Chapter 4
Temporal and Multidimensional Intuitionistic Fuzzy Logics

The first results in temporal intuitionistic fuzzy logic appeared in 1990 (see [1]) on the basis of ideas from [2]. However, the first example for their application was only proposed as early as 15 years later, in [3]. The concept of the temporal IFL was extended to the concept of multidimensional intuitionistic fuzzy logic in a series of papers of the author together with E. Szmidt and J. Kacprzyk.

4.1 Temporal Intuitionistic Fuzzy Logic

Let A be a formula and V be a truth-value function, which maps to A the ordered pair

$$V(A, t) = \langle \mu(A, t), \nu(A, t) \rangle,$$

where $\mu(A, t), \nu(A, t) \in [0, 1]$,

$$\mu(A, t) + \nu(A, t) \leq 1,$$

and $t \in T$ is a fixed time-moment, where T is a fixed set which we shall call "time-scale" and it is strictly ordered by the relation "<".

Let

$$T'(t) = \{t' \mid t' \in T \& t' < t\},$$

$$T''(t) = \{t'' \mid t'' \in T \& t'' > t\}.$$

In [1], for a given formula A and a time-moment t, the author defined the temporal intuitionistic fuzzy operators P, F, H, G, which are analogues of the operators from [2], and for which:

- $V(P(A, t)) = P(V(A), t) = \langle \mu(A, t'), \nu(A, t') \rangle,$

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where $t' \in T'$ satisfies the conditions:

(a) $\mu(A, t') - \nu(A, t') = \max_{t^* \in T'}(\mu(A, t^*) - \nu(A, t^*))$,
(b) if there exists more than one such element of T', then, t' is maximal;

- $V(F(A, t)) = \langle \mu(A, t''), \nu(A, t'') \rangle$,
  where $t'' \in T''$ satisfies the conditions:

(a) $\mu(A, t'') - \nu(A, t'') = \max_{t^* \in T''}(\mu(A, t^*) - \nu(A, t^*))$,
(b) if there exists more than one such element of T'', then, t'' is minimal;

- $V(H(A, t)) = \langle \mu(A, t'), \nu(A, t') \rangle$,
  where $t' \in T'$ satisfies the conditions:

(a) $\mu(A, t') - \nu(A, t') = \min_{t^* \in T'}(\mu(A, t^*) - \nu(A, t^*))$,
(b) if there exists more than one such element of T', then, t' is maximal;

- $V(G(A, t)) = \langle \mu(A, t''), \nu(A, t'') \rangle$,
  where $t'' \in T''$ satisfies the conditions:

(a) $\mu(A, t'') - \nu(A, t'') = \min_{t^* \in T''}(\mu(A, t^*) - \nu(A, t^*))$,
(b) if there exists more than one such element of T'', then, t'' is minimal.

In each of these four definitions, if time intervals T' or T'' are infinite, operations "max" and "min" must be changed with operations "sup" and "inf", respectively.

Theorem 4.1.1 *For every formula A and for every time-moment t:*

(a) $V(H(A, t)) = V(\neg(P(\neg A), t)))$;
(b) $V(G(A, t)) = V(\neg(F(\neg A), t)))$.

Proof (a) Let the formula A and the time-moment t be given. Then,

$$V(\neg(F(\neg A), t))) = \langle \mu(A, t'), \nu(A, t') \rangle$$

where t' is the maximal element of T' for which:

$$\nu(A, t') - \mu(A, t') = \max_{t* \in T'}(\nu(A, t*) - \mu(A, t*)).$$

Therefore, t' is the maximal element of T' for which:

$$\mu(A, t') - \nu(A, t') = \min_{t* \in T'}(\mu(A, t*) - \nu(A, t*)),$$

i.e.,

$$\langle \mu(A, t'), \nu(A, t') \rangle = V(H(A, t)).$$

This completes the proof. Assertion (b) is proved similarly. □

Theorem 4.1.2 *For every two formulas A and B, for every time-moment t for implication* $\rightarrow_4$*:*

(a) $H(A \rightarrow_4 B, t) \rightarrow_4 (P(A, t) \rightarrow_4 P(B, t))$;
(b) $G(A \rightarrow_4 B, t) \rightarrow_4 (F(A, t) \rightarrow_4 F(B, t))$;
(c) $\neg(P(\neg(A \rightarrow_4 B), t)) \rightarrow_4 (P(A, t) \rightarrow_4 P(B, t))$;
(d) $\neg(F(\neg(A \rightarrow_4 B), t)) \rightarrow_4 (F(A, t) \rightarrow_4 F(B, t))$.

are IFTs.

Proof (a) Let the formulas A and B, and the time-moment t be given. Then,

$$H(A \rightarrow_4 B, t) \rightarrow_4 (P(A, t) \rightarrow_4 P(B, t))$$

$$= H(\langle \max(\nu(A), \mu(B)), \min(\mu(A), \nu(B)), t\rangle)$$

$$\rightarrow_4 (\langle \mu(A, t_1), \nu(A, t_1)\rangle \rightarrow_4 \langle \mu(B, t_2), \nu(B, t_2)\rangle)$$

(where t_1 and t_2 are both maximal elements of T' for which the maximums of $\mu(A, t_1) - \nu(A, t_1)$ and of $\mu(B, t_2) - \nu(B, t_2)$ are achieved)

$$= \langle \max(\nu(A, t'), \mu(B, t')), \min(\mu(A, t'), \nu(B, t'))\rangle$$

$$\rightarrow \langle \max(\nu(A, t_1), \mu(B, t_2), \min(\mu(A, t_1), \nu(B, t_2))\rangle$$

(where t' is the maximal element of T' for which the minimum of $\max(\nu(A, t'), \mu(B, t')) - \min(\mu(A, t'), \nu(B, t'))$ is reached)

$$= \langle \max(\nu(A, t_1), \mu(B, t_2)), \min(\mu(A, t'), \nu(B, t')),$$

$$\min(\mu(A, t_1), \nu(B, t_2), \max(\nu(A, t'), \mu(B, t'))\rangle.$$

Then, we consider the expression:

$$X = \max(\nu(A, t_1), \mu(B, t_2)), \min(\mu(A, t'), \nu(B, t'))$$

$$- \min(\mu(A, t_1), \nu(B, t_2), \max(\nu(A, t'), \mu(B, t')).$$

1. If for $\overline{t_2} \in T'$: $\mu(B, \overline{t_2}) \geq \nu(B, \overline{t_2})$, then:

$$X \geq \mu(B, \overline{t_2}) - \nu(B, \overline{t_2}) \geq 0.$$

2. If for $\overline{t_2} \in T'$: $\mu(B, \overline{t_2}) < \nu(B, \overline{t_2})$, then:
   2.1. If for $\overline{t_1} \in T'$: $\nu(A, \overline{t_1}) \geq \mu(A, \overline{t_1})$, then:

$$X \geq \nu(A, \overline{t_1}) - \mu(A, \overline{t_1}) \geq 0;$$

   2.2. If for $\overline{t_1} \in T'$: $\nu(A, \overline{t_1}) < \mu(A, \overline{t_1})$, then:
   2.2.1. If for $\overline{t'} \in T'$: $\min(\mu(A, t'), \nu(B, t')) \geq \max(\nu(A, t'), \mu(B, t'))$, then

$$X \geq \min(\mu(A, t'), \nu(B, t')) - \max(\nu(A, t'), \mu(B, t')) \geq 0,$$

   2.2.2 Otherwise, for $t' \in T'$: $\min(\mu(A, t'), \nu(B, t')) < \max(\nu(A, t'), \mu(B, t'))$, then

$$X \geq \min(\mu(A, t_0), \nu(B, t_0)) - \min(\mu(A, t_0), \nu(B, t_0)) = 0.$$

Therefore, in all cases $X \geq 0$, i.e., (a) is valid.

Assertions (b)–(d) are proved analogically. □

A geometrical interpretation of the temporal IFL is given on Fig. 4.1.

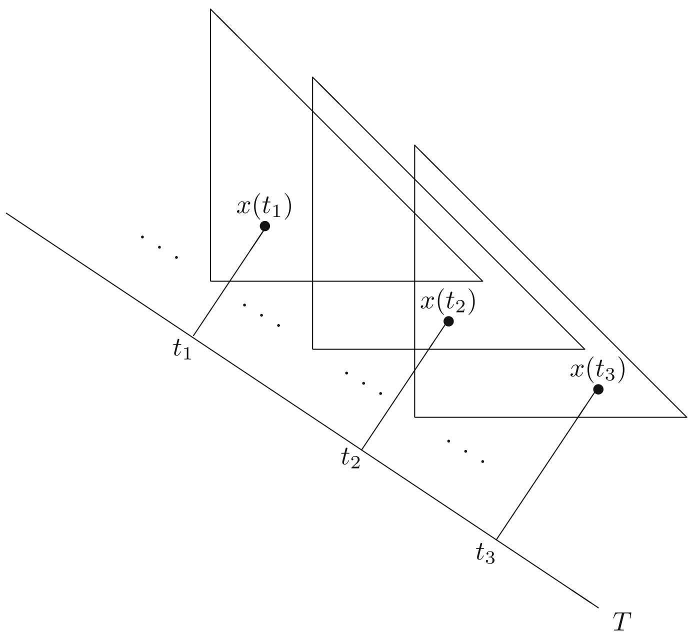

Fig. 4.1 Second geometrical interpretation of the temporal IFL

4.2 Multidimensional Intuitionistic Fuzzy Logics

Let $E, Z_1, Z_2, \ldots, Z_n$ be fixed finite linearly ordered sets.

By analogy with intuitionistic fuzzy multi-dimensional sets, introduced by E. Szmidt, J. Kacprzyk and the author in [4–7], here, following the paper [8] of the same authors and I. Georgiev, for a predicate P of the variables $x, z_1, z_2, \ldots, z_n$, ordered in the present form, we define an intuitionistic fuzzy evaluation function V for P in the form

$$V(P(x, z_1, z_2, \ldots, z_n)) = \langle \mu_P(x, z_1, z_2, \ldots, z_n), \nu_P(x, z_1, z_2, \ldots, z_n)\rangle,$$

where $x \in E$ is a (basic) variable, $z_1 \in Z_1$, $z_2 \in Z_2, \ldots, z_n \in Z_n$ are additional variables, $\mu_P(x, z_1, z_2, \ldots, z_n) \in [0, 1]$, $\nu_P(x, z_1, z_2, \ldots, z_n) \in [0, 1]$ and

$$\mu_P(x, z_1, z_2, \ldots, z_n) + \nu_P(x, z_1, z_2, \ldots, z_n) \leq 1.$$

Here, $\mu_P(x, z_1, z_2, \ldots, z_n)$ and $\nu_P(x, z_1, z_2, \ldots, z_n)$ are the degrees of validity and non-validity of $P(x, z_1, z_2, \ldots, z_n)$, respectively.

In the particular case, when $n = 1$, we obtain the case of temporal IFL (see [1]).

Having in mind the results from [4], we can define the following $(n + 1)$-dimensional intuitionistic fuzzy quantifiers:

(a) (partial) standard quantifier

$$V(\exists(x, z_1, z_2, \ldots, z_n)P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \max_{y\in E} \mu_P(y, z_1, z_2, \ldots, z_n), \min_{y\in E} \nu_P(y, z_1, z_2, \ldots, z_n)\right\rangle,$$

$$V(\forall(x, z_1, z_2, \ldots, z_n)P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \min_{y\in E} \mu_P(y, z_1, z_2, \ldots, z_n), \max_{y\in E} \nu_P(y, z_1, z_2, \ldots, z_n)\right\rangle$$

(b) (partial) i-quantifiers

$$V(\exists^i(x, z_1, z_2, \ldots, z_n)P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \max_{t_i\in Z_i} \mu_P(x, z_1, z_2, \ldots, z_{i-1}, t_i, z_{i+1}, \ldots, z_n), \right.$$

$$\left. \min_{t_i\in Z_i} \nu_P(x, z_1, z_2, \ldots, z_{i-1}, t_i, z_{i+1}, \ldots, z_n)\right\rangle,$$

$$V(\forall^i (x, z_1, z_2, \ldots, z_n) P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \min_{t_i \in Z_i} \mu_P(x, z_1, z_2, \ldots, z_{i-1}, t_i, z_{i+1}, \ldots, z_n), \right.$$

$$\left. \max_{t_i \in Z_i} \nu_P(x, z_1, z_2, \ldots, z_{i-1}, t_i, z_{i+1}, \ldots, z_n) \right\rangle$$

(c) general additional quantifier

$$V(\exists^a (x, z_1, z_2, \ldots, z_n) P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \max_{t_1 \in Z_1} \ldots \max_{t_n \in Z_n} \mu_P(x, t_1, t_2, \ldots, t_n), \min_{t_1 \in Z_1} \ldots \min_{t_n \in Z_n} \nu_P(x, t_1, t_2, \ldots, t_n) \right\rangle,$$

$$V(\forall^a (x, z_1, z_2, \ldots, z_n) P(x, z_1, z_2, \ldots, z_n)$$

$$= \left\langle \min_{t_1 \in Z_1} \ldots \min_{t_n \in Z_n} \mu_P(x, t_1, t_2, \ldots, t_n), \max_{t_1 \in Z_1} \ldots \max_{t_n \in Z_n} \nu_P(x, t_1, t_2, \ldots, t_n) \right\rangle$$

(d) general quantifier

$$V(\exists^g (x, z_1, z_2, \ldots, z_n) P(x, z_1, z_2, \ldots, z_n))$$

$$= \left\langle \max_{y \in E} \max_{t_1 \in Z_1} \ldots \max_{t_n \in Z_n} \mu_P(y, t_1, t_2, \ldots, t_n), \right.$$

$$\left. \min_{y \in E} \min_{t_1 \in Z_1} \ldots \min_{t_n \in Z_n} \nu_P(y, t_1, t_2, \ldots, t_n) \right\rangle,$$

$$V(\forall^g (x, z_1, z_2, \ldots, z_n) P(y, z_1, z_2, \ldots, z_n)$$

$$= \left\langle \min_{y \in E} \min_{t_1 \in Z_1} \ldots \min_{t_n \in Z_n} \mu_P(y, t_1, t_2, \ldots, t_n), \right.$$

$$\left. \max_{y \in E} \max_{t_1 \in Z_1} \ldots \max_{t_n \in Z_n} \nu_P(y, t_1, t_2, \ldots, t_n) \right\rangle$$

Theorem 4.2.1 *For each of the five pairs of quantifiers, the following equalities hold:*

$$\begin{aligned}
&V(\neg\exists(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\forall(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n)),\\
&V(\neg\forall(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\exists(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n)),\\
&V(\neg\exists^i(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\forall^i(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n)),\\
&V(\neg\forall^i(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\exists^i(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n)),\\
&V(\neg\exists^a(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\forall^a(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n),\\
&V(\neg\forall^a(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\exists^a(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n),\\
&V(\neg\exists^g(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\forall^g(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n),\\
&V(\neg\forall^g(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))\\
&= V(\exists^g(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n).
\end{aligned}$$

Proof Let us check the validity of the first equality.

$$V(\neg\exists(x, z_1, z_2, \dots, z_n)\neg P(x, z_1, z_2, \dots, z_n))$$

$$= \neg\exists(x, z_1, z_2, \dots, z_n)\neg\langle\mu_P(x, z_1, z_2, \dots, z_n), \nu_P(x, z_1, z_2, \dots, z_n)\rangle$$

$$= \neg\exists(x, z_1, z_2, \dots, z_n)\langle\nu_P(x, z_1, z_2, \dots, z_n), \mu_P(x, z_1, z_2, \dots, z_n)\rangle$$

$$= \neg\langle\max_{y\in E}\nu_P(y, z_1, z_2, \dots, z_n), \min_{y\in E}\mu_P(y, z_1, z_2, \dots, z_n)\rangle$$

$$= \langle\min_{y\in E}\mu_P(y, z_1, z_2, \dots, z_n), \max_{y\in E}\nu_P(y, z_1, z_2, \dots, z_n)\rangle$$

$$= V(\forall(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n)).$$

The other equalities are proved in the same manner. □

An important problem arises.

Open Problem 21. Which other negations, different from the ones defined in Sect. 1.4, also satisfy these equalities?

For a finite linearly ordered set X, $i \in \{1, \dots, 185\}$, $j \in \{1, 2, 3\}$ and $(n+1)$-dimensional predicate P we define

$$\overset{i,j}{\min_{x\in X}} P(x, y_1, \dots, y_n)$$

$$= P(x_1, y_1, \dots, y_n) \wedge_{i,j} P(x_2, y_1, \dots, y_n) \wedge_{i,j} \dots \wedge_{i,j} P(x_m, y_1, \dots, y_n),$$

$$\overset{i,j}{\max_{x\in X}} P(x, y_1, \dots, y_n)$$

$$= P(x_1, y_1, \dots, y_n) \vee_{i,j} P(x_2, y_1, \dots, y_n) \vee_{i,j} \dots \vee_{i,j} P(x_m, y_1, \dots, y_n),$$

where $x_1 < x_2 < \dots < x_m$ are the elements of X, listed in ascending order. Both operations produce a predicate of $y_1, \dots, y_n$.

Let us define

$$\overline{Q}_{i,j} = \begin{cases} \exists_{i,j}, & \text{if } Q_{i,j} \text{ is } \forall_{i,j} \\ \forall_{i,j}, & \text{if } Q_{i,j} \text{ is } \exists_{i,j} \end{cases}.$$

After these remarks, we continue with definitions of new quantifiers over the $(n+1)$-dimensional predicate P. Let for $1 \le i \le 185$ and for $1 \le j \le 3$: $Q_{i,j} \in \{\forall_{i,j}, \exists_{i,j}\}$. Let

$$\underset{x\in X}{Q_{i,j}} = \begin{cases} \max^{i,j}_{x\in X}, & \text{if } Q_{i,j} = \exists_{i,j} \\ \min^{i,j}_{x\in X}, & \text{if } Q_{i,j} = \forall_{i,j} \end{cases},$$

Then, for $i, i_1, \dots, i_n \in \{1, 2, \dots, 185\}$ and $j, j_1, \dots, j_n \in \{1, 2, 3\}$, we define:

(e) general Q-additional quantifiers

$$V((Q^1_{i_1,j_1}, Q^2_{i_2,j_2}, \dots, Q^n_{i_n,j_n})(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n))$$

$$= \left\langle \underset{t_1\in Z_1}{Q^1_{i_1,j_1}} \dots \underset{t_n\in Z_n}{Q^n_{i_n,j_n}} \mu_P(x, t_1, t_2, \dots, t_n), \underset{t_1\in Z_1}{\overline{Q}^1_{i_1,j_1}} \dots \underset{t_n\in Z_n}{\overline{Q}^n_{i_n,j_n}} \nu_P(x, t_1, t_2, \dots, t_n) \right\rangle,$$

(f) general Q-quantifier

$$V((Q_{i,j}, Q^1_{i_1,j_1}, Q^2_{i_2,j_2}, \dots, Q^n_{i_n,j_n})(x, z_1, z_2, \dots, z_n)P(x, z_1, z_2, \dots, z_n))$$

$$= \left\langle \underset{y\in E}{Q_{i,j}} \underset{t_1\in Z_1}{Q^1_{i_1,j_1}} \dots \underset{t_n\in Z_n}{Q^n_{i_n,j_n}} \mu_P(y, t_1, t_2, \dots, t_n), \right.$$

$$\left. \underset{y\in E}{\overline{Q}_{i,j}} \underset{t_1\in Z_1}{\overline{Q}^1_{i_1,j_1}} \dots \underset{t_n\in Z_n}{\overline{Q}^n_{i_n,j_n}} \nu_P(y, t_1, t_2, \dots, t_n) \right\rangle,$$

Theorem 4.2.2 *For each of the three quantifiers, the following equalities hold:*

$$V(\neg_1(Q^1_{i_1,j_1}, Q^2_{i_2,j_2}, \ldots, Q^n_{i_n,j_n})(x, z_1, z_2, \ldots, z_n)\neg_1 P(x, z_1, z_2, \ldots, z_n))$$

$$= V((\overline{Q}^1_{i_1,j_1}, \overline{Q}^2_{i_2,j_2}, \ldots, \overline{Q}^n_{i_n,j_n})(x, z_1, z_2, \ldots, z_n)P(x, z_1, z_2, \ldots, z_n))$$

$$V(\neg_1(Q_{i,j}, Q^1_{i_1,j_1}, Q^2_{i_2,j_2}, \ldots, Q^n_{i_n,j_n})(x, z_1, z_2, \ldots, z_n)\neg_1 P(x, z_1, z_2, \ldots, z_n))$$

$$= V((\overline{Q}_{i,j}, \overline{Q}^1_{i_1,j_1}, \overline{Q}^2_{i_2,j_2}, \ldots, \overline{Q}^n_{i_n,j_n})(x, z_1, z_2, \ldots, z_n)P(x, z_1, z_2, \ldots, z_n)).$$

Open Problem 22. Which of the equalities from Theorems 4.2.1 and 4.2.2. are valid for the new quantifiers?

The so defined multidimensional intuitionistic fuzzy quantifiers can obtain different applications in the area of artificial intelligence. For example, we can use them in procedures for decision making and for intercriteria analysis, in rules of intuitionistic fuzzy expert systems, and others.

All these multidimensional intuitionistic fuzzy quantifiers are first-order.

The author believe that in a near future possibilities for defining second and higher-order multidimensional intuitionistic fuzzy quantifiers will arise and some properties for standard predicates, discussed in [9–20] will be studied for the multidimensional intuitionistic fuzzy quantifiers.

In future, we will study the possibility to change the condition "Let $E, Z_1, Z_2, \ldots, Z_n$ be fixed finite linearly ordered sets" with which Sect. 4.2 started. When the properties of the new intuitionistic fuzzy conjunctions and disjunctions are studied, probably, we will be able to change this condition with the condition "Let $E, Z_1, Z_2, \ldots, Z_n$ be fixed finite partially ordered sets". So, the new constructions will give additional possibilities for application in some areas of the artificial intelligence.

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