(z_i^I\right)^{\tau}\Big[P_i\Gamma_{i11}(C_{i21}z_i^{I_2}) + \Gamma_{i11}^{\tau}(C_{i21}z_i^{I_2})P_i\Big] z_i^I$$
$$+2\left(z_i^I\right)^{\tau} P_i T_{i1} H_i(W z^I)\Big\}, \tag{3.64}$$

where (3.56) is used above. From Assumption 3.9 and the inequality

$$\|W z^I\| \le \|W_{11} z_1^I\| + \|W_{12} z_2^I\| + \cdots + \|W_{1N} z_N^I\|$$

it follows that

$$\dot{V} \le \sum_{i=1}^{N}\Big\{-\lambda_{\min}(Q_i)\|z_i^I\|^2 + \Big\|P_i\Gamma_{i11}(C_{i21}z_i^{I_2}) + \Gamma_{i11}^{\tau}(C_{i21}z_i^{I_2})P_i\Big\|\ \|z_i^I\|^2$$
$$+2\|z_i^I\|\ \|P_i T_{i1}\|\ \eta_i(W z^I)\Big\}$$
$$\le \sum_{i=1}^{N}\Big\{-\lambda_{\min}(Q_i)\|z_i^I\|^2 + \|R_i(C_{i21}z_i^{I_2})\|\ \|z_i^I\|^2\Big\}$$
$$+2\sum_{i=1}^{N}\Big\{\|z_i^I\|\|PT_{i1}\|\beta_i(W z^I)\sum_{j=1}^{N}\|W_{1j}\|\ \|z_j^I\|\Big\}$$
$$= -\sum_{i=1}^{N}\Big\{\lambda_{\min}(Q_i) - \|R_i(C_{i21}z_i^{I_2})\| - 2\|P_i T_{i1}\|\ \|W_{1i}\|\beta_i(W z^I)\Big\}\|z_i^I\|^2$$
$$+2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\|PT_{i1}\|\ \|W_{1j}\|\ \beta_i(W z^I)\|z_i^I\|\ \|z_j^I\|$$
$$= -\frac{1}{2}\left[\ \|z_1^I\|\ \ \|z_2^I\|\ \cdots\ \|z_N^I\|\ \right](M^{\tau} + M)\begin{bmatrix}\|z_1^I\|\\ \|z_2^I\|\\ \vdots\\ \|z_N^I\|\end{bmatrix}. \tag{3.65}$$

Since by assumption $M^{\tau} + M > 0$ in $\Omega \setminus \{0\}$, the result follows. #

*Remark 3.11* The result given above is in accordance with the fact that sliding mode control is insensitive to matched uncertainty. It also shows that the stability of the sliding mode (3.62) is closely related to the mismatched uncertain interconnections and mismatched nonlinearities.

*Remark 3.12* It should be noted that the function matrix $M$ in Theorem 3.5 is only connected with the sliding mode which is therefore of reduced order. The sliding mode is globally stable if all the bounds on the mismatched uncertainty and mismatched nonlinearity are zero when restricted to the sliding surface. Furthermore, the function matrix $R_i$ reduces to a null matrix if $p = m$. Compared with the results given in [196, 203, 214, 215], the limitations on $M$ here are much weaker.

### 3.3.5 Decentralised Output Feedback Sliding Mode Control

The objective is now to design a decentralised output feedback sliding mode control such that the system state is driven to the sliding surface (3.50). Traditionally, the reachability condition [38, 173] is described by

$$S^\tau(t)\dot{S}(t) < 0$$

for small scale systems with switching surfaces $S(t)$. However, for the interconnected system (3.42)–(3.43), the corresponding condition is described by

$$\sum_{i=1}^{N} \frac{\sigma_i^\tau(x_i)\dot{\sigma}_i(x_i)}{\|\sigma_i(x_i)\|} < 0, \tag{3.66}$$

where $\sigma_i(x_i)$ is defined by (3.51). For details see [69].

In order to fully use system output information, consider the output matrix $C_i$. Comparing System (3.42)–(3.43) with (3.52)–(3.53), it follows that

$$C_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} T_i = C_{i2} \begin{bmatrix} 0 & I_p \end{bmatrix} T_i, \qquad i = 1, 2, \ldots, N, \tag{3.67}$$

where $C_{i2}$ is given by (3.54). Then

$$x_i = T_i^{-1} T_i x_i = T_i^{-1} \begin{bmatrix} (T_i x_i)^I \\ (T_i x_i)^{II} \end{bmatrix} = T_i^{-1} \begin{bmatrix} (T_i x_i)^I \\ C_{i2}^{-1} y_i \end{bmatrix}, \tag{3.68}$$

where $T_i x_i \equiv: \begin{bmatrix} (T_i x_i)^I \\ (T_i x_i)^{II} \end{bmatrix}$ with $(T_i x_i)^I$ being the first $n_i - p_i$ components of $T_i x_i$.

The objective is now to try to satisfy the composite reachability condition (3.66). Consider System (3.42)–(3.43) in domain $\Theta \equiv: \Theta_1 \times \Theta_2 \times \cdots \times \Theta_N$, where $\Theta_i \in \mathbb{R}^{n_i}$ and explicitly

$$\Theta_i \equiv: \left\{ x_i \mid x_i \in \mathbb{R}^{n_i}, \quad \|(T_i x_i)^I\| \le \mu_i \right\}, \qquad i = 1, 2, \ldots, N \tag{3.69}$$

for some positive constant $\mu_i$ for $i = 1, 2, \ldots, N$. Let

$$F_iC_i(A_i+\Gamma_i(y_i))T_i^{-1} \equiv: \begin{bmatrix} \Upsilon_i^I(y_i) & \Upsilon_i^{II}(y_i) \end{bmatrix}, \tag{3.70}$$

where $\Upsilon_i^I$ is the first $n_i - p_i$ columns of the matrix $F_iC_i(A_i+\Gamma(y_i))T_i^{-1}$ for $i = 1, 2, \ldots, N$. Since

$$F_iC_iB_i = F_iC_iT_i^{-1}T_iB_i = F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} = \begin{bmatrix} 0 & F_{i2} \end{bmatrix} \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} = F_{i2}B_{i2}$$

it follows $F_iC_iB_i$ is nonsingular due to the nonsingularity of $F_{i2}$ and $B_{i2}$ for $i = 1, 2, \ldots, N$.

The following control law is proposed

$$u_i = -(F_iC_iB_i)^{-1}\frac{F_iy_i}{\|F_iy_i\|}\Bigg[\|\Upsilon_i^{II}(y_i)C_{i2}^{-1}y_i\| + \frac{\varepsilon_i}{2}\|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i} + \|F_iC_iB_i\|\rho_i(y_i) + k_i(y_i)\Bigg] \tag{3.71}$$

for $i = 1, 2, \ldots, N$, where $\varepsilon_i > 0$ is an adjustable constant; $F_i$ and $\rho_i$ are defined by (3.51) and Assumption 3.9, respectively; and $k_i(y_i)$ is the control gain to be designed later. Obviously, the control law (3.71) depends only on output information and is decentralised.

**Theorem 3.6** *Consider the nonlinear interconnected system (3.42)–(3.43). Under Assumptions 3.6–3.9, the decentralised sliding mode control (3.71) drives the system (3.42)–(3.43) to the composite sliding surface (3.50) and maintains a sliding motion in the domain* $\Theta \setminus \{0\}$ *if the control gain function* $k_i(y_i)$ *satisfies*

$$\sum_{i=1}^{N} k_i(y_i) - \sum_{i=1}^{N} \eta_i(x)\|F_iC_i\| > 0, \tag{3.72}$$

*where* $F_i$ *and* $\eta_i$ *are determined by (3.51) and Assumption 3.9 respectively and* $\Theta$ *is defined by (3.69).*

*Proof* It is observed from the analysis above, all that needs to be proved is that the composite reachability condition (3.66) is satisfied.

From (3.51) and Assumption 3.8, the sliding mode dynamics may be described by

$$\dot{\sigma}_i(x_i) = F_iC_i[A_i + \Gamma_i(y_i)]x_i + F_iC_iB_i[u_i + \Delta g_i(x_i)] + F_iC_iH_i(x), \tag{3.73}$$

for $i = 1, 2 \ldots, N$. Substituting (3.71) into (3.73), it follows that

$$\sum_{i=1}^{N} \frac{\sigma_i^\tau(x_i)\dot{\sigma}_i(x_i)}{\|\sigma_i(x_i)\|} = \sum_{i=1}^{N} \frac{(F_i y_i)^\tau}{\|F_i y_i\|} \Bigg\{ F_i C_i [A_i + \Gamma_i(y_i)] x_i - \frac{F_i y_i}{\|F_i y_i\|} \Bigg[ \Bigg( \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i} \Bigg) + \|F_i C_i B_i\| \rho_i(y_i) + k_i(y_i) \Bigg] + F_i C_i B_i \Delta g_i(x_i) + F_i C_i H_i(x) \Bigg\}. \tag{3.74}$$

From (3.68) and Young's inequality $ab \leq \frac{\varepsilon}{2} a^2 + \frac{b^2}{2\varepsilon}$ for $\varepsilon > 0$, for $i = 1, 2, \ldots, N$

$$\begin{aligned} \|F_i C_i [A_i + \Gamma_i(y_i)] x_i\| &= \left\| F_i C_i [A_i + \Gamma_i(y_i)] T_i^{-1} \begin{bmatrix} (T_i x_i)^I \\ C_{i2}^{-1} y_i \end{bmatrix} \right\| \\ &= \|\Upsilon_i^I(y_i)(T_i x_i)^I + \Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| \\ &\leq \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\|(T_i x_i)^I\|^2}{2\varepsilon_i}. \end{aligned} \tag{3.75}$$

Then it follows from (3.75) that in the domain $\Theta_i$

$$\begin{aligned} &\frac{(F_i y_i)^\tau}{\|F_i y_i\|} \left\{ F_i C_i [A_i + \Gamma_i(y_i)] x_i - \frac{(F_i y_i)^\tau}{\|F_i y_i\|} \left( \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i} \right) \right\} \\ &= \frac{(F_i y_i)^\tau}{\|F_i y_i\|} F_i C_i [A_i + \Gamma_i(y_i)] x_i - \frac{(F_i y_i)^\tau F_i y_i}{\|F_i y_i\|^2} \left[ \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i} \right] \\ &\leq \|F_i C_i [A_i + \Gamma_i(y_i)] x_i\| - \left[ \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i} \right] \\ &\leq 0, \end{aligned} \tag{3.76}$$

where $i = 1, 2, \ldots, N$. From Assumption 3.9,

$$\begin{aligned} \frac{(F_i y_i)^\tau}{\|F_i y_i\|} \Bigg[ -\frac{F_i y_i}{\|F_i y_i\|} \|F_i C_i B_i\| \rho_i(y_i) + F_i C_i B_i \Delta g_i(x_i) \Bigg] &= -\|F_i C_i B_i\| \rho_i(y_i) + \frac{(F_i y_i)^\tau}{\|F_i y_i\|} F_i C_i B_i \Delta g_i(x_i) \\ &\leq -\|F_i C_i B_i\| \rho_i(y_i) + \|F_i C_i B_i\| \, \|\Delta g_i(x_i)\| \\ &\leq 0. \end{aligned} \tag{3.77}$$

Substituting (3.76) and (3.77) into (3.74)

$$\begin{aligned}\sum_{i=1}^{N}\frac{\sigma_i^\tau(x_i)\dot{\sigma}_i(x_i)}{\|\sigma_i(x_i)\|} &\le -\Bigg(\sum_{i=1}^{N}k_i(y_i)-\sum_{i=1}^{N}\frac{(F_iy_i)^\tau}{\|F_iy_i\|}F_iC_iH_i(x)\Bigg)\\ &\le -\Bigg(\sum_{i=1}^{N}k_i(y_i)-\sum_{i=1}^{N}\|F_iC_iH_i(x)\|\Bigg)\\ &\le -\Bigg[\sum_{i=1}^{N}k_i(y_i)-\sum_{i=1}^{N}\eta_i(x)\|F_iC_i\|\Bigg].\end{aligned}$$

Then, if $k_i(y_i)$ is chosen to satisfy (3.72), it follows that in the domain $\Theta$

$$\sum_{i=1}^{N}\frac{\sigma_i^\tau(x_i)\dot{\sigma}_i(x_i)}{\|\sigma_i(x_i)\|}<0.$$

Hence, the result follows. #

*Remark 3.13* It should be noted that Inequality (3.72) can always be satisfied in the domain $\Theta$ with $\mu_i < +\infty$ for $i = 1, 2, \ldots, N$ if

$$\eta_i(x) \le \sum_{j=1}^{N}\xi_{ji}(x_j)$$

for some continuous $\xi_{ij}$ with $i, j = 1, 2, \ldots, N$. In this case one conservative choice of $k_i(y_i)$ is

$$k_i(y_i) > \sum_{j=1}^{N}\|F_jC_j\|\xi_{ij}\Big(T_i^{-1}\text{col}\left((T_ix_i)^I, C_{i2}^{-1}y_i\right)\Big)$$

for $i = 1, 2, \ldots, N$.

*Remark 3.14* From the structure of the control law in (3.71) and the reachability condition (3.72), it can be concluded that the reaching condition is satisfied theoretically in any compact domain of the origin if high gain control is allowed. Generally speaking, the larger the domain that is required, the higher the required control gain. This is in contrast with the work in [196, 203, 214, 215], where some stringent restrictions on the interconnections and uncertainties are necessary (which can only be satisfied in a small domain about the origin). It shows that in this regard, sliding mode control possesses good robustness, not only to the uncertainties present in the isolated subsystem, but also to the effects of the interconnections.

From Theorems 3.5 and 3.6 above, the following conclusions can be obtained directly:

**Corollary 3.1** *For system (3.42)–(3.43), suppose that Assumptions 3.6–3.9 are satisfied, and $M^\tau + M$ with $M$ defined by Theorem 3.5 is positive definite. Then,*

*(i) The sliding mode dynamics (3.62) are globally asymptotically stable.*
*(ii) The closed-loop system composed of (3.42)–(3.43) and the control law*

$$u_i(y_i) = -(F_i C_i B_i)^{-1} \frac{F_i y_i}{\|F_i y_i\|} \Big[ \|\Upsilon_i^{II}(y_i) C_{i2}^{-1} y_i\| + \|F_i C_i B_i\| \rho_i(y_i) + k_i(y_i) \Big] \tag{3.78}$$

*is globally asymptotically stable if $\Upsilon_i^I(y_i) = 0$ and*

$$\|H_i(x)\| \leq \sum_{j=1}^{N} \vartheta_{ji}(y_j)$$

*for some continuous $\vartheta_{ij}$ with $i = 1, 2, \ldots, N$.*

*Proof* (i) From the structure of the Lyapunov function (3.63), the result is obtained directly from Theorem 3.5.

(ii) From the proof of Theorem 3.6, it can be seen that with the control law (3.71), the expressions

$$\frac{\varepsilon_i}{2} \|\Upsilon_i^I(y_i)\|^2 + \frac{\mu_i^2}{2\varepsilon_i}$$

are introduced mainly to cancel the effects of $\Upsilon_i^I(y_i)(T_i x_i)^I$. Obviously, this is unnecessary if $\Upsilon_i^I(y_i) = 0$. In this case, (3.71) becomes (3.78). Under these circumstances it is only necessary to choose $k_i(y_i)$ such that

$$k_i(y_i) > \sum_{j=1}^{N} \|F_j C_j\| \vartheta_{ij}(y_i), \qquad i = 1, 2, \ldots, N.$$

Using the same reasoning as in Theorem 3.6, it can be seen that the corresponding reachability condition is satisfied. Therefore, the controlled trajectories of system (3.42)–(3.43) are driven to the sliding surface (3.50) globally and remain on the surface thereafter. Combining with (i), the result (ii) is obtained immediately. #

It should be noted that in the local case, the domain $\Theta$ given in (3.69) is universal. However, for a specific practical problem, other ways could also be employed to estimate the domain $\Theta$ such that it is as large as possible by combining (3.72) with other requirements.

## 3.4 Case Studies—Implementation of Static Output Control

In this section, case studies on a mass–spring system, coupled inverted pendulums and flight control systems will be used to demonstrate the results developed in Sects. 3.2 and 3.3.

### *3.4.1 Application to a Forced Mass–Spring System*

Consider a mass–spring system with a hardening spring, linear viscous friction and an external force described by (see, [91])

$$m\ddot{s} + c\dot{s} + ks + ka^2 s^3 = u + \Delta g, \tag{3.79}$$

where $s$ denotes the displacement from the reference position, $m$ is the mass of the object sliding on a horizontal surface, $k$ is the spring constant and $u$ is an external force. The term $\Delta g$ includes all uncertainties present in the system. Let

$$x = \text{col}(x_1, x_2) = (s, \dot{s})$$

be the system state. As in [179], the system output is assumed to be

$$y = x_1 + x_2.$$

The parameters are chosen as in [91], pages 172–173. Then, the system is described by

$$\dot{x} = \begin{bmatrix} x_2 \\ -(1 + x_1^2)x_1 - x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + \Delta g(x)) \tag{3.80}$$

$$y = x_1 + x_2. \tag{3.81}$$

It is assumed that

$$\|\Delta g\| \leq 0.1|\sin(y)|\, \|x\|.$$

Then, consider an output feedback control

$$u_1^I(y) = -y + 0.1\sin(y). \tag{3.82}$$

For the closed-loop system composed of (3.82) and (3.80)–(3.81), construct a Lyapunov function candidate

$$V = x^{\tau}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} x + \frac{1}{2}x_1^4.$$

Let

$$\begin{array}{ll} \alpha_1(\nu) = \nu^2, & \alpha_2(\nu) = 8.6056\nu + 2\nu^3, \\ r = 0.2\sqrt{2} & \phi_1(y) = |\sin y| \\ \phi_2(x) = 0.1\|x\|, & \varepsilon = 0.5. \end{array}$$

By similar reasoning as in [91], it is observed that Assumptions 3.1 and 3.2 are both satisfied. Further from

$$\frac{\partial V}{\partial x}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2(x_1 + x_2) = 2y.$$

It is clear that Assumption 3.3 is satisfied with

$$R(y) = 2y.$$

By direct computation, it is observed that the conditions of Theorem 3.1 are satisfied with

$$\alpha_3(\nu) = 1.35\nu^2.$$

Then, consider System (3.80)–(3.81) in $\overline{\mathscr{B}}_r$ with $r = 0.2\sqrt{2}$. The system can be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + \Delta g(x) - x_1^3) \tag{3.83}$$

$$y = [1 \quad 1]x. \tag{3.84}$$

Let

$$\begin{array}{ll} \lambda = \{-1\}, & W = [1 \quad -1]^{\tau} \\ W^g = [1 \quad 0], & B^g = [1 \quad 1]. \end{array}$$

Then, it is observed that Assumption 3.5 is satisfied. According to [220], the sliding surface is chosen as

$$\sigma(x) = [1 \quad 1]x = y, \qquad (F = 1).$$

Since all nonlinearities are matched and the sliding mode dynamics are completely insensitive to matched uncertainty, Theorem 3.2 is satisfied. From Theorem 3.3, the following control can stabilise the system (3.83)–(3.84)

$$u^{II} = -\frac{y}{|y|}\left(\frac{1}{2}(0.1|\sin y| + 0.5)^2 + K\right), \tag{3.85}$$

where $K = 4.5$. Then, it follows from [114] that $\overline{\mathscr{B}}_r$ is an estimate of the domain of attraction of the closed-loop system formed by (3.83) and the control (3.85).

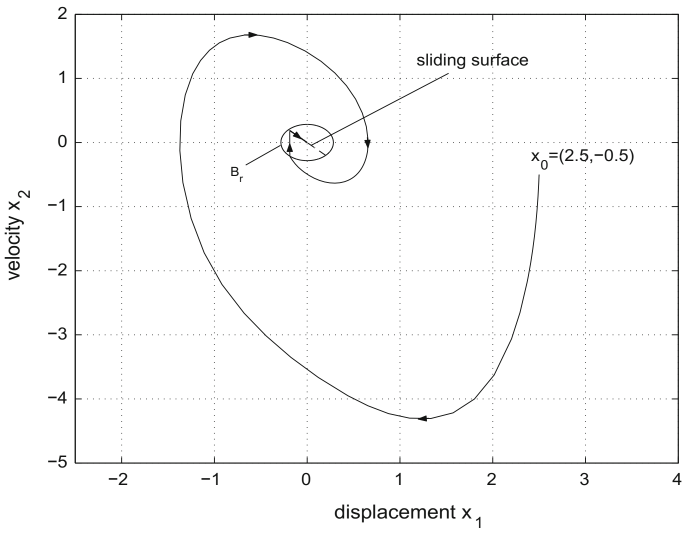


**Fig. 3.1** The $x_1$–$x_2$ phase plane portrait

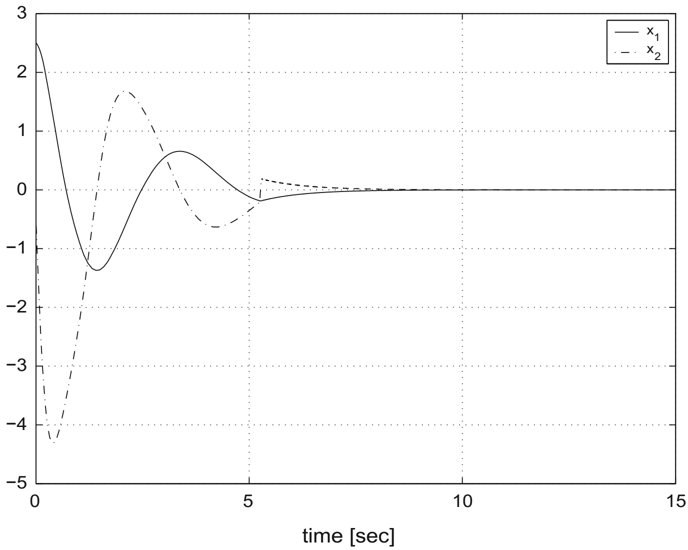


**Fig. 3.2** The time response of displacement and velocity

Using Theorem 3.4, it follows that System (3.80)–(3.81) is globally stabilised by the control (3.40) with $u^I(\cdot)$ and $u^{II}$ defined by (3.82) and (3.85), respectively. Simulation results with the initial condition (2.5, −0.5) are given in Figs. 3.1 and 3.2 which demonstrate that the proposed results are effective.

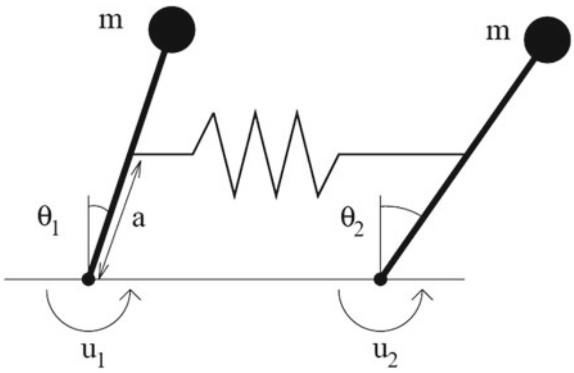


**Fig. 3.3** Coupled inverted pendula

## *3.4.2 Application to Control of Coupled Inverted Pendula*

Consider the system given in Fig. 3.3 formed from two identical inverted pendula which are connected by a spring and subject to two distinct inputs $u_1$ and $u_2$ (see, e.g., [59]).

A salient feature of the system is that the point of attachment of the spring ($a$) can change along the full length ($l$) of the pendula. The input to each pendulum is the torque $u_i$ applied at the pivot point. The two payloads are assumed to be both known and equal to $m$. Let $x_i = \text{col}(x_{i1}, x_{i2}) = \text{col}(\theta_i, \theta_i - \omega_i)$ for $i = 1, 2$, where $\omega_i := \dot{\theta}_i$ is the corresponding angle velocity. From [59], the dynamic equations of the pendula can be described by

$$\dot{x}_1 = \begin{bmatrix} 1 & -1 \\ 1 - \frac{g}{l} & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \frac{ka^2}{ml^2} x_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{ka^2}{ml^2} & 0 \end{bmatrix} x_2 \quad (3.86)$$

$$\dot{x}_2 = \begin{bmatrix} 1 & -1 \\ 1 - \frac{g}{l} & -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ -\frac{ka^2}{ml^2} x_{21} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{ka^2}{ml^2} & 0 \end{bmatrix} x_1, \quad (3.87)$$

where $k$ and $g$ are the spring and gravity constants, and $a$ is an uncertain parameter bounded by $l$. As in [59], assume the only measurable variable is

$$y_i = \begin{bmatrix} -2 & 1 \end{bmatrix} x_i, \qquad i = 1, 2.$$

The parameters are chosen as

$$\frac{g}{l} = 1, \qquad \frac{1}{ml^2} = 1, \qquad \frac{k}{m} = 1.$$

Comparing (3.42)–(3.43), it follows that

$$A_1 = A_2 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \qquad B_1 = B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$C_1 = C_2 = \begin{bmatrix} -2 & 1 \end{bmatrix}, \qquad f_1(y_1) = f_2(y_2) = 0$$

$$\Delta g_1(x_1) = \begin{bmatrix} 0 \\ \left(\frac{a}{l}\right)^2 x_{11} \end{bmatrix}, \qquad \Delta g_2(x_2) = \begin{bmatrix} 0 \\ \left(\frac{a}{l}\right)^2 x_{21} \end{bmatrix}$$

and

$$H_{12}(x) = \begin{bmatrix} 0 \\ -\left(\frac{a}{l}\right)^2 x_{21} \end{bmatrix}, \qquad H_{21}(x) = \begin{bmatrix} 0 \\ -\left(\frac{a}{l}\right)^2 x_{11} \end{bmatrix}.$$

As a result of these definitions,

$$\|\Delta g_i(x_i)\| \leq |x_{i1}|, \qquad \|H_{ij}(x_j)\| \leq |x_{j1}| \quad (i \neq j)$$

for $i, j = 1, 2$. Because both subsystems are square there is no design freedom in the choice of sliding mode dynamics. So without loss of generality choose

$$F_1 = F_2 = 1.$$

Using the algorithm given in [37, 38], it follows that Assumption 3.7 is satisfied and using

$$T_1 = T_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} A_{111} & A_{112} \\ A_{121} & A_{122} \end{bmatrix} = \begin{bmatrix} A_{211} & A_{212} \\ A_{221} & A_{222} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Then, the composite sliding surface is given by

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}^{\tau} = 0.$$

Obviously, the sliding mode is asymptotically stable because all uncertainties and interconnections of System (3.86)–(3.87) are matched. One choice of control law is

$$u_i = y_i + \frac{y_i}{|y_i|} k_i(y_i), \qquad i = 1, 2. \tag{3.88}$$

It is observed that in the domain

$$\Theta = \{(x_{11}, x_{12}, x_{21}, x_{22}) \mid |x_{11}| + |x_{21}| \leq \mu\}.$$

Theorem 3.6 is satisfied if the control gain

$$k_i(y_i) > \mu$$

for $i = 1, 2$, where $\mu$ is a positive constant, and therefore the closed-loop system is asymptotically stable. For simulation purposes, let

$$\mu = 2.3 \quad \text{and} \quad k_i(y_i) = 2.5, \qquad i = 1, 2.$$

With the chosen parameter settings, simulations with initial state

$$x_0 = \left(-1.0,\ -3.5,\ 1.2,\ 5.0\right)$$

are shown in Figs. 3.4 and 3.5. The effectiveness of the proposed control approach is demonstrated by the simulation results.

*Remark 3.15* It should be noted that in this case study, the triple $(A_i, B_i, C_i)$ is not observable. In fact in the work presented in Sect. 3.3, it is not required that the nominal linear system is observable. This is in contrast with the work described in [219].

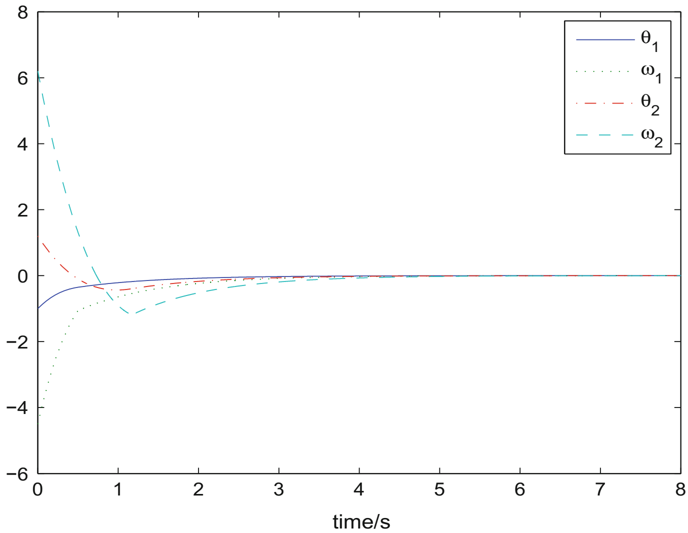


**Fig. 3.4** The time responses of the angles $\theta_1$ and $\theta_2$ and the angular velocities $\omega_1$ and $\omega_2$

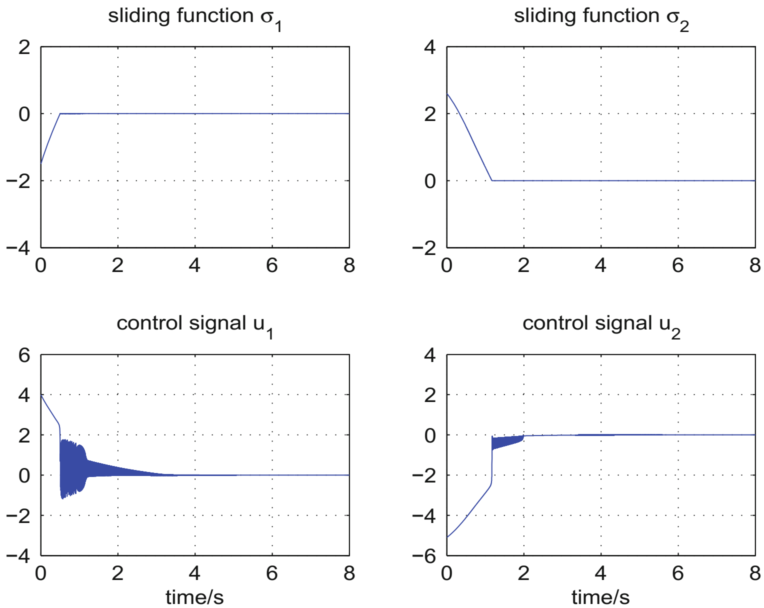


**Fig. 3.5** The time responses of the sliding function $\sigma_1$ and $\sigma_2$ (*upper*) and the control signals $u_1$ and $u_2$ (*bottom*)

### *3.4.3 Application to Flight Control Systems*

Consider a lateral flight control system described in [190]. As in [177], the objective is to design a decentralised scheme to control an (integrated) aircraft system. Let

$$\widehat{x} = \mathrm{col}(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \ldots, \widehat{x}_7),$$

where $\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_7$ denote roll rate, bank angle, aileron deflection, sideslip angle, yaw rate, washout filter output and rudder deflection, respectively.

According to [190], the nominal aircraft lateral mode at the cruising flight condition is described by

$$A = \begin{bmatrix} -1.5880 & 0 & -0.8830 & -4.9670 & -0.7930 & 0 & -0.9710 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -25.0000 & 0 & 0 & 0 & 0 \\ 0.0348 & 0.0353 & 0 & -0.1613 & 1.0000 & 0 & -0.0523 \\ 0.0057 & 0 & 0 & -5.4460 & -0.3860 & 0 & -2.1850 \\ 0.0057 & 0 & 0 & -5.4460 & -0.3860 & -0.5000 & -2.1850 \\ 0 & 0 & 0 & 0 & 0 & 0 & -20.0000 \end{bmatrix}$$

$$B^\tau = \begin{bmatrix} 0 & 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix},$$

where the (lateral) dynamic coefficients represent a Boeing 707 aircraft cruising at an altitude of 8 km at Mach Speed 0.8. Now, Let

$$x_1 =: \mathrm{col}(x_{11}, x_{12}, x_{13}) =: \mathrm{col}(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$$
$$x_2 =: \mathrm{col}(x_{21}, x_{22}, x_{23}, x_{24}) =: \mathrm{col}(\widehat{x}_4, \widehat{x}_5, \widehat{x}_6, \widehat{x}_7).$$

In this way, the system can be rewritten as

$$\begin{aligned}\dot{x}_1 &= \begin{bmatrix} -1.588 & 0 & -0.883 \\ 1.000 & 0 & 0 \\ 0 & 0 & -25.000 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix} u_1 - \begin{bmatrix} 4.967x_{21} + 0.793x_{22} + 0.971x_{24} \\ 0 \\ 0 \end{bmatrix} \\ &\quad + \Delta H_1(x) \\ y_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_1\end{aligned}$$

and

$$\begin{aligned}\dot{x}_2 &= \begin{bmatrix} -0.161 & 1.000 & 0 & -0.052 \\ -5.446 & -0.386 & 0 & -2.185 \\ -5.446 & -0.386 & -0.500 & -2.185 \\ 0 & 0 & 0 & -20.000 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 20 \end{bmatrix} u_2 \\ &\quad + \begin{bmatrix} 0.035x_{11} + 0.035x_{12} \\ 0.006x_{11} \\ 0.006x_{11} \\ 0 \end{bmatrix} + \Delta H_2(x) \\ y_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_2,\end{aligned}$$

where $y_1 = \mathrm{col}(y_{11}, y_{12})$ and $y_2 = \mathrm{col}(y_{21}, y_{22})$ are system outputs and $\Delta H_i(x)$ represents all the uncertainties in the $i$-th subsystem for $i = 1, 2$. According to [190],

$$\|\Delta H_i(x)\| \le \beta_i \|x\|$$

where $\beta_i$ is some positive constant for $i = 1, 2$.

Choose

$$K_1 = -0.6 \quad \text{and} \quad K_2 = -1.3.$$

Then using the algorithm given in [37, 38], it follows that Assumption 3.7 is satisfied and choosing

$$T_1 = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & -0.6000 & 1.0000 \end{bmatrix}$$

yields

$$\begin{bmatrix} A_{111} & A_{112} \\ A_{121} & A_{122} \end{bmatrix} = \left[\begin{array}{cc|c} -1.5880 & -0.5298 & -0.8830 \\ 1.0000 & 0 & 0 \\ \hline -0.6000 & -15.0000 & -25.0000 \end{array}\right]$$

with

$$F_1 = \begin{bmatrix} -0.6 & 1 \end{bmatrix} \quad \text{and} \quad C_{12} = \begin{bmatrix} 1 & 0 \\ 0.6 & 1 \end{bmatrix}$$

whilst

$$T_2 = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & -1.3000 & 1.0000 \end{bmatrix}$$

gives

$$\begin{bmatrix} A_{211} & A_{212} \\ A_{221} & A_{222} \end{bmatrix} = \left[\begin{array}{ccc|c} -0.1613 & 1.0000 & -0.0680 & -0.0523 \\ -5.4460 & -0.3860 & -2.8405 & -2.1850 \\ -5.4460 & -0.3860 & -3.3405 & -2.1850 \\ \hline 7.0798 & 0.5018 & -21.6574 & -17.1595 \end{array}\right]$$

with

$$F_2 = \begin{bmatrix} -1.3000 & 1.0000 \end{bmatrix} \quad \text{and} \quad C_{22} = \begin{bmatrix} 1 & 0 \\ 1.3 & 1 \end{bmatrix}.$$

Let

$$\beta_1 = 0.035 \qquad \text{and} \qquad \beta_2 = 0.0628.$$

By direct computation, it can be shown that the sliding mode is globally stable. Now consider the system in the domain $\Theta = \Theta_1 \times \Theta_2$, where

$$\begin{aligned} \Theta_1 &\equiv: \{x_1 \mid x_{12}, x_{13} \in \mathbb{R}, |x_{11}| \le \mu_1\} \\ \Theta_2 &\equiv: \{x_2 \mid x_{23}, x_{24} \in \mathbb{R}, |x_{21}| + |x_{22}| \le \mu_2\}. \end{aligned}$$

Choose the control gains as

$$k_1(y_1) = 6(\mu_1 + \|y_1\|), \qquad k_2(y_2) = 0.1(\mu_2 + \|y_2\|). \tag{3.89}$$

It can be shown that the reachability condition is satisfied in the domain $\Theta$. Therefore, under the control law

$$\begin{aligned} u_1 &= -\frac{-0.6y_{11} + y_{12}}{25|0.6y_{11} - y_{12}|}\left[25|y_{12}| + 0.18\varepsilon_1 + \frac{\mu_1^2}{2\varepsilon_1} + k_1(y_1)\right] \\ u_2 &= -\frac{-1.3y_{21} + y_{22}}{20|1.3y_{21} - y_{22}|}\left[|2.2536y_{21} + 0.1597y_{22}| + 14.9383\varepsilon_2 + \frac{\mu_2^2}{2\varepsilon_2} + k_2(y_2)\right], \end{aligned}$$

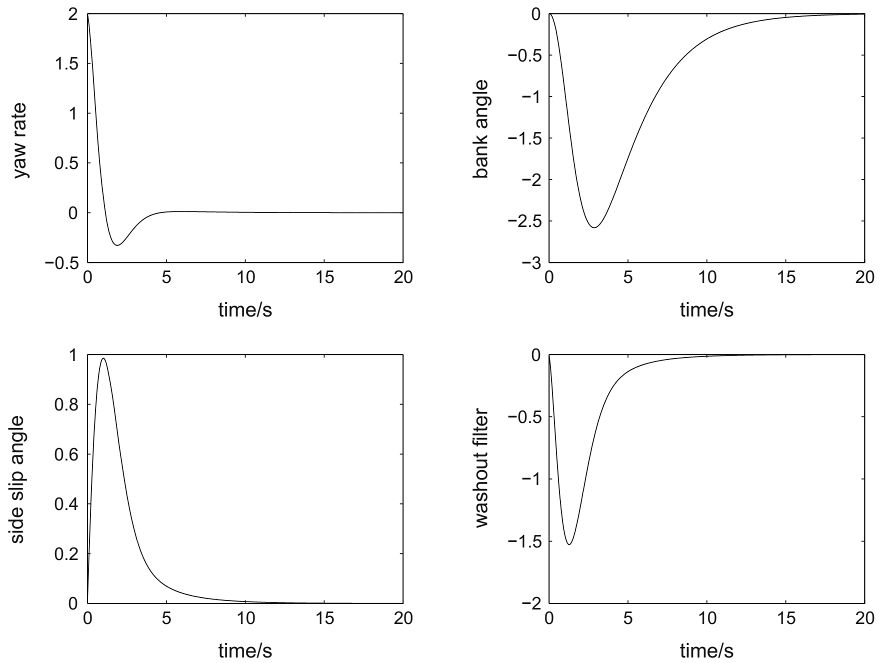


**Fig. 3.6** The time responses of some variables of aircraft system

where $k_1$, $k_2$ are defined in (3.89), the corresponding closed-loop system is asymptotically stable.

For simulation purposes, the parameters are chosen as

$$\varepsilon_1 = \varepsilon_2 = 0.5, \qquad \mu_1 = \mu_2 = 5$$

and the initial conditions are $x_0 = (0, 0, 0, 0, 2, 0, 0)$. Simulation results are shown in Fig. 3.6 and are as expected.

## 3.5 Summary

This chapter has discussed static output feedback control design for both nonlinear systems and interconnected systems. The results developed for nonlinear systems are global. It is expected that the approach proposed in Sect. 3.2 will form the basis for a new technique for further exploring the problem of global stabilisation using static output feedback control for fully nonlinear systems. The study in Sect. 3.3 provides an approach to design a decentralised static output feedback sliding mode control for nonlinear interconnected systems when mismatched uncertainties are involved. Based on the method proposed by Edwards and Spurgeon [38], a composite sliding surface is formed for nonlinear interconnected systems with linear nominal subsystems. The control law eliminates the major limitations of [219]. In certain situations, global results have also been obtained for the interconnected systems in Sect. 3.3. The case studies in Sect. 3.4 have illustrated the approaches developed and simulation results have demonstrated that the approaches developed in Sects. 3.2 and 3.3 are effective and feasible for practical design.

# Chapter 4
# Dynamical Output Feedback Variable Structure Control

The focus of this chapter is dynamical output feedback controller design for nonlinear systems with nonlinear disturbances using sliding mode techniques. Nonlinear control systems with both minimum phase and nonminimum phase nominal systems are considered.

## 4.1 Introduction

As dynamical feedback can use both information about the designed dynamics and system outputs for control design, the limitation on the considered system can be relaxed when compared with the static output feedback methods discussed in Chap. 3. However, extra hardware and/or software is required to build the dynamical compensator for implementation and the dimension of the corresponding closed-loop systems may increase by a factor of two—which may produce other problems.

In view of this, this chapter considers the development of control schemes to provide asymptotic stabilisation based on designed dynamical systems. The considered systems are permitted to have mismatched disturbances/uncertainties. In contrast to Chap. 3, all the uncertainties involved in this chapter are bounded by nonlinear functions of the system state variables, instead of the output variables. The bounding functions are assumed to be known and thus it is possible to use them for sliding mode control design and system analysis to reduce the effects of the uncertainties.

In Sect. 4.2, a sliding surface is designed based on the approach which has been outlined in Sect. 2.6. This sliding surface is independent of any observer design. A sliding mode control is synthesised based on estimated states from an observer and the system outputs. The controller design and the observer design are separated. The designed control can be used with any observer but the developed approach requires that the considered system is minimum phase.

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In Sect. 4.3, a dynamical compensator is designed first. Then, the considered system and the error dynamics between the system and the compensator form an augmented system. A sliding surface is designed for the augmented system based on the equivalent control approach. Therefore, the sliding motion analysis is closely related to the designed compensator and limitations on the compensator are necessary to guarantee the stability of the sliding motion and the satisfaction of a reachability condition. However, it is not required that the nominal system is minimum phase.

Applications to control the High Incidence Research Model (HIRM) aircraft are given in Sect. 4.4. Both longitudinal and lateral aircraft dynamics based on different trim values of Mach and Height are employed in the simulation study.

## 4.2 Control of Nonlinear Systems with Matched and Mismatched Uncertainties

This section considers a class of nonlinear systems with minimum phase nominal systems.

### *4.2.1 Introduction*

There has been significant work which focuses on output feedback control [25, 95, 96, 157, 200, 219]. In the approach proposed by Zak and Hui [219] geometric conditions were presented for the existence of a sliding mode and an associated design algorithm was also derived.

This section considers a class of nonlinear systems involving both matched and mismatched uncertainties. No statistical information is required about the uncertainties. The bounds on the matched and mismatched uncertainties, which take more general forms, are both fully used in the observer design and in the sliding mode control design. By employing the sliding surface prescribed in Sect. 2.6, the stability of the sliding mode is shown first. Then, an asymptotic observer is established to estimate the state variables based on a constrained Lyapunov equation. Further, provided that the observer has been well designed, a variable structure control is proposed using the estimated state and system output to stabilise the considered systems asymptotically.

### *4.2.2 System Description and Preliminaries*

Consider the system

$$\dot{x}(t) = Ax(t) + B(u + \Delta g(x,t)) + \Delta f(x,t) + \Phi(x) \tag{4.1}$$

$$y(t) = Cx(t), \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ $(m \le p < n)$ are state variables, inputs and outputs, respectively; the triple $(A, B, C)$ comprises constant matrices of appropriate dimensions with $B$ and $C$ both being of full rank; $\Delta g(x, t)$ and $\Delta f(x, t)$ are the matched and the mismatched uncertainties, respectively, which are continuous in their arguments; the known nonlinear vector $\Phi(x)$ is sufficiently smooth with $\Phi(0) = 0$. Since $\Phi(x)$ is smooth and $\Phi(0) = 0$, there exists a matrix $H(x) \in \mathbb{R}^{n\times n}$ such that

$$\Phi(x) = H(x)x.$$

The following basic assumptions are imposed on the system (4.1)–(4.2):

**Assumption 4.1** $\operatorname{rank}(CB) = m$.

Under Assumption 4.1, it can be shown from [37] that a coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the triple $(A, B, C)$ with respect to the new coordinates $\widetilde{x}$ has the following structure:

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} 0 \\ \widetilde{B}_2 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} 0 & \breve{T} \end{bmatrix}, \tag{4.3}$$

where $\widetilde{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $\widetilde{B}_2 \in \mathbb{R}^{m\times m}$ is nonsingular and $\breve{T} \in \mathbb{R}^{p\times p}$ is orthogonal.

**Assumption 4.2** For the triple $(\widetilde{A}_{11}, \widetilde{A}_{12}, C_1)$ with $C_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)} \end{bmatrix}$, there exists a matrix $K$ such that $\widetilde{A}_{11} - \widetilde{A}_{12}KC_1$ is stable.

*Remark 4.1* Assumptions 4.1 and 4.2 are based on the linear part of the nominal system. They guarantee the existence of the output sliding surface (see Sect. 2.6). A necessary condition for this is that the triple $(A, B, C)$ is minimum phase [38].

**Assumption 4.3** The pair $(A, C)$ is observable, and the nonlinear function $\Phi(x)$ is Lipschitz in its defined domain.

In view of the observability of $(A, C)$, there exists a gain $L \in \mathbb{R}^{n\times p}$ such that $A - LC$ is Hurwitz stable. Therefore, for $Q > 0$, there exists a unique $P > 0$ satisfying the Lyapunov equation

$$(A - LC)^\tau P + P(A - LC) = -Q. \tag{4.4}$$

**Assumption 4.4** There exist known continuous functions $\xi_1$, $\xi_2$ and $\gamma$ in a domain $\Omega \times \mathbb{R}^+$ $(0 \in \Omega \subset \mathbb{R}^n)$ such that

$$\|\Delta g(x, t)\| \le \xi_1(y, t)\xi_2(x, t), \tag{4.5}$$

$$\Delta f(x, t) = E\Delta\eta(x, t) \tag{4.6}$$

with

$$\|\Delta\eta(x, t)\| \le \gamma(x, t)\|y\|,$$

where $\xi_2$ and $\gamma$ are both Lipschitz about $x$ in $\Omega$ uniformly for $t \in \mathbb{R}^+$, and $E$ is a structural matrix for $\Delta f(x,t)$.

*Remark 4.2* The approach proposed by Zak and Hui [219] has the following stringent assumptions as pointed out in [95, 96]:

- the uncertainty is bounded by a function of the output $y$;
- there exists a matrix $M$ such that $SA = MC$, where $S$ is the sliding matrix.

An example in [95] shows that the former condition is quite restrictive, and another example and some analysis in [219] also show that the latter limitation is relatively strong.

The objective is to propose a control scheme such that the two limitations in Remark 4.2 are eliminated. In view of this, a sliding surface will be designed and the stability of the corresponding sliding motion will be studied first. Then an asymptotic observer will be provided, and based on the estimated state from the observer and the system output, a sliding mode controller will be described to complete the task.

### 4.2.3 Stability Analysis of the Sliding Mode

Under Assumptions 4.1 and 4.2, it follows from Sect. 2.6 that there exists a matrix $F \in \mathbb{R}^{m\times p}$ to form a sliding surface

$$\sigma(x) =: Sx = FCx = 0 \tag{4.7}$$

for the triple $(A, B, C)$ given in (4.1)–(4.2), and a coordinate transformation

$$z = Tx \tag{4.8}$$

such that the triple $(A, B, C)$ has the following structure:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C = [0 \quad C_2],$$

where $A_{11}$ is stable, and the matrices $B_2 \in \mathbb{R}^{m\times m}$ and $C_2 \in \mathbb{R}^{p\times p}$ are both nonsingular.

Now, consider the sliding surface (4.7) for System (4.1)–(4.2). From Sect. 2.6, the existence of the sliding surface (4.7) is guaranteed by Assumptions 4.1 and 4.2. The objective now is to derive the sliding mode dynamics and analyse its stability.

It is straightforward to see that in the new coordinates $z$ defined in (4.8), system (4.1)–(4.2) is described by

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \big(u + \Delta g(T^{-1}z, t)\big) + T\Delta f(T^{-1}z, t) + \begin{bmatrix} R(z) & * \\ * & * \end{bmatrix} z \tag{4.9}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{4.10}$$

where $A_{11}$ is stable, $R(\cdot) \in \mathbb{R}^{(n-m)\times(n-m)}$ and the $*$'s are subblocks of $TH(\cdot)T^{-1}$ which play no part in the subsequent analysis. Furthermore,

$$F \begin{bmatrix} 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & F_2 \end{bmatrix},$$

where $F_2 \in \mathbb{R}^{m\times m}$ is nonsingular.

*Remark 4.3* It should be emphasised that the regular form in (2.22)–(2.23) or system (4.9)–(4.10) can be obtained from a systematic algorithm together with any output feedback pole placement algorithm of choice. To check the output feedback stabilisability of a triple is far from trivial problem [165] but it is well studied. Further details of the algorithms and how to determine the switching surface (4.7) are available in [37, 38].

Since $A_{11}$ is stable, for any $Q_1 > 0$, the following Lyapunov equation has a unique solution $P_1 > 0$ such that

$$A_{11}^{\tau} P_1 + P_1 A_{11} = -Q_1. \tag{4.11}$$

Partition $z = \text{col}(z_1, z_2)$ with $z_1 \in \mathbb{R}^{n-m}$. It follows from Sect. 2.6 that in the new coordinate $z$, the switching function $FCx$ can be described by

$$F \begin{bmatrix} 0 & C_2 \end{bmatrix} z = F_2 z_2,$$

where $F_2$ is nonsingular. From the nonsingularity of $F_2$, it follows that the sliding surface (4.7) becomes $z_2 = 0$. Then, when System (4.9)–(4.10) is restricted to the sliding surface $z_2 = 0$, the sliding mode takes the following form:

$$\dot{z}_1 = A_{11} z_1 + \Delta\Psi(z_1, t) + R(z_1, 0) z_1, \tag{4.12}$$

where $\Delta\Psi(z_1, t)$ is the first $(n-m)$ components of $T\Delta f(T^{-1}z, t)|_{z_2=0}$. From Assumption 4.4, it is easy to assert the existence of a continuous function $\chi(z_1, t)$ depending on $\gamma$ and $T$ such that

$$\|\Delta\Psi(z_1, t)\| \le \chi(z_1, t)\|z_1\|. \tag{4.13}$$

**Theorem 4.1** *Consider System (4.1)–(4.2). Under Assumptions 4.1–4.4, the sliding mode (4.12) is asymptotically stable if there exists a neighbourhood of the origin such that*

$$\chi(z_1, t)\|P_1\| + \|P_1 R(z_1, 0)\| < \frac{1}{2}\lambda_{\min}(Q_1), \tag{4.14}$$

*where $P_1$ and $Q_1$ satisfies (4.11) and $\chi$ is determined by (4.13).*

*Proof* For the sliding mode system (4.12), consider the Lyapunov function candidate $V_1 = z_1^\tau P_1 z_1$. The time derivative of $V_1$ along the trajectories of the dynamic system (4.12) is given by

$$\dot{V}_1 \mid_{(4.12)} = -z_1^\tau Q_1 z_1 + 2z_1^\tau P_1 \Delta\Psi(z_1, t) + 2z_1^\tau P_1 R(z_1, 0)z_1,$$

where (4.11) is used above. Further, from (4.13), it follows that

$$\begin{aligned}\dot{V}_1 \mid_{(4.12)} &\leq -\lambda_{\min}(Q_1)\|z_1\|^2 + 2\chi(z_1, t)\|P_1\|\,\|z_1\|^2 + 2\|P_1 R(z_1, 0)\|\,\|z_1\|^2 \\ &\leq -2\Big(\frac{1}{2}\lambda_{\min}(Q_1) - \chi(z_1, t)\|P_1\| - \|P_1 R(z_1, 0)\|\Big)\|z_1\|^2.\end{aligned}$$

From the condition (4.14), it is observed that $\dot{V}_1 \mid_{(4.12)}$ is negative definite. Hence, the conclusion follows. #

*Remark 4.4* It is observed from (4.12) that the matched uncertainty does not affect the stability of the sliding mode. However, the mismatched uncertainty affects the behaviour of the dynamics of the sliding mode directly. It is therefore necessary to impose some constraints on the mismatched part. The limitation (4.14) on the bound of the mismatched component is used to guarantee the stability of the sliding mode.

*Remark 4.5* In most cases, Theorem 4.1 is local due to (4.14). From the proof above, however, a global conclusion is available if condition (4.14) is satisfied globally. Specifically, Theorem 1 is global if the bounds on all the mismatched uncertainties degenerate to the linear case as in [95, 157].

### *4.2.4 Variable Structure Observer Design*

In this section, an asymptotic variable structure observer is established to estimate the state variables of the system (4.1)–(4.2). The following assumption is required:

**Assumption 4.5** There exist constant matrices $\Gamma$ and $\Upsilon$ such that the solution of the Lyapunov equation (4.4) satisfies the following constraints:

(i) $B^\tau P = \Gamma C$;
(ii) $E^\tau P = \Upsilon C$,

where $E$ is defined as in (4.6).

*Remark 4.6* It should be noted that if there exists a matrix $L$ such that $(A - LC, B, C)$ is passive, then Assumption 4.5 (i) is satisfied with $\Gamma = I$. Similarly, if $(A - LC, E, C)$ is passive, then, Assumption 4.5 (ii) is also satisfied. Similar conditions are adopted by Yan et al. and many other authors (see [214] and reference therein).

Construct the following dynamical system associated with the system (4.1)–(4.2):

$$\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) + B\left(u + \Xi_1(\hat{x}, y, t)\right) + \Pi_2(\hat{x}, y, t) + \Phi(\hat{x}), \quad (4.15)$$

where $L$ is determined by (4.4), and $\Pi_1$ and $\Pi_2$ are defined by

$$\Pi_1(\hat{x}, y, t) =: \frac{\Gamma(y - C\hat{x})}{\|\Gamma(y - C\hat{x})\|}\xi_1(y, t)\xi_2(\hat{x}, t), \quad (4.16)$$

$$\Pi_2(\hat{x}, y, t) =: E\frac{\Upsilon(y - C\hat{x})}{\|\Upsilon(y - C\hat{x})\|}\gamma(\hat{x}, t)\|y\|. \quad (4.17)$$

The following result can be presented:

**Theorem 4.2** *Consider the dynamical system (4.15). Under Assumptions 4.3–4.5, there exists a nonnegative constant $\alpha_1$ and a positive constant $\alpha_2$ such that the state estimation error satisfies*

$$\|x - \hat{x}\| \le \alpha_1 \exp\{-\alpha_2 t\}$$

*if there exists a neighbourhood of the origin $\Omega_q \subset \Omega$ such that*

$$\sup_{x \in \Omega_q, t \in \mathbb{R}^+} \left\{\mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\| + \mathscr{L}_\gamma\|y\|\,\|\Upsilon C\| + \mathscr{L}_\Phi\|P\|\right\} < \frac{1}{2}\lambda_{\min}(Q). \quad (4.18)$$

*Proof* Let $e = x - \hat{x}$. It follows from (4.1) and (4.15) that the state estimation error equation is described by

$$\begin{aligned}\dot{e}(t) = (A - LC)e(t) + B\left(-\Pi_1(\hat{x}, y, t) + \Delta g(x, t)\right) - \Pi_2(\hat{x}, y, t)\\ +\Delta f(x, t) + \Phi(x) - \Phi(\hat{x}).\end{aligned} \quad (4.19)$$

For System (4.19), consider a Lyapunov function candidate

$$V = e^\tau Pe.$$

Then, the time derivative of $V$ along the trajectories of System (4.19) is given as

$$\begin{aligned}\dot{V}\mid_{(4.19)} = -e^\tau Qe + 2e^\tau PB\big(-\Pi_1(\hat{x}, y, t) + \Delta g(x, t)\big)\\ + 2e^\tau P\big(-\Pi_2(\hat{x}, y, t) + \Delta f(x, t)\big) + 2e^\tau P\big(\Phi(x) - \Phi(\hat{x})\big).\end{aligned} \quad (4.20)$$

From Assumptions 4.4 and 4.5, and (4.16), it follows that

$$\begin{aligned} & e^\tau PB\big(-\Pi_1(\hat{x}, y, t) + \Delta g(x, t)\big) \\ & \le -\|\Gamma Ce\|\xi_1(y, t)\xi_2(\hat{x}, t) + \|\Gamma Ce\|\xi_1(y, t)\xi_2(x, t) \\ & \le \mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\|\,\|e\|^2. \end{aligned}$$

Therefore,

$$e^\tau PB\Big(-\Pi_1(\hat{x}, y, t) + \Delta g(x, t)\Big) \le \mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\|\,\|e\|^2. \tag{4.21}$$

By the same reasoning as above, it is observed from (4.17) and Assumptions 4.4 and 4.5 that

$$e^\tau P\Big(-\Pi_2(\hat{x}, y, t) + \Delta f(x, t)\Big) \le \|y\|\mathscr{L}_\gamma\|\Upsilon C\|\,\|e\|^2. \tag{4.22}$$

From Assumption 4.3, $\Phi(x)$ is Lipschitz, and thus

$$e^\tau P\big(\Phi(x) - \Phi(\hat{x})\big) \le \mathscr{L}_\Phi\|P\|\,\|e\|^2. \tag{4.23}$$

Substituting (4.21)–(4.23) into (4.20), it follows that

$$\begin{aligned} \dot{V}\,|_{(4.19)} & \le -e^\tau Qe + 2\Big(\mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\| + \mathscr{L}_\gamma\|y\|\,\|\Upsilon C\| + \mathscr{L}_\Phi\|P\|\Big)\|e\|^2 \\ & \le -2\Big(\frac{1}{2}\lambda_{\min}(Q) - \mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\| - \mathscr{L}_\gamma\|y\|\,\|\Upsilon C\| \\ & \quad -\mathscr{L}_\Phi\|P\|\Big)\|e\|^2. \end{aligned} \tag{4.24}$$

Define

$$\kappa = \lambda_{\min}(Q) - 2\sup_{x\in\Omega, t\in\mathbb{R}^+}\big\{\mathscr{L}_{\xi_2}\xi_1(y, t)\|\Gamma C\| + \mathscr{L}_\gamma\|y\|\,\|\Upsilon C\| + \mathscr{L}_\Phi\|P\|\big\}.$$

It follows from (4.24) and (4.18) that $\kappa > 0$ and

$$\dot{V} \le -\kappa\|e\|^2 \le -\frac{\kappa}{\lambda_{\max}(P)}e^\tau Pe = -\frac{\kappa}{\lambda_{\max}(P)}V.$$

Consequently,

$$V(t) \le V(0)\exp\left\{-\frac{\kappa}{\lambda_{\max}(P)}t\right\}$$

and using the inequality

$$\|e\| \le \sqrt{\frac{V}{\lambda_{\min}(P)}}$$

it follows that

$$\|e(t)\| \le \sqrt{\frac{V(0)}{\lambda_{\min}(P)}} \exp\left\{-\frac{\kappa}{2\lambda_{\max}(P)}t\right\}.$$

Hence, choosing

$$\alpha_1 = \sqrt{\frac{V(0)}{\lambda_{\min}(P)}} \quad \text{and} \quad \alpha_2 = \frac{\kappa}{2\lambda_{\max}(P)}$$

the conclusion follows. #

*Remark 4.7* It should be noted that the injection signals defined by (4.16) and (4.17) may be discontinuous, and thus the classical solution of the dynamical system (4.15) may no longer exist due to the discontinuity in the right-hand side. In this case, a solution of equation (4.15) is defined in the Filippov sense [46]. Formally, the injection signals in (4.16) and (4.17) have only been defined for motion away from the switching surface. In practice, ideal motion along the switching surface cannot be achieved and so this does not pose a problem.

*Remark 4.8* From the proof above, it is clear that the functions $\Pi_1$ and $\Pi_2$ are introduced to reject the effect of the uncertainties $\Delta g(x,t)$ and $\Delta f(x,t)$, respectively. It is obvious that Assumption 4.5 (i) is unnecessary if $\Delta g = 0$. Similarly, Assumption 4.5 (ii) is redundant if the system does not suffer from any mismatched uncertainty. Theorem 4.2 shows that under some conditions, the observer error converges to zero exponentially.

*Remark 4.9* The observer defined in (4.15) with the injection signals in (4.16) and (4.17) may be discontinuous. In practical implementation, the system trajectories may not stay on the associated manifolds and consequently chattering may occur. One way of overcoming this is to introduce a boundary layer about the manifolds: a detailed discussion along these lines is available in [13].

### 4.2.5 Sliding Mode Control Design

In this section, a control is to be designed based on the output and the estimated states so that the system is driven to the sliding surface and forced to remain there.

Consider the following output feedback sliding mode controller:

$$\begin{aligned} u = -(SB)^{-1}\Big\{ & S(A\hat{x} + \Phi(\hat{x})) + \tfrac{Fy}{\|Fy\|}\Big(\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) \\ & + \|SE\|\,\|y\|\gamma(\hat{x},t) + k(y,t)\Big)\Big\}, \end{aligned} \tag{4.25}$$

where $\hat{x}$ is given by (4.15), and the control gain $k(y,t)$ is to be developed to satisfy the reachability condition

$$\sigma^\tau(x)\dot{\sigma}(x) < -\beta\|\sigma(x)\| \tag{4.26}$$

with $\beta$ a positive constant.

**Theorem 4.3** *Suppose that (4.18) is satisfied. Under Assumptions 4.1–4.5, system (4.1)–(4.2), driven by the control law (4.25), converges to the sliding surface (4.7) and remains on it if the control gain $k(y,t)$ is chosen such that*

$$\begin{aligned} k(y,t) > \alpha_1 \exp\{-\alpha_2 t\} \big(&\|SA\| + \|S\|\mathscr{L}_\Phi + \xi_1(y,t)\mathscr{L}_{\xi_2}\|SB\| \\ &+\mathscr{L}_\gamma\|y\|\,\|SE\|\big) + \beta, \end{aligned} \tag{4.27}$$

*where $\beta$ is chosen as a positive constant, $\alpha_1$ and $\alpha_2$ are given as in Theorem 4.2, and $\xi_1$, $\xi_2$ and $\gamma$ are defined by Assumption 4.4.*

*Proof* From (4.1)–(4.2) and (4.7), it is observed that the sliding dynamics can be written as

$$\dot{\sigma}(x) = S\Big(Ax + B(u + \Delta g(x,t)) + \Delta f(x,t) + \Phi(x)\Big). \tag{4.28}$$

Then, from (4.25) and (4.28), it follows that

$$\begin{aligned} \sigma^\tau(x)\dot{\sigma}(x) = {} & -(Sx)^\tau \frac{Fy}{\|Fy\|}k(y,t) + \big((Sx)^\tau SB\Delta g(x,t) \\ & -(Sx)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t)\big) \\ & +\big((Sx)^\tau S\Delta f(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SE\|\,\|y\|\gamma(\hat{x},t)\big) \\ & +\big((Sx)^\tau SAx - (Sx)^\tau SA\hat{x}\big) \\ & +\big((Sx)^\tau S\Phi(x) - (Sx)^\tau S\Phi(\hat{x})\big). \end{aligned} \tag{4.29}$$

From Assumption 4.4, $S = FC$ and Theorem 4.2, it follows that

$$\begin{aligned} & (Sx)^\tau SB\Delta g(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) \\ & \le \|Fy\|\,\|SB\|\xi_1(y,t)\xi_2(x,t) - (Fy)^\tau \frac{Fy}{\|Fy\|}\|SB\|\xi_1(y,t)\xi_2(\hat{x},t) \\ & = \|Fy\|\,\|SB\|\xi_1(y,t)\big(\xi_2(x,t) - \xi_2(\hat{x},t)\big) \\ & \le \alpha_1\|Fy\|\,\|SB\|\xi_1(y,t)\mathscr{L}_{\xi_2}\exp\{-\alpha_2 t\} \end{aligned} \tag{4.30}$$

and

$$
\begin{aligned}
&(Sx)^\tau S\Delta f(x,t) - (Sx)^\tau \frac{Fy}{\|Fy\|}\|SE\|\,\|y\|\gamma(\hat{x},t) \\
&= (Fy)^\tau SE\Delta\eta(x,t) - (Fy)^\tau \frac{Fy}{\|Fy\|}\|SE\|\,\|y\|\gamma(\hat{x},t) \\
&\le \|Fy\|\,\|SE\|\,\|y\|\gamma(x,t) - \|Fy\|\,\|SE\|\,\|y\|\gamma(\hat{x},t) \\
&\le \alpha_1\|Fy\|\,\|SE\|\,\|y\|\mathscr{L}_\gamma \exp\{-\alpha_2 t\}.
\end{aligned} \tag{4.31}
$$

Further,

$$
(Sx)^\tau SAx - (Sx)^\tau SA\hat{x} = (Fy)^\tau SA(x-\hat{x}) \le \alpha_1\|Fy\|\,\|SA\|\exp\{-\alpha_2 t\} \tag{4.32}
$$

and

$$
(Sx)^\tau S\Phi(x) - (Sx)^\tau S\Phi(\hat{x}) \le \alpha_1\|Fy\|\,\|S\|\mathscr{L}_\Phi \exp\{-\alpha_2 t\}. \tag{4.33}
$$

Substituting (4.30)–(4.33) into (4.29), it yields

$$
\begin{aligned}
\sigma^\tau(x)\dot{\sigma}(x) \le -\|Fy\|\Big\{k(y,t) - \alpha_1\exp\{-\alpha_2 t\}\big(\|SA\| + \|S\|\mathscr{L}_\Phi \\
+\xi_1(y,t)\mathscr{L}_{\xi_2}\|SB\| + \|y\|\mathscr{L}_\gamma\|SE\|\big)\Big\}.
\end{aligned} \tag{4.34}
$$

Then by (4.27) it follows that $\sigma^\tau(x)\dot{\sigma}(x) < -\beta\|\sigma(x)\|$ if $\sigma(x) \neq 0$. Hence the result follows. #

*Remark 4.10* It should be noted that the bounds on the matched and mismatched uncertainties are both used in the control analysis and design. The uncertain nonlinearity $\Delta f(x,t)$ and the known nonlinearity $\Phi(x)$ are dealt with separately throughout the section. Therefore, conservatism is reduced as seen from the proof of the theorems above. The class of system considered includes those discussed in previous work [95, 157, 219] as special cases.

## 4.3 Control of Nonminimum Phase Systems with Nonlinear Disturbances

This section considers the stabilisation of a class of nonlinear systems, using sliding mode techniques, where the nominal systems are allowed to be nonminimum phase.

### *4.3.1 Introduction*

In many cases, the disturbance suffered by practical systems does not act in the input channel. Unlike the matched case, any mismatched disturbance impinges on the sliding mode dynamics and affects the behaviour of the sliding mode directly. Based on the work in [219], some dynamical output feedback control schemes are proposed in [96, 157], where it is required that the bound on the mismatched uncertainty is a linear function of $\|x\|$. An approach given by [25] also requires that the mismatched uncertainty is linear due to the limitation of the LMI technique used. However, disturbances experienced by practical systems may be structurally unknown or have nonlinear bounds [155]. In view of this, output feedback sliding mode control is considered in [200, 205], where more general mismatched uncertainty is considered. Unfortunately, in [37, 96, 157, 200, 205, 219] it is required that the system under consideration is minimum phase and relative degree one. The presence of right half-plane transmission zeros also limits the application of many existing results [25, 37, 38, 42, 96, 157, 200, 205, 219].

Results on output feedback stabilisation of nonminimum phase systems have appeared in the literature (see, e.g., [31, 32, 80, 169]). A control scheme is proposed for nonminimum phase systems without disturbances in [31]. The method is extended to systems with uncertainties possessing linear bounds using LMI techniques in [32]. Using geometric approaches and Lyapunov techniques, output feedback stabilisation results have been obtained for SISO systems in [80, 169] but the nominal system needs to have a special structure. Uncertainty is not dealt with in [169] while only parametric uncertainty is considered in [80]. Recently, a robust stabilisation scheme for nonminimum phase nonlinear systems has been proposed based on high-gain observers in [128], where the considered system is required to have a relative degree which guarantees that the considered system can be partially linearisable.

Sliding mode control can deal with uncertainty with unknown structure. During the sliding mode, a reduction in system order occurs, and this makes it possible to reduce the conservatism in the stability analysis of the reduced order sliding motion when compared with a direct Lyapunov approach [214, 215]. This has motivated some authors to apply sliding mode techniques to nonminimum phase systems and some interesting results have been obtained; see, e.g., [152, 160] and the references therein. However, most results focus on the tracking problem for specific signals. Thus sliding mode stabilisation of nonminimum phase systems based only on output information is essentially an open problem.

In this section, sliding mode techniques are employed to study a robust output feedback stabilisation problem for a class of systems in the presence of a nonlinear disturbance. The approach allows the nominal system to be nonminimum phase. The disturbance considered is mismatched and has a nonlinear bound. The strong limitation that the nominal system is relative degree one, employed in Sect. 4.2, is eliminated. The effect of the disturbance is rejected using the disturbance bound in the controller design.

### 4.3.2 System Description and Preliminaries

Consider the system

$$\dot{x} = Ax + Bu + f(x, t) \tag{4.35}$$

$$y = Cx, \tag{4.36}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state variables, input and output, respectively; $A$, $B$, $C$ are constant matrices with appropriate dimensions, and $B$ is assumed to be full rank. The function $f(x,t)$ is a mismatched nonlinear disturbance which is continuous in its arguments. Without loss of generality it is assumed that $C$ is full row rank. The problem which is considered in this section is one of robust stabilisation using only the measured output.

The following further assumptions are imposed on the system (4.35)–(4.36).

**Assumption 4.6** The nonlinear disturbance $f(x,t)$ has a structural decomposition:

$$f(x,t) = E\varDelta\xi(x,t),$$

where $E$ is a known constant matrix, and

$$\|\varDelta\xi(x,t)\| \le \zeta(x,t) \le \eta(x,t)\|x\|,$$

where $\zeta(x,t)$ is Lipschitz with respect to $x$ in the domain $\varOmega \subset \mathbb{R}^n$ (including the origin) and uniformly about $t \in \mathbb{R}^+$, and $\eta(x,t)$ is continuous in $\varOmega \times \mathbb{R}^+$.

*Remark 4.11* In Assumption 4.6, the matrix $E$ is employed to describe the structural characteristics of the nonlinear disturbance $f(x,t)$. The term $\varDelta\xi(x,t)$ represents unstructured uncertainty with known bounds and this will be used in the control design later. It is not necessary to assume $E = B$: this implies that the nonlinear disturbance in this section is allowed to be mismatched.

**Assumption 4.7** The matrix pair $(A, B)$ is controllable and $(A, C)$ is observable.

From Assumption 4.7, there exists a matrix $L$ such that $A - LC$ has $n$ eigenvalues which lie in the open left-half plane. Then, for any $Q_1 > 0$, the Lyapunov equation

$$(A - LC)^T P_1 + P_1(A - LC) = -Q_1 \tag{4.37}$$

has a unique solution $P_1 > 0$.

**Assumption 4.8** There exist (known) matrices $F$ and $P_1$ such that $E^T P_1 = FC$ holds, where $E$ is given by Assumption 4.6, and $P_1$ satisfies (4.37).

*Remark 4.12* It should be noted that a similar condition to Assumption 4.8 has also been imposed in [29, 215]. This constraint can be viewed from a system theoretic point of view as a requirement for the map from the uncertainty $\varDelta\xi$ to the linear combination of the output $Fy$ to be passive. Thus the constraint $E^T P_1 = FC$ is a structural characteristic associated with the system $(A, B, E)$ and is independent of the choice of coordinate system. The associated condition for nonlinear case has been used in Chap. 3 (see Assumption 3.3).

### 4.3.3 Dynamical Compensator Design

Based on the assumptions above, the following dynamical system is constructed for the system (4.35)–(4.36):

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) + \Phi(\hat{x}, y, t), \tag{4.38}$$

where $\hat{x} \in \mathbb{R}^n$, $L \in \mathbb{R}^{n\times p}$ satisfying (4.37) is the design gain, and

$$\Phi(\hat{x}, y, t) = \begin{cases} E\frac{F(y-C\hat{x})}{\|F(y-C\hat{x})\|}\zeta(\hat{x}, t), & F(y - C\hat{x}) \neq 0 \\ 0, & F(y - C\hat{x}) = 0 \end{cases}, \tag{4.39}$$

where the matrices $E$ and $F$ satisfy Assumption 4.8, and the function $\zeta$ is given in Assumption 4.6.

**Theorem 4.4** *Suppose that Assumptions 4.6–4.8 are satisfied. Then, for the system (4.35)–(4.36) and (4.38), in $\Omega \times \mathbb{R}^+$, the following are true:*

*(i)* $(x - \hat{x})^T P_1\Big(f(x, t) - \Phi(\cdot)\Big) \leq \mathscr{L}_\zeta \|FC\| \, \|x - \hat{x}\|^2$;

*(ii) there exist positive constants $\alpha_1$ and $\alpha_2$ such that*

$$\|x - \hat{x}\| \leq \alpha_1 \exp\{-\alpha_2 t\}$$

*if the symmetric positive definite matrix $Q_1$ satisfies*

$$\lambda_{\min}(Q_1) > 2\mathscr{L}_\zeta \|FC\|,$$

*where $P_1$ and $Q_1$ satisfy (4.37).*

*Proof* For convenience, let $e = x - \hat{x}$. It follows from (4.35) and (4.38) that

$$\dot{e} = (A - LC)e + f(x, t) - \Phi(\hat{x}, y, t). \tag{4.40}$$

If $FCe \neq 0$, then using $E^\tau P_1 = FC$:

$$\begin{aligned}(x - \hat{x})^T P_1\left(f(x, t) - \Phi(\hat{x}, y, t)\right) &= (FCe)^T \Delta\xi(x, t) - \frac{(FCe)^T FCe}{\|FCe\|}\zeta(\hat{x}, t) \\ &\leq \|FCe\|\zeta(x, t) - \|FCe\|\zeta(\hat{x}, t) \\ &\leq \mathscr{L}_\zeta \|FC\| \, \|x - \hat{x}\|^2.\end{aligned}$$

otherwise if $FCe = 0$ then

$$(x - \hat{x})^T P_1 \left( f(x,t) - \Phi(\hat{x}, y, t) \right) = 0 \leq \mathscr{L}_\zeta \|FC\| \, \|x - \hat{x}\|^2$$

and hence conclusion (i) follows.

For the system in (4.40), consider a Lyapunov function candidate

$$V_1 = e^T P_1 e.$$

Then, the time derivative of $V_1$ along the trajectories of System (4.40) is given by

$$\dot{V}_1 = -e^T Q_1 e + 2e^T P_1 \left( f(x,t) - \Phi(\hat{x}, y, t) \right),$$

where (4.37) is used to obtain the above. From conclusion (i), it follows that

$$\begin{aligned} \dot{V}_1 &\leq -(\lambda_{\min}(Q_1) - 2\mathscr{L}_\zeta \|FC\|)\|e\|^2 \\ &\leq -\frac{\lambda_{\min}(\mathrm{Q}_1) - 2\mathscr{L}_\zeta \|\mathrm{FC}\|}{\lambda_{\max}(P_1)} e^T P_1 e \\ &= -2\alpha_2 V_1, \end{aligned} \tag{4.41}$$

where

$$\alpha_2 =: \left( \lambda_{\min}(Q_1) - 2\mathscr{L}_\zeta \|FC\| \right) / \left( 2\lambda_{\max}(P_1) \right) > 0$$

if

$$\lambda_{\min}(Q_1) > 2\mathscr{L}_\zeta \|FC\|.$$

It follows that

$$V_1(t) \leq V_1(t_0) \exp\{2\alpha_2 t_0\} \exp\{-2\alpha_2 t\}. \tag{4.42}$$

Since $V_1 \geq \lambda_{\min}(P_1)\|e\|^2$, from (4.42),

$$\|e\| \leq \sqrt{\frac{V_1(t)}{\lambda_{\min}(P_1)}} \leq \underbrace{\sqrt{\frac{V_1(t_0)}{\lambda_{\min}(P_1)}} \exp\{\alpha_2 t_0\}}_{\alpha_1} \exp\{-\alpha_2 t\}. \tag{4.43}$$

Hence, conclusion (ii) follows. #

*Remark 4.13* Conclusion (ii) of Theorem 4.4 shows that under certain conditions the dynamical compensator (4.38) is an exponential observer of the system (4.35)–(4.36). The proof above also shows how to determine the values of $\alpha_1$ and $\alpha_2$.

*Remark 4.14* In this section, discontinuous terms are introduced into both the observer and the control law. The 'solutions' of the resulting systems are understood to mean solutions in the sense of Filippov [46].

Generally speaking, sliding mode control design is composed of two steps as stated in Sect. 1.3.1: The first step is the establishment of the switching surface such that the associated reduced order system has the desired performance. The second is the development of a sliding mode controller which drives the system to the sliding surface and maintains a stable sliding motion thereafter. The subsequent study will follow this procedure.

### *4.3.4 Stability of the Reduced Order Sliding Motion*

In this section, the objective is first to design a sliding surface based on the system output $y$ and the estimated state $\hat{x}$ given by (4.38). Then, the stability of the reduced order sliding motion will be analysed.

Let $N \in \mathbb{R}^{(n-p)\times n}$ be any matrix such that $\begin{bmatrix} C^T & N^T \end{bmatrix}$ is nonsingular. Now, for System (4.35)–(4.36) with the compensator (4.38), consider the switching function described by

$$\sigma(y, \hat{x}) = S_1 y + S_2 N \hat{x}, \tag{4.44}$$

where the matrices $S_1 \in \mathbb{R}^{m\times p}$ and $S_2 \in \mathbb{R}^{m\times(n-p)}$ are both design parameters.

Let $S \in \mathbb{R}^{m\times n}$ be any matrix such that $\det(SB) \neq 0$ and the $n-m$ nonzero eigenvalues of $(I - B(SB)^{-1}S)A$ are in the open left-half plane. Such a matrix $S$ can be designed using any existing state feedback sliding mode design methodology [38]. Let

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix} = S \begin{bmatrix} C \\ N \end{bmatrix}^{-1}. \tag{4.45}$$

Such a definition is well-defined by choice of $N$. Then, by construction $S_1$ and $S_2$ are such that

- $(S_1 C + S_2 N)B = SB$ is nonsingular;
- $A_{eq} =: A - B((S_1 C + S_2 N)B)^{-1}(S_1 C + S_2 N)A = (I - B(SB)^{-1}S)A$ has $n-m$ eigenvalues which lie in the open left-half plane.

*Remark 4.15* It should be emphasised that these constructions do not require that $(A, B, C)$ is minimum phase or that $\text{rank}(CB) = m$. These requirements underpin most of the output feedback results for uncertain systems developed in [38] and so this greatly increases the class of systems for which the robust output feedback sliding mode approach is applicable.

It follows that in the $(x, e)$ coordinate system with $e = x - \hat{x}$, the system (4.35)–(4.36) and (4.38) can be described by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} f \\ f - \Phi \end{bmatrix} \tag{4.46}$$

$$y = Cx \tag{4.47}$$

and the sliding surface can be written as

$$\begin{bmatrix} S_1C + S_2N & -S_2N \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = 0. \tag{4.48}$$

From [38, 173], it is observed that the equivalent control for the system (4.46) is given by

$$\begin{aligned} u_{eq} = &-(SB)^{-1}\Big((S_1C + S_2N)Ax - S_2N(A - LC)e + S_1Cf(x,t) \\ &+ S_2N\Phi(x - e, y, t)\Big). \end{aligned} \tag{4.49}$$

When the system (4.46)–(4.47) is restricted to the sliding surface (4.48), the sliding mode dynamics are described by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = &\begin{bmatrix} A_{eq} & B(SB)^{-1}S_2N(A - LC) \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \\ &\begin{bmatrix} \Big(I_n - B(SB)^{-1}S_1C\Big) f(x,t) - B(SB)^{-1}S_2N\Phi(x - e, y, t) \\ f(x,t) - \Phi(x - e, y, t) \end{bmatrix} \end{aligned} \tag{4.50}$$

which is the so-called equivalent system.

*Remark 4.16* From the structure of the matrix,

$$\begin{bmatrix} A_{eq} & B(SB)^{-1}S_2N(A - LC) \\ 0 & A - LC \end{bmatrix}$$

it follows that $A_{eq}$ has $2n - m$ eigenvalues which lie in the open left-half plane since $A - LC$ has $n$ negative eigenvalues. If $E \in \text{span}\{B\}$, then $\Phi \in \text{span}\{B\}$ and in this case the corresponding sliding mode is asymptotically stable.

In this section, the focus is the mismatched case, where $E \notin \text{span}\{B\}$. It is observed from sliding mode theory that mismatched uncertainty can enter the sliding mode and may affect/destroy its performance directly. Therefore, some restrictions on the mismatched disturbance $f(x,t)$ are necessary to guarantee the stability of the reduced order sliding motion associated with (4.50).

The matrix $S_1C + S_2N$ is full row rank, and thus there exist nonsingular matrices $T_1 \in \mathbb{R}^{n \times n}$ and $T_2 \in \mathbb{R}^{m \times m}$ such that

$$T_2(S_1C + S_2N)T_1 = \begin{bmatrix} I_m & 0 \end{bmatrix}. \tag{4.51}$$

Introduce a nonsingular transformation $z = T_1^{-1}x$. It follows that in the new coordinate system $(z, e)$, the equivalent system (4.50) becomes

$$\begin{bmatrix} \dot{z} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} T_1^{-1} A_{eq} T_1 & T_1^{-1} B(SB)^{-1} S_2 N(A - LC) \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} z \\ e \end{bmatrix} + \begin{bmatrix} T_1^{-1} \left( I_n - B(SB)^{-1} S_1 C \right) f - T_1^{-1} B(SB)^{-1} S_2 N \Phi \\ f - \Phi \end{bmatrix}. \quad (4.52)$$

From (4.51), it follows that

$$(S_1 C + S_2 N)x - S_2 N e = T_2^{-1} z_1 - S_2 N e,$$

where $z = \mathrm{col}(z_1, z_2)$ with $z_1 \in \mathbb{R}^m$. The sliding surface (4.48) in the new coordinates becomes

$$z_1 = T_2 S_2 N e. \quad (4.53)$$

Partition

$$T_1^{-1} A_{eq} T_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } T_1^{-1} B(SB)^{-1} S_2 N(A - LC) = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \quad (4.54)$$

where $A_{11} \in \mathbb{R}^{m \times m}$, $D_1 \in \mathbb{R}^{m \times n}$. Then, it follows from (4.53) and (4.52) that the corresponding sliding mode dynamics are described by

$$\begin{bmatrix} \dot{z}_2 \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{22} & A_{21} T_2 S_2 N + D_2 \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} z_2 \\ e \end{bmatrix} + \begin{bmatrix} f_2 \\ \Psi \end{bmatrix}, \quad (4.55)$$

where $z_2 \in \mathbb{R}^{n-m}$, $f_2(z_2, e, t)$ is the last $n - m$ components of

$$\left[ T_1^{-1} \left( I_n - B(SB)^{-1} S_1 C \right) f - T_1^{-1} B(SB)^{-1} S_2 N \Phi \right]_{z_1 = T_2 S_2 N e} \quad (4.56)$$

and

$$\Psi(z_2, e, y, t) =: [f(T_1 z, t) - \Phi(T_1 z - e, y, t)]_{z_1 = T_2 S_2 N e} \quad (4.57)$$

with $\Phi$ defined by (4.39).

It is observed from Remark 4.16 and the relationship between (4.52) and (4.55) that $A_{22}$ is stable. This implies that for any $Q_2 > 0$, the Lyapunov equation

$$A_{22}^T P_2 + P_2 A_{22} = -Q_2 \quad (4.58)$$

has a unique solution $P_2 > 0$. From Assumption 4.6 and (4.39), it follows that

$$\begin{aligned} & \left\| T_1^{-1} \left( I_n - B(SB)^{-1} S_1 C \right) f(\cdot) - T_1^{-1} B(SB)^{-1} S_2 N \Phi(\cdot) \right\| \\ & \leq \| T_1^{-1} \left( I_n - B(SB)^{-1} S_1 C \right) \| \, \|E\| \eta \|T_1 z\| \\ & \quad + \| T_1^{-1} B(SB)^{-1} S_2 N \| \, \|E\| \eta (\|T_1 z\| + \|e\|). \end{aligned} \quad (4.59)$$

Then, from the definition of $f_2$, (4.56), (4.59) and the inequality

$$\left\| [T_1 z]_{z_1 = T_2 S_2 N e} \right\| = \left\| T_1 \begin{bmatrix} T_2 S_2 N e \\ z_2 \end{bmatrix} \right\| \leq \|T_1\| \left( \|T_2 S_2 N\| \, \|e\| + \|z_2\| \right)$$

it is straightforward to see that there exist functions $\chi_1$ and $\chi_2$ (dependent on $\eta$, $T_1$, $T_2$, $S_1$ and $S_2$) such that

$$\|f_2(z_2, e, t)\| \leq \chi_1(z_2, e, t) \, \|z_2\| + \chi_2(z_2, e, t) \|e\|. \tag{4.60}$$

Then, the following conclusion is ready to be presented.

**Theorem 4.5** *Suppose Assumptions 4.6–4.8 are satisfied. Then, the reduced order sliding motion dynamics (4.55) are asymptotically stable if there exists a domain*

$$\Omega' = \left\{ (z_2, e, t) \mid \|z_2\| \leq d, \, \|e\| \leq d, t \in \mathbb{R}^+ \right\}$$

*for some positive constant $d$ such that in $\Omega' \setminus \{0\}$, the matrix*

$$M = \begin{bmatrix} \lambda_{\min}(Q_2) - 2\lambda_{\max}(P_2)\chi_1 & -\Pi \\ -\Pi & \lambda_{\min}(Q_1) - 2\|FC\|\mathscr{L}_\zeta \end{bmatrix},$$

*is positive definite, where*

$$\Pi := \|P_2(A_{21} T_2 S_2 N + D_2)\| + \lambda_{\max}(P_2)\chi_2$$

*and*

$$\chi_1 = \chi_1(z_2, e, t) \quad and \quad \chi_2 = \chi_2(z_2, e, t)$$

*satisfy (4.60).*

*Proof* It is only required to prove that the system (4.55) is asymptotically stable. For system (4.55), consider the Lyapunov function candidate

$$V(z_2, e) = e^T P_1 e + z_2^T P_2 z_2.$$

Then, the time derivative of $V$ along the trajectories of the dynamic System (4.55) is given as

$$\dot{V} = -e^T Q_1 e - z_2^T Q_2 z_2 2 z_2^T P_2 (A_{21} T_2 S_2 N + D_2) e + 2 z_2^T P_2 f_2(\cdot) + 2 e^T P_1 \Psi(\cdot). \tag{4.61}$$

From conclusion (i) of Theorem 4.4 and the definition (4.57) of $\Psi$, it follows that

$$2e^T P_1 \Psi(z_2, e, y, t) \leq \mathscr{L}_\zeta \|FC\| \, \|e\|^2. \tag{4.62}$$

Then, substituting (4.62) into (4.61), it follows from (4.60) that

$$\begin{aligned}\dot{V} &= -\Big(\lambda_{\min}(Q_1) - 2\|FC\|\mathscr{L}_\zeta\Big)\|e\|^2 - \Big(\lambda_{\min}(Q_2) - 2\lambda_{\max}(P_2)\chi_1\Big)\|z_2\|^2 \\ &\quad +2\Big(\|P_2(A_{21}T_2S_2N + D_2)\| + \lambda_{\max}(P_2)\chi_2\Big)\|z_2\|\,\|e\| \\ &= -\begin{bmatrix}\|z_2\| & \|e\|\end{bmatrix} M \begin{bmatrix}\|z_2\| \\ \|e\|\end{bmatrix}.\end{aligned}$$

Hence, the conclusion follows from $M > 0$. #

*Remark 4.17* It should be pointed out that $M > 0$ implies that

$$\lambda_{\min}(Q_1) > 2\|FC\|\mathscr{L}_\zeta$$

which, together with Assumptions 4.6–4.8, guarantees the convergence of the compensator (4.38).

*Remark 4.18* Note that the sliding motion only depends on the partial state variable $\mathrm{col}(z_2, e)$ instead of $\mathrm{col}(z, e)$, and thus $M$ is independent on the variables $z_1$.

### 4.3.5 Sliding Mode Control Design

The objective now is to find a control law such that the reachability condition

$$\sigma^T(y, \hat{x})\dot{\sigma}(y, \hat{x}) \leq -\beta\|\sigma(y, \hat{x})\| \tag{4.63}$$

is satisfied for some positive constant $\beta$, where $\sigma(\cdot)$ is the sliding function given in (4.44). If condition (4.63) is satisfied by some control, then the system (4.35)–(4.36) is driven to the sliding surface by the control and maintained in a sliding mode.

Based on the estimated state $\hat{x}$ given by (4.38) and the system output $y$, the following sliding mode control is proposed:

$$u = -(SB)^{-1}\left\{(S_1C + S_2N)A\hat{x} + S_2NL(y - C\hat{x}) + \frac{\sigma}{\|\sigma\|}K(\hat{x}, y, t)\right\}, \tag{4.64}$$

where $K(\hat{x}, y, t)$ is a control gain and is to be determined.

**Theorem 4.6** *Suppose Assumptions 4.6–4.8 are satisfied together with the assumptions of Theorem 4.4. Then, the control in (4.64) drives the system (4.35)–(4.36) to the sliding surface and maintains a sliding motion if*

$$\begin{aligned}K(\hat{x}, y, t) \geq \Big(\|S_1CE\| + \|S_2NE\|\Big)\zeta(\hat{x}, t) + \alpha_1\Big(\mathscr{L}_\zeta\|S_1CE\| \\ +\|S_1CA\|\Big)e^{-\alpha_2 t} + \beta,\end{aligned} \tag{4.65}$$

*where $\zeta$ is given in Assumption 4.6, $\alpha_1$ and $\alpha_2$ are given in Theorem 4.4, and $\beta$ is a positive constant.*

*Proof* From the analysis above, it is necessary to prove that the reachability condition (4.63) is satisfied when applying the control given in (4.64).

It follows from (4.44), (4.35) and (4.38) that

$$\begin{aligned}\dot{\sigma}(y,\hat{x}) = (S_1C + S_2N)A\hat{x} + S_2NLCe + (SB)u + S_1Cf(x,t)\\ +S_2N\Phi(\cdot) + S_1CAe.\end{aligned} \tag{4.66}$$

By applying the control (4.64) into (4.66), it follows that

$$\dot{\sigma} = -\frac{\sigma}{\|\sigma\|}K(\hat{x},y,t) + S_1Cf(x,t) + S_2N\Phi(\cdot) + S_1CA(x-\hat{x}).$$

Therefore,

$$\sigma^T\dot{\sigma} \le -\|\sigma\|\left\{K(\hat{x},y,t) - S_1Cf(\cdot) - S_2N\Phi\ - S_1CAe\right\}. \tag{4.67}$$

From Assumption 4.6:

$$\begin{aligned}S_1Cf(\cdot) &\le \|S_1CE\|\Big(\zeta(x,t) - \zeta(\hat{x},t)\Big) + \|S_1CE\|\zeta(\hat{x},t)\\ &\le \mathscr{L}_\zeta\|S_1CE\|\|e\| + \|S_1CE\|\zeta(\hat{x},t).\end{aligned} \tag{4.68}$$

From the definition of $\Phi$ in (4.39) and Assumption 4.6

$$S_2N\Phi(\hat{x},y,t) \le \|S_2NE\|\zeta(\hat{x},t). \tag{4.69}$$

Substituting (4.68) and (4.69) into (4.67), and using conclusion (ii) of Theorem 4.4:

$$\begin{aligned}\sigma^T\dot{\sigma} &\le -\|\sigma\|\Big\{K(\hat{x},y,t) - \Big(\|S_1CE\| + \|S_2NE\|\Big)\zeta(\hat{x},t) -\\ &\quad \Big(\mathscr{L}_\zeta\|S_1CE\| + \|S_1CA\|\Big)\|e\|\Big\}\\ &= -\|\sigma\|\Big\{K(\hat{x},y,t) - \Big(\|S_1CE\| + \|S_2NE\|\Big)\zeta(\hat{x},t)\\ &\quad -\alpha_1\Big(\mathscr{L}_\zeta\|S_1CE\| + \|S_1CA\|\Big)\exp\{-\alpha_2 t\}\Big\}.\end{aligned}$$

Then, by (4.65), it follows that the reaching condition is satisfied. #

The results in Theorems 4.5 and 4.6 together show that the corresponding closed-loop systems are asymptotically stable.

## 4.4 Case Study: Control of HIRM Aircraft

In this section, both the longitudinal and lateral dynamics of the High Incidence Research Model (HIRM) aircraft are to be employed to test the results developed in Sects. 4.2 and 4.3 to demonstrate the approaches proposed.

### *4.4.1 Control of Longitudinal Dynamics*

Consider the simplified longitudinal dynamics of the HIRM aircraft at the trim values Mach: 0.8, Height: 5000 ft taken from [115]:

$$A = \begin{bmatrix} -0.0318 & 0.0831 & -0.0008 & -0.0367 \\ -0.0716 & -1.4850 & 0.9848 & 0 \\ -0.2797 & -5.6725 & -1.0253 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} \tag{4.70}$$

$$B = \begin{bmatrix} 0.0120 & -0.0071 \\ -0.3058 & -0.0223 \\ -22.4293 & 7.8777 \\ 0 & 0 \end{bmatrix} \tag{4.71}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.72}$$

This system has four states $\text{col}(x_1, x_2, x_3, x_4) = x : (v - v_0)/v_0$ with $v$ and $v_0 = 267.51$, respectively, the current airspeed (m/s) and the desired airspeed (m/s), angle of attack (rad), pitch rate (rad/s) and pitch angle (rad), two inputs $\text{col}(u_1, u_2) = u$: symmetrical tailplane deflection (rad) and symmetrical canard deflection (rad), and three outputs $\text{col}(y_1, y_2, y_3) = y$: $(v - v_0)/v_0$, angle of attack (rad) and pitch rate (rad/s). Suppose that the aircraft flies at constant altitude and the associated engine thrust is constant. Based on the model given in [76], the nonlinear term is described by

$$\Phi(x) = \begin{bmatrix} 0 & \frac{F_e}{M}(\sin x_2)/(1 + x_1) & 0 & 0 \end{bmatrix}^\tau,$$

where the parameters $F_e$ and $M$ are the engine thrust and the aircraft mass, respectively, and their values are chosen from [115]. Further suppose

$$\Delta f(x, t) = \begin{bmatrix} 0.010\ \Delta\eta(x, t) & -0.0512\Delta\eta(x, t) & 0 & 0 \end{bmatrix}^\tau,$$

where

$$\|\Delta\eta(x, t)\| \le 0.001\|y\| \sin^2 x_4$$

which is the uncertainty caused by the aerodynamic drag and the modelling error from the lift term.

From (4.70)–(4.72), it is straightforward to see that $\text{rank}(CB) = 2$ and so Assumption 4.1 is satisfied. Then, from the algorithm given by Edwards and Spurgeon [38], the coordinate transformation $\widetilde{x} = \widetilde{T}x$ with

$$\widetilde{T} = \begin{bmatrix} 0 & 0 & 0 & 1.0000 \\ 0.9998 & -0.0222 & 0.0008 & 0 \\ -0.0222 & -0.9997 & 0.0136 & 0 \\ -0.0005 & 0.0136 & 0.9999 & 0 \end{bmatrix}$$

yields the canonical form (4.3) as follows

$$\widetilde{A} = \left[\begin{array}{cc|cc} 0 & 0.0008 & 0.0136 & 0.9999 \\ -0.0366 & -0.0329 & -0.1109 & -0.0220 \\ \hline 0.0008 & 0.0364 & -1.4200 & -0.9792 \\ 0.0000 & -0.1550 & 5.6828 & -1.0891 \end{array}\right] \tag{4.73}$$

$$\widetilde{B} = \left[\begin{array}{cc} 0 & 0 \\ -0.0000 & -0.0000 \\ \hline 0.0000 & 0.1297 \\ -22.4313 & 7.8767 \end{array}\right] \tag{4.74}$$

$$\widetilde{C} = \left[\begin{array}{c|ccc} 0 & 0.9998 & -0.0222 & -0.0005 \\ 0 & -0.0222 & -0.9997 & 0.0136 \\ 0 & 0.0008 & 0.0136 & 0.9999 \end{array}\right]. \tag{4.75}$$

It is straightforward to check that Assumption 4.2 is satisfied with the choice $K^\tau = [-10 \quad -25]$. Let

$$\gamma(x,t) = 0.001 \sin^2 x_4$$
$$\phi(x_2) = \begin{cases} \frac{\sin x_2}{x_2} & x_2 \neq 0 \\ 1 & x_2 = 0 \end{cases}$$

$$\Upsilon = \begin{bmatrix} -0.0293 & -0.1327 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.0010 \\ -0.0512 \\ 0 \\ 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 2.1682 & 0.0940 & -0.0008 \\ -0.0714 & -1.2850 & 0.9848 \\ -0.2797 & -5.6725 & -0.5253 \\ -32.7346 & -0.3223 & 1.0000 \end{bmatrix}$$

$$H(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{F\phi(x_2)}{m(x_1+1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $\Phi(x) = H(x)x$, and the Lyapunov equation (4.4) has the solution

$$P = \begin{bmatrix} 203.1738 & 4.5427 & 0 & 13.6395 \\ 4.5427 & 2.6815 & 0 & 0.2666 \\ 0 & 0 & 1.0000 & 0 \\ 13.6395 & 0.2666 & 0 & 1.1442 \end{bmatrix}.$$

Then, consider the domain

$$\Omega_q = \left\{ (x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{ll} |x_1| < 0.1869, & x_2 > -0.1745 \\ |x_3| < 0.1745, & x_4 < 0.5236 \end{array} \right\}.$$

By direct verification, Assumptions 4.3–4.5 are all satisfied. Then using the algorithm given by Edwards and Spurgeon in [38],

$$F = \begin{bmatrix} -10.0197 & -0.7777 & 0.0053 \\ -24.9944 & 0.5687 & 0.9790 \end{bmatrix} \tag{4.76}$$

and

$$\widehat{T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.9998 & -0.0222 & 0.0008 & 0 \\ -10.0197 & -0.7777 & 0.0053 & 0 \\ -24.9944 & 0.5687 & 0.9790 & 0 \end{bmatrix}. \tag{4.77}$$

Therefore, the designed sliding surface $FCx = 0$ is well-defined, and the canonical form (2.22)–(2.23) can be obtained as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 25.1347 & 0.0136 & 0.9999 \\ -0.0366 & -1.6924 & -0.1109 & -0.0220 \\ \hline 0.3673 & -21.7200 & -0.3107 & -0.7592 \\ 0.9163 & 71.7542 & 8.4561 & -0.5390 \end{array}\right]$$

$$\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0.0000 & 0.1297 \\ -22.4313 & 7.8767 \end{array}\right]$$

$$\begin{bmatrix} 0 & C_2 \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & 0.7643 & -0.0222 & -0.0005 \\ 0 & -9.6780 & -0.9997 & 0.0136 \\ 0 & 25.1347 & 0.0136 & 0.9999 \end{array}\right].$$

Let $Q_1 = I_2$. Then, the solution of the Lyapunov equation (4.11) is

$$P_1 = \begin{bmatrix} 1.2145 & 13.6428 \\ 13.6428 & 202.9155 \end{bmatrix}.$$

It follows that $\chi(z_1, t) = 0.0017$ and

$$R(z_1, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0.000215\phi(x_2)/x_1 \end{bmatrix}.$$

Direct computation shows the conditions of Theorems 4.1 and 4.2 are both satisfied in domain $\Omega_q$. With the given $L$ and $E$ above, the compensator (4.15) is well-defined. Then according to (4.25), design the control

$$\begin{aligned} u = -(FCB)^{\tau} \Big\{ & FCA\hat{x} + \begin{bmatrix} -0.0006 \\ 0.0005 \end{bmatrix} (\sin \hat{x}_2)/(1 + \hat{x}_1) \\ & + \frac{Fy}{\|Fy\|}(0.0617\|y\| \sin^2 \hat{x}_4 + K(y, t)) \Big\}, \end{aligned} \tag{4.78}$$

where the matrices $A$, $B$, $C$ are defined in (4.70)–(4.72) and the matrix $F$ is given in (4.76). The scalar gain

$$K(y, t) = \alpha_1 \exp\{-\alpha_2 t\}(8.5842 + 1.2349 \times 10^{-4}\|y\|) + \beta.$$

In order to illustrate the effectiveness of the designed controller in (4.78), simulations on the corresponding closed-loop system are carried out to show its performance. For simulation purposes, the initial condition is chosen as

$$\text{col}(x_0, \hat{x}_0) = (0, 0, 0.1, 0, 0, 0, 0.1, 0)$$

and

$$\beta = 0.1, \qquad \alpha_1 = 0.2099, \qquad \alpha_2 = 50.7754.$$

The results in Figs. 4.1 and 4.2 show the effectiveness of the controller.

### *4.4.2 Control of Lateral Dynamics*

Consider the lateral dynamics of the HIRM aircraft at the trim values of Mach 0.2 and height 5000 ft taken from [94]. The system matrices are given by

$$A = \begin{bmatrix} -0.0080 & 0.4100 & -0.9047 & 0.1334 \\ -7.3235 & -0.4278 & 2.6462 & 0 \\ -0.1460 & -0.0247 & -0.1544 & 0 \\ 0 & 1.0000 & 0.4558 & 0 \end{bmatrix} \tag{4.79}$$

$$B = \begin{bmatrix} 0.0181 & -0.0094 \\ -3.1026 & 0.4024 \\ -0.4096 & -0.0833 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.80}$$

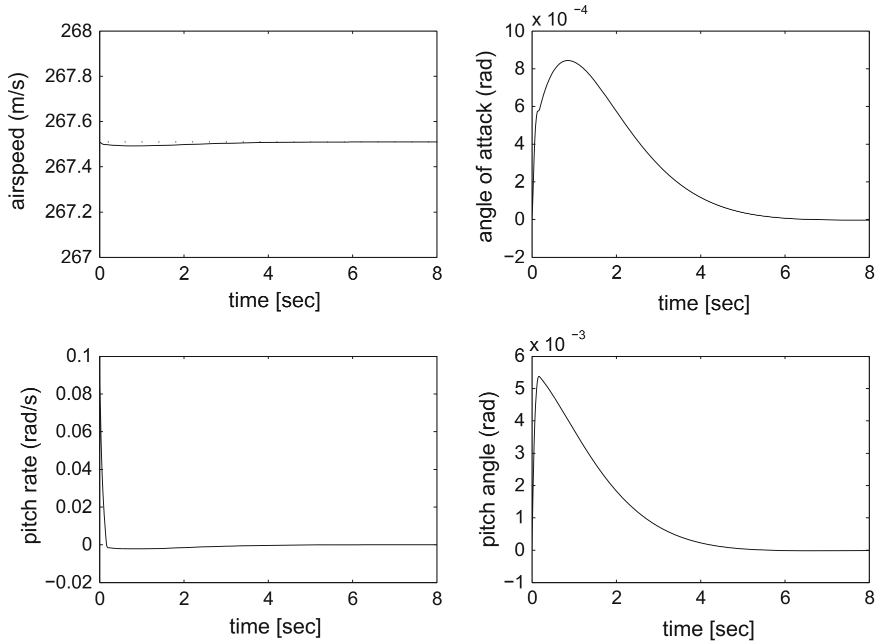


**Fig. 4.1** The time response of the simplified system (4.70)–(4.72) of HIRM

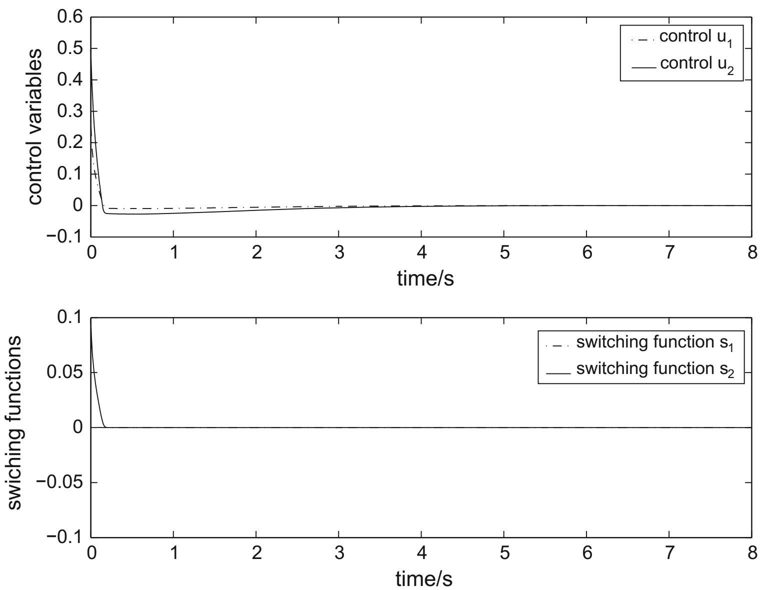


**Fig. 4.2** The response of control (*top*) and the sliding functions (*bottom*)

The system has four states $\mathrm{col}(x_1, x_2, x_3, x_4) = x$: sideslip angle (rad), roll rate (rad/s), yaw rate (rad/s) and bank angle (rad), two inputs $\mathrm{col}(u_1, u_2) = u$: differential tailplane deflection (rad) and differential canard deflection (rad), and two outputs $\mathrm{col}(y_1, y_2) = y$: sideslip angle (rad) and yaw rate (rad/s). This system has zeros at 32.3257 and −0.3254 and thus $(A, B, C)$ is nonminimum phase. Suppose that the system suffers from a perturbation

$$f(x,t) = \begin{bmatrix} 0 \\ -0.0341\Delta\xi_1(x,t) + 0.01851\Delta\xi_2(x,t) \\ -0.9217\Delta\xi_1(x,t) + 0.5\Delta\xi_2(x,t) \\ 0.05\Delta\xi_1(x,t) - 0.02713\Delta\xi_2(x,t) \end{bmatrix}, \tag{4.81}$$

where the unknown signal

$$\Delta\xi(x,t) =: \begin{bmatrix} \Delta\xi_1(x,t) \\ \Delta\xi_2(x,t) \end{bmatrix}$$

satisfies

$$\|\Delta\xi(x,t)\| \le \frac{1}{9}(\sin^2 x_4 + |x_1|).$$

Consider the system (4.79)–(4.80) in the presence of the perturbation (4.81). It is observed that $(A, C)$ is observable. Choose

$$L = \begin{bmatrix} 4.7197 & -0.9285 \\ -7.7057 & 2.6708 \\ -0.1659 & 0.3401 \\ 8.8887 & 0.3748 \end{bmatrix}.$$

Then, $A - LC$ is stable and for $Q_1 = I_4$, the solution of Lyapunov equation (4.37) is

$$P_1 = \begin{bmatrix} 7.0232 & -1.5951 & -0.1564 & -3.7481 \\ -1.5951 & 1.7522 & -0.0159 & 0.9032 \\ -0.1564 & -0.0159 & 1.0178 & 0.0825 \\ -3.7481 & 0.9032 & 0.0825 & 2.1379 \end{bmatrix}.$$

Let

$$E = \begin{bmatrix} 0 & 0 \\ -0.0341 & 0.01851 \\ -0.9217 & 0.5 \\ 0.05 & -0.02713 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0.01115 & -0.9335 \\ -0.00605 & 0.5064 \end{bmatrix} \tag{4.82}$$

with

$$\zeta(x,t) = \frac{1}{9}(\sin^2 x_4 + |x_1|), \quad \eta(x,t) = \frac{1}{9}(|\sin x_4| + 1), \quad \mathscr{L}_\zeta = \frac{1}{3}. \tag{4.83}$$

It is straightforward to check that Assumptions 4.6–4.8 and the inequality

$$\lambda_{\min}(Q_1) > 2\mathscr{L}_{\zeta}\|FC\|$$

are satisfied globally. The compensator from (4.38)–(4.39) is now completely specified. Let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.84}$$

from which it is obvious that $\begin{bmatrix} C^T & N^T \end{bmatrix}$ is invertible.

The ideal sliding mode dynamics is defined by the poles $\{-4, -1\}$ with the associated eigenvectors specified to have the structure

$$\begin{bmatrix} * & 1 & 0 & * \end{bmatrix}^{\tau} \quad \text{and} \quad \begin{bmatrix} * & 0 & 1 & * \end{bmatrix}^{\tau},$$

where $*$ denotes that the corresponding entry of the eigenvector is arbitrary. This choice of eigenstructure will ensure the rolling and yawing motions of the aircraft are decoupled. According to the algorithm given by [38], the corresponding switching surface is determined as

$$S = \begin{bmatrix} -0.6710 & -0.0640 & 0.7378 & 0.0347 \\ 0.2227 & 0.2550 & 0.1812 & 0.9233 \end{bmatrix}$$

which yields the following nonzero eigenvalues of $A_{eq}$, $\{-4, -1\}$ with corresponding eigenvectors:

$$\begin{bmatrix} -0.1083 \; 1 \; 0 \; -0.2500 \end{bmatrix}^{\tau} \quad \text{and} \quad \begin{bmatrix} 1.0760 \; 0 \; 1 \; -0.4558 \end{bmatrix}^{\tau},$$

respectively. It follows from (4.45) that

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -0.6710 & 0.7378 & -0.0640 & 0.0347 \\ 0.2227 & 0.1812 & 0.2550 & 0.9233 \end{array}\right]. \tag{4.85}$$

Choosing $T_2 = 0.1 I_2$ and

$$T_1 = \begin{bmatrix} 6.7105 & 2.2270 & 7.0224 & -0.8345 \\ 0.6401 & 2.5495 & -2.5268 & -9.3116 \\ -7.3783 & 1.8124 & 6.2723 & -1.7129 \\ -0.3469 & 9.2334 & -2.2272 & 3.1087 \end{bmatrix}$$

after some algebra, it can be shown that

$$A_{21} = \begin{bmatrix} 0.8600 & 0.2369 \\ -0.2598 & 1.2556 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -1.2070 & -0.7628 \\ -0.7579 & -3.7930 \end{bmatrix}$$

and

$$D_2 = \begin{bmatrix} 0.5960 & -0.0565 & -0.0049 & 0 \\ 2.8395 & -0.2830 & -0.0239 & 0 \end{bmatrix}.$$

Choose $Q_2 = I_2$, then the solution of the Lyapunov equation (4.58) is

$$P_2 = \begin{bmatrix} 0.4742 & -0.0949 \\ -0.0949 & 0.1508 \end{bmatrix}.$$

By direct computation, it follows that $\chi_1$ and $\chi_2$ can be chosen as $4.8833e-004$ and 0.0034, respectively, and

$$M = \begin{bmatrix} 0.9995 & -0.3754 \\ -0.3754 & 0.2919 \end{bmatrix}$$

is positive definite and thus the conditions of Theorem 4.5 are satisfied globally. Now, from (4.65), $K(\hat{x}, y, t)$ is chosen as

$$K(\hat{x}, y, t) = 0.0933\left(\sin^2 \hat{x}_4 + |\hat{x}_1|\right) + 0.9010\alpha_1 e^{-\alpha_2 t} + \beta. \tag{4.86}$$

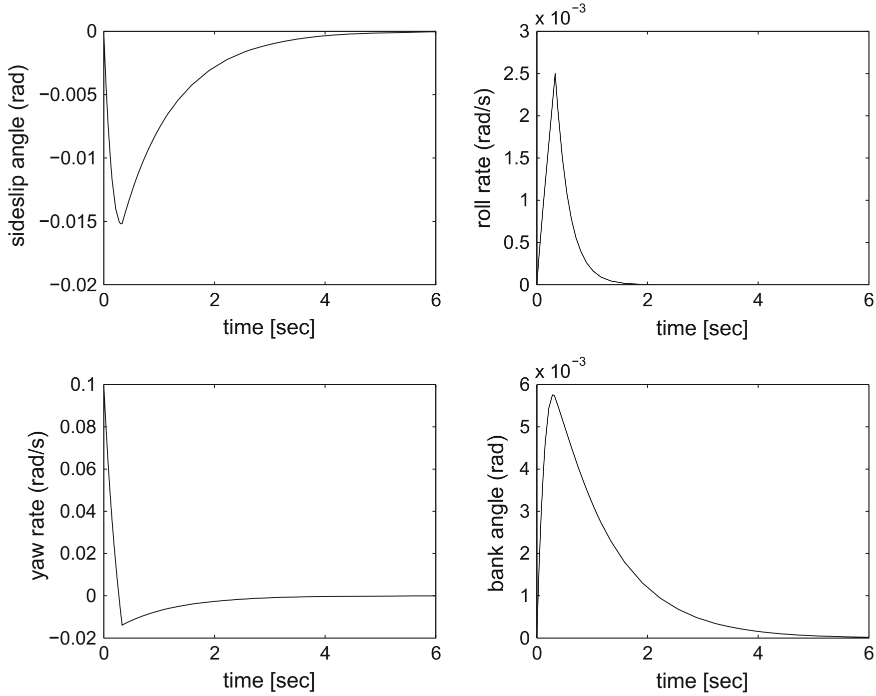


**Fig. 4.3** The time responses of the simplified system (4.79)–(4.81) of HIRM

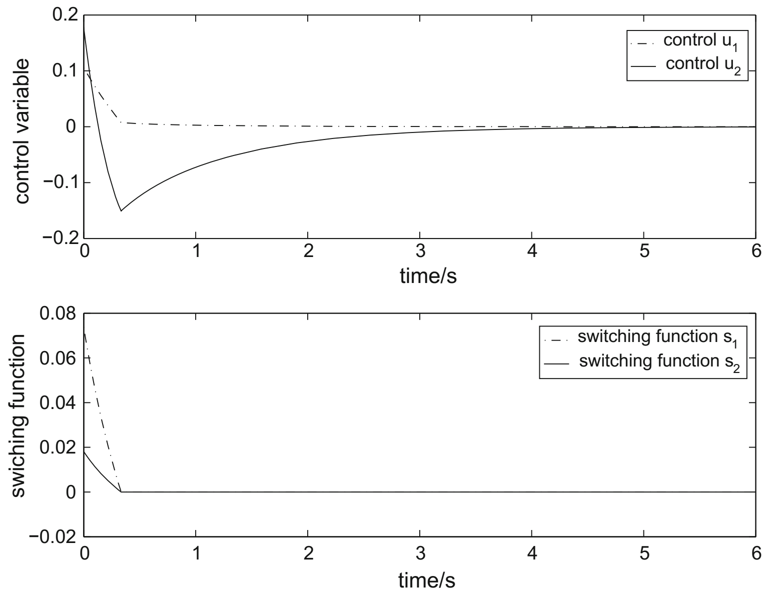


**Fig. 4.4** The evolution of the control signal (*top*) and the corresponding sliding functions (*bottom*)

This completes the controller design.

For simulation, the initial condition is chosen as

$$\mathrm{col}(x_0, \hat{x}_0) = (0, 0, 0.1, 0, 0, 0, 0.1, 0)$$

while

$$\beta = 0.1, \qquad \alpha_1 = 0.3101 \qquad \text{and} \qquad \alpha_2 = 1.3855.$$

The results in Figs. 4.3 and 4.4 show the effectiveness of the controller. A perturbation of the yaw subsystem has very little effect on the bank angle and roll rate responses.

## 4.5 Summary

This chapter has studied sliding mode control design for nonlinear and linear systems with nonlinear disturbances using dynamical output feedback.

In Sect. 4.2, the sliding surface employed is as originally proposed by Edwards and Spurgeon in [38]. A nonlinear asymptotic observer has been proposed which has been shown to give exponential state estimation error convergence based on the solution to a constrained Lyapunov equation. Using the estimated states and the system output, a dynamic variable structure control has been developed which has been shown to satisfy the reachability condition. In Sect. 4.3, a dynamical sliding mode control strategy based on the equivalent control approach has been developed which is applicable to a class of nonminimum phase systems with mismatched disturbances. Compared with the Lyapunov technique, the results obtained in this section have lower conservatism using the property that the sliding mode is a reduced order system. In comparison with the geometric approach, this method does not require the nominal system to have a special structure. Moreover, the approach proposed in Sect. 4.3 is constructive which makes it convenient to use for practical controller design.

The schemes developed in Sects. 4.2 and 4.3 have been used to control a simplified nonlinear HIRM aircraft model. A nonlinear model of the longitudinal dynamics has been considered under the assumption that the engine thrust is fixed. The uncertainty caused by the aerodynamic drag and the error present in the modelling of the aircraft lift has been considered. Control of the lateral dynamics of a HIRM aircraft model has also been considered at the trim values of Mach 0.2 and Height 5000 ft. Simulation results have been given which demonstrate the practicality of the proposed schemes and their effectiveness in achieving robust closed-loop performance.

# Chapter 5
# Reduced-Order Compensator-Based Feedback Control of Large-Scale Systems

This chapter will focus on control design for a class of large-scale interconnected systems using reduced-order compensators. Large-scale systems with nonminimum phase isolated subsystems and similar structure are considered in Sects. 5.2 and 5.3, respectively.

## 5.1 Introduction

It is well known that full-order compensator-based feedback control doubles the order of the dynamical systems. For large-scale systems, the implications of such growth in system order for both design and implementation are severe. However, dynamical output feedback can be employed to reduce the limitations on the system class when compared with static output feedback control. In connection with this, control of large-scale interconnected systems using reduced-order compensators is considered in this chapter.

Decentralised sliding mode controllers are synthesised for a class of nonlinear large-scale systems in Sect. 5.2. It is shown that minimum phaseness of the nominal isolated subsystems is not required if dynamical feedback is employed. Then, control design for nonlinear large-scale systems with similar structure is considered in Sect. 5.3. The study shows that exploiting similar structure can greatly simplify the controller design and system analysis. Section 5.4 provides examples to demonstrate the developed results by simulation.

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Communications and Control Engineering, DOI 10.1007/978-3-319-48962-9_5

## 5.2 Decentralised Sliding Mode Control for Nonminimum Phase Interconnected Systems

In this section, a class of interconnected systems with nonlinear interconnections and nonlinear disturbances is considered. A continuous nonlinear reduced-order compensator is established by exploiting the structure of the uncertainties and using sliding mode techniques. The proposed approach allows both the isolated nominal subsystem and the nominal interconnected system to be nonminimum phase, and the uncertainties to be mismatched with nonlinear bounds.

### *5.2.1 Introduction*

Sliding mode techniques are employed for the stabilisation of a class of nonlinear interconnected systems. Mismatched uncertainties and nonlinear interconnections are both considered, and the bounds on the uncertainties take more general forms as in [200, 215]. Using the structure of the uncertainties, a continuous reduced-order compensator is proposed based on the constrained Lyapunov equations. Then, a sliding surface is proposed in the augmented space formed by the compensator and system output. Using an equivalent control approach and local coordinate transformations, the sliding mode dynamics are established and the stability is analysed. A robust decentralised output feedback sliding mode control scheme is synthesised such that the interconnected system can be driven to the predesigned sliding surface. This approach allows both the nominal isolated subsystem and the whole nominal system to be nonminimum phase. It should be emphasised that methods to deal with nonlinear interconnections are a key issue in the control of interconnected systems. So far nearly all associated work treated such interconnections as a disturbance and then used an extra stability margin to reject the effect of the interconnections. By dealing with uncertain interconnections and known interconnections separately, the conservatism is reduced to some extent as claimed in [214, 215]. However, the interconnections are still treated as a disturbance since the interconnections are not used in the control design. It is shown that by employing sliding mode techniques, the interconnections are directly used in the control design, which together with the fact that the sliding mode dynamics are reduced-order systems, reduces the conservatism and enhances the robustness. A simulation for a HIRM aircraft system is used to show the effectiveness of the proposed control schemes in Sect. 5.4.

### 5.2.2 Interconnected System Description

Consider a nonlinear interconnected system composed of $N$ subsystems as follows:

$$\dot{x}_i = A_i x_i + B_i u_i + \Delta f_i(x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} \Big( H_{ij}(x_j) + \Delta H_{ij}(x_j, t) \Big), \tag{5.1}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N, \tag{5.2}$$

where $x_i \in \Omega_i \subset \mathbb{R}^{n_i}$ $(0 \in \Omega_i)$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the $i$-th subsystem, respectively, with $m_i < n_i$; The triple $(A_i, B_i, C_i)$ comprises constant matrices of appropriate dimensions with $B_i$ and $C_i$ of full rank; $\Delta f_i$ is the mismatched uncertainty of the $i$-th isolated subsystem, whilst the terms

$$\sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}(x_j) \quad \text{and} \quad \sum_{\substack{j=1\\ j\neq i}}^{N} \Delta H_{ij}(x_j, t)$$

are the known and the uncertain interconnections of the i-th subsystem with $H_{ij}(0) = 0$. The functions are all assumed to be continuous in their arguments.

*Remark 5.1* It should be pointed out that matched uncertainty does not affect the stability of sliding mode. As for the reachability, there are many standard techniques (see, e.g., [35, 36, 38, 69, 97, 200]) which can be applied to deal with matched uncertainty. In view of this, matched uncertainty is not considered in systems (5.1)–(5.2).

Without loss of generality, suppose that the nonlinear functions $H_{ij}(\cdot)$ can be decomposed as

$$H_{ij}(x_j) = \Phi_{ij}(x_j)x_j, \qquad i \neq j, \quad i, j = 1, 2 \ldots, N, \tag{5.3}$$

where $\Phi_{ij}(\cdot)$ are continuous. The decomposition (5.3) is always true for $H_{ij}(\cdot)$ satisfying $H_{ij}(0) = 0$ which are smooth enough in their domain of definition.

In order to facilitate the analysis, the following framework is used throughout this section:

- All equations and inequalities involving the indexes $i$ and/or $j$ are satisfied for all $i, j = 1, 2, \ldots, N$ $(i \neq j)$;
- The considered domain is $x = \text{col}(x_1, x_2, \ldots, x_N) \in \Omega \equiv: \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ with $x_i \in \Omega_i \subset \mathbb{R}^{n_i}$;
- Output matrices $C_i$ have the structure $C_i = \begin{bmatrix} I_{p_i} & 0 \end{bmatrix}$.

It should be emphasised that the framework above does not reduce the generality of the work. First, the considered system is nonlinear, and thus the obtained conclusion is, generally speaking, local but sometimes a global result is available. In addition, since $C_i$ is full row rank, there always exist nonsingular matrices $\mathscr{T}_i \in \mathbb{R}^{n_i \times n_i}$ such that $C_i \mathscr{T}_i = [I_{p_i} \quad 0]$. Then the new system with state variables $z_i = \mathscr{T}_i^{-1} x_i$ has the output matrix $\begin{bmatrix} I_{p_i} & 0 \end{bmatrix}$.

The following assumptions are imposed on the nonlinear large-scale interconnected system (5.1)–(5.2):

**Assumption 5.1** The matrix pairs $(A_i, B_i)$ and $(A_i, C_i)$ are controllable and detectable, respectively, and the function $H_{ij}(x_j)$ $(i \neq j)$ satisfies Lipschitz condition in the considered domain.

In view of the detectability of the matrix pair $(A_i, C_i)$, there exists a matrix $L_i$ such that $(A_i - L_i C_i)$ is stable and thus for any $Q_i > 0$ the following Lyapunov equation has a unique solution $P_i > 0$

$$(A_i - L_i C_i)^\tau P_i + P_i (A_i - L_i C_i) = -Q_i. \tag{5.4}$$

**Assumption 5.2** The uncertainties have structural decompositions of the following form:

$$\Delta f_i(x_i, t) = D_i \Delta \widetilde{f}_i(x_i, t), \qquad \Delta H_{ij}(x_j, t) = E_{ij} \Delta \widetilde{H}_{ij}(x_j, t), \tag{5.5}$$

where $D_i$ and $E_{ij}$ $(i \neq j)$ are constant matrices, and

$$\|\Delta \widetilde{f}_i(x_i, t)\| \leq \rho_i(y_i, t)\gamma_i(x_i, t), \qquad \|\Delta \widetilde{H}_{ij}(x_j, t)\| \leq \vartheta_{ij}(y_j, t)\zeta_{ij}(x_j, t), \tag{5.6}$$

where

$$\gamma_i \leq \widetilde{\gamma}_i(x_i, t)\|x_i\| \qquad \text{and} \qquad \zeta_{ij} \leq \widetilde{\zeta}_{ij}(x_j, t)\|x_j\|, \quad (i \neq j)$$

are Lipschitz with $\widetilde{\gamma}_i(\cdot)$ and $\widetilde{\zeta}_{ij}(\cdot)$ continuous.

*Remark 5.2* Assumption 5.1 is a basic requirement for the nominal system of the interconnected system (5.1)–(5.2). Assumption 5.1 limits the uncertainties affecting the isolated subsystems and the interconnections. The matrices $D_i$ and $E_{ij}$ are employed to describe the structure of the uncertainties $\Delta f_i$ and $\Delta H_{ij}$, respectively.

**Assumption 5.3** There exist matrices $G_i$ and $F_{ij}$ $(i \neq j)$ such that

$$D_i^\tau P_i = G_i C_i, \qquad E_{ij}^\tau P_i = F_{ij} C_i, \tag{5.7}$$

where $P_i$ satisfies (5.4); $D_i$ and $E_{ij}$ $(i \neq j)$ satisfy (5.5).

*Remark 5.3* Assumption 5.3 implies that the Lyapunov equations (5.4) obey the constraint (5.7). Similar limitations have been used in [24, 214, 215].

In this section, the interconnected system given in (5.1)–(5.2) is such that the nominal system and/or any isolated nominal subsystem can be nonminimum phase. The objective is to use the sliding mode techniques to develop an output feedback control scheme based on a continuous reduced-order compensator such that the corresponding closed-loop system is asymptotically stable.

### 5.2.3 Reduced-Order Compensator Design

Consider System (5.1)–(5.2). Following the partition of $C_i = \begin{bmatrix} I_{p_i} & 0 \end{bmatrix}$, the system can be rewritten by

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{bmatrix} = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} + \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix} u_i + \begin{bmatrix} D_{i1} \\ D_{i2} \end{bmatrix} \Delta \widetilde{f}_i(x_i, t) + \begin{bmatrix} \sum_{\substack{j=1 \\ j\neq i}}^{N} \left( H_{ij1}(x_j) + E_{ij1} \Delta \widetilde{H}_{ij}(x_j, t) \right) \\ \sum_{\substack{j=1 \\ j\neq i}}^{N} \left( H_{ij2}(x_j) + E_{ij2} \Delta \widetilde{H}_{ij}(x_j, t) \right) \end{bmatrix} \tag{5.8}$$

$$y_i = x_{i1}, \tag{5.9}$$

where $x_i = \text{col}(x_{i1}, x_{i2})$ with $x_{i1} \in \mathbb{R}^{p_i}$, $A_{i1} \in \mathbb{R}^{p_i \times p_i}$, $B_{i1} \in \mathbb{R}^{p_i \times m_i}$; $D_{i1}$, $E_{ij1}$ and $H_{ij1}$ are the first $p_i$ rows of $D_i$, $E_{ij}$ and $H_{ij}(x_j)$, respectively.

Partition $P_i$, $Q_i$ and $L_i$ conformably with the decomposition (5.8)–(5.9) as

$$P_i = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i2}^{\tau} & P_{i3} \end{bmatrix}, \qquad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^{\tau} & Q_{i3} \end{bmatrix}, \qquad L_i = \begin{bmatrix} L_{i1} \\ L_{i2} \end{bmatrix}. \tag{5.10}$$

Then, construct a reduced-order dynamical compensator

$$\begin{aligned} \dot{\hat{z}}_{i2} = {} & (A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2}) \hat{z}_{i2} + \left( P_{i3}^{-1} P_{i2}^{\tau} (A_{i1} - A_{i2} P_{i3}^{-1} P_{i2}^{\tau}) + A_{i3} - A_{i4} P_{i3}^{-1} P_{i2}^{\tau} \right) y_i \\ & + \left( P_{i3}^{-1} P_{i2}^{\tau} B_{i1} + B_{i2} \right) u_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big\{ P_{i3}^{-1} P_{i2}^{\tau} H_{ij1}(y_j, \hat{\nu}_j) \\ & + H_{ij2}(y_j, \hat{\nu}_j) \Big\} \Big|_{\hat{\nu}_j = \hat{z}_{j2} - P_{j3}^{-1} P_{j2} y_j}, \end{aligned} \tag{5.11}$$

where $\hat{z}_{i2} \in \mathbb{R}^{n_i - p_i}$. The following conclusion can be drawn:

**Theorem 5.1** *Let $\hat{x}_{i2} = -P_{i3}^{-1} P_{i2}^{\tau} y_i + \hat{z}_{i2}$ with $\hat{z}_{i2}$ given by (5.11). Then, under Assumptions 5.1–5.3 there exist positive constants $\alpha_1$ and $\alpha_2$ such that*

$$\| x_{i2}(t) - \hat{x}_{i2}(t) \| \leq \alpha_1 \exp\{-\alpha_2 t\} \tag{5.12}$$

*if $W^T + W$ is a positive definite matrix with $W = (w_{ij})_{N \times N}$ defined by*

$$w_{ij} = \begin{cases} \lambda_{\min}(Q_{i3}), & i = j \\ -2 \left( \| P_{i2} \| \mathscr{L}_{H_{ij1}} + \| P_{i3} \| \mathscr{L}_{H_{ij2}} \right), & i \neq j \end{cases},$$

*where $H_{ij1}$ and $H_{ij2}$ are, respectively, the first $p_i$ and the last $n_i - p_i$ components of $H_{ij}(x_j)$, and $P_{i2}$, $P_{i3}$ and $Q_{i3}$ are defined by (5.10).*

*Proof* From Assumption 5.3, $C_i = [I_{p_i} \quad 0]$ and the partition (5.10) of $P_i$, it is observed that

$$P_{i2}^{\tau} D_{i1} + P_{i3} D_{i2} = 0, \qquad P_{i2}^{\tau} E_{ij1} + P_{i3} E_{ij2} = 0 \quad (i \neq j). \tag{5.13}$$

Introduce a nonsingular coordinate transformation $z_i = \hat{T}_i x_i$ defined by

$$\hat{T}_i : \begin{cases} z_{i1} = x_{i1} \\ z_{i2} = P_{i3}^{-1} P_{i2}^{\tau} x_{i1} + x_{i2} \end{cases}. \tag{5.14}$$

Since (5.13) implies

$$P_{i3}^{-1} P_{i2}^{\tau} D_{i1} + D_{i2} = 0 \quad \text{and} \quad P_{i3}^{-1} P_{i2}^{\tau} E_{ij1} + E_{ij2} = 0$$

then it follows from (5.8)–(5.9) that in the new coordinates $z = \text{col}(z_{i1}, \ldots, z_{iN})$, System (5.1)–(5.2) is described by

$$\begin{aligned} \dot{z}_{i1} &= \left(A_{i1} - A_{i2} P_{i3}^{-1} P_{i2}^{\tau}\right) z_{i1} + A_{i2} z_{i2} + B_{i1} u_i + D_{i1} \Delta \widetilde{f}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \Big\{ H_{ij1}(x_j) \\ &\quad + E_{ij1} \Delta \widetilde{H}_{ij}(t, x_j) \Big\} \qquad (5.15) \\ \dot{z}_{i2} &= \left(P_{i3}^{-1} P_{i2}^{\tau} (A_{i1} - A_{i2} P_{i3}^{-1} P_{i2}^{\tau}) + A_{i3} - A_{i4} P_{i3}^{-1} P_{i2}^{\tau}\right) z_{i1} + (A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2}) z_{i2} \\ &\quad + \left(P_{i3}^{-1} P_{i2}^{\tau} B_{i1} + B_{i2}\right) u_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \Big\{ P_{i3}^{-1} P_{i2}^{\tau} H_{ij1}(y_j, \nu_j) \\ &\quad + H_{ij2}(y_j, \nu_j) \Big\}_{\nu_j = z_{j2} - P_{j3}^{-1} P_{j2}^{\tau} z_{j1}} \qquad (5.16) \\ y_i &= z_{i1}. \qquad (5.17) \end{aligned}$$

From (5.14), (5.17) and $\hat{x}_{i2} = -P_{i3}^{-1} P_{i2}^{\tau} y_i + \hat{z}_{i2}$, it follows that

$$x_{i2} - \hat{x}_{i2} = x_{i2} + P_{i3}^{-1} P_{i2}^{\tau} y_i - \hat{z}_{i2} = z_{i2} - \hat{z}_{i2}.$$

Therefore, it is only required to prove that

$$\|z_{i2} - \hat{z}_{i2}\| \leq \alpha_1 \exp\{-\alpha_2 t\}$$

for some positive constants $\alpha_1$ and $\alpha_2$.

Let $e_i = z_{i2} - \hat{z}_{i2}$. From (5.11), (5.16) and $y_i = z_{i1}$,

$$\dot{e}_i = \left(A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2}\right) e_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \Big\{ P_{i3}^{-1} P_{i2}^{\tau} \left[H_{ij1}(y_j, \nu_j) - H_{ij1}(y_j, \hat{\nu}_j)\right] + H_{ij2}(y_j, \nu_j) - H_{ij2}(y_j, \hat{\nu}_j)\Big\}, \tag{5.18}$$

where

$$\nu_j = z_{j2} - P_{j3}^{-1} P_{j2}^{-1} y_j \qquad \text{and} \qquad \hat{\nu}_j = \hat{z}_{j2} - P_{j3}^{-1} P_{j2}^{-1} y_j.$$

For System (5.18), consider a Lyapunov function candidate

$$V_1 = \sum_{i=1}^{N} e_i^{\tau} P_{i3} e_i.$$

Then, the time derivative of $V_1$ along the trajectories of System (5.18) is described by

$$\begin{aligned} \dot{V}_1 \mid_{(5.18)} = \sum_{i=1}^{N} e_i^{\tau} \Big( & \left(A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2}\right)^{\tau} P_{i3} + P_{i3} \left(A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2}\right)\Big) e_i \\ & + 2 \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} e_i^{\tau} \Big( P_{i2}^{\tau} \left[H_{ij1}(y_j, \nu_j) - H_{ij1}(y_j, \hat{\nu}_j)\right] \\ & + P_{i3} \left[H_{ij2}(y_j, \nu_j) - H_{ij2}(y_j, \hat{\nu}_j)\right]\Big). \end{aligned} \tag{5.19}$$

From (5.4), (5.10) and $C_i = [I_{p_i} \quad 0]$, it follows that

$$\left(P_{i3}^{-1} P_{i2}^{\tau} A_{i2} + A_{i4}\right)^{\tau} P_{i3} + P_{i3} \left(P_{i3}^{-1} P_{i2}^{\tau} A_{i2} + A_{i4}\right) = -Q_{i3}. \tag{5.20}$$

Since Assumption 5.2 implies that both $H_{ij1}$ and $H_{ij2}$ are Lipschitz in their domain of definition, $\mathscr{L}_{H_{ij1}}$ and $\mathscr{L}_{H_{ij2}}$ are well-defined. Then, substituting (5.20) into (5.19), it is observed from $\nu_i - \hat{\nu}_i = e_i$ that

$$\begin{aligned} \dot{V}_1 &\le -\sum_{i=1}^{N} e_i^{\tau} Q_{i3} e_i + 2 \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} \left( \|P_{i2}\| \mathscr{L}_{H_{ij1}} + \|P_{i3}\| \mathscr{L}_{H_{ij2}} \right) \|e_i\| \, \|e_j\| \\ &\le -\frac{1}{2} [\|e_1\| \; \|e_2\| \cdots \|e_N\|](W + W^{\tau})[\|e_1\| \; \|e_2\| \cdots \|e_N\|]^{\tau} \\ &\le -\frac{\lambda_{\min}(W + W^{\tau})}{2 \max_i \{\lambda_{\max}(P_{i3})\}} V_1. \end{aligned}$$

This implies that

$$V_1 \le (V_1 \mid_{t=0}) \exp\left\{-\frac{\lambda_{\min}(W+W^\tau)}{2\max_i\{\lambda_{\max}(P_{i3})\}}t\right\}.$$

Let

$$\alpha_1 > \sqrt{\frac{V_1 \mid_{t=0}}{\max_i\{\lambda_{\min}(P_{i3})\}}} \quad \text{and} \quad \alpha_2 \ge \frac{\lambda_{\min}(W+W^\tau)}{2\max_i\{\lambda_{\max}(P_{i3})\}}.$$

Then, from

$$\min_i\{\lambda_{\min}(P_{i3})\}\|e_i\|^2 \le e_i^\tau P_{i3} e_i \le \sum_{i=1}^N e_i^\tau P_{i3} e_i = V_1$$

the conclusion follows. □

*Remark 5.4* The inequalities (5.12) show that $\hat{x}_{i2}$ converges to $x_{i2}$ exponentially. It should be noted that the proof of Theorem 5.1 is constructive and shows how the constants $\alpha_1$ and $\alpha_2$ can be determined.

*Remark 5.5* From the proof of Theorem 5.1, it follows that in the elements of matrix $W$, $\mathscr{L}_{H_{ij1}}$ and $\mathscr{L}_{H_{ij2}}$ represent the Lipchitz constants only associated with the variables $x_{i2}$. Therefore, $\mathscr{L}_{H_{ij1}}$ and $\mathscr{L}_{H_{ij2}}$ can be replaced by the Lipschitz constants of the functions $H_{ij1}(y_j, x_{j2})$ and $H_{ij2}(y_j, x_{j2})$ corresponding to the variables $x_{j2}$, respectively, which can reduce conservatism.

### *5.2.4 Sliding Mode Design and Stability Analysis*

For System (5.1)–(5.2), consider the following sliding surface:

$$\sigma \equiv: \mathrm{col}(\sigma_1, \sigma_2, \ldots, \sigma_N) = 0, \tag{5.21}$$

where

$$\sigma_i(y_i, \hat{x}_{i2}) = S_{i1} y_i + S_{i2}\hat{x}_{i2}, \tag{5.22}$$

where $\hat{x}_{i2}$ is the compensator state given by Theorem 5.1, and $S_{i1} \in \mathbb{R}^{m_i \times p_i}$ and $S_{i2} \in \mathbb{R}^{m_i \times (n_i - p_i)}$ are the design parameters.

As in the proof of Theorem 5.1, let $e_i = x_{i2} - \hat{x}_{i2}$ and define $S_i = [S_{i1} \quad S_{i2}]$. In the new coordinate system $(x_i, e_i)$, the sliding function matrices

$$\sigma_i = \left[\, S_{i1} \; S_{i2} \,\right] x_i - S_{i2} e_i = S_i x_i - S_{i2} e_i. \tag{5.23}$$

The matrices $S_i$ can be chosen using any existing state feedback sliding mode design approach for the pairs $(A_i, B_i)$ such that

(i) the matrices $S_i B_i$ are nonsingular;
(ii) the matrices $A_{eqi} \equiv: A_i - B_i (S_i B_i)^{-1} S_i A_i$ have $n_i - m_i$ eigenvalues which lie in the open left-half plane.

*Remark 5.6* It should be noted that the two requirements above can always be satisfied by choosing appropriate parameters $S_i$. In fact, condition (i) is satisfied from the fact that $B_i$ is full column rank. Condition (ii) is satisfied from the controllability of $(A_i, B_i)$ in Assumption 5.1. Algorithms detailing how to design the parameters $S_i$ satisfying conditions (i) and (ii) are available in [38].

The following analysis is based on the assumption that the design parameters $S_i$ satisfying the conditions (i) and (ii) have been well chosen.

During a sliding motion, both $\sigma_i = 0$ and $\dot{\sigma}_i = 0$. From (5.1), (5.18), (5.23) and $\dot{\sigma}_i = 0$, the equivalent control [173] necessary to maintain sliding is given by

$$\begin{aligned} u_{ieq} = -(S_i B_i)^{-1} \Bigg\{ & S_i A_i x_i - S_{i2} \left( A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \right) e_i + S_i \Delta f_i(\cdot) + \sum_{\substack{j=1 \\ j \neq i}}^{N} S_i \Big( H_{ij}(x_j) \\ & + \Delta H_{ij}(t, x_j) \Big) - S_{i2} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Big( P_{i3}^{-1} P_{i2}^{\tau} \left( H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \right) \\ & + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \Big) \Bigg\}. \end{aligned} \tag{5.24}$$

*Remark 5.7* It should be noted that the equivalent control $u_{ieq}$ in (5.24) cannot be used as the applied control since the uncertainties are explicitly involved. Here, it is introduced to derive the sliding dynamics and analyse their stability.

When System (5.1)–(5.2) is restricted to the sliding surface (5.21), it follows by applying (5.24) to System (5.1) that the associated sliding mode dynamics can be described by

$$\begin{aligned} \begin{bmatrix} \dot{x}_i \\ \dot{e}_i \end{bmatrix} = & \begin{bmatrix} A_{eqi} & B_i (S_i B_i)^{-1} S_{i2} \left( A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \right) \\ 0 & A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \end{bmatrix} \begin{bmatrix} x_i \\ e_i \end{bmatrix} \\ & + \begin{bmatrix} \left( I_{n_i} - B_i (S_i B_i)^{-1} S_i \right) \Big( \Delta f_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \left( H_{ij} + \Delta H_{ij} \right) \Big) \\ 0 \end{bmatrix} \\ & + \begin{bmatrix} B_i (S_i B_i)^{-1} S_{i2} \\ I_{n_i - p_i} \end{bmatrix} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Big\{ P_{i3}^{-1} P_{i2}^{\tau} \left( H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2}) \right) \\ & + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}) \Big\}, \end{aligned} \tag{5.25}$$

where $A_{eqi} = A_i - B_i (S_i B_i)^{-1} S_i A_i$. Since $S_i B_i$ is nonsingular, matrix $S_i$ is full row rank and thus there exist nonsingular matrices $T_{i1} \in \mathbb{R}^{n_i \times n_i}$ and $T_{i2} \in \mathbb{R}^{m_i \times m_i}$ such that

$$T_{i2}S_iT_{i1} = \left[\, I_{m_i} \ 0 \,\right]. \tag{5.26}$$

In order to further analyse the stability of sliding mode (5.25), it is required to derive the reduced-order representation. The coordinate transformation $\mathrm{col}(\xi_i, \eta_i) = T_{i1}^{-1}x_i$ is introduced, where $\xi_i \in \mathbb{R}^{m_i}$ and $T_{i1}$ is determined by (5.26). Then, noticing condition (ii), it is observed that in the new coordinates $(\xi_i, \eta_i, e_i)$, System (5.25) is described by

$$\begin{aligned}\begin{bmatrix} \dot{\xi}_i \\ \dot{\eta}_i \\ \dot{e}_i \end{bmatrix} &= \begin{bmatrix} 0 & 0 & * \\ \widetilde{A}_{i1} & \widetilde{A}_{i2} & \widetilde{A}_{i3} \\ 0 & 0 & A_{i4} + P_{i3}^{-1}P_{i2}^{\tau}A_{i2} \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \\ e_i \end{bmatrix} + \begin{bmatrix} * \\ \Delta f_{i1}(t, x_i) \\ 0 \end{bmatrix} \\ &\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} \begin{bmatrix} * \\ \Pi_{ij}(\cdot) \\ 0 \end{bmatrix} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \begin{bmatrix} * \\ \Theta_{ij1}(\cdot) \\ \Theta_{ij2}(\cdot) \end{bmatrix},\end{aligned} \tag{5.27}$$

where $\widetilde{A}_{i2} \in \mathbb{R}^{(n_i-m_i)\times(n_i-m_i)}$, $\widetilde{A}_{i3} \in \mathbb{R}^{(n_i-m_i)\times(n_i-p_i)}$, and

$$\begin{bmatrix} 0 & 0 \\ \widetilde{A}_{i1} & \widetilde{A}_{i2} \end{bmatrix} = T_{i1}^{-1}A_{eqi}T_{i1}. \tag{5.28}$$

The notation $*$ denotes items which are not used in the subsequent analysis; $\Delta f_{i1}$, $\Pi_{ij}$ and $\Theta_{ij1}$ are the last $n_i - m_i$ components of

$$T_{i1}^{-1}\left(I_{n_i} - B_i(S_iB_i)^{-1}S_i\right)\Delta f_i$$

$$T_{i1}^{-1}\left(I_{n_i} - B_i(S_iB_i)^{-1}S_i\right)(H_{ij} + \Delta H_{ij})$$

and

$$\begin{aligned}T_{i1}^{-1}B_i(S_iB_i)^{-1}S_{i2}\{P_{i3}^{-1}P_{i2}^{\tau}&\left(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2})\right) \\ &+H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2})\},\end{aligned}$$

respectively, and

$$\Theta_{ij2} = P_{i3}^{-1}P_{i2}^{\tau}\left(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2})\right) + H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2}). \tag{5.29}$$

From (5.26), it follows that

$$\sigma_i = S_ix_i - S_{i2}e_i = T_{i2}^{-1}\left[\, I_{m_i} \ 0 \,\right]\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} - S_{i2}e_i = T_{i2}^{-1}\xi_i - S_{i2}e_i.$$

This implies that in the new coordinate system $(\xi_i, \eta_i, e_i)$, $\sigma_i = 0$ can be depicted by $\xi_i = T_{i2}S_{i2}e_i$. Consequently, when the system (5.25) is restricted to the sliding surface (5.21), it can be described in coordinate system $(\xi_i, \eta_i, e_i)$ by

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{e}_i \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{i2} & \widetilde{A}_{i3} + \widetilde{A}_{i1} T_{i2} S_{i2} \\ 0 & A_{i4} + P_{i3}^{-1} P_{i2}^{\tau} A_{i2} \end{bmatrix} \begin{bmatrix} \eta_i \\ e_i \end{bmatrix} + \begin{bmatrix} \Delta f_{i1} \\ 0 \end{bmatrix} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \begin{bmatrix} \Pi_{ij} \\ 0 \end{bmatrix} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \begin{bmatrix} \Theta_{ij1} \\ \Theta_{ij2} \end{bmatrix}. \tag{5.30}$$

From condition (ii) and (5.28), it is observed that the matrix $\widetilde{A}_{i2}$ has $n_i - m_i$ negative eigenvalues. This implies that for any $\widetilde{Q}_i > 0$, the Lyapunov equations

$$\widetilde{A}_{i2}^{\tau} \widetilde{P}_i + \widetilde{P}_i \widetilde{A}_{i2} = -\widetilde{Q}_i \tag{5.31}$$

have unique solutions $\widetilde{P}_i > 0$.

From (5.3) and Assumption 5.2, it is observed that there exist continuous functions $\varphi_{i1}, \varphi_{i2}, \psi_{ij}$ and $\chi_{ij}$ such that

$$\|\widetilde{P}_i \Delta f_{i1}\| \leq \varphi_{i1}(\eta_i, e_i, t)\|\eta_i\| + \varphi_{i2}(\eta_i, e_i, t)\|e_i\| \tag{5.32}$$

$$\|\widetilde{P}_i \left(\Pi_{ij} + \Theta_{ij1}\right)\| \leq \psi_{ij}(\eta_j, e_j, t)\|\eta_j\| + \chi_{ij}(\eta_j, e_j, t)\|e_j\|, \tag{5.33}$$

where $\widetilde{P}_i$ satisfies (5.31).

*Remark 5.8* Since $T_{i1}$ and $T_{i2}$ in (5.26) can be easily obtained from linear system theory, the sliding mode dynamics (5.27) are obtained directly from System (5.25) through the coordinate transformation $\text{col}(\xi_i, \eta_i) = T_{i1}^{-1} x_i$. Then from $\xi_i = T_{i2} S_{i2} e_i$, the reduced-order sliding mode dynamics (5.30) can be obtained as well. With $T_{i1}$, and $T_{i2}$, the inequalities (5.32) and (5.33) can easily be established using Assumption 5.2.

**Theorem 5.2** *Under Assumptions 5.1–5.3, the sliding mode dynamics (5.30) are asymptotically stable if there exists a domain of origin in* $\text{col}(\eta_i, e_i) \in \mathbb{R}^{2n_i - m_i - p_i}$ *such that the matrix* $M^{\tau} + M$ *is positive definite with* $M \in \mathbb{R}^{2N \times 2N}$ *defined by*

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

*where*

$$M_{11} := \begin{bmatrix} \lambda_{\min}(\widetilde{Q}_1) - 2\varphi_{11} & -2\psi_{12} & \cdots & -2\psi_{1N} \\ -2\psi_{21} & \lambda_{\min}(\widetilde{Q}_2) - 2\varphi_{21} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2\psi_{(N-1)N} \\ -2\psi_{N1} & \cdots\cdots & -2\psi_{N(N-1)} & \lambda_{\min}(\widetilde{Q}_N) - 2\varphi_{N1} \end{bmatrix}$$

$$M_{12} := \begin{bmatrix} -2(\varphi_{12}+\varpi_1) & -2\chi_{12} & \cdots & -2\chi_{1N} \\ -2\chi_{21} & -2(\varphi_{22}+\varpi_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2\chi_{(N-1)N} \\ -2\chi_{N1} & \cdots\cdots & -2\chi_{N(N-1)} & -2(\phi_{N2}+\varpi_N) \end{bmatrix}$$

$$M_{21} := \begin{bmatrix} -2(\varphi_{12}+\varpi_1) & -2\chi_{12} & \cdots & -2\chi_{1N} \\ -2\chi_{21} & -2(\varphi_{22}+\varpi_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2\chi_{(N-1)N} \\ -2\chi_{N1} & \cdots\cdots & -2\chi_{N(N-1)} & -2(\varphi_{N2}+\varpi_N) \end{bmatrix}$$

$$M_{22} := \begin{bmatrix} \lambda_{\min}(Q_{13}) & -2\kappa_{12} & \cdots & -2\kappa_{1N} \\ -2\kappa_{21} & \lambda_{\min}(Q_{23}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & -2\kappa_{(N-1)N} \\ -2\kappa_{N1} & \cdots\cdots & -2\kappa_{N(N-1)} & \lambda_{\min}(Q_{N3}) \end{bmatrix},$$

*where* $\varphi_{i1}$, $\varphi_{i2}$, $\psi_{ij}$ *and* $\chi_{ij}$ *are determined by (5.32) and (5.33), and*

$$\kappa_{ij} :\equiv (\|P_{i2}\|\mathscr{L}_{H_{ij1}} + \|P_{i3}\|\mathscr{L}_{H_{ij2}}) \quad \text{and} \quad \varpi_i :\equiv \|\widetilde{P}_i(\widetilde{A}_{i3} + \widetilde{A}_{i1}T_{i2}S_{i2})\|,$$

*where* $P_{i2}$ *and* $P_{i3}$ *are defined in (5.11), and the* $\widetilde{P}_i$ *satisfy (5.31) for* $i = 1, 2, \ldots, N$.

*Proof* For System (5.30), construct the candidate Lyapunov function

$$V = \sum_{i=1}^{N} \left(\eta_i^\tau \widetilde{P}_i \eta_i + e_i^\tau P_{i3} e_i\right).$$

Then, the time derivative of $V$ along the trajectories of System (5.30) is given by

$$\begin{aligned} \dot{V}\,|_{(5.30)} = & -\sum_{i=1}^{N} \eta_i^\tau \widetilde{Q}_i \eta_i + 2\sum_{i=1}^{N} \eta_i^\tau \widetilde{P}_i \left(\widetilde{A}_{i3} + \widetilde{A}_{i1}T_{i2}S_{i2}\right) e_i \\ & +2\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \eta_i^\tau \widetilde{P}_i \left(\Pi_{ij} + \Theta_{ij1}\right) \\ & -\sum_{i=1}^{N} e_i^\tau Q_{i3} e_i + 2\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} e_i^\tau P_{i3}\Theta_{ij2} + 2\sum_{i=1}^{N} \eta_i^\tau \widetilde{P}_i \Delta f_{i1} \\ \leq & -\sum_{i=1}^{N} \left(\lambda_{\min}(\widetilde{Q}_i)\|\eta_i\|^2 + \lambda_{\min}(Q_{i3})\|e_i\|^2\right) \end{aligned}$$

$$+2\sum_{i=1}^{N}\left\|\widetilde{P}_i\left(\widetilde{A}_{i3}+\widetilde{A}_{i1}T_{i2}S_{i2}\right)\right\|\|e_i\|\|\eta_i\|$$
$$+2\sum_{i=1}^{N}\left\|\widetilde{P}_i\Delta f_{i1}\right\|\|\eta_i\|+2\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\{\left\|\widetilde{P}_i\left(\Pi_{ij}+\Theta_{ij1}\right)\right\|\|\eta_i\|\right.$$
$$\left.+\left\|P_{i3}\Theta_{ij2}\right\|\|e_i\|\right\}, \tag{5.34}$$

where (5.20) and (5.31) are used above. From (5.29), it follows that

$$\begin{aligned}\left\|P_{i3}\Theta_{ij2}\right\| &= \left\|P_{i2}^{\tau}\left(H_{ij1}(y_j,x_{j2})-H_{ij1}(y_j,\hat{x}_{j2})\right)+P_{i3}\left(H_{ij2}(y_j,x_{j2})-H_{ij2}(y_j,\hat{x}_{j2})\right)\right\|\\ &\leq (\|P_{i2}\|\mathscr{L}_{H_{ij1}}+\|P_{i3}\|\mathscr{L}_{H_{ij2}})\|e_j\|=\kappa_{ij}\|e_j\|.\end{aligned} \tag{5.35}$$

Then, from (5.35), (5.32) and (5.33) it is obtained that

$$\begin{aligned}\dot{V}\,|_{(5.30)} \leq &-\sum_{i=1}^{N}(\lambda_{\min}(\widetilde{Q}_i)-2\varphi_{i1})\|\eta_i\|^2-\sum_{i=1}^{N}\lambda_{\min}(Q_{i3})\|e_i\|^2\\ &+2\sum_{i=1}^{N}(\varphi_{i2}+\varpi_i)\|\eta_i\|\|e_i\|\\ &+2\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\{\psi_{ij}\|\eta_i\|\|\eta_j\|+\chi_{ij}\|\eta_i\|\|e_j\|+\kappa_{ij}\|e_i\|\|e_j\|\right\}\\ =&\ \frac{1}{2}\left[\|\eta_1\|\cdots\|\eta_N\|\,\|e_1\|\cdots\|e_N\|\right](M^{\tau}+M)\left[\|\eta_1\|\cdots\|\eta_N\|\,\|e_1\|\cdots\|e_N\|\right]^{\tau}.\end{aligned}$$

Hence, the conclusion follows by the positive definiteness of $M^{\tau}+M$. □

*Remark 5.9* It should be noted that the matrix $M$ in Theorem 5.2 only involves the partial state variables $(\eta_i, e_i)$. Actually, it is only required that $M+M^{\tau}$ is positive definite in a range of the origin in the sliding surface. This is in comparison with the work [214, 215], where it is required that the associated matrix is positive definite in the domain of the origin of the whole state space.

### *5.2.5 Sliding Mode Controller Synthesis*

For the system (5.1)–(5.2) with the designed composite sliding surface (5.21), construct the following sliding mode control:

$$u_i = -(S_i B_i)^{-1}\Big\{(S_{i1}A_{i1} + S_{i2}A_{i3})\, y_i + (S_{i1}A_{i2} + S_{i2}A_{i4})\, \hat{x}_{i2}$$
$$+ \Big(\|S_i D_i\|\rho_i(y_i, t)\gamma_i(y_i, \hat{x}_{i2}, t) + K_i(y_i, t) + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\big(\|S_j E_{ji}\|\vartheta_{ji}(y_i, t)\zeta_{ji}(y_i, \hat{x}_{i2}, t)$$
$$+\|S_j H_{ji}(y_i, \hat{x}_{i2})\|\big)\Big)\frac{\sigma_i}{\|\sigma_i\|}\Big\}, \tag{5.36}$$

where $\sigma_i$ is defined by (5.22), and $K_i(y_i, t)$ is the control gain to be determined later. Obviously, the control law is decentralised and only depends on the $\hat{x}_{i2}$ and the system output $y_i$.

The objective now is to show that the control (5.36) can drive the system (5.1)–(5.2) to the sliding surface (5.21) and maintain a sliding motion on it thereafter. From sliding mode theory, it is only required to prove that the composite reachability condition (see [69])

$$\sum_{i=1}^{N}\frac{\sigma_i^{\tau}(y_i, \hat{x}_{i2})\dot{\sigma}_i(y_i, \hat{x}_{i2})}{\|\sigma_i(y_i, \hat{x}_{i2})\|} < 0. \tag{5.37}$$

is satisfied, where $\sigma_i(y_i, \hat{x}_{i2})$ defined by (5.22) is the sliding function for the $i$-th subsystem.

**Theorem 5.3** *Under Assumptions 5.1–5.3 with (5.12) satisfied, the controllers (5.36) drive the system (5.1)–(5.2) to the composite sliding surface (5.21) and maintain a sliding motion thereafter if the control gains $K_i$ are chosen such that*

$$K_i(y_i, t) > \alpha_1 \exp\{-\alpha_2 t\}\Big\{\|S_{i1}A_{i2} + S_{i2}A_{i4}\| + \mathscr{L}_{\gamma_i}\|S_i D_i\|\rho_i(y_i, t)$$
$$+\|S_{i2}P_{i3}^{-1}(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4})\| + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\Big(\|S_j\|\mathscr{L}_{H_{ji}} + \|S_j E_{ji}\vartheta_{ji}\|\mathscr{L}_{\zeta_{ji}}$$
$$+\|S_{i2}P_{i3}^{-1}\|\big(\|P_{i2}\|\mathscr{L}_{H_{ij1}} + \|P_{i3}\|\mathscr{L}_{H_{ij2}}\big)\Big)\Big\}, \tag{5.38}$$

*where the constants $\alpha_1$ and $\alpha_2$ are both determined by (5.12).*

*Proof* It is observed from the proof of Theorem 5.1 that the error dynamics in (5.18) can be rewritten by

$$\dot{e}_i = P_{i3}^{-1}\big(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4}\big)e_i + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}P_{i3}^{-1}\Big\{P_{i2}^{\tau}\big(H_{ij1}(y_j, x_{j2}) - H_{ij1}(y_j, \hat{x}_{j2})\big)$$
$$+P_{i3}\big(H_{ij2}(y_j, x_{j2}) - H_{ij2}(y_j, \hat{x}_{j2})\big)\Big\}. \tag{5.39}$$

From (5.23), (5.1) and (5.39)

$$\begin{aligned}\dot{\sigma}_i &= S_i A_i x_i + S_i B_i u_i + S_i \Delta f_i + \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N} S_i \left(H_{ij}(x_j) + \Delta H_{ij}(x_j,t)\right)\\ &\quad - S_{i2}P_{i3}^{-1}\left(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4}\right)e_i - \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N} S_{i2}P_{i3}^{-1}\Big\{P_{i2}^{\tau}\left(H_{ij1}(y_j,x_{j2}) - H_{ij1}(y_j,\hat{x}_{j2})\right)\\ &\quad + P_{i3}\left(H_{ij2}(y_j,x_{j2}) - H_{ij2}(y_j,\hat{x}_{j2})\right)\Big\}. \end{aligned} \tag{5.40}$$

Then, substituting $u_i$ in (5.36) into (5.40),

$$\begin{aligned}\sum_{i=1}^{N}\frac{\sigma_i^{\tau}\dot{\sigma}_i}{\|\sigma_i\|} &= \sum_{i=1}^{N}\frac{\sigma_i^{\tau}}{\|\sigma_i\|}\Big\{\Big(S_iA_ix_i - (S_{i1}A_{i1} + S_{i2}A_{i3})y_i - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2}\Big)\\ &\quad - S_{i2}P_{i3}^{-1}(P_{i2}^{\tau}A_{i2} + P_{i3}A_{i4})e_i\Big\} + \sum_{i=1}^{N}\left(\frac{\sigma_i^{\tau}}{\|\sigma_i\|}S_i\Delta f_i(t,x_i)\right.\\ &\quad \left. - \|S_iD_i\|\rho_i(t,y_i)\gamma_i(y_i,\hat{x}_{i2},t)\right) - \sum_{i=1}^{N}K_i + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\sigma_i^{\tau}}{\|\sigma_i\|}\Big(S_i\left[H_{ij} + \Delta H_{ij}\right]\\ &\quad - \frac{\sigma_i}{\|\sigma_i\|}\left[\|S_jH_{ji}(y_i,\hat{x}_{i2})\| + \|S_jE_{ji}\|\vartheta_{ji}(y_i,t)\zeta_{ji}(y_i,\hat{x}_{i2},t)\right]\Big)\\ &\quad - \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} S_{i2}P_{i3}^{-1}\Big\{P_{i2}^{\tau}[H_{ij1}(y_j,x_{j2}) - H_{ij1}(y_j,\hat{x}_{j2})]\\ &\quad + P_{i3}\left[H_{ij2}(y_j,x_{j2}) - H_{ij2}(y_j,\hat{x}_{j2})\right]\Big\}. \end{aligned} \tag{5.41}$$

Using the previous partition of $A_i$ in (5.8) and

$$S_i = [S_{i1} \quad S_{i2}]$$

it follows that

$$\begin{aligned}&S_iA_ix_i - (S_{i1}A_{i1} + S_{i2}A_{i3})y_i - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2}\\ &= [S_{i1} \quad S_{i2}]\begin{bmatrix}A_{i1} & A_{i2}\\ A_{i3} & A_{i4}\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\end{bmatrix} - (S_{i1}A_{i1} + S_{i2}A_{i3})x_{i1} - (S_{i1}A_{i2} + S_{i2}A_{i4})\hat{x}_{i2}\\ &= (S_{i1}A_{i2} + S_{i2}A_{i4})e_i. \end{aligned} \tag{5.42}$$

From Assumption 5.2

$$\begin{aligned}\tfrac{\sigma_i^{\tau}}{\|\sigma_i\|}S_i\Delta f_i &- \|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t)\\ &\leq \|S_iD_i\|\,\|\Delta\widetilde{f}_i\| - \|S_iD_i\|\rho_i(y_i,t)\gamma_i(y_i,\hat{x}_{i2},t)\\ &\leq \rho_i(y_i,t)\mathscr{L}_{\gamma_i}\|S_iD_i\|\,\|e_i\| \end{aligned} \tag{5.43}$$

and further from the fact $\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} a_{ij} = \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} a_{ji}$,

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} \frac{\sigma_i^\tau}{\|\sigma_i\|}\Bigg(S_i\left[H_{ij}(x_j)+\Delta H_{ij}(x_j,t)\right]-\frac{\sigma_i}{\|\sigma_i\|}[\|S_j H_{ji}(y_i,\hat{x}_{i2})\| \\
&+\|S_j E_{ji}\|\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)]\Bigg) \\
&=\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\{\frac{\sigma_i^\tau}{\|\sigma_i\|}S_i H_{ij}(x_j)-\|S_j H_{ji}(y_i,\hat{x}_{i2})\|+\frac{\sigma_i^\tau}{\|\sigma_i\|}S_i E_{ij}\Delta\widetilde{H}_{ij}\right. \\
&\left.-\|S_j E_{ji}\|\,\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)\right\} \\
&\le\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\{\mathscr{L}_{H_{ij}}\|S_i\|\,\|e_j\|+\|S_j E_{ji}\|\,\vartheta_{ji}(y_i,t)\zeta_{ji}(x_i,t)-\|S_j E_{ji}\|\,\vartheta_{ji}\zeta_{ji}(y_i,\hat{x}_{i2},t)\right\} \\
&\le\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left(\|S_j\|\mathscr{L}_{H_{ji}}+\|S_j E_{ji}\|\vartheta_{ji}(y_i,t)\mathscr{L}_{\zeta_{ji}}\right)\|e_i\|.
\end{aligned} \tag{5.44}$$

Substituting (5.42)–(5.44) into (5.41), it follows from (5.12) that

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\sigma_i^\tau\dot{\sigma}_i}{\|\sigma_i\|}\le\sum_{i=1}^{N}\Bigg\{&\alpha_1\exp\{-\alpha_2 t\}\Bigg(\|S_{i1}A_{i2}+S_{i2}A_{i4}\|+\mathscr{L}_{\gamma_i}\rho_i\|S_i D_i\| \\
&+\|S_{i2}P_{i3}^{-1}(P_{i2}^\tau A_{i2}+P_{i3}A_{i4})\|+\sum_{\substack{j=1\\j\neq i}}^{N}\Big[\|S_j\|\mathscr{L}_{H_{ji}}+\|S_j E_{ji}\|\vartheta_{ji}\mathscr{L}_{\zeta_{ji}} \\
&+\|S_{i2}P_{i3}^{-1}\|\left(\|P_{i2}\|\mathscr{L}_{H_{ij1}}+\|P_{i3}\|\mathscr{L}_{H_{ij2}}\right)\Big]\Bigg)-K_i(y_i,t)\Bigg\}.
\end{aligned} \tag{5.45}$$

Hence, from (5.38) the conclusion follows. □

*Remark 5.10* From (5.36), it is observed that the known interconnections $H_{ij}(x_j)$ and the bounds on the uncertain interconnections $\vartheta_{ij}(y_j,t)\zeta_{ij}(x_j,t)$ are both used in the control design directly, which can reduce conservatism and avoid unnecessary control energy consumption. This is in contrast to the existing work [69, 97, 124, 196, 170, 200, 214, 215]. From Theorem 5.3, it is observed that the reachability condition is always satisfied if the control gains $K_i(y_i,t)$ are allowed to be arbitrarily large. This shows the robustness of the sliding mode control.

## 5.3 Reduced-Order Control Design for Interconnected Systems with Similar Structure

In this section, reduced-order dynamical control schemes are proposed for a class of nonlinear interconnected systems with similar structure.

### *5.3.1 Introduction*

As claimed earlier, reduced-order control design is very important specifically for large-scale interconnected systems. In view of this, a form of nonlinear reduced-order controller is presented by Chen [23] for a class of nonlinear large-scale systems. However, it is required that the nominal subsystem is stabilisable via static output feedback and practical stability instead of asymptotic stability is achieved. The interconnection is treated as a disturbance bounded by the norm of the state variables in [23]. It is worth noting that in existing work on decentralised control, the system structure has not been explored for tackling uncertainty, which renders unavoidable restrictions on uncertainty and also higher design conservatism.

In this section, a class of nonlinear large-scale interconnected systems with matched and mismatched uncertainties is considered. No statistical information about the uncertainties is imposed while only their bounds are assumed to be known. The bounding functions of the uncertain interconnections take more general forms than those previously considered. By exploiting the system structure of similarity, the proposed nonlinear reduced-order control schemes allow more general forms of uncertainties. Specifically, based on a constrained Lyapunov equation, the effect of the matched uncertainties is cancelled completely. This allows the uncertainties to take arbitrarily large values. Further, the general known interconnections are treated separately from the uncertain interconnections as in [196, 203, 214], which is in contrast with other existing results where all interconnections are treated as disturbances. Therefore, the proposed control schemes possess the advantage of better robustness. Since a reduced-order controller is designed, the order of the interconnected system is reduced, which avoids the dimensionality problems associated with large-scale systems, and makes system analysis and implementation easier. It should be noticed that the amount of computation for solving the Lyapunov equation is reduced greatly by exploiting the similar structure and the reduced-order property, and this is particularly important especially for large-scale systems with high-order subsystems.

### 5.3.2 System Description and Assumptions

Consider a nonlinear large-scale interconnected system described by

$$\dot{x}_i = A_i x_i + f_i(x_i) + B_i[u_i + \Delta\Psi_i(x_i)] + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}(x_j) + \Delta H_i(x), \tag{5.46}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N, \tag{5.47}$$

where $x_i \in \Omega_i \subset \mathbb{R}^n$, $u_i$, $y_i \in \mathbb{R}^m$ are the state vector, input and output of the $i$-th subsystem, respectively, $A_i$, $B_i$, $C_i$ are constant matrices of appropriate dimensions with $C_i$ of full row rank, $f_i(x_i)$ is a continuous nonlinear function with $f_i(0) = 0$, $\Delta\Psi_i(x_i)$ is the matched uncertainty of the $i$-th isolated subsystem, the term

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}(x_j), \qquad H_{ij}(0) = 0 \quad \text{for} \quad j \neq i$$

is the known interconnection while $\Delta H_i(x)$ includes all interconnected uncertainty, which are all continuous in their arguments.

First, the similar structure considered for the interconnected systems (5.46)–(5.47) is defined mathematically as follows:

**Definition 5.1** System (5.46)–(5.47) is said to be a *similar interconnected large-scale system* via static output feedback or possesses a *similar structure* if there exist matrices $E_i$ and nonsingular matrices $D_i$ for $i = 1, 2, \ldots, N$ such that

$$D_1^{-1}(A_1 + B_1E_1C_1)D_1 = D_2^{-1}(A_2 + B_2E_2C_2)D_2 = \cdots = D_N^{-1}(A_N + B_NE_NC_N)D_N = A \tag{5.48}$$

$$D_1^{-1}B_1 = D_2^{-1}B_2 = \cdots = D_N^{-1}B_N = B \tag{5.49}$$

$$C_1D_1 = C_2D_2 = \cdots = C_ND_N = C, \tag{5.50}$$

where $A$, $B$ and $C$ are constant matrices. Then, $(E_i, D_i)$ is called *a similar transformation parameter (STP)* of the $i$-th subsystem for $i = 1, 2, \ldots, N$.

*Remark 5.11* System (5.46)–(5.47) possessing STP $(E_i, D_i)$ implies that its nominal subsystems are all equivalent via static output feedback. This is an extension of the systems dealt with in [110, 163, 214]. However, it should be emphasised that the method proposed in this work is not only applicable to the interconnected system with such a similar structure but can also be extended to more general large-scale systems because the introduced similar structure is only for the simplification of the technical presentation.

It should be noted that the similar large-scale system introduced here possesses nominal subsystems which are equivalent to one another through static output feedback, and the corresponding STP may be obtained by linear system theory.

*Remark 5.12* In manufacturing processes, in order to produce the same or similar engineering components in batches, many identical or similar machines (nominal subsystems) are interconnected to form a large-scale system which is referred to as a similar interconnected system in Sect. 5.3.

The following assumptions are imposed on the system (5.46)–(5.47) with the similar structure (5.48)–(5.50).

**Assumption 5.4** The nonlinear function $H_{ij}(x_j)$ $(j \neq i)$ is Lipschitz in $\Omega_i$ with Lipschitz constants $\mathscr{L}_{H_{ij}}$, and it has the following decomposition:

$$H_{ij}(x_j) = \Gamma_{ij}(x_j)x_j, \tag{5.51}$$

with $\Gamma_{ij}(x_j) \in \mathbb{R}^{n\times n}$ for $i, j = 1, 2, \ldots, N$ with $j \neq i$.

*Remark 5.13* Generally speaking, the function matrix $\Gamma_{ij}$ satisfying (5.51) is not unique. A similar decomposition is also used in [3, 214, 225]. Since $H_{ij}(0) = 0$, it follows that the constraint (5.51) for $H_{ij}$ is not strong. In fact, (5.51) is satisfied as long as $H_{ij}(x_j)$ is sufficiently smooth. □

**Assumption 5.5** There exist known continuous functions $\rho_i(\cdot)$ with $\rho_i(0) = 0$ and $\gamma_i(\cdot)$ such that for $x_i \in \Omega_i$

(i) $\|\Delta\Psi_i(x_i)\| \leq \rho_i(y_i)$,
(ii) $\|\Delta H_i(x)\| \leq \gamma_i(x)\|x\|$,

for $i = 1, 2, \ldots, N$.

**Assumption 5.6** The matrix pairs $(A, B)$ and $(A, C)$ are, respectively, stabilisable and detectable.

System (5.46)–(5.47) possessing STP $(E_i, D_i)$ implies that its nominal subsystems are all equivalent via static output feedback. It should be noted that Assumption 5.6 does not imply the realisation $(A, B, C)$ is output feedback stabilisable. In fact, it is not required that any nominal subsystem of (5.46)–(5.47) is output feedback stabilisable in this work: this is in comparison with the previous work [23].

Under the transformation

$$x_i = D_i\overline{x}_i$$

and the feedback

$$u_i = E_iC_ix_i + v_i, \qquad i = 1, 2, \ldots, N$$

System (5.46)–(5.47) in the new coordinates $\overline{x} =: \mathrm{col}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N)$ is described by

$$\dot{\overline{x}}_i = A\overline{x}_i + D_i^{-1} f_i(D_i\overline{x}_i) + B\,[v_i + \Delta\Psi_i(D_i\overline{x}_i)] + D_i^{-1}\sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}(D_j\overline{x}_j)$$
$$+D_i^{-1}\Delta H_i(D\overline{x}), \tag{5.52}$$
$$y_i = C\overline{x}_i, \qquad i = 1, 2, \ldots, N, \tag{5.53}$$

where $D = \mathrm{diag}\{D_1, D_2, \ldots, D_N\}$, and $C$ is of full row rank as is each of the $C_i$.

Under Assumption 5.6, it is observed that there exist matrices $K$ and $L$ such that, for any $Q > 0$ and $S > 0$, the following Lyapunov equations

$$(A - BK)^{\tau} P + P(A - BK) = -Q \tag{5.54}$$
$$(A - LC)^{\tau} R + R(A - LC) = -S \tag{5.55}$$

have unique solutions $P > 0$ and $R > 0$, respectively.

**Assumption 5.7** There exist matrices $F_1$ and $F_2$ with appropriate dimensions such that

(i) $B^{\tau} P = F_1 C$;
(ii) $B^{\tau} R = F_2 C$,

where $F_1$ is nonsingular.

*Remark 5.14* It should be noted that similar conditions as Assumption 5.7 have also been imposed in [24, 29, 214]. Note that if there exists a matrix $K$ such that the system $(A - BK, B, C)$ is passive, then Assumption 5.7 (i) is satisfied by $F_1 = I$ [68]. Similarly, if $(A - LC, B, C)$ is passive, then Assumption 5.7 (ii) is satisfied by $F_2 = I$.

**Assumption 5.8** The nonlinear function $f_i(x_i)$ of the $i$-th isolated subsystems is Lipschitz in its definition domain with Lipschitz constant $\mathscr{L}_{f_i}$, and satisfies

$$\|f_i(x_i)\| \leq \beta_i(\|y_i\|),$$

where the bounding function $\beta_i(\cdot)$ with $\beta_i(0) = 0$ is known continuous and differentiable at the origin for $i = 1, 2, \ldots, N$.

From $\beta_i(0) = 0$ and the differentiability of $\beta_i(\cdot)$ at the origin, it is observed that there exists a continuous function $\xi_i(\cdot)$ such that for $i = 1, 2, \ldots, N$

$$\beta_i(\|y_i\|) = \xi_i(\|y_i\|)\|y_i\|, \tag{5.56}$$

In fact, $\xi_i(\cdot)$ may be chosen as

$$\xi_i(\tau) = \begin{cases} \dfrac{d\beta}{d\tau}(0), & \tau = 0 \\ \dfrac{1}{\tau}\beta(\tau), & \tau \neq 0 \end{cases}$$

for $i = 1, 2, \ldots, N$.

### 5.3.3 Preliminaries

In this section, some preliminaries are presented which are used in the later development.

Consider the system (5.52)–(5.53). Since $C$ is of full row rank, without loss of generality, it is assumed that

$$C = [I_m \;\; 0]$$

with $I_m$ the identity matrix of dimension $m \times m$. Next, introduce the following decompositions conformable with the matrix $C$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \tag{5.57}$$

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \qquad L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tag{5.58}$$

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\tau & P_3 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^\tau & Q_3 \end{bmatrix}, \tag{5.59}$$

$$R = \begin{bmatrix} R_1 & R_2 \\ R_2^\tau & R_3 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & S_2 \\ S_2^\tau & S_3 \end{bmatrix}, \tag{5.60}$$

where $P$, $Q$, $R$, $S$, $K$ and $L$ are given as in (5.54)–(5.55), and $A_{11}$, $K_1$, $L_1$, $P_1$, $Q_1$, $R_1$, $S_1 \in \mathbb{R}^{m\times m}$. Then, it is observed that System (5.52)–(5.53) can be rewritten as

$$\begin{aligned} \dot{\overline{x}}_i^I &= A_{11}\overline{x}_i^I + A_{12}\overline{x}_i^{II} + \overline{f}_i^I(\overline{x}_i) + B_{11}(v_i + \Delta\Psi_i(D_i\overline{x}_i)) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \overline{H}_{ij}^I(\overline{x}_j) \\ &\quad + \Delta\overline{H}_i^I(\overline{x}), \end{aligned} \tag{5.61}$$

$$\dot{\overline{x}}_i^{II} = A_{21}\overline{x}_i^I + A_{22}\overline{x}_i^{II} + \overline{f}_i^{II}(\overline{x}_i) + B_{21}(v_i + \Delta\Psi_i(D_i\overline{x}_i)) + \sum_{\substack{j=1\\ j\neq i}}^{N} \overline{H}_{ij}^{II}(\overline{x}_j)$$
$$+\Delta\overline{H}_i^{II}(\overline{x}), \tag{5.62}$$
$$y_i = \overline{x}_i^I, \qquad i = 1, 2, \ldots, N, \tag{5.63}$$

where $\overline{x}_i = \mathrm{col}(\overline{x}_i^I, \overline{x}_i^{II})$ with $\overline{x}_i^I \in \mathbb{R}^m$ and $\overline{x}_i^{II} \in \mathbb{R}^{n-m}$, while $\overline{f}_i^I(\overline{x}_i)$ and $\overline{f}_i^{II}(\overline{x}_i)$, $\overline{H}_{ij}^I(\overline{x}_j)$ and $\overline{H}_{ij}^{II}(\overline{x}_j)$, and $\Delta\overline{H}_i^I(\overline{x})$ and $\Delta\overline{H}_i^{II}(\overline{x})$ are the first $m$ and last $n-m$ components of $D_i^{-1}f_i(D_i\overline{x}_i)$, $D_i^{-1}H_{ij}(D_j\overline{x}_j)$ and $\Delta H_i(\mathrm{diag}\{D_1, D_2, \ldots, D_N\}\overline{x})$, respectively.

**Lemma 5.1** *Assume that there exist matrices L such that the Lyapunov equation (5.55) is solvable. Then,*

$$(A_{22} + R_3^{-1}R_2^\tau A_{12})^\tau R_3 + R_3(A_{22} + R_3^{-1}R_2^\tau A_{12}) = -S_3,$$

*where $A_{12}$, $A_{22}$, $S_3$, and $R_i$ with $i = 2, 3$ are defined by (5.57)–(5.60).*

*Proof* By exploiting the block matrix representation (5.57)–(5.60), it is observed that (5.55) can be rewritten as

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} I_m & 0 \end{bmatrix}\right)^\tau \begin{bmatrix} R_1 & R_2 \\ R_2^\tau & R_3 \end{bmatrix} + \begin{bmatrix} R_1 & R_2 \\ R_2^\tau & R_3 \end{bmatrix}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} I_m & 0 \end{bmatrix}\right)$$
$$= -\begin{bmatrix} S_1 & S_2 \\ S_2^\tau & S_3 \end{bmatrix}.$$

and therefore,

$$A_{12}^\tau R_2 + A_{22}^\tau R_3 + R_2^\tau A_{12} + R_3 A_{22} = -S_3.$$

It follows that

$$(A_{12}^\tau R_2 R_3^{-1} + A_{22}^\tau)R_3 + R_3(R_3^{-1}R_2^\tau A_{12} + A_{22}) = -S_3.$$

Then, by noting that $R_3^{-\tau} = R_3^{-1}$, the result follows. #

Next, introduce the nonsingular coordinate transformation

$$z_i =: \begin{bmatrix} z_i^I \\ z_i^{II} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ R_3^{-1}R_2^\tau & I_{n-m} \end{bmatrix}\begin{bmatrix} \overline{x}_i^I \\ \overline{x}_i^{II} \end{bmatrix} =: \mathscr{E}\overline{x}_i = \mathscr{E}D_i^{-1}x_i =: T_i x_i. \tag{5.64}$$

From Assumption 5.7, it follows that

$$R_2^\tau B_{11} + R_3 B_{21} = 0, \qquad R_2 B_{21} + R_1^\tau B_{11} = F_2^\tau. \tag{5.65}$$

Then, from Assumptions 5.6 and 5.7, it follows that in the new coordinates $z = \mathrm{col}(z_1, z_2, \ldots, z_N)$, the system (5.52)–(5.53) or the system (5.61)–(5.63) is described by

$$\dot{z}_i = \widetilde{A} z_i + \mathscr{E} D_i^{-1} f_i(T_i^{-1} z_i) + \widetilde{B}\left[v_i + \Delta\Psi_i(T_i^{-1} z_i)\right] + \mathscr{E} D_i^{-1} \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}(T_j^{-1} z_j)$$

$$+ \mathscr{E} D_i^{-1} \Delta H_i(T^{-1} z), \tag{5.66}$$

$$y_i = \widetilde{C} z_i, \qquad i = 1, 2, \ldots, N, \tag{5.67}$$

where $z_i = \mathrm{col}(z_i^I, z_i^{II}) \in \overline{\Omega}_i =: \left\{T_i^{-1} x_i \mid x_i \in \Omega_i\right\}$ for $i = 1, 2, \ldots, N$, $T = \mathrm{diag}\{T_1,\ T_2,\ \ldots,\ T_N\}$, and

$$\widetilde{A} = \begin{bmatrix} A_{11} - A_{12} R_3^{-1} R_2^\tau & A_{12} \\ R_3^{-1} R_2^\tau A_{11} + A_{21} - (R_3^{-1} R_2^{-\tau} A_{12} + A_{22}) R_3^{-1} R_2^\tau & R_3^{-1} R_2^\tau A_{12} + A_{22} \end{bmatrix} \tag{5.68}$$

$$\widetilde{B} = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} I_m & 0 \end{bmatrix}. \tag{5.69}$$

**Lemma 5.2** *If the Lyapunov equation (5.54) is satisfied, then*

$$(\widetilde{A} - \widetilde{B} K \mathscr{E}^{-1})^\tau \mathscr{E}^{-\tau} P \mathscr{E}^{-1} + \mathscr{E}^{-\tau} P \mathscr{E}^{-1} (\widetilde{A} - \widetilde{B} K \mathscr{E}^{-1}) = -\mathscr{E}^{-\tau} Q \mathscr{E}^{-1},$$

*where $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are defined by (5.68)–(5.69), and $\mathscr{E}$ is as given in (5.64).*

*Proof* From the relationship between the system (5.52)–(5.53) and the system (5.66)–(5.67), it is observed that

$$\widetilde{A} = \mathscr{E} A \mathscr{E}^{-1}, \qquad \widetilde{B} = \mathscr{E} B, \qquad \widetilde{C} = C \mathscr{E}^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix}, \tag{5.70}$$

Then, from (5.54), it follows that

$$\begin{aligned} &(\widetilde{A} - \widetilde{B} K \mathscr{E}^{-1})^\tau \mathscr{E}^{-\tau} P \mathscr{E}^{-1} + \mathscr{E}^{-\tau} P \mathscr{E}^{-1} (\widetilde{A} - \widetilde{B} K \mathscr{E}^{-1}) \\ &= (\mathscr{E} A \mathscr{E}^{-1} - \mathscr{E} B K \mathscr{E}^{-1})^\tau \mathscr{E}^{-\tau} P \mathscr{E}^{-1} + \mathscr{E}^{-\tau} P \mathscr{E}^{-1} (\mathscr{E} A \mathscr{E}^{-1} - \mathscr{E} B K \mathscr{E}^{-1}) \\ &= \mathscr{E}^{-\tau} (A - BK)^{-\tau} P \mathscr{E}^{-1} + \mathscr{E}^{-\tau} P (A - BK) \mathscr{E}^{-1} \\ &= -\mathscr{E}^{-\tau} Q \mathscr{E}^{-1}. \end{aligned} \tag{5.71}$$

Hence the result follows. #

Some of the characteristics of the considered similar structure have been shown above. With the preliminaries provided in this section, it is now possible to consider the controller design and present the main results.

### 5.3.4 Reduced-Order Controller Design

In this section, a reduced-order output feedback control scheme is presented for the interconnected similar system.

First, observe that the system (5.66)–(5.67) can be rewritten as

$$\begin{aligned}
\dot{z}_i^I &= \Big[A_{11} - A_{12}R_3^{-1}R_2^\tau\Big]z_i^I + A_{12}z_i^{II} + \overline{f}_i^I(\mathscr{E}z_i) + B_{11}(v_i + \Delta\Psi_i(T_i^{-1}z_i)) \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}^I(z_j) + \Delta M_i^I(z), \qquad (5.72) \\
\dot{z}_i^{II} &= \Big[R_3^{-1}R_2^\tau(A_{11} - A_{12}R_3^{-1}R_2^\tau) + A_{21} - A_{22}R_3^{-1}R_2^\tau\Big]z_i^I + (A_{22} + R_3^{-1}R_2^\tau A_{12})z_i^{II} \\
&\quad + \big[\, R_3^{-1}R_2^\tau \;\; I_{n-m} \,\big] D_i^{-1} f_i(T_i^{-1}z_i) + \sum_{\substack{j=1\\ j\neq i}}^{N}\Big[R_3^{-1}R_2^\tau M_{ij}^I(z_j) + M_{ij}^{II}(z_j)\Big] \\
&\quad + R_3^{-1}R_2^\tau \Delta M_i^I(z) + \Delta M_i^{II}(z), \qquad (5.73) \\
y_i &= z_i^I, \qquad i = 1, 2, \ldots, N, \qquad (5.74)
\end{aligned}$$

where $\overline{f}_i^I(\mathscr{E}z_i)$ denotes the first $m$ components of $D_i^{-1} f_i(T_i^{-1}z_i)$ and

$$M_{ij}(z_j) := \begin{bmatrix} M_{ij}^I(z_j) \\ M_{ij}^{II}(z_j) \end{bmatrix} \equiv T_i H_{ij}(T_j^{-1}z_j), \qquad (5.75)$$

$$\Delta M_i(z) := \begin{bmatrix} \Delta M_i^I(z) \\ \Delta M_i^{II}(z) \end{bmatrix} \equiv T_i \Delta H_i(T^{-1}z) \qquad (5.76)$$

with $M_{ij}^I(z_j)$ and $\Delta M_i^I(z)$, respectively, the first $m$ components of $M_{ij}(z_j)$ and $\Delta M_i(z)$.

Now, for the system (5.66)–(5.67) or (5.72)–(5.74), construct a reduced-order controller described by

$$\begin{aligned}
\dot{\hat{z}}_i^{II} &= (A_{22} + R_3^{-1}R_2^\tau A_{12})\hat{z}_i^{II} + \Big[R_3^{-1}R_2^\tau(A_{11} - A_{12}R_3^{-1}R_2^\tau) + A_{21} - A_{22}R_3^{-1}R_2^\tau\Big]y_i \\
&\quad + \big[\, R_3^{-1}R_2^\tau \;\; I_{n-m} \,\big] D_i^{-1} f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{N}\Big[R_3^{-1}R_2^\tau M_{ij}^I(y_j, \hat{z}_j^{II}) + M_{ij}^{II}(y_j, \hat{z}_j^{II})\Big] \qquad (5.77)
\end{aligned}$$

$$v_i = -K\mathscr{E}^{-1}\begin{bmatrix} y_i \\ \hat{z}_i^{II} \end{bmatrix} + \mu_i(y_i) + \zeta_i(y_i), \qquad i = 1, 2, \ldots, N, \tag{5.78}$$

where $\hat{z}_i^{II} \in \mathbb{R}^{n-m}$, $R_i$ $(i = 1, 2, 3)$, $A_{ij}$ $(i, j = 1, 2)$ and $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ are defined in (5.57)–(5.60), and $\mu_i(y_i)$ and $\zeta_i(y_i)$ for $i = 1, 2, \ldots, N$ are defined by

$$\mu_i(y_i) = \begin{cases} -\dfrac{F_1 y_i}{\|F_1 y_i\|}\rho_i(y_i), & F_1 y_i \neq 0 \\ 0, & F_1 y_i = 0 \end{cases}, \tag{5.79}$$

$$\zeta_i(y_i) = -\frac{1}{2\varepsilon}\overline{\sigma}^2\left(PD_i^{-1}\right)\xi_i^2(\|y_i\|)F_1^{-\tau}y_i, \tag{5.80}$$

where $F_1$ satisfies Assumption 5.7, and $\varepsilon$ is a positive constant.

From the nonsingularity of $F_1$, it is known that $F_1 y_i = 0$ if and only if $y_i = 0$ for $i = 1, 2, \ldots, N$. Then, $\mu_i(y_i)$ is continuous in its domain of definition due to

$$0 \leq |\mu_i(y_i)| \leq \rho_i(y_i)$$

with $\rho_i(0) = 0$ and the continuity of $\rho_i$ for $i = 1, 2, \ldots, N$. It follows that the dynamical output feedback controller (5.77)–(5.78) is continuous.

*Remark 5.15* The system (5.61)–(5.63) is transformed to the system (5.72)–(5.74) through the transformation

$$z_i = \mathscr{E}\overline{x}_i.$$

It should be noted that one of the important differences between the two systems is that the control variable $v_i$ disappears in (5.73). This greatly simplifies the control design since $v_i$ no longer appears in the dynamical error equation and does not affect the dynamic part (5.77) of the controller (5.77)–(5.78). Thus it is possible to focus on designing the control $v_i$ to reduce or even cancel the effects of uncertainties. □

The main result can now be presented.

**Theorem 5.4** *Suppose that system (5.46)–(5.47) possesses a similar structure with STP $(E_i, D_i)$ for $i = 1, 2, \ldots, N$. Then, under Assumptions 5.4–5.8, there exists a continuous reduced-order controller to stabilise the system (5.46)–(5.47) if there exist a positive constant $\varepsilon$ and a neighbourhood $\Omega'$ of the origin in $\Omega$ such that $W^\tau + W > 0$ in $\Omega' \setminus \{0\}$, where $W = [w_{ij}]_{2N\times 2N}$ is defined by*

$$w_{ij} = \begin{cases} \underline{\lambda}(Q) - \varepsilon - 2\overline{\sigma}(PD_i^{-1})\overline{\sigma}(D_i)\gamma_i, & 1 \le i \le N, \quad i = j \\ \underline{\lambda}(S_3) - 2\overline{\sigma}(D_i)\overline{\sigma}(\begin{bmatrix} R_2^\tau & R_3 \end{bmatrix} D_i^{-1})\mathscr{L}_{f_i}, & N+1 \le i \le 2N, \quad i = j \\ -2\overline{\lambda}(P)\overline{\sigma}(\Gamma ij) - 2\overline{\sigma}(PD_j^{-1})\overline{\sigma}(D_j)\gamma_i, & 1 \le i, j \le N, \quad j \ne i \\ -2\left[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\right]\mathscr{L}_{T_{i-N}H_{(i-N)(j-N)}}, & N+1 \le i, j \le 2N, \\ & j \ne i \\ -2\overline{\sigma}(PBK_2) - 2[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)]\overline{\sigma}(T_i)\overline{\sigma}(D_i)\gamma_i, & j = N+i, \quad 1 \le i \le N \\ -2\left[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\right]\overline{\sigma}(T_{j-N})\overline{\sigma}(D_i)\gamma_{j-N} & j \ne N+i, \quad 1 \le i \le N \\ & N \le j \le 2N \\ -2\overline{\sigma}(PBK_2) - 2[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)]\overline{\sigma}(T_i)\overline{\sigma}(D_j)\gamma_j, & i = N+j, \quad 1 \le j \le N \\ -2\left[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\right]\overline{\sigma}(T_j)\overline{\sigma}(D_{i-N})\gamma_{i-N} & i \ne N+j, \quad 1 \le j \le N, \\ & N \le i \le 2N \end{cases}$$

*with $P$, $Q$, $R_2$, $R_3$ and $S_3$ as given in (5.57)–(5.60).*

*Proof* From the analysis above, it is observed that the system (5.46)–(5.47) is equivalent to the system (5.66)–(5.67) through feedback

$$u_i = E_i C_i x_i + v_i$$

and the nonsingular transformation

$$z_i = T_i x_i$$

for $i = 1, 2, \ldots, N$. Therefore, it is only required to prove that System (5.66)–(5.67) is stabilisable using a reduced-order control.

By applying (5.77)–(5.78) to the system (5.66)–(5.67), the following closed-loop system is obtained

$$\dot{z}_i = \widetilde{A} z_i + \mathscr{E} D_i^{-1} f_i(T_i^{-1} z_i) + \widetilde{B}\Big[ - K\mathscr{E}^{-1} \begin{pmatrix} y_i \\ \hat{z}_i^{II} \end{pmatrix} + \mu_i(y_i) + \zeta_i(y_i) + \Delta\Psi_i(T_i^{-1} z_i)\Big] + \sum_{\substack{j=1 \\ j \ne i}}^{N} \mathscr{E} D_i^{-1} H_{ij}(T_j^{-1} z_j) + \mathscr{E} D_i^{-1} \Delta H_i(T^{-1} z) \tag{5.81}$$

$$\dot{\hat{z}}_i^{II} = (A_{22} + R_3^{-1} R_2^\tau A_{12})\hat{z}_i^{II} + \Big[R_3^{-1} R_2^\tau (A_{11} - A_{12} R_3^{-1} R_2^\tau) + A_{21} - A_{22} R_3^{-1} R_2^\tau\Big] y_i + \Big[ R_3^{-1} R_2^\tau \;\; I_{n-m} \Big] D_i^{-1} f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) + \sum_{\substack{j=1 \\ j \ne i}}^{N} \Big[R_3^{-1} R_2^\tau M_{ij}^I(y_j, \hat{z}_j^{II}) + M_{ij}^{II}(y_j, \hat{z}_j^{II})\Big] \tag{5.82}$$

$$y_i = z_i^I, \qquad i = 1, 2, \ldots, N. \tag{5.83}$$

For the system (5.81)–(5.83), consider the Lyapunov function candidate:

$$\begin{aligned} & V(z_1, z_2, \ldots, z_N, \hat{z}_1^{II}, \hat{z}_2^{II}, \ldots, \hat{z}_N^{II}) \\ &= \textstyle\sum_{i=1}^N z_i^\tau \mathscr{E}^{-\tau} P \mathscr{E}^{-1} z_i + \sum_{i=1}^N (z_i^{II} - \hat{z}_i^{II})^\tau R_3 (z_i^{II} - \hat{z}_i^{II}), \end{aligned}$$

where $P$ and $R_3$ are given, respectively, by (5.54) and (5.60).

Let $e_i = z_i^{II} - \hat{z}_i^{II}$. Noticing that $y_i = z_i^I$, it is easy to see from (5.73) and (5.82) that

$$\begin{aligned} \dot{e}_i &= (A_{22} + R_3^{-1} R_2^\tau A_{12}) e_i + \left[\, R_3^{-1} R_2^\tau \;\; I_{n-m} \,\right] D_i^{-1} \left( f_i(T_i^{-1} z_i) - f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) \right) + \\ & \sum_{\substack{j=1 \\ j \neq i}}^N \left\{ \left[ R_3^{-1} R_2^\tau M_{ij}^I(y_j, z_j^{II}) + M_{ij}^{II}(y_j, z_j^{II}) \right] - \left[ R_3^{-1} R_2^\tau M_{ij}^I(y_j, \hat{z}_j^{II}) + M_{ij}^{II}(y_j, \hat{z}_j^{II}) \right] \right\} \\ & + R_3^{-1} R_2^\tau \Delta M_i^I(z) + \Delta M_i^{II}(z). \end{aligned} \tag{5.84}$$

In addition, it follows from (5.70) and $y_i = z_i^I$ that

$$\mathscr{E}^{-\tau} P \mathscr{E}^{-1} \widetilde{B} K \mathscr{E}^{-1} \begin{bmatrix} y_i \\ \hat{z}_i^{II} \end{bmatrix} \tag{5.85}$$

$$= \mathscr{E}^{-\tau} P B K \mathscr{E}^{-1} \begin{bmatrix} y_i \\ z_i^{II} \end{bmatrix} + \mathscr{E}^{-\tau} P B K \mathscr{E}^{-1} \left\{ \begin{bmatrix} y_i \\ \hat{z}_i^{II} \end{bmatrix} - \begin{bmatrix} z_i^I \\ z_i^{II} \end{bmatrix} \right\}$$

$$= \mathscr{E}^{-\tau} P B K \mathscr{E}^{-1} z_i + \mathscr{E}^{-\tau} P B K_2 e_i. \tag{5.86}$$

Therefore, the time derivative of $V$ along the trajectories of System (5.81)–(5.82) is given by

$$\begin{aligned} & \dot{V}\,|_{(5.81)-(5.82)} \\ &= -\sum_{i=1}^N z_i^\tau \mathscr{E}^{-\tau} Q \mathscr{E}^{-1} z_i + 2 \sum_{i=1}^N z_i^\tau \mathscr{E}^{-\tau} P B \left[ K_2 e_i + \Delta\Psi_i(T_i^{-1} z_i) + \mu_i(y_i) \right] \\ & + 2 \sum_{i=1}^N z_i^\tau \mathscr{E}^{-\tau} P \left[ D_i^{-1} f_i(T_i^{-1} z_i) + B \zeta_i(y_i) \right] \\ & + 2 \sum_{i=1}^N z_i^\tau \mathscr{E}^{-\tau} P D_i^{-1} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^N H_{ij}(T_j^{-1} z_j) + \Delta H_i(T^{-1} z) \right\} \\ & + 2 \sum_{i=1}^N e_i^\tau \left[\, R_2^\tau \;\; R_3 \,\right] D_i^{-1} \left( f_i(T_i^{-1} z_i) - f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) \right) - \sum_{i=1}^N e_i^\tau S_3 e_i + \end{aligned}$$

$$2\sum_{i=1}^{N} e_i^\tau R_3 \sum_{\substack{j=1\\ j\neq i}}^{N}\left\{R_3^{-1}R_2^\tau\left(M_{ij}^I(y_j, z_j^{II}) - M_{ij}^I(y_j, \hat{z}_j^{II})\right) + \left[M_{ij}^{II}(y_j, z_j^{II}) - M_{ij}^{II}(y_j, \hat{z}_j^{II})\right]\right\}$$

$$+2\sum_{i=1}^{N} e_i^\tau R_3\left[R_3^{-1}R_2^\tau \Delta M_i^I(z) + \Delta M_i^{II}(z)\right], \tag{5.87}$$

where Lemmas 5.1, 5.2 and (5.86) are employed in the implication above.

From the structure (5.79) of $\mu_i(y_i)$ and Assumption 5.7, it is observed that

(i) if $F_1 y_i = 0$, then, for $i = 1, 2, \ldots, N$

$$\begin{aligned} z_i^\tau \mathscr{E}^{-\tau} PB\left[\Delta\Psi_i(T_i^{-1}z_i) + \mu_i(y_i)\right] &= (F_1 C\mathscr{E}^{-1}z_i)^\tau \Delta\Psi_i(T_i^{-1}z_i) \\ &= (F_1 y_i)^\tau \Delta\Psi_i(T_i^{-1}z_i) \\ &= 0 \end{aligned}$$

(ii) if $F_1 y_i \neq 0$, then, by Assumption 5.5 it follows that for $i = 1, 2, \ldots, N$

$$\begin{aligned} & z_i^\tau \mathscr{E}^{-\tau} PB\left[\Delta\Psi_i(T_i^{-1}z_i) + \mu_i(y_i)\right] \\ &= (F_1 C\mathscr{E}^{-1}z_i)^\tau \Delta\Psi_i(T_i^{-1}z_i) - (F_1 C\mathscr{E}^{-1}z_i)^\tau \frac{F_1 y_i}{\|F_1 y_i\|}\rho_i(y_i) \\ &\leq \|F_1 y_i\|\rho_i(y_i) - \frac{(F_1 y_i)^\tau F_1 y_i}{\|F_1 y_i\|}\rho_i(y_i) \\ &= 0. \end{aligned}$$

Therefore, for $i = 1, 2, \ldots, N$,

$$z_i^\tau \mathscr{E}^{-\tau} PB[K_2 e_i + \Delta\Psi_i(T_i^{-1}z_i) + \mu_i(y_i)] \leq \overline{\sigma}(PBK_2)\|\mathscr{E}^{-1}z_i\|\,\|e_i\|. \tag{5.88}$$

From (5.56), (5.80), the inequality $2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$ with $\varepsilon > 0$, and Assumptions 5.7 and 5.8, it follows that

$$\begin{aligned} & z_i^\tau \mathscr{E}^{-\tau} PD_i^{-1} f_i(T_i^{-1}z_i) + z_i^\tau \mathscr{E}^{-\tau} PB\zeta_i(y_i) \\ &= (\mathscr{E}^{-1}z_i)^\tau PD_i^{-1} f_i(T_i^{-1}z_i) + (\mathscr{E}^{-1}z_i)^\tau C^\tau F_1^\tau \zeta_i(y_i) \\ &\leq \|\mathscr{E}^{-1}z_i\|\overline{\sigma}(PD_i^{-1})\beta_i(\|y_i\|) + (C\mathscr{E}^{-1}z_i)^\tau F_1^\tau \zeta_i(y_i) \\ &\leq \frac{1}{2}\varepsilon\|\mathscr{E}^{-1}z_i\|^2 + \frac{1}{2\varepsilon}\overline{\sigma}^2\left(PD_i^{-1}\right)\beta_i^2(\|y_i\|) - y_i^\tau F_1^\tau\left(\frac{1}{2\varepsilon}F_1^{-\tau}y_i\overline{\sigma}^2(PD_i^{-1})\xi_i^2(\|y_i\|)\right) \\ &= \frac{1}{2}\varepsilon\|\mathscr{E}^{-1}z_i\|^2 + \frac{1}{2\varepsilon}\overline{\sigma}^2\left(PD_i^{-1}\right)\xi_i^2(\|y_i\|)\|y_i\|^2 - \frac{1}{2\varepsilon}y_i^\tau y_i\overline{\sigma}^2\left(PD_i^{-1}\right)\xi_i^2(\|y_i\|) \\ &= \frac{1}{2}\varepsilon\|\mathscr{E}^{-1}z_i\|^2, \end{aligned} \tag{5.89}$$

where $\varepsilon$ is a positive constant.

From Assumptions 5.5 and 5.6,

$$
\begin{aligned}
& z_i^\tau \mathscr{E}^{-\tau} P D_i^{-1} \left( \sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(T_j^{-1} z_j) + \Delta H_i(T^{-1} z) \right) \\
&= (\mathscr{E}^{-1} z_i)^\tau \left( \sum_{\substack{j=1 \\ j \neq i}}^{N} P D_i^{-1} \Gamma_{ij}(T_j^{-1} z_j) D_j \mathscr{E}^{-1} z_j + P D_i^{-1} \Delta H_i(T^{-1} z) \right) \\
&\leq \|\mathscr{E}^{-1} z_i\| \left( \sum_{\substack{j=1 \\ j \neq i}}^{N} \overline{\sigma}(P D_i^{-1} \Gamma_{ij}(T_j^{-1} z_j) D_j) \|\mathscr{E}^{-1} z_j\| + \overline{\sigma}(P D_i^{-1}) \gamma_i(x) \|T^{-1} z\| \right) \\
&\leq \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[ \overline{\lambda}(P) \overline{\sigma}(\Gamma_{ij}) + \overline{\sigma}(P D_j^{-1}) \overline{\sigma}(D_j) \gamma_i(x) \right] \|\mathscr{E}^{-1} z_i\| \, \|\mathscr{E}^{-1} z_j\| \\
&\quad + \overline{\sigma}(P D_i^{-1}) \overline{\sigma}(D_i) \gamma_i(x) \|\mathscr{E}^{-1} z_i\|^2, \qquad (5.90)
\end{aligned}
$$

where the inequality

$$
\|T^{-1} z\| \leq \sum_{j=1}^{N} \overline{\sigma}(D_j) \|\mathscr{E}^{-1} z_j\|
$$

is used in the last implication. In addition, from Assumption 5.7, it follows that

$$
\begin{aligned}
e_i^\tau \left[ R_2^\tau \;\; R_3 \right] D_i^{-1} \left( f_i(T_i^{-1} z_i) - f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) \right) \\
\leq \|e_i\| \overline{\sigma} \left( \left[ R_2^\tau \;\; R_3 \right] D_i^{-1} \right) \mathscr{L}_{f_i} \overline{\sigma}(D_i) \left\| \begin{bmatrix} 0 \\ z_i^{II} - \hat{z}_i^{II} \end{bmatrix} \right\| \\
= \overline{\sigma}(D_i) \overline{\sigma} \left( \left[ R_2^\tau \;\; R_3 \right] D_i^{-1} \right) \mathscr{L}_{f_i} \|e_i\|^2. \qquad (5.91)
\end{aligned}
$$

As $M_{ij}(z_j)$ is Lipschitz in $\overline{\Omega}_j$, it follows that for $i = 1, 2, \ldots, N$,

$$
\begin{aligned}
& e_i^\tau R_3 \sum_{\substack{j=1 \\ j \neq i}}^{N} \left\{ R_3^{-1} R_2^\tau \left( M_{ij}^I(y_i, z_j^{II}) - M_{ij}^I(y_i, \hat{z}_j^{II}) \right) + \left( M_{ij}^{II}(y_i, z_j^{II}) - M_{ij}^{II}(y_i, \hat{z}_j^{II}) \right) \right\} \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[ e_i^\tau R_2^\tau \left( M_{ij}^I(y_i, z_j^{II}) - M_{ij}^I(y_i, \hat{z}_j^{II}) \right) + e_i^\tau R_3 \left( M_{ij}^{II}(y_i, z_j^{II}) - M_{ij}^{II}(y_i, \hat{z}_j^{II}) \right) \right] \\
&\leq \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[ \overline{\sigma}(R_2) \mathscr{L}_{M_{ij}^I} \|z_j^{II} - \hat{z}_j^{II}\| \, \|e_i\| + \overline{\lambda}(R_3) \mathscr{L}_{M_{ij}^{II}} \|z_j^{II} - \hat{z}_j^{II}\| \, \|e_i\| \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{j=1\\ j\neq i}}^{N} \left[\overline{\sigma}(R_2)\mathscr{L}_{M_{ij}^{I}} + \overline{\lambda}(R_3)\mathscr{L}_{M_{ij}^{II}}\right] \|e_i\|\,\|e_j\| \\
&\le \sum_{\substack{j=1\\ j\neq i}}^{N} \left[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\right] \mathscr{L}_{T_i H_{ij}} \|e_i\|\,\|e_j\|. \qquad (5.92)
\end{aligned}
$$

From Assumption 5.5, it follows that for $i = 1, 2, \ldots, N$,

$$
\begin{aligned}
e_i^{\tau} R_3 &\Big[ R_3^{-1} R_2^{\tau} \Delta M_i^{I}(z) + \Delta M_i^{II}(z) \Big] \\
&= e_i^{\tau} R_2^{\tau} \Delta M_i^{I}(z) + e_i^{\tau} R_3 \Delta M_i^{II}(z) \\
&\le \Big[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\Big] \|\Delta M_i(z)\|\,\|e_i\| \\
&= \Big[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\Big] \|T_i \Delta H_i(T^{-1}z)\|\,\|e_i\| \\
&\le \sum_{j=1}^{N} \Big[\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\Big] \overline{\sigma}(T_i)\overline{\sigma}(D_j)\gamma_i(x) \|\mathscr{E}^{-1} z_j\|\,\|e_i\|. \qquad (5.93)
\end{aligned}
$$

Substituting (5.88)–(5.93) into (5.87) yields

$$
\begin{aligned}
&\dot{V}\ |_{(5.77)-(5.78)} \\
&\le -\sum_{i=1}^{N} \Big[ \Big(\underline{\lambda}(Q) - \varepsilon - 2\overline{\sigma}(P D_i^{-1})\overline{\sigma}(D_i)\gamma_i(x)\Big) \|\mathscr{E}^{-1} z_i\|^2 + \underline{\lambda}(S_3)\|e_i\|^2 \Big] \\
&\quad +2\sum_{i=1}^{N} \Big[ \overline{\sigma}(PBK_2) + \big(\overline{\sigma}(R_2) + \overline{\lambda}(R_3)\big)\, \overline{\sigma}(T_i)\overline{\sigma}(D_i)\gamma_i(x) \Big] \|\mathscr{E}^{-1} z_i\|\,\|e_i\| \\
&\quad +2\sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} \Big\{ [\overline{\lambda}(P)\overline{\sigma}(\Gamma_{ij}) + \overline{\sigma}(P D_j^{-1})\overline{\sigma}(D_j)\gamma_j(x)] \|\mathscr{E}^{-1} z_i\|\,\|\mathscr{E}^{-1} z_j\| \\
&\quad +2\sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} [\overline{\sigma}(R_2) + \lambda_{\max}(R_3)]\, \mathscr{L}_{T_i H_{ij}} \|e_i\|\,\|e_j\| \\
&\qquad +2\sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} [\overline{\sigma}(R_2) + \lambda_{\max}(R_3)] \overline{\sigma}(T_j)\overline{\sigma}(D_i)\gamma_j(x) \|\mathscr{E}^{-1} z_i\|\,\|e_j\| \\
&= -\frac{1}{2} Y^{\tau}(W^{\tau} + W) Y,
\end{aligned}
$$

where $Y = (\|\mathscr{E}^{-1}z_1\|, \|\mathscr{E}^{-1}z_2\|, \ldots, \|\mathscr{E}^{-1}z_N\|, \|e_1\|, \|e_2\|, \ldots, \|e_N\|)^\tau$. From the positive definitiveness of $W^\tau + W$, it follows that System (5.72)–(5.74) is asymptotically stable. Hence the result follows. #

*Remark 5.16* It should be noted that in the case of dynamical output feedback control, the effect of uncertainty, even matched uncertainty, is difficult to cancel completely. In this section, by exploiting Assumption 5.7 and the transformation (5.64), the effect of the matched uncertainties in isolated subsystems is cancelled completely in the proposed control design. Thus the bounding functions of the uncertainties appearing in isolated subsystems are allowed to be arbitrarily large. This greatly reduces the conservatism of the control design and improves the robustness of the designed system. □

From the relationship between state variables $x$ and $z$, it is easily observed from Theorem 5.4 that under Assumptions 5.4–5.8, the continuous reduced-order dynamical output feedback control for the system (5.46)–(5.47) may be presented as follows:

$$\begin{aligned}
\dot{\hat{z}}_i^{II} &= (A_{22} + R_3^{-1}R_2^\tau A_{12})\hat{z}_i^{II} + \Big[R_3^{-1}R_2^\tau(A_{11} - A_{12}R_3^{-1}R_2^\tau) + A_{21} - A_{22}R_3^{-1}R_2^\tau\Big]y_i \\
&\quad + \left[\, R_3^{-1}R_2^\tau \;\; I_{n-m} \,\right] D_i^{-1} f_i(T_i^{-1}(y_i, \hat{z}_i^{II})) \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{N} \Big[R_3^{-1}R_2^\tau M_{ij}^{I}(y_j, \hat{z}_j^{II}) + M_{ij}^{II}(y_j, \hat{z}_j^{II})\Big] \qquad (5.94)\\
u_i &= E_i y_i - K\mathscr{E}^{-1}\begin{bmatrix} y_i \\ \hat{z}_i^{II}\end{bmatrix} + \mu_i(y_i) + \zeta_i(y_i), \qquad (5.95)\\
y_i &= C_i x_i, \qquad (5.96)
\end{aligned}$$

where $\mu_i$ and $\zeta_i$ are, respectively, defined by (5.79) and (5.80) for $i = 1, 2, \ldots, N$.

It should be noticed that Assumptions 5.7 and 5.8 will be redundant if there is no uncertainty in the isolated subsystems of (5.46)–(5.47) or if all uncertainty is treated as uncertain interconnections. In this case, a result which is easier to verify can be obtained but at the cost of more conservatism. For the sake of simplification of notation, consider the following system:

$$\begin{aligned}
\dot{z}_i &= Az_i + Bu_i + \sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}(z_j) + \Delta M_i(z), \qquad (5.97)\\
y_i &= Cz_i, \qquad i = 1, 2, \ldots, N, \qquad (5.98)
\end{aligned}$$

where $z_i \in \mathbb{R}^n$, $u_i, y_i \in \mathbb{R}^m$, the terms

$$\sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}(z_j) \qquad \text{and} \qquad \Delta M_i(z)$$

are the continuous known interconnections and uncertain interconnections, $A$ and $B$ have the same block decomposition as (5.57), and

$$C = \begin{bmatrix} I_m & 0 \end{bmatrix}.$$

Then, the system (5.97)–(5.98) may be rewritten as

$$\dot{z}_i^I = A_{11}z_i^I + A_{12}z_i^{II} + B_1 u_i + \sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}^I(z_j) + \Delta M_i^I(z), \tag{5.99}$$

$$\dot{z}_i^{II} = A_{21}z_i^I + A_{22}z_i^{II} + B_2 u_i + \sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}^{II}(z_j) + \Delta M_i^{II}(z), \tag{5.100}$$

$$y_i = z_i^I, \qquad i = 1, 2, \ldots, N, \tag{5.101}$$

Consider the system (5.99)–(5.101). Let $A_{22}$ be Hurwitz stable. Then, for any positive definite matrix $\bar{S} \in R^{(n-m)\times(n-m)}$, there exists a unique $\bar{R} > 0$ such that

$$A_{22}^{\tau}\bar{R} + \bar{R}A_{22} = -\bar{S}. \tag{5.102}$$

Then, the following result can be presented.

**Theorem 5.5** *Consider the system (5.97)–(5.98). Suppose that*

*(i)* $(A, B)$ *is detectable and* $A_{22}$ *is Hurwitz stable;*
*(ii)* $\|\Delta M_i\| \leq \gamma_i(z)\|z\|$ *with* $i = 1, 2, \ldots, N$ *for* $z \in \Omega$*;*
*(iii)* $M_{ij}^{II}(y_j, z_j^{II})$ *is Lipschitz about* $z_j^{II}$ *with Lipschitz constant* $\mathscr{L}_{M_{ij}^{II}}$ *and*

$$M_{ij}(z_j) = \Gamma_{ij}(z_j)z_j.$$

*Then, the system is stabilised by the reduced-order control*

$$\dot{\hat{z}}_i^{II} = A_{22}\hat{z}_i^{II} + A_{21}y_i + B_2 u_i + \sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}^{II}(y_j, \hat{z}_j^{II}), \tag{5.103}$$

$$u_i = -K_1 y_i - K_2\hat{z}_i^{II}, \qquad i = 1, 2, \ldots, N \tag{5.104}$$

*if there exists a positive constant* $\alpha$ *and a neighbourhood* $\Omega'$ *of the origin in* $\Omega$ *such that in* $\Omega' \setminus \{0\}$*,*

$$W_1 + W_1^{\tau} > 0 \qquad \text{and} \qquad W_2 + W_2^{\tau} > 0,$$

*where* $W_1 = (w_{ij}^1)_{N\times N}$, $W_2 = (w_{ij}^2)_{N\times N}$ *and for* $i, j = 1, 2, \ldots, N$

$$w_{ij}^1 = \begin{cases} \underline{\lambda}(Q) - \frac{1}{2\alpha}\lambda_{\max}(\overline{R})\gamma_i^2 - \lambda_{\max}(P)\gamma_i, & i = j \\ -\bar{\sigma}(P\Gamma_{ij}) - \lambda_{\max}(P)\gamma_i, & i \neq j \end{cases},$$
$$w_{ij}^2 = \begin{cases} \underline{\lambda}(\bar{S}) - \dfrac{1}{2}\alpha\lambda_{\max}(\overline{R}), & i = j \\ -\lambda_{\max}(\bar{R})\mathscr{L}_{M_{ij}^{II}}, & i \neq j \end{cases}$$

*with P, Q and* $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ *as given in (5.54).*

*Proof* Let $e_i = z_i^{II} - \hat{z}_i^{II}$ with $i = 1, 2, \ldots, N$. It follows from (5.100) and (5.103) that

$$\dot{e}_i = A_{22}e_i + \sum_{j=1}^{N}\left[M_{ij}^{II}(z_j) - M_{ij}^{II}(\hat{z}_j)\right] + \varDelta M_i^{II}(z).$$

Then, it is observed from conditions (ii) and (iii) that

$$\begin{aligned} z_i^\tau\ P & \left[\textstyle\sum_{\substack{j=1\\ j\neq i}}^{N} M_{ij}(z_j) + \varDelta M_i(z)\right] \\ & \leq \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N} \overline{\sigma}(P\Gamma_{ij}(z_j))\|z_i\|\,\|z_j\| + \sum_{j=1}^{N}\lambda_{\max}(P)\gamma_i(z)\|z_i\|\,\|z_j\| \end{aligned}$$

and

$$\begin{aligned} e_i^\tau\bar{R} & \Big[\textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\left(M_{ij}^{II}(y_j, z_j^{II}) - M_{ij}^{II}(y_j, \hat{z}_j^{II})\right) + \varDelta M_i^{II}(z)\Big] \\ & \leq \textstyle\sum_{\substack{j=1\\ j\neq i}}^{N}\lambda_{\max}(\bar{R})\mathscr{L}_{M_{ij}^{II}}\|e_i\|\,\|e_j\| + \frac{1}{2}\lambda_{\max}(\bar{R})\Big[\alpha\|e_i\|^2 + \frac{1}{\alpha}\gamma_i^2(z)\sum_{j=1}^{N}\|z_j\|^2\Big]. \end{aligned}$$

Now, for the closed-loop system resulting from (5.99)–(5.101) and (5.103)–(5.104), consider the Lyapunov function candidate

$$V = \sum_{i=1}^{N}\left(z_i^\tau P z_i + e_i^\tau \bar{R} e_i\right).$$

By similar arguments as those in Theorem 5.4, the result can be established directly. #

*Remark 5.17* It is worth noting that the proof of both Theorems 5.4 and 5.5 is constructive and gives explicit control design procedures. The conditions in Theorem 5.5 are easier to verify when compared with those of Theorem 5.4. But it is at the cost of more conservatism. It should be noted that the adjustable parameter $\alpha$ in Theorem 5.5 may be used to reduce the conservatism. □

## 5.4 Simulation Examples

In this section, the approaches proposed in Sects. 5.3 and 5.4 are illustrated by simulation. Case studies on the high incidence research model (HIRM) aircraft system and a coupled inverted pendula system are presented.

### 5.4.1 Case Study on HIRM Aircraft

Consider the dynamics of the HIRM aircraft at the trim values of Mach 0.2 and height 6000 ft taken from [115]. The system matrices are given by

$$A = \begin{bmatrix} -0.0836 & -4.9911 & -0.7190 & -9.8066 & 0.0981 & 0 & 0 & 0 \\ -0.0038 & -0.2788 & 0.9772 & 0 & 0 & 0 & 0 & 0 \\ -0.0030 & -0.5466 & -0.2960 & 0 & 0 & 0.0013 & -0.0007 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & -0.0013 \\ 0 & 0.0072 & 0 & 0 & -0.0113 & 0.4223 & -0.8991 & 0.1330 \\ 0 & 0.0861 & -0.0011 & 0 & -7.5964 & -0.3604 & 2.2864 & 0 \\ 0 & 0.3320 & 0.0005 & 0 & -0.2419 & -0.0313 & -0.1489 & 0 \\ 0 & 0 & 0 & 0.0016 & 0 & 1.0000 & 0.4732 & 0 \end{bmatrix} \tag{5.105}$$

$$B = \begin{bmatrix} -1.2536 & -2.4223 & 0 & 0 \\ 0.0008 & -0.0761 & 0 & 0 \\ 0.5247 & -1.4718 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -0.0006 & -0.0091 & 0.0175 \\ 0.0126 & 0.0498 & 0.2502 & -3.0279 \\ -0.0001 & 0.0233 & -0.0762 & -0.3929 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.106}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.107}$$

This system has eight states: airspeed (m/s), angle of attack (rad), pitch rate (rad/s), pitch angle (rad), sideslip angle (rad), roll rate (rad/s), yaw rate (rad/s) and bank angle (rad); four inputs: symmetrical canard deflection (rad), symmetrical tailplane deflection (rad), differential tailplane deflection (rad) and differential canard deflection (rad); four outputs: pitch rate (rad/s), airspeed (m/s), pitch angle (rad) and bank angle (rad). The system has a zero at 1.9938 and poles at $0.0425 \pm 1.7614i$ and $0.0107 \pm 0.1399i$, and thus the $(A, B, C)$ is nonminimum phase and unstable.

As in the work described in [148, 177], a decentralised control strategy will be used to control the integrated HIRM aircraft. Introduce the coordinate transformation matrix

$$\begin{bmatrix} 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0314 & 0.0402 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.0003 & 0.0003 & 0 & 0 & 0 & 0 & 1 \\ 0 & -0.0205 & -0.0131 & 0 & 0 & 1 & 0 & 0 \\ 0 & -0.0096 & -0.0122 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

In the new coordinate system, the system (5.105)–(5.107) can be described in the form of (5.1)–(5.2) as follows:

$$A_1 = \left[\begin{array}{cc|c} -0.2960 & -0.0202 & -0.0220 \\ -0.7190 & -0.2404 & -0.2006 \\ \hline 24.8696 & -0.1251 & -0.1220 \end{array}\right]$$

$$B_1 = \left[\begin{array}{cc} 0.5247 & -1.4718 \\ -1.2536 & -2.4223 \\ \hline 1.0000 & 0.0000 \end{array}\right], \qquad C_1 = \begin{bmatrix} I_2 & 0_{2\times1} \end{bmatrix}$$

$$A_2 = \left[\begin{array}{cc|ccc} 0 & -0.0013 & 0 & 0 & 0 \\ 0.0016 & 0 & 1.0000 & 0.4732 & 0 \\ \hline -0.1010 & 0 & -0.3604 & 2.2864 & -7.5954 \\ -0.0011 & 0 & -0.0313 & -0.1489 & -0.2419 \\ 0.0001 & 0.1330 & 0.4223 & -0.8991 & -0.0113 \end{array}\right]$$

$$B_2 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 0.2502 & -3.0279 \\ -0.0762 & -0.3929 \\ -0.0091 & 0.0175 \end{array}\right], \qquad C_2 = \begin{bmatrix} I_2 & 0_{2\times3} \end{bmatrix},$$

where the nominal isolated subsystem $(A_1, B_1, C_1)$ is stable and minimum phase with two inputs: symmetrical canard deflection (rad) and symmetrical tailplane deflection (rad); two outputs: pitch rate (rad/s) and airspeed (m/s), and $(A_2, B_2, C_2)$ is unstable and nonminimum phase with two inputs: differential tailplane deflection (rad) and differential canard deflection (rad); two outputs: pitch angle (rad) and bank angle (rad). The interconnection terms are

$$H_{12} = \begin{bmatrix} 0.0013x_{23} - 0.0007x_{24} \\ -9.8066x_{21} + 0.0981x_{25} \\ 7.6624x_{21} - 0.0767x_{25} \end{bmatrix}$$

$$H_{21} = \begin{bmatrix} x_{11} \\ -0.0251x_{12} - 0.0189x_{13} \\ \hline 0.3103x_{11} - 0.0204x_{12} - 0.0278x_{13} \\ 0.2964x_{11} + 0.0086x_{12} + 0.0121x_{13} \\ -0.0077x_{11} + 0.0003x_{12} + 0.0058x_{13} \end{bmatrix}.$$

Suppose that the system sufferers from disturbances

$$\Delta H_{12}(x_2, t) = \begin{bmatrix} \Delta \widetilde{H}_2(x_2, t) \\ \Delta \widetilde{H}_1(x_2, t) \\ -0.5846\Delta \widetilde{H}_1(x_2, t) - 0.0652\Delta \widetilde{H}_2(x_2, t) \end{bmatrix}$$

$$\Delta f_2(x_2, t) = \begin{bmatrix} 0.01\Delta \widetilde{f}(x_2, t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where the unknown signals satisfy

$$\|\Delta \widetilde{f}\| \le \exp\{-2 - t\}|x_{25}|^3$$

$$\left\| \begin{bmatrix} \Delta \widetilde{H}_1 \\ \Delta \widetilde{H}_2 \end{bmatrix} \right\| \le \sin^2 x_{22} + |x_{23}|.$$

The domain $\Omega$ is given by

$$\Omega = \left\{ (x_{11}, x_{12}, x_{13}, x_{21}, \ldots, x_{25}) \;\middle|\; \begin{matrix} x_{ij} \in \mathbb{R}\ (i \neq 2 \text{ and } j \neq 5) \\ |x_{25}| < 0.53 \end{matrix} \right\}$$

which implies that the limitation on the pitch angle is from $-30°$ to $+30°$.

Assumption 5.1 is satisfied. Choose the 'compensator' gains as

$$L_1 = \begin{bmatrix} 99.6998 & -0.3800 \\ -0.3800 & 99.8798 \\ \hline 18.1370 & -58.7751 \end{bmatrix}$$

$$L_2 = \left[\begin{array}{cc} 100.0000 & -0.0004 \\ -0.0004 & 103.8401 \\ \hline -0.1040 & 349.8709 \\ -0.0019 & 87.7352 \\ 0.0004 & -78.7022 \end{array}\right].$$

The corresponding solutions of Lyapunov equation (5.4) for $Q_1 = 100I_3$ and $Q_2 = 100I_5$ are

$$P_1 = \left[\begin{array}{cc|c} 1.3847 & 7.9342 & 13.5392 \\ 7.9342 & 71.6549 & 121.4193 \\ \hline 13.5392 & 121.4193 & 207.6899 \end{array}\right]$$

$$P_2 = \left[\begin{array}{cc|ccc} 0.5000 & 0.0029 & -0.0004 & -0.0002 & 0.0015 \\ 0.0029 & 237.7960 & -46.2122 & -20.6196 & 84.5487 \\ \hline -0.0004 & -46.2122 & 18.8694 & -11.5992 & 6.2753 \\ -0.0002 & -20.6196 & -11.5992 & 72.7900 & 3.2090 \\ 0.0015 & 84.5487 & 6.2753 & 3.2090 & 138.2835 \end{array}\right].$$

Let the matrices which determine the structure of the uncertainties in (5.5) be

$$D_2 = \begin{bmatrix} 0.01 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -0.5846 & -0.0652 \end{bmatrix}$$

and the bounds on the uncertainties from (5.6) are

$$\gamma_2 = \exp\{-2 - t\}\,|x_{25}|^3 \qquad \text{and} \qquad \zeta_{12} = \sin^2 x_{22} + |x_{23}|.$$

It is straightforward to check that Assumption 5.2 is satisfied with

$$G_2 = \begin{bmatrix} 0.005 & 0 \end{bmatrix}, \qquad F_{12} = \begin{bmatrix} 0.0189 & 0.6709 \\ 0.5021 & 0.0189 \end{bmatrix}.$$

The dynamical compensator (5.11) is completely specified and by direct computation,

$$W^\tau + W = \begin{bmatrix} 200 & -68.1072 \\ -68.1072 & 200 \end{bmatrix}$$

is positive definite. Thus from Theorem 5.1, the partial state estimate $\hat{x}_{i2}$ satisfies (5.12). Suppose the desired eigenvalues of $A_{eq1}$ and $A_{eq2}$ are $\{-0.25\}$ and $\{-3 \pm 2i, -2\}$, respectively. According to the algorithm given by Edwards and Spurgeon [38], the designed sliding surfaces are given by

$$S_1 = \left[\, S_{11}\ S_{12} \,\right] = \left[\begin{array}{cc|c} 0.4326 & 0.8128 & -0.3901 \\ 0.9016 & -0.3901 & 0.1872 \end{array}\right], \tag{5.108}$$

$$S_2 = \left[\, S_{21}\ S_{22} \,\right] = \left[\begin{array}{cc|ccc} -0.7659 & 0.2488 & -0.0121 & 0.3246 & -0.4961 \\ -0.6430 & -0.2955 & 0.0146 & -0.3868 & 0.5911 \end{array}\right]. \tag{5.109}$$

It follows that (5.26) is satisfied if

$$T_{11} = \begin{bmatrix} -4.3264 & 9.0157 & 0.0000 \\ -8.1282 & -3.9005 & 4.3264 \\ 3.9005 & 1.8718 & 9.0157 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

and

$$T_{21} = \begin{bmatrix} 7.6586 & -6.4301 & 0.0011 & -0.0030 & 0.0045 \\ -2.4877 & -2.9552 & 0.1900 & -5.0491 & 7.7168 \\ 0.1213 & 0.1460 & 9.9974 & 0.0693 & -0.1059 \\ -3.2457 & -3.8677 & 0.0691 & 8.1610 & 2.8106 \\ 4.9606 & 5.9111 & -0.1056 & 2.8106 & 5.7044 \end{bmatrix}$$

$$T_{22} = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Let

$$\widetilde{Q}_1 = 1 \qquad \text{and} \qquad \widetilde{Q}_2 = I_3.$$

The solutions of the Lyapunov equations (5.31) are

$$\widetilde{P}_1 = 2, \qquad \widetilde{P}_2 = \begin{bmatrix} 0.0934 & -0.0245 & 0.0630 \\ -0.0245 & 0.3594 & -0.3211 \\ 0.0630 & -0.3211 & 1.1631 \end{bmatrix}.$$

By direct computation

$$M^{\tau} + M = \begin{bmatrix} 2.0000 & -0.4700 & -0.0410 & -0.1242 \\ -0.4700 & 1.4693 & -0.1242 & -0.3698 \\ -0.0410 & -0.1242 & 200.0000 & -55.8311 \\ -0.1242 & -0.3698 & -55.8311 & 200.0000 \end{bmatrix}$$

which can be shown to be positive definite. Therefore, the requirements of Theorem 5.2 are satisfied. According to (5.36), the controller is now completely specified. For simulation purposes, suppose that only the bank angle has an initial deviation 0.1 rad while

$$\alpha_1 = 0.3359 \qquad \text{and} \qquad \alpha_2 = 0.286$$

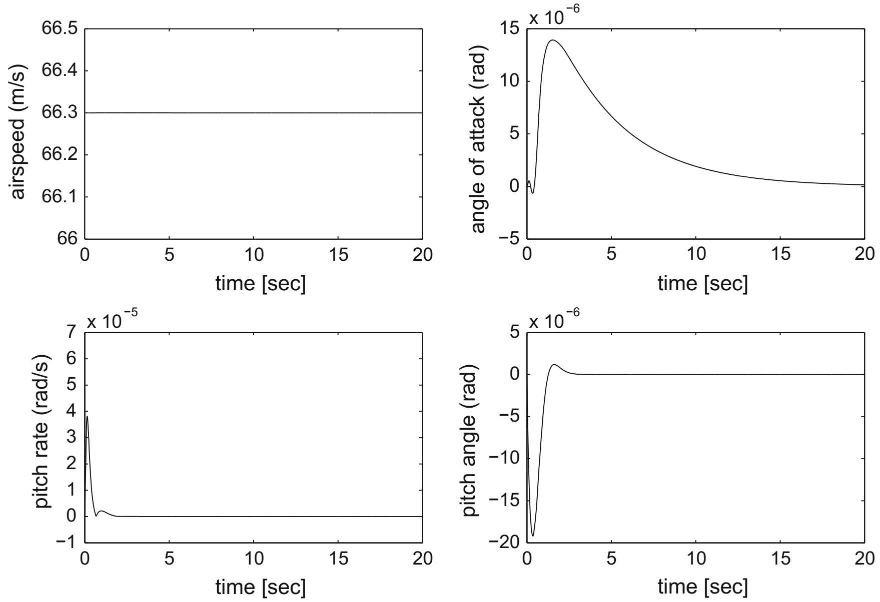


**Fig. 5.1** The response of the nonlinear HIRM aircraft system: airspeed (m/s), angle of attack (rad), pitch rate (rad/s) and pitch angle (rad)

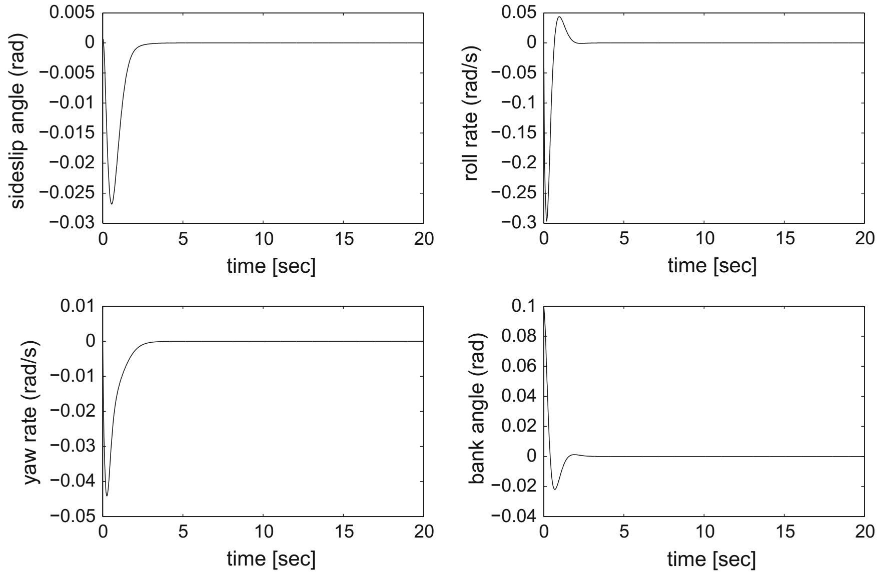


**Fig. 5.2** The response of the nonlinear HIRM aircraft system: sideslip angle (rad), roll rate (rad/s), yaw rate (rad/s) and bank angle (rad)

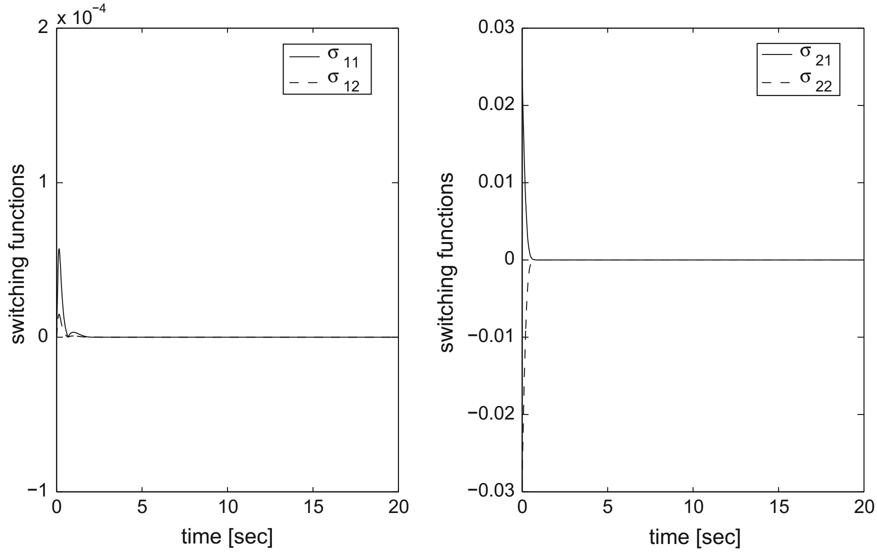


**Fig. 5.3** The response of the switching functions

The simulation results in Figs. 5.1 and 5.2 show the effectiveness of the controller. The airspeed is almost unaffected by the system control variables. This is in accordance with practice since airspeed is mainly controlled by engine thrust. Figure 5.3 shows the time responses of the switching functions.

### *5.4.2 Case Study on Coupled Inverted Pendula*

Consider two identical inverted pendula coupled by a moving spring and subject to two distinct inputs (see [59]). The salient feature of the system is that the position $a$ of the spring can change along the full length $l$ of the pendula. The input to each pendulum is the torque $u_i$ applied at the pivot point. The two payloads are supposed to be both known and equal to $m$ [59]. Let

$$x_i = \text{col}(x_{i1}, x_{i2}) = \text{col}(\theta_i, \theta_i - \dot{\theta}_i)$$

for $i = 1, 2$. Then, according to [59], the dynamic equation of the pendula is described by

$$\dot{x}_1 = \begin{bmatrix} 1 & -1 \\ 1-\frac{g}{l} & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka^2}{ml^2} & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ -\frac{ka^2}{ml^2} & 0 \end{bmatrix} x_2 \tag{5.110}$$

$$\dot{x}_2 = \begin{bmatrix} 1 & -1 \\ 1-\frac{g}{l} & -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} u_2 + \begin{bmatrix} 0 & 0 \\ -\frac{ka^2}{ml^2} & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka^2}{ml^2} & 0 \end{bmatrix} x_2, \tag{5.111}$$

where $k$ and $g$ are the spring and gravity constants. The system output is chosen as

$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i \quad \text{for} \quad i = 1, 2.$$

In order to illustrate our scheme, assume that the uncertainty in the interconnections is represented by making $a$ an unknown function of the state vector, that is

$$|a| \leq \varpi(x).$$

According to [59], the parameters are chosen as

$$g = l = m = k = 1.$$

It is obvious that (5.110)–(5.111) is a similar interconnected system. Comparing (5.97)–(5.98), it follows that

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\Phi_1(x_1) = \Phi_2(x_2) = 0, \quad M_{12} = M_{21} = 0,$$
$$\Delta M_i(x) = \text{diag}\left\{ \begin{bmatrix} 0 & 0 \\ (-1)^{i+1}\frac{ka^2}{ml^2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ (-1)^{i}\frac{ka^2}{ml^2} & 0 \end{bmatrix} \right\} x, \quad (i = 1, 2).$$

Let

$$K = \begin{bmatrix} 115 & -47.5 \end{bmatrix}, \quad Q = 2I, \quad \bar{S} = 2, \quad \alpha = 2.$$

It follows that

$$P = \begin{bmatrix} 4.9525 & -0.0518 \\ -0.0518 & 0.0217 \end{bmatrix}$$

and

$$\bar{R} = 1, \qquad \gamma_1 = \gamma_2 = \varpi^2(x).$$

By direct computation,

$$W_1 = \begin{bmatrix} 2 - 1.25\varpi^4(x) - 4.5531\varpi^2(x) & -4.5531\varpi^2(x) \\ -4.5531\varpi^2(x) & 2 - 1.25\varpi^4(x) - 4.5531\varpi^2(x) \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, it is observed that in domain

$$\Omega' = \{(x_{11}, x_{12}, x_{21}, x_{22}) \mid \varpi(x) \leq 0.46\}$$

all conditions of Theorem 5.5 are satisfied, and thus the system (5.110)–(5.111) is stabilised by the reduced-order controller

$$\dot{\hat{x}}_{12} = -\hat{x}_{12} - u_1, \tag{5.112}$$

$$\dot{\hat{x}}_{22} = -\hat{x}_{22} - u_2, \tag{5.113}$$

$$u_i = -115y_i + 47.5\hat{x}_{i2}, \qquad i = 1, 2. \tag{5.114}$$

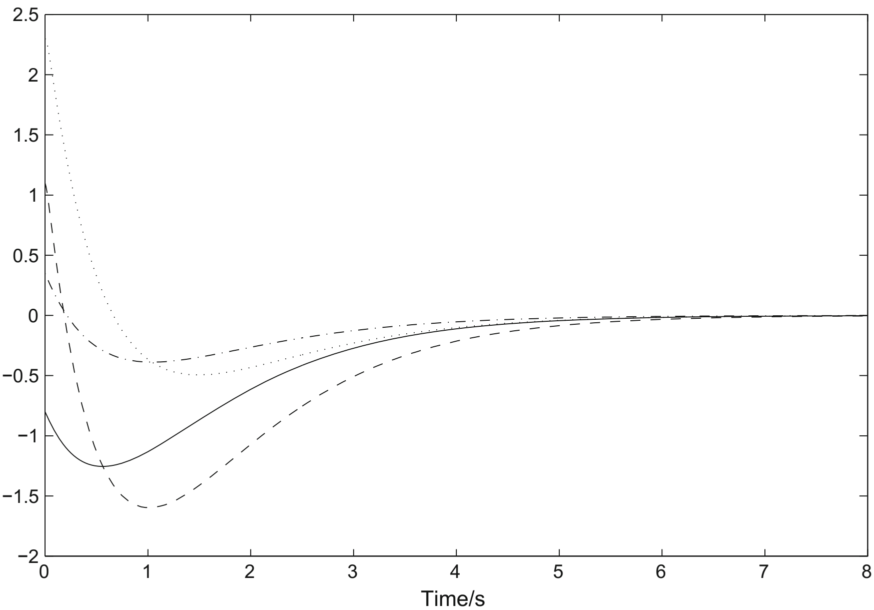


**Fig. 5.4** Evolution of state variables of System (5.110)–(5.111) under control (5.112)–(5.114): ——— $x_{11}(t)$; − − − − − $x_{12}(t)$; · − · − · − · $x_{21}(t)$; · · · · · · · · · · · $x_{22}(t)$

With the chosen parameter settings, a simulation with initial state

$$x_0 = (-0.8,\ 1.0,\ 0.35,\ 2.3)$$

is shown in Fig. 5.4.

### 5.4.3 A Numerical Simulation Example

In this section, a numerical example is given to demonstrate the result presented in Theorem 5.4.

Consider the interconnected system composed of two second-order subsystems described by

$$\begin{aligned}\dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_1 + \begin{bmatrix} 0.2x_{11} \\ 0.1x_{11}\sin x_{12} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (u_1 + \Delta\Psi_1(x_1)) \\ &\quad + \Delta H_1(x) && (5.115)\\ \dot{x}_2 &= \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (u_2 + \Delta\Psi_2(x_2)) + \frac{1}{8}\begin{bmatrix} 0.5x_{11}^2 \\ x_{12} \end{bmatrix} && (5.116)\\ y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 && (5.117)\\ y_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_2, && (5.118)\end{aligned}$$

where $x_i = \text{col}(x_{i1}, x_{i2})$ for $i = 1, 2$, and $x = \text{col}(x_1, x_2)$ are the system state variables, $u_i$, $y_i$ are, respectively, the system inputs and outputs of the $i$-th subsystem, and the uncertainties satisfy

$$\begin{aligned}|\Delta\Psi_1(x_1)| &\le y_1^2 \sin^2 y_1\\ |\Delta\Psi_2(x_2)| &\le |y_2| e^{y_2}\\ \|\Delta H_1(x)\| &\le 0.1(x_{12} + x_{21})^2 \sin \tfrac{\|x\|}{5}.\end{aligned}$$

Let

$$\begin{cases} u_1 = v_1 \\ u_2 = -[1 \quad 0]x_2 + v_2 \end{cases}.$$

It follows that

$$\begin{aligned}\dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_1 + \begin{bmatrix} 0.2x_{11} \\ 0.1x_{11}\sin x_{12} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (v_1 + \Delta\Psi_1(x_1)) \\ &\quad + \Delta H_1(x) && (5.119)\\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (v_2 + \Psi_2(x_2)) + \frac{1}{8}\begin{bmatrix} 0.5x_{11}^2 \\ x_{12} \end{bmatrix} && (5.120)\end{aligned}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 \tag{5.121}$$
$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_2. \tag{5.122}$$

Therefore, the system (5.115)–(5.118) is a similar interconnected system with the *STP* $(E_i, D_i)$ for $i = 1, 2$ defined by

$$D_1 = D_2 = I, \qquad E_1 = 0, \qquad E_2 = -1.$$

Now, choose

$$K = \begin{bmatrix} 1 & \frac{1}{8} \end{bmatrix}, \qquad L = 0, \qquad S = 2I$$

and

$$Q = \begin{bmatrix} 6 & 0.25 \\ 0.25 & 2 \end{bmatrix}.$$

It follows that

$$P = R = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

By direct calculation, in the domain

$$\Omega = \{x \mid |x_{11}| \le 2, x_{12}, x_{21}, x_{22} \in \mathbb{R}\}$$

the corresponding parameters are given by

$$\begin{aligned} &F_1 = F_2 = 2, \qquad \xi_1(r) = \sqrt{5}/10, \quad \xi_2(r) = 0 \\ &\rho_1(r) = r^2 \sin^2 r, \ \rho_2(r) = |r|e^r \end{aligned}$$

$$\begin{aligned} &\gamma_1 = 0.02(x_{12} + x_{21})^2, \ \gamma_2 = 0, \quad \Gamma_{12} = 0 \\ &\Gamma_{21} = \begin{bmatrix} \frac{1}{16}x_{11} & 0 \\ 0 & 1/8 \end{bmatrix}, \quad \mathscr{L}_{f_1} = 0.3, \quad \mathscr{L}_{f_2} = 0 \\ &\mathscr{L}_{T_1 H_{12}} = 0, \qquad \mathscr{L}_{T_2 H_{21}} = \frac{1}{8}, \qquad K_2 = \frac{1}{8}. \end{aligned}$$

Then,

$$W = \begin{bmatrix} 1.9844 - \varepsilon - 0.04(2+\sqrt{2})\vartheta & -0.04(2+\sqrt{2})\vartheta & -\frac{1}{2} - 0.08\sqrt{\frac{3+\sqrt{5}}{2}}\vartheta & 0 \\ -\frac{2+\sqrt{2}}{4} & 1.9844 - \varepsilon & -0.08\sqrt{\frac{3+\sqrt{5}}{2}}\vartheta & -\frac{1}{2} \\ -\frac{1}{2} - 0.08\sqrt{\frac{3+\sqrt{5}}{2}}\vartheta & -0.08\sqrt{\frac{3+\sqrt{5}}{2}}\vartheta & 2 - 0.6\sqrt{3} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 2 \end{bmatrix},$$

where $\vartheta = (x_{12} + x_{21})^2$. It can be verified that $W^\tau + W$ is positive definite in domain

$$\Omega' = \{(x_{11}, x_{12}, x_{21}, x_{22}) \mid \ |x_{11}| \le 2,\ |x_{12} + x_{21}| \le 1.3,\ x_{22} \in \mathbb{R}\}$$

for the chosen $\varepsilon = 0.01$. Therefore, from Theorem 5.4, System (5.115)–(5.118) is stabilised by the reduced-order control

$$\dot{\omega}_1 = -\omega_1 + 0.2y_1 + 0.1y_1 \sin(\omega_1 - y_1) \tag{5.123}$$

$$\dot{\omega}_2 = -\omega_2 + \frac{1}{8}\omega_1 + \frac{1}{8}y_1^2 - \frac{1}{8}y_1 \tag{5.124}$$

$$u_1 = -\frac{1}{8}\omega_1 - y_1|y_1| \sin^2 y_1 - \left(\frac{67}{8} + \frac{5}{2}\sqrt{2}\right) y_1 \tag{5.125}$$

$$u_2 = -\frac{1}{8}\omega_2 - \frac{15}{8}y_2 - y_2 e^{y_2}. \tag{5.126}$$

The state responses corresponding to the initial condition

$$x(0) = (-1.8, 2.5, -3.5, 4.0)$$

are shown in Fig. 5.5.

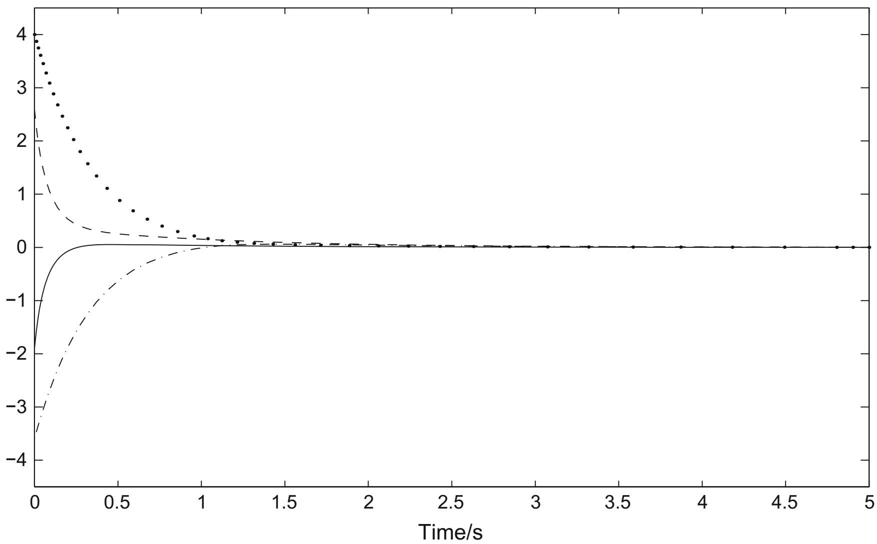


**Fig. 5.5** Evolution of state variables of System (5.115)–(5.118) under the control (5.123)–(5.126): ________ $x_{11}(t)$; − − − − − $x_{12}(t)$; · − · − · − · $x_{21}(t)$; · · · · · · · · · · · $x_{22}(t)$

## 5.5 Summary

In this chapter, reduced-order compensator-based controllers have been synthesised for nonlinear interconnected systems in which the uncertainties are mismatched and have nonlinear bounds. Section 5.2 has presented a dynamical decentralised output feedback control strategy using sliding mode techniques. The notion of the equivalent control and local coordinate transformations are exploited to establish and analyse the stability of the reduced-order sliding mode. The known interconnections are used in the control design which insures the composite reachability condition can be satisfied by the designed controllers. Some remarks have illustrated the advantages of the proposed control scheme. Therefore, the developed results can be applied to a wide class of systems. Section 5.3 has presented two reduced-order control schemes to stabilise a class of nonlinear interconnected systems with mismatched uncertainty. Based on the constrained Lyapunov equations, the conservatism of the developed result has been greatly reduced. The role of coordinate transformations is clearly shown in the analysis.

It has been shown that dynamical feedback can remove the requirements of minimum phaseness of the nominal isolated subsystems. It has also been demonstrated that the system structure plays an important role in simplifying the analysis and design of large-scale interconnected systems. The methods proposed in this chapter may be extended to a wider class of interconnected systems.

# Chapter 6
# Delay Dependent Output Feedback Control

In this chapter, nonlinear time-delay systems are considered. Under the assumption that all the time-delays experienced by the system are completely known, static output feedback control, reduced-order observer-based control and decentralised control is investigated using sliding mode techniques. The Lyapunov–Razumikhin approach will be employed to deal with the time-delay. Simulation examples are provided to demonstrate the developed theoretical results.

## 6.1 Introduction

In the mathematical modelling of a real system, it is often assumed that the system's future behaviour depends only on the current state. However, such an assumption is not always true due to the existence of time-delay elements such as material or information transfer. If the delay is neglected, sometimes the model cannot reflect the system sufficiently well, which may lead to poor performance. This has motivated the study of time-delay systems [141].

Time-delay widely exists in the real world. In economic systems, delays appear in a natural way since the decisions and the effects (caused by the decision) are separated by some time interval. In communication systems, data transmission is always accompanied by a nonzero time between the receiver and transmitter of a message or a signal. Such systems belong to the class of differential-difference equations which are infinite-dimensional when compared with ordinary differential equations.

Sometimes the delay may greatly affect the system performance: for example, a small delay may destabilise a system while a large delay may stabilise a system; chaotic behaviour may appear if the delayed state involves a nonlinear function but in other cases, chaotic systems may be stabilised by a delayed feedback [130]. This shows that time-delay systems are complicated especially when the delay exists in

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nonlinear terms. Therefore, the study of time-delay systems is pertinent and valuable both from a theoretical and applications perspective.

As early as the eighteenth century, time-delay systems have been studied (see the survey paper [66]). Since Krasovskii extended Lyapunov theory to time-delay systems [93] and Razumikhin proposed a method to avoid the use of functionals in the Lyapunov stability analysis [140], great progress has been made, but most of the early work focused on the analysis of unforced time-delay systems. In more recent years, the advancement of control theory has motivated the study of time-delay and control systems. For systems affected by time-delay, contributions have considered cases where the delay may appear in the system state, input, output and disturbances experienced by the system [53, 63, 98]. A variety of control approaches such as sliding mode control, $H_\infty$ control, backstepping techniques and adaptive control, etc. have been applied to the control of systems with time-delay and many important results have been achieved [53, 60, 63, 125].

It is well known that sliding mode control, as one of the discontinuous control approaches, is completely robust to so-called matched disturbances [38, 174]. This has motivated the application of sliding mode techniques to time-delay systems with disturbances [53, 63, 84, 112, 132, 154]. It should be noted that most of the existing results are based on the fact that all of the system state variables are accessible [53, 63]. However, system state variables are often not fully available. Therefore, it is essential to study output feedback control which is more convenient for real implementation.

In the following sections, both static output feedback control and dynamical output feedback control schemes will be formulated for a class of nonlinear time-delay systems, and a decentralised static output control strategy is proposed for a class of interconnected time-delay systems.

## 6.2 Static Output Feedback Control of Time-Delay Systems

Many results associated with different types of delay such as state delay, input delay, and output delay have been produced [98, 154, 184]. Most results for linear time-delay control systems will finally produce linear matrix inequalities (LMIs) [52, 101, 164]. When a time-delay system is nonlinear and has nonlinear uncertainties, the problem becomes more complicated. Although many results have been obtained for time-delay systems, the problem of output feedback control of time-delay nonlinear systems is much less mature.

It has been established that when compared with state feedback, the static output feedback control problem is much more difficult, even for systems without delay [165]. Much less attention has been paid to time-delay systems with delayed disturbances using static output feedback sliding mode control and only a very limited literature is available [84, 112]. A sliding mode control is given in [154] where an output tracking problem is considered. Luo et al. studied a class of time-delay systems where static and dynamic output feedback strategies are both considered [112] but it is required that all of the uncertainty is matched. Janardhanan and Bandyopadhyay [84] proposed a static output sliding mode control scheme for time-delay systems involving a class of linear discrete-time system.

In this section, a static output feedback sliding mode control strategy is proposed to stabilise a class of time-varying delay systems with time-delayed nonlinear disturbances. Both matched and mismatched uncertainties are considered where the bounds on the uncertainties involving time-delay are employed in the control design. A sliding surface is designed and the system structure is analysed and employed in the stability analysis of the sliding motion using the Lyapunov–Razumikhin approach. Then, a sliding mode control with time-delay based on only output information is proposed to drive the system to the designed sliding surface in finite time and maintain a sliding motion on it thereafter. As in [200, 214], the bounds on the unstructured disturbances are allowed to be nonlinear, but unlike [200, 214] time-varying delay exists in the system considered, the disturbances and the bounds on the disturbances.

### 6.2.1 Preliminaries

To develop the time-delay framework, first consider the linear system

$$\dot{x} = Ax + Bu \tag{6.1}$$

$$y = Cx, \tag{6.2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ are the states, inputs and outputs respectively with $m \le p < n$. The triple $(A, B, C)$ comprises constant matrices of appropriate dimensions, where both $B$ and $C$ are of full rank.

For System (6.1)–(6.2), it is assumed that $\text{rank}(CB) = m$. A coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the system triple $(A, B, C)$ with respect to the new coordinates $\widetilde{x}$ has the following structure:

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} 0 \\ \widetilde{B}_2 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} 0 & \widetilde{C}_2 \end{bmatrix}, \tag{6.3}$$

where $\widetilde{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $\widetilde{B}_2 \in \mathbb{R}^{m\times m}$ is nonsingular and $\widetilde{C}_2 \in \mathbb{R}^{p\times p}$ is orthogonal. Assumed that system $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ with $\widetilde{C}_1$ defined by

$$\widetilde{C}_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{p-m} \end{bmatrix} \tag{6.4}$$

is output feedback stabilizable, i.e., there exists a matrix $K \in \mathbb{R}^{m\times(p-m)}$ such that

$$\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$$

is stable. A necessary condition for $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ to be stabilizable is that the invariant zeros of $(A, B, C)$ lie in the open left-half plane.

For the convenience, the following definition is introduced:

**Definition 6.1** The matrix triple $(A, B, C)$ or linear system (6.1)–(6.2) is called normalisable if there exists a nonsingular transformation $z = Tx$ such that in the new coordinate system $z$, the system (6.1)–(6.2) has the following form

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 \tag{6.5}$$

$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u \tag{6.6}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{6.7}$$

where $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^m$, $A_{11}$ is stable, and $B_2 \in \mathbb{R}^{m\times m}$ and $C_2 \in \mathbb{R}^{p\times p}$ are nonsingular. Then, (6.5)–(6.7) is called the regular form of System (6.1)–(6.2).

From Lemma 2.6, the following result is obtained directly.

**Lemma 6.1** *System (6.1)–(6.2) is normalisable if*

*(i)* $\text{rank}(CB) = m$;
*(ii) for the triple* $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ *defined by (6.3) and (6.4), there exists a matrix* $K$ *such that* $\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ *is stable.*

*Remark 6.1* Lemma 6.1 gives a sufficient condition under which System (6.1)–(6.2) is normalisable. If the conditions (i) and (ii) in Lemma 6.1 hold, then the regular form (6.5)–(6.7) can be obtained from a systematic algorithm [38] together with any output feedback pole placement algorithm of choice.

### *6.2.2 System Description and Problem Formulation*

Consider a time-varying delay system with time-delayed disturbance described by

$$\begin{aligned}\dot{x}(t) = {} & Ax(t) + A_0x(t - d(t)) + B\,(u(t) + g(t, x(t), x(t - d(t)))) \\ & + f(t, x(t), x(t - d(t)))\end{aligned} \tag{6.8}$$

$$y(t) = Cx(t) \tag{6.9}$$

where $x \in \Omega \subset \mathbb{R}^n$ ($\Omega$ is a neighbourhood of the origin), $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are system states, inputs and outputs respectively with $m \le p < n$. The matrices $A$, $A_0$, $B$ and $C$ represents constant matrices of appropriate dimensions with $B$ and $C$ of full rank. The vectors $g(\cdot)$ and $f(\cdot)$ represent the matched and mismatched disturbances affecting the system respectively. The known function $d(t)$ is a time-varying delay which is assumed to be continuous, nonnegative and bounded in $\mathbb{R}^+$, that is

$$\overline{d} := \sup_{t\in\mathbb{R}^+}\{d(t)\} < \infty.$$

The initial condition for the system is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. It is assumed that the nonlinear functions $g(\cdot)$ and $f(\cdot)$ are smooth enough in their domain of definition such that the system has unique continuous solutions for the given initial condition.

First, it is necessary to impose some basic assumptions on the system (6.8)–(6.9):

**Assumption 6.1** The triple $(A, B, C)$ is normalisable, and $\mathrm{Im}(A_0) \subset \mathrm{Im}(B)$.

*Remark 6.2* Assumption 6.1 is a limitation on the linear part of System (6.8)–(6.9). It guarantees that the triple $(A, B, C)$ can be transformed to the regular form (6.5)–(6.7). The assumption

$$\mathrm{Im}(A_0) \subset \mathrm{Im}(B)$$

means that the time-delay term $A_0 x(t - d(t))$ is matched and thus it will not affect the sliding motion.

**Assumption 6.2** There exist known continuous nonnegative functions $\rho_i(\cdot)\mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^p \mapsto \mathbb{R}^+$ with $i = 1, 2$ and $\varpi(\cdot) : \mathbb{R}^+ \times \mathbb{R}^p \times \mathbb{R}^p \mapsto \mathbb{R}^+$ such that for $t \in \mathbb{R}^+$, and $x(t),\ x(t-d) \in \Omega$

$$\|f(t, x(t), x(t-d(t)))\| \le \rho_1(t, y(t), y(t-d(t)))\|x(t)\| + \rho_2\big(t, y(t), y(t-d(t))\big)\|x(t-d(t))\| \tag{6.10}$$

$$\|g(t, x(t), x(t-d(t)))\| \le \varpi(t, y(t), y(t-d(t))). \tag{6.11}$$

*Remark 6.3* Assumption 6.2 requires that the uncertainties are bounded by some known continuous functions. It is an extension of the existing results [84, 112, 186] where it is required that the uncertainty is bounded by the linear combination of $\|y(t)\|$ and $\|y(t-d)\|$ which is called the linear growth condition.

The objective now is to design a static output feedback control with time-delay of the form

$$u = u(t, y(t), y(t-d)) \tag{6.12}$$

based on sliding mode techniques such that the closed-loop system formed by the control (6.12) and the system (6.8)–(6.9) is uniformly asymptotically stable in a domain of the origin even in the presence of the disturbances. Notably the control (6.12) only depends on the system output $y(t)$ and time-delay $d(t)$. Since $d(t)$ is assumed to be known, the term $y(t - d(t))$ is available and thus the control (6.12) is called static output feedback control with time-delay.

*Remark 6.4* As in the work in [10, 112, 125, 144], the delay experienced by the system is assumed to be known, which may limit its application. However, in some important industrial systems such as flow through pipes and web-forming processes, the delay existing in the process is known, and can thus be employed in the control design and/or the compensator design [144]. Furthermore, the approach proposed in [33] enables the time-delay to be identified in some cases even when the delay is unknown.

### 6.2.3 Sliding Motion Analysis and Control Design

The main results will be presented in this section. From Sect. 6.2.2, $\mathrm{Im}(A_0) \subset \mathrm{Im}(B)$ (Assumption 6.1), i.e., there exists a matrix $D \in \mathbb{R}^{m\times n}$ such that $A_0 = BD$. Then, from Sects. 6.2.1 and 2.6, it follows that under Assumption 6.1 there exists a coordinate transformation $z = Tx$ such that in the new coordinate system $z$, System (6.8)–(6.9) is described by

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + f_1(t, z(t), z(t - d(t))) \tag{6.13}$$

$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2DT^{-1}z(t - d(t)) + B_2(u + g(t, T^{-1}z(t), T^{-1}z(t - d(t))) + f_2(t, z(t), z(t - d(t))) \tag{6.14}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{6.15}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ is stable, $B_2 \in \mathbb{R}^{m\times m}$ and $C_2 \in \mathbb{R}^{p\times p}$ are nonsingular, and

$$\begin{bmatrix} f_1(t, z(t), z(t - d(t))) \\ f_2(t, z(t), z(t - d(t))) \end{bmatrix} := T\left[f(t, x(t), x(t - d(t)))\right]_{x=T^{-1}z} \tag{6.16}$$

where $f_1(\cdot) \in \mathbb{R}^{n-m}$ and $f_2(\cdot) \in \mathbb{R}^m$.

*Remark 6.5* Since the coordinate transformation matrix $T$ can be obtained using a systematic approach given in [38], the system in (6.13)–(6.15) is well defined and can be directly obtained from System (6.8)–(6.9).

Based on the analysis in Sect. 2.6, consider the following sliding surface for system: (6.8)–(6.9)

$$S = \{x \mid FCx = 0\}, \tag{6.17}$$

where $F$ is defined in (2.21). Then from Lemma 2.6, it follows that in $z$ coordinate system, the sliding surface (6.17) can be described by equation

$$z_2 = 0. \tag{6.18}$$

Then from the regular form (6.13)–(6.15), the sliding dynamics associated with the sliding surface (6.17) are described by

$$\dot{z}_1 = A_{11}z_1 + [f_1(t, z(t), z(t - d(t))]_{z_2(t)=0} \tag{6.19}$$

where $z_1 \in \mathbb{R}^{n-m}$ are the sliding mode state variables and $A_{11}$ is stable. It is clear that the mismatched disturbance affects the sliding motion directly. Obviously System (6.19) which describes the sliding motion involves time-delay. The following further assumption is required:

**Assumption 6.3** There exist known continuous functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ such that

$$\left\| [f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0} \right\| \leq \phi_1(t, z_1(t), \|z_1(t-d)\|)\|z_1(t)\| + \phi_2(t, z_1(t), \|z_1(t-d)\|)\|z_1(t-d(t))\|, \quad (6.20)$$

where the functions $\phi_1(t, r_1, r_2)$ and $\phi_2(t, r_1, r_2)$ are both nondecreasing w.r.t. $r_2$.

*Remark 6.6* Assumption 6.3 is a limitation to the mismatched disturbance. It implies that when a sliding motion takes place, the uncertainty $f_1$ can be bounded by a known continuous function of states $z_1(t)$ and $z_1(t-d(t))$. It should be noted that Assumption 6.3 is unnecessary if the disturbance $f(\cdot)$ in (6.8) does not include time-delay [200].

Since the matrix $A_{11}$ in (6.19) is stable, it follows that for any $Q > 0$ ($Q \in \mathbb{R}^{m\times m}$), there exists an unique matrix $P > 0$ such that

$$A_{11}^T P + PA_{11} = -Q. \quad (6.21)$$

For the later analysis, the following lemma is presented:

**Lemma 6.2** *If Assumption 6.3 holds, then there exist known continuous functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ such that*

$$\left\| P^{\frac{1}{2}}[f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0} \right\| \leq \psi_1(t, z_1(t), \|z_1(t-d)\|)\|P^{\frac{1}{2}}z_1(t)\| + \psi_2(t, z_1(t), \|z_1(t-d)\|)\|P^{\frac{1}{2}}z_1(t-d(t))\| \quad (6.22)$$

*where the functions $\psi_1(t, r_1, r_2)$ and $\psi_2(t, r_1, r_2)$ are both nondecreasing w.r.t. variables $r_2$.*

*Proof* It follows from the fact

$$\|z_1(t)\| \leq \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\| \quad (6.23)$$

$$\|z_1(t-d(t))\| \leq \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t-d(t))\| \quad (6.24)$$

that under Assumption 6.3

$$\begin{aligned}&\left\| P^{\frac{1}{2}}[f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0} \right\| \\ &\quad\leq \lambda_{\max}(P^{\frac{1}{2}})\big(\phi_1(t, z_1(t), \|z_1(t-d)\|)\|z_1(t)\| \\ &\qquad + \phi_2(t, z_1(t), \|z_1(t-d)\|)\|z_1(t-d(t))\|\big)\end{aligned}$$

$$\leq \lambda_{\max}(P^{\frac{1}{2}})\Big(\phi_1\,(t, z_1(t), \|z_1(t-d(t))\|)\,\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|$$
$$+\,\phi_2\,(t, z_1(t), \|z_1(t-d(t))\|)\,\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t-d(t))\|\Big). \quad (6.25)$$

Let

$$\psi_1(t, r_1, r_2) = \lambda_{\max}(P^{\frac{1}{2}})\lambda_{\max}(P^{-\frac{1}{2}})\phi_1(t, r_1, r_2)$$

and

$$\psi_2(t, r_1, r_2) = \lambda_{\max}(P^{\frac{1}{2}})\lambda_{\max}(P^{-\frac{1}{2}})\phi_2(t, r_1, r_2).$$

Then it follows that (6.20) is true and the functions $\psi_1(t, r_1, r_2)$ and $\psi_2(t, r_1, r_2)$ are both nondecreasing w.r.t. variable $r_2$ since $\phi_1(t, r_1, r_2)$ and $\phi_2(t, r_1, r_2)$ are both nondecreasing w.r.t. variable $r_2$. Hence the conclusion follows. □

The following Theorem gives a sufficient condition under which the sliding motion is asymptotically stable.

**Theorem 6.1** *Under Assumption 6.3, the sliding mode dynamics (6.19) are uniformly asymptotically stable if there exists a domain* $\Omega_0 = \{z_1 \mid z_1 \in \mathbb{R}^{n-m}\}$ *of the origin in* $T(\Omega)$ *and a constant* $\zeta > 1$ *such that for any* $z_1(t) \in \Omega_0$ *and* $t \in \mathbb{R}^+$

$$\gamma := \lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) - \sup_{t\in\mathbb{R}^+, z_1(t)\in\Omega_0}\{\Theta(t, z_1(t))\} > 0, \quad (6.26)$$

*where*

$$\Theta(t, z_1(t)) := \psi_1\left(t, z_1(t), \zeta\lambda_{\max}(P^{-\frac{1}{2}})\,\|P^{\frac{1}{2}}z_1(t)\|\right)$$
$$+\,\zeta\,\psi_2\left(t, z_1(t), \zeta\lambda_{\max}(P^{-\frac{1}{2}})\,\|P^{\frac{1}{2}}z_1(t)\|\right) \quad (6.27)$$

*and where* $\psi_1(\cdot)$ *and* $\psi_2(\cdot)$ *satisfy (6.22), and* $P$ *and* $Q$ *satisfy (6.21).*

*Proof* For System (6.19), consider as a Lyapunov function candidate $V(z_1(t)) = (z_1(t))^T P z_1(t)$. It follows from (6.20) and (6.21) that the time derivative of $V$ along the trajectories of System (6.19) is given as

$$\begin{aligned}
&\dot{V}(z_1(t))\,|_{(6.20)}\\
&= (z_1(t))^T\left(A_{11}^T P + PA_{11}\right)z_1(t) + 2(z_1(t))^T P[f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0}\\
&= -(z_1(t))^T P^{\frac{1}{2}}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)P^{\frac{1}{2}}z_1(t)\\
&\quad + 2(z_1(t))^T P^{\frac{1}{2}}P^{\frac{1}{2}}[f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0}\\
&\leq -\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|^2 + \|P^{\frac{1}{2}}z_1\|\Big(\psi_1(t, z_1(t), \|z_1(t-d)\|)\|P^{\frac{1}{2}}z_1(t)\|\\
&\quad + \psi_2(t, z_1(t), \|z_1(t-d)\|)\|P^{\frac{1}{2}}z_1(t-d(t))\|\Big)
\end{aligned} \quad (6.28)$$

where Lemma 6.2 has been used to obtain the above. Since $\psi_1(t, r_1, r_2)$ and $\psi_2(t, r_1, r_2)$ are both nondecreasing w.r.t. variable $r_2$, it follows from (6.23)–(6.24) that

$$\psi_1(t, z_1(t), \|z_1(t-d)\|) \le \psi_1\left(t, z_1(t), \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t-d(t))\|\right) \tag{6.29}$$

$$\psi_2(t, z_1(t), \|z_1(t-d)\|) \le \psi_2\left(t, z_1(t), \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t-d(t))\|\right). \tag{6.30}$$

When

$$\|P^{\frac{1}{2}} z_1(t+\theta)\| \le \zeta \|P^{\frac{1}{2}} z_1(t)\|$$

for any $\theta \in [-\overline{d}, 0]$ and some $\zeta > 1$, by substituting (6.29) and (6.30) to (6.28), it follows that

$$\begin{aligned}
&\dot{V}(z_1(t)) \mid_{(6.20)} \\
&\le -\lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&\quad + \psi_1\left(t, z_1(t), \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t-d(t))\|\right)\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&\quad + \psi_2\left(t, z_1(t), \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t-d(t))\|\right)\|P^{\frac{1}{2}} z_1(t)\| \, \|P^{\frac{1}{2}} z_1(t-d(t))\| \\
&\le -\lambda_{\min}\left(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}\right)\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&\quad + \psi_1\left(t, z_1(t), \zeta\lambda_{\max}(P^{-\frac{1}{2}})\,\|P^{\frac{1}{2}} z_1(t)\|\right)\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&\quad + \zeta\psi_2\left(t, z_1(t), \zeta\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t)\|\right)\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&= -\left(\lambda_{\min}(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}}) - \Theta(t, z_1(t))\right)\|P^{\frac{1}{2}} z_1(t)\|^2 \\
&\le -\gamma\|P^{\frac{1}{2}} z_1(t)\|^2.
\end{aligned} \tag{6.31}$$

Hence, the conclusion follows directly from Lemma 2.5. #

*Remark 6.7* From Theorem 6.1, it follows that the stability of the sliding motion is completely robust to the matched uncertainty $g(\cdot)$ but is affected by the mismatched uncertainty $f(\cdot)$. Since the sliding mode is a reduced-order system, it is clear that only $f_1(\cdot)$ affects the sliding mode and thus in the proposed configuration the limitation on the mismatched uncertainty is weaker than in other work [60, 125, 186] where a similar limitation is imposed on $f(\cdot)$ instead of $f_1(\cdot)$.

*Remark 6.8* From the proof of Theorem 6.1, it follows that in order to establish the stability of the sliding motion, it is necessary to estimate

$$P^{\frac{1}{2}}[f_1(t, z(t), z(t-d(t)))]_{z_2(t)=0}$$

which involves uncertainty and a time-varying delay. It should be pointed out that when the structure of $f_1$ is available, (6.22) may give a less conservative bound than (6.20).

Theorem 6.1 above has shown that, under appropriate conditions, the sliding motion on sliding surface (6.17) is stable. The objective now is to design a controller to drive the system to the sliding surface in finite time. Comparing the linear part of System (6.8)–(6.9) with the linear part of System (6.13)–(6.15), it follows that

$$CT^{-1} = [0 \quad C_2], \qquad TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

where $C_2 \in \mathbb{R}^{p\times p}$ and $B_2 \in \mathbb{R}^{m\times m}$ are nonsingular. From the discussion in Sect. 2.6, it follows that

$$FCB = FCT^{-1}TB = F\underbrace{[0 \quad C_2]}_{\widehat{C}}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = [0 \quad F_2]\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = F_2B_2$$

is nonsingular since both $F_2 \in \mathbb{R}^{m\times m}$ and $B_2 \in \mathbb{R}^{m\times m}$ are nonsingular. Partition the matrices $AT^{-1}$, $A_0T^{-1}$ and $T$ as

$$AT^{-1} := \begin{bmatrix} \varLambda_1 & \varLambda_2 \end{bmatrix}, \qquad A_0T^{-1} := \begin{bmatrix} \varUpsilon_1 & \varUpsilon_2 \end{bmatrix}, \qquad T := \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{6.32}$$

where $\varLambda_1 \in \mathbb{R}^{n\times m}$ and $\varUpsilon_1 \in \mathbb{R}^{n\times m}$ are the first $m$ columns of $AT^{-1}$ and $A_0T^{-1}$ respectively; and $T_1 \in \mathbb{R}^{m\times n}$ and $T_2 \in \mathbb{R}^{(n-m)\times n}$ are the first $m$ and the last $n-m$ rows of $T$. Then, from the analysis above,

$$Tx = \begin{bmatrix} T_1x \\ T_2x \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} T_1x \\ C_2^{-1}y \end{bmatrix}. \tag{6.33}$$

Now, consider the system (6.8)–(6.9) in $\varOmega_1 \times \varOmega_2$ where

$$\varOmega_1 := \{x(t) \mid \|T_1x\| \le \mu_1\} \subset \varOmega \tag{6.34}$$

$$\varOmega_2 := \{x(t-d(t)) \mid \|T_1x(t-d(t))\| \le \mu_2\} \subset \varOmega \tag{6.35}$$

and $T_1$ is defined in (6.32). Then the following output feedback sliding mode controller with time-delay is proposed for the system

$$u = -k(t, y(t), y(t-d(t)))(FCB)^{-1}\text{sgn}(Fy(t)) \tag{6.36}$$

where sgn is the usual signum function and the scalar function $k(\cdot)$ is defined by

$$\begin{aligned} &k(t, y(t), y(t-d(t))) \\ &\quad = \|\varLambda_1\|\mu_1 + \|\varLambda_2C_2^{-1}y\| + \|FCB\|\varpi(t, y(t), y(t-d(t))) \\ &\qquad +, \|FC\|\,\|T^{-1}\|\Big(\rho_1(t, y(t), y(t-d(t)))\left(\mu_1 + \|C_2^{-1}y(t))\|\right) \end{aligned}$$

$$+\rho_2\big(t, y(t), y(t-d(t))\big)\left(\mu_2 + \|C_2^{-1}y(t-d(t))\|\right)\Big) + \eta \tag{6.37}$$

for some $\eta > 0$ where matrices $\Lambda_1$ and $\Lambda_2$ are defined by (6.32), the positive constants $\mu_1$ and $\mu_2$ are given in (6.34)–(6.35), and the functions $\varpi(\cdot)$, $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are given in Assumption 6.2.

*Remark 6.9* It is clear to see that the sliding mode controller (6.36) with $k(\cdot)$ defined by (6.37) is well defined since the matrix $FCB$ is nonsingular and the functions $\varpi(\cdot)$, $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are assumed to be known. Obviously, the proposed control only depends on the time $t$, the known time-delay $d(t)$ and the system output $y(t)$.

**Theorem 6.2** *Consider System (6.8)–(6.9) in $\Omega_1 \times \Omega_2$. Under Assumptions 6.1 and 6.2, the controller (6.36) with the gain $k(\cdot)$ defined by (6.37) drives the system (6.8)–(6.9) to the sliding surface (6.17) in finite time and maintains a sliding motion on it thereafter.*

*Proof* Let $\sigma(x) := FCx$. Then the sliding surface (6.17) can be described by equation $\sigma(x) = 0$. From (6.8) and (6.36), it follows that

$$\begin{aligned}
&\sigma^T\dot{\sigma}\\
&\quad = \sigma^T(x)FC\Big(Ax(t) + A_0x(t-d(t)) + B\left(u(t) + g(t, x(t), x(t-d(t)))\right)\\
&\qquad + f(t, x(t), x(t-d(t)))\Big)\\
&\quad \le \|\sigma(x)\|\Big(\|FC(Ax(t) + A_0x(t-d))\| + \|FCB\|\,\|g(t, x(t), x(t-d(t)))\|\\
&\qquad + \|FC\|\,\|f(t, x(t), x(t-d(t)))\|\Big) - k(t, y(t), y(t-d(t)))\|\sigma(x)\|
\end{aligned} \tag{6.38}$$

where the fact that

$$\sigma^T(x)\mathrm{sgn}(\sigma(x)) \ge \|\sigma(x)\|$$

is used. From (6.33) it follows that in $\Omega_1 \times \Omega_2$ defined by (6.34)–(6.35),

$$\|Tx(t)\| \le \mu_1 + \|C_2^{-1}y(t)\| \tag{6.39}$$

$$\|Tx(t-d(t))\| \le \mu_2 + \|C_2^{-1}y(t-d(t))\|. \tag{6.40}$$

From (6.32) and (6.33),

$$\begin{aligned}
&FC(Ax(t) + A_0x(t-d(t)))\\
&\quad = FC\left(AT^{-1}Tx(t) + A_0T^{-1}Tx(t-d(t))\right)\\
&\quad = FC\left(\begin{bmatrix}\Lambda_1 & \Lambda_2\end{bmatrix}\begin{bmatrix}T_1x\\ C_2^{-1}y\end{bmatrix} + \begin{bmatrix}\Upsilon_1 & \Upsilon_2\end{bmatrix}\begin{bmatrix}T_1x(t-d)\\ C_2^{-1}y(t-d)\end{bmatrix}\right)\\
&\quad = FC\Lambda_1T_1x + FC\Lambda_2C_2^{-1}y + FC\Upsilon_1T_1x(t-d) + FC\Upsilon_2C_2^{-1}y(t-d).
\end{aligned}$$

Therefore, from (6.39)–(6.40),

$$\|FC(Ax(t)+A_0x(t-d(t)))\| \leq \|FC\Lambda_1\|\mu_1+\|FC\Lambda_2C_2^{-1}y(t)\| \\ +\|FC\Upsilon_1\|\mu_2+\|FC\Upsilon_2C_2^{-1}y(t-d)\|. \quad (6.41)$$

From (6.10) and (6.39)–(6.40)

$$\begin{aligned}
&\|f(t,x(t),x(t-d(t)))\| \\
&\quad \leq \rho_1(t,y(t),y(t-d(t)))\,\left\|T^{-1}\right\|\,\|Tx(t)\| \\
&\qquad +\rho_2\big(t,y(t),y(t-d(t))\big)\,\left\|T^{-1}\right\|\,\|Tx(t-d(t))\| \\
&\quad \leq \left\|T^{-1}\right\|\,\Big(\rho_1(t,y(t),y(t-d(t)))\,\big(\mu_1+\|C_2^{-1}y(t)\|\big) \\
&\qquad +\rho_2\big(t,y(t),y(t-d(t))\big)\,\big(\mu_2+\|C_2^{-1}y(t-d(t))\|\big)\Big).
\end{aligned} \quad (6.42)$$

Substituting (6.11), (6.41), (6.42) and (6.37) into (6.38), yields

$$\sigma^T(x)\dot{\sigma}(x) \leq -\eta\|\sigma(x)\|.$$

This shows that the reachability condition [38, 174] is satisfied and thus the conclusion follows. #

Theorems 6.1 and 6.2 together show that the closed-loop system formed by applying control (6.36) with $k(\cdot)$ defined by (6.37) to System (6.8)–(6.9) is uniformly asymptotically stable.

*Remark 6.10* In this section, coordinate transformations are employed to derive the regular form and the sliding mode dynamics, enabling the stability of the sliding motion to be analysed. It should be noted that only static output feedback control is considered in this section. In the control design, a state transformation (6.33) is introduced to separate the known parts $C_2^{-1}y(t)$ and $C_2^{-1}y(t-d(t))$ from $Tx(t)$ and $Tx(t-d(t))$ respectively, so that they can be used in the control design to reduce conservatism and avoid unnecessary control action. This ensures the conclusion holds, possibly in an unbounded domain, since the constraint (6.34)–(6.35) only corresponds to a subset of the state variables.

## 6.3 Reduced-Order Observer-Based Sliding Mode Control

In this section, a stabilisation problem for a class of nonlinear time-delay systems with time-delay disturbances is considered based on a reduced-order observer. Both the observer and the controller are based on a sliding mode approach and the proposed scheme is robust.

## 6.3.1 Introduction

A static output feedback sliding mode control scheme has been proposed in Sect. 6.2. However, strong conditions are necessarily imposed on the considered system given the system dynamical information is not available. Therefore, static output control may not be appropriate for some systems. This has motivated the study of dynamical output feedback control in which not only the system output, but also additional dynamics are employed [112, 133].

In this section, a reduced-order observer is designed for a time-delay nonlinear system using structural characteristics. Based on the estimated states and system outputs, a sliding surface is proposed and the associated sliding mode dynamics, which are nonlinear and time-delayed, are derived using an equivalent control approach and an appropriate coordinate transformation. A sufficient condition is developed, based on the Lyapunov–Razumikhin approach, such that the sliding motion is uniformly asymptotically stable. A sliding mode control law dependent only on the time, the system output and the designed reduced-order dynamical system states, is developed to drive the system to the sliding surface in finite time and maintains a sliding motion thereafter.

Although the delay is assumed to be known, it is allowed to be time varying and the limitation that the time derivative of the delay is less than unity, which is required when the Lyapunov–Karasovskii approach is employed, is not necessary.

## 6.3.2 System Description

Consider a time-varying delay system

$$\dot{x} = Ax + Bu + f\big(t, x, x_d\big) + E\Delta f(t, x, x_d) \tag{6.43}$$

$$y = Cx \tag{6.44}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are states, inputs and outputs respectively; $x_d := x(t - d)$ denotes the delayed state, for simplicity, where $d := d(t)$ is the time-varying delay which is assumed to be known, nonnegative and bounded in $\mathbb{R}^+$, and thus

$$\bar{d} := \sup_{t \in \mathbb{R}^+} \{d(t)\} < \infty.$$

The initial condition associated with the delay is given by

$$x(t) = \phi(t), \qquad t \in [-\bar{d}, 0] \tag{6.45}$$

where $\phi(t) \in \Theta$ with $\Theta$ the admissible initial condition set defined by

$$\Theta = \{\phi(t) \mid \phi(t) \in \mathscr{C}_{[-\bar{d},0]},\ \|\phi(t)\| \le l_0\} \tag{6.46}$$

for some constant $l_0 > 0$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $E \in \mathbb{R}^{n\times q}$ $(m \le p < n)$ are constant with both $B$ and $C$ of full rank. The vector functions $f(\cdot) \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathbb{R}^q$ are, respectively, the known nonlinear term and the nonlinear uncertainty experienced by the system. It is assumed that all the nonlinear functions are smooth enough such that the unforced system has unique continuous solutions.

**Assumption 6.4** The pair $(A, B)$ is controllable and the pair $(A, C)$ is observable.

From the fact that $(A, C)$ is observable, there exists a matrix $L_0$ such that $A - L_0C$ has $n$ eigenvalues which lie in the open left-half plane. Then, the inequality

$$(A - L_0C)^T P_0 + P_0(A - L_0C) < 0 \tag{6.47}$$

is solvable for $P_0 > 0$.

**Assumption 6.5** The matrix equation

$$E^T P_0 = FC \tag{6.48}$$

is solvable for $F \in \mathbb{R}^{q\times p}$ where $P_0$ satisfies (6.47).

*Remark 6.11* Assumption 6.5 implies that (6.47) obeys the constraint (6.48), which can be considered as the standard Constrained Lyapunov Problem (CLP) [57]. Similar limitations have been employed in [24, 26, 206, 214]. Several algorithms to solve the CLP have been discussed in [41, 57]. In particular, recently in [41], it has been shown that with special parameterisations of the variables $L_0$, $P_0$ and $F$, this observer version of the CLP can be written in the form of strict LMIs and solved using any of the convex solvers commonly available.

In this section, local results are sought for the time-delayed system (6.43)–(6.44). In order to avoid difficulties in describing the domain under consideration, the term "a neighbourhood of the origin" is used to express the domain throughout the section. Also, since the output matrix $C$ is full row rank, without loss of generality, it is assumed that $C = [I_p \quad 0]$ at the outset.

**Assumption 6.6** The nonlinear function $f\big(t, x, x_d\big)$ is Lipschitz with respect to $x$ and $x_d$ uniformly for $t \in \mathbb{R}^+$, and has the decomposition

$$f\big(t, x, x_d\big) = \Phi(t, x)x_d \tag{6.49}$$

where $\Phi(\cdot) \in \mathbb{R}^{n\times n}$ is continuous and Lipschitz with respect to $x$ and uniformly for $t \in \mathbb{R}^+$.

**Assumption 6.7** The uncertainty satisfies

$$\|\Delta f(t, x, x_d)\| \le \rho(t, x)\|x_d\|$$

where the function $\rho(\cdot)$ is known continuous and Lipschitz with respect to $x$ uniformly for $t \in \mathbb{R}^+$.

*Remark 6.12* Assumptions 6.6 and 6.7 require that the nonlinear terms $f(\cdot)$ and $\Delta f(\cdot)$ are affine about the delayed variable $x_d$, which limits the class of nonlinear functions $f(\cdot)$ and uncertainties $\Delta f(\cdot)$. However, they include the linear situation as a special case. Moreover, the assumption that the delay is not involved in the functions $\Phi(\cdot)$ and $\rho(\cdot)$ is employed mainly for the simplification of the presentation of the subsequent analysis, and is not an inherent limitation. The approach proposed can be extended to the case when $\Phi(\cdot)$ and $\rho(\cdot)$ have time-delay in the form of $\Phi(t, x, x_d)$ and $\rho(t, x, x_d)$.

### 6.3.3 Reduced-Order Observer Design

In this section, a reduced-order dynamical system is proposed and a sufficient condition in terms of simple LMIs is developed.

Partition System (6.43)–(6.44) in a compatible way to $C = [I_p \quad 0]$. The system can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} f_1(t, x, x_d) \\ f_2(t, x, x_d) \end{bmatrix} + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \Delta f(t, x, x_d) \tag{6.50}$$

$$y = x_1 \tag{6.51}$$

where $x = \mathrm{col}(x_1, x_2)$ with $x_1 \in \mathbb{R}^p$, $A_1 \in \mathbb{R}^{p\times p}$, $B_1 \in \mathbb{R}^{p\times m}$; $f_1(\cdot)$ is the first $p$ components of $f(\cdot)$ and $E_1$ is the first $p$ rows of the matrix $E$.

Consider Inequality (6.47). Partition $P_0$ and $L_0$ conformably with the decomposition (6.50)–(6.51) as

$$P_0 = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \quad \text{and} \quad L_0 = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}. \tag{6.52}$$

Applying the partition above to the matrix equation $E^T P_0 = FC$ in Assumption 6.5, it follows from $C = [I_p \quad 0]$ that

$$E_1^T P_2 + E_2^T P_3 = 0$$

and thus from $P_3 > 0$,

$$P_3^{-1} P_2^T E_1 + E_2 = 0.$$

Then, introduce a nonsingular coordinate transformation $w = \hat{T}x$ defined by

$$\begin{aligned} w_1 &= x_1 \\ w_2 &= P_3^{-1}P_2^T x_1 + x_2 \end{aligned} \tag{6.53}$$

where $w = \text{col}(w_1, w_2)$ with $w_1 \in \mathbb{R}^p$. Then, in the new coordinates system $w$, the system (6.50)–(6.51) is described by

$$\dot{w}_1 = (A_1 - A_2P_3^{-1}P_2^T)w_1 + A_2w_2 + B_1u + F_1(t, w, w_d) + E_1\Delta F(t, w, w_d) \tag{6.54}$$

$$\begin{aligned} \dot{w}_2 = {} & (P_3^{-1}P_2^T(A_1 - A_2P_3^{-1}P_2^T) + A_3 - A_4P_3^{-1}P_2^T)w_1 \\ & + (A_4 + P_3^{-1}P_2^T A_2)w_2 + \begin{bmatrix} P_3^{-1}P_2^T & I_{n-p} \end{bmatrix} Bu \\ & + \begin{bmatrix} P_3^{-1}P_2^T & I_{n-p} \end{bmatrix} F(t, w, w_d) \end{aligned} \tag{6.55}$$

$$y = w_1 \tag{6.56}$$

where $w_{1d} \in \mathbb{R}^p$, $w_d = \text{col}(w_{1d}, w_{2d})$,

$$F(t, w, w_d) := \big[f(t, x, x_d)\big]_{x=\hat{T}^{-1}w} \tag{6.57}$$

$$\Delta F(t, w, w_d) := \big[\Delta f(t, x, x_d)\big]_{x=\hat{T}^{-1}w} \tag{6.58}$$

and $F_1(\cdot)$ in (6.54) is the first $p$ components of the vector $F(\cdot)$ in (6.57). Construct a dynamical system

$$\begin{aligned} \dot{\hat{w}}_2 = {} & (A_4 + P_3^{-1}P_2^T A_2)\hat{w}_2 + \big(P_3^{-1}P_2^T(A_1 - A_2P_3^{-1}P_2^T) + A_3 - A_4P_3^{-1}P_2^T\big)\, y \\ & + \big[P_3^{-1}P_2^T \quad I_{n-p}\big] Bu + \big[P_3^{-1}P_2^T \quad I_{n-p}\big] F(t, y, \hat{w}_2, y_d, \hat{w}_{2d}) \end{aligned} \tag{6.59}$$

where $\hat{w}_2 \in \mathbb{R}^{n-p}$ and $F(\cdot)$ is defined in (6.57). The initial condition associated with the delay $d(t)$ is given by

$$\hat{w}_2(t) = [P_3^{-1}P_2^T \quad I_{n-p}]\phi(t)$$

for $t \in [-\bar{d}, 0]$ where $\phi(t)$ is given in (6.45).

**Theorem 6.3** *Suppose that Assumptions 6.4–6.7 are satisfied in a neighbourhood of the origin. Then, there exist positive constants $\alpha_1$ and $\alpha_2$ such that in the neighbourhood of the origin*

$$\|w_2(t) - \hat{w}_2(t)\| \le \alpha_1 \exp\{-\alpha_2 t\} \tag{6.60}$$

*if there exists a constant $\varepsilon > 0$ such that*

$$\bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + 2P_3 < 0 \tag{6.61}$$

$$\varepsilon \mathscr{L}_f^2 I_{n-p} - P_3 \le 0 \tag{6.62}$$

*hold where*

$$\bar{A} = \begin{bmatrix} A_2^T & A_4^T \end{bmatrix}^T, \qquad \bar{P} = \begin{bmatrix} P_2^T & P_3 \end{bmatrix}. \tag{6.63}$$

*Proof* Let $e(t) = w_2(t) - \hat{w}_2(t)$. From (6.55) and (6.59),

$$\dot{e} = \left(A_4 + P_3^{-1} P_2^T A_2\right) e + \begin{bmatrix} P_3^{-1} P_2^T & I_{n-p} \end{bmatrix} \delta(F, \hat{F}) \tag{6.64}$$

where the symbol

$$\delta(F, \hat{F}) := F(t, w, w_d) - F(t, y, \hat{w}_2, y_d, \hat{w}_{2d}) \tag{6.65}$$

is introduced to simplify the notation. Since $P_0 > 0$, it follows from (6.52) that $P_3 > 0$. For System (6.64), consider a Lyapunov function candidate

$$V_1 = e^T(t) P_3 e(t).$$

Then, the time derivative of $V_1$ along the trajectories of System (6.64) is described by

$$\dot{V}_1 = e^T(t) \left(\bar{A}^T \bar{P}^T + \bar{P}\bar{A}\right) e(t) + 2e^T(t)\bar{P}\delta(F, \hat{F}) \tag{6.66}$$

where $\bar{A}$ and $\bar{P}$ are defined in (6.63). Exploiting Young's inequality $2X^T Y \leq \frac{1}{\varepsilon} X^T X + \varepsilon Y^T Y$ for any scalar $\varepsilon > 0$,

$$\begin{aligned} 2e^T(t)\bar{P}\delta(F, \hat{F}) &\leq \frac{1}{\varepsilon} e^T(t)\bar{P}\bar{P}^T e(t) + \varepsilon \|\delta(F, \hat{F})\|^2 \\ &= \frac{1}{\varepsilon} e^T(t)\bar{P}\bar{P}^T e(t) + \varepsilon \mathscr{L}_f^2 \left(e^T(t)e(t) + e_d^T(t)e_d(t)\right) \end{aligned} \tag{6.67}$$

where Lemma A.3 in Appendix A.2 is used. Substituting (6.67) into (6.66),

$$\dot{V}_1 \leq e^T(t)\left(\bar{A}^T \bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + \varepsilon \mathscr{L}_f^2 I_{n-p}\right) e(t) + \varepsilon \mathscr{L}_f^2 e_d^T(t) e_d(t). \tag{6.68}$$

If (6.61) holds, then there exists a constant $\gamma > 0$ such that

$$\Gamma := -\left(\bar{A}^T \bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + 2P_3 + \gamma P_3\right) > 0. \tag{6.69}$$

Consequently, when $V_1(e(t-d)) \leq (1+\gamma)V_1(e(t))$ for any $d \in [0, \bar{d}]$, the inequality

$$(1+\gamma)e^T(t)P_3 e(t) - e_d^T(t) P_3 e_d(t) \geq 0$$

holds and thus from (6.62),

$$\begin{aligned}\dot{V}_1 &\leq e^T\Big(\bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + \varepsilon\mathscr{L}_f^2 I_{n-p}\Big)e + \varepsilon\mathscr{L}_f^2 e_d^T e_d \\ &\quad + (1+\gamma)e^T P_3 e - e_d^T P_3 e_d \\ &\leq e^T\Big(\bar{A}^T\bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + 2P_3 + \gamma P_3\Big)e \\ &\leq -e^T\Gamma e \leq -\tfrac{\lambda_{\min}(\Gamma)}{\lambda_{\max}(P_3)}V_1 \end{aligned} \tag{6.70}$$

where $\Gamma$ is defined in (6.69). Inequality (6.70) implies that

$$V_1(e(t)) \leq e(0)^T P_3 e(0)\exp\{-\tfrac{\lambda_{\min}(\Gamma)}{\lambda_{\max}(P_3)}t\}. \tag{6.71}$$

Notably, there exists a neighbourhood of the origin such that for any initial value $w_2(0)$ in the neighbourhood of the origin, there exists a constant $b > 0$ and the initial value $\hat{w}_2(0)$ for the designed dynamical system (6.59) such that

$$\|e(0)\| = \|w_2(0) - \hat{w}_2(0)\| \leq b.$$

Then, from (6.71)

$$\lambda_{\min}(P_3)\|e(t)\|^2 \leq b^2\lambda_{\max}(P_3)\exp\{-\tfrac{\lambda_{\min}(\Gamma)}{\lambda_{\max}(P_3)}t\}. \tag{6.72}$$

Let

$$\alpha_1 = \sqrt{\lambda_{\max}(P_3)/\lambda_{\min}(P_3)}\, b \ \text{ and } \alpha_2 = \tfrac{1}{2}\tfrac{\lambda_{\min}(\Gamma)}{\lambda_{\max}(P_3)}. \tag{6.73}$$

Then it is clear from (6.72) that (6.60) holds. Hence the conclusion follows. □

*Remark 6.13* A necessary condition for Inequality (6.61) to hold is that

$$\bar{A}^T\bar{P}^T + \bar{P}\bar{A} < 0. \tag{6.74}$$

However, from partitions (6.50) and (6.52) and the fact that $C = [I_p \quad 0]$, it follows that

$$(A - L_0C)^T P_0 + P_0(A - L_0C) = \begin{bmatrix} * & * \\ * & \bar{A}^T\bar{P}^T + \bar{P}\bar{A} \end{bmatrix}$$

where $*$ represents entries which do not need to be shown. It follows from (6.47) that (6.74) holds.

**Corollary 6.1** *Let $\hat{x}_2 := \hat{w}_2 - P_3^{-1}P_2^T y$ where $\hat{w}_2$ is the state of System (6.59). Then, under the conditions of Theorem 6.3, the inequality*

$$\|x_2(t) - \hat{x}_2(t)\| \leq \alpha_1\exp\{-\alpha_2 t\}$$

*holds in a neighbourhood of the origin, where the constants $\alpha_1$ and $\alpha_2$ are those in (6.60).*

*Proof* From transformation (6.53) and $y = x_1$,

$$\begin{aligned} x_2 - \hat{x}_2 &= w_2 - P_3^{-1} P_2^T x_1 - \hat{w}_2 + P_3^{-1} P_2^T y \\ &= w_2 - \hat{w}_2. \end{aligned}$$

Hence, the conclusion follows from Theorem 6.3. □

### 6.3.4 Sliding Surface Design

In this section, a sliding surface based on the system output $y$ and the estimated state $\hat{x}_2$ given in Corollary 6.1 will be designed.

From sliding mode control theory, for a controllable pair $(A, B)$, there exists a matrix $S \in \mathbb{R}^{m \times n}$ such that

- the matrix $SB$ is nonsingular;
- the matrix $A - B(SB)^{-1}SA$ has $n - m$ eigenvalues which lie in the open left-half plane.

Such a matrix $S$ can be designed using any existing state feedback sliding mode control design methodology [38, 174]. Then, partition the matrix $S$ as $S = [S_1 \quad S_2]$ where $S_1 \in \mathbb{R}^{m \times p}$ and $S_2 \in \mathbb{R}^{m \times (n-p)}$. Choose a switching function as

$$\sigma(y, \hat{x}_2) = S_1 y + S_2 \hat{x}_2 \tag{6.75}$$

then, the sliding surface, which is defined in an augmented space, is described by

$$\left\{ (x_1, x_2, \hat{x}_2) \mid S_1 y + S_2 \hat{x}_2 = 0 \right\}. \tag{6.76}$$

In coordinates $(x, e)$ where $e = x_2 - \hat{x}_2$, System (6.43) and the error dynamics (6.64) can be written as

$$\dot{x} = Ax + Bu + f(t, x, x_d) + E\Delta f(t, x, x_d) \tag{6.77}$$

$$\dot{e} = \left(A_4 + P_3^{-1} P_2^T A_2\right) e + \left[P_3^{-1} P_2^T \quad I_{n-p}\right] \delta_e(f, \hat{f}) \tag{6.78}$$

$$y = x_1 \tag{6.79}$$

where

$$\delta_e(f, \hat{f}) := f(t, x, x_d) - f(t, y, x_2 + e, y_d, x_{2d} + e_d). \tag{6.80}$$

From $x_2 - \hat{x}_2 = e$, it follows that

$$S_1 y + S_2 \hat{x}_2 = S_1 x_1 + S_2 x_2 - S_2 e = Sx - S_2 e.$$

Therefore, in the augmented space $(x, e)$, the sliding surface (6.76) can be described by

$$\left\{ (x, e) \mid Sx - S_2 e = 0 \right\}. \tag{6.81}$$

### 6.3.5 Equivalent System and Sliding Mode Dynamics

A sufficient condition will be developed to guarantee the uniform asymptotic stability of the corresponding sliding motion in this section.

Consider System (6.77)–(6.79) with the sliding surface given in (6.81). During a sliding motion, $S\dot{x} - S_2\dot{e} = 0$. From equivalent control theory [38, 174], it follows that

$$SAx + SBu_{eq} + Sf(\cdot) + SE\Delta f(\cdot) - S_2\big(A_4 + P_3^{-1}P_2^T A_2\big)e \\ -S_2P_3^{-1}\begin{bmatrix}P_2^T & P_3\end{bmatrix}\delta_e(f,\hat{f}) = 0 \tag{6.82}$$

where $u_{eq}$ is the equivalent control. Since $SB$ is nonsingular by design, $u_{eq}$ can be described by

$$u_{eq} = -(SB)^{-1}SAx - (SB)^{-1}S(f(\cdot) + E\Delta f(\cdot)) + (SB)^{-1}S_2\big(A_4 + P_3^{-1}P_2^T A_2\big)e \\ + (SB)^{-1}S_2P_3^{-1}\begin{bmatrix}P_2^T & P_3\end{bmatrix}\delta_e(f,\hat{f}). \tag{6.83}$$

The equivalent control $u_{eq}$, involves the uncertainty $\Delta f(\cdot)$, and is not the real control signal applied to the plant but is used to analyse the sliding mode dynamics. It can be thought of as the average effect of the applied discontinuous injection signal. Substituting $u_{eq}$ in (6.83) into Eqs. (6.77)–(6.78), the equivalent system governing the sliding motion is given by

$$\dot{x} = (A - B(SB)^{-1}SA)x + B(SB)^{-1}S_2(A_4 + P_3^{-1}P_2^T A_2)e \\ + \big(I_n - B(SB)^{-1}S\big)\big(f(t,x,x_d) + E\Delta f(\cdot)\big) \\ + B(SB)^{-1}S_2P_3^{-1}\begin{bmatrix}P_2^T & P_3\end{bmatrix}\delta_e(f,\hat{f}) \tag{6.84}$$

$$\dot{e} = (A_4 + P_3^{-1}P_2^T A_2)e + \begin{bmatrix}P_3^{-1}P_2^T & I_{n-p}\end{bmatrix}\delta_e(f,\hat{f}) \tag{6.85}$$

$$y = x_1 \tag{6.86}$$

where $\delta_e(f,\hat{f})$ is defined in (6.80). From sliding mode control theory, the reduced-order sliding mode dynamics are given by Eqs. (6.84)–(6.85) limited to the sliding surface (6.81). Since by design $SB$ is nonsingular, the matrix $S$ is full row rank, and thus there exists a nonsingular matrix $T$ such that

$$ST = [I_m \quad 0] \tag{6.87}$$

where $T$ can be obtained using elementary column operations. Introduce a coordinate transformation $z = T^{-1}x$. Then, in the new coordinates $(z, e)$, System (6.84)–(6.85) can be described by

$$\begin{aligned}\dot{z} &= T^{-1}(A - B(SB)^{-1}SA)Tz + T^{-1}B(SB)^{-1}S_2(A_4 + P_3^{-1}P_2^TA_2)e \\ &\quad + T^{-1}\left(I_n - B(SB)^{-1}S\right)\left(f(t, Tz, Tz_d) + E\Delta f(t, Tz, Tz_d)\right) \\ &\quad + T^{-1}B(SB)^{-1}S_2P_3^{-1}\begin{bmatrix}P_2^T & P_3\end{bmatrix}[\delta_e(f, \hat{f})]_{x=Tz} \\ \dot{e} &= (A_4 + P_3^{-1}P_2^TA_2)e + P_3^{-1}\begin{bmatrix}P_2^T\ P_3\end{bmatrix}[\delta_e(f, \hat{f})]_{x=Tz}.\end{aligned}$$

Partitioning appropriately

$$\begin{aligned}T^{-1}(A - B(SB)^{-1}SA)T &= \begin{bmatrix}D_1 & D_2 \\ D_3 & D_4\end{bmatrix} \\ T^{-1}B(SB)^{-1}S_2(A_4 + P_3^{-1}P_2^TA_2) &= \begin{bmatrix}M_1 \\ M_2\end{bmatrix} \\ T^{-1}\left(I_n - B(SB)^{-1}S\right) &= \begin{bmatrix}N_1 \\ N_2\end{bmatrix} \\ T^{-1}B(SB)^{-1}S_2P_3^{-1}[P_2^T \quad P_3] &= \begin{bmatrix}H_1 \\ H_2\end{bmatrix}\end{aligned}$$

where $D_4 \in \mathbb{R}^{(n-m)\times(n-m)}$, $M_2 \in \mathbb{R}^{(n-m)\times n}$, $H_2 \in \mathbb{R}^{(n-m)\times n}$ and $N_2 \in \mathbb{R}^{(n-m)\times n}$. Then, System (6.88)–(6.88) can be rewritten as

$$\begin{aligned}\dot{z}_1 &= D_1z_1 + D_2z_2 + M_1e + N_1f(t, Tz, Tz_d) + N_1E\Delta f(t, Tz, Tz_d) \\ &\quad + H_1[\delta_e(f, \hat{f})]_{x=Tz} \qquad (6.88)\\ \dot{z}_2 &= D_3z_1 + D_4z_2 + M_2e + N_2f(t, Tz, Tz_d) + N_2E\Delta f(t, Tz, Tz_d) \\ &\quad + H_2[\delta_e(f, \hat{f})]_{x=Tz} \qquad (6.89)\\ \dot{e} &= (A_4 + P_3^{-1}P_2^TA_2)e + P_3^{-1}\begin{bmatrix}P_2^T\ P_3\end{bmatrix}[\delta_e(f, \hat{f})]_{x=Tz} \qquad (6.90)\end{aligned}$$

where $z = \mathrm{col}(z_1, z_2)$ with $z_1 \in \mathbb{R}^m$ and $z_2 \in \mathbb{R}^{n-m}$. Comparing System (6.88)–(6.90) with System (6.84)–(6.85), it follows that the matrix

$$\left[\begin{array}{cc|c}D_1 & D_2 & M_1 \\ D_3 & D_4 & M_2 \\ \hline 0 & 0 & A_4 + P_3^{-1}P_2^TA_2\end{array}\right] =: A_{eq} \qquad (6.91)$$

has $2n - p - m$ eigenvalues which lie in the open left-half plane. From (6.87) and $z = T^{-1}x$,

$$Sx - S_2e = z_1 - S_2e.$$

Then, in the new coordinates $(z, e)$, the sliding surface (6.81) is described by

$$\left\{(z_1, z_2, e) \mid z_1 - S_2e = 0\right\}. \qquad (6.92)$$

Partition $T$ as $T = [T_1 \quad T_2]$ where $T_1 \in \mathbb{R}^{n\times m}$ and $T_2 \in \mathbb{R}^{n\times(n-m)}$. On the sliding surface (6.92),

$$x = [T_1 \quad T_2]\begin{bmatrix} S_2 e \\ z_2 \end{bmatrix} = T_1 S_2 e + T_2 z_2. \tag{6.93}$$

Thus, by using (6.93), the reduced-order sliding mode dynamics is the equivalent system (6.88)–(6.90) restricted to the sliding surface (6.92), which is described by

$$\begin{aligned} \dot{z}_2 = {} & D_4 z_2 + (M_2 + D_3 S_2)e + N_2 f(t, T_1 S_2 e + T_2 z_2, T_1 S_2 e_d + T_2 z_{2d}) \\ & + N_2 E \Delta f(t, T_1 S_2 e + T_2 z_2, T_1 S_2 e_d + T_2 z_{2d}) + H_2 \delta_T(f, \hat{f}) \end{aligned} \tag{6.94}$$

$$\dot{e} = (A_4 + P_3^{-1} P_2^T A_2)e + P_3^{-1}\left[P_2^T \quad P_3\right]\delta_T(f, \hat{f}) \tag{6.95}$$

where

$$\delta_T(f, \hat{f}) := [f(t, x, x_d) - f(t, y, x_2 + e, y_d, x_{2d} + e_d)]_{x = T\left[\begin{smallmatrix} S_2 e \\ z_2 \end{smallmatrix}\right]} \tag{6.96}$$

### 6.3.6 Stability of Sliding Motion

A sufficient condition which guarantees the stability of the sliding motion will be proposed in terms of a set of matrix inequalities.

**Theorem 6.4** *Suppose that Assumptions 6.4–6.7 are satisfied in a neighbourhood of the origin. Then, System (6.43)–(6.44) has a uniformly asymptotically stable sliding motion with respect to the sliding surface (6.76) if there exists a matrix $P > 0$ such that in a neighbourhood of the origin in the state space $(z_2, e)$, the matrix inequality*

$$W := \begin{bmatrix} X_1 & \Lambda & PN_2\Phi(\cdot)T_2 & PN_2\Phi(\cdot)T_1 S_2 \\ * & X_2 & 0 & 0 \\ * & * & X_3 & \varepsilon_1 \rho^2(\cdot) T_2^T T_1 S_2 \\ * & * & * & X_4 \end{bmatrix} < 0$$

*holds with $q_0 := \inf\{\lambda_{\max}(W(\cdot))\} < 0$, where*

$$\Lambda := P\left(M_2 + D_3 S_2 + \frac{1}{\varepsilon_2} H_2 \bar{P}^T\right).$$

*In the above $W$ is symmetric, the $*$s represent the corresponding symmetric entries, and*

$$\begin{aligned} X_1 &:= D_4^T P + PD_4 + \tfrac{1}{\varepsilon_1} PN_2 EE^T N_2^T P + \tfrac{1}{\varepsilon_2} PH_2 H_2^T P + q_2 P \\ X_2 &:= \bar{A}^T \bar{P}^T + \bar{P}\bar{A} + \tfrac{1}{\varepsilon_2}\bar{P}\bar{P}^T + \varepsilon_2 \mathcal{L}_f^2 I_{n-p} + q_2 P_3 \\ X_3 &:= \varepsilon_1 \rho^2(\cdot) T_2^T T_2 - P \\ X_4 &:= \varepsilon_1 \rho^2(\cdot)(T_1 S_2)^T T_1 S_2 + \varepsilon_2 \mathcal{L}_f^2 I_{n-p} - P_3 \end{aligned}$$

*where the constants $q_2 > 1$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, $P_3$ is defined in (6.52), and $\bar{A}$ and $\bar{P}$ are defined by (6.63).*

*Proof* The analysis above shows that (6.94)–(6.95) represents the sliding mode dynamics in the coordinates $(z, e)$. It is only required to prove that System (6.94)–(6.95) is uniformly asymptotically stable. Consider a Lyapunov function candidate

$$V(z_2(t), e(t)) = z_2^T(t)Pz_2(t) + e^T(t)P_3e(t). \tag{6.97}$$

Then, the time derivative of $V(\cdot)$ along the trajectories of the dynamic system (6.94)–(6.95) is given as

$$\begin{aligned}\dot{V} &= z_2^T(D_4^TP + PD_4)z_2 + 2z_2^TP(M_2 + D_3S_2)e + 2z_2^TPN_2f(\cdot)\\ &\quad + 2z_2^TPN_2E\Delta f(\cdot) + e^T\left(\bar{A}^T\bar{P}^T + \bar{P}\bar{A}\right)e\\ &\quad + 2\left(e^T\bar{P} + z_2^TPH_2\right)\delta_T(f, \hat{f})\end{aligned} \tag{6.98}$$

where $\bar{A}$ and $\bar{P}$ are defined in (6.63). From (6.49),

$$\begin{aligned}z_2^TPN_2f(\cdot) &= z_2^TPN_2\Phi(t, T_1S_2e + T_2z_2)(T_1S_2e_d\\ &\quad + T_2z_{2d}) = z_2^TPN_2\Phi(\cdot)T_1S_2e_d + z_2^TPN_2\Phi(\cdot)T_2z_{2d}.\end{aligned} \tag{6.99}$$

From Assumption 6.7 and Young's inequality $2X^TY \le \frac{1}{\varepsilon}X^TX + \varepsilon Y^TY$ for any $\varepsilon > 0$, it follows that

$$\begin{aligned}2z_2^TPN_2E\Delta f(\cdot) &\le \frac{1}{\varepsilon_1}z_2^TPN_2E(PN_2E)^Tz_2\\ &\quad + \varepsilon_1\|\Delta f(t, T_1S_2e + T_2z_2, T_1S_2e_d + T_2z_{2d})\|^2\\ &\le \frac{1}{\varepsilon_1}z_2^TPN_2E(PN_2E)^Tz_2 + \varepsilon_1\rho^2(\cdot)\big(e_d^T(T_1S_2)^TT_1S_2e_d\\ &\quad + 2z_{2d}^TT_2^TT_1S_2e_d\big)\end{aligned} \tag{6.100}$$

and from (A.6) in Appendix A.2,

$$\begin{aligned}2\left(e^T\bar{P} + z_2^TPH_2\right)\delta_T(f, \hat{f}) &\le \frac{1}{\varepsilon_2}(e^T\bar{P}\bar{P}^Te + z_2^TPH_2(PH_2)^Tz_2 + 2z_2^TPH_2\bar{P}^Te)\\ &\quad + \varepsilon_2\mathscr{L}_f^2(e^Te + e_d^Te_d).\end{aligned} \tag{6.101}$$

If

$$V(z_{2d}, e_d) \le q_2V(z_2, e)$$

for some $q_2 > 1$, then from the definition of $V(\cdot)$ in (6.97)

$$q_2\left(z_2^T(t)Pz_2(t) + e^T(t)P_3e(t)\right) - z_{2d}^T(t)Pz_{2d}(t) - e_d^T(t)P_3e_d(t) \ge 0. \tag{6.102}$$

Substituting (6.99)–(6.101) into (6.98) and employing (6.102),

$$\begin{aligned}
\dot{V} \leq\ & z_2^T(D_4^TP + PD_4)z_2 + 2z_2^TP(M_2 + D_3S_2)e + 2z_2^TPN_2\Phi(\cdot)T_1S_2e_d \\
& + 2z_2^TPN_2\Phi(\cdot)T_2z_{2d} + \frac{1}{\varepsilon_1}z_2^TPN_2E(PN_2E)^Tz_2 \\
& + \varepsilon_1\rho^2(\cdot)\Big(e_d^T(T_1S_2)^TT_1S_2e_d + z_{2d}^TT_2^TT_2z_{2d} + 2z_{2d}^TT_2^TT_1S_2e_d\Big) \\
& + e^T\left(\bar{A}^T\bar{P}^T + \bar{P}\bar{A}\right)e + \frac{1}{\varepsilon_2}\Big(e^T\bar{P}\bar{P}^Te + z_2^TPH_2(PH_2)^Tz_2 + 2z_2^TPH_2\bar{P}^Te\Big) \\
& + \varepsilon_2\mathscr{L}_f^2(e^Te + e_d^Te_d) + q_2\left(z_2^T(t)Pz_2(t) + e^T(t)P_3e(t)\right) \\
& - z_{2d}^T(t)Pz_{2d}(t) - e_d^T(t)P_3e_d(t) \\
=\ & YWY^T \leq q_0\left(\|z_2\|^2 + \|e\|^2\right)
\end{aligned}$$

where $Y := [z_2\ \ e\ \ z_{2d}\ \ e_d]$, and $q_0 < 0$ is used in the last inequality. Hence, by applying Lemma A.1 in Appendix A.1, the conclusion follows. #

*Remark 6.14* The necessary condition for $X_1 < 0$ is that $D_4^TP + PD_4 < 0$ is solvable for $P > 0$. Since (6.74) can be rewritten as

$$\left(P_3^{-1}P_2^TA_2 + A_4\right)^T P_3 + P_3\left(P_3^{-1}P_2^TA_2 + A_4\right) < 0$$

which shows that $A_4 + P_3^{-1}P_2^TA_2$ is stable because $P_3 > 0$, and $A_4 + P_3^{-1}P_2^TA_2$ has $n - p$ eigenvalues in the open left-half plane. Since $A_{eq}$ defined in (6.91) has $2n - p - m$ eigenvalues which lies in the open left-half plane, the matrix

$$\begin{bmatrix} D_4 & M_2 + D_3S_2 \\ 0 & A_4 + P_3^{-1}P_2^TA_2 \end{bmatrix}$$

associated with the sliding mode dynamics in (6.94)–(6.95), has $2n - p - m$ eigenvalues which lie in the open left-half plane. Therefore $D_4$ has $n - m$ eigenvalues which lie in the open left-half plane, which in turn implies that the matrix inequality

$$D_4^TP + PD_4 < 0$$

is solvable for $P > 0$.

*Remark 6.15* Since the sliding motion is completely robust to the matched contribution, it should be emphasised that the function $\Phi(\cdot)$ in Theorem 6.4 can be replaced by $\Phi_1(\cdot)$ if $f(\cdot)$ in System (6.43) can be expressed as

$$f(\cdot) = Bf_0(\cdot) + \Phi_1(\cdot)x_d$$

which may reduce conservatism.

### 6.3.7 Sliding Mode Control Design

In this section, a control law will be proposed such that the reachability condition [38, 174]

$$\sigma^T(y,\hat{x}_2)\dot{\sigma}(y,\hat{x}_2) \le -\eta\|\sigma(y,\hat{x}_2)\| \tag{6.103}$$

is satisfied for some constant $\eta > 0$ where $\sigma(\cdot)$ is the sliding function designed in (6.75).

Consider System (6.43)–(6.44) in a neighbourhood of the origin defined by

$$\{x \mid \|x\| \le l_1\}.$$

Let

$$l := \max\{l_1, l_0\}$$

where $l_0$ is given in (6.46). The following control law is proposed:

$$\begin{aligned}u = -(SB)^{-1}\Big(&\big(S_1A_1 - S_2P_3^{-1}P_2^TA_2P_3^{-1}P_2^T + S_2(A_3 - A_4P_3^{-1}P_2^T)\big)y\\&+ S_2\left(A_4 + P_3^{-1}P_2^TA_2\right)\hat{w}_2 + (S_1 - S_2P_3^{-1}P_2^T)A_2\hat{x}_2\\&+ l\rho(t,y,\hat{x}_2)\|(S_1 - S_2P_3^{-1}P_2^T)E_1\|\\&+ Sf(t,y,\hat{x}_2,y_d,\hat{x}_{2d})\Big) - k(\cdot)(SB)^{-1}\frac{\sigma(y,\hat{x}_2)}{\|\sigma(y,\hat{x}_2)\|}\end{aligned} \tag{6.104}$$

where the function $k(\cdot)$ is the control gain to be determined later.

**Theorem 6.5** *Suppose Assumptions 6.4–6.7 are satisfied in a neighbourhood of the origin. Then, the control given in (6.104) drives the system (6.43)–(6.44) to the sliding surface (6.76) and maintains a sliding motion if in the neighbourhood of the origin, the control gain $k(\cdot)$ satisfies*

$$\begin{aligned}k(\cdot) \ge \alpha_1\exp\{-\alpha_2 t\}\Big(&\left\|(S_1 - S_2P_3^{-1}P_2^T)A_2\right\| + l\left\|(S_1 - S_2P_3^{-1}P_2^T)E\right\|\mathscr{L}_\rho\\&+ \left\|S_1 - S_2P_3^{-1}P_2^T\right\|\left(1 + \exp\{\alpha_2\bar{d}\}\right)\mathscr{L}_f\Big) + \eta\end{aligned} \tag{6.105}$$

*where $\alpha_1$ and $\alpha_2$ are given in Theorem 6.3, and $\eta$ is a positive constant.*

*Proof* From the analysis above, all that needs to be proved is that the reachability condition (6.103) is satisfied when applying the control in (6.104) to System (6.43)–(6.44).

From (6.75), (6.50)–(6.51), (6.59) and using the fact that

$$\hat{x}_2 = \hat{w}_2 - P_3^{-1}P_2^Ty$$

in Corollary 6.1,

$$\begin{aligned}\dot{\sigma}(y,\hat{x}) = {} & (S_1 - S_2P_3^{-1}P_2^T)\,(A_1y + A_2x_2 + E_1\Delta f(\cdot)) + S_2\Big(\big(A_4 + P_3^{-1}P_2^TA_2\big)\,\hat{w}_2 \\ & + \big(P_3^{-1}P_2^T(A_1 - A_2P_3^{-1}P_2^T) + A_3 - A_4P_3^{-1}P_2^T\big)y\Big) \\ & + (S_1 - S_2P_3^{-1}P_2^T)B_1u + S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]Bu \\ & + (S_1 - S_2P_3^{-1}P_2^T)f_1(\cdot) + S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]F(t,y,\hat{w}_2,y_d,\hat{w}_{2d}).\end{aligned} \tag{6.106}$$

It is clear that

$$\begin{aligned}& (S_1 - S_2P_3^{-1}P_2^T)B_1u + S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]Bu \\ & \quad = S_1B_1u - S_2P_3^{-1}P_2^TB_1u + \left[S_2P_3^{-1}P_2^T \;\; S_2\right]\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u. \\ & \quad = S_1B_1u + S_2B_2u = SBu.\end{aligned} \tag{6.107}$$

By similar reasoning, and from (6.57),

$$\begin{aligned}& (S_1 - S_2P_3^{-1}P_2^T)f_1(\cdot) + S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]F(\cdot) \\ & \quad = Sf(\cdot) - S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]\delta(f,\hat{f})\end{aligned} \tag{6.108}$$

where

$$\delta(f,\hat{f}) := f(t,x,x_d) - f(t,y,\hat{x}_2,y_d,\hat{x}_{2d}). \tag{6.109}$$

Substituting (6.107) and (6.108) into (6.106) yields:

$$\begin{aligned}\dot{\sigma}(y,\hat{x}) = {} & \big(S_1A_1 - S_2P_3^{-1}P_2^TA_2P_3^{-1}P_2^T + S_2(A_3 - A_4P_3^{-1}P_2^T)\big)y \\ & + (S_1 - S_2P_3^{-1}P_2^T)A_2x_2 + (S_1 - S_2P_3^{-1}P_2^T)E_1\Delta f(\cdot) \\ & + S_2\left(A_4 + P_3^{-1}P_2^TA_2\right)\hat{w}_2 + SBu \\ & + Sf(t,x,x_d) - S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]\delta(f,\hat{f}).\end{aligned} \tag{6.110}$$

Note

$$S\delta(f,\hat{f}) - S_2\left[P_3^{-1}P_2^T \;\; I_{n-p}\right]\delta(f,\hat{f}) = \left[S_1 - S_2P_3^{-1}P_2^T \;\; 0\right]\delta(f,\hat{f})$$

and in the neighbourhood of the origin,

$$\|x_d\| \le l.$$

It follows by applying control (6.104) to (6.110) that

$$\begin{aligned}\sigma^T(\cdot)\dot{\sigma}(\cdot) \le {} & \|\sigma(\cdot)\|\Big(\left\|(S_1 - S_2P_3^{-1}P_2^T)A_2\right\|\,\|x_2 - \hat{x}_2\| \\ & + l\|(S_1 - S_2P_3^{-1}P_2^T)E_1\|\left(\rho(t,x) - \rho(t,y,\hat{x}_2)\right)\end{aligned}$$

$$
\begin{aligned}
&+ \|S_1 - S_2 P_3^{-1} P_2^T\| \mathscr{L}_f (\|x_2 - \hat{x}_2\| + \|x_{2d} - \hat{x}_{2d}\|)\Big) - k(\cdot)\|\sigma(\cdot)\| \\
\le\ & \alpha_1 \exp\{-\alpha_2 t\}\Big( \big\|(S_1 - S_2 P_3^{-1} P_2^T) A_2\big\| + l\|(S_1 - \\
& S_2 P_3^{-1} P_2^T) E\| \mathscr{L}_\rho + \|S_1 - S_2 P_3^{-1} P_2^T\|(1 \\
& + \exp\{\alpha_2 \bar{d}\}) \mathscr{L}_f \Big) \|\sigma(\cdot)\| - k(\cdot)\|\sigma(\cdot)\|.
\end{aligned}
\tag{6.111}
$$

From the choice of $k(\cdot)$ in (6.105), it follows that

$$
\sigma^T(\cdot)\dot{\sigma}(\cdot) \le -\eta \|\sigma(\cdot)\|.
$$

Hence the conclusion follows. □

Theorem 6.5 shows that the control in (6.104) drives System (6.43)–(6.44) to the sliding surface (6.76) and maintains a sliding motion thereafter. Theorem 6.4 has shown that the sliding motion is uniformly asymptotically stable under certain conditions. From sliding mode control theory, the closed-loop system formed by applying the control (6.104) to System (6.43)–(6.44) is uniformly asymptotically stable.

*Remark 6.16* From (6.76) and/or (6.81), it follows that the sliding surface is designed in the augmented space $(x, \hat{x}_2)$ and/or $(x, e)$. The formulae in (6.76) and/or (6.81) clearly show that it is not required that $e(t) = 0$ when the sliding motion takes place. This implies that in the reduced-order sliding motion stability analysis, it is not required that $e(t) = 0$. Also, in the reachability analysis, only the condition that

$$
\|e(t)\| \le \alpha_1 \exp\{-\alpha_2 t\}
$$

is employed. It should be pointed out that here the fact that $\|e(t)\|$ is bounded is enough to guarantee reachability for some sufficiently large control, and it is not required that the estimation error $e(t)$ is zero.

*Remark 6.17* As in much of the existing work [101, 131, 141, 184], the delay is assumed to be known for the purpose of the observer design, which makes the observer-based control scheme delay dependent.

## 6.4 Decentralised Static Output Feedback Sliding Mode Control

In this section, a class of interconnected time-varying delay systems is considered where both the known and unknown interconnections have time-delays. A decentralised static output feedback sliding mode control, which is dependent on the time-delay, is synthesised to control the interconnected system such that the corresponding closed-loop system is globally uniformly asymptotically stable.

### 6.4.1 Introduction

The phenomenon of time-delay is often encountered in interconnected engineering systems. Mohmond and Bingulac [118] considered a class of interconnected systems where a delay does not appear in the interconnection terms. However, the interconnection between two or more physical systems is often accompanied by phenomena such as material transfer, energy transfer and information transfer. From a mathematical point of view, these transfer phenomena can be represented by delay elements [127]. This has motivated the study of time-delay interconnected systems [22, 118, 187, 218].

It should be noted that most of the existing results for time-delay interconnected systems are based on the assumption that all system states are available, and the associated decentralised output feedback results are scarce. Specifically, when delays are included, only a few results are available [72, 113, 135, 227]. A class of nonlinear interconnected systems with triangular structure is considered in [72], and an interconnected system composed of a set of single-input single-output subsystems with dead zone input is considered in [227]. Park et al. proposed a decentralised control approach for large-scale discrete-delay systems in [135]. However, in [135, 172, 227], the control schemes are based on dynamical output feedback. A decentralised model reference adaptive control scheme is proposed in [113] where the interconnections are linear and matched.

Recently, various control approaches have been employed to deal with time-delay control problems of interconnected systems. Adaptive control is usually powerful for systems possessing parametric uncertainty [113], while backstepping approaches require the considered systems to have a special structure [227]. Sliding mode control is completely robust to so-called matched uncertainty, and can be used successfully to deal with mismatched uncertainty [208]. Since the sliding mode dynamics are reduced order, it is possible to reduce the conservatism in the stability analysis of the sliding motion. This has motivated the application of sliding mode techniques to time-delay systems [53, 112, 132, 154, 208]. However most of the published work focuses on centralised control systems. In [200], a decentralised static output feedback control scheme is proposed using sliding mode control but the results obtained are local, and time-delays are not considered. Results on applying sliding mode techniques to time-delayed interconnected systems are very few. In the limited available literature [28, 56, 195], it is required that all system state variables are available and the interconnection terms are linear and matched. The assumption that all the terms involving time-delay are matched in [28, 156, 195] means the reduced-order sliding mode is no longer a time-delay system. To date, a global decentralised static output feedback sliding mode based control scheme for interconnected systems with mismatched time-delay interconnections has not been formulated.

In this section, a class of nonlinear interconnected systems with time-varying delays is considered, where the time-delay appears not only in the isolated subsystems, but also in the interconnections. The interconnections are separated into matched and mismatched components and are dealt with separately to reduce the conservatism. Using appropriate coordinate transformations, sliding mode dynamics are derived which are reduced-order time-delayed interconnected systems. Sufficient conditions are developed using a Lyapunov–Razumikhin approach such that the sliding motion is uniformly globally asymptotically stable. Unlike existing related work [28, 156, 195], the derived sliding mode dynamics here involve time-delays due to the delayed mismatched interconnections. A decentralised static output feedback control strategy is proposed to drive the system to the composite sliding surface.

### 6.4.2 System Description and Problem Formulation

Consider a time-varying delayed interconnected system composed of $n$, $n_i$-th order subsystems

$$\dot{x}_i = A_i x_i + B_i \left(u_i + G_i(t, x_i, x_{id_i})\right) + \sum_{\substack{j=1\\ j\neq i}}^{n} \left(H_{ij} y_{jd_j} + \Delta H_{ij}(t, x_j, x_{jd_j})\right) \tag{6.112}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, n, \tag{6.113}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ with $m_i \leq p_i < n_i$ are the state variables, inputs and outputs of the $i$-th subsystem respectively. The triples $(A_i, B_i, C_i)$ and $H_{ij} \in \mathbb{R}^{n_i \times p_j}$ $(i \neq j)$ represent known constant matrices of appropriate dimensions with $B_i$ and $C_i$ of full rank. The vector $G_i(\cdot)$ is a matched nonlinearity in the $i$th subsystem. The terms

$$\sum_{\substack{j=1\\ j\neq i}}^{n} H_{ij} y_{jd_j} \qquad \text{and} \qquad \sum_{\substack{j=1\\ j\neq i}}^{n} \Delta H_{ij}\left(t, x_j, x_{jd_j}\right)$$

are, respectively, known interconnections and uncertain interconnections in the $i$-th subsystem. The symbols

$$x_{id_i} := x_i(t - d_i) \qquad \text{and} \qquad y_{id_i} := y_i(t - d_i)$$

represent delayed states and delayed outputs respectively, where $d_i := d_i(t)$ is the time-varying delay which is assumed to be known, continuous, nonnegative and bounded in $\mathbb{R}^+$, that is

$$\overline{d}_i := \sup_{t \in \mathbb{R}^+} \{d_i(t), i = 1, \ldots, n\} < \infty.$$

The initial conditions are given by

$$x_i(t) = \phi_i(t) \qquad t \in [-\overline{d}_i, 0]$$

where the $\phi_i(\cdot)$ are continuous in $[-\overline{d}_i, 0]$ for $i = 1, 2, \ldots, n$. It is assumed that all the nonlinear functions are smooth enough such that the unforced system has a unique continuous solution.

Since the sliding motion is insensitive to matched uncertainty, it is useful to deal with matched interconnections and mismatched interconnections separately to reduce the conservatism. In view of this, consider the following decomposition of the interconnection terms:

$$H_{ij} = H_{ij}^a + H_{ij}^b \tag{6.114}$$

$$\Delta H_{ij}(\cdot) = \Delta H_{ij}^a(t, x_j, x_{jd_j}) + \Delta H_{ij}^b(t, x_j, x_{jd_j}) \tag{6.115}$$

where

$$H_{ij}^a = B_i D_{ij} \tag{6.116}$$

$$\Delta H_{ij}^a(t, x_j, x_{jd_j}) = B_i \Delta\Theta_{ij}(t, x_j, x_{jd_j}) \tag{6.117}$$

for some $D_{ij} \in \mathbb{R}^{m_i \times p_i}$ and $\Delta\Theta_{ij}(\cdot) \in \mathbb{R}^{m_i}$ where $\Delta\Theta_{ij}(\cdot)$ is uncertain for $i \neq j,\ i, j = 1, 2, \ldots, n$.

*Remark 6.18* From basic matrix theory, the decompositions in (6.114) and (6.115) which satisfy (6.116) and (6.117) respectively, can be obtained in the following way. Consider the $l$-th column vector $H_{ij}^{(l)}$ of matrix $H_{ij}$ for $l = 1, 2, \ldots, p_j$. For a given matrix $B_i$, decompose

$$H_{ij}^{(l)} = (H_{ij}^{(l)})^a + (H_{ij}^{(l)})^b$$

such that

$$(H_{ij}^{(l)})^a \in \text{Im}(B_i) \qquad \text{and} \qquad (H_{ij}^{(l)})^b \in (\text{Im}(B_i))^{\perp}$$

where $(\text{Im}(B_i))^{\perp}$ is the orthogonal complimentary space of $\text{Im}(B_i)$. Then

$$H_{ij}^a = \left[ (H_{ij}^{(1)})^a \ (H_{ij}^{(2)})^a \ \cdots \ (H_{ij}^{(p_j)})^a \right]$$

$$H_{ij}^b = \left[ (H_{ij}^{(1)})^b \ (H_{ij}^{(2)})^b \ \cdots \ (H_{ij}^{(p_j)})^b \right]$$

will be a choice for the decomposition (6.114). The decomposition for the uncertain interconnections $\Delta H_{ij}(t, x_j, x_{jd_j})$ can be obtained in the same way.

**Assumption 6.8** There exist known nonnegative continuous functions $g_i(\cdot)$, $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ such that for $i, j = 1, 2, \ldots, n$

$$\|G_i(t, x, x_{id_i})\| \leq g_i(t, y_i, y_{id_i}) \tag{6.118}$$

$$\|\Delta\Theta_{ij}(t, x_j, x_{jd_j})\| \leq \alpha_{ij}(t, y_j, y_{jd_j})\|y_{jd_j}\|, \qquad (i \neq j) \tag{6.119}$$

$$\|\Delta H_{ij}^b(t, x_j, x_{jd_j})\| \leq \beta_{ij}(t, y_j, \|y_{jd_j}\|)\|y_{jd_j}\|, \qquad (i \neq j) \tag{6.120}$$

where $\beta_{ij}(\cdot, \cdot, r)$ is nondecreasing with respect to the variable $r$ in $\mathbb{R}^+$.

*Remark 6.19* Assumption 6.8 is a limitation on the uncertainties that can be tolerated by the system. Similar to [227], it is required that the interconnections can be described or bounded by functions of the system outputs. Unlike [227], time-delays are involved in the interconnections; the bounds on the uncertain interconnections are nonlinear and the results obtained are global.

It follows from (6.115), (6.117), (6.119) and (6.120) that there exist known continuous functions $\rho_{ij}(\cdot)$ such that

$$\|\Delta H_{ij}(t, x_j, x_{jd_j})\| \leq \rho_{ij}(t, y_j, y_{jd_j})\|y_{jd_j}\|, \qquad i \neq j,\ i, j = 1, 2, \ldots, n. \tag{6.121}$$

**Assumption 6.9** $\text{rank}(C_i B_i) = m_i$ for $i = 1, 2, \ldots, n$.

From [38], Assumption 6.9 implies that there exists a nonsingular linear coordinate transformation: $\tilde{x}_i = \tilde{T}_i x_i$ such that the triple $(A_i, B_i, C_i)$ with respect to the new coordinates has the structure

$$\widetilde{A}_i = \begin{bmatrix} \widetilde{A}_{i1} & \widetilde{A}_{i2} \\ \widetilde{A}_{i3} & \widetilde{A}_{i4} \end{bmatrix}, \qquad \widetilde{B}_i = \begin{bmatrix} 0 \\ \widetilde{B}_{i2} \end{bmatrix}, \qquad \widetilde{C}_i = \begin{bmatrix} 0 & \widetilde{C}_{i2} \end{bmatrix} \tag{6.122}$$

where $\widetilde{A}_{i1} \in \mathbb{R}^{(n_i - m_i)\times(n_i - m_i)}$, $\widetilde{B}_{i2} \in \mathbb{R}^{m_i \times m_i}$ is nonsingular and $\widetilde{C}_{i2} \in \mathbb{R}^{p_i \times p_i}$ is orthogonal.

**Assumption 6.10** The triple $(\widetilde{A}_{i1}, \widetilde{A}_{i2}, \Xi_i)$ is output feedback stabilisable, where

$$\Xi_i := \begin{bmatrix} 0_{(p_i - m_i)\times(n_i - p_i)} & I_{p_i - m_i} \end{bmatrix}, \qquad i = 1, 2, \ldots, n$$

*Remark 6.20* In the regular form (6.122), the matrices $\widetilde{A}_{ij}$ with $j = 1, 2, 3, 4$, $\widetilde{B}_{i2}$, and $\widetilde{C}_{i2}$ are dependent on the coordinate transformation used. However, if Assumption 6.9 holds, the satisfaction or otherwise of Assumption 6.10 does not depend on the coordinate transformation employed. Hence, Assumptions 6.9 and 6.10 together, describe inherent properties of the triple $(A_i, B_i, C_i)$.

From Sect. 2.6, Assumptions 6.9 and 6.10 guarantee that there exist a coordinate transformation $x_i \mapsto z_i = T_i x_i$ which will transform the triple $(A_i, B_i, C_i)$ to the following form in the new coordinate system $z_i$,

$$\begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix}, \qquad \begin{bmatrix} 0 & C_{i2} \end{bmatrix} \tag{6.123}$$

where $A_{i1}$ is stable and both $B_{i2} \in \mathbb{R}^{m_i \times m_i}$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ are nonsingular.

Note, the analysis above implies that $m_i < p_i$. However, if $m_i = p_i$, Assumption 6.10 can be replaced by the assumption that $\tilde{A}_{i1}$ is stable. The associated discussion becomes simpler in this case, and details are available in Sect. 5.3 in [38].

*Remark 6.21* Assumptions 6.9 and 6.10 are limitations on the triples $(A_i, B_i, C_i)$. They ensure the existence of the output sliding surface. Assumption 6.10 requires $(\widetilde{A}_{i1}, \widetilde{A}_{i2}, \Xi_i)$ instead of $(A_i, B_i, C_i)$ as in [227] to be output feedback stabilisable. The former is related to a system with order $n_i - m_i$, while the latter is an $n_i$-th order system. Sometimes this reduced order problem is more amenable to solution: for example, if the matrix triple is related to a single-input and two-output system, the output feedback problem reduces to a classical "root-locus" investigation [41].

The objective of this section is to design a variable structure control law of the form

$$u_i = u_i(t, y_i, y_{id_i}), \qquad i = 1, 2, \dots, n \tag{6.124}$$

using sliding mode techniques such that the associated closed-loop system formed by applying the control law in (6.124) to the interconnected system (6.112)–(6.113), is globally uniformly asymptotically stable, even in the presence of the uncertainties and time-delay. It is clear that the control $u_i$ in (6.124) depends on only the time $t$, the $i$-th subsystem output $y_i$ and delayed output $y_{id_i}$. Since the $d_i := d_i(t)$ for $i = 1, \dots, n$ are assumed to be known, the term $y_{id_i}$ is available for design. The control in (6.124) is called a delay dependent decentralised static output feedback control.

### *6.4.3 Sliding Mode Stability Analysis*

A composite output sliding surface is presented for System (6.112)–(6.113) and the associated sliding mode dynamics are derived. Then the stability of the sliding motion is investigated.

It has been shown in Sect. 2.6 that under Assumptions 6.9 and 6.10, there exist matrices $F_i \in \mathbb{R}^{m_i \times p_i}$ such that

$$F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} = \begin{bmatrix} 0 & F_{i2} \end{bmatrix} \tag{6.125}$$

where $F_{i2} \in \mathbb{R}^{m_i \times m_i}$ is any nonsingular matrix which does not affect the sliding motion.

For the interconnected system (6.112)–(6.113), consider the composite sliding surface defined by

$$\Big\{ \mathrm{col}(x_1, x_2, \dots, x_n) \;\Big|\; S_i(x_i) = 0,\ i = 1, 2, \dots n \Big\} \tag{6.126}$$

where

$$S_i(x_i) := F_i C_i x_i = F_i y_i, \qquad i = 1, 2, \dots, n \tag{6.127}$$

and the matrices $F_i$ satisfying (6.125) can be obtained using the algorithm given in [38].

Under Assumptions 6.9 and 6.10, it follows from the analysis in Sect. 6.4.2 that there exists a coordinate transformation $x_i \mapsto z_i = T_i x_i$ for $i = 1, 2, \ldots, n$ such that in the new coordinates $z = \text{col}(z_1, z_2, \ldots, z_n)$, the system (6.112)–(6.113) can be described by

$$\dot{z}_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} z_i + \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} \Big( u_i + G_i(t, T_i^{-1} z_i, T_i^{-1} z_{id_i}) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \big( D_{ij} y_{jd_j} + \Delta\Theta_{ij}(t, T_j^{-1} z_j, T_j^{-1} z_{jd_j}) \big) \Big) + \sum_{\substack{j=1 \\ j \neq i}}^{n} T_i \Big( H_{ij}^b y_{jd_j} + \Delta H_{ij}^b (t, T_j^{-1} z_j, T_j^{-1} z_{jd_j}) \Big) \tag{6.128}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, n \tag{6.129}$$

where $A_{i1} \in \mathbb{R}^{n_i - m_i}$ is stable; both $B_{i2} \in \mathbb{R}^{m_i \times m_i}$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ are nonsingular; $H_{ij}^a(\cdot)$ and $H_{ij}^b(\cdot)$, $\Delta\Theta_{ij}(\cdot)$ and $\Delta H_{ij}^b(\cdot)$ are given in (6.114), (6.117) and (6.115) respectively. Since $A_{i1}$ is stable, it follows that for any $Q_i > 0$, the Lyapunov equation

$$A_{i1}^T P_i + P_i A_{i1} = -Q_i, \qquad i = 1, 2, \ldots, n \tag{6.130}$$

has a unique solution $P_i > 0$ for $i = 1, \ldots n$. For convenience, partition

$$T_i \equiv: \begin{bmatrix} T_{i1} \\ T_{i2} \end{bmatrix}, \qquad T_i^{-1} \equiv: [W_{i1} \quad W_{i2}] \tag{6.131}$$

where $T_{i1} \in \mathbb{R}^{(n_i - m_i) \times n_i}$ and $W_{i1} \in \mathbb{R}^{n_i \times (n_i - m_i)}$. It is clear that System (6.128)–(6.129) can be rewritten as

$$\dot{z}_{i1} = A_{i1} z_{i1} + A_{i2} z_{i2} + \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{n} T_{i1} \Big( H_{ij}^b y_{jd_j} + \Delta H_{ij}^b (t, T_j^{-1} z_j, T_j^{-1} z_{jd_j}) \Big) \tag{6.132}$$

$$\dot{z}_{i2} = A_{i3} z_{i1} + A_{i4} z_{i2} + B_{i2} \Big( u_i + G_i(\cdot) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \big( D_{ij} y_{jd_j} + \Delta\Theta_{ij}(t, T_j^{-1} z_j, T_j^{-1} z_{jd_j}) \big) \Big) + \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{n} T_{i2} \Big( H_{ij}^b y_{jd_j} + \Delta H_{ij}^b (t, T_j^{-1} z_j, T_j^{-1} z_{jd_j}) \Big) \tag{6.133}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, n, \tag{6.134}$$

where $z_i := \text{col}(z_{i1}, z_{i2})$ with $z_{i1} \in \mathbb{R}^{n_i - m_i}$ and $z_{i2} \in \mathbb{R}^{m_i}$.

From (6.127), (6.134) and (6.125),

$$S_i(x_i) = F_i y_i = F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i = [0 \quad F_{i2}] \begin{bmatrix} z_{i1} \\ z_{i2} \end{bmatrix} = F_{i2} z_{i2}$$

where $F_{i2} \in \mathbb{R}^{m_i \times m_i}$ is nonsingular. Therefore, in the new coordinates $z$, the sliding surface (6.126) can be described by

$$\Big\{ \operatorname{col}\big(z_1, z_2, \ldots, z_n\big) \mid z_{i2} = 0,\ i = 1, 2, \ldots, n \Big\}. \tag{6.135}$$

Further, partition the output distribution matrix in (6.134) as

$$[0 \quad C_{i2}] = \Big[ \underbrace{0 \quad C_{i21}}_{C_{is}} \Big| \; C_{i22} \Big]. \tag{6.136}$$

where $C_{is} := [0 \quad C_{i21}] \in \mathbb{R}^{p_i \times (n_i - m_i)}$ and $C_{i22} \in \mathbb{R}^{p_i \times m_i}$. When the system in (6.132)–(6.133) is constrained to the sliding surface, $z_{i2} = 0$, from partition (6.131)

$$T_i^{-1} z_i = W_{i1} z_{i1} \qquad \text{and} \qquad T_i^{-1} z_{id_i} = W_{i1} z_{i1d_i}.$$

From (6.136), the $i$-th subsystem output $y_i$ is described by

$$y_{is} := [0 \quad C_{i21}] z_{i1} = C_{is} z_{i1}, \qquad i = 1, 2, \ldots, n. \tag{6.137}$$

Then, from the structure of System (6.132)–(6.133), the sliding mode dynamics of System (6.112)–(6.113) associated with the sliding surface (6.126) is described by

$$\dot{z}_{i1} = A_{i1} z_{i1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} T_{i1} \Big( H_{ij}^b C_{js} z_{j1d_j} + \Delta H_{ij}^b (t, W_{j1} z_{j1}, W_{j1} z_{j1d_j}) \Big) \tag{6.138}$$

where $W_{i1}$ is defined by the partition (6.131) for $i = 1, 2, \ldots, n$. Obviously, System (6.138) is a reduced-order interconnected system with dimension $\sum_{i=1}^{n} (n_i - m_i)$ when compared with the system (6.112)–(6.113) which has dimension $\sum_{i=1}^{n} n_i$.

**Theorem 6.6** *Assume that Assumptions 6.8–6.10 hold. Then, for the sliding surface in (6.126), the sliding motion associated with the time-delay interconnected system (6.112)–(6.113), is governed by dynamical system (6.138). Moreover, the sliding motion is globally uniformly asymptotically stable if*

*(i) the $n_i \times n_i$ symmetric matrix*

$$N_i := Q_i - P_i T_{i1} \Big( \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\varepsilon_j} H_{ij}^b C_{js} P_j^{-1} C_{js}^T (H_{ij}^b)^T \Big) T_{i1}^T P_i - q \textstyle\sum_{\substack{j=1 \\ j \neq i}}^{n} \varepsilon_j P_j > 0$$

*for some $q > 1$ and $\varepsilon_i > 0$ for $i = 1, 2, \ldots, n$*

*(ii) the $n \times n$ matrix function $M + M^T > 0$ where $M = [m_{ij}(\cdot)]_{n \times n}$ is defined by*

$$m_{ij}(\cdot) := \begin{cases} \lambda_{\min}(N_i), & i = j \\ -\gamma_j \beta_{ij}(t, C_{is} z_{i1} \gamma_j \|C_{js}\| \, \|z_{j1}\|) \|P_i T_{i1}\| \, \|C_{js}\|, & i \neq j \end{cases}$$

*for some* $\gamma_i > 1$ *for* $i, j = 1, 2, \ldots, n$ *and* $\mu := \inf_{z_{11},\cdots,z_{n1}} \{\lambda_{\min}(M + M^T)\} > 0.$

*Proof* The analysis above has shown that for System (6.112)–(6.113), the reduced-order sliding mode dynamics associated with the sliding surface (6.126) are given by the system described in (6.138). All that remains to be proved is that (6.138) is globally uniformly asymptotically stable.

For System (6.138), consider the Lyapunov function candidate

$$V(z_{11}, z_{21}, \ldots, z_{n1}) = \sum_{i=1}^{n} z_{i1}^T P_i z_{i1}$$

where $P_i > 0$ satisfies (6.130) for $i = 1, 2, \ldots, n$. Then, the time derivative of $V(\cdot)$ along the trajectories of System (6.138) is given by

$$\begin{aligned} \dot{V}\,|_{(6.138)} = &-\sum_{i=1}^{n} z_{i1}^T Q_i z_{i1} + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} z_{i1}^T P_i T_{i1} H_{ij}^b C_{js} z_{j1d_j} \\ &+ 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} z_{i1}^T P_i T_{i1} \Delta H_{ij}^b(\cdot) \end{aligned} \tag{6.139}$$

where (6.130) has been used to obtain the expression in (6.139).

From Lemma B.1 (see Appendix B.1), it follows that for any $\varepsilon_j > 0$,

$$\begin{aligned} 2z_{i1}^T P_i T_{i1} H_{ij}^b C_{js} z_{j1d_j} \le\ & \frac{1}{\varepsilon_j} z_{i1}^T P_i T_{i1} H_{ij}^b C_{js} P_j^{-1} (T_{i1} H_{ij}^b C_{js})^T P_i z_{i1} \\ &+ \varepsilon_j z_{j1d_j}^T P_j z_{j1d_j}, \qquad (i \neq j). \end{aligned} \tag{6.140}$$

Since the $z_{i1}$ for $i = 1, 2, \ldots, n$ are independent of each other, it is clear that

$$V(z_{11d_1}, z_{21d_2}, \ldots, z_{n1d_n}) \le qV(z_{11}, z_{21}, \ldots, z_{n1})$$

for $q > 1$, is equivalent to

$$z_{i1d_i}^T P_i z_{i1d_i} \le q z_{i1}^T P_i z_{i1}, \qquad i = 1, 2, \ldots, n \tag{6.141}$$

which implies[1]

$$\|z_{i1d_i}\| \le \gamma_i \|z_{i1}\|, \qquad i = 1, 2, \ldots, n \tag{6.142}$$

---

[1]It is clear that Inequality (6.141) implies that $\lambda_{\min}(P_i)\|z_{i1d_i}\|^2 \le q\lambda_{\max}(P_i)\|z_{i1}\|^2$ from which it follows $\|z_{i1d_i}\| \le q\sqrt{\lambda_{\max}(P_i)/\lambda_{\min}(P_i)}\ \|z_{i1}\|$. This shows that one choice for $\gamma_i$ in (6.142) is $\gamma_i = q\sqrt{\lambda_{\max}(P_i)/\lambda_{\min}(P_i)}$ for $i = 1, 2, \ldots, n$ where $q$ can be chosen as any constant bigger than 1.

for some positive constants $\gamma_i > 1$ with $i = 1, 2, \ldots, n$. Further from (6.137) and (6.142),

$$\|y_{isd_i}\| = \|C_{is}z_{i1d_i}\| \leq \gamma_i \|C_{is}\| \, \|z_{i1}\|. \tag{6.143}$$

Since $\beta_{ij}(t, y_{js}, r)$ is nondecreasing with respect to the variable $r \in \mathbb{R}^+$, from (6.120), (6.143) and (6.137)

$$P_i T_{i1} \varDelta H_{ij}^b(\cdot) \leq \gamma_j \beta_{ij}(t, C_{js}z_{j1}, \gamma_j \|C_{js}\| \, \|z_{j1}\|) \|P_i T_{i1}\| \, \|C_{js}\| \, \|z_{j1}\| \quad (i \neq j). \tag{6.144}$$

Therefore, from (6.139), (6.140) and (6.144), when

$$V(z_{11d_1}, z_{21d_2}, \ldots, z_{n1d_n}) \leq qV(z_{11}, z_{21}, \ldots, z_{n1})$$

it follows that

$$\begin{aligned}
\dot{V}\mid_{(6.138)} \leq & -\sum_{i=1}^n z_{i1}^T Q_i z_{i1} + \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \left( \frac{1}{\varepsilon_j} z_{i1}^T P_i T_{i1} H_{ij}^b C_{js} P_j^{-1} (T_{i1} H_{ij}^b C_{js})^T P_i z_{i1} \right) \\
& + \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \varepsilon_j z_{j1}^T P_j z_{j1} \\
& + 2 \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \gamma_j \beta_{ij}(t, C_{js}z_{j1}, \gamma_j \|C_{js}\| \, \|z_{j1}\|) \, \|P_i T_{i1}\| \, \|C_{js}\| \, \|z_{j1}\| \, \|z_{i1}\| \\
\leq & -\sum_{i=1}^n z_{i1}^T N_i z_{i1} \\
& + 2 \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \gamma_j \beta_{ij}(t, C_{js}z_{j1}, \gamma_j \|C_{js}\| \, \|z_{j1}\|) \|P_i T_{i1}\| \, \|C_{js}\| \, \|z_{i1}\| \, \|z_{j1}\| \\
= & -\frac{1}{2} [\|z_{11}\| \; \|z_{21}\| \; \cdots \; \|z_{n1}\|](M + M^T) \begin{bmatrix} \|z_{11}\| \\ \vdots \\ \|z_{n1}\| \end{bmatrix} \\
\leq & -\frac{1}{2} \inf_{z_{11}, \cdots, z_{n1}} \{\lambda_{\min}(M + M^T)\} \sum_{i=1}^n \|z_{i1}\|^2 \\
= & -\frac{1}{2} \mu \|z\|^2
\end{aligned} \tag{6.145}$$

where $z = \mathrm{col}(z_1, z_2, \ldots, z_n)$. Since $\mu > 0$, the conclusion follows from Lemma A.1 (See Appendix A.1). $\nabla$

*Remark 6.22* If $m_{r_0} = p_{r_0}$ for the $r_0$-th ($1 \le r_0 \le n$) subsystem, then

$$y_{r_0} = z_{r_0 2}$$

and thus from (6.137), $y_{r_0 s} = 0$. In this case, from the proof of Theorem 6.6, condition (i) in Theorem 6.6 can be weakened to be

$$N_i = Q_i - P_i T_{i1}\Big(\sum\nolimits_{\substack{j=1\\ j\neq i, r_0}}^{n} \tfrac{1}{\varepsilon_j} H_{ij}^b C_{js} P_j^{-1} C_{js}^T (H_{ij}^b)^T\Big) T_{i1}^T P_i - q \sum\nolimits_{\substack{j=1\\ j\neq i, r_0}}^{n} \varepsilon_j P_j > 0,$$

for $i = 1, 2, \ldots, n$.

### 6.4.4 Reachability Analysis

The objective in this section is to design a decentralised static output feedback sliding mode control such that the system states are driven to the sliding surface (6.126). For the interconnected system (6.112)–(6.113), the reachability condition is described by (see, [69])

$$\sum_{i=1}^{n} \frac{S_i^T(x_i)\dot{S}_i(x_i)}{\|S_i(x_i)\|} < 0 \tag{6.146}$$

where the switching function $S_i(\cdot)$ is defined by (6.127). In order to develop a global reachability condition based on static output feedback control, the following condition is imposed on the System (6.112)–(6.113).

**Assumption 6.11** The matrix equation

$$\Gamma_i C_i = F_i C_i A_i$$

is solvable for $\Gamma_i$ with $i = 1, 2, \ldots, N$.

*Remark 6.23* Assumption 6.11 is required to guarantee global reachability, and is unnecessary when only the local case is considered. Similar conditions have been employed by Hui and Zak in [219].

**Theorem 6.7** *Consider the time-delay interconnected system (6.112)–(6.113). Under Assumptions 6.8–6.11, there exists a global delay dependent static output feedback decentralised control law which drives the system (6.112)–(6.113) to the composite sliding surface (6.126), and maintains a sliding motion on it thereafter.*

*Proof* Since the triple in (6.123) is obtained from $(A_i, B_i, C_i)$ using the transformation $z_i = T_i x_i$, it follows that for $i = 1, 2, \ldots, n$

$$T_i B_i = \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix}, \qquad C_i T_i^{-1} = \begin{bmatrix} 0 & C_{i2} \end{bmatrix}$$

where both $B_{i2} \in \mathbb{R}^{m_i \times m_i}$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ are nonsingular. Then, from (6.125), it follows that

$$F_i C_i B_i = F_i \,[0 \quad C_{i2}] \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} = F_{i2} B_{i2}$$

which shows that $F_i C_i B_i$ is nonsingular because $F_{i2}$ and $B_{i2}$ are nonsingular. Construct a variable structure control

$$u_i = -(F_i C_i B_i)^{-1} \Big\{ \Gamma_i y_i + \Big( \|F_i C_i B_i\| g_i(t, y_i, y_{id_i}) + \eta_i + \sum_{\substack{j=1\\ j \neq i}}^{n} \|F_j C_j H_{ji}\| \|y_{id_i}\| + \sum_{\substack{j=1\\ j \neq i}}^{n} \|F_j C_j\| \, \rho_{ji}(\cdot) \|y_{id_i}\| \Big) \, \mathrm{sgn}(F_i y_i) \Big\} \tag{6.147}$$

where $g_i(\cdot)$ and $\rho_{ij}(\cdot)$ $(i \neq j)$ are defined by (6.118) and (6.121) respectively; $\eta_i$ can be chosen as any positive constant for $i, j = 1, 2, \ldots, n$, and the symbol $\mathrm{sgn}(\cdot)$ denotes the usual signum vector function. From (6.112), (6.127), (6.121) and Assumption 6.11

$$\begin{aligned} \sum_{i=1}^{n} \frac{S_i^T(x_i) \dot{S}_i(x_i)}{\|S_i(x_i)\|} \leq & \sum_{i=1}^{n} \frac{(F_i y_i)^T}{\|F_i y_i)\|} \Gamma_i y_i + \sum_{i=1}^{n} \frac{(F_i y_i)^T}{\|F_i y_i)\|} F_i C_i B_i \left( u_i + G_i(t, x_i, x_{id_i}) \right) \\ & + \sum_{i=1}^{n} \sum_{\substack{j=1\\ j \neq i}}^{n} \|F_i C_i H_{ij}\| \, \|y_{jd_j}\| \\ & + \sum_{i=1}^{n} \sum_{\substack{j=1\\ j \neq i}}^{n} \rho_{ij}(t, y_j, y_{jd_j}) \|F_i C_i\| \, \|y_{jd_j}\|. \end{aligned} \tag{6.148}$$

Substituting the control from (6.147) into (6.148), it follows from (6.118) that

$$\begin{aligned} & \sum_{i=1}^{n} \frac{S_i^T(x_i) \dot{S}_i(x_i)}{\|S_i(x_i)\|} \\ \leq & \sum_{i=1}^{n} \Big( - \frac{(F_i y_i)^T \mathrm{sgn}(F_i y_i)}{\|F_i y_i\|} \|F_i C_i B_i\| g_i(t, y_i, y_{id_i}) + \|F_i C_i B_i\| \, \|G_i(t, x_i, x_{id_i})\| \Big) \\ & - \sum_{i=1}^{n} \frac{(F_i y_i)^T \mathrm{sgn}(F_i y_i)}{\|F_i y_i\|} \eta_i - \sum_{i=1}^{n} \frac{(F_i y_i)^T \mathrm{sgn}(F_i y_i)}{\|F_i y_i\|} \Big( \sum_{\substack{j=1\\ j \neq i}}^{n} \|F_j C_j H_{ji}\| \Big) \|y_{id_i}\| \\ & + \sum_{i=1}^{n} \Big( \sum_{\substack{j=1\\ j \neq i}}^{n} \|F_j C_j H_{ji}\| \Big) \, \|y_{id_i}\| \end{aligned}$$

$$-\sum_{i=1}^{n}\frac{(F_iy_i)^T\mathrm{sgn}(F_iy_i)}{\|F_iy_i\|}\sum_{\substack{j=1\\j\neq i}}^{n}\rho_{ji}(t,y_i,y_{id_i})\|F_iC_i\|\,\|y_{jd_j}\|$$
$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\rho_{ij}(t,y_j,y_{jd_j})\|F_jC_j\|\,\|y_{jd_j}\|\leq-\sum_{i=1}^{n}\eta_i<0$$

where the conclusions (i) and (ii) of Lemma B.2 (See Appendix B.2) are employed to achieve the expressions above. Hence the conclusion follows. ∇

From sliding mode control theory, Theorems 6.6 and 6.7 together show that the proposed decentralised output feedback control law in (6.147) uniformly asymptotically stabilises the system (6.112)–(6.113) globally.

*Remark 6.24* From the proof in Theorem 6.7, when the interconnections can be expressed in a superposition form as $\sum_{\substack{j=1\\j\neq i}}^{n}\Phi_{ij}(\cdot)$ where $\Phi_{ij}(\cdot)$ can be expressed in terms of (or bounded by) functions of known information: $t$, $y_j$ and $y_{jd_j}$, the interconnection effects can be canceled completely by designing an appropriate decentralised control law to guarantee reachability.

*Remark 6.25* If there exists a term, for example, $\hat{H}_{ik}x_k$ $(i\neq k)$ in the i-th interconnections, the approach proposed here can be applied directly. In this case, condition (i) in Theorem 6.6 should be replaced by

$$N_i:=Q_i-P_iT_{i1}\Big(\sum_{\substack{j=1\\j\neq i}}^{n}\frac{1}{\varepsilon_j}H_{ij}^bC_{js}P_j^{-1}C_{js}^T(H_{ij}^b)^T\Big)T_{i1}^TP_i-q\sum_{\substack{j=1\\j\neq i}}^{n}\varepsilon_jP_j$$
$$-\Big((\hat{H}_{ik}^bT_{i1})^TP_i+P_iT_{i1}\hat{H}_{ik}^b\Big)>0.$$

Also, a condition that there exists a matrix $\Gamma_{ik}$ such that

$$\Gamma_{ik}C_k=F_iC_i\hat{H}_{ik} \tag{6.149}$$

needs to be added to Assumption 6.11. The example in Sect. 6.6 will be used to illustrate this.

## 6.5 Numerical Simulation Examples

In this section, numerical examples with simulation are provided to illustrate the control schemes proposed in Sects. 6.2–6.4.

### 6.5.1 Static Output Feedback Control

Consider the time-varying delay system with delayed disturbance described by

$$
\begin{aligned}
\dot{x} &= \underbrace{\begin{bmatrix} -10 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_0} x(t-d(t)) \\
&\quad + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{B} \Big( u(t) + g(t, x(t), x(t-d(t))) \Big) + \\
&\underbrace{\begin{bmatrix} \sqrt{2}\beta_1(t, x(t), x(t-d(t)))x_1(t) + \beta_2(t, x(t), x(t-d(t)))x_1(t-d(t)) \\ 0 \\ \beta_1(t, x(t), x(t-d(t)))x_3(t) + \beta_2(t, x(t), x(t-d(t)))x_3(t-d(t))) \end{bmatrix}}_{f(t,x(t),x(t-d(t)))}
\end{aligned} \tag{6.150}
$$

$$
y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C} x \tag{6.151}
$$

where $x = \mathrm{col}(x_1, x_2, x_3)$, $u$ and $y = \mathrm{col}(y_1, y_2)$ are respectively the state variables, the inputs and the outputs of the system. The unknown functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are time-delayed disturbances which are assumed to satisfy

$$
\begin{aligned}
|\beta_1(\cdot)| &\le (y_2(t))^2 \, |\sin y_1(t-d(t))| \\
|\beta_2(\cdot)| &\le |y_1(t-d(t))| \sin^2 y_1(t) + (y_2(t))^2.
\end{aligned}
$$

The matched delayed disturbance $g(\cdot)$ has unknown structure but satisfies

$$
\|g(\cdot)\| \le \underbrace{y_2^4(t) \sin^2 y_1(t-d(t))}_{\varpi(\cdot)}.
$$

The domain considered here is

$$
\Omega = \left\{ (x_1, x_2, x_3) \mid x_2 \in \mathbb{R}, \quad \frac{1}{2}x_1^2 + x_3^2 < 12 \right\}.
$$

Obviously

$$\|f(t,x(t),x(t-d(t)))\| \le \underbrace{\sqrt{2}(y_2(t))^2\,|\sin y_1(t-d(t))|}_{\rho_1(\cdot)}\,\|x(t)\| + \underbrace{\left(|y_1(t-d(t))|\sin^2 y_1(t) + (y_2(t))^2\right)}_{\rho_2(\cdot)}\,\|x(t-d(t))\|$$

and Assumption 6.2 holds.

Clearly $CB = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ is full rank. According to the algorithm given in [37], the coordinate transformation $\widetilde{x} = \widetilde{T}x$ with

$$\widetilde{T} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

transforms the triple $(A, B, C)$ into the following from

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = \left[\begin{array}{cc|c} -10 & 0 & -1 \\ 0 & -5 & 1 \\ \hline -1 & 0 & 0 \end{array}\right], \tag{6.152}$$

$$\begin{bmatrix} 0 \\ \widetilde{B}_2 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ \hline -1 \end{array}\right], \qquad \begin{bmatrix} 0 & \breve{T}_2 \end{bmatrix} = [0\ \ I_2]\,. \tag{6.153}$$

It is clear that the triple $(A, B, C)$ is output feedback normalisable with the choice $K = 0$ due to the stability of $\widetilde{A}_{11}$. Further $\mathrm{Im}(A_0) \subset \mathrm{Im}(B)$ since $A_0 = BD$ with $D = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$. Therefore Assumption 6.1 is satisfied.

Since (6.152)–(6.153) already has the regular form (6.5)–(6.7), it follows that

$$\begin{array}{lll} T = \widetilde{T}, & A_{11} = \widetilde{A}_{11} & \\ A_{12} = \widetilde{A}_{12}, & A_{21} = \widetilde{A}_{21} & \\ A_{22} = \widetilde{A}_{22}, & B_2 = \widetilde{B}_2, & C_2 = \widetilde{C}_2 = I_2. \end{array}$$

Let $Q = 10I_2$. It follows that the Lyapunov equation (6.21) has a unique solution

$$P = \mathrm{diag}\{0.5, 1\}$$

and thus

$$P^{\frac{1}{2}} = \mathrm{diag}\left\{\frac{\sqrt{2}}{2}, 1\right\}.$$

According to [37], choose $F = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The designed sliding surface from (6.17) is then described by

$$S(x) = \{(x_1, x_2, x_3) \mid \ y_2 = 0\}. \tag{6.154}$$

By direct computation, it follows from (6.16) that

$$f_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \beta_1(\cdot) P^{\frac{1}{2}} z_1(t) + \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \beta_2(\cdot) P^{\frac{1}{2}} z_1(t - d(t)). \tag{6.155}$$

When a sliding motion takes place, $y_2(t) = 0$, and thus

$$\beta_1(\cdot) = 0, \qquad |\beta_2(\cdot)\| \le |y_1(t - d(t))| \sin^2 y_1(t).$$

Then,

$$\begin{aligned} &\left\| P^{\frac{1}{2}} [f_1(t, z(t), z(t - d(t)))]_{z_2(t)=0} \right\| \\ &\quad \le \|z_1(t - d(t))\| \, (\sin y_1(t))^2 \left\| P^{\frac{1}{2}} z_1(t - d(t)) \right\|. \end{aligned} \tag{6.156}$$

By comparing (6.156) with (6.22), it follows that

$$\psi_1(\cdot) = 0$$

$$\psi_2(\cdot) = \|z_1(t - d(t))\| (\sin y_1(t))^2.$$

Therefore,

$$\Theta(t, z_1(t)) = \zeta^2 \sqrt{2} (\sin y_1(t))^2 \|P^{\frac{1}{2}} z(t)\|.$$

Let $\zeta = 1.01$. By direct computation, $\gamma > 0.0026 > 0$ in $T(\Omega)$, and thus the conditions of Theorem 6.1 hold in the domain $T(\Omega)$. From (6.36) and (6.37), the control is given as follows:

$$\begin{aligned} u = &\Big(10.0499\mu_1 + \sqrt{(y_2(t))^2 + (y_2(t) - 5y_1(t))^2} + y_2^4(t) \sin^2 y_1(t - d(t)) \\ &+ \sqrt{2} (y_2(t))^2 \, |\sin y_1(t - d(t))| \, (\mu_1 + \|y\|) + \Big( |y_1(t - d(t))| \sin^2 y_1(t) \\ &+ (y_2(t))^2 \Big) (\mu_2 + \|y(t - d(t))\|) + \eta \Big) \operatorname{sgn}(y_2(t)). \end{aligned} \tag{6.157}$$

For implementation purposes, choose

$$\mu_1 = \mu_2 = 2 \qquad \text{and} \qquad \eta = 1.$$

The time-varying delay $d(t)$ is chosen as

$$d(t) = 2 + \sin t.$$

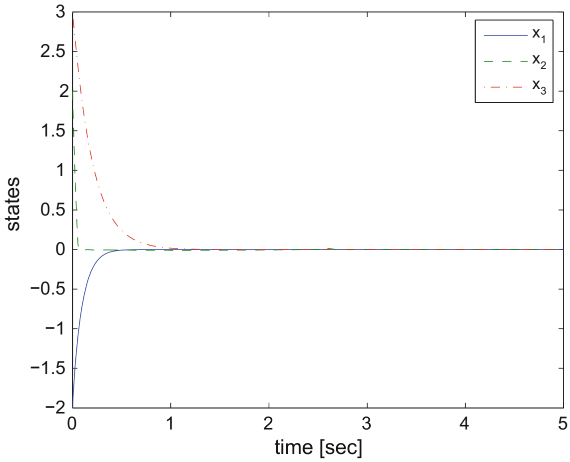


**Fig. 6.1** The time responses of the state variables of System (6.150)–(6.151) under control (6.157)

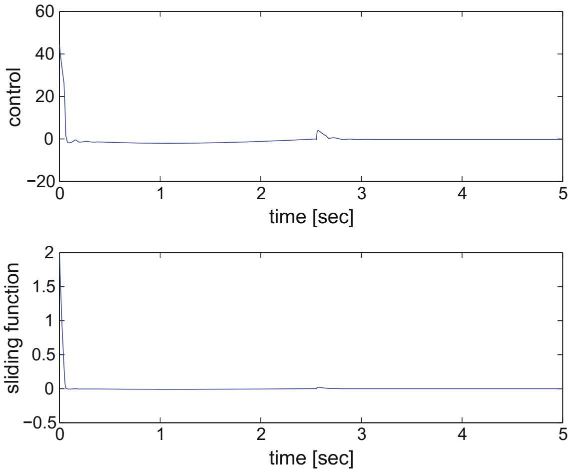


**Fig. 6.2** The time responses of control signal (6.157) (*upper*) and sliding function (6.154) (*lower*)

A simulation with the initial condition

$$\phi(t) = \mathrm{col}(\cos(t), 1, -2\sin(t))$$

is shown in Figs. 6.1 and 6.2 and confirms that the proposed approach is effective.

### 6.5.2 Reduced-Order Observer-Based Feedback Control

Consider a time-delay system described by

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{B} u + \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}}_{E} \Delta f(\cdot)$$
$$+ \underbrace{\begin{bmatrix} x_{11}^2 x_{13d} \\ x_{11}^2 x_{13d} + |x_{12}| x_{12d} \exp\{-t-2\} \\ -x_{11}^2 x_{13d} \end{bmatrix}}_{f(t,x,x_d)} \tag{6.158}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{C} x \tag{6.159}$$

where $x = \mathrm{col}(x_{11}, x_{12}, x_{13}) \in \mathbb{R}^3$, $u_1 \in \mathbb{R}$ and $y = \mathrm{col}(y_{11}, y_{12}) \in \mathbb{R}^2$ are, respectively, the state variables, input and outputs of the system. The uncertainty is assumed to satisfy

$$\|\Delta f(t, x, x_d)\| \le \underbrace{0.3|x_{12}|\ \sin^2(t)}_{\rho(\cdot)} \|x_d\|.$$

The considered domain is

$$\Omega := \big\{(x_{11}, x_{12}, x_{13}) \,|\, |x_{11}| < 5,\ |x_{12}| < 1.16, |x_{13}| < 5\big\}.$$

It is clear that Assumptions 6.4 and 6.7 hold, and

$$f(\cdot) = \underbrace{\begin{bmatrix} 0 & 0 & x_{11}^2 \\ 0 & |x_{12}| \exp\{-t-2\} & x_{11}^2 \\ 0 & 0 & -x_{11}^2 \end{bmatrix}}_{\Phi(\cdot)} x_d$$
$$= x_{11}^2 x_{13d} B + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & |x_{12}| \exp\{-t-2\} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Phi_1(\cdot)} x_d \tag{6.160}$$

which shows that Assumption 6.6 holds. Let

$$L_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}^T.$$

It is straightforward to check that $A - L_0 C$ is stable and Assumption 6.5 holds with

$$P_0 = 0.5 I_3$$

and

$$F = \mathrm{diag}\{0.25, -0.25\}.$$

Let $\varepsilon = 0.8$. It follows that the conditions in Theorem 6.3 are satisfied with $\gamma = 0.01$, and thus the reduced-order observer (6.59) is well defined and

$$\alpha_1 = 5, \qquad \alpha_2 = 0.1825.$$

According to the algorithm given in [38], the sliding surface is chosen as

$$S := [S_1 \;\; S_2] = [0.1961 \;\; -0.9806 \;\; | \;\; 0] \tag{6.161}$$

and the coordinate transformation matrix $T$ is

$$T = [T_1 \;\; T_2] = \left[\begin{array}{r|rr} 0.1961 & 0.9806 & 0 \\ -0.9806 & 0.1961 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Then,

$$ST = [1 \;\; 0 \;\; 0]$$

and

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} := \left[\begin{array}{r|rr} 0 & 0 & 0 \\ \hline [2mm]1 & -1.5 & 0 \\ -0.9806 & 1.4709 & -1 \end{array}\right]$$

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} := 10^{-15} \left[\begin{array}{r} -0.0833 \\ \hline 0.1249 \\ -0.1061 \end{array}\right]$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} := 10^{-15} \left[\begin{array}{rrr} 0 & 0 & 0.0833 \\ \hline 0 & 0 & -0.1249 \\ 0 & 0 & 0.1061 \end{array}\right]$$

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} := \left[\begin{array}{rrr} 0 & 0 & 0 \\ \hline 1.2748 & -1.2748 & 0 \\ -0.2500 & 1.2500 & \leftarrow 1 \end{array}\right].$$

When the system is constrained to the sliding surface, it follows from (6.161) that

$$x_{12} = 0.9806 S_2 e + 0.1961 z_{12} = 0.1961 z_{12} \tag{6.162}$$

where $z =: \mathrm{col}(z_{11}, z_{12}, z_{13})$. From Remark 6.15, the function matrix $\Phi(\cdot)$ in Theorem 6.4 can be replaced by $\Phi_1(\cdot)$ in (6.160). Let

$$q_2 = 1.01, \qquad \varepsilon_1 = 2, \qquad \text{and} \qquad \varepsilon_2 = 0.8.$$

By direct computation, it follows that the matrix $W$ defined in Theorem 6.4 is given by

$$\begin{bmatrix} -0.2848 & 0.3063 & 0 & -0.0833\varpi & 0 & 0 \\ 0.3063 & -0.4794 & 0 & 0.0490\varpi & 0 & 0 \\ * & * & -0.1825 & 0 & 0 & 0 \\ * & * & * & -0.3768 & -0.2942 & 0 \\ * & * & * & -0.2942 & -0.2550 & 0 \\ * & * & * & * & * & -0.1080 \end{bmatrix}$$

where

$$\varpi := |x_{12}| \exp\{-t-2\}$$

and $*$ represents the corresponding symmetric entries, which is negative definite in the considered domain $\Omega$ with maximum eigenvalue $q_0 = -0.0125$. Therefore, the associated sliding motion is uniformly asymptotically stable. Further, by direct computation, $\mathscr{L}_\rho$ and $\mathscr{L}_{\Phi_1}$ can be given by

$$\mathscr{L}_\rho = \sin^2 t \qquad \text{and} \qquad \mathscr{L}_{\Phi_1} = \exp\{-t-2\}.$$

Then the $k(\cdot)$ satisfying (6.105) can be determined directly and thus the controller (6.104) is well defined. For simulation purposes, the uncertainty $\Delta f(\cdot)$ is chosen as

$$\Delta f(\cdot) = \begin{bmatrix} 0.27x_{12}\sin^2(t)\|x_d\| \\ -0.12x_{12}\sin^2(t)\|x_d\| \end{bmatrix}.$$

Let $\eta = 2$, and $x_0 = (-3.5, -1, 4.5)$ with $\hat{w}_{20} = 2$. Assume the delay is

$$d(t) = 5 + \sin(t)$$

and the initial condition associated with the delay is

$$\phi(t) = [\cos(t)\ 0\ 1 + 2\sin(t)]^T.$$

The simulation results in Fig. 6.3 demonstrate that the proposed approach is effective.

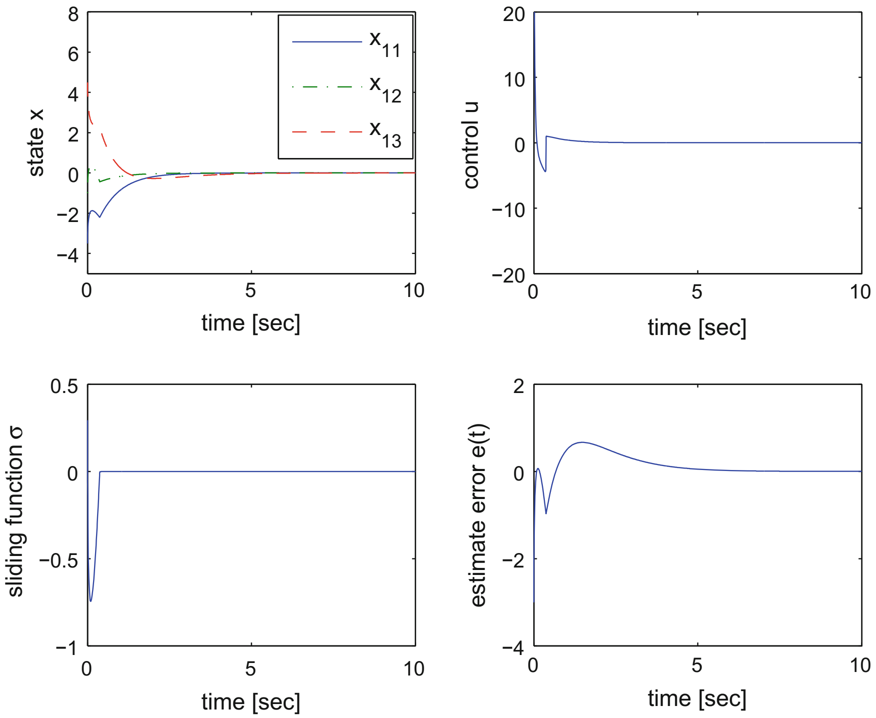


**Fig. 6.3** Evolution of states, control, sliding function and estimate error (e(t):=$x_{13} - \hat{x}_{13}$)

### *6.5.3 Decentralised Output Feedback Control*

Consider an interconnected time-delay system composed of two third-order subsystems and described by

$$\dot{x}_1 = \underbrace{\begin{bmatrix} -8 & 0 & 1 \\ 0 & -8 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_1} \Big( u_1 + \underbrace{(x_{11} + x_{12})^2 x_{13d_1} \sin x_{12d_1} \Delta g_1(\cdot)}_{G_1(\cdot)} \Big)$$
$$+ \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 4 & 1 \end{bmatrix}}_{H_{12}} y_{2d_2} \tag{6.163}$$
$$\dot{x}_2 = \underbrace{\begin{bmatrix} -6 & 0 & 1 \\ 0 & -6 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_2} \Big( u_2 + \underbrace{\Delta g_2(\cdot)}_{G_2(\cdot)} \Big)$$

$$+\underbrace{\begin{bmatrix} 4(x_{11d_1}+x_{12d_1})(\sin x_{13})^2\Delta h_1(\cdot) \\ 4(\sin x_{13})^2 x_{13d_1}(\Delta h_1(\cdot))^2 \\ (x_{11}+x_{12})\,x_{13d_1}(\sin x_{11d_1})^2\Delta h_2(\cdot) \end{bmatrix}}_{\Delta H_{21}(\cdot)} \tag{6.164}$$

$$y_1 = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C_1} x_1 \tag{6.165}$$

$$y_2 = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C_2} x_2 \tag{6.166}$$

where $x_1 = \text{col}(x_{11}, x_{12}, x_{13}) \in \mathbb{R}^3$ and $x_2 = \text{col}(x_{21}, x_{22}, x_{23}) \in \mathbb{R}^3$, $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}$, and $y_1 = \text{col}(y_{11}, y_{12}) \in \mathbb{R}^2$ and $y_2 = \text{col}(y_{21}, y_{22}) \in \mathbb{R}^2$ are, respectively, the state variables, inputs and outputs of the system. The uncertainties $\Delta g_1(\cdot)$, $\Delta g_2(\cdot)$, $\Delta h_1(\cdot)$ and $\Delta h_2(\cdot)$ are assumed to satisfy

$$\begin{aligned} &|\Delta g_1(\cdot)| \le (x_{13d_1})^2 \sin^2(x_{11}+x_{12}) \\ &|\Delta h_1(\cdot)| \le 1 \\ &|\Delta g_2(\cdot)| \le |x_{21d_2}+x_{22d_2}|\, x_{23}^2\, \sin^2 x_{22} \\ &|\Delta h_2(\cdot)| \le \|y_{1d_1}\|. \end{aligned}$$

In this example,

$$\Delta H_{12} = 0 \quad \text{and} \quad H_{21} = 0.$$

The decompositions (6.114) and (6.115) can be given as

$$H_{12}^a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 1 \end{bmatrix}, \qquad H_{12}^b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.167}$$

$$\Delta H_{12}^a = \Delta H_{12}^b = 0, \qquad H_{21}^a = H_{21}^b = 0 \tag{6.168}$$

$$\Delta H_{21}^a = B_2\,\underbrace{(x_{11}+x_{12})\,x_{13d_1}(\sin x_{11d_1})^2\Delta h_2(\cdot)}_{\Theta_{21}(\cdot)} \tag{6.169}$$

$$|\Theta_{21}| \le \underbrace{|y_{11}y_{12d_1}|}_{\alpha_{21}(\cdot)}\ \|y_{1d_1}\| \tag{6.170}$$

$$\Delta H_{21}^b = \begin{bmatrix} 4(x_{11d_1}+x_{12d_1})(\sin x_{13})^2\Delta h_1(\cdot) \\ 4(\sin x_{13})^2 x_{13d_1}(\Delta h_1(\cdot))^2 \\ 0 \end{bmatrix} \tag{6.171}$$

$$\|\Delta H_{21}^b(\cdot)\| \le 4\sin^2 y_{12}\sqrt{y_{11d_1}^2 + y_{12d_1}^2} = \underbrace{4\sin^2 y_{12}}_{\beta_{21}(\cdot)}\,\|y_{1d_1}\|. \tag{6.172}$$

It is straightforward to check that Assumption 6.8 is satisfied with

$$\begin{aligned}
&g_1(\cdot) = y_{11}^2 |y_{12d_1}|^3 \sin^2 y_{11}, && g_2(\cdot) = |y_{21d_2}| y_{22}^2 \\
&\alpha_{12}(\cdot) = 0, && \alpha_{21}(\cdot) = |y_{11} y_{12d_1}| \\
&\beta_{12}(\cdot) = 0, && \beta_{21}(\cdot) = 4\sin^2 y_{12} \\
&\rho_{12}(\cdot) = 0, && \rho_{21}(\cdot) = \sqrt{16\sin^4 y_{12} + (y_{11} y_{12d_1})^2}.
\end{aligned}$$

Assumption 6.9 can also be shown to hold. Using the algorithm given in [38], it follows that Assumption 6.10 is satisfied with

$$\tilde{K}_1 = \tilde{K}_2 = 0$$

and the coordinate transformations $z_i = T_i x_i$ for $i = 1, 2$ are given by

$$T_1 = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and the sliding surface matrices are

$$F_1 = [0 \quad 1] \quad \text{and} \quad F_2 = [0 \quad 1].$$

Let

$$Q_1 = Q_2 = I_2.$$

The corresponding solutions to the Lyapunov equations (6.130) are

$$P_1 = \begin{bmatrix} 0.0625 & 0 \\ 0 & 0.0625 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0.0833 & 0 \\ 0 & 0.0833 \end{bmatrix}.$$

Further, choose $p = \gamma_1 = \gamma_2 = 1.1$. By direct computation,

$$N_1 = \begin{bmatrix} 0.9108 & -0.0083 \\ -0.0083 & 0.9049 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0.9375 & 0 \\ 0 & 0.9375 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.8991 & 0 \\ -0.5185\sin^2 y_{12} & 0.9375 \end{bmatrix}.$$

It can be verified that the conditions in Theorem 6.6 are satisfied globally. So from Theorem 6.6, the sliding motion associated with the sliding surface

$$\{(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}) \mid x_{13} = 0,\ x_{23} = 0\}$$

is globally uniformly asymptotically stable. By direct computation, Assumption 6.11 is satisfied with

$$\Gamma_1 = \Gamma_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

From Theorem 6.7, the decentralised control law

$$u_1 = -y_{11} - \Big(y_{11}^2 |y_{12d_1}|^3 \sin^2 y_{11} + \sqrt{16\sin^4 y_{12} + (y_{11}y_{12d_1})^2} + 20\Big)\operatorname{sgn}(y_{12})$$
$$u_2 = -y_{21} - \big(|y_{21d_2}|y_{22}^2 + 2.0616\|y_{2d_2}\| + 16\big)\operatorname{sgn}(y_{22})$$

stabilises the system (6.163)–(6.164) globally.

For simulation purposes, assume the delays are

$$d_1(t) = 2 - \sin(t)$$

and

$$d_2(t) = 1 - 0.5\cos(t).$$

The initial conditions are

$$\phi_1(t) = \operatorname{col}(\cos(t), 0, -2\sin(t))$$

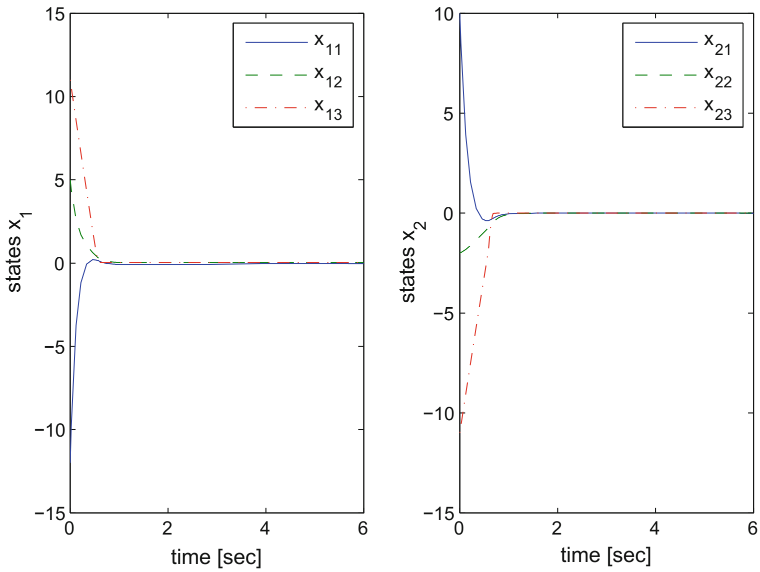


**Fig. 6.4** The evolution of the state variables of the system (6.163)–(6.164)

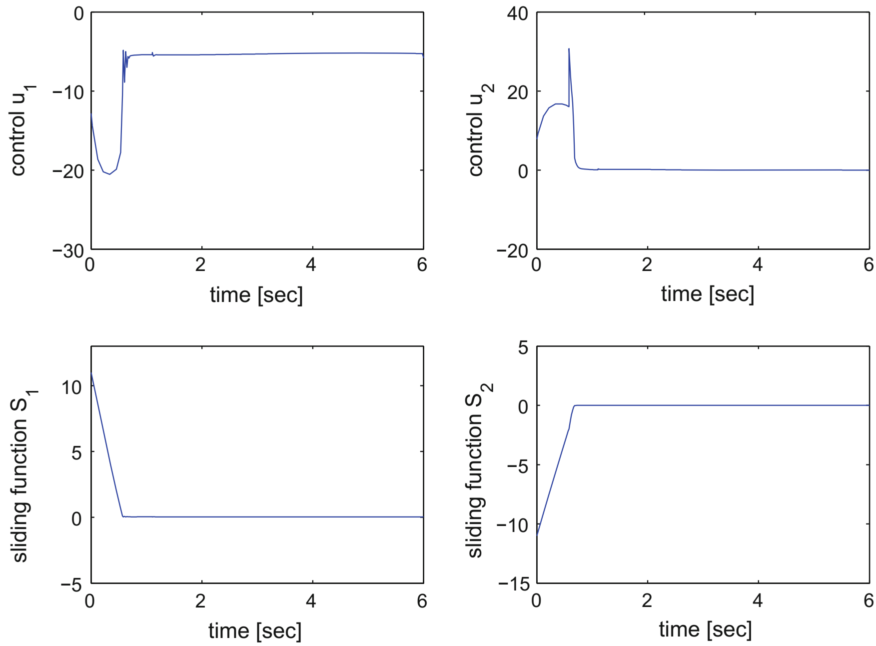


**Fig. 6.5** The time responses of the control signal and sliding functions

and

$$\phi_2(t) = \text{col}(0, 1, 2 + \sin(t)).$$

The simulation results in Figs. 6.4 and 6.5 demonstrate that the proposed approach is effective.

*Remark 6.26* In the example above, the interconnection terms have been decomposed into matched and mismatched parts as in (6.167)–(6.172). These are subsequently 'processed' separately. If all the interconnections in the example are taken as mismatched, then, the conditions of Theorem 6.6 will not be satisfied. This shows that the decomposition in (6.114) and (6.115) can be used to reduce conservatism.

## 6.6 Application to River Pollution Control

In this section, the decentralised control scheme developed in Sect. 6.4 will be applied to a river pollution control problem.

Consider a two-reach model of a river pollution control problem [111]. It is assumed that the concentration of biochemical oxygen demand (BOD) for the first subsystem is perturbed by a time-delay. Then, based on the approach given in [15], the system can be described by

$$\dot{x}_1 = \underbrace{\begin{bmatrix} -1.32\delta & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_1} x_2 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_1} \Big( u_1 + \underbrace{\begin{bmatrix} -13.2(1-\delta) \\ 0 \end{bmatrix} y_{1d_1}}_{G_1(\cdot)} \Big) + \Delta H_{12}(\cdot) \tag{6.173}$$

$$\dot{x}_2 = \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}_{B_2} (u_2 + G_2(\cdot)) + \underbrace{\begin{bmatrix} 0.9\delta \\ 0 \end{bmatrix}}_{H_{21}} y_{1d_1} + \underbrace{\begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}}_{\hat{H}_{21}} x_1 + \underbrace{\begin{bmatrix} -0.9\delta y_1 \\ 0 \end{bmatrix}}_{\Delta H_{21}(\cdot)} \tag{6.174}$$

$$y_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_1} x_1, \qquad y_2 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_2} x_2 \tag{6.175}$$

where $x_1 := \text{col}(x_{11}, x_{12})$ and $x_2 := \text{col}(x_{21}, x_{22})$. The variables $x_{i1}$ and $x_{i2}$ represent the concentration of the BOD and the concentration of dissolved oxygen respectively, and the controls $u_i$ are the BOD of the effluent discharge into the river for $i = 1, 2$. The constant $\delta \in [0, 1]$ is the retarded coefficient. The uncertainties $G_2(\cdot)$ and $\Delta H_{12}(\cdot)$ are added to illustrate the obtained results, and are assumed to satisfy

$$|G_2(\cdot)| \le (y_{2d_2})^2 \sin^2 y_2$$
$$\|\Delta H_{12}\| \le \exp\{y_2 - 2\} |y_{2d_2} \sin t|.$$

Let

$$\delta = 0.5, \quad g_1 = 6.6|y_{1d_1}| \qquad g_2 = (y_{2d_2})^2 \sin^2 y_2$$
$$\alpha_{12} = \beta_{12} = \rho_{12} = \exp\{y_2 - 2\}|\sin t|, \; H_{12} = 0$$
$$H_{21} = \underbrace{\begin{bmatrix} 0.45 \\ 0 \end{bmatrix}}_{H_{21}^a} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{H_{21}^b}, \qquad H_{21}^a = \underbrace{0.45}_{D_{21}} B_2$$
$$\Delta H_{21} = \underbrace{\begin{bmatrix} -0.45 y_1 \\ 0 \end{bmatrix}}_{\Delta H_{21}^a} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\Delta H_{21}^b}, \qquad \Delta H_{21}^a = \underbrace{-0.45 y_1}_{\Delta \Theta_{21}} B_2$$

$$\alpha_{21} = \rho_{21} = 0.45, \qquad \beta_{21} = 0$$
$$\hat{H}_{21} = \underbrace{\begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix}}_{\hat{H}_{21}^a} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0.9 \end{bmatrix}}_{\hat{H}_{21}^b}, \; \hat{H}_{21}^a = B_2 \underbrace{\begin{bmatrix} 9 & 0 \end{bmatrix}}_{\hat{D}_{21}}.$$

It is straightforward to check that Assumptions 6.8 and 6.9 hold. Since both the subsystems are square, Assumption 6.10 is replaced by the condition that $\tilde{A}_{i1}$ is stable [38]. The coordinate transformation matrices $T_i$ are chosen as

$$T_1 = T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} = \begin{bmatrix} -1.2 & -0.32 \\ 0 & -0.66 \end{bmatrix}, \qquad \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} = \begin{bmatrix} -1.2 & -0.32 \\ 0 & -1.32 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ B_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & C_{12} \end{bmatrix} = \begin{bmatrix} 0 & C_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let $Q_1 = Q_2 = 2.4$. Then, $P_1 = P_2 = 1$. The sliding matrices are given by

$$F_1 = F_2 = 1$$

and the sliding functions are

$$S_1 = y_1 \qquad \text{and} \qquad S_2 = y_2.$$

By direct computation, Assumption 6.11 holds with

$$\Gamma_1 = -0.66$$

and

$$\Gamma_2 = -1.32$$

and Eq. (6.149) is satisfied with

$$\Gamma_{21} = 0.9.$$

Following Remark 6.25, it is directly verified that the conditions in Theorem 6.6 hold with

$$p = \gamma_1 = \gamma_2 = 1.01.$$

Then, System (6.173)–(6.175) is stabilisable globally by the control

$$u_1 = -10\left\{-0.66y_1 + 0.9y_1 + \left(1.56|y_{1d_1}| + 1\right)\operatorname{sgn}(y_1)\right\} \tag{6.176}$$

$$u_2 = -10\Big\{ -1.32y_2 + (0.1(y_{2d_2}\sin y_2)^2 + 5 + \exp\{y_2 - 2\}|\sin t|)\operatorname{sgn}(y_2)\Big\} \tag{6.177}$$

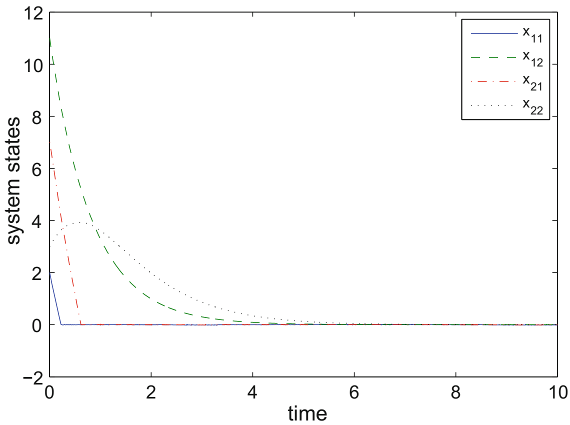


**Fig. 6.6** The evolution of the state variables of the system (6.173)–(6.175) under control (6.176)–(6.177)

where the term $0.9y_1$ in Eq. (6.176) comes from the interconnection $\hat{H}_{21}x_1$. For simulation purposes, the delays are chosen as

$$d_1(t) = 3 - 2\sin(t) \qquad \text{and} \qquad d_2(t) = 1$$

and the delay related initial conditions are chosen as

$$\phi_1(t) = \text{col}(2\cos t, 1) \qquad \text{and} \qquad \phi_2(t) = \text{col}(0, 1 - \sin(t)).$$

The simulation results in Fig. 6.6 show the effectiveness of the results obtained.

## 6.7 Summary

In this chapter, time-varying state delay systems have been considered. Section 6.2 presents static output feedback control strategies. Approaches to deal with nonlinear matched and mismatched disturbances have been proposed when time-varying delay is involved in the nonlinear bounds on the disturbances. Specifically, in the reduced-order observer-based control scheme proposed in Sect. 6.3, coordinate transformations are employed to explore the system structure. The features of sliding mode control are used to enhance the robustness of the scheme. This reduced-order observer scheme has benefits for implementation. Section 6.4 has shown a decentralised static output feedback sliding mode control law to globally stabilise a class of time-varying delay interconnected systems. It has been shown that the effects of the interconnections can be canceled completely in the reachability analysis by designing an appropriate decentralised control law if the interconnections have the form given in Remark 6.24. Several examples have been presented to illustrate the results.

# Chapter 7
# Delay Independent Output Feedback Control

This chapter will introduce time-delay independent control design for nonlinear systems using output feedback control, and then study large-scale interconnected systems.

## 7.1 Introduction

Control design for nonlinear time-delay systems has been discussed in Chap. 6 but the control algorithm required that all the time-delays are precisely known, and thus can be used by the control algorithm. However, in many cases, the time-delays may not be known precisely. Specifically it is very difficult to identify the time-delay if it is time varying. Therefore, it is of interest to study control of time-delay systems when the delay is unknown. It should be noted that a time-delay dependent result is usually less conservative than a corresponding time-delay independent result but requires that information on the time-delay is available, which may be limiting. However, the delay dependent controller usually explicitly depends on the time-delay and thus needs memory to store historical data in implementation, which needs more resources.

In Sects. 7.2 and 7.3, local stabilisation is considered for affine nonlinear control systems with uncertainties involving time-varying delay. It is not assumed that the nominal system is either linearisable or partially linearisable. A static output feedback variable structure control is synthesised to stabilise the system uniformly asymptotically, and a control strategy to enforce exponential stability is also derived in Sect. 7.2. In Sect. 7.3, an appropriate transformation is introduced to express the affine nonlinear system in regular form which facilitates both design and analysis. Then, sufficient conditions are developed based on the Lyapunov–Razumikhin approach such that the sliding motion is uniformly asymptotically or exponentially stable. A static out-

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put feedback control law, independent of the time-delay, is synthesised to guarantee reachability.

Section 7.4 focuses on the stabilisation problem for a class of large-scale systems with nonlinear interconnections. A decentralised static output feedback variable structure control is synthesised and the stability of the corresponding closed-loop system is analysed. A set of conditions is developed to guarantee that the considered large-scale interconnected systems are stabilised uniformly asymptotically. Section 7.5 provides some examples to demonstrate the results developed in Sects. 7.2–7.4.

## 7.2 Lyapunov Technique-Based Variable Structure Control for Nonlinear Systems

This section presents time-delay independent variable structure output feedback controllers to stabilise affine nonlinear systems based on Lyapunov techniques.

### *7.2.1 Introduction*

It is well known that linear dynamical systems cannot adequately describe many phenomena commonly observed in the real world. In order to incorporate complex phenomena, it is necessary to investigate nonlinear systems as a means to more accurately model real systems. Compared with linear control systems, the study of nonlinear systems is relatively immature—largely due to their complexity. In the control paradigm, state or output feedback is used to form a closed-loop system to improve the dynamic performance. Compared with state feedback, output feedback is much more difficult because effectively only a subset of the state variables are available for design and controller implementation. However, in reality, usually only a subset of measurements is available for use by the controller. One way to circumvent this problem is to design an observer to measure/estimate system states [159, 180, 210], and then use the estimated states to replace the true states to form the feedback loop. However, the separation principle usually does not hold for nonlinear systems, which implies that approaches using the true state to design the control law may produce completely different results when estimated states are used for implementation. Although observer based or dynamical output feedback control has been extensively applied [139, 208], extra devices or hardware are necessary which may be too expensive to implement. Therefore, there is a need to consider static output feedback control approaches for nonlinear systems.

In recent decades, static output feedback control has been widely used in control design (see, e.g., [34, 84, 112, 147, 207]). However, the systems considered in much of the existing work are either largely linear or delay free. A class of linear time-delay systems is considered in [34]. By introducing an artificial delay, a static output feedback control scheme is provided in [147] where the considered system is linear and delay free. A class of linear time-delay systems with nonlinear disturbance is considered in [207] where it is required that the time-varying delay is precisely known. Luo et al. considered a class of time-delay systems where both static and dynamic output feedback strategies are studied [112] but it is required that all of the uncertainties are matched. Janardhanan and Bandyopadhyay [84] proposed a static output feedback sliding mode control scheme for time-delay systems where a class of linear discrete-time systems is considered. Recently, a class of nonlinear time-delay systems with uncertainties bounded by nonlinear functions has been considered in [208] but it is required that the time-delay is exactly known and observer-based output feedback is used. Some interesting results on stabilisation of nonlinear time-delay systems have been developed in [136] but state feedback is employed. The problem of static output feedback stabilisation for nonlinear systems is full of challenge especially when the system experiences both uncertainties and time-delay.

Control systems with time-delay disturbances have been widely studied (see, e.g., [71, 188]). Much of the existing work studying systems with disturbances is based on the fact that the nominal system is stable or has desired performance [71, 188, 207, 221]. In this section, a class of affine nonlinear control systems with nonlinear uncertainties which involve time-varying delay is considered. Similar to the work in [71, 188, 207, 221], it is assumed that the nominal system is output feedback stabilisable with an output feedback control law having been well designed. Then, a robust static output feedback variable structure control is synthesised such that the corresponding closed-loop system is uniformly asymptotically stable in the presence of uncertainties and time-delay. Both matched and mismatched uncertainties are considered, and the accessible bounds on the uncertainties are employed in the control design to reduce the effects of the uncertainties to enhance the robustness of the designed controller. Furthermore, a set of sufficient conditions is developed to guarantee that the closed-loop system is exponentially stable. The designed control does not depend on the time-delay and thus it is not required that the time-delay is known. It is not required that either the system is square, or the nominal system is linearisable or partially linearisable.

### *7.2.2 System Description and Analysis*

Consider an affine nonlinear system described by

$$\dot{x} = f(x) + g(x)(u + \Delta g(x, x_d)) + \Delta f(x, x_d) \tag{7.1}$$

$$y = h(x) \tag{7.2}$$

where $x \in \mathscr{X} \subset \mathbb{R}^n$ ($\mathscr{X}$ is a neighbourhood of the origin), $u \in \mathbb{R}^m$, $y \in \mathscr{Y} \subset \mathbb{R}^p$ are system state variables, inputs and outputs respectively. The functions $f(x)$ and $g(x)$

with $f(0) = 0$ are both known functions, and $\Delta g(x, x_d)$ and $\Delta f(x, x_d)$ are matched and mismatched uncertainties respectively—which include all the disturbances and modelling errors. The symbol $x_d := x(t - d(t))$ denotes the delayed states where $d(t)$ is a time-varying delay which is assumed to be continuous, nonnegative in $\mathbb{R}^+$ and

$$\overline{d} := \sup_{t\in\mathbb{R}^+} \{d(t)\} < \infty.$$

The initial condition relating to the time-delay is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0] \tag{7.3}$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. It is assumed that all the nonlinear terms are smooth enough such that the existence and uniqueness of solutions of the unforced system is guaranteed.

For System (7.1)–(7.2), the system

$$\dot{x} = f(x) + g(x)u \tag{7.4}$$

$$y = h(x) \tag{7.5}$$

is called the corresponding nominal system. The following assumptions are imposed on the system (7.1)–(7.2).

**Assumption 7.1** There exist known continuous nonnegative functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\beta(\cdot)$ such that

$$\|\Delta g(x, x_d)\| \leq \alpha_1(y) + \alpha_2(y)\alpha_3(x, \|x_d\|) \tag{7.6}$$

$$\|\Delta f(x, x_d)\| \leq \beta(x, \|x_d\|) \tag{7.7}$$

where $\beta(x, r)$ is a class $\mathscr{K}I$ function.

*Remark 7.1* The mathematical definition of the class $\mathscr{K}I$ function has been provided in Definition 2.5 in Sect. 2.2). Assumption 7.1 provides limitations on the matched and mismatched uncertainties, and requires that the bounds on the uncertainties are known. It should be noted that the bounds on both the matched uncertainty $\Delta g(x, x_d)$ and mismatched uncertainty $\Delta f(x, x_d)$ are nonlinear and subject to the time-delay with general forms. This is in comparison with the existing work in [75, 112, 132, 207, 221] where the bounds are functions of system outputs or satisfy a linear growth condition.

**Assumption 7.2** There exists a continuous function $u_1(y)$ in $\mathscr{Y}$, a $C^1$ function $V(x) : \mathbb{R}^n \mapsto \mathbb{R}$ and a continuous function $M(\cdot) \in \mathbb{R}^{1\times m}$ defined in $\mathscr{Y}$ such that in the considered domain $\mathscr{X}$,

$$c_1(\|x\|) \le V(x) \le c_2(\|x\|) \tag{7.8}$$

$$\frac{\partial V}{\partial x}(f(x) + g(x)u_1(y)) \le -c_3(\|x\|) \tag{7.9}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4(\|x\|) \tag{7.10}$$

$$\frac{\partial V}{\partial x} g(x) = M(y) \tag{7.11}$$

where $c_1(\cdot)$, $c_2(\cdot)$, $c_3(\cdot)$ and $c_4(\cdot)$ are class $\mathscr{K}$ functions in $\mathbb{R}^+$, and

$$\frac{\partial V}{\partial x} =: \left( \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n} \right).$$

*Remark 7.2* Assumption 7.2 limits the nominal system (7.4)–(7.5) where the conditions (7.8)–(7.10) guarantee that the nominal system is output feedback stabilised by $u = u_1(y)$. The condition (7.11) provides a constraint on the Lyapunov function $V(\cdot)$. In order to form a static output feedback control scheme, the limitation (7.11) is required. It should be noted that for a linear system $(A, B, C)$ with Lyapunov function

$$V = x^T P x$$

where $P > 0$ satisfies

$$(A - BKC)^T P + P(A - BKC) < 0$$

condition (7.11) degenerates to the condition that there exists a matrix $F$ such that the matrix equation

$$B^T P = FC$$

holds, which is the well known constrained Lyapunov problem (CLP) [41, 57]. It is clear to see that the conditions (7.8)–(7.10) together with the limitation (7.11) is an extension of the CLP. The CLP has been extended to the nonlinear case in [196] but it is required that the system considered is square. The current consideration has extended the CLP to the non-square nonlinear case which renders all previous settings as special cases in this regard.

**Assumption 7.3** The inequality

$$\|x(t + \theta)\| \le \gamma(\|x\|)$$

holds for some continuous nonnegative function $\gamma(\cdot)$ if

$$V(x(t + \theta)) \le w(V(x(t))), \qquad \theta \in [-\bar{d}, 0]$$

for some function $w(\cdot)$ defined in $\mathbb{R}^+$ satisfying $w(r) > r$ in $r \in \mathbb{R}^+$.

*Remark 7.3* It should be noted that Assumption 7.3 is a limitation on the function $V(\cdot)$ given in Assumption 7.2. It is not required if there is no time-delay involved in the system. Since the considered systems are nonlinear, Lyapunov functions may have various forms/structures for different systems. Therefore, there is no general way to check Assumption 7.3. However, if $V(\cdot)$ has a quadratic form, then Assumption 7.3 is satisfied automatically. Moreover, Lemma C.1 and Remark C.1 in Appendix C present classes of functions which satisfy Assumption 7.3.

In this section, it is assumed that the nominal system (7.4)–(7.5) is output feedback stabilisable and the control $u_1(\cdot)$ in Assumption 7.2 has been well defined. The objective is to design a variable structure control such that the corresponding closed-loop system is uniformly asymptotically stable in the presence of time-delayed uncertainties. The local case will be considered. For ease of exposition, the domain considered may not be specifically stated in the subsequent analysis unless it is necessary, but each variable's dimension will be clearly shown.

### 7.2.3 Asymptotic Stabilisation Control Synthesis

A static output feedback control will be synthesised such that the corresponding closed-loop system is uniformly asymptotically stable. Then conditions for exponential stabilisation will follow.

For System (7.1)–(7.2), consider an output feedback control law

$$u(y) = u_1(y) + u_2(y) \tag{7.12}$$

where the function $u_1(\cdot)$ is given in Assumption 7.2, and $u_2(\cdot)$ is defined by

$$u_2(y) = \begin{cases} -M^T(y)\Big(\frac{1}{\|M(y)\|}\alpha_1(y) + \frac{1}{2\varepsilon}\alpha_2^2(y)\Big), & M(y) \neq 0 \\ 0, & M(y) = 0 \end{cases} \tag{7.13}$$

where $M(\cdot) \in \mathbb{R}^{1\times m}$ is given in (7.11), $\varepsilon$ is an adjustable positive constant, and $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are determined by (7.6).

The control (7.12) is called a variable structure control due to the term $u_2(\cdot)$ which usually prescribes a different structure between the outside and the inside of the surface $M(y) = 0$. The following result is ready to be presented.

**Theorem 7.1** *Under Assumptions 7.1–7.3, the closed-loop system formed by applying the control (7.12)–(7.13) to the system (7.1)–(7.2) is uniformly asymptotically locally stable if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that*

$$c_3(\|x\|) - \frac{\varepsilon}{2}\alpha_3^2(x, \gamma(\|x\|)) - c_4(\|x\|)\beta(x, \gamma(\|x\|)) \geq \alpha(\|x\|) \tag{7.14}$$

*for some positive constant* $\varepsilon$*, where* $c_3(\cdot)$ *and* $c_4(\cdot)$ *satisfy Assumption 7.2 and* $\alpha_3(\cdot)$ *and* $\beta(\cdot)$ *satisfy Assumption 7.1.*

*Proof* The closed-loop system obtained by applying the control (7.12)–(7.13) to System (7.1)–(7.2) is described by

$$\dot{x} = f(x) + g(x)\left(u_1(y) + u_2(y) + \Delta g(x, x_d)\right) + \Delta f(x, x_d) \tag{7.15}$$

$$y = h(x). \tag{7.16}$$

For System (7.15)–(7.16), consider the Lyapunov function candidate $V(x)$ in Assumption 7.2. The time derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}\,|_{(7.15)-(7.16)} &= \frac{\partial V}{\partial x}\left(f(x) + g(x)u_1(y)\right) + \frac{\partial V}{\partial x} g(x)\big(u_2(y) \\ &\quad + \Delta g(x, x_d)\big) + \frac{\partial V}{\partial x}\Delta f(x, x_d) \\ &\le -c_3(\|x\|) + \frac{\partial V}{\partial x} g(x)\big(u_2(y) + \Delta g(x, x_d)\big) \\ &\quad + \frac{\partial V}{\partial x}\Delta f(x, x_d) \end{aligned} \tag{7.17}$$

where (7.9) is used to obtain the last inequality. From the definition of $u_2(\cdot)$ in (7.13), Inequality (7.6) and Eq. (7.11), it follows that

(i) if $M(y) = 0$, then

$$\frac{\partial V}{\partial x} g(x)\left(u_2(y) + \Delta g(x, x_d)\right) = M(y)\left(u_2(y) + \Delta g(x, x_d)\right) = 0;$$

(ii) if $M(y) \neq 0$, then

$$\begin{aligned} &\frac{\partial V}{\partial x} g(x)\left(u_2(y) + \Delta g(x, x_d)\right) \\ &\le \frac{\partial V}{\partial x} g(x) u_2(y) + \left\|\frac{\partial V}{\partial x} g(x)\right\| \|\Delta g(x, x_d)\| \\ &\le -M(y)M^T(y)\left(\frac{1}{\|M(y)\|}\alpha_1(y) + \frac{1}{2\varepsilon}\alpha_2^2(y)\right) \\ &\quad + \|M(y)\|\left(\alpha_1(y) + \alpha_2(y)\alpha_3(x, \|x_d\|)\right) \\ &= -\|M(y)\|\alpha_1(y) - \frac{1}{2\varepsilon}\|M(y)\|^2\alpha_2^2(y) + \|M(y)\|\alpha_1(y) \\ &\quad + \|M(y)\|\alpha_2(y)\alpha_3(x, \|x_d\|) \\ &= -\frac{1}{2\varepsilon}\|M(y)\|^2\alpha_2^2(y) + \|M(y)\|\alpha_2(y)\alpha_3(x, \|x_d\|). \end{aligned} \tag{7.18}$$

Then from the special case of Young's inequality $ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$, it follows that for any $\varepsilon > 0$

$$\|M(y)\|\alpha_2(y)\,\alpha_3(x, \|x_d\|) \le \frac{1}{2\varepsilon}\alpha_2^2(y)\|M(y)\|^2 + \frac{\varepsilon}{2}\alpha_3^2(x, \|x_d\|). \tag{7.19}$$

Further from (7.19), (7.18) and the analysis in (i) and (ii) above, it is straightforward to see that

$$\frac{\partial V}{\partial x} g(x)\,(u_2(y) + \Delta g(x, x_d)) \le \frac{\varepsilon}{2}\alpha_3^2(x, \|x_d\|). \tag{7.20}$$

From (7.7) and (7.10),

$$\frac{\partial V}{\partial x}\Delta f(x, x_d) \le \left\|\frac{\partial V}{\partial x}\right\| \|\Delta f(x, x_d)\| \le c_4(\|x\|)\beta(x, \|x_d\|). \tag{7.21}$$

Substituting (7.20) and (7.21) into (7.17) yields

$$\dot{V} \le -c_3(\|x\|) + \frac{\varepsilon}{2}\alpha_3^2(x, \|x_d\|) + c_4(\|x\|)\beta(x, \|x_d\|). \tag{7.22}$$

In order to apply the Razumikhin approach, suppose that for any $d(t) \in [0,\ \bar{d}]$,

$$V(x(t+\theta)) \le w\,(V(x(t)))$$

where $w(\cdot)$ satisfies $w(r) > r$ for $r \in \mathbb{R}^+$. Then from Assumption 7.3, for any $d(t) \in [0,\ \bar{d}]$,

$$\|x_d\| \le \gamma(\|x\|). \tag{7.23}$$

From the result (i) of Lemma 2.3 in Sect. 2.2 and the condition that $\alpha_3(\cdot)$ is a class $\mathscr{KI}$ function, the function $\alpha_3^2(\cdot)$ belongs to class $\mathscr{KI}$. Since $\beta(\cdot)$ is also a class $\mathscr{KI}$ function, if $V(x(t+\theta)) \le w\,(V(x(t)))$, then (7.23) holds and thus

$$\alpha_3(x, \|x_d\|) \le \alpha_3(x, \gamma(\|x\|)) \tag{7.24}$$

$$\beta(x, \|x_d\|) \le \beta(x, \gamma(\|x\|)). \tag{7.25}$$

Applying (7.24) and (7.25) to (7.22), it follows from (7.14) that

$$\begin{aligned}\dot{V} &\le -c_3(\|x\|) + \frac{\varepsilon}{2}\alpha_3^2(x, \gamma(\|x\|)) + c_4(\|x\|)\beta(x, \gamma(\|x\|)) \\ &\le -\alpha(\|x\|).\end{aligned}$$

Hence the conclusion follows from the Razumikhin Theorem 2.5 in Sect. 2.5. ∇

*Remark 7.4* As for much of the existing work for fully nonlinear systems [71, 136, 211], the conditions developed in this section are sufficient and there is no general way to check the conditions due to the complexity of nonlinear systems. However, Theorem 7.1 has provided a way of dealing with nonlinear systems with time-delay disturbances. Moreover, the results developed are not dependent on time-delay and thus it is not required that the time-delay is known.

### *7.2.4 Exponential Stabilisation Control Synthesis*

Based on the analysis in Sect. 7.2.4, an output feedback control law will be synthesised for System (7.1)–(7.2) in this section such that the corresponding closed-loop system is exponentially stable. The Assumptions 7.1 and 7.2 are now strengthened to the following corresponding Assumptions 7.4 and 7.5.

**Assumption 7.4** The uncertainties $\Delta g(x, x_d)$ and $\Delta f(x, x_d)$ satisfy

$$\|\Delta g(x, x_d)\| \le \alpha_1(y) + \alpha_2(y)\|x_d\| \tag{7.26}$$

$$\|\Delta f(x, x_d)\| \le \beta_1(\|x\|) + \beta_2(\|x_d\|) \tag{7.27}$$

where $\alpha_1 : \mathscr{Y} \mapsto \mathbb{R}^+$ and $\alpha_2 : \mathscr{Y} \mapsto \mathbb{R}^+$ are known continuous and nonnegative functions, and $\beta_1 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\beta_2 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are known nonnegative and $\mathscr{K}C^1$ functions.

From Assumption 7.4, $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are $\mathscr{K}C^1$ functions, and it follows from result (ii) of Lemma 2.3 in Sect. 2.2 that there exist continuous functions $\beta_3 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\beta_4 : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that

$$\beta_1(r) = \beta_3(r)r \tag{7.28}$$

$$\beta_2(r) = \beta_4(r)r. \tag{7.29}$$

It should be noted that one of the choices to obtain $\beta_3(\cdot)$ and $\beta_4(\cdot)$ has been given in the proof of Lemma 2.3 in Sect. 2.2.

**Assumption 7.5** There exists a continuous function $u_a : \mathscr{Y} \mapsto \mathbb{R}^m$, a function matrix $M(\cdot) \in \mathbb{R}^{1\times n}$ defined in $\mathscr{Y}$ and a $C^1$ function $U(x) : \mathbb{R}^n \mapsto \mathbb{R}$ such that in the considered domain $\mathscr{X}$,

$$\kappa_1\|x\|^2 \le U(x) \le \kappa_2\|x\|^2 \tag{7.30}$$

$$\frac{\partial U}{\partial x}(f(x) + g(x)u_a(y)) \le -\kappa_3\|x\|^2 \tag{7.31}$$

$$\left\|\frac{\partial U}{\partial x}\right\| \le \kappa_4\|x\| \tag{7.32}$$

$$\frac{\partial U}{\partial x}g(x) = M(y) \tag{7.33}$$

for some positive constants $\kappa_i$ for $i = 1, 2, 3, 4$, where

$$\frac{\partial U}{\partial x} =: \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \ldots, \frac{\partial U}{\partial x_n}\right)$$

*Remark 7.5* The conditions (7.30)–(7.32) in Assumption 7.5 imply that the nominal system (7.4)–(7.5) is stabilisable by $u = u_a(y)$. Suppose that $f(x)$ and $g(x)$ are $C^1$ functions. Then it follows from [91] that Assumption 7.5 is satisfied if the nominal system (7.4)–(7.5) is exponentially stabilisable by a $C^1$ function $u = u_a(y)$, that is

$$\dot{x} = f(x) + g(x)u_a(y) \tag{7.34}$$

is exponentially stable.

The following result is now ready to be presented.

**Theorem 7.2** *Under Assumptions 7.4 and 7.5, System (7.1)–(7.2) is locally exponentially stabilised by the control*

$$u(\cdot) = u_a(y) + u_2(y) \tag{7.35}$$

*where $u_a(\cdot)$ satisfies Assumption 7.5 and $u_2(\cdot)$ is defined in (7.13), if there exist constants $\varepsilon > 0$ and $q > 1$ such that*

$$\mu := \kappa_3 - \frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1} - \nu > 0 \tag{7.36}$$

*where*

$$\nu := \sup_{x\in\mathscr{X}} \left\{\kappa_4\beta_3(\|x\|) + \kappa_4\sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\left(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\right)\right\} \tag{7.37}$$

*and where the constants $\kappa_1$ and $\kappa_2$ satisfy (7.30), $k_3$ and $\kappa_4$ satisfy (7.31) and (7.32) respectively, and $\beta_3(\cdot)$ and $\beta_4(\cdot)$ are defined by (7.28) and (7.29) respectively.*

*Proof* For the closed-loop system formed by applying the control (7.35) into (7.1)–(7.2), consider the Lyapunov function candidate $U(x)$ defined in Assumption 7.5. The time derivative of $U(x)$ along the trajectories of the closed-loop system is described by

$$\begin{aligned}\dot{U} &= \frac{\partial U}{\partial x}(f(x) + g(x)u_a(y)) + \frac{\partial U}{\partial x}g(x)\big(u_2(y) + \Delta g(x, x_d)\big)\\ &\quad + \frac{\partial U}{\partial x}\Delta f(x, x_d)\\ &\le -\kappa_3\|x\|^2 + \frac{\partial U}{\partial x}g(x)\big(u_2(y) + \Delta g(x, x_d)\big) + \frac{\partial U}{\partial x}\Delta f(x, x_d).\end{aligned} \tag{7.38}$$

Following the analysis on Inequality (7.20) in the proof of Theorem 7.1, it is straightforward to see that

$$\frac{\partial U}{\partial x} g(x)\,(u_2(y) + \Delta g(x, x_d)) \le \frac{\varepsilon}{2}\|x_d\|^2 \tag{7.39}$$

holds for any constant $\varepsilon > 0$.

From (7.27) and (7.32),

$$\frac{\partial U}{\partial x}\Delta f(x, x_d) \le \kappa_4\beta_1(\|x\|)\|x\| + \kappa_4\beta_2(\|x_d\|)\|x\|. \tag{7.40}$$

Substituting (7.39) and (7.40) into (7.38),

$$\dot{U} \le -\kappa_3\|x\|^2 + \frac{\varepsilon}{2}\|x_d\|^2 + \kappa_4\beta_1(\|x\|)\|x\| + \kappa_4\beta_2(\|x_d\|)\|x\|. \tag{7.41}$$

If the inequality $U(x(t+\theta)) \le qU(x(t)$ holds for $q > 1$ and $\theta \in [-\bar{d},\ 0]$, then, it follows from (7.30) that for any $d(t) \in [0,\ \bar{d}]$

$$\kappa_1\|x_d\|^2 \le U(x_d) \le qU(x(t)) \le q\kappa_2\|x\|^2$$

and thus

$$\|x_d\| \le \sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|. \tag{7.42}$$

Since $\beta_2(\cdot)$ is class $\mathscr{K}C^1$, it follows from the nondecreasing property of $\beta_2(\cdot)$ and equation (7.29) that

$$\beta_2(\|x_d\|) \le \beta_2\left(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\right) = \sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\left(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\right)\|x\|.$$

Therefore, when

$$U(x(t+\theta)) \le qU(x(t)$$

for $q > 1$ and $\theta \in [-\bar{d}, 0]$,

$$\begin{aligned}
\dot{U} &\le -\kappa_3\|x\|^2 + \frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1}\|x\|^2 + \kappa_4\beta_3(\|x\|)\|x\|^2 + \kappa_4\sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\left(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\right)\|x\|^2 \\
&= -\left(\kappa_3 - \frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1} - \nu\right)\|x\|^2 \\
&= -\mu\|x\|^2 \\
&\le -\frac{\mu}{\kappa_2}U(x)
\end{aligned} \tag{7.43}$$

where (7.30) is used to obtain the last inequality, and $\mu$ and $\nu$ are defined by (7.36) and (7.37) respectively. From Inequality (7.43),

$$\begin{aligned}\|x(t)\| &\le \sqrt{\frac{1}{\kappa_1}U(x(t))}\\ &\le \sqrt{\frac{U(x(0))}{\kappa_1}\exp\left\{-\frac{\mu}{\kappa_2}t\right\}}\\ &= \sqrt{\frac{U(x(0))}{\kappa_1}}\exp\left\{-\frac{\mu}{2\kappa_2}t\right\}\end{aligned}$$

which implies that System (7.1)–(7.2) is exponentially stable.

Hence the conclusion follows. $\nabla$

*Remark 7.6* It should be noted that a similar assumption to Assumption 7.3 has been removed in the study of exponentially stabilisation because the relevant condition has been guaranteed by condition (7.30). This can be seen from the derived inequality (7.42) which shows that Assumption 7.3 has been satisfied if (7.30) holds.

## 7.3 Sliding Mode Technique-Based Variable Structure Control for Nonlinear Systems

This section presents time-delay independent variable structure output feedback controllers for nonlinear systems using sliding mode techniques.

### *7.3.1 Introduction*

Nearly all real systems are nonlinear in nature and are subject to nonlinear disturbances in which time-delay is often encountered [65, 185, 226]. Nonlinear time-delay systems have been extensively studied [74, 185, 189, 212, 213].

In all of the associated existing results for time-delay systems, it is required that the bounds on the disturbances satisfy a linear growth condition (i.e., linear functions of $\|x\|$ and/or $\|x(t-d)\|$). Recently, the bounds on the disturbances/uncertainties have been extended to the nonlinear case for time-delay systems [210]. However, the designed control explicitly depends on the time-delay which requires the time-delay to be perfectly known. More recently, Pepe and Ito proposed an interesting sliding mode control scheme for nonlinear time-delay systems in [137] but it requires that all the system states are available to the controller. As pointed out in [74], most of the existing sliding mode controllers depend on time-delay, and thus require that the time-delay is known and hence require memory, which is difficult to implement in real systems especially for the case of time-varying delay. Although a memoryless control was proposed for a class of linear systems with nonlinear disturbance in [74], it is required that the nonlinear disturbances are matched and it is again assumed that all the system states are available.

LMI techniques have been widely applied to linear time-delay systems [65, 105], and provide a systematic design approach. However, it is difficult to find a systematic design approach for nonlinear systems because nonlinear systems are very complex and exhibit very rich phenomena. It is desirable to develop a set of conditions such that the controlled nonlinear system exhibits desired performance levels. In this section, a robust stabilisation problem is formulated for a class of nonlinear systems with time-varying delay disturbances. Not only are the disturbances nonlinear functions of the state variables, but the nominal system is nonlinear as well. It is not required that the nominal systems are linearisable or partially linearisable. The disturbances involved are matched and mismatched, and are bounded by nonlinear functions of the state and delayed state variables. By employing an appropriate coordinate transformation, the system is firstly transformed to regular form, which facilitates both analysis and design. Based on the Lyapunov–Razumikhin approach, sufficient conditions are derived to guarantee that the sliding motion is uniformly asymptotically stable, or exponentially stable, irrespective of the disturbances and time-delay. A static output feedback sliding mode control law is then proposed to drive the system to the sliding surface in finite time and maintain a sliding motion on it thereafter. The developed control is independent of the time-delay and thus does not require memory for implementation.

## 7.3.2 System Description and Problem Formulation

Consider a class of nonlinear systems with disturbances described by

$$\dot{x} = f(t,x) + g(t,x)\big(u + \phi(t,x,x_d)\big) + \psi(t,x,x_d) \tag{7.44}$$

$$y = h(x) \tag{7.45}$$

where $x \in \Omega \subset \mathbb{R}^n$ ($\Omega$ is a neighbourhood of the origin), $u \in \mathbb{R}^m$ and $y \in \Omega_y \subset \mathbb{R}^p$ are, respectively, the state variables, inputs and outputs with $m \le p < n$. It is assumed that the matrix function $g(\cdot) \in \mathbb{R}^{n\times m}$ is known and has full column rank; the nonlinear vectors $f(\cdot) \in \mathbb{R}^n$ and $h(\cdot) \in \mathbb{R}^p$ with $h(0) = 0$ are known. The terms $\phi(\cdot)$ and $\psi(\cdot)$ represent the matched and mismatched disturbances respectively. The notation $x_d := x(t-d)$ represents delayed states where $d := d(t)$ denotes the time-varying delay which is continuous, nonnegative and bounded in $\mathbb{R}^+ := \{t \mid t \ge 0\}$, and thus

$$\bar{d} := \sup_{t\in\mathbb{R}^+}\{d(t)\} < \infty.$$

The initial condition relating to the time-delay is given by

$$x(t) = \zeta(t), \qquad t \in [-\bar{d}, 0] \tag{7.46}$$

where $\zeta(\cdot) \in \Theta$ and $\Theta$ is the admissible initial condition set defined by

$$\Theta := \left\{\zeta(t) \mid \zeta(\cdot) \in \mathscr{C}_{[-\bar{d},0]}, \quad \|\zeta(t)\| \le q_1\right\} \tag{7.47}$$

for some constant $q_1 > 0$. It is assumed that all the nonlinear functions are smooth enough, which guarantees that the unforced system has a unique continuous solution in $t \in [0, +\infty)$.

Suppose that the Jacobian matrix of $h(x)$ is full row rank in $\Omega$. Then, there exist $n - p$ smooth functions $\delta_i(x)$ defined in the domain $\Omega$ for $i = 1, \ldots, n - p$ such that the Jacobian matrix of the vector function

$$T(x) := \left[\, \delta_1(x) \cdots \delta_{n-p}(x) \; h^T(x) \,\right]^T$$

is nonsingular in the domain $\Omega$. This implies that $T(x)$ forms a diffeomorphism in $\Omega$. Let

$$z := [\delta_1(x), \ldots, \delta_{n-p}(x)]^T.$$

The diffeomorphism $T(\cdot)$ defines a new coordinate system:

$$T : \; x \mapsto \mathrm{col}(z, y) = T(x). \tag{7.48}$$

Further, it is assumed that the input distribution function matrix $g(t, x)$ satisfies

$$\left[\frac{\partial T(x)}{\partial x} g(t, x)\right] = \begin{bmatrix} 0 \\ G(t, y) \end{bmatrix} \tag{7.49}$$

where $G(t, y) \in \mathbb{R}^{m\times m}$ is nonsingular in $\mathbb{R}^+ \times \Omega_y$ because $g(\cdot)$ has full column rank.

*Remark 7.7* There is no systematic way to find a diffeomorphism $T(\cdot)$ satisfying (7.49). However, for some special cases, it is possible to find the associated diffeomorphism. Consider System (7.44)–(7.45) when

$$f(t, x) = f(x)$$

and

$$g(t, x) = g(x) := [g_1(x), g_2(x), \ldots, g_m(x)]$$

where $g_i(\cdot) \in \mathbb{R}^n$ for $i = 1, 2, \ldots, m$ and the output $y \in \mathbb{R}$ is a scalar. From [121], there exists a diffeomorphism $\bar{T}(\cdot)$ with $\bar{T}(0) = 0$ such that

$$\left[\frac{\partial \bar{T}(x)}{\partial x} g(x)\right] = g_0(y) \tag{7.50}$$

if

(i) $\text{rank}\left\{d(L_f^j h(x)) \mid j = 0, 1, \ldots, n-1\right\} = n$;

(ii) $[ad_f^i \omega, ad_f^j \omega] = 0$ for $i, j = 0, 1, \ldots, n-1$;

(iii) $[g_i, ad_f^j \omega] = 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n-2$, where $\omega$ is the vector field satisfying

$$\left\langle \begin{bmatrix} dh \\ \vdots \\ d(L_f^{n-2}h) \\ d(L_f^{n-1}h) \end{bmatrix}, \quad \omega \right\rangle = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(Here the notation is the same as in [121, 123]). Then, the diffeomorphism $\bar{T}$ can be obtained by

$$\begin{aligned} z_1 &= L_f h(x), \\ z_2 &= L_f^2 h(x) \\ &\vdots \\ z_{n-1} &= L_f^{n-1} h(x) \end{aligned}$$

where $y = h(x)$. Further, from Sect. 5.4 in the Ref. [123], the results can be extended to the multi-output case. If the first $n - m$ rows of $g_0(y)$ in (7.50) are a linear combination of the last $m$ rows of $g_0(y)$, then a linear coordinate transformation $T_L$ can be obtained using basic matrix theory and the transformation $T := T_L \circ \bar{T}$ will satisfy Eq. (7.49).

Then, in the new coordinates $(z, y)$ defined by (7.48), the system (7.44)–(7.45) can be described by

$$\begin{bmatrix} \dot{z} \\ \dot{y}_1 \end{bmatrix} = F_1(t, z, y_1, y_2) + \Psi_1(t, z, y_1, y_2, z_d, y_{1d}, y_{2d}) \tag{7.51}$$

$$\dot{y}_2 = F_2(t, z, y) + G(t, y)\big(u + \Phi(t, z, y, z_d, y_d)\big) + \Psi_2(t, z, y, z_d, y_d) \tag{7.52}$$

$$y = \text{col}(y_1, y_2) \tag{7.53}$$

where $z \in \mathbb{R}^{n-p}$, $y_1 \in \mathbb{R}^{p-m}$ and $y_2 \in \mathbb{R}^m$ form the system states in the new coordinate system, and $u \in \mathbb{R}^m$ and $y := \text{col}(y_1, y_2) \in \mathbb{R}^p$ are the inputs and outputs respectively. In Eqs. (7.51)–(7.52),

$$y_d := y(t-d), \qquad z_d := z(t-d)$$

and

$$\begin{bmatrix} F_1(\cdot) \\ F_2(\cdot) \end{bmatrix} := \left[\frac{\partial T}{\partial x} f(t,x)\right]_{x=T^{-1}(z,y)} \tag{7.54}$$

$$\Psi(\cdot) := \begin{bmatrix} \Psi_1(\cdot) \\ \Psi_2(\cdot) \end{bmatrix} := \left[\frac{\partial T}{\partial x} \psi(t,x,x_d)\right]_{x=T^{-1}(z,y)} \tag{7.55}$$

$$\Phi(\cdot) := [\phi(t,x,x_d)]_{x=T^{-1}(z,y)} . \tag{7.56}$$

In the new coordinates $(z, y_1, y_2)$, the domain $\Omega$ is transferred to

$$\Omega_T := \Omega_z \times \Omega_{y_1} \times \Omega_{y_2} := \{(z, y_1, y_2) = T(x) \mid x \in \Omega\}$$

where $z \in \Omega_z$, $y_1 \in \Omega_{y_1}$ and $y_2 \in \Omega_{y_2}$. The system (7.51)–(7.53) has two properties:

- it is in the usual regular form for sliding mode design;
- the system outputs $y$ together with the variable $z$ form the full state $(z, y)$.

The first feature is very useful for the constructive application of the sliding mode paradigm to practical design, whilst the second feature provides the opportunity to employ the system output to reduce conservatism in static output feedback control design.

In the subsequent analysis, the stabilisation problem for System (7.51)–(7.53) will be considered. The objective is to design a static output feedback sliding mode control law

$$u = u(t, y) \tag{7.57}$$

which depends only on time $t$ and the output $y$, but is independent of the delay $d(t)$, such that the closed-loop system formed by applying the control to the system (7.51)–(7.52) is uniformly asymptotically stable irrespective of the disturbances and time-delay. The local case will be treated in this section. In order to avoid unnecessary notation in describing the local region, the domain may not be specifically stated, but each variable's dimension will be clearly shown.

### *7.3.3 Sliding Motion Analysis and Control Synthesis*

In this section, a sliding surface is proposed first and then a sliding mode control is synthesised.

#### 7.3.3.1 Stability of the Sliding Motion

For System (7.51)–(7.53), choose the switching function as

$$s(z, y) := y_2. \tag{7.58}$$

Then, the output sliding surface is described by

$$\{\mathrm{col}(z, y_1, y_2) \mid y_2 = 0\}\,. \tag{7.59}$$

Since System (7.51)–(7.52) is in regular form, it follows from sliding mode control theory that the corresponding sliding motion is dominated by System (7.51) which, when limited to the sliding surface (7.59), can be described in a compact form by

$$\dot{X} = F_{1s}(t, X) + \Psi_{1s}(t, X, X_d) \tag{7.60}$$

where $X := \mathrm{col}(z, y_1) \in \mathbb{R}^{n-m}$ and $X_d := X(t-d)$ are the states and the delayed states of the sliding mode dynamics. The considered domain is $X \in \Omega_X := \Omega_z \times \Omega_{y_1}$, and

$$F_{1s}(t, X) := F_1(t, z, y_1, 0)$$
$$\Psi_{1s}(t, X, X_d) := \Psi_1(t, z, y_1, 0, z_d, y_{1d}, 0)$$

where, from (7.55), the term $\Psi_{1s}(\cdot)$ is caused by the mismatched disturbance $\psi(\cdot)$. The initial value related to the time-delay for the system (7.60) can be obtained from (7.46) based on the coordinate transformation (7.48).

The objective now is to analyse the stability of System (7.60) in the domain $\Omega_X$. The following assumptions are imposed on System (7.60):

**Assumption 7.6** There exists a $C^1$ function $V(\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n-m} \mapsto \mathbb{R}^+$ and class $\mathscr{K}$ functions $r_i(\cdot)$ for $i = 1, \ldots, 4$ such that for any $X \in \Omega_X$ the following inequalities hold

(i) $r_1^2(\|X\|) \le V(t, X) \le r_2^2(\|X\|)$,

(ii) $\dfrac{\partial V}{\partial t} + \left(\dfrac{\partial V}{\partial X}\right)^T F_{1s}(t, X) \le -r_3^2(\|X\|)$,

(iii) $\left\|\dfrac{\partial V}{\partial X}\right\| \le r_4(\|X\|)$

where $\dfrac{\partial V}{\partial X} := \left[\dfrac{\partial V}{\partial X_1}\ \dfrac{\partial V}{\partial X_2}\ \cdots\ \frac{\partial V}{\partial X_{n-m}}\right]^T$ and $\mathrm{col}(X_1, X_2, \ldots, X_{n-m}) := X$.

*Remark 7.8* Assumption 7.6 implies that the nominal part of System (7.60), i.e., the dynamics

$$\dot{X} = F_{1s}(t, X)$$

is asymptotically stable to guarantee that the sliding mode (7.60) is asymptotically stable. It should be noted that the fact that the system

$$\dot{X} = F_{1s}(t, X)$$

is asymptotically stable does not imply that either the nominal uncontrolled system in (7.44) or the nominal system in (7.51) is stable. From the fact that any class $\mathscr{K}$

function $r(\cdot)$ can be expressed as

$$r(\tau) = \alpha^2(\tau)$$

where $\alpha(\cdot)$ is a class $\mathscr{K}$ function, it is straightforward to see that Assumption 7.6 is the same as the corresponding conditions in the converse Lyapunov Theorem (see Sect. 2.3.3) if the Jacobian matrix of $F_{1s}(\cdot)$ is bounded in a neighbourhood of the origin (see, e.g., Theorem 4.16 in [91]). Here, the functions $r_i^2(\cdot)$ instead of $r_i(\cdot)$ for $i = 1, 2, 3$ are used in Assumption 7.6 to simplify notation.

**Assumption 7.7** The disturbance $\Psi_{1s}(\cdot)$ in (7.60) satisfies

$$\|\Psi_{1s}(t, X, X_d)\| \le \eta\,(t, \|X\|, \|X_d\|) \tag{7.61}$$

where $\eta(\cdot, \tau_1, \tau_2)$ is a known class $\mathscr{W}\mathscr{S}$ function[1] w.r.t. the variables $\tau_1$ and $\tau_2$.

Under Assumption 7.7, the function $\eta(\cdot)$ has a decomposition as

$$\eta\,(\cdot) = \eta_1(t, \|X\|, \|X_d\|)\|X\| + \eta_2(t, \|X\|, \|X_d\|)\|X_d\| \tag{7.62}$$

where the scalar functions $\eta_1(\cdot, \cdot, \tau)$ and $\eta_2(\cdot, \cdot, \tau)$ are nondecreasing with respect to the variable $\tau$ in $\mathbb{R}^+$.

**Assumption 7.8** For any $\theta \in [-\bar{d},\ 0]$, the inequality

$$V(t + \theta, X(t + \theta)) \le \rho(V(t, X(t)))$$

where $\rho(\cdot)$ is a continuous nondecreasing function satisfying $\rho(\tau) > \tau$ for $\tau > 0$, implies that there exists a $\mathscr{K}C^1$ function $b_0(\cdot) > 0$ such that

$$\|X(t + \theta)\| \le b_0(\|X(t)\|).$$

*Remark 7.9* Assumption 7.8, which is related to the time-delay, is a limitation on the function $V(\cdot)$ given in Assumption 7.6. If time-delay is not involved, Assumption 7.8 is unnecessary. A class of functions satisfying Assumption 7.8 is identified in Lemma C.1 and Remark C.1 in the Appendix C.

Since $b_0(\cdot)$ is a class $\mathscr{K}C^1$ function, there exists a continuous function $b(\cdot)$ such that

$$b_0(\|X\|) = b(\|X\|)\|X\|, \qquad X \in \Omega_X. \tag{7.63}$$

[1]The definition of the class of $\mathscr{W}\mathscr{S}$ function can be found in Sect. 2.1.

The function $b(\cdot)$ defined by (see [202])

$$b(r) = \begin{cases} \dfrac{b_0(r)}{r}, & r \in (0, +\infty) \\ \dfrac{d(b_0(r))}{dr}, & r = 0 \end{cases}$$

is continuous and satisfies Eq. (7.63). Note, if the functions $r_i(\cdot)$ for $i = 1, 2, 3, 4$ in Assumption 7.6 are strengthened to class $\mathscr{K}C^1$ functions, then there exist continuous functions $\vartheta_i(\cdot)$ such that for any $X \in \Omega_X$

$$r_i(\|X\|) = \vartheta_i(\|X\|)\|X\|, \qquad i = 1, 2, 3, 4. \tag{7.64}$$

The following result is ready to be presented.

**Theorem 7.3** *Suppose that $r_i(\cdot)$ for $i = 1, 2, 3, 4$ in Assumption 7.6 are class $\mathscr{K}C^1$ functions. Then, under Assumptions 7.6–7.8, the system (7.51)–(7.52) has a uniformly asymptotic stable sliding motion associated with the sliding surface (7.59) if there exists a continuous nondecreasing function $w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying*

$$w(\tau) > 0 \qquad for \qquad \tau > 0$$

*such that for $X \in \Omega_X$ and $t \in \mathbb{R}^+$,*

$$\begin{aligned} \vartheta_3^2(\|X\|) - \ & \vartheta_4(\|X\|)\eta_1\Big(t, \|X\|, b_0(\|X\|)\Big) \\ & -\vartheta_4(\|X\|)b(\|X\|)\eta_2\Big(t, \|X\|, b_0(\|X\|)\Big) \geq w(\|X\|) \end{aligned} \tag{7.65}$$

*where $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are given in (7.62), and $b_0(\cdot)$ and $b(\cdot)$ satisfy (7.63).*

*Proof* From the analysis above, System (7.60) is the sliding mode dynamics relating to the sliding surface (7.59). It remains to be proved that System (7.60) is uniformly asymptotically stable.

Under the conditions that $r_i(\cdot)$ are class $\mathscr{K}C^1$ functions, the inequalities in (7.64) hold. Consider the Lyapunov candidate function $V(\cdot)$ defined in Assumption 7.6 for System (7.60). The time derivative of $V(\cdot)$ along the trajectory of System (7.60) is given by

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial X}\right)^T (F_{1s}(t, X) + \Psi_{1s}(t, X, X_d)) \\ &\leq \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial X}\right)^T F_{1s}(t, X) + \left\| \left(\frac{\partial V}{\partial X}\right)^T \right\| \|\Psi_{1s}(t, X, X_d)\| \\ &\leq -r_3^2(\|X\|) + r_4(\|X\|)\eta(t, \|X\|, \|X_d\|) \\ &= -\vartheta_3^2(\|X\|)\|X\|^2 + \vartheta_4(\|X\|)\eta_1(\cdot)\|X\|^2 \\ &\quad +\vartheta_4(\|X\|)\eta_2(\cdot)\|X\|\|X_d\| \end{aligned} \tag{7.66}$$

where (7.62) and (7.64) are used.

If there exists a function $\rho(\cdot)$ defined in $\mathbb{R}^+$ which satisfies $\rho(\tau) > \tau$ for $\tau > 0$, such that for any $\theta \in [-\bar{d},\ 0]$,

$$V(t+\theta, X(t+\theta)) \leq \rho(V(t, X(t))) \tag{7.67}$$

then, from Assumption 7.8, there exists a class $\mathscr{K}C^1$ function such that for any $\theta \in [-\bar{d}\ 0]$

$$\|X(t+\theta)\| \leq b_0(\|X(t)\|) = b(\|X\|)\|X\| \tag{7.68}$$

where (7.63) is used above. Since both $\eta_1(\cdot, \cdot, \tau)$ and $\eta_2(\cdot, \cdot, \tau)$ are nondecreasing w.r.t. the variable $\tau \in \mathbb{R}^+$, it is clear that if (7.67) holds, then from (7.68) and (7.66),

$$\begin{aligned}
\dot{V} &\leq -\vartheta_3^2(\|X\|)\|X\|^2 + \vartheta_4(\|X\|)\eta_1\left(t, \|X\|, b_0(\|X\|)\right)\|X\|^2 \\
&\quad +\vartheta_4(\|X\|)b(\|X\|)\eta_2(t, \|X\|, b_0\big(\|X\|\big))\|X\|^2 \\
&= -\Big(\vartheta_3^2(\|X\|) - \vartheta_4(\|X\|)\eta_1(t, \|X\|, b_0(\|X\|)) \\
&\quad -\vartheta_4(\|X\|)b(\|X\|)\eta_2(t, \|X\|, b_0(\|X\|))\Big)\|X\|^2 \\
&\leq -w(\|X\|)\|X\|^2
\end{aligned}$$

where (7.65) is employed above. Hence the conclusion follows from the well known Razumikhin Theorem 2.5. $\nabla$

Consider the case when the system

$$\dot{X} = F_{1s}(t, X)$$

is exponentially stable. In this case, it follows from [91] that there exist constants $\gamma_i > 0$ such that Assumption 7.6 is satisfied with

$$r_i(\tau) = \gamma_i \tau, \qquad i = 1, 2, 3, 4. \tag{7.69}$$

The following result can be obtained.

**Corollary 7.1** *Suppose that Assumption 7.6 holds with $r_i(\cdot)$ satisfying (7.69). Then under Assumption 7.7, the sliding motion of System (7.51)–(7.52) associated with the sliding surface (7.59) is exponential stable if for $X \in \Omega_X$ and $t \in \mathbb{R}^+$, the inequality*

$$\begin{aligned}
\mu := \sup_{X \in \Omega_X} &\left\{\eta_1\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big) + \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\eta_2\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big)\right\} \\
&< \frac{\gamma_3^2}{\gamma_4}
\end{aligned} \tag{7.70}$$

*holds for some* $\nu > 1$ *where* $\gamma_3$ *and* $\gamma_4$ *satisfy (7.69), and* $\eta_1(\cdot)$ *and* $\eta_2(\cdot)$ *are given in (7.62).*

*Proof* From condition (i) of Assumption 7.6 and (7.69), for any constant $\nu > 1$ and any $\theta \in [-\bar{d},\ 0]$, if

$$V(t+\theta, X(t+\theta)) \le \nu V(t, X(t))$$

then

$$\begin{aligned}\gamma_1^2 \|X(t+\theta)\|^2 &\le V(t+\theta, X(t+\theta))\\ &\le \nu V(t, X(t)) \le \nu \gamma_2^2 \|X(t)\|^2.\end{aligned}$$

This implies

$$\|X(t+\theta)\| \le \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X(t)\|.$$

Choose

$$b_0(\tau) = \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\tau \qquad \text{and} \qquad \rho(\tau) = \nu\tau.$$

It is straight forward to see that Assumption 7.8 holds. Then following the proof of Theorem 7.3, it follows that if

$$V(t+\theta, X(t+\theta)) \le \nu V(t, X(t))$$

holds, then

$$\begin{aligned}\dot{V} &\le -\Bigg(\gamma_3^2 - \gamma_4\,\eta_1\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big)\\ &\quad -\gamma_4\,\frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\eta_2\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big)\Bigg)\|X\|^2\\ &\le -\gamma_4\Bigg(\frac{\gamma_3^2}{\gamma_4} - \sup_{X\in\Omega_X}\Bigg\{\eta_1\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big)\\ &\quad +\frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\eta_2\Big(t, \|X\|, \frac{\gamma_2}{\gamma_1}\sqrt{\nu}\,\|X\|\Big)\Bigg\}\Bigg)\|X\|^2\\ &= -\gamma_4\Bigg(\frac{\gamma_3^2}{\gamma_4} - \mu\Bigg)\|X\|^2.\end{aligned}$$

From (7.70),

$$\gamma_4\Bigg(\frac{\gamma_3^2}{\gamma_4} - \mu\Bigg) > 0.$$

Hence the conclusion follows. $\nabla$

*Remark 7.10* Corollary 7.1 shows that Assumption 7.8 is unnecessary if

$$\dot{X} = F_{1s}(t, X)$$

is exponentially stable. From (7.61) and (7.62), the limitation on the mismatched uncertainty $\Psi_1(\cdot)$ is reflected through (7.65) or (7.70). Since $X := (z, y_1) \in \mathbb{R}^{n-m}$ represent the partial state variables of $x \in \mathbb{R}^n$ and

$$\Psi_{1s}(t, X, X_d) := \Psi_1(t, z, y_1, 0, z_d, y_{1d}, 0)$$

where from (7.55), $\Psi_1(\cdot)$ is only a subcomponent of the uncertainty $\Psi(\cdot)$, it is straightforward to see that the conservatism of (7.65) and (7.70) is reduced when compared with the results in [210, 214], where the limitation is on $\Psi(\cdot)$ in $x \in \mathbb{R}^n$. Similar to the work in [89, 224], the assumptions and conditions are imposed on the transformed system, which usually reduces conservatism due to the favourable structure of the transformed system, although the relationship between these assumptions and the original system may not be transparent.

Since the input distribution matrix $G(t, y)$ of System (7.51)–(7.52) is nonsingular, it is straightforward to see that both terms $\Phi(\cdot)$ and $\Psi_2(\cdot)$ are matched disturbances. Theorem 7.3 and Corollary 7.1 show that only $\Psi_1(\cdot)$, which is the mismatched disturbance, affects the sliding motion. This is consistent with the well known fact that the sliding mode is insensitive to matched uncertainty.

#### 7.3.3.2 Reachability Analysis

Consider System (7.51)–(7.53). The following assumptions are required.

**Assumption 7.9** The disturbances $\Phi(\cdot)$ and $\Psi_2(\cdot)$ in (7.52) satisfy

$$\|\Phi(t, z, y, z_d, y_d)\| \le \varpi_1(t, z, y, z_d, y_d), \tag{7.71}$$

$$\|\Psi_2(t, z, y, z_d, y_d)\| \le \varpi_2(t, z, y, z_d, y_d) \tag{7.72}$$

for some known functions $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ which satisfy the generalised Lipschitz condition w.r.t. the variables $z$, $z_d$ and $y_d$ uniformly for $t \in \mathbb{R}^+$ and $y \in \Omega_y$.

**Assumption 7.10** The nonlinear function $F_2(t, z, y)$ in (7.52) satisfies the generalised Lipschitz condition w.r.t. the variables $z$ uniformly for $t \in \mathbb{R}^+$ and $y \in \Omega_y$.

From Assumptions 7.9 and 7.10, the following inequalities hold

$$|\varpi_1(t, z, y, z_d, y_d) - \varpi_1(t, 0, y, 0, 0)| \le \mathscr{L}_{\varpi_1 1}(t, y)\|z\| + \mathscr{L}_{\varpi_1 2}(t, y)\|z_d\| + \mathscr{L}_{\varpi_1 3}(t, y)\|y_d\| \tag{7.73}$$

$$|\varpi_2(t, z, y, z_d, y_d) - \varpi_2(t, 0, y, 0, 0)| \le \mathscr{L}_{\varpi_2 1}(t, y)\|z\| + \mathscr{L}_{\varpi_2 2}(t, y)\|z_d\| + \mathscr{L}_{\varpi_2 3}(t, y)\|y_d\| \tag{7.74}$$

$$\|F_2(t, z, y) - F_2(t, 0, y)\| \le \mathscr{L}_{F_2 1}(t, y)\|z\| \tag{7.75}$$

where $\mathcal{L}_{\star i}$ for $i = 1, 2, 3$ are the generalised Lipschitz bounds of the function $\star$ with respect to $z$, $z_d$ and $y_d$ uniformly for $(t, y) \in \mathbb{R}^+ \times \Omega_y$. Consider System (7.51)–(7.53) in

$$\left\{(z, y) \mid \|z\| \leq q_2,\ y \in \Omega_y\right\} \subset \Omega_T$$

where $q_2$ is a positive constant and $\Omega_y := \Omega_{y_1} \times \Omega_{y_2}$. Let

$$q := \max\{q_1, q_2\} \tag{7.76}$$

where $q_1$ is defined in (7.47). Construct the control law

$$\begin{aligned} u(t, y) = &-G^{-1}(t, y)F_2(t, 0, y) - G^{-1}(t, y)\big(\|G(t, y)\|\varpi_1(t, 0, y, 0, 0) \\ &+\varpi_2(t, 0, y, 0, 0)\big)\text{sgn}(y_2) \ - G^{-1}(t, y)k(t, y)\text{sgn}(y_2) \end{aligned} \tag{7.77}$$

where $F_2(\cdot)$ is given in (7.52) and $\varpi_1(\cdot)$ and $\varpi_2(\cdot)$ satisfy (7.71). sgn$(\cdot)$ is the usual signum function, and $k(\cdot)$ is the control gain to be determined later. The $G(t, y)$ is given in (7.49), which is nonsingular in $\mathbb{R}^+ \times \mathcal{Y}$, and thus the control $u(\cdot)$ in (7.77) is well defined. The control $u(\cdot)$ in (7.77) is only dependent on the time $t$ and output $y$ but independent of the time-delay $d(t)$. It is called a memoryless static output feedback control.

**Theorem 7.4** *Consider the nonlinear system (7.51)–(7.52). Under Assumptions 7.9 and 7.10, System (7.51)–(7.52) can be driven to the sliding surface (7.59) in finite time and maintains a sliding motion on it thereafter by means of the control (7.77) if the control gain $k(\cdot)$ is chosen as*

$$\begin{aligned} k(t, y) := q\big(&\mathcal{L}_{F_2 1}(t, y) + \|G(t, y)\|\big(\mathcal{L}_{\varpi_1 1}(t, y) + \mathcal{L}_{\varpi_1 2}(t, y) + \mathcal{L}_{\varpi_1 3}(t, y)\big) \\ &+\mathcal{L}_{\varpi_2 1}(t, y) + \mathcal{L}_{\varpi_2 2}(t, y) + \mathcal{L}_{\varpi_2 3}(t, y)\big) + \rho \end{aligned} \tag{7.78}$$

*for any constant $\rho > 0$, where $q > 0$ is defined in (7.76), and $\varpi_1(\cdot)$, $\varpi_2(\cdot)$ and $F_2(\cdot)$ satisfy (7.73), (7.74) and (7.75) respectively.*

*Proof* From Eq. (7.52),

$$y_2^T \dot{y}_2 = y_2^T \big(F_2(t, z, y) + G(t, y)\big(u + \Phi(t, z, y, z_d, y_d)\big) + \Psi_2(t, z, y, z_d, y_d)\big).$$

Substituting the control $u(\cdot)$ in (7.77) into this equation yields

$$\begin{aligned} y_2^T \dot{y}_2 = y_2^T &\Big(F_2(t, z, y) - F_2(t, 0, y)\Big) + y_2^T G(t, y)\Phi(\cdot) \\ &-\|G(t, y)\|\varpi_1(t, 0, y, 0, 0)y_2^T \text{sgn}(y_2) + y_2^T \Psi_2(\cdot) \\ &-\varpi_2(t, 0, y, 0, 0)y_2^T \text{sgn}(y_2) - k(t, y)y_2^T \text{sgn}(y_2). \end{aligned} \tag{7.79}$$

Under Assumption 7.10,

$$y_2^T\Big(F_2(t,z,y)-F_2(t,0,y)\Big)\le \|y_2\|\,\|F_2(t,z,y)-F_2(t,0,y)\|$$
$$\le q\mathscr{L}_{F_2 1}(t,y)\|y_2\| \tag{7.80}$$

and from the fact that $s^T\text{sgn}(s)\ge\|s\|$ for any vector $s$ (see, Lemma 1 in [199]), it follows that under Assumption 7.9,

$$\begin{aligned}
&y_2^T G(t,y)\Phi(\cdot)-\|G(t,y)\|\,\varpi_1(t,0,y,0,0)y_2^T\text{sgn}(y_2)\\
&\le \|y_2\|\,\|G(t,y)\|\,\varpi_1(t,z,y,z_d,y_d)-\|y_2\|\|G(t,y)\|\,\varpi_1(t,0,y,0,0)\\
&=\|y_2\|\,\|G(t,y)\|\big(\varpi_1(t,z,y,z_d,y_d)-\varpi_1(t,0,y,0,0)\big)\\
&\le \|y_2\|\,\|G(t,y)\|\big(\mathscr{L}_{\varpi_1 1}(t,y)\|z\|+\mathscr{L}_{\varpi_1 2}(t,y)\|z_d\|+\mathscr{L}_{\varpi_1 3}(t,y)\|y_d\|\big)\\
&\le q\|y_2\|\,\|G(\cdot)\|\big(\mathscr{L}_{\varpi_1 1}(t,y)+\mathscr{L}_{\varpi_1 2}(t,y)+\mathscr{L}_{\varpi_1 3}(t,y)\big)
\end{aligned} \tag{7.81}$$

where (7.73) is employed above. By similar reasoning as in (7.81),

$$\begin{aligned}
y_2^T\Psi_2(t,z,y,z_d,y_d)-\varpi_2(t,0,y,0,0)y_2^T\text{sgn}(y_2)&\\
\le \|y_2\|\Big(\varpi_2(t,z,y,z_d,y_d)-\varpi_2(t,0,y,0,0)\Big)&\\
\le q\|y_2\|\Big(\mathscr{L}_{\varpi_2 1}(\cdot)+\mathscr{L}_{\varpi_2 2}(\cdot)+\mathscr{L}_{\varpi_2 3}(\cdot)\Big).&
\end{aligned} \tag{7.82}$$

Substituting (7.80)–(7.82) into (7.79) yields

$$\begin{aligned}
y_2^T\dot y_2 &\le q\Big(\mathscr{L}_{F_2 1}(t,y)+\|G(t,y)\|\big(\mathscr{L}_{\varpi_1 1}(t,y)+\mathscr{L}_{\varpi_1 2}(t,y)+\mathscr{L}_{\varpi_1 3}(t,y)\big)\\
&\quad+\mathscr{L}_{\varpi_2 1}(t,y)+\mathscr{L}_{\varpi_2 2}(t,y)+\mathscr{L}_{\varpi_2 3}(t,y)\Big)\|y_2\|-k(t,y)\|y_2\|\\
&=-\rho\|y_2\|
\end{aligned}$$

where (7.78) is employed above.

Hence the conclusion follows from $s(\cdot)=y_2$ in (7.58). ∇

Following the transformation (7.48), the disturbance $\psi(\cdot)$ in (7.44) is split into $\Psi_1(\cdot)$ in (7.51) and $\Psi_2(\cdot)$ in (7.52). The former only affects the sliding mode dynamics while the latter only impacts on the reachability. From sliding mode control theory, Theorem 7.4 together with Theorem 7.3 (or Corollary 7.1) shows that the associated closed-loop system is uniformly asymptotically (or exponentially) stable.

*Remark 7.11* Assumptions 7.7 and 7.9 are imposed on $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ respectively based on their different effects. This has improved the robustness when compared with the existing work (see, e.g., [210, 214]) where both Assumptions 7.7 and 7.9 are required for $\psi(\cdot)$. Assumption 7.7 and Eq. (7.62) show that $\Psi_{1s}(\cdot)$ must vanish at $X=0$ and $X_d=0$ to guarantee that the sliding motion is asymptotically stable. However, here it is not required that the matched disturbance vanishes at the origin.

It should be noted that in the control design, the parameter $q$ should be selected properly. This depends on $q_1$ and $q_2$ through Eq. (7.76). From (7.47), $q_1$ can be determined by the admissible initial conditions relating to the delay, and $q_2$ can be estimated by Inequality (7.65). In addition, the results developed in this section include systems with linear nominal dynamics as a special case. The design paradigm can be applied to the systems discussed in [74, 75, 209] by appropriate modification.

## 7.4 Decentralised Output Feedback Control for Nonlinear Interconnected Systems

This section considers the stabilisation problem for a class of nonlinear large-scale interconnected systems with uncertainties based on a memoryless static output feedback control scheme.

### *7.4.1 Introduction*

Many practical systems are often modelled as dynamical equations composed of interconnections between a collection of lower dimensional subsystems. A decentralised control approach is adapted to reduce information requirements. These have motivated the study of this section.

Time-delay is an important factor in the effective of control of a large scale interconnected system. The interconnections between two or more physical systems are often accompanied by phenomena such as material transfer, energy transfer and information transfer, which, from a mathematical point of view, can be represented by delay elements [126]. This has motivated the study of large scale time-delay interconnected systems, and many results have been achieved [2, 70, 217].

However most of the existing results consider situations where all the system states are available. The associated decentralized output feedback results for time delay large-scale interconnected systems are very few [73, 227]. An output feedback decentralised control scheme is given in [120] for the case of discrete interconnected systems. A class of nonlinear interconnected systems with triangular structure is considered in [227], and a large-scale system composed of a set of single input single output subsystems with dead zone input is considered in [227]. In both [73, 227], the control schemes are based on dynamical output feedback which increases the computation greatly due to the associated closed-loop system possessing possibly twice the order of the actual plant.

In many of the existing control schemes, controllers are explicitly dependent on time-delay [6, 208] and/or limitations on the rate of change of the time-delay must be imposed [51]. In addition, as pointed out in [227], most of the existing variable structure controllers for nonlinear systems use knowledge of the delay explicitly and hence require memory, which is difficult to implement in practice especially for the case of time-varying delay. Although a memoryless control for a class of linear systems was proposed based on a back-stepping approach in [227], the nonlinear uncertainty is required to be matched and it is assumed that all the system states are available. A memoryless sliding mode control scheme is given in [195], but all the nonlinear terms are assumed to be matched and are without time-delay. This renders the associated sliding mode dynamics to be delay free and thus there is no delay involved in the stability analysis of the sliding mode dynamics. More recently, a class of nonlinear time-delay interconnected systems is considered by [210]. However, it is required that the time-delay is precisely known and each subsystem is square. Further, the results given in [210] do not render themselves suitable for extension to the non-square case.

In this section, a variable structure control is synthesised to stabilise a class of large-scale time-delay systems with nonlinear interconnections. The bounds on the uncertainties are nonlinear and involve time-delay states. A decentralised variable structure control scheme using only output information is proposed which is independent of time-delay. Based on the Lyapunov–Razumikhin approach, sufficient conditions are derived such that the closed-loop system formed by the designed control and the large-scale interconnected systems is uniformly asymptotically stable. Limitation on the rate of change of the time-delay is unnecessary. A compensator, which increases the required computation levels for large-scale interconnected systems, is not required either. Further study shows that the effects of the known interconnections can be largely rejected if they are separated into matched and mismatched parts and dealt with separately. Unlike the work in [210], it is not required that the subsystem is square, and it is not required that the time-delay is known. Thus the controller does not require memory.

## 7.4.2 System Description and Basic Assumptions

Consider a time-varying delay interconnected system composed of $n$ $n_i$-th order subsystems described by

$$\dot{x}_i = A_i x_i + B_i \big(u_i + \xi_i\big(t, x_i, x_{id_i}\big)\big) + F_i(x) + \psi_i(t, x, x_d) \tag{7.83}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, n, \tag{7.84}$$

where $x := \text{col}(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the state variables, inputs and outputs of the $i$-th subsystem respectively. The triples $(A_i, B_i, C_i)$ represent constant matrices of appropriate dimensions with $B_i$ and $C_i$ of full rank. The functions $\xi_i(\cdot)$ are matched uncertainties in the $i$-th subsystem. The function vectors $F_i(x) \in \mathbb{R}^{n_i}$ are known and analytic with $F_i(0) = 0$. The terms $\psi_i(t, x, x_d)$ are uncertain interconnections of the $i$-th subsystem. The symbols

$$x_{id_i} := x_i(t - d_i) \qquad \text{and} \qquad x_d := \text{col}(x_{1d_1}, x_{2d_2}, \ldots, x_{nd_n})$$

are the delayed states, and the symbols $d_i := d_i(t)$ denote the time-varying delays which are assumed to be known, nonnegative and bounded in $\mathbb{R}^+$, that is

$$\overline{d}_i := \sup_{t\in\mathbb{R}^+}\{d_i(t)\} < \infty, \qquad i = 1, 2, \ldots, n.$$

The initial conditions associated with the time-delays are given by

$$x_i(t) = \phi_i(t), \qquad t \in [-\overline{d}_i, 0]$$

where $\phi_i(\cdot)$ are continuous in $[-\overline{d}_i, 0]$ for $i = 1, 2, \ldots, n$. It is assumed that all the nonlinear functions are smooth enough such that the unforced interconnected system has a unique continuous solution.

In this section, the local case will be considered. In order to simplify the description, the considered domain will not be stated unless it is necessary. However, each variable's dimension will be clearly identified. Note, if all the relevant conditions hold globally, the developed results can be extended to the global case.

Since the function vectors $F_i(x)$ are known and analytic with $F_i(0) = 0$, there exist analytic function matrices $\Phi_i(\cdot) \in \mathbb{R}^{n_i \times \sum_{i=1}^n n_i}$ such that (see [3])

$$F_i(x) = \Phi_i(x)x, \qquad i = 1, 2, \ldots, n.$$

Partition the matrices $\Phi_i(\cdot)$ as

$$\Phi_i(x) := \left[\, \Phi_{i1}(x)\ \Phi_{i2}(x)\ \cdots\ \Phi_{in}(x) \,\right]$$

where $\Phi_{ij}(\cdot) \in \mathbb{R}^{n_i \times n_j}$ for $i, j = 1, 2, \ldots, n$. It follows from $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$ that the interconnection terms $F_i(x)$ can be expressed by

$$F_i(x) = \sum_{j=1}^{n} \Phi_{ij}(x)x_j. \tag{7.85}$$

It should be noted that the matrices $\Phi_{ij}(\cdot)$ which satisfy Eq. (7.85) are not unique. One way to find the matrices $\Phi_i(\cdot)$ and thus $\Phi_{ij}(\cdot)$ is presented in [197].

System (7.83)–(7.84) can be considered as being generated by interconnecting its isolated subsystems. The following basic assumptions are required.

**Assumption 7.11** There exist known continuous functions $\rho_i(\cdot)$, $\varpi_i(\cdot)$, $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ such that for $i, j = 1, 2, \ldots, n$

$$\|\xi_i(t, x_i, x_{id_i})\| \le \rho_i(t, y_i) + \varpi_i(t, y_i)\|x_{id_i}\| \tag{7.86}$$

$$\|\psi_i(t, x, x_d)\| \le \sum_{j=1}^{n} \alpha_{ij}(t, x)\|x_j\| + \sum_{j=1}^{n} \beta_{ij}(t, x)\|x_{jd_j}\|. \tag{7.87}$$

*Remark 7.12* Assumption 7.11 describes the limitations on the uncertainties that can be tolerated by the system. It is not required that the interconnections are described or bounded by functions of the system outputs as in [142, 214]. Furthermore, unlike [118, 142, 214], the bounds on the uncertain interconnections are nonlinear and involve the time-delay state variables.

**Assumption 7.12** The triples $(A_i, B_i, C_i)$ are output feedback stabilisable for $i = 1, 2, \ldots, n$.

Assumption 7.12 is fundamental and implies that there exist matrices $K_i$ such that for any $Q_i > 0$, the equations

$$-Q_i = (A_i - B_i K_i C_i)^T P_i + P_i(A_i - B_i K_i C_i) < 0, \qquad i = 1, 2, \ldots, n \quad (7.88)$$

have unique solutions $P_i > 0$.

**Assumption 7.13** There exist matrices $E_i$ such that

$$B_i^T P_i = E_i C_i \quad (7.89)$$

where the matrices $P_i$ satisfy (7.88) for $i = 1, 2, \ldots, n$.

*Remark 7.13* Assumption 7.12 together with Assumption 7.13 describes a structural characteristic associated with the nominal isolated subsystems $(A_i, B_i, C_i)$ which is the standard Constrained Lyapunov Problem (CLP) (see, e.g., [57]). A similar limitation has been imposed by many authors [24, 26, 57]. Necessary and sufficient conditions for solving the CLP can be found in [41, 57].

The objective now is, under the assumption that all the isolated subsystems are output feedback stabilisable, to design a variable structure control law of the form

$$u_i = u_i(t, y_i), \qquad i = 1, 2, \ldots, n \quad (7.90)$$

such that the closed-loop system formed by applying the control law in (7.90) to the large-scale interconnected system (7.83)–(7.84), is uniformly asymptotically stable even in the presence of the uncertainties and time-delays. Since the control $u_i$ in (7.90) are only dependent on the time $t$ and the $i$-th subsystem's output $y_i$, and are independent of time-delay, they constitute a delay independent decentralised static output feedback control.

### *7.4.3 Decentralised Output Feedback Control Design*

A decentralised output feedback controller which is independent of the time-delay will be proposed for the large-scale interconnected system (7.83)–(7.84).

Consider the control law

$$u_i = -K_i y_i - \frac{1}{2\varepsilon_i^a} E_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i), \qquad i = 1, 2, \ldots, n \tag{7.91}$$

where $K_i \in \mathbb{R}^{m_i \times p_i}$ are design parameters satisfying (7.88), $\varpi_i(\cdot)$ are given in (7.86), $\varepsilon_i^a > 0$ are constant and the terms $u_i^a(\cdot)$ are defined by

$$u_i^a(\cdot) := \begin{cases} -\frac{E_i y_i}{\|E_i y_i\|}\rho_i(t, y_i), & E_i y_i \neq 0 \\ 0, & E_i y_i = 0 \end{cases} \tag{7.92}$$

where $E_i$ satisfy (7.89). Clearly each element $u_i$ is decentralised because it is only dependent on the time $t$ and the local output $y_i$. Thus the $u_i$ in (7.91) are called decentralised output feedback variable structure controllers in Sect. 7.4.

The following result is now ready to be presented:

**Theorem 7.5** *Under Assumptions 7.11–7.13, the closed-loop system formed by applying the control (7.91)–(7.92) to System (7.83)–(7.84) is uniformly asymptotically stable if*

$$\gamma := \inf_x \left\{\lambda_{\min}\left(W^T(\cdot) + W(\cdot)\right)\right\} > 0$$

*where $W(\cdot) = [w_{ij}(\cdot)]_{2n\times 2n}$ is a function matrix defined by*

$$w_{ij}(\cdot) = \begin{cases} \lambda_{\min}(Q_i) - q\lambda_{\max}(P_i) - 2\|P_i\Phi_{ii}(x)\| \\ \qquad - 2\alpha_{ii}(t, x_i)\|P_i\| & 1 \le i = j \le n \\ \lambda_{\min}(P_i) - \varepsilon_i^a, & n+1 \le i = j \le 2n \\ -2\|P_i\Phi_{ij}(x)\| - 2\alpha_{ij}(t, x)\|P_i\|, & i \neq j \ \ \textit{and} \ \ 1 \le i, j \le n \\ -2\beta_{i(j-n)}(t, x_{j-n})\|P_i\|, & 1 \le i \le n, \ \textit{and} \ j > n \\ -2\beta_{(i-n)j}(t, x_j)\|P_{i-n}\|, & i > n, \ \textit{and} \ 1 \le j \le n \\ 0, & \textit{otherwise} \end{cases}$$

*for constants $q > 1$ and $\varepsilon_i^a > 0$, where $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ satisfy (7.87) for $i, j = 1, 2, \ldots, n$.*

*Proof* Applying the control (7.91)–(7.92) to System (7.83)–(7.84) and considering Eq. (7.85), the corresponding closed-loop system can be described by

$$\dot{x}_i = A_i x_i + B_i\Big(-K_i C_i x_i - \frac{1}{2\varepsilon_i^a} E_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i) + \xi_i\big(t, x_i, x_{id_i})\Big)$$
$$+ \sum_{j=1}^n \Phi_{ij}(x) x_j + \psi_i(t, x, x_d) \tag{7.93}$$

where $u_i^a(\cdot)$ are given by (7.92) and $\Phi_{ij}(\cdot)$ satisfy Eq. (7.85) for $i, j = 1, 2, \ldots, n$. For System (7.93), consider the Lyapunov function candidate

$$V(x(t)) = V(x_1(t), x_2(t), \ldots, x_n(t)) = \sum_{i=1}^{n} x_i^T(t) P_i x_i(t) \tag{7.94}$$

where $P_i > 0$ satisfy Eq. (7.88) for $i = 1, 2, \ldots, n$. Then, the time derivative of $V(\cdot)$ along the trajectories of System (7.93) is given by

$$\begin{aligned}
\dot{V} = &-\sum_{i=1}^{n} x_i^T Q_i x_i + 2\sum_{i=1}^{n} x_i^T P_i B_i \Big( -\frac{1}{2\varepsilon_i^a} E_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i) \Big) \\
&+2\sum_{i=1}^{n} x_i^T P_i B_i \xi_i(t, x_i, x_{id_i}) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i \Phi_{ij}(x) x_j \\
&+2\sum_{i=1}^{n} x_i^T P_i \psi_i(t, x, x_d).
\end{aligned} \tag{7.95}$$

From (7.86), (7.89) and Young's inequality, it follows that for any $\varepsilon_i^a > 0$

$$\begin{aligned}
x_i^T P_i B_i \xi_i(t, x_i, x_{id_i}) &= (E_i y_i)^T \xi_i(t, x_i, x_{id_i}) \\
&\le \|E_i y_i\| \rho_i(t, y_i) + \|E_i y_i\| \varpi_i(t, y_i) \|x_{id_i}\| \\
&\le \|E_i y_i\| \rho_i(t, y_i) + \frac{1}{2\varepsilon_i^a} \|E_i y_i\|^2 \varpi_i^2(t, y_i) + \frac{\varepsilon_i^a}{2} \|x_{id_i}\|^2.
\end{aligned} \tag{7.96}$$

From (7.89) and the definition of $u_i^a(\cdot)$ in (7.92), it follows that

(i) if $E_i y_i = 0$, then $u_i^a(\cdot) = 0$, and thus

$$x_i^T P_i B_i u_i^a(t, y_i) + \|E_i y_i\| \rho_i(t, y_i) = 0$$

(ii) if $E_i y_i \neq 0$, from the definition of $u_i^a(\cdot)$ in (7.92),

$$\begin{aligned}
&x_i^T P_i B_i u_i^a(t, y_i) + \|E_i y_i\| \rho_i(t, y_i) \\
&\quad \le -(E_i y_i)^T \frac{E_i y_i}{\|E_i y_i\|} \rho_i(t, y_i) + \|E_i y_i\| \rho_i(t, y_i) \\
&\quad = 0.
\end{aligned}$$

Therefore, from (i) and (ii) above,

$$x_i^T P_i B_i u_i^a(t, y_i) + \|E_i y_i\| \rho_i(t, y_i) \le 0, \qquad i = 1, 2, \ldots, n. \tag{7.97}$$

Further, from (7.89),

$$
\begin{aligned}
&-\frac{1}{2\varepsilon_i^a} x_i^T P_i B_i E_i y_i \varpi_i^2(t, y_i) + \frac{1}{2\varepsilon_i^a}\|E_i y_i\|^2 \varpi_i^2(t, y_i) \\
&= -\frac{1}{2\varepsilon_i^a} x_i^T C_i^T E_i^T E_i y_i \varpi_i^2(t, y_i) + \frac{1}{2\varepsilon_i^a}\|E_i y_i\|^2 \varpi_i^2(t, y_i) \\
&= -\frac{1}{2\varepsilon_i^a} (E_i y_i)^T E_i y_i \varpi_i^2(t, y_i) + \frac{1}{2\varepsilon_i^a}\|E_i y_i\|^2 \varpi_i^2(t, y_i) = 0. \qquad (7.98)
\end{aligned}
$$

Therefore, from (7.96), (7.97) and (7.98)

$$
\begin{aligned}
&\sum_{i=1}^{n} x_i^T P_i B_i \Big( -\frac{1}{2\varepsilon_i^a} E_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i) \Big) + \sum_{i=1}^{n} x_i^T P_i B_i \xi_i(t, x_i, x_{id_i}) \\
&\le -\sum_{i=1}^{n} \frac{1}{2\varepsilon_i^a} x_i^T P_i B_i E_i y_i \varpi_i^2(t, y_i) + \sum_{i=1}^{n} x_i^T P_i B_i u_i^a(t, y_i) + \sum_{i=1}^{n} \|E_i y_i\| \rho_i(t, y_i) \\
&\quad + \sum_{i=1}^{n} \frac{1}{2\varepsilon_i^a} \|E_i y_i\|^2 \varpi_i^2(t, y_i) + \sum_{i=1}^{n} \frac{\varepsilon_i^a}{2} \|x_{id_i}\|^2 \\
&\le \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i^a \|x_{id_i}\|^2. \qquad (7.99)
\end{aligned}
$$

From (7.87),

$$
\begin{aligned}
x_i^T P_i \psi_i(t, x, x_d) &\le \|x_i\| \, \|P_i\| \sum_{j=1}^{n} \big( \alpha_{ij}(t, x) \|x_j\| + \beta_{ij}(t, x) \|x_{jd_j}\| \big) \\
&= \sum_{i=1}^{n} \Big( \alpha_{ij}(t, x) \|P_i\| \, \|x_i\| \, \|x_j\| \\
&\qquad + \beta_{ij}(t, x) \|P_i\| \, \|x_i\| \|x_{jd_j}\| \Big). \qquad (7.100)
\end{aligned}
$$

Applying (7.99) and (7.100) to Eq. (7.95) yields

$$
\begin{aligned}
\dot{V} &\le -\sum_{i=1}^{n} x_i^T Q_i x_i + \sum_{i=1}^{n} \varepsilon_i^a \|x_{id_i}\|^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^T P_i \Phi_{ij}(x) x_j \\
&\quad + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \Big( \alpha_{ij}(t, x) \|P_i\| \, \|x_i\| \, \|x_j\| + \beta_{ij}(t, x) \|P_i\| \, \|x_i\| \|x_{jd_j}\| \Big). \qquad (7.101)
\end{aligned}
$$

From the definition of $V(\cdot)$ in (7.94), it is clear that

$$
V(x_{1d_1}, x_{2d_2}, \ldots, x_{nd_n}) \le q V(x_1, x_2, \ldots, x_n), \qquad (q > 1)
$$

implies that

$$\begin{aligned}
& q\sum_{i=1}^{n}\lambda_{\max}(P_i)\|x_i\|^2-\sum_{i=1}^{n}\lambda_{\min}(P_i)\|x_{id_i}\|^2 \\
\geq\ & q\sum_{i=1}^{n}x_i^T P_i x_i-\sum_{i=1}^{n}x_{id_i}^T P_i x_{id_i} \\
\geq\ & 0.
\end{aligned} \tag{7.102}$$

Therefore, when

$$V(x_{1d_1},\ldots,x_{nd_n})\leq qV(x_1,\ldots,x_n)$$

it follows from (7.102) and (7.101) that

$$\begin{aligned}
\dot{V}\leq & -\sum_{i=1}^{n}\lambda_{\min}(Q_i)\|x_i\|^2+\sum_{i=1}^{n}\varepsilon_i^a\|x_{id_i}\|^2+2\sum_{i=1}^{n}\sum_{j=1}^{n}\|P_i\Phi_{ij}(x)\|\,\|x_i\|\,\|x_j\| \\
& +2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\alpha_{ij}(t,x)\|P_i\|\,\|x_i\|\,\|x_j\|+\beta_{ij}(t,x)\|P_i\|\,\|x_i\|\|x_{jd_j}\|\right) \\
& +q\sum_{i=1}^{n}\lambda_{\max}(P_i)\|x_i\|^2-\sum_{i=1}^{n}\lambda_{\min}(P_i)\|x_{id_i}\|^2 \\
\leq & -\sum_{i=1}^{n}\Big(\lambda_{\min}(Q_i)-q\lambda_{\max}(P_i)\Big)\|x_i\|^2-\sum_{i=1}^{n}\Big(\lambda_{\min}(P_i)-\varepsilon_i^a\Big)\|x_{id_i}\|^2 \\
& +2\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\|P_i\Phi_{ij}(x)\|+\alpha_{ij}(t,x)\|P_i\|\Big)\|x_i\|\,\|x_j\| \\
& +2\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}(t,x)\|P_i\|\,\|x_i\|\,\|x_{jd_j}\| \\
= & -\frac{1}{2}Y\left(W^T(\cdot)+W(\cdot)\right)Y^T \\
\leq & -\frac{1}{2}\lambda_{\min}\left(W^T(\cdot)+W(\cdot)\right)(\|x\|^2+\|x_d\|^2) \\
\leq & -\frac{1}{2}\gamma\|x\|^2
\end{aligned}$$

where

$$Y:=\left[\,\|x_1\|\ \cdots\ \|x_n\|\ \|x_{1d_1}\|\ \cdots\ \|x_{nd_n}\|\,\right].$$

Hence, by applying Lemma A.1 in the Appendix A.1, the conclusion follows from $\gamma>0$. □

*Remark 7.14* From inequalities (7.87) in Assumption 7.11, the bounds on the uncertain interconnections are dependent on the system states, and thus they cannot be employed in the control design since static output feedback is used. The effects of such interconnections have been reflected through $\alpha_{ij}(t, x)$ and $\beta_{ij}(t, x)$ in the matrix $W$ in Theorem 7.5. From Lemma A.1 in Appendix A.1, it is straightforward to see that the result in Theorem 7.5 can be extended to the global case if

$$\gamma := \inf_x \left\{\lambda_{\min}\left(W^T(\cdot) + W(\cdot)\right)\right\} > 0$$

holds globally.

It is well known that one of the main challenges for large-scale interconnected systems is to deal with interconnections. It is assumed that the function matrices $\Phi_{ij}(\cdot)$ in the decomposition (7.85) are only dependant on the $i$-th system's outputs $y_i$, that is

$$\Phi_{ij}(x) = \Phi_{ij}(y_i), \qquad i, j = 1, 2, \ldots, n.$$

In this case, the known interconnections $\Phi_i(x)$ in System (7.83) are described by

$$\Phi_i(x) = \sum_{i=1}^{n} \Phi_{ij}(y_i)x_j, \qquad i = 1, 2, \ldots, n \tag{7.103}$$

where $\Phi_{ij}(\cdot) \in \mathbb{R}^{n_i \times n_j}$. It is clear to see that the expressions (7.103) include linear interconnections as a special case in which the matrices $\Phi_{ij}(\cdot)$ are constant.

In order to reduce the effects of the interconnections, the objective now is to separate the interconnections into matched and mismatched contributions, and then try to reject the effects of the accessible parts by appropriate additive control elements. Denote the $l$-th column vector of the matrix $\Phi_{ij}(y_i)$ by $\Phi_{ij}^{(l)}(y_i)$ for $l = 1, 2, \ldots, n_j$. For the given input matrices $B_i$, it is assumed that $\text{Im}(B_i)$ represents the image of the matrix $B_i$, and $(\text{Im}(B_i))^{\perp}$ denotes the orthogonal complimentary space of $\text{Im}(B_i)$. Using basic matrix theory, decompose the vector $\Phi_{ij}^{(l)}(y_i)$ as

$$\Phi_{ij}^{(l)}(y_i) = (\Phi_{ij}^{(l)}(y_i))^a + (\Phi_{ij}^{(l)}(y_i))^b$$

such that

$$(\Phi_{ij}^{(l)}(y_i))^a \in \text{Im}(B_i) \qquad \text{and} \qquad (\Phi_{ij}^{(l)}(y_i))^b \in (\text{Im}(B_i))^{\perp}$$

for $l = 1, 2, \ldots, n_j$. Let

$$\Phi_{ij}^a(y_i) := \left[(\Phi_{ij}^{(1)}(y_i))^a \ (\Phi_{ij}^{(2)}(y_i))^a \ \cdots \ (\Phi_{ij}^{(n_j)}(y_i))^a\right]$$

$$\Phi_{ij}^b(y_i) := \left[(\Phi_{ij}^{(1)}(y_i))^b \ (\Phi_{ij}^{(2)}(y_i))^b \ \cdots \ (\Phi_{ij}^{(n_j)}(y_i))^b\right].$$

It is straightforward to see that $\Phi_{ij}(y_i)$ has the following decomposition

$$\Phi_{ij}(y_i) = \Phi_{ij}^a(y_i) + \Phi_{ij}^b(y_i), \qquad i, j = 1, 2, \ldots, n \tag{7.104}$$

where

$$\Phi_{ij}^a(y_i) = B_i \widetilde{\Phi}_{ij}(y_i), \qquad i, j = 1, 2, \ldots, n \tag{7.105}$$

for some $\widetilde{\Phi}_{ij}(y_i) \in \mathbb{R}^{m_i \times n_j}$.

Then, consider the following control law:

$$u_i = -K_i y_i - \frac{1}{2\varepsilon_i^a} E_i y_i \varpi_i^2(t, y_i) + u_i^a(t, y_i) + u_i^b(t, y_i), \qquad i = 1, 2, \ldots, n \tag{7.106}$$

where $K_i$ and $u_i^a(\cdot)$ are given in (7.91), and the additive control element $u_i^b(\cdot)$ is defined by

$$u_i^b(\cdot) = \begin{cases} -\dfrac{E_i y_i}{\|E_i y_i\|^2} \displaystyle\sum_{j=1}^{n} \left( \frac{1}{2\varepsilon_i^b} \|(E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)\|^2 \right), & E_i y_i \neq 0 \\ 0, & E_i y_i = 0 \end{cases} \tag{7.107}$$

where $\widetilde{\Phi}_{ij}(y_i)$ satisfy (7.105). It should be noted that the control (7.106) is generated by adding the term (7.107) to the control (7.91).

**Corollary 7.2** *Assume that the interconnections of System (7.83)–(7.84) can be expressed in (7.103). Then, under Assumptions 7.11–7.13, the closed-loop system formed by applying the control (7.106) to the system (7.83)–(7.84) is uniformly asymptotically stable if*

$$\inf_x \left\{ \lambda_{\min} \left( \Gamma^T(\cdot) + \Gamma(\cdot) \right) \right\} > 0$$

*where the matrix $\Gamma(\cdot) = [\Gamma_{ij}(\cdot)]_{2n \times 2n}$ is defined by*

$$\Gamma_{ij}(\cdot) = \begin{cases} \lambda_{\min}(Q_i) - q\lambda_{\max}(P_i) - 2\alpha_{ii}(t, x_i)\|P_i\| \\ \qquad - \sum_{j=1}^{n} \varepsilon_j^b, & 1 \leq i = j \leq n \\ -2\|P_i \Phi_{ij}^b(y_i)\| - 2\alpha_{ij}(t, x)\|P_i\|, & i \neq j \ \text{ and } \ 1 \leq i, j \leq n \\ \quad w_{ij}(\cdot), & \text{otherwise} \end{cases}$$

*for some constants*

$$\varepsilon_j^b > 0 \quad \text{and} \quad q > 1$$

*where the functions $w_{ij}(\cdot)$ are defined in Theorem 7.5 and the matrices $\Phi_{ij}^b(\cdot)$ are defined in (7.104).*

*Proof* From (7.103), (7.104) and (7.105),

$$\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + \sum_{i=1}^{n} x_i^T P_i \Phi_i(x)$$
$$= \sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i B_i \widetilde{\Phi}_{ij}(y_i)x_j + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i \Phi_{ij}^b(y_i)x_j. \quad (7.108)$$

Based on the structure of the control in (7.107), consider the following two cases:
(i) if $E_i y_i = 0$, then from (7.89) and (7.107),

$$\begin{aligned}
&\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i B_i \widetilde{\Phi}_{ij}(y_i)x_j \\
&\quad = \sum_{i=1}^{n} (E_i y_i)^T u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} (E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)x_j \\
&\quad = 0
\end{aligned}$$

(ii) if $E_i y_i \neq 0$, then from (7.89), the definition of $u_i^b(\cdot)$ in (7.107) and Young's inequality,

$$\begin{aligned}
&\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i B_i \widetilde{\Phi}_{ij}(y_i)x_j \\
&\quad = \sum_{i=1}^{n} (E_i y_i)^T u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} (E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)x_j \\
&\quad \le \sum_{i=1}^{n} (E_i y_i)^T u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} \left( \frac{1}{2\varepsilon_i^b} \|(E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)\|^2 + \frac{\varepsilon_i^b}{2}\|x_j\|^2 \right) \\
&\quad = -\sum_{i=1}^{n} (E_i y_i)^T \frac{E_i y_i}{\|E_i y_i\|^2} \left( \sum_{j=1}^{n} \frac{1}{2\varepsilon_i^b} \|(E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)\|^2 \right) \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2\varepsilon_i^b} \|(E_i y_i)^T \widetilde{\Phi}_{ij}(y_i)\|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\varepsilon_i^b}{2}\|x_j\|^2 \\
&\quad = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\varepsilon_j^b}{2} \right) \|x_i\|^2.
\end{aligned}$$

From the analysis in (i) and (ii) above, it follows that

$$\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i B_i \widetilde{\Phi}_{ij}(y_i)x_j \le \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\varepsilon_j^b}{2} \right) \|x_i\|^2. \quad (7.109)$$

By applying (7.109) to (7.108),

$$
\begin{aligned}
2\sum_{i=1}^{n} x_i^T P_i B_i u_i^b(t, y_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i \Phi_i(x) \\
\leq 2\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i \Phi_{ij}^b(y_i) x_j + \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\varepsilon_j^b\right)\|x_i\|^2.
\end{aligned}
\tag{7.110}
$$

Hence, the conclusion follows by following the proof of Theorem 7.5. □

*Remark 7.15* From the proof of Corollary 7.2, it is clear to see that using the decomposition (7.104), the nonlinear term

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^T P_i B_i \widetilde{\Phi}_{ij}(y_i) x_j
$$

which results from the matched interconnections can be largely rejected by the designed control (7.107) by choosing the positive parameters $\varepsilon_j^b$ small enough, although this approach may result in high gain control. The numerical example in Sect. 7.5.3 will show that the conservatism can be reduced by employing the additive term (7.107).

## 7.5 Simulation Examples

Examples are presented in this section to demonstrate the results obtained from the approaches developed in Sects. 7.2–7.4.

### *7.5.1 Case Study—A Mass–Spring System*

Consider a mass–spring system which experiences a hardening spring, linear viscous friction and an external force described by (see [91])

$$
m\ddot{s} + c\dot{s} + ks + ka^2 s^3 = u \tag{7.111}
$$

where $s$ denotes the displacement from the reference position, $m$ is the mass of the object sliding on a horizontal surface, $k$ is the spring constant and $u$ is an external force which is considered as the control input. The term

$$
ks + ka^2 s^3
$$

is used to model the restoring force for the hardening spring. Let $x = \mathrm{col}(x_1, x_2) = (s, \dot{s})$. Then,

$$\dot{x}_1 = x_2 \qquad \text{and} \qquad \dot{x}_2 = \ddot{s}.$$

From Eq. (7.111),

$$\begin{aligned}\dot{x}_2 &= -\frac{c}{m}x_2 - \frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 + u\\ &= -\left(\frac{k}{m} + \frac{k}{m}a^2x_1^2\right)x_1 - \frac{c}{m}x_2 + u.\end{aligned}$$

The model is parameterised as in [91] (see, pages 172–173) (alternatively choose $m = c = k = a = 1$). Then, the system is described by

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ -(1+x_1^2)x_1 - x_2 \end{bmatrix}}_{f(\cdot)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g(\cdot)}(u + \Delta g(x, x_d)) + \Delta f(x, x_d) \tag{7.112}$$

$$y = x_1 + x_2 \tag{7.113}$$

where $y$ is the system output. As in [180], the system output here is chosen as the linear combination of the position and velocity, which guarantees that the nominal system is output feedback stabilisable. This may occur in some real systems such as certain remote control applications where the number of transmission and receive lines/frequencies are limited [180]. The uncertainties $\Delta f(x, x_d)$ and $\Delta g(x, x_d)$ are not an inherent property of the system but are specifically added to illustrate the results obtained in Sect. 7.2. It is assumed that

$$\|\Delta g(x, x_d)\| \le \underbrace{1 + y\sin^2 y}_{\alpha_1(\cdot)} + \underbrace{y^2}_{\alpha_2(\cdot)}\underbrace{\|x_d\|}_{\alpha_3(\cdot)} \tag{7.114}$$

$$\|\Delta f(x, x_d)\| \le \underbrace{0.01\|x\|\,\|x_d\|^2}_{\beta(\cdot)}. \tag{7.115}$$

Then, consider an output feedback control

$$u_1(y) = -y \tag{7.116}$$

and a Lyapunov function candidate

$$V = x^\tau \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} x + \frac{1}{2}x_1^4.$$

It is straightforward to see that $V(\cdot)$ is a continuous positive definite function. By direct computation,

$$0.697\|x\|^2 \le V(x) \le 4.303\|x\|^2 + \frac{1}{2}\|x\|^4 \tag{7.117}$$

$$\frac{\partial V}{\partial x}(f(x) + g(x)u_1(y)) = -2x_1^2 - 2x_2^2 - 2x_1^4 \le -2\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le 8.606\|x\| + 2\|x\|^3$$

$$\frac{\partial V}{\partial x}g(x) = 2x_1 + 2x_2 = 2y = M(y).$$

Let

$$\begin{array}{ll} c_1(r) = 0.697r^2, & c_2(r) = 4.303r^2 + \frac{1}{2}r^4, \\ c_3(r) = 2r^2, & c_4(r) = 8.6056r + 2r^3. \end{array}$$

It is clear to see that both Assumptions 7.1 and 7.2 are satisfied. Further, assume that

$$V(x(t+\theta)) \le qV(x(t))$$

for any $\theta \in [-\bar{d}, 0]$ and $q > 1$. From Inequality (7.117), it follows that for any $d \in [0,\ \bar{d}]$,

$$0.697\|x_d\|^2 \le V(x(t-d)) \le qV(x(t)) \le 4.303q\|x\|^2 + \frac{1}{2}q\|x\|^4.$$

Then,

$$\|x_d\|^2 \le 6.2q\|x\|^2 + 0.72q\|x\|^4.$$

This implies that Assumption 7.3 is satisfied with

$$\gamma(r) = \sqrt{6.2qr^2 + 0.72qr^4}. \tag{7.118}$$

By direct computation, it is observed that the conditions of Theorem 7.1 are satisfied in the domain

$$\mathscr{X} := \{x \mid \|x\| \le 1\}$$

with

$$\alpha(r) = 0.01r^2$$

if the parameters $q$ and $\varepsilon$ are chosen in the shaded open set shown in Fig. 7.1.

Based on Theorem 7.1, the mass–spring system (7.112)–(7.113) is stabilised by the control

$$u = u_1 + u_2$$

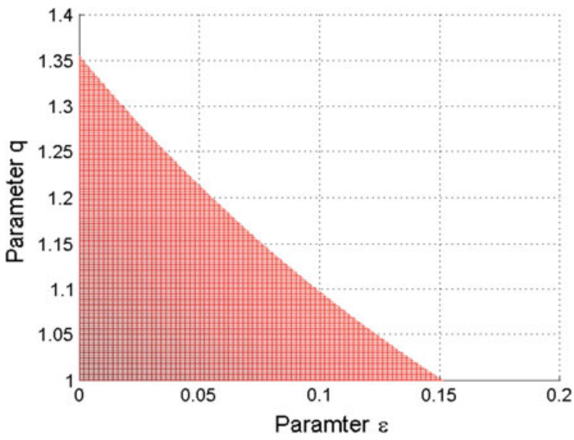


**Fig. 7.1** The estimated admissible region for the parameters $\varepsilon$ and $q$

where

$$u_1 = -y \tag{7.119}$$

$$u_2 = \begin{cases} -\left(\frac{y}{|y|}\left(1 + y\sin^2 y\right) + \frac{1}{\varepsilon}y^5\right), & y \neq 0 \\ 0, & y = 0 \end{cases}. \tag{7.120}$$

For simulation purposes, choose

$$\varepsilon = 0.1 \qquad \text{and} \qquad q = 1.05.$$

The time-delay is chosen as

$$d(t) = 2 - 1.5\cos(t)$$

and the initial condition relating to the delay defined in (7.3) is chosen as $\phi(t) = \sin t$. The simulation results in Fig. 7.2 are as expected.

*Remark 7.16* From Fig. 7.2, it is clear to see that chattering occurs in the control signal. This is a result of the discontinuous control $u_2(\cdot)$ in (7.120). One way of overcoming this drawback is to introduce a boundary layer about the discontinuous surfaces (see [13]).

*Remark 7.17* Considering Assumption 7.3, it is straight forward to see that in the simulation example,

$$w(r) = qr \qquad \text{and} \qquad \gamma(r) = \sqrt{4.303qr^2 + \frac{1}{2}qr^4}.$$

By direct computation, Assumption 7.3 is satisfied for any $q > 1$. The parameter $q$ is embedded in the inequality (14) through the function $\gamma(\cdot)$. From the expression

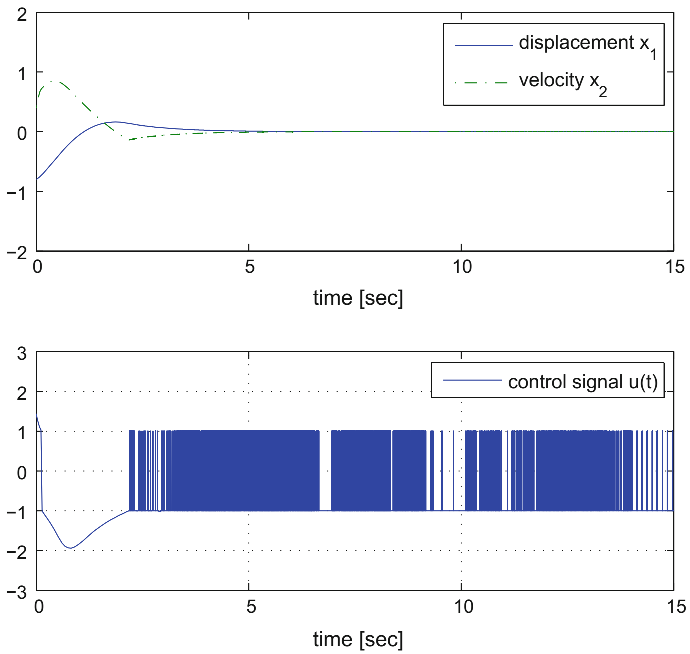


**Fig. 7.2** The time response of state variables of mass–spring system (7.112)–(7.113) (*Upper*) and the control signal (*Bottom*)

for $\gamma(\cdot)$ in (7.118), inequality (14) implies a limitation on the parameters $q$, $\varepsilon$ and the domain size. In this example, any parameter pair $(\varepsilon, q)$ lying in the shaded area in Fig. 7.1 will satisfy the conditions in Theorem 7.1 in the domain

$$\mathscr{X} := \{x \mid \|x\| \leq 1\}.$$

For simulation purposes, $(\varepsilon, q) = (0.1, 1.01)$ is chosen which lies in the shaded area in Fig. 7.1.

## 7.5.2 *Sliding Mode Control Synthesis*

Consider a nonlinear system with time-delay disturbances

$$\dot{x} = \underbrace{\begin{bmatrix} -6x_2^2x_3^2 - 4x_2^2 - 2x_1 \\ -3x_2x_3^2 - 3x_2 + \dfrac{1}{16}\left(x_2^2 - x_1\right)^2 \\ 3x_2^2x_3 - 3x_3 - \dfrac{1}{4}\left(x_2^2 - x_1\right)\exp\{-t\}\cos(x_3 t) \end{bmatrix}}_{f(\cdot)}$$

$$+\underbrace{\begin{bmatrix} -4\left(x_3^2 \sin^2 t + 1\right) \\ 0 \\ 0 \end{bmatrix}}_{g(\cdot)} \left(u + \phi(t, x, x_d)\right) + \psi(t, x, x_d) \tag{7.121}$$

$$y =: \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T = \begin{bmatrix} x_3 & \frac{1}{4}\left(x_2^2 - x_1\right) \end{bmatrix}^T \tag{7.122}$$

where $x = \text{col}(x_1, x_2, x_3) \in \mathbb{R}^3$, $u \in \mathbb{R}$ and $y \in \mathbb{R}^2$ are, respectively, the states, input and outputs, and $\phi(\cdot) \in \mathbb{R}$ and $\psi(\cdot) \in \mathbb{R}^3$ are the matched and mismatched disturbances respectively, satisfying

$$\begin{aligned} |\phi(\cdot)| &\leq (|x_{2d}| + |x_3| + 1)\exp\{-t\} \\ \|\psi(\cdot)\| &\leq \frac{1}{\sqrt{1 + 4x_2^2}}\left(|x_{2d}x_2| \sin^2(tx_{3d}) + \frac{\left|x_2^2 - x_1\right| \left|x_{2d}^2 - x_{1d}\right|}{16}\right). \end{aligned} \tag{7.123}$$

Let

$$T : \begin{cases} z = x_2 \\ y_1 = x_3 \\ y_2 = \frac{1}{4}(x_2^2 - x_1) \end{cases}. \tag{7.124}$$

The Jacobian matrix of $T$ is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{1}{2}x_2 & -1 \end{bmatrix}$$

which is nonsingular, and thus the transformation

$$T(x_1, x_2, x_3) \mapsto (z, y_1, y_2)$$

defined in (7.124) is a diffeomorphism. It is clear that

$$\begin{aligned} \frac{\partial T(x)}{\partial x} g(t, x) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{4} & \frac{1}{2}x_2 & -1 \end{bmatrix} \begin{bmatrix} -4\left(x_3^2 \sin^2 t + 1\right) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \left(y_1^2 \sin^2 t + 1\right) \end{bmatrix}. \end{aligned}$$

The system (7.121)–(7.122) in the new coordinates $(z, y_1, y_2)$ can be described by

$$\begin{bmatrix} \dot{z} \\ \dot{y}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -3zy_1^2 - 3z + y_2^2 \\ 3z^2 y_1 - 3y_1 - y_2 \exp\{-t\} \cos(y_1 t) \end{bmatrix}}_{F_1(\cdot)} + \Psi_1(\cdot) \tag{7.125}$$

$$\dot{y}_2 = \underbrace{-2y_2 + \frac{1}{2} z y_2^2}_{F_2(\cdot)} + \underbrace{(1 + y_1^2 \sin^2 t)}_{G(\cdot)} \big(u + \Phi(\cdot)\big) + \Psi_2(\cdot) \tag{7.126}$$

where the disturbances

$$\Phi(\cdot) := \Big[\phi(t, x, x_d)\Big]_{\substack{x_1 = z^2 - 4y_2 \\ x_2 = z \\ x_3 = y_1}}$$

$$\begin{bmatrix} \Psi_1(\cdot) \\ \Psi_2(\cdot) \end{bmatrix} := \Big[J_T(x)\psi(t, x, x_d)\Big]_{\substack{x_1 = z^2 - 4y_2 \\ x_2 = z \\ x_3 = y_1}}$$

where $\Psi_1(\cdot) \in \mathbb{R}^2$ and $\Psi_2(\cdot) \in \mathbb{R}^1$. By direct computation, it follows from (7.123) that

$$\begin{aligned} |\Phi(\cdot)| &\le \underbrace{(|z_d| + |y_1| + 1)\exp\{-t\}}_{\varpi_1(\cdot)} \\ \|\Psi_1(\cdot)\| &\le |z_d z| \sin^2(t y_{1d}) + |y_{2d} y_2| \\ |\Psi_2(\cdot)| &\le \underbrace{\frac{1}{4}(|z_d z| \sin^2(t y_{1d}) + |y_{2d} y_2|)}_{\varpi_2(\cdot)}. \end{aligned}$$

Choose the switching function $s(z, y) := y_2$. Then, the sliding mode dynamics are described by

$$\dot{X} = \underbrace{\begin{bmatrix} -3zy_1^2 - 3z \\ 3z^2 y_1 - 3y_1 \end{bmatrix}}_{F_{1s}(\cdot)} + \Psi_{1s}(t, X, X_d) \tag{7.127}$$

where $X = \text{col}(z, y_1)$. Substituting $y_2 = 0$ in the bound on $\Psi_1(\cdot)$, it follows that

$$\|\Psi_{1s}(t, X, X_d)\| \le |z_d z| \sin^2(t y_1) \le \underbrace{|z_d z|}_{\eta}$$

and thus Eq. (7.62) holds with

$$\eta_1(\cdot) = 0 \qquad \text{and} \qquad \eta_2(\cdot) = |z|.$$

This implies that Assumption 7.7 is satisfied. Construct a candidate Lyapunov function

$$V(t) = z^2(t) + y_1^2(t).$$

It is straightforward to check that Assumption 7.6 holds and the $r_i(\cdot)$ satisfy (7.69) with

$$\gamma_1 = \gamma_2 = 1, \qquad \gamma_3 = \sqrt{6} \qquad \text{and} \qquad \gamma_4 = 2.$$

Let $\nu = 1.1$. By direct computation, all the conditions in Corollary 7.1 hold in the domain

$$\Omega_X = \{(z, y_1) \big| \, |z| \le 2.8, \;\; y_1 \in \mathbb{R}\}$$

which guarantees that the sliding motion is exponentially stable. Then, from (7.77), the designed control is

$$\begin{aligned} u(t, y) = & \frac{2y_2}{1 + y_1^2 \sin^2 t} - (|y_1| + 1) \exp\{-t\}\text{sgn}(y_2) \\ & - \frac{k(t, y)}{1 + y_1^2 \sin^2 t} \text{sgn}(y_2) \end{aligned} \tag{7.128}$$

where, according to (7.78), the control gain function $k(\cdot)$ is chosen as

$$k(t, y) = q\left(\frac{1}{2} y_2^2 + (1 + y_1^2 \sin^2 t)(\exp\{-t\} + 0.75)\right) + \rho.$$

From Corollary 7.1 and Theorem 7.4, the closed-loop system formed by applying the control (7.128) to the system (7.121), is exponentially stable.

For simulation purposes, the initial states are chosen as $x_0 = (-16, \;2, \;5)$. The delay is chosen as

$$d(t) = 2 - \sin(t)$$

and the initial condition relating to the delay is given by

$$\zeta(t) = \left[\, \sin(t) \;\; 0 \;\; 1 + \cos(t) \,\right]^T, \qquad t \in [-3, 0].$$

The constants $q$ and $\rho$ are chosen as

$$q = 3 \qquad \text{and} \qquad \rho = 1.$$

The simulation in Figs. 7.3 and 7.4 demonstrates the effectiveness of the proposed approach: Fig. 7.3 shows the time response of the closed-loop system states and Fig. 7.4 presents the designed control signal and sliding function.

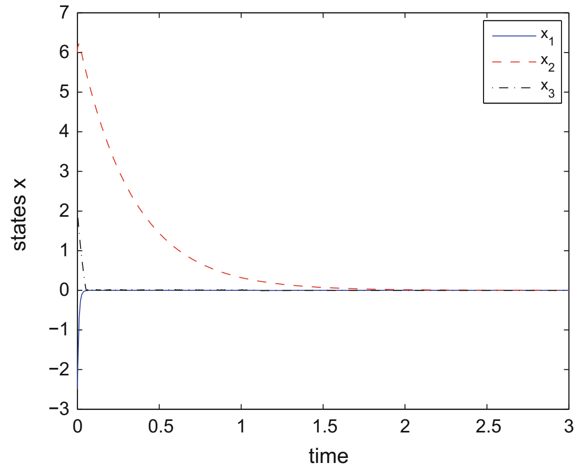


**Fig. 7.3** The time response of System (7.121)–(7.122) under the control (7.128)

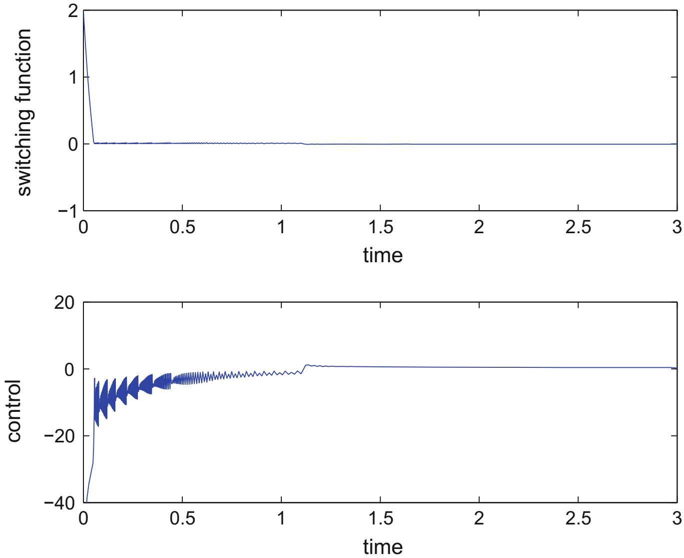


**Fig. 7.4** The time response of the control signal (7.128) (*bottom*), and the switching function (*upper*)

### 7.5.3 A Time-Delay Nonlinear Interconnected System

In order to illustrate the results obtained, consider an interconnected system described by

$$\dot{x}_1 = \underbrace{\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 2 & 2 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{B_1} (u_1 + \xi_1(t, x_1, x_{1d_1})) + \underbrace{\begin{bmatrix} 0.1x_{12}x_{22} \\ -0.1x_{21} \\ (5x_{21} - 5x_{22})x_{12} \end{bmatrix}}_{F_1(x)} + \psi_1(t, x, x_d) \tag{7.129}$$

$$\dot{x}_2 = \underbrace{\begin{bmatrix} 10 & 15 \\ -30 & 1 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B_2} \Big(u_2 + \xi_2(t, x_2, x_{2d_2})\Big) + \underbrace{\begin{bmatrix} 0.1x_{11} + 2x_{12} - 6x_{13} \\ (x_{22} - x_{21})(-2x_{12} + 6x_{13}) \end{bmatrix}}_{F_2(x)} + \psi_2(t, x, x_d) \tag{7.130}$$

$$y_1 = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C_1} x_1, \tag{7.131}$$

$$y_2 = \underbrace{[-1 \quad 1]}_{C_2} x_2, \tag{7.132}$$

where $x_1 := \text{col}(x_{11}, x_{12}, x_{13}) \in \mathbb{R}^3$ and $x_2 := \text{col}(x_{21}, x_{22}) \in \mathbb{R}^2$ are states, $y_1 = \text{col}(y_{11}, y_{12}) \in \mathbb{R}^2$ and $y_2 \in \mathbb{R}^1$ are outputs, and $u_1, u_2 \in \mathbb{R}^1$ are inputs. The uncertainties $\xi_i(\cdot)$ and the uncertain interconnections $\psi_i(\cdot)$ for $i = 1, 2$ satisfy

$$\|\xi_1(t, x_1, x_{1d_1})\| \le \underbrace{(2 + y_{11})^2 \sin^4(y_{12}t)}_{\rho_1(t, y_1)} + \underbrace{|y_{12}y_{11} \sin t|}_{\varpi_1(t, y_1)} \|x_{1d_1}\|$$

$$\|\xi_2(t, x_2, x_{2d_2})\| \le \underbrace{3|y_2| \exp\{-t\}}_{\rho_2(t, y_2)} + \underbrace{y_2^2 |\sin t|}_{\varpi_2(t, y_2)} \|x_{1d_2}\|$$

$$\|\psi_1(t, x, x_d)\| \le \underbrace{\frac{1}{3}|x_{11} \cos x_{22}|}_{\alpha_{12}(t, x)} \|x_2\| + \underbrace{\frac{1}{4}|x_{12}| \sin^2 t}_{\beta_{12}(t, x)} \|x_{2d_2}\|$$

$$\psi_2(t, x, x_d) = 0$$

where the bounds on $\psi_1(\cdot)$ imply that

$$\alpha_{11}(\cdot) = \beta_{11}(\cdot) = 0$$

and the fact that $\psi_2(\cdot) = 0$ shows that

$$\alpha_{21}(\cdot) = \alpha_{22}(\cdot) = \beta_{21}(\cdot) = \beta_{22}(\cdot) = 0.$$

The interconnections $F_1(\cdot)$ and $F_2(\cdot)$ can be expressed in (7.103) as follows:

$$F_1(\cdot) = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Phi_{11}(y_1)} x_1 + \underbrace{\begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \\ 5y_{11} & -5y_{11} \end{bmatrix}}_{\Phi_{12}(y_1)} x_2$$

$$F_2(\cdot) = \underbrace{\begin{bmatrix} 0.1 & 2 & -6 \\ 0 & -2y_2 & 6y_2 \end{bmatrix}}_{\Phi_{21}(y_2)} x_1 + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Phi_{22}(y_2)} x_2.$$

It is straightforward to see that the decompositions (7.104) and (7.105) hold with

$$\tilde{\Phi}_{11}(y_1) = 0, \qquad \Phi_{11}^b(y_1) = 0$$

$$\Phi_{12}(y_1) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{B_1} \underbrace{\begin{bmatrix} 2.5y_{11} & -2.5y_{11} \end{bmatrix}}_{\tilde{\Phi}_{12}(y_1)} + \underbrace{\begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \\ 0 & 0 \end{bmatrix}}_{\Phi_{12}^b(y_1)}$$

$$\Phi_{21}(y_2) = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B_2} \underbrace{\begin{bmatrix} 0 & 2y_2 & -6y_2 \end{bmatrix}}_{\tilde{\Phi}_{21}(y_2)} + \underbrace{\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Phi_{21}^b(y_2)}$$

$$\tilde{\Phi}_{22}(y_2) = 0, \qquad \Phi_{22}^b(y_2) = 0.$$

Let

$$K_1 = [1 \quad 3], \qquad K_2 = -8,$$
$$Q_1 = 8I_3, \qquad Q_2 = I_2.$$

Then the solutions to the Eqs. (7.88) and (7.89) are

$$P_1 = I_3, \qquad P_2 = \begin{bmatrix} 1.25 & 0.25 \\ 0.25 & 1.25 \end{bmatrix},$$
$$E_1 = [0 \quad 2], \; E_2 = -1$$

Let

$$\varepsilon_i = \varepsilon_i^a = \varepsilon_i^b = 0.1 \quad (i = 1, 2) \qquad \text{and} \qquad q = 1.01.$$

Based on the parameters above, the control given in (7.107) is well defined. By direct computation,

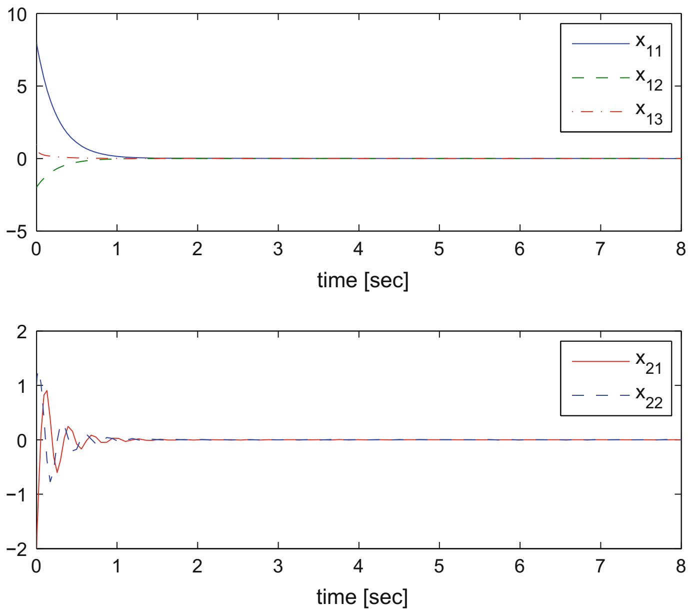


**Fig. 7.5** The time response of the state variables of System (7.129)–(7.132)

$$\Gamma = \begin{bmatrix} 6.7900 & -0.2 - \frac{2}{3}|x_{11}\cos x_{22}| & 0 & -0.6667 - \frac{1}{2}|x_{12}|\sin^2 t \\ -0.2550 & 4.2850 & 0 & 0 \\ 0 & -0.6667 - \frac{1}{2}|x_{12}|\sin^2 t & 0.9000 & 0 \\ 0 & 0 & 0 & 0.9000 \end{bmatrix}$$

which is positive definite in the domain

$$\Omega = \{(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}) \mid |x_{11}| \le 10.5,\ |x_{12}| \le 3.1,\ x_{13}, x_{21}, x_{22} \in \mathbb{R}\}.$$

Hence from Corollary 7.2, the system (7.129)–(7.132) is stabilised by the control (7.91). Simulation results presented in Figs. 7.5 and 7.6 show the results obtained are effective.

*Remark 7.18* Consider a comparison of the matrix $W$ in Theorem 7.5 and the matrix $\Gamma$ in Corollary 7.2. By direct computation it follows that

$$W = \begin{bmatrix} 6.9900 & -14.1435 - \frac{2}{3}|x_{11}\cos x_{22}| & 0 & -0.6667 - \frac{1}{2}|x_{12}|\sin^2 t \\ -17.8891 & 4.4850 & 0 & 0 \\ 0 & -0.6667 - \frac{1}{2}|x_{12}|\sin^2 t & 0.9000 & 0 \\ 0 & 0 & 0 & 0.9000 \end{bmatrix}.$$

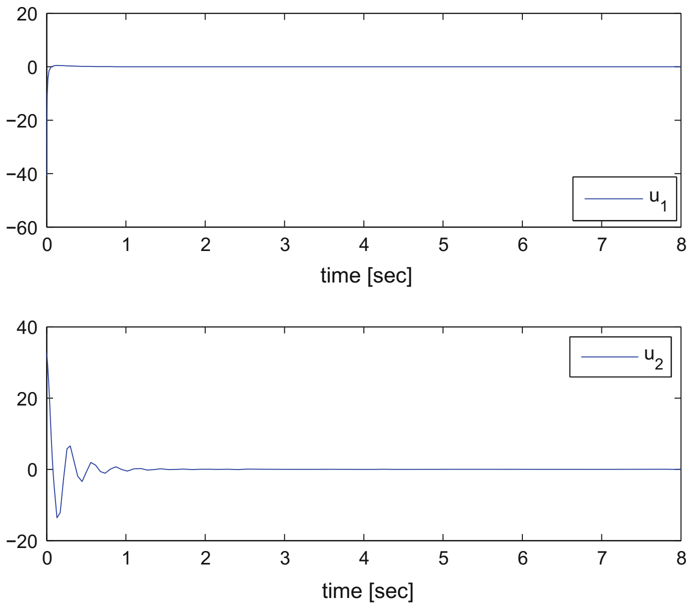


**Fig. 7.6** The time response of the control signals

It is straightforward to check that $W + W^T$ is not positive definite even if $x_{11} = 0$ and $x_{12} = 0$, and thus Theorem 7.5 cannot be applied to the system (7.129)–(7.132). This confirms the result stated in Remark 7.15.

## 7.6 Summary

In this chapter, time-delay independent control schemes have been proposed based on the Lyapunov–Razumikhin approach. Local stabilisation results using static output feedback control are achieved for a class of nonlinear systems with time-delay disturbances. The Lyapunov direct method is used in Sect. 7.2 and sliding mode techniques are employed in Sect. 7.3. Although some of the developed conditions may not be straightforward to check in Sects. 7.2 and 7.3, a framework is provided to stabilise nonlinear systems with time-varying delay disturbances using a delay independent output feedback controller.

A control strategy for a class of interconnected systems with time varying delays has been presented in Sect. 7.4. Where it is not required that the subsystems are square. The proposed controllers are decentralised, independent of the time-delay and based only on output information, which is convenient for implementation.

It should be noted that the Lyapunov–Razumikhin approach may lead to a conservative result but the developed results are delay independent and are independent of the upper bound $\bar{d}$ on time-delay $d(t)$. This implies that it is not required that the time-delay is known, or the bound on time-delay is known. The limitation on the rate of change of the time-varying delay is not required, as is required using the Lyapunov–Krasovskii approach. Therefore, the developed results can be applied to a wide class of nonlinear systems. A tutorial case study of a mass–spring system and simulation examples have shown the effectiveness and feasibility of the proposed control strategies.

# Chapter 8
# Sliding Mode Observer-Based Fault Detection and Isolation

This chapter considers the problem of robust fault detection and isolation (FDI) for a class of nonlinear systems using sliding mode observers. The focus is fault reconstruction and estimation for system faults and sensor faults in the presence of nonlinear uncertainties.

## 8.1 Introduction

It is well known that automatic control systems can reduce the consumption of energy and save manpower, and thus they are widely applied in industry. However such systems are subject to malfunctions and errors because the human operator interaction is reduced/removed in these systems. Furthermore, unexpected variations in the external surroundings or normal wear and tear of components can also make the system faulty. The effect of the faults can cause catastrophic accidents if not detected in time. Therefore, fault detection and isolation (FDI) techniques are of practical significance.

Faults are classified, according to their physical locations, into system faults, actuator faults and sensor faults. In recent decades, the study of FDI has made many significant advances [40, 50, 78, 150, 223]. Compared with actuators, sensors are passive elements in the sense that they only provide operational information about the system, and do not affect the system behaviour directly, and thus have been less studied when compared with the study of actuator FDI.

Obviously, autonomous systems, where the human operator is removed from the loop, are more dependent on the increasing numbers of sensors to acquire system information. This, in turn, makes systems more vulnerable to faults in sensors. The potential for faults in the sensors becomes even more critical when they are applied to the automatic control of a system, where the effects of malfunctions may be devastating [61].

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The main task of FDI is to indicate that something is wrong and determines which subsystem or component has a fault. The advancement of modelling techniques provides the possibility exploiting model-based FDI approaches which have been considered a very effective method for FDI both in theory and practice [20].

Various model-based approaches have been developed [20], and in particular observer-based techniques have obtained much attention. The comprehensive survey paper [49] provides an overview of observer-based approaches, and since then many results have been established (see e.g., [20, 50, 151, 194] and the references therein). Subsequently, some control inspired approaches, for instance, sliding mode techniques [40], modern differential geometric approaches [138] and adaptive control [223] have been successfully incorporated with the observer-based FDI approach. It should be noted that adaptive control is usually only powerful for overcoming linear parametric uncertainty whilst modern geometric approaches tend to require strong geometric conditions on the systems considered. Sliding mode techniques, however, have good robustness and are completely insensitive to so-called matched uncertainty [38, 174]. It has been shown that sliding mode techniques can be used to deal with both structured and mismatched uncertainty [200]. Therefore, the application of sliding mode techniques for robust FDI offers good potential.

This chapter considers fault detection and isolation (FDI) issues for nonlinear systems with uncertainties, using an equivalent output error injection approach. A particular design of sliding mode observer is presented for which the parameters can be obtained using LMI techniques. In Sect. 8.2, an estimation approach is presented to estimate system faults where the estimation error is dependent on the bounds on the uncertainty. For a special class of uncertainty, a fault reconstruction scheme is proposed where the reconstructed signal can approximate the fault signal to any accuracy even in the presence of the class of uncertainties. Section 8.3 considers sensor FDI for nonlinear systems. A nonlinear diffeomorphism is introduced to explore the system structure and a simple filter is presented to 'transform' the sensor fault into a pseudo-actuator fault scenario. A sliding mode observer is designed to reconstruct the sensor fault precisely if the system does not experience any uncertainty, and to estimate the sensor fault when uncertainty exists. Finally, case studies on a robotic arm system and a mass–spring system are presented to show the effectiveness of the proposed FDI approaches.

## 8.2 Nonlinear Robust Fault Reconstruction and Estimation

### *8.2.1 Introduction*

A sliding mode observer was used for FDI as early as 1993 [161]. More recently Edwards et al. [40] proposed an approach based on the concept of equivalent output injection in which the resulting reconstruction signal can approximate the fault to any required accuracy. This is called 'precise' fault reconstruction in this book. Later, it was extended by Tan and Edwards in [166] where sensor faults were considered. However, uncertainty was not considered in these early papers.

It is well known that the observer-based approach is very dependent on the system model. However in practice a precise and accurate model for a real system is often not available due to unknown exogenous disturbances and/or time-varying parameters (component ageing). Modelling uncertainty can cause false and missed alarms. Hence, it is very important to consider robustness when implementing FDI schemes.

An FDI scheme for a class of linear systems with uncertainty was proposed by Tan and Edwards [167] which focused on minimising the $\mathscr{L}_2$ gain between the uncertainty and the fault reconstruction signal by using linear matrix inequalities (LMI). Jiang et al. [85] proposed a fault estimation scheme for a class of systems with uncertainty but the proposed signal is only an estimate of the fault. A robust fault detection method for nonlinear systems with disturbances was considered in [48] where strict geometric conditions are exploited and the disturbance can effectively be considered as linear parametric uncertainty.

It should be emphasised that 'precise' fault reconstruction is very challenging for nonlinear systems especially in the presence of uncertainty. The notion of 'precise' fault reconstruction has been considered in [40, 166] for systems which are linear without uncertainty. When uncertainty is considered, all the results concerning sliding mode observer-based fault reconstruction only provide an estimate of the fault signal. To establish an approach for fault reconstruction in nonlinear systems or to find conditions under which 'precise' fault reconstruction is possible is valuable and meaningful. Moreover, since FDI is required to take place on-line in real engineering systems, this requires the reconstruction fault signal to be based only on the available measured information.

In this section, a class of nonlinear uncertain systems is considered where the uncertainty is allowed to have a nonlinear bound. In order to reduce the conservativeness, appropriate coordinate transformations are introduced to exploit the system structure. A sufficient condition based on LMIs is presented for the existence and stability of a robust sliding mode observer. Then, fault estimation and fault reconstruction methods are presented using the equivalent output injection approach proposed by Edwards et al. [40]. It is shown that under certain geometric conditions associated with the uncertainty structure matrix and the fault distribution matrix, 'precise' fault reconstruction is available for a class of nonlinear systems by exploiting the features of the sliding motion and the structure of the uncertainty. The proposed reconstruction signal converges to the fault with arbitrary accuracy even in the presence of uncertainty. If the geometric condition does not hold, then a strategy is presented to estimate the fault signal, where the estimation error depends on the bounds on the uncertainty. An optimization procedure is presented which provides a tight bound of the effect of the uncertainty on the estimate. The proposed sliding mode observer design procedure is constructive and the design parameters can be obtained using LMI techniques. The associated reconstruction and estimation signals are only based on the plant input and output information which can be obtained on-line. This makes the FDI scheme practically implementable.

### *8.2.2 System Analysis and Preliminaries*

Consider a nonlinear system described by

$$\dot{x} = Ax + G(x, u) + E\Psi(x, u, t) + Df(y, u, t) \tag{8.1}$$

$$y = Cx, \tag{8.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state variables, inputs and outputs respectively; $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times r}$, $D \in \mathbb{R}^{n\times q}$ and $C \in \mathbb{R}^{p\times n}$ $(q \le p < n)$ are constant matrices with $D$ and $C$ both full rank, the known nonlinear term $G(x, u)$ is Lipschitz with respect to $x$ uniformly for $u \in \mathscr{U}$ (here $\mathscr{U}$ is an admissible control set), and the unknown nonlinear term $\Psi(x, u, t)$ represents modelling uncertainties and disturbances experienced by the system. The unknown function $f(y, u, t) \in \mathbb{R}^q$ represents an actuator fault which satisfies

$$\|f(y, u, t)\| \le \rho(y, u, t) \tag{8.3}$$

where the function $\rho(y, u, t)$ is known. All the functions involved in the system (8.1)–(8.2) are assumed to be continuous in their arguments. Furthermore, it is assumed that the faults are associated with the actuators of the system—hence the direct dependence of the signal $f(\cdot)$ on the control signal $u(t)$. It will be assumed that the system in (8.1)–(8.2) is under feedback control and the signals $u(t)$ are (smooth) functions of the states $x(t)$. In the absence of faults, it is assumed that the controller has been well designed so that $x(t)$ is close to its required operating point. If a fault occurs it is assumed that $x(t)$ lies in a bounded compact set for at least a finite time $t_f > 0$, starting from the onset of the fault, which allows time for detection to take place.

First, some basic assumptions will be imposed on the system (8.1)–(8.2).

**Assumption 8.1** There exists a known function $\xi(x, u, t)$ which is Lipschitz about $x$ uniformly for $t \in \mathbb{R}^+$ and $u \in \mathscr{U}$ such that

$$\|\Psi(x, u, t)\| \le \xi(x, u, t). \tag{8.4}$$

**Assumption 8.2** $\text{rank}(C[E \;\; D]) = \text{rank}([E \;\; D])$.

*Remark 8.1* In Assumption 8.1, the bound on the uncertain term $\Psi(x, u, t)$ takes a more general nonlinear form (as in [200, 214]) when compared with the work in [85, 167]. Assumption 8.2 is a limitation on the uncertain distribution matrix and the fault distribution matrix, and implies that $\text{rank}([E \;\; D]) \le p$.

From Lemma D.1 in Appendix D, it can be assumed without loss of generality that System (8.1)–(8.2) has the form

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + G_1(x, u) \tag{8.5}$$

$$\dot{x}_2 = A_3 x_1 + A_4 x_2 + G_2(x, u) + E_2 \Psi(x, u, t) + D_2 f(y, u, t) \tag{8.6}$$

$$y = C_2 x_2 \tag{8.7}$$

where $x = \mathrm{col}(x_1, x_2)$ with $x_1 \in \mathbb{R}^{n-p}$, $G_1(x, u)$ and $G_2(x, u)$ are the first $n-p$ and the last $p$ components of $G(x, u)$ respectively, and $E_2$ and $D_2$ are defined in (D.2).

**Assumption 8.3** All the invariant zeros of the matrix triple $(A, [E \;\; D], C)$ lie in the open left-half plane.

The Assumptions 8.2 and 8.3 amount to relative degree one minimum phase conditions, which are necessary for the design when the system experiences uncertainty or disturbance. For non-square systems, generally $(A, [E \;\; D], C)$ will not possess any invariant zeros and so Assumption 8.3 will be trivially satisfied for many systems.

As pointed out in Remark D.1, the coordinate transformation used to obtain the regular form from Lemma D.1 in Appendix D can be obtained directly from matrix theory. Therefore, System (8.5)–(8.7) is well defined and available from System (8.1)–(8.2) by using basic linear algebra. The following analysis will focus on the system (8.5)–(8.7) which has an amenable structure for subsequent developments.

### *8.2.3 A Sliding Mode Observer Design*

In this section, a robust sliding mode observer will be proposed using the system structure characteristics shown in Sect. 8.2.2.

Consider System (8.5)–(8.7). Introduce a coordinate transformation $z = Tx$ where

$$T := \begin{bmatrix} I_{n-p} & L \\ 0 & I_p \end{bmatrix} \tag{8.8}$$

where $L$ has the structure given in (D.8) in Appendix D. Then, from the analysis in Sect. 8.2, it follows that in the new coordinate System $z$, System (8.5)–(8.7) has the following form

$$\begin{aligned} \dot{z}_1 &= (A_1 + LA_3) z_1 + (A_2 + LA_4 - (A_1 + LA_3)L)\, z_2 \\ &\quad + \left[ I_{n-p} \;\; L \right] G(T^{-1} z, u) \end{aligned} \tag{8.9}$$

$$\begin{aligned} \dot{z}_2 &= A_3 z_1 + (A_4 - A_3 L)\, z_2 + G_2(T^{-1} z, u) + E_2 \Psi(T^{-1} z, u, t) \\ &\quad + D_2 f(y, u, t) \end{aligned} \tag{8.10}$$

$$y = C_2 z_2 \tag{8.11}$$

where $z := \mathrm{col}(z_1, z_2)$ with $z_1 \in \mathbb{R}^{n-p}$. The structure in (8.9) occurs because

$$LE_2 = 0 \quad \text{and} \quad LD_2 = 0$$

from (D.8) and (D.2) in Appendix D.

For System (8.9)–(8.11), consider a dynamical system

$$\begin{aligned}\dot{\hat{z}}_1 &= (A_1 + LA_3)\hat{z}_1 + (A_2 + LA_4 - (A_1 + LA_3)L)\, C_2^{-1} y \\ &\quad + \left[\, I_{n-p}\ L \,\right] G(T^{-1}\hat{z}, u) \qquad (8.12)\\ \dot{\hat{z}}_2 &= A_3\hat{z}_1 + (A_4 - A_3 L)\,\hat{z}_2 - K(y - C_2\hat{z}_2) + G_2(T^{-1}\hat{z}, u) \\ &\quad + \nu(t, u, y, \hat{y}, \hat{z}) \qquad (8.13)\\ \hat{y} &= C_2\hat{z}_2 \qquad (8.14)\end{aligned}$$

where $\hat{z} := \mathrm{col}(\hat{z}_1, C_2^{-1} y)$ and $\hat{y}$ is the output of the dynamical system. Note that $\hat{z}$ *does not* represent the state estimate $\mathrm{col}(\hat{z}_1, \hat{z}_2)$. It is merely used as a piece of convenient notation in the developments which follow. The gain matrix $K$ is chosen such that

$$C_2(A_4 - A_3 L)C_2^{-1} + C_2 K$$

is symmetric negative definite (clearly this is always possible since $C_2$ is nonsingular). The function $\nu$ is defined by

$$\nu := k(\cdot)C_2^{-1}\frac{y - \hat{y}}{\|y - \hat{y}\|}, \qquad \text{if } \ y - \hat{y} \neq 0 \qquad (8.15)$$

where $k(\cdot)$ is a positive scalar function to be determined later.

Let

$$e_1 = z_1 - \hat{z}_1, \qquad \text{and} \qquad e_y = y - \hat{y} = C_2(z_2 - \hat{z}_2).$$

Then from (8.9)–(8.11) and (8.12)–(8.14), the state estimation error dynamical system is described by

$$\begin{aligned}\dot{e}_1 &= (A_1 + LA_3)e_1 + \left[\, I_{n-p}\ L \,\right]\Big(G(T^{-1}z, u) - G(T^{-1}\hat{z}, u)\Big) \qquad (8.16)\\ \dot{e}_y &= C_2 A_3 e_1 + \Big(C_2(A_4 - A_3 L)C_2^{-1} + C_2 K\Big)\, e_y + C_2\big(G_2(T^{-1}z, u) \\ &\quad - G_2(T^{-1}\hat{z}, u)\big) + C_2 E_2 \Psi(T^{-1}z, u, t) + C_2 D_2 f(y, u, t) - C_2\nu \qquad (8.17)\end{aligned}$$

where $\hat{z} = \mathrm{col}(\hat{z}_1, C_2^{-1} y)$ and $\nu$ are defined by (8.15).

*Remark 8.2* In Eq. (8.13), a gain matrix $K \in \mathbb{R}^{p\times p}$ is introduced to guarantee that the nominal linear system matrix of the state estimation error dynamical system (8.16)–(8.17) given by

$$\begin{bmatrix} A_1 + LA_3 & 0 \\ C_2 A_3 & C_2(A_4 - A_3 L)C_2^{-1} + C_2 K \end{bmatrix} \qquad (8.18)$$

is stable. It is obvious from the structure (8.18) that such a $K$ always exists since $A_1 + LA_3$ is stable under Assumption 8.3, and $C_2$ is nonsingular.

From (8.8) to (8.11), it follows that

$$T^{-1}z - T^{-1}\hat{z} = \begin{bmatrix} I_{n-p} & -L \\ 0 & I_p \end{bmatrix} \begin{bmatrix} z_1 - \hat{z}_1 \\ z_2 - C_2^{-1}y \end{bmatrix} = \begin{bmatrix} e_1 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad \left\| T^{-1}z - T^{-1}\hat{z} \right\| = \|e_1\|. \tag{8.19}$$

For System (8.16)–(8.17), consider a sliding surface

$$S = \left\{ (e_1, e_y) \mid e_y = 0 \right\}. \tag{8.20}$$

Then, the following conclusion is ready to be presented:

**Proposition 8.1** *Under Assumptions 8.1–8.3, the sliding motion of System (8.16)–(8.17) associated with the surface (8.20) is asymptotically stable if the matrix inequality*

$$\bar{A}^T \bar{P}^T + \bar{P}\bar{A} + \frac{1}{\varepsilon}\bar{P}\bar{P}^T + \varepsilon\,(\mathscr{L}_G)^2\, I_{n-p} + \alpha P < 0 \tag{8.21}$$

*is solvable for $\bar{P}$ where*

$$\bar{P} := P \left[\, I_{n-p} \;\; L \,\right], \qquad \bar{A} := \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} \tag{8.22}$$

*with $P > 0$, $\varepsilon$ and $\alpha$ are positive constants, $\mathscr{L}_G$ is the Lipschitz constant for $G(x, u)$ with respect to $x$ and the matrix $L$ has the structure (D.8) given in Appendix D.*

*Proof* The analysis above has shown that (8.16) represents the sliding dynamics when restricted to the sliding surface (8.20). Therefore, it is only required to prove that (8.16) is asymptotically stable.

Consider a Lyapunov candidate function

$$V = e_1^T P e_1.$$

The time derivative of $V$ along the trajectories of System (8.16) is given by

$$\begin{aligned} \dot{V}\,|_{(8.16)} &= e_1^T \Big( (A_1 + LA_3)^T P + P(A_1 + LA_3) \Big) e_1 + 2e_1^T P \left[\, I_{n-p} \;\; L \,\right] \\ &\quad \cdot \Big( G(T^{-1}z, u) - G(T^{-1}\hat{z}, u) \Big) \\ &= e_1^T \Big( \bar{A}^T \bar{P}^T + \bar{P}\bar{A} \Big) e_1 + 2\left(\bar{P}^T e_1\right)^T \Big( G(T^{-1}z, u) - G(T^{-1}\hat{z}, u) \Big). \end{aligned}$$

From the well known inequality that $2X^T Y \le \varepsilon X^T X + \frac{1}{\varepsilon} Y^T Y$ for any scalar $\varepsilon > 0$, it follows that

$$\dot{V}\mid_{(8.16)} \leq e_1^T\Big(\bar{P}\bar{A}+\bar{A}^T\bar{P}^T\Big)e_1+\varepsilon e_1^T\bar{P}\bar{P}^Te_1 \\ +\frac{1}{\varepsilon}\Big(G(T^{-1}z,u)-G(T^{-1}\hat{z},u)\Big)^T\Big(G(T^{-1}z,u)-G(T^{-1}\hat{z},u)\Big).$$

From (8.19),

$$\|G(T^{-1}z,u)-G(T^{-1}\hat{z},u)\| \leq \mathscr{L}_G\|e_1\|. \tag{8.23}$$

Consequently

$$\begin{aligned}\dot{V}\mid_{(8.16)} &\leq e_1^T\Big(\bar{P}\bar{A}+\bar{A}^T\bar{P}^T\Big)e_1+\varepsilon e_1^T\bar{P}\bar{P}^Te_1+\frac{1}{\varepsilon}(\mathscr{L}_G)^2\|e_1\|^2 \\ &= e_1^T\Big(\bar{P}\bar{A}+\bar{A}^T\bar{P}^T+\varepsilon\bar{P}\bar{P}^T+\frac{1}{\varepsilon}(\mathscr{L}_G)^2 I\Big)e_1 \\ &\leq -\alpha e_1^T P e_1 = -\alpha V\end{aligned} \tag{8.24}$$

where (8.21) has been used to obtain the last inequality. #

*Remark 8.3* Note that Inequality (8.21) can be transformed into the following LMI problem: for a given scalar $\alpha > 0$, find matrices $P$ and $Y$ and a scalar $\varepsilon$ such that

$$\begin{bmatrix} PA_1+A_1^TP+YA_3+A_3^TY^T+\alpha P+\varepsilon(\mathscr{L}_G)^2 & P & Y \\ P & -\varepsilon I_{n-p} & 0 \\ Y^T & 0 & -\varepsilon I_p\end{bmatrix} < 0 \tag{8.25}$$

where $Y := PL$ with $P > 0$, which can be solved by LMI techniques. If $\mathscr{L}_G$ is known, then for a given $\alpha$, the problem of finding $P$, $Y$ and $\varepsilon$ to satisfy (8.25) is a standard LMI feasibility problem. Alternatively, an optimization problem can be posed which is to find $P$, $Y$ and $\varepsilon$ which maximises $\mathscr{L}_G$ in (8.25). This is a convex eigenvalue optimization problem and can be solved using standard LMI algorithms [56].

Since $\dot{V}(t) \leq -\alpha V(t)$ in (8.24), it follows that there exists a positive scalar $M$ such that

$$\|e_1(t)\| \leq M\|e_1(0)\|\exp\left\{-\frac{\alpha}{2}t\right\} \tag{8.26}$$

where a choice is $M := \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$. Based on Inequality (8.26), introduce a dynamic system given by

$$\dot{\hat{w}}(t) = -\frac{1}{2}\alpha\hat{w}(t). \tag{8.27}$$

Then, for any value $e_1(0)$, choose $\hat{w}(0)$ such that

$$\|e_1(0)\| \leq \hat{w}(0)/M.$$

Then, it is straightforward to see that the available solution $\hat{w}(t)$ to Eq. (8.27) is an upper bound on the size of the corresponding state estimation error $e_1(t)$; specifically

$$\|e_1(t)\| \leq \hat{w}(t), \qquad \forall t \geq 0.$$

Proposition 8.1 has shown that the sliding mode associated with the sliding surface $S$ given in (8.20) is stable if the matrix inequality (8.21) is solvable. The objective is now to determine the scalar gain function $k(\cdot)$ in (8.15) such that the system can be driven to the surface $S$ in finite time and a sliding motion can be maintained.

The following conclusion is ready to be presented:

**Proposition 8.2** *Under Assumptions 8.1–8.3, System (8.16)–(8.17) is driven to the sliding surface (8.20) in finite time and remain on it if the gain $k(\cdot)$ in (8.15) is chosen to satisfy*

$$k(t,u,y,\hat{z}) \geq (\|C_2A_3\| + \|C_2\|\mathscr{L}_G + \|C_2E_2\|\mathscr{L}_\xi)\hat{w}(t) + \|C_2E_2\|\xi(T^{-1}\hat{z},u,t) \\ + \|C_2D_2\|\rho(y,u,t) + \eta \tag{8.28}$$

*where $\eta$ is a positive constant and $\hat{w}$ is the solution to the differential equation (8.27).*

*Proof* Let $\widetilde{V}(e_y) = e_y^T e_y$. From the expression for the output estimation error in (8.17), it follows that

$$\dot{\widetilde{V}}(e_y) = e_y^T\left(C_2(A_4 - A_3L)C_2^{-1} + C_2K + \left(C_2(A_4 - A_3L)C_2^{-1} + C_2K\right)^T\right)e_y \\ +2e_y^T\Big(C_2A_3e_1 + C_2\left(G_2(T^{-1}z,u) - G_2(T^{-1}\hat{z}_y,u)\right) + C_2E_2\Psi(T^{-1}z,u,t) \\ +C_2D_2f(y,u,t) - C_2\nu\Big). \tag{8.29}$$

Since by design the matrix

$$C_2(A_4 - A_3L)C_2^{-1} + C_2K$$

is symmetric negative definite, it follows that

$$\left(C_2(A_4 - A_3L)C_2^{-1} + C_2K\right)^T + C_2(A_4 - A_3L)C_2^{-1} + C_2K < 0. \tag{8.30}$$

By applying (8.3) and (8.30) to (8.29), it follows from (8.19) that

$$\dot{\widetilde{V}} \leq 2\|e_y\|\,(\|C_2A_3\| + \|C_2\|\mathscr{L}_G)\,\|e_1\| + 2\|e_y\|\Big(\|C_2E_2\|\big(\xi(T^{-1}\hat{z},u,t) \\ +\mathscr{L}_\xi\|e_1\|\big) + \|C_2D_2\|\rho(y,u,t)\Big) - 2e_y^TC_2\nu. \tag{8.31}$$

From the arguments above $\|e_1\| \leq \hat{w}$, substituting the $\nu$ given in (8.15) into (8.31)

$$\dot{\widetilde{V}} = 2\|e_y\|\Big((\|C_2A_3\| + \|C_2\|\mathscr{L}_G)\hat{w} + \|C_2E_2\| \left(\xi(T^{-1}\hat{z}, u, t) + \mathscr{L}_\xi\hat{w}\right) \\ +\|C_2D_2\|\rho(\cdot)\Big) - 2k(\cdot)\|e_y\|. \tag{8.32}$$

From (8.28) and (8.32) it follows that

$$\dot{\widetilde{V}} \leq -2\eta\|e_y\| \leq -2\eta\widetilde{V}^{\frac{1}{2}}.$$

This shows that a reachability condition is satisfied (see [174] for example). It follows that $\widetilde{V} = 0$ in finite time and consequently a sliding motion is achieved and maintained after some finite time $t_s > 0$. Hence the conclusion follows. #

From sliding mode control theory, it follows that Propositions 8.1 and 8.2 have shown that (8.12)–(8.13) are sliding mode observer of System (8.9)–(8.11), where $\hat{y}$ defined by (8.14) is called the observer output which will be used in the FDI.

### 8.2.4 Robust Fault Detection and Estimation

It is assumed in this section that the sliding mode observer given in Sect. 8.2.3 has been designed. The objective is to reconstruct/estimate the system fault by using the observer output and the estimated states given by the observer.

The following fault reconstruction results can be obtained using Lemma D.3 presented in Appendix D:

**Theorem 8.1** *Consider System (8.1)–(8.2) satisfying Assumptions 8.1–8.3. Assume that the matrix inequality (8.21) is solvable and $k(\cdot)$ is chosen to satisfy (8.28). Then*

*(i) there exists a continuous function $d(t)$ satisfying $\lim_{t\to\infty} d(t) = 0$ such that*

$$\left\|\hat{f}(t) - f(y(t), u(t), t)\right\| \leq \left\|D_2^+E_2\right\| \xi(x, u, t) + d(t) \tag{8.33}$$

*where $D_2^+$ is any left pseudo inverse of $D_2$ (which exists since $D_2$ is full column rank), and*

$$\hat{f}(t) = k(\cdot)D_2^+C_2^{-1}\frac{y - \hat{y}}{\|y - \hat{y}\| + \sigma_1\exp\{-\sigma_2 t\}} \tag{8.34}$$

*where $\sigma_1$ and $\sigma_2$ are both positive constants;*

*(ii) $\lim_{t\to\infty}\left\|\tilde{f}(t) - f(y(t), u(t), t)\right\| = 0$ if $Im(E_{22}) \cap Im(D_{22}) = \{0\}$, where*

$$\tilde{f}(t) = k(\cdot)H_2^{-1}W_2\left[\,0_{\tilde{q}\times(p-\tilde{q})}\;\; I_{\tilde{q}}\,\right]C_2^{-1}\frac{y - \hat{y}}{\|y - \hat{y}\| + \sigma_1\exp\{-\sigma_2 t\}} \tag{8.35}$$

*where $W_2$ denotes the last $q$ rows of matrix $W$ in (D.9) in Appendix D, and $\sigma_1$ and $\sigma_2$ are both positive constants.*

*Proof* From the features of the sliding mode observer designed in Sect. 8.2.3, it follows that in finite time the error dynamics (8.16)–(8.17) will be driven to the sliding surface (8.20) and a sliding motion is maintained thereafter. During the sliding motion

$$e_y = 0 \quad \text{and} \quad \dot{e}_y = 0. \tag{8.36}$$

Since $C_2$ is nonsingular, it follows from (8.17) and (8.36) that

$$\phi\left(e_1, z, \hat{z}, u\right) + E_2\Psi(T^{-1}z, u, t) + D_2 f(y, u, t) - \nu_{eq} = 0 \tag{8.37}$$

where $\nu_{eq}$ denotes the equivalent output error injection signal required to maintain the sliding motion [38, 174], and $\phi(\cdot)$ is defined by

$$\phi\left(e_1, z, \hat{z}, u\right) := A_3 e_1 + G_2(T^{-1}z, u) - G_2(T^{-1}\hat{z}, u). \tag{8.38}$$

From (8.23), it follows that

$$\|\phi\left(e_1, z, \hat{z}, u\right)\| \le (\|A_3\| + \mathscr{L}_{G_2})\|e_1(t)\| \to 0 \qquad (t \to \infty) \tag{8.39}$$

since $\lim_{t\to\infty} e_1(t) = 0$. In order to reconstruct/estimate the fault signal $f(y, u, t)$, it is necessary to recover the equivalent output error injection signal $\nu_{eq}$. Here the approach given in Edwards et al. [40] will be employed to produce $\nu_{eq}$. From (8.15), the equivalent output error injection signal $\nu_{eq}$ in (8.37) can be approximated by

$$\nu_{eq} \simeq k(\cdot)C_2^{-1}\frac{y - \hat{y}}{\|y - \hat{y}\| + \sigma_1 \exp\{-\sigma_2 t\}} \tag{8.40}$$

where $\sigma_1$ and $\sigma_2$ are both positive constants.

(i) Multiplying both sides of Eq. (8.37) by the pseudo inverse $D_2^+$, it follows from $D_2^+ D_2 = I_q$ that

$$f(y, u, t) = D_2^+\nu_{eq} - D_2^+ E_2\Psi(T^{-1}z, u, t) - D_2^+\phi(e_1, z, \hat{z}, u). \tag{8.41}$$

Then, substituting the arbitrarily close approximation to $\nu_{eq}$ from (8.40) into the equation above, it follows from Assumption 8.1 and the definition of $\hat{f}$ in (8.34) that

$$\begin{aligned}\left\|\hat{f}(t) - f(y, u, t)\right\| &\le \|D_2^+ E_2\|\,\|\Psi(T^{-1}z, u, t)\| + d(t)\\ &\le \|D_2^+ E_2\|\xi(x, u, t) + d(t)\end{aligned}$$

where

$$d(t) := \|D_2^+\phi(e_1(t), z(t), \hat{z}(t), u(t))\|$$

is continuous in $\mathbb{R}^+$. From (8.39), obviously $\lim_{t\to\infty} d(t) = 0$.

(ii) By applying the structural properties of $E_2$ and $D_2$ in (D.2) given in Appendix D.1 to (8.37), it follows that

$$\nu_{eq2} = \phi_2(e_1, z, \hat{z}, u) + \begin{bmatrix} E_{22} & D_{22} \end{bmatrix} \begin{bmatrix} \Psi(T^{-1}z, u, t) \\ f(y, u, t) \end{bmatrix} \tag{8.42}$$

where $\nu_{eq2}$ and $\phi_2$ denote the last $\tilde{q}$ components of $\nu_{eq}$ and $\phi$, respectively. Since by assumption

$$\mathrm{Im}(E_{22}) \cap \mathrm{Im}(D_{22}) = \{0\}$$

it follows from Lemma D.3 in Appendix D that (D.9) holds. Multiplying both sides of (8.42) by $W$ yields

$$W\nu_{eq2} = W\phi_2(e_1, z, \hat{z}, u) + \begin{bmatrix} H_1\Psi(T^{-1}z, u, t) \\ H_2 f(y, u, t) \end{bmatrix}. \tag{8.43}$$

Let $W_2$ denote the last $q$ rows of $W$. It follows from (8.43) that

$$\begin{aligned} f(y, u, t) &= H_2^{-1} W_2 \left(\nu_{eq2} - \phi_2(e_1, z, \hat{z}, u)\right) \\ &= H_2^{-1} W_2 \Big( \begin{bmatrix} 0_{\tilde{q}\times(p-\tilde{q})} & I_{\tilde{q}} \end{bmatrix} \nu_{eq} - \phi_2(e_1, z, \hat{z}, u) \Big). \end{aligned}$$

Then, from (8.40) and the definition of $\tilde{f}$ in (8.35),

$$\begin{aligned} \tilde{f}(t) := k(\cdot) H_2^{-1} W_2 \begin{bmatrix} 0_{\tilde{q}\times(p-\tilde{q})} I_{\tilde{q}} \end{bmatrix} C_2^{-1} \frac{y - \hat{y}}{\|y - \hat{y}\| + \sigma_1 \exp\{-\sigma_2 t\}} \\ \to f(y, u, t) \end{aligned}$$

when $t \to \infty$. Hence the conclusion follows. #

It should be noted that the parameters $\sigma_1$ and $\sigma_2$ in (8.34) and (8.35) determine the degree of approximation to ideal sliding which is attained. Typically, $\sigma_1$ would be small while $\sigma_2$ would usually be large.

*Remark 8.4* From (8.40), it is clear that $\nu_{eq}$ depends only on known system information: the system output $y$, the observer output $\hat{y}$ and the system input $u$. From (8.34) and (8.35), the estimated signal $\hat{f}$ and the reconstructed signal $\tilde{f}$ are both only dependent on available information and thus the fault detection schemes can be implemented on-line.

*Remark 8.5* In the precise reconstruction situation, detection is inherent in the approach since the reconstruction signal reflects the fault faithfully. When precise reconstruction is not possible, detection is more difficult since $\hat{f}(t)$ may become nonzero as a result of the uncertainty and not because faults are present. However, provided the size of the error bounds $\|D^{+}E_2\Psi(\cdot)\|$ are small compared to the size of the faults that need to be detected, then by setting appropriate thresholds, a level of detection (and isolation) can still be achieved.

Theorem 8.1 shows that $\tilde{f}$ is a precise reconstruction of the fault $f$ if

$$\mathrm{Im}(E_{22}) \cap \mathrm{Im}(D_{22}) = \{0\}.$$

If this condition is not satisfied, then an estimation signal $\hat{f}$ is available to estimate the fault $f$. From (8.33), one way to reduce the effect of the uncertainty on the fault estimation signal is to choose an appropriate matrix $D_2^+$ such that $\|D_2^+ E_2\|$ is minimised. Let

$$\mathscr{D} = \left\{ X \in \mathbb{R}^{q \times p} \mid X D_2 = I_q \right\}.$$

It follows that $\mathscr{D}$ can be parameterised as

$$\mathscr{D} = \left\{ (D_2^T D_2)^{-1} D_2^T + \mathscr{M} D_2^N) \mid \mathscr{M} \in \mathbb{R}^{q \times (p-q)} \right\}$$

where $D_2^N \in \mathbb{R}^{(p-q) \times p}$ is such that the columns of $(D_2^N)^T$ span the null-space of $D_2^T$ which implies that

$$D_2^N D_2 = 0.$$

Consequently, for any $D_2^+ \in \mathscr{D}$,

$$D_2^+ E_2 = (D_2^T D_2)^{-1} D_2^T E_2 + \mathscr{M} D_2^N E_2$$

and so the objective is to solve the following optimization problem

$$\min_{\mathscr{M} \in \mathbb{R}^{q \times (p-q)}} \left\{ \left\| \left( D_2^T D_2 \right)^{-1} D_2^T E_2 + \mathscr{M} D_2^N E_2 \right\| \right\}.$$

This is well defined and can be easily solved using an LMI optimization approach (Alternatively, the analytic theory as described in (p. 43, [228]) could be employed).

## 8.3 Sensor FDI for a Class of Nonlinear Systems

### *8.3.1 Introduction*

Sensor faults are incorrect readings due to malfunctions in the sensor components or transducers, such as broken wires, resulting in the loss of effectiveness, or more subtly, unknown biases at the sensor outputs as a result of poor calibration or even unexpected changes in the dynamic characteristics of the transducers. Since the signals from sensors often carry the most important information in automated/feedback control systems, the state of health of the sensors is, therefore, very important for the reliable operation of the entire system. This has motivated the study of sensor FDI.

Sensor redundancy [17] is an obvious solution to the sensor fault problem, where multiple sensors are installed to measure the same quantity. The main problem of this approach is the extra equipment and maintenance costs and the additional space required to accommodate the equipment. In [222], an isolation scheme for sensor faults is proposed using an adaptive estimator. A sensor fault FDI strategy for a linear discrete time system was discussed in [103] using a structural vector-based approach. By using sliding mode techniques, continuous time systems were considered in [40, 166, 167] where it is required that the systems are linear. However, most real systems are more accurately modelled by nonlinear equations.

It is well known that one approach for dealing with nonlinear systems is to linearize around some operating point by using approximation techniques [21, 104]. However, the linear system obtained in this way is valid only in a neighbourhood of the operating point and tends to suffer from poor detection or high false alarm rates due to the error of approximation. Furthermore, when a large region of the state space is required to be considered, the linearisation method may not be sufficient. Therefore, it is necessary to study nonlinear systems.

In this section, sensor FDI is studied for a class of nonlinear systems with uncertainty. The sensor fault considered is modelled as an additive fault. A diffeomorphism is first used to explore the system structure and no approximation is employed. By designing an appropriate filter, the sensor fault can be modelled as a pseudo-actuator fault. Then, using the transformed system structure and the characteristics of the designed filter, a sliding mode observer is presented to reconstruct the sensor fault precisely if no uncertainty exists in the system. A sensor fault estimation scheme is also proposed when the system is affected by uncertainty, in which case the estimation error depends on the bound on the uncertainty.

The reconstruction/estimation schemes which are proposed in this section can be implemented online. It is not required that the system is linear/linearizable, and the minimum phase limitation required in [40, 198] is removed. Therefore, this work is applicable to a wide-class of systems.

### 8.3.2 Problem Formulation

Consider a nonlinear system described by

$$\dot{x}(t) = F(x(t), u(t)) + \Delta F(x(t)) \tag{8.44}$$

$$y(t) = h(x(t)) + Df_s(t), \qquad x_0 = x(0) \tag{8.45}$$

where $x \in \Omega \subset \mathbb{R}^n$ (and $\Omega$ is a neighbourhood of $x_0$), $u = \text{col}(u_1, u_2, \ldots, u_m) \in \mathscr{U} \in \mathbb{R}^m$, and $y = \text{col}(y_1, y_2, \ldots, y_p) \in \mathbb{R}^p$ are the state variables, inputs and outputs, respectively, where $\mathscr{U}$ is an admissible control set. $F(x, u)$ is a known smooth vector field in $\Omega \times \mathcal{U}$ and the known function $h : \Omega \mapsto \mathbb{R}^p$ is smooth; $D \in \mathbb{R}^{p\times q}$ ($q \le p$) is a known sensor fault distribution matrix which is full column rank; the unknown vector function $\Delta F(x(t))$ models all the uncertainties and disturbances affecting the system and $f_s(t) \in \mathbb{R}^q$ is a sensor fault satisfying

$$\|f_s(t)\| \le \rho(t) \tag{8.46}$$

where $\rho(t)$ is a known continuous function. It is assumed that $f_s$ is unknown and $f_s(t) = 0$ when there is no fault. Therefore, the function $f_s(\cdot)$ is defined in $t \in \mathbb{R}^+$.

In this work, the fact that $\mathcal{U}$ is an admissible control set means that for any $u(t) \in \mathcal{U}$, the corresponding closed-loop system (8.44) has a unique solution lying in the domain $\Omega$.

**Definition 8.1** Consider System (8.44)–(8.45). The differential and algebraic equations

$$\dot{x}(t) = F(x(t), u(t)) \tag{8.47}$$

$$y(t) = h(x(t)), \qquad x_0 = x(0) \tag{8.48}$$

are called the nominal system associated with (8.44)–(8.45).

For convenience, the nominal system (8.47)–(8.48) is also denoted by a vector pair $(F(x, u), h(x))$.

This section considers the problem of reconstructing (or estimating) the sensor faults $f_s(t)$ for System (8.44)–(8.45). A sliding mode observer will be established and then, based on the observer, a signal $\hat{f}$, which only depends on available information, will be given such that

(i) the function $\hat{f}_s$ is a precise reconstruction of the sensor fault $f_s(t)$, i.e.,

$$\lim_{t\to\infty} \|\hat{f}_s(t) - f_s(t)\| = 0$$

if there is no uncertainty;

(ii) the inequality

$$\|\hat{f}_s(t) - f_s(t)\| \le \xi(t)$$

holds if the system experiences some uncertainty, where $\xi(t)$ is the estimation error which usually depends on the bound on the uncertainty.

### *8.3.3 System Analysis and Assumptions*

In order to solve the problem proposed in Sect. 8.3.2, it is required to impose assumptions to the system considered. Then based on these assumptions, the system is transformed to a new system which facilitates the design.

**Assumption 8.4** The nonlinear systems represented by the pair $(F(x,u), h(x))$ has an uniform observability index $\{r_1, r_2, \ldots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in the domain $\Omega \times \mathscr{U}$.

Construct a nonlinear transformation $T : x \mapsto z$ as follows:

$$z_{i1} = h_i(x) \tag{8.49}$$
$$z_{i2} = L_{F(x,u)} h_i(x) \tag{8.50}$$
$$\vdots$$
$$z_{ir_i} = L_{F(x,u)}^{r_i-1} h_i(x) \tag{8.51}$$

where $z_i := \mathrm{col}(z_{i1}, z_{i2}, \ldots, z_{ir_i})$ for $i = 1, 2, \ldots, p$ and $z := \mathrm{col}(z_1, z_2, \ldots, z_p)$.

*Remark 8.6* Under Assumption 8.4, it follows from Definition 2.8 that $M(x,u)$ has rank $p$ in $\Omega \times \mathscr{U}$, implying all the $z_i$ are independent of the control $u$, which combined with the restriction $\sum_{i=1}^{p} r_i = n$ means the corresponding Jacobian matrix of $T(x)$ is nonsingular. Therefore, equations (8.49)–(8.51) are a diffeomorphism in the domain $\Omega$, and $z = \mathrm{col}(z_1, z_2, \ldots, z_p)$ forms a new coordinate system which can be obtained by direct computation from (8.49)–(8.51).

Since $L_{F(x,u)}^{j} h_i(x)$ is independent of $u$ for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i - 1$, it follows by direct computation that for $i = 1, 2, \ldots, p$

$$\begin{aligned}
\dot{z}_{i1} &= \tfrac{\partial h_i}{\partial x} F(x,u) = L_{F(x,u)} h_i(x) = z_{i2} \\
\dot{z}_{i2} &= \tfrac{\partial \left(L_{F(x,u)} h_i(x)\right)}{\partial x} F(x,u) = L_{F(x,u)}^{2} h_i(x) = z_{i3} \\
&\vdots \\
\dot{z}_{ir_{i-1}} &= L_{F(x,u)}^{r_i-1} h_i(x) = z_{ir_i} \\
\dot{z}_{ir_i} &= L_{F(x,u)}^{r_i} h_i(x).
\end{aligned}$$

Therefore, in the new coordinates $z$ defined by (8.49)–(8.51), the system (8.44)–(8.45) has the following form

$$\dot{z} = Az + B\Phi(z,u) + \Psi(z) \tag{8.52}$$
$$y = Cz + Df_s(t) \tag{8.53}$$

where

$$A = \mathrm{diag}\{A_1, \ldots, A_p\}, \quad B = \mathrm{diag}\{B_1, \ldots, B_p\} \quad \text{and} \quad C = \mathrm{diag}\{C_1, \ldots, C_p\}$$

where $A_i \in \mathbb{R}^{r_i \times r_i}$, $B_i \in \mathbb{R}^{r_i \times 1}$ and $C_i \in \mathbb{R}^{1 \times r_i}$ for $i = 1, 2, \ldots, p$ are defined by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{8.54}$$

and

$$\Phi(z,u) := \begin{bmatrix} \phi_1(z,u) \\ \phi_2(z,u) \\ \vdots \\ \phi_p(z,u) \end{bmatrix} := \begin{bmatrix} L_{F(x,u)}^{r_1} h_1(x) \\ L_{F(x,u)}^{r_2} h_2(x) \\ \vdots \\ L_{F(x,u)}^{r_p} h_p(x) \end{bmatrix}_{x=T^{-1}(z)} \tag{8.55}$$

$$\Psi(z) := \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \\ \vdots \\ \psi_p(z) \end{bmatrix} := \left[ \frac{\partial T(x)}{\partial x} \Delta F(x) \right]_{x=T^{-1}(z)} \tag{8.56}$$

where $\phi_i : T(\Omega) \times \mathscr{U} \mapsto \mathbb{R}$ and $\psi_i : T(\Omega) \times \mathscr{U} \mapsto \mathbb{R}^{r_i}$ for $i = 1, 2, \ldots, p$.

*Remark 8.7* It should be noted that System (8.52)–(8.53) is still a nonlinear system but possesses a structure which is convenient for later analysis. In this section, it is not required that the system (8.44)–(8.45) is linearizable. It is also not required that the nonlinear function $\Phi(z,u)$ can be expressed as a function of $u$ and $y$ (in comparison with the work in [122]). Also, there is no approximation employed above and this makes the transformations valid in the whole domain $\Omega$ instead of just a small neighbourhood of $x_0$ as in [21, 104].

Choose the constants $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir_i}$ such that all the roots of the polynomial algebraic equations

$$\lambda^{r_i} + \alpha_{i(r_i-1)}\lambda^{r_i-1} + \cdots + \alpha_{i1}\lambda + \alpha_{i0} = 0 \tag{8.57}$$

lie in the open left-half plane for $i = 1, 2, \ldots, p$. Then, from (8.54), it follows that $(A - B\Lambda)$ is stable where

$$\Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2, \ldots, \Lambda_p\} \tag{8.58}$$

with $\Lambda_i \in \mathbb{R}^{1\times r_i}$ defined by

$$\Lambda_i = [\alpha_{i0}\ \alpha_{i1} \cdots \alpha_{i(r_i-1)}] \tag{8.59}$$

which satisfy (8.57) for $i = 1, 2, \ldots, p$.

**Assumption 8.5** The nonlinear function $\Phi(\cdot)$ in (8.55) can be expressed as

$$\Phi(z, u) = -\Lambda z + \Gamma(z, u) \tag{8.60}$$

where for any $z, \hat{z} \in T(\Omega)$ and $u \in \mathscr{U}$

$$\|\Gamma(z, u) - \Gamma(\hat{z}, u)\| \le \mathscr{L}(u)\|z - \hat{z}\| \tag{8.61}$$

where $\mathscr{L}(\cdot)$ is a continuous function defined on $\mathscr{U}$.

*Remark 8.8* Assumption 8.5 is a limitation on the nonlinear term $\Phi(\cdot)$. If the Jacobian matrix of $F(x, u)$ in (8.44), evaluated at $(x_0, u_0)$ $(u_0 \in \mathscr{U})$ is stable, then Assumption 8.5 is likely to be satisfied in a neighbourhood of $(x_0, u_0)$.

### *8.3.4 Sliding Mode Observer Synthesis*

In this section, the main results will be presented. The special case when $\Delta F(x, u) = 0$ is considered first, and the study of the uncertain system (8.44)–(8.45) when $\Delta F(x, u) \neq 0$ follows.

Suppose Assumption 8.4 holds. Then, from the analysis in Sect. 8.3.2, it follows that in the new coordinates $z$ defined by the diffeomorphism (8.49)–(8.51), System (8.44)–(8.45) can be described by (8.52)–(8.53). For System (8.52)–(8.53), the following linear filter is introduced

$$\dot{z}_a = A_a z_a + B_a y \tag{8.62}$$

where $z_a \in \mathbb{R}^p$ is the filter state, $A_a \in \mathbb{R}^{p\times p}$ and $B_a \in \mathbb{R}^{p\times p}$ are constant matrices which are design parameters to be defined later; and $y$ is the output of System (8.52)–(8.53). The matrix $A_a$ must be Hurwitz stable, but for simplicity in the subsequent analysis it will be assumed that $A_a$ is symmetric negative definite. This is not a stringent assumption since $A_a$ is a design parameter. Then, under Assumption 8.5, the following augmented system can be obtained

$$\dot{z} = (A - B\Lambda)z + B\Gamma(z, u) + \Psi(z) \tag{8.63}$$

$$\dot{z}_a = B_a C z + A_a z_a + B_a D f_s(t) \tag{8.64}$$

$$y_a = C_a z_a \tag{8.65}$$

where $z \in T(\Omega) \subset \mathbb{R}^n$, $C_a \in \mathbb{R}^{p\times p}$ is orthogonal (where one simple choice is to let $C_a = I_p$), $\Gamma(\cdot)$ is determined by (8.60) and finally $\Psi(\cdot)$ is defined in (8.56) and involves the uncertainty.

It is observed that the sensor fault in System (8.44)–(8.45) has been transformed into a pseudo-actuator fault in System (8.63)–(8.65). Now, consider the following dynamical system

$$\dot{\hat{z}} = (A - B\Lambda)\hat{z} + B\Gamma(\hat{z}, u) \tag{8.66}$$
$$\dot{\hat{z}}_a = B_a C\hat{z} + A_a\hat{z}_a + \nu(t, y_a, \hat{y}_a) \tag{8.67}$$
$$\hat{y}_a = C_a\hat{z}_a \tag{8.68}$$

where

$$\nu := k(t)C_a^T \frac{y_a - \hat{y}_a}{\|y_a - \hat{y}_a\|} \tag{8.69}$$

and the scalar gain $k(t)$ is to be designed later.

Let $e(t) := z(t) - \hat{z}(t)$ and $e_a(t) := z_a(t) - \hat{z}_a(t)$. It follows from (8.63)–(8.65) and (8.66)–(8.68) that the error dynamics can be described by

$$\dot{e} = (A - B\Lambda)e + B\left(\Gamma(z, u) - \Gamma(\hat{z}, u)\right) + \Psi(z) \tag{8.70}$$
$$\dot{e}_a = B_a Ce + A_a e_a + B_a Df_s(t) - \nu(t, y_a, \hat{y}_a) \tag{8.71}$$

where $\Gamma(\cdot)$ is determined by (8.60), $\Psi(\cdot)$ is the uncertain term which is defined by (8.56) and $\nu(\cdot)$ is given by (8.69).

## 8.3.5 Sensor FDI for the Nominal Case

In this section, the special case $\Delta F \equiv 0$ is considered, which implies that the system under consideration does not experience any uncertainty. In this case, the corresponding augmented system is the same as (8.63)–(8.65) except $\Psi(\cdot) \equiv 0$ in (8.63), and thus the corresponding error dynamical system (8.70)–(8.71) is described by

$$\dot{e} = (A - B\Lambda)e + B\left(\Gamma(z, u) - \Gamma(\hat{z}, u)\right) \tag{8.72}$$
$$\dot{e}_a = B_a Ce + A_a e_a + B_a Df_s(t) - \nu(t, y_a, \hat{y}_a) \tag{8.73}$$

The objective now is to develop a condition under which (8.66)–(8.68) is a sliding mode observer of the system (8.63)–(8.65) with $\Psi(\cdot) \equiv 0$ in (8.63), and can be employed to reconstruct the fault signal $f_s(t)$.

From the analysis above, the following conclusion is ready to be presented:

**Proposition 8.3** *Suppose Assumption 8.5 holds. Then, System (8.72) is stable if there exists a matrix $P > 0$ such that*

$$(A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 PBB^T P + \frac{1}{\varepsilon_1}(\mathcal{L}(u))^2 I_n < 0 \tag{8.74}$$

*for all $u \in \mathcal{U}$ where $\varepsilon_1$ is a positive constant, $\Lambda$ is defined by (8.58) and $\mathcal{L}(u)$ satisfies (8.61).*

*Proof* For System (8.72), consider a Lyapunov function candidate

$$V = e^T(t)Pe(t)$$

where $P > 0$ is a solution for the matrix inequality (8.74). The time derivative of $V$ along the trajectories of System (8.72) is given by

$$\begin{aligned}\dot{V}\mid_{(8.72)} \le\, & e^T(t)\left((A - B\Lambda)^T P + P(A - B\Lambda)\right)e(t) \\ & +2e(t)^T PB\left(\Gamma(z,u) - \Gamma(\hat{z},u)\right).\end{aligned} \tag{8.75}$$

From the fact $2X^TY \le \varepsilon_1 X^TX + \frac{1}{\varepsilon_1}Y^TY$, it follows that

$$\begin{aligned}\dot{V}\mid_{(8.72)} &\le e^T(t)\left((A - B\Lambda)^T P + P(A - B\Lambda)\right)e(t) + \varepsilon_1(B^TPe(t))^TB^TPe(t) \\ &\quad +\frac{1}{\varepsilon_1}\left(\Gamma(z,u) - \Gamma(\hat{z},u)\right)^T\left(\Gamma(z,u) - \Gamma(\hat{z},u)\right) \\ &\le e^T(t)\left((A - B\Lambda)^T P + P(A - B\Lambda)\right)e(t) + \varepsilon_1 e^T(t)PBB^TPe(t) + \\ &\quad \frac{1}{\varepsilon_1}(\mathscr{L}(u))^2\|z - \hat{z}\|^2 \\ &= e^T\left((A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 PBB^TP + \frac{1}{\varepsilon_1}(\mathscr{L}(u))^2 I_n\right)e\end{aligned}$$

where (8.61) is used to establish the second inequality. Hence the conclusion follows from (8.74). △

It should be noted that

- Proposition 8.3 implies that $e(t)$ is bounded, and thus

$$\sup_{0\le t<\infty}\{\|e(t)\|\} \le b \tag{8.76}$$

  for some finite positive scalar $b$;
- because of the scalar $\varepsilon_1$ in (8.74) which provides additional design freedom, without loss of generality it can be assumed that $P > I_n$ rather than just being positive definite.

Consider a sliding surface

$$\mathscr{S} = \{\mathrm{col}(e, e_a) \mid e_a = 0\}. \tag{8.77}$$

Proposition 8.3 implies that the sliding mode dynamics of the error system (8.72)–(8.73) associated with the sliding surface (8.77) is stable. According to sliding mode theory, in order to guarantee the stability of the observer it is only required to prove that the error system can be driven to the sliding surface in finite time by choosing an appropriate gain $k(t)$ in (8.69). In view of this, the following conclusion is presented:

**Proposition 8.4** *If Inequality (8.76) holds, then the error system (8.72)–(8.73) is driven to the sliding surface (8.77) if $k(\cdot)$ in (8.69) is chosen to satisfy*

$$k(t) \geq \|B_a C\| b + \|B_a D\| \rho(t) + \eta \tag{8.78}$$

*where $\eta$ is a positive constant.*

*Proof* From Eq. (8.73), it follows that

$$e_a^T \dot{e}_a = e_a^T B_a C e + e_a^T A_a e_a + e_a^T B_a D f_s(t) - e_a^T \nu(t, y_a, \hat{y}_a).$$

Since $A_a < 0$ it follows that $e_a^T A_a e_a \leq 0$. Since $C_a$ is orthogonal,

$$\|y_a - \hat{y}_a\| = \sqrt{(C_a e_a)^T C_a e_a} = \|e_a\|. \tag{8.79}$$

Then, from (8.76), (8.69) and (8.46)

$$\begin{aligned} e_a^T \dot{e}_a &\leq e_a^T B_a C e + e_a^T B_a D f_s(t) - k(t) e_a^T C_a^T \frac{y_a - \hat{y}_a}{\|y_a - \hat{y}_a\|} \\ &\leq \|e_a\| \, \|B_a C\| b + \|e_a\| \, \|B_a D\| \rho(t) - k(t)(C_a e_a)^T \frac{C_a e_a}{\|e_a\|} \\ &= (\|B_a C\| b + \|B_a D\| \rho(t) - k(t)) \, \|e_a\| \end{aligned} \tag{8.80}$$

where (8.79) is used to obtain the second inequality. Then, it follows from (8.80) and (8.78) that

$$e_a^T \dot{e}_a \leq -\eta \|e_a\|.$$

This means that the reachability condition is satisfied [174], and a sliding motion on $\mathscr{S}$ is attained in finite time. △

*Remark 8.9* It should be stressed that the dynamics of the error system $e(t)$ in (8.70), which represents the reduced order sliding motion associated with the sliding surface (8.77), must be stable so that the term $B_a C e$ in Eq. (8.71) vanishes with time. This makes it possible to reconstruct/estimate the sensor fault.

*Remark 8.10* It is tempting from (8.78) to select $B_a = 0$. However, this is not possible since if $B_a = 0$, it follows from (8.64) that the sensor fault term will also disappear and thus it cannot be reconstructed.

From sliding mode theory, Propositions 8.3 and 8.4 have shown that (8.66)–(8.68) is an observer of System (8.63)–(8.65) when $\Psi(\cdot) \equiv 0$ in (8.63). The objective is now to establish a reconstruction signal for the sensor fault $f_s(t)$ based on the sliding mode observer (8.66)–(8.68).

Since the fault distribution matrix $D$ is assumed to be full column rank, there exists a nonsingular matrix $N \in \mathbb{R}^{p \times p}$ such that

$$ND = \begin{bmatrix} 0_{(p-q)\times q} \\ D_1 \end{bmatrix} \tag{8.81}$$

where $D_1 \in \mathbb{R}^{q\times q}$ is nonsingular. The matrix $N$ can be obtained from QR decomposition. Then, from the analysis above, it follows that a sliding motion takes place in finite time, and during the sliding motion

$$e_a = 0 \quad \text{and} \quad \dot{e}_a = 0$$

and thus from (8.73)

$$B_a Ce + B_a Df_s(t) - \nu_{eq} = 0 \tag{8.82}$$

where $\nu_{eq}$ is the equivalent output error injection which plays the same role as the equivalent control in sliding mode control [38, 174]. The equivalent output injection signal represents the average behaviour of the discontinuous function $\nu$ defined by (8.69), which is necessary to maintain an ideal sliding motion.

In order to reconstruct the sensor fault, the design parameter $B_a$ in filter (8.62) is chosen as $B_a = N$ where $N$ is given by (8.81). It follows that

$$B_a D = \begin{bmatrix} 0_{(p-q)\times q} \\ D_1 \end{bmatrix} \tag{8.83}$$

where $D_1$ is nonsingular. From (8.82) and (8.83)

$$\begin{bmatrix} 0_{q\times(p-q)} & I_q \end{bmatrix} B_a Ce + D_1 f_s(t) - \begin{bmatrix} 0_{q\times(p-q)} & I_q \end{bmatrix} \nu_{eq} = 0$$

and since $D_1$ is nonsingular, it follows that

$$\begin{aligned} f_s(t) &= -\begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1} \end{bmatrix} \left(B_a Ce - \nu_{eq}\right) \\ &= -\begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1} \end{bmatrix} B_a Ce + \begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1} \end{bmatrix} \nu_{eq}. \end{aligned} \tag{8.84}$$

Now, it is required to recover the equivalent output error injection $\nu_{eq}$. Considering the structure of $\nu(\cdot)$ in (8.69), it follows from [40] that by choosing an appropriate positive constant scalar $\sigma$, $\nu_{eq}$ can be approximated to any accuracy by

$$\nu_\sigma = k(t) C_a^T \frac{(y_a - \hat{y}_a)}{\|y_a - \hat{y}_a\| + \sigma} \tag{8.85}$$

where $k(\cdot)$ satisfies (8.78). Let

$$\hat{f}_s(t) := \begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1} \end{bmatrix} \nu_\sigma \tag{8.86}$$

where $\nu_\sigma$ is defined by (8.85) and $D_1$ is given by (8.81). Then from (8.84) and (8.86),

$$f_s(t) - \hat{f}_s(t) = -\left[0_{q\times(p-q)}\ D_1^{-1}\right] B_a C e + \left[0_{q\times(p-q)}\ D_1^{-1}\right](\nu_{eq} - \nu_\sigma)$$

where $\lim_{t\to\infty} e(t) = 0$. Therefore, $\hat{f}_s$ defined by (8.86) is a reconstruction of the sensor fault $f_s(t)$ since $\|\nu_{eq} - \nu_\sigma\|$ can be made arbitrarily small by choice of $\sigma$.

*Remark 8.11* From (8.85) and (8.86), it is clear that the reconstruction signal $\hat{f}_s$ given by (8.86) is only dependent on $y_a$ and $\hat{y}_a$ which can be obtained on-line. Therefore, the fault reconstruction scheme is convenient for real implementation.

### *8.3.6 Sensor FDI for Systems with Uncertainty*

Here, it is assumed $\Delta F(x(t)) \neq 0$. This means that the system under consideration is affected by uncertainties or disturbances. In this case, the corresponding dynamical error equation is given by (8.70)–(8.71).

**Assumption 8.6** The uncertain function $\Psi(z)$ defined by (8.56) satisfies

$$\sqrt{\Psi(z)^T P \Psi(z)} \leq \frac{1}{2} d, \qquad \forall z \in T(\Omega)$$

where the s.p.d. matrix $P$ satisfies (8.74) with $P > I_n$ and $d$ is a known constant.

*Remark 8.12* Assumption 8.6 is a limitation on the magnitude of the uncertainty $\Psi(\cdot)$. It can be written as

$$\|P^{\frac{1}{2}} \Psi(z)\| \leq \frac{1}{2} d$$

which is just a special weighted norm for $\Psi(\cdot)$. Assumption 8.6 can, therefore, be interpreted as a requirement that the uncertainty $\Psi(\cdot)$ is bounded (in the special norm) and its bound is known. It is clear that Assumption 8.6 holds if $\Psi(\cdot)$ is bounded in the domain $T(\Omega)$.

Define

$$Q := (A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 P B B^T P + \tfrac{1}{\varepsilon_1}(\mathscr{L}(u))^2 I_n \tag{8.87}$$

where $\varepsilon_1$ is the positive constant associated with (8.74).

**Proposition 8.5** *Assume that the matrix inequality (8.74) is solvable for $P > 0$. Then under Assumptions 8.5 and 8.6, for any scalar $\varepsilon_2 > 0$ there exists a time $T_1$ such that for $t \geq T_1$, $e(t)$ will enter the set*

$$\mathscr{B} = \left\{ e \mid e^T P e \leq \left(\tfrac{d+\varepsilon_2}{\alpha}\right)^2 \right\} \tag{8.88}$$

*and remains there for all subsequent time, where the positive constant*

$$\alpha := -\lambda_{max}(P^{-1}Q)$$

*where the matrix $Q$ is defined in (8.87).*

*Proof* Consider $V = e(t)^T Pe(t)$ as a potential Lyapunov function for System (8.70). Since (8.74) is solvable, the matrix $Q$ defined by (8.87) is symmetric negative definite and so $\alpha := -\lambda_{max}(P^{-1}Q)$ is a positive quantity. By the same reasoning as in the proof of Proposition 8.3, it follows from (8.87) and Assumptions 8.5 and 8.6 that

$$\begin{aligned}\dot{V}\mid_{(8.70)} &\leq e^T(t)Qe(t) + 2e(t)^T P\Psi(z)\\ &= e^T(t)P^{1/2}P^{-1/2}QP^{-1/2}P^{1/2}e(t) + 2e(t)^T P^{1/2}P^{1/2}\Psi(z)\\ &\leq \lambda_{max}(P^{-1/2}QP^{-1/2})V + \sqrt{V}d\end{aligned} \tag{8.89}$$

since $V = e^T P^{1/2}P^{1/2}e = \|P^{1/2}e\|^2$. Further, since

$$\lambda_{max}(P^{-1/2}QP^{-1/2}) = \lambda_{max}(P^{-1}Q)$$

from the standard properties of eigenvalues, Inequality (8.89) can be written as

$$\dot{V}\mid_{(8.70)} \leq \left(d - \alpha\sqrt{V}\right)\sqrt{V}.$$

It follows that for any $\varepsilon_2 > 0$, if $e(t) \notin \mathscr{B}$, $d - \alpha\sqrt{V} < -\varepsilon_2$ and so

$$\dot{V}\mid_{(8.70)} \leq -\varepsilon_2\sqrt{V}. \tag{8.90}$$

This implies that System (8.70) is uniformly ultimate bounded with respect to $\mathscr{B}$: i.e., $e(t)$ will enter the ball $\mathscr{B}$ defined in (8.88) after a finite time $T_1$ and remain in it thereafter. Hence the conclusion follows. #

It should be noted that (8.73) is exactly the same as (8.71). Therefore, by the same reasoning as in Sect. 8.3.5, System (8.70)–(8.71) will be driven to the sliding surface (8.77) in finite time, and a sliding motion maintained on it, if the function $\nu$ is designed as in (8.69) and $k(\cdot)$ satisfies (8.78). The main difference is that in this case when uncertainty exists, the sliding motion is ultimately bounded instead of asymptotically stable. By combining Proposition 8.5, it follows that (8.66)–(8.68) is an approximate observer of System (8.63)–(8.65) when uncertainty is considered. Similar to the analysis in Sect. 8.3.5, it follows that (8.84) is true when a sliding motion takes place. It follows that

$$\hat{f}_s(t) = \begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1}\end{bmatrix}\nu_\sigma \tag{8.91}$$

is an estimation of the sensor fault $f_s$ where $\nu_\sigma$ is given by (8.85) and $D_1$ is defined by (8.81). From (8.84) and (8.91),

$$\|f_s - \hat{f}_s\| \le \left\|D_1^{-1}\right\| \Big(\|B_a\| \, \|C\| \, \|e(t)\| + \|\nu_{eq} - \nu_\sigma\|\Big).$$

Since $\nu_{eq}$ can be approximated by $\nu_\sigma$ to any accuracy by choosing an appropriate $\sigma$, it follows that for any $\varepsilon > 0$ there exists a time $T_2$ such that for $T > T_2$

$$\|\nu_{eq} - \nu_\sigma\| < \tfrac{1}{\|D_1^{-1}\|}\varepsilon. \tag{8.92}$$

From the fact that the matrix $C$ from (8.54) has the property $\|C\| = 1$ and $P > I_n$, it follows that

$$V = e^T P e \ge \|e\|^2$$

and

$$\mathscr{B} \subset \left\{ e \mid \|e\| < \tfrac{(d+\varepsilon_2)}{\alpha} \right\}. \tag{8.93}$$

Hence by combining with Proposition 8.5, the sensor fault estimation error

$$\|f_s(t) - \hat{f}_s(t)\| \le \left\|D_1^{-1}\right\| \, \|B_a\| \tfrac{d+\varepsilon_2}{\alpha} + \varepsilon \tag{8.94}$$

for all $t > T := \max\{T_1, T_2\}$, where the scalars $\varepsilon_2$ and $\varepsilon$ are arbitrary small positive constants. Clearly, from (8.94), the estimation error is closely connected with the uncertain bound $d$.

*Remark 8.13* Sensor fault estimation has been considered in [40, 166, 167]. Slowly varying sensor faults are considered in [40] but more general ones are considered in [160, 167]. However, in these papers, only linear systems are considered. In [198], actuator fault reconstruction was developed but a minimum phase condition was required for the system. In this section, the corresponding minimum phase limitation has been removed which makes the work applicable to a wider class of systems.

*Remark 8.14* The sensor faults considered in this chapter are modelled as an additive disturbance. Fault detection is concerned with identifying a problem in the monitored system while fault isolation is the determination of which component is faulty. If the system is not affected by any uncertainty/disturbance, then a 'precise' reconstruction signal has been proposed in this section, i.e., after some time the reconstruction signal can duplicate the fault precisely. In this situation it is clear to see which channel has the fault through the reconstruction signal. This implies that the solution of the isolation problem is inherent in the approach. If the system is subject to uncertainty, the results developed in this section only represent an estimation of the fault. In this case, an appropriate threshold is required to be established for fault isolation. Its accuracy will be limited by the size of the bound on the uncertainty compared to the size of the fault signals to be detected.

In the following, an approach based on LMI techniques is presented to determine the design parameters. Suppose $\beta \in \mathbb{R}$ is such that $\mathscr{L}(u) \le \beta$ for all $u \in \mathscr{U}$. Also suppose $\Lambda$ has been chosen so that $(A - B\Lambda)$ is stable (clearly this is a necessary condition for (8.74) to have a positive definite solution for $P$). Consider the matrix inequality

$$(A - B\Lambda)^T P + P(A - B\Lambda) + PBB^T P + \beta^2 I_n + \frac{1}{\gamma} P < 0 \tag{8.95}$$

where $\gamma \in \mathbb{R}$ is a positive scalar. If (8.95) is satisfied for some matrix $P > 0$, then from the definition of $Q$ in (8.87) where $\varepsilon_1$ is chosen as 1, it follows that

$$P^{-1/2} Q P^{-1/2} + \frac{1}{\gamma} I_n < 0$$

which implies

$$\lambda_{\max}(P^{-1}Q) < -\frac{1}{\gamma}$$

and so

$$\gamma > -1/\lambda_{\max}(P^{-1}Q) = \frac{1}{\alpha}.$$

Consequently minimising $\gamma$, subject to solving (8.95) for $P$, decreases the radius of the ultimate boundedness set $\mathscr{B}$ from (8.93). A plausible convex optimization problem is to minimise $\gamma$ with respect to $P$ and $X$ subject to

$$\begin{bmatrix} (A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 \beta^2 I_n + X & PB \\ B^T P & -\varepsilon_1 I_p \end{bmatrix} < 0 \tag{8.96}$$

$$I_n < P \tag{8.97}$$

$$P < \gamma X \tag{8.98}$$

where $X \in \mathbb{R}^{n \times n}$ is a s.p.d 'slack' variable. This is a well-posed convex optimization problem and can be solved using LMI techniques. From the Schur complement, if (8.96) is satisfied then

$$(A - B\Lambda)^T P + P(A - B\Lambda) + PBB^T P + \beta^2 I_n < -X$$

and since from (8.98) $-X < -\frac{1}{\gamma} P$, it follows that (8.95) is satisfied.

### 8.3.7 Procedure for Sensor Fault Reconstruction/Estimation

Based on the analysis above, a design procedure is summarised as follows:

Step 1. Check that the system (8.44)–(8.45) has uniform observability indices $\{r_1, r_2, \ldots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in $\Omega \times \mathscr{U}$;

Step 2. Find the diffeomorphism $T$ defined by (8.49)–(8.51). Then compute the transformed system (8.52)–(8.53);
Step 3. Choose constants $\alpha_{ij}$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i$ such that all the roots of the polynomials (8.57) lie in the left-half plane;
Step 4. Check Assumption 8.5 and compute the functions $\Gamma(z, u)$ satisfying (8.60) and $\mathscr{L}(u)$ satisfying (8.61);
Step 5. Choose $A_a < 0$, $B_a = N$ satisfying (8.81) and $C_a$ orthogonal. Then, establish the filter (8.62) and the observer (8.66)–(8.68);
Step 6. Using an LMI package, find the solution $P$ of the matrix inequalities (8.96)–(8.98), then $Q$ can be obtained from (8.87);
Step 7. Choose the gain $k(\cdot)$ to satisfy (8.78) and establish the observer (8.66)–(8.68);
Step 8. According to (8.86) compute the reconstruction/estimation signal $\hat{f}_s$. (The estimation error can be obtained from (8.94)).

If the system under consideration satisfies the conditions proposed in this section, then the procedure described above can be employed to reconstruct/estimate the sensor fault signal.

## 8.4 Case Studies

The FDI schemes developed in Sects. 8.2 and 8.3 will be applied to a robot arm system and a mass–spring system, respectively.

### *8.4.1 Fault Reconstruction and Estimation for a Robot System*

Consider a single-link flexible joint robot system, where the system nonlinearities come from the joint flexibility modelled as a stiffened torsional spring, and the gravitational force. The dynamical model for the robot can be described by [44]:

$$\dot{\theta}_1 = \omega_1 \tag{8.99}$$

$$\dot{\omega}_1 = \frac{1}{J_1}(\kappa_1(\theta_2 - \theta_1) + \kappa_2(\theta_2 - \theta_1)^3) - \frac{B_v}{J_1}\omega_1 + \frac{K_\tau}{J_1}u \tag{8.100}$$

$$\dot{\theta}_2 = \omega_2 \tag{8.101}$$

$$\begin{aligned}\dot{\omega}_2 = &-\frac{1}{J_2}(\kappa_1(\theta_2 - \theta_1) + \kappa_2(\theta_2 - \theta_1)^3) - \frac{m_l g h}{J_2}\sin\theta_2 \\ &+\Psi(\theta_1, \omega_1, \theta_2, \omega_2, t)\end{aligned} \tag{8.102}$$

where $\theta_1$ and $\omega_1$ are the motor position and velocity respectively; $\theta_2$ and $\omega_2$ are the link position and velocity respectively; $J_1$ is the inertia of the DC motor, $J_2$ is the inertia of the link, $2h$ is the length of the link while $m_l$ represents its mass, $B_v$ is the viscous friction, $\kappa_1$ and $\kappa_2$ both are positive constants and $K_\tau$ is the amplifier gain. The domain considered here is

$$\{(\theta_1, \omega_1, \theta_2, \omega_2) \mid |\theta_2 - \theta_1| < 2.8, |\omega_1| \le 50\}.$$

It is assumed that the motor position, motor velocity and the sum of link velocity and link position are measured. The quantity $\Psi(\cdot)$ satisfying

$$|\Psi(x, u, t)| \le |\omega_1 \sin \omega_2| \exp\{-t\}$$

represents the uncertainty affecting the system, which has been added to illustrate the results obtained in this section and is not a feature of [44].

Suppose that a fault $f$ occurs in the input channel in the robot system. Then, the fault distribution matrix $D$ will be equal to the input matrix. According to [44], suitable values for the parameters are:

$$\begin{aligned}
&J_1 = 3.7 \times 10^{-3}\,\text{kgm}^2\\
&J_2 = 9.3 \times 10^{-3}\,\text{kgm}^2\\
&h = 1.5 \times 10^{-1}\,\text{m},\\
&m = 0.21\,\text{kg}, \quad B_v = 4.6 \times 10^{-2}\,\text{m}\\
&\kappa_1 = \kappa_2 = 1.8 \times 10^{-1}\,\text{Nm/rad},\\
&K_\tau = 8 \times 10^{-2}\,\text{Nm/V}.
\end{aligned}$$

Let

$$x = \text{col}(x_1, x_2, x_3, x_4) := (\theta_1, \omega_1, \theta_2, \omega_2).$$

Then, the robot system can be described in the form (8.1)–(8.2) as follows

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.0486 & -12.4324 & 0.0486 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0194 & 0 & -0.0194 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0.0194(x_3 - x_1)^3 + 21.6216u \\ 0 \\ 0.0486(x_3 - x_1)^3 - 83.4324 \sin x_3 \end{bmatrix}}_{G(x,u)}$$

$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{E} \Psi(x, u, t) + \underbrace{\begin{bmatrix} 0 \\ 21.6216 \\ 0 \\ 0 \end{bmatrix}}_{D} f(y, u, t)$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} x.$$

Introduce a coordinate transformation $z = Tx$ with $T$ defined by

$$T = \begin{bmatrix} -1.4142 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -0.01 \end{bmatrix}.$$

It follows that

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{c|ccc} -1 & 0 & 1.4142 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 8.0496 & -0.0486 & -11.4324 & 0 \\ 0.0137 & 0.0194 & 0 & 0 \end{array}\right]$$

$$D_2 = \begin{bmatrix} 0 \\ \hline D_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \hline -21.6216 \\ 0 \end{bmatrix}, \tag{8.103}$$

$$E_2 = \begin{bmatrix} 0 \\ \hline E_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \\ -1 \end{bmatrix} \tag{8.104}$$

$$C_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{8.105}$$

and

$$G(T^{-1}z, u) = \begin{bmatrix} 0 \\ 0 \\ -21.6216u - 0.0486(z_2 + 0.7071z_1)^3 \\ 0.0002(z_2 + 0.7071z_1)^3 + 0.3319 \sin z_2 \end{bmatrix}.$$

Let $\alpha = 0.5$. From the LMI synthesis, the optimal value of the Lipschitz gain $\mathscr{L}_G = 0.7499$ when $L = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\varepsilon = 1.9989$, and $P = 1.5$ and the conditions of Proposition 8.1 are satisfied. Then, by choosing

$$K = \begin{bmatrix} 0 & 1.1 & 1 \\ 10.2324 & -0.0486 & 0 \\ 0 & 0.0194 & -1 \end{bmatrix}$$

by direct computation, it follows that

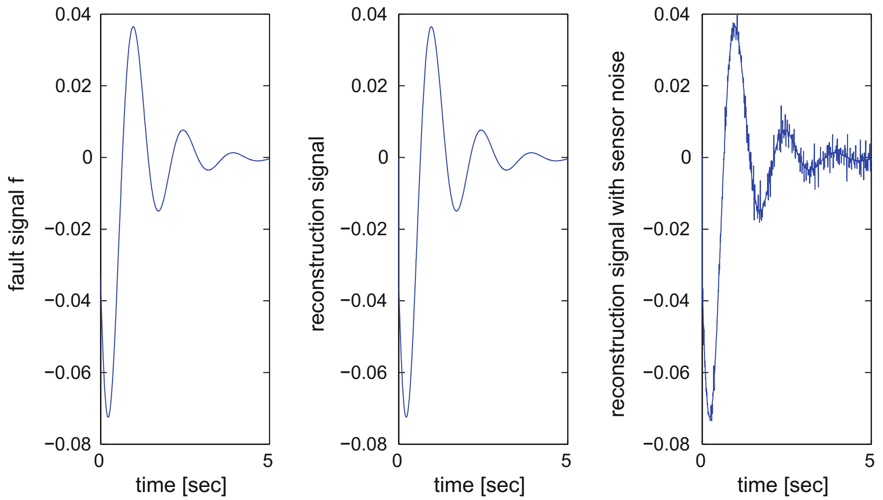


**Fig. 8.1** The fault reconstruction for the robot system with the fault signal $f = 0.5 \sin u$ (*Left* fault signal; *Middle* reconstruction signal; *Right* reconstruction signal with sensor noise)

$$C_2(A_4 - A_3 L)C_2^{-1} + C_2 K = \begin{bmatrix} -1.2 & 0 & 0 \\ 0 & -1.1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and thus (8.30) is true. The design of the observer (8.12)–(8.13) is now completely specified. From (8.103)–(8.105), $\mathrm{Im}(E_{22}) \cap \mathrm{Im}(D_{22}) = \{0\}$. A suitable choice of the decoupling matrix is

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with $H_2 = -21.6216$. Then, for any fault $f$, from Theorem 8.1, the signal $\tilde{f}$ obtained from (8.35) is a reconstruction of the fault.

For simulation purposes, a linear state feedback controller

$$u = [-0.6797 \ \ 13.0863 \ \ 0.2836 \ \ -35.8391]x$$

has been introduced to stabilise the system. In the first case, the fault signal is $f(t) = 0.5 \sin u$ which does not affect the stability of the system. The associated simulation is shown in Fig. 8.1. In the second case the fault signal is $f(t) = \sin u$ which destroys the system stability. The corresponding simulation is shown in Fig. 8.2. The simulations show that the signal $\tilde{f}$ can reconstruct the fault perfectly even if the faults destroy the stability of the system. However, in the second simulation the reconstruction properties will eventually be lost over time as the states of the plant become unbounded. It also shows that in the presence of sensor noise the reconstruction scheme is still effective because in this case the reconstruction can still preserve

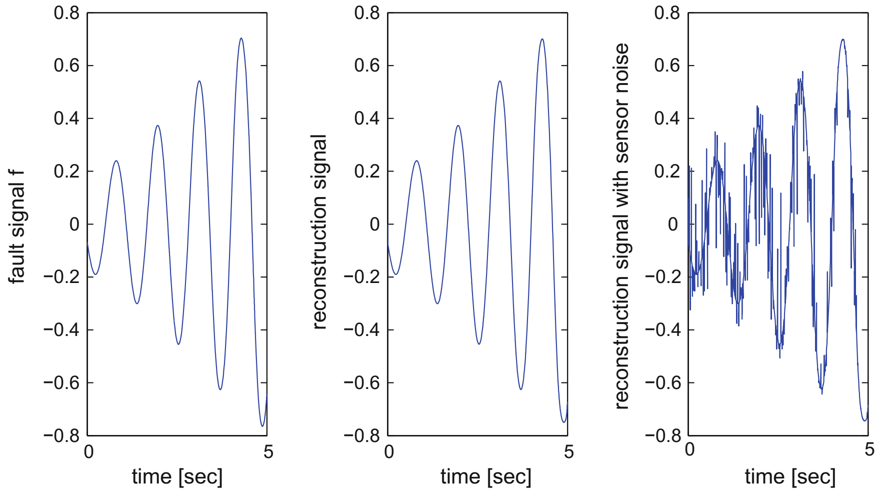


**Fig. 8.2** The fault reconstruction for the robot system with the fault $f = \sin u$ (*Left* fault signal; *Middle* reconstruction signal; *Right* reconstruction signal with sensor noise)

the fault signal shape and the effect of the fault is clearly visible in the detection signal.

### *8.4.2 Sensor FDI for a Mass–Spring System*

Consider a mass–spring system with a hardening spring, linear viscous friction and an external force described by

$$M\ddot{x} + c\dot{x} + \mu x + \mu a^2 x^3 = u \tag{8.106}$$

where $x$ denotes the displacement from a reference position, $M$ is the mass of the object sliding on a horizontal surface, $\mu$ is the spring constant, $a$ represents a coefficient which is associated with the hardening properties of the spring and $u$ is the control signal which represents an external force applied to the system (see, [91], pp. 8–9). Let $z = \text{col}(z_1, z_2) = (x, \dot{x})$. The system output is assumed to be $y = z_1$. The parameters are chosen as in ([91], pp. 172–173). Then, the system is described in the form of (8.52)–(8.53) as follows:

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{(-z_1 - z_2 - z_1^3 + u)}_{\Phi(x,u)} + \Psi(z) \tag{8.107}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} z + D f_s(t) \tag{8.108}$$

where $D = 1$ and the term $Df_s(z)$ is the sensor fault added to illustrate the results developed in Sect. 8.3. It is assumed that $\Psi(\cdot)$ includes all uncertainties present in the system and in this case is assumed to satisfy

$$|\Psi(z)| \leq 0.1 \sin^2 y.$$

This function has been added to demonstrate the results which have been developed and is not a feature of [91]. The domain considered in this example is

$$\Omega = \{(z_1, z_2) \mid |z_1| < 0.44, z_2 \in \mathbb{R}\}.$$

Let $\Lambda = [-1 \quad -1]$. It follows that Assumption 8.5 holds with

$$\Gamma(z, u) = -z_1^3 + u$$

which satisfies (8.61) in $\Omega$ with $\mathscr{L}(u) = 0.5808$. Choose $\varepsilon_1 = \gamma = 1$. It follows that the LMIs (8.96)–(8.98) have a solution

$$P = \begin{bmatrix} 1.2047 & 0.2428 \\ 0.2428 & 1.2881 \end{bmatrix}$$

and $\beta = 0.6 > \mathscr{L}(u)$. Therefore, the conditions of Proposition 8.3 are satisfied in the domain $\Omega$.

Choose $A_a = -1$. Obviously $D_1$ can be chosen as

$$D_1 = D = 1, \qquad \text{and} \qquad B_a = 1.$$

Then the filter is described by

$$\begin{aligned} \dot{x}_a &= -x_a + y \\ y_a &= x_a.s. \end{aligned}$$

If $k(t)$ is chosen to satisfy (8.78), it follows that $\hat{f}_s = \nu_\sigma$ is a reconstruction for $f_s(t)$ if $\Psi(\cdot) = 0$ and an estimate of the fault $f_s(t)$ if $\Psi(\cdot) \neq 0$. The simulation in Figs. 8.3, 8.4 and 8.5 shows that the approach is effective. The middle figure in Fig. 8.3 shows that the reconstruction signal reproduces the fault faithfully if no uncertainty is present in the system and thus the sensor fault can be detected easily from the reconstruction. The lower figure in Fig. 8.3 considers the case when

$$\Psi = [0.6 \;\; 0.6]^T \sin^2 y.$$

It shows that the estimation signal still reproduces the fault to a reasonable extent when uncertainty $\Psi(\cdot)$ is present. In the presence of uncertainty, it follows that for any $\varepsilon > 0$, there exists a time $T_1$ such that

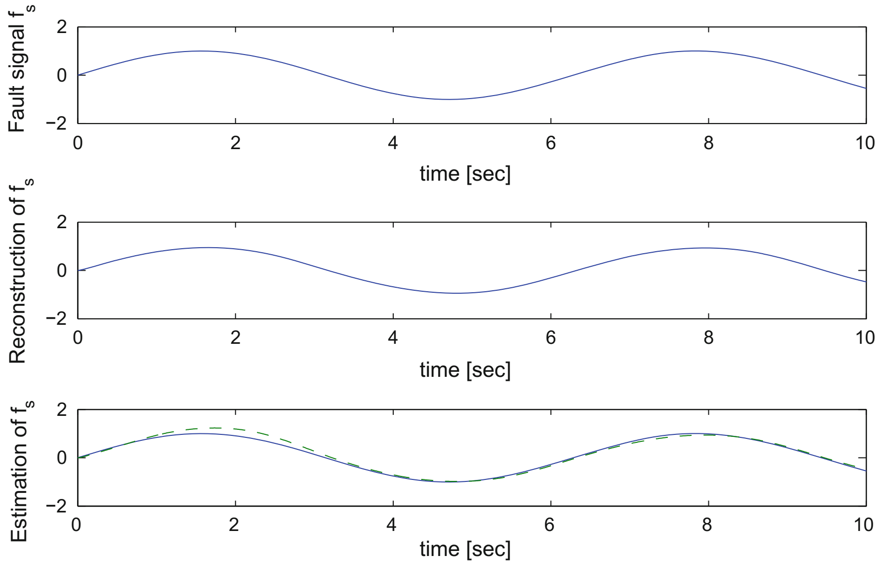


**Fig. 8.3** Sensor fault reconstruction/estimation for Mass–Spring System (8.107)–(8.108) (*Upper* fault signal; *Middle* reconstruction signal; *Bottom* Estimation signal where the *dashed line* is the estimation signal and the *solid line* is the fault signal

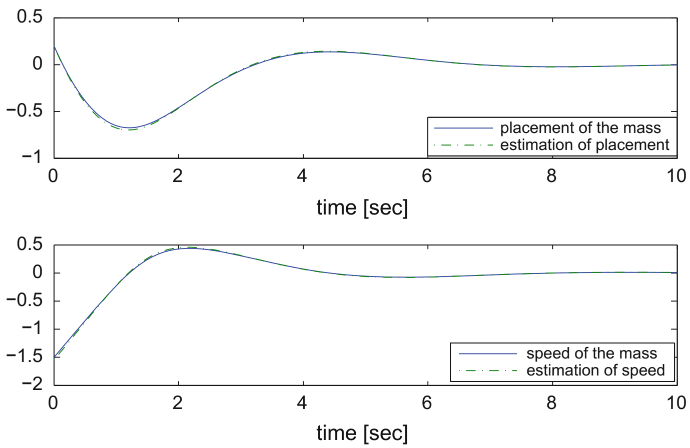


**Fig. 8.4** State estimation of Mass–Spring System (8.107)–(8.108) without uncertainty (*Upper* placement of the mass and its estimation; *Bottom* speed of the mass and its estimation

$$\|f_s(t) - \hat{f}(t)\| \leq \frac{0.0444 + \varepsilon}{0.0554} + \varepsilon, \qquad t > T_1.$$

In this case, an appropriate threshold is required to be established for fault detection. The error bound given above is conservative and the performance which is achieved

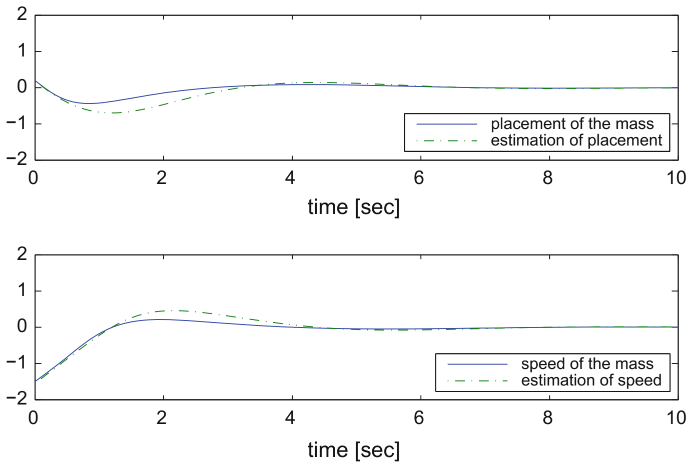


**Fig. 8.5** State estimation of Mass–Spring System (8.107)–(8.108) with uncertainty (*Upper* placement of the mass and its estimation; *Bottom* speed of the mass and its estimation

by the scheme is considerably better. Figures 8.4 and 8.5 shows the state estimation of the Mass–Spring system (8.107)–(8.108) with and without uncertainty respectively.

## 8.5 Summary

Approaches for robust fault estimation and reconstruction for a class of nonlinear systems have been proposed in this chapter based on a sliding mode observer. The observer design parameters can be obtained using LMI techniques in a systematic way. Under appropriate conditions, the reconstruction signal can approximate the fault signal to any accuracy even in the presence of uncertainties. The proposed FDI scheme is straightforward to implement in real systems and can be applied to a reasonably wide class of systems.

For the sensor fault case, the nonlinear system has been transformed to a system with special structure to facilitate the design, and an augmented system has been established by designing a simple filter to process the outputs. A sliding mode observer has been proposed for the augmented system to estimate the system states. Based on the observer, sensor FDI schemes are presented for the system with/without uncertainty.

Case studies have shown that the proposed FDI schemes are effective. The simulation examples show how to use the reconstruction signal to detect a sensor fault.

# Chapter 9
# Application of Decentralised Sliding Mode Control to Multimachine Power Systems

In this chapter, a robust stabilisation problem for multimachine power systems is considered using only output information. The power system is formed from an interconnected set of lower order systems through a network transmission which is nonlinear and has an associated nonlinear bound. Under some mild conditions, a decentralised sliding mode control scheme is developed. Simulation results for a three-machine power system are presented to show the effectiveness of the proposed method.

## 9.1 Introduction

The demand for electrical energy has increased greatly with the development of technology. Various complex power systems have been built to satisfy this demand. These systems are often modelled as dynamic equations composed of the interconnection of a set of lower dimensional subsystems through a network transmission.

The complexity of the multimachine power system comes from its high dimensionality (if there are many generators), strong nonlinearity (each motor behaves nonlinearly) and strong interconnection between the subsystems (all the generators usually interact with each other), which makes traditional linear centralised control schemes difficult to implement. In fact, multimachine power systems are often widely distributed in space, and thus the information transfer among subsystems may be very difficult due to high cost, or even impossible due to practical limitations. These factors motivate the development of decentralised control which can avoid such shortcomings.

Power systems are important and many stabilising control schemes have been proposed for such systems. In [109], using modern geometric methods, Lu and Sun proposed a nonlinear control scheme for a multimachine power system. However, the approach is based on a mathematical model with fixed structure and without

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uncertainty. Wang et al. [183] studied a class of single-machine systems, which was later extended to multimachine power systems in [182].

Decentralised control is an effective approach for the control of large-scale interconnected systems (see, for example [204, 214]), and many authors have successfully applied these techniques to multimachine power systems. Based on estimated states, a decentralised control strategy is presented for multimachine power systems in [19]. Recently, robust decentralised controllers have been designed for multimachine systems in [67] exploiting the systems lower triangular structure. In [67], however, parametric uncertainty is not considered and only matched interconnections are dealt with. Xie et al. [193] have developed a control scheme to deal with parametric uncertainty using LMIs. However, it should be pointed out that in all these results it is required that the interconnections are bounded by linear functions of the norm of the system state. Furthermore the uncertainty structure is not used in the control design, which may result in unnecessary conservatism. All the results mentioned above [19, 67, 193] are state variable based.

However, usually, all the system state variables are not fully available. Sometimes it may be possible to use an observer to estimate unknown states, but unfortunately, this approach not only requires more hardware resources, but also makes the dimension of the corresponding closed-loop system increase. This may cause further difficulties, especially for large-scale power systems and thus it should be avoided if possible. Therefore, it is pertinent to study decentralised control for multimachine power systems using static output feedback.

In this chapter, as in previous work [108, 193], only the excitation control problem is considered. Not only are nonlinear interconnections considered, parametric disturbances are dealt with as well. Furthermore, the interconnections are allowed to be nonlinear and have nonlinear bounds. Mismatched uncertain interconnections are dealt with and parametric uncertainties in the direct axis transient short circuit time constants, which affect the subsystem input distribution matrix, are considered. By using the approach outlined in Sect. 2.5, an output sliding surface is synthesised which has stable sliding dynamics when the system is restricted to the surface. The approach used in this chapter is practical when compared with previous theoretical output feedback sliding mode control strategies which impose some strong geometric conditions on the nominal subsystems. A robust decentralised sliding mode controller is proposed, using only system output information, such that the system can reach the sliding surface in a finite time. Robustness is enhanced by using the sliding mode technique and conservatism is reduced by fully using system output information and the available structure of the uncertainties.

The proposed approach can deal with interconnection terms and parametric disturbances with large magnitude. It also allows significant nonlinearity to be present in the interconnection terms. Furthermore, the obtained results hold in a large region of the origin if the control gain is high enough. This allows the operating point of the multimachine power system to vary to satisfy different load demands. Finally, simulation results for a three-machine power system are presented to illustrate the control scheme.

## 9.2 Dynamical Model for Multimachine Power Systems

The exciter is one of the main control systems which directly affect the performance of multimachine power systems. It can be approximately depicted by Fig. 9.1.

The classical model of power systems was given by Bergen [8] (see e.g., Sect. 1.5.5). Based on the model in Bergen [8], multimachine power systems consisting of $N$ synchronous generators interconnected through a transmission network can be modelled, as in [67, 108, 182, 193], by:

$$\dot{x}_i = (A_i + \Delta A_i)x_i + (B_i + \Delta B_i)v_{fi} + M_i(x) + \Delta M_i(x) \tag{9.1}$$

$$y_i = C_i x_i, \qquad i = 1, 2, \ldots, N, \tag{9.2}$$

where $x = \mathrm{col}(x_1, x_2, \ldots, x_N)$ with

$$x_i = \mathrm{col}(x_{i1}, x_{i2}, x_{i3}) := \mathrm{col}\big(\delta_i - \delta_{i0}, \omega_i, P_{ei} - P_{mi0}\big)$$

for $i = 1, 2, \ldots, N$; $v_{fi} \in \mathbb{R}$ and $y_i \in \mathbb{R}^{p_i}$ are the input and the output of the $i$-th subsystems respectively; $C_i \in \mathbb{R}^{p_i \times 3}$ with $p_i \leq 3$ is the system output matrix; $M_i(x)$ is the interconnection term; and $\Delta M_i(x)$ includes the network transmission disturbance, the torque disturbance acting on the rotating shaft, the electromagnetic disturbances entering the excitation winding and other unstructural uncertainties.

The nominal system and input distribution matrices are

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{D_i}{2H_i} & -\frac{\omega_0}{2H_i} \\ 0 & 0 & -\frac{1}{T'_{doi}} \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T'_{doi}} \end{bmatrix}. \tag{9.3}$$

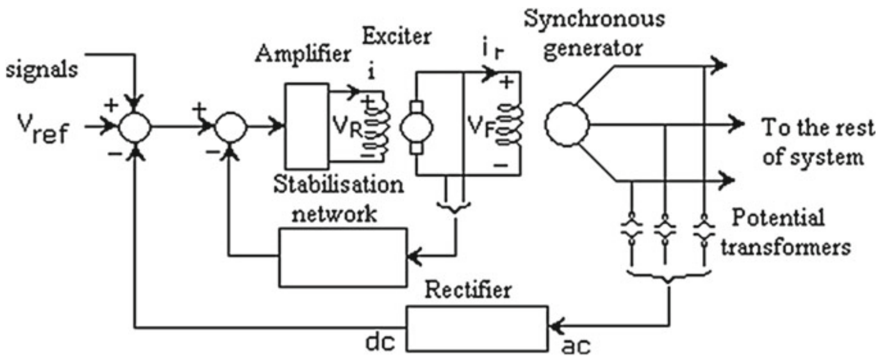


**Fig. 9.1** Excitation system

The uncertainty is described by

$$\Delta A_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta_i \end{bmatrix}, \qquad \Delta B_i = \begin{bmatrix} 0 \\ 0 \\ -\theta_i \end{bmatrix}, \tag{9.4}$$

where

$$\theta_i = \frac{1}{T'_{doi}} - \frac{1}{T'_{doi} + \Delta T'_{doi}}. \tag{9.5}$$

The interconnection term is given as

$$M_i(x) = \begin{bmatrix} 0 \\ 0 \\ \Phi_i(x) \end{bmatrix}, \tag{9.6}$$

where

$$\Phi_i(x) = E'_{qi} \sum_{j=1}^{N} \dot{E}'_{qj} B_{ij} \sin(\delta_i - \delta_j) - E'_{qi} \sum_{j=1}^{N} E'_{qj} B_{ij} \cos(\delta_i - \delta_j)\omega_j. \tag{9.7}$$

The input control variables are

$$v_{fi} = I_{qi} K_{ci} u_{fi} - (x_{di} - x'_{di}) I_{qi} I_{di} - P_{mi0} - T'_{doi} Q_{ei} \omega_i, \tag{9.8}$$

where $u_{fi}$ is the actual input of the amplifier of the $i$-th generator for $i = 1, 2, \ldots, N$. The physical meanings of all the symbols used above are shown in Appendix E.1.

In this work, $P_{mi} = P_{mi0} = constant$ since only excitation control is considered. It should be noted that direct feedback linearisation compensation for the power system representation has been used to obtain the system model (9.1)–(9.2) as described in [182]. The feedback transformation (9.8) is nonsingular since $I_{qi} K_{ci} \neq 0$ for a generator working in the normal region.

From the work in [67]:

$$|\Phi_i(x)| \leq \sum_{j=1}^{N} \left( \gamma_{ij}^{I} |\sin \delta_j| + \gamma_{ij}^{II} |\omega_j| \right), \tag{9.9}$$

where the constants $\gamma_{ij}^{I}$ and $\gamma_{ij}^{II}$ are defined by

$$\gamma_{ij}^{I} = \frac{4}{|T'_{doj}|_{min}} |P_{ei}|_{max} \tag{9.10}$$

$$\gamma_{ij}^{II} = |Q_{ei}|_{max}. \tag{9.11}$$

Therefore, for $i = 1, 2, \ldots, N$

$$\|M_i(x)\| = |\Phi_i(x)| \le \sum_{j=1}^{N} \left(\gamma_{ij}^{I} |\sin x_{j1}| + \gamma_{ij}^{II} |x_{j2}|\right). \tag{9.12}$$

*Remark 9.1* From (9.7) and (9.12), it is observed that the interconnections $M_i(x)$ are nonlinear and their bounds also take nonlinear forms instead of constants as in the work described in [67]. By using the nonlinear bounds, a control scheme with reduced conservatism will result.

## 9.3 Sliding Motion Analysis and Control Design

In this section, a sliding surface will be synthesised using the approach proposed by Edwards and Spurgeon [37, 38]. Then, under some mild conditions, the stability of the sliding mode dynamics is analysed and a decentralised output feedback sliding mode control strategy is proposed to guarantee that the system (9.1)–(9.2) can reach the sliding surface in finite time and remain on it thereafter.

### *9.3.1 Basic Assumptions*

Some basic assumptions are imposed on the system (9.1)–(9.2).

**Assumption 9.1** The matrices $C_i$ and $B_i$ satisfy $C_i B_i \neq 0$ for $i = 1, 2, \ldots, N$.

From Sect. 2.6, it follows that Assumption 9.1 implies that there exists a nonsingular linear coordinate transformation such that the triple $(A_i, B_i, C_i)$ with respect to the new coordinates has the structure

$$\widetilde{A}_i = \begin{bmatrix} \widetilde{A}_{i1} & \widetilde{A}_{i2} \\ \widetilde{A}_{i3} & \widetilde{A}_{i4} \end{bmatrix}, \qquad \widetilde{B}_i = \begin{bmatrix} 0 \\ \widetilde{b}_i \end{bmatrix}, \qquad \widetilde{C}_i = \begin{bmatrix} 0 & \widetilde{C}_{i2} \end{bmatrix}, \tag{9.13}$$

where $\widetilde{A}_{i1} \in \mathbb{R}^{2\times 2}$, $\widetilde{b}_i \in \mathbb{R}$ and $\widetilde{C}_{i2} \in \mathbb{R}^{p_i \times p_i}$ for $i = 1, 2, \ldots, N$. Furthermore $\widetilde{b}_i \neq 0$ and $\det(\widetilde{C}_{i2}) \neq 0$.

**Assumption 9.2** The triple $(\widetilde{A}_{i1}, \widetilde{A}_{i2}, \Xi_i)$ is output feedback stabilisable, where the matrix pair $(\widetilde{A}_{i1}, \widetilde{A}_{i2})$ is given by (9.13) and the matrix $\Xi_i =: \begin{bmatrix} 0_{(p_i-1)\times(n_i-p_i)} & I_{p_i-1} \end{bmatrix}$ for $i = 1, 2, \ldots, N$.

Under Assumptions 9.1 and 9.2, Edwards and Spurgeon [37, 38] show that there exists a coordinate transformation $x_i \mapsto z_i = T_i x_i$, where

$$T_i = \begin{bmatrix} I & 0 \\ -K_i \Xi_i & I \end{bmatrix}$$

such that in the new coordinates system $(A_i, B_i, C_i)$ has the following structure

$$\begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \quad \begin{bmatrix} 0 & C_{i2} \end{bmatrix}, \tag{9.14}$$

where $A_{i1} = \widetilde{A}_{i1} - \widetilde{A}_{i2} K_i \Xi_i$ is stable, $b_i \neq 0$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ is nonsingular.

*Remark 9.2* Assumptions 9.1 and 9.2 are limitations on the isolated nominal subsystems. They ensure the existence of the output sliding surface. Notably, Assumption 9.2 requires $(\widetilde{A}_{i1}, \widetilde{A}_{i2}, \Xi_i)$ instead of $(A_i, B_i, C_i)$ to be output feedback stabilisable. This is in contrast with other output feedback control results for interconnected systems (see, for example [196, 214]). It should be emphasised that all the matrices in (9.13) and (9.14) can be obtained directly from $(A_i, B_i, C_i)$ using the algorithm given in [37, 38].

**Assumption 9.3** There exist positive constants $\alpha_i < 1$ and known continuous functions $\beta_{ij}(x_j)$ such that

$$|T'_{doi}\theta_i| \leq \alpha_i \tag{9.15}$$

$$\|\Delta M_i(x)\| \leq \sum_{j=1}^{N} \beta_{ij}(x_j)\|x_j\|. \tag{9.16}$$

for $i, \ j = 1, 2, \ldots, N$.

*Remark 9.3* Assumption 9.3 is a limitation on the uncertainties that can be tolerated by the system. From the work in [108, 193], these assumptions are fundamental and reasonable. The structural requirement on the interconnection bounds in (9.16) is not essential because it can be easily extended to a more general case (see for example [215]).

## 9.3.2 Stability of Sliding Motion

Based on the assumptions above, the stability of the sliding mode is analysed in this section. Suppose Assumptions 9.1 and 9.2 are satisfied. From Sect. 2.6, there exist matrices

$$F_i = \begin{bmatrix} K_1 & 1 \end{bmatrix} \widetilde{C}_{i2}^{-1} \tag{9.17}$$

such that for $i = 1, 2, \ldots, N$ the system

$$\dot{x}_i = A_i x_i + B_i v_{fi}$$

when restricted to

$$F_i C_i x_i = 0$$

is stable, where $F_i C_i x_i = 0$ is called the *switching surface*. Consider the composite sliding surface for the interconnected system (9.1)–(9.2) as

$$S(x) = 0 \tag{9.18}$$

with $S(x) =: \mathrm{col}(S_1(x_1), S_2(x_2), \ldots S_N(x_N))$ and

$$S_i(x_i) = F_i C_i x_i = F_i y_i, \tag{9.19}$$

where the $F_i$ can be obtained from the algorithm given in [37, 38]. Next the stability of the system (9.1)–(9.2) when restricted to the sliding surface (9.18) will be considered.

From the structure of $\Delta A_i$ in (9.4), it follows that

$$T_i \Delta A_i T_i^{-1} z_i = T_i \begin{bmatrix} 0 \\ 0 \\ \theta_i x_{i3} \end{bmatrix} = (T_i B_i) T'_{doi} \theta_i (P_{ei} - P_{mi0}). \tag{9.20}$$

In the new coordinates $z = \mathrm{col}(z_1, z_2, \ldots, z_N)$, System (9.1)–(9.2) has the following form

$$\dot{z}_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} z_i + \begin{bmatrix} 0 \\ b_i \end{bmatrix} \Big( (1 - T'_{doi}\theta_i) v_{fi} + \theta_i T'_{doi} (P_{ei} - P_{mi0}) + T'_{doi} \Phi_i(x) \Big) + T_i \Delta M_i(x) \tag{9.21}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, N, \tag{9.22}$$

where $A_{i1}$ is stable, $b_i \neq 0$ and $C_{i2} \in \mathbb{R}^{p_i \times p_i}$ is nonsingular with

$$F_i \begin{bmatrix} 0_{p_i \times (n_i - p_i)} & C_{i2} \end{bmatrix} = \begin{bmatrix} 0_{1 \times 2} & f_i \end{bmatrix}, \tag{9.23}$$

where $f_i \neq 0$ is a real constant.

Since $A_{i1}$ is stable for $i = 1, \ldots N$, for any $\Lambda_i > 0$, the following Lyapunov equation has a unique solution $\Pi_i > 0$ such that

$$A_{i1}^{\tau} \Pi_i + \Pi_i A_{i1} = -\Lambda_i, \qquad i = 1, 2, \ldots, N. \tag{9.24}$$

For convenience, partition

$$T_i =: \begin{bmatrix} T_{i1} \\ T_{i2} \end{bmatrix}, \qquad T_i^{-1} =: \begin{bmatrix} W_{i1} & W_{i2} \end{bmatrix}, \tag{9.25}$$

where $T_{i1} \in \mathbb{R}^{2 \times 3}$ and $W_{i1} \in \mathbb{R}^{3 \times 2}$. Then, System (9.21)–(9.22) can be rewritten as

$$\dot{z}_{i1} = A_{i1}z_{i1} + A_{i2}z_{i2} + T_{i1}\varDelta M_i(T^{-1}z) \tag{9.26}$$

$$\dot{z}_{i2} = A_{i3}z_{i1} + A_{i4}z_{i2} + (1 - T'_{doi}\theta_i)v_{fi} + \theta_i T'_{doi}\varDelta P_{ei} + T'_{doi}\varPhi_i(x) + T_{i2}\varDelta M_i(T^{-1}z) \tag{9.27}$$

$$y_i = \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i, \qquad i = 1, 2, \ldots, N, \tag{9.28}$$

where $T^{-1} =: \text{diag}\left\{T_1^{-1}, T_2^{-1}, \ldots, T_N^{-1}\right\}$, $z_{i1} \in \mathbb{R}^2$ and $z_{i2} \in \mathbb{R}$. Now, consider the sliding surface (9.19) in the new coordinate system. From (9.23),

$$F_i \begin{bmatrix} 0 & C_{i2} \end{bmatrix} z_i = f_i z_{i2}$$

and since $f_i \neq 0$ it follows that the sliding surface (9.18) becomes

$$z_{i2} = 0, \qquad i = 1, 2, \ldots, N. \tag{9.29}$$

When System (9.26)–(9.28) is restricted to the sliding surface (9.29), it has the following form

$$\dot{z}_{i1} = A_{i1}z_{i1} + T_{i1}\varDelta M_i(Wz_1), \qquad i = 1, 2, \ldots, N, \tag{9.30}$$

where $z_1 =: \text{col}(z_{11}, 0, z_{21}, 0, \ldots, z_{N1}, 0)$, and $W =: \text{diag}\{W_{11}, 0, W_{21}, 0, \ldots, W_{N1}, 0\}$.

**Theorem 9.1** *For System (9.1)–(9.2), suppose Assumptions 9.1–9.3 are satisfied. Then, the sliding mode is asymptotically stable if there exists a domain $\varOmega \subset \mathbb{R}^{N\times(n-m)}$ including the origin, such that*

$$L^\tau + L > 0$$

*in $\varOmega \setminus \{0\}$, where $L \in \mathbb{R}^{N\times N}$ is given element-wise by*

$$L_{ij} = \begin{cases} \lambda_{min}(\varLambda_i) - 2\|\varPi_i T_{i1}\| \, \|W_{i1}\|\beta_{ii}(W_{i1}z_{i1}, 0), & i = j \\ -2\|\varPi_i T_{i1}\| \, \|W_{j1}\|\beta_{ij}(W_{j1}z_{j1}, 0), & i \neq j, \end{cases}$$

*where $\varPi_i$ and $\varLambda_i$ are defined in (9.24), and $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue of the matrix for $i, \ j = 1, 2 \ldots, N$.*

*Proof* From the analysis above, all that needs to be proved is that System (9.30) is asymptotically stable. For System (9.30), consider the Lyapunov function candidate

$$V = \sum_{i=1}^{N} (z_{i1})^\tau \, \varPi_i z_{i1}.$$

The time derivative of $V$ along the trajectories of System (9.30) is given by

$$\dot{V}\ |_{(9.30)} = \sum_{i=1}^{N}\Big\{-(z_{i1})^{\tau}\,\Lambda_i z_{i1} + 2\,(z_{i1})^{\tau}\,\Pi_i T_{i1}\Delta M_i(Wz_1)\Big\}, \tag{9.31}$$

where (9.24) is used to obtain the first term in the bracket. From Assumption 9.3

$$\begin{aligned}
\dot{V} &\le \sum_{i=1}^{N}\Big\{-\lambda_{min}(\Lambda_i)\|z_{i1}\|^2 + 2\|z_{i1}\|\,\|\Pi_i T_{i1}\|\sum_{j=1}^{N}\|\Delta M_i(Wz_1)\|\Big\}\\
&\le -\sum_{i=1}^{N}\lambda_{min}(\Lambda_i)\|z_{i1}\|^2 + 2\sum_{i=1}^{N}\Big\{\|z_{i1}\|\,\|\Pi_i T_{i1}\|\sum_{j=1}^{N}\beta_{ij}(W_{j1}z_{j1},0)\|W_{j1}\|\,\|z_{j1}\|\Big\}\\
&= -\sum_{i=1}^{N}\Big\{\lambda_{min}(\Lambda_i) - 2\beta_{ii}(W_{i1}z_{i1},0)\|\Pi_i T_{i1}\|\,\|W_{i1}\|\Big\}\|z_{i1}\|^2\\
&\quad +2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\beta_{ij}(W_{j1}z_{j1},0)\|\Pi_i T_{i1}\|\,\|W_{j1}\|\,\|z_{i1}\|\,\|z_{j1}\|\\
&= -\frac{1}{2}\big[\|z_{11}\|\ \|z_{21}\|\ \cdots\ \|z_{N1}\|\big](L^{\tau}+L)\begin{bmatrix}\|z_{11}\|\\ \|z_{21}\|\\ \vdots\\ \|z_{N1}\|\end{bmatrix}.
\end{aligned} \tag{9.32}$$

Then, the conclusion follows since $L^{\tau} + L > 0$ for $\mathrm{col}(z_{11}, z_{21}, \ldots, z_{N1}) \in \Omega \setminus \{0\}$. #

It should be emphasised that in Theorem 9.1, $L^{\tau} + L > 0$ only depends on the partial state variables $z_{i1}$ instead of the entire state variables $z_i$ (actually $x_i$). This is in contrast with the work [196, 214, 215]. As such, this result is less conservative.

Theorem 9.1 presents a condition under which the sliding mode dynamics is asymptotically stable. The next objective is to design a decentralised output feedback sliding mode control law such that the system state is driven to and maintained on the sliding surface.

### *9.3.3 Sliding Mode Control Synthesis*

Traditionally, the reachability condition (see for example [38, 173]) is described by

$$S^{\tau}(t)\dot{S}(t) < 0$$

for small-scale systems with switching function $S(t)$. However, for the interconnected system (9.1)–(9.2), the corresponding condition is described by

$$\sum_{i=1}^{N} \frac{S_i^\tau(x_i)\dot{S}_i(x_i)}{\|S_i(x_i)\|} < 0, \tag{9.33}$$

where $S_i(x_i)$ is defined by (9.19). For details see [69]. This condition is called composite reachability condition for the interconnected systems.

From (9.12) and (9.16), for $i = 1, 2, \ldots, N$

$$\begin{aligned}\|M_i(x) + \Delta M_i(x)\| &\le \sum_{j=1}^{N} \left(\gamma_{ij}^{I} |\sin x_{j1}| + \gamma_{ij}^{II} |x_{j2}|\right) + \sum_{j=1}^{N} \beta_{ij}(x_j)\|x_j\| \\ &=: \sum_{j=1}^{N} \eta_{ij}(x_j).\end{aligned} \tag{9.34}$$

In order to fully use system output information, consider the output matrix $C_i$. Comparing System (9.1)–(9.2) with (9.21)–(9.22), it follows that

$$C_i = \begin{bmatrix}0 & C_{i2}\end{bmatrix} T_i = C_{i2}\begin{bmatrix}0 & I_{p_i}\end{bmatrix} T_i, \qquad i = 1, 2, \ldots, N, \tag{9.35}$$

where $C_{i2}$ is nonsingular and satisfies (9.23). Splitting $T_i x_i$ into two components $(T_i x_i)_1 \in \mathbb{R}^{(3-p_i)}$ and $(T_i x_i)_2$, it follows that

$$x_i = T_i^{-1} T_i x_i = T_i^{-1} \begin{bmatrix}(T_i x_i)_1 \\ (T_i x_i)_2\end{bmatrix} = T_i^{-1} \begin{bmatrix}(T_i x_i)_1 \\ C_{i2}^{-1} y_i\end{bmatrix}. \tag{9.36}$$

Further, let

$$F_i C_i A_i T_i^{-1} =: \begin{bmatrix}\varUpsilon_{i1} & \varUpsilon_{i2}\end{bmatrix} \tag{9.37}$$

$$F_i C_i \begin{bmatrix}0 \\ T_{i,3}^{-1}\end{bmatrix} =: \begin{bmatrix}\varGamma_{i1} & \varGamma_{i2}\end{bmatrix}, \tag{9.38}$$

where $T_{i,3}^{-1}$ denotes the third row of the matrix $T_i^{-1}$, $\varUpsilon_{i1} \in \mathbb{R}^{1\times(3-p_i)}$ and $\varGamma_{i1} \in \mathbb{R}^{1\times(3-p_i)}$ for $i = 1, 2, \ldots, N$. Since

$$F_i C_i B_i = F_i C_i T_i^{-1} T_i B_i = F_i \begin{bmatrix}0 \; C_{i2}\end{bmatrix} \begin{bmatrix}0 \\ b_i\end{bmatrix} = \begin{bmatrix}0 \; f_i\end{bmatrix} \begin{bmatrix}0 \\ b_i\end{bmatrix} = f_i b_i$$

it follows that $F_i C_i B_i$ is nonsingular due to $f_i \neq 0$ and $b_i \neq 0$ for $i = 1, 2, \ldots, N$.

The objective is to satisfy the composite reachability condition (9.33). Consider System (9.1)–(9.2) in the domain $\mathscr{D} =: \mathscr{D}_1 \times \mathscr{D}_2 \times \cdots \times \mathscr{D}_N$, where $\mathscr{D}_i \in \mathbb{R}^3$ and explicitly

$$\mathscr{D}_i =: \left\{ x_i \mid x_i \in \mathbb{R}^3, \quad \|(T_i x_i)_1\| \le \mu_i \right\}, \qquad i = 1, 2, \ldots, N \tag{9.39}$$

for some positive constant $\mu_i$.

Then, the following control law is proposed for $i = 1, 2, \ldots, N$

$$v_{fi} = -\frac{1}{1-\alpha_i}(F_i C_i B_i)^{-1}\text{sign}(F_i y_i)\Big[\|\varUpsilon_{i2} C_{i2}^{-1} y_i\| + \frac{\alpha_i}{T'_{doi}}\|\varGamma_{i2} C_{i2}^{-1} y_i\| + k_i(y_i)\Big], \tag{9.40}$$

where $\text{sign}(\cdot)$ represents the signum function, $F_i$ is defined by (9.19) and can be designed by the approach in [37, 38], $\alpha_i$ is determined by Assumption 9.3, and $k_i(y_i) \ge 0$ is a control gain to be designed later. Obviously, the control law (9.40) depends only on system outputs and is decentralised.

**Theorem 9.2** *Consider the nonlinear interconnected system (9.1)–(9.2). Under Assumptions 9.1–9.3, the decentralised sliding mode control (9.40) drives the system (9.1)–(9.2) to the composite sliding surface (9.18) and maintains a sliding motion in the domain $\mathscr{D}$ if the control gain function $k_i(y_i)$ satisfies*

$$k_i(y_i) > \left(\|\varUpsilon_{i1}\| + \frac{\alpha_i}{T'_{doi}}\|\varGamma_{i1}\|\right)\mu_i + \sum_{j=1}^{N}\|F_j C_j\|\eta_{ji}(x_i), \tag{9.41}$$

*where $F_i$ and $\eta_{ji}$ are determined by (9.19) and (9.34) respectively for $i, j = 1, 2, \ldots, N$ and $\mathscr{D}$ is defined by (9.39).*

*Proof* It is necessary to prove that the composite reachability condition (9.33) is satisfied.

From (9.19), (9.37), (9.38) and the structures of $B_i$ and $\varDelta B_i$, the sliding mode dynamics of the system (9.1)–(9.2) can be described by

$$\begin{aligned}\dot{S}_i(x_i) &= F_i C_i (A_i + \varDelta A_i) x_i + F_i C_i (B_i + \varDelta B_i) v_{fi} + F_i C_i \left[M_i(x) + \varDelta M_i(x)\right] \\ &= (\varUpsilon_{i1} + \theta_i \varGamma_{i1})(T_i x_i)_1 + (\varUpsilon_{i2} + \theta_i \varGamma_{i2}) C_{i2}^{-1} y_i + F_i C_i B_i \left(1 - \theta_i T'_{doi}\right) v_{fi} \\ &\quad + F_i C_i \left[M_i(x) + \varDelta M_i(x)\right]\end{aligned} \tag{9.42}$$

for $i = 1, 2 \ldots, N$. Substituting (9.40) into (9.42), it follows that

$$\begin{aligned}&\sum_{i=1}^{N}\frac{S_i^\tau(x_i)\dot{S}_i(x_i)}{\|S_i(x_i)\|} \\ &\quad = \sum_{i=1}^{N}\frac{(F_i y_i)^\tau}{\|F_i y_i\|}\Bigg\{(\varUpsilon_{i2} + \theta_i \varGamma_{i2}) C_{i2}^{-1} y_i - \frac{1-\theta_i T'_{doi}}{1-\alpha_i}\text{sign}(F_i y_i)\Big(\|\varUpsilon_{i2} C_{i2}^{-1} y_i\| \\ &\qquad + \frac{\alpha_i}{T'_{doi}}\|\varGamma_{i2} C_{i2}^{-1} y_i\|\Big) + (\varUpsilon_{i1} + \theta_i \varGamma_{i1})(T_i x_i)_1 + F_i C_i \left[M_i(x) + \varDelta M_i(x)\right] \\ &\qquad - \frac{1-\theta_i T'_{doi}}{1-\alpha_i}\text{sign}(F_i y_i) k_i(y_i)\Bigg\}.\end{aligned} \tag{9.43}$$

From Assumption 9.3,

$$1-\theta_i T'_{doi} \geq 1-\left|\theta_i T'_{doi}\right| \geq 1-\alpha_i > 0. \tag{9.44}$$

Then, for $i = 1, 2, \ldots, N$

$$\begin{aligned}
&\frac{(F_i y_i)^\tau}{\|F_i y_i\|}\left\{(\Upsilon_{i2}+\theta_i\Gamma_{i2})\,C_{i2}^{-1}y_i - \frac{1-\theta_i T'_{doi}}{1-\alpha_i}\operatorname{sign}(F_i y_i)\Big(\|\Upsilon_{i2}C_{i2}^{-1}y_i\|\right.\\
&\quad\left.+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i2}C_{i2}^{-1}y_i\|\Big)\right\}\\
&=\frac{(F_i y_i)^\tau}{\|F_i y_i\|}\Big(\Upsilon_{i2}C_{i2}^{-1}y_i+\theta_i\Gamma_{i2}C_{i2}^{-1}y_i\Big)-\frac{1-\theta_i T'_{doi}}{1-\alpha_i}\Big(\|\Upsilon_{i2}C_{i2}^{-1}y_i\|+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i2}C_{i2}^{-1}y_i\|\Big)\\
&\leq \|\Upsilon_{i2}C_{i2}^{-1}y_i\|+|\theta_i|\,\|\Gamma_{i2}C_{i2}^{-1}y_i\|-\|\Upsilon_{i2}C_{i2}^{-1}y_i\|-\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i2}C_{i2}^{-1}y_i\|\\
&\qquad\qquad\qquad = \left(|\theta_i|-\frac{\alpha_i}{T'_{doi}}\right)\|\Gamma_{i2}C_{i2}^{-1}y_i\|\\
&\qquad\qquad\qquad \leq 0,
\end{aligned} \tag{9.45}$$

and from (9.34)

$$\begin{aligned}
&\frac{(F_i y_i)^\tau}{\|F_i y_i\|}\left\{(\Upsilon_{i1}+\theta_i\Gamma_{i1})\,(T_i x_i)_1 + F_i C_i\left[M_i(x)+\Delta M_i(x)\right]\right.\\
&\quad\left.-\frac{1-\theta_i T'_{doi}}{1-\alpha_i}\operatorname{sign}(F_i y_i)k_i(y_i)\right\}\\
&=\frac{(F_i y_i)^\tau}{\|F_i y_i\|}\Big[(\Upsilon_{i1}+\theta_i\Gamma_{i1})\,(T_i x_i)_1+F_i C_i\left[M_i(x)+\Delta M_i(x)\right]\Big]-\frac{1-\theta_i T'_{doi}}{1-\alpha_i}k_i(y_i)\\
&\leq\left(\|\Upsilon_{i1}\|+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i1}\|\right)\|(T_i x_i)_1\|+\|F_i C_i\|\,\|M_i(x)+\Delta M_i(x)\|-k_i(y_i)\\
&\leq\left(\|\Upsilon_{i1}\|+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i1}\|\right)\|(T_i x_i)_1\|+\|F_i C_i\|\sum_{j=1}^{N}\eta_{ij}(x_j)-k_i(y_i),
\end{aligned} \tag{9.46}$$

where (9.44) is used to establish the first inequality.

Now, substituting (9.45) and (9.46) into (9.43), in the domain $\mathscr{D}$

$$\begin{aligned}
&\sum_{i=1}^{N}\frac{S_i^\tau(x_i)\dot{S}_i(x_i)}{|S_i(x_i)|}\\
&\quad\leq\sum_{i=1}^{N}\left\{\Big(\|\Upsilon_{i1}\|+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i1}\|\Big)\mu_i+\|F_i C_i\|\sum_{j=1}^{N}\eta_{ij}(x_j)-k_i(y_i)\right\}\\
&\quad=\sum_{i=1}^{N}\left\{\Big[\Big(\|\Upsilon_{i1}\|+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i1}\|\Big)\mu_i+\sum_{j=1}^{N}\|F_j C_j\|\eta_{ij}(x_i)\Big]-k_i(y_i)\right\}.
\end{aligned} \tag{9.47}$$

Then, if $k_i(y_i)$ is chosen to satisfy (9.41), it follows that in the domain $\mathscr{D}$

$$\sum_{i=1}^{N} \frac{S_i^{\tau}(x_i)\dot{S}_i(x_i)}{|S_i(x_i)|} < 0.$$

Hence, the result follows. #

*Remark 9.4* It should be noted that Inequality (9.41) can be satisfied globally only in some specific cases. However, it can always be satisfied in the arbitrarily large domain $\mathscr{D}$ with $\mu_i < \infty$ for $i = 1, 2, \ldots, N$ if the control gain $k_i(y_i)$ is sufficiently high. In fact, one conservative choice of $k_i(y_i)$ is

$$k_i(y_i) > \left(\|\Upsilon_{i1}\| + \frac{\alpha_i}{T'_{doi}}\|\Gamma_{i1}\|\right)\mu_i + \sum_{j=1}^{N}\|F_j C_j\| \max_{x_i \in \mathscr{D}_i}\left\{\eta_{ji}\left((T_i x_i)_1, C_{i2}^{-1} y_i\right)\right\}$$

for $i = 1, 2, \ldots, N$.

*Remark 9.5* From the analysis above, it is observed that there is no special requirement on the interconnections $M_i(x_i)$ for $i = 1, 2, \ldots, N$. Only their bounds are assumed to be known. This shows that the approach is applicable to the multimachine power system which has high nonlinearity and coupling.

*Remark 9.6* From (9.8) and (9.40), the designed excitation control for the original multimachine power system is as follows

$$\begin{aligned} u_{fi} = &-\frac{1}{I_{qi}K_{ci}}\Bigg[\frac{1}{1-\alpha_i}(F_i C_i B_i)^{-1}\mathrm{sign}(F_i y_i)\Big(\|\Upsilon_{i2}C_{i2}^{-1}y_i\| \\ &+\frac{\alpha_i}{T'_{doi}}\|\Gamma_{i2}C_{i2}^{-1}y_i\| + k_i(y_i)\Big) + (x_{di} - x'_{di})I_{qi}I_{di} + P_{mi0} \\ &+T'_{doi}Q_{ei}\omega_i\Bigg], \quad i = 1, 2, \ldots, N. \end{aligned} \tag{9.48}$$

*Remark 9.7* According to sliding mode control theory, Theorems 9.1 and 9.2 show that the closed-loop system resulting from the designed control law (9.48) and system (9.1)–(9.2) is asymptotically stable. Moreover, under Assumptions 9.1–9.3, the multimachine power system is globally stabilised by (9.48) if for $i,\ j = 1, 2, \ldots, N$,

(i) $L^T + L > 0$ is satisfied globally;
(ii) $\Upsilon_{i1} = 0$ and $\Gamma_{i1} = 0$;
(iii) $\eta_{ij}(x_i)$ is bounded by a function of $y_i$.

## 9.4 Simulation for the Three-Machine Power System

Consider the three-machine power system shown in Fig. 9.2, where the generator 3 is an infinite busbar being used as a reference.

This system is also called the two-machine infinite bus power system (see [67]). The simulation parameters listed in Appendix E.2 are chosen as in [67, 193]. Then it follows that

$$\begin{array}{ll} |P_{e1}|_{\max} = |Q_{e1}|_{\max} = 1.4, & |P_{e2}|_{\max} = |Q_{e2}|_{\max} = 1.5 \\ |T'_{do1}|_{\min} = 6.21\,\text{s}, & |T'_{do2}|_{\min} = 7.614\,\text{s} \end{array}.$$

As in [193], take

$$\Delta T'_{doi} = 0.1 T'_{doi}$$

for $i = 1, 2$. With the chosen value of $\Delta T'_{doi}$, it follows that Eq. (9.15) is satisfied for

$$\alpha_1 = \alpha_2 = 0.1.$$

In addition, assume

$$\|\Delta M_1\| = \|\Delta M_2\| \leq (x_{13} - 0.0025 x_{11})^2 \|x_1\|^2 + 0.006\|x_2\|.$$

Then, from (9.1)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.625 & -39.27 \\ 0 & 0 & -0.1449 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1449 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

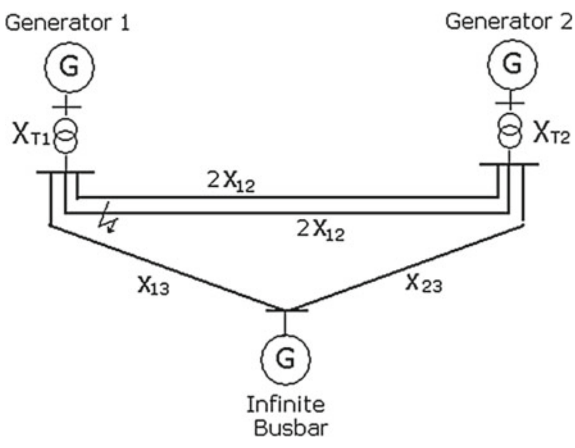


**Fig. 9.2** A three-machine power system

and

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.2941 & -30.8 \\ 0 & 0 & -0.1256 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1256 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

where $C_1$ and $C_2$ are assumed to be the system output matrices.

Obviously, Assumption 9.1 is satisfied. Let

$$K_1 = -0.0025, \qquad K_2 = -0.0008.$$

Then according to the algorithm given by Edwards and Spurgeon [37, 38], it can be verified that Assumption 9.2 is satisfied, and the appropriate transformation matrices (9.25) are given by

$$T_1 = \left[\frac{T_{11}}{T_{12}}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline -0.0025 & 0 & 1 \end{array}\right], \qquad T_2 = \left[\frac{T_{21}}{T_{22}}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline -0.0008 & 0 & 1 \end{array}\right]$$

and consequently

$$W_{11} = \begin{bmatrix} 0 & 1.0000 \\ 1.0000 & 0 \\ 0 & 0.0025 \end{bmatrix}, \qquad W_{21} = \begin{bmatrix} 0 & 1.0000 \\ 1.0000 & 0 \\ 0 & 0.0008 \end{bmatrix}.$$

In the new $z_i$ coordinate system, the special representation of the triple in (9.14) takes the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} = \left[\begin{array}{cc|c} -0.6250 & -0.0982 & -39.27 \\ 1.0000 & 0 & 0 \\ \hline -0.0025 & -0.0004 & -0.1449 \end{array}\right]$$

$$\begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} = \left[\begin{array}{cc|c} -0.2941 & -0.0246 & -30.8000 \\ 1.0000 & 0 & 0 \\ \hline -0.0008 & -0.0001 & -0.1256 \end{array}\right]$$

and

$$C_{12} = \begin{bmatrix} 0.0025 & 1 \\ 1 & 0 \end{bmatrix} \qquad C_{22} = \begin{bmatrix} 0.0008 & 1 \\ 1 & 0 \end{bmatrix}.$$

The associated switching functions matrices from (9.17) are

$$F_1 = \begin{bmatrix} 1 & -0.0025 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & -0.0008 \end{bmatrix}.$$

Choosing $\Lambda_1 = I_2$, $\Lambda_2 = 0.1 I_2$ and solving the Lyapunov equations (9.24) yields

$$\Pi_1 = \begin{bmatrix} 8.9466 & 5.0916 \\ 5.0916 & 4.0608 \end{bmatrix}$$

and

$$\Pi_2 = \begin{bmatrix} 7.0810 & 2.0325 \\ 2.0325 & 0.7720 \end{bmatrix}.$$

Since in the sliding surface

$$x_{13} - 0.0025x_{11} = 0$$

it is straightforward to see that

$$\beta_{11}(W_{11}z_{11}, 0) = 0, \qquad \text{and} \qquad \beta_{21}(W_{11}z_{11}, 0) = 0$$

and further

$$\beta_{12}(W_{21}z_{21}, 0) = 0.006, \qquad \text{and} \qquad \beta_{22}(W_{21}z_{21}, 0) = 0.006.$$

By direct computation,

$$L + L^\tau = \begin{bmatrix} 2.0000 & -0.1458 \\ -0.1458 & 0.0157 \end{bmatrix} > 0.$$

Then, from Theorem 9.1 the designed sliding mode is globally asymptotic stable. From Theorem 9.2, the three-machine power system is stabilised by the control law

$$\begin{aligned} v_{f1}(y_1) = &-\frac{1}{0.9 \times 0.1449}\text{sign}(y_{11} - 0.0025y_{12})\Big(0.1449|y_{11}| \\ &+\frac{1}{69}|y_{11} - 0.0025y_{12}| + k_1(y_1)\Big) \end{aligned} \tag{9.49}$$

$$\begin{aligned} v_{f2}(y_2) = &-\frac{1}{0.9 \times 0.1256}\text{sign}(y_{21} - 0.0008y_{22})\Big(0.1256|y_{21}| \\ &+\frac{10}{796}|y_{21} - 0.0008y_{22}| + k_2(y_2)\Big) \quad , \end{aligned} \tag{9.50}$$

where

$$k_1(y_1) = 2.9025\mu_1 + 1.8036|\sin y_{12}| + (y_{11} - 0.0025y_{12})(y_{11}^2 + y_{12}^2 + \mu_1^2) + 0.5,$$
$$k_2(y_2) = 2.9008\mu_2 + 1.471|\sin y_{22}| + 0.012\sqrt{y_{21}^2 + y_{22}^2 + \mu_2^2} + 0.5.$$

The original control signals $u_{f1}$ and $u_{f2}$ can be obtained from (9.48).

For simulation purposes, let

$$\mu_1 = \mu_2 = 5.$$

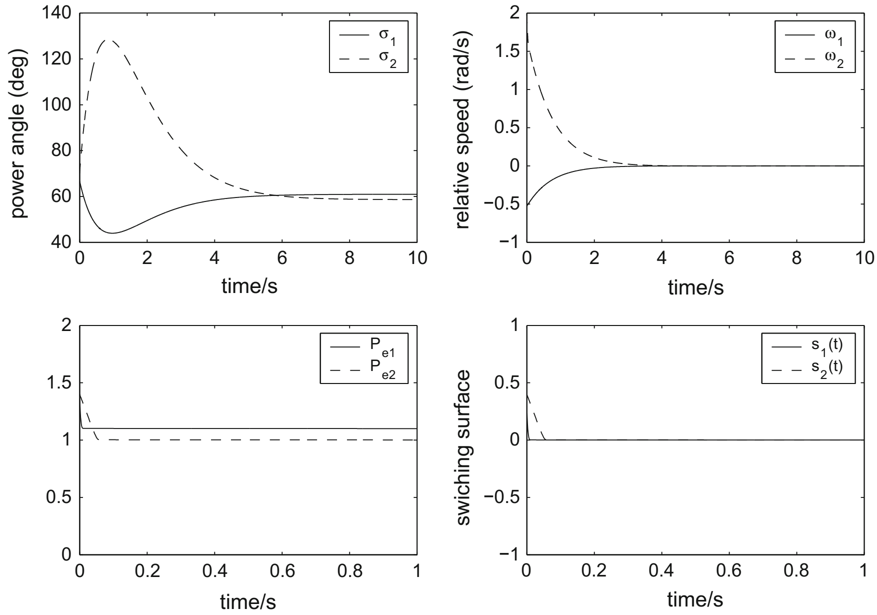


**Fig. 9.3** The time responses of the three-machine power system under control (9.49)–(9.50)

The operating point is chosen as

$$\delta_{10} = 60.98^\circ, \qquad \delta_{20} = 58.62^\circ$$
$$\omega_{10} = \omega_{20} = 0 \text{ r/s}, \; P_{m10} = 1.1 \text{ p.u.}, \qquad P_{m20} = 1.0 \text{ p.u.}$$

Simulation results with initial conditions

$$x_0 = (0.05, -0.5, 0.3, 0.1, 2, 0.4)$$

are presented in Fig. 9.3 to verify that the results are as effective as expected.

## 9.5 Summary

This chapter has presented a sliding mode control strategy to stabilise multimachine power systems using only static output feedback. A composite sliding surface is initially formed and then a decentralised control scheme is synthesised which guarantees the reachability condition holds for the whole interconnected system. The developed results are convenient for practical design due to their static output feedback nature. Significant matched uncertainty and nonlinearities in the interconnection terms can be accommodated. Simulation shows that the results are effective.

# Chapter 10
# Concluding Remarks

Study of complex control systems has received much attention in order to satisfy the increasing requirement for system performance in the modern world. This book has presented some of the recent research work of the authors along with associated fundamentals in the area of variable structure control. There is great interest in the area of variable structure control, as high robustness is pursued by engineers working in a wide variety of application areas. The book has included various feedback frameworks including static output feedback control design, dynamical output feedback control design and reduced-order compensator-based feedback control for complex systems. Both time-delay dependent and independent control schemes have been presented for complex systems in the presence of time-delays. Centralised control for nonlinear systems and decentralised control for interconnected systems have been considered. Sliding mode observer-based fault detection and isolation strategies have also been discussed. Many examples and case studies with simulations have been provided to demonstrate the theoretical results, which also help readers to understand and apply the theoretical results provided in this book.

This book has focused on enhancing robustness to uncertainties and reducing conservatism of the theoretical results. All uncertainties considered in this book are nonlinear and bounded by nonlinear functions of the system states and/or delayed states, or outputs and/or delayed outputs. This is in comparison with other relevant work in which it is required that bounds on uncertainties satisfy linear growth condition [75, 84, 112, 132, 186, 201]. Both static and dynamic output feedback controllers are designed to stabilise complex control systems: the former is convenient for practical design but the developed results are usually conservative; the latter usually results in low conservatism but requires more resources in real implementation. All time-delays involved in this book are time varying, and the Lyapunov–Razumikhin approach is employed to deal with the time-delay. There is no limitation to the rate of change of the time-delay. Reconstruction/estimation for both system faults and sensor faults is considered using sliding mode observers. The results presented in this book are based on rigorous underpinning theory, but with wide practical applications.

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It should be pointed out that nearly all of the designed controllers in this book are variable structure which usually results in discontinuous systems and thus chattering may occur. Chattering may be harmful because it leads to low control accuracy and high wear of moving mechanical parts although chattering is tolerable for some systems such as power electronics. In order to overcome/attenuate chattering, the boundary layer approach was proposed in [13]. This provides a smooth control signal at the cost of control accuracy. The other choice is to apply higher order sliding mode techniques which achieve finite time convergence and yield continuous closed-loop systems [5, 9, 99, 153]. This area has not been considered in this book.

Since the systems considered in this book are complex and all developed results are mathematically rigorous, the proposed control schemes, fault detection and isolation strategies are complex and thus may be difficult to implement in real systems. How to implement the various theoretical control schemes presented in this book is a challenge for researchers and control engineers. Even from a theoretical point of view, the study of complex systems is far from mature. All of the existing results are for a limited class of complex nonlinear systems and nearly all of the obtained conditions are sufficient. The degree of conservatism in the results is important: how large is the class of systems and how conservative are the conditions. It should be noted that for many known nonlinear unforced systems, it is very difficult to know whether the nonlinear system is stable or not. The stabilisation problem for nonlinear control systems with uncertainties, delay and/or coupling is even more challenging. It is worth noting that this book, like most of the existing efforts on complex systems, focused on reduction of conservatism or enhancement of robustness provided the nominal systems have the desired performance or assuming that the controllers/observers have been well designed for the nominal system.

In the real world, there are many phenomena which need to be explored. Thus complex models are required to describe various phenomena, which will increase the complexity of the research. It is impossible to find a systematic way to study all complex systems as has been done for linear systems. Recall, at the beginning of the book, it was mentioned, from a general point of view, that nonlinearity, uncertainty/modelling error, time-delay and interconnection are sources of complexity. Some specific examples and remarks to help readers to further understand the complexity caused by these sources are now provided. This motivates suggestions for possible future work.

**Nonlinearity** is one of the main characteristics of complex systems. The systems studied in this book are either nonlinear or have nonlinear uncertainty (bounded by nonlinear functions). The behaviour of a nonlinear system is usually very hard to predict or control even for a specific nonlinear system. Instead of studying the nonlinear system itself, the book has focused on developing less conservative results to tolerate/reject the effects of uncertainties by using available information about the uncertainties. In this way the systems considered have the desired performance even in the presence of uncertainties provided that the corresponding nominal systems have the desired performance. Although studies on linear systems have become very mature, many ideas/results for linear systems cannot be extended to nonlinear cases. In connection with this, the following simple example is provided.

*Example 10.1* It is well-known that a simple linear system

$$\dot{x} = Ax \tag{10.1}$$

where $x \in \mathbb{R}^n$ is the system state and $A \in \mathbb{R}^{n\times n}$ is a constant matrix, is asymptotically stable if all the eigenvalues of the matrix $A$ lie in the left-half plane. However, this is not true for nonlinear systems. Consider the following 2nd-order nonlinear system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1 & \frac{1}{x_2^2(t)} \\ 0 & -1 \end{bmatrix}}_{A(x)} x(t) \tag{10.2}$$

where $x = \text{col}(x_1, x_2) \in \mathbb{R}^2$ is the system state and the initial condition $x_0$ is given by

$$x_0 := \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}. \tag{10.3}$$

It is clear to see that both the eigenvalues of the matrix $A(x)$ in (10.2) are negative for any $x \in \mathbb{R}^2 \setminus \{0\}$. However, the solution to the Eq. (10.1) with respect to the initial condition $x_0$ is not stable.

It can be seen that $x_2 = \frac{1}{2}e^{-t}$. Then using the integrating factor approach, it follows that $x_1 = e^t$. Therefore, the solution to System (10.1) with initial condition $x_0 = \text{col}(1, \frac{1}{2})$ is

$$x_1(t) = e^t$$
$$x_2(t) = \frac{1}{2}e^{-t}$$

which is not stable.

*Remark 10.1* Example 10.1 shows that a nonlinear system

$$\dot{x} = A(x)x \tag{10.4}$$

where $x \in \mathbb{R}^n$ and $A(x) \in \mathbb{R}^{n\times n}$, may not be stable even if all the eigenvalues of the matrix $A(x)$ are negative in the considered domain. In order to guarantee the stability of nonlinear system (10.4), extra conditions are required: detailed discussion is available in [3]. This is true for linear time-varying systems as well, that is, a time-varying system $\dot{x}(t) = A(t)x(t)$ may not be asymptotically stable even if for any $t \in \mathbb{R}$, all the real parts of the eigenvalues of matrix $A(t)$ lie on the open left-half plane. Like the well-known modern differential geometric approach for nonlinear systems proposed by Isidori [79], to explore new tools to study nonlinear control systems is interesting and challenging.

**An interconnected system** can be considered as a system composed of many lower order subsystems interacting with each other, for which decentralised strategies are preferred. It is well-known that even if all the isolated subsystems are stable/controllable/observable, the whole interconnected system may not be stable/controllable/observable, which implies that the interconnections affect the performance of the whole interconnected system. This book has shown that if the interconnections or the bounds on the uncertain interconnections have a 'superposition' property, their effects can be reduced/cancelled by designing a proper sliding mode controller even if only a decentralised scheme is employed. To deal with interconnections between the isolated subsystems is one of the main tasks for an interconnected system specifically when decentralised strategies are considered. The following example shows how much interconnection terms affect the whole system performance.

*Example 10.2* Consider the following nonlinear interconnected system

$$\dot{x}_1 = f(x_1) + \psi(x_1, x_2) \tag{10.5}$$

$$\dot{x}_2 = Ax_2 + Bu \tag{10.6}$$

where $x = \text{col}(x_1, x_2)$ with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, and $u \in \mathbb{R}^m$ are system states and control, respectively, the matrices $A$ and $B$ are constant with appropriate dimensions, and the term $\psi(\cdot)$ satisfies $\psi(x_1, 0) = 0$.

System (10.5) and (10.6) given in Example 10.2 can be considered as an interconnected system consisting of two subsystems where the interconnection exists in only the first subsystem which is the term $\psi(x_1, x_2)$ in (10.5). The study in [146] disclosed that even if the subsystem

$$\dot{x}_1 = f(x_1)$$

is globally asymptotically stable and the matrix pair $(A, B)$ is stabilisable, it is the interconnection term $\psi(x_1, x_2)$ which determines whether the whole system (10.5) and (10.6) are stabilisable or not.

*Remark 10.2* Example 10.2 shows that interconnections not only affect the whole system performance but sometimes they may dominate the whole interconnected system performance. This clearly demonstrates that the interconnections between subsystems greatly increase the complexity of the problem. How to employ the structure and the possible known information about the interconnection terms to design decentralised controllers to reduce/reject the effects of interconnections on the whole system is always significant for complex interconnected systems.

**Uncertainties** in a control system may destroy the system performance completely. A stable controlled system may become unstable if an uncertainty is added to the system. To enhance the performance of a control system, it is necessary to consider uncertainties experienced by the system when controllers are designed. This book has considered various uncertainties, and controllers have been designed to reduce/reject the effects of the uncertainties when their bounds are known. The following example shows that an asymptotically stable controlled system will become unstable even if a 'small' uncertainty is added to the controlled system.

*Example 10.3* Consider the following simple control system

$$\dot{x} = f(x) + u \tag{10.7}$$

where $x \in \mathbb{R}$ and $u \in \mathbb{R}$ are system state and control, respectively, and $f(x)$ is continuous in $\mathbb{R}$. Assume $x_0 = x(0)$ represents the initial condition.

The scalar system (10.7) can have any desired performance by designing an appropriate controller. It is straight forward to see that System (10.7) is globally stabilised by the controller, for example,

$$u = -x - f(x). \tag{10.8}$$

Now, consider the system

$$\dot{x} = f(x) + u + 2x^2 e^{-t}, \qquad x_0 = 2 \tag{10.9}$$

where $x_0 = 2$ is the initial condition. System (10.9) can be considered as a new system obtained by adding a nonlinear term $2x^2e^{-t}$ to the system (10.7) which can be considered as a disturbance on System (10.7). The term $2x^2e^{-t}$ has the following properties:

- It vanishes at the origin $x = 0$;
- It includes an exponentially damping factor $e^{-t}$.

However, System (10.9) cannot be stabilised by the controller (10.8). Actually the corresponding closed-loop system obtained by applying the controller (10.8) to system (10.9) is

$$\dot{x} = -x + 2x^2 e^{-t}, \qquad x(0) = 2\,. \tag{10.10}$$

Letting $z = 1/x$, the system (10.10) can be expressed as a standard first order linear differential equation. Then, using the integrating factor method, the solution of System (10.10) is given by

$$x = \frac{2}{-e^t + 2e^{-t}}\,. \tag{10.11}$$

It is clear to see that $x(t) \to \infty$ when $t \to \frac{1}{2}\ln 2$ and thus it is not stable.

*Remark 10.3* The example above shows that a 'small' uncertainty may destroy system performance. Other examples are available in [90]. This book has provided many results to deal with various uncertainties using bounds on uncertainties to enhance robustness. If bounds on the uncertainties are not available, some other approaches may be required to identify/estimate the bounds on uncertainties [54, 134].

**Time-delay** widely exists in reality. It should be noted that sometimes even a small delay may greatly affect the performance of a system; a stable system may become unstable, or chaotic behaviour may appear due to the delay in the system [130]. Time-delay usually results in unpredictable results and thus increases the complexity of the research. This book has considered both delay dependent and delay independent control design. Delay dependent control needs the time-delay to be known so that it can be used in the design and thus the obtained results are usually less conservative when compared with delay independent control. However, delay independent control can be applied to the case when the delay is unknown. The following example shows that a globally stabilised control systems may not be stabilised globally if there is a delay in the input channel.

*Example 10.4* Consider the 2nd order nonlinear control system

$$\dot{x}_1 = x_1 + x_1^4 x_2 \tag{10.12}$$

$$\dot{x}_2 = u(t) \tag{10.13}$$

where $\mathrm{col}(x_1, x_2) \in \mathbb{R}^2$ is the state and $u \in \mathbb{R}$ is input. It is easy to check, using the Lyapunov function $V = x_1^2 + x_2^2$, that the system (10.12) and (10.13) are stabilisable by feedback

$$u = -x_2 - x_1^5\,. \tag{10.14}$$

However, if the input has a constant delay $\tau > 0$, then System (10.12) and (10.13) are changed to the following time-delay systems

$$\dot{x}_1 = x_1 + x_1^4 x_2 \tag{10.15}$$

$$\dot{x}_2 = u(t - \tau). \tag{10.16}$$

It is shown in [125] that the closed-loop system formed by applying the control (10.14) to System (10.15) and (10.16) is not globally asymptotically stable.

*Remark 10.4* In this book, only state delay is considered, and both delay dependent and delay independent results have been provided. However, input delay and output delay were not considered. Example 10.4 shows that a delay in the input channel may destroy the performance of the controlled system. It is interesting to study complex systems in the presence of input delay and/or output delay based on the skills and knowledge provided in this book in the future.

It should be noted that there are many sources of complexity in control systems and only a few of them have been considered in this book. The examples and remarks have shown that nonlinearities, uncertainties/disturbances, time-delay and interconnections, make the behaviour of systems very difficult to predict and significantly increase the complexity of the research greatly. However, in order to describe various phenomena existing in the real world, and also satisfy increasing requirements for system performance, it is necessary to consider complex systems from the point of view of both theoretical research and practical application. It is helpful and feasible to build a research framework for a class of complex systems. Study on complex systems is an ongoing task for control researchers and engineers.

# Appendix A
# Results Used in Sect. 6.3

This section will provide some results employed in Sect. 6.3.

## A.1 Quadratic Lypunov Function for Time-Delay Systems

A result relating to quadratic Lyapunov functions for time-delay systems which is developed from the Razumikhin Theorem in Sect. 2.5, will be presented here.

Consider a time-delay system

$$\dot{x}(t) = f(t, x(t - d(t))) \tag{A.1}$$

with an initial condition

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where the function vector $f : \mathbb{R}^+ \times \mathscr{C}_{[-\overline{d},0]} \mapsto \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}_{[-\overline{d},0]}$) into bounded sets in $\mathbb{R}^n$; $d(t)$ is the time-delay and $\overline{d} := \sup_{t \in \mathbb{R}^+}\{d(t)\} < \infty$.

**Lemma A.1** *Consider System (A.1). If there exists a function $V_0(x) = x^T P x$ with $P > 0$ such that for $d \in [-\overline{d}, 0]$, the time derivative of $V_0$ along the solution of System (A.1) satisfies*

$$\dot{V}_0(t, x) \leq -q_1 \|x\|^2 \tag{A.2}$$

*if*

$$V_0(x(t - d)) \leq q_2 V_0(x(t))$$

*for some $q_1 > 0$ and $q_2 > 1$, then System (A.1) is uniformly asymptotic stable. Further, if all the conditions hold globally, then System (A.1) is globally uniformly asymptotic stable.*

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*Proof* Since $P > 0$, it is clear that

$$\lambda_{\min}(P)\|x\|^2 \leq V_0(x) \leq \lambda_{\max}(P)\|x\|^2.$$

Let

$$\zeta_1(\tau) = \lambda_{\min}(P)\tau^2$$

and

$$\zeta_2(\tau) = \lambda_{\max}(P)\tau^2.$$

It is straightforward to see that both $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ are class $\mathscr{K}_\infty$ functions and

$$\zeta_1(\|x\|) \leq V_0(x) \leq \zeta_2(\|x\|), \quad x \in \mathbb{R}^n.$$

Therefore, System (A.1) is uniformly asymptotically stable

Further, let

$$\zeta_3(\tau) = -q_1\tau^2 \qquad \text{and} \qquad \xi(\tau) = q_2\tau.$$

It is clear from $q_1 > 0$ and $q_2 > 1$ that for $\tau > 0$.

$$\xi(\tau) > \tau \qquad \text{and} \qquad \zeta_3(\tau) > 0.$$

Hence, the conclusion follows from (A.2) and Theorem 2.5. ∇

## A.2 Transformation and Lipschitz Condition

Two results associated with coordinate transformations will be presented. From the fact that $f(t, x, x_d)$ satisfies Lipschitz condition with respect to $x$ and $x_d$ in Assumption 6.6, it follows that for any $x$, $\hat{x}$ and $x_d$, $\hat{x}_d$ in the domain considered, and any $t \in \mathbb{R}^+$

$$\|f(t, x, x_d) - f(t, \hat{x}, \hat{x}_d)\| \leq \mathscr{L}_f \left\| \begin{bmatrix} x - \hat{x} \\ x_d - \hat{x}_d \end{bmatrix} \right\|. \tag{A.3}$$

**Lemma A.2** *Assume that $f(\cdot)$ satisfies (A.3) in a neighbourhood of the origin. Then there exists a neighbourhood of the origin such that for any $(w_1, w_2)$, $(\hat{w}_1, \hat{w}_2)$, $(w_{1d}, w_{2d})$ and $(\hat{w}_{1d}, \hat{w}_{d2})$ in the neighbourhood of the origin and $t \in \mathbb{R}^+$*

$$\left\|\delta(F, \hat{F})\right\| \leq \mathscr{L}_f \left\| \begin{bmatrix} w_2 - \hat{w}_2 \\ w_{2d} - \hat{w}_{2d} \end{bmatrix} \right\| \tag{A.4}$$

*where $\delta(F, \hat{F})$ is defined in (6.65).*

*Proof* For convenience, let

$$\mu(\xi_1, \xi_2) := \xi_2 - P_3^{-1} P_2^T \xi_1.$$

From (A.3), the definition of $F(\cdot)$ in (6.57), $y = x_1$ in (6.51) and the transformation (6.53), it follows that

$$\begin{aligned} \left\| \delta(F, \hat{F}) \right\| &= \| f\left(t, w_1, \mu(w_1, w_2), w_{1d}, \mu(w_{1d}, w_{2d})\right) \\ &\quad - f(t, w_1, \mu(w_1, \hat{w}_2), w_{1d}, \mu\left(w_{1d}, \hat{w}_{2d}\right)) \| \\ &\leq \mathscr{L}_f \left\| \begin{bmatrix} 0 \\ \mu(w_1, w_2) - \mu(w_1, \hat{w}_2)\| \\ 0 \\ \mu(w_{1d}, w_{2d}) - \mu(w_{1d}, \hat{w}_{2d}) \end{bmatrix} \right\| \\ &= \mathscr{L}_f \left\| \begin{bmatrix} w_2 - \hat{w}_2 \\ w_{2d} - \hat{w}_{2d} \end{bmatrix} \right\|. \end{aligned}$$

Hence the conclusion follows. ∇

**Lemma A.3** *Assume that the transformation $z = T_0 x$ is nonsingular. If (A.3) holds, then*

$$\left\| \left[ f(t, x_1, x_2, x_{1d}, x_{2d}) - f(t, x_1, \hat{x}_2, x_{1d}, \hat{x}_{2d}) \right]_{x=T_0^{-1}z} \right\|^2 \leq \mathscr{L}_f^2 \left(e^T e + e_d^T e_d\right) \tag{A.5}$$

*where $x_1 \in \mathbb{R}^p$, $\hat{x}_2 \in \mathbb{R}^{n-p}$ is given by Corollary 6.1, $e := x_2 - \hat{x}_2$ and $e_d := x_{2d} - \hat{x}_{2d}$.*

*Proof* Let $\hat{x} := \begin{pmatrix} x_1 \\ \hat{x}_2 \end{pmatrix}$. Then $\hat{x}_d = \begin{pmatrix} x_{1d} \\ \hat{x}_{2d} \end{pmatrix}$ and it follows from (A.3) that

$$\begin{aligned} &\left\| \left[ f(t, x_1, x_2, x_{1d}, x_{2d}) - f(t, x_1, \hat{x}_2, x_{1d}, \hat{x}_{2d}) \right]_{x=T_0^{-1}z} \right\|^2 \\ &= \left\| \left[ f(t, x, x_d) - f(t, \hat{x}, \hat{x}_d) \right]_{x=T_0^{-1}z} \right\|^2 \\ &\leq \mathscr{L}_f^2 \left\| \begin{bmatrix} x - \hat{x} \\ x_d - \hat{x}_d \end{bmatrix} \right\|^2 \\ &= \mathscr{L}_f^2 \left\| \begin{bmatrix} 0 \\ e \\ 0 \\ e_d \end{bmatrix} \right\|^2 \\ &= \mathscr{L}_f^2 \left(e^T e + e_d^T e_d\right). \end{aligned}$$

Hence the conclusion follows. ∇

Lemma A.3 shows that if (A.3) holds in a neighbourhood of the origin, then the Inequality (A.5) holds in the neighbourhood of the origin. Specifically, (A.5) holds during the sliding motion, which implies that

$$\|\delta_T(f, \hat{f})\|^2 \leq \mathscr{L}_f^2 \left(e^T e + e_d^T e_d\right) \tag{A.6}$$

where $\delta_T(f, \hat{f})$ is defined in (6.96).

# Appendix B
# Results Used in Sect. 6.4

This Appendix summarises the results used in Sect. 6.4.

## B.1 An Inequality

**Lemma B.1** *Let the matrix* $N_1 \in \mathbb{R}^{m\times n}$ *and suppose the vectors* $x \in \mathbb{R}^m$ *and* $y \in \mathbb{R}^n$. *Then, the inequality*

$$x^T N_1 y \leq \frac{1}{2\varepsilon} x^T N_1 N_2^{-1} N_1^T x + \frac{\varepsilon}{2} y^T N_2 y$$

*holds for any symmetric positive definite matrix* $N_2 \in \mathbb{R}^{n\times n}$ *and any positive constant* $\varepsilon$.

*Proof* For any $n \times n$ matrix $N_2 > 0$, $N_2^{\frac{1}{2}}$ is well defined and $N_2^{\frac{1}{2}} > 0$. Let vector

$$\vartheta := \sqrt{\frac{1}{2\varepsilon}} N_2^{-\frac{1}{2}} N_1^T x - \sqrt{\frac{\varepsilon}{2}} N_2^{\frac{1}{2}} y.$$

By direct computation, it follows that

$$\vartheta^T \vartheta = \frac{1}{2\varepsilon} x^T N_1 N_2^{-1} N_1^T x - x^T N_1 y + \frac{\epsilon}{2} y^T N_2 y.$$

Hence the conclusion follows from the fact that $\vartheta^T \vartheta \geq 0$. $\nabla$

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## B.2 Properties of the Signum Function and Summation

This section will provide an inequality for the signum function $\text{sgn}(x)$ when $x$ is a vector, and an equality relating to summation.

**Lemma B.2** *Assume $H_{ij} \in \mathbb{R}^{n_i \times p_j}$ with $n_i$ and $p_j$ positive integers, and $x = \text{col}(x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, n$. Then*

*(i)* $\|x\| \le x^T sgn(x)$ *where* $sgn(\cdot)$ *denotes the usual vector signum function;*

*(ii)* $\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ij} x_j = \sum_{i=1}^{n} \left( \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ji} \right) x_i.$

*Proof* [1]
(i) Let $x_i = \text{col}(x_{i1}, x_{i2}, \ldots, x_{in_i})$. It follows that for $i = 1, 2, \ldots, N$

$$\begin{aligned} x_i^T \text{sgn}(x_i) &= \begin{bmatrix} x_{i1} \ x_{i2} \ \ldots \ x_{in_i} \end{bmatrix} \begin{bmatrix} \text{sgn}(x_{i1}) \\ \text{sgn}(x_{i2}) \\ \vdots \\ \text{sgn}(x_{in_i}) \end{bmatrix} \\ &= |x_{i1}| + |x_{i2}| + \cdots + |x_{in_i}| \ge \|x_i\|. \end{aligned}$$

Therefore

$$\begin{aligned} x^T \text{sgn}(x) &= \begin{bmatrix} x_1^T \ x_2^T \ \ldots \ x_N^T \end{bmatrix} \begin{bmatrix} \text{sgn}(x_1) \\ \text{sgn}(x_2) \\ \vdots \\ \text{sgn}(x_N) \end{bmatrix} \\ &= x_1^T \text{sgn} x_1 + x_2^T \text{sgn} x_2 + x_N^T \text{sgn} x_N \\ &\ge \|x_1\| + \|x_2\| + \cdots + \|x_N\| \ge \|x\|. \end{aligned}$$

Hence conclusion (i) follows.
(ii) From the fact that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij} x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} H_{ij} x_j$$

it follows that

$$\begin{aligned} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ij} x_j &= \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij} x_j - H_{11} x_1 - H_{22} x_2 - \cdots - H_{nn} x_n \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} H_{ij} x_j - \sum_{j=1}^{n} H_{jj} x_j \\ &= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} H_{ij} x_j - H_{jj} x_j \right) \\ &= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} H_{ij} - H_{jj} \right) x_j = \sum_{i=1}^{n} \left( \sum_{\substack{j=1 \\ j\neq i}}^{n} H_{ji} \right) x_i. \end{aligned}$$

Hence the conclusion follows. ∇

[1] The proof of conclusion (i) is also available in [199].

# Appendix C
# Identification of a Class of Functions

This appendix will show a result used to identify a class of functions with special properties, which is employed in Sect. 7.2.

**Lemma C.1** *Let $\bar{d} > 0$, $\xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a class $\mathscr{K}$ function and $P \in \mathbb{R}^{n\times n}$ be symmetric positive definite. There exists $k > 1$ such that*

$$k\xi(r) \le \xi(c_0 r)$$

*in $r \in \mathbb{R}^+$ for some constant $c_0 > 0$. Then,*

*(i) the function $V(x) := \xi(x^T P x)$ $(x \in \mathbb{R}^n)$ is positive definite;*
*(ii) there exists a nondecreasing continuous function $w : R^+ \mapsto \mathbb{R}^+$ satisfying $w(r) > r$ for $r > 0$ such that for any $\theta \in [-\bar{d},\ 0]$,*

$$\|x(t+\theta)\| \le \eta\|x\| \quad \text{if} \quad V(x(t+\theta)) \le w\,(V(x(t)))$$

*where $\eta$ is a positive constant.*

*Proof* (i) From the definition of the class $\mathscr{K}$ function, $\xi(\cdot)$ is strictly increasing in $\mathbb{R}^+$ with $\xi(0) = 0$. Then from $P > 0$ and thus $x^T P x \ge 0$, it follows that if $x \ne 0$

$$V(x) = \xi(x^T P x) > \xi(0) = 0 \quad \text{and} \quad V(0) = 0 \Longleftrightarrow x = 0.$$

This shows that the function $V(x)$ is positive definite.

(ii) Let

$$w(r) := kr, \qquad (r \in \mathbb{R}^+).$$

It is straightforward to see from $k > 1$ that $w(\cdot)$ is nondecreasing and continuous satisfying $w(r) > r$ for $r > 0$. From the fact that $\xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is strictly increasing in $\mathbb{R}^+$ and $k\xi(r) \le \xi(c_0 r)$, it follows that when

$$V(x(t+\theta)) \le w(V(x)) = kV(x)$$

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for any $\theta \in [-\bar{d}, 0]$,

$$\begin{aligned}
&\xi\left(x^T(t+\theta)Px(t+\theta)\right) \leq k\xi\left(x^T Px\right) \leq \xi(c_0 x^T Px) \\
&\Longrightarrow\ x^T(t+\theta)Px(t+\theta) \leq c_0 x^T Px \\
&\Longrightarrow\ \underline{\lambda}(P)\|x(t+\theta)\|^2 \leq x^T(t+\theta)Px(t+\theta) \leq c_0 x^T Px \leq c_0\overline{\lambda}(P)\|x\|^2 \\
&\Longrightarrow\ \|x(t+\theta)\| \leq \sqrt{c_0\overline{\lambda}(P)/\underline{\lambda}(P)}\|x\|.
\end{aligned}$$

where $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ denote the maximum and minimum eigenvalues of the matrix $P$ respectively. Hence the conclusion follows by choosing

$$\eta \geq \sqrt{c_0\overline{\lambda}(P)/\underline{\lambda}(P)}$$

∇

*Remark C.1* It is straightforward to check that the function

$$V(x) = (x^T Px)^\delta$$

where the constant $\delta > 0$ and the matrix $P > 0$, satisfies the condition of Lemma C.1. Thus,

$$V(x) = (x^T Px)^\delta$$

have the properties stated in both (i) and (ii) of Lemma C.1. Furthermore, if $\delta \geq 1$, then the function $V(\cdot)$ is differentiable.

# Appendix D
# Lemmas for Chap. 8

This Appendix will present three lemmas used in Chap. 8.

## D.1 System Structure

**Lemma D.1** *Let* rank$[E \;\; D] = \tilde{q}$. *Then under Assumption 8.2 there exists a coordinate system in which the triple* $(A, [E \;\; D], C)$ *has the following structure:*

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad \begin{bmatrix} 0_{(n-p)\times r} & 0_{(n-p)\times q} \\ E_2 & D_2 \end{bmatrix}, \quad \begin{bmatrix} 0_{p\times(n-p)} & C_2 \end{bmatrix}\right) \tag{D.1}$$

*where* $A_1 \in \mathbb{R}^{(n-p)\times(n-p)}$, $C_2 \in \mathbb{R}^{p\times p}$ *is nonsingular and*

$$E_2 = \begin{bmatrix} 0_{(p-\tilde{q})\times r} \\ E_{22} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0_{(p-\tilde{q})\times q} \\ D_{22} \end{bmatrix} \tag{D.2}$$

*where* $E_{22} \in \mathbb{R}^{\tilde{q}\times r}$, *and the matrix* $D_{22} \in \mathbb{R}^{\tilde{q}\times q}$ *is offull rank. The sub-blocks* $A_1$ *and* $A_3$ *when partitioned have the following structure*

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0_{(n-p-l)\times l} & A_{22} \end{bmatrix}, \qquad \text{and} \qquad A_3 = \left[\begin{array}{cc} 0_{(p-\tilde{q})\times l} & A_{31} \\ \hline \multicolumn{2}{c}{A_{32}} \end{array}\right] \tag{D.3}$$

*where* $A_{11} \in \mathbb{R}^{l\times l}$ *and* $A_{31} \in \mathbb{R}^{(p-\tilde{q})\times(n-p-l)}$ *for some integer* $l \geq 0$. *The pair* $(A_{22}, A_{31})$ *is completely observable. The eigenvalues of* $A_{11}$ *are the invariant zeros of the triple* $(A, [E \;\; D], C)$.

*Proof* Without loss of generality, it can be assumed that the output matrix has the following form

$$C = \begin{bmatrix} 0 & I_p \end{bmatrix}. \tag{D.4}$$

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Partition $[E \;\; D]$ in a compatible way with $C$ as

$$[E \;\; D] = \begin{bmatrix} \tilde{E}_1 & \tilde{D}_1 \\ \tilde{E}_2 & \tilde{D}_2 \end{bmatrix} \tag{D.5}$$

where $\tilde{E}_1 \in \mathbb{R}^{(n-p)\times r}$ and $\tilde{D}_1 \in \mathbb{R}^{(n-p)\times q}$. It follows from (D.4) and (D.5) that $C[E \;\; D] = [\tilde{E}_2 \;\; \tilde{D}_2]$. From Assumption 8.2, $\text{rank}([\tilde{E}_2 \;\; \tilde{D}_2]) = \tilde{q}$, which implies that there exists a matrix $\tilde{X}$ such that

$$[\tilde{E}_1 \;\; \tilde{D}_1] = \tilde{X}[\tilde{E}_2 \;\; \tilde{D}_2] = [\tilde{X}\tilde{E}_2 \;\; \tilde{X}\tilde{D}_2]. \tag{D.6}$$

Furthermore, from the fact that $\text{rank}([\tilde{E}_2 \;\; \tilde{D}_2]) = \tilde{q} \leq p$, it follows that there exists a nonsingular matrix $\tilde{T} \in \mathbb{R}^{p\times p}$ such that

$$\tilde{T}[\tilde{E}_2 \;\; \tilde{D}_2] = \left[\begin{array}{c|c} 0_{(p-\tilde{q})\times r} & 0_{(p-\tilde{q})\times q} \\ E_{22} & D_{22} \end{array}\right] = [E_2 \;\; D_2] \tag{D.7}$$

where $E_{22} \in \mathbb{R}^{\tilde{q}\times r}$ and $D_{22} \in \mathbb{R}^{\tilde{q}\times q}$ is of full rank, i.e., has column rank $q$. Construct a nonsingular matrix

$$T_X = \begin{bmatrix} I_{n-p} & -\tilde{X} \\ 0 & \tilde{T} \end{bmatrix}.$$

Then, from (D.5)–(D.7), it follows that

$$T_X[E \;\; D] = \begin{bmatrix} 0_{(n-p)\times r} & 0_{(n-p)\times q} \\ 0_{(p-\tilde{q})\times r} & 0_{(p-\tilde{q})\times q} \\ E_{22} & D_{22} \end{bmatrix} = \begin{bmatrix} 0_{(n-p)\times r} & 0_{(n-p)\times q} \\ E_2 & D_2 \end{bmatrix}$$

$$CT_X^{-1} = [0_{p\times(n-p)} \;\; \tilde{T}^{-1}].$$

Obviously $D_{22}$ is of full rank since $D$ is of full rank. Letting $C_2 = \tilde{T}^{-1}$ gives the structure of the input and output distribution matrices in (D.1) and $E_2$ and $D_2$ have the structures in (D.2).

By the same reasoning as used in [38], and employing further changes of coordinates, the system matrix can be forced to have the structure given in (D.3) whilst preserving the structures of the input and output distribution matrices. #

*Remark D.1* It should be noted that the matrix $[E \;\; D]$ may not be full rank. Therefore, the conclusion in Lemma D.1 above cannot be directly obtained from Lemma 6.1 of [38]. From [38], it is straightforward to see that the nonnegative integer $l$ in Lemma D.1 above denotes the number of the invariant zeros of the triple $(A, [E \;\; D], C)$. It should be emphasised that the transformation employed in Lemma D.1 can be easily obtained from linear system theory.

## D.2 An Equivalent Condition

**Lemma D.2** *Under Assumption 8.3, there exists a matrix* $L \in \mathbb{R}^{(n-p)\times p}$ *which has a structure*

$$L = [L_1 \ \ 0_{(n-p)\times\tilde{q}}] \tag{D.8}$$

*with* $L_1 \in \mathbb{R}^{(n-p)\times(n-\tilde{q})}$ *such that* $A_1 + LA_3$ *is stable where* $A_1$ *and* $A_3$ *are given by (D.3).*

*Proof* From the fact that $(A_{22}, A_{31})$ is observable in Lemma D.1, it follows that there exists a matrix $L_{12} \in \mathbb{R}^{(n-p-l)\times(p-\tilde{q})}$ such that $A_{22} + L_{12}A_{31}$ is stable. Let

$$L = \left[\begin{array}{c|c} L_{11} & 0_{l\times\tilde{q}} \\ L_{12} & 0_{(n-p-l)\times\tilde{q}} \end{array}\right] := \left[\, L_1 \mid 0_{(n-p)\times\tilde{q}} \,\right].$$

Then, from the partition (D.3), it follows that

$$A_1 + LA_3 = \begin{bmatrix} A_{11} & A_{12} + L_{11}A_{31} \\ 0 & A_{22} + L_{12}A_{31} \end{bmatrix}.$$

From Lemma D.1 and Assumption 8.3, $A_{11}$ is stable. Therefore, $A_1 + LA_3$ is stable due to the stability of $A_{22} + L_{12}A_{31}$. #

**Lemma D.3** *Let* $E_{22}$ *and* $D_{22}$ *be given by (D.2). Then, the following conditions are equivalent:*

*(i)* $Im(E_{22}) \cap Im(D_{22}) = \{0\}$;
*(ii) there exists a nonsingular matrix* $W \in \mathbb{R}^{\tilde{q}\times\tilde{q}}$ *such that*

$$W[E_{22} \ \ D_{22}] = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \tag{D.9}$$

*where* $H_1 \in \mathbb{R}^{(\tilde{q}-q)\times r}$ *and* $H_2 \in \mathbb{R}^{q\times q}$ *is nonsingular.*

*Proof* "(ii) $\Longrightarrow$ (i)" Let $\eta \in \mathrm{Im}(E_{22}) \cap \mathrm{Im}(D_{22})$. Then, there exist $\eta_1 \in \mathbb{R}^r$ and $\eta_2 \in \mathbb{R}^q$ such that

$$E_{22}\eta_1 = D_{22}\eta_2 = \eta. \tag{D.10}$$

From (D.9) and (D.10),

$$W\eta = WE_{22}\eta_1 = \begin{bmatrix} H_1\eta_1 \\ 0 \end{bmatrix}$$

$$W\eta = WD_{22}\eta_2 = \begin{bmatrix} 0 \\ H_2\eta_2 \end{bmatrix}.$$

This implies that $W\eta = 0$ and thus $\eta = 0$ since $W$ is nonsingular. Hence (i) is true.

"(i) $\Longrightarrow$ (ii)" Since $D_{22}$ is of full rank, there exists a nonsingular matrix $W$ such that

$$WD_{22} = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \tag{D.11}$$

where $H_2$ is nonsingular. Then, partition $WE_{22}$ in a compatible way to (D.11) as

$$WE_{22} = \begin{bmatrix} H_1 \\ H_3 \end{bmatrix}. \tag{D.12}$$

From the fact that $\mathrm{Im}(E_{22}) \cap \mathrm{Im}(D_{22}) = \{0\}$ and $W$ is nonsingular, it is observed that

$$\mathrm{Im}(WE_{22}) \cap \mathrm{Im}(WD_{22}) = \{0\}.$$

Then, from (D.11), (D.12) and the nonsingularity of $H_2$, it is observed that $H_3 = 0$. Therefore,

$$W[E_{22} \;\; D_{22}] = [WE_{22} \;\; WD_{22}] = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}.$$

Hence the conclusion (ii) follows. #

# Appendix E
# Notation and Parameters for Multimachine Power Systems

This Appendix will show the notation and parameters used in the Sect. 1.5.5 and Chap. 9 associated with multimachine power systems.

## E.1 Notation for Multimachine Power Systems in Sect. 1.5.5

The following table shows the notation used in Sect. 1.5.5 and Chap. 9:

| | |
|---|---|
| $\delta_i$ | generator power angle [rad] |
| $P_{ei}$ | electrical power [p.u.] |
| $\omega_i$ | relative speed [rad/s] |
| $\omega_0$ | synchronous machine speed [rad/s] |
| $D_i$ | per unit damping constant |
| $H_i$ | inertia constant [s] |
| $E'_{qi}$ | transient EMF in the quadrature axis [p.u.] |
| $T'_{doi}$ | direct axis transient short circuit time constant [s] |
| $x_{di}$ | direct axis reactance [p.u.] |
| $x'_{di}$ | direct axis transient reactance [p.u.] |
| $B_{ij}$ | $i$-th row and $j$-th column element of nodal susceptance matrix at internal nodes after eliminating all physical buses [p.u.] |
| $I_{qi}$ | quadrature axis current [p.u.] |
| $K_{ci}$ | gain of the excitation amplifier [p.u.] |
| $u_{fi}$ | input of the SCR amplifier [p.u.] |
| $Q_{ei}$ | reactive power [p.u.] |
| $I_{di}$ | direct axis current [p.u.] |
| $P_{mi0}$ | mechanical input power [p.u.] |

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## E.2 Parameters Used in the Simulation in Sect. 9.4

The following table shows the value of the parameter values used in the simulation in Sect. 9.4:

| parameter | unit | Generator 1 | Generator 2 |
|---|---|---|---|
| $x_d$ | p.u. | 1.863 | 2.36 |
| $x'_d$ | p.u. | 0.257 | 0.319 |
| $x_{ad}$ | p.u. | 1.712 | 1.712 |
| $T'_{do}$ | s | 6.9 | 7.96 |
| $X_T$ | p.u. | 0.129 | 0.11 |
| $H$ | s | 4 | 5.1 |
| $D$ | p.u. | 5 | 3 |
| $K_c$ | | 1 | 1 |
| $\omega_0$ | rad/s : | 314.159 | |
| $X_{12} = 0.55$p.u., | | $X_{13} = 0.53$p.u., | $X_{23} = 0.60$p.u. |

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