Chapter 3
Intuitionistic Fuzzy Modal Logics

The first step of the development of the idea of intuitionistic fuzziness (see [1]), was related to introducing an intuitionistic fuzzy interpretation of the classical (standard) modal operators *"necessity"* and *"possibility"* (see, e.g., [2–5]). In the period 1988–1993, we defined eight new operators, extending the first two ones. In the end of last and in the beginning of this century, a lot of new operators were introduced. Here, we discuss the most interesting ones of them and study their basic properties.

3.1 Intuitionistic Fuzzy Classical Modal Operators

For the formula A for which $V(A) = \langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, following [1], we define the two modal operators "necessity" and "possibility":

$$V(\square A) = \langle a, 1 - a \rangle,$$

$$V(\lozenge A) = \langle 1 - b, b \rangle,$$

respectively.

It is suitable to define the evaluation function V so that:

$$V(\square A) = \square V(A),$$

$$V(\lozenge A) = \lozenge V(A).$$

Two different geometrical interpretations of both operators are given in Figs. 3.1, 3.2, 3.3 and 3.4, respectively.

Fig. 3.1 Second geometrical
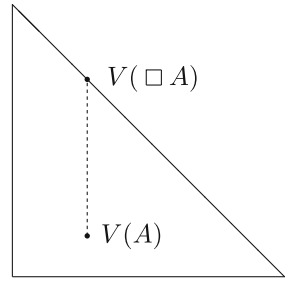interpretation of operator □

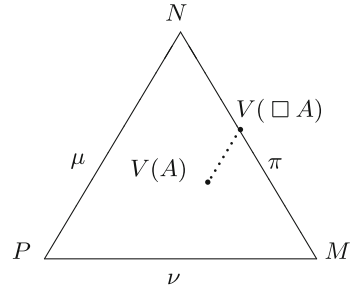
Fig. 3.2 Third geometrical
interpretation of operator □

Fig. 3.3 Second geometrical
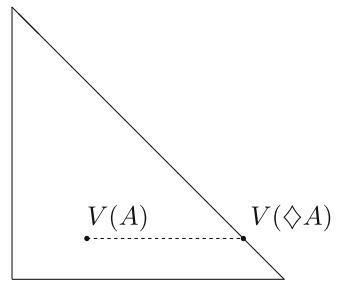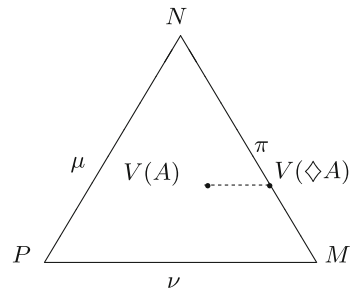interpretation of operator ◇

Fig. 3.4 Third geometrical
interpretation of operator ◇

It is obvious, that if p is a tautology, then, $\Box\, p$ and $\Diamond\, p$ are also tautologies. Moreover,

$$V(\Box\, p) \leq V(p) \leq V(\Diamond\, p),$$

where relation " $\leq$ " is defined as in (1.1.9).

Let everywhere below:

$$V(p) = \langle a, b \rangle,\ V(q) = \langle c, d \rangle,\ V(r) = \langle e, f \rangle,$$

where $a, b, c, d, e, f \in [0, 1], a + b \leq 1, c + d \leq 1, e + f \leq 1$.

Here, some of the most important assertions, related to the two classical (standard) modal operators, are formulated and proved for the intuitionistic fuzzy case.

First, following [1], we see that the basic properties of the (standard) modal operators in their intuitionistic fuzzy interpretations for every formula A are:

$$V(\Box\, \Box\, A) = V(\Box\, A),$$

$$V(\Box\, \Diamond\, A) = V(\Diamond\, A),$$

$$V(\Diamond\, \Box\, A) = V(\Box\, A),$$

$$V(\Diamond\, \Diamond\, A) = V(\Diamond\, A).$$

In classical modal logic, expressions

$$V(\neg\, \Box\, A) = V(\Diamond\, \neg A) \tag{3.1.1}$$

$$V(\Box\, A) = V(\neg\Diamond\, \neg A) \tag{3.1.2}$$

$$V(\neg\Diamond\, A) = V(\Box\, \neg A) \tag{3.1.3}$$

$$V(\Diamond\, A) = V(\neg\, \Box\, \neg A) \tag{3.1.4}$$

are (in some sense) equivalent. In the intuitionistic fuzzy case, similarly with De Morgan's Laws, the situation is different.

Theorem 3.1.1 *For every formula A,*

(a) *expression (3.1.1) is a tautology and an IFT for negations* $\neg_1, \neg_2, \neg_6, \neg_8, \neg_{13},$
    $\neg_{14}, \neg_{35}, \ldots, \neg_{38}, \neg_{40}, \neg_{42}, \ldots, \neg_{46}, \neg_{48}, \neg_{50}, \neg_{53},$
(b) *expression (3.1.2) is a tautology and an IFT for negations* $\neg_1, \neg_8, \neg_{53},$
(c) *expression (3.1.3) is a tautology and an IFT for negations* $\neg_1, \neg_3, \ldots, \neg_6, \neg_{11},$
    $\neg_{14}, \neg_{35}, \ldots, \neg_{37}, \neg_{39}, \neg_{41}, \ldots, \neg_{45}, \neg_{47}, \neg_{49}, \neg_{50}, \neg_{51},$
(d) *expression (3.1.4) is a tautology and an IFT for negations* $\neg_1, \neg_3, \neg_4.$

This is the first case, when a given expression is a tautology in all cases when it is an IFT. As we saw in the two previous chapters, only small number of IFTs are tautologies.

Theorem 3.1.2 *For every formula A, each of the expressions*

$$\Box A \rightarrow A,$$

$$A \rightarrow \Diamond A,$$

$$\Box A \rightarrow \Diamond A$$

is

(a) *a tautology for implications* $\rightarrow_2, \rightarrow_3, \rightarrow_5, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{15}, \rightarrow_{20}, \rightarrow_{23}, \rightarrow_{24},$ $\rightarrow_{27}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{34}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{42}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{49}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{57},$ $\rightarrow_{62}, \rightarrow_{63}, \rightarrow_{65}, \rightarrow_{68}, \rightarrow_{69}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{83}, \rightarrow_{84}, \rightarrow_{88}, \rightarrow_{92}, \rightarrow_{93},$ $\rightarrow_{97}, \rightarrow_{176}, \ldots, \rightarrow_{185},$

(b) *an IFT for implications* $\rightarrow_1, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{15}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20}, \ldots,$ $\rightarrow_{24}, \rightarrow_{27}, \ldots, \rightarrow_{38}, \rightarrow_{40}, \rightarrow_{42}, \rightarrow_{44}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \rightarrow_{57}, \rightarrow_{59}, \ldots, \rightarrow_{66},$ $\rightarrow_{68}, \rightarrow_{69}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \ldots, \rightarrow_{85}, \rightarrow_{88}, \ldots, \rightarrow_{94}, \rightarrow_{97},$ $\ldots, \rightarrow_{139}, \rightarrow_{141}, \rightarrow_{146}, \ldots, \rightarrow_{170}, \rightarrow_{176}, \ldots, \rightarrow_{185}.$

Theorem 3.1.3 *For every two formulas A and B, each of the equalities*

$$V(\Diamond(A \vee B)) = V(\Diamond A \vee \Diamond B) \tag{3.1.5}$$

$$V(\Box(A \wedge B)) = V(\Box A \wedge \Box B), \tag{3.1.6}$$

holds for the disjunction and conjunction, defined by (1.1.4) and (1.1.5).

Open Problem 12 Which other disjunctions and conjunctions, existing of which is discussed in Sect. 1.7, satisfy (3.1.5) and (3.1.6)?

Theorem 3.1.4 *For every two formulas A and B, each of the expressions*

$$\Box A \vee \Box B \rightarrow \Box(A \vee B)$$

$$\Diamond(A \wedge B) \rightarrow \Diamond A \wedge \Diamond B$$

is

(a) *a tautology for implications* $\rightarrow_2, \rightarrow_3, \rightarrow_5, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{15}, \rightarrow_{20},$ $\rightarrow_{23}, \rightarrow_{24}, \rightarrow_{27}, \rightarrow_{31}, \rightarrow_{32}, \rightarrow_{34}, \rightarrow_{37}, \rightarrow_{40}, \rightarrow_{42}, \rightarrow_{47}, \ldots, \rightarrow_{49}, \rightarrow_{52},$ $\rightarrow_{55}, \rightarrow_{57}, \rightarrow_{62}, \rightarrow_{63}, \rightarrow_{65}, \rightarrow_{68}, \rightarrow_{69}, \rightarrow_{74}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{83}, \rightarrow_{84}, \rightarrow_{88},$ $\rightarrow_{92}, \rightarrow_{93}, \rightarrow_{97}, \rightarrow_{176}, \ldots, \rightarrow_{185},$

(b) *an IFT for implications* $\rightarrow_1, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{15}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20}, \ldots,$
$\rightarrow_{24}, \rightarrow_{27}, \ldots, \rightarrow_{38}, \rightarrow_{40}, \rightarrow_{42}, \rightarrow_{44}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \rightarrow_{57}, \rightarrow_{59}, \ldots, \rightarrow_{66},$
$\rightarrow_{68}, \rightarrow_{69}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{74}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \ldots, \rightarrow_{85}, \rightarrow_{88}, \ldots, \rightarrow_{94}, \rightarrow_{97},$
$\ldots, \rightarrow_{139}, \rightarrow_{141}, \rightarrow_{146}, \ldots, \rightarrow_{170}, \rightarrow_{176}, \ldots, \rightarrow_{185}.$

Theorem 3.1.5 *For every two formulas A and B, the expression*

$$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

is

(a) *a tautology for implications* $\rightarrow_2, \rightarrow_3, \rightarrow_5, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{20}, \rightarrow_{24},$
$\rightarrow_{25}, \rightarrow_{27}, \rightarrow_{29}, \rightarrow_{47}, \ldots, \rightarrow_{49}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{60}, \rightarrow_{69}, \rightarrow_{77},$
$\rightarrow_{79}, \rightarrow_{81}, \rightarrow_{92}, \rightarrow_{93}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{177}, \rightarrow_{179}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{184},$
(b) *an IFT for implications* $\rightarrow_1, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{14}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20},$
$\rightarrow_{21}, \rightarrow_{24}, \rightarrow_{25}, \rightarrow_{27}, \ldots, \rightarrow_{29}, \rightarrow_{46}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \rightarrow_{57}, \ldots, \rightarrow_{61}, \rightarrow_{64},$
$\rightarrow_{66}, \rightarrow_{69}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{75}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \ldots, \rightarrow_{81}, \rightarrow_{91}, \ldots, \rightarrow_{94}, \rightarrow_{97},$
$\ldots, \rightarrow_{102}, \rightarrow_{108}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{120}, \ldots, \rightarrow_{128}, \rightarrow_{134}, \ldots, \rightarrow_{137}, \rightarrow_{139},$
$\rightarrow_{141}, \rightarrow_{147}, \rightarrow_{149}, \ldots, \rightarrow_{154}, \rightarrow_{156}, \rightarrow_{158}, \ldots, \rightarrow_{162}, \rightarrow_{165}, \ldots, \rightarrow_{167},$
$\rightarrow_{169}, \rightarrow_{177}, \rightarrow_{179}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{184}.$

Proof (a) Let, for example, the implication be considered in the variant $\rightarrow_{11}$. Then,

$$V(\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B))$$
$$= \Box\langle 1 - (1-c)\mathrm{sg}(a-c), d\,\mathrm{sg}(a-c)\mathrm{sg}(d-b)\rangle \rightarrow (\langle a, 1-a\rangle \rightarrow \langle c, 1-c\rangle)$$
$$= \langle 1 - (1-c)\mathrm{sg}(a-c), (1-c)\mathrm{sg}(a-c)\rangle$$
$$\rightarrow \langle 1 - (1-c)\mathrm{sg}(a-c), (1-c)\mathrm{sg}(a-c)^2\rangle$$
$$= (1 - (1 - (1 - (1-c)\mathrm{sg}(a-c))))\mathrm{sg}(a-c)$$
$$\mathrm{sg}((1 - (1-c)\mathrm{sg}(a-c)) - 1 - (1-c)\mathrm{sg}(a-c)),$$
$$(1-c)\mathrm{sg}(a-c)\mathrm{sg}(((1-c)\mathrm{sg}(a-c)) - (1-c)\mathrm{sg}(a-c))^2\rangle$$
$$= (1 - (1-c)\mathrm{sg}(a-c))\mathrm{sg}(a-c)\mathrm{sg}(0), (1-c)\mathrm{sg}(a-c)\mathrm{sg}(0)\rangle$$
$$= \langle 1, 0\rangle.$$

This completes the proof. □

Theorem 3.1.6 *For every two formulas A and B, the expression*

$$(\Box(A \rightarrow B) \wedge \Box A) \rightarrow \Box B$$

is

(a) *a tautology for implications* $\rightarrow_2, \rightarrow_3, \rightarrow_8, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{15}, \rightarrow_{20}, \rightarrow_{24},$
$\rightarrow_{25}, \rightarrow_{27}, \rightarrow_{29}, \rightarrow_{47}, \rightarrow_{48}, \rightarrow_{52}, \rightarrow_{55}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{60}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81},$
$\rightarrow_{92}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{177}, \rightarrow_{179},$
(b) *an IFT for implications* $\rightarrow_1, \ldots, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{15}, \rightarrow_{17}, \rightarrow_{18}, \rightarrow_{20},$
$\rightarrow_{21}, \rightarrow_{24}, \rightarrow_{25}, \rightarrow_{27}, \ldots, \rightarrow_{29}, \rightarrow_{46}, \ldots, \rightarrow_{53}, \rightarrow_{55}, \rightarrow_{57}, \ldots, \rightarrow_{61}, \rightarrow_{64},$

$\rightarrow_{66}, \rightarrow_{69}, \rightarrow_{71}, \ldots, \rightarrow_{73}, \rightarrow_{75}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \ldots, \rightarrow_{81}, \rightarrow_{91}, \ldots, \rightarrow_{94},$
$\rightarrow_{96}, \ldots, \rightarrow_{102}, \rightarrow_{106}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \ldots, \rightarrow_{128}, \rightarrow_{134}, \ldots, \rightarrow_{138},$
$\rightarrow_{151}, \rightarrow_{158}, \rightarrow_{161}, \rightarrow_{166}, \rightarrow_{167}, \rightarrow_{169}, \rightarrow_{177}, \rightarrow_{179}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{184}.$

Theorem 3.1.7 *For every two formulas A and B, the expression*

$$\Box A \rightarrow (\Box (A \rightarrow B) \rightarrow \Box B)$$

is

(a) *a tautology for implications* $\rightarrow_3, \rightarrow_5, \rightarrow_{11}, \rightarrow_{14}, \rightarrow_{20}, \rightarrow_{25}, \rightarrow_{27}, \rightarrow_{29},$
   $\rightarrow_{48}, \rightarrow_{49}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{60}, \rightarrow_{77}, \rightarrow_{79}, \rightarrow_{81}, \rightarrow_{97}, \rightarrow_{99}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{184},$
(b) *an IFT for implications* $\rightarrow_1, \rightarrow_3, \ldots, \rightarrow_7, \rightarrow_9, \rightarrow_{11}, \ldots, \rightarrow_{14}, \rightarrow_{17}, \rightarrow_{18},$
   $\rightarrow_{20}, \rightarrow_{21}, \rightarrow_{25}, \rightarrow_{27}, \ldots, \rightarrow_{29}, \rightarrow_{46}, \rightarrow_{48}, \ldots, \rightarrow_{51}, \rightarrow_{53}, \rightarrow_{57}, \rightarrow_{58}, \rightarrow_{60},$
   $\rightarrow_{61}, \rightarrow_{64}, \rightarrow_{66}, \rightarrow_{71}, \rightarrow_{72}, \rightarrow_{75}, \ldots, \rightarrow_{77}, \rightarrow_{79}, \ldots, \rightarrow_{81}, \rightarrow_{91}, \ldots, \rightarrow_{94},$
   $\rightarrow_{97}, \ldots, \rightarrow_{102}, \rightarrow_{107}, \ldots, \rightarrow_{113}, \rightarrow_{118}, \rightarrow_{120}, \ldots, \rightarrow_{122}, \rightarrow_{124}, \ldots, \rightarrow_{128},$
   $\rightarrow_{134}, \ldots, \rightarrow_{137}, \rightarrow_{151}, \rightarrow_{158}, \rightarrow_{161}, \rightarrow_{166}, \rightarrow_{167}, \rightarrow_{169}, \rightarrow_{181}, \rightarrow_{182}, \rightarrow_{184}.$

An interesting open problem here is the following.

Open Problem 13 Determine for which pair of implications $(\rightarrow_i, \rightarrow_j)$ and for every two formulas A and B, the following equality is valid:

$$V(A \rightarrow_i B) = V(\Box (A \rightarrow_j B)).$$

We remind that for each evaluation function V and for each formula A such that $V(x) = \langle a, b \rangle$, A is "intuitionistic fuzzy sure" (IF-sure), iff $a \geq 0.5 \geq b$.
   Let for the variable x, for which $V(x) = \langle a, b \rangle$, it is valid that
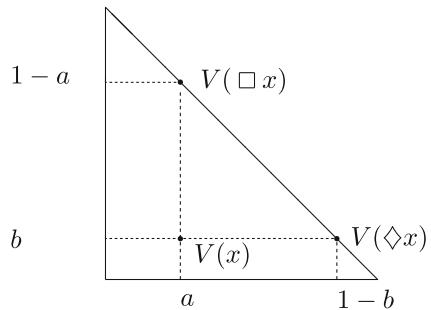
$$V(\Box x) = \langle a, 1 - a \rangle,$$

$$V(\Diamond x) = \langle 1 - b, b \rangle$$

(see Fig. 3.5).

Fig. 3.5 Second geometrical interpretation of operators $\Box$ and $\Diamond$

Following and extending [6], we give the following theorem.

Theorem 3.1.8 (Locality of intuitionistic fuzzy sure) *For every formula A, if* $A(\Box x)$ *and* $A(\Diamond x)$ *are IF-sure, then, for every y, for which*

$$V(\Box x) \le V(y) \le V(\Diamond x), \tag{3.1.7}$$

$A(y)$ *is IF-sure.*

Proof Let

$$V(A(\Box x)) = \langle a(x), b(x) \rangle = \langle a, b \rangle,$$

$$V(A(\Diamond x)) = \langle c(x), d(x) \rangle = \langle c, d \rangle,$$

$$V(A(y)) = \langle \alpha(y), \beta(y) \rangle = \langle \alpha, \beta \rangle.$$

We shall prove the assertion by induction on the complexity of the formula A. Let A be a variable, i.e., $A(x) = x$. Then,

$$a(\Box x) \ge 0.5 \ge b(\Box x) = 1 - a(\Box x)$$

$$1 - d(\Diamond x) = c(\Diamond x) \ge 0.5 \ge d(\Diamond x)$$

and from (3.1.7) it follows that

$$\alpha - \beta \ge a - b \ge 0,$$

i.e., $A(y)$ is IF-sure.

Let $A = P \wedge Q$, where for P and Q the assertion is valid. Then,

$$\alpha - \beta = \min(\mu(P(y)), \mu(Q(y))) - \max(\nu(P(y)), \nu(Q(y))) \ge 0.5 - 0.5 = 0.$$

For $A = P \vee Q$, the check is similar.

Let $A(x) = \forall z P(x, z)$, where the assertion is valid for $P(x, z)$. For every y, for which (3.1.7) is valid, and for every z:

$$\mu(P(y, z)) \ge 0.5 \ge \nu(P(y, z))$$

by assumption. Then,

$$\alpha - \beta = \min \mu(P(x, z)) - \max(P(\nu(x, z))) \ge 0.5 - 0.5 = 0.$$

For $A(x) = \exists z P(x, z)$ the assertion is proved analogically.

Let $A(x) = \Box P(x)$, where the assertion is valid for $P(x)$. Then, we obtain directly, that

$$\alpha - \beta \geq a - b \geq 0.5 - 0.5 = 0.$$

When $A(x) = \Diamond P(x)$ the assertion is proved analogically. $\Box$

Corollary 3.1.1 (Locality of intuitionistic fuzzy truth) *For every formula A, if* $A(\Box x)$ *and* $A(\Diamond x)$ *are tautologies, then, for every y, for which (3.1.7) is valid,* $A(y)$ *is a tautology.*

The case of IFTs is more complicated.

It can be established that if A and B are IFTs, then, $A \wedge B$ is not necessarily an IFT. For example, let $p \vee \neg p$ and $q \vee \neg q$ for different propositions p and q be IFTs, then

$$\begin{aligned} V(p \vee \neg p) &= \langle \max(\mu(p), \mu(\neg p)), \min(\nu(p), \nu(\neg p)) \rangle \\ &= \langle \max(\mu(p), \nu(p)), \min(\nu(p), \mu(\neg p)) \rangle. \end{aligned}$$

Nevertheless, the form

$$A = (p \vee \neg p) \wedge (q \vee \neg q)$$

is not always an IFT. Take e.g.:

$$V(p) = \langle 0.4, 0.4 \rangle,$$

$$V(q) = \langle 0.2, 0.2 \rangle.$$

Then,

$$V(p \vee \neg p) = \langle 0.4, 0.4 \rangle,$$

$$V(q \vee \neg q) = \langle 0.2, 0.2 \rangle,$$

but

$$V(A) = \langle 0.2, 0.4 \rangle.$$

A Conjunctive Normal Form (CNF) A is of the sort $D_1 \wedge \cdots \wedge D_m$, where $D_i = l_{i,1} \vee \cdots \vee l_{i,k_i}$ is a clause of literals. A literal is either a propositional variable (e.g., p) or a negated variable (in the same case – $\neg p$). The literals p and $\neg p$ are called opposite. Two clauses C and D are called *connected* if they contain a common variable occurring in opposite literals (e.g., p in C and $\neg p$ in D).

Lemma 3.1.1 *A clause C is an IFT if and only if it is a classical two-valued tautology iff C contains a pair of opposite literals.*

Lemma 3.1.2 *A conjunction of two literals C and D, which are IFTs, is an IFT iff they are connected.*

Proof If C and D are IFTs and are connected, then, $C \wedge D$ is an IFT. Consider an arbitrary V. Let $C = p \vee A$, $D = \neg p \vee B$, where $V(A) = \langle a, b \rangle$, $V(B) = \langle c, d \rangle$ and let $V(p) = \langle \mu, \nu \rangle$. Then,

$$\max(\mu, a) \geq \mu \geq \min(\mu, d)$$
$$\max(\nu, c) \geq \nu \geq \min(\nu, b)$$
$$\max(\mu, a) \geq \min(\nu, b)$$
$$\max(\nu, c) \geq \min(\mu, d)$$

and from

$$V(C \wedge D) = \langle \min(\max(\mu, a), \max(\nu, c)), \max(\min(\nu, b), \min(\mu, d)) \rangle$$

and

$$\min(\max(\mu, a), \max(\nu, c)) \geq \max(\min(\nu, b), \min(\mu, d))$$

it follows that $C \wedge D$ is an IFT.

Let C and D be two IFT clauses. Let us define the following evaluation function W: For variables p which occur in both positive and negative literals in C let $W(p) = \langle 0.2, 0.2 \rangle$. For a variable q that appears in both the positive and the negative forms in D: $W(q) = \langle 0.4, 0.4 \rangle$. Note that the sets of such variables are disjoint. For variables which occur positively in C or D, let W be $\langle 0.2, 0.4 \rangle$ and for variables occurring negatively in C or $D - \langle 0.4, 0.2 \rangle$. It is a simple check which shows that $W(C \wedge D) = \langle 0.2, 0.4 \rangle$. Thus, the conjunction of C and D is not an IFT. $\qquad\square$

A CNF A is called *totally connected* if every pair of clauses C, D in it is connected.

Lemma 3.1.3 *A CNF A is an IFT iff all clauses in it are IFTs and A is totally connected.*

Proof Assume that all clauses of A are IFTs and that A is totally connected. If we assume that for a some evaluation function W:

$$W(A) = \langle \mu, \nu \rangle$$

is such that $\mu < \nu$, then, it can be easily seen that there is a pair of clauses C and D of A such that $C \wedge D$ is already not an IFT (due to W) – but this is impossible by Lemma 3.1.2. In the opposite direction: if at least two clauses in A are not connected, then their conjunction will not be an IFT, hence, A will not be an IFT either. $\qquad\square$

Theorem 3.1.9 (Locality of IFT) *For every connected formula A, if $A(\square x)$ and $A(\diamond x)$ are IFTs, then, for every y, for which (3.1.7) is valid, $A(y)$ is an IFT.*

Proof The proof is similar to the proof of Theorem 3.1.8, but in the case when formula A is a conjunction of two connected formulas, we use Lemma 3.1.2 $\qquad\square$

Now, we discuss the basic relations between the quantifiers $\forall$ and $\exists$ and the two modal operators $\Box$ and $\Diamond$.

The equalities from [7], Sect. 1.6 can be transformed for both modal operators and both quantifiers, as follows.

Theorem 3.1.10 *Let A be a formula and x be a variable. Then,*

(a) $V(\forall x \Box A) = V(\Box \forall x A)$,
(b) $V(\exists x \Box A) = V(\Box \exists x A)$,
(c) $V(\forall x \Diamond A) = V(\Diamond \forall x A)$,
(d) $V(\exists x \Diamond A) = V(\Diamond \exists x A)$.

Proof Let us check the validity of (a):

$$
\begin{aligned}
&V(\forall x \Box A) \\
&= \langle \min_x \mu(A), \max_x(1 - \mu(A)) \rangle \\
&= \langle \min_x \mu(A), 1 - \min_x \mu(A) \rangle \\
&= V(\Box \forall x A).
\end{aligned}
$$

Equalities (b)–(d) are proved analogically. $\Box$

Theorem 3.1.11 *Let A be a formula and x be a variable. Then,*

(a) $V(\Box \exists x \Box A) = V(\Diamond \exists x \Box A) = V(\neg \Box \forall x \Diamond \neg A) = V(\neg \Diamond \forall x \Diamond \neg A)$,
(b) $V(\Box \exists x \Diamond A) = V(\Diamond \exists x \Diamond A) = V(\neg \Box \forall x \Box \neg A) = V(\neg \Diamond \forall x \Box \neg A)$,
(c) $V(\Box \forall x \Box A) = V(\Diamond \forall x \Box A) = V(\neg \Box \exists x \Diamond \neg A) = V(\neg \Diamond \exists x \Diamond \neg A)$,
(d) $V(\Box \forall x \Diamond A) = V(\Diamond \forall x \Diamond A) = V(\neg \Box \exists x \Box \neg A) = V(\neg \Diamond \exists x \Box \neg A)$,
(e) $V(\Box \exists x \Box \neg A) = V(\Diamond \exists x \Box \neg A) = V(\neg \Box \forall x \Diamond A) = V(\neg \Diamond \forall x \Diamond A)$,
(f) $V(\Box \exists x \Diamond \neg A) = V(\Diamond \exists x \Diamond \neg A) = V(\neg \Box \forall x \Box A) = V(\neg \Diamond \forall x \Box A)$,
(g) $V(\Box \forall x \Box \neg A) = V(\Diamond \forall x \Box \neg A) = V(\neg \Box \exists x \Diamond A) = V(\neg \Diamond \exists x \Diamond A)$,
(h) $V(\Box \forall x \Diamond \neg A) = V(\Diamond \forall x \Diamond \neg A) = V(\neg \Box \exists x \Box A) = V(\neg \Diamond \exists x \Box A)$.

Proof Let us check the validity of (a)

$$
\begin{aligned}
&V(\Box \exists x \Box A) \\
&= \Box \exists x \Box V(A) \\
&= \Box \exists x \langle \mu(A), 1 - \mu(A) \rangle \\
&= \Box \langle \max_x \mu(A), \min_x(1 - \mu(A)) \rangle \\
&= \langle \max_x \mu(A), 1 - \max_x \mu(A) \rangle;
\end{aligned}
$$

$$V(\lozenge \exists x \, \square \, A)$$
$$= \lozenge \langle \max_{x}(\mu(A)), \min_{x}(1 - \mu(A)) \rangle$$
$$= \langle 1 - \min_{x}(1 - \mu(A)), \min_{x}(1 - \mu(A)) \rangle$$
$$= \langle \max_{x}(\mu(A)), 1 - \max_{x}(\mu(A)) \rangle;$$

$$V(\neg \, \square \, \forall x \lozenge \neg A) = \neg \, \square \, \forall x \lozenge \langle \nu(A), \mu(A) \rangle = \neg \, \square \, \forall x \langle 1 - \mu(A), \mu(A) \rangle$$
$$= \neg \, \square \, \langle \min_{x}(1 - \mu(A)), \max_{x}(\mu(A)) \rangle$$
$$= \neg \langle \min_{x}(1 - \mu(A)), 1 - \min_{x}(1 - \mu(A)) \rangle$$
$$= \langle 1 - \min_{x}(1 - \mu(A)), \min_{x}(1 - \mu(A)) \rangle$$
$$= \langle \max_{x} \mu(A), 1 - \max_{x} \mu(A) \rangle$$
$$= V(\neg \lozenge \forall x \lozenge \neg A)$$
$$= \neg \lozenge \langle \min_{x}(1 - \mu(A)), \max_{x} \mu(A) \rangle$$
$$= \neg \langle 1 - \max_{x} \mu(A), \max_{x} mu(A) \rangle$$
$$= \langle \max_{x} \mu(A), 1 - \max_{x} \mu(A) \rangle.$$

Equalities (b)–(h) are proved analogically. $\qquad \square$

Let for a fixed formula A and for a variable x:

$$S(A) = \{ \square \exists x \, \square \, A, \lozenge \, \exists x \, \square \, A, \neg(\square \forall x \lozenge \, \neg A), \neg(\lozenge \, \forall x \lozenge \, \neg A) \},$$

$$T(A) = \{ \square \exists x \lozenge \, A, \lozenge \exists x \lozenge \, A, \neg(\square \forall x \, \square \, \neg A), \neg(\lozenge \, \forall x \, \square \, \neg A) \},$$

$$U(A) = \{ \square \forall x \, \square \, A, \lozenge \, \forall x \, \square \, A, \neg(\square \exists x \lozenge \, \neg A), \neg(\lozenge \, \exists x \lozenge \, \neg A) \},$$

$$V(A) = \{ \square \forall x \lozenge \, A, \lozenge \forall x \lozenge \, A, \neg(\square \exists x \, \square \, \neg A), \neg(\lozenge \, \exists x \, \square \, \neg A) \},$$

$$W(A) = \{ \square \exists x \, \square \, \neg A, \lozenge \, \exists x \, \square \, \neg A, \neg(\square \forall x \lozenge \, A), \neg(\lozenge \, \forall x \lozenge \, A) \},$$

$$X(A) = \{ \square \exists x \lozenge \, \neg A, \lozenge \, \exists x \lozenge \, \neg A, \neg(\square \forall x \, \square \, A), \neg(\lozenge \, \forall x \, \square \, A) \},$$

$$Y(A) = \{ \square \forall x \, \square \, \neg A, \lozenge \, \forall x \, \square \, \neg A, \neg(\square \exists x \lozenge \, A), \neg(\lozenge \, \exists x \lozenge \, A) \},$$

$$Z(A) = \{ \square \forall x \lozenge \, \neg A, \lozenge \, \forall x \lozenge \, \neg A, \neg(\square \exists x \, \square \, A), \neg(\lozenge \, \exists x \, \square \, A) \}.$$

Having in mind Theorem 3.1.11, we can prove the following theorem.

Theorem 3.1.12 *Let A be a formula and x be a variable. Then,*

(a) if $P \in S(A)$ and $Q \in T(A)$, then, $V(P) \leq V(\exists x A) \leq V(Q)$,
(b) if $P \in U(A)$ and $Q \in V(A)$, then, $V(P) \leq V(\forall x A) \leq V(Q)$,
(c) if $P \in W(A)$ and $Q \in X(A)$, then, $V(P) \leq V(\forall x A) \leq V(Q)$,
(d) if $P \in Y(A)$ and $Q \in Z(A)$, then, $V(P) \leq V(\exists x A) \leq V(Q)$,

where $V(X) \leq V(Y)$ for the formulas X and Y if and only if $\mu(X) \leq \mu(Y)$ and $\nu(X) \geq \nu(Y)$.

Finally, following [8], we discuss another modal operator, without an analogue in modal logic.

For the formula A, for which $V(A) = \langle a, b \rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, we define the new modal operator "$\bigcirc$" by:

$$V(\bigcirc A) = \left\langle \frac{a}{a + b}, \frac{b}{a + b} \right\rangle.$$

Obviously, the pair $\langle \frac{a}{a+b}, \frac{b}{a+b} \rangle$ is an intuitionistic fuzzy pair and more particularly – a fuzzy pair, because

$$\frac{a}{a + b} + \frac{b}{a + b} = 1.$$

The new operator has the following more interesting properties.

Theorem 3.1.13 *For every formula A:*

(a) $\bigcirc \Box A = \Box A$,
(b) $\bigcirc \Diamond A = \Diamond A$,
(c) $\Box \bigcirc A = \bigcirc A$,
(d) $\Diamond \bigcirc A = \bigcirc A$,
(e) $\bigcirc \bigcirc A = \bigcirc A$.

Proof (e) For formula A we obtain:

$$V(\bigcirc \bigcirc A) = \bigcirc \left\langle \frac{a}{a+b}, \frac{b}{a+b} \right\rangle = \left\langle \frac{\frac{a}{a+b}}{\frac{a}{a+b} + \frac{b}{a+b}}, \frac{\frac{b}{a+b}}{\frac{a}{a+b} + \frac{b}{a+b}} \right\rangle = V(\bigcirc A).$$

This completes the proof. The rest assertions in (a)–(d) are proved analogically. □

Theorem 3.1.14 *For every formula A:*

(a) Only negation $\neg_1$ satisfies equality

$$\neg \bigcirc \neg A = \bigcirc A,$$

(b) Only negations $\neg_1, \neg_2, \neg_{11}, \neg_{18}, \neg_{53}$ satisfy equality

$$\neg \bigcirc A = \bigcirc \neg A.$$

Theorem 3.1.15 *For every two formulas A and B and for the disjunction and conjunction defined by (1.1.4) and (1.1.5):*

$$V(\bigcirc(A \wedge B)) \leq V(\bigcirc A \wedge \bigcirc B),$$

$$V(\bigcirc(A \vee B)) \geq V(\bigcirc A \vee \bigcirc B).$$

Proof Let A and B be two formulas, so that $V(A) = \langle a, b \rangle$, $V(B) = \langle c, d \rangle$, $a, b, c, d \in [0, 1]$ and $a + b \leq 1$, $c + d \leq 1$. Then,

$$V(\bigcirc(A \wedge B)) = \bigcirc(\langle a, b \rangle \wedge \langle c, d \rangle) = \bigcirc\langle \min(a, c), \max(b, d) \rangle$$

$$= \left\langle \frac{\min(a, c)}{\min(a, c) + \max(b, d)}, \frac{\max(b, d)}{\min(a, c) + \max(b, d)} \right\rangle.$$

and

$$V(\bigcirc A \wedge \bigcirc B) = \bigcirc\langle a, b \rangle \wedge \bigcirc\langle c, d \rangle$$

$$= \left\langle \frac{a}{a + b}, \frac{b}{a + b} \right\rangle \wedge \left\langle \frac{c}{c + d}, \frac{d}{c + d} \right\rangle$$

$$= \left\langle \min\left(\frac{a}{a + b}, \frac{c}{c + d} \right), \max\left(\frac{b}{a + b}, \frac{d}{c + d} \right) \right\rangle.$$

First, we prove the validity of the following inequality. For every three real numbers $a, b, c \in [0, 1]$, if $a \geq c$, then:

$$\frac{a}{a + b} \geq \frac{c}{c + b}. \tag{3.1.8}$$

Obviously, the inequality is valid in the form of an equality, when $a = c$. Let $a > c$. Then, sequentially, we obtain:

$$\frac{a}{a + b} - \frac{c}{c + b} = \frac{ab - bc}{(a + b)(c + b)} = \frac{b(a - c)}{(a + b)(c + b)} > 0,$$

i.e., (3.1.6) is valid.

Now, we check the validity of inequality

$$\min\left(\frac{a}{a + b}, \frac{c}{c + d} \right) \geq \frac{\min(a, c)}{\min(a, c) + \max(b, d)}. \tag{3.1.9}$$

Let

$$X \equiv \min\left(\frac{a}{a + b}, \frac{c}{c + d} \right) - \frac{\min(a, c)}{\min(a, c) + \max(b, d)}.$$

If $a \geq c$, then, we obtain:

$$X = \min\left(\frac{a}{a+b}, \frac{c}{c+d}\right) - \frac{c}{c+\max(b,d)}.$$

If $\frac{a}{a+b} \geq \frac{c}{c+d}$, then,

$$X = \frac{c}{c+d} - \frac{c}{c+\max(b,d)} \geq 0.$$

If $\frac{a}{a+b} \leq \frac{c}{c+d}$, then, from (3.1.6),

$$X = \frac{a}{a+b} - \frac{c}{c+\max(b,d)} \geq \frac{c}{b+c} - \frac{c}{c+\max(b,d)} \geq 0.$$

Let $a < c$. Then,

$$X = \min\left(\frac{a}{a+b}, \frac{c}{c+d}\right) - \frac{a}{a+\max(b,d)}.$$

If $\frac{a}{a+b} \geq \frac{c}{c+d}$, then, from (1), we obtain

$$X = \frac{c}{c+d} - \frac{a}{a+\max(b,d)} \geq \frac{c}{c+d} - \frac{a}{a+d} \geq 0.$$

If $\frac{a}{a+b} \leq \frac{c}{c+d}$, then,

$$X = \frac{a}{a+b} - \frac{a}{a+\max(b,d)}.$$

In the same way, we can prove that

$$\frac{\max(b,d)}{\min(a,c)+\max(b,d)} \geq \left(\frac{b}{a+b}, \frac{d}{c+d}\right),$$

i.e., the first inequality is checked.

The second inequality in the theorem, as well as the Theorem 3.1.16 are proved by analogy.   □

Theorem 3.1.16 *For every predicate P:*

$$V(\bigcirc \exists x P(x)) \geq V(\exists x \bigcirc P(x)),$$

$$V(\bigcirc \forall x P(x)) \leq V(\forall x \bigcirc P(x)).$$

3.2 Extensions of the Intuitionistic Fuzzy Modal Operators

In this section, we introduce the first group of extended intuitionistic fuzzy modal operators.

First, by analogy with the IFS-operators from [7, 9], in the period 1988–1993, we define eight new modal operators.

Let A be a fixed formula for which $V(A) = \langle a, b \rangle$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in [0, 1]$.

We define operators D_α, $F_{\alpha,\beta}$, $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, $J^*_{\alpha,\beta}$ and $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}$ by:

$$V(D_\alpha(A)) = \langle a + \alpha.(1 - a - b), b + (1 - \alpha).(1 - a - b) \rangle,$$
$$V(F_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - b), b + \beta.(1 - a - b) \rangle, \text{ for } \alpha + \beta \leq 1,$$
$$V(G_{\alpha,\beta}(A)) = \langle \alpha.a, \beta.b \rangle,$$
$$V(H_{\alpha,\beta}(A)) = \langle \alpha.a, b + \beta.(1 - a - b) \rangle,$$
$$V(H^*_{\alpha,\beta}(A)) = \langle \alpha.a, b + \beta.(1 - \alpha.a - b) \rangle,$$
$$V(J_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - b), \beta.b \rangle,$$
$$V(J^*_{\alpha,\beta}(A)) = \langle a + \alpha.(1 - a - \beta.b), \beta.b \rangle,$$
$$V(X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(A)) = \langle \alpha.a + \beta.(1 - a - \gamma.b), \delta.b + \varepsilon.(1 - \eta.a - b) \rangle,$$

for

$$\alpha + \varepsilon - \varepsilon.\eta \leq 1 \tag{3.2.1}$$

$$\beta + \delta - \beta\gamma \leq 1, \tag{3.2.2}$$

$$\beta + \varepsilon \leq 1. \tag{3.2.3}$$

In [10], it was mentioned that the third condition (3.2.3) was omitted in [7, 9]. It was introduced to the definition in [10], because without it, e.g., for constant $U^* = \langle 0, 0 \rangle$ we obtain
$$X_{0,1,0,0,1,0}(U^*) = \langle 1, 1 \rangle,$$

which is impossible.

Obviously,

$$\Box A = D_0(A),$$

$$\Diamond A = D_1(A),$$

$$D_\alpha(A) = F_{\alpha,1-\alpha}(A),$$

$$\Box A = X_{1,0,r,1,1,1}(A),$$

$$\Diamond A = X_{1,1,1,1,0,r}(A),$$

$$D_\alpha(A) = X_{1,\alpha,1,1,1-\alpha,1}(A),$$

$$F_{\alpha,\beta}(A) = X_{1,\alpha,1,1,\beta,1}(A), \text{ for } \alpha + \beta \leq 1,$$

$$G_{\alpha,\beta}(A) = X_{\alpha,0,r,\beta,0,r}(A),$$

$$H_{\alpha,\beta}(A) = X_{\alpha,0,r,1,\beta,1}(A),$$

$$H^*_{\alpha,\beta}(A) = X_{\alpha,0,r,\beta,0,\alpha}(A),$$

$$J_{\alpha,\beta}(A) = X_{1,\alpha,1,\beta,0,r}(A),$$

$$J^*_{\alpha,\beta}(A) = X_{1,\alpha,\beta,\beta,0,r}(A),$$

where r is an arbitrary real number in [0, 1].
Let us define for every formula A:

$$V(D_\alpha(A)) = D_\alpha(V(A)),$$

$$V(F_{\alpha,\beta}(A)) = F_{\alpha,\beta}(V(A)),$$

$$V(G_{\alpha,\beta}(A)) = G_{\alpha,\beta}(V(A)),$$

$$V(H_{\alpha,\beta}(A)) = H_{\alpha,\beta}(V(A)),$$

$$V(H^*_{\alpha,\beta}(A)) = H^*_{\alpha,\beta}(V(A)),$$

$$V(J_{\alpha,\beta}(A)) = J_{\alpha,\beta}(V(A)),$$

$$V(J^*_{\alpha,\beta}(A)) = J^*_{\alpha,\beta}(V(A)),$$

$$V(X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(A)) = X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(V(A)).$$

To every formula A, the evaluation function V assigns for $D_\alpha(A)$ a point from the segment between $V(\square A)$ and $V(\lozenge A)$ depending on the value of the argument $\alpha \in [0, 1]$ (see Fig. 3.6). As in the case of some of the above operations, this construction needs auxiliary elements which are shown in Fig. 3.6.

To every formula A, the evaluation function V assigns for $F_{\alpha,\beta}(A)$ a point from the triangle with vertices $V(A)$, $V(\square A)$ and $V(\lozenge A)$, depending on the value of the arguments $\alpha, \beta \in [0, 1]$ for which $\alpha + \beta \leq 1$ (see Fig. 3.7).

To every formula A, the evaluation function V assigns for $G_{\alpha,\beta}(A)$ a point in the rectangle whose vertices are the point $V(A)$ and points with coordinates, $\langle pr_1 V(A), 0\rangle$, $\langle 0, pr_2 V(A)\rangle$ and $\langle 0, 0\rangle$, where $pr_i p$ is the i-th projection ($i = 1, 2$) (see Fig. 3.8).

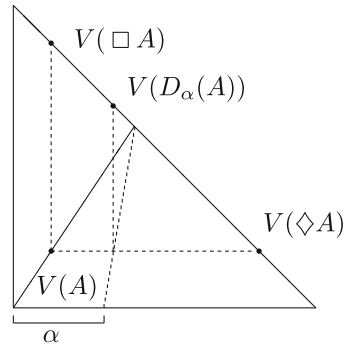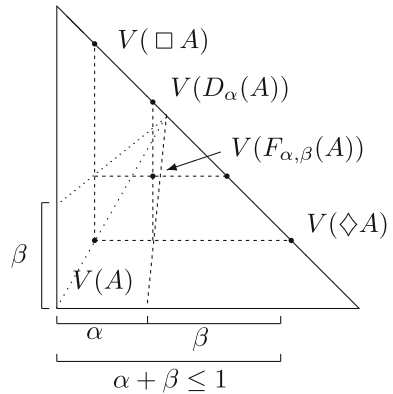**Fig. 3.6** Second geometrical interpretation of operator D_α

Fig. 3.7 Second geometrical interpretation of operator $F_{\alpha,\beta}$

To every formula A, the evaluation function V assigns for $K_{\alpha,\beta}(A)$ a point $V(H_{\alpha,\beta}(A))$ from the rectangle whose vertices are the points with coordinates $\langle 0, pr_2V(A)\rangle$ and $\langle 0, pr_2V(\Box A)\rangle$ and vertices $V(\Box A)$ and $V(A)$, depending on the value of the parameters $\alpha, \beta \in [0, 1]$ (see Fig. 3.9).

To every formula A, the evaluation function V assigns for $J_{\alpha,\beta}(A)$ a point $V(J_{\alpha,\beta}(A))$ from the rectangle whose vertices are the points with coordinates $\langle pr_1V(\Diamond A), 0\rangle$, $\langle pr_1V(A), 0\rangle$ and vertices $V(A)$ and $V(\Diamond A)$, depending on the value of the parameters $\alpha, \beta \in [0, 1]$ (see Fig. 3.10).

To every formula A, the evaluation function V assigns for $H^*_{\alpha,\beta}(A)$ a point $V(H^*_{\alpha,\beta}(A))$ from the trapezoid with vertices with coordinates $\langle 0, pr_2V(A)\rangle$ and $\langle 0, 1\rangle$ and vertices $V(\Box A)$ and $V(A)$, depending on the value of the parameters $\alpha, \beta \in [0, 1]$ (see Fig. 3.11).

To every formula A, the evaluation function V assigns for $J^*_{\alpha,\beta}(A)$ a point $V(J^*_{\alpha,\beta}(A))$ from the trapezoid with vertices with coordinates $\langle 1, 0\rangle$ and $\langle pr_1V(A), 0\rangle$ and vertices $V(A)$ and $V(\Diamond A)$, depending on the value of the parameters $\alpha, \beta \in [0, 1]$ (see Fig. 3.12).

Fig. 3.8 Second geometrical
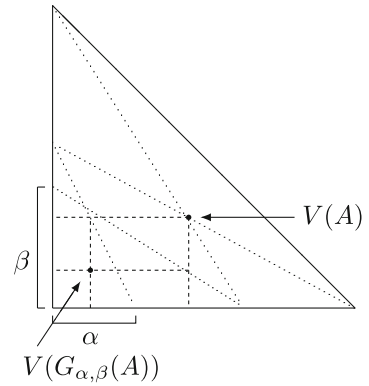interpretation of operator $G_{\alpha,\beta}$

$$V(G_{\alpha,\beta}(A))$$

Fig. 3.9 Second geometrical
interpretation of operator $H_{\alpha,\beta}$
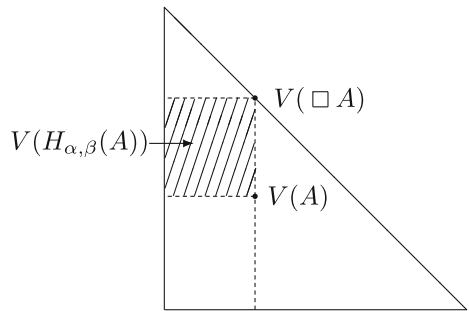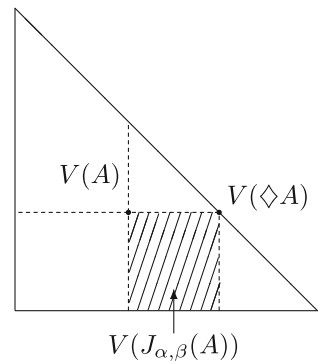

Fig. 3.10 Second geometrical
interpretation of operator $J_{\alpha,\beta}$

   Below, we formulate and prove assertions, that by the moment have been checked
only for some intuitionistic fuzzy implications and negation, which gives rise to the
following important open problem for solving in future.

Open Problem 14 Check these assertions for all the remaining implications and
negations.

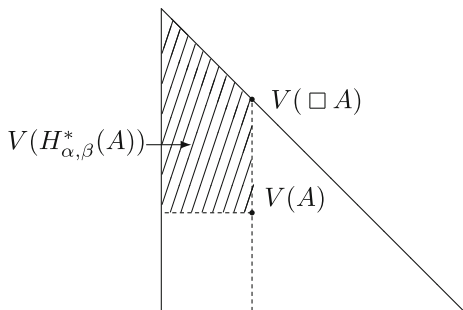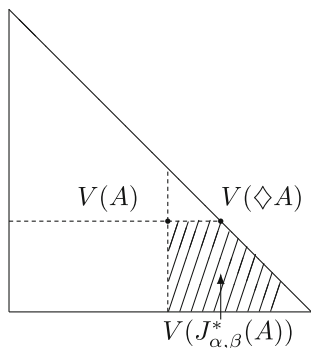**Fig. 3.11** Second geometrical interpretation of operator $H_{\alpha,\beta}^*$

$V(H_{\alpha,\beta}^*(A))$
$V(\square A)$
$V(A)$

Fig. 3.12 Second geometrical interpretation of operator $J_{\alpha,\beta}^*$

$V(A)$ $V(\lozenge A)$
$V(J_{\alpha,\beta}^*(A))$

Theorem 3.2.1 *If formula A is a tautology, then:*

(a) *for every two* $\alpha, \beta \in [0,1] : D_\alpha(A), F_{\alpha,\beta}(A)$, *for* $\alpha + \beta \leq 1, J_{\alpha,\beta}(A)$ *and* $J_{\alpha,\beta}^*(A)$ *are tautologies,*

(b) $G_{\alpha,\beta}(A), H_{\alpha,\beta}(A), H_{\alpha,\beta}^*(A)$, *are IFTs,*

(c) *for every* $\alpha, \varepsilon, \eta$ *so that* $\alpha \geq \varepsilon(1 - \eta), X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(A)$ *is an IFT.*

Proof (b) From $V(A) = \langle 1, 0 \rangle$ it follows that

$$V(G_{\alpha,\beta}(A)) = \langle \alpha, 0 \rangle,$$

$$V(H_{\alpha,\beta}(A)) = V(H_{\alpha,\beta}^*(A)) = \langle \alpha, \beta(1 - 1 - 0) \rangle = \langle \alpha, 0 \rangle,$$

i.e., $G_{\alpha,\beta}(A), H_{\alpha,\beta}(A)$ and $H_{\alpha,\beta}^*(A)$ are IFTs. The other cases are proved by analogy. □

Theorem 3.2.2 *If formula A is an IFT, then:*

(a) *for* $\alpha \geq 0.5$ $D_\alpha(A)$ *is an IFT,*

(b) *for* $\alpha \geq \beta$ *and* $\alpha + \beta \leq 1$ $F_{\alpha,\beta}(A)$ *is an IFT,*

(c) *for* $\alpha \geq \beta$ $G_{\alpha,\beta}(A)$ *is an IFT,*

(d) $J_{\alpha,\beta}(A), J_{\alpha,\beta}^*(A)$ *are IFTs,*

(e) *for* $\alpha \geq \delta, \beta \geq \varepsilon, \eta \geq \gamma$ $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(A)$ *is an IFT.*

It can be easily seen that for every $a, b, \alpha, \beta, \gamma, \delta, \varepsilon, \eta \in [0, 1]$ and $a + b \leq 1$, if

$$O \in \{D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}, X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}\},$$

then

$$O(\langle a, b\rangle) \to_4 \langle a, b\rangle \quad \text{and} \quad O(\langle a, b\rangle) \to_{11} \langle a, b\rangle$$

are IFTs. For example,

$$
\begin{aligned}
&V(F_{\alpha,\beta}(\langle a, b\rangle) \to_{11} \langle a, b\rangle) \\
&= \langle a + \alpha.(1 - a - b), b + \beta(1 - a - b)\rangle \to_{11} \langle a, b\rangle \\
&= \langle 1 - (1 - a).\mathrm{sg}(\alpha(1 - a - b)), b.\mathrm{sg}(\alpha(1 - a - b)).\mathrm{sg}(b + \beta.(1 - a - b) - b)\rangle, \\
&= \langle 1 - (1 - a).\mathrm{sg}(\alpha(1 - a - b)), b.\mathrm{sg}(\alpha(1 - a - b)).\mathrm{sg}(\beta.(1 - a - b))\rangle
\end{aligned}
$$

$$
= \begin{cases}
\langle 1, 0\rangle, & \text{if } \alpha = 0 \text{ or } a + b = 1 \\[2mm]
\langle a, b\rangle, & \text{if } \alpha > 0 \text{ and } a + b < 1
\end{cases},
$$

i.e., $F_{\alpha,\beta}(\langle a, b\rangle) \to_{11} \langle a, b\rangle$ is an IFT.

On the other hand, we can see that there are cases in which for the different implications, different conditions must hold for the validity of some expression. The following two assertions serve as examples.

Theorem 3.2.3 *For every* $a, b, \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \alpha', \beta', \gamma', \delta', \varepsilon', \eta' \in [0, 1]$ *and* $a + b \leq 1$, *if* $O \in \{D, F, G, H, H^*, J, J^*\}$, *then:*

(a) $O_{\alpha,\beta}(\langle a, b\rangle) \to_{11} O_{\alpha',\beta'}(\langle a, b\rangle)$ *is a tautology for* $\alpha \leq \alpha'$,
(b) $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(\langle a, b\rangle) \to_{11} X_{\alpha',\beta',\gamma',\delta',\varepsilon',\eta'}(\langle a, b\rangle)$ *is a tautology for* $\alpha \leq \alpha', \beta \leq \beta'$ *and* $\beta\gamma \geq \beta'\gamma'$.

Proof (b) Let $\alpha \leq \alpha', \beta \leq \beta'$ and $\beta.\gamma \geq \beta'.\gamma'$. Then,

$$
\begin{aligned}
&V(X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(\langle a, b\rangle) \to_{11} X_{\alpha',\beta',\gamma',\delta',\varepsilon',\eta'}(\langle a, b\rangle)) \\
&= \langle \alpha a + \beta(1 - a - \gamma b), \delta b + \varepsilon(1 - \eta a - b)\rangle \\
&\quad \to_{11} \langle \alpha'a + \beta'(1 - a - \gamma'b), \delta'b + \varepsilon'(1 - \eta'a - b)\rangle \\
&= \langle 1 - (1 - (\alpha'a + \beta'(1 - a - \gamma'b)))\mathrm{sg}(\alpha a + \beta(1 - a - \gamma b) \\
&\quad -(\alpha'a + \beta'(1 - a - \gamma'b))), (\delta'b + \varepsilon'(1 - \eta'a - b)) \\
&\quad .\mathrm{sg}(\alpha a + \beta(1 - a - \gamma b) - (\alpha'a + \beta'(1 - a - \gamma'b))) \\
&\quad .\mathrm{sg}(\delta'b + \varepsilon'(1 - \eta'a - b) - (\delta b + \varepsilon(1 - \eta a - b)))\rangle \\
&= \langle 1, 0\rangle,
\end{aligned}
$$

because

$$
\begin{aligned}
&\alpha a + \beta(1 - a - \gamma b) - (\alpha'a + \beta'(1 - a - \gamma'b) \\
&= a(\alpha - \alpha') + (1 - a)(\beta - \beta') - b(\beta\gamma - \beta'\gamma') \leq 0
\end{aligned}
$$

and therefore, $\mathrm{sg}(\alpha a + \beta(1 - a - \gamma b) - (\alpha' a + \beta'(1 - a - \gamma' b))) = 0$. The other cases are proved similarly. $\qquad\square$

Theorem 3.2.4 *For every* $a, b, \alpha, \beta, \gamma, \delta, \varepsilon, \eta, \alpha', \beta', \gamma', \delta', \varepsilon', \eta' \in [0, 1]$ *and* $a + b \leq 1$, *if* $O \in \{D, F, G, H, H^*, J, J^*\}$, *then*

(a) $O_{\alpha,\beta}(\langle a, b\rangle) \to_4 O_{\alpha',\beta'}(\langle a, b\rangle)$ *is an IFT for* $\alpha \leq \alpha'$ *or* $\beta \geq \beta'$,

(b) $X_{\alpha,\beta,\gamma,\delta,\varepsilon,\eta}(\langle a, b\rangle) \to_4 X_{\alpha',\beta',\gamma',\delta',\varepsilon',\eta'}(\langle a, b\rangle)$ *is an IFT for* $\alpha \leq \alpha', \beta \leq \beta'$ *and* $\gamma \geq \gamma'$ *or for* $\delta \geq \delta', \varepsilon \geq \varepsilon'$ *and* $\eta \leq \eta'$.

Open Problem 15 Which other implications satisfy Theorems 3.2.3 and 3.2.4?

There are properties that, probably, are specific for $\neg_1$. For example, the following assertion is valid for $\neg_1$, but for $0 < \varepsilon \leq \eta \leq 1$ it is not valid for $\neg_{45,\varepsilon,\eta}$ (by definition, $\neg_1$ coincides with $\neg_{45,0,0}$).

Theorem 3.2.5 *For every* $a, b, \alpha, \beta \in [0, 1]$ *and* $a + b \leq 1$:

(a) $V(F_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 F_{\beta,\alpha}(\neg_1\langle a, b\rangle))$, *for* $\alpha + \beta \leq 1$,
(b) $V(G_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 G_{\beta,\alpha}(\neg_1\langle a, b\rangle))$,
(c) $V(H_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 J_{\beta,\alpha}(\neg_1\langle a, b\rangle))$,
(d) $V(J_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 H_{\beta,\alpha}(\neg_1\langle a, b\rangle))$,
(e) $V(H^*_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 J^*_{\beta,\alpha}(\neg_1\langle a, b\rangle))$,
(f) $V(J^*_{\alpha,\beta}(\langle a, b\rangle)) = V(\neg_1 H^*_{\beta,\alpha}(\neg_1\langle a, b\rangle))$.

Open Problem 16 Which other negations satisfy Theorems 3.2.5?

Let for every two formulas A and B: $A \leftarrow B$ iff $B \to A$.

Theorem 3.2.6 *For every two formulas* A *and* B, *for every two real numbers* $\alpha, \beta \in [0, 1]$ *and for implication* $\to_4$:

(a) $F_{\alpha,\beta}(A \wedge B) \to_4 F_{\alpha,\beta}(A) \wedge F_{\alpha,\beta}(B)$, *for* $\alpha + \beta \leq 1$,
(b) $F_{\alpha,\beta}(A \vee B) \leftarrow_4 F_{\alpha,\beta}(A) \vee F_{\alpha,\beta}(B)$, *for* $\alpha + \beta \leq 1$,
(c) $G_{\alpha,\beta}(A \wedge B) = G_{\alpha,\beta}(A) \wedge G_{\alpha,\beta}(B)$,
(d) $G_{\alpha,\beta}(A \vee B) = G_{\alpha,\beta}(A) \vee G_{\alpha,\beta}(B)$,
(e) $H_{\alpha,\beta}(A \wedge B) \to_4 H_{\alpha,\beta}(A) \wedge H_{\alpha,\beta}(B)$,
(f) $H_{\alpha,\beta}(A \vee B) \leftarrow_4 H_{\alpha,\beta}(A) \vee H_{\alpha,\beta}(B)$,
(g) $J_{\alpha,\beta}(A \wedge B) \leftarrow_4 H_{\alpha,\beta}(A) \wedge H_{\alpha,\beta}(B)$,
(h) $H_{\alpha,\beta}(A \vee B) \to_4 H_{\alpha,\beta}(A) \vee H_{\alpha,\beta}(B)$,
(i) $H^*_{\alpha,\beta}(A \wedge B) \to_4 H^*_{\alpha,\beta}(A) \wedge H^*_{\alpha,\beta}(B)$,
(j) $H^*_{\alpha,\beta}(A \vee B) \leftarrow_4 H^*_{\alpha,\beta}(A) \vee H^*_{\alpha,\beta}(B)$,
(k) $J^*_{\alpha,\beta}(A \wedge B) \leftarrow_4 H^*_{\alpha,\beta}(A) \wedge H^*_{\alpha,\beta}(B)$,
(l) $H^*_{\alpha,\beta}(A \vee B) \to_4 H^*_{\alpha,\beta}(A) \vee H^*_{\alpha,\beta}(B)$

are IFTs.

Proof (a) For the formulas A and B:

$$V(F_{\alpha,\beta}(A \wedge B) \rightarrow_4 F_{\alpha,\beta}(A) \wedge F_{\alpha,\beta}(B))$$
$$= \langle \min(a,c) + \alpha(1 - \min(a,c) - \max(b,d)), \max(b,d) +$$
$$\beta(1 - \min(a,c) - \max(b,d)) \rangle \rightarrow \langle \min(a + \alpha(1 - a - b), c$$
$$+ \alpha(1 - c - d)), \max(b + \beta(1 - a - b), d + \beta(1 - c - d)) \rangle$$
$$= \langle \max(\max(b,d) + \beta(1 - \min(a,c) - \max(b,d)), \min(a +$$
$$\alpha(1 - a - b), c + \alpha(1 - c - d))), \min(\min(a,c) + \alpha(1 - \min(a,c)$$
$$- \max(b,d)), \max(b + \beta(1 - a - b), d + \beta(1 - c - d))) \rangle$$

and

$$\max(\max(b,d) + \beta(1 - \min(a,c) - \max(b,d)), \min(a +$$
$$\alpha(1 - a - b), c + \alpha(1 - c - d))) - \min(\min(a,c) + \alpha(1 -$$
$$\min(a,c) - \max(b,d)), \max(b + \beta(1 - a - b), d + \beta(1 - c - d)))$$
$$\geq \max(b,d) + \beta(1 - \min(a,c) - \max(b,d))$$
$$- \max(b + \beta(1 - a - b), d + \beta(1 - c - d)) \geq 0,$$

i.e.,

$$F_{\alpha,\beta}(A \wedge B) \rightarrow_4 F_{\alpha,\beta}(A) \wedge F_{\alpha,\beta}(B)$$

is an IFT.

Formulas (b)–(l) are checked by analogy. $\square$

Open Problem 17 Which other conjunctions and disjunctions (whenever be defined, following the ideas from Sect. 1.7) have similar properties?

Theorem 3.2.7 *For every predicate A, and for every $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$:*

(a) $V(\exists x F_{\alpha,\beta}(P(x))) \leq V(F_{\alpha,\beta} \exists x P(x))$,
(b) $V(\forall x F_{\alpha,\beta}(P(x))) \geq V(F_{\alpha,\beta} \forall x P(x))$.

Corollary 3.2.1 *For every predicate A, and for every $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$:*

(a) $V(\exists x D_{\alpha}(P(x))) \leq V(D_{\alpha} \exists x P(x))$,
(b) $V(\forall x D_{\alpha}(P(x))) \geq V(D_{\alpha} \forall x P(x))$.

Theorem 3.2.8 *For every formula A and for every $\alpha, \beta, \gamma, \delta \in [0,1]$ such that $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$:*

$$V(F_{\alpha,\beta}(F_{\gamma,\delta}(A))) = V(F_{\alpha+\gamma-\alpha\gamma-\alpha\delta, \beta+\delta-\beta\gamma-\beta\delta}(A)).$$

Proof Let for $\alpha, \beta, \gamma, \delta \in [0,1]$: $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$. Let $V(A) = \langle a, b \rangle$. Then,

$$V(F_{\alpha,\beta}(F_{\gamma,\delta}(A)))$$
$$= F_{\alpha,\beta}(\langle a + \gamma(1 - a - b), b + \delta(1 - a - b)\rangle)$$
$$= \langle a + \gamma(1 - a - b) + \alpha(1 - a - \gamma(1 - a - b) - b - \delta(1 - a - b)),$$
$$\quad b + \delta(1 - a - b) + \beta(1 - a - \gamma(1 - a - b) - b - \delta(1 - a - b))\rangle$$
$$= \langle a + (\alpha + \gamma - \alpha\gamma - \alpha\delta)(1 - a - b),$$
$$\quad b + (\beta + \delta - \beta\gamma - \beta\delta)(1 - a - b)\rangle$$
$$= V(F_{\alpha+\gamma-\alpha\gamma-\alpha\delta,\beta+\delta-\beta\gamma-\beta\delta}(A)).$$

This completes the proof.   □

Corollary 3.2.2 *For every predicate A, and for every $\alpha \in [0, 1]$:*

(a) $V(D_\alpha(D_\beta(A))) = V(D_\beta(A))$,
(b) $V(D_\alpha(F_{\beta,\gamma}(A))) = V(D_{\alpha+\beta-\alpha\beta-\alpha\gamma}(A))$, *for $\beta + \gamma \leq 1$,*
(c) $V(F_{\alpha,\beta}(D_\gamma(A))) = V(D_\gamma(A))$.

Theorem 3.2.9 *For every formula A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$:*

$$V(G_{\alpha,\beta}(G_{\gamma,\delta}(A))) = V(G_{\alpha\gamma,\beta\delta}(A)).$$

Theorem 3.2.10 *For every formula A and for every $\alpha, \beta \in [0, 1]$:*

(a) $V(\square F_{\alpha,\beta}(A)) \geq V(F_{\alpha,\beta} \square A)$ *for $\alpha + \beta \leq 1$,*
(b) $V(\lozenge F_{\alpha,\beta}(A)) \leq V(F_{\alpha,\beta} \lozenge A)$ *for $\alpha + \beta \leq 1$,*

(c) $V(\square G_{\alpha,\beta}(A)) \leq V(G_{\alpha,\beta} \square A)$,
(d) $V(\lozenge G_{\alpha,\beta}(A)) \geq V(G_{\alpha,\beta} \lozenge A)$,

(e) $V(\square H_{\alpha,\beta}(A)) \leq V(H_{\alpha,\beta} \square A)$,
(f) $V(\lozenge H_{\alpha,\beta}(A)) \geq V(H_{\alpha,\beta} \lozenge A)$,

(g) $V(\square J_{\alpha,\beta}(A)) \geq V(J_{\alpha,\beta} \square A)$,
(h) $V(\lozenge J_{\alpha,\beta}(A)) \leq V(J_{\alpha,\beta} \lozenge A)$,

(i) $V(\square H^*_{\alpha,\beta}(A)) \leq V(H^*_{\alpha,\beta} \square A)$,
(j) $V(\lozenge J^*_{\alpha,\beta}(A)) \geq V(J^*_{\alpha,\beta} \lozenge A)$.

Theorem 3.2.11 *For every A and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$:*

$$G_{\alpha,\beta}(G_{\gamma,\delta}(A)) = G_{\alpha\gamma,\beta\delta}(A).$$

Theorem 3.2.12 *For every formula A and for every $\alpha, \beta \in [0, 1]$:*

(a) $V(H_{\alpha,\beta}(G_{\gamma,\delta}(A))) \leq V(G_{\gamma,\delta}(H_{\alpha,\beta}(A)))$,
(b) $V(J_{\alpha,\beta}(G_{\gamma,\delta}(A))) \leq V(G_{\gamma,\delta}(J_{\alpha,\beta}(A)))$,
(c) $V(H^*_{\alpha,\beta}(G_{\gamma,\delta}(A))) \leq V(G_{\gamma,\delta}(H^*_{\alpha,\beta}(A)))$,
(d) $V(J^*_{\alpha,\beta}(G_{\gamma,\delta}(A))) \leq V(G_{\gamma,\delta}(J^*_{\alpha,\beta}(A)))$.

Theorem 3.2.13 *Let A be a formula and x be a variable. Then, for every $\alpha, \beta \in [0, 1]$:*

$$V(\forall x G_{\alpha,\beta}(A)) = V(G_{\alpha,\beta}(\forall x A)),$$

$$V(\exists x G_{\alpha,\beta}(A)) = V(G_{\alpha,\beta}(\exists x A)).$$

We finish with an assertion for the operator X, following [11].

Theorem 3.2.14 *For every two IFPs $\langle \mu, \nu \rangle$ and $\langle \rho, \sigma \rangle$, there are real numbers $a, b, c, d, e, f \in [0, 1]$ satisfying (3.2.1)–(3.2.3), such that*

$$V(X_{a,b,c,d,e,f}(\langle \mu, \nu \rangle)) = \langle \rho, \sigma \rangle.$$

Proof Let $\mu, \nu, \rho, \sigma \in [0, 1]$, such that $\mu + \nu \le 1, \rho + \sigma \le 1$. We search for $a, b, c, d, e, f \in [0, 1]$ that satisfy (3.2.1)–(3.2.3) and for which

$$\langle \rho, \sigma \rangle = V(X_{a,b,c,d,e,f}(\langle \mu, \nu, \rangle)) = \langle a\mu + b(1 - \mu - c\nu), d\nu + e(1 - f\mu - \nu) \rangle,$$

i.e.,

$$\rho = a\mu + b(1 - \mu - c\nu),$$

$$\sigma = d\nu + e(1 - f\mu - \nu).$$

We discuss nine cases.

Case 1. $\pi = \mu = 0$. Then, $\nu = 1$. We put

$$a = c = e = f = 0, \ b = \rho, \ d = \sigma.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{0,\rho,0,\sigma,0,0}(\langle \mu, \nu, \rangle) = \langle 0 + \rho(1 - 0 - 0 \times 1), \sigma \times 1 + 0(1 - 0 \times \mu - 1) \rangle = \langle \rho, \sigma \rangle.$$

Case 2. $\pi = \nu = 0$. Then, $\mu = 1$. We put

$$a = \rho, \ b = c = d = f = 0, \ e = \sigma.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{\rho,0,0,0,\sigma,0}(\langle \mu, \nu, \rangle) = \langle \rho + 0 \times (1 - 0 \times 1 - 0), 0 + \sigma(1 - 0 \times 1 - 1) \rangle = \langle \rho, \sigma \rangle.$$

When $\pi = 0$ and $\mu, \nu > 0$, there are three (sub)cases. It is important to mention that now $\mu, \nu < 1$.

Case 3. $\rho > \mu$. Then, from $\pi = 0$ it follows that $\mu = 1 - \nu$, and hence $\sigma \le 1 - \rho < 1 - \mu = \nu$. So, we put

$$a = 1, \ b = \frac{\rho - \mu}{1 - \mu}, c = e = f = 0, \ d = \frac{\sigma}{\nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{1,\frac{\rho-\mu}{1-\mu},0,\frac{\sigma}{\nu},0,0}(\langle\mu,\nu,\rangle) = \left\langle\mu + \frac{\rho-\mu}{1-\mu}(1-\mu), \frac{\sigma}{\nu}\nu\right\rangle = \langle\rho,\sigma\rangle.$$

Case 4. $\sigma > \nu$. Then, from $\pi = 0$ it follows again that $\mu = 1 - \nu$, and hence $\rho \leq 1 - \sigma < 1 - \nu = \mu$. So, we put

$$a = \frac{\rho}{\mu}, \ b = c = f = 0, \ d = 1, e = \frac{\sigma-\nu}{1-\nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{\frac{\rho}{\mu},0,0,1,\frac{\sigma-\nu}{1-\nu},0,0}(\langle\mu,\nu,\rangle) = \left\langle\frac{\rho}{\mu}\mu + 0, \nu + \frac{\sigma-\nu}{1-\nu}(1-\nu)\right\rangle = \langle\rho,\sigma\rangle.$$

Case 5. $\rho \leq \mu$ and $\sigma \leq \nu$. Then, we put

$$a = \frac{\rho}{\mu}, \ b = c = e = f = 0, \ d = \frac{\sigma}{\nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{\frac{\rho}{\mu},0,0,\frac{\sigma}{\nu},0,0}(\langle\mu,\nu,\rangle) = \left\langle\frac{\rho}{\mu}\mu + 0, \frac{\sigma}{\nu}\nu + 0\right\rangle = \langle\rho,\sigma\rangle.$$

When $\pi > 0$, then, $\mu, \nu < 1$.

Case 6. $\rho > \mu$ and $\sigma > \nu$. Then, we put

$$a = c = d = f = 1, \ b = \frac{\rho-\mu}{\pi}, \ e = \frac{\sigma-\nu}{\pi}.$$

Then, conditions (3.2.1)–(3.2.3) are valid, because:

$$a + e - e.f = 1 + e - e = 1 \leq 1,$$

$$b + d - b.c = d = \frac{\sigma-\nu}{\pi} \leq 1,$$

$$b + e = \frac{\rho-\mu}{\pi} + \frac{\sigma-\nu}{\pi} = \frac{\rho+\sigma-\mu-\nu}{\pi} \leq 1.$$

All other checks are done in a similar way. Now,

$$X_{1,\frac{\rho-\mu}{\pi},1,1,\frac{\sigma-\nu}{\pi},1}(\langle\mu,\nu,\rangle) = \left\langle\mu + \frac{\rho-\mu}{\pi}\pi, \nu + \frac{\sigma-\nu}{\pi}\pi\right\rangle = \langle\rho,\sigma\rangle.$$

Case 7. $\rho > \mu$ and $\sigma \leq \nu$. Then, as in Case 3, we put

$$a = 1, \ b = \frac{\rho - \mu}{1 - \mu}, c = e = f = 0, \ d = \frac{\sigma}{\nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{1, \frac{\rho-\mu}{1-\mu}, 0, \frac{\sigma}{\nu}, 0, 0}(\langle \mu, \nu, \rangle) = \left\langle \mu + \frac{\rho - \mu}{1 - \mu}(1 - \mu), \frac{\sigma}{\nu}\nu \right\rangle = \langle \rho, \sigma \rangle.$$

Case 8. $\rho \leq \mu$ and $\sigma > \nu$. Then, as in point 4, we put

$$a = \frac{\rho}{\mu}, \ b = c = f = 0, \ d = 1, e = \frac{\sigma - \nu}{1 - \nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{\frac{\rho}{\mu}, 0, 0, 1, \frac{\sigma-\nu}{1-\nu}, 0, 0}(\langle \mu, \nu, \rangle) = \left\langle \frac{\rho}{\mu}\mu + 0, \nu + \frac{\sigma - \nu}{1 - \nu}(1 - \nu) \right\rangle = \langle \rho, \sigma \rangle.$$

Case 9. $\rho \leq \mu$ and $\sigma \leq \nu$. Then, as in Case 5, we put

$$a = \frac{\rho}{\mu}, \ b = c = e = f = 0, \ d = \frac{\sigma}{\nu}.$$

Then, conditions (3.2.1)–(3.2.3) are valid and

$$X_{\frac{\rho}{\mu}, 0, 0, \frac{\sigma}{\nu}, 0, 0}(\langle \mu, \nu, \rangle) = \left\langle \frac{\rho}{\mu}\mu + 0, \frac{\sigma}{\nu}\nu + 0 \right\rangle = \langle \rho, \sigma \rangle.$$

Therefore, the theorem is proved.   □

A modification of the above theorem is the following theorem.

Theorem 3.2.15 *For every two formulas A, B there exists an operator $Y \in \{F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}\}$ and there exist real numbers $\alpha, \beta \in [0, 1]$ such that*

$$V(A) = V(Y_{\alpha,\beta}(B)).$$

Proof Let everywhere $V(A) = \langle a, b \rangle$ and $V(B) = \langle c, d \rangle$, where $a, b, c, d \in [0, 1]$ and $a + b \leq 1$ and $c + d \leq 1$.

The following 9 cases are possible for a, b, c and d.

Case 1. $a = c$ Then, for Y :
         and   if $Y = F$, then, $\alpha = \beta = 0$;
         $b = d$ if $Y = G$, then, $\alpha = \beta = 1$;
                 if $Y = H$ or $Y = H^*$, then, $\alpha = 1$ and $\beta = 0$;
                 if $Y = J$ or $Y = J^*$ then, $\alpha = 0$ and $\beta = 1$.

Case 2. $a > c$ Then, $Y = F$ and $\alpha = \dfrac{a-c}{1-c-d}$ and $\beta = 0$

and (we shall note that $1 - c - d > 1 - a - b \geq 0$)

$b = d$ or $Y = J$ or $Y = J^*$, and α has the above form and $\beta = 1$.

Case 3. $a < c$ Then, $Y = G$ and $\alpha = \dfrac{a}{c}$ and $\beta = 1$ (we note

and $that\, c > a \geq 0$)

$b = d$ or $Y = J$ or $Y = J^*$, and α has the above form and $\beta = 0$.

Case 4. $a = c$ $Then, Y = F$ and $\alpha = 0$ and $\beta = \dfrac{b-d}{1-c-d}$

and (we note that $1 - c - d > 1 - a - b \geq 0$)

$b > d$ or $Y = H$ or $Y = H^*$, and $\alpha = 1$ and β has the above form.

Case 5. $a > c$ Then, $Y = F$ and $\alpha = \dfrac{a-c}{1-c-d}$ and

and $\beta = \dfrac{b-d}{1-c-d}$ (we note that

$b > d$ $1 - c - d > 1 - a - b \geq 0$)

Case 6. $a < c$ Then, there are two subcases:

and

$b > d$.

6.1. $if\ b \leq 1 - c,$ then, $Y = H$ and $\alpha = \dfrac{a}{c}$ and

$$\beta = \frac{b-d}{1-c-d}$$

or $Y = H^*$ and $\alpha = \dfrac{a}{c}$ and $\beta = \dfrac{b-d}{1-a-d}$

(we note that $1 - a - d > 1 - c - d \geq b - d > 0$ and $c > a \geq 0$)

6.2. $if\ b > 1 - c,$ then, $Y = H^*$ and $\alpha = \dfrac{a}{c}$ and

$$\beta = \frac{b-d}{1-a-d} \text{ (we note that } 1 - a - d \geq b - d > 0$$

and $c > a \geq 0$)

Case 7. $a = c$ Then, $Y = G$ and $\alpha = 1$ and $\beta = \dfrac{b}{d}$ (we note

and $that\ d > b \geq 0$)

$b < d$.

or $Y = J$ or $Y = H^*$, and $\alpha = 0$ and β has the above form.

Case 8. $a > c$ Then, there are two subcases:
            and
            $b < d$.

    8.1. $if a \leq 1 - d$, then, $Y = J$ and $\alpha = \dfrac{a - c}{1 - c - d}$ and

        $\beta = \dfrac{b}{d}$

        or $Y = J^*$ and $\alpha = \dfrac{a - c}{1 - b - c}$ and $\beta = \dfrac{b}{d}$ (we

        note that $1 - c - b > 1 - c - d \geq a - c > 0$
        and $d > b \geq 0$)

    8.2. $if a > 1 - d$, then, $Y = J^*$ and $\alpha = \dfrac{a - c}{1 - b - c}$

        and $\beta = \dfrac{b}{d}$ (we note that $1 - c - b \geq a - c > 0$

        and $d > b \geq 0$)

Case 9. $a < c$ Then, $Y = G$ and $\alpha = \dfrac{a}{c}$ and $\beta = \dfrac{b}{d}$ (we shall
            and   note that $c > a \geq 0$ and $d > b \geq 0$).
            $b < d$.

This completes the proof. $\square$

3.3 Second Type of Intuitionistic Fuzzy Modal Operators

In this section, following [7, 9, 12–17], several different modal-like operators of
second type are defined and the consequences of their generalizations are discussed.
We formulate the properties of these operators which hold for them but do not hold
for their extensions.

The following two operators of modal type are similar to the operators in Sect. 3.1.
Let for formula A: $V(A) = \langle \mu, \nu \rangle$. Then,

$$V(\boxplus A) = \left\langle \frac{\mu}{2}, \frac{\nu + 1}{2} \right\rangle, \tag{3.3.1}$$

$$V(\boxtimes A) = \left\langle \frac{\mu + 1}{2}, \frac{\nu}{2} \right\rangle. \tag{3.3.2}$$

All of their properties are valid for their first extensions. For a given real number
$\alpha \in [0, 1]$ and formula A,

$$V(\boxplus_\alpha A) = \langle \alpha\mu, \alpha\nu + 1 - \alpha \rangle, \tag{3.3.3}$$

$$V(\boxtimes_\alpha A) = \langle \alpha\mu + 1 - \alpha, \alpha\nu \rangle. \tag{3.3.4}$$

Fig. 3.13 Second
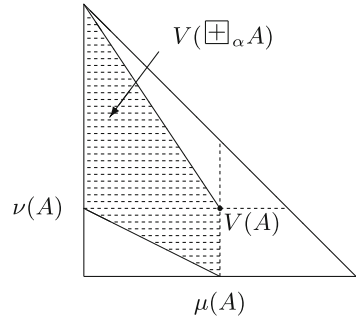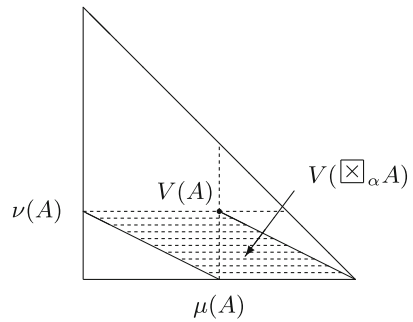geometrical interpretation of
operator $\boxplus_\alpha$

Fig. 3.14 Second
geometrical interpretation of
operator $\boxtimes_\alpha$

Obviously,

$$0 \leq \alpha\mu + \alpha\nu + 1 - \alpha = 1 - \alpha(1 - \mu - \alpha\nu) \leq 1.$$

For every formula A,

$$V(\boxplus_{0.5}A) = V(\boxplus A),$$

$$V(\boxtimes_{0.5}A) = V(\boxtimes A).$$

Therefore, the new operators "$\boxplus_\alpha$" and "$\boxtimes_\alpha$" are generalizations of $\boxplus$ and $\boxtimes$, respectively. Their geometrical interpretations are given in Figs. 3.13 and 3.14, respectively.

For every formula A, for every $\alpha \in [0, 1]$:

(a) $V(\boxplus_\alpha A) \leq V(A) \leq V(\boxtimes_\alpha A)$,
(b) $V(\neg_1 \boxplus_\alpha \neg_1 A) = V(\boxtimes_\alpha A)$,
(c) $V(\boxplus_\alpha \boxplus_\alpha A) \leq V(\boxplus_\alpha A)$,
(d) $V(\boxtimes_\alpha \boxtimes_\alpha A) \leq V(\boxtimes_\alpha A)$.

For every formula A, and for every two real numbers $\alpha, \beta \in [0, 1]$,

(a) $V(\boxplus_\alpha \boxplus_\beta A) = V(\boxplus_\beta \boxplus_\alpha A)$,

(b) $V(\boxtimes_\alpha \boxtimes_\beta A) = V(\boxtimes_\beta \boxtimes_\alpha A)$,

(c) $V(\boxtimes_\alpha \boxplus_\beta A) \geq V(\boxplus_\beta \boxtimes_\alpha A)$.

For every formula A, and for every three real numbers $\alpha, \beta, \gamma \in [0, 1]$,

(a) $V(\boxplus_\alpha D_\beta(A)) = V(D_\beta(\boxplus_\alpha A))$,

(b) $V(\boxplus_\alpha F_{\beta,\gamma}(A)) = V(F_{\beta,\gamma}(\boxplus_\alpha A))$, where $\beta + \gamma \leq 1$,

(c) $V(\boxplus_\alpha G_{\beta,\gamma}(A)) \leq V(G_{\beta,\gamma}(\boxplus_\alpha A))$,

(d) $V(\boxplus_\alpha H_{\beta,\gamma}(A)) = V(H_{\beta,\gamma}(\boxplus_\alpha A))$,

(e) $V(\boxplus_\alpha H^*_{\beta,\gamma}(A)) = V(H^*_{\beta,\gamma}(\boxplus_\alpha A))$,

(f) $V(\boxplus_\alpha J_{\beta,\gamma}(A)) = V(J_{\beta,\gamma}(\boxplus_\alpha A))$,

(g) $V(\boxplus_\alpha J^*_{\beta,\gamma}(A)) = V(J^*_{\beta,\gamma}(\boxplus_\alpha A))$,

(h) $V(\boxtimes_\alpha D_\beta(A)) = V(D_\beta(\boxtimes_\alpha A))$,

(i) $V(\boxtimes_\alpha F_{\beta,\gamma}(A)) = V(F_{\beta,\gamma}(\boxtimes_\alpha A))$, where $\beta + \gamma \leq 1$,

(j) $V(\boxtimes_\alpha G_{\beta,\gamma}(A)) \leq V(G_{\beta,\gamma}(\boxtimes_\alpha A))$,

(k) $V(\boxtimes_\alpha H_{\beta,\gamma}(A)) = V(H_{\beta,\gamma}(\boxtimes_\alpha A))$,

(l) $V(\boxtimes_\alpha H^*_{\beta,\gamma}(A)) = V(H^*_{\beta,\gamma}(\boxtimes_\alpha A))$,

(m) $V(\boxtimes_\alpha J_{\beta,\gamma}(A)) = V(J_{\beta,\gamma}(\boxtimes_\alpha A))$,

(n) $V(\boxtimes_\alpha J^*_{\beta,\gamma}(A)) = V(J^*_{\beta,\gamma}(\boxtimes_\alpha A))$.

The second extension was introduced in [17] by K. Dencheva. She extended the last two operators to the forms:

$$V(\boxplus_{\alpha,\beta} A) = \langle \alpha\mu, \alpha\nu + \beta \rangle, \tag{3.3.5}$$

$$V(\boxtimes_{\alpha,\beta} A) = \langle \alpha\mu + \beta, \alpha\nu \rangle, \tag{3.3.6}$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$.

Obviously, for every formula A,

$$V(\boxplus A) = V(\boxplus A_{0.5,0.5}),$$

$$V(\boxtimes A) = V(\boxtimes A_{0.5,0.5}),$$

$$V(\boxplus A_\alpha) = V(\boxplus A_{\alpha,1-\alpha}),$$

$$V(\boxtimes A_\alpha) = V(\boxtimes A_{\alpha,1-\alpha}).$$

For every formula A, and for every $\alpha, \beta, \alpha + \beta \in [0, 1]$,

(a) $V(\neg \boxplus_{\alpha,\beta} \neg A) = V(\boxtimes_{\alpha,\beta} A)$,

(b) $V(\neg \boxtimes_{\alpha,\beta} \neg A) = V(\boxplus_{\alpha,\beta} A)$.

For every formula A, and for every $\alpha, \beta \in [0, 1]$, each of the inequalities

(a) $V(\boxplus_{\alpha,\beta} \boxplus_{\alpha,\beta} A) \leq V(\boxplus_{\alpha,\beta} A)$,

(b) $V(\boxtimes_{\alpha,\beta} \boxtimes_{\alpha,\beta} A) \geq V(\boxtimes_{\alpha,\beta} A)$,

is valid if and only if $\alpha + \beta = 1$.

For every formula A, and for every $\alpha \in [0, 1]$,

$$V(\boxplus_{\alpha,\beta} \boxtimes_{\alpha,\beta} A) = V(\boxtimes_{\alpha,\beta} \boxplus_{\alpha,\beta} A) \ \text{ iff } \ \beta = 0.$$

For every formula A, and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta, \gamma + \delta \in [0, 1]$,

$$V(\boxplus_{\alpha,\beta} \boxtimes_{\gamma,\delta} A) \leq V(\boxtimes_{\gamma,\delta} \boxplus_{\alpha,\beta} A).$$

Now, the third extension of the above operators is as follows:

$$V(\boxplus_{\alpha,\beta,\gamma} A) = \langle \alpha\mu, \ \beta\nu + \gamma \rangle, \tag{3.3.7}$$

$$V(\boxtimes_{\alpha,\beta,\gamma} A) = \langle \alpha\mu + \gamma, \ \beta\nu \rangle, \tag{3.3.8}$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

Obviously, for every formula A,

$$V(\boxplus A) = V(\boxplus A_{0.5,0.5,0.5}),$$

$$V(\boxtimes A) = V(\boxtimes A_{0.5,0.5,0.5}),$$

$$V(\boxplus A_\alpha) = V(\boxplus A_{\alpha,\alpha,1-\alpha}),$$

$$V(\boxtimes A_\alpha) = V(\boxtimes A_{\alpha,1-\alpha}),$$

$$V(\boxplus A_{\alpha,\beta}) = V(\boxplus A_{\alpha,\alpha,\beta}),$$

$$V(\boxtimes A_{\alpha,\beta}) = V(\boxtimes A_{\alpha,\alpha,\beta}).$$

For every formula A, and for every $\alpha, \beta, \gamma \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma \leq 1$,

(a) $V(\neg \boxplus_{\alpha,\beta,\gamma} \neg A)) = V(\boxtimes_{\beta,\alpha,\gamma} A)$,

(b) $V(\neg \boxtimes_{\alpha,\beta,\gamma} \neg A)) = V(\boxplus_{\beta,\alpha,\gamma} A)$.

For every formula A, and for every $\alpha, \beta, \gamma \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma \leq 1$,

(a) $V(\boxplus_{\alpha,\beta,\gamma} \boxplus_{\alpha,\beta,\gamma} A) \leq V(\boxplus_{\alpha,\beta,\gamma} A)$ is valid iff $\beta + \gamma = 1$,

(b) $V(\boxtimes_{\alpha,\beta,\gamma} \boxtimes_{\alpha,\beta,\gamma} A) \geq V(\boxtimes_{\alpha,\beta,\gamma} A)$ is valid iff $\alpha + \gamma = 1$.

For every formula A, and for every $\alpha, \beta, \alpha + \beta \in [0, 1]$,

$$V(\boxplus_{\alpha,\beta,\gamma} \boxtimes_{\alpha,\beta,\gamma} A) = V(\boxtimes_{\alpha,\beta,\gamma} \boxplus_{\alpha,\beta,\gamma} A) \text{ iff } \gamma = 0.$$

For every formula A, and for every $\alpha, \beta, \gamma \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma \leq 1$, the four properties

(a) $V(\boxplus_{\alpha,\beta,\gamma} \square A) = V(\square \boxplus_{\alpha,\beta,\gamma} A)$,

(b) $V(\boxtimes_{\alpha,\beta,\gamma} \square A) = V(\square \boxtimes_{\alpha,\beta,\gamma} A)$,

(c) $V(\boxplus_{\alpha,\beta,\gamma} \Diamond A) = V(\Diamond \boxplus_{\alpha,\beta,\gamma} A)$,

(d) $V(\boxtimes_{\alpha,\beta,\gamma} \Diamond A) = V(\Diamond \boxtimes_{\alpha,\beta,\gamma} A)$,

are valid iff $\alpha = \beta$ and $\alpha + \gamma = 1$.

For every two formulas A and B, and for every $\alpha, \beta, \gamma \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma \leq 1$,

(a) $V(\boxplus_{\alpha,\beta,\gamma}(A \wedge B)) = V(\boxplus_{\alpha,\beta,\gamma} A \wedge \boxplus_{\alpha,\beta,\gamma} B)$,

(b) $V(\boxtimes_{\alpha,\beta,\gamma}(A \wedge B)) = V(\boxtimes_{\alpha,\beta,\gamma} A \wedge \boxtimes_{\alpha,\beta,\gamma} B)$,

(c) $V(\boxplus_{\alpha,\beta,\gamma}(A \vee B)) = V(\boxplus_{\alpha,\beta,\gamma} A \vee \boxplus_{\alpha,\beta,\gamma} B)$,

(d) $V(\boxtimes_{\alpha,\beta,\gamma}(A \vee B)) = V(\boxtimes_{\alpha,\beta,\gamma} A \vee \boxtimes_{\alpha,\beta,\gamma} B)$,

For every predicate A, and for every $\alpha, \beta, \gamma \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma \leq 1$,

(a) $V(\boxplus_{\alpha,\beta,\gamma} \exists x A) = V(\exists x \boxplus_{\alpha,\beta,\gamma} A)$,

(b) $V(\boxtimes_{\alpha,\beta,\gamma} \exists x A) = V(\exists x \boxtimes_{\alpha,\beta,\gamma} A)$,

(c) $V(\boxplus_{\alpha,\beta,\gamma} \forall x A) = V(\forall x \boxplus_{\alpha,\beta,\gamma} A)$,

(d) $V(\boxtimes_{\alpha,\beta,\gamma} \forall x A) = V(\forall x \boxtimes_{\alpha,\beta,\gamma} A)$.

A natural extension of the last two operators is the operator

$$V(\boxdot_{\alpha,\beta,\gamma,\delta} A) = \langle \alpha\mu + \gamma, \beta\nu + \delta \rangle, \tag{3.3.9}$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\max(\alpha, \beta) + \gamma + \delta \leq 1$.

It is the fourth type of operator from the currently discussed group.

Obviously, for every formula A,

$$V(\boxplus A) = V(\boxdot A_{0.5,0.5,0,0.5}),$$

$$V(\boxtimes A) = V(\boxdot A_{0.5,0.5,0.5,0}),$$

$$V(\boxplus A_\alpha) = V(\boxed{\bullet} A_{\alpha,\alpha,0,1-\alpha}),$$

$$V(\boxtimes A_\alpha) = V(\boxed{\bullet} A_{\alpha,\alpha,1-\alpha,0}),$$

$$V(\boxplus A_{\alpha,\beta}) = V(\boxed{\bullet} A_{\alpha,\alpha,0,\beta}),$$

$$V(\boxtimes A_{\alpha,\beta}) = V(\boxed{\bullet} A_{\alpha,\alpha,\beta,0}),$$

$$V(\boxplus A_{\alpha,\beta,\gamma}) = V(\boxed{\bullet} A_{\alpha,\beta,0,\gamma}),$$

$$V(\boxtimes A_{\alpha,\beta,\gamma}) = V(\boxed{\bullet} A_{\alpha,\beta,\gamma,0}).$$

For every formula A, and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma + \delta \le 1$:

(a) $V(\neg \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} \neg A) = V(\boxed{\bullet}_{\beta,\alpha,\delta,\gamma} A),$
(b) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(\boxed{\bullet}_{\varepsilon,\zeta,\eta,\theta} A)) = V(\boxed{\bullet}_{\alpha\varepsilon,\beta\zeta,\alpha\eta+\gamma,\beta\theta+\delta} A),$
(c) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} \square A) \ge V(\square \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A),$
(d) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} \lozenge A) \le V(\lozenge \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A).$

For every pair of formulas A and B, and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma + \delta \le 1$,

(a) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(A \wedge B)) = V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A \wedge \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} B),$
(b) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}(A \vee B)) = V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A \vee \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} B).$

In [18], G. Çuvalcioğlu introduced the operator $E_{\alpha,\beta}$ by

$$V(E_{\alpha,\beta}(A)) = \langle \beta(\alpha\mu + 1 - \alpha), \alpha(\beta\nu + 1 - \beta) \rangle, \qquad (3.3.10)$$

where $\alpha, \beta \in [0, 1]$, and he studied some of its properties. Obviously,

$$V(E_{\alpha,\beta}(A)) = V(\boxed{\bullet}_{\alpha\beta,\alpha\beta,(1-\alpha)\beta,(1-\beta)\alpha} A).$$

For every predicate A, and for every $\alpha, \beta, \gamma, \delta \in [0, 1]$ for which $\max(\alpha, \beta) + \gamma + \delta \le 1$,

(a) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} \exists x A) = V(\exists x \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A),$
(b) $V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} \forall x A) = V(\forall x \boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A).$

A new (potentially final?) extension of the above operators is the operator

$$V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A) = \langle \alpha\mu - \varepsilon\nu + \gamma, \beta\nu - \zeta\mu + \delta \rangle, \qquad (3.3.11)$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1, \tag{3.3.12}$$

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0. \tag{3.3.13}$$

Assume that in the particular cases when $\alpha - \zeta > -\varepsilon$, $\beta = \delta = 0$ and $\delta - \zeta < \beta$, $\gamma = \varepsilon = 0$, the inequalities

$$\gamma \geq \varepsilon \quad \text{and} \quad \beta + \delta \leq 1$$

hold. Obviously, for every IFS A,

$$V(\boxplus A) = V(\boxed{\circ}_{0.5,0.5,0,0.5,0,0}A),$$

$$V(\boxtimes A) = V(\boxed{\circ}_{0.5,0.5,0.5,0,0,0}A),$$

$$V(\boxplus_\alpha A) = V(\boxed{\circ}_{\alpha,\alpha,0,1-\alpha,0,0}A),$$

$$V(\boxtimes_\alpha A) = V(\boxed{\circ}_{\alpha,\alpha,1-\alpha,0,0,0}A),$$

$$V(\boxplus_{\alpha,\beta} A) = V(\boxed{\circ}_{\alpha,\alpha,0,\beta,0,0}A),$$

$$V(\boxtimes_{\alpha,\beta} A) = V(\boxed{\circ}_{\alpha,\alpha,\beta,0,0,0}A),$$

$$V(\boxplus_{\alpha,\beta,\gamma} A) = V(\boxed{\circ}_{\alpha,\beta,0,\gamma,0,0}A),$$

$$V(\boxtimes_{\alpha,\beta,\gamma} A) = V(\boxed{\circ}_{\alpha,\beta,\gamma,0,0,0}A),$$

$$V(\boxed{\bullet}_{\alpha,\beta,\gamma,\delta} A) = V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,0,0}A),$$

$$V(E_{\alpha,\beta} A) = V(\boxed{\circ}_{\alpha\beta,\alpha\beta,\beta(1-\alpha),\alpha(1-\beta)}A).$$

For every formula A, and for every $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ for which (3.3.12) and (3.3.13) are valid, the equality

$$V(\neg\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}\neg A) = V(\boxed{\circ}_{\beta,\alpha,\delta,\gamma,\zeta,\varepsilon}A)$$

holds.

For every formula A, and for every $\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \varepsilon_2, \zeta_2 \in$ [0, 1] for which conditions that are similar to (3.3.12) and (3.3.13) are valid, the equality

$$V(\boxed{\circ}_{\alpha_1,\beta_1,\gamma_1,\delta_1,\varepsilon_1,\zeta_1}(\boxed{\circ}_{\alpha_2,\beta_2,\gamma_2,\delta_2,\varepsilon_2,\zeta_2}A))$$

$$=V(\boxed{\circ}_{\alpha_1\alpha_2+\varepsilon_1\zeta_2,\beta_1\beta_2+\zeta_1\varepsilon_2,\alpha_1\gamma_2-\varepsilon_1\delta_2+\gamma_1,\beta_1\delta_2-\zeta_1\gamma_2+\delta_1,\alpha_1\varepsilon_2+\varepsilon_1\beta_2,\beta_1\zeta_2+\zeta_1\alpha_2}A)$$

holds.

It must be noted that the equalities

$$V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A \wedge B)) = V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \wedge \boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B)$$

and

$$V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A \vee B)) = V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A \vee \boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}B),$$

which are valid for operator $\boxed{\bullet}_{\alpha,\beta,\gamma,\delta}$, are not always valid for $\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$.

Open Problem 18 Check the validity of the above formulas for the case of all intuitionistic fuzzy conjunctions, disjunctions, implications and negations.

Following [9], we formulate and prove the following:

Theorem 3.3.1 *Operators* $X_{a,b,c,d,e,f}$ *and* $\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ *are equivalent.*

Proof Let $a, b, c, d, e, f \in [0, 1]$ and satisfy (6.24) and (6.25). Let

$$\alpha = a - b, \quad \beta = d - e, \quad \gamma = b, \quad \delta = e, \quad \varepsilon = bc, \quad \zeta = ef.$$

Also, let

$$X \equiv \alpha\mu - \varepsilon\nu + \gamma = (a - b)\mu - bc\nu + b,$$

$$Y \equiv \beta\nu - \zeta\mu + \delta = (d - e)\nu - ef\mu + e.$$

Then,

$$X \geq (a - b).0 - bc.1 + b = b(1 - c) \geq 0,$$

$$X \leq (a - b).1 - bc.0 + b = a \leq 1,$$

$$Y \geq (d - e).0 - ef.1 + e = e(1 - f) \geq 0,$$

$$Y \leq (d - e).1 - ef.0 + e = d \leq 1$$

and

$$X + Y = (a - b)\mu - bc\nu + b + (d - e)\nu - ef\mu + e$$

$$= (a - b - ef)\mu + (d - e - bc)\nu + b + e$$

$$\leq (a - b - ef)\mu + (d - e - bc)(1 - \mu) + b + e$$

$$= d - e - bc + b + e + (a - b - ef - d + e + bc)\mu$$

$$\leq d - bc + b + a - b - ef - d + e + bc$$

$$= a - ef + e = \alpha + \gamma - \zeta + \delta \leq 1$$

(from (3.3.12)).

Thus, we obtain

$$V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta} A) = \langle \alpha\mu - \varepsilon\nu + \gamma, \beta\nu - \zeta\mu + \delta \rangle$$

$$= \langle x, (a - b)\mu - bc\nu + b, (d - e)\nu - ef\mu + e \rangle$$

$$= \langle x, a\mu + b(1 - \mu - c\nu),$$

$$d\nu + e(1 - f\mu - \nu) \rangle$$

$$= V(X_{a,b,c,d,e,f}(A)).$$

Conversely, let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ and satisfy (3.3.12) and (3.3.13). From (3.3.13) it follows that for $\alpha = \beta = \delta = \zeta = 0 :\ \varepsilon \leq \gamma$, while for $\alpha = \beta = \gamma = \varepsilon = 0 :\ \zeta \leq \delta$; from (3.3.12) it follows that for $\beta = \delta = \varepsilon = \zeta = 0 :\ \alpha + \gamma \leq 1$, while for $\alpha = \gamma = \varepsilon = \zeta = 0 :\ \beta + \delta \leq 1$. Then, let

$$a = \alpha + \gamma \ (\leq 1),$$

$$b = \gamma,$$

$$c = \frac{\varepsilon}{\gamma} \ (\leq 1),$$

$$d = \beta + \delta \ (\leq 1),$$

$$e = \delta,$$

$$f = \frac{\zeta}{\delta} \ (\leq 1).$$

Let

$$X \equiv a\mu + b(1 - \mu - c\nu) = (\alpha + \gamma)\mu + \gamma\left(1 - \mu - \frac{\varepsilon}{\gamma}\nu\right),$$

$$Y \equiv d\nu + e(1 - f\mu - \nu) = (\beta + \delta)\nu + \delta\left(1 - \frac{\zeta}{\delta}\mu - \nu\right).$$

Then, we obtain,

$$0 \leq \gamma - \varepsilon \leq X = \alpha\mu + \gamma - \varepsilon\nu \leq \alpha + \gamma \leq 1,$$

$$0 \leq \delta - \zeta \leq Y = \beta\nu + \delta - \zeta\mu \leq \beta + \delta \leq 1,$$

$$X + Y = \alpha\mu + \gamma - \varepsilon\nu + \beta\nu + \delta - \zeta\mu$$

$$= (\alpha - \zeta)\mu - (\beta - \varepsilon)\nu + \gamma + \delta$$

$$\leq (\alpha - \zeta)\mu - (\beta - \varepsilon)(1 - \mu) + \gamma + \delta$$

$$= (\alpha - \zeta + \beta - \varepsilon)\mu - \beta + \gamma + \delta + \varepsilon$$

$$\leq \alpha - \zeta + \beta - \varepsilon - \beta + \gamma + \delta + \varepsilon$$

$$= \alpha - \zeta + \gamma + \delta$$

$$\leq \max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leq 1$$

(from (3.3.12)).

Then, we obtain

$$V(X_{a,b,c,d,e,f}(A))$$

$$= \langle a\mu + b(1 - \mu - c\nu), d\nu + e(1 - f\mu - \nu)\rangle$$

$$= \langle(\alpha + \gamma)\mu + \gamma(1 - \mu - \frac{\varepsilon}{\gamma}\nu), (\beta + \delta)\nu + \delta(1 - \frac{\zeta}{\delta}\mu - \nu)\rangle$$

$$= \langle(\alpha + \gamma)\mu + \gamma - \gamma\mu - \varepsilon\nu, (\beta + \delta)\nu + \delta - \zeta\mu - \delta\nu\rangle$$

$$= \langle\alpha\mu - \varepsilon\nu + \gamma, \beta\nu - \zeta\mu + \delta\rangle$$

$$= V(\boxed{\circ}_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}A).$$

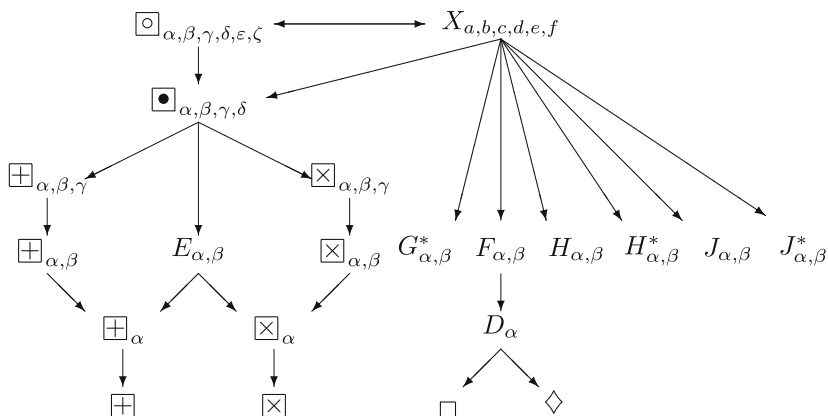Therefore, the two operators are equivalent.

Fig. 3.15 Relations among modal operators

Finally, we construct Fig. 3.15 in which

$$\text{``} X \longrightarrow Y''$$

denotes that operator X represents operator Y, while the converse is not valid, and

$$\text{``} X \longleftrightarrow Y''$$

denotes that each of the operators represents the other.

Following [19], we introduce the following new operator from modal type, that is a modification of the above discussed operators. It has the form

$$V(\otimes_{\alpha,\beta,\gamma,\delta} A) = \langle \alpha a + \gamma b, \beta a + \delta b \rangle,$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\alpha + \beta \leq 1, \gamma + \delta \leq 1$.

First, we check that the new operator generates an intuitionistic fuzzy pair. Indeed,

$$0 \leq \alpha a + \gamma b \leq a + b \leq 1,$$

$$0 \leq \beta a + \delta b \leq a + b \leq 1$$

and

$$0 \leq \alpha a + \gamma b + \beta a + \delta b$$

$$= (\alpha + \beta)a + (\gamma + \delta)b$$

$$\leq a + b \leq 1.$$

Second, it is easy to see that

$$\otimes_{1,0,0,1} A = A,$$

$$\otimes_{0,1,1,0} A = \neg_1 A.$$

Theorem 3.3.2 *For every formula A, for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$ and for negation $\neg_1$,*

$$V(\neg_1 \otimes_{\alpha,\beta,\gamma,\delta} \neg_1 A) = V(\otimes_{\delta,\gamma,\beta,\alpha} A).$$

Proof We obtain sequentially that

$$V(\neg_1 \otimes_{\alpha,\beta,\gamma,\delta} \neg_1 A)$$

$$= \neg_1 \otimes_{\alpha,\beta,\gamma,\delta} \langle b, a \rangle$$

$$= \neg_1 \langle \alpha b + \gamma a, \beta b + \delta a \rangle$$

$$= \langle \beta b + \delta a, \alpha b + \gamma a \rangle$$

$$= \otimes_{\delta,\gamma,\beta,\alpha} A.$$

This completes the proof. □

Theorem 3.3.3 *For every two formulas A and B an for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$,*

(a) $V(\otimes_{\alpha,\beta,\gamma,\delta}(A \vee B)) = V(\otimes_{\alpha,\beta,\gamma,\delta} A \vee \otimes_{\alpha,\beta,\gamma,\delta} B),$
(b) $V(\otimes_{\alpha,\beta,\gamma,\delta}(A \wedge B)) = V(\otimes_{\alpha,\beta,\gamma,\delta} A \wedge \otimes_{\alpha,\beta,\gamma,\delta} B).$

Proof For (a), first, we obtain that

$$V(\otimes_{\alpha,\beta,\gamma,\delta}(A \vee B)) = \otimes_{\alpha,\beta,\gamma,\delta} \langle \max(a, c), \min(b, d) \rangle$$

$$= \langle \alpha \max(a, c) + \gamma \min(b, d), \beta \max(a, c) + \delta \min(b, d) \rangle.$$

Second, we obtain that

$$V(\otimes_{\alpha,\beta,\gamma,\delta} A \vee \otimes_{\alpha,\beta,\gamma,\delta} B)$$

$$= \langle \alpha a + \gamma b, \beta a + \delta b \rangle \vee \langle \alpha c + \gamma d, \beta c + \delta.d \rangle$$

$$= \langle \max(\alpha a + \gamma b, \alpha c + \gamma d), \min(\beta a + \delta b, \beta c + \delta d) \rangle$$

Let

$$X \equiv \max(\alpha a + \gamma b, \alpha c + \gamma d) - \alpha \max(a, c) - \gamma \min(\nu_A(x), d)$$

Now, for a, c, b, d we must study the following four cases.

Case 1. $a \geq c, b \geq d$:

$$X = \max(\alpha a + \gamma b, \alpha c + \gamma d) - \alpha a - \gamma d$$

$$\geq \alpha a + \gamma b - \alpha a - \gamma d \geq 0$$

Case 2. $a \geq c, b < d$:

$$X = \max(\alpha a + \gamma b, \alpha c + \gamma d) - \alpha a - \gamma b \geq 0$$

Case 3. $a < c, b \geq d$:

$$X = \max(\alpha a + \gamma b, \alpha c + \gamma d) - \alpha c - \gamma d \geq 0$$

Case 4. $a < c, b < d$:

$$X = \max(\alpha a + \gamma b, \alpha c + \gamma d) - \alpha c) - \gamma b$$

$$\geq \alpha c + \gamma d - \alpha c) - \gamma b \geq 0$$

Assertion (b) is proved analogously.   □

The proofs of the next assertion follow by analogy.

Theorem 3.3.4 *For every formula A and for every four real numbers $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha + \beta \leq 1, \gamma + \delta \leq 1$:*

(a) $V(\Box \otimes_{\alpha,\beta,\gamma,\delta} A) \leq V(\otimes_{\alpha,\beta,\gamma,\delta} \Box A)$,
(b) $V(\otimes_{\alpha,\beta,\gamma,\delta} \Diamond A) \leq V(\Diamond \otimes_{\alpha,\beta,\gamma,\delta} A)$.

Theorem 3.3.5 *Let A be a formula, such that $V(A) = \langle \mu, \nu \rangle$ and let $a, b, c, d, e, f, g, h \in [0, 1]$, so that $a + b, c + d, e + f, g + h \in [0, 1]$. Then,*

$$V(\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A))) = V(\otimes_{ae+bg,af+bh,ce+dg,cf+dh}(A)). \qquad (3.3.14)$$

Proof Let formula A and the real numbers a, b, c, d, e, f, g, h satisfy the conditions for operator $\otimes$. Then

$$V(\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A)))$$

$$= \otimes_{e,f,g,h} \langle x, a\mu + c\nu, b\mu + d\nu \rangle$$

$$= \langle x, ae\mu + ce\nu + bg\mu + dg\nu, af\mu + cf\nu + bh\mu + dh\nu \rangle$$

$$= \langle x, (ae + bg)\mu + (ce + dg)\nu, (af + bh)\mu + (cf + dh)\nu$$

$$= V(\otimes_{ae+bg,af+bh,ce+dg,cf+dh}(A)).$$

Therefore, (3.3.14) is valid. $\square$

Theorem 3.3.6 *Let A be a formula, such that $V(A) = \langle \mu, \nu \rangle$ and let $a, d, e, h \in (0, 1]$, $b, c, f, g \in (0, 1]$, so that $a + b, c + d, e + f, g + h \in [0, 1]$ and*

$$bg = cf, \tag{3.3.15}$$

$$ag + ch = ce + dg. \tag{3.3.16}$$

Then,
$$V(\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A))) = V(\otimes_{a,b,c,d}(\otimes_{e,f,g,h}(A))). \tag{3.3.17}$$

Proof Let formula A and the real numbers a, b, c, d, e, f, g, h satisfy the conditions of the theorem. First, we see, that from (3.3.15) and (3.3.16) it follows:

$$af + bh - be - df = af + \frac{cf}{g}h - \frac{cf}{g}e - df = \frac{f}{g}(ag + ch - ce - dg) = 0.$$

But, by the above conditions, $f, g > 0$. Therefore,

$$af + bh - be - df = 0,$$

i.e.,
$$af + bh = be + df. \tag{3.3.18}$$

Now,
$$V(\otimes_{e,f,g,h}(\otimes_{a,b,c,d}(A))) = \otimes_{e,f,g,h}\langle a.\mu + c.\nu, b.\mu + d.\nu \rangle$$

$$= \langle a.e.\mu + c.e.\nu + b.g.\mu + d.g.\nu, a.f.\mu + c.f.\nu + b.h.\mu + d.h.\nu \rangle$$

(from (3.3.16) and (3.3.18))

$$= \langle a.e.\mu + a.g.\nu + c.f.\mu + c.h.\nu, b.e.\mu + b.g.\nu + d.f.\mu + d.h.\nu \rangle$$

$$= \otimes_{a,b,c,d}\langle e.\mu + g.\nu, f.\mu + h.\nu \rangle$$

$$= V(\otimes_{a,b,c,d} \otimes_{e,f,g,h} (A)).$$

Therefore, (3.3.17) is valid. $\square$

Theorem 3.3.7 *Let A be a formula,* $\alpha, \beta, \gamma, \delta \in [0, 1]$*, so that* $\alpha + \beta, \gamma + \delta \in [0, 1]$*,* $a, b, c, d, e, f \in [0, 1]$ *so that* $a + e - e.f \leq 1$*,* $b + d - b.c \leq 1$ *and* $b + e \leq 1$*. Then,*

(a) $\boxed{\circ}_{a,b,c,d,e,f}(\otimes_{\alpha,\beta,\gamma,\delta}(A)) = \boxed{\circ}_{a\alpha-e\beta,b\beta-f\alpha,c,d,e\delta-a\gamma,f\gamma-b\delta}(A),$

(b) $\otimes_{\alpha,\beta,\gamma,\delta}(\boxed{\circ}_{a,b,c,d,e,f}(A)) = \boxed{\circ}_{a\alpha-f\gamma,a\beta-f\delta,c\alpha+d\gamma,c\beta+d\delta,e\alpha-b\gamma,e\beta-b\delta}(A).$

The following two open problems are interesting:

Open Problem 19 Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be represented by the extended modal operators?

Open Problem 20 Can operator $\otimes_{\alpha,\beta,\gamma,\delta}$ be used for representation of some types of modal operators?

3.4 Intuitionistic Fuzzy Level Operators

Following [9], here we introduce the following two intuitionistic fuzzy level operators for each formula A with evaluation $V(A) = \langle a, b \rangle$:

$$V(P_{\alpha,\beta}(A)) = P_{\alpha,\beta}(V(A)) = \langle \max(a, \alpha), \min(b, \beta) \rangle,$$

$$V(Q_{\alpha,\beta}(A)) = Q_{\alpha,\beta}(V(A)) = \langle \min(a, \alpha), \max(b, \beta) \rangle.$$

We must note, that for every formula A

$$V(P_{\alpha,\beta}(A)) = V(A) \vee \langle \alpha, \beta \rangle$$

and

$$V(Q_{\alpha,\beta}(A)) = V(A) \wedge \langle \alpha, \beta \rangle.$$

Theorem 3.4.1 *For every formula A and for every* $\alpha, \beta, \gamma, \delta \in [0, 1]$*, such that* $\alpha + \beta \leq 1$*,* $\gamma + \delta \leq 1$*:*

(a) $V(\neg P_{\alpha,\beta}(\neg A)) = V(Q_{\beta,\alpha}(A));$

(b) $V(P_{\alpha,\beta}(Q_{\gamma,\delta}(A))) = V(Q_{\max(\alpha,\gamma),\min(\beta,\delta)}(P_{\alpha,\beta}(A)));$

(c) $V(Q_{\alpha,\beta}(P_{\gamma,\delta}(A))) = V(P_{\min(\alpha,\gamma),\max(\beta,\delta)}(Q_{\alpha,\beta}(A)));$

(d) $V(P_{\alpha,\beta}(P_{\gamma,\delta}(A))) = V(P_{\max(\alpha,\gamma),\min(\beta,\delta)}(A));$

(e) $V(Q_{\alpha,\beta}(Q_{\gamma,\delta}(A))) = V(Q_{\min(\alpha,\gamma),\max(\beta,\delta)}(A)).$

Proof (b) Let A be a formula. Then,

$$
\begin{aligned}
&V(P_{\alpha,\beta}(Q_{\gamma,\delta}(A)))\\
&= V(P_{\alpha,\beta}(\langle\min(\gamma, a), \max(\delta, b)\rangle))\\
&= \langle\max(\alpha, \min(\gamma, a)), \min(\beta, \max(\delta, b))\rangle\\
&= \langle\min(\max(\alpha, \gamma), \max(\alpha, a)), \max(\min(\beta, \delta), \max(\beta, a))\rangle\\
&= Q_{\max(\alpha,\gamma),\min(\beta,\delta)}(\langle\max(\alpha, a), \max(\beta, b)\rangle)\\
&= V(Q_{\max(\alpha,\gamma),\min(\beta,\delta)}(P_{\alpha,\beta}(A))).
\end{aligned}
$$

This completes the proof.   □

Theorem 3.4.2 *For every two formulas A and B, and for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:*

(a) $V(P_{\alpha,\beta}(A \wedge B)) = V(P_{\alpha,\beta}(A)) \wedge V(P_{\alpha,\beta}(B))$,
(b) $V(P_{\alpha,\beta}(A \vee B)) = V(P_{\alpha,\beta}(A)) \vee V(P_{\alpha,\beta}(B))$,
(c) $V(Q_{\alpha,\beta}(A \wedge B)) = V(Q_{\alpha,\beta}(A)) \wedge V(Q_{\alpha,\beta}(B))$,
(d) $V(Q_{\alpha,\beta}(A \vee B)) = V(Q_{\alpha,\beta}(A)) \vee V(Q_{\alpha,\beta}(B))$.

Proof (a) Let A and B be two formulas. Then,

$$
\begin{aligned}
&V(P_{\alpha,\beta}(A \wedge B))\\
&= P_{\alpha,\beta}(\langle\min(a, c), \max(b, d)\rangle)\\
&= \langle\max(\alpha, \min(a, c)), \min(\beta, \max(b, d))\rangle\\
&= \langle\min(\max(\alpha, a), \max(\alpha, c)), \max(\min(\beta, b), \min(\beta, d))\rangle\\
&= V(P_{\alpha,\beta}(A) \wedge P_{\alpha,\beta}(B)).
\end{aligned}
$$

This completes the proof.   □

Theorem 3.4.3 *Let A be a formula and x be a variable. Then, for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:*

$$
\begin{aligned}
&\text{(a)}\quad V(\exists x\, P_{\alpha,\beta}(A)) = V(P_{\alpha,\beta}(\exists x A)),\\
&\text{(b)}\quad V(\forall x\, Q_{\alpha,\beta}(A)) = V(Q_{\alpha,\beta}(\forall x A)).
\end{aligned}
$$

Proof (a) Let A be a formula. Then,

$$
\begin{aligned}
&V(\exists x\, P_{\alpha,\beta}(A))\\
&= \exists x \langle\max(\alpha, \mu(A)), \min(\beta, \nu(A))\rangle\\
&= \langle\max_x(\max(\alpha, \mu(A))), \min_x(\min(\beta, \nu(A)))\rangle\\
&= \langle\max(\alpha, \max_x(\mu(A))), \min(\beta, \min_x(\nu(A)))\rangle\\
&= V(P_{\alpha,\beta}(\exists x A)).
\end{aligned}
$$

This completes the proof. Assertion (b) is proved by analogy.   □

3.5 Pseudo-fixed Points of the Intuitionistic Fuzzy Operators and Quantifiers

Let $\mathcal{S}$ be a set of propositions (or more general, formulas) and let $V : \mathcal{S} \to [0, 1] \times [0, 1]$, be defined for every $A \in \mathcal{S}$ as in Sect. 1.1.

Let for operator Y and for IFP $\langle a, b \rangle$:

$$Y(\langle a, b \rangle) = \langle a, b \rangle.$$

Then, we call that the IFP is a fixed point for operator Y. But, when operator Y is defined over elements of $\mathcal{S}$, i.e., when for formula A

$$V(Y(A)) = \langle \mu(Y(A)), \nu(Y(A)) \rangle,$$

then we will call that A is a pseudo-fixed point for operator Y. In this case, the equality

$$\langle \mu(Y(A)), \nu(Y(A)) \rangle = \langle \mu(A), \nu(A) \rangle \qquad (3.5.1)$$

holds (see [20]).

Obviously, if (3.5.1) is valid for IFP $V(A) = \langle a, b \rangle$, then, $\langle a, b \rangle$ is a fixed point for operator Y.

Below, we determine all pseudo-fixed points of all quantifiers and operators, defined in Chaps. 2 and 3.

Theorem 3.5.1 *For all $\alpha, \beta \in [0, 1]$ the pseudo-fixed point(s) of:*

(a) $\exists$ are all elements $A \in S$ for which $V(A) = \langle 1, 0 \rangle$,
(b) $\forall$ are all elements $A \in S$ for which $V(A) = \langle 0, 1 \rangle$,
(c) $\exists_\mu$ are all elements $A \in S$ for which

$$\mu(A) = \sup_{x \in S} \mu(x)$$

and in the more general case, all elements $A \in S$ for which $V(A) = \langle 1, 0 \rangle$,
(d) $\exists_\nu$ are all elements $A \in S$ for which

$$\nu(A) = \inf_{x \in S} \nu(x)$$

and in the more general case, all elements $A \in S$ for which $V(A) \in [0, 1] \times \{0\}$,
(e) $\forall_\mu$ are all elements $A \in S$ for which

$$\mu(A) = \inf_{x \in S} \mu(x)$$

and in the more general case, all elements $A \in S$ for which $V(A) \in \{0\} \times [0, 1]$,

(f) $\forall_\nu$ *are all elements* $A \in S$ *for which*

$$\nu(A) = \sup_{x \in \mathcal{S}} \nu(x)$$

and in the more general case, all elements $A \in S$ *for which* $V(A) = \langle 0, 1 \rangle$,

(g) $\Box, \Diamond, \bigcirc$ *are all elements* $A \in S$ *for which* $\mu(A) + \nu(A) = 1$,

(h) D_α *are all elements* $A \in S$ *for which* $\mu(A) + \nu(A) = 1$,

(i) $F_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) + \nu(A) = 1$ *and* $\alpha + \beta \leq 1$,

(j) $G_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) = \nu(A) = 0$,

(k) $H_{\alpha,\beta}, H^*_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) = 0$ *and* $\nu(A) = 1$,

(l) $J_{\alpha,\beta}, J^*_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) = 1$ *and* $\nu(A) = 0$,

(m) $\boxplus, \boxplus_\alpha$ *are all elements* $A \in S$ *for which* $\mu(A) = 0$ *and* $\nu(A) = 1$,

(n) $\boxtimes, \boxtimes_\alpha$ *are all elements* $A \in S$ *for which* $\mu(A) = 1$ *and* $\nu(A) = 0$,

(o) $\boxplus_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) = 0, \nu(A) = 1$ *and* $\alpha + \beta = 1$,

(p) $\boxtimes_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\mu(A) = 1, \nu(A) = 0$ *and* $\alpha + \beta = 1$,

(q) $P_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $\alpha \leq \mu(A) = 1$ *and* $0 \leq \nu(A) \leq \beta$,

(r) $Q_{\alpha,\beta}$ *are all elements* $A \in S$ *for which* $0 \leq \mu(A) = \alpha$ *and* $\beta \leq \nu(A) \leq 1$.

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